

RECENT DEVELOPMENTS IN GENERALIZED ANALYTIC FUNCTIONS AND THEIR APPLICATIONS

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Generalized Analytic Functions And Their Applications

Tbilisi, Georgia, 12 – 14 September 2011



Edited by **G.Giorgadze** (Tbilisi State University)

**Recent Developments
in
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Recent Developments In Generalized Analytic Functions And Their Applications. Ed. G.Giorgadze, 2011. Proceedings cover all topics of the International Conference on Generalized Analytic Functions and Their Applications organized by complex analysis group at the I.Vekua Institute of Applied Mathematics of the Tbilisi State University: functional analytic methods for first order elliptic systems on complex plane and Riemann surfaces, boundary value problems for analytic and generalized analytic functions and vectors, quasiconformal mappings, Fuchsian systems and holomorphic bundles on Riemann surfaces, analytic differential equations in complex domain, applications of generalized analytic functions in mathematical physics.

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Preface

According to the observation made by I. Vekua and L. Bers in the 1950s, it is possible to extend the main theorems of analytic functions to a wider class of functions than the space of analytic functions. Such different and unique approach of Vekua and Bers towards the issue gave us a new function space attributed with the best properties, the elements of which are today known as the generalized analytic functions (И.Веква, *Обобщенные аналитические функции*. Москва, 1959) or the pseudo-analytic functions (L.Bers, *Theory of Pseudo-Analytic Functions*, New York, 1953).

I. Vekua's concept which is based on the theory of first order linear elliptic systems on a complex plane has proved to be fruitful not only in terms of the theory of functions, but for its application in the related fields of science. Rewriting of elliptic systems in a complex form and presenting of the solution analytically as proposed by Vekua, widened the range of application of the boundary value problems of the analytic function theory, threw new light on the possibility of extending the theory on vector functions; using the generalized analytic functions it became also possible to make a complete analysis of the Beltrami equation, which in its turn is an important tool of classification of analytic manifolds, classical theory of mathematical physics - theory of shells and the modern branches, conformal and topological field theories and Yang-Mills theory.

Nowadays many leading mathematical centers conduct processing and extension of the generalized analytic functions theory or its methods, as well as the study of mathematical physics using this theory. Among them is the I. Vekua Institute of Applied Mathematics of the Tbilisi State University, where in various periods outstanding researches were carried out in the department of "Complex Analysis and Applications" founded by I.Vekua himself, which was chaired by G.Manjavidze later. Currently the scientific group of complex analysis of the Institute (G. Akhalaia, G. Giorgadze, E. Gordadze, V. Jikia, N. Kaldani, G. Makatsaria, N. Manjavidze) is focusing on irregular elliptic systems on Riemann surfaces and Riemann-Hilbert boundary value problem for such systems, namely: the qualitative research of systems of singular elliptic differential equations; the study of local and global influence of isolated and nonisolated singularities of Carleman-Bers-Vekua systems in corresponding spaces, the investigation of the space of solutions of elliptic systems (pseudo-analytic, polyanalytic) in the neighborhood of the singular point, including the case, when the singular point of the equation

is a branched point of the solution; classification of elliptic systems with respect to the singular points; obtaining analogs of Riemann-Hurwitz formula on compact Riemann surfaces.

In case of Fuchsian systems, the invariants (total Chern number, splitting type) of induced holomorphic bundles with meromorphic connections are naturally related to the numerical invariants (index, partial indices) of Riemann-Hilbert boundary value problem. In this context it is interesting to investigate elliptic systems with the first order poles together with factorization of piecewise constant matrix-function in various weighted spaces and boundary value problem of linear conjugation for general open curves in Lebesgue spaces with variable exponent. Note that the boundary values of Cauchy-type integral may have different behavior in different points of curve, which is better pointed by the variable exponent than by the constant one. The boundary value problem of linear conjugation for Q -holomorphic vector-functions in particular case when Q matrix satisfies the so-called commutative condition has not been studied until now. The investigation space of holomorphic sections of an induced holomorphic bundles as well as the space of solutions of the boundary value problem of linear conjugation applying sheaf theory methods is also a very attractive and perspective problem.

It is well known that the matrix Riemann-Hilbert boundary value (linear conjugation) problem is naturally connected with the problem of factorization of matrix-functions. The latter leads to an important geometric interpretation of the holomorphic vector bundle on the Riemann sphere. The space of solutions of boundary value problems coincides with the space of holomorphic sections of the bundle. The partial indices of matrix functions and the index of the problem correspondingly, represent the splitting type and the total Chern number of holomorphic bundle. Such an approach allows one to state the Riemann-Hilbert problem and to solve it on compact Riemann surfaces, to replace the matrix function by a loop in a Lie group, to calculate the partial indices in two- and three-dimensional cases in terms of numerical invariants of deformation of the holomorphic bundle.

The Riemann-Hilbert monodromy problem (the Hilbert 21st problem) is to construct the Fuchsian system by the marked points and given non-degenerate matrices, for which the marked points are the poles of the system and the monodromy matrices coincide with given matrices. This problem was first posed by Riemann in one of his last works and was solved by Hilbert in one particular

case. He inserted the general problem in his famous problems list. Till the 1990-s it was considered that Plemelj solved the 21st Hilbert problem. Despite the fact that some authors observed the mistake in the solution proposed by Plemelj, it was considered that this inaccuracy could be rectified and the final result would be correct. A.Bolibruh constructed the counter example and showed that this is not true and the solution of the 21st Hilbert problem strongly depends on the apriori given matrices, i.e. on the monodromy representation. This means that solution of the problem depends on the complex/conformal structure of Riemann surface with marked points. In turn the complex structure of Riemann surface is defined by the Beltrami equation. For these reasons we believe describing the solvability of Riemann-Hilbert monodromy problem in terms of the Beltrami differential is an interesting problem.

As is well known, the complex functions on punctured Riemann sphere can be described in terms of moduli space of polygonal linkages. The arising interplay between the Riemann-Hilbert monodromy problem and moduli spaces of linkages seems interesting and promising.

The Riemann-Hilbert boundary value problem (the problem of linear conjugation) was posed by Riemann in the same work as one of the methods for solving of the monodromy problem. Plemelj followed the same way. He reduced the boundary value problem with piecewise constant boundary matrix to the problem with continuous boundary matrix-function and after solving this problem he constructed the desired system of equations. Likely the inaccuracy made by Plemelj was the result of his insufficient manipulation techniques on matrix differential and integral equations. Later on such techniques were developed by I.Lappo-Danilevski for the 21st Hilbert problem and by N.Muskhelishvili and N. Vekua in their joint work for the problem of linear conjugation. Since then these two problems are studied independently.

In spite of the fact that the problem of linear conjugation was studied for the generalized analytic functions and vectors and the progress achieved in research of singular elliptic systems, the connection between Riemann-Hilbert problem with singular elliptic systems has not been noted until now. In our view this is a very important problem of the theory of generalized analytic functions.

The Riemann-Hilbert monodromy problem induces holomorphic bundle with meromorphic connections and permits to give the necessary and sufficient

solvability conditions for the 21st Hilbert problem. In some weak form the analogous result applies to Riemann surfaces and functions with values in complex Lie groups. In our opinion, the statement and investigation of the monodromy problem for some subclasses of elliptic systems is one of the most relevant and important problems for the study of solution space of elliptic systems, as well as for constructing the L_p -connections and investigation of the moduli spaces. The results obtained in this direction make it possible to extend the range of application of elliptic systems in mathematical physics and technique: elasticity theory, conformal, gauge, topological field theories and quantum computation.

Leading scientists of the field worldwide have been invited to discuss the above stated problems in the framework of the conference organized by the complex analysis group at the I. Vekua Institute of Applied Mathematics. This book completely covers the subjects of the conference. Here are presented the authors of four generations, among them are the students of the Tbilisi State University. The collection of works was ready to be published when we learned about the tragic decease of Prof. A. Timofeev. His last work is included in the proceedings in memory of our dear colleague and friend.

We would like to particularly underline the great contribution of the author of classical works in the theory of generalized analytic functions Prof. B. Bojarski to the support and promotion of the conference. The book contains one of his works representing one of the best models of transparency of statements, history of the flow of scientific thinking in time and didactics.

We would like to thank the distinguished experts, authors of important contributions to the topic V.Adukov, H. Begehr, V. Kravchenko, S.Krushkal, V.Palamodov, S.Plaksa, S.Rogosin, R.Saks, W. Tutschke for kind consideration and moral support.

G.Giorgadze, G.Khimshiashvili
Tbilisi, September 2011

Functional classes for generalized Beltrami systems*

G. Akhalaia¹ and N. Manjavidze²

Abstract

The functional classes of generalized analytic vectors for generalized Beltrami systems are introduced and investigated. Some properties of these classes which turned to be useful in order to solve the discontinuous boundary value problems are established.

In the work of Bojarski [3] was shown that the methods of generalized analytic functions are admitting further generalization on case of the first order elliptic systems the complex form of which is the following

$$\partial_{\bar{z}}\omega - Q(z)\partial_z\omega + A\omega + B\bar{\omega} = 0, \quad (1)$$

$\partial_z \equiv \frac{1}{2}(\partial_x - i\partial_y)$, $Q(z)$, $A(z)$, $B(z)$ are given square matrices of order n , $Q(z)$ is a matrix of the special quasi-diagonal form, $Q(z) \in W_p^1(C)$, $p > 2$, $|q_{ii}| \leq q_0 < 1$, $Q(z) \equiv 0$ outside of some circle, A , B are bounded measurable matrices.

In these works by the full analogy with the theory of generalized analytic functions are given the formulas of general representation of regular solutions of the system (1), the so-called generalized analytic vectors. On this basis the boundary value problems of Riemann-Hilbert and linear-conjugation in case of Holder-continuous coefficients are considered. These results and some further development of the theory of generalized analytic vectors are presented in the monograph [4].

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Consider the first order system of partial differential equations in the complex plane C

$$w_{\bar{z}} = Q(z) w_z, \quad (2)$$

where Q is abovementioned matrix.

Following Hile [5] if Q is self-commuting in C , which means

$$Q(z_1) Q(z_2) = Q(z_2) Q(z_1),$$

for any $z_1, z_2 \in C$ and $Q(z)$ has eigenvalues with the modulus less than 1 then the system (2) is called generalized Beltrami system. Solutions of this equation are called Q -holomorphic vectors. Under the solution in some domain D we understand so-called regular solution [7]. Equation (2) is to be satisfied almost everywhere in D .

The matrix valued function $\Phi(z)$ is a generating solution of the system (2) if it satisfies the following properties [5]:

- (i) $\Phi(z)$ is a C^1 solution of (2) in C ;
- (ii) $\Phi(z)$ is self-commuting and commutes with Q in C ;
- (iii) $\Phi(t) - \Phi(z)$ is invertible for all z, t in C , $z \neq t$;
- (iv) $\Phi(z)$ is invertible for all z in C .

We call the matrix

$$V(t, z) = \partial_t \Phi(t) [\Phi(t) - \Phi(z)]^{-1}$$

the generalized Cauchy kernel for the system (2).

Let now Γ be a union of simple closed non-intersecting Liapunov-smooth curves, bounding finite or infinite domain. If Γ is one closed curve then D^+ denotes the finite domain; if Γ consists of several curves then by D^+ we denote the connected domain with the boundary Γ , on these curves the positive direction is chosen such, that when moving to this direction D^+ remains left; the complement of open set $D^+ \cup \Gamma$ in the whole plane denote by D^- .

Consider the following integral

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} V(t, z) d_Q t \varphi(t), \quad (3)$$

where $\varphi(t) \in L(\Gamma)$, $d_Q(t) = I dt + Q d\bar{t}$, I is an identity matrix. It is evident, that (3) is Q -holomorphic vector everywhere outside of Γ , $\Phi(\infty) = 0$. We

call the integral (3) the generalized Cauchy-Lebesgue type integral for the system (2) with the jump line Γ .

The boundary values of $\Phi(z)$ on Γ are given by the formulas:

$$\Phi^\pm = \pm \frac{1}{2} \varphi(t) + \frac{1}{2\pi i} \int_{\Gamma} V(\tau, t) d_Q \tau \mu(\tau). \quad (4)$$

The formulas (4) are to be fulfilled almost everywhere on Γ , provided that Φ^\pm are angular boundary values of the vector $\Phi(z)$ and the integral in (4) is to be understood in the sense of Cauchy principal value.

Theorem 1 *Let $\Phi(z)$ be a Q -holomorphic vector on the plane cut along Γ , $\Phi(\infty) = 0$. Let $\Phi(z)$ have the finite angular boundary values Φ^\pm . The vector $\Phi(z)$ is represented by the Cauchy-Lebesgue type integral (3) if and only if the following equality*

$$\frac{1}{\pi i} \int_{\Gamma} V(t, t_0) d[\Phi^+(t) - \Phi^-(t)] = \Phi^+(t_0) + \Phi^-(t_0) \quad (5)$$

is fulfilled almost everywhere on Γ .

Introduce some classes of Q -holomorphic vectors. Let

$$\rho(t) = \prod_{k=1}^r |t - t_k|^{\rho_k}, \quad -\frac{1}{p} < \rho_k < \frac{1}{p^*} \quad p^* = \frac{p}{p-1}, \quad (6)$$

$$k = 1, \dots, r.$$

t_k are some fixed points on Γ .

We say that the Q -holomorphic vector $\Phi(z)$ belongs to the class

$$E_p(D^+, \rho, Q) | E_p(D^-, \rho, Q), p > 1,$$

if $\Phi(z)$ is represented by generalized Cauchy-Lebesgue type integral in the domain D^+ (D^-) with the density from the class $L_p(\Gamma, \rho) = \langle \varphi | \rho \varphi \in L_p(\Gamma) \rangle$. It follows from (6) that $E_p(D^\pm, \rho, Q) \subseteq E_{1+\varepsilon}(D^\pm, Q)$ for sufficiently small positive ε .

The following theorems are valid [1,2]:

Theorem 2 *If $Q \in E_p(D^\pm, \rho, Q)$ then it can be represented by generalized Cauchy-Lebesgue integral with respect to its angular boundary values.*

Theorem 3 Let Q -holomorphic vector $\Phi(z)$ is represented by generalized Cauchy-Lebesgue type integral in the domain D^+ (D^-) with the summable density. If the angular boundary values Φ^+ (Φ^-) belong to the class $L_p^n(\Gamma, \rho, Q)$ for some weight function (6) then $\Phi(z) \in E_p(D^+, \rho, Q)$ ($\Phi(z) \in E_p(D^-, \rho, Q)$).

Theorem 4 Let D be a domain of the complex plane bounded by the union of simple closed non-intersecting Liapunov curves $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_m$, $\Gamma_1, \dots, \Gamma_m$ are situated outside of each other but inside of Γ_0 . If $Q \in E_p(D, \rho, Q)$ then it admits the following representation

$$\Phi(z) = \frac{1}{\pi i} \int_{\Gamma} V(t, z) d_Q t \mu(t) + i C, \quad (7)$$

where $\mu(t) \in L_p(\Gamma, \rho)$ is real vector, $i C$ is real constant vector. The vector $\mu(t)$ is defined on Γ_j , $j \geq 1$, uniquely within the constant vector, $\mu(t)$ on Γ_0 and the constant vector C are defined by $\Phi(z)$ uniquely.

Theorem 5 Let D be the domain defined as in above theorem. If $\Phi(z) \in E_{1+\varepsilon}(D, Q)$ and $\operatorname{Re} \Phi^+(t)(\operatorname{Im} \Phi^+(t))$ belongs to the class $L_p(\Gamma, \rho)$, $p > 1$, ρ has the form (6), then $\operatorname{Im} \Phi^+(t)(\operatorname{Re} \Phi^+(t))$ also belongs to the class $L_p(\Gamma, \rho)$.

On the basis of introduced and investigated weighted Cauchy-Lebesgue classes for the generalized analytic vectors can be considered the discontinuous boundary value problems of generalized analytic vectors since they are natural classes for such problems. Similarly, as in case of analytic functions[6], we mean the problems when the desired vectors in considered case have the angular boundary values almost everywhere on boundary Γ and the boundary conditions are fulfilled almost everywhere on Γ . In this connection given coefficients of the boundary conditions are piecewise-continuous non-singular matrices.

For example in our view the Riemann-Hilbert type discontinuous boundary value problems can be solved by means of these classes. Reducing these problems to the corresponding singular integral systems one can establish the solvability criterions and index formulas of corresponding functional classes. While the investigation of such problems are appearing some difficulties connected with the fact that the Liouville theorem is not valid in general as well as the unique-ness theorem. In most cases these difficulties may be successfully avoided.

References

- [1] Akhalaia G., Manjavidze N. The Cauchy-Lebesgue type integrals for the generalized Beltrami systems. *Seminar of I. Vekua Institute of Applied Mathematics, Reports*, **35** (2009), 52–55.
- [2] Akhalaia G., Makatsaria G., Manjavidze N., On some qualitative issues for the first order elliptic systems in the plane. *Progress in Analysis and Its Application. Proc. 7th Int. ISAAC Congress, Imperial College, London, UK, 13-18 July 2009*, pp.67-74
- [3] Bojarski B., Theory of generalized analytic vectors. (Russian) *Ann. Polon. Math.* **17** (1966), 281–320.
- [4] Gilbert R., Buchanan J., First order elliptic systems. A function theoretic approach. *Mathematics in Science and Engineering*, 163. *Academic Press, Inc., Orlando, Fl.*, 1983.
- [5] Hile G. N., Function theory for generalized Beltrami systems. *Contemporary Math.* **11** (1982), 101–125.
- [6] Khvedelidze B. V., Linear discontinuous boundary value problems of the theory of functions, singular integral equations and some applications. (Russian) *Proc. Tbilisi Math. Inst.* **23** (1957), 3–158.
- [7] Vekua I., Generalized analytic functions. *Pergamon, Oxford*, 1962.

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On a principle in the theory of complex polynomials implying Gauss-Lucas theorem

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Abstract

A phenomenon for an arbitrary complex polynomial P is revealed showing that any cluster of zeros of P (even of very few zeros) attracts, in a sense, zeros of $P^{(k)}$. The results imply Gauss-Lucas theorem and are closely connected with Crace-Heawood's theorem and Walsh's two circle theorem.

In what follows D is a given convex domain whose boundary ∂D has continuous curvature.

We define the following *distance* $\Delta \equiv \Delta(D, z_1, z_2, \dots, z_N)$ of a given set of points $z_1, z_2, \dots, z_N \in \mathbb{C}$ with respect to $\bar{D} := D \cup \partial D$ as

$$\Delta := N - \frac{1}{2\pi} \sum_{i=1}^N \alpha(z_i),$$

where $\alpha(z_i) = 2\pi$ for any $z_i \in \bar{D}$ and for any $z_i \notin \bar{D}$ the magnitude $\alpha(z_i)$ is two times the angle under which the domain D is seen from the point z_i .

Comment 1. First we observe that $\Delta \in [0, N)$ meanwhile $\Delta = 0$ means that all points z_1, z_2, \dots, z_N lie in \bar{D} , (that is these points are "maximally close" to \bar{D}) and $\Delta \rightarrow N$ means that all these points z_1, z_2, \dots, z_N tend to infinity (that is these points are "maximally far from \bar{D} "). In general the closer (in average) the set z_1, z_2, \dots, z_N to the domain D the smaller is Δ .

In terms of this distance we establish a phenomenon for arbitrary complex polynomial $P(z)$ which, speaking qualitatively, can be expressed as follows:

the clusters of zeros of $P(z)$ attracts the sets of zeros of the derivatives $P^{(k)}(z)$.

Let $P(z)$ be polynomial of degree n . Speaking about zeros of P and $P^{(k)}$ we always count them according to their multiplicity. Denote by n_D the number of zeros of P in \bar{D} , by $z_2^{(k)}, \dots, z_{n-k}^{(k)}$ the zeros of $P^{(k)}(z)$ and by $\Delta^{(k)}(P)$ the magnitude Δ written for these zeros.

Theorem 1 (on mutual locations of zeros of P and $P^{(k)}$). *For an arbitrary polynomial $P(z)$ of degree n , arbitrary integer $k \in [1, \dots, n-1]$ and arbitrary convex domain D whose boundary has continuous curvature and does not pass through any zero of $P^{(k)}$, $k = 0, 1, \dots, n-1$, we have*

$$\Delta^{(k)} \leq n - n_D. \quad (1)$$

Theorem 1 gives an upper bounds for $\Delta^{(k)}$, what, in turn, means that for the zeros $z_1^{(k)}, z_2^{(k)}, \dots, z_{n-k}^{(k)}$ are, to certain extend, close to the domain D provided that $k \geq n - n_D$, see Comment 1. Speaking qualitatively we can express the observed phenomenon as follows: zeros of polynomials attract zeros of its derivatives. In the extremal case when $n_D = n$ we have $\Delta^{(k)} = 0$ (see comment 1) so that inequality (1) implies the following

Corollary 1 (Gauss-Lukas' theorem). Any \bar{D} involving all zeros of P involves also all zeros of $P' \text{ }^1$.

We see that the Gauss-Lucas theorem reveals in fact the same phenomenon (closeness of the zeros of P' to the zeros of P) but in the mentioned extremal case merely, when $n_D = n$

In another particular case we have the following well known Grace-Heawood theorem asserting that *any polynomial P of the degree n satisfying $P(-1) = P(1) = 0$ has a zero of P' in $|z| \leq \cot(\pi/n)$* . Notice that $\cot(\pi/n) \geq 5$ in the case when $n > 4$. Thus in this main case ($n > 4$) the Grace-Heawood theorem deals "silently" with the polynomials non admitting on the segment $J := \{(x, y) : x \in [-1, 1], y = 0\}$ multiple zeros of P in -1 and 1 and also non admitting zeros of P' on J .

We put now the following much wider problems of the same spirit.

Problem 1. Assume P has n_J zeros on J and $P^{(k)}$ has $n_J^{(k)}$ zeros on J . How close to J will be other zeros of $P^{(k)}$?

¹In the Gauss-Lukas' theorem \bar{D} is the convex hull of all zeros of P (unlike our case when D is simply arbitrary convex domain). We will not touch anymore this negligible difference.

Notice that Problem 1 widens in different directions the previous Grace-Heawood's setting: $k \geq 1$, J may involve more than two zeros of P and J may involve also zeros of $P^{(k)}$.

In turn this problem is a very particular case of the following

Problem 2. Assume P has n_D zeros on \bar{D} and $P^{(k)}$ has $n_D^{(k)}$ zeros on \bar{D} . How close to \bar{D} will be other zeros of $P^{(k)}$?

Notice that we can take as D an ellipsis whose small diameter tends to zero and respectively D tends to J so that the Problem 2 is a much wider version of the Problem 1.

Theorem 1 permits gives solution of the Problem 2. Indeed. Assume that $n_D - n_D^{(k)} - k > 0$, where $n_D^{(k)}$ stands clearly for the number of points $z_i^{(k)} \in \bar{D}$. Due to inequality (1) we have $n - k - \frac{1}{2\pi} \sum_{i=1}^{n-k} \alpha(z_i^{(k)}) \leq n - n_D$ or, taking into account that $\frac{1}{2\pi} \sum_{i=1}^{n-k} \alpha(z_i^{(k)}) = n_D^{(k)} + \frac{1}{2\pi} \sum_{i=1}^{n-n_D^{(k)}-k} \alpha(z_i^{(k)}(*))$, where $z_i^{(k)}(*)$ stands for $z_i^{(k)} \notin \bar{D}$. From here

$$\frac{1}{2\pi} \sum_{i=1}^{n-n_D^{(k)}-k} \alpha(z_i^{(k)}(*)) \geq n_D - n_D^{(k)} - k. \quad (2)$$

Consider the domain $G := G(n_D, n_D^{(k)}, k)$ determined by

$$\alpha(z) \leq \frac{2\pi [n_D - n_D^{(k)} - k]}{n - n_D^{(k)} - k} \quad (3)$$

and observe that either all $z_i^{(k)}(*) \notin \bar{D}$ lie on the boundary ∂G (then we have equality both in (3) and in (2)) or we have a point $z_i^{(k)}(*) \notin \bar{D}$ which lies inside G . Thus in any case we have a point $z_i^{(k)}(*) \notin \bar{D}$ which lies in \bar{G} and we obtain the following

Corollary 2 (solution of the Problem 2). Assume that $n_D - n_D^{(k)} - k > 0$. Then there is a zero of $P^{(k)}$ lying in $\overline{G(n_D, n_D^{(k)}, k)} \setminus \bar{D}$.

Now if we assume that our domain D_h is an ellipse centered at the origin whose large diameter is the above defined segment J with endpoints at $(-1, 0)$ and $(1, 0)$ and whose small diameter is equal to h . Assume that $n_{D_h} - n_{D_h}^{(k)} - k > 0$. Notice that when h tends to zero the ellipse D_h tend to J . Then n_{D_h} and $n_{D_h}^{(k)}$ become respectively the number n_J of zeros of P and the number

$n_J^{(k)}$ of zeros of $P^{(k)}$ lying on the diameter J . The corresponding domain G_h tends then to a limit domain $G^* := G^*(n_J, n_J^{(k)}, k)$ (which is of view in the right figure). Notice that this case (zeros lie on a segment) is widely studied in the theory of polynomials.

Corollary 2 implies then

Corollary 3 (solution of the Problem 1). Assume that $n_J - n_J^{(k)} - k > 0$. Then there is a zero of $P^{(k)}$ lying in $\overline{G^*(n_J, n_J^{(k)}, k)} \setminus J$.

Now we show that a very particular case this corollary closely relates to the above mentioned Grace-Heawood theorem. Indeed as it explained above Grace-Heawood theorem deals with the case when $k = 1$, $n_J = 2$ and $n_J^{(k)} = 0$ which determines corresponding particular type domain $G^*(2, 0, 1)$. Corollary 3 implies then the following corollary in the spirit of Grace-Heawood theorem.

Corollary 3₁. Any polynomial P of the degree n with $P(1) = P(-1) = 0$ has a zero of P' in the closure of $G^*(2, 0, 1)$.

It should be stressed however, that unlike Grace-Heawood theorem dealing with the case $n_J = 2$, $n_J^{(k)} = 0$, $k = 1$, Corollary 3 deals with much larger combination of values n_J , $n_J^{(k)}$ and k .

Comments. Thus we see, that both Gauss-Lucas theorem and Grace-Heawood theorem also show a closeness of locations of the zeros of P' to certain cluster of the zeros of P : maximal cluster of the zeros (n zeros) in the case of Gauss-Lucas theorem and minimal cluster of zeros (two zeros merely) in the case of Grace-Heawood theorem. Of the same nature is known Walsh's two circle theorem, which deals with the case when all zeros of P lie in two different disks. However it should be stressed that Theorem 1 and Corollaries 2 and 3 deal with arbitrary clusters of zeros and thus we have a general phenomenon which can be expressed as follows: any cluster of some zeros of a given complex polynomial P attracts, in a sense, zeros of $P^{(k)}$.

Theorems 1 we derive from the following result related to arbitrary meromorphic functions.

Theorem 2. For an arbitrary meromorphic function $w(z)$ in a given simply connected domain d with continuous curvature $c(s)$, $s \in \partial D$, and arbitrary integer k we have

$$n(D, 0, w) - n(D, \infty, w) \leq \frac{1}{2\pi} \int_0^1 \left| (\arg(w^{(k)}(z(t)))'_t \right| dt + \frac{k}{2\pi} \int_{\partial D} |c| ds, \quad (3)$$

Proofs of Theorems 1-2. We derive first Theorems 1 and 2 from Theorem 3. It is known that for any convex D with continuous curvature

$$\int_{\partial D} |c| ds = 2\pi. \quad (4)$$

Also when w is a polynomial P we have $n(D, \infty, P) = 0$ and under the hypotheses of Theorem 1 we have also

$$\int_0^1 \left| (\arg(P^{(k)}(z(t)))'_t \right| dt \leq \sum_{i=1}^{n-k} \int_{\partial D} \left| d \arg \left(z - z_i^{(k)} \right) \right|.$$

so that applying (3) we have

$$\begin{aligned} n_D := n(D, 0, w) &\leq \frac{1}{2\pi} \sum_{i=1}^{n-k} \int_{\partial D} \left| d \arg \left(z - z_i^{(k)} \right) \right| + \frac{k}{2\pi} \int_{\partial D} |c| ds = \\ &\frac{1}{2\pi} \sum_{i=1}^{n-k} \int_{\partial D} \left| d \arg \left(z - z_i^{(k)} \right) \right| + k. \end{aligned}$$

This can be rewritten as

$$n - k - \frac{1}{2\pi} \sum_{i=1}^{n-k} \int_{\partial D} \left| d \arg \left(z - z_i^{(k)} \right) \right| \leq n - n_D$$

and observing that the left hand side here is $\Delta^{(k)}$ we obtain Theorem 1.

To prove Theorem 2 we make use the argument variation principle:

$$n(D, 0, w) - n(D, \infty, w) = \frac{1}{2\pi} \int_{w \in \partial D} d \arg w.$$

If the boundary curve of ∂D is written in the form $z(t) = x(t) + iy(t)$, $t \in [0, 1)$, we have

$$\frac{1}{2\pi} \int_{w \in \partial D} d \arg w \leq \frac{1}{2\pi} \int_0^1 \left| (\arg w(z(t)))'_t \right| dt.$$

The last integral (total variation of $\arg z$ on ∂D) can be estimated by the total boundary rotation on ∂D , that by $\int_0^1 \left| (\arg(w(z(t)))'_t \right| dt$: due to Theorem

11.5.6 in [3] (2002), see pages 387-389, for any smooth loop γ in $\mathbb{C} \setminus 0$ the total variation of $\arg z$ on γ does not exceed the total boundary rotation on γ ². Applying this to the curve $w(z(t))$, $t \in [0, 1)$, after some simple calculations we get Theorem 3.

References

- [1] Barsegian G., *On geometric structure of image of disks under mappings by meromorphic functions*, Math. Sbornic, v. 106(148), n. 1, 1978, p.35-43.
- [2] Barsegian G., *Some interrelated results in different branches of geometry and analysis*, p.3-33, in book "Further progress in analysis. Proceedings of the 6th International ISAAC Congress, Ankara, Turkey, 13 – 18 August 2007", editors Begehr H., Celebi O. and Gilbert R., World Scientific, 2009.
- [3] Sheill-Small R., *Complex Polynomials*, (Ser. Cambridge Studies in Advanced Mathematics, No. 75, 2002).

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²Similar result has been established long ago in 1978 [1] and in much more general setting: for estimating the sum of total variation of $\arg(z - a_\nu)$ on γ , where $a_\nu \in \mathbb{C}$, $\nu = 1, 2, \dots, q$. The result in [1] was formulated for curves $w(\{z : |z| = r\})$ merely but the proof was valid for any smooth loop γ and for $q = 1$ (as in [3]). More details, applications and the most general wording can be found in [2].

On the proximity property of meromorphic functions.

Initiating some novel studies in the complex analysis

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Abstract

The classical Nevanlinna theory (1920s) and Ahlfors theory (1935) describe numbers of a -points of meromorphic functions. These theories are considered as some culminations of numerous other studies related to the numbers of a -points of different classes of meromorphic functions. The next stage was obviously studying of geometric locations of these a -points. At the end of 1970s so called proximity (closeness) property of a -points of meromorphic functions has been established describing the locations of the a -points (instead of numbers merely). Moreover, it turned out that this property implies the key conclusions of Nevanlinna-Ahlfors theories. In the present paper we give a new simplified wording of this property.

1 Introduction

In what follows we denote by $w(z)$ a given meromorphic function in the plane. The proximity property a -points of w has been established in different forms: first in 1978 as a consequence of some results related to Gamma-lines [2], and then as a further development of Ahlfors' theory [3] (1983), [4], (1985). The property describes the geometry of a -points of w by revealing the following phenomenon: for a given set of pairwise different values a_1, a_2, \dots, a_q , $q > 2$, the a_1, a_2, \dots, a_q -points of w locate mainly close to each other for different

$a_i \neq a_j$. Notice that the classical Nevanlinna-Ahlfors theories ([5], [1],) deal with the numbers of a -points merely and their key conclusion looks rather similar: the numbers of a_1, a_2, \dots, a_q -points of w are mainly close for different $a_i \neq a_j$. By the way this conclusion follows also from the proximity property. Thus the proximity property initiates investigations of locations of a -points and implies the known results related to the numbers of a -points.

However, the version of this property presented in [3] is much more complicated than the key conclusions of Nevanlinna-Ahlfors theories which this property implies. This is why we offer here another simplified (but more weak) version of this property which is easy to understand even for the very beginners.

2 The key results of Ahlfors theory

Let $w(z)$ be a meromorphic function in \mathbb{C} and let a_1, a_2, \dots, a_q , be distinct complex values. We make use standard notations: $\bar{n}(r, a_\nu)$ is the number of a -points (multiplicities are not counted) in $D(r) := \{z : |z| < r\}$ and $n_0(r, a_\nu)$ is the number of simple a -points in $D(r)$, $A(r)$ is Ahlfors-Shimizu characteristic, that is

$$A(r) = \frac{1}{\pi} \int \int_{D(r)} \frac{|w'|^2}{(1 + |w|^2)^2} d\sigma.$$

In what follows we denote by E some sets of finite logarithmic measure on axis; they are different in different cases.

