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BOUNDARY VALUE PROBLEMS OF ELECTROELASTICITY WITH CONCENTRATED SINGULARITIES

T. BUCHUKURI AND D. YANAKIDI

ABSTRACT. We investigate the solutions of boundary value problems of linear electroelasticity, having growth as a power function in the neighbourhood of infinity or in the neighbourhood of an isolated singular point. The number of linearly independent solutions of this type is established for homogeneous boundary value problems.

რეზიუმე. გამოკვლეულია წრფივი ელექტროდრეკადობის სასაზღვრო ამოცანების ამონახსნები, რომელთაც უსასრულობის მიდამოში გააჩნია ხარისხოვანი ზრდის რიგი. დადგენილია ასეთი ტიპის წრფივად დამოუკიდებელი ამონახსნების რიცხვი ერთგვაროვანი სასაზღვრო ამოცანებისთვის.

The basic equations of the static state of an electroelastic medium are written in terms of displacement and electric potential components as follows [1, 2]:

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + e_{kij}\frac{\partial^2 \varphi}{\partial x_j \partial x_k} + F_i = 0, \tag{1}$$

$$-e_{ikl}\frac{\partial^2 u_k}{\partial x_i \partial x_l} + \mathcal{E}_{ik}\frac{\partial^2 \varphi}{\partial x_i \partial x_k} = 0, \quad i = 1,2,3, \tag{2}$$

where $u = (u_1, u_2, u_3)$ is a displacement vector, φ is an electric field potential, c_{ijkl}, e_{kij}, $\mathcal{E}_{ik}$, are constants, $F = (F_1, F_2, F_3)$ is mass force. It is assumed that the constants c_{ijkl}, e_{kij} , $\mathcal{E}_{ik}$ satisfy the conditions

$$\begin{gathered} c_{ijkl} = c_{jikl} = c_{klij}, \\ e_{kij} = e_{kji}, \quad \mathcal{E}_{ij} = \mathcal{E}_{ji}, \quad i,j,k,l = 1,2,3. \end{gathered} \tag{3}$$

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System (1), (2) can be written in the matrix form. We introduce the operator

$$A(\partial x) = \|A_{ij}(\partial x)\|_{4\times 4}, \tag{4}$$

$$A_{ik}(\partial x) = c_{ijkl}\frac{\partial^2}{\partial x_j \partial x_l}, \quad i,k = 1,2,3,$$

$$A_{i4}(\partial x) = e_{kij}\frac{\partial^2}{\partial x_k \partial x_j}, \quad i = 1,2,3,$$

$$A_{4k}(\partial x) = -e_{ikl}\frac{\partial^2}{\partial x_i \partial x_l}, \quad k = 1,2,3,$$

$$A_{44}(\partial x) = \mathcal{E}_{ik}\frac{\partial^2}{\partial x_i \partial x_k}.$$

Introducing the four-component vectors $U = (U_1, U_2, U_3, U_4) = (u_1, u_2, u_3, \varphi)$ and $\chi = (F_1, F_2, F_3, 0)$, system (1), (2) is rewritten as

$$A(\partial x)U + \chi = 0. \tag{5}$$

It is easy to show that the operator $A(\partial x)$ is a second order homogeneous operator of the elliptic type.

Assume that an electroelastic medium occupies a bounded domain Ω^+ of the three-dimensional space $\mathbb{R}^3$. Let $\Omega^- = \mathbb{R}^3\setminus\Omega^+$, $S = \partial\Omega^+ = \partial\Omega^-$.

Assume that the surface S is partitioned into four parts: S_{11}, S_{12}, S_{13}, S_{14}, where $S_{1i} \cap S_{1j} = \varnothing$, $i \neq j$ and $\cup_{i=1}^4 S_{1i} = S$. Also assume that we have another partitioning of S into two parts: S_{21} and S_{22}; $S_{21} \cap S_{22} = \varnothing$, $S_{21} \cup S_{22} = S$.

We shall consider a boundary value problem for system (5) when the following conditions are given: displacements on the part S_{11} of the boundary S, boundary mechanical stresses on the part S_{12}, normal components of the displacement vector and tangential components of the mechanical stress vector on the part S_{13}, and normal components of the boundary stress vector and tangential components of the displacement vector on the part S_{14}. These conditions can be written in the form

$$u_i\Big|_{S_{11}}(y) = f_i(y), \quad i = 1,2,3; \tag{6}$$

$$\tau_{ji}n_j\Big|_{S_{12}}(y) = g_i(y), \quad i = 1,2,3; \tag{7}$$

$$u_i n_i\Big|_{S_{13}}(y) = f^{(n)}(y), \quad (\tau_{ji}n_j - n_k\tau_{jk}n_j)\Big|_{S_{13}}(y) = g_i^{(\tau)}(y), i = 1,2,3, \tag{8}$$

$$n_i\tau_{ji}n_j\Big|_{S_{14}}(y) = g^{(n)}(y), \quad (u_i - n_k u_k n_i)\Big|_{S_{14}}(y) = \\ = f_i^{(\tau)}(y), i = 1,2,3. \tag{9}$$

Here f_i, g_i, $f^{(n)}$, $g^{(n)}$, $f_i^{(\tau)}$, $g_i^{(\tau)}$ are the known functions.

In additition to the above "mechanical" boundary conditions we should also be given "electric" conditions

$$\varphi\Big|_{S_{21}}(y) = \psi(y), \tag{10}$$

$$\mathcal{D}_i n_i\Big|_{S_{22}}(y) = h(y). \tag{11}$$

In the above formulas τ_{ji} denotes the mechanical stress tensor, $\mathcal{D}_i$ the electric induction vector. These values are related with the unknown displacement vector U and the electric potential φ by the relations

$$\tau_{ij} = \frac{1}{2}c_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) + e_{kij}\frac{\partial \varphi}{\partial x_k}, \tag{12}$$

$$\mathcal{D}_i = \frac{1}{2}e_{ikl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) - \mathcal{E}_{ik}\frac{\partial \varphi}{\partial x_k}. \tag{13}$$

We shall apply the term "the basic internal regular boundary value problem of electrostatics" to the following problem:

Find in the domain Ω^+ the four-component vector $U = (u, \varphi)$ of the class $C^2(\Omega^+) \cap C^1(\bar{\Omega}^+)$ which is a solution of system (5) and satisfies conditions (6)-(11). Denote this problem by $(\mathcal{E})^+$.

The external boundary value problem $(\mathcal{E})^-$ is formulated absolutely in the same manner. In that case the vector U is sought for in the domain Ω^-.

From the basic problem $(\mathcal{E})^\pm$ one can obtain, as the particular case, various problems. Denote by $(p,q)^\pm$ the problem $(\mathcal{E})^\pm$ when $S_{1p} = S$ ($S_{1i} = \varnothing$, if $i \neq p$) and $S_{2q} = S$ ($S_{2i} = \varnothing$, if $i \neq q$).

Denote by $(\mathcal{E})_0^\pm$ the problem $(\mathcal{E})^\pm$ with the homogeneous boundary conditions $f_i = 0$, $g_i = 0$, $f_i^{(\tau)} = 0$, $g_i^{(\tau)} = 0$, $f^{(n)} = 0$, $g^{(n)} = 0$, $\psi = 0$, $h = 0$, when $\chi = 0$. The notation $(p,q)_0^\pm$ has the same meaning as above.

The following uniqueness theorem is valid:

Theorem 1. *If $U = (u, \varphi)$ is a solution of the problem $(\mathcal{E})_0^\pm$, then it has the form*

$$U_i(x) = \varepsilon_{ijk}a_i x_k + b_i, \quad \varphi = \varphi_0, \quad i = 1,2,3,$$

where φ_0, a_i, b_i are arbitrary constants and ε_{ijk} is the Levy-Civita symbol. In that case if $S_{11} \neq \varnothing$ and is not a subset of some plane, then $a_i = 0$, $b_i = 0$, $(i = 1,2,3)$; if $S_{21} \neq \varnothing$, then $\varphi_0 = 0$.

The proof is based on the Green formula

$$\sum_{i,k=1}^{4} \int_{\Omega^+} U_i(x) A_{ik}(\partial x) V_k(x)\, dx = \sum_{i,k=1}^{4} \int_S U_i(y) T_{ik}(\partial y, n) V_k(y)\, d_y S - \int_{\Omega^+} E(U,V)(x)\, dx, \tag{14}$$

where $T_{ik}(\partial y, n)$ are the components of the boundary stress operator

$$\begin{aligned} T_{ik}(\partial y, n) &= c_{ijkl} n_j(y) \frac{\partial}{\partial y_l}, \quad i,k = 1,2,3, \\ T_{i4}(\partial y, n) &= e_{kij} n_j(y) \frac{\partial}{\partial y_k}, \quad i = 1,2,3, \\ T_{4k}(\partial y, n) &= -e_{ikl} n_i(y) \frac{\partial}{\partial y_l}, \quad k = 1,2,3, \\ T_{44}(\partial y, n) &= \mathcal{E}_{ik} n_i(y) \frac{\partial}{\partial y_k}; \end{aligned} \tag{15}$$

$$\begin{aligned} E(U,V) &= c_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial v_k}{\partial x_k} + e_{kij} \frac{\partial u_i}{\partial x_j} \frac{\partial \psi}{\partial x_k} - \\ &\quad - e_{ikl} \frac{\partial \varphi}{\partial x_i} \frac{\partial v_k}{\partial x_l} + \mathcal{E}_{ik} \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_k}, \\ U &= (u, \varphi), \quad V = (v, \varphi). \end{aligned} \tag{16}$$

If $U = (u, \varphi)$ is a solution of the problem $(\mathcal{E})_0^+$, then

$$\forall y \in S \ : \ \sum_{i,k=1}^{4} U_i(y) T_{ik}(\partial y, n) U_k(y) = 0. \tag{17}$$

Applying (17), from the formula (14) we conclude that

$$\forall x \in \Omega^+ \ : \ E(U,U) = 0, \tag{18}$$

where U is a solution of the problem $(\mathcal{E})_0^+$. From (18) we obtain

$$\begin{aligned} \forall x \in \Omega^+ \ &: \ \frac{\partial u_i(x)}{\partial x_j} + \frac{\partial u_j(x)}{\partial x_i} = 0, \\ &\frac{\partial \varphi(x)}{\partial x_i} = 0, \quad i,j = 1,2,3. \end{aligned} \tag{19}$$

This formula immediately implies that all the statements of Theorem 1 are valid.

Theorem 2. *If U is a solution of the problem $(\mathcal{E})_0^-$ and satisfies the conditions*

$$U_i(x) = O(|x|^{-1}), \quad i = 1,2,3,4, \\ \frac{\partial U_i(x)}{\partial x_j} = o(|x|^{-1}), \quad j = 1,2,3, \tag{20}$$

in the neighbourhood of a point at infinity, then

$$\forall x \in \Omega^- \ : \ U(x) = 0.$$

The theorem is proved by the reasoning used in proving Theorem 1 for the domain $\Omega_r \equiv \Omega^- \setminus \overline{B(0,r)}$ where $B(0,r)$ is the ball with centre at the point 0 and radius r and with passage to the limit as $r \to \infty$.

We can formulate Theorem 2 more precisely, since it turns out that conditions (20) can be considerably weakened.

The results of [3] imply, as the particular case,

Lemma 1. *Let U be a solution of the system*

$$A(\partial x)U = 0 \tag{5_0}$$

of the class $C^2(\Omega^-)$ in the domain Ω^- and one of the conditions below

$$\lim_{r\to\infty} r^{-(p+4)} \int\limits_{B(0,r)\setminus B(0,r/2)} |U(y)|\,dy = 0, \tag{21}$$

$$U(y) = o(|y|^{p+1}), \quad |y| \to \infty, \tag{22}$$

$$\int\limits_{\Omega^-} \frac{|U(y)|}{1+|y|^{p+4}}\,dy < +\infty \tag{23}$$

be fulfilled for some nonnegative integer number p. Then for any nonnegative integer q we have the representation

$$\forall x \in \Omega^- \ : \ U_j(x) = \sum_{|\alpha|\le p} c_j^{(\alpha)} x^\alpha + \sum_{|\beta|\le q} d_k^{(\beta)} \partial^\beta \Phi_{jk}(x) + \psi_j(x), \\ j = 1,2,3,4, \tag{24}$$

$c_j^{(\alpha)} = const$, $d_k^{(\beta)} = const$, $\psi_j \in C^2(\Omega^-)$, and in the neighbourhood of infinity

$$\partial^\gamma \psi_j(x) = O(|x|^{-2-|\gamma|-q}). \tag{25}$$

Here $\Phi = \|\Phi_{jk}\|_{4\times 4}$ is the matrix of fundamental solutions of equations (5_0).

Theorems 2 and 3 imply directly the following uniqueness theorem:

Theorem 3. *Let U be a solution of the problem $(\mathcal{E})_0^-$ and satisfy at infinity the condition*

$$U(x) = o(1). \tag{26}$$

Then

$$\forall x \in \Omega^- \; : \; U(x) = 0.$$

Consider now the boundary value problem: Find in the domain Ω^- the solution U of the Problem $(\mathcal{E})^-$, satisfying at infinity the condition

$$U(x) = o(|x|^{p+1}). \tag{27}$$

This problem will be denoted by $(\mathcal{E}_p)^-$ and the corresponding homogeneous problem by $(\mathcal{E}_p)_0^-$.

Let $K(p)$ be the number of linearly independent polynomial solutions of system (5_0) with a degree not higher that p. Repeating the reasoning given in [4] for an equation of classical elasticity we can readily prove that

$$K(p) = 4\left[\binom{p+2}{2} + \binom{p+1}{2}\right] = 4(p+1)^2. \tag{28}$$

Now it is easy to prove the following

Lemma 2. *Let the homogeneous problem $(\mathcal{E})_0^-$ has a solution satisfying condition* (26) *(it will be trivial by virtue of Theorem 3). Then the homogeneous problem $(\mathcal{E}_p)_0^-$ has at most $K(p) = 4(p+1)^2$ linearly independent solutions.*

Proof. Let $U^{(1)}, \ldots, U^{(r)}$ $(r > 4(p+1)^2)$ be solutions of the homogeneous problem $(\mathcal{E}_p)_0^-$. By virtue of Lemma 1 $U^{(i)} = P^{(i)} + V^{(i)}$ where $P^{(i)}$ is a polynomial solution of system (5_0) of a degree not higher than p and $V^{(i)}$ is a solution of system (5_0) satisfying condition (26). Then by the condition of the lemma there exist numbers c_i not all equal to zero such that $\sum_{i=1}^r c_i P^{(i)} = 0$. Consider the vector $W \equiv \sum c_i U^{(i)} = \sum c_i V^{(i)}$. W is a linear combination of $U^{(i)}$ and hence will be a solution of the homogeneous problem $(\mathcal{E}_p)_0^-$, but at the same time W is a linear combination of solutions $V^{(i)}$, therefore satisfying condition (26), and hence, on account of Theorem 3, $W = 0$. Thus solutions $U^{(i)}$ are linearly dependent. ■

Lemma 2 immediately yields

Corollary 1. *If the nonhomogeneous problem* $(\mathcal{E})^-$ *has the unique solution for any* f_i, g_i, $f^{(n)}$, $g^{(n)}$, $f_i^{(\tau)}$, $g_i^{(\tau)}$, ψ, *and* h *belonging to the class* C^∞, *then the homogeneous problem* $(\mathcal{E}_p)_0^-$ *has exactly* $K(p) = 4(p+1)^2$ *linearly independent solutions, while the nonhomogeneous problem* $(\mathcal{E}_p)^-$ *has the solution* U *for arbitrary boundary data and this solution is represented as*

$$U = U_0 + U^{(p)},$$

where U_0 *is the solution of the problem* $(\mathcal{E})^-$ *satisfying condition* (26) *and* $U^{(p)}$ *is an arbitrary solution of the problem* $(\mathcal{E}_p)_0^-$.

The problem $(\mathcal{E})^+$ is treated with sufficient completeness in [5]. This paper also contains the proof of the existence of a generalized solution in Sobolev spaces. Using the well-known regularization theorems [6], from these results we easily obtain the existence of classical solutions for sufficiently smooth S and boundary data. In particular, we have

Lemma 3. *Let the boundary* S *of the domain* Ω^+ *and the boundary data belong to the class* $C^\infty(\Omega^+)$. *Then:*

problems $(1.1)^+$ *and* $(4.1)^+$ *have the unique solution of the class* $C^\infty(\Omega^+)$;

the problem $(3.1)^+$ *has the unique solution of the class* $C^\infty(\bar{\Omega}^+)$ *if* S *is not the rotation surface;*

the problem $(2.1)^+$ *has a solution of the class* $C^\infty(\bar{\Omega}^+)$ *if and only if the conditions*

$$\int_S g_i(y)\, d_yS = 0, \quad i = 1,2,3, \tag{29}$$

$$\int_S \varepsilon_{ijk} y_j g_k(y)\, d_yS = 0, \quad i = 1,2,3, \tag{30}$$

are fulfilled to within a term of the form $U = (u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, 0)$ *where*

$$U_i^{(0)}(x) = \varepsilon_{ijk} x_j a_k + b_i, \tag{31}$$

a_i *and* b_i *are arbitrary constants* $(i = 1,2,3)$;

problems $(1.2)^+$ *and* $(4.2)^+$, *and also problem* $(3.2)^+$ *if* S *is not the rotation surface, have a solution if and only if*

$$\int_S h(y)\, d_yS = 0; \tag{32}$$

the solution is defined to within a term of the form $U = (0,0,0,\varphi_0)$ *where* φ_0 *is an arbitrary number;*

problem $(2.2)^+$ *has a solution if and only if conditions* (29), (30), (32) *are fulfilled; the solution is defined to within a term of the form* $U = (u, \varphi_0)$ *where* u *is written as* (31) *and* φ_0 *is an arbitrary number.*

We are interested in investigating not smooth solutions of the problems $(\mathcal{E})^+$, but such solutions that at some given points have singularities not higher than given power orders.

Let $x^{(1)}, \dots, x^{(r)}$ be points lying in the domain Ω^+, $M_r \equiv \{x^{(1)}, \dots, x^{(r)}\}$.

Consider the problem with concentrated singularities: Find the solution U of equation (5) which belongs to the class $C^2(\Omega^+ \backslash M_r) \cap C^1(\bar{\Omega}^+ \backslash M_r)$, satisfies the boundary conditions (6)-(11) and, in the neighbourhood of the point $x^{(i)}$, the condition

$$|U(x)| \leq \frac{c}{|x - x^{(i)}|^{p_i}}, \quad i = 1, \dots, r,$$

where p_i are given nonnegative numbers. Denote this problem by $(\mathcal{E})^p_{cs}$.

The investigation of this problem is largely based on one proposition following from the theorem proved in a more general situation in [3].

Lemma 4. *Let* $\Omega \subset \mathbb{R}^3$, $y \in \Omega$, U *be a solution of* (5_0) *of the class* $C^2(\Omega \backslash \{y\})$ *in the domain* $\Omega \backslash \{y\}$ *and, for some* $c > 0$ *and* $p \geq 0$,

$$|U(x)| \leq \frac{c}{|x - y|^p}.$$

Then

$$U_j(x) = U_j^{(0)}(x) + \sum_{k=1}^{4} \sum_{|\alpha| \leq [p]-1} a_k^{(\alpha)} \partial^\alpha \Phi_{jk}(x - y), \quad j = 1, \dots, 4,$$

where $U^{(0)}$ *is the solution of system* (5_0) *of the class* $C^2(\Omega)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ *is the multiindex,* $[p]$ *is the integer part of the number* p, $a_k^{(\alpha)} = const$, $\Phi = \|\Phi_{jk}\|_{4\times 4}$ *is the matrix of fundamental solutions of equation* (5).

Using this lemma, by a reasoning analogous to that from [4], we prove

Lemma 5. *Let* $\mathcal{F}_p$ *be a finite-dimensional space stretched onto the system of vectors* $\{\partial^\alpha \Phi^{(k)}(\cdot - x^{(i)});\ k = 1, 2, 3, 4;\ |\alpha| \leq [p_i] - 1\}$, *where* $\Phi^{(k)} = (\Phi_{1k}, \Phi_{2k}, \Phi_{3k}, \Phi_{4k})$. *Then*

$$\dim \mathcal{F}_p = 4 \sum_{i=1}^{r} [p_i]^2.$$

Proof. We assume $V = (v, \varphi)$ and introduce the notation

$$\epsilon_{ij}^{(V)} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad E^{(V)} = -\frac{\partial \varphi}{\partial x_k},$$

$$\tau_{ij}^{(V)} = c_{ijkl}\epsilon_{kl}^{(V)} - e_{kij}E_k^{(V)}, \quad \mathcal{D}_i^{(V)} = e_{ikl}\epsilon_{kl}^{(V)} + \mathcal{E}_{ik}E_k^{(V)}.$$

Let $\{\omega^{(k)}; k = 1, \dots, \dim \mathcal{F}_p\}$ be base spaces $\mathcal{F}$. Denote by $U^{(k)}$ a solution of system (5_0) satisfying the boundary conditions (6)-(11) for

$$f_i = \omega_i^{(k)}, \quad g_i = \tau_{ji}^{(\omega^{(k)})} n_j, \quad f^{(n)} = \omega_i^{(k)} n_i,$$

$$g_i^{(\tau)} = \tau_{ji}^{(\omega^{(k)})} n_j - n_l \tau_{jl}^{(\omega^{(k)})} n_j n_i, \quad g^{(n)} = n_i \tau_{ji}^{(\omega^{(k)})} n_j,$$

$$f_i^{(\tau)} = \tau_{ji}^{(\omega^{(k)})} n_j - n_l \tau_{jl}^{(\omega^{(k)})} n_j n_i, \quad \psi = \omega_4^{(k)}, \quad h = \mathcal{D}_i^{(\omega^{(k)})} n_i.$$

Consider the vectors $V^{(k)} = U^{(k)} - \omega^{(k)}$. Obviously, the vector $V^{(k)}$ is a solution of the homogeneous problem $(\mathcal{E}_p)_0^+$. We shall prove that the system of vectors

$$\left\{V^{(k)},\ \psi^{(i)};\ k = 1, \dots, \dim \mathcal{F}_p;\ i = 1, \dots, q\right\},$$

where $\{\psi^{(i)}\}$ a linearly independent system of solutions of the homogeneous problem $(\mathcal{E})_0^+$, is linearly independent. Indeed, if

$$\sum_{k=1}^{\dim \mathcal{F}_p} c_k V^{(k)} + \sum_{i=1}^{q} d_i \psi^{(i)} = 0,$$

then

$$\sum_{k=1}^{\dim \mathcal{F}_p} c_k \omega^{(k)} = \sum_{k=1}^{\dim \mathcal{F}_p} c_k U^{(k)} + \sum_{i=1}^{q} d_i \psi^{(i)}.$$

By virtue of (28) $\sum c_k \omega^{(k)} = 0$ and therefore $c_k = 0$ and $ud_i = 0$. Now from Lemma 2 we obtain the proof of Theorem 4. ■

Theorem 4. *If the problem $(\mathcal{E})^+$ has a solution for any boundary data of class C^∞, then the homogeneous problem $(\mathcal{E}_0)_{cs}^p$ has exactly $4\sum_{i=1}^r [p_i]^2 + q$ linearly independent solutions, where q is the number of linearly independent solutions of the problem $(\mathcal{E}_0)^+$ and $[p_i]$ is the integer part of the number p_i.*

This theorem readily implies

Corollary 2. *The homogeneous problems $(1.1)_{cs}^p$ and $(4.1)_{cs}^p$ have exactly $4\sum_{i=1}^r [p_i]^2$ linearly independent solutions and if the boundary S is not the surface of rotation, then Problem $(3.1)_{cs}^p$, too, has the same number of linearly independent solutions.*

Similar theorems hold for the other problems as well.

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PASSAGE OF THE LIMIT THROUGH THE DOUBLE DENJOY INTEGRAL

N. CHELIDZE

ABSTRACT. The conditions are given for passage of the limit through the double Denjoy integral defined by V.G. Chelidze.

რეზიუმე. მოცემულია ზღვარზე გადასვლის პირობები დანჟუას ორჯერად ინტეგრალში, რომელიც შემოღებული იყო ვლ. ჭელიძის მიერ.

As is well-known, there exist various definitions of the double Denjoy integral (see [1,2,5]). Conditions for passage of limits through these integrals have not yet been studied. The object of this paper is to investigate the conditions for passage of the limit through the double Denjoy integral defined by V.G. Chelidze (see [7]).

Here we shall use the well-known terms (see, for example, [8]). We recall only a few definitions.

Definition 1 (see [8], pp. 127-128). A function $f : R_0 \to \mathbb{R}$ with $R_0 = [a,b] \times [c,d]$ is said to be Denjoy integrable (D-integrable) on R_0 or, briefly, $f \in D(R_0)$ if there exists a generalized absolutely continuous function F (ACG-function) on R_0, briefly, $F \in ACG(R_0)$ (see [8], pp. 99-100), with an approximate derivative ([8], p. 103) equal to f a.e. The function F is called a D-primitive of f and $\Delta(f; R_0) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$ is called the D-integral of f on R_0 which is written as

$$\Delta(F, R_0) = (D) \iint\limits_{R_0} f(x,y)dxdy.$$

A function $f : R_0 \to \mathbb{R}$ is called D-integrable on a measurable [1]

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[1]Measurability is meant in the Lebesgue sense and $|A|$ will stand for the Lebesgue measure of the set A.

subset $E \subset R_0$ (briefly $f \in D(E)$), if $f_E(x,y) \in D(R_0)$ where

$$f_E(x,y) = \begin{cases} f(x,y) & (x,y) \in E, \\ 0 & (x,y) \notin E. \end{cases}$$

Definition 2. A compact interval I is said to be of type 2 with respect to E if at least one pair of the opposite vertices of I lies in E.

Lemma 1. *Let E_1 be a compact subset of $[a,b]$ whose contiguous intervals are $\{r_i\}_i$. If $f \in D(E)$, $f \in D(\bar{R}_i)$, $i = \overline{1,\infty}$, where $E = E_1 \times [c,d]$, $R_i = r_i \times [c,d]$, and*

$$\sum_{i=1}^{\infty} O(D; f, \bar{R}_i) < \infty, \tag{1}$$

where

$$O[D; f, \bar{R}_i] = \sup_{\rho \subset \bar{R}_i} \left\{ \left|(D) \iint_{\rho} f(t,\tau)dt\,d\tau\right| \right\}$$

and ρ stands for any measurable subset of $\bar{R}_i$. Then $f \in D(R_0)$ and for each subsegment $R \subset R_0$ we have

$$(D) \iint_R f(t,\tau)dt\,d\tau = (D) \iint_{R \cap E} f(t,\tau)dt\,d\tau + \sum_i \iint_{R \cap \bar{R}_i} f(t,\tau)dt\,d\tau.$$

Proof. Let us consider the function

$$F(x,y) = \sum_i (D) \iint_{\bar{R}_i \cap I(x,y)} f(t,\tau)dt\,d\tau,$$

where $I(x,y) = [a,x;c,y]$.

We shall show that F is continuous on R_0, i.e., F is continuous at any point (x_0,y_0) of R_0; for $\varepsilon > 0$ there must exist a $\delta(\varepsilon) > 0$ such that the inequality

$$|F(x_0+k,\ y_0+l) - F(x_0,y_0)| < \varepsilon.$$

must take place for all k and l, $|k| < \delta(\varepsilon)$, $|l| < \delta(\varepsilon)$. Then

$$\left|F(x_0+k,\ y_0+l) - F(x_0,y_0)\right| = \left|\sum_i (D) \iint_{\bar{R}_i \cap I(x_0+k,y_0+l)} f(t,\tau)dt\,d\tau - \right.$$

$$\left. - \sum_i (D) \iint_{\bar{R}_i \cap I(x_0,y_0)} f(t,\tau)dt\,d\tau\right| \le \sum_i \left|(D) \iint_{\bar{R}_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\right|,$$

where

$$P_{x_0,y_0} = [a,x_0;y_0,y_0+l] \cup [x_0,x_0+k;c,y_0+l].$$

To estimate the last sum let us consider two subsystems of intervals from the system $\{R_i\}_{i=\overline{1,\infty}}$: $I' = \{R'_i\}$ and $I'' = \{R''_i\}$ such that system I' satisfies the conditions:

$\bar{R}'_i \cap P_{x_0,y_0} \neq \varnothing$ and $i > N$

and system I'' the conditions

$\bar{R}''_i \cap P_{x_0,y_0} \neq \varnothing$ and $i \leq N$.

For the first system for all $\varepsilon > 0$ there is an $N_0 = N_0(\varepsilon)$ such that if $N > N_0$, then by the condition (1) we get the estimate

$$\sum_{i>N} \left|(D) \iint_{\bar{R}'_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\right| < \sum_{i>N} O[D; f, \bar{R}'_i] < \frac{\varepsilon}{2}.$$

Now consider system I''. Since $i \leq N$, this system is finite. Let N^* be the number of members of this system. Then for $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $|P_{x_0,y_0}| < \delta$ and

$$|P_{x_0,y_0} \cap \bar{R}''_i| < |P_{x_0,y_0}| < \delta.$$

Hence it follows

$$(D) \iint_{\bar{R}''_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau < \frac{\varepsilon}{2N^*}.$$

Now

$$\sum_{i \leq N} \left|(D) \iint_{\bar{R}''_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\right| \leq \frac{\varepsilon}{2N^*} \cdot N^* = \frac{\varepsilon}{2}.$$

Finally, we obtain

$$|F(x_0 + k, y_0 + l) - F(x_0, y_0)| \leq \sum_i \left|(D) \iint_{\bar{R}_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\right| =$$

$$= \sum_{i>N} \left|(D) \iint_{\bar{R}'_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\right| +$$

$$+ \sum_{i \leq N} \left|(D) \iint_{\bar{R}''_i \cap P_{x_0,y_0}} f(t,\tau)dt\,d\tau\right| \leq \varepsilon.$$

Thus F is continuous on R_0.

Now for the fixed i let us consider any finite system of pairwise disjoint compact intervals of type 2 with respect to R_i, say, $\{S_k\}_{k=1}^n$. If

$$\sum_{k=1}^n |S_k| < \delta,$$

then, by the conditions of the lemma,

$$\sum_k |\Delta(F;S_k)| = \sum_k \Big| \iint_{S_k} f(t,\tau)dt\,d\tau \Big| < \varepsilon.$$

Therefore $F \in ACG\,(\bar{R}_i)_{i=\overline{1,\infty}}$. Now we shall show that F is absolutely continuous on E (or, briefly, is AC, written as $F \in AC(E)$) (see Definition 3 or [8], p. 97). To this end we define the function G as follows:

$$G(x,y) = \begin{cases} 0, & \text{when} \quad (x,y) \in E, \\ \frac{1}{|\bar{R}_i|}(D) \iint_{\bar{R}_i} f(t,\tau)dt\,d\tau, & \text{when} \quad (x,y) \in R_i, \ \ i = \overline{1,\infty}. \end{cases}$$

We show that G is Lebesgue integrable on R_0.

$$\iint_{R_0} |G(x,y)|dx\,dy = \sum_i \iint_{\bar{R}_i} |G(x,y)dx\,dy =$$

$$= \sum_i \iint_{\bar{R}_i} \Big(\frac{1}{|\bar{R}_i|}(D) \Big| \iint_{\bar{R}_i} f(t,\tau)dt\,d\tau \Big| \Big) dx\,dy =$$

$$= \sum_i \Big| (D) \iint_{\bar{R}_i} f(t,\tau)dt\,d\tau \Big| \iint_{\bar{R}_i} \frac{dx\,dy}{|\bar{R}_i|} =$$

$$= \sum_i \Big| (D) \iint_{\bar{R}_i} f(t,\tau)dt\,d\tau \Big| < C.$$

Put

$$\Psi(x,y) = \int_a^x \int_c^y G(t,\tau)dt\,d\tau.$$

Since G is summable, for any $\varepsilon > a$ there is a $\delta > 0$ such that $|e| < \delta$ implies

$$\Big| \iint_e G(t,\tau)dt\,d\tau \Big| < \varepsilon \tag{2}$$

for each measurable subset $e \subset R_0$.

If we consider any finite system of pairwise disjoint compact intervals of type 2 with respect to E, say, $\{I_k\}_{k=1}^m$, and if $\sum_{k=1}^m |I_k| < \delta$, then

$$\sum_{k=1}^m \Big| \iint_{I_k} G(t,\tau)dt\,d\tau \Big| < \varepsilon.$$

We shall now show that Ψ is AC on E. Assume $I_k = [\alpha_k, \beta_k; \gamma_k, \delta_k]$. Then

$$\sum_{k=1}^m |\Delta\Psi(I_k)| = \sum_{k=1}^m \Big| \iint_{I_k} G(t,\tau)dt\,d\tau \Big| < \varepsilon.$$

It is not difficult to check that $F(x,y) = \Psi(x,y)$ when $(x,y) \in E$. Since F is ACG on each R_i and $F \in AC(E)$, it follows that F is ACG on R_0.

Moreover,

$DF(x,y) = D\Psi(x,y) = G(x,y) = 0$ almost everywhere on E

and

$DF(x,y) = f(x,y)$ almost everywhere on $R_0 \setminus E = \bigcup_i R_i$.

Thus F is a D-primitive of φ_1 defined as follows:

$$\varphi_1(x,y) = \begin{cases} f(x,y) & \text{when} \quad (x,y) \in \bigcup_i R_i, \\ 0 & \text{when} \quad (x,y) \in E. \end{cases}$$

On the other hand, we consider

$$\varphi_2(x,y) = \begin{cases} 0 & \text{when} \quad (x,y) \in \bigcup_i R_i, \\ f(x,y) & \text{when} \quad (x,y) \in E. \end{cases}$$

Since $f \in D(E)$, we have $\varphi_2 \in D(R_0)$ and $f \in D(R_0)$, $f(x,y) = \varphi_1(x,y) + \varphi_2(x,y)$.

Finally, we obtain the equality

$$(D)\iint_R f(t,\tau)dt\,d\tau = (D)\iint_R \varphi_1(t,\tau)dt\,d\tau + (D)\iint_R \varphi_2(t,\tau)dt\,d\tau =$$
$$= (D)\iint_{R\cap E} \varphi_1(t,\tau)dt\,d\tau + (D)\iint_{R\cap(\bigcup_i \bar{R}_i)} \varphi_1(t,\tau)dt\,d\tau +$$
$$+(D)\iint_{R\cap E} \varphi_2(t,\tau)dt\,d\tau + (D)\iint_{R\cap(\bigcup_i \bar{R}_i)} \varphi_2(t,\tau)dt\,d\tau =$$
$$(D)\iint_{R\cap E} f(t,\tau)dt\,d\tau + (D)\iint_{R\cap(\bigcup_i \bar{R}_i)} f(t,\tau)dt\,d\tau =$$
$$(D)\iint_{R\cap E} f(t,\tau)dt\,d\tau + \sum_i (D)\iint_{R\cap \bar{R}_i} f(t,\tau)dt\,d\tau,$$

where R is a subsegment of R_0. ■

Lemma 2. *Let F_n be continuous on R_0, $n \in N$, and $\lim_{n\to\infty} F_n = F$, where F is also continuous. Then*

$$\lim_{n\to\infty} O[F_n; R_0] \geq O(F; R_0).$$

Proof. Since F is continuous, there exist points (x_1,y_1), (x_2,y_2) on R_0 such that $F(x_2,y_2) - F(x_1,y_1) = O(F; R_0)$. On the other hand, for any $\varepsilon > 0$ there is an $N_0 = N_0(\varepsilon) > 0$ such that

$$F_n(x_i,y_i) - \varepsilon \leq F(x_i,y_i) \leq F_n(x_i,y_i) + \varepsilon, \quad i = 1,2,$$

for all $n > N_0$.

It follows that

$$F(x_2,y_2) - F(x_1,y_1) \leq F_n(x_2,y_2) - F_n(x_1,y_1) + 2\varepsilon$$

and so

$$O(F;R_0) \leq O(F_n;R_0) + 2\varepsilon$$

from which the result is immediate. ■

Definition 3. Let $F_n : R_0 \to R$ be continuous, $n \in N$. Then $\{F_n;\ n \in N\}$ is said to be a sequence of uniformly A_x-functions on the set $P \subset [a,b]$ if $P = \bigcup_k P_k$ and F_n is uniformly AC on each $H_k = P_k \times [c,d]$ (see [9]), i.e., for all $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for any finite system of pairwise disjoint compact intervals of type 2 with respect to H_k, say, $\{R_i\}$, $i = \overline{1,m}$, the inequality

$$\sum_{i=1}^{m} |R_i| < \delta$$

implies

$$\sum_{i=1}^{m} |\Delta(F_n;R_i)| < \varepsilon$$

for all $n \in N$.

