ON FIBER STRONG SHAPE THEORY

Supervisor Prof. Vladimer Baladze

This dissertation is submitted for the degree of Academic Doctor of Mathematics
## Contents

- **Introduction** 1

- **1. Fiber Strong Shape Deformation Retractions and Fibrant Spaces** 22
  - 1.0 On Fiberwise Topological Preliminaries and Auxiliary Facts 22
  - 1.1 On fiber Borsuk pairs 32
  - 1.2 On Fiber SSDR-maps and Fibrant Spaces 37

- **2. Fiber Strong Shape Classifications of Compact Metrizable Spaces** 51
  - 2.1 On Fiber Strong Shape Category of Compact Metrizable Spaces 51
  - 2.2 On Fiber Strong Shape Equivalences of Compact Metrizable Spaces 59

- **3. Fiber Strong Shape Theory of Arbitrary Topological Spaces** 71
  - 3.1 Resolution and Strong Expansions of Spaces over $B_0$ 71
  - 3.2 On Fiber Strong Shape Category for Arbitrary Topological Spaces 83

- **Conclusion** 93

- **Bibliography** 94
Introduction

Shape Theory is an important and rich branch of Geometric Topology. Its methods can be successfully applied to the study of problems of Topology as well as other branches of Mathematics. Shape theory is meaningful extension of homotopy theory of spaces having homotopy type of ANR-spaces, polyhedras and simplicial complexes to the categories of more general spaces.

The shape theories that satisfy the main results of classical shape theory play essential role in modern topology. Their quantity and importance are systematically growing in the process of the research of the various problems of topology (Homology theory, Homotopy theory, Retracts theory, Shape Theory, Dynamical Systems, C*-algebra and others).

At the begining shape theory was constructed by K.Borsuk ([Bo2]-[Bo4]) for the category of compact metric spaces. S.Mardešić and J.Segal extended Borsuk’s shape theory to the category of compact Hausdorff spaces ([M-S1]-[M-S3]). After that R.H.Fox spread Borsuk’s theory on the category of metrizable spaces [Fo]. The another generalization of shape theory was described by B.J.Ball and R.B.Sher in [Ba-Sh], where they constructed proper shape theory for the category of locally compact separable spaces and proper maps. Besides, B.J.Ball investigated the proper shape theory for the category of locally compact metrizable spaces and proper maps [Ba]. The proper shape theory for the category of locally compact paracompact spaces and proper
maps was developed by V.Baladze \cite{B7}. Shape classifications of paracompact and \( p \)-paracompact spaces was described by A.Šostak \cite{S} and S. Mardešić and A.Šostak \cite{M-S}. Shape theory for the category of arbitrary topological spaces was developed by K.Morita \cite{Mor} and S. Mardešić \cite{M1}.

The categorical aspects of shape theory were studied by J.M. Cordier and T.Porter \cite{Co-P}.

The shape type extensions of uniform homotopy theory of absolute neighbourhood uniform retracts, equivariant homotopy theory of equivariant absolute neighbourhood retracts and \( n \)-homotopy theory of absolute neighbourhood retracts were constructed and investigated by several authors.

Uniform shape theory for the category of uniform spaces introduced by Agaronian and Smirnov \cite{A-S}, V.Baladze \cite{B8, B9, B11}, V.Baladze and L. Turmanidze \cite{B-Tu1, B-Tu2}, D. Doičinov \cite{Do1-Do3}, Nguyen Anh Kiet \cite{Ki}, T. Miyata \cite{Mi1-Mi2}, T. Miyata and J. Segal \cite{Mi-S}, T. Miyata and T. Watanabe \cite{Mi-W}, Nguyen To Nhu \cite{Nh}.

The origins of equivariant shape theory of spaces with action topological group can be traced back to papers by S. A. Antonyan, R. Jimenez and S. de Neymet \cite{An-J-N}, S.A. Antonian and S. Mardešić \cite{An-M}, Z. Čerin \cite{C3}, P.S. Gevorgian \cite{G1-G3} and Yu. M. Smirnov \cite{Sm1-Sm3}. In the solution of problems of equivariant shape theory important role plaid the methods and results of papers \cite{An1-An4, An-J-N, An-M}.

The \( n \)-shape theory was constructed by A. Chigogidze \cite{Ch1, Ch2} for the category of compact metric spaces. His results have been expanded on the category locally compact separable spaces and proper maps by Y. Akaike \cite{Ak1, Ak2}, Y. Akaike and K. Sakai \cite{Ak-Sa} and K. Sakai \cite{Sa}. The \( n \)-shape theory for the category of arbitrary compact Hausdorff spaces was investigated by R. Jimenez and L.R. Rubin \cite{Ji-R}.
There are several approaches to the fiber shape theory for spaces over a fixed space $B$ and continuous maps. The fiber shape theory is an extension of fiber of homotopy theory of ANR$_{B_0}$-spaces ([Dol], [Y$_2$]) and ANR-maps ([U], [N-S]).

Fiber shape theory for the category of compact metric spaces over fixed space $B_0$ and fiber-preserving maps was introduced by H. Kato ([K$_1$]–[K$_4$]) and M. Clap and L. Montejano ([Cl-Mo]). In papers ([Y$_1$]–[Y$_4$]) T. Yagasaki considered and investigated fiber shape theory of category metric spaces over $B_0$ and fiber-preserving maps. Fiber shape theories for arbitrary spaces over $B_0$, maps of metric spaces, and maps of topological spaces developed in papers V. Baladze ([B$_2$]–[B$_6$], [B$_11$]), Z. Čerin ([Č$_2$]) and D. A. Edwards and P. T. Mc. Auley ([E-A]).

Together with the classical shape theory and its variants there exists an important branch of modern geometric topology, so called strong shape theory, which besides the applications in topology (general topology, algebraic topology, geometric topology) ([M$_3$], [Md]), has also applications in other branches of mathematics (dynamical systems, C*-algebras) ([H], [D]).

Strong shape theory for different categories of spaces was investigated by several authors. For the category of compact metric spaces equivalent strong shape theories were introduced by F. W. Bauer ([Bau]), A. Calder and H. M. Hastings ([Ca-H]), F. W. Cathey ([C$_1$]), J. Dydak and J. Segal ([Dy-S]), D. A. Edwards and H. M. Hastings ([E-H]), Y. Kodama and J. Ono ([Ko-O]), Yu. T. Lisica ([L$_3$]) and J. B. Quigly ([Q]).

Strong shape theory for the category of general topological spaces and arbitrary categories was constructed by M. Batanin ([Bat]), F. W. Bauer ([Bau]), J. Dydak and S. Nowak ([Dy-N$_1$, Dy-N$_2$]), Yu. T. Lisica ([L$_3$]), Yu. T. Lisica and S. Mardešić ([L-M]), Z. Miminoshvili ([Mim]) and L. Stramaccia ([St]). In the papers are solved several serious problems of Topology ([M$_3$]).

For the present period of the shape theory development it is characteristic to design
and research different versions of strong shape theory.

Strong shape theory based on the notion of equivariant homotopy constructed by V.Baladze for metric $G$-spaces and A.Bykov and M.Texis for compact metric $G$-spaces.

Strong shape theory based on the notion of $n$-homotopy was developed by Y.Iwamoto and K.Sakai.

Fiberwise topology is a new direction of topology developed on the basis of General Topology, Algebraic Topology and Geometric Topology. Fiberwise topology occupies a central place in topology today. It’s methods were played important role in the solutions some problems of Differential Geometry, Lie Groups and Dynamical Systems, so establishment of new properties and characteristics of fiber spaces has more important significance.

The aspects of Algebraic topology and Homotopy topology for fiberwise topology were studied by James and James and Crabb. The investigation of fiberwise topology in the view of general topology was developed by F.Cammaroto, B.Pasymkov, D.Buhagiar and T.Miya (Bu-Miw-Pa, Ca-Pa).

The problem of construction of strong shape theory for fiberwise topology is one of interesting problems. As the strong shape theory arises from homotopy theory, so fiber strong shape theory arises from fiberwise homotopy theory. To develop of the fiber strong shape theory is natural. It is hoped that this may stimulate further research of fiberwise topology, in particular, of fiberwise homotopy theory.

The main goal of the dissertation work is to develop the strong shape theory of fiber topology, the investigation of the problem of construction of fiber strong shape classification for compact metric spaces and general topological spaces, the search of necessary and sufficient conditions the fulfillment of which will imply that the shape morphisms will be fiber strong shape equivalences. The aim of the thesis is also to
transfer main focus on the geometric interpretation of fiber strong shape theory.

We begin with a short description of results of the thesis by chapters. The dissertation work consists of Introduction, three Chapters and Bibliography.

Introduction shortly describes the history of strong shape theory and the results obtained in dissertation work. Chapter 1 provides a survey of fiberwise topology, beginning with basis theory and proceeding to a selection and specialized topics of fiberwise homotopy theory and fiberwise retracts theory. Chapter 1 also includes the fiber Borsuk’s pairs, strong shape deformation maps and fibrant spaces. In chapter 2 are defined fiber cotelescopes, constructed fiber strong shape category and established the characterizations of fiber strong shape equivalences of compact metrizable spaces. In Chapter 3 studied fiber strong ANR $B_0^0$-expansion and investigated the fiber strong shape theory for arbitrary spaces. At the end of dissertation is given the bibliography of used references.

Now we give a short survey of obtained results.

In the section 1.1 of Chapter 1 are obtained some results concerning the characterizations of Borsuk’s pairs over $B_0$ used in other sections of work.

The properties of fiberwise Borsuk’s pairs are described in the following propositions.

**Theorem 1.1.1.** A map $i : (A, \pi_A) \to (X, \pi_X)$ over $B_0$ is a cofibration over $B_0$ if and only if the map $j : (\text{Cyl}(i), \pi_{\text{Cyl}(i)}) \to (X \times I, \pi_{X \times I})$ over $B_0$ is retractible.

**Corollary 1.1.2.** A closed pair $(X, A)$ of space $X$ over $B_0$ and its closed subspace $A$ is a Borsuk pair over $B_0$ if and only if the subspace $(X \times \{0\}) \cup (A \times I) \subset X \times I$ is a retract over $B_0$ of $X \times I$.

**Corollary 1.1.3.** For each closed Borsuk’s pair $(X, A)$ over $B_0$ and for every space $Y$ over $B_0$ the pair $(X \times Y, A \times Y)$ over $B_0$ is a closed Borsuk’s pair over $B_0$. 
Corollary 1.1.4. If $(X, A)$ is a Borsuk’s pair over $B_0$ and $A$ is a closed subspace of locally compact Hausdorff space $X$ then for each space $Y$ over $B_0$ the map $i^* : Y^X \rightarrow Y^A$ is a cofibration over $B_0$.

Theorem 1.1.5. A pair $(X, A)$ of space $(X, \pi_X)$ over $B_0$ and its closed subspace $(A, \pi_{X|A})$ is a Borsuk pair over $B_0$ if and only if there exist a map $\psi : X \rightarrow I$ and a fiber homotopy $G : (X \times I, \pi_{X \times I}) \rightarrow (X, \pi_X)$ with respect $A$ such that $A = \psi^{-1}(0)$, $G(x, 0) = x$ and $G(x, t) \in A$ when $\psi(x) < t$.

Theorem 1.1.6. Let $(X, A)$ be a Borsuk pair over $B_0$. Then $(X \times I, (X \times \{0\}) \cup (A \times I) \cup X \times \{1\})$ is the Borsuk pair over $B_0$.

Theorem 1.1.7. Let $(X, A)$ be a Borsuk pair over $B_0$. Then each deformation retraction $r : (X, \pi_X) \rightarrow (A, \pi_{X|A})$ over $B_0$ is a strong deformation retraction over $B_0$.

Theorem 1.1.8. A closed pair $(X, A)$ of spaces over $B_0$ is a Borsuk pair over $B_0$ if and only if $\tilde{A} = (X \times \{0\}) \cup (A \times I)$ is a strong deformation retract over $B_0$ of $(X \times I, \pi_{X \times I})$.

Corollary 1.1.9. Let $(X, A)$ be a closed Borsuk pair over $B_0$. Then the subspace $(A, \pi_A)$ is a strong deformation retraction over $B_0$ of $(X, \pi_X)$ if and only if the inclusion $i : (A, \pi_A) \rightarrow (X, \pi_X)$ is a fiber homotopy equivalence.

In section 1.2 of Chapter 1 are given definitions and various concepts associated to fiber SSDR-maps and fibrant spaces and established their properties.

All spaces in Section 1.2 are metrizable. Here the basic definition is the following

Definition 1.2.1. Let $(X, \pi_X) \in \text{ob}(\mathcal{M}_{B_0})$ and let $A$ be a closed subspace of $X$. The subspace $(A, \pi_{X|A})$ over $B_0$ is called a shape strong deformation retract over $B_0$ of $(X, \pi_X)$ if there exists an embedding $\alpha : (X, \pi_X) \hookrightarrow (Y, \pi_Y) \in \text{AR}_{B_0}$ over $B_0$ satisfying the following condition:

for any pair of neighbourhoods $U$ and $V$ of $\alpha(X)$ and $\alpha(A)$ respectively in $(Y, \pi_Y)$,
there is a homotopy $H : (X \times I, \pi_{X \times I}) \to (U, \pi_{Y \mid U})_{\text{rel}} B_0$ such that $H(x, 0) = \alpha(x)$ and $H(x, 1) \in V$ for each $x \in X$.

This definition involves that if an embedding $\alpha : (X, \pi_X) \to (M, \pi_M)$ over $B_0$ satisfies the conditions of definition 1.2.1, then these conditions hold for any closed embedding $\beta : (X, \pi_X) \to (Z, \pi_Z) \in \text{AR}_{B_0}$.

A closed embedding $i : (A, \pi_A) \to (X, \pi_X)$ over $B_0$ is called $\text{SSDR}_{B_0}$-map if $i$ embeds $(A, \pi_A)$ in $(X, \pi_X)$ as a shape strong deformation retract over $B_0$ of $(X, \pi_X)$.

The notion of $\text{SSDR}_{B_0}$-map generalizes the notion of $\text{SDR}_{B_0}$-map.

One of main results of section 2.1 of Chapter 1 is the following

**Theorem 1.2.2.** Let $(X, \pi_X) \in M_{B_0}$ and $A$ be a closed subspace of $X$. Then the following conditions are equivalent:

a) $i : (A, \pi_{X \mid A}) \hookrightarrow (X, \pi_X)$ is an $\text{SSDR}$-map over $B_0$;

b) for any map $f : (A, \pi_{X \mid A}) \to (Y, \pi_Y) \in \text{ANR}_{B_0}$ over $B_0$, there is an extension $\tilde{f} : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ such that $\tilde{f} \cdot i = f$ and any two such extensions over $B_0$ are fiber homotopic with respect $i_A$;

c) for any commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & E \\
\downarrow{i} & & \downarrow{p} \\
X & \xrightarrow{\pi_X} & B \\
\pi_{X \mid A} & & \pi_B \\
\end{array}
\]

where $p : (E, \pi_E) \to (B, \pi_B)$ is a fibration over $B_0$ and $(E, \pi_E)$ and $(B, \pi_B)$ are $\text{ANR}_{B_0}$-spaces, there exists a map $\tilde{F} : (X, \pi_X) \to (E, \pi_E)$ over $B_0$ such that $\tilde{F} \cdot i = f$. 
and $p \cdot \tilde{F} = F$.

d) for any commutative diagram of maps over $B_0$

![Diagram](image)

there exists a filler $H : (X, \pi_X) \to (P^K, \pi_{PK})$ over $B_0$ provided $P \in \text{ANR}_{B_0}$ and $L$ is a subcomplex of a finite CW-complex $K$ with an inclusion map $j : L \hookrightarrow K$.

This result is plaining assertional role in whole work.

In Chapter 1 also introduced definition and investigation of fibrant spaces over $B_0$.

**Definition 1.2.3.** A space $(Y, \pi_Y)$ over $B_0$ is called a fibrant space over $B_0$ if for every SSDR-map $i : (A, \pi_{X|A}) \to (X, \pi_X)$ over $B_0$ and every map $f : (A, \pi_{X|A}) \to (Y, \pi_Y)$ over $B_0$, there is a map $F : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ such that $F \cdot i = f$, i.e. the following diagram commutes:

![Diagram](image)

The class of fibrant spaces over $B_0$ is sufficiently large. It contains the class of absolute neighbourhood retracts over $B_0$ (Theorem 1.2.4).
Apart from this result here is proved that, if \((Y, \pi_Y)\) is a fibrant space over \(B_0\) and \(K\) is a compact metric space, then \((Y^K_{B_0}, \pi_Y^K_{B_0})\) also is a fibrant space over \(B_0\) (Theorem 1.2.3).

The result of Chapter 1 are summarized in the propositions which systematically are used in next parts of work.

**Theorem 1.2.6.** Let \(Y = (((Y_n, \pi_{Y_n}), p_{n,n+1}, N^+)\) be an inverse system of fibrant spaces over \(B_0\) and fibrations over \(B_0\). Then the fiber limit space \(Y = \lim \leftarrow Y\) is a fibrant space over \(B_0\) and the natural projections \(p_n : (Y, \pi_Y) \to (Y_n, \pi_{Y_n})\) are fibrations over \(B_0\).

**Theorem 1.2.7.** Let \(f : (X, \pi_X) \to (Y, \pi_Y)\) be a map over \(B_0\). If \((X, \pi_X), (Y, \pi_Y) \in \text{ANR}_{B_0}\), then \(\text{coCyl}_{B_0}(f) \in \text{ANR}_{B_0}\).

**Theorem 1.2.8.** Let \(f : (X, \pi_X) \to (Y, \pi_Y)\) be a map over \(B_0\) of fibrant spaces over \(B_0\). Then the \(\text{coCyl}_{B_0}(f)\) over \(B_0\) is a fibrant space over \(B_0\).

The section 2.1 of Chapter 2 are begined to study of fiber cotelescope \(\text{coTel}(X)\) of inverse sequence \(X = \{(X_n, \pi_{X_n}, q_{n+1}^n, N^+)\}\) over \(B_0\).

The detailed descriptions of constructions given here allows us to prove the following main

**Theorem 2.1.1.** Let \(X = (((X_n, \pi_{X_n}), q_{n+1}^n, N^+)\) be an inverse sequence consisting of fibrant spaces over \(B_0\) and maps over \(B_0\). Then the cotelescope \(\text{coTel}_{B_0}(X)\) is a fibrant space over \(B_0\). If all \((X_n, \pi_{X_n})\) members of the inverse system \(X\) are ANR\(_{B_0}\)-spaces, then \(\text{coTel}_{B_0}(X)\) is a fibrant space over \(B_0\) too.

There exists the unique natural embedding \(i_q : (X, \pi_X) \to (\text{coTel}_{B_0}(X), \pi_{\text{coTel}_{B_0}(X)})\) over \(B_0\) such that \(\tilde{q}_n \cdot i_q = i_n \cdot q_n\) for each \(n \geq 0\).

In order to define the fiber strong shape classification in dissertation work are offered the notion of fiber resolution, which is a special case of the definition of resolution
over $B_0$ given in \([B_4]\) by V.Baladze.

**Definition 2.1.2.** An inverse sequence $X = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ is called resolution over $B_0$ of compact space $(X, \pi_X)$ over $B_0$ if

\begin{enumerate}[(a)]
  \item $(X, \pi_X) = \lim \leftarrow X$;
  \item the family $q = \{q_n : (X, \pi_X) \to (X_n, \pi_{X_n})\}_{n \in N^+}$ satisfies the following condition: for each $n \in N^+$ and open neighbourhood $U$ of $q_n(X)$ in $(X, \pi_{X_n})$ there exists $m \geq n$ such that $q_m^n(X_m) \subseteq U$.
\end{enumerate}

If all the $(X_n, \pi_{X_n}) \in \text{ANR}_{B_0}$, then $q$ is called an ANR$_{B_0}$-resolution over $B_0$.

One of the crucial point of the methods developed in Chapter 2 is the theorem of existence of fiber resolution.

**Theorem 2.1.3.** For each compact metrizable space $(X, \pi_X)$ over $B_0$ there exists an ANR$_{B_0}$-resolution $q : (X, \pi_X) \to X$ over $B_0$.

From this results follows that for every fiber resolution of compact metrizable space $(X, \pi_X)$ over $B_0$ there corresponds an fiber fibrant estension of $(X, \pi_X)$, namely the fiber cotelescope of this fiber resolution. The following result plays essential role in constructions given in work.

**Theorem 2.1.4.** Let $(X, \pi_X)$ be a compact metrizable space over $B_0$. If $q : (X, \pi_X) \to X = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ is a resolution over $B_0$ of $(X, \pi_X)$, then there exists an infinite strong deformation

$$D : \text{coTel}_{B_0}(X) \times [0, \infty) \to \text{coTel}_{B_0}(X)$$

of $\text{coTel}_{B_0}(X)$ over $B_0$ onto $i_q(X)$. In particular, the map $i_q : (X, \pi_X) \to \text{coTel}_{B_0}(X)$ is an SSDR-map over $B_0$.

The effect of Theorem 2.1.1, Theorem 2.1.3 and Theorem 2.1.4 is given by the following result. Let $\tilde{X} = \text{coTel}_{B_0}(X)$.
Theorem 2.1.5. For each compact metrizable space \((X, \pi_X)\) over \(B_0\) there is a fibrant extension \(i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})\) over \(B_0\). In particular, if \(q : (X, \pi_X) \to X = ((X_n, \pi_{X_n}), q^n_{n+1}, N^+)\) is an ANR\(_{B_0}\)-resolution over \(B_0\), then the embedding \(i_q : (X, \pi_X) \to (\text{coTel}_{B_0}(X), \pi_{\text{coTel}_{B_0}(X)})\) is a fibrant extension over \(B_0\).

The results obtained in dissertation work yield that the fiber strong shape theory is coarser than the fiber homotopy theory, but is finer that the fiber shape theory.

The main aim of Chapter 2 is the construction of fiber strong shape theory for compact metrizable spaces over a fixed base space \(B_0\), using the fiber versions of cotelescop and fibrant space.

The fiber strong shape category here constructed is the full image of functor reflector from the fiber homotopy category \(H(\text{CM}_{B_0})\) of compact metrizable spaces over \(B_0\) in the fiber homotopy category \(H(\text{F}_{B_0})\) of fiber fibrant spaces.

The Theorems 2.1.1, 2.1.3, 2.1.4 and 2.1.5 and routine diagram-choicing, as in the analogous situation in category theory, yield the following

Theorem 2.1.6. Let \(i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})\) be a fibrant extension over \(B_0\) of space \((X, \pi_X) \in \text{CM}_{B_0}\). Then the morphism \([i_X]_{B_0} : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})\) of category \(H(\text{CM}_{B_0})\) is an \(H(\text{F}_{B_0})\)-reflection.

The family \(\{i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})\}_{(X, \pi_X) \in \text{ob}(H(\text{CM}_{B_0}))}\) induces the \(H(\text{F}_{B_0})\)-reflector

\[ R : H(\text{CM}_{B_0}) \to H(\text{F}_{B_0}) \]

that is a functor given by formula

\[ R((X, \pi_X)) = (\tilde{X}, \pi_{\tilde{X}}), (X, \pi_X) \in \text{ob}(H(\text{CM}_{B_0})) \]

and satisfying the condition:

for each map \(f : (X, \pi_X) \to (Y, \pi_Y)\) over \(B_0\) of compact metrizable spaces the
is commutative. For the map $f$ over $B_0$ there exists a unique up to fiber homotopy map $\tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Y}, \pi_{\tilde{Y}})$ over $B_0$ such that the following diagram commutes

In this case the pair $(i_X, i_Y) : f \to \tilde{f}$ is called a fibrant extension over $B_0$ of map $f$.

**Definition 2.1.7.** The fiber strong shape category $\text{SSH}_{B_0}$ of compact metrizable spaces over $B_0$ is full image of the reflector $R : \text{H}(\text{CM}_{B_0}) \to \text{H}(\text{F}_{B_0})$.

There is a commutative diagram

$$
\begin{array}{c}
\text{SSH}_{B_0} \\
\downarrow J_R \\
\text{SSH}_{B_0}
\end{array}
\xrightarrow{R} 
\begin{array}{c}
\text{H}(\text{CM}_{B_0}) \\
\downarrow \text{H}(\text{F}_{B_0})
\end{array}
$$
Note that, for each

\[(X, \pi_X), (Y, \pi_Y) \in \text{ob}(H(CM_{B_0}))\]

\[\text{ob}(SSH_{B_0}) = \text{ob}(H(CM_{B_0}))\]

\[\text{Mor}_{SSH_{B_0}}((X, \pi_X), (Y, \pi_Y)) = \{([\tilde{X}, \pi_{\tilde{X}}], [\tilde{Y}, \pi_{\tilde{Y}}])_{B_0}\},\]

\[SS_{B_0}((X, \pi_X)) = (X, \pi_X)\]

and for a fibrant extension \((i_X, i_Y) : f \to \tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Y}, \pi_{\tilde{Y}})\) over \(B_0\) of each map \(f : (X, \pi_X) \to (Y, \pi_Y)\) over \(B_0\)

\[SS_{B_0}([f]_{B_0}) = R([f]_{B_0}) = [\tilde{f}]_{B_0}.\]

According to J. Dydak and S. Novak [Dy-N_1] in section 2.2 of Chapter 2 defined fiber strong shape equivalence.

**Definition 2.2.1.** A map \(f : (X, \pi_X) \to (Y, \pi_Y)\) over \(B_0\) is a fiber shape equivalence if for each \(ANR_{B_0}\)-space \((P, \pi_P)\) induces a bijection \(f^* : [Y, P]_{B_0} \to [X, P]_{B_0}\). A fiber shape equivalence \(f\) is called a fiber strong shape equivalence if for any two maps \(g, h : (Y, \pi_Y) \to (P, \pi_P) \in ANR_{B_0}\) over \(B_0\) and a fiber homotopy \(H : (X \times I, \pi_{X \times I}) \to (P, \pi_P)\) over \(B_0\) joining \(g f\) and \(h f\), \(H\) is fiber homotopic rel \(X \times \{0, 1\}\) to \(H' (f \times 1_I)\), where \(H' : (Y \times I, \pi_{Y \times I}) \to (P, \pi_P)\) is a fiber homotopy between \(g\) and \(h\).

The notion of fiber double mapping cylinder is very useful and simple geometric object. It is a comfortable tool for investigation of fiber strong shape theory.

The double mapping cylinder \(dCyl_{B_0}(f)\) over \(B_0\) of map \(f : (X, \pi_X) \to (Y, \pi_Y)\) over \(B_0\) is the subspace \(X \times I \cup Cyl_{B_0}(f) \times \{0, 1\}\) of space \(Cyl_{B_0}(f) \times I\) over \(B_0\).

