Andria Razmadze Mathematical Institute

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INITIAL AND CHARACTERISTIC PROBLEMS FOR NONSTRICTLY HYPERBOLIC QUASILINEAR EQUATIONS

01.01.02 – Differential Equations

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General Characteristics

Actuality. The investigation of mixed type partial differential equations is closely connected with many practical and mathematical models that correspond to different physical processes. Such equations are usually nonlinear, and each class of these equations demands an individual approach. Nonlinear equations are sometimes reduced to linear ones on the basis of important assumptions and limitations.

It is known that elliptic and hyperbolic equations with discrete coefficients of minor order members can be reduced to parabolically degenerated equations by simple transformation.

Based on fundamental investigation of the aforementioned and some other facts, F. Tricomi has established the theory of boundary value problems for mixed type linear equations, which was later developed in the papers of G. Fon Karman, M. Lavrentiev, M. Keldish, A. Bitsadze and others scientists.

First works on nonlinear mixed type equations belong to J. Gvazava, I. Mayorov and M.Aliev. Subsequently, R. Bitsadze, M. Menteshashvili, O. Menshikh, A. Podgaev, O. Joxadze and others took an interest in these issues. They investigated many different problems using methods of nonlinear analysis.

In the present paper, we consider second order mixed type quasilinear hyperbolic equations with admissible parabolic degeneracy. The general solution of these equations can be represented by the superposition of two arbitrary functions. As is known, the construction of general integral or general solution is possible only for special class of equations.

It is noteworthy that essential difficulties emerge when classical problems for nonlinear equations, especially those of hyperbolic types, are posed. In the linear case the set of solutions is presented by well-studied groups whose specific character considerably facilitate the investigation of the problems posed. In general, it is impossible to transfer them directly to nonlinear equations due to the dependence of characteristic manifolds on unknown solutions. Despite the correct formulation of the problem, it is not always possible to establish its solvability when data supports consist of characteristic arcs in the mixed or characteristic problems.

Aim of the Thesis. To investigate Cauchy initial problem and nonlinear versions of Darboux and Goursat characteristic problems for mixed type quasilinear hyperbolic equation with admissible parabolic degeneracy; to obtain the conditions of their solvability, to establish the explicit forms of solutions, and describe their domain of definition.

Research methodology is based on the classical method of characteristics. On the basis of the analysis of characteristic systems, we construct the general integral and general solution, making it possible to formulate and investigate initial and characteristic problems. Nonlinear analogues of mean value property are determined.

Research objects. Mixed type quasilinear hyperbolic equations with parabolic degeneracy.

Scientific innovation. The Cauchy initial problem and nonlinear analogue of Darboux problem are studied. Several nonlinear versions of Goursat problems are investigated as well, including on the basis of nonlinear modification of mean value property. The case of parabolic degeneracy is considered and the structures of domain of definition of solutions are described.

Main results of the Thesis.

i) The general integral and general solution of parabolically degenerated quasilinear hyperbolic equation are constructed;
ii) Cauchy initial problem studied and the domain of definition of its solution found;
iii) The First Darboux problem and several versions of Goursat characteristic problem investigated;
iv) The solvability of the Goursat problem in a class of both regular and general solutions proved on the basis of the nonlinear analogue of the mean value property.

Approval of the Thesis. The main results of the thesis were reported at the Enlarged Sessions of the Seminars of I. Vekua Mathematical Institute of Tbilisi State University (1995, 1997, 1999), at the International Symposium in Differential Equations and Mathematical Physics (DEMPh 1997), at the Symposium in Differential Equations and Mathematical Physics held at A.Razmadze Mathematical Institute (DEMPh 2003).

Size and structure of the Thesis. The thesis consists of an introduction and three Chapters presented on 67 printed pages. There is indicated the list of used literature, which consists 41 references, and the list of four papers published by the author of the thesis.

Contents of the Thesis

The Introduction presents proofs of the topicality of problems discussed in the Thesis and the argument of the Thesis.

In Chapter 1 of the Thesis we consider the second order quasilinear equation

$$L(u) \equiv 2y\left(u_y - 2y\right)u_{xx} + \left(u_y - 2yu_x - 2y\right)u_{xy} - u_x u_{yy} = 2u_x\left(u_x - 1\right). \qquad (1)$$

Two real characteristic directions

$$2ydy = dx, \qquad \left(u_y - 2y\right)dy + u_x dx = 0 \qquad (2)$$

correspond to differential operator L. The first characteristic family of equation (1) is defined by independent variables; the second one depends on first order derivatives of an unknown solution.