Lemma 3. *If $f_n \in D(R_0)$, $n \in N$, $E \subset [a,b]$ is a closed set and the sequence*

$$F_n(x,y) = \int_a^x \int_c^y f_n(t,\tau)dt\,d\tau, \quad n \in N,$$

is uniformly AC on $E' = E \times [\gamma,\delta]$, then for all $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for all measurable sets, e, $e \subset E$ and $|e| < \delta$, we have

$$\left| \iint_{e'} f_n(t,\tau)dt\,d\tau \right| < \varepsilon, \quad n = \overline{1,\infty}, \quad \textit{where } e' = e \times [\gamma,\delta].$$

Proof. By the conditions of the lemma, if $\varepsilon > 0$, then there is a $\delta > 0$ such that for any finite system $\{R_k\}$, $k = \overline{1,m}$, of disjoint compact intervals of type 2 with respect to E' the inequality

$$\sum_{k=1}^{m} |R_k| < \delta$$

implies

$$\sum_{k=1}^{m} |F_n(R_k)| = \sum_{k=1}^{m} \left| \iint_{R_k} f_n(t,\tau)dt\,d\tau \right| < \varepsilon \tag{3}$$

for all $n \in N$.

Let now e be a measurable subset of E with $|e| < \delta$. Then there are open sets G_m, $m \in N$, such that $G_m \supset G_{m+1} \supset e$, $m \in N$, and $\lim_{m\to\infty} |G_m| = |e|$. It can further be assumed that

$$G_m = \bigcup_{k=1}^{\infty} r_k^{(m)}, \quad r_k^{(m)} \cap r_e^{(m)} = \varnothing, \quad k \neq l,$$

for each m, where $r_k^{(m)}$ are open intervals. We set

$$e_m = E \cap G_m = \bigcup_k (E \cap r_k^{(m)}).$$

Denote by $\rho_{k,j}^{(m)}$, $j \in N$, contiguous intervals of $E \cap r_k^{(m)}$ on $r_k^{(m)}$. Then

$$G_m \setminus e_m = \bigcup_k \bigcup_j \rho_{k,j}^{(m)}.$$

The endpoints of each $\rho_{k,j}^{(m)}$ lie on E. Moreover, for $\delta = \delta(\varepsilon) > 0$ there is an $N_0(\delta) = N_0 > 0$ such that if $m > N_0$, then

$$|G_m| < \delta.$$

If $m > N_0$, from (3) we obtain

$$\begin{gathered}\sum_k \left| \iint_{r_k^{(m)} \times [\gamma,\delta]} f_n(t,\tau)\,dt\,d\tau \right| < \varepsilon, \\ \sum_k \sum_j \left| \iint_{\rho_{k,j}^{(m)} \times [\gamma,\delta]} f_n(t,\tau)\,dt\,d\tau \right| < \varepsilon \end{gathered} \tag{4}$$

for all $n \in N$.

Now, by Lemma 1, for $m > N_0$ we have

$$\iint_{r_k^{(m)} \times [\gamma,\delta]} f_n(t,\tau)\,dt\,d\tau =$$

$$= \iint_{(r_k^{(m)} \cap E) \times [\gamma,\delta]} f_n(t,\tau)\,dt\,d\tau + \sum_j \iint_{\rho_{k,j}^{(m)} \times [\gamma,\delta]} f_n(t,\tau)\,dt\,d\tau, \quad n = \overline{1,\infty}.$$

Performing the summation of the last expression over k, from (4) we obtain

$$\left| \iint_{e_m \times [\gamma,\delta]} f_n(t,\tau)\,dt\,d\tau \right| < 2\varepsilon, \quad n = \overline{1,\infty},$$

and so

$$\left| \iint_{e'} f_n(t,\tau)\,dt\,d\tau \right| < 2\varepsilon, \quad \overline{1,\infty}. \quad \blacksquare$$

Note that the following lemma of P. Romanovskii is very essential for our investigation.

Lemma 4. *Let $\mathfrak{F}$ be a nonempty system of open subintervals of the open interval $]a,b[$, having the following four properties:*

1) *If $]\alpha,\beta[\in\mathfrak{F}$ and $]\beta,\gamma[\in\mathfrak{F}$, then $]\alpha,\gamma[\in\mathfrak{F}$;*

2) *If $]\alpha,\beta[\in\mathfrak{F}$, then every subinterval of $]\alpha,\beta[$ also belongs to $\mathfrak{F}$;*

3) *If every proper open subinterval of $]\alpha,\beta[$ belongs to $\mathfrak{F}$, then $]\alpha,\beta[\in\mathfrak{F}$;*

4) *If all contiguous intervals on $[a,b]$ of a nonempty perfect set E from $]a,b[$ belong to $\mathfrak{F}$, then $\mathfrak{F}$ contains an interval $]\alpha,\beta[$ such that $]\alpha,\beta[\cap E\neq\varnothing$.*

Then $]a,b[\in\mathfrak{F}$.

Theorem. *If $f_n\in D(R_0)$, $\lim_{n\to\infty}f_n=f$ almost everywhere on R_0 and the sequence*

$$F_n(x,y)=\int_a^x\int_c^y f_n(t,\tau)dt\,d\tau,\quad n=\overline{1,\infty},$$

is uniformly continuous on R_0 and uniformly A_x on $[a,b]$, then $f\in D(R_0)$ and

$$\lim_{n\to\infty}\iint_{R_0}f_n(t,\tau)dt\,d\tau=\iint_{R_0}f(t,\tau)dt\,d\tau.$$

Proof. The subinterval $]\alpha,\beta[\subset]a,b[$ will be said to lie in the family of subintervals $\mathfrak{F}$ if for $]\alpha,\beta[$ and all all compact subintervals we have

$$\lim_{n\to\infty}\int_\alpha^\beta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau=\int_\alpha^\beta\int_\gamma^\delta f(t,\tau)dt\,d\tau,\qquad(*)$$

where $[\gamma,\delta]\subset[c,d]$.

The theorem will be proved if we show that $\mathfrak{F}$ satisfies the conditions of Romanovskii's lemma. The condition of the theorem implies that there exists a portion P such that $\{F_n\}_{n\geq1}$ is uniformly AC on $P\times[c,d]$. Then due to Vitali's theorem on passage of limits through integrals and by virtue of some properties of the Denjoy integral there exists an interval $[\alpha^*,\beta^*,\gamma^*,\delta^*]$ for which the equation (*) is fulfilled. Thus $\mathfrak{F}\neq\varnothing$.

Now assume that $]\alpha,\beta[\in\mathfrak{F}$ and $]\beta,\eta[\in\mathfrak{F}$. Then $]\alpha,\eta[\in\mathfrak{F}$. Indeed,

$$\lim_{n\to\infty}\int_\alpha^\eta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau =$$

$$=\lim_{n\to\infty}\left(\int_\alpha^\beta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau+\int_\beta^\eta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau\right)=$$

$$=\int_\alpha^\beta\int_\gamma^\delta f(t,\tau)dt\,d\tau+\int_\beta^\eta\int_\gamma^\delta f(t,\tau)dt\,d\tau=\int_\alpha^\eta\int_\gamma^\delta f(t,\tau)dt\,d\tau,$$

i.e., $]\alpha,\eta[\in\mathfrak{F}$.

Condition 2) follows from the definition of the family $\mathfrak{F}$.

Now let us consider condition 3). It is assumed that every open proper subinterval of $]\alpha,\beta[$ is contained in $\mathfrak{F}$. Then we must show that $]\alpha,\beta[\in\mathfrak{F}$.

Consider $]\lambda_m,\mu_m[\in\mathfrak{F}$, $]\lambda_m,\mu_m[\subset]\alpha,\beta[$, $\lambda_m\downarrow\alpha$, $\mu_m\uparrow\beta$, $m\to\infty$.

Since $\{F_n\}_{n\geq 1}$ is uniformly continuous on R_0, for all $\varepsilon>0$ there is an $N_0=N_0(\varepsilon)$ such that if $m>N_0$, then

$$\int_{\lambda_m}^{\mu_m}\int_\gamma^\delta f_n(t,\tau)dt\,d\tau-\varepsilon\leq\int_\alpha^\beta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau\leq$$

$$\leq\int_{\lambda_m}^{\mu_m}\int_\gamma^\delta f_n(t,\tau)dt\,d\tau+\varepsilon,$$

where $[\gamma,\delta]\subset[c,d]$ is arbitrary.

Since $]\lambda_m,\mu_m[\in\mathfrak{F}$, we have

$$\int_{\lambda_m}^{\mu_m}\int_\gamma^\delta f(t,\tau)dt\,d\tau-\varepsilon\leq\varliminf_{n\to\infty}\int_\alpha^\beta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau\leq$$

$$\leq\varlimsup_{n\to\infty}\int_\alpha^\beta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau\leq\int_{\lambda_m}^{\mu_m}\int_\gamma^\delta f(t,\tau)dt\,d\tau+\varepsilon. \tag{5}$$

Hence, $\varepsilon>0$ being arbitrary, (5) implies that

$$\lim_{n\to\infty}\int_{\lambda_m}^{\mu_m}\int_\gamma^\delta f(t,\tau)dt\,d\tau \tag{6}$$

exists and

$$\lim_{n\to\infty}\int_\alpha^\beta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau=\int_\alpha^\beta\int_\gamma^\delta f(t,\tau)dt\,d\tau.$$

Thus $]\alpha,\beta[\in\mathfrak{F}$ and condition 3) is satisfied.

Finally, we shall check condition 4). It is assumed that $E\subset[a,b]$ is a closed set. Since the sequence $\{F_n\}_{n\geq 1}$ is uniformly A_x on $[a,b]$, there exists a portion $P=E\cap[\alpha,\beta]$ such that $\{F_n\}_{n\geq 1}$ is uniformly AC on $P\times[c,d]$.

Let $\{I_k=]\alpha_k,\beta_k[,\ k=\overline{1,\infty}\}$ be the contiguous intervals of P on $]\alpha,\beta[$ which are the members of $\mathfrak{F}$. By Lemma 1, where $I_k\times[\gamma,\delta]$ are the contiguous intervals of $P\times[\gamma,\delta]$, $k=\overline{1,\infty}$, we have

$$\int_\alpha^\beta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau=\iint_{P\times[\gamma,\delta]}f_n(t,\tau)dt\,d\tau+$$

$$+\sum_k\iint_{I_k\times[\gamma,\delta]}f_n(t,\tau)dt\,d\tau. \tag{7}$$

By Lemma 3 and Vitali's theorem on passage of limits through integrals

$$\lim_{n\to\infty}\iint_{P'}f_n(t,\tau)dt\,d\tau=\iint_{P'}f(t,\tau)dt\,d\tau, \tag{8}$$

where $P'=P\times[\gamma,\delta]$.

Since $\{F_n\}_{n\geq 1}$ is uniformly AC on P' for all $\varepsilon>0$, there is an $N_0=N_0(\varepsilon)$ such that if $k>N_0$, then

$$\sum_{k>N_0}O(F_n;I_k\times[\gamma,\delta])<\varepsilon. \tag{9}$$

Since $I_k\in\mathfrak{F}$, (9) implies

$$\sum_{k>N_0}O(F;I_k\times[\gamma,\delta])<\varepsilon,\sum_{k=1}^{\infty}O(F;I_k\times[\gamma,\delta])<+\infty. \tag{10}$$

Thus $f\in D([\alpha,\beta;\gamma,\delta])$ and from (7), (8), (10) we obtain

$$\lim_{n\to\infty}\int_\alpha^\beta\int_\gamma^\delta f_n(t,\tau)dt\,d\tau=\int_\alpha^\beta\int_\gamma^\delta f(t,\tau)dt\,d\tau,$$

i.e., $]\alpha,\beta[\in\mathfrak{F}$ and $]\alpha,\beta[\cap E\neq\varnothing$. Therefore $\mathfrak{F}$ satisfies all conditions of Romanovskii's lemma and $]a,b[\in\mathfrak{F}$. ∎

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GENERALIZED SIERPINSKI SETS

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ABSTRACT. The notion of a Sierpinski topological space is introduced and some properties of such spaces connected with Borel measures are considered.

რეზიუმე. ნაშრომში შემოღებულია სერპინსკის ტოპოლოგიური სივრცის ცნება და განხილულია ამ სივრცეების ბორელის ზომებთან დაკავშირებული თვისებები.

We assume that all topological spaces E to be considered below possess the following property: any singleton in E is a Borel subset of E. In particular, all Hausdorff topological spaces possess this property.

We say that a topological space E is a Luzin space if each σ-finite continuous, i.e., difffused, measure defined on the Borel σ-algebra of E is identically zero.

We say that a topological space E is a Sierpinski space if E contains none of Luzin spaces with the cardinality equal to card(E).

The classical Luzin set on the real line R gives us a nontrivial example of an uncountable Luzin topological space (see, for example, [1]). Similarly, the classical Sierpinski set on R gives us a nontrivial example of an uncountable Sierpinski topological space (see [1]). Therefore Luzin topological spaces (accordingly, Sierpinski topological spaces) may be considered as generalized Luzin sets (accordingly, as generalized Sierpinski sets). Some properties of Luzin and Sierpinski topological spaces are investigated in [2] and [3]. In this paper we investigate some other properties of Sierpinski topological spaces.

It is obvious that the cardinality of any Luzin topological space is strictly majorized by the first measurable (in the broad sense) cardinal number. The following simple proposition gives a characterization of measurable (in the broad sense) cardinal numbers in the terms of Sierpinski topological spaces.

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Proposition 1. *Let E be the main base set. Then the next two relations are equivalent:*

1) card(E) *is a measurable (in the broad sense) cardinal number;*

2) *the topological space (E, T) is a Sierpinski space for every topology T on the set E.*

Proof. Indeed, if card(E) is a measurable (in the broad sense) cardinal number, then for each set $X \subset E$ with card(X) = card(E) there exists a probability continuous measure defined on the family of all subsets of X. Therefore for any topology T on the set E the space (E, T) contains none of Luzin spaces with the cardinality equal to card(E). Thus we see that in this situation the topological space (E, T) is a Sierpinski space. Conversely, let us assume that the topological space (E, T) is a Sierpinski space for every topology T on the set E. If we set

$$T = \text{a discrete topology on } E,$$

then we immediately find that card(E) is a measurable (in the broad sense) cardinal number.

It is not difficult to verify that if E is a Sierpinski topological space and X is a subspace of E with card(X) = card(E), then X too is a Sierpinski topological space. ■

Proposition 2. *Let E be a topological space and let*

$$E = \bigcup_{i \in I} E_i,$$

where $(E_i)_{i \in I}$ is a finite family of Sierpinski subspaces of E. Then the topological space E is a Sierpinski space. In particular, the topological sum of any finite family of Sierpinski spaces is also a Sierpinski space.

Proof. Let X be an arbitrary subspace of the space E with card(X) = card(E). Assume that X is a Luzin subspace of the space E. Since the set of indices I is finite, there exists an index $i \in I$ such that the equality

$$\text{card}(E_i \cap X) = \text{card}(E_i) = \text{card}(E)$$

is fulfilled.

Let us consider the set $E_i \cap X$. This set, being a subset of the Luzin topological space X, is also the Luzin space. At the same time, this set is a subspace of the topological space E_i. Therefore we see that the Siepinski topological space E_i contains the Luzin topological space $E_i \cap X$ with the same cardinality, which is impossible. The obtained contradiction proves the proposition. ■

We shall ascertain below that the result of Proposition 2, generally speaking, does not hold for topological sums of infinite families of Sierpinski spaces.

Let E be the main base set whose cardinality is not cofinal with the least infinite cardinal number $\omega = \omega_0$. We set

$$T(E) = \{X \subset E : \mathrm{card}(E \setminus X) < \mathrm{card}(E)\} \cup \{\varnothing\}.$$

It is not difficult to verify that $T(E)$ is a topology in E and the topological space $(E, T(E))$ is a Sierpinski space. It is further easy to ascertain that if $\mathrm{card}(E) = \omega_{\xi+1}$, then any subset of $(E, T(E))$, having the cardinality ω_ξ, is discrete. Hence it follows that if $\mathrm{card}(E) = \omega_{\xi+1}$ and the cardinal number ω_ξ is not measurable (in the broad sense), then any subset of the space $(E, T(E))$, having the cardinality ω_ξ, is a Luzin topological space.

Let us now consider a countable disjoint familiy of sets $(E_n)_{n\in\omega}$ where

$$(\forall n)(n \in \omega \Rightarrow \mathrm{card}(E_n) = \omega_{n+1}).$$

Provide each set E_n with the topology $T(E_n)$ defined above and denote by E the space which is the topological sum of the family of spaces $(E_n, T(E_n))_{n\in\omega}$. We assert that E is not a Sierpinski space. Indeed, each space $(E_n, T(E_n))$ contains a discrete subspace L_n with the cardinality equal to ω_n. By the well-known result of Ulam the cardinal number ω_n is not measurable (in the broad sense). Therefore L_n is a Luzin space. Thus the space E contains the topological sum L of the countable family $(L_n)_{n\in\omega}$ of Luzin spaces. Obviously, L itself is a Lusin space (we have the general fact by virtue of which the topological sum of the family of Luzin spaces is a Luzin space if the cardinality of the set of indices of this family is not measurable in the broad sense). Finally, it is absolutely clear that

$$\mathrm{card}(L) = \mathrm{card}(E) = \omega_\omega$$

and thus the topological space E is not a Sierpinski space.

Similar arguments are used to prove

Proposition 3. *Let us assume that all cardinal numbers are not measurable in the broad sense (this assumption does not contradict the standard axioms of the modern set theory). Then for any infinite cardinal number a there exists a family $(E_i)_{i\in I}$ of topological spaces such that*

1) $\mathrm{card}(I) = a$;

2) *each E_i is a Sierpinski topological space;*

3) *the topological sum of the family $(E_i)_{i\in I}$ is not a Sierpinski space.*

On the other hand, we would like to note here that by Proposition 1 if $(E_i)_{i \in I}$ is a family of Sierpinski spaces and $\mathrm{card}(I)$ is a measurable (in the broad sense) cardinal number, then the topological sum $(E_i)_{i \in I}$ too is a Sierpinski space.

Example 1. Let E be the main base set and Φ be a filter in E. Consider the topology T_Φ on E associated with the filter Φ. This topology is defined by the equality

$$T_\Phi = \{X : X \in \Phi\} \cup \{\varnothing\}.$$

It is clear that the topology $T(E)$ considered above is the particular case of the topology associated with the filter. Assume that for the filter Φ the following conditions hold:

1) for each element $x \in E$ the set $E \setminus \{x\}$ belongs to the filter Φ;

2) the filter Φ is countably complete (ω_1-complete), i.e., Φ is closed with respect to intersections of arbitrary countable families of its elements.

Now we can see that the topological space (E, T_Φ) is not a Luzin space, since in this situation there exists a two-valued probability continuous Borel measure on E. Assume also that the cardinality of E is not measurable in the broad sense. Then the said topological space (E, T_Φ) is a Sierpinski space if and only if the relation $X \cap Y \neq \varnothing$ is fulfilled for each set $X \subset E$ with $\mathrm{card}(X) = \mathrm{card}(E)$ and for each set $Y \in \Phi$.

Let now E be a topological space. Assume that E is not a Luzin space. Then there naturally arises the question: does the space E contain at least one Sierpinski subspace? It will be shown below that the answer to this question is negative even in the case when the cardinality of the space E is equal to the first uncountable cardinal number ω_1. For this we need one auxiliary assertion.

Lemma. *Let ω_α be an arbitrary uncountable regular initial order number provided with the standard order topology and X be an arbitrary unbounded subset of ω_α. Then there exists an unbounded nonstationary (in ω_α) set $Y \subset X$.*

Proof. The required set $Y \subset X$ can be readily constructed by the method of transfinite recursion. Indeed, using the regularity of the ordinal number ω_α, it is possible to define by transfinite recursion the family of sets $(Z_\xi)_{\xi < \omega_\alpha}$ satisfying the following conditions:

1) for any index $\xi < \omega_\alpha$ the set Z_ξ is an open bounded interval in the ordered set ω_α;

2) the intersection $Z_\xi \cap X$ is not empty for any index $\xi < \omega_\alpha$;

3) for any indices $\xi < \omega_\alpha$ and $\zeta < \omega_\alpha$ such that $\xi < \zeta$, the right end-point of the interval Z_ξ is strictly less that the left end-point of the interval Z_ζ.

Having constructed the family $(Z_\xi)_{\xi<\omega_\alpha}$ we choose for each ordinal number $\xi < \omega_\alpha$, an element y_ξ from the nonempty intersection $Z_\xi \cap X$ and set

$$Y = \bigcup_{\xi<\omega_\alpha} \{y_\xi\}.$$

Now it is not difficult to verify that the set Y is the required one, since it is unbounded and nonstationary in ω_α and is certainly entirely contained in the original set X. ■

Example 2. Let us consider the first uncountable ordinal number ω_1 provided with its order topology. Let Φ be the filter in the topological space ω_1 generated by the family of all unbounded closed subsets of this space. Further, let T_Φ be the topology in the set ω_1 associated with the filter Φ. Then the topological space (ω_1, T_Φ) is not a Luzin space. Indeed, if μ is the standard two-valued probability continuous Borel measure in the space ω_1, then the domain of definition of the measure $\bar{\mu}$ (of the usual completion of μ) coincides with the Borel σ-algebra of the topological space (ω_1, T_Φ). Thus we see that there exist nontrivial σ-finite Borel measures on the space (ω_1, T_Φ). Let us ascertain that at the same time the space (ω_1, T_Φ) does not contain any Sierpinski subspace. Let X be an arbitrary subspace of the space (ω_1, T_Φ). Without loss of generality it can be assumed that the cardinality of X is equal to ω_1. Then, according to the above lemma, there exists a nonstationary (in ω_1) set $Y \subset X$ with the cardinality also equal to ω_1. Note now that the subspace Y of the space X is discrete. Taking into consideration the fact that by the result of Ulam the cardinal number ω_1 is not measurable in the broad sense, we see that Y is a Luzin subspace of the space X. Finally, we ascertain that the topological space X is not a Sierpinski space. Therefore our topological space (ω_1, T_Φ) does not contain Seirpinski subspaces.

A similar example can evidently be constructed for an arbitrary uncountable regular nonmeasurable (in the broad sense) cardinal number ω_α.

For our further consideration we need two simple notions from the general set theory.

Let E be a base set and D be a family of subsets of E covering E. We put

$$\text{cov}(D) = \inf\{\text{card}(D') : D' \subset D \text{ and } D' \text{ covers } E\}.$$

It is easy to see that the cardinal number $\mathrm{cov}(D)$ is an invariant with respect to bijective mappings. Therefore $\mathrm{cov}(D)$ can be regarded as some cardinal-valued characteristic of the given family D.

Let E be a base set again and D be an arbitrary family of subsets of E. We put

$$\mathrm{cof}(D) = \inf\{\mathrm{card}(D') : D' \subset D \text{ and for any set } X \in D \text{ there exists a set } Z \in D' \text{ such that } X \subset Z\}.$$

The cardinal number $\mathrm{cof}(D)$ is also an invariant with respect to bijective mappings and therefore can be regarded as another cardinal-valued characteristic of the given family D. If D is an ideal of subsets of E and D' is a subfamily of D such that

$$(\forall X)(X \in D \Rightarrow (\exists Z)(Z \in D' \& X \subset Z)),$$

then we say that D' is a base of the ideal D. Clearly, this notion is dual to the well-known notion of a filter base.

We have the following proposition which gives us some sufficient conditions for the existence, in the topological space E, of a Sierpinski subspace with the cardinality equal to $\mathrm{card}(E)$.

Proposition 4. *Let E be a topological space, μ be a nonzero σ-finite continuous Borel measure in E and $D(\mu)$ be the countably additive ideal of subsets of E generated by all μ-measure-zero sets. It is also assumed that the next two conditions are fulfilled for the ideal $D(\mu)$:*

1) *if* $\mathrm{card}(I) < \mathrm{card}(E)$, *then the union of any family $(X_i)_{i\in I} \subset D(\mu)$ does not coincide with the space E (in other words, the equality* $\mathrm{cov}(D(\mu)) = \mathrm{card}(E)$ *is true);*

2) *there exists a base of the ideal $D(\mu)$ with the cardinality less than or equal to the cardinality of E (in other words, the inequality* $\mathrm{cof}(D(\mu)) \leq \mathrm{card}(E)$ *is true).*

Then the topological space E contains some Sierpinski subspace S with $\mathrm{card}(S) = \mathrm{card}(E)$.

Proof. Let ω_α be the initial ordinal number corresponding to the cardinality of the given topological space E. Fix any base of the ideal $D(\mu)$ with the cardinality less than or equal to ω_α. It is obvious that we can represent this base as a familiy of sets $(Z_\xi)_{\xi<\omega_\alpha}$. Now, using the method of transfinite recursion, let us define the ω_α-sequence $(s_\xi)_{\xi<\omega_\alpha}$ of elements of the given space E. Assume that for an ordinal number $\xi < \omega_\alpha$ we have already defined the partial ξ-sequence $(s_\zeta)_{\zeta<\xi}$

of elements of E. Consider the set

$$\Big(\bigcup_{\zeta<\xi} Z_\zeta\Big) \cup \Big(\bigcup_{\zeta<\xi} \{s_\zeta\}\Big).$$

Since the condition 1) holds, this set does not coincide with the space E. Therefore there exists an element $s_\xi \in E$ which does not belong to the mentioned set. In this manner we shall construct the ω_α-sequence $(s_\xi)_{\xi<\omega_\alpha}$ and, having done so, set

$$S = \bigcup_{\xi<\omega_\alpha} \{s_\xi\}.$$

It remains for us to verify that the set S is a Sierpinski subspace of the space E. Indeed, let X be an arbitrary subset of S such that

$$\mathrm{card}(X) = \mathrm{card}(S) = \mathrm{card}(E) = \omega_\alpha.$$

Then the procedure of the construction of the set S immediately implies that the set X is not contained in any of the sets Z_ξ $(\xi < \omega_\alpha)$. Thus we have the inequality $\mu^*(X) > 0$ where μ^* denotes an outer measure associated with the given measure μ. The latter fact leads to a conclusion that the topological space X is not a Luzin subspace. Therefore the constructed topological space S is the Sierpinski space. ■

One may easily note here that if $\mathrm{card}(E) = \omega_1$, then the condition 1) in the formulation of Proposition 4 becomes superfluous. The next simple example, on the other hand, shows that if $\mathrm{card}(E) = \omega_2$, then this condition plays an essential role.

Example 3. Assume that the relation

$$2^{\omega_1} = \omega_2$$

is fulfilled. Take two disjoint sets E_1 and E_2 such that

$$\mathrm{card}(E_1) = \omega_1, \quad \mathrm{card}(E_2) = \omega_2.$$

Identify the set E_1 with the ordinal number ω_1 and equip E_1 with the topology T_Φ (see Example 2). Further, equip the set E_2 with a discrete topology. Denote by E the topological sum of two spaces E_1 and E_2. Now it is not difficult to verify that the topological space E satisfies the following conditions:

a) $\mathrm{card}(E) = \omega_2$;

b) there exists a probability continuous Borel measure μ in the space E such that $\mathrm{cof}(D(\mu)) \le \omega_2$;

c) the space E does not contain any Sierpinski subspace.

Example 4. Let ω_α be an arbitrary uncountable regular initial ordinal number not measurable in the broad sense. Equip ω_α with its order topology and denote by Φ the filter in ω_α generated by the family of all closed unbounded subsets of ω_α. Further, set $E = \omega_\alpha$ and equip E with the topology T_Φ. Then the topological space E satisfies the following conditions:

a) $\mathrm{card}(E) = \omega_\alpha$;

b) there exists a probability continuous Borel measure μ in the space E such that $\mathrm{cov}(D(\mu)) = \omega_\alpha$;

c) the space E does not contain any Sierpinski subspace.

Thus Example 4 shows us that the condition 2) in the formulation of Proposition 4 also plays an essential role and cannot be omitted.

Example 5. Let E be a metric space with the cardinality continuum 2^ω. It is clear that if the cardinality continuum is measurable in the broad sense, then any subspace X of E with $\mathrm{card}(X) = \mathrm{card}(E)$ is a Sierpinski topological space (in particular, E is not a Luzin topological space). Now let us consider some situation when the cardinality continuum is not measurable in the broad sense. Assume that Martin's axiom holds and that the given metric space E is not a Luzin space. Then one can establish that there always exists a subset S of E satisfying the next two conditions:

a) $\mathrm{card}(S) = \mathrm{card}(E) = 2^\omega$;

b) S is a Sierpinski subset of the space E.

The existence of such a space $S \subset E$ is proved in [3]. Note that the proof is essentially based on the following well-known fact: every σ-finite Borel measure defined in the space E is concentrated on some separable subspace of E (in other words, for every σ-finite Borel measure given in the space E there exists a separable support in E).

Example 6. Let E be again a metric space with the cardinality continuum 2^ω. Assume that the continuum hypothesis holds and that E is not a first category space. Then there always exists a subset L of E satisfying the next two conditions:

a) $\mathrm{card}(L) = \mathrm{card}(E) = 2^\omega$;

b) L is a Luzin subspace of the space E.

The existence of such a set $L \subset E$ is also proved in [3]. To establish this result we need to consider two cases. We begin by assuming that the space E is not separable. Then there exists a discrete set $L \subset E$ such that

$$\mathrm{card}(L) = \omega_1 = 2^{\omega}.$$

As we know, the least uncountable cardinal number ω_1 is not measurable in the broad sense. Therefore the subspace L of E is a Luzin topological space with the cardinality continuum. Now assume E to be a separable metric space. In that case every σ-finite continuous Borel measure defined in the space E is concentrated on a first category subset of E. Using the classical transfinite construction due to Luzin (see, for instance, [1]), we can define a set $L \subset E$ with the cardinality continuum such that we shall have the inequality

$$\mathrm{card}(L \cap X) \leq \omega$$

for each first category set $X \subset E$.

Now it is not difficult to verify that such a set L is a Luzin subspace of our space E.

From the foregoing arguments it also follows that any nonseparable metric space E contains an uncountable Luzin subspace.

Finally, let us formulate one unsolved problem concerning Sierpinski topological spaces.

Problem. Give a characterization of Sierpinski spaces in purely topological terms.

Remark. In the literature the term "Luzin topological space" is used in other senses as well. For example, Luzin topological spaces in the sense of N. Bourbaki coincide with Borel subsets of complete separable metric spaces while in the modern set-theoretical topology Luzin topological spaces are uncountable Hausdorff spaces which contain neither isolated points nor uncountable first category subsets.

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ON SOME PROPERTIES OF SOLUTIONS OF SECOND ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The properties of solutions of the equation $u''(t) = p_1(t)u(\tau_1(t)) + p_2(t)u'(\tau_2(t))$ are investigated where $p_i : [a, +\infty[\to R$ $(i = 1, 2)$ are locally summable functions, $\tau_1 : [a, +\infty[\to R$ is a measurable function and $\tau_2 : [a, +\infty[\to R$ is a nondecreasing locally absolutely continuous one. Moreover, $\tau_i(t) \geq t$ $(i = 1, 2)$, $p_1(t) \geq 0$, $p_2^2(t) \leq (4 - \varepsilon)\tau_2'(t)p_1(t)$, $\varepsilon = const > 0$ and $\int_a^{+\infty}(\tau_1(t) - t)p_1(t)dt < +\infty$. In particular, it is proved that solutions whose derivatives are square integrable on $[a, +\infty[$ form a one-dimensional linear space and for any such solution to vanish at infinity it is necessary and sufficient that $\int_a^{+\infty} tp_1(t)dt = +\infty$.

რეზიუმე. გამოკვლეულია $u''(t) = p_1(t)u(\tau_1(t)) + p_2(t)u'(\tau_2(t))$ განტოლების ამონახსნების თვისებები, სადაც $p_i : [a, +\infty[\to R$ $(i = 1, 2)$ ლოკალურად ჯამებადი, $\tau_1 : [a, +\infty[\to R$ ზომადი, ხოლო $\tau_2 : [a, +\infty[\to R$ ლოკალურად აბსოლუტურად უწყვეტი და არაკლებადი ფუნქციებია, ამასთან $\tau_i(t) \geq t$ $(i = 1, 2)$, $p_1(t) \geq 0$, $p_2^2(t) \leq (4 - \varepsilon)\tau_2'(t)p_1(t)$, $\varepsilon = const > 0$ და $\int_a^{+\infty}(\tau_1(t) - t)p_1(t)dt < +\infty$. კერძოდ ნაჩვენებია, რომ ამონახსნები, რომელთა წარმოებულები კვადრატით ჯამებადია $[a, +\infty[$ შუალედში, ქმნიან ერთგანზომილებიან წრფივ სივრცეს, ამასთან ყოველი ასეთი ამონახსნის უსასრულობაში ქრობადობისათვის აუცილებელი და საკმარისია, რომ $\int_a^{+\infty} tp_1(t)dt = +\infty$.

Consider the differential equation

$$u''(t) = p_1(t)u(\tau_1(t)) + p_2(t)u'(\tau_2(t)), \tag{1}$$

where $p_i : [a, +\infty[\to R$ $(i = 1, 2)$ are locally summable functions, $\tau_i : [a, +\infty[\to R$ $(i = 1, 2)$ are measurable functions and

$$\tau_i(t) \geq t \quad \text{for} \quad t \geq a \quad (i = 1, 2). \tag{2}$$

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We say that a solution u of the equation (1) is a *Kneser-type* solution if it satisfies the inequality

$$u'(t)u(t) \le 0 \quad \text{for} \quad t \ge a_0$$

for some $a_0 \in [a, +\infty[$. A set of such solutions is denoted by K. By W we denote a space of solutions of (1) that satisfy

$$\int_a^{+\infty} u'^2(t)dt < +\infty.$$

The results of [1, 2] imply that if

$$p_1(t) \ge 0 \quad \text{for} \quad t \ge a$$

and the condition

$$(i) \quad \tau_i(t) \equiv t, \quad (i = 1, 2), \quad \int_a^{+\infty} |p_2(t)|dt < +\infty,$$

or

$$(ii) \quad p_2(t) \le 0, \quad \text{for} \quad t \ge 0 \quad \int_a^{+\infty} sp_1(s)ds < +\infty,$$

$$\int_a^{+\infty} \frac{s}{\tau_2(s)}|p_2(s)|ds < +\infty,$$

holds, then $W \supset K$ and K is a one-dimensional linear space. The case when the conditions (i) and (ii) are violated, the matter of dimension of K and W and their interconnection has actually remained unstudied. An attempt is made in this note to fill up this gap to a certain extent.

Theorem 1. *Let* $\tau_i(t) \ge t$ $(i = 1, 2)$, $p_1(t) \ge 0$ *for* $t \ge a$,

$$\int_a^{+\infty} \Big[\tau_1(t) - t\Big]p_1(t)dt < +\infty, \tag{3}$$

and let τ_2 *be a nondecreasing locally absolutely continuous function satisfying*

$$p_2^2(t) \le (4 - \varepsilon)\tau_2'(t)p_1(t) \quad \textit{for} \quad t \ge a\,, \tag{4}$$

where $\varepsilon = const > 0$. *Then*

$$W \subset K, \quad \dim W = 1. \tag{5}$$

Before proceeding to the proof of the theorem we shall give two auxiliary statements.

Lemma 1. *Let the conditions of Theorem* 1 *be fulfilled and let* $a_0 \in [a, +\infty[$ *be large enough for the equality*

$$\int_{a_0}^{+\infty} \Big[\tau_1(s) - s\Big] p_1(s) ds \le 4\delta^2, \tag{6}$$

where $\delta = \frac{1}{4}[2-(4-\varepsilon)^{1/2}]$, *to hold. Then any solution* u *of the equation* (1) *satisfies*

$$\delta \int_t^x \Big[u'^2(s) + p_1(s) u^2(s)\Big] ds \le u'(x)u(x) - u'(t)u(t) +$$
$$+(1-\delta) \int_x^{\tau(x)} u'^2(s) ds \quad \textit{for} \quad a_0 \le t \le x < +\infty\,, \tag{7}$$

where $\tau(x) = \operatorname{ess\,sup}_{a_0 \le t \le x}[\max_{1\le i\le 2} \tau_i(x)]$. *Moreover, if* $u \in W$, *then*

$$u'(t)u(t) \le -\delta \int_t^{+\infty} \Big[u'^2(s) + p_1(s) u^2(s)\Big] ds \quad \textit{for} \quad t \ge a_0 \tag{8}$$

and

$$2\delta \int_t^{+\infty} (s-t)\Big[u'^2(s) + p_1(s)u^2(s)\Big] ds \le u^2(t) \quad \textit{for} \quad t \ge a_0. \tag{9}$$

Proof. Let u be any solution of the equation (1). Then

$$-u''(t)u(t) + p_1(t)u^2(t) = p_1(t)u(t) \int_{\tau_1(t)}^t u'(s)ds -$$
$$-p_2(t)u'(\tau_2(t))u(t).$$

Integrating this equality from t to x, we obtain

$$u'(t)u(t) - u'(x)u(x) + \int_t^x \Big[u'^2(s) + p_1(s)u^2(s)\Big] ds =$$
$$= \int_t^x \Big[p_1(s)u(s) \int_{\tau_1(s)}^s u'(y)dy\Big] ds - \int_t^x p_2(s)u'(\tau_2(s))u(s)ds.$$

However, in view of (4) and (6),

$$\int_t^x \Big[p_1(s)u(s)\int_{\tau_1(s)}^s u'(y)dy\Big]ds \le \delta\int_t^x p_1(s)u^2(s)ds +$$

$$+\frac{1}{4\delta}\Big[\int_t^x \Big[\tau_1(s)-s\Big]p_1(s)ds\Big]\Big[\int_t^{\tau(x)} u'^2(s)ds\Big] \le$$

$$\le \delta\int_t^x p_1(s)u^2(s)ds + \delta\int_t^{\tau(x)} u'^2(s)ds \quad \text{for} \quad a_0 \le t \le x < +\infty$$

and

$$-\int_t^x p_2(s)u'(\tau_2(s))u(s)ds \le$$

$$\le 2(1-2\delta)\int_t^x \Big[p_1(s)u^2(s)\Big]^{1/2}\Big[\tau_2'(s)u'^2(\tau_2(s))\Big]^{1/2}ds \le$$

$$\le (1-2\delta)\int_t^x p_1(s)u^2(s)ds + (1-2\delta)\int_t^x \tau_2'(s)u'^2(\tau_2(s))ds \le$$

$$\le (1-2\delta)\int_t^x p_1(s)u^2(s)ds + (1-2\delta)\int_t^{\tau(x)} u'^2(s)ds$$

$$\text{for} \quad a_0 \le t \le x < +\infty.$$

Therefore

$$u'(t)u(t) - u'(x)u(x) + \int_t^x \big[u'^2(s) + p_1(s)u^2(s)\big]ds \le$$

$$\le (1-\delta)\int_t^x \Big[u'^2(s) + p_1(s)u^2(s)\Big]ds +$$

$$+(1-\delta)\int_x^{\tau(x)} u'^2(s)ds \quad \text{for} \quad a_0 \le t \le x < +\infty$$

and thus the inequality (7) holds.