Using the notion of fiber double mapping cylinder are given the characterizations
of fiber strong shape morphisms. Here are found necessary and sufficient conditions under which a map over $B_0$ is a fiber strong shape equivalence. Using the properties of fiber function spaces here are proved the following results.

One of main results is the following

**Theorem 2.2.3.** Let $f : (X, \pi_X) \rightarrow (Y, \pi_Y)$ be a map over $B_0$. The following conditions are equivalent:

1). $f$ is a fiber strong shape equivalences;

2). for a given space $(Z, \pi_Z)$ over $B_0$ containing $(X, \pi_X)$ as a closed subspace over $B_0$, every map $g : (Z, \pi_Z) \rightarrow (P, \pi_P) \in \text{ANR}_{B_0}$ over $B_0$ extends to $(Z \cup \text{Cyl}_{B_0}(f), \pi_{Z \cup \text{Cyl}_{B_0}(f)})$ and every map

$$H : (Z \times I \cup \text{dCyl}_{B_0}(f), \pi_{Z \times I \cup \text{dCyl}_{B_0}(f)}) \rightarrow (P, \pi_P) \in \text{ANR}_{B_0}$$

over $B_0$ extends to $((Z \cup \text{Cyl}_{B_0}(f)) \times I, \pi_{(Z \cup \text{Cyl}_{B_0}(f)) \times I})$;

3). if $(X, \pi_X)$ is a closed subspace of $(Z, \pi_Z)$, then the fiber inclusions

$$i : (Z, \pi_Z) \rightarrow (Z \cup \text{Cyl}_{B_0}(f), \pi_{Z \cup \text{Cyl}_{B_0}(f)})$$

and

$$j : (Z \times I \cup \text{dCyl}_{B_0}(f), \pi_{Z \times I \cup \text{dCyl}_{B_0}(f)}) \rightarrow ((Z \cup \text{Cyl}_{B_0}(f)) \times I, \pi_{(Z \cup \text{Cyl}_{B_0}(f)) \times I})$$

are fiber shape equivalences;

4). if $(X, \pi_X)$ is a closed subspace of $(Z, \pi_Z)$, then the fiber inclusion

$$i : (Z, \pi_Z) \rightarrow (Z \cup \text{Cyl}_{B_0}(f), \pi_{Z \cup \text{Cyl}_{B_0}(f)})$$
is a fiber strong shape equivalence;

5). if \((X, \pi_X)\) is a closed subspace of \((Z, \pi_Z)\), then the fiber inclusion
\[ i : (Z, \pi_Z) \to (Z \cup \text{Cyl}_{B_0}(f), \pi_{Z\cup\text{Cyl}_{B_0}(f)}) \]
is a fiber shape equivalence;

6). the fiber inclusions
\[ k : (X, \pi_X) \to (\text{Cyl}_{B_0}(f), \pi_{\text{Cyl}_{B_0}(f)}) \]
and
\[ l : (d\text{Cyl}_{B_0}(f), \pi_{d\text{Cyl}_{B_0}(f)}) \to (\text{Cyl}_{B_0}(f) \times I, \pi_{\text{Cyl}_{B_0}(f) \times I}) \]
are fiber shape equivalences;

7). every map \(g : (X, \pi_X) \to (P, \pi_P) \in \text{ANR}_{B_0}\) over \(B_0\) extends to \((\text{Cyl}_{B_0}(f), \pi_{\text{Cyl}_{B_0}(f)})\)
and every map \(H : (d\text{Cyl}_{B_0}(f), \pi_{d\text{Cyl}_{B_0}(f)}) \to (P, \pi_P) \in \text{ANR}_{B_0}\) over \(B_0\) extends to
\((\text{Cyl}_{B_0}(f) \times I, \pi_{\text{Cyl}_{B_0}(f) \times I})\).

The consequences of theorem 2.2.3 are the following propositions.

**Corollary 2.2.4.** Let \((X, \pi_X)\) be a space over \(B_0\) and \(A \subset X\). The fiber inclusion
\[ i : (A, \pi_{X|A}) \to (X, \pi_X) \]
is a fiber strong shape equivalence if and only if \(i\) and \(j : (X \times \{0\} \cup A \times I \cup X \times \{1\}, \pi_{X \times \{0\} \cup A \times I \cup X \times \{1\}}) \to (X \times I, \pi_{X \times I})\)
are fiber shape equivalences.

**Corollary 2.2.5.** Let \(f : (X, \pi_X) \to (Y, \pi_Y)\) be a fiber homotopy equivalence. Then \(f\) is a fiber strong shape equivalence.

**Corollary 2.2.6.** If \(g : (X, \pi_X) \to (Y, \pi_Y)\) is fiber homotopic to a fiber strong shape equivalence \(f : (X, \pi_X) \to (Y, \pi_Y)\), then \(g\) is a fiber strong shape equivalence.

**Theorem 2.2.7.** Let \(f : (X, \pi_X) \to (Y, \pi_Y)\) and \(g : (Y, \pi_Y) \to (Z, \pi_Z)\) be fiber
strong shape equivalences. Then the composition \( g \circ f : (X, \pi_X) \to (Z, \pi_Z) \) is a fiber strong shape equivalence.

**Theorem 2.2.8.** Let \( f : (X, \pi_X) \to (Y, \pi_Y) \) and \( g : (Y, \pi_Y) \to (Z, \pi_Z) \) be maps over \( B_0 \) such that \( g \circ f \) is a fiber strong shape equivalence. If one of \( f \) and \( g \) is a fiber strong equivalence, then both \( f \) and \( g \) are fiber strong shape equivalences.

**Corollary 2.2.9.** Let \( f : (X, \pi_X) \to (Y, \pi_Y) \) be a fiber shape equivalence. If \((X, \pi_X)\) has the fiber homotopy type of an ANR \( B_0 \), then \( f \) is a fiber strong shape equivalence.

The next theorem show that in terms of fiber double cylinders it is possible to describe fiber strong shape isomorphisms of category \( SSH_{B_0} \).

**Theorem 2.2.10.** A closed fiber embedding \( i : (A, \pi_{X|A}) \to (X, \pi_X) \) is a fiber strong shape equivalence if and only if \( i \) is a SSDR-map over \( B_0 \).

**Theorem 2.2.11.** Let \( f : (X, \pi_X) \to (Y, \pi_Y) \) be a map over \( B_0 \) of compact metrizable spaces over \( B_0 \) and \((i_X, i_Y) : f \to \tilde{f} \) a fibrant extension over \( B_0 \) of \( f \). Then \( f \) is a fiber strong shape equivalence if and only if \( \tilde{f} \) is a fiber homotopy equivalence.

**Corollary 2.2.12.** A map \( f \) over \( B_0 \) of compact metrizable spaces over \( B_0 \) is a fiber strong shape equivalence in the sense of Definition 2.2.1 if and only if \( SS_{B_0}([f]_{B_0}) \) is an isomorphism of the category \( SSH_{B_0} \).

In the Chapter 3 is constructed and developed a fiber strong shape theory for arbitrary spaces over fixed metrizable space \( B_0 \). The approach given here is based on the method of Mardešić-Lisica and instead of resolutions, introduced by Mardešić, their fiber preserving analogues are used. The fiber strong shape theory yields the classification of spaces over \( B_0 \) which is coarser than the classification of spaces over \( B_0 \) induced by fiber homotopy theory, but is finer than the classification of spaces over \( B_0 \) given by usual fiber shape theory.

The construction of fiber strong shape category uses the notion of fiber strong \( ANR_{B_0} \)-expansion of space over \( B_0 \). Fiber strong expansions of spaces over \( B_0 \) are mor-
phisms of category \( \text{pro} - \text{Top}_{B_0} \) from spaces over \( B_0 \) to inverse systems of spaces over \( B_0 \), which satisfy a stronger version of fiber homotopy conditions of ANR\(_{B_0}\)-expansion defined by V. Baladze (\([B_4]\), \([B_{10}]\)).

In the section 3.1 it is proved that fiber resolutions of spaces over \( B_0 \) induce fiber strong expansions of spaces over \( B_0 \). In order to construct the fiber strong shape category \( \text{SSH}_{B_0} \) is used this result.

The essential role in section 3.1 play the following notions and results.

Let \( U = \{U_\alpha\}_{\alpha \in \mathcal{A}} \) be a covering of a space \((Y, \pi_Y)\) over \( B_0 \). We say that the fiber preserving maps \( f, g : (X, \pi_X) \to (Y, \pi_Y) \) are \( U \)-near, if for every \( x \in X \) there exists a \( U_\alpha \in U \) such that, \( f(x), g(x) \in U_\alpha \). We say that a fiber preserving homotopy \( H : (X \times I, \pi_{X \times I}) \to (Y, \pi_Y) \) which connects \( f \) and \( g \), is a \( U \)-homotopy if for every \( x \in X \) there exists a \( U_\alpha \in U \) such that \( H(x, t) \subseteq U_\alpha \) for all \( t \in I \).

**Proposition 3.1.1** (Comp. \([B_5]\), Proposition 7) Let \((Y, \pi_Y)\) be an ANR\(_{B_0}\)-space. Then every open covering \( U \) of \((Y, \pi_Y)\) admits an open covering \( V \) of \((Y, \pi_Y)\) such that, whenever any two f.p. maps \( f, g : (X, \pi_X) \to (Y, \pi_Y) \) from an arbitrary space \((X, \pi_X)\) over \( B_0 \) into the space \((Y, \pi_Y)\) over \( B_0 \) are \( V \)-near, then there exists f.p. \( U \)-homotopy \( H : (X \times I, \pi_{X \times I}) \to (Y, \pi_Y) \) which connects \( f \) and \( g \). Moreover, if for a subset \( A \subseteq X \), \( f|_A = g|_A \), then \( H \) is f.p. homotopy rel\( A \).

**Definition 3.1.4.** (V. Baladze, see \([B_4]\), \([B_6]\), \([B_{10}]\)) Let \((X, \pi_X)\) be a topological space over \( B_0 \), \( X = (\{(X_\alpha, \pi_{X_\alpha}), p_{\alpha \alpha'}, \mathcal{A}\}) \) an inverse system in \( \text{Top}_{B_0} \) and \( p = (p_\alpha) : (X, \pi_X) \to X \) a morphism of \( \text{pro} - \text{Top}_{B_0} \). We call \( p \) an expansion over \( B_0 \) of the space \((X, \pi_X)\) over \( B_0 \) provided it has the following properties:

\( E_{B_0}1 \). For every ANR\(_{B_0}\)-space \((P, \pi_P)\) over \( B_0 \) and f.p. map \( f : (X, \pi_X) \to (P, \pi_P) \) there is an index \( \alpha \in \mathcal{A} \) and a f. p. map \( h : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P) \) such that \( h p_\alpha \simeq f \).

\( E_{B_0}2 \). If \( f, f' : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P) \) are f. p. maps, \((P, \pi_P) \in \text{ANR}_{B_0}\) and \( f p_\alpha \simeq f' p_\alpha \), then there is an index \( \alpha' \geq \alpha \) such that \( f p_{\alpha \alpha'} \simeq f' p_{\alpha \alpha'} \).
Definition 3.1.5. A morphism \( p : (X, \pi_X) \to ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathcal{A}) \) is called a strong expansion over \( B_0 \) provided it satisfies condition \( E_{B_0}1 \) and the following condition:

SE\(_{B_0}2\). Let \((P, \pi_P)\) be an ANR\(_{B_0}\)-space, let \( f_0, f_1 : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P), \alpha \in \mathcal{A} \) be f.p. maps and let \( S : (X \times I, \pi_{X \times I}) \to (P, \pi_P) \) be a f.p. homotopy such that

\[
S(x, 0) = f_0p_\alpha(x), \quad x \in X,
\]
and

\[
S(x, 1) = f_1p_\alpha(x), \quad x \in X.
\]

Then there exists a \( \alpha' \geq \alpha \) and a f.p. homotopy \( H : (X_\alpha', I, \pi_{X_\alpha' \times I}) \to (P, \pi_P) \), such that

\[
H(x, 0) = f_0p_{\alpha\alpha'}(z), \quad z \in X_{\alpha'},
\]

\[
H(x, 1) = f_1p_{\alpha\alpha'}(z), \quad z \in X_{\alpha'},
\]

\[
H(p_{\alpha' \times I}) \simeq S(\text{rel}(X \times \partial I)).
\]

Every strong expansion over \( B_0 \) is an expansion over \( B_0 \).

If all \((X_\alpha, \pi_{X_\alpha}) \in \text{ANR}_{B_0}\), then \( p \) is called an ANR\(_{B_0}\)-expansion and strong ANR\(_{B_0}\)-expansion, respectively.

The main results of section 3.1 is the following theorem.

Theorem 3.1.6. Let \((X, \pi_X)\) be a topological space over \( B_0 \). Then every resolution \( p : (X, \pi_X) \to X \) over \( B_0 \) induces a strong ANR\(_{B_0}\)-expansion over \( B_0 \).

Corollary 3.1.7 Every ANR\(_{B_0}\)-resolution over \( B_0 \) induces ANR\(_{B_0}\)-expansion over \( B_0 \).

Corollary 3.1.8 Every space \((X, \pi_X)\) over \( B_0 \) admits a cofinite strong ANR\(_{B_0}\)-
expansion over $B_0$.

In the proof of Theorem 3.1.6 are used the following lemmas.

**Lemma 3.1.9.** Let $(X, \pi_X)$ be a topological space over metrizable space $B_0$, let $(P, \pi_P), (P', \pi_{P'})$ be ANR$_{B_0}$-spaces, let $f : (X, \pi_X) \to (P', \pi_{P'})$, $h_0, h_1 : (P', \pi_{P'}) \to (P, \pi_P)$ be f.p. maps and let $S : (X \times I, \pi_{X \times I}) \to (P, \pi_P)$ be a f.p. homotopy such that

$$S(x, 0) = h_0 f(x), \quad x \in X,$$

$$S(x, 1) = h_1 f(x), \quad x \in X.$$  

Then there exists an ANR$_{B_0}$-space $(P'', \pi_{P''})$, f.p. maps $f' : (X, \pi_X) \to (P'', \pi_{P''})$, $h : (P'', \pi_{P''}) \to (P', \pi_{P'})$ and a f.p. homotopy $K : (P'' \times I, \pi_{P'' \times I}) \to (P, \pi_P)$ such that

$$hf' = f,$$

$$K(z, 0) = h_0 h(z), \quad z \in P''$$

$$K(z, 1) = h_1 h(z), \quad z \in P''$$

$$K(f' \times 1_I) = S.$$  

**Lemma 3.1.10.** Let $p : (X, \pi_X) \to X$ be a resolution over $B_0$ and let $\alpha, (P, \pi_P), f_0, f_1$ and $(F, \pi_F)$ be as in SE$_{B_0, 2})$. Then for every open covering $\mathcal{U}$ of $(P, \pi_P)$, there exist a $\alpha' \geq \alpha$ and a f.p. homotopy $H : (X_{\alpha'} \times I, \pi_{X_{\alpha'} \times I}) \to (P, \pi_P)$ such that

$$H(y, 0) = f_0 p_{\alpha\alpha'}(y), \quad y \in X_{\alpha'}$$

$$H(y, 1) = f_1 p_{\alpha\alpha'}(y), \quad y \in X_{\alpha'}$$

$$(S, H(1 \times p_{\alpha'})) \leq \mathcal{U}.$$
In the section 3.2 of Chapter 3 is constructed fiber coherent prohomotopy category \( \mathbf{CPHTop}_{B_0} \). The fiber coherent prohomotopy category \( \mathbf{CPHTop}_{B_0} \) has as objects inverse systems \( X = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathscr{A}) \) of topological spaces over \( B_0 \) and f.p. maps over directed cofinite index sets. The morphisms are f.p. coherent homotopy classes \([f] : X \to Y\) of f.p. coherent maps \( f : X \to Y\) of such systems. Composition is defined by composing representatives, which are special f.p. coherent maps.

There exist the functors \( C : \mathbf{pro-Top}_{B_0} \to \mathbf{CPHTop}_{B_0} \) and \( E : \mathbf{CPHTop}_{B_0} \to \mathbf{pro-HTop}_{B_0} \). The composition \( E \circ C : \mathbf{pro-Top}_{B_0} \to \mathbf{pro-HTop}_{B_0} \) is the functor induced by the f.p. homotopy functor \( H : \mathbf{Top}_{B_0} \to \mathbf{HTop}_{B_0} \).

The objects of fiber strong shape category \( \mathbf{SSH}_{B_0} \) are all topological spaces over \( B_0 \). The morphisms of category \( \mathbf{SSH}_{B_0} \) are defined by the following way.

Let \( p : (X, \pi_X) \to X \) and \( q : (Y, \pi_Y) \to Y \) be an ANR\(_{B_0}\)-resolutions of \( (X, \pi_X) \) and \( (Y, \pi_Y) \), respectively. Let \([f] : X \to Y\) be a some morphism of category \( \mathbf{CPHTop}_{B_0} \).

Let \( p' : (X, \pi_X) \to X', q' : (Y, \pi_Y) \to Y', [f'] : X' \to Y' \) be another triple of fiber resolutions of spaces \( (X, \pi_X) \) and \( (Y, \pi_Y) \) over \( B_0 \) and morphism of category \( \mathbf{CPHTop}_{B_0} \).

The triples \((p, q, [f])\) and \((p', q', [f'])\) are called equivalent if \([f'] [i] = [j] [f]\), where \([i] : X \to X'\) and \([j] : Y \to Y'\) are isomorphisms of category \( \mathbf{CPHTop}_{B_0} \).

The fiber strong shape morphisms \( F : (X, \pi_X) \to (Y, \pi_Y) \) are the equivalence classes of triples \((p, q, [f])\) with respect to the defined equivalence relation.

By symbol \( \mathbf{ssh}_{B_0}((X, \pi_X)) \) is denoted the equivalence class of topological space \( (X, \pi_X) \) and call the fiber strong shape of \( (X, \pi_X) \).

In the sections 3.2 are constructed a fiber strong shape functor \( \mathbf{SS}_{B_0} : \mathbf{HTop}_{B_0} \to \mathbf{SSH}_{B_0} \) and a functor \( S : \mathbf{SSH}_{B_0} \to \mathbf{SH}_{B_0} \) into V.Baladze fiber shape category \([B_4]\).

One of main results of this section is the following.

Theorem 3.2.5 There exists the following commutative diagram
where $S_{B_0}$ is V. Baladze fiber shape functor $[B_4]$.

**Corollary 3.2.6.** Let $(X, \pi_X)$ and $(Y, \pi_Y)$ be topological spaces over $B_0$. If $ssh_{B_0}((X, \pi_X)) = ssh_{B_0}((Y, \pi_Y))$, then $sh_{B_0}((X, \pi_X)) = sh_{B_0}((Y, \pi_Y))$. 
Chapter 1

Fiber Strong Shape Deformation
Retractions and Fibrant Spaces

Chapter 1 provides a survey of fiberwise topology, beginning with basis theory and proceeding to a selection and specialized topics of fiberwise homotopy theory and fiberwise retracts theory. Chapter 1 also include investigation of fiber Borsuk’s pairs, strong shape deformation maps and fibrant spaces.

1.0 On Fiberwise Topological Preliminaries and Auxiliary Facts

In this section we introduce the basic notations and results which we use in the next.

Let $R : \mathcal{X} \to \mathcal{L}$ be a functor. The full image of functor $R$ is a category $\text{fim}R$ and a factorization of $R$
where \( E \) is the identity on objects of category \( \mathcal{K} \) and \( J \) is fully faithful.

Let \( \mathcal{L} \) be a full subcategory of \( \mathcal{P} \). Then an element \( \tau : X \to Y \) of \( \text{Mor}_{\mathcal{P}}(X,Y) \) with \( Y \in \text{ob}(\mathcal{L}) \) is called \( \mathcal{L} \)-reflection of \( X \) if the function \( \tau^\# : \mathcal{L}(Y,L) \to \mathcal{P}(X,L) \) is bijective for each \( L \in \text{ob}(\mathcal{L}) \). Let \( \mathcal{K} \subseteq \mathcal{P} \) be a subcategory of \( \mathcal{P} \) and let \( \{ \tau_X : X \to RX \}_{X \in \text{ob}(\mathcal{K})} \) be a family of \( \mathcal{L} \)-reflections, where \( R \) is a function mapping objects of \( \mathcal{K} \) to objects of \( \mathcal{L} \). It is clear that the function \( R \) extends to a functor \( R : \mathcal{K} \to \mathcal{L} \). By definition, for each element \( f : A \to X \) of \( \text{Mor}_{\mathcal{K}}(A,X) \), \( Rf \) is a morphism \( Rf : RA \to RX \) for which \( (Rf) \cdot \tau_A = \tau_X \cdot f \). Defined functor is called a reflection or reflector of \( \mathcal{K} \) in \( \mathcal{L} \).

For a given fixed object \( B_0 \) of category \( \mathcal{K} \) by \( \mathcal{K}_{B_0} \) denote the following category. The objects of \( \mathcal{K}_{B_0} \) are pairs \((X, \pi_X)\) consisting of object \( X \in \text{ob}(\mathcal{K}) \) and morphism \( \pi_X : X \to B \) from \( \text{Mor}_{\mathcal{K}}(X,Y) \), called the projection.

The morphisms of \( \mathcal{K}_{B_0} \) are morphisms \( f : X \to Y \) of \( \mathcal{K} \) with property \( \pi_X = \pi_Y \cdot f \). These morphisms are called morphisms over \( B_0 \).

We will denote by \( \text{Top} \), \( \text{M} \) and \( \text{CM} \) the categories of topological spaces, metrizable spaces and compact metrizable spaces, respectively. Consequently, for fixed objects \( B_0 \) of given categories there exist the categories \( \text{Top}_{B_0} \), \( \text{M}_{B_0} \) and \( \text{CM}_{B_0} \). In this categories the notion of fiber homotopy is defined.

For each object \((X, \pi_X)\) of some category of spaces over \( B_0 \) the pair \((X \times Z, \pi_{X \times Z})\), where \( Z \) is a space and \( \pi_{X \times Z} \) is the projection given by formula

\[
\pi_{X \times Z}(x, z) = \pi_X(x), \quad (x, z) \in X \times Z,
\]

is the space over \( B_0 \). Note that the natural projection \( p_X : X \times Z \to X \) is the map over \( B_0 \).

Let \( Y^Z \) be the function space with compact-open topology. Consider the subspace
1.0. On Fiberwise Topological Preliminaries and Auxiliary Facts

$Y^Z_{B_0}$ of the space $Y^Z$:

$Y^Z_{B_0} = \{ f \in Y^Z : \pi_Y \cdot f = \text{const} \}$.

Let $\pi_{Y^Z_{B_0}} : Y^Z_{B_0} \to B_0$ be a map given by

$\pi_{Y^Z_{B_0}}(f) = \pi_Y(f(z)), z \in Z$.

Consequently, the pair $(Y^Z_{B_0}, \pi_{Y^Z_{B_0}})$ is a space over $B_0$.

By exponential law there exists a homeomorphism map over $B_0$

$E : (Y, \pi_Y)^{(X \times Z, \pi_{X \times Z})} \to (Y^Z_{B_0}, \pi_{Y^Z_{B_0}})^{(X, \pi_X)}$

given by formula

$(E(H)(x))(z) = H(x, z), H : (X \times Z, \pi_{X \times Z}) \to (Y, \pi_Y), x \in X, z \in Z$.

Let $f, g : (X, \pi_X) \to (Y, \pi_Y)$ be maps over $B_0$ and $I = [0, 1]$. A fiber homotopy from $f$ to $g$ is called a map $H : (X \times I, \pi_{X \times I}) \to (Y, \pi_Y)$ over $B_0$ such that $H_0 = f$ and $H_1 = g$.

A fiber homotopy $H$ from $f$ to $g$, $H : f \simeq g$, we also call a homotopy over $B_0$. The fiber homotopy class of fiber map $f$ is denoted by $[f]_{B_0}$. We write $[X, Y]_{B_0}$ for the set of all fiber homotopy classes. By $H(\text{Top}_{B_0})$, $H(\text{M}_{B_0})$ and $H(\text{CM}_{B_0})$ we denote the fiber homotopy categories of categories $\text{Top}_{B_0}$, $\text{M}_{B_0}$ and $\text{CM}_{B_0}$, respectively.

By exponential law a homotopy $H : (X \times I, \pi_{X \times I}) \to (Y, \pi_Y)$ over $B_0$ induces a map $E(H) : (X, \pi_X) \to (Y^I_{B_0}, \pi_{Y^I_{B_0}})$ over $B_0$, where $Y^I_{B_0} = \{ f : I \to Y | \pi_Y \cdot f = \text{const} \}$ and $\pi_{Y^I_{B_0}}$ is a map defined by formula

$\pi_{Y^I_{B_0}}(f) = \pi_Y(f(t)), t \in I, f \in Y^I_{B_0}$. 
Now we give definitions of some maps which are used in the next.

By $\omega_0 : Y^I_{B_0} \rightarrow (Y, \pi_Y)$ and $\omega_1 : Y^I_{B_0} \rightarrow (Y, \pi_Y)$ we denote maps given by formulas

\[ \omega_0(\varphi) = \varphi(0), \varphi \in Y^I_{B_0}, \]

\[ \omega_1(\varphi) = \varphi(1), \varphi \in Y^I_{B_0}, \]

respectively.

For each $t \in I$ there exist the embedding maps $\sigma_t : (X, \pi_X) \rightarrow (X \times I, \pi_{X \times I})$ over $B_0$ given by formula

\[ \sigma_t(x) = (x, t), x \in X. \]

Let $j : L \rightarrow K$ be a map. By $j^* : P^K_{B_0} \rightarrow P^L_{B_0}$ we denote a map over $B_0$ given by formula

\[ j^*(u) = u \cdot j, u \in P^K_{B_0}. \]

There exists a fiber homotopy functor $H : \text{Top}_{B_0} \rightarrow \text{H(Top}_{B_0)}$ given by formulas

\[ H(f) = [f]_{B_0}, f \in \text{Mor}_{\text{Top}_{B_0}}(X, Y) \]

and

\[ H((X, \pi_X)) = (X, \pi_X), (X, \pi_X) \in \text{ob(\text{Top}_{B_0})}. \]

Let $A \subset X$ and $\pi_A = \pi_{X|A}$. A map $r : (X, \pi_X) \rightarrow (A, \pi_A)$, over $B_0$ is a fibrewise retraction over $B_0$, if $r \cdot i = 1_A$ and, in addition, $i \cdot r \simeq_{B_0} 1_A$, then $r$ is called a fibrewise deformation retraction, or deformation retraction over $B_0$.