Theorem A (Ahlfors' second fundamental theorem, [1]). *For any meromorphic function $w(z)$ in C and distinct complex values a_1, a_2, \dots, a_q , $q > 2$, we have*

$$\sum_{\nu=1}^q \bar{n}(r, a_\nu) \geq (q - 2)A(r) - o(A(r)), \quad r \rightarrow \infty, \quad r \notin E. \quad (1)$$

For $q > 4$ we have

$$\sum_{\nu=1}^q n_0(r, a_\nu) \geq (q - 4)A(r) - o(A(r)), \quad r \rightarrow \infty, \quad r \notin E. \quad (2)$$

Integration of (1) and (2) gives corresponding key results in Nevanlinna theory, see [1] or [5], chapter 13.

Below we give a new

Simplified version of the proximity property a -points of meromorphic functions.

This is simply an extraction from [3].

We refer the set c_i , $i = 1, 2, \dots$ as a set of *proper cluster* of a_1, a_2, \dots, a_q -points of w if different c_i have no common points, any c_i consists of a_1, a_2, \dots, a_q -points and involves each a_ν -point for any $\nu = 1, 2, \dots, q$ not more that one time; here multiple points are counted only one time. For any cluster c we denote by $n(c)$ the number of elements in c ; here multiple points are counted only one time. Notice that due to the definition $n(c) \leq q$ for any c . Denote by $d(c)$ the diameter of c , that is maximal distance between elements in c . Notation $[x]'$ stands for the greatest integer not exceeding x ,

Theorem 1 (Simplified proximity phenomenon). *Let $w(z)$ be a meromorphic function in \mathbb{C} , $a_1, a_2, \dots, a_q \in \overline{\mathbb{C}}$, $q > 2$, be a set of distinct complex values, $\varphi(r)$ be a monotone function tending to $+\infty$ (as slowly as we please). Then in any $D(r)$ there are $[A(r)]'$ proper clusters $c_i(r)$ of a_1, a_2, \dots, a_q -points of w , $i = 1, 2, \dots, [A(r)]'$, such that*

$$\sum_{i=1}^{[A(r)]'} n(c_i(r)) \geq (q-2)A(r) - o(A(r)), \quad r \rightarrow \infty, \quad r \notin E \quad (3)$$

and for any $i = 1, 2, \dots, [A(r)]'$

$$d(c_i(r)) \leq \varphi(r) \frac{A^{1/2}(r)}{r}. \quad (4)$$

Sharpness. Notice that $\sum_{i=1}^{[A(r)]'} n(c_i) \leq \sum_{\nu=1}^q \bar{n}(r, a_\nu)$, consequently (3) implies (*), that is implies Ahlfors' second fundamental theorem which is sharp. Sharpness of (4) (up to the multiplier $\varphi(r)$) can be easily checked for the double periodic functions.

3 A new aspect in distribution of a -points.

Notice that Theorem B deals with essentially new aspects which were not touched in Nevanlinna-Ahlfors' theories. In inequality (3) we deal with essentially the same a_1, a_2, \dots, a_q -points (as in the classics) but regrouped into some clusters, which have, in average from $q-2$ till q points. In addition inequality (4) shows that all a_1, a_2, \dots, a_q -points occurring in each cluster

should be close to each other. Thus we obtain the proximity (or closeness) phenomenon for meromorphic functions which qualitatively can be expressed as follows: these set a_1, a_2, \dots, a_q -points consists of $[A(r)]'$ proper clusters of close to each other different a_1, a_2, \dots, a_q -points. Notice that for w with lower order greater than 2 the diameters $d(c_i(r))$ of the clusters occurring in $D(r)$ tend to zero when r tend to infinity. Respectively tend to zeros distances between a_1, a_2, \dots, a_q -points occurring in each of these clusters. The more large is characteristic of w (or what is the same the lower order of w) the more stronger tend to zero these distances.

Theorem 1 generalizes inequality (1) of Theorem A. The following result generalizes similarly inequality (2) dealing with simple a -points merely.

Theorem 2 (Simplified proximity phenomenon for simple a -points merely). *Let $w(z)$ be a meromorphic function in \mathbb{C} , $a_1, a_2, \dots, a_q \in \tilde{\mathbb{C}}$, $q > 4$, be a set of distinct complex values, $\varphi(r)$ be a monotone function tending to $+\infty$ (as slowly as we please). Then in any $D(r)$ there are $[A(r)]'$ proper clusters $c_i^0(r)$ of simple a_1, a_2, \dots, a_q -points of w , $i = 1, 2, \dots, [A(r)]'$, such that*

$$\sum_{i=1}^{[A(r)]'} n(c_i^0(r)) \geq (q-4)A(r) - o(A(r)), \quad r \rightarrow \infty, \quad r \notin E$$

and for any $i = 1, 2, \dots, [A(r)]'$

$$d(c_i^0(r)) \leq \varphi(r) \frac{A^{1/2}(r)}{r}.$$

References

- [1] Ahlfors L., *Zur Theorie Überlagerungsflächen*, Acta Soc. Sci. Fenn., v.1, n.9, 1930, p. 1-40.
- [2] Barsegian G. *New results in the theory of meromorphic functions*, Dokl. Acad. Nauk SSSR, v.238, n.4, 1978, p.777-780.
- [3] Barsegian G., *Proximity property of a -points of meromorphic functions*, Math. Sbornic, vol. 120(162), n. 1, 1983, p. 42-63, (in Russian); Transl. in "Math. USSR Sbornik", n. 1, 1984, p. 39-60.

- [4] Barsegian G., *Proximity property of meromorphic functions and the structure of the univalent domains on Riemann surfaces*, Izvestia Acad. Nauk Armenii, Matematika, v. 20, n. 5, (1985), 375–400; v. 20, n. 6, (1985), 407–425, (in Russian); Transl. in "Journal of Contemporary Math. Analysis", Armenian Acad. of Sci., Allerton Press.
- [5] Nevanlinna R., *Eindeutige analytische Funktionen*, Springer Verlag in Berlin, 1936.

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On the Beltrami equation*

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Abstract

We prove that the quasiregular mappings given by the (normalized) principal solution of the linear Beltrami equation (1) and the principal solution of the quasilinear Beltrami equation are inverse to each other. This basic fact is deduced from the Liouville theorem for generalized analytic functions. It essentially simplifies the known proofs of the measurable Riemann mapping theorem and its holomorphic dependence on parameters.

The first global, i.e. defined in the full complex plane \mathbb{C} and expressed by an explicit analytical formula, solution of the Beltrami equation

$$w_{\bar{z}} - q(z)w_z = 0 \tag{1}$$

was given by Vekua in the years 1953-54 and it appeared in the first issue of Doklady for 1955 [33]. Vekua in [33] was given by Vekua in the years 1953-54 and it appeared in the first issue of Doklady for 1955 [33].

Vekua in [32] considered the equation (1) with compactly supported $q(z)$, $q(z) \equiv 0$ for $|z| > R$, for some finite R , satisfying the uniform ellipticity condition

$$|q(z)| \leq q_0 < 1, q_0 \text{ constant.} \tag{2}$$

In [33] he considers the class of solutions of (1) represented by the Cauchy complex potential $T\omega$ in the form

$$\omega(z) = -\frac{1}{\pi} \int_C \frac{\omega(\zeta)d\sigma_\zeta}{\zeta - z} + \phi(z) \equiv T\omega + \phi(z) \tag{3}$$

*Editor comment: this paper is short version author's work [4]

where $\omega(\zeta)$ is a complex density, $\omega \in L_p(\mathbb{C}), p > 1$, and $\phi(z)$ is an entire function.

The function $w = w(z)$ is a $W_{loc}^{1,p}(\mathbb{C})$ solution of (1) iff the density ω is a solution of the singular integral equation

$$\omega - q(z)S\omega = q(z)\phi'(z) \equiv h(z), \quad (4)$$

with the singular integral

$$S\omega = -\frac{1}{\pi} \int_C \frac{\omega(\zeta)}{(\zeta - z)^2} d\sigma_\zeta \quad (5)$$

understood in the sense of the Cauchy principal value.

It was probably Vekua who first introduced the singular integral operator S to the study of elliptic equations in the plane. It appeared as early as 1952-53 in connection with the study of general boundary value problems, specifically the Poincaré boundary value problem, in the theory of generalized analytic functions (GAF-for short), which was defined and developed in Vekua's famous paper [31]. See also [32] and the Ph.D. dissertation [5], [6], prepared in 1953-54.

Later the operator S was called the Hilbert transform.

The main role of the operator S in the Vekua school was to transform the derivative $w_{\bar{z}}$ into w_z for compactly supported smooth functions $w \in C_0^\infty(\mathbb{C})$,

$$S(w_{\bar{z}}) = w_z = \frac{\partial}{\partial z} T(w_{\bar{z}}).$$

Since for $w \in C_0^\infty(\mathbb{C})$, the entire function $\phi(z)$ in (3) reduces to $\phi(z) \equiv 0$, integration by parts then gives $\|w_{\bar{z}}\|_{L^2} = \|w_z\|_{L^2}$ and

$$\|Sw_{\bar{z}}\|_{L^2} = \|w_z\|_{L^2} \quad (6)$$

where $\|\cdot\|_{L^2}$ denotes the L^2 norm of square integrable functions. In consequence S extends as a unitary isometry to the Hilbert space $L^2(\mathbb{C})$.

For further reference we also note the equivalent description in terms of the Fourier transform

$$\widehat{S\omega}(\xi) = \frac{\xi}{\bar{\xi}} \widehat{\omega}(\xi) \text{ for every } \xi \in \mathbb{C} - \{0\}. \quad (7)$$

The uniform ellipticity condition (2) gives the L^2 norm estimate for the operator qS ,

$$\|qS\|_{L^2} \leq q_0 < 1 \quad (8)$$

and immediately leads to the unconditional solvability of the integral equation (4) in the space $L^2(\mathbb{C})$ by the Neumann series or successive approximations.

Lemma 1 *For arbitrary measurable dilatation $q(z)$, satisfying (2), the integral equation (4) has a unique solution in $L^2(\mathbb{C})$ given by the formula*

$$\omega = (1 - qS)^{-1}h. \quad (9)$$

This means that the differential Beltrami equation (1) with the compactly supported coefficient $q = q(z)$ has a unique solution in the Sobolev space $W_{loc}^{1,2}(\mathbb{C})$, admitting a holomorphic extension of the form (3) outside the support of q . Actually, by [32], any generalized (weak) solution of (1) in the space $W_{loc}^{1,2}(\mathbb{C})$, can be obtained by the described process. However, below, for the purposes of the theory of quasiconformal mappings, we are interested in very special solutions of (1) only. As a convolution type operator S commutes with differential operators. Moreover, it preserves the Hölder-Zygmund classes $C^{k,\alpha}$ with $0 < \alpha < 1$ and $C^\infty(\mathbb{C}) \cap L^2(\mathbb{C})$. In particular, this implies that for $q(z)$ and h C^∞ -smooth, or of the class $C^{k,\alpha}$ the uniquely determined L^2 solutions of the integral equation (4) are as smooth as the data q and h allow. We formulate this as

Lemma 2 *For compactly supported C^∞ -smooth dilatation $q(z)$ the weak $W_{loc}^{1,2}(\mathbb{C})$ solutions of the Beltrami equation (1) are C^∞ -smooth.*

The proof of Lemma 2 is rather direct, relying on the classical tools of standard potential theory and is described in detail in Vekua's book [34].

By the Calderón-Zygmund theorem [16] the operator S acts also as a bounded operator in $L^p(\mathbb{C})$ for each p , $1 < p < \infty$, and its norm \mathcal{A}_p is continuous at $p = 2$. Thus

$$\mathcal{A}_p q_0 < 1 \text{ for } 2 \leq p < 2 + \epsilon \quad (10)$$

and the equation $\omega - qS\omega = h$ is uniquely solvable:

$$\omega = (I - qS)^{-1}h, \omega \in L^p, \quad (11)$$

for any $h \in L^p$ and p satisfying (10), what we henceforth assume. In particular, for any measurable dilatation $q(z)$ the L^2 solution ω of equation

(4) is actually in some $L^p, p > 2$. Thus, in other words, the $W_{loc}^{1,2}(\mathbb{C})$ solutions of (1) belong to $W_{loc}^{1,2}(\mathbb{C}), p > 2$. In particular, they are continuous (α -Hölder, $\alpha = 1 - \frac{2}{p} > 0$).

For $\phi(z) \equiv z, h(z) \equiv q(z)$, formula (3) gives a particular solution of the Beltrami equation (1)

$$w \equiv z - \frac{1}{\pi} \int_C \frac{\omega(\zeta)}{\zeta - z} d\sigma_\zeta, \quad (12)$$

where ω is the unique solution of the equation

$$\omega - q(z)S\Omega = q(z). \quad (13)$$

Following Vekua [33], see also [7], we call (12) the principal solution of the Beltrami equation. A fundamental issue of the theory of elliptic equations and planar quasiconformal mappings was the understanding that the formulae (12)-(13) give a univalent solution of the uniformly elliptic Beltrami equation (1)-(2) realizing a homeomorphic quasiconformal mapping of the complex plane with the assigned measurable complex dilatation $q(z)$ (the Measurable Riemann Mapping Theorem). This was achieved in 1954, and published in the first months of 1955 in [7], [8], [33], by the collaborative efforts of Vekua and the author. Let us briefly recall the main steps. The existence of $W_{loc}^{1,2}(\mathbb{C})$ solutions was clear from the outset and the problem essentially reduced to the L^2 isometry of the operator S and the classical properties of the complex potential $T : L^2 \rightarrow W_{loc}^{1,2}(\mathbb{C})$, described in [31], [32]. The idea of applying the Calderón-Zygmund theorem [16] and, thus, extending the range of admissible parameters p to the interval $2 - \epsilon < p < 2 + \epsilon$ for some positive ϵ , due to the author [7], [34], immediately allowed us to consider $W_{loc}^{1,2}(\mathbb{C})$ solutions, $p > 2$, and, by the Sobolev imbedding theorems, or classical properties of the complex potentials $T\omega$, α -Hölder continuous solutions with $\alpha = 1 - \frac{2}{p} > 0$.

This we formulate in

Proposition 1 *The Beltrami equation (1) with an arbitrary measurable dilatation $q(z)$, satisfying (2) and compactly supported, always admits the solution of the form (12) in the Sobolev class $W_{loc}^{1,2}(\mathbb{C})$, $p > 2$. Moreover, the norms $\|w_z - 1\|_{L^p}$, $\|w_{\bar{z}}\|_{L^p}$ of this solution are uniformly bounded by quantities depending only on q_0 in (2) and $\|q\|_{L^p}$ (or the support of $|q|$).*

Not necessarily homeomorphic solutions of the Beltrami equations are known as quasiregular mappings. By formulas (3) and (4) above they are relatively easy to construct. The proof that univalent solutions exist at all, the more so, that the solutions (12) are homeomorphisms onto, is much more subtle. In the papers [7], [33] it splits into

Proposition 2 *If the dilatation $q(z)$ is sufficiently smooth, then the mapping (12) is a homeomorphism onto, i.e., it is a quasiconformal mapping of the complex plane.*

and

Proposition 3 *For arbitrary measurable dilatation $q(z)$, satisfying condition (2), the formulae (12)-(13) realize a quasiconformal mapping of the complex plane with the assigned dilatation almost everywhere.*

In the context of papers [7], [33], Proposition 2 was proved by Vekua in [33] for the class of Hölder continuous dilatations, though for the purposes of [7], where the first complete proof of Proposition 3 was given, it is enough to have Proposition 2 for dilatations of much higher regularity, say of class C^∞ , only.

In [33] a local version of Proposition 2 is proved first (Proposition 4 below). The global version of Proposition 2 is obtained by some general, global, geometric monodromy type argument recalled below.

The idea of the present paper is to prove Proposition 2 without appealing to Proposition 4, but by direct construction of a quasiregular mapping, i.e. $W_{loc}^{1,2}(\mathbb{C})$, $p > 2$, solution of some other Beltrami equation, a quasilinear one, and such that the constructed mapping is actually the two-sided inverse to the mapping given by formula (12).

The proof of the implication **Proposition 2** \Rightarrow **Proposition 3** proceeds as in [7] (it was repeated in [9], [34]).

In view of the approximating procedure described in [7] it is, obviously, enough to consider the Beltrami equation (1) with dilatation $q(z)$ of arbitrary high smoothness (even C^∞). To this aim we consider, parallel to equation (1), the quasilinear equation for the mappings $z = z(w)$ of the image plane C_w in (1) to the source plane C_z

$$\frac{\partial z}{\partial \bar{w}} + q(z) \frac{\bar{\partial} z}{\partial w} = 0. \quad (14)$$

We call it the conjugate (quasilinear) Beltrami equation. Now we are interested in a particular solution of (14) of the form

$$\psi(w) = w + T\tilde{\omega} \equiv w - \frac{1}{\pi} \int_C \frac{\tilde{\omega}(\zeta)}{\zeta - w} d\sigma_\zeta \quad (15)$$

with $\tilde{\omega} \in L^p$ for some $p > 2$.

(15) is a solution of (14) of the Sobolev class $W_{loc}^{1,2}(\mathbb{C})$ iff the complex density $\tilde{\omega}$ is a solution of the singular integral equation

$$\tilde{\omega} + \tilde{q}(w)\overline{S}\tilde{\omega} = -\tilde{q}(w) \quad (16)$$

with $\tilde{q}(w) \equiv q(\psi(w))$. Hence $\psi(w) - w$ is in the class $W^{1,p}$ and $\psi(w)$ is the solution of the conjugate Beltrami equation

$$\frac{\partial\psi}{\partial\bar{w}} + \tilde{q}(w)\frac{\partial\bar{\psi}}{\partial w} = 0 \quad (17)$$

with $\tilde{q}(w)$ at least Hölder continuous with exponent $\alpha = 1 - \frac{2}{p} > 0$. In the terminology adopted above the mapping $\psi(w)$ is a quasiregular mapping of the complex plane C_w into the plane C_z .

Considered as an operator equation for the unknown density $\tilde{\omega}(w)$, (16) is a highly nonlinear operator equation. However, its solvability in L^p spaces is easily controlled.

Lemma 3 *The quasilinear conjugate Beltrami equation with smooth dilatation $q(z)$ always admits a solution of type (15) in some $W_{loc}^{1,2}(\mathbb{C})$. Equivalently, the nonlinear equation (16) always admits a solution $\tilde{\omega}$ in $L_p(\mathbb{C}_w)$ (compactly supported) for some $p > 2$.*

The solution (15) of (14) is unique. We will comment on the proof of Lemma 3 later. Here we remark only that in [7], [8] and [9] there is a plentiful of theorems of the type of Lemma 3 and their proofs are constructed along, the, now standard, procedure based on Banach or LeraySchauder fixed point theorems and a priori estimates directly deduced from the linear uniformly elliptic equations (4) and (16) written in the form

$$\tilde{\omega} + \tilde{q}(w)\overline{S}\tilde{\omega} = h, h \in L^p. \quad (18)$$

For our proof of Proposition 2 we shall need the Liouville theorem for generalized analytic functions (GAF) of Vekua, introduced in [31], and in his earlier works and thoroughly described in [9] and [34].

Lemma 4 *Let $w = w(z)$, in $W^{1,2}(\mathbb{C})$ be a (generalized) solution of the equation*

$$w_{\bar{z}} - q(z)w_z = Aw \quad (19)$$

with the coefficient $q(z)$: measurable, compactly supported and satisfying uniform ellipticity condition (2), and $A \in L^p(\mathbb{C})$ for some $p > 2$. For simplicity assume also that A is compactly supported. If w vanishes at ∞ , i.e. $|z||w(z)| < C$ for all z , then $w \equiv 0$.

For details see [9] and [34].

Corollary 1 *The conclusion of Lemma 4 holds also for mappings $w = w(z)$ in $W^{1,2}(\mathbb{C})$, $w(\infty) = 0$, satisfying the inequality*

$$|w_{\bar{z}} - q_1(z)w_z - q_2(z)\overline{w_z}| \leq A(z)|w(z)| \quad (20)$$

if the coefficients q_1, q_2 have compact support and satisfy the uniform ellipticity condition

$$|q_1(z)| + |q_2(z)| \leq q_0 < 1, q_0\text{-const.} \quad (21)$$

and $A \in L_p(\mathbb{C})$, $p > 2$, vanishes for $|z|$ big enough.

Lemma 4, the Corollary and the proof above, are given here only for the completeness of the presentation. They could be simply referred to Theorems 4.1 and 4.2 and the remark in Sections 4.1 and 4.2 in [9].

The important concept of GAF, corresponding to the system (19), and discussed in [31] and [34] under the term: generalized constants (or generalized units), is also useful in the global theory of the Beltrami equation (1).

Lemma 5 *In the conditions of Lemma 4 the equation (19) has a unique solution defined in the full complex plane $v = v(z)$, $z \in \mathbb{C}$, regular at $z \rightarrow \infty$, and such that $v(\infty) = \infty$. This solution does not vanish for any $z \in \mathbb{C}$,*

$$v(z) \neq 0.$$

Lemma 5, as Lemma 4 above, could be also referred to [9].

Corollary 2 *The derivative w_z of the principal solution (12) of the Beltrami equation with smooth dilatation $q(z)$ ($q(z) \in W^{1,p}$, $p > 2$, is enough) is a generalized constant for equation (19). In particular,*

$$w_z \equiv 1 + S\omega \neq 0, \text{ for all } z \in \mathbb{C} \quad (22)$$

Corollary 2 immediately implies the following

Proposition 4 *In the conditions of Proposition 2 the principal (quasiregular) solution (12) is a local homeomorphism.*

In [33] Vekua deduced Proposition 2 from Proposition 4 by appealing to the "argument principle" for local homeomorphisms of the complex plane. It was also well known that the monodromy theorem for open mappings of the Riemann sphere S^2 or the closed plane $\widehat{\mathbb{C}}$ may also be used to deduce Proposition 2 from Proposition 4. Even reference to the famous uniformization theorem was exploited sometimes!

Though all the above statements are "well known", "well understood" and "intuitively obvious" for geometers and practicing complex analysts, neither of them can be considered "elementary".

Lemma 6 and Proposition 5 below reduce the proof of Proposition 2 to the Liouville theorem for Vekua's generalized analytic functions in the extended complex plane $\widehat{\mathbb{C}}$: our Lemma 4 and Corollary 1.

Let us now consider the Beltrami equation (1) with a smooth compactly supported dilatation $q(z)$ and the conjugate Beltrami quasilinear equation (14).

Lemma 6 *Let $\chi = \chi(z)$ be the normalized (principal) solution (12) of equation (1) and $\psi = \psi(w)$ the principal solution (15) of the quasilinear equation (14). Consider the composed mappings*

$$\tilde{\phi}(w) = \chi \circ \psi(w), \tilde{\phi} : \mathbb{C}_w \rightarrow \mathbb{C}_w, \phi(w) = \chi \circ \psi(w), \phi : \mathbb{C}_w \rightarrow \mathbb{C}_w, \quad (23)$$

Then $\tilde{\phi} = \tilde{\phi}(w)$ is a solution of the Cauchy-Riemann equation

$$\frac{\partial \tilde{\phi}}{\partial \bar{w}} = 0 \quad (24)$$

and ϕ satisfies the inequality

$$|\phi_{\bar{z}} - \tilde{q}(z)(\phi_z - \bar{\phi}_{\bar{z}})| \leq A(z)|\phi(z) - z| \quad (25)$$

with a bounded, compactly supported function $A(z)$ and

$$\tilde{q}(z) \equiv \frac{q(z)}{1 + |q(z)|^2}.$$

Lemma 6 and its proof have a nice geometric interpretation in terms of Lavrentiev fields (characteristics), see below, and also [13] and [35], [28].

Lemmas 4 and 6 lead to Proposition 2 with a completely new direct proof. We formulate it as

Proposition 5 *The normalized solutions (12) and (15) of the smooth Beltrami equation and the conjugate quasilinear equation are homeomorphisms of the complex planes $C_z \rightarrow C_w$ inverse to each other, i.e. the formulas hold*

$$\chi(w) \circ \psi(w) \equiv w \text{ and } \psi \circ \chi(z) \equiv z. \quad (26)$$

We also have an important corollary.

Corollary 3

$$J_\chi \cdot J_\psi \equiv 1 \quad (27)$$

where $J_\chi = |\chi_z|^2 - |\chi_{\bar{z}}|^2$ and $J_\psi = |\psi_w|^2 - |\psi_{\bar{w}}|^2$ are the Jacobians. In particular,

$$J_\chi \neq 0 \text{ and } J_\psi \neq 0 \quad (28)$$

at every point. Actually $J_\chi \geq c_\chi > 0$ and $J_\psi \geq c_\psi > 0$ for positive constants (in general, dependent on the mapping).

(28) is also a direct consequence of Lemma 5 above.

In the foundational study of Vekua and the author on Beltrami equations described in [33], [34] and [7], [9] formulas of the type (26) and (15) play an important role. However there, given $\chi = \chi(z)$ defined by (12), for a sufficiently smooth dilatation $q(z)$, it is first proved (Proposition 2) that $\chi = \chi(z)$ is a homeomorphism and the formulas (26) are used to define $\psi = \psi(w)$. Only in the next step $\psi(w)$ is shown to be a solution of the conjugate Beltrami quasilinear equation (14) which may be represented in the form (15).

These facts are the cornerstones of the theory of Beltrami equations in the plane as developed and described in detail in [33] and [7]. In [7], [9] there are also established the basic a priori estimates in the L^p and $W^{1,p}$ norms, for $2 < p < 2 + \epsilon$, for normalized solutions of the Beltrami equation (1) and their inverses $\psi = \psi(w)$ in terms of the dilatation $q(z)$. These estimates are preserved under various limiting processes, $q_n \rightarrow q$ for $n \rightarrow \infty$, even in the space of bounded measurable functions with topology defined by almost everywhere convergence as long as the uniform ellipticity condition (2) is

uniformly fulfilled: $\sup_{z,n} |q_n| \leq q_0 < 1$. For the proofs of all these facts and their important consequences we refer to [7]. See also [9] and [34].

After 1955-57, most of the numerous publications (and all monographs) on the generalized Riemann mapping theorem and the existence problems for quasiconformal mappings were repeating, with rather slight modifications, the analytical methods of Vekua and his school or, at least, were essentially relying on these arguments. In many cases, embarrassingly enough, these works contain only marginal, if any, references to the sources.

Douady's latest proof in [17] relies on the L^2 solution, expressed in terms of the Fourier transform (7) above, of the singular integral equation (4), (13) and on the the $L^p, p > 2$, a priori estimates as above, he returns to the Grötzsch-Lavrentiev-Morrey-Ahlfors results on the uniform Hölder estimates for QC-maps-reappearing in all compactness arguments of the elliptic theory of p.d.e.'s and quasiconformal mappings with bounded dilatation-and proposes a rather long way to go, with many references to exterior results, before achieving the existence theorem. See also the comments of Kra & Earle in the second edition of [1].

In contrast, the deep theory of Lavrentiev [21], [22] and his followers: Volkovyskii [35], Belinskii [3], Pesin [27], using mainly direct geometric methods, contains many far reaching new ideas, so far only partially exploited and waiting, see [13], [28], for a modern, up-to-date presentation.

We leave aside the extensively growing and important research on mappings of finite distortion and their various generalizations.

The work of Vekua and his school on the solutions of the Beltrami equation yielded much more than the previous methods due to Lichtenstein [25], Lavrentiev [21] or Morrey [26], where, in various forms, the Riemann mapping theorem for QC-maps was proved.

The explicit representation formulas of Vekua's school and related a priori estimates for global mapping problems, created a powerful and flexible tool and a method to attack many local and global problems, inaccessible in any preceding theory. The study of quasiconformal extensions of holomorphic univalent functions and of the theory of deformations of planar quasiconformal mappings is hardly conceivable without these tools. They serve as a solid foundation for the development of important applications of the theory inside as well as outside the planar elliptic p.d.e. theory. The long list of the first ones starts with the deep results of Vinogradov and Danilyuk on basic boundary value problems for general elliptic equations and generalized analytic functions described in Vekua's monograph [34]. For the latter,

i.e. applications outside the GAF, it is enough to mention the deep and beautiful ideas and constructions of the Ahlfors-Bers school in the theory of Teichmüller spaces, moduli spaces and Kleinian groups or the results in complex holomorphic dynamics [19], [1] (the 2006 edition).

It is necessary to stress here that the explicit formulas (12) and (22) written in the form

$$w_{\bar{z}} = \omega = (1 - qS)^{-1}q$$

and

$$w_z - 1 = S\omega = S(1 - qS)^{-1}q$$

show that the derivatives $w_{\bar{z}}$ and w_z of the principal solution (12) depend holomorphically, in the general functional sense, on the complex dilatation q . This functional dependence, naturally, implies that, if the dilatation $q(z)$ itself depends on some parameters t , holomorphically, real analytically, smoothly or just continuously, then the principal solutions (12) depend holomorphically, smoothly. . . etc., as the case may be, on these parameters. We will give some more comments on this topic later.

In [34] the existence of homeomorphic solutions of the complex Beltrami equation is also discussed in the compactified complex plane $\widehat{\mathbb{C}}$, identified with the Riemann sphere S^2 . In this case, for the general measurable dilatation satisfying the condition (2) only, the homeomorphic solution cannot be in general represented by formula (12). However, as shown in [34], the principal homeomorphism can be constructed by the composition of two homeomorphisms of type (12) obtained by splitting the complex dilatation $q(z) = q_1 + q_2$ with $q_1(z)$ and $q_2(\frac{1}{\bar{z}})$ compactly supported, and a simple natural change of variables.

Also the behaviour of the complex dilatation $q_w = \frac{w_{\bar{z}}}{w_z}$ under composition of quasiconformal mappings $f = w \circ v^{-1}$ is discussed in [9] and the simple, but important, formula

$$q_f = \left\{ \frac{q_w - q_v \frac{v_z}{v_{\bar{z}}}}{1 - \bar{q}_v q_w \frac{v_z}{v_{\bar{z}}}} \right\} \circ v^{-1}$$

appears and is used, at some crucial points, in [9].

Let us return to comments on the proof of Lemma 3: Consider the convex set Σ of mappings of the form (15) parametrized by the densities $\tilde{\omega} \in L^p(\mathbb{C}_w)$ for some fixed admissible $p > 2$. For $z = z(w) \in \Sigma$ consider the principal

solution $\psi(w)$ of the conjugate linear Beltrami equation

$$\frac{\partial\psi}{\partial\bar{w}} + \tilde{q}(z)\frac{\overline{\partial\psi}}{\partial w} = 0. \quad (29)$$

with $\tilde{q}(w) \equiv q(z(w))$

This defines the nonlinear map $\psi = F(z)$ of Σ into Σ . Since (29) is again a Beltrami equation in the w -plane, with the same uniform ellipticity estimate as (1), Lemma 1 and Proposition 1 hold and a priori estimates follow. Hence F is compact and the fixed point of F is the required solution of the quasilinear equation (14). For many analogous arguments see [9], [10], [11].

It follows also from the above a priori estimates (Proposition 1) that the iteration process

$$\frac{\partial z_{n+1}}{\partial\bar{w}} + q(z_n(w))\frac{\overline{\partial z_{n+1}}}{\partial w} = 0 \quad (30)$$

defined on the class of principal (quasiregular) solutions defines uniquely the compact sequence of mappings in Σ converging in $W_{loc}^{1,p}(\mathbb{C}_w)$ to the required solution of the quasilinear equation (14)

$$z_n(w) \rightarrow z(w) \quad (31)$$

converge locally uniformly and weakly in $W_{loc}^{1,p}$ to the (unique) solution of the equation (14). Summing up, we can state that, with the proof of Lemma 3 reduced to (30) and (31), our Lemmata 1-6 and Propositions 1-5 described above, together with the paper [7], give a complete, fully self-contained (i.e., not requiring references to any earlier analytical or geometrical results), detailed and thus "elementary" proof of the basic theorems on the existence and structural properties of solutions of the planar measurable Beltrami equation.

ADDITIONAL COMMENTS. The concept of the principal solution of form (12) or its slight generalization

$$w(z) = az - \frac{1}{\pi} \int_C \frac{\omega(\zeta)}{\zeta - z} d\sigma_\zeta, \quad a\text{-complex constant}, \quad (32)$$

is meaningful for the general Beltrami equation

$$w_{\bar{z}} + q(z)w_z - q_1(z)\bar{w}_z = 0 \quad (33)$$

with the uniform ellipticity condition

$$|q(z)| + |q_1(z)| \leq q_0 < 1, \quad q_0\text{-const.} \quad (34)$$

These equations correspond to Lavrentiev's quasiconformal mappings [21], [22], with "two pairs of characteristics" [35], [29], [9], [13], and in Vekua's school they have been considered from the outset [8], [9], [34], [28].

The infinitesimal geometric meaning of a differentiable transformation $w = w(z)$ at a point z_0 is defined by the linear tangent map

$$Dw(z)(\xi) = w_z(z_0)\xi + w_{\bar{z}}\bar{\xi} \quad (35)$$

It transforms ellipses in the tangent plane at z_0 into ellipses in the tangent plane at the image point $w(z_0)$.

Ellipses centred at z are defined up to a similarity transformation by the ratio $p \geq 1$ of their semiaxes and, if $p > 1$, the angle $\theta \bmod \pi$ between majoraxis and the positive z -axis, and denoted by $\mathcal{E}(p, \theta, z)$ or $\mathcal{E}_h(p, \theta, z)$ where h is the length of the minoraxis. The pair (p, θ) is called the characteristic of the infinitesimal ellipse, and the family $\mathcal{E}_h(p, \theta, z), h > 0, z \in G$, is a field of infinitesimal ellipses (Lavrentiev field). A homeomorphism $w = w(z)$ is said to map the infinitesimal ellipse $\mathcal{E}(p, \theta, z)$ onto $\mathcal{E}(p_1, \theta_1; z)$ if the tangent map $Dw(z)$ transforms $\mathcal{E}(p, \theta; z)$ onto $\mathcal{E}(p_1, \theta_1; z)$.