Suppose now that $u \in W$. Then, as one can easily verify,

$$\liminf_{x\to+\infty} |u'(x)u(x)| = 0.$$

So (7) immediately implies (8). Integrating both sides of (8) from t to $+\infty$, we obtain the estimate (9). ■

Lemma 2. *Let the conditions of Lemma* 1 *be fulfilled and there exist* $b \in]a_0, +\infty[$ *such that*

$$p_i(t) = 0 \quad \textit{for} \quad t \geq b \ \ (i = 1, 2). \tag{10}$$

Then for any $c \in R$ *there exists a unique solution of the equation* (1) *satisfying*

$$u(a_0) = c, \quad u'(t) = 0 \quad \textit{for} \quad t \geq b. \tag{11}$$

Proof. In view of (2) and (10), for any $\alpha \in R$ the equation (1) has a unique solution $v(\cdot; \alpha)$ satisfying

$$v(t; \alpha) = \alpha \quad \text{for} \quad b \leq t < +\infty.$$

Moreover,

$$v(t; \alpha) = \alpha v(t; 1).$$

On the other hand, by Lemma 1 the function $v(\cdot; 1) : [a_0, +\infty[\to R$ is non increasing and $v(a_0; 1) \geq 1$. Therefore the function $u(\cdot) = \frac{c}{v(a_0;1)} v(a_0; \cdot)$ is a unique solution of (1), (11). ■

Proof of Theorem 1. First of all we shall prove that for any $c \in R$ the equation (1) has at least one solution satisfying

$$u(a_0) = c, \quad \int_{a_0}^{+\infty} u'^2(s)ds < +\infty. \tag{12}$$

For any natural k put

$$p_{ik}(t) = \begin{cases} p_i(t) & \text{for} \quad a_0 \leq t \leq a_0 + k \\ 0 & \text{for} \quad t > a_0 + k \end{cases} \quad (i = 1, 2). \tag{13}$$

According to Lemma 2, for any k the equation

$$u''(t) = p_{1k}(t)u(\tau_1(t)) + p_{2k}(t)u'(\tau_2(t))$$

has a unique solution u_k satisfying

$$u_k(a_0) = c, \quad u_k'(t) = 0 \quad \text{for} \quad t \geq a + k. \tag{14}$$

On the other hand, by Lemma 1

$$|u_k(t)| \leq |c| \quad \text{for} \quad t \geq a_0, \quad 2\delta \int_{a_0}^{+\infty} (s - a_0)u_k'^2(s)ds \leq c^2. \tag{15}$$

Taking (2) and (13)–(15) into account, it is easy to show that the sequences $\left(u_k\right)_{k=1}^{+\infty}$ and $\left(u'_k\right)_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous on each closed subinterval of $[a_0,+\infty[$. Therefore, by the Arzela-Ascoli lemma, we can choose a subsequence $\left(u_{k_m}\right)_{m=1}^{+\infty}$ out of $\left(u_k\right)_{k=1}^{+\infty}$, which is uniformly convergent alongside with $\left(u'_{k_m}\right)_{m=1}^{+\infty}$ on each closed subinterval of $[a,+\infty[$. By (13)–(15) the function $u(t) = \lim_{m\to+\infty} u_{k_m}(t)$ for $t \geq a$ is a solution of the problem (1), (12).

We have thus proved that $\dim W \geq 1$. On the other hand, by Lemma 1 any solution $u \in W$ satisfies (8) and is therefore a Kneser-type solution. To complete the proof it remains only to show that $\dim W \leq 1$, i.e., that the problem (1), (12) has at most one solution for any $c \in R$. Let u_1 and u_2 be two artbitrary solutions of this problem and

$$u_0(t) = u_2(t) - u_1(t).$$

Since $u_0 \in W$ and $u_0(a_0) = 0$, by Lemma 1

$$2\int_{a_0}^{+\infty}(s-a_0)u_0'^2(s)ds = 0 \quad \text{and} \quad u_0(t) = 0 \quad \text{for} \quad t \geq a_0,$$

i.e., $u_1(t) \equiv u_2(t)$. ■

Remark 1. The condition (4) of Theorem 1 cannot be replaced by the condition

$$p_2^2(t) \leq (4+\varepsilon)\tau_2'(t)p_1(t) \quad \text{for} \quad t \geq a. \tag{16}$$

Indeed, consider the equation

$$u''(t) = \frac{1}{(4+\varepsilon)t^2}u(t) - \frac{1}{t}u'(t), \tag{17}$$

satisfying all conditions of Theorem 1 except (4), instead of which the condition (16) is fulfilled. On the other hand, the equation (17) has the solutions

$$u_i(t) = t^{\lambda_i} \quad (i = 1,2),$$

where

$$\lambda_i = (-1)^i(4+\varepsilon)^{-\frac{1}{2}} \quad (i = 1,2).$$

Clearly, $u_i \in W$ $(i = 1,2)$. Therefore in our case instead of (5) we have

$$K \subset W, \quad \dim W = 2.$$

Corollary 1. *Let the conditions of Theorem 1 be fulfilled. Let, moreover,*

$$p_2(t) \le 0 \quad \textit{for} \quad t \ge a. \tag{18}$$

Then

$$K = W, \quad \dim K = 1. \tag{19}$$

Proof. Let $u \in K$. Then by virtue of (18) and the non-negativity of p_1 there exists $t_0 \in [a, +\infty[$ such that

$$u(t)u'(t) \le 0, \quad u''(t)u(t) \ge 0 \quad \text{for} \quad t \ge t_0.$$

Hence

$$\int_{t_0}^{+\infty} u'^2(s)ds \le |u(t_0)u'(t_0)|.$$

Therefore $u \in W$. Thus we have proved that $W \supset K$. This fact, together with (5), implies (19). ■

A solution u of the equation (1) will be called *vanishing at infinity* if

$$\lim_{t \to +\infty} u(t) = 0. \tag{20}$$

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. Then for any solution $u \in W$ to vanish at infinity it is necessary and sufficient that*

$$\int_a^{+\infty} s p_1(s)ds = +\infty. \tag{21}$$

Proof. Let $u \in W$. Then by Lemma 1 $u^2(t) \ge \eta$ for $t \ge a_0$, where $\eta = \lim_{t\to+\infty} u^2(t)$, and $\int_{a_0}^{+\infty}(s - a_0)p_1(s)u^2(s)ds \le u^2(a_0)/2\delta$. Hence it follows that (21) implies $\eta = 0$, i.e., u is a vanishing solution at infinity.

To complete the proof it is enough to establish that if

$$\int_a^{+\infty} s p_1(s)ds < +\infty, \tag{22}$$

then any nontrivial solution $u \in W$ tends to a nonzero limit as $t \to +\infty$. Let us assume the contrary: the equation (1) has a nontrivial solution $u \in W$ vanishing at infinity. Then by Lemma 1

$$u(t)u'(t) \le 0, \quad \rho(t) \le \eta^2 u^2(t) \quad \text{for} \quad t \ge a_0, \tag{23}$$

where

$$\rho(t) = \int_t^{+\infty} (s-t)\Big[u'^2(s) + p_1(s)u^2(s)\Big]ds, \quad \eta = (2\delta)^{-\frac{1}{2}}.$$

On the other hand, by (4), (20) and (22) we have

$$|u(t)| = \left| \int_t^{+\infty} (s-t)\Big[p_1(s)u(\tau_1(s)) + p_2(s)u'(\tau_2(s))\Big]ds \right| \le$$

$$\le \left[\int_t^{+\infty} (s-t)p_1(s)ds\right]^{1/2} \left[\int_t^{+\infty} (s-t)p_1(s)u^2(\tau_1(s))ds\right]^{1/2} +$$

$$+2\int_t^{+\infty} (s-t)[p_1(s)]^{1/2}[\tau_2'(s)]^{1/2}|u'(\tau_2(s)|ds \le$$

$$\le \left[\int_t^{+\infty} (s-t)p_1(s)\right]^{1/2} \left[\int_t^{+\infty} (s-t)p_1(s)u^2(\tau_1(s))ds\right]^{1/2} +$$

$$+2\left[\int_t^{+\infty} (s-t)p_1(s)ds\right]^{1/2} \left[\int_t^{+\infty} (s-t)\tau_2'(s)u'^2(\tau_2(s))ds\right]^{1/2}$$

$$\text{for} \quad t \ge a_0.$$

Hence by (2) and (23) we find

$$|u(t)| \le \left[\int_t^{+\infty} (s-t)p_1(s)ds\right]^{1/2} \left[\int_t^{+\infty} (s-t)p_1(s)u^2(s)ds\right]^{1/2} +$$

$$+2\left[\int_t^{+\infty} (s-t)p_1(s)ds\right]^{1/2} \left[\int_t^{+\infty} (s-t)u'^2(s)ds\right]^{1/2} \le$$

$$\le 3\eta\left[\int_t^{+\infty} (s-t)p_1(s)ds\right]^{1/2} |u(t)| \quad \text{for} \quad t \ge a_0$$

and therefore $u(t) = 0$ for $t \ge a_1$, where a_1 is a sufficiently large number. By virtue of (2) the last equality implies

$$u(t) = 0 \quad \text{for} \quad t \ge a.$$

But this is impossible, since by our assumption u is a nontrivial solution. The obtained contradiction proves the theorem. ■

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TWO-WEIGHTED ESTIMATES FOR SOME INTEGRAL TRANSFORMS IN THE LEBESGUE SPACES WITH MIXED NORM AND IMBEDDING THEOREMS

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ABSTRACT. Two-weighted inequalities are proved for anisotropic potentials. These estimates are used to obtain the refinements of the well-known imbedding theorems in the scale of weighted Lebesgue spaces.

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Two-weighted inequalities are obtained for anisotropic potentials in Lebesgue spaces with mixed norm. These estimates are used to prove imbedding theorems for different metrics and different dimensions for weighted spaces of anisotropic Bessel potentials.

Nonweighted cases were previously treated in [1–3]. One-weighted estimates for isotropic Bessel potentials can be found in [4].

1. A measurable almost everywhere positive function $\varrho : \mathbb{R}^1 \to \mathbb{R}^1$ will be called a weight function. Let $w = (w_1, w_2, \dots, w_n)$ be a vector-function where w_i $(i = 1, 2, \dots, n)$ is a weight function. By definition a measurable function $f(x) = f(x_1, x_2, \dots, x_n)$ given on the n-dimensional space $\mathbb{R}^n$ belongs to L^p_w, $p=(p_1, p_2, \dots, p_n)$, $1 < p_i < \infty$ $(1 = 1, 2, \dots, n)$, if the norm

$$\|f\|_{L^p_w(\mathbb{R}^n)} =$$

$$=\left(\int\limits_{-\infty}^{\infty} w_1^{p_1}(x_1)dx_1\left(\int\limits_{-\infty}^{\infty} w_2^{p_2}(x_2)dx_2\dots\left(\int\limits_{-\infty}^{\infty} f(x)w_n^{p_n}(x_n)dx_n\right)^{\frac{p_{n-1}}{p_n}}\right)\dots\right)^{\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}}$$

is finite.

We shall introduce a class of pairs of weight functions.

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For a given number r, $1 < r < \infty$, we write $r' = \frac{r}{r-1}$.

Definition 1. A pair of weight functions (ϱ, σ) given on $\mathbb{R}^1$ will be said to belong to the class $G_\beta^{s,r}$, $0 < \beta < 1$, $1 < r < s < \infty$, if the conditions

$$\sup \Big(\int_I \varrho^s(t)dt\Big)^{\frac{1}{s}} \Big(\int_{\mathbb{R}^1} \frac{\sigma^{-r'}(t)}{(|I| + |t_I - t|)^{\beta r'}} dt\Big)^{\frac{1}{r'}} < \infty, \qquad (1.1)$$

$$\sup \Big(\int_I \sigma^{-r'}(t)dt\Big)^{\frac{1}{r'}} \Big(\int_{\mathbb{R}^1} \frac{\varrho^s(t)}{(|I| + |t_I - t|)^{s\beta}} dt\Big)^{\frac{1}{s}} < \infty, \qquad (1.2)$$

are fulfilled, where the supremum is taken over all bounded one-dimensional intervals I, with centre and length, t_I and $|I|$ respectively.

In the sequel we shall proceed from

Theorem A [5–7]. *The fractional integral*

$$I_\gamma(f)(x) = \int_{-\infty}^{\infty} \frac{f(\tau)}{|t-\tau|^{1-\gamma}} d\tau, \quad 0 < \gamma < 1,$$

generates a continuous operator from $L_\sigma^r(\mathbb{R}^1)$ into $L_\varrho^s(\mathbb{R}^1)$ if and only if $(\varrho, \sigma) \in G_{1-\gamma}^{s,r}$.

Let numbers $a_j > 0$ $(j = 1, 2, \dots, n)$ be given. For $x=(x_1, x_2, \dots, x_n)$ we set

$$|x|_a = \Big(\sum_{i=1}^{n} |x|^{\frac{2}{a_j}}\Big)^{\frac{1}{2}}.$$

It is obvious that for $a_j = 1$ $(j = 1, 2, \dots, n)$ we obtain an usual Euclidian distance.

Theorem 1. *Let $w = (w_1, w_2, \dots, w_n)$, $v = (v_1, v_2, \dots, v_n)$, where w_i and v_i $(i = 1, 2, \dots, n)$ are weight functions given on $\mathbb{R}^n$. We set*

$$\mathcal{K}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|_a^\mu} dy,$$

where

$$\mu = \sum_{j=1}^{n} a_j(1-\gamma_j), \quad 0 < \gamma_j < 1 \ (j = 1, 2, \dots, n).$$

If $1 < p_i < q_i < \infty$, $(v_i, w_i) \in G_{1-\gamma_i}^{q_i, p_i}$ $(i = 1, 2, \dots, n)$, then there exists a positive number c such that the inequality

$$\|\mathcal{K}f\|_{L_v^q} \le c\|f\|_{L_w^p}$$

holds for any $f \in L^p_w$.

The proof of Theorem 1 will be based on several lemmas. The first lemma is a weighted analogue of the well-known Hardy–Littlewood inequality (see [8], Theorem 382).

Lemma 1. *Let* $(\varrho, \sigma) \in G^{s,r}_{1-\gamma}$, $1 < r < s < \infty$, $0 < \gamma < 1$. *Then there exists a constant* $c > 0$, *such that the inequality*

$$\left|\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\varphi(x)\psi(y)}{|x-y|^{1-\gamma}}\,dx\,dy\right| \le c\|\varphi\|_{L^r_\sigma}\|\psi\|_{L^{s'}_{1/\varrho}} \tag{1.3}$$

holds for any arbitrary $\varphi \in L^r_\sigma(\mathbb{R}^1)$ *and* $\psi \in L^{s'}_{1/\varrho}(\mathbb{R}^1)$.

Proof. By virtue of the Hölder inequality we have

$$\left|\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\varphi(x)\psi(y)}{|x-y|^{1-\gamma}}\,dx\,dy\right| \le$$

$$\le \|\varphi\sigma\|_{L^r}\left(\int\limits_{\mathbb{R}^1}\left(\int\limits_{\mathbb{R}^1}\frac{|\psi(y)|dy}{|x-y|^{1-\gamma}}\right)^{r'}\frac{1}{\sigma^{r'}(x)}dx\right)^{\frac{1}{r'}}.$$

From the condition $(\varrho, \sigma) \in G^{s,r}_{\gamma-1}$ readily follows that $(\frac{1}{\sigma}, \frac{1}{\varrho}) \in G^{r',s'}_{1-\gamma}$. Using Theorem A we obtain the estimate

$$\left|\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\varphi(x)\psi(y)}{|x-y|^{1-\gamma}}\,dx\,dy\right| \le c\|\varphi\|_{L^r_\sigma}\cdot\|\psi\|_{L^{s'}_{1/\varrho}}. \quad ■$$

Bellow we shall set $\varrho = (\frac{1}{v_1}, \frac{1}{v_2}, \dots, \frac{1}{v_n})$ for $v = (v_1, v_2, \dots, v_n)$.

Lemma 2. *Let* $1 < p_i < q_i < \infty$, $(v_i, w_i) \in G^{q_i,p_i}_{1-\gamma_i}$, $0 < \gamma_i < 1$. *Then there exists a positive constant c such that*

$$\left|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\varphi(x)\psi(y)}{|x-y|^{\mu}_{a}}\,dx\,dy\right| \le c\|\varphi\|_{L^p_w}\cdot\|\psi\|_{L^{q'}_\varrho}$$

for arbitrary $\varphi \in L^p_w(\mathbb{R}^n)$ *and* $\psi \in L^{q'}_\varrho(\mathbb{R}^n)$.

Proof. We shall apply the reduction technique to the one-dimensional case.

Let $x' = (x_1, x_2, \dots, x_{n-1})$, $y' = (y_1, y_2, \dots, y_{n-1})$ and

$$\mu' = \sum_{j=1}^{n-1} a_j(1-\gamma_j), \quad a' = (a_1, a_2, \dots, a_{n-1}).$$

It readily follows that

$$|x-y|_a^{\mu} = |x-y|_a^{\mu'}|x-y|_a^{a_n(1-\gamma_n)} \geq |x'-y'|_{a'}^{\mu'}|x_n-y_n|^{1-\gamma_n}.$$

Therefore by Lemma 1 we obtain

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\varphi(x)\psi(y)}{|x-y|_a^{\mu}}dx\,dy\right| \leq$$

$$\leq \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{dx'dy'}{|x'-y'|_a^{\mu'}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\varphi(x)|\,|\psi(y)|}{|x_n-y_n|^{1-\gamma_n}}dx_n dy_n \leq$$

$$\leq c \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{F(x')H(y')}{|x'-y'|_{a'}^{\mu'}}dx'dy',$$

where

$$F(x') = \left(\int_{\mathbb{R}^1} |\varphi(x)|^{p_n} w_n^{p_n}(x_n)dx_n\right)^{\frac{1}{p_n}},$$

$$H(x') = \left(\int_{\mathbb{R}^1} |\psi(y)|^{q_n'} v_n^{-q_n'}(y_n)dy_n\right)^{\frac{1}{q_n'}}.$$

Further reduction leads us to the proof of Lemma 2. ■

Proof of Theorem 1. By the property of the norm and also by Lemma 2 we have

$$\|\mathcal{K}f\|_{L_v^q} = \sup\left|\int_{\mathbb{R}^n} \mathcal{K}f(x)g(x)dx\right|,$$

where the least upper bound is taken over all functions g for which

$$\|g\|_{L_\varrho^{q'}} \leq 1, \quad \varrho = \left(\frac{1}{v_1}, \frac{1}{v_2}, \dots, \frac{1}{v_n}\right).$$

Next, by Lemma 2 we have

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(y)g(x)}{|x-y|_a^{\mu}}dx\,dy\right| \leq c\|f\|_{L_w^p} \cdot \|g\|_{L_\varrho^{q'}} \leq c\|f\|_{L_w^p}. \ ■$$

Theorem 2. *Let* $1 \leq m \leq n$, $v_+ = (v_1, v_2, \dots, v_m)$, $w_+ = (w_1, w_2, \dots, w_m)$, $w = (w_1, w_2, \dots, w_m, 1, \dots, 1)$, $1 < p_i < q_i < \infty$ $(i = 1, 2, \dots, m)$, $q_+ = (q_1, q_2, \dots, q_m)$, $p_+ = (p_1, p_2, \dots, p_m)$, $1 < p_i < \infty$ $(i = m+1, \dots, n)$.

Next we set

$$\mu = \sum_{i=1}^{m} a_j(1-\gamma_j) + \sum_{j=m+1}^{n} \frac{a_j}{p_j'}, \tag{1.4}$$

where $a_j > 0$, $0 < \gamma_j < 1$ $(j = 1, 2, \dots, m)$

If $(v_i, w_i) \in G^{q_i, p_i}_{1-\gamma_i}$ $(i = 1, 2, \dots, m)$, *then there exists a constant* $c > 0$ *such that for any* $f \in L^p_w(\mathbb{R}^n)$ *and arbitrary* $(x^0_{m+1}, \dots, x^0_n)$ *the function*

$$h(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x - y|^\mu_a} dy$$

belongs to the space $L^{q^+}_{v_+}(\mathbb{R}^m)$ *and the inequality holds*

$$\|h\|_{L^{q^+}_{v_+}} \le c\|f\|_{L^p_w},$$

where the constant c *is independent of* f.

The proof of Theorem 2 is based on the following

Lemma 3. *Let the conditions of Theorem 2 be fulfilled. If* $\varrho = (\frac{1}{v_1}, \frac{1}{v_2}, \dots, \frac{1}{v_n})$, $q'_+ = (q'_1, q'_2, \dots, q'_m)$, *then there exists a constant* $c > 0$ *such that for all* $f : \mathbb{R}^n \to \mathbb{R}^1$ *and* $g : \mathbb{R}^m \to \mathbb{R}^1$ *we have*

$$\left| \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} \frac{g(x)f(y)}{|x - y|^\mu_a} \right| \le c\|f\|_{L^p_w(\mathbb{R}^n)} \cdot \|g\|_{L^{q'_+}_{\varrho_+}(\mathbb{R}^m)}. \tag{1.5}$$

Proof. Let $y' = (y_1, y_2, \dots, y_m)$, $y'' = (y_{m+1}, \dots, y_n)$, $p_+ = (p_1, p_2, \dots, p_m)$, $p_- = (p_{m+1}, \dots, p_n)$, $p'_- = (p'_{m+1}, \dots, p'_n)$.

Obviously

$$\left| \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} \frac{g(x)f(y)}{|x - y|^\mu_a} dx\, dy \right| \le$$

$$\le \int\limits_{\mathbb{R}^m} |g(x)| \left(\int\limits_{\mathbb{R}^m} \left(\int\limits_{\mathbb{R}^{n-m}} \frac{|f(y)| dy''}{|x - y|^\mu_a} \right) dy' \right) dx. \tag{1.6}$$

By virtue of the Hölder inequality

$$\int\limits_{\mathbb{R}^{n-m}} \frac{|f(y)| dy''}{|x - y|^\mu_a} \le \|f\|_{L^{p_-}(\mathbb{R}^{n-m})} \cdot \| \, |x - y|^{-\mu}_a \|_{L^{p'_-}(\mathbb{R}^{n-m})}. \tag{1.7}$$

We introduce some notation:

$$\varphi(y') = \|f(y', \cdot)\|_{L^{p_-}(\mathbb{R}^{n-m})},$$

$$\varphi_1(x, y') = \| \, |x - y|^{-\mu}_a \|_{L^{p'_-}(\mathbb{R}^{n-m})},$$

$$T = \left(\sum_{j=1}^m |x_j - y_j|^{\frac{2}{a_j}} \right)^{\frac{1}{2}}.$$

Let us prove that there exists a positive number c_1 such that

$$\varphi_1(x,y') \le c_1 T^{\left(\sum_{j=m+1}^n \frac{a_j}{p'_j}-\mu\right)}. \tag{1.8}$$

We have

$$\varphi_1(x,y') = \Big\| \Big(T^2 + \sum_{j=m+1}^n |x_j-y_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}} \Big\|_{L^{p'_-}(\mathbb{R}^{n-m})} =$$

$$= \Big\| \Big(T^2 + \sum_{j=m+1}^n |y_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}} \Big\|_{L^{p'_-}(\mathbb{R}^{n-m})}.$$

The change of the variable $y_j = T^{a_j} t_j$ in the latter expression leads to the equality

$$\varphi(x,y') = T^{\sum_{j=m+1}^n \frac{a_j}{p'_j}-\mu} \Big\| \Big(1 + \sum_{j=m+1}^n |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}} \Big\|_{L^{p'_-}(\mathbb{R}^{n-m})}.$$

Now it is obvious that

$$\Big(1 + \sum_{j=m+1}^n |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}} = \Big(1 + \sum_{j=m+1}^n |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{1}{2}\sum_{k=m+1}^n \frac{a_k}{p'_k}+\sum_{i=1}^m a_i(1-\gamma_i)} =$$

$$= \prod_{j=m+1}^n \Big(1 + \sum_{j=m+1}^n |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{1}{2}(\frac{a_j}{p'_j}+\varepsilon)} \le \prod_{j=m+1}^n \Big(1 + |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{1}{2}(\frac{a_j}{p'_j}+\varepsilon)},$$

where $\varepsilon > 0$.

Therefore

$$\Big\| \Big(1 + \sum_{j=m+1}^n |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}} \Big\|_{L^{p'_-}(\mathbb{R}^{n-m})} \le$$

$$\le \Big\| \prod_{j=m+1}^n \Big(1 + |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{1}{2}(\frac{a_j}{p'_j}+\varepsilon)} \Big\|_{L^{p'_-}(\mathbb{R}^{n-m})}.$$

On the other hand,

$$\int_{\mathbb{R}^1} \frac{dt_j}{\Big(1 + |t_j|^{\frac{2}{a_j}}\Big)^{\frac{1}{2}(\frac{a_j}{p'_j}+\varepsilon)p'_j}} \le \int_{\mathbb{R}^1} \frac{dt_j}{(1+|t_j|)^{1+\frac{\varepsilon p'_j}{a_j}}} < \infty.$$

Thus we have proved the estimate (1.8). It implies

$$\varphi(x,y') \le \Big(\sum_{j=1}^m |x_j-y_j|^{\frac{2}{a_j}}\Big)^{-\frac{1}{2}\sum_{j=1}^m a_j(1-\gamma_j)} \le$$

$$\le c_1 \prod_{j=1}^m |x_j-y_j|^{(\gamma_j-1)}. \tag{1.9}$$

Using the generalized Hölder inequality and Lemma 1 with (1.6), (1.7), (1.8) and (1.9) we obtain:

$$\left|\int_{\mathbb{R}^m} dx \int_{\mathbb{R}^n} \frac{g(x)f(y)}{|x-y|_a^{\mu}} dy\right| \le c_2 \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|g(x)|\varphi(y')}{\prod_{j=1}^m |x_j - y_j|^{1-\gamma_j}} dx\, dy \le$$

$$\le c_3 \|\varphi(y')\|_{L^{p+}_{w_+}(\mathbb{R}^m)} \|g\|_{L^{q'}_{\varrho}(\mathbb{R}^m)}.$$

This implies that

$$\left|\int_{\mathbb{R}^m} dx \int_{\mathbb{R}^n} \frac{g(x)f(y)}{|x-y|_a^{\mu}} dy\right| \le c_3 \|f\|_{L^p_w(\mathbb{R}^n)} \|g\|_{L^{q+}_{\varrho}(\mathbb{R}^m)}. \quad \blacksquare$$

Proof of Theorem 2. Using the standard arguments, the validity of Theorem 2 readily follows from Lemma 3. ■

2. In this paragraph we shall prove the imbedding theorems for different metrics and different dimensions for weighted spaces of anisotropic Bessel potentials.

Definition 2 (see[2]). Let $r = (r_1, r_2, \dots, r_n)$, $p = (p_1, p_2, \dots, p_n)$, $r_j > 0$ $(j = 1, 2, \dots, n)$. It will be said that $f \in L^{p,r}_w(\mathbb{R}^n)$ if

$$f(x) = \int_{\mathbb{R}^n} G_r(x-y) g(y) dy,$$

where G_r is the anisotropic Bessel–Macdonald kernel and $g \in L^p_w(\mathbb{R}^n)$. By the definition,

$$\|f\|_{L^{p,r}_w} = \|g\|_{L^p_w}.$$

The kernel G_r is characterized by its Fourier transform as follows (see[2])

$$(2\pi)^{\frac{n}{2}} \widehat{G}_2(\lambda) = [1 + \sigma^2(\lambda)]^{-\frac{r^*}{2}}$$

where the function $\sigma(\lambda)$ is determined by the equation

$$\sum_{j=1}^n \frac{\lambda_j^2}{\sigma^{2a_j}} = 1, \quad a_j = \frac{r^*}{r_j}, \quad \frac{1}{r^*} = \frac{1}{n}\sum_{j=1}^n \frac{1}{r_j}.$$

The kernel G_r obeys, along each j – the coordinate direction, the estimates

$$|G_r(x)| \le c|x_j|^{-r_j\left(\sum_{i=1}^n \frac{1}{r_i} - 1\right)}. \tag{2.1}$$

Now we shall prove the imbedding theorem of different metrics.

Theorem 3. *Let* $1 < p_j < q_j < \infty$, $(v_i, w_i) \in G^{q_i,p_i}_{1-\gamma_i}$ $(i = 1, 2, \dots, n)$. *Put*

$$\varkappa = 1 - \sum_{j=1}^{n} \frac{\gamma_j}{r_j}$$

and

$$\varrho = \varkappa r, \quad r = (r_1, r_2, \dots, r_n).$$

Then each function $f \in L^{p,r}_{w}(\mathbb{R}^n)$ *belongs to the space* $L^{q,\varrho}_{v}(\mathbb{R}^n)$ *and the inequality*

$$\|f\|_{L^{q,\varrho}_v} \leq c\|f\|_{L^{p,r}_w}$$

holds, where the constant c is independant of f.

Proof. We have

$$f(x) = \int_{\mathbb{R}^n} G_\varrho(x-y)h(y)dy,$$

where

$$h(x) = \int_{\mathbb{R}^n} G_{r(1-\varkappa)}(x-y)g(y)dy$$

and $g \in L^p_w(\mathbb{R}^n)$.

Now it will be shown that $h \in L^q_v(\mathbb{R}^n)$. Due to (2.1) we have

$$|G_{r(1-\varkappa)}(x)| \leq c|x_i|^{-r_j(1-\varkappa)\left[\sum_{i=1}^n \frac{1}{r_i(1-\varkappa)} - 1\right]},$$

or

$$|G_{r(1-\varkappa)}(x)| \leq c|x_i|^{-r_j \sum_{i=1}^n \frac{1-\gamma_i}{r_i}}.$$

Let $a_j = \frac{1}{r_j}$ and

$$\max_{1\leq j\leq n} |x_j|^{\frac{1}{a_j}\sum_{i=1}^n a_i(1-\gamma_i)} = |x_{j_0}|^{\frac{1}{a_{j0}}\sum_{i=1}^n a_i(1-\gamma_i)}.$$

As can be easily verified,

$$\left(\sum_{i=1}^{n} |x_j|^{\frac{2}{a_j}}\right)^{\frac{1}{2}\sum_{i=1}^n a_i(1-\gamma_i)} \leq c_1|x_{j_0}|^{\frac{1}{a_{j0}}\sum_{i=1}^n a_i(1-\gamma_i)}.$$

Therefore

$$|G_{r(1-\varkappa)}(x)| \leq c_2|x|_a^{-\mu}, \tag{2.2}$$

where

$$\mu = \sum_{j=1}^{n} a_j(1-\gamma_j).$$

Hence

$$|h(x)| \leq \int_{\mathbb{R}^n} |g(y)|\,|x-y|_a^{-\mu}dy.$$

Applying Theorem 1, we obtain

$$\|h\|_{L_v^q} \le c_3 \|f\|_{L_w^p},$$

which implies

$$\|f\|_{L_v^{q,\varrho}} \le \|f\|_{L_w^{p,r}}. \quad \blacksquare$$

Using Theorem 2 one may prove an imbedding theorem of different dimensions in a similar manner.

Theorem 4. *Let* $1 < p_i < \infty$ $(i = 1,2,\dots,n)$, $1 < p_i < q_i < \infty$ $(i = 1,2,\dots,m)$, $1 \le m \le n$. *It is also assumed that* $(v_i, w_i) \in G_{1-\gamma_i}^{q_i,p_i}$, $0 < \gamma_i < 1$ $(i = 1,2,\dots)$.

If

$$\varkappa = 1 - \sum_{j=1}^{m} \frac{\gamma_j}{r_j} - \sum_{j=m+1}^{n} \frac{1}{r_j p_j}, \tag{2.3}$$

then for an arbitrary function f *from the space* $L_w^{p,r}(\mathbb{R}^n)$ *the function*

$$F(x_1, x_2, \dots, x_m) = f(x_1, x_2, \dots, x_m, \overset{0}{x}_{m+1}, \dots, \overset{0}{x}_n)$$

belongs to the space $L_v^{q,\varrho}(\mathbb{R}^m)$ *and the inequality*

$$\|F\|_{L_v^{q,\varrho}(\mathbb{R}^m)} \le c\|f\|_{L_w^{p,r}(\mathbb{R}^n)}$$

holds where the constant c *is independent of* f.

Proof. In the case under consideration the kernel $G_{(1-\varkappa)r}$ admits the estimate

$$|G_{(1-\varkappa)r}(x)| \le c|x|_a^{-\mu},$$

where $a = (a_1, a_2, \dots, a_n)$, $a_i = \frac{1}{r_j}$ and

$$\mu = \sum_{i=1}^{m} \frac{1-\gamma_i}{r_i} + \sum_{j=m+1}^{n} \frac{1}{r_j p_j'}. \tag{2.4}$$

Hence we can apply Theorem 2. The rest of the proof is as for the preceding theorem. ■

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LINEAR DYNAMICAL SYSTEMS OF HIGHER GENUS

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ABSTRACT. A class of linear systems which after ordinary linear systems are in a certain sense the simplest ones, is associated with every algebraic function field. From the standpoint developed in the paper ordinary linear systems are associated with the rational function field.

რეზიუმე. ალგებრულ ფუნქციათა ყოველ ველთან დაკავშირებულია კლასი წრფივი სისტემებისა, რომლებიც გარკვეული აზრით უმარტივესია ჩვეულებრივი წრფივი სისტემების შემდეგ. ნაშრომში განვითარებული თვალსაზრისის მიხედვით ეს უკანასკნელნი წარმოადგენენ რაციონალურ ფუნქციათა ველთან დაკავშირებულ წრფივ სისტემებს.

§ 0. INTRODUCTION.

As is well known, there is a close relationship between linear systems and the rational function field. The subject of the paper is to study new linear systems which are closely connected with arbitrary algebraic function fields.

The idea of introducing linear systems of "higher genus" is due to R.Hermann [7]. He tries to describe them in terms of linear spaces of infinite dimension. Our approach is different and uses vector bundles (of finite rank) over algebraic function fields.

In what follows we shall assume that the reader is familiar with the elementary concepts of commutative algebra such as a discrete valuation, a Dedekind domain, a maximal ideal, an exact sequence of modules and a localization. In the appendix we give all necessary concepts and facts from the theory of algebraic function fields.

Throughout the paper, k will denote a ground field, and m and p input and output numbers, respectively. We fix once and for all:

an algebraic function field R over k;

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a discrete valuation v of R trivial on k and such that its residue field coincides with k;

a function s such that $v(s) = -1$.

The simplest example $(\star)$ is given by the following data:

$R = k(z)$ where z is an indeterminate;

v = discrete valuation determined by the formula $v(f/g) = \deg g - \deg f$ if $f, g \in k[z]$, $g \neq 0$;

$s = z$.

Let X denote the set of all places of R. Recall that each place x gives a discrete valuation ord_x of R trivial on k, and that this correspondence is bijective. Denote by ∞ the place corresponding to v and call it the infinite place. Let $\mathcal{O}$ denote the standard vector bundle over R. For any divisor D the associated vector bundle is denoted by $\mathcal{O}(D)$. For each integer n let us write $\mathcal{O}(n)$ instead of $\mathcal{O}(n\infty)$. Let A denote the ring of functions which are regular outside from ∞. This is a Dedekind domain. Its maximal ideals are in the one-to-one correspondence with places distinct from ∞. Finally, let K denote the divisor of the differential ds and g the genus of R.