A subspace $A$ of metrizable space $X$ over $B_0$ is called a fibrewise neighborhood retract of $X$ if there exist an open neighborhood $U$ of $A$ in $X$ and a fibrewise retraction $r : U \rightarrow A$. 

A deformation retraction \( r : (X, \pi_X) \to (A, \pi_A) \) over \( B_0 \) is called a strong deformation retraction over \( B_0 \), or SDR\(_{B_0}\)-map over \( B_0 \), if \( i \cdot r \simeq 1_{X \text{rel} A} \).

Note that for each fiber homotopy equivalence \( f : (X, \pi_X) \to (Y, \pi_Y) \) the subspace \((X, \pi_X) \subset (\text{Cyl}(f), \pi_{\text{Cyl}(f)})\) over \( B_0 \) is a strong deformation retract over \( B_0 \) of \( \text{Cyl}(f) \).

Let \( A \) be a closed subset of a space \((X, \pi_X) \in \mathbf{M}_{B_0}\) over \( B_0 \). We say that the map \( D : (X \times [0, +\infty), \pi_{X \times [0, +\infty)}) \to (X, \pi_X) \) over \( B_0 \) is an infinite strong deformation of \((X, \pi_X)\) onto \((A, \pi_X|_A)\) if \( D(x, 0) = x \) for all \( x \in X \), \( D(a, t) = a \) for all \( a \in A, t \in [0, +\infty) \) and for any open neighbourhood \( U \) of \( A \) in \( X \) there exists a \( \lambda \in [0, +\infty) \) such that \( D(X \times [\lambda, \infty)) \subseteq U \).

We also use the following notions. Let \( B_0 \) be a fixed metrizable space. A space \((Y, \pi_Y) \in \text{ob}(\mathbf{M}_{B_0})\) is an absolute retract over \( B_0 \), \((Y, \pi_Y) \in \text{AR}_{B_0}\) (an absolute neighbourhood retract over \( B_0 \), \((Y, \pi_Y) \in \text{ANR}_{B_0}\)), if it has the following property: for any closed embedding \( i : (Y, \pi_Y) \to (X, \pi_X) \in \text{ob}(\mathbf{M}_{B_0}) \) over \( B_0 \) there exists a fiberwise retraction \( r : (X, \pi_X) \to (i(Y), \pi_X|_{i(Y)}) \) (an open neighbourhood \( U \) of \( i(Y) \) in \( X \) and a fiberwise retraction \( r : (U, \pi_X|_U) \to (i(Y), \pi_X|_{i(Y)}) \)).

The space \((Y, \pi_Y) \in \text{ob}(\mathbf{M}_{B_0})\) is an absolute extensor over \( B_0 \), \( Y \in \text{AE}_{B_0}\) (an absolute neighbourhood extensor over \( B_0 \), \((Y, \pi_Y) \in \text{ANE}_{B_0}\)), if it has the following property: for any space \((X, \pi_X) \in \text{ob}(\mathbf{M}_{B_0})\) over \( B_0 \) and any closed subspace \( A \subseteq X \), every map \( f : (A, \pi_X|_A) \to (Y, \pi_Y) \) over \( B_0 \) has an extension \( \tilde{f} : (X, \pi_X) \to (Y, \pi_Y) \) over \( B_0 \) (\( \tilde{f} : (U, \pi_X|_U) \to (Y, \pi_Y) \), where \( U \) is an open neighbourhood of \( A \) in \( X \)).

The next results are routine generalizations of the results of the retracts theory.

A metrizable space over \( B_0 \) is an A(N)R\(_{B_0}\)-space if and only if it is an A(N)E\(_{B_0}\)-space \([Y_2]\).

For every metric space \((X, \pi_{B_0}) \) over \( B_0 \) there exist fibrepreserving closed embedding into ANR\(_{B_0}\)-space \((M, \pi_M)\) with weight \( w(M) \leq \max(w(X), w(B_0), R_0) \) \([B_4]\).
1.0. On Fiberwise Topological Preliminaries and Auxiliary Facts

The space $Y_{B_0}$ of maps $\varphi : Z \to Y$ from compact metrizable space $Z$ into ANR$_{B_0}$-space $Y$, with compact-open topology and property $\pi_Y \cdot \varphi = \text{const}$ is ANR$_{B_0}$-space $[B_4]$.

Let $(Y, \pi_Y) \in \text{ANR}_{B_0}$ and $A \subset X$ be a closed subspace of $(X, \pi_X) \in \text{ob}(M_{B_0})$. Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a map over $B_0$, and let $H : (A \times I, \pi_{X \times I|A \times I}) \to (Y, \pi_Y)$ be a homotopy over $B_0$ of map $f|_A$. Then there exists an extension of $\tilde{H}$ to a homotopy over $B_0$ of $f$ itself $[Y_2]$.

Apart from this result in $[Y_2]$ the following proposition is shown:

Let $f, g : (X, \pi_X) \to (Y, \pi_Y) \in \text{ANR}_{B_0}$ be maps over $B_0$ from metric space over $B_0$ and let $H : (A \times I, \pi_{A \times I}) \to (Y, \pi_Y)$ be a homotopy over $B_0$ between restrictions on a closed subspace $A$ of $X$ of maps $f$ and $g$. Then there exist an open neighborhood $U$ of $A$ in $X$ and a homotopy $\tilde{H} : (U \times I, \pi_{X \times I|U \times I}) \to (Y, \pi_Y)$ over $B_0$ between restrictions $f|_U$ and $g|_U$ such that $\tilde{H}|_{A \times I} = H$.

A map $i : (A, \pi_A) \to (X, \pi_X)$ over $B_0$ is called a cofibration over $B_0$ if for each commutative diagram

![Diagram](image)

where all maps are maps over $B_0$ and $\omega_0 \cdot F = \tilde{f} \cdot i$, there exists a map $\tilde{F} : (X : \pi_X) \to (Y_{B_0}, \pi_{Y_{B_0}})$ over $B_0$ such that $F = \tilde{F} \cdot i$ and $\omega_0 \cdot \tilde{F} = \tilde{f}$.

The map $p : (E, \pi_E) \to (B, \pi_B)$ over $B_0$ is called a fibration over $B_0$ if for each commutative diagram
where all maps are maps over $B_0$ and $p \cdot \tilde{f} = F \cdot \sigma_0$, there exists a map $\tilde{F} : (X \times I, \pi_{X \times I}) \to (E, \pi_E)$ over $B_0$ such that $F = p \cdot \tilde{F}$ and $\tilde{F} \cdot \sigma_0 = \tilde{f}$.

The cylinder $\text{Cyl}_{B_0} (f)$ of a map $f : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ is the pair consisting of cylinder $\text{Cyl}(f)$ of map $f : X \to Y$ and projection $\pi_{\text{Cyl}(f)} : \text{Cyl}(f) \to B$ given by formulas

$$\pi_{\text{Cyl}(f)} ([x, t]) = \pi_X (x), [x, t] \in \text{Cyl}(f),$$

$$\pi_{\text{Cyl}(f)} (y) = \pi_Y (y), y \in Y \subset \text{Cyl}(f).$$

There exists a commutative diagram
where $\sigma_1$, $(\sigma_1)_f$ and $f^\#$ are maps given by formulas

$$
\sigma_1(x) = (x, 1), \quad x \in X,
$$
$$
(\sigma_1)_f(y) = [y], \quad y \in Y,
$$
$$
f^\#((x, t)) = [x, t], \quad (x, t) \in X \times I.
$$

Let $j : (\text{Cyl}(f), \pi_{\text{Cyl}(f)}) \to (Y \times I, \pi_{Y \times I})$ be a map over $B_0$ defined by formulas

$$
j[(x, t)] = (f(x), t), \quad (x, t) \in X \times I,
$$
$$
j(y) = (y, 0), \quad y \in Y.
$$

It is easy to see that the map $i : (X, \pi_X) \to (\text{Cyl}(f), \pi_{\text{Cyl}(f)})$ over $B_0$ is a cofibration over $B_0$ and the retraction map $r : (\text{Cyl}(f), \pi_{\text{Cyl}(f)}) \to (Y, \pi_Y)$ over $B_0$ given by formulas

$$
r([x, t]) = [x, 1], \quad [x, t] \in \text{Cyl}(f),
$$
$$
r(y) = y, \quad y \in Y \subset \text{Cyl}(f).
$$

is a fiber homotopy equivalence.

A map $f : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ is a cofibration over $B_0$ if and only if the map $j$ over $B_0$ is a retractionable map, i.e. there exists a retraction $r : (X \times I, \pi_{X \times I}) \to (\text{Cyl}(f), \pi_{\text{Cyl}(f)})$ over $B_0$.

The cocylinder over $B_0$ of map $f : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$, denoted by $\text{coCyl}_{B_0}(f)$, is the subspace of cocylinder $\text{coCyl}(f)$ consisting of pairs $(u, x)$, where $u \in Y_{B_0}^f$, $x \in X$ and $u(1) = f(x)$, i.e. $\pi_Y \cdot u = \text{const}$. The subspace $\text{coCyl}_{B_0}(f)$ is a space over $B_0$ with the projection $\pi_{\text{coCyl}_{B_0}(f)} : \text{coCyl}_{B_0}(f) \to B_0$ given by formula

$$
\pi_{\text{coCyl}_{B_0}(f)}(u, x) = \pi_Y(f(x)), (u, x) \in \text{coCyl}_{B_0}(f).
$$
There exists a commutative diagram

\[
\begin{array}{ccc}
\text{coCyl}_{B_0}(f) & \xrightarrow{\omega_1^\#} & X \\
\downarrow f_{\omega_1} & & \downarrow f \\
Y \rightarrow_{B_0} Y' & \xrightarrow{\omega_1} & Y',
\end{array}
\]

where \(\omega_1^\#\), \(f_{\omega_1}\) and \(\omega_1\) are maps given by formulas

\[
\begin{align*}
\omega_1^\#(u, x) &= x, & (u, x) \in \text{coCyl}_{B_0}(f), \\
f_{\omega_1}(u, x) &= u, & (u, x) \in \text{coCyl}_{B_0}(f), \\
\omega_1(u) &= u(1), & u \in Y'_{B_0}.
\end{align*}
\]

Let \(p : \text{coCyl}_{B_0}(f) \to Y\) be a map defined as follows:

\[p(u, x) = u(0), (u, x) \in \text{coCyl}_{B_0}(f).\]

It is clear that \(p\) is a map over \(B_0\). Observe that \(p = \omega_0 \cdot f_{\omega_1}\). Note that the map \(p : (\text{coCyl}_{B_0}(f), \pi_{\text{coCyl}_{B_0}(f)}) \to (Y, \pi_Y)\) is a fibration over \(B_0\).

Let \(0_y : I \to Y\) be the constant path in point \(y \in Y\). The pair \((0_{f(x)}, x)\) belongs to \(\text{coCyl}_{B_0}(f)\) because \(0_{f(x)}(1) = f(x)\).

Let \(i : X \to \text{coCyl}_{B_0}(f)\) be a map defined by formula

\[i(x) = (0_{f(x)}, x), x \in X.\]
Now we define a map $r : \text{coCyl}_{B_0}(f) \to X$ by formula

$$r(u, x) = x, \ (u, x) \in \text{coCyl}_{B_0}(f).$$

Then $r \cdot i = 1_X$ and $i \cdot r \simeq 1_X$. Hence, $X$ is embeddable in $\text{coCyl}_{B_0}(f)$ and it is strong deformation retract over $B_0$ of $\text{coCyl}_{B_0}(f)$. Thus, $i$ is a homotopy equivalence over $B_0$ and there exists a factorization

$$
\begin{array}{ccc}
(X, \pi_X) & \overset{f}{\longrightarrow} & (Y, \pi_Y) \\
\downarrow{i} & & \downarrow{p} \\
\text{coCyl}_{B_0}(f), & &
\end{array}
$$

i.e. $f = p \cdot i$. Indeed,

$$f(x) = 0_{f(x)}(0) = p(0_{f(x)}, x) = (p \cdot i)(x).$$

The map $r : (\text{coCyl}_{B_0}(f), \pi_{\text{coCyl}_{B_0}(f)}) \to (X, \pi_X)$ over $B_0$ is a shrinkable fibration over $B_0$ with respect to $i : (X, \pi_X) \to (\text{coCyl}_{B_0}(f), \pi_{\text{coCyl}_{B_0}(f)})$, if $r \cdot i = 1_X$ and $i \cdot r \simeq 1_{\text{coCyl}_{B_0}(f)} \text{rel} i(X)$.

It is easy to see that if in the pull-back diagram of maps over $B_0$

$$
\begin{array}{ccc}
E' & \overset{f'}{\longrightarrow} & E \\
\downarrow{\pi_{E'}} & & \downarrow{\pi_E} \\
B' & \overset{\pi_B}{\longrightarrow} & B \\
\downarrow{i'} & & \downarrow{i} \\
B_0 & \overset{q'}{\longrightarrow} & B_0 \\
\downarrow{q} & & \downarrow{q} \\
\text{E}' & \overset{f}{\longrightarrow} & \text{E}
\end{array}
$$
$q$ is a shrinkable fibration over $B_0$ with respect to $i : (B, \pi_B) \to (E, \pi_E)$, then $q'$ is also a shrinkable fibration over $B_0$ with respect to a uniquely defined embedding $i' : (B', \pi_{B'}) \to (E', \pi_{E'})$ over $B_0$ such that $f' \cdot i' = i \cdot f$.

1.1 On fiber Borsuk pairs

A pair $(X, A)$ consisting of a space $(X, \pi_X)$ over $B_0$ and subspace $A \subset X$ is a pair of Borsuk over $B_0$ or fiber Borsuk pair, if the inclusion $i : (A, \pi_{X|A}) \to (X, \pi_X)$ over $B_0$ is a cofibration over $B_0$. Note that a closed pair $(X, A)$ is a pair of Borsuk over $B_0$, if $X \times 0 \cup A \times I$ is a retract over $B_0$ of $X \times I$.

First we prove some propositions about cofibrations over $B_0$ and Borsuk’s pairs over $B_0$.

**Theorem 1.1.1.** A map $i : (A, \pi_A) \to (X, \pi_X)$ over $B_0$ is a cofibration over $B_0$ if and only if the map $j : (\text{Cyl}(i), \pi_{\text{Cyl}(i)}) \to (X \times I, \pi_{X \times I})$ over $B_0$ is fiberwise retractible.

**Proof.** Let $F : (A \times I, \pi_{A \times I}) \to (Y, \pi_Y)$ be a homotopy over $B_0$ and let $f_0 = F \cdot \sigma_0$. Consider an extension map $\tilde{f} : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ of $f_0$. The maps $F$ and $\tilde{f}$ over $B_0$ induce a map $g : \text{Cyl}_{B_0}(i) \to Y$ over $B_0$ such that

$$g([a, t]) = F(a, t), \quad [(a, t)] \in \text{Cyl}_{B_0}(i),$$
$$g(x) = \tilde{f}(x), \quad x \in \text{Cyl}_{B_0}(f).$$

The pair $(F, \tilde{f})$ is cone over $(\sigma_0, i)$. It is clear that $g$ is a morphism of the cone $(i_#, (\sigma_0)_i)$ into the cone $(F, \tilde{f})$. Consequently, if there is a retraction $r : (X \times I, \pi_{X \times I}) \to (\text{Cyl}_{B_0}(i), \pi_{\text{Cyl}_{B_0}(i)})$ over $B_0$, then the composition $\tilde{F} = g \cdot r : (X \times I, \pi_{X \times I}) \to (Y, \pi_Y)$
is a homotopy over $B_0$ of map $f$ over $B_0$ because $\tilde{F} \cdot j = g$ and, hence,

$$\tilde{F}(x,0) = (\tilde{F} \cdot j)(x) = g(x) = \bar{f}(x), \quad x \in X.$$ 

Note that

$$\tilde{F}((i(a),t)) = (\tilde{F} \cdot j)([(a,t)]) = g[(a,t)] = F((a,t))$$

for each pair $(a,t) \in (A \times I, \pi_{A \times I})$.

Thus, if the map $j$ over $B_0$ is retractible, then the map $i$ over $B_0$ is a cofibration.

Now conversely assume that the map $i : (A, \pi_A) \to (X, \pi_X)$ over $B_0$ is a cofibration over $B_0$. Then there exists a retraction $r : (X \times I, \pi_{X \times I}) \to (\text{Cyl}_{B_0}(i), \pi_{\text{Cyl}_{B_0}(i)})$ over $B_0$ such that

$$r((x,0)) = x, \quad (x,0) \in X \times I,$$

$$r(i(a),t) = [(a,t)], \quad a \in A, t \in I.$$ 

Thus, $r$ is a retraction of $j$. 

\[\square\]

**Corollary 1.1.2.** A closed pair $(X, A)$ of space $X$ over $B_0$ and its closed subspace $A$ is a Borsuk pair if and only if the subspace $(X \times \{0\}) \cup (A \times I) \subset X \times I$ is a retract over $B_0$ of $X \times I$. 

\[\square\]

**Corollary 1.1.3.** For each closed Borsuk’s pair $(X, A)$ over $B_0$ and for every space $Y$ over $B_0$ the pair $(X \times Y, A \times Y)$ over $B_0$ is a closed Borsuk’s pair over $B_0$. 

\[\square\]

**Corollary 1.1.4.** If $(X, A)$ is a Borsuk’s pair over $B_0$ and $A$ is a closed subspace of locally compact Hausdorff space $X$ then for each space $Y$ over $B_0$ the map $i^* : Y^X \to Y^A$ is a cofibration over $B_0$. 

\[\square\]

**Theorem 1.1.5.** A pair $(X, A)$ of space $(X, \pi_X)$ over $B_0$ and its closed subspace $(A, \pi_X|_A)$ is a Borsuk pair over $B_0$ if and only if there exist a map $\psi : X \to I$ and
1.1. On fiber Borsuk pairs

A fiber homotopy \( G : (X \times I, \pi_{X \times I}) \to (X, \pi_X) \) with respect \( A \) such that \( A = \psi^{-1}(0) \), \( G(x, 0) = x \) and \( G(x, t) \in A \) when \( \psi(x) < t \).

**Proof.** Let \((X, A)\) be a Borsuk pair over \(B_0\). By Corollary 1.1.2 there exists a retraction \( r : X \times I \to \tilde{A} = (X \times \{0\}) \cup (A \times I) \) over \( B_0 \). Let \( r((x, t)) = (\tilde{r}(x, t), \rho(x, t)) \), where \( \tilde{r}(x, t) \in X \) is first coordinate of \( r(x, t) \) and \( \rho(x, t) \in I \) is the projection in \( I \) of point \( r(x, t) \).

Let \( \psi : X \to I \) be a function given by

\[
\psi(x) = \max\{t - \rho(x, t) \mid x \in X\}.
\]

Note that \( A = \psi^{-1}(0) \). Besides, if \( \psi(x) < t \), then \( \rho(x, t) > 0 \) and consequently, \( \tilde{r}(x, t) \in A \).

Let \( G = \tilde{r} : (X \times I, \pi_{X \times I}) \to (X, \pi_X) \). It is clear that \( G(x, 0) = \tilde{r}(x, 0) = x \), \( G(x, t) \in A \) for \( \psi(x) < t \) and

\[
\pi_X(x) = \pi_{X \times I}(x, t) = \pi_A(\tilde{r}(x, t), \rho(x, t)) = \pi_X(\tilde{r}(x, t)),
\]

i.e. \( \tilde{r} \) is a map over \( B_0 \).

Now assume that hold the conditions of theorem. Then the map \( r : (X \times I, \pi_{X \times I}) \to (X, \pi_X) \) given by

\[
\begin{align*}
r(x, t) = \begin{cases} (G(x, t), 0), & t \leq \psi(x) \\ (G(x, t), t - \psi(x)), & t \geq \psi(x). \end{cases}
\end{align*}
\]

is a retraction over \( B_0 \). Consequently, \((X, A)\) is a Borsuk pair over \( B_0 \). \( \square \)

**Theorem 1.1.6.** Let \((X, A)\) be a Borsuk pair over \(B_0\). Then \((X \times I, (X \times \{0\}) \cup (A \times I) \cup X \times \{1\})\) is the Borsuk pair over \( B_0 \).
Proof. For simplicity by $X_A$ denote the set $(X \times \{0\}) \cup (A \times I) \cup (X \times \{1\})$. By Theorem 1.1.5 there exist a function $\varphi : X \to I$ and a fiber homotopy $g_t : U \to X_{rel A}$ from $U = X \setminus \varphi^{-1}(0)$ to $X$ such that $\varphi^{-1}(0) = A$, $g_0(x) = x$ and $g_t(x) \in A$ for each $x \in U$.

The function $\psi : X \times I \to I$ defined by

$$\psi(x, t) = 2 \min(2\varphi(x), \tau, 1 - \tau), \quad (x, \tau) \in X \times I$$

has property $\psi^{-1}(0) = X_A$.

Let $V = X \times I \setminus \psi^{-1}(1)$ be a set consisting of points $(x, \tau) \in X \times I$ for which $\tau \neq \frac{1}{2}$ or $\psi(x) = \frac{1}{4}$.

The maps $h_t : V \to V \times I$ given by formulas

$$h_t(x, \tau) = \begin{cases} (x, \tau(1 - t)), & 2\tau \leq \varphi(x); \\ (g(x, \frac{2t}{\varphi(x)} - 1)t, \tau(1 - t)), & \varphi(x) \leq 2\tau \leq \min(2\varphi(x), 1); \\ (g(x, t), (\tau - 2\varphi(x))t + \tau), & \varphi(x) \leq \tau \leq \min(2\varphi(x), \frac{1}{2}); \\ (g(x, t), \tau), & 2\varphi(x) \leq \tau \leq 1 - 2\varphi(x); \\ (g(x, t), \tau + (2\varphi(x) + \tau - 1)t), & \max(1 - 2\varphi(x), \frac{1}{2}) \leq \tau \leq 1 - \varphi(x); \\ (g(x, \frac{2(1-\tau)}{\varphi(x)} - 1)t, \tau + t - \tau t), & \max(2(1 - \varphi(x)), 1) \leq 2\tau \leq 2 - \varphi(x); \\ (x, \tau + t - \tau t), & 2 - \varphi(x) \leq 2\tau \\ \end{cases}$$

have properties

$$h_0(x, \tau) = (x, \tau), h_1(x, \tau) \in X_A, (x, \tau) \in X \times I$$

and

$$h_t(x, \tau) = (x, \tau), (x, \tau) \in X_A.$$

Thus, the pair $(X \times I, (X \times \{0\}) \cup (A \times I) \cup (X \times \{1\}))$ is the Borsuk pair over
Theorem 1.1.7. Let $(X, A)$ be a Borsuk pair over $B_0$. Then each deformation retraction $r : (X, \pi_X) \to (A, \pi_{X|A})$ over $B_0$ is a strong deformation retraction over $B_0$.

Proof. Let $F : (X \times I, \pi_{X \times I}) \to (X, \pi_X)$ be a fiber homotopy between $i \cdot r : (X, \pi_X) \to (X, \pi_X)$ and $1_X : (X, \pi_X) \to (X, \pi_X)$. By Theorem 1.1.6 the pair $(X \times I, X_A)$ is the Borsuk pair over $B_0$.

Hence, there exists a fiber homotopy given by

$$F_r(x, t) = \begin{cases} F((i \cdot r)(x), \tau), & t = 0, \\ F(x, t + (1 - t)\tau), & x \in A, t \in I, \\ x, & t = 1. \end{cases}$$

Note that $F_0(x, t) = F(x, t)$ and $F_1 : X \times I \to X\text{rel}A$ is a fiber homotopy between $1_X$ and $i \cdot r$.

Theorem 1.1.8. A closed pair $(X, A)$ of spaces over $B_0$ is a Borsuk pair over $B_0$ if and only if $\tilde{A} = (X \times \{0\}) \cup (A \times I)$ is a strong deformation retract over $B_0$ of $(X \times I, \pi_{X \times I})$.

Proof. Let $(\tilde{A}, \pi_{\tilde{A}})$ be a strong deformation retract over $B_0$ of $(X \times I, \pi_{X \times I})$. By Corollary 1.1.2 the pair $(X, A)$ is a Borsuk pair over $B_0$.

Consequently, as the product $(X \times I, \pi_{X \times I})$ is deformable in $X \times \{0\}$ and hence, in $\tilde{A}$, by Corollary 1.1.2 there exists a retraction $r : (X \times I, \pi_{X \times I}) \to (\tilde{A}, \pi_{\tilde{A}})$ over $B_0$. This retraction is deformation retraction over $B_0$. By Theorem 1.1.7 $r$ is a strong deformation retraction over $B_0$. Let $r(x, t) = (\tilde{r}(x, t), \rho(x, t))$, where $x \in X$, $t \in I$ and $\tilde{r}(x, t) \in X$, $\rho(x, t) \in I$. 

$\Box$
The deformation \( g_\tau : X \times I \to X \times I \) defined by formula

\[
g_\tau(x, t) = (\bar{r}(x, (1 - \tau)t), (1 - \tau)\rho(x, t) + \tau t), \ x \in X, t \in I
\]

is deformation over \( B_0 \) and it satisfies the following conditions:

\[
g_0 = i \cdot r, \\
g_1 = 1_X, \\
g_\tau(x, t) = (x, t), \quad (x, t) \in \tilde{A}.
\]

\[\square\]

**Corollary 1.1.9.** Let \((X, A)\) be a closed Borsuk pair over \( B_0 \). Then the subspace \((A, \pi_A)\) is a strong deformation retraction over \( B_0 \) of \((X, \pi_X)\) if and only if the inclusion \(i : (A, \pi_A) \to (X, \pi_X)\) is a fiber homotopy equivalence. \[\square\]

### 1.2 On Fiber SSDR-maps and Fibrant Spaces

In this section we give the definition and discuss various concepts which are associated to SSDR-maps over \( B_0 \). The following provides a shape version of SDR-map over \( B_0 \).

All spaces in Section 1.2 are metrizable.

Here the basic definition is the following

**Definition 1.2.1.** Let \((X, \pi_X) \in ob(M_{B_0})\) and let \( A \) be a closed subspace of \( X \). The subspace \((A, \pi_{X|A})\) over \( B_0 \) is called a shape strong deformation retract over \( B_0 \) of \((X, \pi_X)\) if there exists an embedding \( \alpha : (X, \pi_X) \hookrightarrow (Y, \pi_Y) \in AR_{B_0} \) over \( B_0 \) satisfying the following condition:

for any pair of neighbourhoods \( U \) and \( V \) of \( \alpha(X) \) and \( \alpha(A) \) respectively in \( (Y, \pi_Y) \),
there is a homotopy $H : (X \times I, \pi_{X \times I}) \to (U, \pi_{Y | U}) \text{rel } B_0$ over $B_0$ such that $H(x, 0) = \alpha(x)$ and $H(x, 1) \in V$ for each $x \in X$.