Analytically this is described in terms of the components $w_{\bar{z}}$ and w_z in the tangent map Dw (35) by the general Beltrami equation (33) where the coefficients q and q_1 are determined by the invertible formulas

$$q(z) = -\frac{p - p^{-1}}{p + p^{-1} + p_1 + p_1^{-1}}e^{2i\theta}, q_1(z) = -\frac{p_1 - p_1^{-1}}{p + p^{-1} + p_1 + p_1^{-1}}e^{2i\theta_1}. \quad (36)$$

In particular the solutions of the Beltrami equation (1) ($q_1 \equiv 0$) map the field of ellipses $\mathcal{E}(p_1, \theta_1; z)$ into infinitesimal circles ($p_1 \equiv 1$) whereas the conjugate Beltrami equations (14), (17) map the infinitesimal discs ($p \equiv 1$) into ellipses ($p_1 \geq 1$).

We should stress that the clue of the Lavrentiev idea is that the source characteristic (p, θ) is mapped into the "target" characteristic (p_1, θ_1) independently of the considered particular solution of the general Beltrami equation (33) as long as the relation source \rightarrow target $w = w(z)$ is preserved. The formulae (36) describe then a pair of distinguished or canonical Lavrentiev fields intrinsic for the Beltrami system considered and a selected, pointwise correspondence $w = w(z)$. Of course any chosen solution $w = w(z)$ at every differentiability point transforms an arbitrary Lavrentiev field of infinitesimal ellipses into "some" Lavrentiev field whose characteristics (p_1, θ_1) at the

image point $w(z)$ depend on the behaviour of the map at neighbouring points, i.e., on the values of the derivatives $w_z, w_{\bar{z}}$ at z .

Lavrentiev in his seminal paper of 1935 [21] defined QC-mappings as homeomorphic mappings of the unit disc D onto itself such that at every point $z \in D$ the infinitesimal ellipse $\mathcal{E}(p(z), \theta(z); z)$ is mapped onto an infinitesimal circle in the sense defined above. He also proved the existence theorem for such mappings by a direct geometric construction without referring to any classical solutions of boundary value problems, e.g. in the Lichtenstein paper [25]. Thus Lavrentiev is probably to be credited for the first direct, self-contained proof of the global Riemann mapping theorem for a rather general class of complex Beltrami equations with continuous coefficients.

The density $\omega(\zeta)$ of the principal solution (32) satisfies the singular integral equation

$$\omega - qS\omega - q_1\bar{S}\omega = aq + \bar{a}q_1 \quad (37)$$

which is uniquely solvable and its L^2 solutions are necessarily in L^p for some $p > 2$. Equations of type (37) are linear over the real field only and were widely applied in [8], [9], [34] and many later works. We state here a direct corollary of the above theory of Beltrami equation (1) which we formulate as

Proposition 6 *The equation (33) has always a unique principal solution of the form (32). If $a \neq 0$ then the principal solution realizes a homeomorphic quasiconformal mapping of the full complex plane $\bar{\mathbb{C}}$.*

For $a = 0$ the principal solution is identically $\equiv 0$ (Liouville theorem).

We stress the fact that, after the theory of Beltrami equations (1) is available as formulated above, no work at all is needed to prove Proposition 6. However, there would be a long way, though that is possible, see [11], before one could construct an analytical inverse mapping to (32) with the help of global solutions of some quasilinear general Beltrami equations (of type (14)). See also [11]. An important class of Beltrami equations (33) appears when the identity mapping $w(z) = z$ is a solution of (33). These are characterized by the formula

$$q(z) + q_1(z) \equiv 0 \quad (38)$$

and already appeared in [35] and [9]. In view of (36) the relation (38) reduces to the formulas $p \equiv p_1$ and $\theta \equiv \theta_1$ and have the beautiful geometric characterization in terms of Lavrentiev characteristics (Lavrentiev fields), see [35], [9] and [13]. In terms of the linear tangent map Dw of the given pointwise

mappings $w = w(z)$ the Lavrentiev field (see [13]) associated with system (33) at point z (in $T_z\mathbb{C}$) is parallel translated to the tangent plane T_w at the image point $w = w(z)$. In [9] the systems (33)-(38) appeared in connection with the uniqueness problem for Riemann mapping corresponding to the general Beltrami system (33). The study of principal solutions (32) as a function of the parameter a in this formula is an interesting topic and should be continued.

The study of the genuinely nonlinear Beltrami equations

$$w_{\bar{z}} = H(z, w, w_z) \quad (39)$$

for some complex valued function $H(z, w, \xi)$ has also been started by the author [11] in connection with the programme of introducing complex analytic methods to the Lavrentiev theory of fully nonlinear (implicit) first order systems (e.g. [22] and many other papers). Lavrentiev's geometric ellipticity concept was interpreted as the Lipschitz condition

$$|H(z, w, \zeta_1) - H(z, w, \zeta_2)| \leq q_0 |\zeta_1 - \zeta_2| \quad (40)$$

with $q_0 = \text{const.} < 1$. Global existence and structure theorems for solutions of principal type (12) were proved, see [11] and [15].

As is well known, the principal solution (12) generates all solutions of the Beltrami equations (1). This is described in the famous so called Stoilow factorisation theorems [30]. This fact is crucial in establishing important structural properties of quasiconformal mappings, like homotopy, factorization into mappings with arbitrary small dilatations, differentiability (though the obtained differentiability results are far from the subtle results of Mensor (1931) on the differentiability of open mappings), etc., etc., see [9], [34], [24]. It gives also the parametrization of planar QC-mappings by L^p solutions of the integral equation (13) (by densities in L^p , which also allows us to introduce the Banach manifold structure into the set of all QC-maps).

Formulas (12) and other related formulas give us a convenient tool to study the dependence of quasiconformal mappings $f(t, z)$ on parameters $t \in P$, P denoting some parameter space. In the simplest case the parameter t may vary in some interval of the real line. Representation formulas (12) allow us to reduce the problem to the study of parameter families of the corresponding complex densities $\omega(t) \equiv \omega(t, z) \equiv \frac{\partial f(t, z)}{\partial \bar{z}}$ in their behaviour under small variations of the parameter, differentiation, etc.

The parameter derivatives $\dot{\omega}, \dot{q}, \dot{h}$ are then the partials $\dot{\omega} = \frac{\partial \omega}{\partial t}$ etc., and are seen to be the solutions of the integral equation

$$\dot{\omega}(t) - q(t)S(\dot{\omega}) = \dot{q}S\omega + \dot{h}, \quad (41)$$

obtained from the integral equation (4) by differentiation with respect to the parameter $t \in P$. Naturally, it is assumed that the coefficient q and the right hand side h in equation (4) are differentiable with respect to the parameter t (see 11, 13).

If the parameter space is, e.g., an open subset of \mathbb{R}^n or \mathbb{C}^n , the partials $\frac{\partial}{\partial t}$ may be replaced by some "total" Fréchet type differential operators, in general denoted by the symbol D_t . Then the equation (41) takes the "general" form

$$\dot{\omega}(t) - q(t)S(\dot{\omega}) = D_t q S \omega + D_t h. \quad (42)$$

In the literature there are many examples of this type of studies [2], [12], [14]. They all rely on the a priori estimates of the solutions of integral equations of type (4), which are, essentially, consequences of our assumptions (10).

In the particular case when t is the complex structure parameter and $D_t \equiv \frac{\partial}{\partial t}$ (41) together with the assumption $\frac{\partial q}{\partial t} = \frac{\partial h}{\partial t} = 0$ leads to the equation

$$\frac{\partial \omega}{\partial \bar{t}} \equiv 0$$

which implies $\frac{\partial f}{\partial \bar{t}} \equiv 0$ and is interpreted as holomorphic dependence of QC-maps on holomorphic parameters. Note that the complex conjugate Beltrami equations (14) and (16) are linear only over the reals and the general differentiations D_t should take this into account. This refers in particular to equations (30), (37) and implies that even for holomorphic in t complex dilatations $q(t, z)$, the inverse mappings $f^{-1}(t, z)$ are holomorphic in \bar{t} instead of t .

For normalized quasiconformal mappings of the complex plane, the unit disc and other "model" domains, explicit formulas of the type (12) allow us to calculate the Gateaux differential of the normalized mappings in their dependence on the infinitesimal variation of the complex dilatation (see [1], [14], and many other papers by Gutlyanskii), revealing the connections of the analytic theory of Beltrami equations with the study of deformations of conformal structures in the geometric function theory of Goluzin and Kufarev.

Some historical remarks scattered in this paper do not pretend to give, in any sense, a full and satisfactory account of the history of research in the

area. It is also clear that in any sufficiently rich and mature mathematical theory progress is, generally speaking, the result of the collective effort of many researchers throughout the years. However, some landmarks can, and perhaps, even should be highlighted and it is proper and useful that this be done responsibly. I am convinced that the Beltrami equations and their applications lack a serious historical presentation though they certainly deserve one, with a view into the past as well as the future.

References

- [1] Ahlfors, L.: Lectures on quasiconformal mappings. - Van Nostrand, Princeton 1966; Second Edition, Univ. Lect. Series 38, Amer. Math. Soc., Providence, 2006.
- [2] Ahlfors, L., and L. Bers: Riemann's mapping theorem for variable metrics. - Ann. of Math. (2) 72, 1960, 385-404.
- [3] Belinskiĭ, P. P.: General properties of quasiconformal mappings. - Nauka, Novosibirsk, 1974 (in Russian).
- [4] Bojarski, B.: On the Beltrami equation, once again: 54 years later. -Annales Academiae Scientiarum Fennicae Mathematica, vol.35, 2010, 59-73
- [5] Bojarski, B.: Boundary value problems for elliptic systems with two independent variables. - Ph.D. dissertation and report, MGU, Moscow, 1954 (in Russian).
- [6] Bojarski, B.: On a boundary value problem for a system of elliptic equations. - Dokl. Akad. Nauk SSSR (N.S.) 102, 1955, 201-204 (in Russian).
- [7] Bojarski, B.: Homeomorphic solutions of the Beltrami systems. - Dokl. Akad. Nauk SSSR (N.S.) 102, 1955, 661-664 (in Russian).
- [8] Bojarski, B.: On solutions of elliptic systems in the plane. - Dokl. Akad. Nauk SSSR (N.S.) 102, 1955, 871-874 (in Russian).
- [9] Bojarski, B.: Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients. - Mat.

- Sb. N.S. 43:85, 1957, 451.503 (in Russian); English transl. in Rep. Univ. Jyväskylä Dept. Math. Stat. 118, 2009, available at www.jyu.fi/science/laitokset/maths/tutkimus/reports.
- [10] Bojarski, B.: General properties of solutions of elliptic systems in the plane. - In: *Issled. sovr. probl. teor. funktsii. kompl. peremen.* (Moscow Conf. 1957), ed. I. A. Markushevich, Gosfizmatizdat, Moscow, 1960, 461-483 (in Russian); Transl. in *Fonctions d'une variable complexe, Probl. contemporaines*, Gauthier-Villars, Paris, 1962, 254-268.
- [11] Bojarski, B.: Quasiconformal mappings and general structural properties of systems of non-linear equations elliptic in the sense of Lavrentiev. - In: *Symposia Mathematica XVIII* (Convegno sulle Trasformazioni Quasiconformi e Questioni Connesse, INDAM, Rome, 1974), Academic Press, London, 1976, 485-499.
- [12] Bojarski, B.: Old and new on Beltrami equation. - In: *Functional Analytic Methods in Complex Analysis and Applications to Partial Differential Equations* (Trieste, 1988), World Scientific, River Edge, 1990.
- [13] Bojarski, B.: Geometry of the general Beltrami equations. - In: *Complex Analysis and Potential Theory*, ed. T. Aliyev Azeroglu and P. M. Tamrazov, World Scientific, Singapore, 2007, 66-83.
- [14] Bojarski, B., and V.Ya. Gutlyanskii: On the Beltrami equation. - In: *Conference Proceedings and Lecture Notes on Analysis I* (Tianjin, 1992), ed. Zhong Li, International Press, Cambridge, MA, 1994, 833.
- [15] Bojarski, B., and T. Iwaniec: Quasiconformal mappings and non-linear elliptic equations I. - *Bull. Pol. Acad. Sci.* 27, 1974, 473-478.
- [16] Calderón, A. P., and A. Zygmund: On the existence of certain singular integrals. - *Acta Math.* 88, 1952, 85-139.
- [17] Douady, A.: Le théorème d'intégrabilité des structures presque complexes (d'après des notes de X. Buff). - In: *The Mandelbrot Set, Theme and Variations*, Cambridge Univ. Press, Cambridge, 2000, 307-324.

- [18] Hamilton, D. H.: Area distortion of quasiconformal mappings. - In: Handbook of Complex Analysis: Geometric Function Theory, vol. 1, Elsevier, Amsterdam, 2002, 147-160.
- [19] Hubbard, J. H.: Teichmüller theory with applications to topology, geometry, and dynamics, vol. I: Teichmüller theory. - Matrix Edition, Ithaca, 2006.
- [20] Kühnau, R. (ed.): Handbook of complex analysis: Geometric function theory, vol. 12. - Elsevier, Amsterdam, 2002-2005.
- [21] Lavrentiev, M. A.: Sur une classe de représentations continues. - Rec. Math. Moscou 42, 1935.
- [22] Lavrentiev, M. A.: A fundamental theorem of the theory of quasiconformal mapping of plane regions. - Izv. Akad. Nauk SSSR 12, 1948 (in Russian).
- [23] Lehto, O.: Univalent functions and Teichmüller spaces. - Springer, Berlin, 1987.
- [24] Lehto, O., and K. I. Virtanen: Quasikonforme Abbildungen. - Grundlehren Math. Wiss. 126, Springer, Berlin, 1965; 2nd ed.: Quasiconformal mappings in the plane, Springer, Berlin, 1973.
- [25] Lichtenstein, L.: Zur Theorie der konformen Abbildungen: Konforme Abbildungen nichtanalytischer singularitätenfreier Flächenstücke auf ebene Gebiete. - Bull. Acad. Sci. Cracovie, 1916, 192-217.
- [26] Morrey, C. B.: On the solutions of quasi-linear elliptic partial differential equations. - Trans. Amer. Math. Soc. 43, 1938, 126-166.
- [27] Pesin, I. N.: Metric properties of Q -quasiconformal mappings. - Mat. Sb. N.S. 40:82, 1956, 281-294.
- [28] Renelt, H.: Elliptic systems and quasiconformal mappings. - John Wiley & Sons, Chichester, 1988.
- [29] Schapiro, Z.: Sur l'existence des représentations quasi-conformes. - Dokl. Akad. Nauk SSSR (N.S.) 30, 1941, 690-692.

- [30] Stoilow, S.: Brief summary of my research work. - In: Analysis and Topology, A volume dedicated to the memory of S. Stolow, World Scientific Publ., River Edge, NJ, 1998.
- [31] Vekua, I. N.: Systems of differential equations of the first order of elliptic type and boundary value problems, with an application to the theory of shells. - Mat. Sb. N.S. 31:73, 1952, 217-314.
- [32] Vekua, I. N.: The general representation of functions of two independent variables admitting a derivative in the sense of S. L. Sobolev. - Dokl. Akad. Nauk SSSR 89, 1953, 773-775 (in Russian).
- [33] Vekua, I. N.: The problem of reducing differential forms of elliptic type to canonical form and the generalized CauchyRiemann system. - Dokl. Akad. Nauk SSSR 100, 1955, 197-200 (in Russian).
- [34] Vekua, I. N.: Obobshchennye analiticheskie funktsii. - Fizmatgiz, Moscow, 1959; English transl.: Generalized analytic functions. - Pergamon Press, Oxford, 1962.
- [35] Volkovysky, L. I.: Quasiconformal mappings. - Lvov, 1954 (in Russian).

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The periodicity of the space of generalized analytic functions

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Abstract

In this paper the spaces of generalized analytic functions $\Omega(a, b)$, $a, b \in L_p, 2$ are considered and it is shown that this spaces as vector spaces on \mathbb{R} have different structures.

Let $F(z), G(z)$ be two complex valued Holder continuous functions defined in some domain such that $Im(\overline{F}G) > 0$. A function $w = \phi F + \psi G$, where ϕ and ψ are real, is called (F, G) pseudoanalytic, if $\phi_{\bar{z}}F + \psi_{\bar{z}}G = 0$. The function $\dot{w} = \phi_z F + \psi_z G$ is called the (F, G) derivatives of w . Every generating pair (F, G) has a successor (F_1, G_1) such that (F, G) derivatives are (F_1, G_1) pseudoanalytic. The successor is not uniquely determined. A generating pair (F, G) is said to have *minimum period* n if there exists generating pairs (F_i, G_i) such that $(F_0, G_0) = (F, G)$, (F_{i+1}, G_{i+1}) is a successor of (F_i, G_i) and $(F_n, G_n) = (F_0, G_0)$. If such an n does not exist, (F, G) is said to have minimum period ∞ .

It is known, that w is pseudanalytic iff w satisfies the following Carleman-Bers-Vekua equation

$$w_{\bar{z}} = aw + b\bar{w}, \quad (1)$$

where the function $a(z, \bar{z}), b(z, \bar{z})$ expressed by the generating pair (F, G) by the following identity

$$a = \frac{\overline{G}F_{\bar{z}} - \overline{F}G_{\bar{z}}}{F\overline{G} - \overline{F}G}, b = \frac{FG_{\bar{z}} - GF_{\bar{z}}}{F\overline{G} - \overline{F}G}. \quad (2)$$

Define also the the quantities

$$A = \frac{\overline{G}F_z - \overline{F}G_z}{F\overline{G} - \overline{F}G}, B = \frac{FG_z - GF_z}{F\overline{G} - \overline{F}G}. \quad (3)$$

The (F, G) -derivative \dot{w} satisfies the following Carlemmn-Bers-Vekua equation

$$\dot{w}_{\bar{z}} = a\dot{w} - B\bar{w} \quad (4)$$

The functions a, b, A, B are called the characteristic coefficients of the generating pair (F, G) .

Proposition 1 [3] *Given functions a, b, A, B are characteristic coefficients of the generating pair if and only if they satisfy the system of differential equations*

$$A_{\bar{z}} = a_z + b\bar{b} - B\bar{B}, \quad B_{\bar{z}} = b_z + (\bar{a} - A)b + (a - \bar{A})B. \quad (5)$$

Proposition 2 [3] *1) The space $\Omega(a, b)$ have period one iff there exist a Function A_0 satisfying the equation*

$$A_{0\bar{z}} = a_0, \quad (A_0 - \bar{A}_0) = \bar{a} - a + \frac{1}{b}(b_z + b_{\bar{z}}) \quad (6)$$

2) The space $\Omega(a, b)$ have period two iff there exist a Functions A_0, A_1, B_0 satisfying the system of equations

$$A_{0\bar{z}} = a_z + b\bar{b} - B_0\bar{B}_0, \quad B_{0\bar{z}} = b_z + (\bar{a} - A_0)b + (a - \bar{A}_0)B_0, \quad (7)$$

$$A_{1\bar{z}} = a_z + b\bar{b} - B_0\bar{B}_0, \quad B_{0z} = b_{\bar{z}} + (A_1 - \bar{a})B_0 + (\bar{A}_1 - a)b. \quad (8)$$

Proposition 3 *Let (F, G) generating pair of (1), then generating pair of the adjoint equation*

$$w_{\bar{z}} = -aw - \bar{B}\bar{w}, \quad (9)$$

is

$$F^* = \frac{2\bar{G}}{F\bar{G} - \bar{F}G}, \quad G^* = \frac{2\bar{F}}{F\bar{G} - \bar{F}G}, \quad (10)$$

We prove, that the characteristic coefficient induced from adjoint generating pair (F^*, G^*) are equal to $-a$ and $-\bar{b}$.

Indeed,

$$\frac{\bar{G}^* F_{\bar{z}}^* - \bar{F}^* G_{\bar{z}}^*}{F^* \bar{G}^* - \bar{F}^* G^*} = \frac{\frac{2F}{D}(\frac{2\bar{G}}{D})_{\bar{z}} - \frac{2G}{D}(\frac{2\bar{F}}{D})_{\bar{z}}}{\frac{2\bar{G}}{D} \frac{2F}{D} - \frac{2G}{D} \frac{2\bar{F}}{D}} = \frac{\frac{4F}{D}(\frac{\bar{G}_{\bar{z}}}{D} - \frac{\bar{G}}{D^2} D_{\bar{z}}) - \frac{4G}{D}(\frac{\bar{F}_{\bar{z}}}{D} - \frac{\bar{F}}{D^2} D_{\bar{z}})}{\frac{4}{DD}(F\bar{G} - \bar{F}G)} =$$

$$= \frac{F\bar{G}_z - G\bar{F}_z}{D} - \frac{F\bar{G} - \bar{F}G}{D^2}D_z, \quad (11)$$

where $D = F\bar{G} - \bar{F}G$, $D = -\bar{D}$, $D_z = F_z\bar{G} + F\bar{G}_z - \bar{F}_zG - \bar{F}G_z$. From (11) we have

$$a_1 = \frac{-F_z\bar{G} - F\bar{G}_z - \bar{F}_zG - \bar{F}G_z + \bar{F}_zG + \bar{F}G_z}{D} = -\frac{F_z\bar{G} - \bar{F}G_z}{D} \implies a = -a_1$$

Analogically to above

$$\begin{aligned} b_{1(F^*, G^*)} &= \frac{F^*G_{G^*}^* - G^*F_z^*}{F^*\bar{G}^* - \bar{F}^*G^*} = \frac{\frac{G}{D}\bar{G}(\frac{F}{D})_z - \frac{G}{D}\bar{F}(\frac{G}{D})_z}{\frac{G}{D}} \\ &= \frac{\bar{D}}{D} \left(\frac{\bar{G}\bar{F}_z}{D} - \frac{\bar{G}\bar{F}}{D^2}D_z - \frac{\bar{F}\bar{G}_z}{D} + \frac{\bar{G}\bar{F}}{D^2}D_z \right) = -\frac{\bar{G}\bar{F}_z - \bar{F}\bar{G}_z}{D}, \end{aligned}$$

therefore

$$\bar{b}_1 = -\frac{GF_z - FG_z}{D} \implies b_1 = -\bar{B}.$$

By definition [2] the pseudoanalytic functions corresponding to (1) satisfies the following holomorphic disc equation

$$\omega_{\bar{z}} = q(z)\bar{\omega}_z, \quad \text{where} \quad q(z) = \frac{F + iG}{F - iG} \quad (12)$$

Proposition 4 *Holomorphic disc equation, corresponding to (9) is*

$$\omega_{\bar{z}} = -\overline{q(z)}\bar{\omega}_z.$$

Indeed, coefficient of holomorphic disc equation, corresponding to (9) expressed by the generating pair (F^*, G^*) of (9) as

$$q_1 = \frac{F^* + iG^*}{F^* - iG^*} \Rightarrow q_1 = \frac{\frac{2\bar{G}}{D} + i\frac{2\bar{F}}{D}}{\frac{2\bar{G}}{D} - i\frac{2\bar{F}}{D}} = \frac{\bar{G} + i\bar{F}}{\bar{G} - i\bar{F}} \Rightarrow q_1 = -\bar{q}.$$

Proposition 5 *If system (1) has the period one, then the system (9) also has period one.*

The proof immediately follows from the proof of the preceding proposition.

Proposition 6 *The generating pair of the space $\Omega(a, 0)$ is (f, if) , where $f \neq 0$ and is solution of the equation $f_{\bar{z}} = -af$.*

Indeed,

$$\begin{aligned} \operatorname{Im}(\bar{f}if) &= i|f|^2; (F\bar{G} - \bar{F}G) = f(-i\bar{f}) - \bar{f}(if) = -2i|f|^2. \\ a_{(f,if)} &= \frac{\bar{f}if_{\bar{z}} - f_{\bar{z}}(-i\bar{f})}{-2i|f|^2} = -\frac{f_{\bar{z}}}{f}, b_{(f,if)} = \frac{fif_{\bar{z}} - f_{\bar{z}}if}{-2i|f|^2} = 0. \end{aligned}$$

Consider the particular cases of this theorem. When f is constant, or is complex analytic, we obtain the space of holomorphic functions $\Omega(0, 0)$.

Proposition 7 *If f is real and $f \neq 0$, then $(f, \frac{i}{f})$ generates the space $\Omega(0, b)$.*

The proof obtained from directly computation:

$$\begin{aligned} \operatorname{Im}(f\frac{i}{f}) &= 1 > 0, \text{ because } \bar{f} = f; a_{(f,\frac{i}{f})} = \frac{-f(\frac{if_{\bar{z}}}{f^2}) - f_{\bar{z}}(-\frac{i}{f})}{-2i} = 0; \\ b_{(f,\frac{i}{f})} &= -\frac{-f(\frac{if_{\bar{z}}}{f^2}) - f_{\bar{z}}(\frac{i}{f})}{-2i} = \frac{f_{\bar{z}}}{f}; \end{aligned}$$

Proposition 8 *From $\omega \in \Omega(a, 0)$ follows, that $\dot{\omega} \in \Omega(a, 0)$.*

By proposition 6 the generating pair of the space $\Omega(a, 0)$ is $(f, \frac{i}{f})$. The function $\dot{\omega}$ satisfies the equation (9), therefore it is necessary to compute B from (3). It is easy, that $B = 0$.

In case, when the function F, G are complex analytic, then from (2) follows, that we obtain the space of holomorphic function $\Omega(0, 0)$, but this space not "isomorphic" to induced from $(1, i)$ generating pair space of holomorphic functions, because at follows from (3), B not equal to zero. From this follows, that this space have period $N > 1$. In [3] shows, that period this space is equal to 2.

Proposition 9 [?] *1) There exist real analytic function b in the a neighborhood of the origin, such that the space $\Omega(0, b)$ has minimum period infinity.*

2) For each positive integer N there exists a real analytic function b in the neighborhood of the origin, such that the space $\Omega(0, b)$ has minimum period N .

A necessary and sufficient condition that $\Omega(a, b)$ generated by (F, G) have the period one, obtained by Bers, and proved, that this conditions is identity $\frac{F}{G} = \tau(y)$. Bers proved also, that if $\frac{F}{G} = \sigma(x)$, then the minimum period is at most two.

Remark. Markushevich observed that every system of linear partial differential equations

$$c_i u_x + d_i v_x = a_i u_y + b_i v_y, \quad i = 1, 2 \quad (13)$$

with sufficiently smooth coefficients $a_1(x, y), \dots, d_2(x, y)$ can be written in a form such that $\frac{\partial c_i}{\partial x} = \frac{\partial a_i}{\partial y}, \frac{\partial d_i}{\partial x} = \frac{\partial b_i}{\partial y}, i = 1, 2$. In this case the integrals

$$U = \int (a_2 u + b_2 v) dx + (c_2 u + d_2 v) dy, \quad V = \int (a_1 u + b_1 v) dx + (c_1 u + d_1 v) dy$$

are path-independent and (u, v) satisfy a system (13_1) which is of the same form as (13)(see [6]) system (13) is said to be embedded into a cycle if there exists a sequence of systems (13), $(13_1), (13_2), \dots$ such that (13_i) is related to (13_{i+1}) as (13) was related to (13_1) . The cycle is called of finite order n if (13_n) is equivalent to (13), of infinite order if there is no such n . In [4] Lukomskaya (a) proves that every (13) can be embedded into a cycle of infinite order, and (b) gives necessary and sufficient conditions in order that the minimum order n_{min} of a cycle beginning with (13) be 1. In [2] states as an open problem the question on the existence of systems with $n_{min} > 2$. We remark that for elliptic systems the natural setting for this problem is the theory of pseudo-analytic functions [2] and finely result in this direction gives Protter [3] solving the so called *periodicity problem* for pseudoanalytic functions.

References

- [1] I.N. Vekua, *Generalized analytic functions*. 2rd Edition, Nauka, 1988.
- [2] L. Bers. Theory of pseudo-analytic functions. - New York University, 1953.
- [3] M.H.Protter. The periodicity problem for pseudoanalytic functions. Ann. of Math. (2) 64, 1956, pp. 154-174

- [4] M. Lukomskaya. On cycles of systems of linear homogeneous differential equations. Mat. Sbornik (N.S)., 29, 1951, pp. 551-558
- [5] M. Lukomskaya. Solution of some systems of partial differential equations by means of inclusion in a cycle. Akad.Nauk SSSR., Prikl.Mat.Meh., vol.17, 1953, pp.745-747
- [6] M. I.G.Petrovskii. On some problems of the theory of the theory of partial differential equations. Uspekhi Matem. Nauk, vol. 1, 1946, pp. 44-70

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On the Darboux transformation for Carleman-Bers-Vekua system

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Abstract

In this paper we used Darboux transformation technique for investigation of stationary Schrödinger two dimensional equation and s.c. main Vekua equation.

1 The basic fact and definitions.

The theory of pseudoanalytic functions have the goal of applying complex analysis methods to systems of partial differential equations which are more general than Cauchy-Riemann systems [1], [2]. Recently in [4] give new application of the theory of pseudoanalytic functions to differential equations of mathematical physics.

The canonical form of a uniformly elliptic linear first-order system for two desired real-valued functions in a domain of the complex plane has the form

$$w_{\bar{z}} = a(z)w + b(z)\bar{w}, \quad (1)$$

which is known as Carleman-Bers-Vekua system. If f is a real valued function, then

$$w_{\bar{z}} = \frac{f_{\bar{z}}}{f}\bar{w} \quad (2)$$

is called the corresponding main Vekua equation. In ([4]) author's applications of pseudoanalytic functions to differential equations of mathematical physics are based on the factorization of a second order differential operator in a product of two first order differential operators whose one of these

two factors leads to a main Vekua equation. In particular it is shown, that if f, h, ψ are real-valued functions, $f, \psi \in C_2(\Omega)$, $\Omega \subset \mathbb{C}$ and besides f is positive particular solution of the two dimensional stationary Schrödinger equation

$$(-\Delta + h)f = 0 \quad (3)$$

in domain $\Omega \subset \mathbb{C}$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is two dimensional Laplace operator, then

$$(\Delta - h)\psi = 4\left(\partial_{\bar{z}} + \frac{f_z}{f}C\right)\left(\partial_{\bar{z}} - \frac{f_z}{f}C\right)\psi = 4\left(\partial_z + \frac{f_{\bar{z}}}{f}C\right)\left(\partial_z - \frac{f_{\bar{z}}}{f}C\right)\psi, \quad (4)$$

where C denotes the operator of complex conjugation.

Let $w = w_1 + iw_2$ be a solution of the equation (2). Then the functions $u = f^{-1}w_1$ and $v = fw_2$ are the solutions of the following conductivity and associated conductivity equations

$$\operatorname{div}(f^2\nabla u) = 0, \text{ and } \operatorname{div}(f^2\nabla v) = 0, \quad (5)$$

respectively. The real and imaginary part of the solution of the equation (2) w_1 and w_2 are solutions of the stationary Schrodinger and associated stationary Schrodinger equations

$$-\Delta w_1 + r_1 w_1 = 0 \quad \text{and} \quad -\Delta w_2 + r_2 w_2 = 0, \quad (6)$$

respectively, where $r_1 = \frac{\Delta f}{f}$, $r_2 = \frac{2(\nabla f)^2}{f^2} - r_1$, $\nabla f = f_x + f_y$ and $(\nabla f)^2 = f_x^2 + f_y^2$.

In other hand it is known that the elliptic equation

$$\partial_z \partial_{\bar{z}} \psi + h\psi = 0 \quad (7)$$

is covariant with respect to the Darboux transformation [3]

$$\psi \rightarrow \psi[1] = \theta(\psi, \psi_1)\psi_1^{-1}, \quad \theta(\psi, \psi_1) = \int_{(z_0, \bar{z}_0)}^{(z, \bar{z})} \Omega, \quad (8)$$

$$h[1] = h + 2\partial_z \partial_{\bar{z}} \ln \psi_1, \quad (9)$$

where ψ_1 is a fixed solution of equation (7), Ω is closed 1-differential form

$$\Omega = (\psi \partial_z \psi_1 - \psi_1 \partial_z \psi) dz - (\psi \partial_{\bar{z}} \psi_1 - \psi_1 \partial_{\bar{z}} \psi) d\bar{z}.$$

Here covariant properties means, that $\psi[1]$ satisfies the following equation

$$\partial_z \partial_{\bar{z}} \psi[1] + h[1]\psi[1] = 0.$$

From the equality $d\Omega = 0$ follows, that the function $\theta(\psi_1, \psi)$ in (8) does not depend on the path of integration.

2 Main result

Theorem 1 *Let $w = w_1 + iw_2$ is the solution of the main Vekua equation*

$$w_{\bar{z}} = \frac{\psi_{1\bar{z}}}{\psi_1} \bar{w}. \quad (10)$$

Then $w_1 = \psi_1$ and $w_2 = -\frac{1}{2}\psi[1]$, where ψ_1 is the real positive solution of the equation

$$-\Delta\psi + h\psi = 0 \quad (11)$$

$h = \frac{\Delta\psi_1}{\psi_1}$ and $\psi[1]$ its Darboux transformation defined by (8), (9).

Conversely, if ψ_1 is the real positive solution of the equation (11) and $\psi[1]$ its Darboux transformation, then the solution of main Vekua equation (10) equal to $w = \psi_1 - \frac{1}{2}\psi[1]$.