The given equation degenerates parabolically at the set of points, where the following condition

$$u_y + 2y\left(u_x - 1\right) = 0 \qquad (3)$$

is fulfilled. Therefore, the equation (1) belongs to the class of mixed type parabolically degenerated hyperbolic equations.

To construct the general integral, we use the classical method of characteristics. In the first paragraph of this Chapter we consider two differential systems of characteristic relations

$$2y\,dy - dx = 0, \quad du_x - \frac{u_x}{u_y - 2y}du_y - \frac{u_x\left(u_x - 1\right)}{y\left(u_y - 2y\right)}\,dx = 0, \qquad (4)$$

$$\left(u_y - 2y\right)dy + u_x\,dx = 0, \quad du_x + \frac{du_y}{2y} - \frac{u_x\left(u_x - 1\right)}{y\left(u_y - 2y\right)}\,dx = 0 \qquad (5)$$

together with the compatibility condition

$$du = u_x dx + u_y dy\,. \qquad (6)$$

In the class of hyperbolic solutions the following proposition is valid:

T h e o r e m. Each characteristic system (4), (6) and (5), (6) of equation (1) is admitting two first integrals, represented by formulas:

$$\xi = x - y^2\,, \quad \xi_1 = \frac{q-2y}{p} + 2y \qquad (7)$$

and

$$\eta = u - y^2\,, \;\; \eta_1 = 2y(p-1) + q\,. \qquad (8)$$

These first integrals are known as characteristic or Riemann invariants. One pair of invariants ξ and ξ_1 is constant along all characteristics, which correspond to the first of relations (2) and define the whole family. A set of these characteristics in the sequel will be called a $\xi -$family. Based on the relation (7) of the above-stated theorem, the equation (1) has an intermediate integral:

$$\frac{q-2y}{p} + 2y = G\left(x - y^2\right),$$

where $G \in C^2\left(R^1\right)$ - is an arbitrary function.

Another pair of invariants η and η_1 is defined by means of the second of relations (2) and the corresponding set of characteristics will be called an $\eta -$family. Similarly to (7), the relations (8) give us the second intermediate integral of the equation (1)

$$2y(p-1) + q = H\left(u - y^2\right)$$

with $H \in C^2\left(R^1\right)$ arbitrary function.

In the second paragraph using Riemann invariants and intermediate integrals we construct the general integral of the equation (1):

$$y = g\left(x - y^2\right) + h\left(u - y^2\right), \qquad (9)$$

where g and h are the arbitrary smooth functions.

It is proved the following

T h e o r e m. The equation (1) is integrated and its general solution is represented by superposition of two arbitrary functions:

$$u = y^2 + f\left[\, y + g\left(x - y^2\right)\right], \qquad (10)$$

where f is also a smooth function.

As we already mentioned, if characteristic roots are equal, the equation (1) degenerates parabolically, and all parabolic solutions of (1) are determined by condition (3). In the equation (1) this condition can be dually considered.

Specifically, replacing the expression $\left(2y - u_y\right)/u_x$ with $2y$ in the given equation, we obtain the following linear parabolic equation

$$4y^2u_{xx} + 4yu_{xy} + u_{yy} = -2(u_x - 1),$$

whose characteristic equation has multiple root

$$\lambda_1 = \lambda_2 = \frac{1}{2y}$$

and general solution is represented by formula:

$$u = x + y\, f_1\left(x - y^2\right) + g_1\left(x - y^2\right),$$

where f_1 and g_1 are arbitrary functions.

And replacing $2y$ by the expression $u_y/(1-u_x)$, we obtain the following quasilinear equation of parabolic type:

$$u_y{}^2u_{xx} - 2u_y(u_x - 1)\, u_{xy} + (u_x - 1)^2 u_{yy} = -2(u_x - 1)^3,$$

whose corresponding multiple characteristic root is

$$\mu_1 = \mu_2 = -\frac{u_x - 1}{u_y}.$$

The general integral of this equation is as follows:

$$x = y^2 - y\varphi(u - x) + \psi(u - x),$$

where φ and ψ are also arbitrary functions.

Hence we can conclude that considering the condition (3) in the equation (1), we receive two linear and nonlinear equations. All these three equations are related to each other. There exist their general solutions. As distinct from equations they do not connect to each other, since the condition (3) does not include the solutions itself but the first derivatives of these solutions. Whereas differentiating these three general solutions, we obtain one and the same result - the condition (3), which determines the class of parabolic solutions of the equation (1).

In Chapter 2 we investigate the Cauchy initial problem and the nonlinear analogue of Darboux characteristic problem.