It is clear that if an embedding $\alpha : (X, \pi_X) \to (M, \pi_M)$ over $B_0$ satisfies the conditions of definition 1.2.1, then these conditions hold for any closed embedding $\beta : (X, \pi_X) \to (Z, \pi_Z) \in \text{AR}_{B_0}$.

A closed embedding $i : (A, \pi_A) \to (X, \pi_X)$ over $B_0$ is called SSDR$_{B_0}$-map if $i$ embeds $(A, \pi_A)$ in $(X, \pi_X)$ as a shape strong deformation retract over $B_0$ of $(X, \pi_X)$.

Note that the notion of SSDR$_{B_0}$-map generalizes the notion of SDR$_{B_0}$-map.

We get the following theorem which is a fiber version of Theorem 1.2 of (C$1$, C$2$).

**Theorem 1.2.2.** Let $(X, \pi_X) \in M_{B_0}$ and $A$ be a closed subspace of $X$. Then the following conditions are equivalent:

a) $i : (A, \pi_{X|A}) \to (X, \pi_X)$ is an SSDR-map over $B_0$;

b) for any map $f : (A, \pi_{X|A}) \to (Y, \pi_Y) \in \text{ANR}_{B_0}$ over $B_0$, there is an extension $\tilde{f} : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ such that $\tilde{f} \cdot i = f$ and any two such extensions over $B_0$ are fiber homotopic with respect $i A$;

c) for any commutative diagram

```
\begin{array}{ccc}
A & \xrightarrow{f} & E \\
\downarrow{i} & \downarrow{iX|A} & \downarrow{\pi_E} \\
X & \xrightarrow{\pi_X} & B \\
\end{array}
```

where $p : (E, \pi_E) \to (B, \pi_B)$ is a fibration over $B_0$ and $(E, \pi_E)$ and $(B, \pi_B)$ are ANR$_{B_0}$-spaces, there exists a map $\tilde{F} : (X, \pi_X) \to (E, \pi_E)$ over $B_0$ such that $\tilde{F} \cdot i = f$ and
p: \tilde{F} = F.

d) for any commutative diagram of maps over $B_0$

\[
\begin{array}{ccc}
A & \xrightarrow{f} & P^K_{B_0} \\
\downarrow{i} & & \downarrow{j^*} \\
X & \xrightarrow{\pi_X} & B_0 \\
\downarrow{H} & & \downarrow{\pi_{P^L}} \\
F & \xrightarrow{\pi_{P^L}} & P^L_{B_0} \\
\end{array}
\]

there exists a filler $H: (X, \pi_X) \to (P^K, \pi_{P^K})$ over $B_0$ provided $P \in \text{ANR}_{B_0}$ and $L$ is a subcomplex of a finite CW-complex $K$ with an inclusion map $j: L \hookrightarrow K$.

**Proof.** We check up the following implications a)$\Rightarrow$b)$\Rightarrow$c)$\Rightarrow$d)$\Rightarrow$a).

a)$\Rightarrow$b). As in the proof of Proposition 2 of [B4] we can show that $(X, \pi_X)$ is a closed subspace of AE$_{B_0}$-space for metric spaces $(M, \pi_M)$ over $B_0$ with weight $w(M) \leq \max(w(X), w(B_0), \aleph_0)$. Here $M = B \times K$, where $K$ is a convex hull of $X$ in a normed vector space $L$. Since $(Y, \pi_Y)$ is ANE$_{B_0}$-space there exist an open neighbourhood $V$ of $A$ in $M$ and extension $\hat{f}: (V, \pi_{M|A}) \to (Y, \pi_Y)$ over $B_0$ of map $f: (A, \pi_{X|A}) \to (Y, \pi_Y)$. By condition a) there exists a homotopy $H: (X \times I, \pi_{X \times I}) \to (M, \pi_M)$ over $B_0$ such that

\[
\begin{align*}
H(x, 0) &= x, & x & \in X, \\
H(x, 1) &\in V, & x & \in X, \\
H(a, t) &= a, & a & \in A, t \in I.
\end{align*}
\]
Let \( \tilde{f} : (X, \pi_X) \to (Y, \pi_Y) \) be a map given by the following formula

\[
\tilde{f} = \hat{f}(H(x, 1)), \quad x \in X.
\]

Note that

\[
(\pi_Y \cdot \tilde{f})(x) = \pi_Y(\hat{f}(x)) = \pi_Y(H(x, 1)) = (\pi_Y \cdot \hat{f})(H(x, 1)) = \\
= \pi_{M|A}(H(x, 1)) = \pi_M(H(x, 1)) = \pi_{X \times I}(x, 1) = \pi_X(x).
\]

Thus, \( \pi_Y \cdot \tilde{f} = \pi_X \) and hence, \( \tilde{f} \) is a map over \( B_0 \). Now show that any two such type extensions are fiber homotopic with respect \( A \). Let \( \tilde{f}_1, \tilde{f}_2 : (X, \pi_X) \to (Y, \pi_Y) \) be extensions over \( B_0 \) of map \( f \). Consider a subspace \( N = X \times \{0\} \cup A \times I \cup X \times \{1\} \) of space \( (M \times I, \pi_{M \times I}) \). Define a map \( F : N \to Y \) over \( B_0 \) by

\[
F(x, 0) = \tilde{f}_1(x), \quad x \in X, \\
F(x, 1) = \tilde{f}_2(x), \quad x \in X, \\
F(x, a) = f(a), \quad a \in A, t \in I.
\]

It is clear that (see Proposition 1.1 of [Y2]) \( M \times I = (B \times K) \times I \approx B \times (K \times I) \in \text{AE}_{B_0} \), because \( K \times I \in \text{AE}(M) \). There exists an extension \( \bar{F} : (W, \pi_{M \times I|W}) \to (Y, \pi_Y) \) over \( B_0 \) of map \( F : (N, \pi_{M \times I|N}) \to (Y, \pi_Y) \) over \( B_0 \) on some open neighbourhood \( W \) of \( N \) in \( M \times I \).

Let \( U \) be an open neighbourhood of \( X \) in \( M \) such that \( U \times \{0\} \subset W \) and \( U \times \{1\} \subset W \). Besides, consider an open neighbourhood \( V \) of \( A \) in \( M \) such that \( V \times I \subset W \). By condition a) it follows the existence of homotopy \( D : (X \times I, \pi_{X \times I}) \to (U, \pi_{M|U}) \) over
B₀ with properties

\[ D(x,0) = x, \quad x \in X, \]
\[ D(x,1) \in V, \quad x \in X, \]
\[ D(a,t) = a, \quad a \in A, t \in I. \]

Let \( F'(x,t) = \bar{F}(D(x,t),0) \), \( F''(x,t) = \bar{F}(D(x,t),1) \) and \( H(x,t) = \bar{F}(D(x,1),t) \).

Note that \( F', F'' \) and \( H \) induce the fiber homotopies:

\[ F' : \tilde{f}_1 \simeq_{B_0} h_1 \text{rel} A, \]
\[ F'' : \tilde{f}_2 \simeq_{B_0} h_2 \text{rel} A, \]
\[ H : h_1 \simeq_{B_0} h_2 \text{rel} A. \]

Therefore, \( \tilde{f}_1 \simeq_{B_0} \tilde{f}_2 \text{rel} A. \)

b) \( \Rightarrow \) c). By condition b) for a space \((E, \pi_E) \in \text{ANE}_{B_0}\) over \(B_0\) there is a map \( \bar{F} : (X, \pi_X) \to (E, \pi_E) \) over \(B_0\) such that \( \bar{F} \cdot i = f \). Note that \( F \cdot i = p \cdot f = p \cdot \bar{F} \cdot i \).

From condition b) also follows the existence of homotopy \( H : F \simeq_{B_0} p \cdot \bar{F} \text{rel} i(A) \) over \(B_0\). Thus there is a fiber homotopy \( \bar{H} : (X \times I, \pi_X \times I) \to (E, \pi_E) \) such that \( p \cdot \bar{H} = H \).

The fiber homotopy \( \bar{H} \) induces a map \( \bar{F} : (X, \pi_X) \to (E, \pi_E) \) over \(B_0\) with properties \( \bar{F} \cdot i = f \) and \( p \cdot \bar{F} = F \).

c) \( \Rightarrow \) d). By proposition 9 of [B₄] the space \( P^K_{B_0} \) and \( P^L_{B_0} \) are ANR\(_{B_0}\) -spaces. Also note that \( j^* : P^K_{B_0} \to P^L_{B_0} \) is a fibration over \(B_0\). Hence, there exists a filler \( H : (X, \pi_X) \to (P^K_{B_0}, \pi_{P^K_{B_0}}) \) over \(B_0\).

d) \( \Rightarrow \) a). Let \((X, \pi_X)\) be a closed subspace over \(B_0\) of \(\text{AR}_{B_0}\)-space \((M, \pi_M)\). Let \( i : (A, \pi_{X \mid A}) \to (X, \pi_X) \) be the inclusion over \(B_0\) of closed set \(A\) of \(X\) given by \( i(a) = a \) for each \(a \in A\).
Consider open neighbourhoods $U$ and $V$ of $X$ and of $A$, respectively, in $M$ such that $V \subseteq U$. Note that $(U, \pi_{M|U}), (V, \pi_{M|V}) \in \text{ANR}_{B_0}$. Let $P = V$, $K = \{\ast\}$, $L = \emptyset$ and let $f : (A, \pi_{X|A}) \rightarrow (V, \pi_{M|V})$ be the inclusion map over $B_0$. By condition d) there exists a map $r : (X, \pi_X) \rightarrow (V, \pi_{M|V})$ over $B_0$ such that $r \cdot i = f$. Now assume that $P = U$, $K = I$ and $L = \{0, 1\}$. Consider a commutative diagram

\[ \begin{array}{ccc}
A & \xrightarrow{f} & U_B^I \\
\downarrow{i} & & \downarrow{i} \\
X & \xrightarrow{H} & U \times_{B_0} U,
\end{array} \]

where

\[
\pi(\omega) = (\omega(0), \omega(1)), \quad \omega \in U_B^I, \\
f(a)(t) = a, a \in A, \quad t \in I, \\
F(x) = (x, r(x)), \quad x \in X.
\]

It is clear that $\pi_{X|A} = \pi_{U_{B_0}} \cdot f$, $\pi_{U_{B_0}} = \pi_{U \times_{B_0} U} \cdot \pi$ and $\pi_X = \pi_{U \times_{B_0} U} \cdot F$. Also note that $U \times_{B_0} U$ and $U_{B_0}$ are ANR$_{B_0}$-spaces.

By condition d) there exists a map $H : (X, \pi_X) \rightarrow (U_B^I, \pi_{U_{B_0}})$ over $B_0$ such that $H \cdot i = f$ and $\pi \cdot H = F$.

Let $D : (X \times I, \pi_{X \times I}) \rightarrow (U, \pi_{M|U})$ be a map over $B_0$ given by formula

\[ D(x, t) = H(x, t), \quad (x, t) \in X \times I. \]
The map $D$ satisfies the conditions of the definition of SSDR-map over $B_0$. \hfill \square

Now we need to introduce definition and investigation of fibrant spaces over $B_0$.

**Definition 1.2.3.** A space $(Y, \pi_Y)$ over $B_0$ is called a fibrant space over $B_0$ if for every SSDR-map $i : (A, \pi_{X|A}) \to (X, \pi_X)$ over $B_0$ and every map $f : (A, \pi_{X|A}) \to (Y, \pi_Y)$ over $B_0$, there is a map $F : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ such that $F \cdot i = f$, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow{\pi_{X|A}} & & \downarrow{\pi_X} \\
B_0 & \xrightarrow{f} & Y \\
\uparrow{\pi_Y} & & \uparrow{\pi_Y} \\
Y & \xrightarrow{F} & Y
\end{array}
$$

We have the following proposition.

**Theorem 1.2.4.** If $Y$ is an ANR$_{B_0}$-space, then $Y$ is a fibrant space over $B_0$.

**Proof.** Let $i : (A, \pi_{X|A}) \to (X, \pi_X)$ be a SSDR$_{B_0}$-map and $(Y, \pi_Y) \in$ ANR$_{B_0}$. By implication a$\Rightarrow$b) for each map $f : (A, \pi_{X|A}) \to (Y, \pi_Y)$ over $B_0$ there exists an extension $\tilde{f} : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ with property $\tilde{f} \cdot i = f$. Thus $Y$ is a fibrant space over $B_0$. \hfill \square

**Theorem 1.2.5.** If $(Y, \pi_Y)$ is a fibrant space over $B_0$ and $Z$ is a compact space, then $(Y_{B_0}, \pi_{Y|B_0})$ also is a fibrant space over $B_0$.

**Proof.** Let $(X, \pi_X)$ be a metric space over $B_0$, $A$ a closed subset of $X$, $i : (A, \pi_{X|A}) \to (X, \pi_X)$ a SSDR$_{B_0}$-map and $f : (A, \pi_{X|A}) \to (Y_{B_0}, \pi_{Y|B_0})$ a map over $B_0$. The map $F : (A \times Z, \pi_{A \times Z}) \to (Y, \pi_Y)$ given by formula

$$
F(a, z) = (f(a))(z), (a, z) \in (A \times Z, \pi_{A \times Z})
$$
is a map over $B_0$. Indeed, for every $(a, z) \in A \times Z$ we have

\[
(\pi_Y \cdot F) = \pi_Y(F(a, z)) = \pi_Y(f(a)(z)) = \pi_{Y|B_0}(f(a)) = \pi_{X|A}(a) = \pi_{A \times Z}(a, z).
\]

Observe that if $i : (A, \pi_{X|A}) \to (X, \pi_X)$ is a SSDR-map over $B_0$, then $i \times 1_Z : (A \times Z, \pi_{A \times Z}) \to (X \times Z, \pi_{X \times Z})$ is a SSDR-map over $B_0$. Indeed, we can assume the pair $(X \times Z, \pi_{X \times Z})$, where $\pi_{X \times Z}((x, z)) = \pi_X(x)$, is embeddable in some $\text{AR}_{B_0}$-space $(M \times N, \pi_{M \times N})$ such that $(X, \pi_X)$ and $Z$ are embeddable in $(M, \pi_M) \in \text{AR}_{B_0}$ and $N \in \text{AR}$, respectively. Let $W$ and $Q$ be open neighbourhoods of $X \times Z$ and $A \times Z$ in an $\text{AR}_{B_0}$-space $M \times N$, respectively. There exist open neighbourhoods $U$ and $V$ of $X$ and $A$ respectively in $M$ such that $U \times Z \subset W$ and $V \times Z \subset Q$. Since $i$ is a SSDR-map over $B_0$ there exists a homotopy $H : (X, \pi_X) \times I \to (U, \pi_{M|U})$ over $B_0$ with properties $H(x, 0) = i(x)$ and $H(x, 1) \in V$.

Let $\tilde{H} : (X \times Z \times I, \pi_{X \times Z \times I}) \to (U \times Z, \pi_{U \times Z})$ be a map given by formula

\[
\tilde{H}(x, z, t) = (H(x, t), z), (x, z) \in X \times Z, t \in I.
\]

Note that $\tilde{H}$ is a map over $B_0$ satisfying the following conditions

\[
\tilde{H}(x, z, 0) = (H(x, 0), z) = (i(x), z) = (i \times 1_Z)(x, z)
\]

and

\[
\tilde{H}(x, z, 1) = (H(x, 1), z) \in V \times Z \subset Q.
\]

Since $(Y, \pi_Y)$ is an fibrant space over $B_0$ there is a map $\bar{F} : (X \times Z, \pi_{X \times Z}) \to (Y, \pi_Y)$.
over $B_0$ such that $\tilde{F} \cdot (i \times 1_Z) = \tilde{f}$, where $\tilde{f} : (A \times Z, \pi_{A \times Z}) \to (Y, \pi_Y)$ be a map over $B_0$ given by formula

$$\tilde{f}(a, z) = (f(a))(z), (a, z) \in A \times Z.$$ 

Let $\tilde{F} : (X, \pi_X) \to (Y_{B_0}, \pi_{Y_{B_0}})$ be a map given by

$$(\tilde{F}(x))(z) = \tilde{F}(x, z), x \in X, z \in Z.$$ 

It is clear that $\tilde{F} \cdot i = f$. 

**Theorem 1.2.6.** Let $Y = ((Y_n, \pi_{Y_n}), p_{n,n+1}, N^+)$ be an inverse system of fibrant spaces over $B_0$ and fibrations over $B_0$. Then the fiber limit space $Y = \lim_{\leftarrow} Y$ is a fibrant space over $B_0$ and the natural projections $p_n : (Y, \pi_Y) \to (Y_n, \pi_{Y_n})$ are fibrations over $B_0$. 

**Proof.** Let $(y_n) \in Y = \lim_{\leftarrow} Y$. It is clear that for each $n < n + 1$

$$\pi_n(y_n) = (\pi_n \cdot p_{n,n+1})(y_{n+1}) = \pi_{n+1}(y_{n+1}).$$

Assume that

$$\pi_Y((y_n)) = \pi_n(y_n), (y_n) \in Y.$$ 

Note that $\pi_{Y_n} \cdot p_n = \pi_Y$. Consequently, $(Y, \pi_Y)$ is a space over $B_0$ and $p_n : Y \to Y_n$ is a map over $B_0$. 

Let $f_n = p_n \cdot f, n \in N$. It is clear that there exists a map $F_1 : X \to Y_1$ over $B_0$ such that $F_1 \cdot i = p_1 \cdot f$. For the commutative diagram
there is a map $F_2 : (X, \pi_X) \to (Y_2, \pi_{Y_2})$ over $B_0$ with properties $F_2 \cdot i = f_2$ and $p_{1,2} \cdot F_2 = F_1$. Inductively we can construct the sequence $\{F_n\}_{n \in N^+}$ of maps $F_n : X \to Y_n$ over $B_0$ for which $p_{n,n+1} \cdot F_{n+1} = F_n$ and $F_n \cdot i = f_n$.

Let $F = \Delta \bigtriangleup_{n \in N^+} F_n : X \to \prod_{n \in N^+} Y_n$ be the diagonal product over $B_0$ of maps $F_n : (X, \pi_X) \to (Y_n, \pi_{Y_n}), n \in N^+$. The map $F$ induces a map over $B_0$ which we again denote by $F : (X, \pi_X) \to (Y, \pi_Y)$. It is clear that $F \cdot i = f$.

Now show that $p_1 : (Y, \pi_Y) \to (Y_1, \pi_{Y_1})$ is fibration over $B_0$. Consider the diagram

There exists a map $H_1^1 : (X \times I, \pi_{X \times I}) \to (Y_2, \pi_{Y_2})$ over $B_0$ such that $H_1^1 \cdot \sigma_0 = p_1 \cdot f = p_{1,2} \cdot (p_2 \cdot f)$. Hence, we can choose a map $H_2^1 : (X \times I, \pi_{X \times I}) \to (Y_2, \pi_{Y_2})$ over $B_0$ for which $H_2^1 \cdot \sigma_0 = p_2 \cdot f$ and $p_{1,2} \cdot H_2^1 = H_1^1$. Thus, inductively we can construct a
sequence $H^1_n, H^2_n, \cdots, H^n_n, \cdots$ of maps $H^1_n : (X \times I, \pi_{X \times I}) \to (Y_n, \pi_{Y_n})$ over $B_0$ such that

$$H^1_n = p_{n,n+1} \cdot H^1_{n+1}, n \in \mathbb{N}^+. \text{ Let } H_1 : (X \times I, \pi_{X \times I}) \to (Y, \pi_Y) \text{ be a map given by}$$

$$H_1 = \Delta_n H^1_n : (X \times I, \pi_{X \times I}) \to (Y, \pi_Y).$$

Finally, we observe that $H^1 \cdot \sigma_0 = f$ and $p_1 \cdot H_1 = H^1_1$.

Analogously, we can prove that $p_2, p_3, \cdots$ maps over $B_0$ are fibration over $B_0$. \(\square\)

**Theorem 1.2.7.** Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a map over $B_0$. If $(X, \pi_X), (Y, \pi_Y) \in \text{ANR}_{B_0}$, then $\text{coCyl}_{B_0}(f) \in \text{ANR}_{B_0}$.

**Proof.** Let $(Z, \pi_Z) \in \text{ob}(M_{B_0})$ and $A$ be a closed subspace of $Z$ and let $g : (A, \pi_{Z|A}) \to \text{coCyl}_{B_0}(f)$ be a map over $B_0$. For the composition $g_2 = \omega^\#_1 \cdot g : (A, \pi_{Z|A}) \to (X, \pi_X)$ there exist a neighbourhood $U$ of $A$ in $Z$ and an extension $\tilde{g}_2 : (U, \pi_{Z|U}) \to (X, \pi_X)$ over $B_0$ of map $\omega^\#_1 \cdot g$ over $B_0$. Note that

$$f \cdot \tilde{g}_2(a) = f(\tilde{g}_2(a)) = f(g_2(a)) = f \cdot \omega^\#_1 \cdot g(a) = \omega_1 \cdot f \cdot g(a) = (f \cdot g(a))(1).$$

The composition $f \cdot g : (A, \pi_{Z|A}) \to (Y^I_{B_0}, \pi_{Y^I_{B_0}})$ induces the map $H : A \times I \to Y$ over $B_0$ given by

$$H(a,t) = ((f \cdot g)(a))(t), (a,t) \in A \times I.$$  

It is clear that for each $a \in A$ and $t \in I$

$$H(a,1) = ((f \cdot g)(a))(1) = (\omega_1 \cdot f \cdot g)(a) = (f \cdot \omega^\#_1 \cdot g)(a) = f \cdot ((\omega^\#_1 \cdot g)(a)) = f(\tilde{g}_2(a)) = (f \cdot \tilde{g}_2)(a) = (f \cdot \tilde{g}_2|_A)(a).$$
Let $G : ((U \times \{0\}) \cup A \times I, \pi_{U \times \{0\} \cup A \times I}) \to (Y, \pi_Y)$ be a map defined by formula

\[ G(u, 1) = f \tilde{g}_2(u), \quad u \in U, \]
\[ G(a, t) = H(a, t), \quad (a, t) \in A \times I. \]

There exists an extension $\tilde{G} : (U \times I, \pi_{U \times I}) \to (Y, \pi_Y)$ over $B_0$ such that

\[ \tilde{G}_{|U \times \{1\}} = f \cdot \tilde{g}_2 \]

and

\[ \tilde{G}_{|A \times I} = H. \]

The map $\tilde{G}$ induces a map $\tilde{g}_1 : (U, \pi_{Z|U}) \to (Y^I_{B_0}, \pi_{Y^I_{B_0}})$ for which

\[ (\tilde{g}_1(u))(t) = \tilde{G}(u, t), \quad u \in U, \quad t \in I. \]

Let $\tilde{g} = \tilde{g}_1 \Delta \tilde{g}_2 : (U, \pi_{Z|U}) \to (\text{coCyl}_{B_0}(f), \pi_{\text{coCyl}_{B_0}(f)})$.

Also note that for each pair $\tilde{g}(u) = (\tilde{g}_1(u), \tilde{g}_2(u))$ holds the condition

\[ \tilde{g}_1(u)(1) = \tilde{G}(u, 1) = f \tilde{g}_2(u), \quad u \in U, \]

i.e. $\tilde{g}(u) \in \text{coCyl}_{B_0}(f)$. Besides,

\[ \tilde{g}(a) = (\tilde{g}_1(a), \tilde{g}_2(a)) = (\tilde{g}_1(a), g_2(a)) = (\tilde{g}_1(a), \omega^\#_1 g(a)), \quad a \in A. \]

Note that $\tilde{g}_1(a)$ is a map $\tilde{g}_1(a) : I \to Y$ such that

\[ \tilde{g}_1(a)(t) = \tilde{G}(a, t) = H(a, t) = ((f_\omega \cdot g)(a))(t), \]
\[ \tilde{g}(a) = (\tilde{g}_1(a), \tilde{g}_2) = (f \omega_1 \cdot g(a), \omega_1 \cdot g(a)) = g(a). \]

**Theorem 1.2.8.** Let \( f : (X, \pi_X) \to (Y, \pi_Y) \) be a map over \( B_0 \) of fibrant spaces over \( B_0 \). Then the \( \text{coCyl}_{B_0}(f) \) over \( B_0 \) is a fibrant space over \( B_0 \).

**Proof.** Let \( g : (A, \pi_{Z|A}) \to (\text{coCyl}_{B_0}(f), \pi_{\text{coCyl}_{B_0}(f)}) \) be a map over \( B_0 \) from a closed subspace \( (A, \pi_{Z|A}) \in \text{ob}(M_{B_0}) \) to the \( (\text{coCyl}_{B_0}(f), \pi_{\text{coCyl}_{B_0}(f)}) \). There exists an extension \( \tilde{g}_2 : (Z, \pi_Z) \to (X, \pi_X) \) over \( B_0 \) of map \( g_2 = \omega_1 \cdot g : (A, \pi_{Z|A}) \to (X, \pi_X) \) over \( B_0 \). Note that from the equivalence a)$\Leftrightarrow$b) of Theorem 1.2.2 it follows that the inclusion \( (X \times \{0\} \cup A \times I, \pi_{X \times \{0\} \cup A \times I}) \to (X \times I, \pi_{X\times I}) \) over \( B_0 \) is an SSDR-map over \( B_0 \).

Let

\[ G : (Z \times \{0\} \cup A \times I, \pi_{Z \times \{0\} \cup A \times I}) \to (Y, \pi_Y) \]

be a map given by formulas

\[ G(z, 1) = f \tilde{g}_2(z), z \in X \]

and

\[ G(a, t) = H(a, t), (a, t) \in A \times I, \]

where \( H : (A \times I, \pi_{A \times I}) \to (Y, \pi_Y) \) is a map over \( B_0 \) given by \( H(a, t) = ((f \omega_1 \cdot g)(a))(t) \).