First part of the theorem follows from (6), (1). Here we prove the second part of theorem. Let ψ is real solution of (11), then in this case the Darboux transformation (8),(9) has the form

$$h[1] = h - 2\Delta\ln\psi_1 \quad \text{and} \quad \psi[1] = 2i\psi_1^{-1} \text{Im} \int (\psi\psi_{1\bar{z}} - \psi_{\bar{z}}\psi_1) d\bar{z}.$$

We seek the solution of the equation (10) in the form $w = \psi + iw_2$. Then

$$\psi_{1\bar{z}} + iw_{2\bar{z}} = \frac{\psi_{1\bar{z}}}{\psi_1} \psi - i \frac{\psi_{1\bar{z}}}{\psi_1} w_2,$$

from this the solution of the corresponding homogenous equation is $w_2 = \frac{C(z)}{\psi_1}$, where $C(z)$ arbitrary holomorphic function. Let an $w_2 = \frac{C(z, \bar{z})}{\psi_1}$ solution of above equation. Then

$$\begin{aligned} \psi_{\bar{z}} + i \frac{C_{\bar{z}}}{\psi_1} - i \frac{\psi_{1\bar{z}}}{(\psi_1)^2} C(z, \bar{z}) &= \frac{\psi_{1\bar{z}}}{\psi_1} \psi - i \frac{\psi_{1\bar{z}}}{(\psi_1)^2} C(z, \bar{z}) \Rightarrow \\ \Rightarrow C_{\bar{z}} &= -i(\psi\psi_{1\bar{z}} - \psi_{\bar{z}}\psi_1) \Rightarrow C(z, \bar{z}) = -i \int (\psi\psi_{1\bar{z}} - \psi_{\bar{z}}\psi_1) d\bar{z} + \tilde{C}(z). \end{aligned}$$

From this we obtain that

$$w_2 = \psi_1^{-1}(b(z) - i \int (\psi\psi_{1\bar{z}} - \psi_{\bar{z}}\psi_1) d\bar{z}).$$

We choose $b(z)$ in last expression such, that w_2 is real. Then $w_2 = \psi_1^{-1} \text{Im} \int (\psi \psi_{1\bar{z}} - \psi_{\bar{z}} \psi_1) d\bar{z}$, from this follows, that $-2iw_2 = \psi[1]$, therefore $w = \psi_1 - \frac{1}{2}\psi[1]$ is the solution of (11).

Here we give new formulation and proof of theorem 1.

Theorem 2 1) Let $W = W_1 + iW_2$ is the solution of the equation $W_{\bar{z}} = \frac{f_{\bar{z}}}{f} \bar{W}$, then W_1 and W_2 related to by Darboux transformation $W_2 = iW_1[1]$ and $W_1 = -iW_1[1]$.

2) If W_1 is a solution the equation $(\Delta - \frac{\Delta f}{f})\psi = 0$, then $W_1 - W_1[1]$ is the solution of the equation $W_{\bar{z}} = \frac{f_{\bar{z}}}{f} \bar{W}$.

3) If W_2 is the solution of the equation $(\Delta + \frac{\Delta f}{f} - 2(\frac{\nabla f}{f})^2)\psi = 0$, then $-iW_2[1] + iW_2$ is a solution of the equation $W_{\bar{z}} = \frac{f_{\bar{z}}}{f} \bar{W}$.

From the theorem 33 [4] follows, that $W_1 + iW_2 = W$ is the solution of the equation $W_{\bar{z}} = \frac{f_{\bar{z}}}{f} \bar{W}$, then

$$W_2 = f^{-1} \bar{A}[if^2 \partial_{\bar{z}}(f^{-1}W_1)] \text{ and } W_1 = -f \bar{A}[i\bar{f}^2 \partial_{\bar{z}}(fW_2)],$$

where $\bar{A}[\phi] = 2\text{Re} \int \phi d\bar{z} = 2\text{Im} \int i\phi d\bar{z}$. Therefore,

$$W_2 = -f^{-1} 2\text{Im} \int f^2 \partial_{\bar{z}}(f^{-1}W_1) d\bar{z} \text{ and } W_1 = f 2\text{Im} \int f^{-2} \partial_{\bar{z}}(fW_2) d\bar{z}.$$

Consider the equation $(\Delta - \frac{\Delta f}{f})\psi = 0$ and take the function f as particular solution of this equation, then by theorem 33 [4] the function W_1 is the solution of this equation. Consider the Darboux transformation W_1 :

$$W_1 \rightarrow W_1[1] = f^{-1} \int \Omega(W_1, f),$$

$$\Omega(W_1, f) = (W_1 f_z - W_{1z} f) dz - (W_1 f_{\bar{z}} - W_{1\bar{z}} f) d\bar{z} = 2i\text{Im}[f^2 \partial_{\bar{z}}(f^{-1}W_1)],$$

$$W_1[1] = f^{-1} 2i\text{Im} \int f^2 \partial_{\bar{z}}(f^{-1}W_1) d\bar{z}.$$

Therefore $W_2 = iW_1[1]$.

New, consider the function $\frac{1}{f}$ as particular solution of the equation $(\Delta - \frac{\Delta f}{f})\psi = 0$, then from theorem 33 [4] follows, that W_2 is a solution of this equation. Consider the Darboux transformation of W_2 :

$$W_2 \rightarrow W_2[1] = (\frac{1}{f})^{-1} \int \Omega(W_2, f^{-1}) = f \int \Omega(W_2, f^{-1}),$$

$$\begin{aligned}\Omega(W_2, f^{-1}) &= (W_2\partial_z(\frac{1}{f}) - W_{2z}\frac{1}{f})dz - (W_2\partial_{\bar{z}}(\frac{1}{f}) - W_{2\bar{z}}\frac{1}{f})d\bar{z} = \\ &= (-W_2\frac{f_{\bar{z}}}{f^2} - W_{2z}\frac{1}{f})dz + \frac{1}{f^2}(W_2f_{\bar{z}} + W_{2\bar{z}}f)d\bar{z} = 2iIm[f^{-2}\partial_{\bar{z}}(fW_2)].\end{aligned}$$

Therefore, $W_1 = -iW_2[1]$.

Remark. In [5] the authors studied intertwining relations, supersymmetry and Darboux transformations for time-dependent generalized Schrodinger equations and obtained intertwiners in an explicit form, it means that it is possible to construct arbitrary-order Darboux transformations for some class of equations. The authors developed a corresponding supersymmetric formulation and proved equivalence of the Darboux transformations with the supersymmetry formalism. In our opinion the method given in this paper it is possible to use in this direction also.

References

- [1] I.N. Vekua, *Generalized analytic functions*. 2nd Edition, Nauka, 1988.
- [2] Bers. Theory of pseudo-analytic functions. - New York University, 1953.
- [3] V.B.Matveev, M.A. Salle. Darboux Transformations and Solitons. Springer, 1991
- [4] V.V. Kravchenko. Applied pseudoanalytic function theory. Birkhaser, 2009
- [5] A.A. Suzko, G. Giorgadze, Phys. Part. Nucl., (2008) Vol. 39, No.4. 1115-1149

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Some properties of the space of generalized analytic functions*

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Abstract

In this paper we investigate the relationship between the holomorphic and conformal structures and the spaces of generalized analytic functions induced from conformal structures.

1 Introduction

Let $s_1, \dots, s_m \in \mathbb{CP}^1$ be some points, with no ∞ among them, $\varrho : \pi_1(\mathbb{CP}^1 \setminus \{s_1, \dots, s_m\}, z_0) \rightarrow \mathrm{GL}_n(\mathbb{C})$ be a representation. The Riemann-Hilbert monodromy problem consists in the following: for the representation ϱ , find such a Fuchsian system $df = \left(\sum_{j=1}^m \frac{A_j}{z-s_j} dz \right) f$, whose monodromy representation coincides with ϱ , where A_j are constant matrices satisfying the condition $\sum_{j=1}^m A_j = 0$ [4], [5].

The configuration of the points depends on the solving Riemann-Hilbert monodromy problem [4] and it is important problem point of view inverse problem of mathematical physics. In this direction obtained important results: first, it is proved that monodromy representation for which Riemann-Hilbert monodromy problem have positive solutions are dense in the space of all representation of the fundamental group of $CP^1 \setminus \{s_1, \dots, s_m\}$, and second, it is known the Shlesinger theorem on the isomonodromic deformation of the singular point of the Fuchsian system. In our opinion is important connect

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independently of configuration of singular points of positive (or negative) solving of the Riemann-Hilbert monodromy problem by Beltrami equation and by deformation complex structures. Here we give some preliminary results of this direction.

The class of general first order elliptic system of partial differential equations

$$\frac{\partial w(z)}{\partial \bar{z}} - \mu_1(z) \frac{\partial w(z)}{\partial z} - \mu_2(z) \frac{\partial \bar{w}}{\partial z} = A(z)w(z) + B(z)\overline{w(z)} + C(z), \quad (1)$$

on the complex plane with natural restrictions on the functions μ_1, μ_2, A, B , contains many well-known equations: Cauchy-Riemann, Beltrami, Carleman-Bers-Vekua, holomorphic disc and other equations, which are obtained from (1) by appropriate choice of the coefficients (see [7], [2], [3]). All these equations are "deformations" of the Cauchy-Riemann equation and the properties of the solution spaces of the corresponding equations are close to the properties of the spaces of analytic functions.

Let X is a smooth oriented surface. A *conformal structure* on X is given by a smooth Riemann metric on X given up to multiplication by a positive smooth function on X . It is known, that for every Riemann metric on a neighborhood of $z \in X$ we can find local coordinates at z such that the metric takes the form $a(x, y)(dx^2 + dy^2)$. This reduction connected to Beltrami equation and deformation of complex structures of the Riemann surfaces.

The results given in this paper on the deformation of complex structures one can use to study deformation of complex structures of Riemann surfaces and deformation of the complex structures for holomorphic vector bundles on the Riemann surfaces. Deformation of complex structures defines by the Beltrami differential and the deformation of complex structures of the holomorphic vector bundles defines connection. Thus the spaces of the connections with the logarithmic singularities and the space of the Fuchsian systems are identifies.

The general construction of the deformation of the spaces by the cocycles is possible to apply for the construction the family of the connections, thus for construction of the isomonodromic deformation of the Fuchsian system for Riemann sphere. It is clear, that by isomonodromic deformation of the singular points of the Fuchsian system topological characteristics induced from monodromy representation do not changed.

Below we consider the relationship between solution spaces of the following equations induced from complex structure:

a) The Carleman-Bers-Vekua equation [7]

$$\frac{\partial w(z)}{\partial \bar{z}} = A(z)w(z) + B(z)\overline{w(z)};$$

b) The Beltrami equation [2]

$$\frac{\partial w(z)}{\partial \bar{z}} = \mu_1(z) \frac{\partial w(z)}{\partial z};$$

c) The holomorphic disc equation [6]

$$\frac{\partial w(z)}{\partial \bar{z}} = \mu_1(z) \frac{\partial \overline{w(z)}}{\partial z}.$$

These equations are invariant with respect to conformal transformations and therefore are correctly defined on Riemann surfaces. The functions A, B define the pair of complex functions (F, G) , satisfying the inequality $Im(\overline{F}G) > 0$ and (F, G) -pseudo-analytic functions are solutions of the Carleman-Bers-Vekua equation and vice versa [1].

2 Almost complex structure

Let X be a two-dimensional connected smooth manifold. By definition two complex atlases U and V are equivalent if their union is a complex atlas. A complex structure on X is an equivalence class of complex atlases. A Riemann surface is a connected surface with a complex structure. A differential 1-form on X with respect to a local coordinate z can be represented in the form $\omega = \alpha dz + \beta d\bar{z}$. Thus, ω has bidegree (1,1) and is a sum of the forms $\omega^{1,0} = \alpha dz$ and $\omega^{0,1} = \beta d\bar{z}$ of bidegree (1,0) and (0,1) respectively. The change of local coordinate $z \rightarrow iz$ induces on the differential forms the mapping given by $\omega \rightarrow i(\alpha dz - \beta d\bar{z}) = i\omega^{1,0} - i\omega^{0,1}$. Denote by J the operator defined on 1-forms by the rule $J\omega = i\omega^{1,0} - i\omega^{0,1}$. This operator does not depend on the change of the local coordinate z and $J^2 = -1$, where 1 denotes the identity operator. Therefore, the splitting $\Lambda^1 = \Lambda^{1,0} + \Lambda^{0,1}$ is the decomposition of the space of differential 1-forms into eigenspaces of $J : T^*(X)_{\mathbb{C}} \rightarrow T^*(X)_{\mathbb{C}}$. On the tangent space TX the operator J acts via $\omega(Jv) = (J\omega)(v)$, for every vector field $v \in TX$. If $z = x + iy$ and taking $v = \frac{\partial}{\partial x}$, one has

$$dz(Jv) = idz\left(\frac{\partial}{\partial x}\right) = i = dz\left(\frac{\partial}{\partial y}\right) \Rightarrow J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

It means that on the basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ of TX the operator J is given by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Therefore, by the complex structure defined from local coordinates defines the operator $J : T^*(X)_{\mathbb{C}} \rightarrow T^*(X)_{\mathbb{C}}$, with the property $J^2 = -1$. This operator is called an almost complex structure.

Conversely, let X be a smooth surface and let $J : T_x(X) \rightarrow T_x(X)$, $x \in X$, be such an operator, i. e. $J^2 = -1$. The pair (X, J) is called a pseudoanalytic surface. As above, by duality it is possible to define J on 1-forms on X . The space of 1-forms Λ^1 decomposes into eigenspaces corresponding to the eigenvalues $\pm i$ of J and $\Lambda^1 = \Lambda_J^{1,0} + \Lambda_J^{0,1}$. In particular, $J\Lambda_J^{1,0} = i\Lambda_J^{1,0}$ and $J\Lambda_J^{0,1} = -i\Lambda_J^{0,1}$.

Let f be a smooth function, then $df \in \Lambda^1$ and decomposes by bidegree as $df = \partial_J f + \bar{\partial}_J f$, where $\partial_J f := (df)_J^{1,0}$ and $\bar{\partial}_J f := (df)_J^{0,1}$. By definition, f is J -holomorphic, if it satisfies the Cauchy-Riemann equation $\bar{\partial}_J f = 0$.

Let (X, J) be a pseudoanalytic surface. In the neighborhood of every point $x \in X$ it is possible to choose the local coordinate in such a way that dz will be of $(1, 0)_J$ -type. Then the decomposition of dz by bidegree is $dz = \omega + \bar{\delta}$, where ω, δ are forms of bidegree $(1, 0)_J$. Because the fibre of $T_J^{1,0}X$ is a one-dimensional complex space, we have $\delta = \mu\bar{\omega}$, where μ is some smooth function $\mu(0) = 0$. From this it follows, that

$$dz = \omega + \mu\bar{\omega} \text{ and } d\bar{z} = \bar{\omega} + \bar{\mu}\omega. \quad (2)$$

Therefore, for every smooth function f in the neighborhood of $x \in X$ we have

$$df = (\partial f + \bar{\mu}\bar{\partial}f)\omega + (\bar{\partial}f + \mu\partial f)\bar{\omega} = \partial_J f + \bar{\partial}_J f.$$

From this it follows, that f is J -holomorphic iff $\bar{\partial}_J f = 0$, i.e.

$$\bar{\partial}f + \mu\partial f = 0. \quad (3)$$

The equation (3) is called the Beltrami equation. Thus a smooth function defined on a pseudoanalytic surface (X, J) is J -holomorphic iff it satisfies the Beltrami equation (3).

Suppose f is J -holomorphic and let $f = \varphi + i\psi$, where φ and ψ are real-valued functions. Consider the complex-valued function w defined by the identity $w = \varphi F + \psi G$, where F, G are complex-valued Hölder continuous functions satisfying the condition $Im(\bar{F}G) > 0$.

Theorem 2.1 *The function $w = \varphi F + \psi G$ is (F, G) -pseudo-analytic.*

Proof. Indeed, $w = \varphi F + \psi G = \frac{iG-F}{2}f + \frac{-iG-F}{2}\bar{f}$, from which it follows, that f is a solution of the Beltrami equation

$$(iG - F)\bar{\partial}f - (iG + F)\partial f = 0$$

iff w is a solution of the Carleman-Bers-Vekua equation

$$\bar{\partial}w + \frac{\overline{F\partial G} - \bar{\partial}F\bar{G}}{F\bar{G} - F\bar{G}}w + \frac{F\bar{\partial}G - \bar{\partial}F\bar{G}}{F\bar{G} - F\bar{G}}\bar{w} = 0.$$

In $D \subset \mathbb{C}$ every metric has the form $\lambda|dz + \mu d\bar{z}|$, where $\lambda > 0$ and the complex function μ satisfies $|\mu| < 1$, from which it follows, that J is defined in a unique way by the 1-form $\omega = dz + \mu d\bar{z}$ on D with properties $J\omega = i\omega$, $J\bar{\omega} = -i\bar{\omega}$. The forms of this type are forms of bidegree $(1,0)$ with respect to J (the space of such forms has been denoted above by $\Lambda_J^{1,0}$). If $\delta \in \Lambda_J^{1,0}$, then $\delta = \alpha\omega + \beta\bar{\omega}$ and it is proportional to ω . Therefore J is determined uniquely up to a constant multiplier by $(1,0)_J$ -holomorphic form ω . Functions holomorphic with respect to J have differentials proportional to ω . Indeed, if $df + iJ(df) = 0$, then $J(df) = idf$ and from the representation $df = \alpha\omega + \beta\bar{\omega}$ we obtain, that $\beta\bar{\omega} = 0$. Because $df = \alpha\omega + \beta\bar{\omega}$, in $D \subset \mathbb{C}$ the Cauchy-Riemann equation with respect to J with base form $\omega = dz + \mu d\bar{z}$ can be represented as the Beltrami equation $\bar{\partial}f = \mu\partial f$. This equation has a solution f such that it is a biholomorphic map from (D, J) to $(f(D), J_{st})$, where J_{st} is the standard conformal complex on \mathbb{C} .

Therefore we have proved the following proposition.

Proposition 2.1 *On simply connected areas there exists only one complex structure and conformal structures are in one-to-one correspondence with complex functions μ with $|\mu| < 1$.*

From this proposition and theorem 2.1 follows the proposition

Proposition 2.2 *There exists a one-to-one correspondence between the space of conformal structures and the space of generalized analytic functions on each simply connected open area of the complex plane.*

3 The equaton of holomorphic discs

Let \mathbb{D} be the unit disc in the complex plane \mathbb{C} with the standard complex structure J_{st} and the coordinate function ζ . J_{st} is uniquely determined by

the form $d\zeta \in \Lambda_{J_{st}}^{1,0}$. The map $\phi : \mathbb{D} \rightarrow X$ of class C^1 is holomorphic iff $\psi^* \Lambda_J^{1,0}(X) \subset \Lambda^{1,0}(\mathbb{D})$. Let z be another coordinate function on \mathbb{D} . We study a local problem, therefore, without loss of generality, it is possible to consider ϕ as a mapping from (\mathbb{D}, J_{st}) to (\mathbb{C}_z, J) , where the complex structure J is defined by $dz = \omega + \mu\bar{\omega}$, $\bar{\omega} \in \Lambda_J^{1,0}$. Therefore we have

$$\zeta \rightarrow z = z(\zeta), z(0) = 0.$$

From (2) we obtain that

$$\omega = \frac{dz - \mu d\bar{z}}{1 - |\mu|^2}.$$

The form ω is J -holomorphic, which means that the form

$$z^*(dz - \mu d\bar{z}) = (\partial_\zeta z - \mu \partial_\zeta \bar{z})d\zeta + (\partial_{\bar{\zeta}} z - \mu \partial_{\bar{\zeta}} \bar{z})d\bar{\zeta}$$

has bidegree $(1, 0)$ on \mathbb{D} , therefore

$$\partial_{\bar{\zeta}} z - \mu \partial_{\bar{\zeta}} \bar{z} = 0.$$

From this after using the identity $\partial_{\bar{\zeta}} \bar{z} = \overline{\partial_\zeta z}$ we obtain, that

$$\partial_{\bar{\zeta}} z = \mu(z) \overline{\partial_\zeta z}. \quad (4)$$

The obtained expression is called the *equation of holomorphic disc*. It is known that f satisfies this equation iff f^{-1} satisfies the corresponding Beltrami equation (see [6]).

Proposition 3.1 *If $\omega = u + iv$ satisfies the equation $\frac{\partial \omega(z)}{\partial \bar{z}} + \mu(z) \overline{\frac{\partial \omega(z)}{\partial z}} = 0$, $|\mu| < 1$ and a and b are holomorphic functions such that $\mu = \frac{a-b}{a+b}$, then $W = au + ibv$ is holomorphic.*

Proof. Indeed,

$$\frac{\partial}{\partial \bar{z}} \left(a \frac{\omega + \bar{\omega}}{2} + ib \frac{\omega - \bar{\omega}}{2} \right) = \frac{a}{2} (\omega_{\bar{z}} + \bar{\omega}_{\bar{z}}) + \frac{b}{2} (\omega_{\bar{z}} - \bar{\omega}_{\bar{z}}) = \omega_{\bar{z}} \left(\frac{a+b}{2} \right) + \bar{\omega}_{\bar{z}} \left(\frac{a-b}{2} \right),$$

therefore if ω is a solution of the equation $\omega_{\bar{z}} + \frac{a-b}{a+b} \bar{\omega}_{\bar{z}} = 0$, then $\partial_{\bar{z}} W = 0$.

From this proposition it follows in particular, that W is (a, ib) -pseudo-analytic.

References

- [1] L. Bers, *Theory of pseudo-analytic functions*. New York University, 1953.
- [2] B.Bojarski, *Generalized solutions of the PD system of the first order and elliptic type with discontinuous coefficients*. Mat.Sbornik **43** (1957), 451–503.
- [3] Bojarski, Bogdan On the Beltrami equation, once again: 54 years later. Ann. Acad. Sci. Fenn. Math. 35 (2010), no. 1, 59-73,
- [4] Bolibruch, A. A. On isomonodromic deformations of Fuchsian systems. J. Dynam. Control Systems 3 (1997), no. 4, 589-604.
- [5] Bolibruch, A. A. On orders of movable poles of the Schlesinger equation. J. Dynam. Control Systems 6 (2000), no. 1, 57-73
- [6] E.M.Chirka, *Riemann surfaces*. Steklov Math.Inst. Moscow, 2006.
- [7] I.N. Vekua, *Generalized analytic functions*. 2rd Edition, Nauka, 1988.

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On the boundary value problem of linear conjugation with a piecewise continuous coefficient on Carleson curves*

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Let Γ is Jordan closed rectifiable curve. Denote by D_{Γ}^{+} and D_{Γ}^{-} domains dividing the extended plane by this curve (we assume that $\infty \in D_{\Gamma}^{-}$).

We say that $\Gamma \in R$, or Γ is Karleson curve, if singular integral

$$S\varphi \equiv \int_{\Gamma} \frac{\varphi(t)}{t-\tau} dt$$

is bounded operator in $L_p(\Gamma)$, $p > 1$.

As usual Cauchy type integral is called

$$\Phi(z) \equiv \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(t)}{t-z} dt \equiv (K\varphi)(z), \quad \varphi \in L_1(\Gamma). \quad (1)$$

We say that $\Phi(z) \in K_p(D_{\Gamma}^{+})$ or $\Phi(z) \in K_p(D_{\Gamma}^{-})$ if representation (1) has a sense in D_{Γ}^{+} , or correspondingly in D_{Γ}^{-} and $\varphi(t) \in L_p(\Gamma)$, $p \geq 1$ and we say, that $\Phi(z) \in \tilde{K}_p(D_{\Gamma}^{-})$, if $\Phi(z) = \Phi_0(z) + P(z)$, where $P(z)$ is polynomial, $\Phi_0(z) \in K_{\Gamma}(D_{\Gamma}^{-})$.

We consider the boundary value problem of linear conjugation: find the function $\Phi(z) \in K_p(D_{\Gamma}^{\pm})$ satisfying condition

$$\Phi^{+}(z) = G(t)\Phi^{-}(z) + g(t), \quad t \in \Gamma, \quad (2)$$

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G and g are given functions, $g(t) \in L_p(\Gamma)$, $G(t)$ is piecewise continuous function, $\Gamma \in R$.

Let's assume $\{t_k\}_{k=1}^n$, $t_k \in \Gamma$, $t_n = t_1$ are the discontinuous points of function $G(t)$. Let's take as a positive direction on Γ direction, which leaves D_Γ^+ domain on the left side. For open arc with the ends a and b we use the notation Γ_{ab} . We consider direction from a to b as positive. In notation $t \rightarrow t_k^\pm$ we mean that tending is inside the arc $\Gamma_{t_k t_{k+1}}$, i.e. $t \in \Gamma_{t_k t_{k+1}}$. Also, $t_k \rightarrow t_k^-$, is $t \in \Gamma_{t_{k-1} t_k}$.

Denote $G(t_k + 0) \equiv \lim_{t \rightarrow t_k^+} G(t)$ and $G(t_k - 0) \equiv \lim_{t \rightarrow t_k^-} G(t)$. Arc $\Gamma_{t_k t_{k+1}}$ sometimes denote by Γ_k , but $\chi(\Gamma_k)$ is characteristic function for Γ_k .

To solve the boundary value problem 2 in above mentioned conditions usually was assumed that the curve will have at least one sided tangents in the points t_k .

It follows from the Seifullaev theorem that, if $\Gamma \in R$, $t_k \in \Gamma$, then limits

$$\overline{\lim}_{t \rightarrow t_k^+} \frac{\arg(t - t_k)}{|\ln |t - t_k||}, \quad \overline{\lim}_{t \rightarrow t_k^-} \frac{\arg(t - t_k)}{|\ln |t - t_k||}, \quad \lim_{t \rightarrow t_k^-} \frac{\arg(t - t_k)}{|\ln |t - t_k||}, \quad \underline{\lim}_{t \rightarrow t_k^-} \frac{\arg(t - t_k)}{|\ln |t - t_k||}$$

are finite.

We assume that

$$\lim_{t \rightarrow t_k^+} \frac{\arg(t - t_k)}{|\ln |t - t_k||} \quad \text{and} \quad \lim_{t \rightarrow t_k^-} \frac{\arg(t - t_k)}{|\ln |t - t_k||} \quad (3)$$

exist. Under $\arg(t - t_k)$ we mean any continuous branch. Our condition (3) is more weak than the usual one and admits infinite number of turns of the curve in t_k points (see for example [3, p. 12]).

For solving problem (2) we represent $G(t)$ as a product of two functions. Denote by

$$\begin{aligned} \omega^{(1)}(t) &\equiv \ln G(t) - \omega^{(2)}(t), \\ \omega^{(2)}(t) &\equiv \sum_{k=1}^n \left[\ln |G(t_k + 0)| + \right. \\ &\quad \left. + \frac{\ln G(t_{k+1} - 0) - \ln G(t_k + 0)}{t_{k+1} - t_k} (t - t_k) \right] \chi(\Gamma_{t_k t_{k+1}}), \quad t_1 = t_n. \end{aligned}$$

It's simple to verify, that

$$\omega^{(2)}(t_k + 0) = \ln G(t_k + 0), \quad \omega^{(2)}(t_k - 0) = \ln G(t_k - 0).$$

Under $\ln G(t)$ we mean continuous branch for which $0 \leq \arg G(t_k + 0) < 2\pi$.

Consider $G(t) = G_1(t) \cdot G_2(t)$, where $G_1(t) = \exp \omega^{(1)}(t)$, $G_2(t) = \exp \omega^{(2)}(t)$. From the well-known results [4], [5] it follows, that

$$X_1(z) \equiv \exp(K_\Gamma \ln G_1)(z) \in E_p(D_\Gamma^\pm), \quad \forall p > 1. \quad (4)$$

Recall that $\Phi(z) \in E_p(D)$, $p > 0$, D is D^+ or D^- , if $\Phi(z)$ is analytic of D and if there exists the sequence of rectifiable curves $\{\Gamma_n\}_{n=1}^\infty$ such that $\Gamma_n \subset D$, $\Gamma_n \rightarrow \Gamma$ and $\sup_n \int_{\Gamma_n} |\Phi(z)|^p dz < \infty$, $\Phi(\infty) = 0$.

Besides $X_1^+ \in W_p(\Gamma)$, $\forall p > 1$, this means, that

$$\|(X_1^+)^{-1} S X_1^+ \varphi\|_{L_p} \leq M_p \|\varphi\|_{L_p}. \quad (5)$$

To investigate the function $\exp(K \ln G_2)(z)$, we need the following

Lemma. *If Γ is closed Jordan curve $\Gamma \in R$, $a \in \Gamma$, A is complex number $A = A_1 + iA_2$, then $\exists \delta > 0$ such that*

$$(z - a)^A \in W_\delta(D^+)$$

(we mean the continuous branch).

Theorem 1. *If $\Gamma \in R$ is Jordan closed curve, then $\exists \delta > 0$ such that*

$$X_2(z) \in E_\delta(D_\Gamma^+).$$

Note, that in the neighborhood of t_k points

$$\exp(K\omega_2)(z) = \Phi_k(z) |z - t_k|^{m_k(z)}, \quad (6)$$

where

$$m_k(z) \equiv \left(\ln |G(t_k - 0)| - \ln |G(t_k + 0)| \right) \Delta_k + \varepsilon_k(z) + \frac{\arg G(t_k + 0) - \arg G(t_k - 0)}{2\pi},$$

but $\Phi_k(z) < M$, $\varepsilon_k(z)$ is a small number depending on the measure of the neighborhood of the points t_k . We can make it arbitrarily small. Consider the numbers \varkappa_k and α_k as follows

$$-\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad (7)$$

\varkappa_k are integer and $m_k(z) - \varepsilon_k(z) = \varkappa_k + \alpha_k$, i.e.

$$\begin{aligned} \ln |G(t_k - 0)| - \ln |G(t_k + 0)| \Delta_k + \frac{\arg G(t_k + 0) - \arg G(t_k - 0)}{2\pi} = \\ = \varkappa_k + \alpha_k. \end{aligned} \quad (8)$$

It's clear, for each p we can take $-\frac{1}{p} \leq \alpha_k < \frac{1}{q}$. We select such p , in order to fulfill (7). From (6) and (7) it's clear, that

$$\begin{aligned} X_2^+(t) &= \prod_{k=1}^n (t - t_k)^{-\varkappa_k} \exp K \ln G_2 \in L_p(\Gamma), \\ (X^{-1})^+(t) &\in L_p(\Gamma), \end{aligned}$$

from which we have

Theorem 2.

$$X_2(z) \equiv \exp(K \ln G_2)(z) \in E_p(D_\Gamma^+).$$

Therefore we obtain

Theorem 3.

$$\prod_{k=1}^n \left(\frac{z - t_k}{z - z_0} \right)^{-\varkappa_k} \exp(K \ln G_2)(z) \in E_p(D_\Gamma^-) + 1.$$

Theorem 2 and Theorem 3 give us

Theorem 4. $X_2(z)$ is factor-function of $G_2(t)$, but

$$X(z) \equiv X_1(z) \cdot X_2(z) = \exp(K \ln G)(z)$$

is factor-function of $G(t)$ in L_p with the index \varkappa , $\varkappa = \sum_{k=1}^n \varkappa_k$.

In classical case factor-function we usually write by means of two equalities, however one can find the same notations (for example in [6])

From abovementioned theorems it follows, that for the problem (2) the classical results are valid, we have the same representation for the solution as well as the solvability conditions. Difference is only in index formula. In our case the index is $\varkappa = \sum_{k=1}^n \varkappa_k$, where \varkappa_k is from formula (8). Hence the index depends on $\ln |G(t_k \pm 0)|$ and from the behavior of the curve in t_k points.

References

- [1] G. David, Opérateurs intégraux singuliers sur certaines courbes du plan complexe. (French) [Singular integral operators over certain curves in the complex plane] *Ann. Sci. École Norm. Sup. (4)* **17** (1984), No. 1, 157–189.
- [2] R. K. Seifullaev, A Riemann boundary value problem on a nonsmooth open curve. (Russian) *Mat. Sb. (N.S.)* 112(154) (1980), No. 2(6), 147–161.
- [3] A. Böttcher, Yu. I. Karlovich, Carleson curves, Muckenhoupt weights, and Toeplitz operators. Progress in Mathematics, 154. *Birkhauser Verlag, Basel*, 1997.
- [4] B. V. Khvedelidze, The method of Cauchy type integrals in discontinuous boundary value problems of the theory of holomorphic functions of a complex variable. (Russian) *Current problems in mathematics, Vol. 7 (Russian)*, pp. 5–162 (errata insert). *Akad. Nauk SSSR Vsesojuz. Inst. Nauchn. i Tehn. Informacii, Moscow*, 1975.
- [5] I. Ts. Gokhberg, N. Ya. Krupnik, Introduction to the theory of one-dimensional singular integral operators. (Russian) *Izdat. "Stinca", Kishinev*, 1973.
- [6] L. I. Chibrikova, The basic boundary value problems for analytic functions. (Russian) *Kazan Univ. Press, Kazan*, 1977.