In the first paragraph of this Chapter we study the following Cauchy problem: suppose $\tau(x)$ and $\nu(x)$ are twice and once continuously differentiable functions correspondingly at the segment $[-1,1]$. Beside that $\nu(x) \neq 0,\ x \in J$ inequality is fulfilled everywhere on given interval, where

$$J = \{(x, y) : -1 \leq x \leq 1,\ y = 0\}.$$

The segment $[-1,1]$ of axis $y = 0$ is selected without loss of generality.

The Cauchy Problem. *Find the regular solution of the equation (1) together with its domain of definition, which satisfies following initial conditions:*

$$u\big|_{y=0} = \tau(x), \quad x \in [-1,1], \tag{11}$$

$$u_y\big|_{y=0} = \nu(x), \quad x \in [-1,1], \tag{12}$$

where $\tau \in C^2[-1,1]$, $\nu \in C^1[-1,1]$ *are given functions.*

The initial functions $\tau(x)$ and $\nu(x)$ and the derivative $\tau'(x)$ determine the characteristic directions at each point of the data support, as well as the behaviour of equation and its parabolic degeneracy. Particularly,

i) if function $\nu \neq 0$ for all points of data support J, then characteristics have different directions on J and the type of the equation (1) is hyperbolic;

ii) if for some x_0 point of the support J, the function ν is equal to zero of any order and $\tau'(x_0) \neq 0$, then two characteristic directions coincide at this point and the equation (1) is parabolic.

We introduce designation:

$$\vartheta'(x) = \frac{\tau'(x)}{\nu(x)}, \; \vartheta'(x) \neq 0, \; x \in [-1,1],$$

where $\vartheta \in C^2[-1,1]$ is an arbitrary function.

Theorem. *If the initial functions satisfy the inequalities*

$$\tau'(x) \neq 0, \; \nu(x) \neq 0, \; \vartheta'(x) \neq 0$$

at the whole segment $[-1,1]$ *and the equation* $\zeta = \vartheta(x)$ *has only one solution* $x = G(\zeta)$, $G(0) = 0$, *then the Cauchy problem is uniquely solvable in the class of regular solutions, which is presented by formula*

$$u = y^2 + \tau\left\{ G\left[y + \int_a^{x-y^2} \frac{\tau'(t)}{\nu(t)} dt \, , \, a \right] \right\}$$

and its domain of definition is bounded by characteristic curves of both families: $x - y^2 = -1$, $x - y^2 = 1$ *and* $y + \vartheta(x - y^2) = \vartheta(-1)$, $y + \vartheta(x - y^2) = \vartheta(1)$.

In the second paragraph of the same Chapter we consider the First Darboux problem. In general the Darboux problem implies the construction of the solution by given values on two different curves coming out of the common point, one of which is characteristic and the other is a free curve. We consider the case, when the common point is the origin, the first curve is the arc of characteristic parabola $x = y^2$, $x \in [0,a]$ and the second one is the segment $[0,b]$ of the axis $y = 0$. Based on the formulation of the problem, this second curve has to be free. For that, in accordance with the structure of the second characteristic root, the characteristic direction $\frac{dy}{dx} = -\frac{u_x}{u_y - 2y}$ does not have to define the parallel direction of the axis Ox. Hence we require the fulfilment of condition

$$u(x,0) \neq const. \tag{13}$$

The First Darboux Problem. We have to define the regular solution of equation (1) together with its domain of definition, if it satisfies the following conditions:

$$\begin{aligned} u\Big|_{y=\sqrt{x}} &= \varphi(x), \; 0 \leq x \leq a, \\ u\Big|_{y=0} &= \psi(x), \; 0 \leq x \leq b, \; a \leq b, \end{aligned} \tag{14}$$

where $\varphi \in C^2[0,a]$, $\psi \in C^1[0,b]$ *are given functions and* $\varphi(0) = \psi(0)$.

It has to be noted that depending on the functions φ and ψ, the problem (1), (14) could become an analogue of either Goursat or Darboux problem. For example, if $\psi = const$, then the problem (1), (14) represents the well-known Goursat problem and is uniquely solvable; if $\varphi(x) = x + c$, then the parabola $x = y^2$ will be the characteristic and the curve of degeneracy of the equation (1) at the same time.