As in the proof of Proposition 1.2.7 we can check up that there exists map \( \tilde{g}_1 : \)
1.2. On Fiber SSDR-maps and Fibrant Spaces

\[(Z, \pi_Z) \to (Y^I, \pi_{Y^I})\] over \(B_0\) such that

\[\tilde{g}_1(z)(1) = f\tilde{g}_2(z), z \in Z.\]

Let \(\tilde{g} = \tilde{g}_1 \Delta \tilde{g}_2 : (Z, \pi_Z) \to (\text{coCyl}_{B_0}(f), \pi_{\text{coCyl}_{B_0}(f)}).\) It is clear that \(\tilde{g}|_A = g.\)

Thus, the pair \((\text{coCyl}_{B_0}(f), \pi_{\text{coCyl}_{B_0}(f)})\) is a fibrant space over \(B_0.\) \hfill \(\square\)
Chapter 2

Fiber Strong Shape Classifications of Compact Metrizable Spaces

In chapter 2 are defined and studied fiber cotelescopes and ANR$_{B_0}$-resolutions, proved the theorem of existence of ANR$_{B_0}$-resolution, constructed fiber strong shape category of compact metrizable spaces and established the characterizations of fiber strong shape equivalences based on the notion of the double mapping cylinder over $B_0$. The constructed fiber strong shape category is the full image of functor reflector from the fiber homotopy category of compact metrizable spaces over $B_0$ in the fiber homotopy category of fiber fibrant spaces.

2.1 On Fiber Strong Shape Category of Compact Metrizable Spaces

First we consider cotelescopes of inverse sequences over $B_0$. Let $X = \{(X_n, \pi_{X_n}), q_n^{n+1}, N^+\}$ be an inverse sequence over $B_0$. For each bonding map $q_n^{n+1} : (X_{n+1}, \pi_{X_{n+1}}) \to (X_n, \pi_{X_n})$ over $B_0$ consider the cocylinder $X_{n,n+1} = \text{coCyl}_{B_0}(q_n^{n+1})$ over $B_0$ of map

51
Let \( q_{n+1}^{n+1} : (X_{n+1}, \pi_{X_{n+1}}) \to (X_n, \pi_{X_n}) \) be a fibration over \( B_0 \) and the shrinkable fibration \( r_{n+1} : X_{n+1,n} \to X_{n+1} \) over \( B_0 \) with respect to the SDR \( i_{n+1} : X_{n+1} \to X_{n,n+1} \).

The cotelescope over \( B_0 \) of the inverse sequence \( X \), denoted by \( \text{coTel}_{B_0}(X) \), is defined as the inverse limit of the diagram \( T(X) \)

\[
\begin{align*}
&\xymatrix{ & (X_{0,1}, \pi_{X_{0,1}}) & (X_{1,2}, \pi_{X_{1,2}}) & (X_{n-1,n}, \pi_{X_{n-1,n}}) \\
& p_0 & B_0 & p_1 & B_0 & r_1 & B_0 & p_2 & B_0 & \cdots & p_{n-1} & B_0 & r_n & \cdots \\
q_0 & (X_0, \pi_{X_0}) & q_1 & (X_1, \pi_{X_1}) & q_2 & (X_2, \pi_{X_2}) & \cdots & q_{n-1} & (X_{n-1,1}, \pi_{X_{n-1,1}}) & q_n & (X_n, \pi_{X_n}) \\
& & & & & & & & & & & & &
}\end{align*}
\]

By definition of cotelescope over \( B_0 \), \( \text{coTel}_{B_0}(X) = \varprojlim T(X) \) is a space over \( B_0 \) of points \( (x_0, \omega_0, x_1, \omega_1, x_2, \omega_2, \cdots) \in \prod_{i=0}^{\infty} (X_i \times_{B_0} X_i) \) for which

\[
\omega_0(0) = x_0, \omega_0(1) = q_0^1(x_1), \omega_1(1) = q_1^2(x_2), \cdots.
\]

Let \( T_n(X) \) be a finite subdiagram consisting of first \( n \) numbers of diagram \( T(X) \) and \( (X_{0,n}, \pi_{X_{0,n}}) = \varprojlim T_n(X) \). Now consider the following diagram
2.1. On Fiber Strong Shape Category of Compact Metrizable Spaces

Note that in this diagram $p_1, p_2, \ldots, p_n$ are fibrations over $B_0$. Hence, the maps $p_{n,m}$ also are fibrations over $B_0$ and $r_{n,m}$ maps are shrinkable fibrations with respect to maps $i_{n,m}$ since each $r_n$ is a shrinkable fibration with respect to $i_n$. Changing $r_n$ by $i_n$, $r_{n,m}$ by $i_{n,m}$ and putting $(\tilde{X}_0, \pi_{\tilde{X}_0}) = (X_0, \pi_{X_0})$, $\tilde{q}^1_0 = p_0, \tilde{i}_0 = 1_{X_0}, \tilde{i}_1 = i_1$, $(\tilde{X}_n, \pi_{\tilde{X}_n}) = (X_0, \pi_{X_0,n}), \tilde{q}^n_{n+1} = p_{n+1}, \tilde{i}_n = i_{n,n+1}, \ldots, i_{n-1,n} \cdot i_n$ for $n > 1$ we obtain the following inverse system $\tilde{X} = ((\tilde{X}_n, \pi_{\tilde{X}_n}), \tilde{q}^{n+1}_n, N^+)$ and commutative diagram

Note that $\text{coTel}_{B_0}(X) = \varprojlim X, \tilde{q}^{n+1}_n : (\tilde{X}_{n+1}, \pi_{\tilde{X}_{n+1}}) \to (\tilde{X}, \pi_{\tilde{X}})$ is a fibration over $B_0$ and $\tilde{i}_n : (X_n, \pi_{X_n}) \to (\tilde{X}_n, \pi_{\tilde{X}_n})$ is SDR$_{B_0}$-map over $B_0$ for each $n \geq 0$. Also note that if all $(X_n, \pi_{X_n})$ are ANR$_{B_0}$-spaces (fibrant spaces over $B_0$), then all $(\tilde{X}_n, \pi_{\tilde{X}_n})$ are ANR$_{B_0}$-spaces (fibrant spaces over $B_0$). In particular, we have obtained the following theorem.
Theorem 2.1.1. Let $X = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ be an inverse sequence consisting of fibrant spaces over $B_0$ and maps over $B_0$. Then the cotelescope $\text{coTel}_{B_0}(X)$ is a fibrant space over $B_0$. If all $(X_n, \pi_{X_n})$ members of the inverse system $X$ are ANRB$_{B_0}$-spaces, then $\text{coTel}_{B_0}(X)$ is a fibrant space over $B_0$ too.

Let $X = \lim \leftarrow X$ and $q = \{q_n\}_{n \in N^+}$, where $q_n : X \to X_n$ are the natural projections over $B_0$. Then SSDR-maps $\tilde{i}_n$ over $B_0$ from the above given diagram induce the unique natural embedding $i_q : (X, \pi_X) \to (\text{coTel}_{B_0}(X), \pi_{\text{coTel}_{B_0}(X)})$ over $B_0$ such that $\tilde{q}_n \cdot i_q = i_q \cdot q_n$ for each $n \geq 0$.

Definition 2.1.2. An inverse sequence $X = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ is called resolution over $B_0$ of compact space $(X, \pi_X)$ over $B_0$ if

a)$(X, \pi_X) = \lim \leftarrow X$;

b)the family $q = \{q_n : (X, \pi_X) \to (X_n, \pi_{X_n})\}_{n \in N^+}$ satisfies the following condition: for each $n \in N^+$ and open neighbourhood $U$ of $q_n(X)$ in $(X, \pi_{X_n})$ there exists $m \geq n$ such that $q_m^n(X_m) \subseteq U$.

If all the $(X_n, \pi_{X_n}) \in \text{ANR}_{B_0}$, then $q$ is called an ANRB$_{B_0}$-resolution over $B_0$.

Note that this definition of resolution over $B_0$ is a special case of the definition of resolution over $B_0$ given in [B4].

Now prove the theorem of existence of resolution over $B_0$ of compact metrizable spaces over $B_0$.

Theorem 2.1.3. For each compact metrizable space $(X, \pi_X)$ over $B_0$ there exists an ANRB$_{B_0}$-resolution $q : (X, \pi_X) \to X$ over $B_0$.

Proof. We can assume that $(X, \pi_X)$ is a closed subspace of some ARB$_{B_0}$-space $(M, \pi_M)$. Indeed, there exists a closed embedding $j = i \triangle \pi_X : (X, \pi_X) \to (M, \pi_M) = (N \times B_0, \pi_{N \times B_0})$, where $i : X \to N$ is an closed inclusion of $X$ into AR-space $N$. Let
${X_n}$ be the union $\bigcup_{x\in X} B(x, \frac{1}{n})$, where $B(x, \frac{1}{n})$ is the open ball in $M$ with center $x$ and radius $\varepsilon = \frac{1}{n}$. For any neighbourhood $U$ of $X$ in $M$ and $x \in X$ there exists $\varepsilon_x$ such that $B(x, \varepsilon_x) \subset U$. There exists a finite set $\{x_1, x_2, ..., x_k\} \subset X$ such that $X \subseteq \bigcup_{i=1}^{k} B(x_i, \varepsilon_{x_i})$.

Let $\varepsilon = \frac{1}{n} \leq min\{\varepsilon_{x_1}, \varepsilon_{x_2}, ..., \varepsilon_{x_k}\}$. It is clear that $X_n = \bigcup_{x\in X} B(x, \frac{1}{n})$ has the property $X_n \subseteq X$. Note that obtained family of neighbourhoods of $X$ in $M$ form an inverse sequence $X = (X_n, q_n^{n+1}, N^+)$ of ANR$_B$-spaces, where $q_n^{n+1}$ is the inclusion of $X_{n+1}$ into $X_n$. Since $X = \bigcap_{n=1}^{\infty} X_n$, we can conclude $(X, \pi_X) = \varprojlim X$.

Therefore, the family $\mathbf{q} = \{q_n\}_{n \in N^+}$ of inclusions $q_n : (X, \pi_X) \to (X_n, \pi_{X_n})$ over $B_0$ form a resolution $\mathbf{q} : (X, \pi_X) \to X$ over $B_0$ of space $(X, \pi_X)$ over $B_0$.

**Theorem 2.1.4.** Let $(X, \pi_X)$ be a compact metrizable space over $B_0$. If $\mathbf{q} : (X, \pi_X) \to X = ((X_n, \pi_{X_n}), q_n^{n+1}, N^+)$ is a resolution over $B_0$ of $(X, \pi_X)$, then there exists an infinite strong deformation

$$D : \text{coTel}_{B_0}(X) \times [0, \infty) \to \text{coTel}_{B_0}(X)$$

of $\text{coTel}_{B_0}(X)$ over $B_0$ onto $i_\mathbf{q}(X)$. In particular, the map $i_\mathbf{q} : (X, \pi_X) \to \text{coTel}_{B_0}(X)$ is an SSDR-map over $B_0$.

**Proof.** Let $\hat{X} = \text{coTel}_{B_0}(X)$. The projections $\hat{q}_i : (\hat{X}, \pi_{\hat{X}}) \to (\hat{X}_i, \pi_{\hat{X}_i})$ over $B_0$ are fibrations over $B_0$ and they have fiber homotopy lifting property.

Hence, there are deformations $\hat{D}_n : (\hat{X} \times I, \pi_{\hat{X} \times I}) \to (\hat{X}_n, \pi_{\hat{X}_n})$ over $B_0$ of $\hat{X}$ onto $F_n = \hat{q}_n^{-1}i_n(X_n)$. The family $\{F_n\}$ is a decreasing family of closed subsets of $\hat{X}$, i.e. for each $n \geq 0$

$$\hat{X} = F_0 \supset F_1 \supset ... \supset F_n \supset F_{n+1} \supset i_\mathbf{q}(X).$$

Since $\mathbf{q}$ is a resolution over $B_0$, then for each neighborhood $\hat{U}$ of $i_\mathbf{q}(X)$ in $(\hat{X}, \pi_{\hat{X}})$ there exists an index $m$ such that $F_m \subset \hat{U}$. There are an index $n$ and neighborhood $\hat{V}$ of $q_n(i_\mathbf{q}(X))$ in $(\hat{X}_n, \pi_{\hat{X}_n})$ such that $\hat{q}_n^{-1}(\hat{V}) \subset \hat{U}$. Let $V = \hat{q}_n^{-1}(\hat{U})$ and $q_n(X) \subset V \subset X_n$. 

2.1. On Fiber Strong Shape Category of Compact Metrizable Spaces

There is an index $m \geq n$ for which $\tilde{q}_m^n(X_m) \subset V$ and $\tilde{q}_m^n(i_m(X_m)) \subset \tilde{V}$. Note that

$$F_m = \tilde{q}_m^{-1}(i_m(X_m)) \subseteq \tilde{q}_n^{-1}(\tilde{q}_m^n(i_m(X_m))) \subseteq \tilde{q}_n^{-1}(\tilde{V}) \subset \tilde{U}.$$ 

The strong deformations $D_i$ over $B_0$ induce the required infinite deformation $D : \text{coTel}_{B_0}(X) \times [0, +\infty) \to \text{coTel}_{B_0}(X)$ over $B_0$.

The next theorem follows directly from Theorems 2.1.1, 2.1.3 and 2.1.4.

**Theorem 2.1.5.** For each compact metrizable space $(X, \pi_X)$ over $B_0$ there is a fibrant extension $i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})$ over $B_0$. In particular, if $q : (X, \pi_X) \to X = ((X_n, \pi_{X_n}), q_{n+1}^{n+1}, N^+) \text{ is an ANR}_{B_0}$-resolution over $B_0$, then the embedding $i_q : (X, \pi_X) \to (\text{coTel}_{B_0}(X), \pi_{\text{coTel}_{B_0}(X)}) \text{ is a fibrant extension over } B_0$.

The purpose of this section is to construct of fiber strong shape theory for compact metrizable spaces over a fixed base space $B_0$, using the fiber versions of cotelescop and fibrant space.

The constructed fiber strong shape category is the full image of functor reflector from the fiber homotopy category of compact metrizable spaces over $B_0$ in the fiber homotopy category of fibrant spaces.

The obtained classification of spaces over $B_0$ demonstrates the advantage of fiber strong shape theory over fiber shape theory. Now define the fiber strong shape category $\text{SSH}_{B_0}$ for compact metrizable spaces over $B_0$ in a quite usual way as the full image of some functor-reflector. Here we consider the reflector of the fiber homotopy category $H(\text{CM}_{B_0})$ of compact metrizable spaces over $B_0$ in the fiber homotopy category of fibrant spaces $H(\text{F}_{B_0})$.

Let $(X, \pi_X) \in \text{ob}(\text{CM}_{B_0})$ and $i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})$ be a fibrant extension over $B_0$. For each map $f : (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$, where $(Y, \pi_Y)$ is a fibrant space over $B_0$, there exists a map $\tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (Y, \pi_Y)$ over $B_0$ such that the following diagram
2.1. On Fiber Strong Shape Category of Compact Metrizable Spaces

\[ \xymatrix{ X \ar[rr]^{i_X} \ar[dr]_{\pi_X} & & \tilde{X} \ar[dl]^{\pi_{\tilde{X}}} \\ & B_0 & } \]

commutes, i.e. \( f = \tilde{f} \cdot i_X \). From Theorem 1.1.2 follows that if \( f \sim f' : (\tilde{X}, \pi_{\tilde{X}}) \to (Y, \pi_Y) \) and \( \tilde{f} \cdot i_X = f' \), then \( \tilde{f} \sim f' \). Hence, the map

\[ [i_X]_{B_0}^\#: [\tilde{X}, Y]_{B_0} \to [X, Y]_{B_0} \]

given by formula

\[ [i_X]_{B_0}^\#([\tilde{f}]_{B_0}) = [\tilde{f} \cdot i_X]_{B_0} \]

is bijective. Thus, we have the following.

**Theorem 2.1.6.** Let \( i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}}) \) be a fibrant extension over \( B_0 \) of space \( (X, \pi_X) \in \text{CM}_{B_0} \). Then the morphism \( [i_X]_{B_0} : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}}) \) of category \( H(\text{CM}_{B_0}) \) is an \( H(F_{B_0}) \)-reflection. \( \square \)

It is clear that the family \( \{i_X : (X, \pi_X) \to (\tilde{X}, \pi_{\tilde{X}})\}_{(X, \pi_X) \in \text{ob}(H(\text{CM}_{B_0}))} \) induces the \( H(F_{B_0}) \)-reflector

\[ R : H(\text{CM}_{B_0}) \to H(F_{B_0}) \]

that is a functor given by formula

\[ R((X, \pi_X)) = (\tilde{X}, \pi_{\tilde{X}}), (X, \pi_X) \in \text{ob}(H(\text{CM}_{B_0})) \]

and satisfying the condition:

for each map \( f : (X, \pi_X) \to (Y, \pi_Y) \) over \( B_0 \) of compact metrizable spaces the
is commutative. Indeed, for the map there exists a unique up to fiber homotopy map \( \tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Y}, \pi_{\tilde{Y}}) \) such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{[i_X]_{B_0}} & \tilde{X} \\
\downarrow{\pi_X} & & \downarrow{\pi_{\tilde{X}}} \\
B_0 & \xrightarrow{[f]_{B_0}} & \tilde{X} \\
\downarrow{\pi_Y} & & \downarrow{\pi_{\tilde{Y}}} \\
Y & \xrightarrow{[i_Y]_{B_0}} & \tilde{Y} \\
\end{array}
\]

In this case the pair \((i_X, i_Y) : f \to \tilde{f}\) is called a fibrant extension over \(B_0\) of map \(f\).

**Definition 2.1.7.** The fiber strong shape category \(\text{SSH}_{B_0}\) of compact metrizable spaces over \(B_0\) is full image of the reflector \(R : H(CM_{B_0}) \to H(F_{B_0})\).

Note that

\[
\text{ob(SSH}_{B_0}} = \text{ob(H(CM}_{B_0}))
\]

and

\[
\text{Mor}_{\text{SSH}_{B_0}}((X, \pi_X), (Y, \pi_Y)) = [(\tilde{X}, \pi_{\tilde{X}}), (\tilde{Y}, \pi_{\tilde{Y}})]_{B_0}, (X, \pi_X), (Y, \pi_Y) \in \text{ob(HCM}_{B_0})).
\]
Besides,

$$SS_{B_0}((X, \pi_X)) = (X, \pi_X)$$

for each $((X, \pi_X) \in ob(HCM_{B_0}))$ and

$$SS_{B_0}([f]_{B_0}) = R([f]_{B_0}) = [\tilde{f}]_{B_0}$$

for a fibrant extension $(i_X, i_Y): f \to \tilde{f}: (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Y}, \pi_{\tilde{Y}})$ over $B_0$ of map $f: (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$.

There is a commutative diagram

\[ \begin{array}{ccc} H(CM_{B_0}) & \xrightarrow{R} & H(F_{B_0}) \\ \downarrow{SS_{B_0}} & & \downarrow{J_R} \\ SSH_{B_0}. & & \end{array} \]

### 2.2 On Fiber Strong Shape Equivalences of Compact Metrizable Spaces

The double mapping cylinder $d\text{Cyl}_{B_0}(f)$ over $B_0$ of map $f: (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ is the subspace $X \times I \cup \text{Cyl}_{B_0}(f) \times \{0, 1\}$ of space $\text{Cyl}_{B_0}(f) \times I$ over $B_0$.

By J. Dydak and S. Nowak in ([Dy-N1], [Dy-N2]) were defined a strong shape equivalence. We give the definition of fiber version of strong shape equivalence.

**Definition 2.2.1.** A map $f: (X, \pi_X) \to (Y, \pi_Y)$ over $B_0$ is a shape equivalence if for each ANR$_{B_0}$-space $(P, \pi_P)$ induces a bijection $f^*: [Y, P]_{B_0} \to [X, P]_{B_0}$. A fiber shape equivalence $f$ is called a fiber strong shape equivalence if for any two maps $g, h: (Y, \pi_Y) \to (P, \pi_P) \in \text{ANR}_{B_0}$ over $B_0$ and a fiber homotopy $H: (X \times I, \pi_{X \times I}) \to (P, \pi_P)$ over $B_0$ joining $gf$ and $hg$, $H$ is fiber homotopic rel $X \times \{0, 1\}$ to $H^* (f \times 1_I)$, where $H^*: (Y \times I, \pi_{Y \times I}) \to (P, \pi_P)$ is a fiber homotopy between $g$ and $h$. 
2.2. On Fiber Strong Shape Equivalences of Compact Metrizable Spaces

Theorem 2.2.2. Let \( f : (X, \pi_X) \rightarrow (Y, \pi_Y) \) be a fiber shape equivalence and let \( g : (\partial I^n \times Y, \pi_{\partial I^n \times Y}) \rightarrow (P, \pi_P) \) be a map over \( B_0 \) such that the composition \( g(1_{I^n} \times f) : (I^n \times X, \pi_{I^n \times X}) \rightarrow (P, \pi_P) \) has an extension onto \( (I^n \times X, \pi_{I^n \times X}) \). Then \( g \) has an extension onto \( (I^n \times X, \pi_{I^n \times X}) \).

Proof. The map \( g : (\partial I^n \times Y, \pi_{\partial I^n \times Y}) \rightarrow (P, \pi_P) \) induce the map over \( B_0 \) from \( (Y, \pi_Y) \) into \( (P, \pi_{\partial I^n \times Y}) \) which we also denoted by \( g : (Y, \pi_Y) \rightarrow (P, \pi_{\partial I^n \times Y}) \).

Let \( h : (X, \pi_X) \rightarrow (P, \pi_{\partial I^n \times Y}) \) be a fiber extension of \( gf \). By condition of theorem \( f \) is a fiber shape equivalence. Hence, there exists a map \( h' : (Y, \pi_Y) \rightarrow (P, \pi_{\partial I^n \times Y}) \) over \( B_0 \) such that \( h' f \cong h \). By \( h' \) again denote map \( h' : (Y \times I, \pi_{Y \times I}) \rightarrow (P, \pi_P) \) over \( B_0 \) induced by \( h' \). From the relation \( h' f \cong h \) and the equality \( h = gf \) it follows that \( h' f \cong g \cdot f \). Hence, \( h' f \cong g \cdot f \). Since the pair \( (I^n \times Y, \partial I^n \times Y) \) has the fiber homotopy extension property \( g \) extends onto \( I^n \times Y \).

Theorem 2.2.3. Let \( f : (X, \pi_X) \rightarrow (Y, \pi_Y) \) be a map over \( B_0 \). The following conditions are equivalent:

1). \( f \) is a fiber strong shape equivalence;

2). for a given space \( (Z, \pi_Z) \) over \( B_0 \) containing \( (X, \pi_X) \) as a closed subspace over \( B_0 \), every map \( g : (Z, \pi_Z) \rightarrow (P, \pi_P) \in \text{ANR}_{B_0} \) over \( B_0 \) extends to \( (Z \cup \text{Cyl}_{B_0}(f), \pi_{Z \cup \text{Cyl}_{B_0}(f)}) \) and every map

\[
H : (Z \times I \cup \text{dCyl}_{B_0}(f), \pi_{Z \times I \cup \text{dCyl}_{B_0}(f)}) \rightarrow (P, \pi_P) \in \text{ANR}_{B_0}
\]

over \( B_0 \) extends to \( ((Z \cup \text{Cyl}_{B_0}(f)) \times I, \pi_{((Z \cup \text{Cyl}_{B_0}(f)) \times I)}) \);

3). if \( (X, \pi_X) \) is a closed subspace of \( (Z, \pi_Z) \), then the fiber inclusions

\[
i : (Z, \pi_Z) \rightarrow (Z \cup \text{Cyl}_{B_0}(f), \pi_{Z \cup \text{Cyl}_{B_0}(f)})
\]
2.2. On Fiber Strong Shape Equivalences of Compact Metrizable Spaces

and

\[ j : (Z \times I \cup dCyl_{B_0}(f), \pi_{Z \times I \cup dCyl_{B_0}(f)}) \rightarrow ((Z \cup Cyl_{B_0}(f)) \times I, \pi_{(Z \cup Cyl_{B_0}(f)) \times I}) \]

are fiber shape equivalence;

4). if \((X, \pi_X)\) is a closed subspace of \((Z, \pi_Z)\), then the fiber inclusion

\[ i : (Z, \pi_Z) \rightarrow (Z \cup Cyl_{B_0}(f), \pi_{Z \cup Cyl_{B_0}(f)}) \]

is a fiber strong shape equivalences;

5). if \((X, \pi_X)\) is a closed subspace of \((Z, \pi_Z)\), then the fiber inclusion

\[ i : (Z, \pi_Z) \rightarrow (Z \cup Cyl_{B_0}(f), \pi_{Z \cup Cyl_{B_0}(f)}) \]

is a fiber shape equivalence;

6). the fiber inclusions

\[ k : (X, \pi_X) \rightarrow (Cyl_{B_0}(f), \pi_{Cyl_{B_0}(f)}) \]

and

\[ l : (dCyl_{B_0}(f), \pi_{dCyl_{B_0}(f)}) \rightarrow (Cyl_{B_0}(f) \times I, \pi_{Cyl_{B_0}(f) \times I}) \]

are fiber shape equivalences;

7). every map \(g : (X, \pi_X) \rightarrow (P, \pi_P) \in ANR_{B_0} over B_0\) extends to \((Cyl_{B_0}(f), \pi_{Cyl_{B_0}(f)})\)
and every map \(H : (dCyl_{B_0}(f), \pi_{dCyl_{B_0}(f)}) \rightarrow (P, \pi_P) \in ANR_{B_0} over B_0\) extends to
\((Cyl_{B_0}(f) \times I, \pi_{Cyl_{B_0}(f) \times I}).\)

Proof. 1)⇒2). Let \(g : (Z, \pi_Z) \rightarrow (P, \pi_P) \in ANR_{B_0} be a map over B_0. Consider the fiberpreserving restriction \(g|_X : (X, \pi_X) \rightarrow (P, \pi_P). This map has a fiber extension
2.2. On Fiber Strong Shape Equivalences of Compact Metrizable Spaces

g' : (Cyl_{B_0}(f), \pi_{Cyl_{B_0}(f)}) \to (P, \pi_P). The maps g' and g induce a map g'' : \(Z \cup Cyl_{B_0}(f), \pi_{Z \cup Cyl_{B_0}(f)}\) \to \(P, \pi_P\) over \(B_0\) which is fiber extension of g.

Let q : (dCyl_{B_0}(1_X), \pi_{dCyl_{B_0}(1_X)}) \to (dCyl_{B_0}(f), \pi_{dCyl_{B_0}(f)}) be the fiber natural projection and let f' : (dCyl_{B_0}(1_X), \pi_{dCyl_{B_0}(1_X)}) \to (dCyl_{B_0}(1_Y), \pi_{dCyl_{B_0}(1_Y)}) be a map over \(B_0\) induced by f. Note that

\[H q \simeq H' f' \text{ rel } X \times \{1\} \times \{0, 1\},\]

where \(H' : H_{[Y \times \{1\} \times \{0\}]_{B_0}} \simeq H_{[Y \times \{1\} \times \{1\}]_{B_0}}\) is a homotopy over \(B_0\). Consequently, the map \(H\) has a fiber extension onto \((Z \cup \text{Cyl}_{B_0}(f) \times I)\).