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Dirichlet problem for holomorphic functions in spaces described by modulus of continuity

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1 Introduction

The following result is used in the study of boundary value problems associated with generalized analytic functions ([1], [2, p. 131]):

Theorem 1 *If a function g is defined on $\partial G = \{z \in \mathbb{C} : |z| = 1\}$ and is Hölder continuous with an exponent λ ($0 < \lambda < 1$), then there exists a unique holomorphic function f in $G = \{z \in \mathbb{C} : |z| < 1\}$ that is continuous in a closed disk and satisfies the conditions*

$$\Re f|_{\partial G} = g(z), \Im f|_{z=z_0} = c,$$

where $z_0 \in \partial G$ is a fixed point. Moreover, f is Hölder continuous with same exponent λ .

There is an interesting question is whether or not analogue of this theorem holds for classes of functions that are more general than Hölder spaces.

It is well known (see e.g. [2]), that the limiting values of the real and imaginary parts of a holomorphic function are expressed by the Hilbert transform formulas

$$\begin{cases} v(e^{i\gamma_0}) = -\frac{1}{2\pi} \int_0^{2\pi} u(e^{i\gamma}) \cot \frac{\gamma - \gamma_0}{2} d\gamma + v_0, v_0 = v(0, 0), \\ u(e^{i\gamma_0}) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\gamma}) \cot \frac{\gamma - \gamma_0}{2} d\gamma + u_0, u_0 = u(0, 0). \end{cases}$$

The singular Hilbert integral on the right-hand side is closely related to the Cauchy improper integral along the unit circle centered at the origin (see e.g. [3]).

Thus, this problem is related to the characterization of spaces that are invariant under the Hilbert and Cauchy integral operators. It is well known that Hölder spaces with exponents $\lambda \in (0, 1)$ and spaces L_p , $1 < p < \infty$, are invariant under these operators.

2 Φ space of moduli of continuity

A function $\mu : (0, l] \rightarrow \mathbb{R}$ is said to belong to Φ if it satisfies the following conditions:

- 1°. $\lim_{t \rightarrow +0} \mu(t) = 0$;
- 2°. $\mu(t)$ is almost increasing, that is there exists such a constant $c = c_\mu$ that for any $t_1, t_2 \in (0, l_0] : t_1 \leq t_2$ implies $\mu(t_1) \leq c\mu(t_2)$;
- 3°. $\sup_{t > 0} \frac{1}{\mu(t)} \int_0^t \frac{\mu(t)}{t} dt = A_\mu < \infty$;
- 4°. $\sup_{t > 0} \frac{t}{\mu(t)} \int_t^{l_0} \frac{\mu(t)}{t^2} dt = B_\mu < \infty$.

Here are examples of functions from space Φ :

1. $\mu(t) = t^\alpha$, $0 < \alpha < 1$;
2. $\mu(t) = t^\alpha \cdot (\ln \frac{1}{t})^p$, $0 < \alpha < 1$, $0 < p$, $t \in (0, \frac{1}{2}]$.

There are also functions that do not belong to Φ , for example, $\mu(t) = \frac{1}{\ln \frac{1}{t}}$.

The solution of the formulated problem is given by the following theorem ([4]).

Theorem 2 *Suppose that g is given on $\partial G = \{z \in \mathbb{C} : |z| = 1\}$ and satisfies the condition*

$$|g(e^{i\theta_1}) - g(e^{i\theta_2})| \leq C\mu(|e^{i\theta_1} - e^{i\theta_2}|),$$

where $\mu \in \Phi$. Then there exists a unique holomorphic function f in $G = \{z \in \mathbb{C} : |z| < 1\}$ that is continuous in a closed disk and satisfies the conditions

$$\Re f|_{\partial G} = g(z), \Im f|_{z=z_0} = c,$$

where $z_0 \in \partial G$ is a fixed point. Moreover, f satisfies

$$|f(z_1) - f(z_2)| \leq A\mu(|z_1 - z_2|)$$

for any $z_1, z_2 \in \overline{G}$.

3 Some generalizations of space Φ .

Let us consider a positive continuous almost decreasing function $\rho : (0, l] \rightarrow \mathbb{R}$ and integer $n \geq 0$. Let us also consider a positive continuous function $\mu : (0, l] \rightarrow \mathbb{R}$ with the property that is $\frac{\mu}{t}$ is almost decreasing. The function μ is said to belong to $\Phi_{A_n}^\rho$ if it satisfies the following conditions:

$$1_n. \lim_{t \rightarrow +0} \mu(t)\rho^n(t) = 0;$$

$$2_n. \mu(t)\rho^{n+1}(t) \text{ is almost increasing};$$

$$3_{A_n}. \exists A_\mu > 0 : \forall x \in (0; l_0] \int_0^x \frac{\mu(t)}{t} \rho^n(t) dt \leq A_\mu \mu(x) \rho^{n+1}(x);$$

$$4_{A_n}. \exists B_\mu > 0 : \forall x \in (0; l_0] \int_x^{l_0} \frac{\mu(t)}{t^2} dt \leq B_\mu \frac{\mu(x)}{x}.$$

Similarly, The function μ is said to belong to $\Phi_{B_n}^\rho$ if it satisfies $1_n, 2_n$ and

$$3_{B_n}. \exists A_\mu > 0 : \forall x \in (0; l_0] \int_0^x \frac{\mu(t)}{t} \rho^{n+1}(t) dt \leq A_\mu \mu(x) \rho^{n+1}(x);$$

$$4_{B_n}. \exists B_\mu > 0 : \forall x \in (0; l_0] \int_x^{l_0} \frac{\mu(t)}{t^2} dt \leq B_\mu \frac{\mu(x)}{x} \rho(x).$$

Spaces $\Phi_{A_n}^\rho, \Phi_{B_n}^\rho$ share some properties with space Φ , moreover, for any majorized ρ and any integer $n \geq 0$ implies that $\Phi_{A_n}^\rho = \Phi_{B_n}^\rho = \Phi$.

For spaces $\Phi_{A_n}^\rho, \Phi_{B_n}^\rho$ analogue of result for space Φ can be proved.

Theorem 3 Suppose that g is given on $\partial G = \{z \in \mathbb{C} : |z| = 1\}$ and satisfies the condition

$$|g(e^{i\theta_1}) - g(e^{i\theta_2})| \leq C\mu(|\theta_1 - \theta_2|)\rho^n(|\theta_1 - \theta_2|),$$

where $n \in \mathbb{Z}$, $n \geq 0$, $\mu \in \Phi_{A_n}^\rho \cup \Phi_{B_n}^\rho$. Then there exists a unique holomorphic function f in $G = \{z \in \mathbb{C} : |z| < 1\}$ that is continuous in a closed disk and satisfies the conditions

$$\Re f|_{\partial G} = g(z), \Im f|_{z=z_0} = c,$$

where $z_0 \in \partial G$ is a fixed point. Moreover, f satisfies

$$|f(z_1) - f(z_2)| \leq C\mu(|z_1 - z_2|)\rho^{n+2}(|z_1 - z_2|)$$

for any $z_1, z_2 \in \overline{G}$.

References

- [1] Tutschke W., *Vorlesungen über partielle Differentialgleichungen. Klassische, funktionalanalytische und komplexe Methoden*, Leipzig: Teubner-Texte zur Mathematik, 1978. 193 p.
- [2] Monakhov V. N., *Boundary Value Problems with Free Boundaries for Elliptic Systems of Equations*, Americal Mathematical Society, Providence, R.I., 1983. 522 p.
- [3] Zabreyko P. P., Koshelev A. I., Krasnosel'skii M. A., Mikhlin S. G., Rakovshchik L. S., Stet'senko V. Y., *Integral equations: A reference text*, Leyden: Noordhoff International Pub., 1975. 443 p.
- [4] Ilchukov A. S., Timofeev A. Y., *Dirichlet problem for holomorphic functions in spaces described by modulus continuity (in Russian)*, Bulletin of Udmurt University. Mathematics. Mechanics. Computer science/ 2010 (1). P. 58–65.

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Dolbeaut's lemma for the functions of the class $L_p^{loc}(\mathbb{C})$, $p > 2$

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Abstract

In this paper the existence of $\frac{\partial}{\partial \bar{z}}$ -primitive of the function of the class $L_p^{loc}(\mathbb{C})$, $p > 2$ is proved.

It is well-known that for the every function $a \in L_{p,2}(\mathbb{C})$, $p > 2$, by means of the integral

$$A(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{a(\zeta)}{\zeta - z} d\xi d\eta, \quad \zeta = \xi + i\eta \quad (1)$$

$\frac{\partial}{\partial \bar{z}}$ primitive [3] with respect of the generalized derivative $\frac{\partial}{\partial \bar{z}}$ in Sobolev on the whole plane is constructed[1], i.e.

$$\frac{\partial A}{\partial \bar{z}} = a. \quad (2)$$

The present work deals with Carleman-Vekua equations with irregular coefficients, therefore it is necessary to investigate the problem of existence of $\frac{\partial}{\partial \bar{z}}$ primitives of the functions, not belonging to the class $L_{p,2}(\mathbb{C})$, $p > 2$.

The following theorem is valid:

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Theorem 1. Every function $a(z)$ of the class $L_p^{\text{loc}}(\mathbb{C})$, $p > 2$, has $\frac{\partial}{\partial \bar{z}}$ primitive function $Q(z)$ on the whole complex plane, satisfying the Holder condition with the exponent $\frac{p-2}{p}$ on each compact subset of the complex plane E ; moreover if $q(z)$ is one $\frac{\partial}{\partial \bar{z}}$ primitives of the function $a(z)$ then all $\frac{\partial}{\partial \bar{z}}$ primitives of this function are given by formula

$$Q(z) = q(z) + \Phi(z), \quad (3)$$

where $\Phi(z)$ is an arbitrary entire function.

Proof. Let a arbitrary function $L_p^{\text{loc}}(\mathbb{C})$, $p > 2$.

Consider two sequences of positive numbers R_n and R'_n , $n = 1, 2, 3, \dots$, satisfying the conditions:

$$R'_1 < R_1; \quad R_{n-1} < R'_n < R_n, \quad n = 2, 3, \dots \quad (4)$$

$$\lim_{n \rightarrow \infty} R_n = +\infty. \quad (5)$$

Consider also the following sequences of the domains of complex planes:

$$\begin{aligned} G_1 &= \{z : |z| < R_1\}, \quad G_n = \{z : R_{n-1} < |z| < R_n\}, \quad n = 2, 3, \dots, \\ G'_n &= \{z : |z| < R'_n\}, \quad n = 1, 2, 3, \dots, \quad D_1 = G_1, \\ D_n &= \{z : |z| < R_n\}, \quad n = 2, 3, \dots \end{aligned}$$

and the boundaries of G_n

$$\gamma_n = \{z : |z| = R_n\}, \quad n = 1, 2, 3, \dots$$

Construct the sequences of the functions:

$$g_n(t) = -\frac{1}{\pi} \iint_{G_{n+1}} \frac{a(\zeta)}{\zeta - t} d\xi d\eta + \frac{1}{\pi} \iint_{G_n} \frac{a(\zeta)}{\zeta - t} d\xi d\eta, \quad (6)$$

$$\zeta = \xi + i\eta, \quad t \in \gamma_n, \quad n = 1, 2, 3, \dots$$

$$F_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{g_k(t)}{t - z} dt, \quad (7)$$

$$z \in E \setminus \gamma_k, \quad k = 1, 2, 3, \dots$$

It is clear that $g_n(t) = (T_{G_{n+1}}a)(t) - (T_{G_n}a)(t)$. Since $a \in L_p^{\text{loc}}(\mathbb{C})$, $p > 2$, then by virtue of the theorem 1.19 from [1] we have $T_{G_n}a, T_{G_{n+1}}a \in C_{\frac{p-2}{p}}(\gamma_n)$, $n = 1, 2, 3, \dots$. Hence it is evident, that every function

$$g_n(t) \in C_{\frac{p-2}{p}}(\gamma_n) \quad (8)$$

and every function $F_k(z)$ is holomorphic for each $z \in \mathbb{C} \setminus \gamma_k$, as they are Cauchy type integrals. Since the function $F_k(z)$ is holomorphic on the circle D_k , it can be expanded in Taylor series on D_k .

Assume it has the following form:

$$F_k(z) = \sum_{j=0}^{\infty} C_j^{(k)} z^j, \quad |z| < R_k, \quad (9)$$

and consider the sequence of positive numbers ε_k , $k = 1, 2, 3, \dots$, for which the series $\sum_{k=1}^{\infty} \varepsilon_k$ converges.

Since the Taylor-series (9) uniformly converges on the closed circle $\overline{G'_k} \subset D_k$, then there exist the natural numbers N_k such that for every natural $n > N_k$ -the following inequality

$$\left| F_k(z) - \sum_{j=0}^n C_j^{(k)} z^j \right| < \varepsilon_k, \quad z \in \overline{G'_k}. \quad (10)$$

is completely defined. In particular, if we assume that $n = N_k + 1$, then for complete defined polynomials

$$f_k(z) = \sum_{j=0}^{N_k+1} C_j^{(k)} z^j, \quad k = 1, 2, 3, \dots \quad (11)$$

the following inequality:

$$|F_k(z) - f_k(z)| < \varepsilon_k, \quad z \in \overline{G'_k}, \quad k = 1, 2, 3, \dots \quad (12)$$

holds.

Consider the function

$$\Phi(z) = \sum_{k=1}^{\infty} (F_k(z) - f_k(z)), \quad z \in \mathbb{C} \setminus \bigcup_{k=1}^{\infty} \gamma_k. \quad (13)$$

Let us fix arbitrary natural number n and let us prove that the function $\Phi(z)$ is a holomorphic function in the domain G_n . Represent the function $\Phi(z)$ as follows

$$\Phi(z) = \sum_{k=1}^n (F_k(z) - f_k(z)) + \sum_{k=n+1}^{\infty} (F_k(z) - f_k(z)), \quad z \in G_n. \quad (14)$$

Using the inequalities (13), we have

$$\begin{aligned} |F_{n+1}(z) - f_{n+1}(z)| &< \varepsilon_{n+1}, & z \in \overline{G'_{n+1}} \\ |F_{n+2}(z) - f_{n+2}(z)| &< \varepsilon_{n+2}, & z \in \overline{G'_{n+2}} \\ \dots\dots\dots & & \dots\dots\dots \\ |F_{n+l}(z) - f_{n+l}(z)| &< \varepsilon_{n+l}, & z \in \overline{G'_{n+l}} \\ \dots\dots\dots & & \dots\dots\dots \end{aligned} \quad (15)$$

(4) implies that the following inclusion

$$G_n \subset \overline{G'_{n+1}} \subset \overline{G'_{n+2}} \subset \dots \subset \overline{G'_{n+l}} \subset \dots \quad (16)$$

take place, thus from (15) we have

$$|F_{n+j}(z) - f_{n+j}(z)| < \varepsilon_{n+j}, \quad z \in G_n, \quad j = 1, 2, 3, \dots \quad (17)$$

Every function $F_k(z)$, $k = 1, 2, 3, \dots$, is holomorphic on the D_k , when $k \geq n+1$, $G_n \subset D_k$. Hence the function $F_k(z)$ is holomorphic on G_n . The function $f_k(z)$ is holomorphic on the whole plane, as it is polynomial. Consequently, $F_k(z) - f_k(z)$ is holomorphic on the domain G_n .

From the inequalities (17) we get that the series $\sum_{k=n+1}^{\infty} (F_k(z) - f_k(z))$ is uniformly convergent on G_n . By virtue of Weierstrass first theorem the series $\sum_{k=n+1}^{\infty} (F_k(z) - f_k(z))$ is holomorphic function on the domain G_n .

Since the function $F_k(z)$, $k = 1, 2, 3, \dots$ are holomorphic functions in every point z except the points of the curve γ_k , then the first summand on the right-hand side of the inequality (14) $\sum_{k=1}^n (F_k(z) - f_k(z))$ is a holomorphic function on the domain G_n . From here the function $\Phi(z)$ is holomorphic on the domain G_n . Since

$$\bigcup_{n=1}^{\infty} G_n = \mathbb{C} \setminus \bigcup_{k=1}^{\infty} \gamma_k, \quad (18)$$

Then $\Phi(z)$ is a holomorphic function in every point z , where $z \in \mathbb{C} \setminus \bigcup_{k=1}^{\infty} \gamma_k$.

Consider along with the function $\Phi(z)$ one more function which is defined on the set $\bigcup_{n=1}^{\infty} G_n$ by the following form:

Let z be a arbitrary point from the set $\bigcup_{n=1}^{\infty} G_n$, then there exists the unique natural number n , such that $z \in G_n$. Denote by

$$H(z) = -\frac{1}{\pi} \iint_{G_n} \frac{a(\zeta)}{\zeta - z} d\xi d\eta, \quad z \in G_n, \quad \zeta = \xi + i\eta. \quad (19)$$

By means of the the functions $\Phi(z)$ and $H(z)$ construct the function

$$Q(z) = \Phi(z) + H(z), \quad z \in \mathbb{C} \setminus \bigcup \gamma_n. \quad (20)$$

$Q(z)$ is a continuous function in every point of the set $\bigcup_{n=1}^{\infty} G_n$. Indeed, let $z \in G_n$, then $H(z) = (T_{G_n} a)(z) \in C_{\frac{p-2}{p}}(\mathbb{C})$.

The function $\Phi(z)$ is a continuous on the domain G_n .

Let us prove, that the function $Q(z)$ is continuously extensible on the whole complex plane \mathbb{C} . In fact, let us fix an arbitrary natural number n and consider the left hand $Q^+(t_0)$ and right hand $Q^-(t_0)$ limits of the function $Q(z)$ in an arbitrary point $t_0 \in \gamma_n$.

Taking into account, that the interior domain of the contour γ_n contains the domain G_n and the exterior domain contains the domain G_{n+1} , let us represent the above mentioned limits by the following form:

$$Q^+(t_0) = \lim_{\substack{z \rightarrow t_0 \\ z \in G_n}} Q(z), \quad (21)$$

$$Q^-(t_0) = \lim_{\substack{z \rightarrow t_0 \\ z \in G_{n+1}}} Q(z). \quad (22)$$

In order to calculate the limits (21)-(22) let us represent the function $Q(z)$ by the following form:

$$\begin{aligned} Q(z) &= \sum_{k=1}^{n-1} (F_k(z) - f_k(z)) + (F_n(z) - f_n(z)) + \\ &+ \sum_{k=n+1}^{\infty} (F_k(z) - f_k(z)) - \frac{1}{\pi} \iint_{G_n} \frac{a(\zeta)}{\zeta - z} d\xi d\eta, \quad z \in G_n, \end{aligned} \quad (23)$$

$$\begin{aligned}
Q(z) &= \sum_{k=1}^{n-1} (F_k(z) - f_k(z)) + (F_n(z) - f_n(z)) + \\
&+ \sum_{k=n+1}^{\infty} (F_k(z) - f_k(z)) - \frac{1}{\pi} \iint_{G_{n+1}} \frac{a(\zeta)}{\zeta - z} d\xi d\eta, \quad z \in G_{n+1}. \quad (24)
\end{aligned}$$

Each function $F_k(z)$, where $k \neq n$, is continuous on the curve γ_n , as the Cauchy -type integral. Therefore the sum $\sum_{k=1}^{n-1} (F_k(z) - f_k(z))$ is continuous on the curve γ_n . In case $k \geq n+1$, the function $F_k(z)$ is holomorphic in the domain G'_{n+1} . i.e. the function $F_k(z) - f_k(z)$ is holomorphic on G'_{n+1} . The series $\sum_{k=n+1}^{\infty} (F_k(z) - f_k(z))$ is uniformly convergent on the domain G'_{n+1} . By virtue of Weirstrass first theorem about the holomorphic functions, the sum $\sum_{k=n+1}^{\infty} (F_k(z) - f_k(z))$ is holomorphic on the domain G'_{n+1} . Hence, the sum $\sum_{k=n+1}^{\infty} (F_k(z) - f_k(z))$ is continuous on the curve γ_n .

It follows from the formula (6), that $g_n(t) = (T_{G_{n+1}}a)(t) - (T_{G_n}a)(t)$. Since $a \in L_p^{\text{loc}}(\mathbb{C})$, $p > 2$, by virtue of theorem 1.19 from ([1]) we have

$$(T_{G_n}a)(t), (T_{G_{n+1}}a)(t) \in C_{\frac{p-2}{p}}(\mathbb{C}), \quad t \in \mathbb{C}.$$

Therefore $g_n(t) \in C_{\frac{p-2}{p}}(\mathbb{C})$.

As far as $F_n(z) = \frac{1}{2\pi i} \int_{\gamma_n} \frac{g_n(t)}{t-z} dt$, using the Sokhotsky-Plemely formulas,

we get

$$\begin{aligned}
F_n^+(t_0) &= \frac{1}{2} g_n(t_0) + \frac{1}{2\pi i} \int_{\gamma_n} \frac{g_n(t)}{t-t_0} dt, \\
F_n^-(t_0) &= -\frac{1}{2} g_n(t_0) + \frac{1}{2\pi i} \int_{\gamma_n} \frac{g_n(t)}{t-t_0} dt.
\end{aligned} \quad (25)$$

From the formulas (21)-(25) and from the above stated, we get

$$\begin{aligned}
Q^+(t_0) &= \sum_{k=1}^{n-1} (F_k(t_0) - f_k(t_0)) + \frac{1}{2} g_n(t_0) + \\
&+ \frac{1}{2\pi i} \int_{\gamma_n} \frac{g_n(t)}{t-t_0} dt - f_n(t_0) +
\end{aligned}$$

$$+ \sum_{k=n+1}^{\infty} (F_k(t_0) - f_k(t_0)) - \frac{1}{\pi} \iint_{G_n} \frac{a(\zeta)}{\zeta - t_0} d\xi d\eta, \quad (26)$$

$$\begin{aligned} Q^-(t_0) &= \sum_{k=1}^{n-1} (F_k(t_0) - f_k(t_0)) - \frac{1}{2} g_n(t_0) + \\ &+ \frac{1}{2\pi i} \int_{\gamma_n} \frac{g_n(t)}{t - t_0} dt - f_n(t_0) + \\ &+ \sum_{k=n+1}^{\infty} (F_k(t_0) - f_k(t_0)) - \frac{1}{\pi} \iint_{G_{n+1}} \frac{a(\zeta)}{\zeta - t_0} d\xi d\eta. \end{aligned} \quad (27)$$

Applying the formulas (26) and (27) we have

$$\begin{aligned} Q^+(t_0) - Q^-(t_0) &= g_n(t_0) - \frac{1}{\pi} \iint_{G_n} \frac{a(\zeta)}{\zeta - t_0} d\xi d\eta + \\ &+ \frac{1}{\pi} \iint_{G_{n+1}} \frac{a(\zeta)}{\zeta - t_0} d\xi d\eta = 0. \end{aligned}$$

Consequently, the function $Q(z)$ is continuously extendable on whole complex plane. We denote by $Q(z)$ the continuously extended function.

Let us prove, that the function $Q(z)$ is $\frac{\partial}{\partial \bar{z}}$ primitive of the function $a(z)$.

Consider arbitrary domain G_n , $n = 1, 2, 3, \dots$. On this domain the function $Q(z)$ is representable by the following form:

$$Q(z) = \Phi(z) - \frac{1}{\pi} \iint_{G_n} \frac{a(\zeta)}{\zeta - z} d\xi d\eta = \Phi(z) + (T_{G_n} a)(z), \quad z \in G_n. \quad (28)$$

It is evident, that G_n on $\frac{\partial \Phi}{\partial \bar{z}} = 0$.

Since $a \in L_p(G_n)$, $p > 2$, therefore by virtue of the theorem 1.13 from [1] we have $\frac{\partial}{\partial \bar{z}} (T_{G_n} a)(z) = a(z)$, $z \in G_n$. Using the equality (28) we get the following equality on the domain G_n :

$$\frac{\partial Q}{\partial \bar{z}} = a(z), \quad z \in G_n. \quad (29)$$

The function $Q(z)$ is a continuous function on the whole complex plane. It is clear that $\mathbb{C} = \left(\bigcup_{n=1}^{\infty} G_n \right) \cup \left(\bigcup_{n=1}^{\infty} \gamma_n \right)$.

From the equality (29) and the stated above we have, that the following equality holds on the whole plane:

$$\frac{\partial Q}{\partial \bar{z}} = a(z), \quad z \in \mathbb{C}. \quad (30)$$

We obtain, that the constructed function $Q(z)$ is $\frac{\partial}{\partial \bar{z}}$ primitive of the function $a(z)$ on the whole plane.

Let us prove, that the function $Q(z)$ satisfies the Hölder -condition with the exponent $\frac{p-2}{p}$ on each compact.

Consider arbitrary compact $D \subset \mathbb{C}$. Consider also the bounded domain G , which contains D . It is easy to see, that $\frac{\partial Q}{\partial \bar{z}} = a(z)$, $z \in G$.

Since $a \in L_p(G)$, $p > 2$, by virtue of the theorem 1.16 from [1] the following equality

$$Q(z) = K(z) + (T_G a)(z), \quad z \in G, \quad (31)$$

is valid, where $K(z)$ is holomorphic function on G .

Using the theorem 1.19 [1], we have $(T_G a)(z) \in C_{\frac{p-2}{p}}(\mathbb{C})$ as far as the function $K(z)$ is a holomorphic on the domain G , therefore $K(z) \in C_{\frac{p-2}{p}}(D)$. It follows from the equality (31), that $Q(z) \in C_{\frac{p-2}{p}}(D)$.

Let $q(z)$ be one of $\frac{\partial}{\partial \bar{z}}$ primitives of the function $a(z)$ and let $\Phi(z)$ be an arbitrary entire function. Consider the function $Q(z) = q(z) + \Phi(z)$. Then the following equality $\frac{\partial Q}{\partial \bar{z}} = \frac{\partial q}{\partial \bar{z}} + \frac{\partial \Phi}{\partial \bar{z}} = \frac{\partial q}{\partial \bar{z}} = a(z)$, $z \in \mathbb{C}$, is valid, since $\frac{\partial \Phi}{\partial \bar{z}} = 0$.

Let $Q(z)$ be an arbitrary $\frac{\partial}{\partial \bar{z}}$ primitive of the function $a(z)$. i.e. the equality $\frac{\partial Q}{\partial \bar{z}} = a(z)$, $z \in \mathbb{C}$ is fulfilled. Because $\frac{\partial q}{\partial \bar{z}} = a(z)$, $z \in \mathbb{C}$, therefore the following equality $\frac{\partial(Q(z)-q(z))}{\partial \bar{z}} = \frac{\partial Q}{\partial \bar{z}} - \frac{\partial q}{\partial \bar{z}} = a(z) - a(z) = 0$, $z \in \mathbb{C}$, is fulfilled. From here the function $\Phi(z) = Q(z) - q(z)$ by virtue of the theorem 1.5 from [1] holomorphic on the whole plane, i.e. $\Phi(z)$ is entire function. The theorem is complete proved.

References

- [1] Vekua I. N. Generalized analytic functions. Second Edition, Nauka, 1988.
- [2] Vekua I. N. On one class of the elliptic systems with singularities. Reprinted from the Proceedings of the International Conference on Functional Analysis and Related Topics. Tokyo April, 1969, p. 142–147.
- [3] Vekua I.N. A general representation of functions of two independent variables admitting derivatives in the Sobolev sense and the problem of primitives. Dokl.Akad.Nauk SSSR, 89, N5, pp.773-775

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Some properties of the generalized power functions*

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Consider the generalized power functions

$$U_{2k}(z, z_0) = \mathcal{R}_{z_0}^{A,B}((z - z_0)^k), U_{2k+1}(z, z_0) = \mathcal{R}_{z_0}^{A,B}(i(z - z_0)^k), \quad (1)$$

$$V_{2k}(z, z_0) = \mathcal{R}_{\infty}^{A,B}((z - z_0)^k), V_{2k+1}(z, z_0) = \mathcal{R}_{\infty}^{A,B}(i(z - z_0)^k), \quad (2)$$

of the Carleman-Bers-Vekua equation

$$\partial_{\bar{z}}w + Aw + B\bar{w} = 0, A, B \in L_{p,2}, p > 2, \quad (3)$$

where $z_0 \neq \infty$, $k = 0, \pm 1, \pm 2, \dots$, $\mathcal{R}_{z_0}^{A,B}$ is the operator (see [1], chapter 3,§3) associating to every analytic function φ and the point $z_0 \in \overline{\mathbb{C}}$ the solution w of the equation (3) satisfying the following conditions:

- 1) the function $\tilde{w}(z, z_0) = \frac{w(z, z_0)}{\varphi}$ is continuous in the domain, where φ is analytic and continuously extendable on \mathbb{C} , moreover $\tilde{w} \in C_{\alpha}(\mathbb{C})$, $\alpha = \frac{p-2}{p}$;
- 2) $\tilde{w}(z, z_0) \neq 0$ on $\overline{\mathbb{C}}$;
- 3) $\tilde{w}(z_0, z_0) = 1$.

The function φ is called an analytic divisor (with respect to the point z_0) of the function $w = \mathcal{R}_{z_0}^{A,B}(\varphi)$; the point z_0 we call the point of the coordination φ and w . When $z_0 = \infty$, the function φ is called the normal analytic divisor of the function w .

These functions are representable in the following form:

$$U_k(z, z_0) = (z - z_0)^{[\frac{k}{2}]} \tilde{U}_k(z, z_0), V_k(z, z_0) = (z - z_0)^{[\frac{k}{2}]} \tilde{V}_k(z, z_0), \quad (4)$$

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where $\tilde{U}_{2k}, \tilde{U}_{2k+1}, \tilde{V}_{2k}$ and \tilde{V}_{2k+1} are the generalized constants (see [1], chapter 3, §4) of the class $\mathcal{A}(A, B)$, $B_k(z) = B(z) \frac{(z-\bar{z}_0)^k}{(z-z_0)^k}$, satisfying the conditions

$$\tilde{U}_{2k}(z_0, z_0) = \tilde{V}_{2k}(\infty, z_0) = \lim_{z \rightarrow \infty} \tilde{V}_{2k}(z, z_0) = 1 \quad (5)$$

$$\tilde{U}_{2k+1}(z_0, z_0) = \tilde{V}_{2k+1}(\infty, z_0) = \lim_{z \rightarrow \infty} \tilde{V}_{2k+1}(z, z_0) = i. \quad (6)$$

Besides $\tilde{U}_k(*, z_0), \tilde{V}_k(*, z_0)$ belong to the class $C_{\frac{p-2}{p}}(\mathbb{C})$ and are satisfying inequality

$$M^{-1} \leq |\tilde{U}_k(z, z_0)| \leq M, M^{-1} \leq |\tilde{V}_k(z, z_0)| \leq M, \quad (7)$$

$$z \in \mathbb{C}, k = 0, \pm 1, \pm 2, \dots,$$

where $M = \exp\{M_p \| |A| + |B| \|_{p,2}, M_p$ is constant depending only on p (see [1], chapter 3, §4).

The generalized power functions U_k and V_k are differing from each other only by coordinated points with their analytic divisors.

It is easy to see that the following equalities hold:

$$U_{2k}(z, z_0) = c_{2k,0} V_{2k}(z, z_0) + c_{2k,1} V_{2k+1}(z, z_0) \quad (8)$$

$$U_{2k+1}(z, z_0) = c_{2k+1,0} V_{2k}(z, z_0) + c_{2k+1,1} V_{2k+1}(z, z_0) \quad (9)$$

where $c_{2k,\alpha}, c_{2k+1,\alpha}, \alpha = 1, 2$ are real constants (z_0 is fixed point), representable by the formulas:

$$c_{2k,0} + ic_{2k,1} = \tilde{U}_{2k}(\infty, z_0) = -\frac{\operatorname{Im} \tilde{V}_{2k+1}(z_0, z_0) - i \operatorname{Im} \tilde{V}_{2k}(z_0, z_0)}{\operatorname{Im} [\tilde{V}_{2k}(z_0, z_0) \overline{\tilde{V}_{2k+1}(z_0, z_0)}]} \quad (10)$$

$$c_{2k+1,0} + ic_{2k+1,1} = \tilde{U}_{2k+1}(\infty, z_0) = \frac{\operatorname{Re} \tilde{V}_{2k+1}(z_0, z_0) - i \operatorname{Re} \tilde{V}_{2k}(z_0, z_0)}{\operatorname{Im} [\tilde{V}_{2k}(z_0, z_0) \overline{\tilde{V}_{2k+1}(z_0, z_0)}]} \quad (11)$$

Note, that the denominator in right-hand sides of the equalities (10) and (11) is not equal to zero. Indeed, assuming contrary we have

$$\tilde{V}_{2k}(z_0, z_0) = c \tilde{V}_{2k+1}(z_0, z_0)$$

where c is a real constant. But the last equality is impossible as the functions $\tilde{U}_{2k}(*, z_0)$ and $c \tilde{U}_{2k+1}(*, z_0)$ are the generalized constants of one and the same class $\mathcal{A}(A, B_k)$ satisfying the conditions

$$\tilde{V}_{2k}(\infty, z_0) = 1, c \tilde{V}_{2k+1}(\infty, z_0) = ic$$

and hence they couldn't have same meanings in any point of the plane.

The equations (10) and (11) are obviously equivalent to the following equations

$$V_{2k}(z, z_0) = \widehat{c}_{2k,0}U_{2k}(z, z_0) + \widehat{c}_{2k,1}U_{2k+1}(z, z_0), \quad (12)$$

$$V_{2k+1}(z, z_0) = \widehat{c}_{2k+1,0}U_{2k}(z, z_0) + \widehat{c}_{2k+1,1}U_{2k+1}(z, z_0), \quad (13)$$

where

$$\widehat{c}_{2k,0} = \frac{1}{\Delta_k}c_{2k+1,0}, \widehat{c}_{2k,1} = -\frac{1}{\Delta_k}c_{2k,1}, \widehat{c}_{2k+1,0} = -\frac{1}{\Delta_k}c_{2k+1,0}, \quad (14)$$

$$\widehat{c}_{2k+1,1} = \frac{1}{\Delta_k}c_{2k,0},$$

$$\Delta_k \equiv c_{2k,0}c_{2k+1,1} - c_{2k,1}c_{2k+1,0} = [Im(\widetilde{V}_{2k+1}(z, z_0)\overline{\widetilde{V}_{2k}(z, z_0)})]^{-1}.$$

The last equality follows directly from the formulas (10),(11).