We define a linear system over (R, v, s) as a quintuple $(V, \mathcal{E}, \theta, B, C)$ consisting of a linear space V over k, a vector bundle $\mathcal{E}$ over R, a morphism $\theta : \mathcal{O} \otimes V \longrightarrow \mathcal{E}$ and linear maps $B : k^m \longrightarrow H^0\mathcal{E}(-1)$, $C : V \longrightarrow k^p$. It is required that the following conditions hold:

(1) $H^1\mathcal{E}(K) = 0$;

(2) θ induces a bijective linear map $V \longrightarrow \mathcal{E}(\infty)$;

(3) the canonical map $H^0\mathcal{O}(K) \otimes V \oplus H^0\mathcal{O}(K)^m \longrightarrow H^0\mathcal{E}(K)$ is surjective.

It is the goal of the paper to show that this definition should lead to an interesting theory.

Let us see what linear systems are in the example $(\star)$. We have: $K = -2\infty$, $H^0\mathcal{O}(-2) = H^1\mathcal{O} = 0$. So the conditions (1) and (3) can be rewritten as $H^1\mathcal{E}(-2) = 0$ and $H^0\mathcal{E}(-2) = 0$, respectively. Vector bundles with the above properties and linear spaces (of finite dimension) are made equivalent by the functors $\mathcal{E} \longrightarrow H^0\mathcal{E}(-1)$ and $W \longrightarrow \mathcal{O}(1) \otimes W$. Next, giving a morphism $\theta : \mathcal{O} \otimes V \longrightarrow \mathcal{O}(1) \otimes W$ is equivalent to giving a pair of linear maps $E, A : V \longrightarrow W$. It follows that in the case of $(\star)$ a linear system can be described in terms of linear algebra, namely, as a sextuple (V, W, E, A, B, C) where V and W are finite-dimensional linear spaces, $E : V \longrightarrow W$ is a bijective linear map and $A : V \longrightarrow W$, $B : k^m \longrightarrow W$, $C : V \longrightarrow k^p$ are arbitrary linear maps. It is easily seen that such sextuples are equivalent to ordinary linear systems. (The equivalence is established by

$$(V, W, A, E, B, C) \longrightarrow (V, E^{-1}A, E^{-1}B, C).\,)$$

Thus linear systems associated with $(\star)$ and ordinary linear systems are the same thing.

The paper is organized as follows.

In §1 we define controllability, observability, transfer functions and Martin–Hermann sheaves. Here we also introduce a category Σ whose objects are triples $(\mathcal{F}, D, N)$ consisting of a coherent sheaf $\mathcal{F}$ generated by global sections, a morphism $D : \mathcal{O}^m \longrightarrow \mathcal{F}$ such that $D(\infty) : k^m \longrightarrow \mathcal{F}(\infty)$ is a bijection and a morphism $N : \mathcal{O}^p \longrightarrow \mathcal{F}$ such that $N(\infty) : k^p \longrightarrow \mathcal{F}(\infty)$ is zero.

In §2 we prove that the category of linear systems is equivalent to the opposite category of Σ. This means that a linear system can be defined as an object of Σ.[1] From this we easily derive Kalman's theorem on realization.

In §3 we define a feedback equivalence and prove the Martin-Hermann theorem which says that two linear systems are feedback equivalent if and only if their Martin–Hermann sheaves are isomorphic. Then we discuss the pole-placement theorem. Unfortunately, we prove it for one input linear systems only.

§ 1. Linear Systems

In this section we do not impose the third condition on linear systems. So by a linear system here we shall mean a quintuple $(V, \mathcal{E}, \theta, B, C)$ where V is a linear space, $\mathcal{E}$ is an effective vector bundle such that $H^1\mathcal{E}(K) = 0$, θ is a morphism of $\mathcal{O} \otimes V$ into $\mathcal{E}$ such that the linear map $\theta(\infty) : V \longrightarrow \mathcal{E}(\infty)$ is bijective, B is a linear map $k^m \longrightarrow H^0\mathcal{E}(-1)$ and C is a linear map $V \longrightarrow k^p$.

1. Let $\sigma = (V, \mathcal{E}, \theta, B, C)$ be a linear system.

Definition. We define the rank of σ as the rank of $\mathcal{E}$ or, what is the same, as the dimension of V. We define the McMillan degree of σ as the degree of $\mathcal{E}$.

Definition. The characteristic sheaf of σ is defined to be the cokernel of θ.

If $\mathcal{C}$ denotes the characteristic sheaf of σ, then, by definition, one has an exact sequence

$$0 \longrightarrow \mathcal{O} \otimes V \longrightarrow \mathcal{E} \longrightarrow \mathcal{C} \longrightarrow 0. \qquad (1)$$

---

[1] Such a definition was in fact the starting point of the paper. One immediately comes to it through Corollary 4 of Theorem 1 from [10].

Definition. The state sheaf of σ is defined to be $\mathcal{C}(K)$, and the pole sheaf of σ is defined to be $\mathcal{E}xt^1(\mathcal{C}, \mathcal{O})$.

Observe that to give a linear map $k^m \longrightarrow H^0\mathcal{E}(-1)$ is to give a morphism $\mathcal{O}^m \longrightarrow \mathcal{E}$, which induces a zero map on the reduced stalks at infinity. (This follows from the exact sequence $0 \longrightarrow H^0\mathcal{E}(-1) \longrightarrow H^0\mathcal{E} \longrightarrow \mathcal{E}(\infty)$.) Likewise, to give a linear map $V \longrightarrow k^p$ is to give a morphism $\mathcal{O} \otimes V \longrightarrow \mathcal{O}^p$. For this reason we shall use the same letters B and C for the corresponding morphisms.

One defines morphisms of linear systems in the obvious way.

2. Let $\sigma = (V, \mathcal{E}, \theta, B, C)$ be a linear system.

Definition. If x is a finite place, then we say that

(a) σ is reachable at x if

$$\mathrm{rk}[\theta(x)\, B(x)] = \dim V,$$

i.e. if the morphism

$$[\theta\, B] : \mathcal{O} \otimes V \oplus \mathcal{O}^m \longrightarrow \mathcal{E}$$

is surjective at x;

(b) σ is observable at x if

$$\mathrm{rk}\begin{bmatrix} \theta(x) \\ C \end{bmatrix} = \dim V,$$

i.e. if the morphism

$$\begin{bmatrix} \theta \\ C \end{bmatrix} : \mathcal{O} \otimes V \longrightarrow \mathcal{E} \oplus \mathcal{O}^p$$

is left invertible at x.

Because θ is bijective at one place, namely at infinity, it should be bijective at all but finitely many places. This implies, in particular, that every linear system is reachable (observable) at all but finitely many places.

For each $N \geq 0$ put

$$\Omega(-N) = H^0\mathcal{O}(K + (N+2)\infty).$$

We then have a composition series

$$H^0\mathcal{O}(K) \subseteq \Omega(0) \subseteq \Omega(-1) \subseteq \Omega(-2) \subseteq \cdots.$$

Let

$$\Omega = \cup \Omega(-N).$$

Ω consists of the sections of $\mathcal{O}(K)$ over the affine open set $X - \{\infty\}$, and therefore, is a projective A-module of rank 1. All successive quotients in the above series are one dimensional linear spaces.

Example. For $(\star)$ we have: $H^0\mathcal{O}(K) = 0$ and

$$\Omega(-N) = \{\text{the space of polynomials in } s \text{ of degree} \le N\}.$$

Therefore $\Omega = k[s]$.

Set

$$\Gamma = R/\Omega.$$

This is an injective A-module. We have

$$R/H^0\mathcal{O}(K) \supseteq R/\Omega(0) \supseteq R/\Omega(-1) \supseteq R/\Omega(-2) \cdots$$

and $\Gamma = \cap R/\Omega(-N)$.

Following R.Kalman we call Ω^m the input module and Γ^p the output module.

Now let $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{P}$ denote the characteristic, the state and the pole sheaves of σ, respectively. These three sheaves may be regarded as A-modules of finite length because their supports do not contain ∞.

Tensoring the morphism $[\theta\, B]$ by $\mathcal{O}(K)$, we obtain a morphism $\mathcal{O}(K) \otimes V \oplus \mathcal{O}(K) \longrightarrow \mathcal{E}(K)$. This gives a morphism $\mathcal{O}^m(K) \longrightarrow \mathcal{S}$ and hence a homomorphism of A-modules

$$I(\sigma) : \Omega^m \longrightarrow \mathcal{S}.$$

Further, dualizing $\begin{bmatrix} \theta(K) \\ C \end{bmatrix}$, we obtain a morphism $\mathcal{E}^* \oplus \mathcal{O}^p \longrightarrow \mathcal{O} \otimes V^*$. This gives a morphism $\mathcal{O}^p \longrightarrow \mathcal{P}$ and hence a homomorphism of A-modules $A^p \longrightarrow \mathcal{P}$. Applying now the functor $\mathrm{Hom}_A(\cdot, \Gamma)$, we get a homomorphism

$$O(\sigma) : \mathcal{S} \longrightarrow \Gamma^p.$$

We call $I(\sigma)$ the input homomorphism and $O(\sigma)$ the output homomorphism of σ.

Proposition 1. *Let σ be as above and x be a finite place. Then*
(a) *σ is reachable at x if and only if $I(\sigma)$ is surjective at x.*
(b) *σ is observable at x if and only if $O(\sigma)$ is injective at x.*

Proof. (a) To say that $I(\sigma)$ is surjective at x is equivalent to saying that the morphism $\mathcal{O}^m \longrightarrow \mathcal{C}$ is surjective at x. The assertion follows now from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_x \otimes V & \longrightarrow & \mathcal{E}_x & \longrightarrow & \mathcal{C}_x & \longrightarrow & 0 \\ & & & & \uparrow & \nearrow & & & \\ & & & & \mathcal{O}_x^m & & & & \end{array}$$

having an exact row.

(b) σ is observable at x if and only if the morphism $[\theta^* \; C^*]$ is surjective at x. From this, as above, it follows that a necessary and sufficient condition for σ to be observable at x is that the homomorphism $\mathcal{O}_x^p \longrightarrow \mathcal{P}_x$ be surjective. Since $\mathcal{P}_x$ is an $\mathcal{O}_x$-module of finite length, this homomorphism gives rise to a homomorphism $\hat{\mathcal{O}}_x^p \longrightarrow \mathcal{P}_x$ where $\hat{}$ denotes the adic completion. Moreover, the surjectivity of the first one is equivalent to that of the second one. Now applying the functor $\mathrm{Hom}(\cdot, R/\Omega_x)$, we complete the proof. (Recall that the above functor is exact, and by the local duality

$$\mathrm{Hom}(\mathcal{P}_x, R/\Omega_x) = \mathcal{S}_x \quad \text{and} \quad \mathrm{Hom}(R/\Omega_x, R/\Omega_x) = \hat{\mathcal{O}}_x.$$

See [5].) ■

3. Let us call a function $f \in R$ strictly proper if $\mathrm{ord}_\infty(f) > 0$.

Definition. A transfer function is a $(p \times m)$-matrix whose entries are strictly proper functions.

A transfer function may be identified with a homomorphism $\mathcal{O}_\infty^m \longrightarrow \mathcal{O}_\infty^p$ which takes values in $t\mathcal{O}_\infty^p$.

Let $\sigma = (V, \mathcal{E}, \theta, B, C)$ be a linear system. We have a sequence of $\mathcal{O}_\infty$-homomorphisms

$$\mathcal{O}_\infty^m \xrightarrow{B} \mathcal{E}_\infty \xrightarrow{\theta^{-1}} \mathcal{O}_\infty \otimes V \xrightarrow{C} \mathcal{O}_\infty^p.$$

Since $B(\infty) : k^m \longrightarrow \mathcal{E}(\infty)$ is zero, the composed linear map

$$k^m \xrightarrow{B(\infty)} \mathcal{E}(\infty) \xrightarrow{\theta^{-1}(\infty)} V \xrightarrow{C} k^p$$

is also zero. This implies that the above composed homomorphism is a transfer function. We denote it by $T(\sigma)$ and call the transfer function of σ.

4. Let Σ denote the category of triples $(\mathcal{F}, D, N)$, where $\mathcal{F}$ is a globally generated coherent sheaf of rank m, D is a morphism of $\mathcal{O}^m$ into $\mathcal{F}$ such that $D(\infty) : k^m \longrightarrow \mathcal{F}(\infty)$ is bijective, and N is a morphism of $\mathcal{O}^p$ into $\mathcal{F}$ such that $N(\infty) : k^p \longrightarrow \mathcal{F}(\infty)$ is zero. Morphisms of this category are defined in the obvious way.

Definition. Let $\sigma = (V, \mathcal{E}, \theta, B, C)$ be a linear system. We let $MH(\sigma)$ denote the cokernel of the morphism

$$\begin{bmatrix} \theta^* \\ B^* \end{bmatrix} \longrightarrow \mathcal{O} \otimes V^* \oplus \mathcal{O}^m$$

(which is clearly injective) and call it the Martin–Hermann sheaf of σ. Next, we let $D(\sigma)$ denote the canonical morphism $\mathcal{O}^m \longrightarrow MH(\sigma)$ and call it the denominator of σ. Finally, we let $N(\sigma)$ denote the composition of C^* and the canonical morphism $\mathcal{O} \oplus V^* \longrightarrow MH(\sigma)$, and call it the numerator of σ.

Definition. Let σ be a linear system. Put

$$FR(\sigma) = (MH(\sigma), D(\sigma), N(\sigma))$$

and call it the fraction representation of σ.

It is easily seen that FR is a contravariant functor from the category of linear systems to the category Σ.

Let $\sigma = (V, \mathcal{E}, \theta, B, C)$ be a linear system and let $FR(\sigma)=(\mathcal{F}, D, N)$.

Proposition 2. *The McMillan degree of σ is equal to* $\deg \mathcal{F}$.

Proof. The proof follows immediately from the exact sequence

$$0 \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O} \otimes V^* \oplus \mathcal{O}^m \longrightarrow \mathcal{F} \longrightarrow 0. \quad \blacksquare \qquad (2)$$

Proposition 3. *The pole sheaf $\mathcal{P}$ of σ is canonically isomorphic to* coker D.

Proof. Dualizing (1) we get an exact sequence

$$0 \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O} \otimes V^* \longrightarrow \mathcal{P} \longrightarrow 0.$$

From this and from (2) follows the statement. ■

Proposition 4. *Let x be a place. Then*

(a) *σ is reachable at x if and only if $\mathcal{F}$ is locally free at x.*

(b) *σ is observable at x if and only if the morphism $[D\ N]$ is surjective at x.*

Proof. (a) We have an exact sequence

$$0 \longrightarrow \mathcal{E}^*_x \longrightarrow \mathcal{O}_x \otimes V^* \oplus \mathcal{O}^m_x \longrightarrow \mathcal{F}_x \longrightarrow 0.$$

σ is reachable at x if and only if the linear map $\mathcal{E}^*(x) \longrightarrow k(x) \otimes V^* \oplus k(x)^m$ is injective. Hence, the assertion follows from Proposition 6 of [2], Ch.2, §3.

(b) Let $\mathcal{P}$ be the pole sheaf. From the proof of Proposition 1 we know that σ is observable at x if and only if $\mathcal{O}_x^p \longrightarrow \mathcal{P}_x$ is surjective. So, the assertion follows from the commutative diagram

$$\begin{array}{ccccccc} \mathcal{O}_x^m & \longrightarrow & \mathcal{F}_x & \longrightarrow & \mathcal{P}_x & \longrightarrow & 0 \\ & & \uparrow & \nearrow & & & \\ & & \mathcal{O}_x^p & & & & \end{array}$$

having an exact row. ■

Proposition 5. *The transfer function of σ is equal to $N_\infty^* \circ D_\infty^{*-1}$.*

Proof. Let L denote the canonical morphism $\mathcal{O} \otimes V^* \longrightarrow \mathcal{F}$. Using the exact sequence

$$0 \longrightarrow \mathcal{E}_\infty^* \longrightarrow \mathcal{O}_\infty \otimes V^* \oplus \mathcal{O}_\infty^m \longrightarrow \mathcal{F}_\infty \longrightarrow 0,$$

one easily verifies that $D_\infty \circ B_\infty^* \circ \theta^{*-1} = L_\infty$. It follows from this that $B_\infty^* \circ \theta^{*-1} \circ C_\infty^* = D_\infty^{-1} \circ N_\infty$. ■

§ 2. Realization Theorem

Lemma 1. *Let $(V, \mathcal{E}, \theta, B, C)$ be a linear system and $\mathcal{F}$ be its Martin-Hermann sheaf. Then*

$$\dim V \leq \dim H^0\mathcal{F}(-1).$$

Proof. By Serre's duality, $H^0\mathcal{E}^* = H^1\mathcal{E}(K) = 0$. Hence, from (2) we get an exact sequence

$$0 \longrightarrow V^* \oplus k^m \longrightarrow H^0\mathcal{F}.$$

The map $V^* \longrightarrow H^0\mathcal{F}$ must have its image in $H^0\mathcal{F}(-1)$ because the composed map $V^* \longrightarrow H^0\mathcal{F} \longrightarrow \mathcal{F}(\infty)$ is zero. Consequently, we have a canonical injective linear map $V^* \longrightarrow H^0\mathcal{F}(-1)$. Furthermore, the composition $k^m \longrightarrow H^0\mathcal{F} \longrightarrow \mathcal{F}(\infty)$ is bijective. ■

Lemma 2. *Under the notations of the previous lemma the following conditions are equivalent:*

(a) $\dim V = \dim H^0\mathcal{F}(-1)$;

(b) *the map* $H^0\mathcal{O}(K) \otimes V \oplus H^0\mathcal{O}(K)^m \longrightarrow H^0\mathcal{E}(K)$ *is surjective;*

(c) *the canonical sequence* $H^0\mathcal{O}(K)^m \longrightarrow H^0\mathcal{S} \longrightarrow V \longrightarrow 0$, *where $\mathcal{S}$ is the state sheaf, is exact.*

Proof. It follows from the proof of the previous lemma that (a) is equivalent to the bijectivity of the linear map $V^* \oplus k^m \longrightarrow H^0\mathcal{F}$. On the other hand, by Serre's duality, (b) is equivalent to the injectivity of the linear map $H^1\mathcal{E}^* \longrightarrow H^1\mathcal{O} \otimes V^* \oplus H^1\mathcal{O}^m$. So, the equivalence (a) $\Leftrightarrow$ (b) follows immediately from the cohomological exact sequence

$$0 \longrightarrow V^* \oplus k^m \longrightarrow H^0\mathcal{F} \longrightarrow H^1\mathcal{E}^* \longrightarrow H^1\mathcal{O} \otimes V^* \oplus H^1\mathcal{O}^m,$$

which can be derived from (2). The equivalence (b) $\Leftrightarrow$ (c) follows immediately from the exact cohomological sequence

$$0 \longrightarrow H^0\mathcal{O}(K) \otimes V \longrightarrow H^0\mathcal{E}(K) \longrightarrow H^0\mathcal{S} \longrightarrow V \longrightarrow 0$$

induced by the exact sequence

$$0 \longrightarrow \mathcal{O}(K) \otimes V \longrightarrow \mathcal{E}(K) \longrightarrow \mathcal{S} \longrightarrow 0. \quad \blacksquare$$

In what follows we restrict attention only to linear systems which satisfy the equivalent conditions of the previous lemma, i.e. to linear systems defined as in Introduction.

Theorem 1. *The functor FR establishes an equivalence of the category of linear systems with the category Σ^{op}.*

Proof. Let $(\mathcal{F}, D, N)$ be an object of Σ. Since $\mathcal{F}$ is generated by global sections, we have, in particular, an exact sequence

$$0 \longrightarrow H^0\mathcal{F}(-1) \longrightarrow H^0\mathcal{F} \longrightarrow \mathcal{F}(\infty) \longrightarrow 0.$$

Because the composed map

$$k^m \longrightarrow H^0\mathcal{F} \longrightarrow \mathcal{F}(\infty)$$

is bijective, this exact sequence splits canonically, i.e. there is a canonical isomorphism

$$H^0\mathcal{F} \simeq H^0\mathcal{F}(-1) \oplus k^m.$$

Furthermore, because $N(\infty)$ is zero, the map H^0N takes values in $H^0\mathcal{F}(-1)$.

Now put

$$\Phi(\mathcal{F}, D, N) = (V, \mathcal{E}, \theta, B, C),$$

where $V = (H^0\mathcal{F}(-1))^*$, $\mathcal{E} = (\ker(\mathcal{O} \otimes H^0\mathcal{F} \longrightarrow \mathcal{F}))^*$, θ is the canonical morphism $\mathcal{O} \otimes V \longrightarrow \mathcal{E}$, B is the canonical linear map $k^m \longrightarrow H^0\mathcal{E}$ and C is the dual map to $k^p \longrightarrow H^0\mathcal{F}(-1)$. It is easy to see that this is a linear system.

Clearly, Φ is a contravariant functor from the category Σ to the category of linear systems, and one checks without difficulty that FR and Φ are inverse to each other.

The theorem is proved. ■

Corollary (Kalman's theorem on realization). *The assignment $\sigma \longrightarrow T(\sigma)$ induces a bijective correspondence between the isomorphism classes of canonical linear systems and the transfer functions.*

(The sense of the word "canonical" is evident.) To prove the corollary we need one lemma.

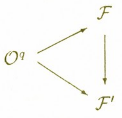
Let $q = m + p$. Let $Grass_m(R^q)$ be the set of m-dimensional subspaces in R^q and let $LFQ_m(\mathcal{O}^q)$ be the set of locally free quotients of $\mathcal{O}^q$ of rank m. The elements of $Grass_m(R^q)$ may be identified with the equivalence classes of $(m \times q)$-matrices of rank m with entries in R. (Two such matrices M_1 and M_2 are equivalent if $M_2 = GM_1$ for some $G \in GL(m, R)$.) The elements of $LFQ_m(\mathcal{O}^q)$ may be identified with the isomorphic classes of pairs $(\mathcal{F}, f)$, where $\mathcal{F}$ is a vector bundle of rank m and f is an epimorphism of $\mathcal{O}^q$ onto $\mathcal{F}$. (Two such pairs $(\mathcal{F}_1, f_1)$ and $(\mathcal{F}_2, f_2)$ are isomorphic if $f_2 = \phi \circ f_1$ for some isomorphism $\phi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$.)

Lemma 3. *There is a natural bijection between the sets $Grass_m(R^q)$ and $LFQ_m(\mathcal{O}^q)$.*

Proof. Let F be a nondegenerate $(m \times q)$-matrix with elements in R and let $f_1, \dots, f_q$ be its columns. Define a vector bundle $\mathcal{F}$ by the formula

$$\mathcal{F} = \Big(R^m, \big(\sum_{i=1}^{q} \mathcal{O}_x f_i\big)\Big).$$

Clearly, all $f_i \in H^0\mathcal{F}$. Hence, we may view the matrix F as a morphism $\mathcal{O}^q \longrightarrow \mathcal{F}$ (of course, surjective). If now $F' = GF$, where $G \in GL(m, R)$, and $\mathcal{F}'$ is the corresponding vector bundle, then G clearly defines an isomorphism of $\mathcal{F}$ onto $\mathcal{F}'$ such that the diagram



commutes.

Thus we have a well-defined map from $Grass_m(R^q)$ to $LFQ_m(\mathcal{O}^q)$.

Conversely, let $\mathcal{F} = (F, (F_x))$ be a vector bundle of rank m, and let $f : \mathcal{O}^q \longrightarrow \mathcal{F}$ be an epimorphism. We then have a surjective R-linear map $R^q \longrightarrow F$. The image of the dual linear map is a linear subspace in R^q $(= R^{q\times 1})$ of dimension m. By the "transposing" of this one we get an element in $Grass_m(R^q)$. It is obvious that if we take a pair isomorphic to $(\mathcal{F}, f)$, we shall come to the same element.

So we have a map from $LFQ_m(\mathcal{O}^q)$ into $Grass_m(R^q)$.

It is not hard to verify that the above two maps are inverse to each other. ■

Proof of the corollary. First note that by

$$T \longrightarrow [I\ T^*] \mod GL(m, R),$$

where I is the identity $(m \times m)$-matrix, one can identify transfer functions with some elements from $Grass_m(R^q)$.

Now let T be a transfer function. Let $\mathcal{F}$ be the vector bundle corresponding to $[I\ T^*]$ as in the proof of the previous lemma, and let $D : \mathcal{O}^m \longrightarrow \mathcal{F}$ and $N : \mathcal{O}^p \longrightarrow \mathcal{F}$ be the morphisms determined by the matrices I and T^*, respectively. Clearly, $\mathcal{F}_\infty = \mathcal{O}^m_\infty$, $D_\infty = I$ and $N(\infty) = 0$. So, $(\mathcal{F}, D, N)$ is a linear system. The proof now can be easily completed. ■

§ 3. Feedback

By a linear system in this section we shall mean a quadruple $(V, \mathcal{E}, \theta, B)$ where V is a linear space over k of finite dimension, $\mathcal{E}$ is a vector bundle over R such that $H^1\mathcal{E}(K) = 0$, θ is a morphism $\mathcal{O} \otimes V \longrightarrow \mathcal{E}$ such that the induced map $V \longrightarrow \mathcal{E}(\infty)$ is bijective, and B is a linear map $k^m \longrightarrow H^0\mathcal{E}(-1)$. We shall assume that the equivalent conditions of Lemma 2 hold. It follows from the proof of Theorem 1 that such a linear system can be defined as a pair $(\mathcal{F}, f)$ where $\mathcal{F}$ is a globally generated coherent sheaf and $f : \mathcal{O}^m \longrightarrow \mathcal{F}$ is such that $f(\infty) : k^m \longrightarrow \mathcal{F}(\infty)$ is bijective.

1. Definition. Two linear systems $(V_1, \mathcal{E}_1, \theta_1, B_1)$ and $(V_2, \mathcal{E}_2, \theta_2, B_2)$ are said to be feedback equivalent if there exist an isomorphism $\phi : \mathcal{E}_2 \longrightarrow \mathcal{E}_1$, a bijective linear map $\alpha : V_2 \longrightarrow V_1$, a linear automorphism $\beta : k^m \longrightarrow k^m$ and a linear map $L : V_2 \longrightarrow k^m$ such that

$$\theta_2 = \phi^{-1}\theta_1\alpha + \phi^{-1}B_1L \quad \text{and} \quad B_2 = \phi^{-1}B_1\beta.$$

Theorem 2 (Martin–Hermann). *Two linear systems are feedback equivalent if and only if their Martin-Hermann sheaves are isomorphic.*

Proof. Let $\sigma_1 = (V_1, \mathcal{E}_1, \theta_1, B_1)$ and $\sigma_2 = (V_2, \mathcal{E}_2, \theta_2, B_2)$ be linear systems, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be their Martin–Hermann sheaves, respectively.

Suppose that σ_1 and σ_2 are feedback equivalent. By definition, we then have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_1^* & \longrightarrow & \mathcal{O} \otimes V_1^* \oplus \mathcal{O}^m \\ {\scriptstyle \phi^*}\big\downarrow & & \big\downarrow \begin{bmatrix} \alpha^* & L^* \\ 0 & \beta^* \end{bmatrix} \\ \mathcal{E}_2^* & \longrightarrow & \mathcal{O} \otimes V_2^* \oplus \mathcal{O}^m \end{array}$$

where ϕ, α, β and L are as above. Since the vertical arrows here are isomorphisms, this diagram yields an isomorphism $\mathcal{F}_1 \simeq \mathcal{F}_2$.

Conversely, suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are isomorphic, and let ψ be any isomorphism of $\mathcal{F}_1$ onto $\mathcal{F}_2$. We then have a commutative diagram

$$\begin{array}{ccc} \mathcal{O} \otimes H^0\mathcal{F}_1 & \longrightarrow & \mathcal{F}_1 \\ {\scriptstyle H^0\psi}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ \mathcal{O} \otimes H^0\mathcal{F}_2 & \longrightarrow & \mathcal{F}_2. \end{array}$$

Since $H^0\mathcal{F}_1 \simeq V_1^* \oplus k^m$ and $H^0\mathcal{F}_2 \simeq V_2^* \oplus k^m$, we can find linear maps $\alpha : V_2 \longrightarrow V_1$, $\beta : k^m \longrightarrow k^m$, $L : V_2 \longrightarrow k^m$ and $G : k^m \longrightarrow V_1$ such that $\begin{bmatrix} \alpha & G \\ L & \beta \end{bmatrix}$ is nonsingular and the following diagram

$$\begin{array}{ccc} \mathcal{O} \otimes V_1^* \oplus \mathcal{O}^m & \longrightarrow & \mathcal{F}_1 \\ \begin{bmatrix} \alpha^* & L^* \\ G^* & \beta^* \end{bmatrix} \big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ \mathcal{O} \otimes V_2^* \oplus \mathcal{O}^m & \longrightarrow & \mathcal{F}_2 \end{array}$$

is commutative. This diagram can be extended to the commutative diagram

$$\begin{array}{ccccc} \mathcal{E}_1^* & \longrightarrow & \mathcal{O} \otimes V_1^* \oplus \mathcal{O}^m & \longrightarrow & \mathcal{F}_1 \\ {\scriptstyle \phi^*}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \psi}, \\ \mathcal{E}_2^* & \longrightarrow & \mathcal{O} \otimes V_2^* \oplus \mathcal{O}^m & \longrightarrow & \mathcal{F}_2 \end{array}$$

where ϕ is an isomorphism of $\mathcal{E}_2$ onto $\mathcal{E}_1$. We thus have

$$[\theta_1\alpha + B_1 L \;\; \theta_1 G + B_1\beta] = [\phi\theta_2 \;\; \phi B_2].$$

It remains to show that $G = 0$. By the above equality,

$$\theta_1(\infty)G + B_1(\infty)\beta = \phi(\infty)B_2(\infty).$$

Since $B_1(\infty)$ and $B_2(\infty)$ are zero, we obtain from this that $\theta_1(\infty)G = 0$; whence $G = 0$. ∎

2. Let $\mathcal{F}$ be a globally generated coherent sheaf of rank m which is nonsingular at infinity. Given an effective divisor D, which does not contain ∞, one can ask whether there exists an injective morphism $f : \mathcal{O}^m \longrightarrow \mathcal{F}$ such that $\chi(\operatorname{coker} f) = D$. (Note that such a morphism will necessarily be bijective at infinity.) This is the pole placement problem (PPP).

Lemma 4. *Let $f : \mathcal{O}^m \longrightarrow \mathcal{F}$ be an injective morphism, where $\mathcal{F}$ is a coherent sheaf of rank m. Let $\mathcal{T}$ be a torsion subsheaf of $\mathcal{F}$ and $\mathcal{F}_1 = \mathcal{F}/\mathcal{T}$. Then*

$$\chi(\operatorname{coker} f) = \chi(\operatorname{coker} f_1) + \chi(\mathcal{T}),$$

where f_1 denotes the canonical morphism from $\mathcal{O}^m$ to $\mathcal{F}_1$.

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}^m & \longrightarrow & \mathcal{O}^m & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_1 & \longrightarrow & 0 \end{array}$$

Applying Proposition 2.10 of [1], we get an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(f_1) \longrightarrow 0.$$

By Proposition 6.9 of [1] from this follows the lemma. ∎

The above lemma means in particular that "one cannot change the unreachable modes by feedback". One sees also that it reduces the PPP to the case when $\mathcal{F}$ is a vector bundle.

Observe that if $\mathcal{F}$ is a vector bundle of rank m and f is an injective morphism of $\mathcal{O}^m$ into $\mathcal{F}$, then the class of the divisor $\chi(\operatorname{coker} f)$ is equal to the Chern class of $\mathcal{F}$.

Thus, the PPP for a globally generated vector bundle $\mathcal{F}$ of rank m can be posed in the following way: Given an effective divisor D which does not contain ∞ and is such that $\operatorname{cl}(D) = \operatorname{ch}(\mathcal{F})$, does there exist an injective morphism $f : \mathcal{O}^m \longrightarrow \mathcal{F}$ with $\chi(\operatorname{coker} f) = D$?

Example. Consider the case $(\star)$. For this case the homomorphism $\deg\colon \operatorname{Cl}(R) \longrightarrow \mathbb{Z}$ is an isomorphism. Hence, the Chern class of a vector bundle can be identified with its degree. Next, effective divisors supported in $X - \{\infty\}$ can be identified with monic polynomials in s. Let now $\mathcal{F}$ be a vector bundle of rank m and degree n. The PPP takes the form: Given a polynomial P in s of degree n, choose a morphism

$f : \mathcal{O}^m \longrightarrow \mathcal{F}$ such that $f(\infty)$ is bijective and $\chi(\operatorname{coker} f) = P$. Notice that the sheaf $\operatorname{coker} f$ being finite and with support in $X - \{\infty\}$ can be identified with a finite $k[s]$-module or, which is the same thing, with a pair (V, F), where V is a linear space over k and F is an endomorphism of V. Clearly, $\chi(\operatorname{coker} f) =$ the characteristic polynomial of F.

We do not know if the answer to the PPP is always affirmative. But we have the following

Theorem 3. *The* PPP *has a solution in the case of one input.*

Proof. See Proposition 7.7 in [6], Ch.2. Here is another proof. By hypothesis, the sheaves $\mathcal{O}(D)$ and $\mathcal{F}$ are isomorphic. Hence the sheaves $\mathcal{O}$ and $\mathcal{F}(-D)$ also are isomorphic. The sheaf $\mathcal{F}(-D)$ is a subsheaf of $\mathcal{F}$, since D is effective. Thus, there exists a monomorphism $f : \mathcal{O} \longrightarrow \mathcal{F}$ whose image is $\mathcal{F}(-D)$. We have: $\operatorname{coker} f = \mathcal{F}/\mathcal{F}(-D)$. Because $\mathcal{F}$ is locally free of rank 1, it follows that $\operatorname{coker} f \simeq \mathcal{O}/\mathcal{O}(-D)$; whence $\chi(\operatorname{coker} f) = D$. ■

The following lemma may be helpful when one attempts to solve the PPP.

Lemma 5. *Let $\mathcal{F}$ be a globally generated vector bundle of rank m. Let $\mathcal{O} \longrightarrow \mathcal{F}$ be an injective morphism such that its cokernel $\mathcal{F}_1$ is a vector bundle too. (Such a morphism always exists.) Assume that the canonical map $H^0\mathcal{F} \longrightarrow H^0\mathcal{F}_1$ is surjective. If the* PPP *is solvable for $\mathcal{F}_1$, then it is solvable for $\mathcal{F}$ as well.*

Proof. Let D be an effective divisor such that $\infty \notin \operatorname{Supp} D$ and $\operatorname{cl}(D) = \operatorname{ch}(\mathcal{F})$. Clearly, $\operatorname{ch}(\mathcal{F}_1) = \operatorname{ch}(\mathcal{F})$. According to our assumption there exists an injective morphism $f_1 : \mathcal{O}^{m-1} \longrightarrow \mathcal{F}_1$ such that $\chi(\operatorname{coker} f_1) = D$. Since $H^0\mathcal{F} \longrightarrow H^0\mathcal{F}_1$ is surjective, f can be lifted to some $f : \mathcal{O}^m \longrightarrow \mathcal{F}$. We then have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}^m & \longrightarrow & \mathcal{O}^{m-1} & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f_1}\downarrow & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_1 & \longrightarrow & 0. \end{array}$$

One can derive easily from it an isomorphism $\operatorname{coker} f \simeq \operatorname{coker} f_1$. ■

Remark. From the above lemma one can deduce at once the classical result on state feedback. Indeed, let $g = 0$. Let $\mathcal{O} \longrightarrow \mathcal{F}$ be any injective morphism with a locally free quotient $\mathcal{F}_1$. We then have an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_1 \longrightarrow 0$$

which yields an exact sequence of cohomologies

$$H^0\mathcal{F} \longrightarrow H^0\mathcal{F}_1 \longrightarrow H^1\mathcal{O}.$$

Since $H^1\mathcal{O} = 0$, we find that the additional condition of Lemma 6 holds automatically. By the induction argument we obtain the desired result.

Appendix

Here we give a brief review of the theory of algebraic function fields (in one variable). For additional information, see [3,4,6,11,12]. (Recall that algebraic function fields are equivalent as objects to nonsingular complete irreducible algebraic curves.)

In what follows, k is a ground field.

An algebraic function field over k is a finitely generated extension of k of transcendence degree 1 or, which is the same thing, a finite extension of a field isomorphic to the rational function field over k in one indeterminate.

Let R be such a field. For simplicity assume that it is separable over k.

A place of R is an equivalence class of nontrivial absolute values of R trivial on k. (Recall that two nontrivial absolute values $|\ |_1$ and $|\ |_2$ of a field are said to be equivalent if they induce the same topology. It is not hard to prove that this holds if and only if $|\ |_2 = |\ |_1^\lambda$ for some $\lambda > 0$. See [8], Ch.XII, Prop. 1.) Denote the set of all places by X. There is a one-to-one correspondence between the places of R and the discrete valuations of R trivial to k. A discrete valuation corresponding to a place x is denoted by ord_x.