T h e o r e m. *If the inequality (13) and the following conditions are fulfilled:*

1. *the functional equation* $\varphi(x) - x = z$ *has the only one inverse* $\Phi(z)$, $\Phi(\varphi(0)) = 0$*;*
2. $\varphi'(x) > 1$*;*
3. $\psi(x) < \varphi(a) - a, \quad x \in [0, b]$,

then the problem (1), (14) has a unique regular solution; this solution is represented explicitly by

$$u = y^2 + \varphi\left\{\left[y + \sqrt{\Phi\left[\psi\left(x - y^2\right)\right]}\right]^2\right\} - \left\{y + \sqrt{\Phi\left[\psi\left(x - y^2\right)\right]}\right\}^2 \qquad (15)$$

formula and is defined in curvilinear quadrangle, which is bounded by the supports of the problem and arcs of $y + \sqrt{\Phi\left[\psi\left(x - y^2\right)\right]} = \sqrt{a}$, $x - y^2 = b$ *characteristic curves.*

T h e o r e m. *If the inequality (13), conditions 1 and 2 of the previous Theorem and* $\psi(x) \geq \varphi(a) - a$, $x \in [0, b]$ *are fulfilled, then the problem (1), (14) is solvable in the class of regular solutions; its unique solution is represented by formula (15) and is defined in curvilinear triangle, which is bounded by the supports of the problem and* $y + \sqrt{\Phi\left[\psi\left(x - y^2\right)\right]} = \sqrt{a}$ *characteristic curves.*

The solution of the Darboux problem can be unbounded as well: if b is infinitely large number, then none of characteristics intersects the parabola $x - y^2 = b$.

Chapter 3 is devoted to the investigation of Goursat characteristic problem. This problem is required to define the solution by its given values on finite or infinite arcs of two different characteristics coming out of the common point. In our case, one of these characteristics is parabola $x = y^2 + \delta$. Another one depends on an unknown solution and can be chosen arbitrarily on condition that it will not have more than one intersection point with the characteristics of $\xi -$family and will not coincide with them anywhere. Suppose that this characteristic is represented explicitly by the equation

$$x = \omega(y), \qquad a \leq y \leq c, \quad \omega(a) = \delta + a^2, \qquad (16)$$

where $\omega \in C^2[a, c]$. Our requirements regarding this characteristic express in the following form in terms of ω function: the equation

$$\omega(y) - y^2 = k \qquad (17)$$

for an arbitrary right-hand side $k > 0$ with respect to y will not have more than one solution and the inequality

$$\omega'(y) - 2y \neq 0 \qquad (18)$$

is fulfilled everywhere in the interval $[a, c]$. Therefore, the equation (17) for any positive $k \in \left[\delta, \omega(c) - c^2\right]$ has a unique continuously differentiable solution:

$$y = W(k), \ W(\delta) = a. \qquad (19)$$

The functions $\omega(y) - y^2 = k$ and $y = W(k)$ are inversed.

The Goursat problem. We have to find the solution of equation (1) together with its domain of definition, if it satisfies the condition

$$u\big|_{x=y^2+\delta} = \upsilon(y), \quad a \le y \le b\,, \;\; \upsilon \in C^2[a\,,b] \tag{20}$$

and the arc of the curve (16) is characteristic of the $\eta-$family.

Theorem. If the conditions (17), (18) are fulfilled and each of the function $\omega(y) - y^2 = k$, $\upsilon(y) - y^2 = t$ has only one inverse: $y = W(k)$, $W(\delta) = a$ and $y = V(t)$, $V[\upsilon(a) - a^2] = a$ correspondingly, then the problem (1), (16), (20) is uniquely solvable in the class of regular solutions; its solution is represented by formula

$$u = y^2 + \upsilon\left\{y - W\left(x - y^2\right) + a\right\} - \left\{y - W\left(x - y^2\right) + a\right\}^2 \tag{21}$$

and defined in curvilinear quadrangle bounded by characteristic curves: $x = \omega(y)$, $x = y^2 + \delta$, $x = y^2 + \omega(y + a - b) + (y + a - b)^2$, $x = y^2 + \omega(c) - c^2$.

In the second paragraph of the same Chapter we are studying the Singular Goursat problem, whose characteristic directions are coincide in a common point. In our case it is the origin. The supports of the problem are: the characteristic parabola

$$x = y^2 \tag{22}$$

and the curve γ, which is represented by equation

$$x = \omega(y), \quad 0 \le y \le b\,, \quad \omega(0) = 0\,, \quad \omega'(0) = 0\,, \tag{23}$$

where $\omega \in C^2[0,b]$ is the given function and following conditions are fulfilled: the equation

$$\omega(y) - y^2 = \zeta \tag{24}$$

for an arbitrary right-hand side $\zeta > 0$ with respect to y is failed to have more than one solution and the inequality (18) is fulfilled everywhere in the interval $[0,b]$. Therefore, the equation (24) for any positive $\zeta \in [0, \omega(b) - b^2]$ has a solution, which belongs to the class $C[0,\omega(b) - b^2] \cap C^1(0,\omega(b) - b^2]$:

$$y = W(\zeta),\; W(0) = 0\,. \tag{25}$$

The functions $\omega(y) - y^2 = \zeta$ and $y = W(\zeta)$ are inversed.