The generalized power functions of the conjugate equation of the equation (3)

$$\partial_{\bar{z}}w' - Aw' - \overline{B}w' = 0, A, B \in L_{p,2}, p > 2, \quad (15)$$

of the lass $\mathcal{A}(-A, -\overline{B})$ denote by U'_k and V'_k , $k = 0, \pm 1, \pm 2, \dots$

It is evident, that all relations established above for the functions U_k and V_k take place for U'_k and V'_k too.

Let as prove the following theorem.

Theorem 1 *Let Γ be a piecewise-smooth simple closed curve surrounding the point $z_0 \neq \infty$. Then the following identities hold*

$$Re \frac{1}{2\pi i} \int_{\Gamma} U_k(z, z_0)U'_m(z, z_0)dz = I_{k,m}, \quad (16)$$

$$Re \frac{1}{2\pi i} \int_{\Gamma} V_k(z, z_0)V'_m(z, z_0)dz = I_{k,m}, \quad (17)$$

where $I_{k,m} = 1$ ($I_{k,m} = -1$), if k and m even (odd) numbers and $[\frac{k}{2}] + [\frac{m}{2}] = -1$; if all remaining cases $I_{k,m} = 0$.

Proof. Denote by $I_{k,m}(U, \Gamma)$ and $I_{k,m}(V, \Gamma)$ the left-hand sides of the identities (16) and (17). From the Green identity (see [1], chapter 3, §9) it follows that for every $R > 0$

$$I_{k,m}(U, \Gamma) = I_{k,m}(U, \Gamma_R), I_{k,m}(V, \Gamma) = I_{k,m}(V, \Gamma_R), \quad (18)$$

where Γ_R is circle with the radius R and the origin in the point z_0 . By virtue of the equalities (4) we have

$$I_{k,m}(U, \Gamma_R) = Re \frac{1}{2\pi i} \int_{\Gamma_R} \chi_{k,m}^{(U)}(z, z_0)(z - z_0)^\alpha dz, \quad (19)$$

$$I_{k,m}(V, \Gamma_R) = Re \frac{1}{2\pi i} \int_{\Gamma_R} \chi_{k,m}^{(V)}(z, z_0)(z - z_0)^\alpha dz, \quad (20)$$

where

$$\chi_{k,m}^{(U)}(z, z_0) = \tilde{U}_k(z, z_0) \tilde{U}'_m(z, z_0), \quad (21)$$

$$\chi_{k,m}^{(V)}(z, z_0) = \tilde{V}_k(z, z_0) \tilde{V}'_m(z, z_0), \quad (22)$$

$$\alpha = \left[\frac{k}{2} \right] + \left[\frac{m}{2} \right].$$

The functions $\chi_{k,m}^{(U)}(*, z_0)$ are $\chi_{k,m}^{(V)}(*, z_0)$ the Hölder continuous and are bounded on the whole complex plane.

When $\alpha > -1$ and $\alpha' < -1$ it follows from identities (19), (20), that

$$\lim_{R \rightarrow 0} I_{k,m}(U, \Gamma_R) = \lim_{R \rightarrow 0} I_{k,m}(V, \Gamma_R) = 0$$

and

$$\lim_{R \rightarrow \infty} I_{k,m}(U, \Gamma_R) = \lim_{R \rightarrow \infty} I_{k,m}(V, \Gamma_R) = 0$$

respectively.

Therefore, by virtue of (18) we get

$$I_{k,m}(U, \Gamma) = I_{k,m}(V, \Gamma) = 0.$$

Let now $\alpha = -1$. Consider three different cases separately.

a) k and m are even numbers. Then from (21),(22),(5), (6) we have:

$$\chi_{k,m}^{(U)}(z_0, z_0) = \lim_{z \rightarrow \infty} \chi_{k,m}^{(V)}(z, z_0) = 1.$$

Taking into account these equations, from (19), (20) we obtain

$$\lim_{R \rightarrow 0} I_{k,m}(U, \Gamma_R) = \lim_{R \rightarrow \infty} I_{k,m}(V, \Gamma_R) = 1.$$

Hence, in considered case the following identity

$$I_{k,m}(U, \Gamma) = I_{k,m}(V, \Gamma) = 1$$

hold.

b) k and m are odd numbers. Then from (4)

$$\chi_{k,m}^{(U)}(z_0, z_0) = \lim_{z \rightarrow \infty} \chi_{k,m}^{(V)}(z_0, z_0) = -1$$

and as in above case

$$I_{k,m}(U, \Gamma) = I_{k,m}(V, \Gamma) = -1;$$

c) k and m are numbers with different purity. In this case

$$\chi_{k,m}^{(U)}(z_0, z_0) = \lim_{z \rightarrow \infty} \chi_{k,m}^{(V)}(z_0, z_0) = i$$

and therefore

$$\lim_{R \rightarrow 0} I_{k,m}(U, \Gamma_R) = \lim_{R \rightarrow 0} I_{k,m}(V, \Gamma) = 0.$$

Hence, by virtue of (19) we have

$$I_{k,m}(U, \Gamma) = I_{k,m}(V, \Gamma) = 0.$$

The theorem is proved.

In particular case, when $B = 0$ in (2) we have

$$U_{2k}(z, z_0) = (z - z_0)^k e^{\omega(z) - \omega(z_0)}, U_{2k+1}(z, z_0) = iU_{2k}(z, z_0);$$

$$V_{2k}(z, z_0) = (z - z_0)^k e^{\omega(z)}, V_{2k+1}(z, z_0) = iV_{2k}(z, z_0);$$

$$U'_{2k}(z, z_0) = (z - z_0)^k e^{\omega(z_0) - \omega(z)}, U'_{2k+1}(z, z_0) = iU'_{2k}(z, z_0);$$

$$V'_{2k}(z, z_0) = e^{-\omega(z)}, V'_{2k+1}(z, z_0) = iV'_{2k}(z, z_0),$$

where $\omega = \frac{1}{p} \iint_{\mathbb{C}} \frac{A(\xi)}{\xi - z} d\sigma_{\xi}$. From above identities follows the following formula:

$$\frac{1}{2\pi i} \int_{\gamma} (z - z_0)^{\alpha} = \begin{cases} 1, & \text{if } \alpha = -1 \\ 0, & \text{if } \alpha \neq -1. \end{cases}$$

References

- [1] I.N. Vekua Generalized analytic functions. Pergamon Press, London, 1962.

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The Cauchy transform and certain non-linear boundary value problem on non-rectifiable arc

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As known, the Riemann–Hilbert boundary value problem is the problem on evaluation of holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ satisfying equality

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), t \in \Gamma,$$

where Γ is given curve on the complex plane \mathbb{C} , $\Phi^\pm(t)$ stand for limit values of Φ at point $t \in \Gamma$ from the left and from the right correspondingly, and functions $G(t)$, $g(t)$ are defined on Γ .

The classical results on this problem concern the case of piecewise–smooth curve Γ , and the classical technique of its solution bases on the properties of the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)dt}{t - z}.$$

The Riemann problem makes sense for non-rectifiable Jordan curves, too. But the curvilinear integral $\int_{\Gamma} \cdot dt$ is not defined for non-rectifiable Γ . This is a reason why the author of the present report solved the Riemann–Hilbert boundary value problem on non-rectifiable curves at the early 80s without using of the Cauchy type integral.

In the present work we describe another way for solution of that problems. We replace the Cauchy type integral by the Cauchy transform of certain distributions with supports on non-rectifiable curve Γ . As a result, we obtain a new instrument for solution of the whole family of problems connected with the Riemann–Hilbert boundary value problem. In the final part of report we apply this technique for solution of a non-linear boundary value problem on non-rectifiable arc.

1 The Cauchy transform

Let ϕ be a distribution with compact support S on the complex plane, i.e., a continuous in customary sense functional on the space $C^\infty(\mathbb{C})$. Then function

$$\text{Cau } \phi(z) = \frac{1}{2\pi i} \langle \phi, \frac{1}{t-z} \rangle$$

is called Cauchy transform of ϕ . It is holomorphic in $\mathbb{C} \setminus S$, and vanishes at infinity point.

Let Γ be a Jordan curve (closed or open) on the complex plane. Generally speaking, it is not rectifiable. We consider a holomorphic in $\mathbb{C} \setminus \Gamma$ function $F(z)$. If it is locally integrable in \mathbb{C} , then we identify it with distribution

$$\langle F, \omega \rangle := \iint_{\mathbb{C}} F(z) \omega(z) dz \wedge d\bar{z}, \omega \in C_0^\infty(\mathbb{C}).$$

Its distributional derivative

$$\left\langle \frac{\partial F}{\partial \bar{z}}, \omega \right\rangle := - \iint_{\mathbb{C}} F(z) \frac{\partial \omega}{\partial \bar{z}} dz \wedge d\bar{z}$$

has compact support Γ .

Let $H_\nu(A)$ stand for the Hölder space on the set $A \subset \mathbb{C}$, i.e., it consists of defined on A functions f satisfying inequality

$$h_\nu(f; A) := \sup \left\{ \frac{|f(t') - f(t'')|}{|t' - t''|^\nu} : t', t'' \in A, t' \neq t'' \right\} < \infty.$$

Its norm is $\|f\|_{H_\nu(A)} := h_\nu(f; A) + \|f\|_{C(A)}$. If the set A is compact, then any function $f \in H_\nu(A)$ is extendable up to a function $f^w \in H_\nu(\mathbb{C})$, and this extension is isometric: $\|f^w\|_{H_\nu(\mathbb{C})} = \|f\|_{H_\nu(A)}$ (the Whitney theorem). We put

$$H_\nu^*(A) := \bigcap_{\nu' > \nu} H_{\nu'}(A).$$

In what follows we need characteristics of dimensional type for non-rectifiable curve Γ . The simplest one is the upper metric dimension (it is called also fractal dimension, Minkowskii dimension and so on). The upper metric dimension $\text{Dm } A$ of compact set $A \subset \mathbb{C}$ equals to

$$\text{Dm } A := \limsup_{\varepsilon \rightarrow 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon},$$

where $N(A, \varepsilon)$ is the least number of disks of radius ε covering A . As known, $1 \leq \text{Dm } \Gamma \leq 2$ for any curve $\Gamma \subset \mathbb{C}$, and $\text{Dm } \Gamma = 1$ if Γ is rectifiable.

Theorem 1 *Let $\text{Dm } \Gamma = d < 2$, $F \in L_{loc}^p(\mathbb{C})$, $p > 1$. Then*

$$\left| \left\langle \frac{\partial F}{\partial \bar{z}}, \omega \right\rangle \right| \leq C \|\omega\|_{H_\nu(\bar{D})}$$

for

$$\nu > 1 - \frac{2-d}{p'}, \frac{1}{p'} + \frac{1}{p} = 1, \quad (1)$$

where D is any finite domain containing Γ , and the constant C does not depend on ω .

In other words, under restriction (1) the distribution $\frac{\partial F}{\partial \bar{z}}$ is extendable up to bounded functional on the closure of $C^\infty(\mathbb{C})$ in $H_\nu(\bar{D})$. This closure contains the space $H_\nu^*(\bar{D})$.

If $f \in H_\nu(D)$, then it belongs to this closure under assumption (1). The product $f\omega$ also belongs it for any $\omega \in C^\infty(\mathbb{C})$. Consequently, there is defined distribution

$$\left\langle \frac{\partial F}{\partial \bar{z}} f, \omega \right\rangle := \left\langle \frac{\partial F}{\partial \bar{z}}, f\omega \right\rangle$$

with support on Γ . If $f \in H_\nu(\Gamma)$, then we apply this definition to its Whitney extension f^w . The main subject of this section is the Cauchy transform

$$\Phi(z) = \text{Cau} \frac{\partial F}{\partial \bar{z}} f(z). \quad (2)$$

Theorem 2 *Let function $F(z)$ have limit values $F^+(t)$ and $F^-(t)$ from the left and from the right correspondingly at any point $t \in \Gamma \setminus E$, where $E \subset \Gamma$. If F is integrable with exponent $p > 2$ in a neighborhood of E , $f \in H_\nu$, and*

$$\nu > \frac{1}{2} \text{Dm } \Gamma, \quad (3)$$

then the function $\Phi(z)$ also has limit values $\Phi^+(t)$ and $\Phi^-(t)$ from the left and from the right correspondingly at any point $t \in \Gamma^\circ$, it is integrable with exceeding two exponent in a neighborhood of E , and

$$\Phi^+(t) - \Phi^-(t) = (F^+(t) - F^-(t))f(t), t \in \Gamma \setminus E. \quad (4)$$

Thus, the function $\text{Cau} \frac{\partial F}{\partial \bar{z}} f(z)$ keeps the main properties of the Cauchy type integral with density $(F^+(t) - F^-(t))f(t)$.

2 The Szegő functions

We apply this result for solution of the following non-linear boundary value problem.

Let Γ be a Jordan arc beginning at point -1 and ending at point $+1$. We seek a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $S(z)$ such that

$$S^+(t)S^-(t) = \rho(t), t \in \Gamma \setminus \{-1, +1\}, \quad (5)$$

where $\rho(t)$ is defined on Γ function, and

$$C^{-1} < |S(z)| < C, z \in \mathbb{C} \setminus \Gamma, C = C(S) > 0. \quad (6)$$

The solution of this problem for smooth arcs is known. In recent works concerning Padé approximations of Markov type functions its solutions are called the Szegő functions. The Cauchy transform with

$$F(z) = \frac{1}{\sqrt{z^2 - 1}}$$

enables us to obtain the following result.

Theorem 3 *Let $\rho(t) \neq 0$, $\rho \in H_\nu(\Gamma)$, and $\nu > \frac{1}{2} \text{Dm } \Gamma$. Then the functions*

$$S(z) = \pm \exp \left(\frac{1}{2} F^{-1}(z) \text{Cau} \left(\frac{\partial F}{\partial \bar{z}} \log \rho \right) (z) \right) \quad (7)$$

are solutions of the boundary value problem (5), (6).

Under certain additional restrictions the problem has not another solutions.

The result of this section is obtained by David B. Kats. We hope that it will be useful for estimations of the rational approximations of holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ functions.

Note 1 *We can replace here the upper metric dimension by new metric dimensions introduced by the author. As a result, the results will be sharper.*

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Complex geometry of quadrilateral linkages*

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Abstract

We present a number of observations on the complex geometry of quadrilateral linkages. In particular, we show that, for each configuration of a planar quadrilateral linkage $Q(a, b, c, d)$ with pairwise distinct side-lengths (a, b, c, d) , the cross-ratio of its vertices belongs to the circle of radius ac/bd centered at point $1 \in \mathbb{C}$.

Moreover, we establish an analog of Poncelet porism for the discrete dynamical system on the planar moduli space of 4-bar linkage defined by the product of diagonal involutions, and discuss some related issues suggested by a beautiful link to the theory of discrete integrable systems discovered by J. Duistermaat.

Finally, we establish a connection between certain extremal problems for configurations of 4-bar linkage and tetrahedra obtained from its configurations using the famous Minkowski theorem on polyhedra with prescribed areas of faces.

1 Introduction

The simplest and prototypical mechanical linkages, *4-bar mechanisms*, were an object of active investigation from various points of view for a long time

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[6], [9], [11] (nowadays they are also called *quadrilateral linkages* [5], [11]). Comprehensive results on the geometry of planar 4-bar mechanisms, including some classical ones from [6], are presented in [9]. Complex geometry provide a natural language for investigation of planar quadrilateral linkages and this topic is known to be related to several deep results of algebraic geometry and function theory, in particular, to Poncelet theorem [8] and theory of elliptic functions [6].

The aim of this paper is to complement the results of [9] and [13] by presenting several new results on the complex geometry of 4-bar linkages which emerged along the lines of [13], [14], [16]. In this context it is natural to consider a polygonal linkage as a purely mathematical object defined by a collection of positive numbers [5].

We begin with recalling the topological structure of $M_N(L)$ for all $N \geq 2$ (Propositions 2.1, 2.2, 2.3). We also introduce the complexified planar moduli space and define *bending flows* on $M_3(Q)$ [11] and *diagonal involutions* on $M_2(Q)$. One of peculiar features of a planar 4-vertex linkage Q (no matter closed or open) is that there is a naturally defined *cross-ratio map* Cr_Q from its planar moduli space into the complex plane. We describe the image of cross-ratio map in the case 4-bar linkage (Proposition 3.1) and briefly discuss related problems for 3-arm.

The next section is concerned with certain natural mappings on planar and spatial moduli spaces of closed 4-vertex linkages (4-bar mechanisms). We recall some results of [12] concerned with the bending flows on the spatial moduli space of 4-bar mechanism and discuss their analogs in the case of planar moduli space suggest several interesting and rather hard problems which we discuss in some detail in the rest of the section. It should be noted that progress in these topics became possible using the link to the theory of discrete integrable systems discovered by J. Duistermaat and described in his last book [7].

In the last section we briefly describe an interesting and apparently unexplored connection between configurations of spatial linkage and convex polyhedra obtained from its configurations based on the Minkowski 1897 theorem, and illustrate it by a simple result concerned with three-dimensional configurations of quadrilateral linkages.

2 Moduli spaces of quadrilateral linkages

We freely use some definitions and constructions from the mathematical theory of linkages, in particular, the concept of N -th *moduli space* of a polygonal linkage for which we refer to [5]. Recall that a quadrilateral (4-bar) linkage $Q = Q(l)$ is defined by a quadruple of positive numbers $l = (a, b, c, d) \in \mathbb{R}_+^4$ called the *side-length vector* of Q . Analogously, (robot) 3-arm is analogously defined by a triple of positive numbers $l \in \mathbb{R}_+^3$. For any polygonal linkage L , its N th configuration space $M_N(L)$ is defined as the set of its configurations (realizations) in \mathbb{R}^N taken modulo the group of orientation preserving isometries of \mathbb{R}^N [5]. For generic side-length vector l , $M_N(L)$ is a smooth compact manifold. For completeness we recall the topological structure of all moduli spaces for 4-bar linkages and 3-arms which is known for a long time (see, e.g., [11]).

Proposition 2.1. *For any l , the moduli space $M_3(A_3(l))$ is homeomorphic to three-sphere S^3 . For any natural $N \geq 4$, the moduli space $M_N(A(l))$ is homeomorphic to the closed 3-dimensional ball B^3 .*

Proposition 2.2. *The complete list of homeomorphy types of planar moduli spaces of a 4-bar linkages is as follows: circle, disjoint union of two circles, bouquet of two circles, two circles with two common points, three circles with pairwise intersections equal to one point.*

Taking into account the stabilization phenomenon [17] one can describe the topology of all moduli spaces as well.

Proposition 2.3. *([17]) For each Q , $M_3(Q)$ is homeomorphic to a two-sphere S^2 . For all $N \geq 4$, $M_N(Q)$ is homeomorphic to a disc (two-dimensional) ball B^2 .*

In order to introduce certain self-mappings of the planar and spatial moduli spaces of a 4-bar linkage, in the rest of this section we always assume that side-lengths l_j are pairwise non-equal. Then $Q(l)$ does not have configurations with coinciding vertices so, for each configuration V of $Q(l)$, both diagonals are non-zero and define two different lines in the ambient space.

Consider first the spatial moduli space. For each configuration V one can rotate it by an angle $\alpha \in [0, 2\pi]$ about either of diagonals which defines two families of homeomorphisms of $M_3(Q(l))$ called *bending flows* [12] which give and action of the plane \mathbb{R}^2 on $M_3(Q(l))$. As is shown in [12], this action is

transitive on $M_3(Q(l))$, in other words, any configuration can be transformed into another one by a sequence of bendings.

Analogous mappings can be considered for planar moduli spaces. Since side-lengths l_i are pairwise non-equal then, for each planar configuration V of such a $Q(l)$, one can construct its reflections in the diagonals v_1v_3 and v_2v_4 respectively. This obviously defines two *diagonal involutions* i_1 and i_2 on $M_2(Q)$ which are the same as bendings by angle π called π -bendings in [12]. If we introduce unit vectors u_i along the sides of V then it is easy to verify that diagonal involutions act by formulas:

$$i_1 : (u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_3^{-1} \frac{l_1 + l_2 u_2}{l_1 + l_2 u_2^{-1}}, u_4^{-1} \frac{l_1 + l_2 u_2}{l_1 + l_2 u_2^{-1}});$$

$$i_2 : (u_1, u_2, u_3, u_4) \mapsto (u_1, u_2^{-1} \frac{l_1 + l_2 u_2}{l_1 + l_2 u_2^{-1}}, u_3^{-1} \frac{l_1 + l_2 u_2}{l_1 + l_2 u_2^{-1}}, u_4).$$

These formulas enable one to investigate many concrete issues related to diagonal involutions. In particular, they do not commute and it is easy to describe how they act on various functions on the moduli space such as the cross-ratio considered in the next section. As usual, for non-commuting involutions it is reasonable to consider their composition $\tau = i_1 \circ i_2$ which in this case is called the *Darboux transformation* of $Q(l)$ [7]. The above formulas show, in particular, that τ has no fixed-points on $M_2(Q(l))$. However τ may have periodic points and their investigation leads to interesting problems which can be better understood by considering the action of τ on *complexification* $M_2^C(Q(L))$ of the planar moduli space which can be introduced as follows [12].

Both above maps are birational transformations of \mathbb{C}^4 . The reduced planar moduli space $\tilde{M}_2 \approx S^1$ can be defined as the quotient of the curve

$$E = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 : u_1 = 1, |u_j| = 1, j = 1, 2, 3, \sum l_j u_j = 0\}$$

by action of the involution $\theta : (u_1, u_2, u_3, u_4) \mapsto (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4)$.

The complexified planar moduli space $M_2^C(Q(L))$ is defined as the complexification E^C of E .

Proposition 2.4. ([6], [9]) *For each generic Q , $\overline{M}_2^C(Q)$ is a nonsingular connected elliptic curve (Riemann surface of genus one) and τ extends to an automorphism τ^C of E^C which has no fixed-points.*

In particular, $\overline{M}_2^C(Q)$ is always homeomorphic to the two-torus T^2 which leads to interesting connections with function theory and Poncelet theorem [8].

3 Cross-ratio map of planar quadrilateral linkage

We need some elementary properties of cross-ratio which can be found in [3]. Consider first a 4-bar linkage $Q = Q(a, b, c, d)$ with smooth $M_2(Q)$ (i.e., without aligned configurations) and assume moreover that side-lengths are pairwise different so that Q has no configurations with coinciding vertices. Then, for each planar configuration $V = (v_1, v_2, v_3, v_4) \in \mathbb{C}^4$ of Q , put

$$Cr(V) = Cr((v_1, v_2, v_3, v_4)) = \frac{v_3 - v_1}{v_3 - v_2} : \frac{v_4 - v_1}{v_4 - v_2}.$$

This obviously defines a continuous (actually, real-analytic) mapping $Cr_Q : M(Q) \rightarrow \mathbb{C}$ and our first aim is to describe its image $\Gamma_Q = \text{Im } Cr_Q$ which is obviously a continuous curve in \mathbb{C} . Taking into account some well-known properties of cross-ratio and moduli space, we were able to establish some qualitative geometric properties of Γ_Q . In the process of discussing observations with Elias Wegert of Bergakademie (Freiberg) in January 2011 we came up with a quite explicit description of $\text{Im } Cr_Q$, which should be considered as our joint result.

Theorem 3.1. *For a quadrilateral linkage Q as above, $\text{Im } Cr_Q$ is a subset of the circle $C(1, ac/bd)$ of radius ac/bd centered at the point $1 \in \mathbb{C}$. The image is always an arc symmetric about the real axis and contains the point $1 + \frac{ac}{bd}$ corresponding to the convex cyclic configuration of Q .*

Proof. After having found this result its proof was immediate by using well-known properties of cross-ratio. First notice that, for each quadruple of points $v_j \in \mathbb{C}$, we have:

$$Cr(v_1, v_2; v_3, v_4) = 1 - Cr(v_1, v_3; v_2, v_4).$$

Now,

$$Cr(v_1, v_3; v_2, v_4) = \frac{v_3 - v_1}{v_3 - v_2} : \frac{v_4 - v_1}{v_4 - v_2},$$

hence

$$|Cr(v_1, v_3; v_2, v_4)| = \frac{|v_2 - v_1|}{|v_2 - v_3|} \cdot \frac{|v_4 - v_1|}{|v_4 - v_3|}.$$

Since the moduli in the r.h.s. are equal to distances between the points v_i which form a configuration of $Q(a, b, c, d)$ we finally get

$$|1 - Cr(v_1, v_2; v_3, v_4)| = \frac{ac}{bd},$$

as was claimed. The remaining statements follow easily from the above remarks. \square

Since the argument of $Cr(v_1, v_3; v_2, v_4)$ can be expressed in terms of the angles between the sides of configuration one can now express the length of $\text{Im } Cr_Q$ in terms of side-length vector $l = (a, b, c, d)$ and characterize those l for which $\text{Im } Cr_Q = C(1, ac/bd)$.

These results suggest further use of complex geometry as follows. For a given configuration of a planar linkage, consider certain geometrically or physically meaningful point defined by the configuration. For example, one can place certain masses or charges at the vertices of configuration and consider the center of mass or the point of stable electrostatic equilibrium of those charges. In both cases one obtains a mapping into \mathbb{C} (complex-valued function) and may study its image and singular points. Notice that such maps make sense for all moduli spaces of polygonal linkage with arbitrary number of sides so they definitely deserve a closer look.

4 Poncelet porism for planar quadrilateral linkages

Now we wish to show how the complex geometry enables one to obtain interesting results about the geometry of mapping $\tau = i_1 i_2 : M_2(Q) \rightarrow M_2(Q)$. One can also obtain quite comprehensive results about the behaviour of the discrete dynamical system τ^n generated by τ . This follows from a beautiful link with the theory of so-called *QRT maps* discovered by J.Duistermaat and developed in [7].

The first result in this direction is a direct analog of Poncelet porism [8]. As is mentioned on the page 512 of [7], this result and related conjectures were discussed in a colloquium talk of the present author in Utrecht on 9.02.2006, which gave an impetus for the study initiated by J.Duistermaat. Later on,

it turned out that this version of Poncelet porism has already been known for G.Darboux [6], for which reason J.Duistermaat suggested to call τ the *Darboux transformation* of a 4-bar linkage $Q = Q(l)$.

It is known that the moduli space $M^2(Q)$ and its complex projectivization $M_2^C(Q)$ are smooth if and only if the side-lengths satisfy a certain genericity condition [9]. This condition, traditionally called the *Grashof condition*, actually means that $Q(l)$ does not have *aligned* configurations [9], or equivalently: there do not exist numbers $e_i = \pm 1$ such that $\sum_{i=1}^4 e_i l_i = 0$.

Theorem 4.1. (*[6], [7]*) *For a 4-bar linkage Q with positive side-lengths satisfying the Grashof condition, one has the following dichotomy: either each configuration is periodic with the same period or the orbit of each configuration is infinite.*

The "raison d'être" of this result is rather simple and elegant: it turns out that τ can be realized as an automorphism of the complexified planar moduli space $M_2^C(Q)$ acting as a translation, which makes the statement evident. For completeness we give an outline of the proof.

Outline of proof. Let us use a rigid motion to place the first two vertices of $Q = Q(a, b, c, d)$ at points $v_1 = (0, 0)$, $v_2 = (a, 0)$ and consider an angular parametrization of $M_2(Q)$ by putting $v_3 = (a + b \cos \phi, b \sin \phi)$, $v_4 = (d \cos \psi, d \sin \psi)$. Then the remaining distance condition $d(v_3, v_4) = c$ is easily seen to be equivalent to

$$c^2 = a^2 + b^2 + d^2 + 2a(b \cos \phi - d \cos \psi) - 2bd(\cos \phi \cos \psi + \sin \phi \sin \psi).$$

Finally, using now the rational parameterizations

$$\cos \phi = \frac{u^2 - 1}{u^2 + 1}, \sin \phi = \frac{2u}{u^2 + 1}, \cos \psi = \frac{-v^2 + 1}{u^2 + 1}, \sin \psi = \frac{-2v}{u^2 + 1}$$

we rewrite the distance equation in the form

$$\begin{aligned} &((a + b + d)^2 - c^2)u^2v^2 + ((a + b - d)^2 - c^2)u^2 + ((a - b + d)^2 - c^2)v^2 + \\ &+ (-a + b + d)^2 - c^2 = 0. \end{aligned}$$

The above curve in the (u, v) -plane is biquadratic and, as explained in [7], if it is smooth as a curve in the Cartesian square of the complex projective line, then it is an elliptic curve. Since we assume that $Q(l)$ satisfies the Grashof condition, this curve is smooth in $\mathbb{P}^1 \times \mathbb{P}^1$ and $M_2^C(Q)$ is an elliptic

curve. It is now easy to directly verify that the Darboux transformation corresponds to the horizontal switch $(u, v) \mapsto (u', v)$ followed by the vertical switch $(u', v) \mapsto (u', v')$ in the (u, v) -plane. Therefore the Darboux transformation on $M_2(Q)$ coincides with the so-called *QRT transformation* of the biquadratic curve above [7]. Then, as is shown in [7], the Darboux transformation acts on it as translation [7], which makes the result obvious. \square

Details of the proof can be found in Section 11.3 of [7]. For us especially inspiring is that this relation to Poncelet porism established in [7] indicates a clear way to solving several natural problems formulated in the aforementioned 2006 talk of the author (see [7]).

5 Tetrahedra and spatial quadrilateral linkages

In this section, we describe an apparently unconventional aspect of polygonal linkages emerging from the *Minkowski 1897 theorem* (see, e.g., [1]). Given a polygonal linkage $L = L(l_1, \dots, l_n)$, for each of its configurations P in \mathbb{R}^N , Minkowski 1897 theorem yields a unique (up to isometry) convex N -dimensional polyhedron $\mu(P)$ with $(N - 1)$ -dimensional faces $F_i, i = 1, \dots, n$ such that F_i is orthogonal to the side $p_i p_{i+1}$ of P and the $(N - 1)$ -dimensional volume of F_i is equal to l_i . In this situation we say that the corresponding polyhedron is a *Minkowski N -polyhedron* (or a *face calibrated polyhedron*) with *face gauge* l (one could also speak of *Minkowski face control*).

If we include into consideration also degenerate polyhedra having all vertices in the same hyperplane of \mathbb{R}^N , and identify congruent configurations, we obtain a certain compact space $\Omega_N(l)$ which can be called the *moduli space of Minkowski N -polyhedra with face gauge l* . As is easy to show, Minkowski theorem in fact implies that $\Omega_N(l)$ can be naturally identified with N -th moduli space $M_N(l)$ of linkage $L(l)$ which immediately yields the topological structure of $\Omega_N(l)$ from the known results on $M_N(l)$. Moreover, this observation suggests a plenty of natural geometric problems of the same type as the one described below. Here we only consider the case when $N = 3$ and $n = 4$. In other words, we discuss relations between spatial configurations quadrilateral $Q(l)$ and Minkowski tetrahedra with face gauge l . More general situations will be considered elsewhere.

In line with [14], [15], we wish to compare the critical points of the two

functions V_o, V^c on moduli spaces $\Omega_3(l)$ and $M_3(l)$. The first one is defined on $\Omega_3(l)$ as the usual oriented volume of tetrahedron, while, for a configuration P of a 4-bar linkage L , we put $V^c(P) = V_o(\text{Conv } P)$, where $\text{Conv } P$ is the convex hull of P , i.e. also a tetrahedron, maybe degenerate. The following formula gives the crucial relation between oriented volume of the convex hull of a spatial configuration P of $Q(l)$ and the oriented volume of its Minkowski transform $\mu(P)$. Its proof follows easily from the determinantal formula for the volume of tetrahedron.

Proposition 5.1. *With the notation as above one has $3V^c = 4V_o^2$.*

This formula enables one to describe the maxima of $V_o(\text{Conv } P)$ on $M_3(Q)$ and V_o on $Q(l)$. For a positive number d , let X_d denote the set of all configurations P of L such that the length of the diagonal p_1p_3 is equal to d . It is then obvious that the maximum of V_c on X_d is attained at configuration for which the dihedral angle between the two faces containing p_1p_3 is equal to $\pi/2$. By the same reasoning, for the V_c -maximal configuration, the dihedral angle by the second diagonal should also be $\pi/2$. It is easy to verify that there always exists a configuration P^* with both dihedral angles by diagonals equal to $\pi/2$ and so the (global) maximum of $V_o(\text{Conv } P)$ is attained at P^* .

Now, Proposition 5.1 obviously implies that the maximum of V^o on $Q(l)$ is attained on the Minkowski transform of the V^c -maximal configuration identified above. From the definition of Minkowski transform follows that each pair of the opposite sides of the V_o -maximal tetrahedron should be orthogonal (perpendicular). As is well known this condition is equivalent to the orthocentricity of tetrahedron in question. In other words, all of its four heights have a common point (called orthocenter). Thus we arrive to the following pleasant result.