2) \Rightarrow 3). Note that \(i_* : [Z \cup \text{Cyl}_{B_0}(f), P]_{B_0} \to [Z, P]_{B_0}\) is the surjection for each \(P \in \text{ANR}_{B_0}\). Prove that \(i_*\) is an injective map.

Let \(g, h : (Z \cup \text{Cyl}_{B_0}(f), \pi_{Z \cup \text{Cyl}_{B_0}(f)}) \to (P, \pi_P)\) be maps over \(B_0\) with some fiber homotopy

\[H : g|_Z \simeq h|_Z.\]

There exists a map \(G : (Z \times I \cup d\text{Cyl}_{B_0}(f), \pi_{Z \times I \cup d\text{Cyl}_{B_0}(f)}) \to (P, \pi_P)\) over \(B_0\) such that

\[G_{(\text{Cyl}_{B_0}(f) \times \{0\})} = g,\]

and

\[G_{(\text{Cyl}_{B_0}(f) \times \{1\})} = h.\]

Let \(G' : (\text{Cyl}_{B_0}(f) \times I, \pi_{\text{Cyl}_{B_0}(f) \times I}) \to (P, \pi_P)\) be a fiber extension of \(G\). Then \(G' : g \simeq h\). Now show that \(j_* : [(Z \cup \text{Cyl}_{B_0}(f)) \times I, P]_{B_0} \to [Z \times I \cup d\text{Cyl}_{B_0}(f), P]_{B_0}\) is a bijection for each \((P, \pi_P) \in \text{ANR}_{B_0}\). Let \(G, H : ((Z \cup \text{Cyl}_{B_0}(f)) \times I, \pi_{(Z \cup \text{Cyl}_{B_0}(f)) \times I}) \to (P, \pi_P)\) be maps over \(B_0\) whose restrictions on subspace \(Z \times I \times \text{dCyl}_{B_0}(f)\) are fiber
homotopic. Notice that

\[ G|_{(Z \cup \text{Cyl}_{B_0}(f)) \times \{0\}} \simeq H|_{(Z \cup \text{Cyl}_{B_0}(f)) \times \{0\}}. \]

Since the inclusion \((Z \cup \text{Cyl}_{B_0}(f)) \times \{0\} \to (Z \cup \text{Cyl}_{B_0}(f)) \times I\) is the fiber inclusion the maps \(G\) and \(H\) over \(B_0\) are homotopic over \(B_0\).

3) \implies 4) Let \(H : (Z \times I, \pi_{Z \times I}) \to (P, \pi_P) \in \text{ANR}_{B_0}\) be a fiber homotopy between restrictions \(g|_Z\) and \(h|_Z\) of maps \(g, h : (Z \cup \text{Cyl}_{B_0}(f), \pi_{Z \cup \text{Cyl}_{B_0}(f)}) \to (P, \pi_P)\) over \(B_0\). There exists an extension map \(G : (Z \times I \cup \text{dCyl}_{B_0}(f), \pi_{Z \times I \cup \text{dCyl}_{B_0}(f)}) \to (P, \pi_P)\) over \(B_0\) of \(H\) such that \(G|_{\text{Cyl}_{B_0}(f) \times \{0\}} = g\) and \(G|_{\text{Cyl}_{B_0}(f) \times \{1\}} = h\). By condition iii) there exists a fiber homotopy extension \(G' : ((Z \cup \text{Cyl}_{B_0}(f)) \times I, \pi_{(Z \cup \text{Cyl}_{B_0}(f)) \times I}) \to (P, \pi_P)\) of \(G\). The pair \(((Z \cup \text{Cyl}_{B_0}(f)) \times I, Z \times I \cup \text{dCyl}_{B_0}(f))\) has the fiber homotopy extension property with respect to any space over \(B_0\) because

\[ (Z \times I \cup \text{dCyl}_{B_0}(f)) \times I \cup \text{Cyl}_{B_0}(f) \times I \times \{0\} \]

is a fiber retract of

\[ (Z \times I \cup \text{dCyl}_{B_0}(f) \cup Y \times I) \times I \cup \text{Cyl}_{B_0}(f) \times I \times \{0\} \]

and

\[ (Z \times I \cup \text{dCyl}_{B_0}(f) \cup Y \times I) \times I \cup \text{Cyl}_{B_0}(f) \times I \times \{0\} \]

is a fiber retract of \((Z \cup \text{Cyl}_{B_0}(f)) \times I \times I\). Consequently, \(G'\) is a fiber homotopy between \(g\) and \(h\) and the restriction of \(G'\) on \(Z \times I\) is equal to \(G\).

4) \implies 5). The verification of this implications is trivial.

5) \implies 6). Let \(Z = X \times I \cup \text{Cyl}_{B_0}(f) \times \{0\}\). By condition v) we infer that the fiber inclusion \(X \times I \cup \text{Cyl}_{B_0}(f) \times \{0\} \to \text{dCyl}_{B_0}(f)\) is a fiber shape equivalence. Consequently,
the fiber inclusion $d\text{Cyl}_{B_0}(f) \to \text{Cyl}_{B_0} \times I$ is shape equivalence. Besides, for $Z = X$ we get that $X \to \text{Cyl}_{B_0}(f)$ is a fiber shape equivalence over $B_0$.

6) $\implies$ 7). This implication is obvious because $(\text{Cyl}_{B_0}(f), X)$ and $(\text{Cyl}_{B_0}(f) \times I, d\text{Cyl}_{B_0}(f))$ have the fiber homotopy extension property with respect to any space over $B_0$.

7) $\implies$ 1). Let $H : (d\text{Cyl}_{B_0}(1_X), \pi_{d\text{Cyl}_{B_0}(1_X)}) \to (P, \pi_P) \in \text{ANR}_{B_0}$ be a fiber homotopy between $gf$ and $hf$, where $g, h : (Y, \pi_Y) \to (P, \pi_P) \in \text{ANR}_{B_0}$ are maps over $B_0$. There exists a map $G : (d\text{Cyl}_{B_0}(f), \pi_{d\text{Cyl}_{B_0}(f)}) \to (P, \pi_P)$ over $B_0$ such that $G_{Y \times \{0\}} = g$ and $G_{Y \times \{1\}} = h$. Let $G' : (\text{Cyl}_{B_0}(f) \times I, \pi_{\text{Cyl}_{B_0}(f) \times I}) \to (P, \pi_P)$ be an extension over $B_0$ of $G$. Using the fiber projection $\pi : X \times I \times I \to \text{Cyl}_{B_0}(f) \times I$ and strong fiber deformation retraction of $X \times I \times I$ onto $X \times \{1\} \times I$ we infer that $H$ is fiber homotopic rel $X \times \{1\} \times \{0, 1\}$ to $H' \times (f \times 1_I)$, where $H' : (d\text{Cyl}_{B_0}(1_Y), \pi_{d\text{Cyl}_{B_0}(1_Y)}) \to (P, \pi_P)$ is a fiber homotopy between $g$ and $h$. Hence, $f$ is a fiber strong shape equivalence. 

**Corollary 2.2.4.** Let $(X, \pi_X)$ be a space over $B_0$ and $A \subset X$. The fiber inclusion $i : (A, \pi_{X|A}) \to (X, \pi_X)$ is a fiber strong shape equivalence if and only if $i$ and $j : (X \times \{0\} \cup A \times I \cup X \times \{1\}, \pi_{X \times \{0\} \cup A \times I \cup X \times \{1\}}) \to (X \times I, \pi_{X \times I})$ are fiber shape equivalences.

**Proof.** Let $f = i$. This corollary is straight consequence of equivalence of conditions 1) and 6). 

**Corollary 2.2.5.** Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a fiber homotopy equivalence. Then $f$ is a fiber strong shape equivalence.

**Proof.** The space $(X, \pi_X)$ is a strong deformation retract over $B_0$ of $\text{Cyl}_{B_0}(f)$. Hence, $(Y \times \{0\}, \pi_{Y \times \{0\}})$ is a strong deformation retract of $d\text{Cyl}_{B_0}(f)$. Thus, the fiber inclusions of $(X, \pi_X)$ into $\text{Cyl}_{B_0}(f)$ and $(d\text{Cyl}_{B_0}(f), \pi_{d\text{Cyl}_{B_0}(f)})$ into $\text{Cyl}_{B_0}(f) \times I$ are fiber homotopy equivalences.
**Corollary 2.2.6.** If \( g : (X, \pi_X) \to (Y, \pi_Y) \) is fiber homotopic to a fiber strong shape equivalence \( f : (X, \pi_X) \to (Y, \pi_Y) \), then \( g \) is a fiber strong shape equivalence.

**Proof.** The cylinder \( \text{Cyl}_{B_0}(g) \) over \( B_0 \) is fiber homotopy equivalence to cylinder \( \text{Cyl}_{B_0}(f) \text{rel} X \).

Hence, for every space \( (M, \pi_M) \) over \( B_0 \) containing \( X \) as a closed set the spaces \( M \cup \text{Cyl}_{B_0}(f) \) and \( Z \cup \text{Cyl}_{B_0}(g) \) over \( B_0 \) are fiber homotopy equivalent with respect \( M \). By equivalence of conditions 1) and 5) of Theorem 3 \( g \) is a fiber strong shape equivalence. \( \square \)

Now prove the following

**Theorem 2.2.7.** Let \( f : (X, \pi_X) \to (Y, \pi_Y) \) and \( g : (Y, \pi_Y) \to (Z, \pi_Z) \) be fiber strong shape equivalences. Then \( g f : (X, \pi_X) \to (Z, \pi_Z) \) fiberpreserving map is a fiber strong shape equivalence.

**Proof.** It is clear that the composition \( g f \) is a fiber shape equivalence. Let \( \varphi, \psi : (Z, \pi_Z) \to (P, \pi_P) \in \text{ANR}_{B_0} \) be fiberpreserving maps and \( H : (X \times I, \pi_{X \times I}) \to (P, \pi_P) \) be a fiber homotopy \( H : \varphi gf \simeq \psi gf \). By condition of theorem there exists a fiberpreserving homotopy \( H' : (Y \times I, \pi_{Y \times I}) \to (P, \pi_P) \) between fiberpreserving maps \( \varphi g \) and \( \psi g \) such that

\[
H \simeq H' (f \times 1_I) \text{rel} X \times \{0, 1\}.
\]

Besides, there is a fiber homotopy \( H'' : (Z \times I, \pi_{Z \times I}) \to (P, \pi_P) \) between \( \varphi \) and \( \psi \) such that

\[
H' \simeq H'' (g \times 1_I) \text{rel} Y \times \{0, 1\}.
\]

Consequently, we have the following fiber homotopy

\[
H \simeq H'' (g f \times 1_I) \text{rel} X \times \{0.1\}.
\]
Theorem 2.2.8. Let \( f : (X, \pi_X) \to (Y, \pi_Y) \) and \( g : (Y, \pi_Y) \to (Z, \pi_Z) \) be maps over \( B_0 \) such that \( gf \) is a fiber strong shape equivalence. If one of \( f \) and \( g \) is a fiber strong equivalence, then both \( f \) and \( g \) are fiber strong shape equivalences.

Proof. By condition of theorem \( f \) and \( g \) are fiber shape equivalences. Let \( H \) be some map over \( B_0 \) from \((\text{dCyl}_{B_0}(f), \pi_{\text{dCyl}_{B_0}(f)})\) into \((P, \pi_P) \in \text{ANR}_{B_0}\). There is a fiber extension \( H' : (X \times I \cup \text{Cyl}_{B_0}(f) \times \{0, 1\} \cup \text{Cyl}_{B_0}(g) \times \{0, 1\}, \pi_{X \times I \cup \text{Cyl}_{B_0}(f) \times \{0, 1\}}) \to (P, \pi_P) \) of \( H \) because the fiber inclusion \((Y, \pi_Y) \to (\text{Cyl}_{B_0}(g), \pi_{\text{Cyl}_{B_0}(g)})\) is a fiber shape equivalence. By Corollary 2.5 of [F] \((\text{Cyl}_{B_0}(f) \cup \text{Cyl}_{B_0}(g), \pi_{\text{Cyl}_{B_0}(f) \cup \text{Cyl}_{B_0}(g)})\) is fiber homotopy equivalent of \((\text{Cyl}_{B_0}(gf), \pi_{\text{Cyl}_{B_0}(gf)})\). Besides, by condition \( gf \) is a fiber strong shape equivalence. Consequently, \( H' \) extends onto \((\text{Cyl}_{B_0}(f) \cup \text{Cyl}_{B_0}(g)) \times I, \) and hence, on \text{Cyl}_{B_0}(f) \times I. Thus, by equivalence 1)\(\Leftrightarrow\)7) \( f \) is a fiber strong shape equivalence.

Let \( H : (Y \times I, \pi_{Y \times I}) \to (P, \pi_P) \in \text{ANR}_{B_0} \) be a fiber homotopy between \( g \varphi \) and \( g \psi \), where \( \varphi, \psi : (Z, \pi_Z) \to (P, \pi_P) \). There is a fiber homotopy \( H'' : (Z \times I, \pi_{Z \times I}) \to (P, \pi_P) \) such that \( H'' : \varphi \simeq \psi, \) \( H''(gf \times 1_I) \simeq H(f \times 1_I)_{\text{rel}X \times \{0, 1\}} \). Let \( G : (Y \times \partial I^2, \pi_{Y \times \partial I^2}) \to (P, \pi_P) \) be a map over \( B_0 \) given by

\[
G(y, 0, t) = H(y, t), \quad y \in Y, t \in I,
\]
\[
G(y, 1, t) = H''(g(y), t), \quad y \in Y, t \in I,
\]
\[
G(y, t, 0) = \varphi g(y), \quad y \in Y, t \in I,
\]
\[
G(y, t, 1) = \psi g(y), \quad y \in Y, t \in I.
\]

Then \( G(f \times 1_I) : (X \times (\partial I^2), \pi_{X \times (\partial I^2)}) \to (P, \pi_P) \) extends onto \( X \times I^2 \). By Theorem
2.2. On Fiber Strong Shape Equivalences of Compact Metrizable Spaces

2.2.2 the map $G$ extends onto $Y \times I^2$. Hence, we have

$$H \cong H''(g \times 1_I)_{rel} Y \times \{0, 1\}.$$ 

\[\square\]

**Corollary 2.2.9.** Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a fiber shape equivalence. If $(X, \pi_X)$ has the fiber homotopy type of an ANR$_{B_0}$, then $f$ is a fiber strong shape equivalence.

**Proof.** Note that there is a map $g : (Y, \pi_Y) \to (X, \pi_X)$ over $B_0$ such that $gf \cong 1_X$. By Theorem 2.2.8, $gf$ is a fiber strong shape equivalence. Since $gf$ is fiber strong shape equivalence and $f$ is fiber shape equivalences, then $f$ and $g$ are fiber strong shape equivalences. \[\square\]

The next Theorem 2.2.10 and Theorem 2.2.11 show that in terms of fiber double cylinders it is possible to describe fiber strong shape isomorphisms of category $SSH_{B_0}$.

**Theorem 2.2.10.** A closed fiber embedding $i : (A, \pi_{X|A}) \to (X, \pi_X)$ is a fiber strong shape equivalence if and only if $i$ is a SSDR-map over $B_0$.

**Proof.** Let $i$ is a SSDR-map over $B_0$. First show that the function $i_* : [X, P]_{B_0} \to [A, P]_{B_0}, (P, \pi_P) \in$ ANR$_{B_0}$ is a bijection. From the equivalence a)$\Rightarrow$ b) of Theorem 2.2.2 follows that $i_*$ is a surjection because for each map $f : (A, \pi_{X|A}) \to (P, \pi_P)$ over $B_0$ there is a map $\tilde{f} : (X, \pi_X) \to (P, \pi_P)$ over $B_0$ such that $\tilde{f}i = f$ and $i_*(\tilde{f}) = f$. The map $i_*$ also is an injection. Indeed, let $g, h : (X, \pi_X) \to (P, \pi_P)$ be two maps over $B_0$ such that $i_*(h) = g = i_*(f)$, i.e. $hi \cong f \cong gi$. By fiber version of Borsuk’s homotopy extension theorem \[\square\] there exists maps $\tilde{f}_1, \tilde{f}_2 : (X, \pi_X) \to (P, \pi_P)$ over $B_0$ such that $\tilde{f}_1|_A = f = \tilde{f}_2|_A$, $\tilde{f}_1 \cong g$ and $\tilde{f}_2 \cong h$. By the implication a)$\Rightarrow$b) we have $\tilde{f}_1 \cong \tilde{f}_2_{rel} i(A)$. Hence, $g \cong h$. One easily sees that $[g]_{B_0} = [h]_{B_0}$. \[\square\]
Let now $H : (A \times I, \pi_{A \times I}) \to (P, \pi_P)$ be a fiber homotopy between $g i$ and $h i$. Since $(P^I, \pi_{P^I}) \in \text{ANR}_{B_0}$, the function $i_* : [X, P^I]_{B_0} \to [A, P^I]_{B_0}$ is a bijection. Consequently, the function $(i \times 1_I)_* : [X \times I, P]_{B_0} \to [A \times I, P]_{B_0}$ is a bijection too. Hence, there exists a map $F : (X \times I, \pi_{X \times I}) \to (P, \pi_P)$ over $B_0$ and a fiber homotopy $S : (i \times 1_I)_*(F) = F (i \times 1_I) \simeq H$. Let $G = S \cup F : X \times I \times \{0\} \cup (A \times I) \times I \to (P, \pi_P)$ be a map over $B_0$ given by formulas

$$G|_{X \times I \times \{0\}} = F,$$

$$G_{A \times I \times I} = S.$$

By Borsuk’s fiber homotopy extension theorem there exists a map $\tilde{G} : (X \times I \times I, \pi_{X \times I}) \to (P, \pi_P)$ over $B_0$ such that $\tilde{G}|_{X \times I \times \{1\}}$ is a fiber homotopy between fiber maps $\tilde{g} : (X, \pi_X) \to (P, \pi_P)$ and $\tilde{h} : (X, \pi_X) \to (P, \pi_P)$ given by formulas

$$\tilde{g}(x) = G(x, 1, 0), x \in X,$$

$$\tilde{h}(x) = \tilde{G}(x, 1, 1), x \in X,$$

$$\tilde{g}|_A = g i,$$

$$\tilde{h}|_A = h i.$$

By the Theorem 3 of [B,T_1], there exist fiber homotopies $T : g \simeq \tilde{g}$ and $Q : \tilde{h} \simeq h$. The combination of given fiber homotopies

$$L = T \cup \tilde{G}_{X \times I \times \{1\}} \cup Q : X \times I \times \{1\} \to (P, \pi_P)$$
is fiber homotopy between $g$ and $h$. Note that

$$L(i \times 1_1) \simeq H \text{rel } A \times \{0, 1\},$$

i.e. $i$ is fiber strong shape equivalence.

Now prove inverse fact. Let $i$ be a fiber strong shape equivalence. Then $i_*$ is a bijection. Consequently, for each map $f : (A, \pi_X|A) \to (P, \pi_P)$ over $B_0$ there is a map $\tilde{F} : (X, \pi_X) \to (P, \pi_P)$ over $B_0$ such that $i_*(\tilde{F}) = \tilde{F} i \simeq f$.

Using Borsuk’s fiber homotopy extension theorem we can conclude that there exists a fiber extension $\tilde{f} : (X, \pi_X) \to (P, \pi_P)$ for which $\tilde{f} \simeq F$.

Let $\tilde{f}_1, \tilde{f}_2 : (X, \pi_X) \to (P, \pi_P)$ be two such fiber extensions of $f$. Then there is a fiber homotopy $H' : \tilde{f}_1 \simeq \tilde{f}_2$, for which

$$(i \times 1_Y) H' \simeq H : f \simeq f \text{rel } A \times \{0, 1\}.$$ 

Hence, by implication b)⇒a) of Theorem 2, $i$ is an SSDR-map over $B_0$. 

\[\square\]

**Theorem 2.2.11.** Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a map over $B_0$ of compact metrizable spaces over $B_0$ and $(i_X, i_Y) : f \to \tilde{f}$ a fibrant extension over $B_0$ of $f$. Then $f$ is a fiber strong shape equivalence if and only if $\tilde{f}$ is a fiber homotopy equivalence.

**Proof.** It is known that $f = p i$, where $i : (X, \pi_X) \to (\text{Cyl}_{B_0}(f), \pi_{\text{Cyl}_{B_0}(f)})$ and $p : (\text{Cyl}_{B_0}(f), \pi_{\text{Cyl}_{B_0}(f)}) \to (Y, \pi_Y)$ are cofibration and fiber homotopy equivalence over $B_0$, respectively. Let $i_{\text{Cyl}_{B_0}(f)} : (\text{Cyl}_{B_0}(f), \pi_{\text{Cyl}_{B_0}(f)}) \to (\tilde{Z}, \pi_{\tilde{Z}})$ be a fibrant extension over $B_0$ of the mapping cylinder of $f$. There exist maps $\tilde{i} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Z}, \pi_{\tilde{Z}})$ and $\tilde{p} : (\tilde{Z}, \pi_{\tilde{Z}}) \to (\tilde{Y}, \pi_{\tilde{Y}})$ over $B_0$ such that

$$i_{\text{Cyl}_{B_0}(f)} i = \tilde{i} i_X,$$
\[ i_Y \cdot p = \tilde{p} \cdot i_{Cy_{B_0}(f)}. \]

Let \( f \) be a fiber strong shape equivalence, in the sense of Definition 2.2.1. Since \( p \) is a fiber homotopy equivalence it is strong shape equivalence. Thus from the equality \( f = p \cdot i \) it follows that \( i \) is a fiber strong shape equivalence. By Theorem 2.2.10 \( i \) is SSDR-map over \( B_0 \). Consequently, \( \tilde{i} \) is a fiber homotopy equivalence. Hence, the composition \( \tilde{p} \cdot \tilde{i} \) is a fiber homotopy equivalence. Note that \( \tilde{p} \cdot \tilde{i} \) and \( \tilde{f} \) are fiber extensions over \( B_0 \) of map \( f \). Therefore \( \tilde{p} \cdot \tilde{i} \simeq \tilde{f} \). It follows that \( \tilde{f} \) is fiber homotopy equivalence over \( B_0 \).

Now prove that if \( \tilde{f} : (\tilde{X}, \pi_{\tilde{X}}) \to (\tilde{Y}, \pi_{\tilde{Y}}) \) is a fiber homotopy equivalence then \( f \) is a fiber strong shape equivalence. Note that for each \( P \in ANR_{B_0} \) the functions \( \tilde{f}_*: [\tilde{Y}, P]_{B_0} \to [\tilde{X}, P]_{B_0}, (i_X)_*: [\tilde{X}, P]_{B_0} \to [X, P]_{B_0} \) and \( (i_Y)_*: [\tilde{Y}, P]_{B_0} \to [Y, P]_{B_0} \) are bijections. Since \( (f)_* \cdot (i_Y)_* = (i_X)_* \cdot \tilde{f}_* \), we conclude \( f_* \) is a bijection too. The space \( P^I_{B_0} \) over \( B_0 \) is an ANR_{B_0}-space. Hence, \( f_* : [Y, P^I_{B_0}]_{B_0} \to [X, P^I_{B_0}]_{B_0} \) is a bijection.

Let \( H : g \cdot f \simeq h \cdot f \) be a fiber homotopy, where \( f, g : (Y, \pi_Y) \to (P, \pi_P) \) are maps over \( B_0 \). Then there exists a map \( H' : (Y \times I, \pi_{Y \times I}) \to (P, \pi_P) \) over \( B_0 \) such that

\[ H'(f \times 1_I) \simeq H. \]

Using the argument of proof of Theorem 2.2.3 for fiber inclusion \( i : (f(X), \pi_Y\mid f(X)) \to (Y, \pi_Y) \) we can construct a fiber homotopy \( \tilde{H} : g \simeq h \) for which \( \tilde{H}(f \times 1_I) \simeq H \). Thus, \( f \) is a fiber strong shape equivalence in the sense of Definition 2.2.1.

**Corollary 2.2.12.** A map \( f \) over \( B_0 \) of compact metrizable spaces over \( B_0 \) is a fiber strong shape equivalence in the sense of Definition 2.2.1 if and only if \( \text{SS}_{B_0}([f]_{B_0}) \) is an isomorphism of the category \( \text{SSH}_{B_0} \). □
Chapter 3

Fiber Strong Shape Theory of Arbitrary Topological Spaces

In the Chapter 3 we construct and develop a fiber strong shape theory for arbitrary spaces over fixed metrizable space $B_0$. Our approach is based on the method of Mardešić-Lisica and instead of resolutions, introduced by Mardešić, their fiber preserving analogues are used. The fiber strong shape theory yields the classification of spaces over $B_0$ which is coarser than the classification of spaces over $B_0$ induced by fiber homotopy theory, but is finer than the classification of spaces over $B_0$ given by usual fiber shape theory.

3.1 Resolution and Strong Expansions of Spaces over $B_0$

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a covering of a space $Y$. We say that the maps $f, g : X \to Y$ are $\mathcal{U}$-near, if for every $x \in X$ there exists a $U_\alpha \in \mathcal{U}$ such that, $f(x), g(x) \in U_\alpha$. We say that a homotopy $H : X \times I \to Y$ which connects $f$ and $g$, is a $\mathcal{U}$-homotopy if for every
$x \in X$ there exists a $U_\alpha \in \mathcal{U}$ such that $H(x, t) \subseteq U_\alpha$ for all $t \in I$.

**Proposition 3.1.1.** (Comp. [B3], Proposition 7) Let $(Y, \pi_Y)$ be a ANR$^0$. Then every open covering $\mathcal{U}$ of $(Y, \pi_Y)$ admits an open covering $\mathcal{V}$ of $Y$ such that, whenever any two f.p. maps $f, g : (X, \pi_X) \to (Y, \pi_Y)$ from an arbitrary space $(X, \pi_X)$ over $B_0$ into the space $(Y, \pi_Y)$ over $B_0$ are $\mathcal{V}$-near, then there exists f.p. $\mathcal{U}$-homotopy $H : (X \times I, \pi_{X \times I}) \to (Y, \pi_Y)$ which connects $f$ and $g$. Moreover, if for a subset $A \subseteq X$, $f|_A = g|_A$, then $H$ is f.p. homotopy rel $A$.