The Goursat problem. We have to define the regular solution of equation (1) together with its domain of definition, if it satisfies the condition

$$u\big|_{x=y^2} = \upsilon(y), \quad 0 \le y \le a\,, \;\; \upsilon \in C^2[0,a] \tag{26}$$

and the arc of γ curve represented by equality (23) is the characteristic of the $\eta-$family.

Theorem. If the conditions (18) and (24) are fulfilled and each of the function $\omega(y) - y^2 = \zeta$, $\upsilon(y) - y^2 = z$ is invertible: $y = W(\zeta), W(0) = 0$ and $y = V(z)$, $V(\upsilon(0)) = 0$ correspondingly, then Goursat problem has only one regular solution, which is represented by formula

$$u = y^2 + \upsilon\left\{y - W\left(x - y^2\right)\right\} - \left\{y - W\left(x - y^2\right)\right\}^2. \tag{27}$$

This solution is defined and continuous everywhere in curvilinear quadrangle bounded by characteristic curves: $x = y^2$, $x = \omega(y)$, $x = \omega(y-a) + 2ay - a^2$, $x = y^2 + \omega(b) - b^2$. *And the curve* $x = y^2$ *is singular for the first order derivatives of this solution.*

We also consider the special case, when all characteristics of $\eta -$family intersect each other and parabola $x = y^2$ envelops them.

In the next paragraph we consider Goursat problem by means of the nonlinear analogue of mean value property. This property allows us to prove the solvability of the problem in the class of regular and of general solutions as well. The mean value property for the given equation is as follows*: the sums of ordinates at the opposite vertices of arbitrary characteristic quadrangle are equal.*

The characteristic curves are presented by (22) and

$$x = \omega(y), \qquad 0 \le y \le b, \quad \omega(0) = 0 \tag{28}$$

relations, where $\omega \in C^2[0,b]$ is the given function. The equation (24) for an arbitrary $\zeta > 0$ with respect to y is failed to have more than one solution and the inequality (18) is fulfilled everywhere in the interval $[0,b]$. In addition, the equation (24), for any positive $\zeta \in [0, \omega(b) - b^2]$ has a unique continuously differentiable solution represented by formula (25). The functions $\omega(y) - y^2 = \zeta$ and $y = W(\zeta)$ are inversed.

The Goursat problem. Find a solution of equation (1) together with its domain of definition, if it satisfies the condition

$$u\Big|_{x=y^2} = \upsilon(y), \quad 0 \le y \le a, \ \upsilon \in C^2[0,a]$$

and the arc of the curve (28) is the characteristic of the $\eta -$ *family.*

As is shown in this paragraph, the curve (28) make it possible to present explicitly all characteristics of $\eta -$family that coming out of the points of parabola $x = y^2$. Hence we can describe entirely the domain of definition of the solution of Goursat problem.

T h e o r e m. If the conditions (18) and (24) are fulfilled and function $\omega(y) - y^2 = \zeta$ *has only one inverse:* $y = W(\zeta)$, $W(0) = 0$, *then Goursat problem is uniquely solvable in the class of regular solutions; its solution is represented by formula (27) and defined in curvilinear quadrangle bounded by following characteristic curves:*

$$x = y^2, \ x = \omega(y), \ x = y^2 + \omega(y-a) + (y-a)^2, \ x = y^2 + \omega(b) - b^2.$$

The main results obtained in the thesis are published in the following papers:

1. M. Klebanskaya, On Cauchy and Goursat problems for the class of quasilinear mixed type equations. // *Tbilisi University Press, Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.*, **10**(1995), No.1, 52-54. [83]

2. M. Klebanskaya, Darboux problem for hyperbolic equation with parabolic degeneracy. // *Tbilisi University Press, Rep. Enlarged Sess. Semin. I. Vekua App. Math,* **12** (1997), No. 1-3, 12-14. [83]
3. M. Klebanskaya, On Singular Nonlinear Goursat Problem. // *Tbilisi University Press, Rep. Enlarged Sess. Semin. I. Vekua App. Math*, **14** (1999), No. 1, 42-45. [83]
4. M. Klebanskaya, On One Nonlinear Analogue of the Mean Value Property and its Application to the Investigation of the Nonlinear Goursat Problem. // *Proc.A.Razmadze Math. Inst.*, **141** (2006), 67-74. [214]