Theorem 5.2. *The maximum of V^o on $Q(l)$ is attained on an orthocentric tetrahedron.*

It is possible to show that the V_o -maximal face gauged tetrahedron unique. an analogous result holds for f -gauged tetrahedra in all dimensions. Similar considerations are applicable in the case of the normalized determinant of configuration considered by M.Atiyah [2]. These and other generalizations of Theorem 5.2 will be considered elsewhere. In general, the connection described above suggests many open problems and plausible conjectures which will be addressed in the further research of the author. We conclude by

adding that most of considerations in this paper make sense for spherical linkages on S^N (cf. [10] for the case of two-dimensional sphere) and it would be interesting to find out what results in the spirit of this paper can be obtained for spherical linkages.

References

- [1] A.D.Aleksandrov, Convex polyhedra, Moscow, 1956. (in Russian)
- [2] M.Atiyah, Configurations of points, Phil. Trans. R. Soc. London A359, 2001, 1375-1387.
- [3] M.Berger, Geometrie, Vol.1, Paris, Cedec, 1984.
- [4] G.Giorgadze, G.Khimshiashvili, Remarks on spherical linkages, Bull. Georgian Natl. Acad. Sci. 4, No.2, 2010, 13-18.
- [5] R.Connelly, E.Demaine, Geometry and topology of polygonal linkages, CRC Handbook of discrete and computational geometry. 2nd. ed., 2004, 197-218.
- [6] G.Darboux, De l'emploi des fonctions elliptiques dans la théorie du quadrilatère plan, Bull. Sci. Math. Astron. 3, 1879, 109-120.
- [7] J.Duistermaat, Discrete integrable systems, Springer, 2010.
- [8] L.Flatto, Poncelet's Theorem, Amer. Math. Soc., Providence, 2009.
- [9] C.Gibson, P.Newstead, On the geometry of the planar 4-bar mechanism, Acta Appl. Math., 1986, 7, 23, 113-135.
- [10] G.Giorgadze, G.Khimshiashvili, Remarks on spherical quadrilateral linkages, Bull. Georgian Natl. Acad. Sci. 4, No.2, 2010, 21-24.
- [11] M.Kapovich, J.Millson, On the moduli spaces of polygons in the Euclidean plane, J. Diff. Geom. 42, No.1, 1995, 133-164.
- [12] M.Kapovich, J.Millson, The symplectic geometry of polygons in Euclidean space, J. Differ. Geom. 44, No.3, 1996, 479-513.

- [13] G.Khimshiashvili, Cyclic polygons as critical points, Proc. I.Vekua Inst. Appl. Math. 3, 2008, 73-78.
- [14] G.Khimshiashvili, Signature formulae and configuration spaces, J. Math. Sci. (N.Y.) 160, No.10, 2009, 727-736.
- [15] G.Khimshiashvili, Extremal problems on configuration spaces, Proc. A.Razmadze Math. Institute 155, 2011, 73-77.
- [16] G.Khimshiashvili, G.Panina, Cyclic polygons are critical points of area. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 360, 2008, 238-245.
- [17] O.Mermoud, M.Steiner, Configuration spaces of weighted graphs in high dimensional Euclidean spaces, Beitr. Algebra Geom. 43, No.1, 2002, 27-31.

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Riemann-Hilbert boundary value problem for generalized analytic functions in Smirnov classes

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Riemann-Hilbert problem for complex holomorphic functions in the classic Smirnov classes was studied in the works: [1] for the Lyapunov boundaries and [2] for domains with Radon boundaries.

Let G is bounded simply connected domain in complex z -plane, $z = x + iy$, $i^2 = -1$, with rectifiable boundary $\Gamma = \partial G$; $\overline{G} = G \cup \Gamma$; $A(z), B(z) \in L_s(\overline{G})$ $s > 2^1$, are given complex functions. Without limiting the generality, we assume that the point $z = 0$ is located inside G .

Further we assume Γ Lyapunov curve or Radon curve without cusp points.

We consider in \overline{G} canonical elliptic system in the complex entry

$$\partial_{\bar{z}}w + A(z)w + B(z)\bar{w} = 0, \quad (1)$$

where $w = w(z) = u(z) + iv(z)$ is unknown complex function, u and v are its real and imaginary parts, $\partial_{\bar{z}} = 1/2(\partial/\partial x + i\partial/\partial y)$ is derivative in the Sobolev sense.

In this work we investigate *Riemann-Hilbert (Hilbert) problem* in the next posing: to find in the domain G solution $w = w(z)$ of the equation (1), $w(z) \in E_p(A, B)$, $p > 1$, [4], [5], whose non-tangent limiting values on Γ satisfies almost everywhere boundary condition

$$\operatorname{Re} \left\{ \overline{\lambda(t)} w(t) \right\} = g(t), \quad (2)$$

where $t = t(s)$, $s \in [0, S]$, is the affix of the point on Γ , $\lambda = \lambda(t)$ is complex measurable function defined on Γ and satisfies conditions $0 < k_0 \leq |\lambda(t)| \leq$

¹We use the notations from the book [3].

$k_1 < \infty$, k_0, k_1 are real constants, $g(t) = g(t(s)) \equiv g(s) \in L_p(\Gamma) \equiv L_p[0, S]$ is real function, defined on Γ .

Following [3, p. 179], we say that the equation

$$\partial_{\bar{z}} w^* - A(z)w^*(z) - \overline{B(z)w^*(z)} = 0, \quad z \in G, \quad (3)$$

is *adjoint* to the equation (1).

Generalizing [3, p. 301] we call the *adjoint* (homogeneous) problem to (2) the problem of finding in G the solution of (3) $w^*(z) \in E_{p'}(-A, -\overline{B})$, $1/p + 1/p' = 1$, which non-tangent limit values on Γ almost everywhere on Γ satisfy boundary condition

$$\operatorname{Re} \{ \lambda(t)t'(s)w^*(t) \} = 0. \quad (4)$$

Following [6, p. 190] (and [1], [2]) we *assume* that we can choose at least one starting point $s = 0$ on Γ so that the function $\omega(t) = \arg \lambda(t)$ satisfies the next condition:

$$\omega(s) = \tilde{\omega}_0(s) + \tilde{\omega}_1(s) + \omega_2(s), \quad (5)$$

where $\tilde{\omega}_0(s)$ is continuous function on $[0, S]$ (at the ends we mean the one-side continuity); $\tilde{\omega}_1(s)$ is the function of finite variation on $[0, S]$; $\omega_2(s)$ is measurable function on $[0, S]$ satisfying the next conditions:

$$|\omega_2(s)| \leq \nu\pi, \quad 0 < \nu < \frac{1}{2p}, \quad 0 < \nu < \frac{1}{2p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (6)$$

Without the loss of generality we assume [6, p. 190] that $\omega(0) = \omega(S)$ and $\tilde{\omega}_1(s)$ is right-side continuous at the point $s = 0$. After these assumptions we can rewrite (5) in the next form [6, p. 190]:

$$\omega(s) = \omega_0(s) + \omega_1(s) + \omega_2(s), \quad (7)$$

where $\omega_2(s)$ is former; $\omega_1(s)$ is the jump function of $\tilde{\omega}_1(s)$, $\{s_k\}$ is no more than countable set of discontinuity points of $\tilde{\omega}_1(s)$:

$$\omega_1(0) = 0, \quad \omega_1(s) = \sum_{0 < s_k < s} h_k + [\tilde{\omega}_1(s) - \tilde{\omega}_1(s-0)], \quad 0 < s \leq S,$$

$h_k = \tilde{\omega}_1(s_k + 0) - \tilde{\omega}_1(s_k - 0)$. The continuous on $[0, S]$ function $\omega_0(s)$ equals to the sum $\tilde{\omega}_0(s) + [\tilde{\omega}_1(s) - \omega_1(s)]$.

The *indexes* \varkappa and $\overset{(p)}{\varkappa}$ of the boundary value problem (1), (2) was defined in [1], [2].

Theorem 1 *If in (7) $\omega_1(s) \equiv 0$ and Γ is Lyapunov curve, then at $\varkappa \geq 0$ homogenous problem (1), (2) (at $g(t) \equiv 0$) has exactly $2\varkappa+1$ linear independent in the real sense solutions in the class $E_p(A, B)$, $p > 1$. Non-homogenous problem is solvable in $E_p(A, B)$ at arbitrary right side $g(t) \in L_p(\Gamma)$ of the boundary condition.*

If $\varkappa < 0$ then the homogenous problem (1), (2) has not in $E_p(A, B)$ non-zero solution and non-homogenous problem has unique solution in $E_p(A, B)$ if and only if $-2\varkappa-1$ (real) conditions on the right side $g(t)$ of the boundary condition (2) are held:

$$\int_{\Gamma} g(s)e^{i\omega(s)}w_k^*(t)t'(s)ds = 0. \quad (8)$$

Here $w_k^(t) \in E_{p'+\varepsilon}(-A, -\overline{B})$, $E_{p+\varepsilon}(-A, -\overline{B})$, $k = 1, \dots, -2\varkappa-1$, is the full system linear independent in the real sense solutions of the adjoint to (1), (2) boundary value problem (3), (4) with index $\varkappa^* = -\varkappa-1 \geq 0$, $\varepsilon > 0$ is little.*

Theorem 2 *If in (7) $\omega_2(s) \equiv 0$, and Γ is Lyapunov curve or Radon curve without cusp points, then at $\overset{(p)}{\varkappa} \geq 0$ homogenous problem (1), (2) (at $g(t) \equiv 0$) has exactly $\overset{(p)}{\varkappa} + 1$ linear independent in the real sense solutions in the class $E_p(A, B)$, $p > 1$. Non-homogenous problem is solvable in $E_p(A, B)$ at arbitrary right side $g(t) \in L_p(\Gamma)$ of the boundary condition.*

If $\overset{(p)}{\varkappa} < 0$, homogenous problem (1), (2) has not non-zero solution in $E_p(A, B)$, $p > 1$, and non-homogenous problem has unique solution if and only if $-\overset{(p)}{\varkappa}-1$ (real) conditions on the right side $g(t)$ of the boundary condition (2) are held:

$$\int_{\Gamma} e^{i\omega(s)}w_k^*(t(s))t'(s)g(s)ds = 0, \quad k = 1, 2, \dots, -\overset{(p)}{\varkappa}-1. \quad (9)$$

Here $\{w_k^(z)\} \in E_{p'}(-A, -\overline{B})$ is the full system linear independent in the real sense solutions of the adjoint to (1), (2) boundary value problem (3), (4).*

It should be noted then if $\overset{(p)}{\varkappa} = -1$, we get $k = 0$. It means uniquely unconditionally solvability of the non-homogenous problem.

The main difficulty is the impossibility in the case of the non-smooth border to reduce the problem by conformal mapping to one in Hardy class of generalized analytic functions.

To solve the problem in this work the special representation for generalized analytic functions of Smirnov classes is built. This representation has independent interest.

Definition 1 We say that for the boundary value problem (2) ((2)) the condition **D** is hold if:

- 1) when Γ is Lyapunov curve, in (7) or $\omega_1(s) \equiv 0$, or $\omega_2(s) \equiv 0$;
- 2) when Γ is Radon curve without cusp points, in (7) $\omega_2(s) \equiv 0$.

Theorem 3 Let the condition **D** is held and or index \varkappa of the boundary value problem (2) is non-negative, or index $\varkappa^{(p)} \geq -1$.

If $w(z) \in E_p(A, B)$, $p > 1$, the next representation takes place:

$$w(z) + P_\lambda w(z) = \Phi(z), \quad (10)$$

where $\Phi(z) \in E_p$ and almost everywhere on Γ

$$\operatorname{Re}\{\bar{\lambda}(t)w(t)\} = \operatorname{Re}\{\bar{\lambda}(t)\Phi(t)\}, \quad t \in \Gamma. \quad (11)$$

If $\Phi(z) \in E_p$, then the relation (10) uniquely defines the function $w(z) \in E_p(A, B)$, satisfying almost everywhere on Γ condition (11). Formula (10) establishes (real) linear isomorphism between Banach spaces $E_p(A, B)$ E_p , and also the operator $P_\lambda : E_p(A, B) \rightarrow L_p(\Gamma)$ is completely continuous.

Remark 1 If G is the unit disk, $\lambda(t) = t^n$, $n \geq 0$ — non-negative integer, then the operator $P_\lambda w$ coincides with the operator $P_n(Aw + B\bar{w})$, constructed by I.N. Vekua in [3, p. 293–296].

The full text of the work will be published in [7].

References

- [1] S. B. Klimentov, Hilbert boundary value problem for holomorphic functions in the Smirnov classes. *Studies on mathematical analysis, differential equations, and there applications. Review of Science. The South of Russia. Mathematical Forum* 4, SMI, Vladikavkaz, 2010, 252–263 (in Russian).

- [2] S. B. Klimentov, Hilbert boundary value problem for holomorphic functions in the Smirnov classes in the domain with Radon boundary. *Izvestia Vuzov. Sev.- Kav. Region. Natural Science*, No. 3, 14–18, 2011 (in Russian).
- [3] I. N. Vekua, Generalized analytic functions. Fizmatgiz, Moscow, 1959 (in Russian). English translation: I. N. Vekua, “Generalized Analytic Functions”. XXIX + 668 S. Oxford/London/New York/Paris 1962. Pergamon Press.
- [4] K. M. Musaev, Some classes of generalized analytic functions. *Izv. Acad. Nauk Azerb. SSR*, No. 2, 40–46, 1971 (in Russian).
- [5] S. B. Klimentov, Smirnov classes of generalized analytic functions. *Izvestia Vuzov. Sev.- Kav. Region. Natural Science*, No. 1, 13–17, 2005 (in Russian).
- [6] I. I. Danilyuk, Non-regular boundary value problems in a plane. Nauka, Moscow, 1975 (in Russian).
- [7] S. B. Klimentov, Riemann-Hilbert Boundary Value Problem for Generalized Analytic Functions in Smirnov Classes. *Global and Stochastic Analysis* **1** (2011), No. 2 (to appear).

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Hypergeometric isomonodromic deformations of Fuchsian systems

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Schlesinger families of Fuchsian systems

Let us consider a Fuchsian system of p equations

$$\frac{dy(z)}{dz} = \left(\sum_{i=1}^n \frac{B_i}{z - a_i^0} \right) y(z). \quad (1)$$

on Riemann sphere $\bar{\mathbb{C}}$ with $n + 1$ singular points $a_1^0, \dots, a_n^0, a_{n+1}^0 = \infty$ and constant matrices B_i of the size $p \times p$. The Schlesinger family of Fuchsian systems for system (1) is following family

$$\frac{dy(z, a)}{dz} = \left(\sum_{i=1}^n \frac{B_i(a)}{z - a_i} \right) y(z, a). \quad (2)$$

with initial conditions $B_i(a^0) = B_i, \quad i = 1, 2, \dots, n$, where

$$a^0 = (a_1^0, a_2^0, \dots, a_n^0) \in \mathbb{C}_*^n,$$

$$\mathbb{C}_*^n = \{(a_1, a_2, \dots, a_n) \mid a_i \neq a_j, \quad i \neq j, \quad i, j = 1, 2, \dots, n\}.$$

Isomonodromic Schlesinger families

The Schlesinger family (2) is an isomonodromic family for system (1) if for points a from enough small polydisk neighborhood $U(a^0) \subset \mathbb{C}_*^n$ of the point $a^0 = (a_1^0, a_2^0, \dots, a_n^0)$ there exist family of fundamental matrices of solutions $Y(z, a)$ such that their monodromy matrices $G_i, \quad i = 1, \dots, n$ corresponding "small loops" going around singular points a_1, \dots, a_n does not depend from a .

Schlesinger systems

In small neighborhood $U(a^0) \subset \mathbb{C}_*^n$ of the point $a^0 = (a_1^0, a_2^0, \dots, a_n^0)$ the sufficient isomonodromic condition on the family (2) is the system of the Schlesinger equations on matrices $B_i(a)$, $i = 1, 2, \dots, n$

$$d B_i(a) = - \sum_{j=1, j \neq i}^n [B_i(a), B_j(a)] \frac{d(a_i - a_j)}{a_i - a_j}. \quad (3)$$

Here $[B_i, B_j] = B_i B_j - B_j B_i$ denote the commutator of matrices.

Some properties of the Schlesinger systems

- Non-linear Pfaff system (3) is integrable in Frobenius sense [2,4] and consequently in sufficient small neighborhood of the a^0 there exist the local solution $B(a) = (B_1(a), B_2(a), \dots, B_n(a))$ of the Pfaff system (3) with every initial value $B(a^0) = (B_1(a^0) = B_1, B_2(a^0) = B_2, \dots, B_n(a^0) = B_n)$.
- As well-known [2] that all eigenvalues of all matrices $B_i(a)$ are constants, that is, don't depend on a .
- The sum

$$\sum_{i=1}^n B_i(a) = -B_\infty$$

is constant matrix.

- Malgrange has proved [5] that the local solution $B(a)$ has meromorphic continuation on whole universal covering \tilde{C}_*^n .
- In general, polar divisor Θ (theta-divisor Malgrange) of the solution $B(a)$ on \tilde{C}_*^n is non-empty and it depends on initial data $B(a^0)$.
- The Malgrange theta-divisor is defined by zeros of the Miwa tau-function $\tau(a)$, which is a solution of the equation

$$d \log \tau(a) = \kappa \sum_{i \neq j, i, j=1}^n \text{tr}(B_i(a) B_j(a)) \frac{d(a_i - a_j)}{a_i - a_j}. \quad (4)$$

Upper-triangular Schlesinger systems

We suppose that all matrices $B_i(a), i = 1, \dots, n$ from the Schlesinger family (2) are upper-triangular matrices $B_i(a) = \Lambda_i + U_i^1(a) + \dots + U_i^{p-1}(a)$. Here Λ_i are constant diagonal matrices. $U_i^1(a)$ have non-zero entries only in first off-diagonal, respectively, $U_i^2(a)$ have non-zero entries in second off-diagonal and so on. Then corresponding Schlesinger system (3) has the following special form

$$\begin{aligned} d\Lambda_i &= 0, \\ dU_i^1 &= - \sum_{j=1, j \neq i}^n ([\Lambda_i, U_j^1] + [U_i^1, \Lambda_j]) \frac{d(a_i - a_j)}{a_i - a_j}, \\ dU_i^{k+1} &= - \sum_{j=1, j \neq i}^n ([\Lambda_i, U_j^k] + [U_i^k, \Lambda_j]) + \sum_{r+s=k+1} [U_i^r, U_j^s] \frac{d(a_i - a_j)}{a_i - a_j}, \\ &k = 2, \dots, p-1. \end{aligned} \quad (3')$$

As example we consider upper-triangular Schlesinger systems for $p=2$. For matrices the size 2×2

$$B_i = \begin{pmatrix} \lambda_1^i & b_i(a) \\ 0 & \lambda_2^i \end{pmatrix}, \quad i = 1, \dots, n.$$

the Schlesinger system (3') has the form

$$db^i(a) = - \sum_{j \neq i} (\lambda_{12}^i b^j(z) - (\lambda_{12}^j b^i(z))) \frac{d(z_i - z_j)}{z_i - z_j}, \quad (3'')$$

where $\lambda_{12}^i = \lambda_1^i - \lambda_2^i$, $i = 1, \dots, n$.

Jordan-Pochhammer systems

We will consider the class of Fuchsian systems on multidimensional linear complex spaces \mathbb{C}^n

$$dy(z) = \left(\sum_{1 \leq i < j \leq n} J_{ij}(\lambda) \frac{d(z_i - z_j)}{z_i - z_j} \right) y(z). \quad (5)$$

Here $y(z)$ is a vector-column with n component $y_i(z)$, $i = 1, \dots, n$ and $J_{ij}(\lambda)$ are the following matrices of the size $n \times n$:

$$J_{ij}(\lambda) = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & \lambda_j & \dots & -\lambda_i & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & -\lambda_j & \dots & \lambda_i & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix} \begin{matrix} i \\ \\ j \\ \\ \\ \end{matrix}$$

Another form of the equation (5) is the equation

$$dy_i(z) = \left(\sum_{j \neq i, j=1}^n (\lambda_j y_i - \lambda_i y_j) \frac{d(z_i - z_j)}{z_i - z_j} \right). \quad (6)$$

The symbol λ in $J_{ij}(\lambda)$ denotes an ordered collection of complex numbers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. The matrices $J_{ij}(\lambda)$ satisfy the relations

$$[J_{ij}(\lambda), J_{ik}(\lambda) + J_{jk}(\lambda)] = 0, \quad 1 \leq i < j < k \leq n; \quad (*)$$

$$[J_{ij}(\lambda), J_{kl}(\lambda)] = 0, \quad \{i, j\} \cap \{k, l\} = \emptyset. \quad (**).$$

The relations (*) and (**) are equivalent to the Frobenius condition of the integrability of system (5) (that is, the system (5) is integrable).

Then any fundamental matrix of solutions $Y(z)$ (coordinates of basis solution $y(z)$ state in a column) of the (5) has a holomorphic continuation on the universal covering and deck transformations corresponding to elements $g \in \pi_1(\mathbb{C}_*^n, a^0)$ act on $Y(z)$ by the rule $Y(g^*z) = Y(z)M(g)$. The constant matrix $M(g)$ is called the monodromy matrix for $g \in \pi_1(\mathbb{C}_*^n, a^0)$.

The entries y_{ij} of the fundamental matrix $Y(z)$ of solutions of the system (5) have the following hypergeometric integral representations [4], [6]

$$y_{ij}(a_1, \dots, a_n) = \lambda_i \int_{\gamma_j} (t - a_1)^{\lambda_1} \dots (t - a_n)^{\lambda_n} \frac{dt}{t - a_i}, \quad (7)$$

where γ_j , $j = 1, \dots, n$ is a basis in homology group

$$H_1(\mathbb{C}P^1 \setminus \{a_1, \dots, a_n, \infty\}, \mathcal{L}_\chi)$$

with local coefficients \mathcal{L}_χ . The local system \mathcal{L}_χ is defined by a representation $\chi : \pi_1(\mathbb{C}P^1 \setminus \{a_1, \dots, a_n, \infty\}, t_0) = F_n \rightarrow \mathbb{C}^*$ of the fundamental group $\pi_1(\mathbb{C}P^1 \setminus \{a_1, \dots, a_n, \infty\}, t_0) = F_n$, that maps generators x_1, \dots, x_n of this free group F_n to non-zero complex numbers $q_1 = e^{-2\pi i \lambda_1}, \dots, q_n = e^{-2\pi i \lambda_n}$.

Hypergeometric solutions

theorem 1. *For upper-triangular matrices $B_i(a)$ the size 2×2 , the Schlesinger system (3'') coincides with the Jordan-Pochhammer system in the form (6) under suitable choice parameters $\lambda_i, i = 1, \dots, n$. Hence upper-triangular solutions of the Schlesinger system (3) are only hypergeometric type solutions and among of entries such solutions does not a new transcendent.*

Corollary 1. *Under conditions of the theorem 1 Malgrange divisor of Schlesinger system (3) is the empty set.*

The statement of corollary is the consequence of linearity and integrability of Jordan-Pochhammer system and properties of their solutions pointed above.

Corollary 2. *In fixed singular points $a \in \mathcal{H} = \cup_{1 \leq i < j \leq n} H_{ij}$ the solutions of Schlesinger system have moderate growth.*

R.R. Gontsov [2] found sufficient conditions on initial Fuchsian systems two order on the Riemann sphere that have only upper-triangular Schlesinger deformations up to the conjugation on constant a matrix. For formulation this result we suppose that coefficients of the system (1) have traces equal to zero (that is, $B_i \in sl_2(\mathbb{C}), i = 1, \dots, n$) then $B_i(a) \in sl_2(\mathbb{C}), i = 1, \dots, n$. Denote $\lambda_i, i = 1, \dots, n$ eigenvalues of matrices B_i and λ_∞ eigenvalue of matrix $B_\infty = -\sum_{i=1}^n B_i$. We suppose also that B_∞ is a diagonal matrix.

theorem 2 (R. Gontsov). *Let the monodromy representation of Fuchsian system (1) be reducible. If for some choice of signs*

$$\varepsilon_i = \pm 1, i \in \{1, 2, \dots, n, \infty\}$$

the sum of eigenvalues $\sum_{i=1}^m \varepsilon_i \lambda_i + \varepsilon_\infty \lambda_\infty$ is equal to zero then after conjugation $CB_i(a)C^{-1}$ by a constant matrix C of any solution $B_i(a), i = 1, 2, \dots, n$ the Schlesinger system (3) it is reduced to upper-triangular form

References

- [1] A.A. Bolibruch, *Inverse monodromy problems in analytic theory of differential equations*. M: MCCME, 2009 (in Russian).
- [2] R.R. Gontsov, *On movable singularities of Garnier systems*. Matem. Zametki. 2010. V.88.nom. 6. S. 858
- [3] T. Kohno, *Linear representations of braid groups and classical Yang-Baxter equations*. Contemporary Math. 1988. V.78. P.339-363.
- [4] M. Kapovich and J. Millson, *Quantization of bending deformations of polygons in \mathbb{E}^3 , hypergeometric integrals and the Gassner representation*. Canad. Math. Bull. 2001. V.44. P.36-60.
- [5] B. Malgrange, *Sur les déformations isomonodromiques.I: Singularités régulières*. Birkhäuser. 1983.(Progress in Math.;V.37). P.401-426.
- [6] A. Varchenko, *Special functions, KZ type equations, and representation theory*. CBMS regional conference series in Mathematics. No. 98. AMS-NSF. 2003.

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About some quasilinear and nonlinear equations of Cauchy-Riemann and Beltrami types

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In this paper the problems of existence and representation of solutions of some quasilinear and nonlinear equations of Cauchy-Riemann and Beltrami types are considered.

1. Let's consider the following quasilinear equation

$$\frac{\partial w}{\partial \bar{z}} = f(z, w), \quad z = x + iy, \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad (1)$$

Where the function $f(z, w)$ is defined in domain $\Omega = \{z \in G, w \in E, |w| < \infty\}$, G is bounded domain of the complex plane E and satisfies the condition:

$$|f(z, w)| \leq A(z)|w|^\alpha, \quad 0 \leq A(z) \in L_p(\bar{G}), \quad p > 2, \quad \alpha = \frac{m}{n} \neq 1, \quad z \in Q \quad (2)$$

The solution of the equations here and below is understood in a generalized sense I.N. Vekua [1].

Theorem 1. *Let function $w(z) \neq 0$ is the solution of the equation (1), where $f[z, w]$ satisfies the condition (2). Then the function*

$$\Phi(z) = \left[w^\alpha + \frac{1}{\alpha} \omega(z) \right]^{\frac{1}{\alpha}} \quad (3)$$

belongs to the class $U_0^*(G)$, where $\omega(z) = \frac{1}{\pi} \iint_G \frac{f(\xi, w)}{w^\alpha} \frac{d\xi d\eta}{\xi - \eta}, \quad \xi = \xi + i\eta$.

By virtue of (2) $\frac{\partial \Phi}{\partial \bar{z}} = 0$. Therefore, our statement is valid.

Theorem 2. *If the equation (1) satisfying the condition (2) in domain G has the generalized solution $w(z)$ then we have the representation of I. N.*

Vekua of the first type:

$$w(z) = [(1 - \alpha)(\Phi(z) + \omega(z))]^{\frac{1}{1-\alpha}}, \quad (5)$$

where $\Phi(z)$ is arbitrary analytic in G function, $\omega(z)$ is defined by the formula (4).

Opposite to the case of natural α in considered case the solution hasn't always the property of a discretization of zero. For example, for the equation

$$\frac{\partial w}{\partial \bar{z}} = 2\sqrt{w}$$

the conditions of the theorem 2 hold. The function $w(z) = (z + \bar{z})^2$ is a solution, but for the domain containing segment of conjugate axis, the zero fill out all this segment.

Let $f(z, w)$ satisfies the additional condition

$$\iint_G \left| \frac{f(\xi, u_1)}{u_1^\alpha} - \frac{f(\xi, u_2)}{u_2^\alpha} \right| d\xi d\eta \leq A_0 \iint_G |u_1 - u_2|^p d\xi d\eta. \quad (6)$$

The operator:

$$Hu = \frac{1}{(\alpha - 1)^{\frac{1}{\alpha-1}}} \left[\Phi(z) + \frac{1}{\pi} \iint_G \frac{f(\xi, u)}{u^\alpha} \frac{d\xi d\eta}{\xi - z} \right]^{\frac{1}{1-\alpha}} \quad (7)$$

is continuously maps $L_p(\bar{G})$ in itself. Then the solutions of the equation (2) one can find applying the method of iteration.

2. We shall consider now nonlinear equation

$$\frac{\partial w}{\partial \bar{z}} = F\left(z, \frac{\partial w}{\partial z}\right), \quad (8)$$

where function

$$F_1(z) \equiv F(z, h(z)) \in L_p(\bar{G}) \quad \text{and} \quad |F(z, h_2) - F(z, h_1)| \leq A_F |h_2 - h_1|. \quad (9)$$

The equation (8) is equivalent to the equation

$$w(z) = -\frac{1}{\pi} \iint_G \frac{F\left(\xi, \frac{\partial w}{\partial \xi}\right)}{\xi - z} d\xi d\eta + \Phi(z) \equiv T_G F_1 + \Phi(z), \quad (10)$$

where $\Phi(z)$ is arbitrary analytic function in \bar{G} .

Let's use the following notation:

$$S_G F \left(z, \frac{\partial w}{\partial z} \right) = S_G F_1(z) = -\frac{1}{\pi} \iint_G \frac{F_1(\xi)}{(\xi - z)^2} d\xi d\eta,$$

$$\frac{\partial w}{\partial \bar{z}} = w^*(z).$$

We obtain nonlinear two-dimensional singular integral equation

$$w^*(z) = \Phi' + S_G F(z, w^*(z)). \quad (11)$$

It is not easy to show, that the operator S_G^* is an operator of compression in L_p , $p \geq 2$. Therefore equation (10) has a unique solution w_0^* .

Then solving the equation $\frac{\partial w}{\partial z} = w_0^*$, we have

$$w(z) = -\frac{1}{\pi} \iint_G \frac{w_0^*(\xi)}{\xi - \bar{z}} d\xi d\eta + \Phi_1(z)$$

where $\Phi_1(z)$ is arbitrary antiholomorphic function in \bar{G} .

Remark. The equation of a type (8) with a right-hand side $F(z, w, \frac{\partial w}{\partial z})$ by was investigated by Wolfgang Tutschke in a different way [4].

References

- [1] Vekua I. N. *Generalized analytic functions*. 2nd ed., Nauka, Moscow 1988; English transl. of 1st ed., Pergamon, London, Paris, Frankfurt; Addison-Wesley, Reading, MA 1962.
- [2] Monakhov V.N. *Boundary value problems with the free boundaries for elliptic system of equations*. Novosibirsk: Nauka. 1977.
- [3] Magomedov G.A. *Fundamentals of the theory of generalized analytical functions by one and many variable*. M.: Minvuz, 2003
- [4] Wolfgang Tutschke. *On the approximation of continuous functions by solutions of partial complex differential equations*, Ann. pol. Math. XLX (1985)

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Some properties of the irregular Elliptic Systems on the Plane*

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Abstract

In this paper we proved, that the solutions of some singular elliptic systems have principally nonanalytic behavior in the neighborhood of fixed singular points.

The solutions of the regular equations [1]

$$\frac{\partial w}{\partial \bar{z}} + Aw + B\bar{w} = 0. \quad (1)$$

can't have the singularities of the pole type of "infinite order" and nontrivial solutions of such equations can't have nonisolated zero and zero of "infinite order" in the points of regularity. The solutions of the irregular equations of the wide class aren't subjected to such exclusions.

The above mentioned general properties of the irregular equations of the form (1) make clear the complexity of investigations. These equations were the subject of investigation of various authors. Among them there are basic works of Vekua I. (see [2],[3]). In this direction of generalized analytic functions the most important results were obtained by Mikhailov L., Vinogradov V., Usmanov Z., Blied N., Schmidt V., Saks R., Tungatarov A., Najhmidinov Kh., Akhmedov R., Begehr H., DaiD. Q., Reissig M., Timofeev A. The references according to this subject are presented in [4] and the present paper we use the notations from the same work.

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To illustrate the possible structures of the solutions of quasiregular equation (1) consider the following simplest one

$$\frac{\partial w}{\partial \bar{z}} + \frac{\lambda \cdot \exp(i\varphi)}{r^2} w = 0, \quad (2)$$

where the complex number $\lambda \neq 0$, r, φ - are polar coordinates of the variable z , $z = r \exp(i\varphi)$ and the main thing is, that the domain G contains the origin $z = 0$. It is clear, that (2) is irregular equation. Checking directly we get, that the function w is contained in the class

$$S_\lambda \equiv \mathfrak{A}^* \left(\frac{\lambda \cdot \exp(i\varphi)}{r^2}, 0, G \right) \quad (3)$$

if and only if w has the form

$$w(z) = \Phi(z) \cdot \exp \left\{ \frac{2\lambda}{r} \right\}, \quad (4)$$

where the function $\Phi \in \mathfrak{A}_0^*(G)$. It follows from (4), that the classes S_λ aren't similar. In order to explain what has been said note, that in case $\operatorname{Re} \lambda = 0$ the module of every function of the class S_λ coincides with the module of analytic function in the domain G . When $\operatorname{Re} \lambda > 0$ in the class S_λ there are neither nontrivial bounded in the neighborhood of the point $z = 0$ functions nor the functions with power growth; i.e. the functions admitting estimate

$$w(z) = O \left(\frac{1}{|z|^\sigma} \right), \quad z \rightarrow 0, \quad (5)$$

for some real number $\sigma > 0$. When $\operatorname{Re} \lambda < 0$ in S_λ there exists extensive subclass, every function of which more rapidly tends to zero then arbitrary positive power of $|z|$, while $z \rightarrow 0$.