**Proof.** We may assume that $(Y, \pi_Y)$ is a closed subset of space $B_0 \times K$, where $K$ is a convex set of normed vector space $L$. Let $\pi : B_0 \times K \to K$ be the map given by the formula $\pi(b, k) = k$ for every $(b, k) \in B_0 \times K$. Since $(Y, \pi_Y)$ is an ANR$^0$, there is an open neighbourhood $(G, \pi_G)$ of $(Y, \pi_Y)$ in $B_0 \times K$ together with a fibrewise retraction $r : (G, \pi_G) \to (Y, \pi_Y)$. Let $\{O_\alpha \times Q_\alpha\}_{\alpha \in \mathcal{A}}$ be a refinement of $r^{-1}(\mathcal{U})$, where $Q_\alpha$ is convex for every $\alpha \in \mathcal{A}$. Then $\mathcal{V} = \{(O_\alpha \times Q_\alpha) \cap Y\}_{\alpha \in \mathcal{A}}$ is an open refinement of the covering $\mathcal{U}$. For any two $\mathcal{V}$-near f.p. maps $f, g : (X, \pi_X) \to (Y, \pi_Y)$ we can define a f.p. homotopy $H : (X \times I, \pi_{X \times I}) \to K$ by formula

$$H'(x, t) = (\pi_X(x), (1 - t)\pi(f(x)) + t\pi(g(x))), \quad (x, t) \in X \times I.$$

Define a f.p. map $H : (X \times I, \pi_{X \times I}) \to (Y, \pi_Y)$ by taking

$$H(x, t) = r(H'(x, t)), \quad (x, t) \in X \times I.$$

Clearly, we have $H_0 = f$, $H_1 = g$ and $H$ is a $\mathcal{U}$-homotopy. Obviously, if $f(x) = g(x)$, for each $x \in A$, then $H(x, t) = f(x) = g(x)$ for every $t \in I$. \qed

An inverse system of the category $\textbf{Top}_{B_0}$ is a collection $X = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathcal{A})$ of space $(X_\alpha, \pi_{X_\alpha})$ over $B_0$ indexed by a directed set $\mathcal{A}$ and f.p. maps $p_{\alpha\alpha'} : (X_{\alpha'}, \pi_{X_{\alpha'}}) \to$
A morphism \( (f_\beta, \varphi) : X \to Y = ((Y_\beta, \pi_{Y_\beta}), q_\beta, \mathcal{B}) \) of inverse systems of the category \( \text{Top}_{B_0} \) consists of a function \( \varphi : \mathcal{B} \to \mathcal{A} \) and of f.p. maps \( f_\beta : (X_{\varphi(\beta)}, \pi_{X_{\varphi(\beta)}}) \to (Y_\beta, \pi_{Y_\beta}), \beta \in \mathcal{B} \), such that whenever \( \beta \leq \beta' \), then there is an index \( \alpha \geq \varphi(\beta), \varphi(\beta') \) for which \( f_\beta p_{\varphi(\beta)} = q_{\beta'} f_{\beta'} p_{\varphi(\beta')} \).

Two morphisms \( (f_\beta, \varphi), (g_\beta, \psi) : X \to Y \) are said to be equivalent, \( f \simeq g \), provided for each \( \beta \in \mathcal{B} \) there is an \( \alpha \in \mathcal{A}, \alpha \geq \varphi(\beta), \psi(\beta) \), such that \( f_\beta p_{\varphi(\beta)\alpha} = g_\beta p_{\psi(\beta)\alpha} \).

Let \( \text{pro} - \text{Top}_{B_0} \) be a category, whose objects are the inverse systems \( X \) of the category \( \text{Top}_{B_0} \) and whose morphisms are the equivalence classes \( f \) of morphisms \( (f_\beta, \varphi) : X \to Y \) with respect to relation \( \simeq_{B_0} \).

A morphism \( p = (p_\alpha) : (X, \pi_X) \to X = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha'}, \mathcal{A}) \) from a rudimentary system \( ((X, \pi_X)) \) to an inverse system \( X \) consists of the f.p. maps \( p_\alpha : (X, \pi_X) \to (X_\alpha, \pi_{X_\alpha}), \alpha \in \mathcal{A} \), such that \( p_\alpha = p_{\alpha'} p_{\alpha'}, \alpha \leq \alpha' \).

**Definition 3.1.2** (V. Baladze, see \([\text{B}_4 - \text{B}_6]\)). Let \( (X, \pi_X) \) be a space over \( B_0 \) and let \( X = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha'}, \mathcal{A}) \) be an inverse system of the category \( \text{Top}_{B_0} \). We say that \( p : (X, \pi_X) \to X \) is a resolution over \( B_0 \) or fiber resolution of the space \( (X, \pi_X) \) over \( B_0 \) provided it satisfies the following two conditions:

**R_{B_0}1.** Let \( (P, \pi_P) \in \text{ANR}_{B_0} \), let \( \mathcal{U} \) be an open covering of \( (P, \pi_P) \) and let \( h : (X, \pi_X) \to (P, \pi_P) \) be a f.p. map. Then there exist an index \( \alpha \in \mathcal{A} \) and a f.p. map \( f : (X_\alpha, \pi_{(P, \pi_P)}) \to (P, \pi_P) \) such that \( h \) and \( f p_\alpha \) are \( \mathcal{U} \)-near.

**R_{B_0}2.** Let \( (P, \pi_P) \in \text{ANR}_{B_0} \) and let \( \mathcal{U} \) be an open covering of \( (P, \pi_P) \). Then there is an open cover \( \mathcal{U}' \) of \( (P, \pi_P) \) with the following property: if \( \alpha \in \mathcal{A} \) and \( f, f' : (X, \pi_X) \to (P, \pi_P) \) are f.p. maps such that the f.p. maps \( f p_\alpha \) and \( f' p_\alpha \) are \( \mathcal{U}' \)-near, then there is an index \( \alpha' \geq \alpha \) such that the f.p. maps \( f p_{\alpha'\alpha} \) and \( f' p_{\alpha'\alpha} \) are \( \mathcal{U} \)-near.
If in a fiber resolution \( p : (X, \pi_X) \to X = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathcal{A}) \) of the space \((X, \pi_X)\) over \(B_0\) each \((X_\alpha, \pi_{X_\alpha})\) is an ANR over \(B_0\), then we say that \( p \) is a fiber ANR\(_B_0\)-resolution.

The next theorem of V.Baladze (\[B_4\]-\[B_6\]) is essential in the construction of the fiber shape category for arbitrary spaces over \(B_0\).

**Theorem 3.1.3.** Every space \((X, \pi_X)\) over a metrizable space \(B_0\) admits an ANR\(_B_0\)-resolution over \(B_0\).

**Definition 3.1.4** (V.Baladze, see \[B_4\]-\[B_6\], \[B_{10}\]). Let \((X, \pi_X)\) be a topological space over \(B_0\), \(X = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathcal{A})\) an inverse system in \(\text{Top}_{B_0}\) and \(p = (p_\alpha) : (X, \pi_X) \to X\) a morphism of \(\text{pro-Top}_{B_0}\). We call \(p\) an expansion over \(B_0\) of the space \((X, \pi_X)\) over \(B_0\) provided it has the following properties:

**E\(_B_0\)1.** For every ANR\(_B_0\)-space \((P, \pi_P)\) over \(B_0\) and f.p. map \(f : (X, \pi_X) \to (P, \pi_P)\) there is an index \(\alpha \in \mathcal{A}\) and a f. p. map \(h : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P)\) such that \(h p_\alpha \cong f_\alpha\).

**E\(_B_0\)2.** If \(f, f' : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P)\) are f. p. maps, \((P, \pi_P) \in \text{ANR}_{B_0}\) and \(f p_\alpha \cong f' p_\alpha\), then there is an index \(\alpha' \geq \alpha\) such that \(f p_{\alpha\alpha'} \cong f' p_{\alpha\alpha'}\).

**Definition 3.1.5.** A morphism \(p : (X, \pi_X) \to ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathcal{A})\) is called a strong expansion over \(B_0\) provided it satisfies condition \(E\(_B_0\)1\) and the following condition:

**SE\(_B_0\)2.** Let \((P, \pi_P)\) be an ANR\(_B_0\)-space, let \(f_0, f_1 : (X_\alpha, \pi_{X_\alpha}) \to (P, \pi_P), \alpha \in \mathcal{A}\) be f.p. maps and let \(F : (X \times I, \pi_{X \times I}) \to (P, \pi_P)\) be a f.p. homotopy such that

\[
S(x, 0) = f_0p_\alpha(x), \quad x \in X, \\
S(x, 1) = f_1p_\alpha(x), \quad x \in X.
\]

Then there exists a \(\alpha' \geq \alpha\) and a f.p. homotopy \(H : (X_\alpha', \pi_{X_\alpha' \times I}) \to (P, \pi_P)\), such that

\[
H(x, 0) = f_0p_{\alpha\alpha'}(z), \quad z \in X_\alpha',
\]
3.1. Resolution and Strong Expansions of Spaces over $B_0$

$$H(x, 1) = f_1 p_{\alpha'}(z), \quad z \in X_{\alpha'},$$

$$H(p_{\alpha'} \times 1_I) \cong S(\text{rel}(X \times \partial I)).$$

It is clear that, every strong expansion over $B_0$ is an expansion over $B_0$.

If all $(X_\alpha, \pi_{X_\alpha}) \in \text{ANR}_{B_0}$, then $p$ is called an $\text{ANR}_{B_0}$-expansion and strong $\text{ANR}_{B_0}$-expansion, respectively.

The main result of section 4.1 is the following theorem.

**Theorem 3.1.6.** Let $(X, \pi_X)$ be a topological space over $B_0$. Then every resolution $p : (X, \pi_X) \to X$ over $B_0$ induces a strong $\text{ANR}_{B_0}$-expansion. \hfill \Box

**Corollary 3.1.7.** Every $\text{ANR}_{B_0}$-resolution over $B_0$ induces $\text{ANR}_{B_0}$-expansion over $B_0$. \hfill \Box

**Corollary 3.1.8.** Every space $(X, \pi_X)$ over $B_0$ admits a cofinite strong $\text{ANR}_{B_0}$-expansion over $B_0$. \hfill \Box

In the proof of Theorem 3.1.6 we need the following lemmas.

**Lemma 3.1.9.** Let $(X, \pi_X)$ be a topological space over metrizable space $B_0$, let $(P, \pi_P), (P', \pi_{P'})$ be $\text{ANR}_{B_0}$-spaces, let $f : (X, \pi_X) \to (P', \pi_{P'})$, $h_0, h_1 : (P', \pi_{P'}) \to (P, \pi_P)$ be f.p. maps and let $S : (X \times I, \pi_{X \times I}) \to (P, \pi_P)$ be a f.p. homotopy such that

$$S(x, 0) = h_0 f(x), \quad x \in X,$$

$$S(x, 1) = h_1 f(x), \quad x \in X.$$

Then there exists an $\text{ANR}_{B_0}$-space $(P'', \pi_{P''})$, f.p. maps $f' : (X, \pi_X) \to (P'', \pi_{P''})$, f.p.
3.1. Resolution and Strong Expansions of Spaces over $B_0$

$h : (P'', \pi_{P''}) \to (P', \pi_{P'})$ and a f.p. homotopy $K : (P'' \times I, \pi_{P'' \times I}) \to (P, \pi_P)$ such that

$$hf' = f,$$

$$K(z, 0) = h_0h(z), \quad z \in P''$$

$$K(z, 1) = h_1h(z), \quad z \in P''$$

$$K(f' \times 1_I) = S.$$

Proof. Let $S : (X \times I, \pi_{X \times I}) \to (P, \pi_P)$ be a map such that $S(x, 0) = (h_0 f)(x)$,

$S(x, 1) = (h_1 f)(x)$ and $\pi_P S = \pi_{X \times I}$. Consider the subspace $C_{B_0}(I, P)$ of the space $C(I, P)$. Let $\pi_{C_{B_0}(I,P)} : C_{B_0}(I,P) \to B_0$ be the map given by $\pi_{C_{B_0}(I,P)}(\varphi) = \pi_P(\varphi(t))$.

Consequently, $C_{B_0}(I,P)$ is a space over $B_0$. The f.p. map $S : (X \times I, \pi_{X \times I}) \to (P, \pi_P)$ defines the map $s : (X, \pi_X) \to C_{B_0}(I,P)$ such that $(s(x))(t) = S(x, t), x \in X, t \in I$. The image of the point $x \in X, s(x) \in C_{B_0}(I,P)$, because $\pi_P s(x) : I \to B_0$ is a constant map. Indeed,

$$(\pi_P s(x))(t) = \pi_P(s(x))(t) = \pi_P(S(x, t)) = \pi_{X \times I}(x, t) = \pi_X(x)$$

for every $t \in I$.

For each $x \in X$ we have

$$(\pi_{C_{B_0}(I,P)} s)(x) = (\pi_{C_{B_0}(I,P)}(s(x)) = \pi_P(s(x))(t) =$$

$$= \pi_P(S(x, t)) = \pi_{X \times I}(x, t) = \pi_X(x).$$

Thus, $\pi_{C_{B_0}(I,P)} s = \pi_X$. Hence, $s : (X, \pi_X) \to C_{B_0}(I,P)$ is a f.p. map. For all $x \in X$ we have

$$(s(x))(0) = S(x, 0) = (h_0 f)(x)$$
3.1. Resolution and Strong Expansions of Spaces over $\mathbf{B}_0$

and

$$(s(x))(1) = S(x, 1) = (h_1 f)(x).$$

Let $P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P) = \{(y, \varphi)|\pi_{P'}(y) = \pi_{C_{\mathbf{B}_0}(I, P)}(\varphi)\}$. The map $f' : (X, \pi_X) \to P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P)$, given by $f'(x) = (f(x), s(x))$, is a f.p. map. Let $\pi_{P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P)} : P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P) \to \mathbf{B}_0$ be a map defined by

$$\pi_{P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P)}(y, \varphi) = \pi_{P'}(y) = \pi_{C_{\mathbf{B}_0}(I, P)}(y).$$

Then we have

$$\pi_{P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P)} f' = \pi_{P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P)}(f(x), s(x)) = \pi_{P'}(f(x)) = \pi_X(x).$$

Thus, $\pi_X = \pi_{P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P)} f'$. It is clear that the first projection $h : P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P) \to (P', \pi_{P'})$ is a f.p. map and $h f' = f$.

We define the subset $(P'', \pi_{P''})$ of $P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P)$ be the following way:

$$P'' = \{(y, \varphi) \in P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P)|\varphi(0) = h_0(y), h_1(y) = \varphi(1)\}.$$ 

Let $K : P' \times_{\mathbf{B}_0} C_{\mathbf{B}_0}(I, P) \times I \to P$ be a map given by formula

$$K((y, \varphi), t) = \varphi(t), y \in P', \varphi \in C_{\mathbf{B}_0}(I, P), t \in P.$$ 

The restriction of $K$ on $(P'' \times I, \pi_{P''} \times I)$ again denote by $K : (P'' \times I, \pi_{P''} \times I) \to (P, \pi_P)$. This map is a f.p. homotopy between $h_0 |_{P''}$ and $h_1 |_{P''}$. 

Indeed, for every \((y, \varphi) \in P''\) and \(t \in I\) we have

\[
K((y, \varphi), 0) = \varphi(0) = h_0(y) = h_0 h(y, \varphi),
\]

\[
K((y, \varphi), 1) = \varphi(1) = h_1(y) = h_1 h(y, \varphi),
\]

\[
\pi_{P' \times B_0} c_{B_0}(I, P)(y, \varphi) = (y, \varphi) = \pi_{P' \times B_0} c_{B_0}(I, P)(y, \varphi) = (y, \varphi),
\]

\[
\pi_{P'}(y) = \pi_{C_{B_0}(I, P)}(\varphi) = \pi_P(\varphi(t)) = \pi_P(K(y, \varphi), t).
\]

Note that for each \(x \in X\) and \(t \in I\)

\[
K(f' \times 1_I)(x, t) = K((f(x), s(x)), t) = (s(x))(t) = (S(x, t)).
\]

Hence, \(K(f' \times 1_I) = S\).

We shall prove that \((P'', \pi_{P''}) \in \text{ANR}_{B_0}\). Now suppose that \(A\) is a closed subspace of a space \(Z\) over \(B_0\) and \(l : A \to P''\) is a map such that \(\pi_A = \pi_{Z | A} = \pi_P l\).

Denote by \(L : (A \times I, \pi_{A \times I}) \to (P, \pi_P)\) the map defined by

\[
L(a, t) = (h' l(a))(t), (a, t) \in A \times I,
\]

where \(h'\) is the second projection \(P' \times_{B_0} C_{B_0}(I, P) \to C_{B_0}(I, P)\). It is clear that \(L\) is a f.p. map. Indeed,

\[
(\pi_P L)(a, t) = \pi_P(L(a, t)) = \pi_P((h' l(a))(t)) = \pi_{C_{B_0}(I, P)}(h'(l(a))) = \pi_A(a) = \pi_{A \times I}(a, t).
\]
The map $L$ is a f.p. homotopy between $h_0 h l$ and $h_1 h l$. Indeed,

$$L(a, 0) = (h' l(a))(0) = h_0 h l(a), \quad a \in A$$

and

$$L(a, 1) = (h' l(a))(1) = h_1 h l(a), \quad a \in A.$$ 

Observe that, since $(P', \pi_{P'}) \in \text{ANR}_{B_0}$ and $h l : (A, \pi_A) \to (P', \pi_{P'})$ is a f.p. map, there is a neighbourhood $U$ of $A$ in $Z$ and there exists a f.p. map $\tilde{l}' : (U, \pi_U) \to (P', \pi_{P'})$ such that $\tilde{l}'|_A = h l$.

There exist a neighbourhood $V$ of $A$ in $U$ and a f.p. homotopy $\tilde{L} : (V \times I, \pi_{V \times I}) \to (P, \pi_P)$ between $h_0 \tilde{l}'|_V$ and $h_1 \tilde{l}'|_V$. Also note that $\tilde{L}(a, t) = L(a, t)$ for each $a \in A$ and $t \in I$. Let $\tilde{l}''$ be a f.p. map $\tilde{l}'' : (V, \pi_V) \to C_{B_0}(I, P)$, given by $(\tilde{l}''(z))(t) = \tilde{L}(z, t)$, $z \in V, t \in I$. For every $a \in A$ we have

$$(\tilde{l}''(a))(t) = \tilde{L}(a, t) = L(a, t) = (h' l(a))(t).$$

Consequently, $\tilde{l}''|_A = h' l$. Now define the f.p. map $\tilde{l} : (V, \pi_V) \to P' \times_{B_0} C_{B_0}(I, P)$ by the formula

$$\tilde{l}(z) = (\tilde{l}, \tilde{l}''), \quad z \in V.$$ 

For each $z \in V$ we have

$$(\tilde{l}''(z))(0) = \tilde{L}(z, 0) = h_0 \tilde{l}(z),$$

$$(\tilde{l}''(z))(1) = \tilde{L}(z, 1) = h_1 \tilde{l}(z).$$

Consequently, $\tilde{l} : (V, \pi_V) \to (P'', \pi_{P''})$ is an extension of the f.p. map $l : (A, \pi_A) \to (P'', \pi_{P''})$. This fact completes the proof of lemma 3.1.9.
Lemma 3.1.10. Let \( p : (X, \pi_X) \to X \) be a resolution over \( B_0 \) and let \( \alpha, P, f_0, f_1 \) and \( F \) be as in \( SE_{B_0}2 \). Then for every open covering \( \mathcal{U} \) of \( (P, \pi_P) \), there exist \( \alpha' \geq \alpha \) and a f.p. homotopy \( H : (X_{\alpha'} \times I, \pi_{X_{\alpha'} \times I}) \to (P, \pi_P) \) such that

\[
H(y, 0) = f_0 p_{\alpha'}(y), \quad y \in X_{\alpha'}
\]
\[
H(y, 1) = f_1 p_{\alpha'}(y), \quad y \in X_{\alpha'}
\]
\[
(S, H(1 \times p_{\alpha'})) \leq \mathcal{U}.
\]

Proof. Let \( \mathcal{U} \) be an open covering of \( (P, \pi_P) \). There exists an open star-refinement \( \mathcal{U}' \) of \( \mathcal{U} \). Now we choose an open covering \( \mathcal{V} \) of \( (P, \pi_P) \) such that the assertions of Proposition 3.1.1 hold for \( \mathcal{U}' \). We can assume that \( \mathcal{V} \) is a star-refinement of \( \mathcal{U}' \). We choose \( \mathcal{V}' \) so that \( \mathcal{V}' \) is a star-refinement of \( \mathcal{V} \) and \( R_{B_0}2 \) holds for \( (P, \pi_P), \mathcal{V} \) and \( \mathcal{V}' \).

Let \( P' = P \times_{B_0} P \). By \( g_0, g_1 : (P', \pi_{P'}) \to (P, \pi_P) \) denote the two projections. Let \( f : (X, \pi_X) \to (P', \pi_{P'}) \) be the diagonal product of f.p. maps \( f_0 p_\alpha : (X, \pi_X) \to (P, \pi_P) \) and \( f_1 p_\alpha : (X, \pi_X) \to (P, \pi_P) \). It is clear that \( g_0 f = f_0 p_\alpha, g_1 f = f_1 p_\alpha, F_0 = g_0 f \) and \( F_1 = g_1 f \).

By the lemma 3.1.9 there exists an ANR\(_{B_0}\) space \( (P'', \pi_{P''}) \), f.p. maps \( f' : (X, \pi_X) \to (P'', \pi_{P''}) \), \( g : (P'', \pi_{P''}) \to (P', \pi_{P'}) \) and a f.p. homotopy \( G : (P'' \times I, \pi_{P'' \times I}) \to (P, \pi_P) \) such that

\[
g f' = f,
\]
\[
G_0 = g_0 g, G_1 = g_1 g,
\]
\[
G(f' \times 1) = F.
\]

We choose for the open covering \( G^{-1}(\mathcal{V}') \) of \( (P'' \times I, \pi_{P'' \times I}) \) a refinement, which is a stacked covering \( \mathcal{V} \) of \( (P'' \times I, \pi_{P'' \times I}) \), given by a locally finite open covering \( \mathcal{W} \) of \( (P'', \pi_{P''}) \) and by finite open coverings \( \mathcal{W}, W \in \mathcal{W} \) of \( I \).
By condition $R_{B_0}1)$ there exists a $\alpha'' \geq \alpha$ and f.p. mapping $h : (X_{\alpha''}, \pi_{X_{\alpha''}}) \rightarrow (P'', \pi_{P''})$ such that 

$$(f', h_{p_{\alpha''}}) \leq \mathcal{W}$$

It is clear that for any $W \in \mathcal{W}$, $W \times 0 \subseteq W \times J$, where $J \in \mathcal{J}_W$ and $W \times J \subseteq G^{-1}(V')$ for some $V' \in V'$. 

Note that

$$g_0g(W) = G_0(W) = G(W \times 0) \subseteq G(W \times J) \subseteq V'.$$

Hence, $g_0g(\mathcal{W})$ refines $\mathcal{V}'$ and $(g_0 g f', g_0 g h_{p_{\alpha''}}) \leq \mathcal{V}'$.

From the equalities

$$g_0 g f' = g_0 f = f_0 p_\lambda = f_0 p_{\alpha' p_{\alpha''}}$$

it follows that

$$(g_0 h_{p_{\alpha''}}, f_0 p_{\alpha' p_{\alpha''}}) \leq \mathcal{V}'$$

We also can claim that

$$(g_1 h_{p_{\alpha''}}, f_1 p_{\alpha' p_{\alpha''}}) \leq \mathcal{V}'$$

By condition $R_{B_0}2)$ there is a $\alpha' \geq \alpha''$ such that

$$(g_0 h_{p_{\alpha''} p_{\alpha'}}, f_0 p_{\alpha'}) \leq \mathcal{V}$$

and

$$(g_1 h_{p_{\alpha''} p_{\alpha'}}, f_1 p_{\alpha'}) \leq \mathcal{V}.$$
Besides, there exist $U'$-f.p. homotopies $K, L : (X_{\alpha'} \times I, \pi_{X_{\alpha'}} \times I) \to (P, \pi_P)$ such that $K_0 = f_0 p_{\alpha' \alpha}, K_1 = g_0 g h p_{\alpha'' \alpha'}, L_0 = f_1 p_{\alpha' \alpha}$ and $L_1 = g_1 g h p_{\alpha'' \alpha'}$.

Note that for any $t \in I$ the pairs $(f'(x), t)$ and $(h p_{\alpha''}(x), t)$ belong to some elements of $\mathcal{V}$ and consequently to $G^{-1}(\mathcal{V}')$ for some $\mathcal{V}' \in \mathcal{V}$. Thus $G(f' \times 1_I)$ and $G(h p_{\alpha''} \times 1_I)$ are $\mathcal{V}'$-near. Hence,

$$(G(f' \times 1_I), G(h p_{\alpha''} \times 1_I)) \leq \mathcal{V}.$$ 

Now we define f.p. homotopy $H : (X_{\alpha'} \times I, \pi_{X_{\alpha'}} \times I) \to (P, \pi_P)$ by formulas

$$H(y, t) = \begin{cases} 
K(y, \frac{t}{\varphi(z)}), & 0 \leq t \leq \varphi(z), \\
G(z, \frac{t-\varphi(z)}{1-2\varphi(z)}), & \varphi \leq t \leq 1 - \varphi(z), \\
L(y, \frac{1-t}{\varphi(z)}), & 1 - \varphi(z) \leq t \leq 1,
\end{cases}$$

where $z = h p_{\alpha'' \alpha'}(y)$ and $\varphi : (P'', \pi_{P''}) \to I$ is a continuous map defined in $\mathbb{M}_2$.

As in $\mathbb{M}_2$ we can prove that for every $(x, t) \in X \times I$, there is a $U \in \mathcal{U}$ such that

$$F(x, t), H(p_{\alpha}(x), t) \in U.$$ 

\[\square\]

**Proof of Theorem 3.1.6**. First prove the following condition.

**E$_{B_0}$1)** Let $\mathcal{U}$ be a open covering of $(P, \pi_P)$. Consider open covering $\mathcal{V}$ as in Proposition 3.1.1. By $\mathbb{R}_{B_0}$1) there exist an index $\alpha \in \mathcal{A}$ and a f.p. mapping $h : (X_{\alpha}, \pi_{X_{\alpha}}) \to (P, \pi_P)$ which satisfies condition $(h p_{\alpha}, f) \leq \mathcal{V}$. Thus, by the choice of $\mathcal{V}$, $f \simeq h p_{\alpha}$. 