A function $f \in R$ is said to be regular at a place x if $\mathrm{ord}_x(f) \geq 0$. The set of regular functions at x, denoted by $\mathcal{O}_x$, is a discrete valuation ring. The residue field $k(x)$ of $\mathcal{O}_x$ is a finite extension of k; one denotes its degree by $d(x)$. A place x is said to be rational if $d(x) = 1$. An affine set is a complement to a nonempty finite set of places of X. If U is an affine set, then the ring of regular functions on U is a Dedekind domain. Its maximal ideals are in a natural one-to-one correspondence with the places in U. The affine sets together with the empty set and the whole space form a topology on X. A constant is a rational function which is algebraic over k or, equivalently, which is regular everywhere. The constants form a finite extension of k. If R possesses at least one rational place, then the constant field coincides with k.

A divisor is an element of the free abelian group $\mathrm{Div}(R)$ generated by places. There is an evident partial order on divisors. One says that a divisor D is effective if $D \geq 0$. If f is a rational function $\neq 0$, then $\text{ord}_x(f) = 0$ for almost all x, and therefore $[f] = \sum \text{ord}_x(f)x$ is a divisor. It is called the principal divisor belonging to f. The quotient group of $\text{Div}(R)$ modulo the principal divisors is called the divisor class group and is denoted by $\text{Cl}(R)$. For any divisor $D = \sum n(x)x$ one puts $\deg D = \sum n(x)d(x)$. Clearly, $\deg : \text{Div}(R) \longrightarrow \mathbb{Z}$ is a homomorphism. An important fact is that the degree of a principal divisor is zero. This makes possible to define $\deg : \text{Cl}(R) \longrightarrow \mathbb{Z}$.

The space of differential forms of R over k is a "universal" R-linear space $\Omega(R/k)$ equipped with a k-linear map $d : R \longrightarrow \Omega(R/k)$ satisfying the condition

$$d(fg) = f\,dg + g\,df; \quad f, g \in R.$$

Since R/k is a finitely generated separable extension of transcendence degree 1, this is a linear space of dimension 1.

Let ω be a nonzero differential form. If x is a place and if π is a uniformizer at x, then $\omega = f\,d\pi$ for some $f \in R$. Put $\text{ord}_x(\omega) = \text{ord}_x(f)$. This definition does not depend on choosing π. For all but finitely many places x one has: $\text{ord}_x(\omega) = 0$. Therefore the formal sum $[\omega] = \sum \text{ord}_x(\omega)x$ is a divisor. It is called the divisor associated to ω.

A vector bundle $\mathcal{E} = (E, (E_x))$ of rank r consists of a linear space E over R of dimension r and of a 'coherent' system (E_x) of $\mathcal{O}_x$-lattices in E (i.e. of free $\mathcal{O}_x$-submodules of E of maximal rank). The coherence means that if $(e_1, \ldots, e_r)$ is a basis of E, then $E_x = \mathcal{O}_x e_1 + \cdots + \mathcal{O}_x e_r$ for almost every x. (See [12], Ch.6.) The elements of E are called the rational sections of $\mathcal{E}$. The elements of $\Gamma(\mathcal{E}) = \cap E_x$ are called the global sections. The simplest example of a vector bundle is $\mathcal{O} = (R, (\mathcal{O}_x))$.

A morphism of a vector bundle $(E, (E_x))$ into a vector bundle $(F, (F_x)$ is a linear map $\theta : E \longrightarrow F$ over R such that $\theta(E_x) \subseteq F_x$ for each x.

Let $\mathcal{E} = (E, (E_x))$ be a vector bundle of rank r. Let $e_1, \ldots, e_r$ be linearly independent rational sections of $\mathcal{E}$. For each place x choose any basis $(e_{x1}, \ldots, e_{xr})$ of E_x and put $[e_1, \ldots, e_r]_x = \text{ord}_x(\det(a_{ij}))$, where $a_{ij} \in R$ are defined by $e_{xi} = \sum a_{ij} e_{xj}$. This is an integer which does not depend on choosing $(e_{x1}, \ldots, e_{xr})$. Clearly, $[e_1, \ldots, e_r]_x = 0$ for almost every x. Therefore $[e_1, \ldots, e_r] = \sum [e_1, \ldots, e_r]_x \cdot x$ is a divisor. The class of this divisor is independent of $e_1, \ldots, e_r$. This is the Chern class of $\mathcal{E}$ denoted by $\text{ch}(\mathcal{E})$.

A vector bundle of rank 1 is called a linear bundle. If $D = \sum n_x \cdot x$ is a divisor, we have a linear bundle $\mathcal{O}(D) = (R, (\{f \mid \operatorname{ord}_x f \geq -n_x\}))$. Every linear bundle is isomorphic to $\mathcal{O}(D)$ for some D. Note that if $\mathcal{L}$ is a linear bundle, then $\mathcal{L} \simeq \mathcal{O}(D)$ if and only if $\operatorname{ch}(\mathcal{L}) = \operatorname{cl}(D)$.

A finite sheaf is a collection of $\mathcal{O}_x$-modules M_x of finite length such that $M_x = 0$ for almost all x. The characteristic divisor of a finite sheaf $\mathcal{M} = (M_x)$ denoted by $\chi(\mathcal{M})$ is defined as the divisor $\sum \operatorname{length}(M_x)x$.

We are going now to define sheaves and their cohomologies. We shall do this under the hypothesis that we are given a fixed nonconstant rational function s.

Let U_1 and U_2 be the sets where s and s^{-1} are respectively regular. These are affine sets, and they cover the whole of X. Denote their intersection by U and put: $A_1 = \mathcal{O}(U)$, $A_2 = \mathcal{O}(U_2)$ and $A = \mathcal{O}(U)$. A sheaf is a quintuple (M_1, M_2, M, r_1, r_2) where M_1, M_2 and M are modules over the rings A_1, A_2 and A, respectively, and $r_1 : M_1 \longrightarrow M$ and $r_2 : M_2 \longrightarrow M$ are homomorphisms over A_1 and A_2, respectively. It is required that the canonical homomorphisms

$$A \otimes_{A_1} M_1 \longrightarrow M \quad \text{and} \quad A \otimes_{A_2} M_2 \longrightarrow M$$

be isomorphisms. Here are

Examples. 1) Let $(E, (E_x))$ be a vector bundle. Then

$$\Big(\bigcap_{x \in U_1} E_x, \bigcap_{x \in U_2} E_x, \bigcap_{x \in U} E_x, j_1, j_2 \Big),$$

where j_1 and j_2 are the canonical inclusions, is a sheaf.

2) Let (M_x) be a finite sheaf. Then

$$\Big(\bigoplus_{x \in U_1} M_x, \bigoplus_{x \in U_2} M_x, \bigoplus_{x \in U} M_x, r_1, r_2 \Big),$$

where r_1 and r_2 are the obvious restriction maps, is a sheaf.

3) Let E be a linear space over R. Then

$$(E, E, E, id, id)$$

is a sheaf. We shall denote it simply by E.

A sheaf is said to be coherent if the modules M_1 and M_2 are of finite type. It is said to be locally free if these modules are projective, and is said to be torsion if they are torsion modules. One can identify coherent locally free sheaves with vector bundles (see Example 1)), and coherent torsion sheaves with finite sheaves (see Example 2)).

For each sheaf $\mathcal{F}$ one defines in the obvious way the space of global sections $\Gamma(\mathcal{F})$, the stalk $\mathcal{F}_x$ and the reduced stalk $\mathcal{F}(x)$ at a point x, the support $\operatorname{Supp} \mathcal{F}$ and the rank $\operatorname{rk}(\mathcal{F})$. (See, for example, [9], §1.1.)

One defines in the standard way subsheaves and quotient sheaves, morphisms, kernels, cokernels and images of morphisms, various operations on sheaves (direct sums, direct limits, tensor products, sheaves $\mathcal{H}om$, dual sheaves), exact sequences of sheaves. For a sheaf $\mathcal{F}$ and a divisor D one usually writes $\mathcal{F}(D)$ for $\mathcal{F} \otimes \mathcal{O}(D)$. One defines (as in [9], §1.1, for example) the functors $\mathcal{E}xt^i(\cdot, \mathcal{O})$, $i = 0, 1$.

If V is a linear space V over k of finite dimension and $\mathcal{F}$ is a coherent sheaf, then $\mathrm{Hom}(\mathcal{O} \otimes V, \mathcal{F}) = \mathrm{Hom}(V, \Gamma(\mathcal{F}))$. In particular, one has a canonical morphism $\mathcal{O} \otimes \Gamma(\mathcal{F}) \longrightarrow \mathcal{F}$. If this morphism is surjective, one says that $\mathcal{F}$ is generated by global sections.

If $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism, then for each point x one has a homomorphism $\phi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ and a linear map $\phi(x) : \mathcal{F}(x) \longrightarrow \mathcal{G}(x)$.

For each sheaf $\mathcal{F} = (M_1, M_2, M, r_1, r_2)$ we introduce k-linear spaces $C^0\mathcal{F} = M_1 \oplus M_2$ and $C^1\mathcal{F} = M$, and define k-linear map $d : C^0\mathcal{F} \longrightarrow C^1\mathcal{F}$ by the formula $d(m_1, m_2) = r_1(m_1) - r_2(m_2)$. We denote the kernel and cokernel of this map by $H^0\mathcal{F}$ and $H^1\mathcal{F}$, respectively, and call them the 0-dimensional and 1-dimensional cohomology spaces of $\mathcal{F}$, respectively.

Clearly, H^0 and H^1 are functors. Here are their principal properties:

(a) For each exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

there is an exact sequence of cohomologies

$$0 \longrightarrow H^0\mathcal{F}_1 \longrightarrow H^0\mathcal{F} \longrightarrow H^0\mathcal{F}_2 \longrightarrow H^1\mathcal{F}_1 \longrightarrow H^1\mathcal{F} \longrightarrow H^1\mathcal{F}_2 \longrightarrow 0;$$

(Moreover, this cohomological sequence is functorial).

(b) H^0 and H^1 commute with direct limits;

(c) $H^0\mathcal{F} = \Gamma(\mathcal{F})$ for each $\mathcal{F}$;

(d) $H^1\mathcal{F} = 0$ for each finite $\mathcal{F}$;

(e) $H^1R = 0$.

(a) follows from Proposition 2.10 of [1]. Other properties are obvious. It is not difficult to prove that the above properties determine H^0 and H^1 uniquely.

The following is the basic result on cohomologies.

Finiteness theorem. *If $\mathcal{F}$ is a coherent sheaf, then*

$$h^i\mathcal{F} = \dim H^i\mathcal{F} < +\infty.$$

The genus of the curve is the number $g = h^1\mathcal{O}$.

The degree of a coherent sheaf $\mathcal{F}$ is defined by the formula

$$\deg \mathcal{F} = h^0\mathcal{F} - h^1\mathcal{F} - \mathrm{rk}(\mathcal{F})(1 - g).$$

The degree is an additive function. This means that if

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

is an exact sequence of coherent sheaves, then

$$\deg \mathcal{F} = \deg \mathcal{F}_1 + \deg \mathcal{F}_2.$$

Note that if $\mathcal{M}$ is a finite sheaf, then $\deg \mathcal{M} = \deg \chi(\mathcal{M})$.

For vector bundles we have the famous

Riemann-Roch theorem. *If $\mathcal{E}$ is a vector bundle, then*

$$\deg \mathcal{E} = \deg \operatorname{ch}(\mathcal{E}).$$

Let K be the divisor of the differential ds. If $\mathcal{E}$ is a vector bundle, set $\check{\mathcal{E}} = \mathcal{E}^*(K)$. Clearly, $\check{\check{\mathcal{E}}} = \mathcal{E}$. We finish with the following important result.

Serre's duality theorem. *For every vector bundle $\mathcal{E}$ there is a non-degenerate canonical pairing*

$$H^0\check{\mathcal{E}} \times H^1\mathcal{E} \longrightarrow k.$$

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ON THE DURRMEYER-TYPE MODIFICATION OF SOME DISCRETE APPROXIMATION OPERATORS

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Abstract. In [10], for continuous functions f from the domain of certain discrete operators L_n the inequalities are proved concerning the modulus of continuity of $L_n f$. Here we present analogues of the results obtained for the Durrmeyer-type modification $\widetilde{L}_n$ of L_n. Moreover, we give the estimates of the rate of convergence of $\widetilde{L}_n f$ in Hölder-type norms

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1. Introduction and Notation

Let I be a finite or infinite interval. Consider a sequence $(J_k)_1^\infty$ of some index sets contained in $Z := \{0, \pm 1, \pm 2, \dots\}$, choose real numbers $\xi_{j,k} \in I$ and fix non-negative functions $p_{j,k}$ continuous on I. Write, formally,

$$L_k f(x) := \sum_{j=J_k} f(\xi_{j,k}) p_{j,k}(x) \quad (x \in I, k \in N := \{1, 2, \dots\}) \qquad (1)$$

for univariate (complex-valued) functions f defined on I. If for $f_0(x) \equiv 1$ on I the values $L_k f_0(x)$ ($x \in I$, $k \in N$) are finite, then $L_k f$ are well-defined for every function f bounded on I. Under appropriate additional assumptions, operators (1) are meaningful also for some locally bounded functions f on infinite intervals I. The fundamental approximation properties of operators (1) in the space $C(I)$ of all continuous functions on I can be deduced, for example, via the general Bohman-Korovkin theorems ([5], Sect. 2.2).

Recently, several authors have investigated relations between the smoothness properties of the functions f and $L_k f$ ([1], [10], [15]). For

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example, taking an arbitrary function $f \in C(I) \cap \mathrm{Dom}(L_n)$, $n \in N$, Kratz and Stadtmüller [10] obtained the following result. Let

$$\sum_{j \in J_k} p_{j,k}(x) \leq c_1 \quad \text{for all} \quad x \in I, \quad k \in N, \tag{2}$$

and let the sum of the above series be independent of x; if, moreover,

$$p'_{j,k} \in C(\overset{\circ}{I}), \quad \sum_{j \in J_k} |(\xi_{j,k} - x)p'_{j,k}(x)| \leq c'_1 \quad \text{for all} \quad x \in \overset{\circ}{I}, \quad k \in N,$$

where c_1, c'_1 are positive constants and $\overset{\circ}{I}$ denotes the interior of I, then the ordinary moduli of continuity of f and $L_n f$ satisfy the inequality

$$\omega(L_n f; \delta) \leq 2(c_1 + c'_1)\omega(f; \delta) \quad (\delta \geq 0).$$

They proved an analogous inequality for the suitable weighted moduli of continuity of f and $L_n f$ when I is an infinite interval and f has the modulus $|f|$ of polynomial growth at infinity. In [12] their result is extended to functions f having $|f|$ of a stronger growth than the polynomial one. [12] also presents some applications of the above-mentioned inequalities in problems of approximation of continuous functions f by $L_n f$ in some Hölder-type norms.

Suppose that for every $j \in J_k$ and every $k \in N$ the integral

$$\int_I p_{j,k}(t)dt$$

coincides with a positive number, say, $1/q_{j,k}$. Denote by $\tilde{L}_k$ the operators given by

$$\tilde{L}_k f(x) \equiv \tilde{L}_k(f)(x) :=$$
$$:= \sum_{j \in J_k} q_{j,k} p_{j,k} \int_I f(t) p_{j,k}(t)dt \quad (x \in I, \quad k \in N) \tag{3}$$

for those measurable (complex-valued) functions f for which the right-hand side of (3) is meaningful. This modification of the classical Bernstein polynomials was first introduced by J.I. Durrmeyer (see [4]). The approximation properties of these polynomials were investigated, for example, in [4], [7], [2]. Some results on the approximation of functions by the Durrmeyer–type modification of the Szász–Mirakyan operators, Baskakov operators or Meyer–König and Zeller operators can be found, for example, in [8], [9], [13], [14], [16].

In this paper we derive Kratz and Stadtmüller type inequalities involving ordinary or weighted moduli of continuity of the functions

f and $\tilde{L}_n f$ on I. Using these inequalities, we obtain estimates of the degree of approximation of f by $\tilde{L}_n f$ in some Hölder-type norms. Theorems 1-3 show that the smoothness properties of $\tilde{L}_n f$ are slightly different from those of $L_n f$.

We adopt the following notation. Given any non-negative function w defined on I and any $x,\ y \in I$, we write $\check{w}(x,y) := \min\{w(x), w(y)\}$.

For an arbitrary function f defined on I we introduce the quantities

$$\|f\|_w := \sup\{|f(x)|w(x) :\ \ x \in I\},$$

$$\Omega_w(f;\delta) := \sup\{|f(x) - f(y)|\check{w}(x,y) :\ x,y \in I, |x-y| \leq \delta\}\ \ (\delta \geq 0).$$

If f is continuous on I and $\|f\|_w < \infty$, we say that $f \in C_w(I)$. The quantity $\Omega_w(f;\delta)$ is called the weighted modulus of continuity of f on I. In case $w(x) = 1$ for all $x \in I$, $\Omega_w(f;\delta)$ becomes $\omega(f;\delta)$ and the symbol $\|f\|$ is used instead of $\|f\|_w$. If the weight w is nondecreasing [nonincreasing] on I, then

$$\Omega_w(f;\delta) := \sup\{|f(x) - f(y)|w(x)\}$$

$$\left[\Omega_w(f;\delta) := \sup\{|f(x) - f(y)|w(y)\}\right],$$

where the supremum is taken over all $x, y \in I$ such that $0 < y - x \leq \delta$.

We denote by W the set of all continuous functions w on I with values not greater than 1, which are positive in the interior of I and satisfy the inequality $\check{w}(x,y) \leq w(t)$ for any three points $x, t, y \in I$ such that $x \leq t \leq y$ (obviously, this inequality holds if, for example, w is nondecreasing, nonincreasing or concave on I). When I is an infinite interval, we indtrouce, in addition, the set Λ of all positive functions η belonging to W such that $\eta(x) \to 0$ as $|x| \to 0$.

Given two weights $w, \eta \in W$, we define a more general modulus of continuity of f on I by

$$\Omega_{w,\eta}(f;\delta) := \sup\{|f(x) - f(y)|\check{w}(x,y)\check{\eta}(x,y)\ :\ x, y \in I,\ |x-y| \leq \delta\}.$$

It reduces to $\Omega_w(f;\delta)$ if $\eta \equiv 1$ on I, and to $\Omega_\eta(f;\delta)$ if $w \equiv 1$ on I. Taking into account that the positive function φ is nondecreasing on the interval $(0,1]$ and has values not greater that 1, we put

$$\|f\|_{w,\eta}^{(\varphi)} := \|f\|_{w\eta} +$$

$$+ \sup\left\{\frac{|f(x) - f(y)|\check{w}(x,y)\check{\eta}(x,y)}{\varphi(|x-y|)}\ :\ x,\ y \in I,\ \ < |x-y| \leq 1\right\}.$$

If this quantity is finite, we call it the Hölder-type norm of f on I. Under the assumption $f \in C_\eta(I)$, $\|f\|_{w,\eta}^{(\varphi)} < \infty$ if and only if there exists a positive constant K such that $\Omega_{w,\eta}(f;\delta) \leq K\varphi(\delta)$ for every

$\delta \in (0,1]$. We write $\|f\|_w^{(\varphi)}$ for $\|f\|_{w,\eta}^{(\varphi)}$ if $\eta \equiv 1$ on I, and $\|f\|_\eta^{(\varphi)}$ if $w \equiv 1$ on I.

Throughout this paper the symbols c_ν ($\nu = 1, 2, \dots$) will mean some positive constants depending only on a given sequence $(L_k)_1^\infty$ and eventually on the considered weights w, η, ρ. The integer part of the real number will be denoted by $[a]$.

2. Smoothness Properties

Let $\tilde{L}_k$, $k \in N$, be the operators defined by (3) such that $\tilde{L}_k f_0(x)$ are finite at every $x \in I$. Put

$$r_k(x) := \sum_{j \in J_k} p_{j,k}(x) - 1 \quad (x \in I, \quad k \in N)$$

and make the standing assumption that all functions $p_{j,k}$ ($j \in J_k$, $k \in N$) are absolutely continuous on every compact interval contained in I. Consider measurable functions f locally bounded on I and belonging to $\mathrm{Dom}(\tilde{L}_n)$ for some $n \in N$. Write, as in Section 1, $\overset{\circ}{I} = \mathrm{Int}\, I$.

Theorem 1. *Suppose that condition* (2) *is satisfied and*

$$\sum_{j \in J_k} q_{j,k} |p'_{j,k}(x)| \int_I |t - x| p_{j,k}(t)\,dt \le \frac{c_2}{w(x)} \tag{4}$$

for $x \in \overset{\circ}{I}$ *and all* $k \in N$, *w being a function of the class* W. *Then*

$$\Omega_w(\tilde{L}_n f; \delta) \le c_3 \omega(f; \delta) + \|f\|_w \omega(r_n; \delta) \quad (\delta \ge 0), \tag{5}$$

where $c_3 = 2(c_1 \|w\| + c_2)$.

Proof. Let $x, y \in I$ $0 < y - x \le \delta$ and let $x_0 := (x+y)/2$. Clearly,

$$\begin{aligned} \tilde{L}_n f(x) - \tilde{L}_n f(y) = \sum_{j \in J_n} q_{j,n} (p_{j,n}(x) - p_{j,n}(y)) \times \\ \times \int_I (f(t) - f(x_0)) p_{j,n}(t)\,dt + f(x_0)(r_n(x) - r_n(y)). \end{aligned} \tag{6}$$

Taking into account (2) and the well-known inequality

$$|f(t) - f(x_0)| \le (1 + [|t - x_0|\delta^{-1}])\omega(f; \delta),$$

we obtain

$$|\tilde{L}_n f(x) - \tilde{L}_n f(y)| \le (2c_1 + A_n(x,y))\omega(f;\delta) + |f(x_0)|\omega(r_n;\delta),$$

where

$$A_n(x,y) := \sum_{j\in J_n} q_{j,n}|p_{j,n}(x) - p_{j,n}(y)|\delta^{-1} \int_{I\setminus I_\delta} |t-x_0|p_{j,n}(t)dt \le$$

$$\le \delta^{-1}\int_x^y \Big(\sum_{j\in J_n} q_{j,n}|p'_{j,n}(s)| \int_{I\setminus I_\delta} |t-x_0|p_{j,n}(t)dt\Big)ds$$

and $I_\delta := I \cap (x_0-\delta, x_0+\delta)$. If $x<s<y$ and $|t-x_0| \ge y-x$, then $|t-x_0| \le 2|t-s|$. Hence, applying (4), we get

$$A_n(x,y) := 2\delta^{-1}\int_x^y \Big(\sum_{j\in J_n} q_{j,n}|p'_{j,n}(s)| \int_I |t-s|p_{j,n}(t)dt\Big)ds \le$$

$$\le 2c_2\delta^{-1}\int_x^y \frac{1}{w(s)}ds\ ,$$

and inequality (5) follows.

The result of Theorem 1 is interesting if $\omega(f;\delta) < \infty$. This holds, for example, for functions $f \in C(I)$ on the compact interval I. If I is an infinite interval, the assumption $\omega(f;\delta) < \infty$ implies the restriction $f(x) = O(|x|)$ as $|x| \to \infty$. So, in this case, it is convenient to use the weighted modulus of continuity $\Omega_\eta(f;\delta)$ with some $\eta \in \Lambda$. If $f \in C_\eta(I)$, then this modulus is a nondecreasing function of δ on the interval $[0,\infty)$. It is easy to verify that, for every $\delta > 0$ and for all $x, y \in I$ there holds the inequality

$$|f(x) - f(y)|\check{\eta}(X,y) \le (1 + [\delta^{-1}|x-y|])\Omega_\eta(f;\delta). \qquad (7)$$

Moreover, in case $\rho \in \Lambda$ and $\rho(x)/\eta(x) \to 0$ as $|x| \to \infty$ we have $\Omega_\rho(f;\delta) \to 0$ as $\delta \to 0+$, whenever $f \in C_\eta(I)$ is uniformly continuous on each finite interval contained in I.

Note that under the assumptions $\eta \in \Lambda$, $f \in C_\eta(I)$ and $\widetilde{L}_k(1/\eta)(x) < \infty$ we have $|L_k f(x)| < \infty$. If, moreover, $\rho \in \Lambda$ and

$$\widetilde{L}_k\Big(\frac{1}{\eta}\Big)(x) \le \frac{c_4}{\rho(x)} \quad \text{for all} \quad x \in I \ \text{ and } \ k \in N \qquad (8)$$

then $\|\widetilde{L}_k f\|_\rho < \infty$.

In the next two theorems it is assumed that I is an infinite interval. ■

Theorem 2. *Let condition* (2) *be satisfied. Suppose, moreover, that there exist functions* $w \in W$, $\rho, \eta \in \Lambda$, $\rho \leq \eta$ *such that* (4), (8) *and*

$$\sum_{j \in J_k} q_{j,k} |p'_{j,k}(x)| \int_I \frac{|t-x|}{\eta(t)} p_{j,k}(t)dt \leq \leq \frac{c_5}{w(x)\rho(x)} \quad \text{for a.e. } x \in \overset{\circ}{I} \text{ and } k \in N \tag{9}$$

hold. Then

$$\Omega_{w,\rho}(\widetilde{L}_n f;\delta) \leq c_6 \Omega_\eta(f;\delta) + \|f\|_{w\rho}\omega(r_n;\delta) \quad (\delta \geq 0), \tag{10}$$

where $c_6 = 2((c_1 + c_4)\|w\| + c_2 + c_5)$.

Proof. Consider $x, y \in I$ such that $0 < y - x \leq \delta$. Retain the symbol x_0 used in the proof of Theorem 1 and start with identity (6). In view of (7),

$$|\widetilde{L}_n f(x) - \widetilde{L}_n f(y)| \leq B_n(x,y)\Omega_\eta(f;\delta) + |f(x_0)||r_n(x) - r_n(y)|,$$

where

$$B_n(x,y) := \sum_{j \in J_n} q_{j,n}|p_{j,n}(x) - p_{j,n}(y)| \int_I (1 + [\delta^{-1}|t - x_0|]) \frac{1}{\check{\eta}(t,x_0)} p_{j,n}(t)dt.$$

Observing that for every $t \in I$

$$\frac{\check{\rho}(x,y)}{\check{\eta}(t,x_0)} \leq 1 + \frac{\check{\rho}(x,y)}{\eta(t)} \tag{11}$$

and applying (2), we obtain

$$B_n(x,y)\check{\rho}(x,y) \leq 2c_1 + \sum_{j \in J_n} q_{j,n}|p_{j,n}(x) - p_{j,n}(y)| \int_I \frac{\check{\rho}(x,y)}{\eta(t)} p_{j,n}(t)dt +$$

$$+\delta^{-1} \sum_{j \in J_n} q_{j,n} \int_x^y |p'_{j,n}(s)|ds \int_{I - I_\delta} \left(1 + \frac{\check{\rho}(x,y)}{\eta(t)}\right) |t - x_0| p_{j,n}(t)dt.$$

Further, the inequality $|t - x_0| \leq 2|t - s|$ $(t \in I \backslash I_\delta,\ x < s < y)$ and assumptions (4), (8), (9) lead to

$$B_n(x,y)\check{\rho}(x,y) \leq 2(c_1 + c_4) + 2\delta^{-1} \int_x^y \frac{c_2 + c_5}{w(s)} ds.$$

The desired estimate is now evident. ∎

For functions f for which $|f|$ is of the polynomial growth at infinity our result can be stated as follows.

Theorem 3. *Let conditions* (2), (4) *be satisfied and let* $\eta(x) = (1 + |x|)^{-\sigma}$ $x \in I$ $\sigma > 0$. *Suppose that inequality* (9) *in which* $\rho = \eta$ *holds. Then*

$$\Omega_{w,\eta}(\widetilde{L}_n f;\delta) \le c_7 \Omega_\eta(f;\delta) + \|f\|_{w\eta}\omega(r_n;\delta) \quad (\delta \ge 0),$$

where $c_7 = 2(c_1 + 2 \cdot 3^\sigma c_1 + c_2 + 2c_5)$.

Proof. To see this it is enough to make a slight modification in the evaluation of the term $B_n(x,y)$ occurring in the proof of Theorem 2. Namely, let us divide the interval I into two sets I_n and $I \backslash I_h$, where $I_h := I \cap (x_0 - h, x_0 + h)$, $h = y - x$. If $t \in I_h$, then $[\delta^{-1}|t - x_0|] = 0$ and

$$\frac{\check{\eta}(x,y)}{\eta(t)} \le 3^\sigma \check{\eta}(x,y)\left(\frac{1}{\eta(x)} + \frac{1}{\eta(y)}\right) \le 2 \cdot 3^\sigma.$$

This inequality, (11) and (2) imply

$$B_n(x,y)\check{\eta}(x,y) \le 2(1 + 2 \cdot 3^\sigma)c_1 +$$

$$+ \sum_{j\in J_n} q_{j,n}|p'_{j,n}(s)ds| \int\limits_{I\backslash I_h} \left(\frac{|t-x_0|}{\delta} + \frac{\check{\eta}(x,y)}{\eta(t)}\left(1 + \frac{|t-x_0|}{y-x}\right)\right)p_{j,n}(t)dt.$$

Observing that $|t - x_0| \le 2|t - s|$, $|t - x_0| \le y - x$ whenever $t \in I \backslash I_h$ $x < s < y$, we obtain, on account of (4) and (9) (with $\rho = \eta$),

$$B_n(x,y)\check{\eta}(x,y) \le 2(1 + 2 \cdot 3^\sigma)c_1 +$$

$$+\frac{2}{\delta}\int\limits_x^y \frac{c_2}{w(s)}ds + 4\frac{\check{\eta}(x,y)}{y-x}\int\limits_x^y \left(\sum_{j\in J_n} q_{j,n}|p'_{j,n}(s)| \int\limits_I \frac{|t-s|}{\eta(t)} p_{j,n}(t)dt\right)ds \le$$

$$\le 2(1 + 2 \cdot 3^\sigma)c_1 + \frac{2}{y-x}\int\limits_x^y \frac{c_2 + c_5}{w(s)}ds\,,$$

Thus

$$B_n(x,y)\check{w}(x,y)\check{\eta}(x,y) \le 2(1 + 2 \cdot 3^\sigma)c_1\|w\| + 2c_2 + 4c_5, \quad \blacksquare$$

Remark 1. For many known operators the functions $r_k(x) \equiv 0$ on I, the quantities

$$\mu_{2,k}(x) := \sum_{j\in J_k} (\xi_{j,k} - x)^2 p_{j,k}(x)$$

are finite at every $x \in I$ and positive in $\overset{\circ}{I}$; moreover,

$$p'_{j,k}(x)\mu_{2,k}(x) = p_{j,k}(x)(\xi_{j,k} - x) \tag{12}$$

for every $x \in \mathring{I}$ and every $k \in N$. In view of identity (12) and the Cauchy-Schwartz inequality the left-hand side of (4) can be estimated from above by $(\tilde{\mu}_{2,k}(x)/\mu_{2,k}(x))^{1/2}$, where

$$\tilde{\mu}_{2,k}(x) := \sum_{j\in J_k} q_{j,k}|p_{j,k}(x)| \int_I (t-x)^2 p_{j,k}(t)dt.$$

Therefore, in this case, assumption (4) can be replaced by

$$\frac{\tilde{\mu}_{2,k}(x)}{\mu_{2,k}(x)} \le \frac{c_2^2}{w^2(x)} \quad \text{for all} \quad x \in \mathring{I}, \quad k \in N. \tag{13}$$

Analogously, the left-hand side of (9) can be estimated by

$$\frac{1}{\mu_{2,k}(x)}\left(\tilde{\mu}_{2,k}(x) \sum_{j\in J_k} q_{j,k}(\xi_{j,k}-x)^2 p_{j,k}(x) \int_I \frac{p_{j,k}(t)}{\eta^2(t)}dt\right)^{1/2}.$$

Hence, if

$$\frac{1}{\mu_{2,k}(x)} \sum_{j\in J_k} q_{j,k}p_{j,k}(x)(\xi_{j,k}-x)^2 \int_I \frac{p_{j,k}(t)}{\eta^2(t)}dt \le \frac{c_8^2}{\rho^2(x)} \tag{14}$$

for all $x \in \mathring{I}$, $k \in N$, then (9) holds with $c_5 = c_2 \cdot c_8$.

Remark 2. Let $w \in W$, $\eta \in \Lambda$. Define the weighted modulus $\Phi_w(f;\delta)$ and $\Phi_{w,\eta}(f;\delta)$ as in Section 1, replacing $\check{w}(x,y)$ by

$$\overline{w}(x,y) := \begin{cases} 0 & \text{if} \quad w(x)=0 \quad \text{or} \quad w(y)=0, \\ \left(\dfrac{1}{w(x)}+\dfrac{1}{w(y)}\right)^{-1} & \text{otherwise,} \end{cases}$$

and $\check{\eta}(x,y)$ by $\overline{\eta}(x,y)$, respectively. Since $\overline{w}(x,y) \le \check{w}(x,y)$ for every pair of points $x,y \in I$, Theorem 1 remains valid for $\Phi_w(\tilde{L}_n f;\delta)$. Further, in this case, inequality (7) becomes

$$|f(x)-f(y)|\overline{\eta}(x,y) \le 2(1+[\delta^{-1}|x-y|])\Phi_\eta(f;\delta).$$

Consequently, under the assumptions of Theorem 2, the modulus $\Phi_{w,\rho}(\tilde{L}_n f;\delta)$ and $\Phi_\eta(f;\delta)$ satisfy inequality (10) with the constant $2c_6$ instead of c_6.

Note that, for the weight $\eta(x) = (1+|x|)^{-\sigma}$ with the parameter $\sigma > 0$, the modulus $\Phi_\eta(f;\delta)$ is equivalent to the one introduced in [10], p. 331 (see also [12]).

3. Approximation Properties

Considering still the functions f as in Section 2 we first estimate the ordinary weighted norm of the difference $\widetilde{L}_n f - f$.

Theorem 4. *Let condition* (2) *be satisfied and let*

$$\rho(x)\widetilde{L}_k\Big(\frac{1}{\eta^2}\Big)(x) \le \frac{c_9}{\eta(x)} \quad \text{for all} \quad x \in I, \ \ k \in N, \tag{15}$$

$$\rho(x)\widetilde{\mu}_{2,k}(x) \le c_{10}\eta(x)\delta_k^2 \quad \text{for all} \quad x \in I, \ \ k \in N, \tag{16}$$

where $(\delta_k)_1^\infty$ *is a sequence of positive numbers,* η *is a positive function on* I *and* ρ *is a non-negative one such that* $\rho \le \eta$. *Then*

$$\|\widetilde{L}_n f - f\|_\rho \le c_{11}\Omega_\eta(f;\delta_n) + \|f\|_\rho \|r_n\|, \tag{17}$$

where $c_{11} = c_1 + (c_1c_9)^{1/2} + (c_9c_{10})^{1/2} + c_{10}$.

Proof. Start with the obvious identity

$$\widetilde{L}_n f(x) - f(x) = \sum_{j\in J_n} q_{j,n}p_{j,n}(x)\int_I (f(t) - f(x))p_{j,n}(t)dt + f(x)r_n(x)$$

and take a positive number δ. In view of (7) and the inequality $(\check{\eta}(x,t))^{-1} \le (\eta(x))^{-1} + (\eta(t))^{-1}$ we have

$$|\widetilde{L}_n f(x) - f(x)| \le \gamma_n(x)\Omega_\eta(f;\delta) + |f(x)|\cdot\|r_n\|,$$

where

$$\gamma_n(x) := \sum_{j\in J_n} q_{j,n}p_{j,n}(x)\int_I (1 + [\delta^{-1}|t-x|])\Big(\frac{1}{\eta(x)} + \frac{1}{\eta(t)}\Big)p_{j,n}(t)dt.$$

Further, by (2), (15) and (16) and the Cauchy-Schwartz inequality we obtain

$$\gamma_n(x)\rho(x) \le c_1 + \widetilde{L}_n\Big(\frac{1}{\eta}\Big)(x)\rho(x) + \delta^{-2}\frac{\rho(x)}{\eta(x)}\widetilde{\mu}_{2,n}(x) +$$

$$+\rho(x)\delta^{-1}\sum_{j\in J_n} q_{j,n}p_{j,n}(x)\int_I \frac{|t-x|}{\eta(t)}p_{j,n}(t)dt \le$$

$$\le c_1 + \Big(c_1\widetilde{L}_n\Big(\frac{1}{\eta^2}\Big)(x)\Big)^{1/2}\rho(x) + c_{10}\delta^{-2}\delta_n^2 +$$

$$+\rho(x)\delta^{-1}(\widetilde{\mu}_{2,n}(x))^{1/2}\Big(\widetilde{L}_n\Big(\frac{1}{\eta^2}\Big)(x)\Big)^{1/2} \le$$

$$\le c_1 + (c_1c_9)^{1/2} + c_{10}\delta^{-2}\delta_n^2 + (c_9c_{10})^{1/2}\delta^{-1}\delta_n.$$

Choosing $\delta = \delta_n$, we get (17) at once. ∎

Remark 3. In the case when $\eta(x) = 1$ for all $x \in I$, the constant c_{11} in (17) is equal to $c_1 + c_{10}$. If we use the modulus $\Phi_\eta(f;\delta)$ (defined in Remark 2) instead of $\Omega_\eta(f;\delta)$, the constant c_{11} should be multiplied by 2.

Passing to approximation in the Hölder-type norm we note that, for an arbitrary $\nu_n \in (0,1]$,

$$\|\widetilde{L}_n f - f\|_{w,\eta}^{(\varphi)} \le \Big(1 + \frac{2}{\varphi(\nu_n)}\Big)\|\widetilde{L}_n f - f\|_{w\eta} + \\ + \sup\Big\{\frac{1}{\varphi(\delta)}\Big(\Omega_{w,\eta}(\widetilde{L}_n f;\delta) + \Omega_{w,\eta}(f;\delta)\Big) \ : \ 0 < \delta \le \nu_n\Big\} \tag{18}$$

(see, for example, [11], [12]). This inequality, Theorem 4 and the estimates obtained in Section 2 allow us to state a few standard results. We will formulate only one of them. Namely, combining inequality (18) with Theorems 1 and 2 gives

Theorem 5. *Let conditions* (2), (4) *be satisfied and let* $(\delta_k)_1^\infty$ *be a sequence of numbers from* $(0,1]$ *for which* (16) *holds with* $\rho = w$ *and* $\eta \equiv 1$ *on* I. *Then*

$$\|\widetilde{L}_n f - f\|_w^{(\varphi)} \le c_{12} \sup\Big\{\frac{\omega(f;\delta)}{\varphi(\delta)} \ : \ 0 < \delta \le \delta_n\Big\} + \|f\|_w \Delta_n^{(\varphi)},$$

where $c_{12} = 3c_1 + 2c_2 + 3c_{10} + (1 + 2c_1)\|w\|$ *and*

$$\Delta_n^{(\varphi)} = 3\|r_n\|/\varphi(\delta_n) + \sup\{\omega(r_n;\delta)/\varphi(\delta) \ : \ 0 < \delta \le \delta_n\}.$$

Remark 4. Clearly, if the assumptions of Theorems 1 – 5 hold for positive integers k belonging to a certain subset N_1 of N, then the corresponding assertions remain valid only if $n \in N_1$.

4. Examples

1) The Bernstein polynomials $B_k f \equiv L_k f$ are defined by (1) with $\xi_{j,k} = j/k$, $p_{j,k} = \binom{k}{j} x^j (1-x)^{k-j}$, $I = [0,1]$, $J_k = \{0,1,2,\dots,k\}$. The corresponding Bernstein-Durrmeyer polynomials $\widetilde{L}_k f \equiv \widetilde{L}_k f$ are of the form (3) in which $q_{j,k} = k+1$ for all $j \in J_k$, $k \in N$. In this case $r_k(x) = 0$ for all $x \in I$, the constant c_1 in (2) equals 1, $\mu_{2,k}(x) = x(1-x)/k$ and equality (12) is true. Since

$$\widetilde{\mu}_{2,k}(x) = \frac{2x(1-x)(k-3)+2}{(k+2)(k+3)} \quad (x \in I, \ \ k \in N)$$

(see [4]), we easily state that condition (13) is satisfied with $c_2 = 1$, $w(x) = (x(1-x))^{1/2}$. Hence, in view of Theorem 1 (and Remark 1), for every $f \in C(I)$ and every $n \in N$,

$$\Omega_w(\widetilde{B}_n f; \delta) \leq 3\omega(f; \delta) \quad (\delta \geq 0)$$

Further, $\widetilde{\mu}_{2,k}(x) \leq \frac{1}{2k}$ for all $x \in I$, $k \in N$ (see [4], p. 327). Therefore (16) holds with $\rho(x) = \eta(x) = 1$ for all $x \in I$, $\delta_k = k^{-1/2}$ and $c_{10} = 1/2$. Thus Theorem 4 gives

$$\|\widetilde{B}_n f - f\| \leq \frac{3}{2}\omega(f; n^{-1/2}) \quad \text{for all} \quad n \in N$$

(cf. [4], Theorem II.2). Also, Theorem 5 applies with $w(x) = (x(1-x))^{1/2}$, $\delta_n = n^{-1/2}$, $c_{12} = 8$ and $\Delta_n^{(\varphi)} = 0$.

2) The Meier-König and Zeller operators $M_k \equiv L_k$ are defined by $\xi_{j,k} = j/(j+k)$, $p_{j,k}(x) = \binom{k+j-1}{j} x^j (1-x)^k$, $x \in I = [0,1)$, $j \in J_n = N_0$, $N_0 := \{0, 1, \ldots\}$. Their Durrmeyer modification $\widetilde{M} \equiv \widetilde{L}_k$ are of the form (3) in which $q_{j,k} = (k+j)(k+j+1)/k$. Condition (2) holds with $c_1 = 1$. Since

$$p'_{j,k}(x)\frac{x(1-x)^2}{k} = p_{j,k+1}(x)\left(\frac{j}{k+j} - x\right)^2 \quad (0 < x < 1),$$

the left-hand side of (4) can be estimated from above by

$$\frac{k}{x(1-x)^2}\left(\left\{\sum_{j=0}^{\infty}\left(\frac{j}{k+j} - x\right)^2 p_{j,k+1}(x)\right\} \times \right.$$
$$\left. \times \left\{\sum_{j=0}^{\infty} q_{j,k} p_{j,k+1}(x) \int_0^1 (t-x)^2 p_{j,k}(t)dt\right\}\right)^{1/2}$$

for all $x \in (0,1)$, $k \in N$. If $k \geq 3$, the expression in the first curly brackets is not greater than $2x(1-x)^2/k$ (see [3]); straightforward calculation shows that the expression in the second ones does not exceed $7(1-x)^2/k$. Thus, for the functions $f \in C(I) \cap \mathrm{Dom}(\widetilde{M}_n)$ and $\widetilde{M}_n f$ $(n \geq 3)$, inequality (5) applies with $c_3 = 10$, $w(x) = x^{1/2}$ and $r_n(x) = 0$ for all $x \in I$.

3) The Baskakov-Durrmeyer operators $\widetilde{U}_{k,c} \equiv \widetilde{L}_k$ (with a parameter $c \in N_0$) are defined by (3) in which $I = [0, \infty)$, $J_k = N_0$, $p_{j,k}(x) = (-1)^j x^j \psi_{k,c}^{(j)}(x)/j!$, $\psi_{k,c}(x) = e^{-kx}$ if $c = 0$, and $\psi_{k,c}(x) = (1+cx)^{-k/c}$ if $c \geq 1$, $q_{j,k} = k - c$ for $k > c$ (see [9]). Now $r_k(x) = 0$ for all $x \in I$,

$k \in N$, $c_1 = 1$, $\mu_{2,k}(x) = x(1+cx)/k$ for all $x \in I$, $k > c$ and condition (12) holds with $\xi_{j,k} = j/k$. Further,

$$\tilde{\mu}_{2,k} = \frac{2x(1+cx)(k+3c)+2}{(k-2c)(k-3c)} \quad \text{for} \quad x \in I, \quad k > 3c.$$

Hence Theorem 1 (via Remarks 1, 4) applies for $n > 3c$, with $w(x) = (x(/(1+x))^{1/2}$, $c_3 = 2(1+c_2)$, $c_2 = (2(1+3c)(1+6c)/(1+c))^{1/2}$.

4) The Szász-Mirakyan-Durrmeyer operators $\tilde{S}_k$ are the special case of operators $\tilde{U}_{k,c}$ defined in 3), with $c = 0$. From 3) we know that, for these operators, conditions (2) and (13) hold with $c_1 = 1$, $c_2 = 2^{1/2}$ and $w(x) = (x/(1+x))^{1/2}$. Consider $f \in C_\eta(I)$ with the weight $\eta(x) = (1+x)^{-\sigma}$ where $\sigma \in N$. It is easy to see that, for $k \geq 2\sigma$,

$$\int_0^\infty \frac{1}{\eta^2(t)} p_{j,k}(t)dt = \frac{k^j}{j!} \int_0^\infty (1+t)^{2\sigma} t^j e^{-kt} dt \leq$$

$$\leq 2^{2\sigma-1}\left(\frac{1}{k} + \frac{k^j}{j!}\int_0^\infty t^{2\sigma+j} e^{-kt} dt\right) = 2^{2\sigma-1}\frac{1}{k}\left(1 + \frac{(2\sigma+j)!}{j!}k^{-2\sigma}\right) \leq$$

$$\leq 2^{2\sigma-1}\frac{1}{k}\left(1 + \left(\frac{j}{k}+1\right)^{2\sigma}\right).$$

Consequently, the left-hand side of (14) is not greater than

$$\frac{2^{2\sigma-1}}{\mu_{2,k}(x)} \sum_{j=0}^\infty \left(\frac{j}{k} - x\right)^2 p_{j,k}(x)\left(1 + 2^{2\sigma-1}\left((1+x)^{2\sigma} + \left(\frac{j}{k} - x\right)^{2\sigma}\right)\right) =$$

$$= 2^{2\sigma-1}\left(1 + 2^{2\sigma-1}(1+x)^{2\sigma}\right) + \frac{4^{2\sigma-1}}{\mu_{2,k}(x)} \sum_{j=0}^\infty \left(\frac{j}{k} - x\right)^{2\sigma+2} p_{j,k}(x) \leq$$

$$\leq c_{13}(1+x)^{2\sigma}$$

(see [10], p. 334). Applying Theorem 3 (together with Remarks 1, 4), we get the estimate

$$\Omega_{w,\eta}(\tilde{S}_n f; \delta) \leq c_{14}\Omega_\eta(f;\delta) \quad (\delta \geq 0, \ \ n \geq 2\sigma). \tag{19}$$

Since $\tilde{\mu}_{2,k}(x) \leq 2(1+x)/k$, conditions (15) and (16) are satisfied with $\rho(x) = (1+x)^{-\sigma-1}$ and $\delta_k = k^{-1/2}$. Consequently, Theorem 4 gives

$$\|\tilde{S}_n f - f\|_\rho \leq c_{15}\Omega_\eta(f; n^{-1/2}) \quad \text{for all} \ \ n \in N.$$

Combining this result and (19) with the general inequality (18), we easily verify that, for $n \geq 2\sigma$,

$$\|\tilde{S}_n f - f\|_{w,\rho}^{(\varphi)} \leq c_{16} \sup\left\{\frac{1}{\varphi(\delta)}\Omega_\eta(f;\delta) \ : \ 0 < \delta \leq n^{-1/2}\right\}.$$

5) The generalized Favard operators $F_k \equiv L_k$ are deefined by (1) with $\xi_{j,k} = j/k$, $J_k = Z$, $I = (-\infty, \infty)$ and

$$p_{j,k}(x) \equiv p_{j,k}(\gamma; x) = (\sqrt{2\pi}k\gamma_k)^{-1} \exp\Big(-\frac{1}{2}\gamma_k^{-2}\Big(\frac{j}{k} - x\Big)^2\Big),$$

$\gamma = (\gamma_k)_1^\infty$ being a positive null sequence satisfying

$$k^2\gamma_k^2 \geq \frac{1}{2}\pi^{-2}\log k \quad \text{for} \quad k \geq 2, \quad \gamma_1^2 \geq \frac{1}{2}\pi^{-2}\log 2$$

(see [6]). Denote by $\widetilde{F}_k$ their Durrmeyer modification of form (3) in which $q_{j,k} = k$ for all $j \in Z$ and $k \in N$. As is known ([6], [12]), for all $x \in I$ and $k \in N$,

$$|r_k(x)| \equiv |r_k(\gamma; x)| = \Big|\sum_{j=-\infty}^{\infty} p_{j,k}(\gamma; x) - 1\Big| \leq 2$$
$$\text{or} \quad |r_k(\gamma; x)| \leq 7\pi\gamma_k.$$

$\mu_{2,k}(x) \equiv \mu_{2,k}(\gamma; x) \leq 51\gamma_k^2$; moreover, $\omega(r_k(\gamma; x)| \leq 16\pi\delta$ for every $\delta \geq 0$ (see [10], p. 336). It is easy to see that

$$\widetilde{\mu}_{2,k}(x) \equiv \widetilde{\mu}_{2,k}(\gamma; x) = \mu_{2,k}(\gamma; x) + \gamma_k^2(1 + r_k(\gamma; x)) \leq 54\gamma_k^2.$$

Observing that

$$p'_{j,k}(\gamma; x) = \gamma_k^{-2}\Big(\frac{j}{k} - x\Big)p_{j,k}(\gamma; x)$$

and applying the Cauchy-Schwartz inequality, we estimate the left-hand side of (4) by

$$k\gamma_k^{-2} \sum_{j=-\infty}^{\infty} \Big|\frac{j}{k} - x\Big|p_{j,k}(\gamma; x) \int_{-\infty}^{\infty} |t - x|p_{j,k}(\gamma; t)dt \leq$$
$$\leq \gamma_k^{-2}(\mu_{2,k}(\gamma; x))^{1/2}(\widetilde{\mu}_{2,k}(\gamma; x))^{1/2},$$

i.e., $w(x) = 1$ for all real x and $c_2 = 52,5$. Thus Theorem 1 yields the estimate

$$\omega(\widetilde{F}_n f; \delta) \leq 111\omega(f; \delta) + 16\pi\delta\|f\| \quad (\delta \geq 0)$$

for every $n \in N$ and every $f \in C(I)$. Clearly, this inequality is interesting if $f \in C(I)$ is bounded on I.

Consider now $f \in C_\eta(I)$ where $\eta(x) = \exp(-\sigma x^2)$ $\sigma > 0$. If $\sigma\gamma_k^2 \geq 3/32$, then

$$\exp(\sigma x^2)\exp\Big(-\frac{1}{2}\gamma_k^{-2}\Big(\frac{j}{k}-x\Big)^2\Big)\exp\Big(-\frac{1}{2}\gamma_k^{-2}\Big(\frac{j}{k}-t\Big)^2\Big) \leq$$
$$\leq \exp(4\sigma x^2)\exp\Big(-\frac{1}{8}\gamma_k^{-2}\Big(\frac{j}{k}-x\Big)^2\Big)\exp\Big(-\frac{1}{8}\gamma_k^{-2}\Big(\frac{j}{k}-t\Big)^2\Big);$$

whence

$$\widetilde{F}_k(1/\eta)(x) \leq 2(1 + r_k(2\gamma; x))\exp(4\sigma x^2).$$

Analogously, one can show that the left-hand side of (9) is not greater than

$$2\gamma_k^{-2}\mu_{2,k}(\gamma; x))^{1/2}(\widetilde{\mu}_{2,k}(2\gamma; x))^{1/2}\exp(4\sigma x^2).$$

provided that $\sigma\gamma_k^2 \leq 3/64$. Further (see [12]),

$$r_k(2\gamma; x) \leq 2/15, \quad \mu_{2,k}(2\gamma; x) \leq 23\gamma_k^2$$

and

$$\widetilde{\mu}_{2,k}(2\gamma; x) = \mu_{2,k}(2\gamma; x) + (2\gamma_k)^2(1 + r_k(2\gamma; x)) \leq \frac{413}{15}\gamma_k^2.$$

Thus Theorem 2 applies with $w(x) \equiv 1$, $\rho(x) = \exp(-4\sigma x^2)$, $c_4 = 68/15$, $c_5 = 75$ (i.e. $c_6 = 271$) and n such that $\sigma\gamma_n^2 \leq 3/64$. In the same way one can show that Theorem 4 is true with $\rho(x) = \rho_1(x) := \exp(-7\sigma x^2)$, $\delta_n = \gamma_n$, $\sigma\gamma_n^2 \leq 3/64$ and a positive absolute constant c_{11}. From these results the estimate of $\|\widetilde{F}_n f - f\|_{\rho_1}^{(\varphi)}$ follows at once via inequality (18).

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FRACTIONAL TYPE OPERATORS IN WEIGHTED GENERALIZED HÖLDER SPACES

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ABSTRACT. Weighted Zygmund type estimates are obtained for the continuity modulus of some convolution type integrals. In the case of fractional integrals this is strengthened to a result on isomorphism between certain weighted generalized Hölder type spaces.

რეზიუმე. გარკვეული სახის კონვოლუციების უწყვეტობის მოდულებისათვის დადგენილია ზიგმუნდის ტიპის შეფასებები, საიდანაც წილადური ინტეგრალების შემთხვევაში დამტკიცებულია თეორემა წონიანი განზოგადებული ჰელდერის ტიპის სივრცეების იზომორფულობის შესახებ.

1. INTRODUCTION

A great number of results is known concerning boundedness of convolution type operators in spaces of summable functions, including the weighted case. In the spaces of continuous functions such as $H_0^\omega(\rho)$ the convolution type operators are less investigated. The goal of this paper is to fill a gap to a certain extent in investigations of such a kind.

We consider here the Volterra convolution type operators

$$K\varphi = \int_a^x k(x-t)\varphi(t)\,dt, \quad a < x < b, \tag{1.1}$$

in the weighted generalized Hölder spaces $H_0^\omega(\rho)$ (see definitions in Sec.2), $-\infty < a < b < \infty$. The kernel $k(x)$ is assumed to be close in a sense to a power function.

The result of the type

$$K : H_0^\omega(\rho) \to H_0^{\omega_1}(\rho) \tag{1.2}$$

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for certain characteristic functions $\omega(h)$ and $\omega_1(h)$ was earlier known in the case of the power kernel $k(x) = x^{\alpha-1}$, $0 < \alpha < 1$, and a power weight function $\rho(t)$. We deal here with arbitrary kernels and weights, i.e. not necessarily power ones.

We introduce a certain class V_λ of kernels and the class w_μ of weight functions for which we manage to give the weighted Zygmund type estimate, that is, to estimate the modulus of continuity $\omega(\rho K_\varphi, h)$ by the modulus of continuity $\omega(\rho\varphi, h)$. This estimate provides the general result of the type (1.2).

In the case of purely power kernel, i.e. in the case of the fractional integration operator

$$I^\alpha_{a+}\varphi = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \quad 0 < \alpha < 1, \tag{1.3}$$

the result (1.2) is extended to isomorphism:

$$I^\alpha_{a+}[H_0^\omega(\rho)] = H_0^{\omega_\alpha}(\rho) \tag{1.4}$$

with $\omega_\alpha(h) = h^\alpha\omega(h)$. This is achieved by the preliminary derivation of Zygmund type estimate for fractional differentiation. The latter is treated in a difference form due to A.Marchaud [9] and G.H.Hardy and J.E.Littlewood [2] (see [17], Sec.13, in this connection):

$$D^\alpha_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(x)}{(x-a)^\alpha} + \alpha\int_a^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt\right], \quad 0<\alpha<1. \tag{1.5}$$

The paper is organized as follows. In Sec.2 we give necessary preliminaries. Sec.3 contains Zygmund type estimates for the operator (1.2) in the case of kernels in V_λ in the non-weighted case first (Theorem 1) and afterwards in the weighted case (Theorem 2). In Theorem 3 we give conditions of Zygmund–Bari–Stechkin type on a characteristic function $\omega(h)$ guaranteeing the result (1.2) for $k(x) \in V_\lambda$ and weighted functions in w_μ. The characteristic function $\omega_1(h)$ in (1.2) proves to be equal to $hk(h)\omega(h)$. We note corollary of Theorem 3 for $k(x) = x^{\alpha-1}(\ln\gamma/x)^\beta$, $\gamma > b-a$.

In Sec.4 we establish the weighted Zygmund type estimate for $D^\alpha_{a+}f$ with a weight function in w_μ (Theorem 4). We prove the assertion $D^\alpha_{a+} : H_0^\omega(\rho) \to H_0^{\omega-\alpha}(\rho)$ with $\omega_{-\alpha}(h) = h^{-\alpha}\omega(h)$ under appropriate assumptions on $\omega(h)$ and $\rho(x)$ (Theorem 5).

As a corollary of Theorems 3 and 5 we give conditions for validity of the second index law of E.R.Love within the framework of the spaces

$H_0^\lambda(x^\mu)$. Finally, in Sec.5 we prove the isomorphism (1.1) (Theorem 6).

Presented theorems generalize the results of the papers [10]–[12],[18], where the power case for both $k(x) = x^{\alpha-1}$ and $\rho(x) = (x-a)^\alpha$ was considered. The presentation of the results of [10] in the non-weighted case can be also found in [18], Sec.13. Note that in [12] the case $\rho(x) = (x-a)^\mu(b-x)^\nu$ was also considered, not contained in the results of the present paper. The origin of the statement (1.4) is the classical result by G.H.Hardy and J.E.Littlewood [2] for the fractional integration concerning the case $\omega(h) = h^\lambda$, $\rho(x) \equiv 1$, $\alpha + \lambda < 1$. (As for the case $\omega(h) = \prod_{k=1}^n |x - x_k|^{\mu_k}$, see [13] and [17], Sec.13.)

We also note the papers [4]–[5] where Zygmund type estimates are given for the fractional integrodifferentiation in the case of L_p-moduli of continuity.

The question we finally note as open is whether $I_{a+}^\alpha[H_0^\omega(\rho)] = H_0^\omega(\rho)$ in the case of purely imaginary α, under the appropriate assumptions on $\omega(h)$ and $\rho(x)$. We refer to the paper [7] by E.R.Love concerning such fractional integrals (see also [17], Sec.2, $n^0 4$).

2. Preliminaries

We follow the papers [14]–[15] in the definitions below. Positive constants which can be different at different places will be denoted by c.

Definition 1. We say that $\psi(x) \in W_\mu = W_\mu([0,l])$ if $\psi(x) \in C([0,l])$, $\psi(0) = 0$, $\psi(x) > 0$ for $x > 0$, $\psi(x)$ is almost increasing, while $\psi(x)/x^\mu$ is almost decreasing and there exists c such that

$$\left|\frac{\psi(x) - \psi(y)}{x - y}\right| \le c\frac{\psi(x^*)}{x^*}, \quad x^* = \max(x,y). \tag{2.1}$$

We remind that a non-negative function $\psi(x)$, $0 \le x \le l$, $0 < l \le \infty$, is called almost increasing (decreasing) if $\psi(x) \le c\psi(y)$ for all $x \le y$ ($x \ge y$, resp.), this notion being due to S.Bernstein.

Definition 2. We say that $\psi(x) \in W_\mu^*$ if $\psi(x) \in W_\mu$ and $\psi(x)/x^{\mu-\varepsilon}$ is almost increasing for all $\varepsilon > 0$.

We shall also need the following modification of the Definition 2.

Definition 3. We say that a non-negative function $k(x)$ on $[0,l]$ belongs to the class V_λ, $\lambda > 0$ if

i) $k(x) \not\equiv 0$, $x^\lambda k(x)$ is almost increasing and $x^\lambda k(x)\Big|_{x=0} = 0$;

ii) there exists ε, $0 < \varepsilon < \lambda$, such that $x^{\lambda-\varepsilon}k(x)$ is almost decreasing;

iii) there exists c such that

$$\left|\frac{k(x)-k(y)}{x-y}\right| \leq c\frac{k(x^*)}{x^*}, \quad x^* = \max(x,y). \tag{2.2}$$

Remark 1. $x^\lambda k(x) \in W_\lambda^* \Rightarrow k(x) \in V_\lambda$ and $k(x) \in V_\lambda \Rightarrow k(x) \in W_\lambda$.

Remark 2. If the almost monotonicity in Definitions 1 and 3 is replaced by the usual monotonicity, then conditions (2.1) and (2.2) are satisfied automatically.

Indeed, let us prove e.g. (2.1), following [14]. If $\varphi(x)/x^\mu$ is decreasing, so that $1-\varphi(x)/\varphi(y) \leq 1 - x^\mu/y^\mu$ for $y \geq x$, then $\varphi(y)-\varphi(x) \leq \frac{y^\mu - x^\mu}{y^\mu}\varphi(y)$. Since $y^\mu - x^\mu \leq c(y-x)y^{\mu-1}$, we obtain (2.1).

Definition 4 ([1]). A non-negative function $\varphi(t)$ on $[0,l]$ belongs to Zygmund class $Z = Z([0,l])$ if

$$\int_0^h \frac{\varphi(t)}{t}\,dt \leq c\varphi(h), \quad 0 \leq h \leq l.$$

Definition 5 ([1]). A non-negative function $\varphi(t)$ belongs to Zygmund class $Z_1 = Z_1([0,l])$ if

$$\int_h^l \frac{\varphi(t)}{t^2}\,dt \leq c\frac{\varphi(h)}{h}.$$

Definition 6. A function $\varphi(x)$ belongs to the generalized Hölder space $H^\omega = H^\omega([a,b])$ if

$$\omega(\varphi,h) \stackrel{\text{def}}{=} \sup_{0<t<h}\ \sup_{x,x+t\in[a,b]} |\varphi(x+t)-\varphi(x)| \leq c\omega(h), \tag{2.3}$$

where $\omega(h)$ is a given positive function on $[0,l]$, $\omega(0)=0$; we set

$$\|\varphi\|_{H^\omega} = \|\varphi\|_C + \sup_{h>0}[\omega(\varphi,h)/\omega(h)].$$

By H_0^ω we denote the subspace of functions in H^ω which vanish at $x=a$.

The function $\omega(h)$ is called a characteristic of the space H^ω.

Definition 7. By $H_0^\omega(\rho)$ we denote the space of functions $f(x)$ such that

$$\rho(x)f(x) \in H_0^\omega, \quad \|f\|_{H_0^\omega(\rho)} = \|\rho f\|_{H_0^\omega},$$

where $\rho(x)$ is a non-negative weight function.

In the sequel we shall use the following inequalities:

1) if $\omega(\varphi, h)$ is the continuity modulus (2.3), then

$$\frac{\omega(\varphi, x)}{x} \le c\frac{\omega(\varphi, y)}{y}, \quad x \ge y; \tag{2.4}$$

2) if $0 < \alpha < 1$, then

$$\frac{\omega(\varphi, h)}{h^\alpha} \le c\int_0^h \frac{\omega(\varphi, t)}{t^{1+\alpha}}\,dt; \tag{2.5}$$

3) if $\psi(x) \in W_\mu$, then

$$\psi(x) \le c\left(\frac{x}{y}\right)^\mu \psi(y), \quad x \ge y; \tag{2.6}$$

4) if $\psi(x) \in W_\mu$ with $0 < \mu < 1$, then (2.1) holds with x^* replaced both by x and y:

$$\left|\frac{\psi(x) - \psi(y)}{x - y}\right| \le c\frac{\psi(x)}{x}, \qquad \left|\frac{\psi(x) - \psi(y)}{x - y}\right| \le c\frac{\psi(y)}{y}; \tag{2.7}$$

5) if $k(x) \in V_\lambda$, then

$$k(x) \le c\left(\frac{y}{x}\right)^\lambda k(y), \quad x \le y, \tag{2.8}$$

and there exists $\varepsilon > 0$ such that

$$k(x) \le c\left(\frac{y}{x}\right)^{\lambda-\varepsilon} k(y), \quad x \ge y; \tag{2.9}$$

6) if $\lambda \le 1$, then

$$|x^\lambda - y^\lambda| \le c(x - y)y^{\lambda-1}, \quad x \ge y > 0, \tag{2.10}$$

and if $\lambda \ge 0$, then

$$|x^\lambda - y^\lambda| \le c(x - y)x^{\lambda-1}, \quad x \ge y > 0. \tag{2.11}$$

Lemma 1. *Let $k(x) \in V_\lambda$, $\lambda > 0$ and let $\omega(x) \ge 0$ be an almost increasing function. Then*

$$\omega(x)k(x) \le c\int_x^l \frac{\omega(t)k(t)}{t}\,dt$$

for $0 < x < l/2$.

Proof. By (2.8) we have

$$\int_x^l \frac{\omega(t)k(t)}{t}\,dt \geq c\omega(x)x^\lambda k(x)\int_x^l \frac{dt}{t^{1+\lambda}} \geq$$

$$\geq c\omega(x)x^\lambda k(x)\int_x^{2x} \frac{dt}{t^{1+\lambda}} = c\omega(x)k(x). \blacksquare$$

3. Mapping Properties of Convolution Operators in the Space $H_0^\omega(\rho)$

The following theorem provides a Zygmund type estimate for the integral (1.1).

Theorem 1. *Let* $k(x) \in V_\lambda$, $0 < \lambda < 1$ *and* $\varphi(x) \in C([a,b])$, $\varphi(a) = 0$. *Then*

$$\omega(K\varphi, h) \leq chk(h)\omega(\varphi, h) + ch\int_h^{b-a} \frac{k(t)\omega(\varphi,t)}{t}\,dt. \tag{3.1}$$

Proof. Let $a = 0$ for simplicity. We denote $g(x) = \varphi(x) - \varphi(0)$ and

$$f(x) = \int_0^x k(x-t)g(t)dt.$$

For all $x, x+h \in [0,b]$ we have

$$f(x+h) - f(x) = \int_{-h}^x [g(x-t) - g(x)]k(t+h)dt -$$

$$-\int_0^x [g(x-t)-g(x)]k(t)dt + g(x)\left[\int_{-h}^x k(t+h)dt - \int_0^x k(t)dt\right].$$

So,

$$|f(x+h) - f(x)| \leq \left|\int_{-h}^0 [g(x-t) - g(x)]k(t+h)dt\right| +$$

$$+\left|\int_0^x [g(x-t)-g(x)][k(t) - k(t+h)]dt\right| +$$

$$+\left|g(x)\int_x^{x+h} k(t)dt\right| = A_1 + A_2 + A_3.$$

Taking (2.8) and increasing of $\omega(\varphi,t)$ into account, we have for A_1:

$$A_1 \le \int_0^h \omega(\varphi,t)k(h-t)dt \le c\omega(\varphi,h)k(h)\int_0^h \left(\frac{h}{h-t}\right)^\lambda dt \le$$
$$\le chk(h)\omega(h). \tag{3.2}$$

For A_2, applying (2.2) and (2.9), we obtain in the case $h \ge x$:

$$A_2 \le ch\int_0^x \frac{\omega(\varphi,t)k(t+h)}{t+h}dt \le chk(h)h^{\lambda-\varepsilon}\int_0^x \frac{\omega(\varphi,t)dt}{(t+h)^{1+\lambda-\varepsilon}} =$$
$$= chk(h)\int_0^{x/h} \frac{\omega(\varphi,ht)dt}{(t+1)^{1+\lambda-\varepsilon}} \le chk(h)\omega(\varphi,h)\int_0^1 \frac{dt}{(t+1)^{1+\lambda-\varepsilon}} \le$$
$$\le chk(h)\omega(\varphi,h). \tag{3.3}$$

In the case $h < x$ we write $A_2 \le \int_0^h + \int_h^x = B_1 + B_2$. For B_1 the estimate (3.3) is valid, while for B_2 we have

$$B_2 \le ch\int_h^b \frac{\omega(\varphi,t)k(t)}{t}dt. \tag{3.4}$$

As regards A_3, we have in the case $h \ge x$:

$$A_3 \le c\omega(\varphi,h)k(x+h)(x+h)^\lambda \int_x^{x+h} \frac{dt}{t^\lambda} \le$$
$$\le c\omega(\varphi,h)k(h)h^\lambda \int_0^{2h} \frac{dt}{t^\lambda} \le c\omega(\varphi,h)hk(h). \tag{3.5}$$

If $h < x$, we use Lemma 1 to obtain

$$A_3 \le c\omega(\varphi,x)hk(x) \le ch\int_x^b \frac{\omega(\varphi,t)k(t)dt}{t} \le$$
$$\le ch\int_h^b \frac{\omega(\varphi,t)k(t)}{t}dt. \tag{3.6}$$

Gathering all the estimates for A_i, $i=1,2,3$, we arrive at (3.1). ■

Theorem 2. *Let* $k(x) \in V_\lambda$, $0<\lambda<1$, $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2$. *Assume that*

i) $\rho(x)\varphi(x) \in C([a,b])$ *and* $\rho(x)\varphi(x)\big|_{x=a} = 0$;

ii) $\int_0^{b-a} t^{-\gamma}\omega(\rho\varphi,t)dt < \infty$, $\gamma = \max(1,\mu)$. *Then the following Zygmund type estimate holds:*

$$\omega(\rho K\varphi,h) \le ch^{\gamma}k(h)\int_0^h \frac{\omega(\rho\varphi,t)}{t^{\gamma}}dt + ch\int_h^{b-a}\frac{\omega(\rho\varphi,t)k(t)}{t}dt, \quad (3.7)$$

if $0 < \mu < 1+\lambda$ *and*

$$\omega(\rho K\varphi,h) \le ch\int_0^h \frac{\omega(\rho\varphi,t)}{t^{\mu}}dt + h\int_h^{b-a}\frac{\omega(\rho\varphi,t)}{t}dt, \quad (3.8)$$

if $1+\lambda \le \mu < 2$.

Proof. Let $\varphi_0(x) = \rho(x)\varphi(x)$ and $a = 0$ for simplicity. We have

$$\rho(x)(K\varphi)(x) = \int_0^x k(x-t)\varphi_0(t)dt + \int_0^x \frac{\psi(x)-\psi(t)}{\psi(t)}k(x-t)\varphi_0(t)dt =$$
$$= f_1(x) + f_2(x).$$

Since $\varphi_0 \in C([0,b])$ and $\varphi_0(0) = 0$, the first term $f_1(t)$ is covered by Theorem 1. To estimate $\omega(f_2,h)$ we represent the difference $f_2(x+h) - f_2(x)$ as

$$f_2(x+h) - f_2(x) = \int_x^{x+h}\frac{\psi(x+h)-\psi(t)}{\psi(t)}\varphi_0(t)k(x+h-t)dt +$$
$$+\int_0^x \frac{\psi(x+h)-\psi(x)}{\psi(t)}\varphi_0(t)k(x+h-t)dt +$$
$$+\int_0^x \frac{\psi(x)-\psi(t)}{\psi(t)}[k(x-t+h)-k(x-t)]\varphi_0(t)dt =$$
$$= I_1 + I_2 + I_3.$$

Estimate of I_1. A) Let $0 < \mu < 1$ at first. Taking (2.4), (2.7) and (2.8) into account, we have

$$|I_1| \le \int_x^{x+h}\frac{(x+h-t)k(x+h-t)\omega(\varphi_0,t)dt}{t} \le$$

$$\le chk(h)\int_x^{x+h}\frac{\omega(\varphi_0,t-x)dt}{t-x} \le chk(h)\int_0^h\frac{\omega(\varphi_0,t)dt}{t}. \quad (3.9)$$

B) If $1 \le \mu < 2$, then by (2.4),(2.6) and (2.8) we obtain

$$|I_1| \le c\int_0^h \frac{(h-t)k(h-t)(x+h)^{\mu-1}}{(x+t)^{\mu-1}}\,\frac{\omega(\varphi_0,t)}{x+t}\,dt \le$$

$$\le chk(h)\int_0^h \frac{(x+h)^{\mu-1}}{t(x+h)^{\mu-1}}\omega(\varphi_0,t)dt. \qquad (3.10)$$

In the case $h < x$ we derive from (3.10)

$$|I_1| \le chk(h)\int_0^h \frac{x^{\mu-1}\omega(\varphi_0,t)dt}{t(x+t)^{\mu-1}} \le chk(h)\int_0^h \frac{\omega(\varphi_0,t)dt}{t}. \qquad (3.11)$$

In the case $h > x$ the inequality (3.10) yields

$$|I_1| \le chk(h)h^{\mu-1}\int_0^h \frac{\omega(\varphi_0,t)dt}{t(x+t)^{\mu-1}} \le$$

$$\le ch^{\mu}k(h)\int_0^h \frac{\omega(\varphi_0,t)dt}{t^{\mu}}. \qquad (3.12)$$

So from (3.9), (3.11) and (3.12) there follows the estimate

$$|I_1| \le ch^{\gamma}k(h)\int_0^h \frac{\omega(\varphi_0,t)dt}{t^{\gamma}}, \quad \gamma = \max(1,\mu). \qquad (3.13)$$

Estimate for I_2. A) Let $0 < \mu < 1$. By (2.6) and (2.7) we have

$$|I_2| \le ch\int_0^x \frac{k(x+h-t)\omega(\varphi_0,t)dt}{t}. \qquad (3.14)$$

In the case $h < x$ we represent (3.14) as $|I_2| \le \int_0^h + \int_h^{(x+h)/2} + \int_{(x+h)/2}^x = I_2' + I_2'' + I_2'''$. It is clear that

$$I_2' \le chk(h)\int_0^h \frac{\omega(\varphi_0,t)dt}{t}. \qquad (3.15)$$

Since $x + h - t \ge t$ in I_2'', we obtain

$$I_2'' \le ch\int_h^b \frac{k(t)\omega(\varphi_0,t)dt}{t}. \qquad (3.16)$$

Further, $x+h-t \leq t$ in I_2''', so by (2.4)

$$I''' \leq ch \int\limits_{(x+h-t)/2}^{x} \frac{k(x+h-t)\omega(\varphi_0, x+h-t)dt}{x+h-t} \leq$$

$$\leq ch \int\limits_{h}^{b} \frac{\omega(\varphi_0,t)k(t)dt}{t}. \tag{3.17}$$

If $h \geq x$,then (3.14) immediately yields

$$|I_2| \leq chk(h) \int\limits_{0}^{h} \frac{\omega(\varphi_0,t)dt}{t}. \tag{3.18}$$

B) Let now $1 \leq \mu < 2$. Taking (2.1) into account, we have

$$|I_2| \leq ch \int\limits_{0}^{x} \frac{(x+h)^{\mu-1}k(x+h-t)\omega(\varphi_0,t)dt}{t^\mu}. \tag{3.19}$$

Hence

$$|I_2| \leq ch^\mu k(h) \int\limits_{0}^{h} \frac{\omega(\varphi_0,t)dt}{t^\mu}$$

in the case $h \geq x$. If $h < x$, we represent (3.19) as

$$|I_2| \leq \int\limits_{0}^{h} + \int\limits_{h}^{(x+h)/2} + \int\limits_{(x+h)/2}^{x} = B_1 + B_2 + B_3.$$

Taking into account that $x+h \leq 2(x+h-t)$ in B_1, in the case $0 < \mu - 1 < \lambda$ we obtain

$$|B_1| \leq ch \int\limits_{0}^{h} \frac{(x+h-t)^{\mu-1}k(x+h-t)\omega(\varphi_0,t)dt}{t^\mu} \leq$$

$$\leq ch^\mu k(h) \int\limits_{0}^{h} \frac{\omega(\varphi_0,t)dt}{t^\mu} \tag{3.20}$$

by (2.9). If $\mu - 1 > \lambda$, the function $t^{\mu-1}k(t)$ is bounded. So

$$B_1 \leq ch \int\limits_{0}^{h} \frac{\omega(\varphi_0,t)dt}{t^\mu}. \tag{3.21}$$

Since $x + h \le 2(x + h - t)$ again, we have

$$B_2 \le ch \int_h^{(x+h)/2} \frac{(x+h-t)^{\mu-1} k(x+h-t)}{t^\mu}\, dt.$$

Here $x + h - t \ge t$, so that

$$B_2 \le ch \int_h^b \frac{k(t)\omega(\varphi_0, t)dt}{t} \tag{3.22}$$

by (2.9), if $\mu - 1 < \lambda$. If $\mu - 1 \ge \lambda$, by boundedness of $t^{\mu-1}k(t)$ we have

$$B_2 \le ch \int_h^b \frac{\omega(\varphi_0, t)dt}{t^\mu}. \tag{3.23}$$

To estimate B_3 we notice that $t \ge x + h - t$ in B_3. So by (2.4) we have

$$B_3 \le ch \int_{x+h-t/2}^{x} \frac{k(x+h-t)\omega(\varphi_0, x+h-t)dt}{x+h-t} \le$$

$$\le ch \int_h^b \frac{k(t)\omega(\varphi_0, t)dt}{t}. \tag{3.24}$$

Thus, I_2 admits the estimate

$$|I_2| \le ch^\gamma k(h) \int_0^h \frac{\omega(\varphi_0, t)dt}{t^\mu} + ch \int_h^b \frac{k(t)\omega(\varphi_0, t)dt}{t}, \tag{3.25}$$
$$\gamma = \max(1, \mu),$$

if $\mu < 1 + \lambda$ and

$$|I_2| \le ch \int_0^h \frac{\omega(\varphi_0, t)dt}{t^\mu} + ch \int_h^b \frac{\omega(\varphi_0, t)dt}{t^\mu} \tag{3.26}$$

if $\mu \ge 1 + \lambda$.