**S$_{B_0}$2)** Let $\mathcal{U}$ be a open covering $\mathcal{U}$ of. Consider a covering $\mathcal{V}$ as in Proposition 3.1.1. By Lemma 3.1.9 there exist a $\alpha' \geq \alpha$ and f.p. homotopy $H : (X_{\alpha'} \times I, \pi_{X_{\alpha'} \times I}) \to (P, \pi_P)$...
which satisfies

\[ H(z, 0) = f_0 p_{\alpha'}(z), \quad z \in X', \]

\[ H(z, 1) = f_1 p_{\alpha'}(z), \quad z \in X', \]

\[ (S, H(1 \times p_{\alpha'})) \leq \mathcal{V}. \]

Consider the spaces \( Z = X \times I \) and \( A = X \times \partial I \) over \( B_0 \) and f.p. mappings \( h_0 = F \) and \( h_1 = H(p_{\alpha'} \times 1) \).

Note that \( h_0|_A = h_1|_A \). Indeed, for each \( x \in X \)

\[ h_0(x, 0) = F(x, 0) = f_0 p_{\alpha}(x) = f_0 p_{\alpha'} p_{\alpha'}(x) = H(p_{\alpha'}(x), 0) = h_1(x, 0). \]

Analogously, for each \( x \in X \) we have

\[ h_0(x, 1) = F(x, 1) = f_1 p_{\alpha}(x) = f_1 p_{\alpha'} p_{\alpha'}(x) = H(p_{\alpha'}(x), 1) = h_1(x, 0). \]

Consequently, \((h_0, h_1) \leq \mathcal{V}\). By Proposition 3.1.1 there exists a f.p. homotopy \( \text{rel}(X \times \partial I) \), which connects \( F \) and \( H(p_{\alpha'} \times 1_I) \).

\[ \square \]

3.2 On Fiber Strong Shape Category for Arbitrary Topological Spaces

Let \( \Delta^n \) be the standard \( n \)-simplex, i.e. the set of all points \( t = \{t = (t_0, t_1, \ldots, t_n) \in R^{n+1}\} \), where \( t_0 \geq 0, \ldots, t_n \geq 0 \) and \( t_0 + \cdots + t_n = 1 \).

For \( n > 0 \) and \( 0 \leq j \leq n \) there exist \( \partial_j^n : \Delta^{n-1} \to \Delta^n \) \( j \)-th face operators and for \( n \geq 0 \) and \( 0 \leq j \leq n \) there exist \( \sigma_j^n : \Delta^{n+1} \to \Delta^n \) \( j \)-th degeneracy operators given by
3.2. On Fiber Strong Shape Category for Arbitrary Topological Spaces

Formulas

\[ \partial^n_j(t_0, \cdots, t_{n-1}) = (t_0, \cdots, t_{j-1}, 0, t_j, \cdots, t_{n-1}), \]

\[ \sigma^n_j(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \cdots, t_{n+1}). \]

Let \( B \) be a directed set. By \( B^n \) denote the set of all sequences \( \beta = (\beta_0, \cdots, \beta_n), \beta_0 \leq \cdots \leq \beta_n \) of elements of \( B \).

For \( n > 0 \) and \( 0 \leq j \leq n \) we consider the \( j \)-th face operator \( d^n_j : B^n \to B^{n-1} \) given by formula

\[ d^n_j(\beta_0, \cdots, \beta_n) = (\beta_0, \cdots, \beta_{j-1}, \beta_{j+1}, \cdots, \beta_n) \]

and for \( n \geq 0 \) and \( 0 \leq j \leq n \) by \( s^n_j \) we denote \( j \)-th degeneracy operator \( s^n_j : B^n \to B^{n+1} \) given by formula

\[ s^n_j(\beta_0, \cdots, \beta_n) = (\beta_0, \cdots, \beta_j, \beta_j, \cdots, \beta_n). \]

For simplicity the images \( d^n_j(\beta) \) and \( s^n_j(\beta) \) we denote by \( \beta_j \) and \( \beta^j \), respectively.

Let \( X = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'} \in A) \) and \( Y = ((Y_\beta, \pi_{Y_\beta}), p_{\beta\beta'} \in B) \) be the objects of category \( \text{pro} - \text{Top}_{B_0} \).

A coherent map \( f : X \to Y \) over \( B_0 \) or fiber preserving (f.p) coherent map consists of function \( \varphi : B^n \to A \) and fiber preserving maps \( f_\beta : X_{\varphi(\beta)} \times \Delta^n \to Y_{\beta_0}, \beta = (\beta_0, \cdots, \beta_n) \in B^n, n \geq 0 \) having the following properties:

i). The function \( \varphi \), which assigns to every \( n \geq 0 \) and \( \beta = (\beta_0, \cdots, \beta_n) \in B^n \) an element \( \varphi(\beta) = \varphi(\beta_0, \cdots, \beta_n) \in A \), satisfies condition:

\[ \varphi(\beta) \geq \varphi(\beta_j), \quad 0 \leq j \leq n, n > 0. \]

ii). For every \( n \geq 0 \) and every \( \beta = (\beta_0, \cdots, \beta_n) \in B^n \) the fiber preserving maps
3.2. On Fiber Strong Shape Category for Arbitrary Topological Spaces

\( f_\beta : (X_{\varphi(\beta)} \times \Delta^n, \pi_{X_{\varphi(\beta)} \times \Delta^n}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}}) \) satisfies condition:

\[
f_\beta(x, \partial^n_j t) = \begin{cases} 
q_{\beta_0 \beta} f_\beta_0 (p_{\varphi(\beta_0)} p_{\varphi(\beta)}(x), t), & j = 0 \\
f_{\beta_j} (p_{\varphi(\beta_j)} p_{\varphi(\beta)}(x), t), & 0 \leq j \leq n,
\end{cases}
\]

where \( x \in X_{\varphi(\beta)}, t \in \Delta^{n-1}, n \geq 0, X_{\varphi(\beta)} \times \Delta^n \) is the space over \( B_0 \) with projection \( \pi_{X_{\varphi(\beta)} \times \Delta^n} : X_{\varphi(\beta)} \times \Delta^n \to B_0 \) given by formula

\[
\pi_{X_{\varphi(\beta)} \times \Delta^n}(x, t) = \pi_{X_{\varphi(\beta)}}(x), \quad x \in X_{\varphi(\beta)}, t \in \Delta^n
\]

and

\[
f_{\beta_j} (p_{\varphi(\beta_j)} p_{\varphi(\beta)}(x), \sigma^n_j(t)) = f_{\beta_j} (x, t), 0 \leq j \leq n, x \in X_{\varphi(\beta)}, t \in \Delta^{n+1}, n \geq 0.
\]

The identity coherent map \( 1_X : X \to X \) over \( B_0 \) is given by formulas:

\[
\varphi(\alpha) = \alpha_n, \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathscr{A}^n,
\]

\[
1_\alpha(x, t) = p_{\alpha_0 \alpha_n}(x), x \in X_{\alpha_n}, t \in \Delta^n, n \geq 0.
\]

A coherent homotopy over \( B_0 \) or fiber preserving (f.p.) homotopy \( F : X \times I \to Y \)

connecting f.p. coherent maps \( f, f' : X \to Y \), is a f.p. coherent map of \( X \times I = ((X_n \times I, \pi_{X_n \times I}), p_{aa'} \times 1_I, \mathscr{A}) \) to \( Y \), given by a function \( \Phi \) and by f.p. maps \( F_\beta : (X_{\varphi(\beta)} \times I \times \Delta_n, \pi_{X_{\varphi(\beta)} \times I \times \Delta_n}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}}) \), which have i) and ii) properties and satisfy the conditions

\[
\Phi(\beta) \geq \varphi(\beta), \varphi'(\beta),
\]

\[
F_\beta(x, 0, t) = f_\beta(p_{\varphi(\beta)} \phi(\beta)(x), t),
\]
3.2. On Fiber Strong Shape Category for Arbitrary Topological Spaces

\[ F_\beta(x, 1, t) = f'(p_{\varphi(\beta)}\Phi(\beta)(x), t), \]

where \( x \in X_{\varphi(\beta)}, t \in \Delta^n, n \geq 0. \)

As in [L-M] we can prove the following

**Proposition 3.2.1.** The f.p. coherent homotopy relation of f.p. coherent maps is an equivalence relation.

A f.p. coherent map \( f : X \to Y \) is called a special f.p. coherent map or a special coherent map over \( B_0 \) if \( \varphi(\beta) = \varphi(\beta_n) \) for each \( \beta \in B^n \) and \( \varphi|_B : B \to A \) is an increasing function.

The composition \( h = g f \) of special f.p. coherent maps over \( B_0 \) is defined as in [L-M].

A special f.p. coherent homotopy connecting two special f.p. coherent maps \( f, f' : X \to Y \) is a f.p. coherent homotopy \( F : X \times I \to Y \) between \( f \) and \( f' \) and at the same time it is a special f.p. coherent map.

Note that if the index set \( B \) of \( Y \) is cofinite, then special f.p. coherent homotopy relation of special f.p. coherent maps is an equivalence relation.

The proofs of the following proposition pass as in [L-M].

**Proposition 3.2.2.** Let \( f, f' : X \to Y, g, g' : Y \to Z \) be special f.p. coherent maps and let \( F, G \) be special f.p. coherent homotopies connecting \( f \) with \( f' \) and \( g \) with \( g' \), respectively. If the index set \( C \) is cofinite, then there is a special f.p. coherent homotopy connecting \( g f \) and \( g' f' \).

**Proposition 3.2.3.** If \( f : X \to Y, g : Y \to Z \) and \( h : Z \to W \) are special f.p. coherent maps of inverse systems of \( \text{Top}_{B_0} \) over cofinite index sets, then there is a special f.p. coherent homotopy connecting \( h(gf) \) with \( (hg)f \).
Proposition 3.2.4. If \( f : X \rightarrow Y \) is a special f.p. coherent map of inverse systems of \( \text{Top}_{B_0} \) over cofinite index sets and \( 1_X \) and \( 1_Y \) are the f.p. coherent identity maps, then there exist special f.p. coherent homotopies connecting \( 1_X \) with \( f \) and \( 1_Y \) with \( f \).

As in [L-M] we can show that whenever the index set \( \mathcal{B} \) of \( Y \) is cofinite, then every f.p. coherent homotopy class \([f] : X \rightarrow Y\) of f.p. coherent maps \( f : X \rightarrow Y \) contains a unique f.p. coherent homotopy class of special f.p. coherent maps. Consequently, in the cofinite case one can define composition of f.p. coherent homotopy classes by composing their special representatives.

Now define the following category. The f.p. coherent prohomotopy category \( \text{CPHTop}_{B_0} \) has as objects inverse systems \( X = ((X_\alpha, \pi_{X_\alpha}), p_{\alpha\alpha'}, \mathcal{A}) \) of topological spaces over \( B_0 \) and f.p. maps over directed cofinite index sets. The morphisms are f.p. coherent homotopy classes \([f] : X \rightarrow Y\) of f.p. coherent maps \( f : X \rightarrow Y \) of such systems. Composition is defined by composing representatives, which are special f.p. coherent maps. Identity morphism of \( X \) is the class, containing the coherent map \( 1_X : X \rightarrow X \).

Now define the functor \( C : \text{pro-Top}_{B_0} \rightarrow \text{CPHTop}_{B_0} \). Let \((f_\beta, \varphi) : X \rightarrow Y\) be a map of inverse systems. We associate with \((f_\beta, \varphi)\) a f.p. coherent map \( f : X \rightarrow Y \). For this aim we extend \( \varphi : \mathcal{B} \rightarrow \mathcal{A} \) to a function \( \varphi \) defined for all \( \beta = (\beta_0, \cdots, \beta_n) \) in such a way that

\[
\varphi(\beta) \geq \varphi(\beta_j), \text{ } 0 \leq j \leq n.
\]

We use the method of induction. Let \( n = 1 \) and \( \beta = (\beta_0, \beta_1) \). Note that

\[
\int_{\beta_0} p_{\varphi(\beta_0) \varphi(\beta_0, \beta_1)} = q_{\beta_0 \beta_1} \int_{\beta_1} p_{\varphi(\beta_1) \varphi(\beta_0, \beta_1)}.
\]
Let \( f_\beta : (X_{\varphi(\beta)} \times \Delta^n, \pi_{X_{\varphi(\beta)} \times \Delta^n}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}}) \) a f.p. mapping defined by

\[
f_\beta(x, t) = f_{\beta_0}p_{\varphi(\beta_0)}(x) \quad x \in X_{\varphi(\beta)}, t \in \Delta^n.
\]

Also note that

\[
f_\beta(x, \partial^n_0 t) = f_{\beta_0}p_{\varphi(\beta_0)}(x) = q_{\beta_0, \beta_1}f_{\beta_1}p_{\varphi(\beta_1)}(x, t)
\]
and

\[
f_\beta(x, \partial^n_j t) = f_{\beta_0}p_{\varphi(\beta_0)}(x) = f_{\beta_j}(p_{\varphi(\beta_j)}(x, t)), \quad 0 < j \leq n,
\]

\[
f_\beta(p_{\varphi(\beta)}(x, \sigma^n_j t)) = f_{\beta_0}p_{\varphi(\beta_0)}(x) = f_{\beta_j}(x, t), \quad 0 \leq j \leq n.
\]

Let \( \varphi' \) be another extension of \( \varphi \). We obtain another f.p. coherent map \( f' \). Note that \( f \) and \( f' \) are f.p. coherently homotopic.

Let \((f_\beta, \varphi), (f'_\beta, \varphi') : X \to Y\) are equivalent morphisms. As in [L-M], we can show that the associated f.p. coherent maps \( f \) and \( f' \) are connected by some f.p. coherent homotopy \( F : X \times I \to Y \).

Thus, to every morphism of \( f : X \to Y \) of \( \text{pro} - \text{Top}_B \) we can associate a morphism \([f] = C(f)\) of \( \text{CPHTop}_B \). If we restrict \( \text{pro} - \text{Top}_B \) to inverse systems over cofinite index sets, then we have defined a functor \( C : \text{pro} - \text{Top}_B \to \text{CPHTop}_B \).

By definition,

\[
C(f) = [f], \quad f \in \text{Mor}_{\text{pro} - \text{Top}_B}(X, Y),
\]

\[
C(X) = X, \quad X \in \text{ob}(\text{pro} - \text{Top}_B).
\]

\( C(1_Y) \) is the f.p. coherent homotopy class of \( 1_Y \). Let \( f : X \to Y \) and \( g : Y \to Z \) be morphism of \( \text{pro} - \text{Top}_B \). As in [L-M], we can prove that \( C(gf) = C(g) C(f) \).

Besides, there exists a functor \( E : \text{CPHTop}_B \to \text{pro} - \text{HTop}_B \). Assume that
for each inverse system $X = (X_\alpha, p_{\alpha\alpha'}, \mathcal{A})$ in $\text{Top}_{B_0}$, $EX = (X_\alpha, [p_{\alpha\alpha'}]_{B_0}, \mathcal{A})$.

Let $f : X \to Y$ be a f.p. coherent map given by $f_\beta$ and $\varphi$. We associate with $f$ the morphism $f : X \to Y$ of $\text{pro} - \text{HTop}_{B_0}$, given by function $\varphi|_{B_0} : B_0 \to \mathcal{A}$ and the fiber homotopy classes over $B_0$, $[f_\beta]_{B_0} : X_{\varphi(\beta_0)} \to Y_{\beta_0}$.

Note that $f$ is a morphism of $\text{pro} - \text{HTop}_{B_0}$. Indeed, for $\beta_0 \leq \beta_1$ and $\alpha = \varphi(\beta_0, \beta_1)$ we have $\alpha \geq \varphi(\beta_0), \varphi(\beta_1)$. Besides, the f.p. map $f_{\beta_0\beta_1} : (X_\alpha \times \Delta^1, \pi_{X_\alpha \times \Delta^1}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}})$ satisfies the conditions

$$f_{\beta_0\beta_1}(x, \partial^1_0(1)) = q_{\beta_0\beta_1} f_{\beta_1}(p_{\varphi(\beta_1)\alpha}(x), 1)$$

and

$$f_{\beta_0\beta_1}(x, \partial^1_1(1)) = f_{\beta_0}(p_{\varphi(\beta_0)\alpha}(x), 1).$$

Thus,

$$[f_{\beta_0}]_{B_0} [p_{\varphi(\beta_0)\alpha}]_{B_0} = [q_{\beta_0\beta_1}]_{B_0} [f_{\beta_1}]_{B_0} [p_{\varphi(\beta_1)\alpha}]_{B_0}.$$

Let $f, f' : X \to Y$ be f.p. coherent homotopic maps. Let $F : X \times I \to Y$ be a f.p. coherent homotopy between $f$ and $f'$, given by $\Phi$ and $F_\beta$. Note that $\Phi(\beta_0) \geq \varphi(\beta_0), \varphi'(\beta_0)$ and $F_{\beta_0} : X_{\Phi(\beta_0) \times I \times \Delta^0} \to Y_{\beta_0}$ is a f.p. map satisfying conditions

$$F_{\beta_0}(x, 0, 1) = f_{\beta_0}(p_{\varphi(\beta_0)\Phi(\beta_0)}(x), 1)$$

and

$$F_{\beta_0}(x, 1, 1) = f'_{\beta_0}(p_{\varphi'(\beta_0)\Phi(\beta_0)}(x), 1).$$

Consequently,

$$[f_{\beta_0}]_{B_0} [p_{\varphi(\beta_0)\Phi(\beta_0)}]_{B_0} = [f'_{\beta_0}]_{B_0} [p_{\varphi'(\beta_0)\Phi(\beta_0)}]_{B_0}. $$
Thus, with $f$ and with $f'$ is associated the same morphism of $\text{pro} - \text{HTop}_{B_0}$. Consequently, it is possible to define a functor $E : \text{CPHTop}_{B_0} \to \text{pro} - \text{HTop}_{B_0}$.

The composition $E \circ C : \text{pro} - \text{Top}_{B_0} \to \text{pro} - \text{HTop}_{B_0}$ is the functor induced by the f.p. homotopy functor $H : \text{Top}_{B_0} \to \text{HTop}_{B_0}$.

A f.p. coherent map $f : X \to Y$ consists of f.p. maps $f_\beta : (X \times \Delta^n, \pi_{X \times \Delta^n}) \to (Y_{\beta_0}, \pi_{Y_{\beta_0}}), \beta = (\beta_0, \cdots, \beta_n) \in \mathcal{B}, n \geq 0$, satisfying the following conditions: for each $x \in X$, $t \in \Delta^{n-1}, n > 0$

$$f_\beta(x, \partial_j^n t) = \begin{cases} q_{\beta_0 \beta_1} f_{\beta_0}(x, t), & j = 0, \\ f_{\beta_j}(x, t), & 0 < j \leq n \end{cases}$$

and for each $x \in X$, $t \in \Delta^{n+1}, n \geq 0$

$$f_\beta(x, \sigma_j^n t) = f_{\beta_i}(x, t), \quad 0 \leq j \leq n.$$

Note that a f.p. coherent map $f : X \to Y$ is always a special f.p. coherent map.

A f.p. coherent homotopy $F : X \times I \to Y$, connecting $f$ and $f'$, is a f.p. coherent map given by $F_\beta$ and satisfying the conditions: for each $x \in X$, $t \in \Delta^n$

$$F_\beta(x, 0, t) = f_\beta(x, t)$$

and

$$F_\beta(x, 1, t) = f'_\beta(x, t).$$

Let $p = (p_\alpha) : X \to X$ be a morphism of $\text{pro} - \text{Top}_{B_0}$. It is clear that with $p$ is associated a unique f.p. coherent map $p : X \to X$ given by formula

$$p_\alpha(x, t) = p_{\alpha_0}(x),$$
where $\alpha = (\alpha_0, \cdots, \alpha_n) \in \mathcal{A}^n$, $x \in X$, $t \in \Delta^n$.

The objects of category $\text{SSH}_{B_0}$ are all topological spaces over $B_0$. The morphisms of category $\text{SSH}_{B_0}$ are defined by the following way.

Let $p : X \to X$ and $q : Y \to Y$ be an ANR$_{B_0}$-resolutions of $X$ and $Y$, respectively. Let $[f] : X \to Y$ be a some morphism of category $\text{CPHTop}_{B_0}$. Let $p' : X \to X'$, $q' : Y \to Y'$, $[f'] : X' \to Y'$ be another triple of fiber resolutions of spaces $X$ and $Y$ over $B_0$ and morphism of category $\text{CPHTop}_{B_0}$.

Now define the following equivalence relation. We say the triples $(p, q, [f])$ and $(p', q', [f'])$ are equivalent if $[f'] [i] = [j] [f]$, where $[i] : X \to X'$ and $[j] : Y \to Y'$ are isomorphisms of category $\text{CPHTop}_{B_0}$.

The fiber strong shape morphisms $F : (X, \pi_X) \to (Y, \pi_Y)$ are the equivalence classes of triples $(p, q, [f])$ with respect to the above defined relation $\sim$.

Let $F : (X, \pi_X) \to (Y, \pi_Y)$ and $G : (Y, \pi_Y) \to (Z, \pi_Z)$ be the fiber strong shape morphisms, defined by triples $(p, q, [f])$ and $(p', q', [g])$, where $p' : (Y, \pi_Y) \to Y'$, $q' : (Z, \pi_Z) \to Z$ and $[g] : Y' \to Z$.

As we know there exists an unique morphism $[h] : Y \to Y'$ of category $\text{CPHTop}_{B_0}$ such that $[h] [q] = [q']$. Note that

$$[j][q] = [q'] = [h] [q].$$

Hence, $[j] = [h]$. Besides, $[g] [j] = [g] [h] [1_Z]$.

Thus, we can assume that the morphisms $F$ and $G$ are given by triples $(p, q, [f])$ and $(q, r, [g])$.

Consequently, we can define the composition $GF : X \to Z$ as the morphism given
3.2. On Fiber Strong Shape Category for Arbitrary Topological Spaces

by triple \((p, r, [g] [f])\).

In the role an identity morphism \(I : X \to X\) we can take the morphism defined by triple \((p, p, [1_X])\).

The obtained category \(SSH_{B_0}\) call the fiber strong shape category.

Let \(X \in \text{ob}(SSH_{B_0})\). By symbol \(ssh_{B_0}(X)\) denote the equivalence class of topological space \((X, \pi_X)\) and call the fiber strong shape of \((X, \pi_X)\).

For each f.p. map \(\varphi : (X, \pi_X) \to (Y, \pi_Y)\) choose ANR\(_B\_0\)-resolutions \(p : (X, \pi_X) \to X\) and \(q : (Y, \pi_Y) \to Y\). There exists a unique morphism \([f] : X \to Y\) of category \(CPHTop_{B_0}\) such that \([q] [\varphi] = [f] [p]\).

We can define a functor \(SS'_{B_0} : \text{Top}_{B_0} \to SSH_{B_0}\). By definition,

\[
SS'(X) = X, \quad X \in \text{ob}(\text{Top}_{B_0})
\]

and

\[
SS'(\varphi) = \Phi, \quad \varphi \in \text{Mor}_{\text{Top}_{B_0}}(X, Y).
\]

Here \(\Phi\) is a fiber strong shape morphism defined by triple \((p, q, [f])\).

As in \([L-M]\) we can prove that functor \(SS'_{B_0}\) induces a functor \(SS_{B_0} : \text{HTop}_{B_0} \to SSH_{B_0}\), which we call the fiber strong shape functor. By definition,

\[
SS_{B_0}(X) = X, X \in \text{ob}(\text{HTop}_{B_0})
\]

and

\[
SS_{B_0}([\varphi]_{B_0}) = SS'(\varphi), [\varphi]_{B_0} \in \text{Mor}_{\text{HTop}_{B_0}}(X, Y).
\]

Let us define a functor \(S : SSH_{B_0} \to SH_{B_0}\). Assume that \(S(X) = X\) for each object \(X \in \text{ob}(SSH_{B_0})\). Let \(F : (X, \pi_X) \to (Y, \pi_Y)\) be a fiber strong shape morphism.
given by a triple \((p, q, [f])\).

Consider the morphism \(E([f])\) as an image of \([f]\) with respect the functor \(E : \text{CPHTop}_{B_0} \to \text{pro} - \text{HTop}_{B_0}\). The triple \((Hp, Hq, E[f])\) generates a fiber shape morphism, which we denote by \(S(F) : (X, \pi_X) \to (Y, \pi_Y)\).

Now we can formulate the following

**Theorem 3.2.5.** There exists a commutative diagram

\[
\begin{array}{ccc}
\text{HTop}_{B_0} & \xrightarrow{S_{B_0}} & \text{SH}_{B_0} \\
\downarrow \quad \quad \quad \downarrow S & & \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\text{SS}_{B_0} & \xrightarrow{S_{B_0}} & \text{SSH}_{B_0}
\end{array}
\]

where \(S_{B_0}\) is \(V.\text{Baladze fiber shape functor \([B_4]\)}\).

\[\square\]

**Corollary 3.2.6.** Let \((X, \pi_X)\) and \((Y, \pi_Y)\) be topological spaces over \(B_0\). If \(\text{ssh}_{B_0}(X) = \text{ssh}_{B_0}(Y)\), then \(\text{sh}_{B_0}(X) = \text{sh}_{B_0}(Y)\).

\[\square\]

**Remark 3.2.7.** Using the methods developed in this paper and papers (\([B_{10}], \ [L-M], \ [M_2], \ [M_3]\)) it is possible to construct fiber strong shape theory for category of arbitrary continuous maps.
Conclusion

The basic achievements made in the thesis are as follows:

1. The study of Borsuks fiber pairs and investigation of their properties.

2. The definition of fiber strong shape deformation retracts, so called SSDR\(_{B_0}\)-maps and investigation of their properties.

3. The definition of fibrant spaces over \(B_0\) and establishment of their properties.

4. The construction of fiber cotelescope of inverse sequence of spaces over \(B_0\) and study of their properties.

5. The construction of fiber strong shape classification of compact metric spaces by means of fiber cotelescope, fibrant spaces over \(B_0\) and fiber resolutions.

6. The characterization of fiber strong shape equivalences by means of double map cylinder.

7. The introduction of a concept of fiber strong ANR\(_{B_0}\)-extension and proof of its existence theorem.

8. The constructions of fiber strong shape category \(SSH_{B_0}\) of general topological spaces, the fiber strong shape functor \(SS_{B_0} : HTop_{B_0} \to SSH_{B_0}\), the functor \(S : SSH_{B_0} \to SH_{B_0}\) with values in \(V\). Baladzes fiber shape category \(SH_{B_0}\) and proof of the equality \(S \cdot SS_{B_0} = S_{B_0}\), where \(S_{B_0} : HTop_{B_0} \to SH_{B_0}\) is V.Baladze fiber shape functor \([B_4]\).
Bibliography


[Bat] M.A. Batanin, Categorical strong shape theory, Cahiers Topologie Gom. Diffen-


[BIBLIOGRAPHY]