Estimate for I_3. Let $0 < \mu < 1$. By (2.2) and (2.4) we have

$$|I_3| \le ch \int_0^x \frac{k(x+h-t)\omega(\varphi_0, t)dt}{t}$$

which coincides with the estimate in (3.14). If $1 \le \mu < 2$, we derive from (2.1),(2.2) and (2.6):

$$|I_3| \le ch \int_0^x \frac{x^{\mu-1}k(x+h-t)\omega(\varphi_0, t)dt}{t^\mu} \le$$

$$\le ch \int_0^x \frac{(x+h)^{\mu-1}k(x+h-t)\omega(\varphi_0, t)dt}{t^\mu}.$$

The latter coincides with (3.19). Gathering estimates for I_1, I_2 and I_3, we obtain (3.7)–(3.8). ■

Theorem 3. *Let* $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2$, $k(t) \in V_\lambda$, $0 < \lambda < 1$. *Assume that*

i) $\mu < \lambda + 1$;

ii) $t^{-\max(0,\mu-1)}\omega(t) \in Z \quad tk(t)\omega(t) \in Z_1$.

Then the operator K *is bounded from* $H_0^\omega(\rho)$ *into* $H_0^{\omega_k}(\rho)$ *with*

$$\omega_k(h) = hk(h)\omega(h).$$

Proof. Let $f = K\varphi$ with $\varphi \in H_0^\omega(\rho)$ and let $a = 0$. To prove that $f \in H_0^{\omega_k}(\rho)$ we remark at first that

$$\int_0^b \frac{\omega(\rho\varphi, t)}{t^\gamma}\, dt < \infty, \quad \gamma = \max(1, \mu). \tag{3.27}$$

Therefore Zygmund type estimate (3.27) concerning the case $0 < \mu < 1 + \lambda$ holds which gives

$$\frac{\omega(\rho f, h)}{hk(h)\omega(h)} \le \|\rho\varphi\|_{H_0^\omega} \left\{ \frac{h^{\gamma-1}}{\omega(h)} \int_0^h \frac{\omega(t)}{t^\gamma}\, dt + \right.$$

$$\left. + \frac{1}{k(h)\omega(h)} \int_h^b \frac{\omega(t)k(t)dt}{t} \right\}. \tag{3.28}$$

Hence by the condition ii) we have

$$\frac{\omega(\rho f, h)}{hk(h)\omega(h)} \le c\|\varphi\|_{H_0^\omega(\rho)}. \tag{3.29}$$

It remains to prove that $\rho(x)f(x)\big|_{x=0} = 0$. After the change of variable $t = x - \xi x$ we have

$$|\rho(x)f(x)| \le x\psi(x)\int_0^1 \frac{|\varphi_0(x - x\xi)|k(x\xi)d\xi}{\psi(x - x\xi)}.$$

Since $\varphi_0(0) = 0$, this yields

$$|\rho(x)f(x)| \le cx\psi(x)\int_0^1 \frac{\omega(\varphi_0(, 1 - \xi)d\xi}{\psi(x - x\xi)}. \tag{3.30}$$

According to (2.6) and (2.8) we see that

$$|\rho(x)f(x)| \le cxk(x)\int_0^1 \frac{\omega(\varphi_0, 1 - \xi)d\xi}{\xi^\lambda(1 - \xi)^\gamma} = c_1xk(x) \to 0 \tag{3.31}$$

as $x \to 0$ in view of (3.27). So $\rho(x)f(x)\big|_{x=0} = 0$. ■

Corollary 1. *The operator* (1.1) *with the kernel* $k(t) = t^{\alpha-1}(\ln\frac{\gamma}{t})^\beta$, $\gamma > b - a$, $0 < \alpha < 1$, $\beta \ge 0$ *is bounded from* $H_0^\omega(\rho)$ *into* $H_0^{\omega_{\alpha,\beta}}(\rho)$, *where* $\rho(x) = \psi(x - a)$, $\psi(x) \in W_\mu$ *and*

$$\omega\alpha, \beta(h) = \omega(h)h^\alpha\Big(\ln\frac{\gamma}{h}\Big)^\beta$$

under the assumption that $0 < \mu < 2 - \alpha$ *and*

$$h^{-\max(0,\mu-1)}\omega(h) \in Z, \quad h^\alpha\Big(\ln\frac{\gamma}{h}\Big)^\beta\omega(h) \in Z_1.$$

In the case $\psi(x) = x^\mu$ and $\omega(h) = h^\lambda$ the assertion of Corollary 1 was proved in [6] (see [17], Theorem 21.2).

Corollary 2. *The operator of the form*

$$\int_x^b k(t - x)\varphi(t)dt$$

is bounded from $H_0^\omega(\rho)$ *into* $H_0^{\omega_k}(\rho)$ *under the assumptions of Theorem 3 if the requirement* $\rho f\big|_{x=a} = 0$ *in the definition of the space* $H_0^\omega(\rho)$ *is replaced by* $\rho f\big|_{x=b} = 0$.

4. Mapping Properties of Fractional Differentiation in the Spaces $H_0^\omega(\rho)$

Now we give Zygmund type estimate for the fractional derivative (1.5).

Theorem 4. *Let* $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2$, *and*

$$\int_0^{b-a} \frac{\omega(\rho f, t)}{t^{\alpha+\mu}}\, dt < \infty, \quad \gamma = \max(1, \mu).$$

Then

$$\omega(\rho D_{a+}^\alpha f, h) \le c h^{\gamma-1} \int_0^h \frac{\omega(\rho f, t)}{t^{\alpha+\gamma}}\, dt. \tag{4.1}$$

Proof. According to (1.5) we have

$$\rho(x)(D_{a+}^\alpha f)(x) = \frac{\rho(x) f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} +$$

$$+\frac{\alpha}{\Gamma(1-\alpha)}\rho(x) \int_0^{x-a} \frac{f(x)-f(x-t)}{t^{1+\alpha}}\, dt. \tag{4.2}$$

We set $a = 0$ and denote

$$\theta(x) = \psi(x) \int_0^x \frac{f(x)-f(x-t)}{t^{1+\alpha}}\, dt.$$

To estimate the difference $\theta(x+h) - \theta(x)$ we represent it in the form

$$\theta(x+h) - \theta(x) = \sum_{k=1}^{8} A_k(x)$$

(as in [13] in the purely power case), where

$$A_1(x) = \left[1 - \frac{\psi(x)}{\psi(x+h)}\right] \int_0^x \frac{g(x+h)-g(y)}{(x+h-y)^{1+\alpha}}\, dy,$$

$$A_2(x) = [\psi(x+h) - \psi(x)] \int_0^{x+h} \frac{g(y)}{(x+h-y)^{1+\alpha}} \left[\frac{1}{\psi(x+h)} - \frac{1}{\psi(y)}\right] dy,$$

$$A_3(x) = \frac{\psi(x)}{\psi(x+h)} \int_x^{x+h} \frac{g(x+h)-g(y)}{(x+h-y)^{1+\alpha}}\, dy,$$

$$A_4(x) = \psi(x) \int_x^{x+h} \frac{g(y)}{(x+h-y)^{1+\alpha}} \left[\frac{1}{\psi(x+h)} - \frac{1}{\psi(y)}\right] dy,$$

$$A_5(x) = \int_0^x [g(x) - g(y)][(x+h-y)^{-1-\alpha} - (x-y)^{-1-\alpha}]\, dy,$$

$$A_6(x) = \psi(x) \int_0^x g(y) \left[\frac{1}{\psi(x)} - \frac{1}{\psi(y)}\right] \left[(x+h-y)^{-1-\alpha} - (x-y)^{-1-\alpha}\right] dy,$$

$$A_7(x) = \frac{1}{\alpha} \frac{\psi(x)}{\psi(x+h)} [g(x+h) - g(x)][h^{-\alpha} - (x+h)^{-\alpha}],$$

$$A_8(x) = \frac{1}{\alpha} \psi(x) g(x) \left[\frac{1}{\psi(x+h)} - \frac{1}{\psi(x)}\right] \left[h^{-\alpha} - (x+h)^{-\alpha}\right].$$

Estimate for A_1. By (2.1) we have

$$|A_1| \le c \frac{h}{x+h} \int_0^{x+h} \frac{\omega(g, x+h-y)}{(x+h-y)^{1+\alpha}} dy = c \frac{h}{x+h} \int_0^{x+h} \frac{\omega(g,t)}{t^{1+\alpha}} dt. \tag{4.3}$$

If $h \ge x$, it is obvious that

$$|A_1| \le c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\, dt. \tag{4.4}$$

If $h < x$ represent (4.3) as $|A_1| \le \int_0^h + \int_h^{x+h} = A_1' + A_1''$. For A_1' the estimate (4.4) holds. As regards A_1'', applying (2.4) and (2.5), we have

$$A_1'' \le ch \int_x^{x+h} \frac{\omega(g,t)}{t} \frac{dt}{t^{1+\alpha}} \le c\omega(g,h) \int_h^{\infty} \frac{dt}{t^{1+\alpha}} \le$$

$$\le c \frac{\omega(g,h)}{h^\alpha} \le c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\, dt. \tag{4.5}$$

Estimate for A_2. A) In the case $0 < \mu < 1$, using (2.7), we obtain

$$|A_2| \le c \frac{h}{x+h} \int_0^{x+h} \frac{\omega(g,y)}{y(x+h-y)^\alpha}\, dy = \int_0^{h/2} + \int_{h/2}^{x+h} = A_2' + A_2''.$$

Obviously,

$$A_2' \le c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\, dt.$$

Using (2.4) and (2.5), we derive the following estimate for A_2'':

$$A_2'' \le c\frac{\omega(g,h)}{x+h}\int_0^{x+h}\frac{dy}{(x+h-y)^\alpha} = c_1\frac{\omega(g,h)}{(x+h)^\alpha} \le c_2\int_0^h\frac{\omega(g,t)}{t^{1+\alpha}}\,dt.$$

B) If $1 \le \mu < 2-\alpha$, taking (2.1) and (2.6) into account, we obtain for A_2:

$$|A_2| \le c\frac{h}{(x+h)^{2-\mu}}\int_0^h\frac{\omega(g,y)dy}{y^\mu(x+h-y)^\alpha} = \int_0^{h/2} + \int_{h/2}^{x+h} = B_1 + B_2.$$

It is obvious that

$$B_1 \le ch^{\mu-1}\int_0^h\frac{\omega(g,t)}{t^{\mu+\alpha}}\,dt. \tag{4.6}$$

As regards B_2, we apply (2.4) and obtain

$$B_2 \le c\frac{\omega(g,h)}{(x+h)^{2-\mu}}\int_{h/2}^{x+h}\frac{dy}{y^{\mu-1}(x+h-y)^\alpha} \le$$

$$\le c\frac{\omega(g,h)}{(x+h)^\alpha}\int_0^1\frac{dt}{t^{\mu-1}(1-t)^\alpha} = c_1\frac{\omega(g,h)}{(x+h)^\alpha}.$$

Using then (2.5), we notice that the estimate for B_2 is the same as in (4.6).

Estimate for A_3. Since $\psi(x)$ is almost increasing, we have

$$|A_3| \le c\int_x^{x+h}\frac{\omega(g,x+h-y)}{(x+h-y)^{1+\alpha}}\,dy = c\int_0^h\frac{\omega(g,t)}{t^{1+\alpha}}\,dt.$$

Estimate for A_4. Let $0 < \mu < 1$ at first. In view of (2.7) we have

$$|A_4| \le c\int_0^h\frac{\omega(g,x+h-t)}{t^\alpha(x+h-t)}\,dt. \tag{4.7}$$

In the case $h < x$ we have $t \le x+h-t$ in (4.7). So by (2.4) we obtain

$$|A_4| \le c\int_0^h\frac{\omega(g,t)}{t^{1+\alpha}}\,dt.$$

If $h \geq x$, we represent (4.7) as $|A_4| \leq \int_0^{(x+h)/2} + \int_{(x+h)/2}^h$. Since $t \leq x+h-t$ and $t \geq x+h-t$ in the first and second terms, respectively, by (2.4) we derive that

$$|A_4| \leq c \int_0^{(x+h)/2} \frac{\omega(g,t)}{t^{1+\alpha}}\,dt + c \int_{(x+h)/2}^{h} \frac{\omega(g,x+h-t)}{(x+h-t)^{1+\alpha}}\,dt \leq$$

$$\leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt.$$

Let now $1 \leq \mu < 2-\alpha$. Using (2.1) and (2.6), we get

$$|A_4| \leq \frac{c}{(x+h)^{1-\mu}} \int_0^h \frac{\omega(g,x+h-t)}{(x+h-t)^{\mu} t^{\alpha}}\,dt.$$

If $h < x$, by (2.4) we have

$$|A_4| \leq c(x+h)^{\mu-1} \int_0^h \frac{\omega(g,t)dt}{(x+h-t)^{\mu-1} t^{1+\alpha}} \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt.$$

If $h \geq x$, then $|A_4| \leq \int_0^{(x+h)/2} + \int_{(x+h)/2}^h$. We use (2.4) in the first term and the inequality $t^\alpha \geq (x+h-t)^\alpha$ in the second. This yields

$$|A_4| \leq \int_0^{(x+h)/2} \frac{\omega(g,t)}{t^{1+\alpha}}\,dt\; c(x+h)^{\mu-1} \int_{(x+h)/2}^{h} \frac{\omega(g,x+h-t)}{(x+h-t)^{\alpha+\mu}}\,dt \leq$$

$$\leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt + ch^{\mu-1} \int_x^{(x+h)/2} \frac{\omega(g,t)}{t^{\alpha+\mu}}\,dt \leq ch^{\mu-1} \int_0^h \frac{\omega(g,t)}{t^{\alpha+\mu}}\,dt.$$

Estimate for A_5. Applying (2.11), we have

$$|A_5| \leq ch \int_0^h \omega(g,x-y) \frac{dy}{(x+h-y)(x-y)^{1+\alpha}} = ch \int_0^x \frac{\omega(g,t)dt}{t^{1+\alpha}(t+h)}.$$

In the case $h \geq x$ it is clear that

$$|A_5| \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt. \tag{4.8}$$

If $h < x$, then $|A_5| \leq \int_0^h + \int_h^x = A_5' + A_5''$ with the same estimate as in (4.8) for A_5'. As regards A_5'', we have

$$A_5'' \leq c\omega(g,h) \int_h^\infty \frac{dt}{t^{1+\alpha}} \leq c\frac{\omega(g,h)}{h^\alpha} \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt \tag{4.9}$$

by (2.4) and (2.5).

Estimate for A_6. A) Let $0 < \mu < 1$ at first. Applying (2.7) and (2.11) we arrive at

$$|A_6| \leq ch \int_0^x \frac{\omega(g,y)dy}{y(x+h-y)(x-y)^\alpha} = \int_0^{x/2} + \int_{x/2}^x = A_6' + A_6''.$$

For A_6' we have

$$A_6' \leq ch \int_0^{x/2} \frac{\omega(g,y)dy}{(y+h)y^{1+\alpha}}. \tag{4.10}$$

If $h < x$, $A_6' \leq \int_0^{h/2} + \int_{h/2}^{x/2} = K_1 + K_2$. It is evident that K_1 admits the same estimate as in (4.9). For K_2 the application of (2.5) provides the same result:

$$K_2 \leq c\omega(g,h) \int_{h/2}^\infty \frac{dt}{t^{1+\alpha}} \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt.$$

If $h \geq x$, then immediately

$$A_6' \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt. \tag{4.11}$$

To estimate A_6' we remark that $y \geq x - y$ so that

$$A_6'' \leq ch \int_{x/2}^x \frac{\omega(g,x-y)dy}{(x-y)^{1+\alpha}(x+h-y)} = ch \int_0^{x/2} \frac{\omega(g,t)dt}{t^{1+\alpha}(t+h)},$$

which is the same as in (4.10) and so A_6'' admits the same estimates as in (4.5).

B) Let $1 \leq \mu < 2 - \alpha$. Using (2.1), (2.6) and (2.11), we have

$$A_6 \leq chx^{\mu-1} \int_0^x \frac{\omega(g,y)dy}{y^\mu(x+h-y)(x-y)^\alpha} = \int_0^{x/2} + \int_{x/2}^x = U_1 + U_2.$$

If $h < x$, we set $U_1 = \int_0^{h/2} + \int_{h/2}^{x/2} = U_1' + U_1''$, whence easy calculations yield the inequality

$$U_1 \le ch \int_0^h \frac{\omega(g,t)dt}{t^{1+\alpha}} + ch^{\mu-1} \int_0^h \frac{\omega(g,t)dt}{t^{\mu+\alpha}} \le$$

$$\le ch^{\mu-1} \int_0^h \frac{\omega(g,t)dt}{t^{\mu+\alpha}}. \tag{4.12}$$

For U_2 by (2.5) we have

$$U_2 \le ch \int_{x/2}^{x} \frac{\omega(g,y)dy}{y(x+h-y)(x-y)^\alpha} \le c\omega(g,h) \int_0^{x/2} \frac{dt}{t^\alpha(t+h)} \le$$

$$\le c\frac{\omega(g,h)}{h^\alpha} \int_0^\infty \frac{d\xi}{\xi^\alpha(1+\xi)} \le c\int_0^h \frac{\omega(g,t)dt}{t^{1+\alpha}}.$$

If $h \ge x$, the estimation of U_1 and U_2 is easy and provides the same as in (4.2). Gathering all the estimates, we obtain

$$|A_6| \le ch^{\gamma-1} \int_0^h \frac{\omega(g,t)}{t^{\alpha+\gamma}}\,dt, \quad \gamma = \max(1,\mu).$$

Estimate for A_7. Applying (2.5) and (2.10) and almost increasing of $\psi(x)$, we easily obtain

$$|A_7| \le c\int_0^h \frac{\omega(g,t)}{t^{1+\alpha}}\,dt. \tag{4.13}$$

Estimate for A_8. Using the inequalities (2.1) and (2.10) for $0 < \mu < 2-\alpha$, we make sure of validity of the estimate (4.13) for A_8 as well.

It remains to consider the first term

$$r(x) = \frac{\psi(x-a)f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} = \frac{g(x)}{\Gamma(1-\alpha)(x-a)^\alpha}$$

in (4.2). Since $g(x) \in H_0^\omega$, we have the estimate

$$|r(x+h) - r(x)| \le c\int_0^h \frac{\omega(g,t)dt}{t^{1+\alpha}}, \tag{4.14}$$

which is derived by direct calculations under the assumptions of the theorem.

Collecting all the estimates for A_i, $i = 1,\dots,8$, and (4.14), we obtain the required inequality (4.1). ∎

Theorem 5. *Let* $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2-\alpha$ *and let*

$$\begin{aligned} &1)\ \omega(t) \neq 0,\ \ t > 0, \\ &2)\ \omega(t)t^{-\alpha}\Big|_{t=0} = 0, \qquad\qquad (4.15)\\ &3)\ \omega(t)t^{1-\alpha-\gamma} \in Z, \quad \gamma = \max(1,\mu). \end{aligned}$$

Then the operator D^α_{a+} *continuously maps* $H_0^\omega(\rho)$ *into* $H_0^{\omega-\alpha}(\rho)$ *with* $\omega_{-\alpha}(h) = h^{-\alpha}\omega(h)$.

Proof. Let $f(x) \in H_0^\omega(\rho)$ and $\varphi(x) = D^\alpha_{a+}f(x)$. To show that

$$\sup_{0<h\le b-a} \frac{h^\alpha \omega(\rho\varphi, h)}{\omega(h)} = c < \infty$$

we observe that the inclusion $\omega(t)t^{1-\alpha-\gamma} \in Z$ implies convergence of the integral $\int_0^{b-a}\omega(t)t^{-\alpha-\gamma}dt$, so Theorem 4 is applicable. Using the estimate (4.1) of Theorem 4, we obtain

$$\frac{h^\alpha\omega(\rho\varphi,h)}{\omega(h)} \le \frac{h^{\alpha+\gamma-1}\int_0^1 \omega(\rho f,t)t^{-\alpha-\gamma}dt}{\omega(h)} \le c\|f\|_{H_0^\omega(\rho)}. \qquad (4.16)$$

It remains to show that $\rho(x)\varphi(x)|_{x=a} = 0$. By (4.2) we have

$$|\rho(x)\varphi(x)| \le \frac{\omega(\rho f, x-a)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{x-a}\frac{\omega(\rho f,t)dt}{t^{1+\alpha}} +$$

$$+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{x-a}\frac{|\psi(x-a)-\psi(x-a-t)|\omega(\rho f, x-a-t)dt}{t^{1+\alpha}\psi(x-a-t)} =$$

$$= D_1 + D_2 + D_3. \qquad (4.17)$$

Here

$$D_1 \le c\|f\|H_0^\omega(\rho)\frac{\omega(x-a)}{(x-a)^\alpha},$$

the condition (4.16) implies $\omega(x)x^{-\alpha}|_{x=0} = 0$. So $\lim_{x\to a} D_1 = 0$. The equality $\lim_{x\to a} D_2 = 0$ is obvious by the existence of the integral in D_2.

For the term D_3 in the case $0 < \mu < 1$ we have by (2.7)

$$D_3 \le c\int_0^{x-a}\frac{\omega(\rho f, x-a-t)dt}{t^\alpha(x-a-t)}.$$

We evaluate this separately for $x-a-t \geq t$ and $x-a-t \leq t$ by means of (2.4):

$$D_3 \leq c \int_0^{(x-a)/2} \frac{\omega(\rho f, t)dt}{t^{1+\alpha}} + c \int_{(x-a)/2}^{x-a} \frac{\omega(\rho f, x-a-t)}{(x-a-t)^{1+\alpha}} dt,$$

whence $\lim_{x\to a} D_3 = 0$

If $1 \leq \mu \leq 2-\alpha$, we use (2.1) and (2.5) and obtain

$$D_3 \leq c \int_0^{x-a} \frac{\omega(\rho f, x-a-t)}{(x-a)^{1-\mu}(x-a-t)^{\mu} t^{\alpha}} = \int_0^{(x-a)/2} + \int_{(x-a)/2}^{x-a}$$

and similar to what we did above we have

$$D_3 \leq c \int_0^{(x-a)/2} \frac{(x-a)^{\mu-1}\omega(\rho f, t)}{t^{\alpha+\mu}} dt +$$

$$+ c \int_{(x-a)/2}^{x-a} \frac{(x-a)^{\mu-1}\omega(\rho f, x-a-t)}{(x-a-t)^{\alpha+\mu}} dt \leq$$

$$\leq (x-a)^{\mu-1} \int_0^{(x-a)/2} \frac{\omega(\rho f, t)dt}{t^{\alpha+\mu}},$$

so that $\lim_{x\to a} D_3 = 0$. Therefore, $\lim_{x\to a} \rho(x)\varphi(x) = 0$. ∎

Corollary 1. *Let* $\varphi(x) \in W_\mu$ $0 < \mu < 2-\alpha$, *and let* $\omega(t)$ *be an almost increasing function on* $[0, b-a]$ *such that*

1) $\omega(0) = 0$, $\omega(t) \neq 0$ *for* $t \in (0, b-a]$;

2) $\omega(t)t^{-1-\gamma} \in Z$, $\gamma = \max(1, \mu)$.

Then the operator D^α_{a+} *of fractional differentiation continuously maps* $H_0^{\omega_\alpha}(\rho)$ *into* H_0^ω *with* $\rho(x) = \varphi(x-a)$, $\omega_\alpha(h) = h^\alpha \omega(h)$.

Another corollary (of Theorems 5 and 3) will be related to the following Love's index law [8]:

$$I^\gamma_{0+} x^\alpha I^\beta_{0+} x^\gamma I^\alpha_{0+} x^\beta f(x) = f(x), \quad \alpha+\beta+\gamma = 0 \tag{4.18}$$

well known in fractional calculus. This corollary will provide conditions guaranteeing validity of (4.18) for functions $f \in H_0^\omega(\rho)$. For simplicity we restrict ourselves with the cases $\omega(x) = x^\lambda$ and $\rho(x) = x^\mu$. The notation

$$\alpha^* = \begin{cases} \alpha, & \alpha \leq 1 \\ 1, & \alpha \geq 1 \end{cases}$$

is used below.

Corollary 2. *Relation (4.18) is valid for all functions $f(x) \in H_0^\lambda(x^\mu)$ and all $\alpha, \beta, \gamma \in R^1$ such that $\alpha + \beta + \gamma = 0$, if the number $\lambda \in (0,1]$ satisfies the conditions*

$$\lambda > -\alpha, (\lambda+\alpha)^* + \beta > 0 \quad [(\lambda+\alpha)^* + \beta]^* + \gamma > 0$$

while the weight exponent μ satisfies the conditions

$$1)\ \mu < (\lambda+\alpha)^* + 1$$
$$2)\ \mu < [(\lambda+\alpha)^* + \beta]^* + 1$$
$$3)\ \mu < \{[(\lambda+\alpha)^* + \gamma\}^* + 1.$$

5. A Theorem on Isomorphism

In Theorem 3 and Corollary 1 of Theorem 5 it was proved that

$$I_{a+}^a : H_0^\omega(\rho) \longrightarrow H_0^{\omega_\alpha}(\rho), \tag{5.1}$$
$$D_{a+}^a : H_0^{\omega_\alpha}(\rho) \longrightarrow H_0^\omega(\rho) \tag{5.2}$$

under the appropriate assumptions on $\omega(h)$ and $\rho(x)$. To derive the assertion $I_{a+}^\alpha[H_0^\omega(\rho)] = H_0^{\omega_\alpha}(\rho)$ it remains to show that any function in $H_0^{\omega_\alpha}(\rho)$ is representable by the fractional integral of a function in $H_0^\omega(\rho)$. This will be the goal of Theorem 6 below. Preliminarily we state two auxiliary assertions we need.

Lemma 2 ([17], p.185; p.231 in English ed.). *In order a function $f(x)$ to be representable as $f = I_{a+}^\alpha\varphi$, $\varphi \in L_p(a,b)$, $-\infty < a < b < \infty$, it is necessary and sufficient that*

i) $f(x)(x-a)^{-\alpha} \in L_p(a,b)$;

ii) $\|\psi_\varepsilon\|_{L_p} \le c < \infty$ with c not depending on ε, where

$$\psi_\varepsilon(x) = \int_a^{x-\varepsilon} \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}\,dt,$$

for $a+\varepsilon < x < b$ and $\psi_\varepsilon(x) = 0$ for $a < x < a+\varepsilon$.

A close version of Lemma 2 can be found in [16]. (See also [3] for another version under additional assumptions that $f \in L_p$ and $1 < p < 1/2\alpha$).

Lemma 3. *Let $\omega(t)t^{-\delta} \in Z$. There exists $p > 1$ such that $\omega(t)t^{-1-\delta} \in L_p(0,1)$.*

Proof. It is known that the inclusion $\varphi(t) \in Z$ implies existence of $\varepsilon \in (0,1)$ such that $t^{-\varepsilon}\varphi(t)$ is almost increasing (see, e.g. [1]). Therefore, there exists $\varepsilon \in (0,1)$ such that $t^{-\varepsilon-\delta}\omega(t)$ is bounded. So $\omega(t) \leq ct^{\delta+\varepsilon}$ and to have a finite L_p-norm for $\omega(t)/t^{1+\delta}$ we must choose $p < \frac{1}{1-\varepsilon}$. ■

Theorem 6 (On isomorphism). *Let $\psi(x) \in W_\mu$, $0 < \mu < 2 - \alpha$ and let $\omega(t)$ be a continuous function such that $\omega(t)t^{1-\gamma} \in Z$, $\omega(t)t^\alpha \in Z_1$,*

$$\gamma = \max(1, \mu), \quad 0 < \alpha < 1.$$

Then the fractional integration operator I_{a+}^α isomorphically maps the weighted space $H_0^\omega(\rho)$ with $\rho(x) = \psi(x-a)$ onto the space $H_0^{\omega_\alpha}(\rho)$ with the same weight and the characteristic $\omega_\alpha(h) = h^\alpha\omega(h)$.

Proof. In view of (5.1)–(5.2) it is sufficient to prove the representability of a function $f \in H_0^{\omega_\alpha}(\rho)$ by a fractional integral. Aiming to apply Lemma 2 we shall prove that there exists $p > 1$ such that conditions i)-ii) of Lemma 2 are satisfied.

The estimate

$$\frac{|f(x)|}{(x-a)^\alpha} \leq c\|f\|_{H_0^{\omega_\alpha}} \frac{\omega(x-a)}{(x-a)^\mu} \tag{5.3}$$

is valid for any $f(x) \in H_0^{\omega_\alpha}(\rho)$. Really, by (2.6) we have

$$\frac{|f(x)|}{(x-a)^\alpha} \leq c\frac{|\rho(x)f(x)|}{(x-a)^{\mu+\alpha}}, \tag{5.4}$$

which immediately provides (5.3).

Since $\omega(x)/x^{\delta-1} \in Z$, from (5.3) and Lemma 3 we conclude that there exists $p > 1$ such that the condition i) of Lemma 2 is satisfied.

For this p we shall show that a constant $c > 0$ exists such that

$$\|\psi_\varepsilon\|_{L_p} \leq c \leq \infty. \tag{5.5}$$

We set $g(x) = f(x)\psi(x-a)$ and have

$$|\psi_\varepsilon(x)| \leq \frac{1}{\psi(x-a)} \int_a^{x-\varepsilon} \frac{|g(x) - g(t)|}{(x-t)^{1+\alpha}}\, dt +$$

$$+ \int_a^{x-\varepsilon} \frac{\left|g(t)\left[\frac{1}{\psi(x-a)} - \frac{1}{\psi(t-a)}\right]\right|}{(x-t)^{1+\alpha}}\, dt.$$

To estimate F_1 we use (2.6) and obtain

$$F_1 \leq \frac{c}{(x-a)^\mu} \int_a^x \frac{\omega(g, x-t)}{(x-t)^{1+\alpha}}\, dt \leq$$

$$\leq \frac{c}{(x-a)^\gamma} \int_a^x \frac{\omega(g, x-t)}{(x-t)^{1+\alpha}}\, dt. \tag{5.6}$$

Since $g(x) \in H_0^{\omega_\alpha}(\rho)$, it is easily proved that (5.6) yields

$$F_1 \leq \frac{c}{(x-a)^\gamma}. \tag{5.7}$$

For F_2 in the case $0 < \mu < 1$ we use (2.6) and (2.7) and obtain

$$F_2 \leq \frac{c}{(x-a)} \int_0^{x-a} \frac{\omega(g, x-t-a)}{t^\alpha (x-t-a)}\, dt = \int_0^{(x-a)/2} + \int_{(x-a)/2}^{x-a}.$$

Hence after simple calculations

$$F_2 \leq \frac{c}{x-a} \int_0^{(x-a)/2} \frac{\omega(t)dt}{t} + \frac{c}{x-a} \int_{(x-a)/2}^{x-a} \frac{\omega(x-t-a)dt}{x-t-a} \leq$$

$$\leq c\frac{\omega(x-a)}{x-a} + \frac{c}{x-a} \int_0^{x-a} \frac{\omega(t)}{t}\, dt. \tag{5.8}$$

Since the condition $\omega(t)t^{1-\mu} \in Z$ with $0 < \mu < 1$ implies $\omega(t) \in Z$, we derive from (5.8) the estimate $F_2 \leq c\omega(x-a)/(x-a)$.

Let now $1 \leq \mu < 2-\alpha$. We use inequalities (2.1) and (2.6) to obtain

$$F_3 \leq \frac{c}{x-a} \int_0^{x-a} \frac{\omega(g, x-t-a)}{t^\alpha (x-t-a)^\mu}\, dt \leq \frac{c}{x-a} \int_0^{x-a} \frac{\omega(x-t-a)dt}{t^\alpha (x-t-a)^{\mu-a}} =$$

$$= \int_0^{(x-a/2} + \int_{(x-a)/2}^{x-a}.$$

Calculations and arguments similar to those in the case $0 < \mu < 1$ give the estimate

$$F_2 \leq c\frac{\omega(x-a)}{(x-a)^\mu}.$$

Therefore,

$$F_2 \leq c\frac{\omega(x-a)}{(x-a)^\gamma}, \quad \gamma = max(1,\mu). \tag{5.9}$$

So, from (5.8) and (5.9) we obtain

$$|\psi_\varepsilon(x)| \leq c\frac{\omega(x-a)}{(x-a)^\gamma} \in L_p.$$

Hence $\|\psi\|_{L_p} \leq c$. ■

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COMPLEXITY OF THE DECIDABILITY OF THE UNQUANTIFIED SET THEORY WITH A RANK OPERATOR

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ABSTRACT. The unquantified set theory MLSR containing the symbols $\cup$, $\setminus$, $=$, $\in$, R ($R(x)$ is interpreted as a rank of x) is considered. It is proved that there exists an algorithm which for any formula Q of the MLSR theory decides whether Q is true or not using the space $c|Q|^3$ ($|Q|$ is the length of Q).

რეზიუმე. დამტკიცებულია, რომ არსებობს ალგორითმი, რომელიც $\cup$, $\setminus$, $=$, $\in$, R ($R(x)$-ის ინტერპრეტაციაა x სიმრავლის რანგი ფონ ნეიმანის აზრით) სიმბოლოების მქონე უკვანტორო სიმრავლეთა MLSR თეორიის ყოველი Q ფორმულისათვის წყვეტს ამოხსნადობის პრობლემას $c|Q|^3$ გამოთვლითი სირთულით (მოცულობის თვალსაზრისით; $|Q|$ აღნიშნავს Q-ს სიგრძეს).

Let MLSR be an unquantified set theory whose language contains the symbols $\cup$, $\setminus$, $=$, $\in$, R, where R denotes a unary functional symbol and $R(x)$ is interpreted as a rank of the set x in the sense of J.von Neumann. The decidability problem for the theory MLSR reduces readily to testing the satisfiability of conjunctions of literals of the following types:

$$(=)\quad x = y \cup z, \quad x = y \setminus z, \quad (\in)\quad x \in y, \quad (R)\quad x = R(y)$$

(the literal $x \notin y$ is equivalent to the formula $x \in z \,\&\, z = z \setminus y$, while the literal $x \neq y$ to the formula $[u \in x \,\&\, u \notin y] \vee [u \in y \,\&\, u \notin x]$, where z, u are new variables). The conjunction Q of literals is sometimes treated as a set of its literals.

Let, further, MLS be an unquantified set theory whose language contains the symbols $\cup$, $\setminus$, $=$, $\in$.

Definition 1. The interpretation α of the MLS language is called the singleton model of a formula φ of this language if it associates a subset of the $\{\varnothing\}$ with each variable and φ is true in α.

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Definition 2. The singleton model α of a set of all literals of the type $(=)$ of the formula Q of the language of MLSR theory is called the place of the conjunction Q.

If α is the place of Q and y is the variable occurring in Q, then $y(\alpha)$ is the value of y in α.

Definition 3. Let y be the variable occurring in Q. The place α of the formula Q is called the place at y if

$$y(\alpha) = \begin{cases} 1, & \text{when } x \in y \text{ occurs in } Q, \\ 0, & \text{when } x \notin y \text{ occurs in } Q. \end{cases}$$

D. Cantone et al. the following result have given in [1]:

Theorem 1. *Let Q be the conjunction of literals of the types $(=, \in, R)$ and $y_1, \dots, y_m$ be its pairwise-distinct variables occurring in Q. Then Q is satisfiable iff there exist*

(i) *a set $\Pi = \{\alpha_1, \dots, \alpha_n\}$ of pairwise-distinct places of Q;*

(ii) *a function $F : \{1, \dots, m\} \to \{1, \dots, n\}$;*

(iii) *a set $W \subseteq \{1, \dots, m\}$;*

(iv) *a sequence of integers $0 = r_0 < r_1 < \dots < r_k = n$ such that the following conditions are fulfilled:*

(a) *$\alpha_{F(i)}$ is the place at y_i for all $i \in \{1, \dots, m\}$;*

(b) *If $y_i \underset{\Pi}{\sim} y_j$ (i.e., $\forall \alpha \in \Pi(y_i(\alpha) = y_j(\alpha))$), then $F(i) = F(j)$ $(i, j \in \{1, \dots, m\})$;*

(c) *If $y_i(\alpha_j) = 1$, then $k_j < k_{F(i)}$ $(i \in \{1, \dots, m\}, j \in \{1, \dots, n\})$, where for every $j \in \{1, \dots, n\}$ k_j denotes the number r_s such that $r_{s-1} < j \leq r_s$;*

(d) *If $y_j = R(y_i)$ is a literal of Q, then $j \in W$;*

(e) *If $y_j = R(y_i)$ is a literal of Q, then $k_{i^*} = k_{j^*}$, where $h^* = \max\{l : y_h(\alpha_l) = 1\}$, $h \in \{1, \dots, m\}$;*

(f) *$\forall i \in W \forall j \in W$ $(y_i \underset{\Pi}{\not\sim} y_j \Rightarrow y_i \hat{\in} y_j \vee y_j \hat{\in} y_i)$, where $y_i \hat{\in} y_j$ denotes $y_j(\alpha_{F(i)}) = 1$;*

(g) *$\forall i \in W \forall j \in W (y_i \hat{\in} y_j \Rightarrow y_i \hat{\subseteq} y_j)$, where $y_i \hat{\subseteq} y_j$ denotes $\forall \alpha \in \Pi$ $(y_i(\alpha) = 1 \Rightarrow y_j(\alpha) = 1)$;*

(h) *If for $i, j \in W$ there exists $h \in \{1, \dots, n\}$ such that $y_i(\alpha_h) = 0$ and $y_j(\alpha_h) = 1$, then $\forall r \in \{1, \dots, n\}(y_i(\alpha_r) = 1 \Rightarrow k_r < k_h)$.*

The proof of the necessity of this theorem makes essential use of the Venn diagram [2]. The construction of the set Π is based on the Venn diagram for m sets. Consequently the number n of places is, in general, of order 2^m. Therefore, despite the fact that Theorem 1 solves the decidability problem for the MLSR theory, the corresponding decision procedure has an exponential computational complexity (by space). Thus is of great interest to find an algorithm solving the decidability problem for the MLSR theory with a polynomial computational complexity (by space). The next theorem shows that such an algorithm can really be constructed.

Theorem 2. *Let the conjunction Q of literals of the types $(=, \in, R)$ is satisfiable and $y_1, \dots, y_m$ be all pairwise-distinct variables of Q. Then there exist:*

(i) *a set $\bar{\Pi} = \{\beta_1, \dots, \beta_{\bar{n}}\}$ of pairwise-distinct places Q;*

(ii) *a function $\bar{F} : \{1, \dots, m\} \to \{1, \dots, \bar{n}\}$;*

(iii) *a set $\bar{W} \subseteq \{1, \dots, m\}$;*

(iv) *a sequence $0 = \bar{r}_0 < \bar{r}_1 < \cdots < \bar{r}_{\bar{k}} = \bar{n}$ of natural numbers such that all the conditions* (a)–(h) *of Theorem* 1 *are fulfilled and $\bar{n} \leq c|Q|^2$, where $|Q|$ is the length of Q.*

Proof. Let Q be satisfiable. By virtue of Theorem 1 there exists: a set $\Pi = \{\alpha_1, \dots, \alpha_n\}$, a function $F : \{1, \dots, m\} \to \{1, \dots, n\}$, a set W and a sequence $r_0, r_1, \dots, r_k$, satisfying the conditions (a)–(h) of this theorem.

Denote by Π^F a set $\{\alpha_{F(i)} : i = 1, \dots, m\}$ of places of Q. $\Pi^F \subseteq \Pi$. Let $< i_1, j_1 >, \dots, < i_\tau, j_\tau >$ be pairs of numbers from $\{1, \dots, m\}$ such that $i_\nu < j_\nu$ and $F(i_\nu) \neq F(j_\nu)$, $\nu = 1, \dots, \tau$. Then by the condition (b) of Theorem 1 $y_{i_\nu} \underset{\Pi}{\not\sim} y_{j_\nu}$, $\nu = 1, \dots, \tau$. Therefore for each pair $< i_\nu, j_\nu >$, $\nu = 1, \dots, \tau$, there exists a place $\alpha_{l_\nu} \in \Pi$ with the lowest index l_ν such that $y_{i_\nu}(\alpha_{l_\nu}) \neq y_{j_\nu}(\alpha_{l_\nu})$. Denote $\tilde{\Pi} = \{\alpha_{l_\nu} : \nu = 1, \dots, \tau\}$. It is clear that $\tau \leq m(m-1)/2$.

For each variable y_i of Q assume that α_{μ_i} is a place from Π with the highest index such that $y_i(\alpha_{\mu_i}) = 1$ (if such a place exists in Π). Denote by Π^R a set of all such places. Let $\Pi_0 = \Pi^F \cup \tilde{\Pi} \cup \Pi^R$. It is clear that $\mathrm{card}(\Pi_0) \leq (m^2 + m)/2$.

Finally, denote by Π^Δ a set of places from the set Π which do not belong to the set Π_0 but whose indices are the ends of intervals $(r_{s-1}, r_s]$ containing the indices of places from the Π_0. Let $\bar{\Pi} = \Pi_0 \cup \Pi^\Delta$. It is obvious that $\mathrm{card}(\Pi^\Delta) \leq \mathrm{card}(\Pi_0)$ and therefore $\mathrm{card}(\bar{\Pi}) \leq m^2 + m$. Let $\bar{n} = \mathrm{card}(\bar{\Pi})$.

Let us now construct a new sequence of integers $0 = \bar{r}_0 < \bar{r}_1 < \cdots < \bar{r}_{\bar{k}} = \bar{n}$. Identify each place with its index i. From the set $\bar{\Pi}$ remove the integers which are not the indices of places from the set $\{1, \dots, n\}$ and enumerate the elements of the set $\bar{\Pi}$, preserving their order by the natural numbers $1, 2, \dots, \bar{n}$. Clearly, in this case all numbers r_i from the set $\bar{\Pi} \subseteq \Pi$ will acquire the corresponding values from the set $\{1, \dots, \bar{n}\}$ which will form a new sequence $0 = \bar{r}_0 < \bar{r}_1 < \dots < \bar{r}_{\bar{k}} = \bar{n}$ of natural numbers.

Finally define the function $\bar{F} : \{1, \dots, m\} \to \{1, \dots, \bar{n}\}$ as follows: $\bar{F}(i)$ is the natural number from $[1, \bar{n}]$ into which the number $F(i)$ turns during the new enumeration of elements of the set $\bar{\Pi}$ ($\bar{F}(i)$ is the index of a place from $\bar{\Pi}$).

Let us check that the set $\bar{\Pi} = \{\beta_1, \dots, \beta_{\bar{n}}\}$ of places of Q, the function $\bar{F}$, the set W and the sequence $0 = \bar{r}_0 < \bar{r}_1 < \dots < \bar{r}_{\bar{k}} = \bar{n}$ satisfy all the conditions (a)–(h) of Theorem 1:

(a) $\beta_{\bar{F}(i)}$ is a place at y_i, $i = 1, \dots, m$, since $\beta_{\bar{F}(i)} = \alpha_{F(i)}$;

(b) If $y_i \underset{\bar{\Pi}}{\sim} y_j$, then on account of $\bar{\Pi} \subseteq \Pi_0 \subseteq \Pi$, we shall also have $y_i \underset{\Pi}{\sim} y_j$. But in that case $F(i) = F(j)$. Therefore $\bar{F}(i) = \bar{F}(j)$;

(c) Let $y_i(\beta_j) = 1$, i.e., $y_i(\alpha_{j'}) = 1$, where $j = 1, \dots, n$. Then j and $F(i)$ belong to different intervals $(r_{s_1-1}, r_{s_1}]$ and $(r_{s_2-1}, r_{s_2}]$, respectively, the first interval preceding the second one, and, since $\alpha_{F(i)} \in \bar{\Pi}$ and $\alpha_{j'} = \beta_j \in \bar{\Pi}$, the ends of both intervals belong to the set of indices of places from $\bar{\Pi}$. But their order in the set $\{1, \dots, n\}$ has remained unchanged during the new enumeration of elements of the set $\bar{\Pi}$, the numbers j and $\bar{F}(i)$ belong to different new intervals corresponding to the sequence $\bar{r}_0, \bar{r}_1, \dots, \bar{r}_{\bar{k}}$. Therefore $k_j < k_{\bar{F}(i)}$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$;

(d) It is obvious;

(e) Let $y_j = R(y_i)$ be a literal of the conjunction Q. Then by Theorem 1 i^* and j^* belong to the same interval $(r_{s-1}, r_s]$, $s \leq n$. But $\Pi^R \subseteq \Pi_0 \subseteq \Pi$ and the intervals were not subdivided during the transformation. Therefore, after the new enumeration the equality $k_{i^*} = k_{j^*}$ has also remained unchanged for the set $\bar{\Pi}$;

(f) Let $y_i \underset{\bar{\Pi}}{\nsim} y_j$, i.e., $\exists \beta \in \bar{\Pi}[y_i(\beta) \neq y_j(\beta)]$. Since $\bar{\Pi} \subseteq \Pi$, we have $\exists \alpha \in \Pi[y_i(\alpha) \neq y_j(\alpha)]$, i.e. $y_i \underset{\Pi}{\nsim} y_j$. For example, $y_i \hat{\in} y_j$ is true by virtue of Theorem 1 (the case when $y_j \hat{\in} y_i$ is true is considered in a similar manner), i.e. $y_j(\alpha_{F(i)}) = 1$. But $\alpha_{F(i)} \in \Pi^F \subseteq \bar{\Pi}$ and $\alpha_{F(i)} = \beta_{\bar{F}(i)}$. Therefore $y_i \hat{\in} y_j$ is true in $\bar{\Pi}$;

(g) Let $y_i \hat{\in} y_j$ hold in $\bar{\Pi}$, i.e. $y_j(\beta_{\bar{F}(i)}) = 1$. But $\beta_{\bar{F}(i)} = \alpha_{F(i)}$, i.e. $y_j(\alpha_{F(i)}) = 1$. Therefore $y_i \hat{\in} y_j$ is true in Π. Then $y_i \hat{\subseteq} y_j$ is true in Π by virtue of Theorem 1, i.e. $\forall \alpha \in \Pi(y_i(\alpha) = 1 \Rightarrow y_j(\alpha) = 1)$. Since $\bar{\Pi} \subseteq \Pi$, the more so $\forall \alpha \in \bar{\Pi}(y_i(\alpha) = 1 \Rightarrow y_j(\alpha) = 1)$, i.e., $y_i \hat{\subseteq} y_j$ is true in $\bar{\Pi}$;

(h) Let $i, j \in W$ and there exists $\bar{h} \in \{1, \dots, \bar{n}\}$ such that $y_i(\beta_{\bar{h}}) = 0$

and $y_j(\beta_{\bar h}) = 1$. But $\beta_{\bar h} = \alpha_h$ for some $h \in \{1,\dots,n\}$, i.e. $y_i(\alpha_h) = 0$, and by Theorem 1 $\forall r \in \{1,\dots,n\}(y_i(\alpha_r) = 1 \Rightarrow k_r < k_h)$. Let $\bar r \in \{1,\dots,\bar n\}$. Then $\beta_{\bar r} = \alpha_r$ for some $r \in \{1,\dots,n\}$. Therefore if $y_i(\beta_{\bar r}) = 1$, then $y_i(\alpha_r) = 1$. Consequently, $k_r < k_h$, i.e. $\alpha_r, \alpha_h \in \bar\Pi$, and r and h belong to different intervals $(r_{s_1-1}, r_{s_1}]$ and $(r_{s_2-1}, r_{s_2}]$, the first interval preceding the second one. The ends of these intervals were not removed from the set $\{1,\dots,n\}$. Therefore the inequality $k_{\bar r} < k_{\bar h}$ has preserved. ■

Corrolary 1. *There exists a Turing machine which for any formula Q of the* MLSR *theory decides whether Q is true or not, using the space $c|Q|^3$.*

Let MLSO_n be an unquantified set theory whose language contains the symbols $\cup$, $\setminus$, $=$, $\in$ and a single-place predicate O_n ($O_n(x)$ is interpreted as follows: x is an ordinal). Since $O_n(x)$ iff $x = R(x)$, we have

Corrolary 2. *There exists a Turing machine which for any formula Q of the* MLSO_n *theory decides whether Q is true or not, using the space $c|Q|^3$.*

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ON PERFECT MAPPINGS FROM $\mathbb{R}$ TO $\mathbb{R}$

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ABSTRACT. Perfect mappings from $\mathbb{R}$ to $\mathbb{R}$ and also mappings close to perfect ones are considered. Some of their properties are given.

რეზიუმე. განიხილება სრულყოფილი და, აგრეთვე, სრულყოფილ ასახვათა მსგავსი ასახვები $\mathbb{R}$-დან $\mathbb{R}$-ში. მოცემულია ასეთ ასახვათა ზოგიერთი თვისება.

This paper deals with some properties of perfect mappings from the space of real numbers with natural topology $\mathbb{R}$ to $\mathbb{R}$. For example, it is shown that no perfect mapping from $\mathbb{R}$ to $\mathbb{R}$ has finite partial limits at $+\infty$ and $-\infty$, that no bounded mapping from $\mathbb{R}$ to $\mathbb{R}$ is perfect, and so on. We also consider mappings from $\mathbb{R}$ to $\mathbb{R}$ (not necessarily continuous) with closed, compact and bounded fibers and mapppings under which the image of any closed subset of $\mathbb{R}$ is closed in $\mathbb{R}$.

Recall that a mapping $f : X \to Y$ of topological spaces is called a closed mapping [1] if f is continuous and the following condition is satisfied:

$(CL\downarrow)$ for every closed subset F of the space X the image $f(F)$ is closed in Y.

Recall also that a mapping $f : X \to Y$ of topological spaces is said to be perfect [1] if X is a Hausdorff space, f is a closed mapping and f satisfies the following condition:

(CM_p^{-1}) for any point $y \in Y$ the fiber $f^{-1}(y)$ is compact.

Note that if $f : X \to Y$ is a mapping from the Hausdorff space X to the space Y (not necessarily continuous), satisfying the conditions $(CL\downarrow)$ and (CM_p^{-1}) (see Example 1 below), then for every compact subspace Z of the space Y the inverse image $f^{-1}(Z)$ is compact. (The proof of this fact repeats the proof of Theorem 3.7.2 [11].) This implies that every function $f : X \to \mathbb{R}$ from the Hausdorff space X to $\mathbb{R}$, satisfying the conditions $(CL\downarrow)$ and (CM_p^{-1}), is a Borel function.

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In the sequel $\mathcal{K}$ will denote the class of mappings (not necessarily continuous) from $\mathbb{R}$ to $\mathbb{R}$.

We shall define the following classes of mappings:

$\mathcal{K}(C) \equiv \{f \in \mathcal{K} | f$ is continuous $\}$;

$\mathcal{K}(CL\downarrow) \equiv \{f \in \mathcal{K}|$ for any closed subset F of $\mathcal{K}$ the image $f(F)$ is closed in $\mathbb{R}\}$;

$\mathcal{K}(CM_p^{-1}) \equiv \{f \in \mathcal{K}|$ for any point $y \in \mathbb{R}$ the fiber $f^{-1}(y)$ is compact $\}$;

$\mathcal{K}(CL_p^{-1}) \equiv \{f \in \mathcal{K}|$ for any point $y \in \mathbb{R}$ the fiber $f^{-1}(y)$ is closed in $\mathbb{R}\}$;

$\mathcal{K}(B_p^{-1}) \equiv \{f \in \mathcal{K}|$ for any point $y \in \mathbb{R}$ the fiber $f^{-1}(y)$ is a bounded subset of $\mathbb{R}\}$.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a closed mapping and let*

$$\lim_{x\to+\infty} f(x) = a \in \mathbb{R} \quad (\lim_{x\to-\infty} f(x) = a \in \mathbb{R}).$$

Then there exists a real number M such that for any $x > M$ (for any $x < M$) we have $f(x) = a$.

Proof. Let $\lim_{x\to+\infty} f(x) = a \in \mathbb{R}$ (the other case is analogous). Assume that our assertion is not true. Then for any real number M there must exist a real number $x_M > M$ such that $f(x_M) \neq a$. In particular, $\exists x_1 : x_1 > 1 : f(x_1) \neq a; \exists x_2 > x_1 + 1 : f(x_2) \neq a$ and so on $\exists x_k : x_{k-1} + 1 : f(x_k) \neq a; \exists x_2 > x_1 + 1 : f(x_2) \neq a$ and so on. Thus we have a sequence $(x_k)_{k\geq 1}$ of real numbers such that $x_k > k$ and $f(x_k) \neq a$. for any natural number k.

It is obvious that $\lim_{k\to\infty} x_k = +\infty$, the set $\{x_k\}_{k=1}^{\infty}$ is closed in $\mathbb{R}$ and $a \in \{f(x_k)\}_{k=1}^{\infty}$. But, by assumption, $\lim_{x\to+\infty} f(x) = a$. Hence, by virtue of the continuity of f, $\lim_{k\to\infty} f(x_k) = a$. Therefore a belongs to the closure of the set $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}$. Since $a \in \{f(x_k)\}_{k=1}^{\infty}$, the set $\{f(x_k)\}_{k=1}^{\infty}$ is not closed in $\mathbb{R}$.

On the other hand, since the set $\{x_k\}_{k=1}^{\infty}$ is closed in $\mathbb{R}$, the set $f(\{x_k\})_{k=1}^{\infty}$ must be closed too by the condition. This is the contradiction. ■

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a closed mapping and let*

$$\lim_{x\to+\infty} f(x) = a \in \mathbb{R} \quad (\lim_{x\to-\infty} f(x) = a \in \mathbb{R}).$$

Then the set $f^{-1}(a)$ is not bounded.

Proof. Let $\lim_{x\to+\infty} f(x) = a \in \mathbb{R}$ ($\lim_{x\to-\infty} f(x) = a \in \mathbb{R}$). Then, by the above theorem, there exists a real number M such that $f^{-1}(a) \supset (M; +\infty)(f^{-1}(a) \supset (-\infty; M))$. Therefore $f^{-1}(a)$ is not bounded. ■

Corollary 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a perfect mapping, then there are no limits of the function f at $+\infty$ and $-\infty$. In particular, no perfect mapping $f : \mathbb{R} \to \mathbb{R}$ has asymptotes.*

Proof. Let $\lim_{x \to +\infty} f(x) = a \in \mathbb{R}$ ($\lim_{x \to -\infty} f(x) = a \in \mathbb{R}$). Then, by the previous corollary, $f^{-1}(a)$ is not bounded. But since f is a perfect mapping, the set $f^{-1}(a)$ must be compact and hence bounded. This is the contradiction. ■

Theorem 2. *Let $f \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(CL_p^{-1})$ (see Example 2 below). Assume that $a \in \mathbb{R}$. If there exists a left-hand limit $\lim_{x \to a-} f(x)$ (respectively, if there exists a right-hand limit $\lim_{x \to a+} f(x)$), then*

$$\lim_{x \to a-} f(x) = f(a) \quad (\textit{respectively, } \lim_{x \to a+} f(x) = f(a)).$$

Proof. Let $\lim_{x \to a-} f(x) = b \in \mathbb{R}$. Assume that $b \neq f(a)$. (The other case is analogous.)

We have two possible cases:

1) $\exists \delta > 0 \quad (\delta \in \mathbb{R}) : \forall x \in (a - \delta; a) : f(x) = b$;

2) $\forall \delta > 0 \quad (\delta \in \mathbb{R}) \ \exists x_\delta \in (a - \delta; a) : f(x_\delta) \neq b$.

Let us consider each of them separately.

Case 1. We have $f^{-1}(b) \supset (a - \delta; a)$. By assumption, $f(a) \neq b$. Therefore $a \notin f^{-1}(b)$. On the other hand, since $f^{-1}(b) \supset (a - \delta; a)$, the point a belongs to the closure of the set $f^{-1}(b)$. Hence the set $f^{-1}(b)$ is not closed. This is the contradiction. Therefore case 1 is impossible.

Case 2. Denote $\frac{|b - f(a)|}{2} \equiv \varepsilon$. Clearly, $\varepsilon > 0$. Since $\lim_{x \to a-} f(x) = b$, there exists a positive real number δ such that for $f(x) \in (f(a) - \varepsilon; f(a) + \varepsilon)$ for any $x \in (a - \delta; a)$. Therefore $b \notin (f(a) - \varepsilon; f(a) + \varepsilon)$.

By the condition $\exists x_1 \in (a - \delta; a) : f(x_1) \neq b$.

Assume that $z_1 \equiv a - \delta$ and $z_2 \equiv \max\{a - \frac{\delta}{2}; x_1\}$. Then there exists $x_2 \in (z_2; a)$ such that $f(x_2) \neq b$.

Let the points $x_1, x_2, \dots, x_{k-1}$ be already constructed. Denote $z_k \equiv \max\{a - \frac{\delta}{k}; x_{k-1}\}$. According to the condition there exists a point x_k such that $x_k \in (z_k; a)$ and $f(x_k) \neq b$.

Therefore for any natural k we have a point $x_k \in \mathbb{R}$ such that $a - \frac{\delta}{k} < x_k < a$ and $f(x_k) \neq b$.

Since $\lim_{k \to \infty}(a - \frac{\delta}{k}) = a$, we have $\lim_{k \to \infty} x_k = a$.

Since $\lim_{x \to a-} f(x) = b$, we have $\lim_{k \to \infty} f(x_k) = b$.

The set $\{a\} \cup \{x_k\}_{k=1}^{\infty}$ is obviously closed in $\mathbb{R}$.

Now let us consider the set $f(\{a\} \cup \{x_k\}_{k=1}^{\infty}) = \{f(a)\} \cup \{f(x_k)\}_{k=1}^{\infty}$. Since for any $k \geq 1$ $\ f(x_k) \neq b$ and, by assumption, $f(a) \neq b$, we obtain $b \notin f(\{a\} \cup \{x_k\}_{k=1}^{\infty})$. On the other hand, since $\lim_{k \to \infty} f(x_k) =$

b, b belongs to the closure of the set $f(\{a\} \cup \{x_k\}_{k=1}^{\infty}) = \{f(a)\} \cup \{f(x_k)\}_{k=1}^{\infty}$. Hence the set $f(\{a\} \cup \{x_k\}_{k=1}^{\infty})$ is not closed in $\mathbb{R}$.

But the set $\{a\} \cup \{x_k\}_{k=1}^{\infty}$ is closed in $\mathbb{R}$ and, by the condition, the set $f(\{a\} \cup \{x_k\}_{k=1}^{\infty})$ must be closed in $\mathbb{R}$. This is the contradiction. ■

Corollary 3. *Let $f \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(CL_p^{-1})$. Then any point of discontinuity of the mapping f must be only of the second kind.*

Theorem 3. *Let $f \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(B_p^{-1})$ (see Example 3 below). Then the function f does not have finite partial limits at $+\infty$ and $-\infty$.*

Proof. Assume, conversely, that there is a sequence of real numbers $x_1, x_2, \ldots, x_n, \ldots$ such that $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} f(x_n)$ exists. Let $\lim_{n\to\infty} f(x_n) = a \in \mathbb{R}$. (The case concerning $-\infty$ is analogous.) Here we have two possible cases:

(a) the set $\{\, n \mid f(x_n) = a \,\}$ is empty or finite;

(b) the set $\{\, n \mid f(x_n) = a \,\}$ is infinite.

Consider each of them separately.

Case (a). There exists a natural number n_0 such that $f(x_n) \neq a$ for any $n \geq n_0$. Then $a \neq f(\{x_n\}_{n=n_0}^{\infty}) = \{f(x_n)\}_{n=n_0}^{\infty}$. But $\lim_{n\to\infty} f(x_n) = a$. Therefore the set $f(\{x_n\}_{n=n_0}^{\infty})$ is not closed in $\mathbb{R}$. On the other hand, since $\lim_{n\to\infty} x_n = +\infty$, the set $\{x_n\}_{n=n_0}^{\infty}$ is closed in $\mathbb{R}$ and, hence, by the condition, the set $f(\{x_n\}_{n=n_0}^{\infty})$ must be closed in $\mathbb{R}$. This is the contradiction.

Case (b). There exists a subsequence $(x_{n_i})_i \geq 1$ of the sequence $(x_n)_{n\geq 1}$ such that $f(x_{n_i}) = a$ for any $i \geq 1$. Since $\lim_{n\to\infty} x_n = +\infty$, we have $\lim_{i\to\infty} x_{n_i} = +\infty$. Hence the set $\{x_{n_i}\}_{i=1}^{\infty}$ is not bounded. On the other hand, $f^{-1}(a) \supset \{x_{n_i}\}_{i=1}^{\infty}$ and therefore the set $f^{-1}(a)$ will not be bounded. This is the contradiction. ■

Remark 1. Corollary 2 is a consequence of Theorem 3.

Corollary 4. *No perfect mapping from $\mathbb{R}$ to $\mathbb{R}$ has finite partial limits at $+\infty$ and $-\infty$.*

Corollary 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that at least one of the following two conditions is satisfied:*

1) $\exists M,\ L \in \mathbb{R} : \forall x > M : |f(x)| < L$;

2) $\exists M,\ L \in \mathbb{R} : \forall x < M : |f(x)| < L$.

Then the function f is not perfect. in particular, no bounded function from $\mathbb{R}$ to $\mathbb{R}$ is perfet.

Proof. If $f : \mathbb{R} \to \mathbb{R}$, satisfies for example, the first condition, then there exist at least two (possibly equal to each other) partial limits of f at $+\infty$: $\overline{\lim}_{x\to+\infty} f(x)$ and $\underline{\lim}_{x\to+\infty} f(x)$. But this contradicts Corollary 4. ■

Corollary 6. *If $f : \mathbb{R} \to \mathbb{R}$ is a perfect mapping, then for any positive real numbers M and L there exist points $x''_M < -M$ and $x'_M > M$ such that $|f(x''_M)| > L$ and $|f(x'_M)| > L$.*

Theorem 4. *Let $f \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(CL_p^{-1})$ (see Example 2 below). Assume that $a \in \mathbb{R}$ and let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $\lim_{n\to\infty} x_n = a$. Assume also that the limit $\lim_{n\to\infty} f(x_n)$ exists. Then $\lim_{n\to\infty} f(x_n) = f(a)$.*

Proof. Assume that $\lim_{n\to\infty} f(x_n) = b \in \mathbb{R}$ and $b \neq f(a)$. The following two cases are possible:

1) $\exists n_0 : \forall n \geq n_0 \quad f(x_n) = b$;
2) $\forall n\ \exists m(n) > n : f(x_{m(n)}) \neq b$.

Case 1. Clearly, $f^{-1}(b) \supset \{x_n\}_{n=n_0}^{\infty}$ and since, by assumption, $f(a) \neq b$, we have $a \notin f^{-1}(b)$. From $f^{-1}(b) \supset \{x_n\}_{n=n_0}^{\infty}$ and $\lim_{n\to\infty} x_n = a$ it follows that the point a belong to the closure of the set $f^{-1}(b)$. hence the set $f^{-1}(b)$ is not closed in $\mathbb{R}$. This is the contradiction.

Case 2. There exists a subsequence $(x_{n_i})_{i\geq 1}$ of the sequence $(x_n)_{n\geq 1}$ such that for any natural $i \geq 1$ we have $f(x_{n_i}) \neq b$. Clearly,

$$\lim_{i\to\infty} f(x_{n_i}) = b.$$

Since $\lim_{n\to\infty} x_n = a$, we have $\lim_{i\to\infty} x_{n_i} = a$. Hence the set $\{a\} \cup \{x_{n_i}\}_{i=1}^{\infty}$ is closed in $\mathbb{R}$. Consider the image

$$f(\{a\} \cup \{x_{n_i}\}_{i=1}^{\infty}) = \{f(a)\} \cup \{f(x_{n_i})\}_{i=1}^{\infty}.$$

Since $f(x_{n_i}) \neq b$ and $f(a) \neq b$ for any $i \geq 1$, we obtain $b \notin \{f(a)\} \cup \{f(x_{n_i})\}_{i=1}^{\infty}$. On the other hand, since $\lim_{i\to\infty} f(x_{n_i}) = b$, b belongs to the closure of the set $\{f(a)\} \cup \{f(x_{n_i})\}_{i=1}^{\infty}$. Hence this set is not closed. This is the contradiction. ■

We shall conclude the paper with several examples:

Example 1. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a function defined by the formula

$$f_1(x) = \begin{cases} \log_{\frac{1}{2}} |x|, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

We shall show that $f_1 \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(CM_p^{-1})$ but $f_1 \notin \mathcal{K}(C)$. Indeed, it is obvious that $f_1 \in \mathcal{K}(CM_p^{-1})$ and $f_1 \notin \mathcal{K}(C)$. Therefore it remains to show that $f_1 \in \mathcal{K}(CL\downarrow)$.

Let F be a closed subset of $\mathbb{R}$ and a be an accumulation point of the set $f(F)$. Then there exists a sequence of real numbers $(y_k)_{k\geq 1}$ such that for every $k \geq 1$ we have $y_k \in f(F)$, $y_k \neq a$, $y_k \neq 0$, and $\lim_{k\to\infty} y_k = a$. Clearly, for every $k \geq 1$ there exists a point $x_k \in F$ such that $f(x_k) = y_k$. We have

$$a = \lim_{k\to\infty} y_k = \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} \log_{\frac{1}{2}} |x_k| \Rightarrow \lim_{k\to\infty} |x_k| = \left(\frac{1}{2}\right)^a .$$

Then either there exists a subsequence $(x_{n_i})_{i\geq 1}$ of the sequence $(x_n)_{n\geq 1}$ such that $\lim_{i\to\infty} x_{n_i} = (\frac{1}{2})^a$ or there exists a subsequence $(x_{n_j})_{j\geq 1}$ of the sequence $(x_n)_{n\geq 1}$ such that $\lim_{j\to\infty} x_{n_j} = -(\frac{1}{2})^a$.

Let $(x_{n_j})_{j\geq 1}$ be a subsequence of the sequence $(x_n)_{n\geq 1}$ such that $\lim_{j\to\infty} x_{n_j} = -(\frac{1}{2})^a$. (If such a subsequence does not exist, one may consider a subsequence $(x_{n_i})_{i\geq 1}$ of the sequence $(x_n)_{n\geq 1}$ with $\lim_{i\to\infty} x_{n_i} = (\frac{1}{2})^a$).

Since F is closed and for $x_{n_j} \in F$, $-(\frac{1}{2})^2 \in F$ for each $j \geq 1$. But $f(-(\frac{1}{2})^a) = a$. Therefore $a \in f(F)$. Hence the set $f(F)$ is closed in $\mathbb{R}$.

Example 2. Let the mapping $f_2 : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f_2(x) = \begin{cases} 1, & \text{for } x \leq 0, \\ \log_{\frac{1}{2}} x, & \text{for } x > 0. \end{cases}$$

We shall show that $f_2 \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(CL_p^{-1})$ but $f_2 \notin \mathcal{K}(C)$ and $f_2 \notin \mathcal{K}(B_p^{-1}$.

It is obvious that $f_2 \notin \mathcal{K}(C)$ and $f_2 \in \mathcal{K}(CL_p^{-1})$.

Since $f_2^{-1}(1) = (-\infty; 0] \cup \{\frac{1}{2}\}$, $f_2 \notin \mathcal{K}(B_p^{-1})$.

Now let us prove that $f_2 \in \mathcal{K}(CL\downarrow)$. Assume that $F \subseteq \mathbb{R}$ is a closed subset of $\mathbb{R}$ and $a \in \mathbb{R}$ is an accumulation point of the set $f(F)$. Then there exists a sequence of real numbers $(y_k)_{k\geq 1}$ such that for any $k \geq 1$ we have $y_k \in f(F)$, $y_k \neq a$, $y_k \neq 1$, and $\lim_{k\to\infty} y_k = a$.

Since for any $k \geq 1$ we have $y_k \neq 1$ and $y_k \in f(F)$, there exists, for each $k \geq 1$, a positive real number x_k such that $x_k \in F$ and $f(x_k) = \log_{\frac{1}{2}} x_k = y_k$. Thus

$$\lim_{k\to\infty} y_k = a \Rightarrow \lim_{k\to\infty} \left(\frac{1}{2}\right)^{y_k} = \left(\frac{1}{2}\right)^a \Rightarrow \lim_{k\to\infty} x_k = \left(\frac{1}{2}\right)^a .$$

Since F is closed and for each $k \geq 1$, we have $x_k \in F$, $(\frac{1}{2})^a \in F$. But $f((\frac{1}{2})^a) = a$. Hence $a \in f(F)$. Therefore the set $f(F)$ is closed in $\mathbb{R}$.

Example 3. Let the mapping $f_3 : \mathbb{R} \to \mathbb{R}$ be determined by the formula

$$f_3(x) = \begin{cases} x, & \text{for } x \in (-\infty; 0] \cup [1; +\infty), \\ 0, & \text{for } x \in (0; 1). \end{cases}$$

We shall show that $f_3 \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(B_p^{-1})$ but $f_3 \notin \mathcal{K}(C)$ and $f_3 \notin \mathcal{K}(CL_p^{-1})$. That $f_3 \notin \mathcal{K}(C)$ and $f_3 \in \mathcal{K}(B_p^{-1})$ is obvious.

Since $f_3^{-1}(0) = [0; 1)$, we have $f_3 \notin \mathcal{K}(CL_p^{-1})$. Let us show that $f_3 \in \mathcal{K}(CL\downarrow)$. For this take any closed subset F of $\mathbb{R}$ and consider an accumulation point a of the set $f(F)$. Since $\overline{f(F)} \subset \overline{f(\mathbb{R})} = (-\infty; 0] \cup [1; +\infty)$, we have $a \in (-\infty; 0] \cup [1; +\infty)$.

If $a \in (-\infty; 0]$ (if $a \in [1; +\infty)$), then there exists a sequence of real numbers $(y_k)_{k\geq 1}$ such that for every $k \geq 1$ we have $y_k \in f(F)$, $y_k \neq a$, $y_k < 0$ (respectively, $y_k > 1$) and $\lim_{k\to\infty} y_k = a$.

Since each $y_k \in f(F)$, for every $k \geq 1$ there exists $x_k \in F$ such that $y_k = f(x_k) = x_k$. Therefore $a = \lim_{k\to\infty} y_k = \lim_{k\to\infty} x_k$. Since F is closed, $a \in F$. But, clearly, $f(a) = a$.

Hence $a \in f(F)$. Therefore $f(F)$ is a closed subset of $\mathbb{R}$.

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ავტორთა საყურადღებოდ

"საქართველოს მეცნიერებათა აკადემიის მაცნე. მათემატიკა" გამოდის **1993** წლის თებერვლიდან ორ თვეში ერთხელ. ჟურნალი აქვეყნებს შრომებს წმინდა და გამოყენებითი მათემატიკის ყველა დარგში. შრომები უნდა შეიცავდნენ ახალ შედეგებს სრული დამტკიცებებით.

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