When $\operatorname{Re} \lambda > 0$, there aren't nontrivial regular solutions of the equation (2) in the point $z = 0$. Indeed, if the function w satisfies the equation (2) in $z = 0$, then it has the form (4), where Φ is analytic function in some neighborhood $V_\rho(0) = \{z : |z| < \rho\}$, $\rho > 0$, of the point $z = 0$. It is clear, that the function ((4)) can't satisfy the following condition

$$\iint_{V_\rho(0)} |w(z)| dx dy < +\infty$$

for every number $\rho > 0$ and for nontrivial analytic function Φ . When $\operatorname{Re} \lambda \leq 0$ the equation (2) has extensive class of regular solutions in $z = 0$. When $\operatorname{Re} \lambda < 0$ the formula (4) gives the regular in $z = 0$ solution not only when the function Φ has the pole of arbitrary power but even in the case when Φ has the essential singular point in $z = 0$, but hasn't rapidly exponential growth in its neighborhood. This growth measure is limited by the multiplier $\exp\left\{\frac{2\lambda}{r}\right\}$. When $\operatorname{Re} \lambda < 0$ the regular (of sufficiently wide class) solutions of the equation (2) have zero of infinite order, i.e. these solutions w satisfy the condition

$$\lim_{z \rightarrow 0} \frac{w(z)}{(z - z_0)^k} = 0, \quad k = 0, 1, 2, \dots$$

In these cases $\operatorname{Re} \lambda < 0$ the equation (2) also has regular solutions w (their class is sufficiently wide) also such that $z = 0$ is limit point of their zeroes.

The equation (2) is the particular case of the equation (4) with the coefficients

$$\begin{aligned} A(z) &= \frac{\lambda \cdot \exp(i\varphi)}{r^\nu} + \frac{A_0(z)}{r^{\nu_1}} + \overline{h(z)}, \\ B(z) &= \frac{B_0(z)}{r^\mu}, \end{aligned} \tag{6}$$

where the real numbers ν, ν_1, μ satisfy the condition

$$\mu \geq 0, \quad \nu \geq \max\{[\mu] + 2, [\nu_1] + 2\}, \tag{7}$$

and the functions

$$h \in \mathfrak{A}_0^*(G), \quad A_0, B_0 \in L_p(G), \quad p > 2 \tag{8}$$

(G is a bounded domain containing the origin).

Most statements formulated above for the model equation (1) can be proved for the equation (1) with the coefficients (6) also.

Picture, described above for the class $\mathfrak{A}^*(A, B, G)$, sharply changes, if we carry the apparently insignificant change in the coefficient $A(z)$ from (6), namely we get a very interesting picture if we consider the coefficients

$$\begin{aligned} A(z) &= \frac{\lambda \cdot \exp(im\varphi)}{|z|^\nu} + \frac{A_0(z)}{|z|^{\nu_1}} + \overline{h(z)}, \\ B(z) &= \frac{B_0(z)}{|z|^\mu}, \end{aligned} \tag{9}$$

where m is a natural number and with respect to other parameters of the functions A, B the above assumptions (7) and (1) are fulfilled.

It is clear, that the equation (1) with the coefficients (9) is quasiregular. Using the relation (see [4])

$$\Phi = w \cdot \exp\{-\Theta\} \quad (10)$$

for this equation first and then applying a modification of the well known principle of Phragmen-Lindelof from function theory we get the following theorem

Theorem 1 *Let the generating pair of the class $\mathfrak{A}^*(A, B, G)$ be of the form (9), the conditions (7) and (8) be fulfilled and*

$$\lambda \neq 0, \quad m > 1, \quad m \neq \nu, \quad (11)$$

then every solution $w \in \mathfrak{A}^(A, B, G)$ satisfying the condition*

$$w(z) = O(\Psi(z)), \quad z \rightarrow 0, \quad (12)$$

for some function $\Psi \in \mathfrak{A}_0^(G)$ is identically zero.*

The essential extension of the Theorem 1 is proved; the existence of the real number $\delta_0 > 0$, such that every solution w of $\mathfrak{A}^*(A, B, G)$, (the generating pair should satisfy the conditions of the Theorem 1) satisfying the following condition

$$w(z) = O\left(\Psi(z) \cdot \exp\left\{\frac{\delta}{|z|^{\nu-1}}\right\}\right), \quad z \rightarrow 0, \quad (13)$$

for some $\delta < \delta_0$, $\Psi \in \mathfrak{A}_0^*(G)$ is identically zero is also proved.

From Theorem 1 (taking as the analytic function $\Psi \equiv 1$) we get immediately the triviality of solution of the class $\mathfrak{A}^*(A, B, G)$ bounded in the neighborhood of the singular for the equation point $z = 0$. Further, let the solution $w \in \mathfrak{A}^*(A, B, G)$ has the power of the growth (5) for some $\sigma > 0$. Taking as $\Psi(z)$ the function

$$\Psi(z) = \frac{1}{z^{[\sigma]+1}}$$

we conclude that $w \equiv 0$.

As the next application of the Theorem 1 consider arbitrary solution $w \in \mathfrak{A}^*(A, B, G)$ (the generating pair should satisfy the conditions of the Theorem 1) and let the analytic function $\Psi \in \mathfrak{A}_0^*(G)$ satisfy the condition

$$\Psi(z) = O(w(z)), \quad z \rightarrow 0. \quad (14)$$

Applying the functions w, Ψ we construct the function

$$W = \frac{\Psi}{w},$$

which is bounded in the neighborhood of $z = 0$.

Direct checking gives, that

$$\frac{\partial W}{\partial \bar{z}} - AW - \frac{B\bar{\Psi}}{\Psi} \left(\frac{W}{\bar{W}} \right)^2 \bar{W} = 0, \quad (15)$$

i.e. W is a quasiregular solution of the quasiregular equation (15). It is evident, that for the coefficients of this equation all conditions of the Theorem 1 are fulfilled and therefore the solution $W \equiv 0$, i.e. $\Psi \equiv 0$.

Summarizing all said we conclude that the following theorem is valid.

Theorem 2 *Let the generating pair of the class $\mathfrak{A}^*(A, B, G)$ be of the form (9) and let the conditions (7), (8), (11) be fulfilled. Then every function $\Psi \in \mathfrak{A}_0^*(G)$ satisfying the condition (14) for some solution $w \in \mathfrak{A}^*(A, B, G)$ is identically zero.*

It follows from the Theorems 1,2 that the quasiregular solutions of the equation (1) of sufficiently wide class in the neighborhood of singular point of the equation doesn't admit the estimation (neither from above nor from below) by the module of the analytic function and therefore the behavior of the solution is non-analytic. But these solutions have one common property - remanding the behavior of the analytic functions in the neighborhood of the essentially singular point. Namely, these solutions have no limit in singular (for the equation) point. Indeed, the unboundedness of every nontrivial function $w \in \mathfrak{A}^*(A, B, G)$ imply that no finite limit exists in the point $z = 0$.

Let as prove the impossibility of the equality

$$\lim_{z \rightarrow 0} w(z) = \infty. \quad (16)$$

In fact, if it takes place the equality (16) then there exists a real number $\rho > 0$ such, that

$$|w(z)| \geq 1, \quad 0 < |z| < \rho,$$

however, this is impossible by virtue of the Theorem 2. Summarizing what has been said we conclude that the following theorem is valid.

Theorem 3 *Let the generating pair of the class $\mathfrak{A}^*(A, B, G)$ be of the form (9) and let the conditions (7), (8), (11) be fulfilled. Then every nontrivial function $w \in \mathfrak{A}^*(A, B, G)$ has no limit (neither finite nor infinite) in singular for the equation point $z = 0$.*

References

- [1] Vekua I. N. Generalized analytic functions. *Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass.* 1962.
- [2] Vekua I. N. Stationary singularities of generalized analytic functions. (Russian) *Dokl. Akad. Nauk SSSR* **145**(1962), 24–26.
- [3] Vekua I. N. On one class of the elliptic systems with singularities. *Proc. Internat. Conf. on Functional Analysis and Related Topics. Tokyo* (1969), 142–147.
- [4] Makatsaria G. Singular points of solutions of some elliptic systems on the plane. *Journal of Mathematical Sciences*. Vol.160, N 6,(2009) pp. 737-744.

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\mathbb{R} –linear and Riemann–Hilbert problems for multiply connected domains

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1 Introduction

Various boundary value problems are reduced to singular integral equations [3, 12, 16]. Only some of them can be solved in closed form. In the present talk, we follow the lines of [8, 9, 10] and describe solution to the \mathbb{R} –linear problem which in a particular case yields the Riemann–Hilbert problem.

These problems can be considered as a generalization of the classical Dirichlet and Neumann problems for harmonic functions. They include as a partial case the mixed boundary value problem. One knows the famous Poisson formula which solves the Dirichlet problem for a disk. The exact solution of the Dirichlet problem for a circular annulus is also known due to Villat–Dini. Formulae from [8, 9, 10] can be considered as a generalization of the Poisson and Villat–Dini formulae to arbitrary circular multiply connected domains. In order to deduce our formulae we first reduce the boundary value problem to the \mathbb{R} –linear problem and solve the later one by use of functional equations. By *functional equations* we mean iterative functional equations with shift into domain. Hence, we do not use traditional integral equations and infinite systems of linear algebraic equations. The solution is given explicitly in terms of the known functions or constants and geometric parameters of the domain.

First, we discuss functional equations and prove the convergence of the method of successive approximations for these equations. Application of the

method of successive approximations yields the solution in the form of the θ_2 -Poincaré series [9]. As a sequence we obtain the almost uniform convergence of the Poincaré series for any multiply connected domain.

2 Riemann–Hilbert problem

Let D be a multiply connected domain on the complex plane whose boundary ∂D consists of n simple closed Lyapunov curves. The positive orientation on ∂D leaves D to the left. The scalar linear Riemann–Hilbert problem for D is stated as follows. Given Hölder continuous functions $\lambda(t) \neq 0$ and $f(t)$ on ∂D . To find a function $\phi(z)$ analytic in D , continuous in the closure of D with the boundary condition

$$Re \overline{\lambda(t)}\phi(t) = f(t), \quad t \in \partial D. \quad (2.1)$$

The problem (2.1) had been completely solved for simply connected domains ($n = 1$). Its solution and general theory of boundary value problems is presented in the classic books by Gakhov [3], Muskhelishvili [12] and Vekua [16]. In 1975 Bancuri [1] had solved the Riemann–Hilbert problem for circular annulus ($n = 2$).

First results concerning the Riemann–Hilbert problem for general multiply connected domains were obtained by Kveselava [5] in 1945. He reduced the problem to an integral equation. From 1952 I.N. Vekua and later Bojarski begun to extensively study this problem. Their results are presented in the book [16]. This Georgian attack to the problem supported by young Polish mathematician were successful. Due to Kveselava, Vekua and Bojarski, we have a theory of solvability of the problem (2.1) based on integral equations and estimations of its defect numbers, l_{\varkappa} , the number of linearly independent solutions and p_{\varkappa} , the number of linearly independent conditions of solvability on $f(t)$, depending on the index (winding number) $\varkappa = wind_{\partial D}\lambda$.

Any multiply connected domain D can be conformally mapped onto a circular multiply connected domain. Hence, it is sufficient to solve the problem (2.1) for a circular domain and after to write the solvability conditions and solution using the conformal mapping. The complete solution to the problem (2.1) for arbitrary circular multiply connected domain had been given in [8, 9, 10].

3 \mathbb{R} -linear problem

The \mathbb{R} -linear problem is stated as follows. Given Hölder continuous functions $a(t) \neq 0$, $b(t)$ and $f(t)$ on ∂D . To find a function $\phi(z)$ analytic in $\cup_{k=1}^n D_k \cup D$, continuous in $D_k \cup \partial D_k$ and in $D \cup \partial D$ with the conjugation condition

$$\phi^+(t) = a(t)\phi^-(t) + b(t)\overline{\phi^-(t)} + f(t), \quad t \in \partial D. \quad (3.1)$$

Here $\phi^+(t)$ is the limit value of $\phi(z)$ when $z \in D$ tends to $t \in \partial D$, $\phi^-(t)$ is the limit value of $\phi(z)$ when $z \in D_k$ tends to $t \in \partial D$. In the case $|a(t)| \equiv |b(t)|$ the \mathbb{R} -linear problem is reduced to the Riemann-Hilbert problem (2.1) [7].

In the case of the smooth boundary ∂D , the problem with constant coefficients

$$\phi^+(t) = a\phi^-(t) + b\overline{\phi^-(t)} + f(t), \quad t \in \partial D. \quad (3.2)$$

is equivalent to the *transmission problem* from the theory of harmonic functions

$$u^+(t) = u^-(t), \quad \lambda^+ \frac{\partial u^+}{\partial n}(t) = \lambda^- \frac{\partial u^-}{\partial n}(t), \quad t \in \partial D. \quad (3.3)$$

Here the real function $u(z)$ is harmonic in D and continuously differentiable in $D_k \cup \partial D_k$ and in $D \cup \partial D$, $\frac{\partial}{\partial n}$ is the normal derivative to ∂D . The conjugation conditions express the perfect contact between materials with different conductivities λ^+ and λ^- . The functions $\phi(z)$ and $u(z)$ are related by the equalities $u(z) = \operatorname{Re} \phi(z)$, $z \in D$, $u(z) = \frac{\lambda^- + \lambda^+}{2\lambda^+} \operatorname{Re} \phi(z)$, $z \in D_k$ ($k = 1, 2, \dots, n$). The coefficients are related by formulae (for details see [10], Sec. 2.12.) $a = 1$, $b = \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+}$.

In 1932, having used the theory of potentials Muskhelishvili [11] (see also [13], p.522) reduced the problem (3.3) to a Fredholm integral equation and proved that it has a unique solution in the case $\lambda^\pm > 0$, the most interesting in applications. In 1933, I. N. Vekua and Ruhadze [14], [15] constructed a solution of (3.3) in closed form for annulus and ellipse (see also papers by Ruhadze quoted in [13]). Hence, the paper [11] published in 1932 is the first result on solvability of the \mathbb{R} -linear problem, [14] and [15] published in 1933 are the first papers devoted to exact solution to the the \mathbb{R} -linear problem for annulus and ellipse. A little bit later Golusin [4] considered the \mathbb{R} -linear problem in the form (3.3) by use of the functional equations for analytic functions. Therefore, Golusin's paper [4] published in 1935 is the first paper which concerns constructive solution to the the \mathbb{R} -linear problem for special circular multiply connected domains. In the further works these first results were not associated to the \mathbb{R} -linear problem even by their authors.

In 1946 Markushevich [6] had stated the \mathbb{R} -linear problem in the form (3.1) and studied it in the case $a(t) = 0$, $b(t) = 1$, $f(t) = 0$. Later Muskhelishvili [12] (p. 455) did not determine whether (3.1) was his problem (3.3) discussed in 1932 in terms of harmonic functions.

In 1960 Bojarski [2] shown that in the case $|b(t)| < |a(t)|$ with $a(t)$, $b(t)$ belonging to the Hölder class $H^{1-\varepsilon}$ with sufficiently small ε , the \mathbb{R} -linear problem (3.1) qualitatively is similar to the \mathbb{C} -linear problem $\phi^+(t) = a(t)\phi^-(t) + f(t)$, $t \in \partial D$. Later Mikhaïlov [7] developed this result to continuous coefficients $a(t)$ and $b(t)$; $f(t) \in \mathcal{L}^p(\partial D)$.

References

- [1] R.D. Bancuri: *On the Riemann–Hilbert problem for doubly connected domains*, Soobsch. AN GruzSSR **80** (1975), n. 4, 549-552.
- [2] B. Bojarski: *On generalized Hilbert boundary value problem*, Soobsch. AN GruzSSR **25** (1960), n. 4, 385-390.
- [3] F.D. Gakhov: *Boundary Value Problems*. Nauka, Moscow, 1977.
- [4] G.M. Golusin: *Solution of plane heat conduction problem for multiply connected domains enclosed by circles in the case of isolated layer*. Math. zb. **42:2** (1935), 191-198.
- [5] D.A. Kveselava: *Riemann–Hilbert problem for multiply connected domain*, Soobsch. AN GruzSSR **6** (1945), n. 8, 581-590.
- [6] A.I. Markushevich: *On a boundary value problem of analytic function theory*. Uch. zapiski MGU **1** (1946), vyp. 100, 20-30.
- [7] L.G. Mikhailov: *New Class of Singular Integral Equations and its Applications to Differential Equations with Singular Coefficients*. AN TadzhSSR, Dushanbe, 1963.
- [8] V.V. Mityushev: *Solution of the Hilbert boundary value problem for a multiply connected domain*. Slupskie Prace Mat.-Przyr. **9a** (1994), 37-69.
- [9] V.V. Mityushev: *Convergence of the Poincaré series for classical Schottky groups*. Proceedings Amer.Math.Soc. **126**(1998), n. 8, 2399-2406.
- [10] V.V. Mityushev, S.V. Rogosin: *Constructive methods to linear and non-linear boundary value problems of the analytic function. Theory and applications*, Chapman & Hall / CRC, Boca Raton etc. 2000.

- [11] N.I.Muskhelishvili: *To the problem of torsion and bending of beams constituted from different materials*, Izv. AN SSSR, (1932), N 7, 907-945.
- [12] N.I. Muskhelishvili: *Singular Integral Equations*. Nauka, Moscow, 1968.
- [13] N.I. Muskhelishvili: *Some Basic Problems of Mathematical Elasticity Theory*. Nauka, Moscow, 1966.
- [14] I.N. Vekua, A.K. Rukhadze: *The problem of the torsion of circular cylinder reinforced by transversal circular beam*. Izv. AN SSSR, 1933, n. 3, 373-386.
- [15] I.N. Vekua, A.K. Rukhadze: *Torsion and transversal bending of the beam compounded by two materials restricted by confocal ellipces*. Prikladnaya Matematika i Mechanika (Leningrad), 1933, **1**, n. 2, 167-178.
- [16] I.N. Vekua: *Generalized Analytic Functions*. Nauka, Moscow, 1988.

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On isomonodromic deformations and integrability concerning linear systems of differential equations

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Abstract

We review the modern theory of isomonodromic deformations, considering linear systems of differential equations. On that background we illustrate the natural relations between such phenomena as integrability, isomonodromy and Painlevé property. The recent advances in the theory of isomonodromic deformations we present show perfect agreement to that approach.

1 Introduction

Using Fuchsian systems and their isomonodromic deformations as a kind of "toy model" we want to illustrate the idea of some deep natural relations existing between the integrability, the isomonodromy and the Painlevé property. Briefly, it claims that the integrability and the isomonodromy are supposed to be somehow equivalent when the Painlevé property usually serves as a sign of some underlying hidden integrability. This concept is valid for a number of classical integrable systems and differential equations. The isomonodromic equivalents are established for such famous models as the KP -KdV equations, the associativity or WDWW system, the Yang-Mills,

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Painlevé-VI and for a lot of others. One can find some elucidation in reviews [1] and [4]. The Painlevé test, in the theory of integrable systems says, that if the equation is integrable then there should exist some reduction procedure transforming it to an equation possessing the Painlevé property. Such reductions are known for all classical integrable systems mentioned above. The Painlevé property is the property for the equation not to have the movable singular points other than simple poles. Historically it was Kowalewskaya who first used the Painlevé property in her researches. Investigating the problem of integrability of the top she had arrived to the idea to find all sets of the parameters for which the equation of motion possesses the Painlevé property and then to try to solve the equation for those cases. She had found that there are three such possibilities for parameters, two were known as the Euler and the Lagrange tops, and for the third one she had found a solution known now as the Kowalewskaya top. This story is a nice illustration of the desirable relation between the integrability, the isomonodromy and the Painlevé property.

Considering Fuchsian systems and looking at the progress at the theory of their deformations we see that first, when only Schlesinger deformation was known this correlated perfectly with the integrability-isomonodromy-Painlevé property picture. The isomonodromy of a Schlesinger deformation was equivalent to a complete integrability of the Schlesinger form distribution and the resulting family possessed the Painlevé property. Later, when non-Schlesinger deformations were found the relations to the Painlevé property seemed to be broken. But recently, after establishing a full classification of the Fuchsian system isomonodromic deformations, this relation was restored by the appearance of new degrees of freedom with a free dynamics. This is a feature of the so called resonant case. Detaching this new degrees of freedom, (which is always possible because of independence of their dynamics from the dynamics of other variables), one can come back to the agreement with the integrability-isomonodromy-Painlevé property concept we formulated above. Recently this results were generalized to the case of the integrable deformations of meromorphic linear systems with the poles of greater orders.

2 Fuchsian systems

Fuchsian systems are the linear systems of differential equations with meromorphic coefficients having no singular points other than first order poles.

They are equivalent to the logarithmic connections on the trivial vector bundle over Riemann sphere.

$$\nabla y = dy - \omega y = dy - \left(\sum_{i=1}^n \frac{B_i}{z - a_i} dz \right) y = 0$$

Being the simplest possible linear systems Fuchsian systems are however a source of a number of classical problems. The main characteristics of the linear differential equation are its singular points, asymptotics and monodromy. For generic Fuchsian system that data set appears to be complete. By monodromy here we mean the branching of the system's fundamental matrix under analytical continuation along the loops encircling singular points ([6] for details). The monodromy map has a number of very interesting and useful properties revealing the geometry and symplectic nature of isomonodromic deformations [2].

Considering Fuchsian system $dy = \omega y$ construction of isomonodromic deformation consist in establishing a family of $\omega(a, t)$ such that $\omega(a_0, t_0) = \omega$ for some initial point (a_0, t_0) and for any fixed $a = a^*, t = t^*$ the monodromy of Fuchsian system $\omega(a^*, t^*)$ is the same as of the system ω (again [6] for precise definitions). The questions of construction of the isomonodromic family for given initial system and establishing whether a given family is isomonodromic naturally arise.

Theorem 2.1 ([7]) $\omega(a, t)$ is isomonodromic iff \exists 1-form Ω such that:

- $d\Omega = \Omega \wedge \Omega$
- $\Omega|_{(a,t)=(a^*,t^*)} = \omega(a^*, t^*)$ for any fixed (a^*, t^*)

Geometrically it means that continuous isomonodromic deformations are in fact deformations of an embedding, one just move the sphere in the space where the global form Ω is defined and take the restriction of Ω to that sphere as ω . Also it gives a nice example of the integrability-isomonodromy relation: the family ω is isomonodromic if and only if some distribution $dy = \Omega y$ is completely integrable.

Now one can ask for some classification or any additional information about forms Ω . Historically the first and the very important result here is the following ansatz proposed by Schlesinger in the beginning of the XX century.

Theorem 2.2 For any given Fuchsian system $dy = \omega(a^0)y$ there exists 1-form Ω_s of the following type:

$$\Omega_s = \sum_{i=1}^n \frac{B_i(a)}{z - a_i} d(z - a_i)$$

satisfying the theorem 2.1 conditions such that $\Omega_s(a^0) = \omega(a^0)$.

In other words, the problem of construction of isomonodromic family is always solvable by Schlesinger type forms. Written in coordinates equation $d\Omega_s = \Omega_s \wedge \Omega_s$ is known as Schlesinger equation, it possesses a number of nice properties, Painlevé property among others, and became a subject of intensive study a long ago. The next statement concerns the universality of Schlesinger deformation.

Theorem 2.3 ω is non-resonant $\Rightarrow \Omega = \Omega_s$

The resonance here means the non-simplicity of the spectrum of one of the residue matrices of the system. If there exist such i, j, k that for some β_i^j, β_i^k in spectrum of B_i there holds $\beta_i^j - \beta_i^k \in \mathbb{N}$, then the point a_i and the entire system are called resonant. Being formulated in analytical terms resonances however have natural geometric interpretation these are special points or even special components of the Fuchsian systems moduli space lattice.

Theorem 2.4 If the only parameters of deformation are the positions of singular points (if $\omega = \omega(a)$) then $\Omega = \Omega_s + \Omega_{res}$, where

$$\Omega_{res} = \sum_{i=1}^n \sum_{j=1}^{r_i} \sum_{k=1}^n \frac{\beta_{ijk}}{(z - a_i)^j} da_k$$

r_i here is the maximal i -resonance of the system (the greatest possible natural difference between the elements of the spectrum of $B_i(a)$), and n is the number of singular points (a_1, \dots, a_n)

This result is due to Andrey Bolibrukh [3]. He also showed that for deformations of that type the Painlevé property is usually broken. Regardless that the isomonodromy property of the Fuchsian family is still equivalent to a complete integrability of some special distribution the general integrability-isomonodromy-Painlevé property picture meet here some problems. The recent advances in the theory we are going to present eliminates this inconvenience and restore the full agreement to concept we discuss.

3 Recent advances

Theorem 3.1 ([6, 5]) *For any (appropriately defined, see [6]) integrable deformation (E, ∇) the solutions $\{B_{ii}(t)\}$ of integrability equations continue holomorphically to $T \setminus \Theta$ and meromorphically to θ . Here T is the deformation space and $\Theta \subset T$ is the theta-divisor – an analytical subset of the deformation space of codimension one.*

In accordance to general concept we mentioned before, the statement above claims that integrable deformations of linear meromorphic connections on trivial vector bundles do possess the Painlevé property not only for Fuchsian systems but for the systems of greater pole orders as well. That is a generalization of famous Malgrange theorem on the Painlevé property of Schlesinger equation, the equation that encodes isomonodromic deformation of logarithmic connections or Fuchsian systems.

The proof is based on vector bundles and connections technique and involves Fredholm operators theory.

Theorem 3.2 ([8, 5]) *The differential 1-form Ω defining the isomonodromic deformation of Fuchsian system $dy = \omega y$ is always constructed as $\Omega = \Omega_s + \Omega_{res} + \Omega_{add}$ where*

$$\Omega_{res} = \sum_{i=1}^n \sum_{j=1}^{r_i} \sum_{k=1}^s \frac{\beta_{ijk}}{(z - a_i)^j} dt_k \quad (1)$$

Here r_i is the maximal i -resonance of the system and s is the dimension of the continuous component of Fuchsian system's moduli space.

The coefficients of the form Ω are meromorphic functions of (a, t) . Any other 1-form $\tilde{\Omega}$ encoding the isomonodromic deformation of the same Fuchsian system is a reduction under $t = t(a)$ condition $\tilde{\Omega} = \Omega|_{t=t(a)}$, where $t(a)$ are some single-valued function on the deformation space.

This statement elucidate the gape in the in the integrability-isomonodromy-Painlevé property conception concerning the deformations of resonant Fuchsian families. It appears that Painlevé property is broken due to the freedom in choosing reducing functions $t = t(a)$ encoding the evolution of Levelt's filtration, the only necessary condition they should satisfy is being single-valued. So choosing the functions of faster then polynomial growth leads to the obeying Painlevé condition.

Proving that result we deal again with vector bundles and especially their gauge transformations. The technique of gauges affecting the splitting type of the bundle and changing its particular indices while the monodromy is preserved was established.

References

- [1] P.Boalch, "Six results on Painlevé VI", *SMF, Séminaires et congrés*, vol. 14
- [2] P.Boalch, "Symplectic manifolds and isomonodromic deformations", *Adv. Math.* 163:2, (2001), 137-201.
- [3] A.Bolibrukh, "On isomonodromic confluences of Fuchsian singularities", *Proc. Steklov Inst. Math.*, 221 (1998), 127-142
- [4] M.Jimbo T.Miwa, Monodromy preserving deformations of linear differential equations with rational coefficients II, *Physica 2D* (1981), 407-448
- [5] R.R. Gontsov, G.F. Helminck, V.A. Poberezhny "On deformation of linear differential systems", *Russian Mathematical surveys*(2011), 66:1, 63-105.
- [6] G.F. Helminck, V.A. Poberezhny "Moving poles of meromorphic linear systems on $\mathbf{P}_1(C)$ in the complex plane", *Theoretical and Mathematical Physics* (2011), 165:3, 1637-1649.
- [7] A.Its V.Novokshenov, "The isomonodromic deformation method in the theory of Painlevé equations", *L.N.Math.*, Springer, Vol. 1191 (1986)
- [8] V.Poberezhny "On the Painlevé property of isomonodromic deformations of Fuchsian systems", *Acta Appl. Math.*, 101:1-3 (2008) 255-263.

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On the Fuchsian systems free from accessory parameters

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Abstract

We consider a system of Fuchsian linear differential equations free from accessory parameters with 3 singular points and study monodromy groups of such systems.

It is well known that for any homomorphism

$$\chi : \pi_1(CP^1 - D, z_0) \rightarrow GL(p, C) \quad (1)$$

of the fundamental group of the complement of a set $D = \{a_1, \dots, a_n\}$ of points of the Riemann sphere CP^1 into the group of complex-valued nondegenerate matrices of order p one can construct a Fuchsian equation

$$y^{(p)} + q_1(z)y^{(p-1)} + \dots + q_p(z)y = 0 \quad (2)$$

with given monodromy (1), whose set D' of singular points coincides with the set $D \cup \{b_1, \dots, b_m\}$. The additional singular points $\{b_1, \dots, b_m\}$ do not contribute to the monodromy and are called false singular points. From (1), by a standard method we construct a vector bundle F' over $CP^1 - D$ with structure group $GL(p; C)$. Let F be Yu. Manins continuation of this bundle to all of CP^1 . Then according to the Birkhoff-Grothendieck theorem we have

$$F \cong O(-k_1) \oplus O(-k_2) \oplus \dots \oplus O(-k_p),$$

where $k_1 \geq \dots \geq k_p$, and $O(-r)$ is the r th power of the Hopf bundle $O(-1)$ on CP^1 . Denote by l the number of the first numbers k_1, \dots, k_p , ($k_1 = \dots = k_l$) that are equal to each other.

It is known, that for any irreducible representation (1) a Fuchsian equation (2) exists with given monodromy (1), the number m of additional false singular points of which satisfies the inequality

$$m \leq [(n-2)p(p-1)]/2 - \sum_{i=1}^p (k_1 - k_i) + 1 - l.$$

Theorem 1[4]. For any Fuchsian equation on the Riemann sphere it is possible to construct a Fuchsian system

$$df = \left(\sum_{i=1}^p \frac{B_i}{z - a_i} dz \right) f \quad (3)$$

with the same singular points and the same monodromy.

While an arbitrary equivalence class of irreducible representations

$$\pi_1(C - \{0, 1\}) \rightarrow GL(2, C) \quad (4)$$

is induced by a certain hypergeometric differential equation

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby(x) = 0, \quad (5)$$

and vice versa, there are some classes of reducible representations which are not induced by (5). In [2] the author sets up the canonical bases with respect to which the twenty reducible classes induced by (5) are realized in a simple form. This includes the necessary connection formulas in the degenerate cases (in which a or b or $c - a$ or $c - b$ is an integer).

Example. Let generators of the representation (4) are

$$G_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, G_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}, G_3 = \begin{pmatrix} 1 & -s_1 - s_2 \\ 0 & 1 \end{pmatrix},$$

where $s_1 s_2 (s_1 + s_2) \neq 0$, then does not exist Fuchsian differential equation with 3 singular points and with representation generated by G_1, G_2, G_3 .

Proposition 1. For any representation in two dimensional case, whose generators are different from G_1, G_2, G_3 realisable as monodromy representation of hypergeometric equations.

Proposition 2[4] 1) For $n = 3$ any irreducible representation (1) of dimension $p = 2$ can be realized as the monodromy of the Gauss equation, i.e., a second-order Fuchsian equation with three singular points.

2) If the representation (1) is realized as the representation of the monodromy of the Fuchsian equation (2) without additional apparent singular points, then it is also realized as the representation of a monodromy of a certain Fuchsian system with the same singular points.

3) Let the representation (1) ($n > 2$) be reducible and let each of the matrices G_i of the monodromy, corresponding to a circuit of the point a_i along a small loop, reduce to a Jordan cell. Then the Fuchsian equation (2) does not exist without additional false singular points, whose monodromy coincides with (1).

Proposition 3. For any three points on the Riemann sphere and for any irreducible representation (1) of dimension $p = 4$ there exists a Fuchsian system on CP^1 with given monodromy (1), whose singular points coincide with three given points. [4]

Example. The Fuchsian system constructed from Gauss equation

$$y'' + \frac{\gamma - (\alpha + \beta + 1)z}{z(1-z)}y' - \frac{\alpha\beta}{z(1-z)}y = 0 \quad (6)$$

has the form

$$df = \left(\left(\begin{array}{cc} 0 & 0 \\ -\alpha\beta & -\gamma \end{array} \right) \frac{dz}{z} + \left(\begin{array}{cc} 0 & 1 \\ 0 & \gamma - (\alpha + \beta) \end{array} \right) \frac{dz}{z-1} \right) f.$$

Fuchsian systems of differential equations on CP^1 , which are free from accessory parameters, have the following important property: their monodromies can be calculated explicitly from their coefficients (for an arbitrary system there is no way to calculate its monodromy in general) [1].

Consider the system of differential equations of the form

$$(xI_n - T) \frac{dY}{dx} = AY \quad (7)$$

on CP^1 of rank n , which called *Okubo normal form*, where $T = t_1 I_{n_1} \oplus \dots \oplus t_p I_{n_p}, t_i \in C (1 \leq i \leq p), t_i \neq t_j (i \neq j), n_1 + \dots + n_p = n$, A is a diagonalizable and $A \in \text{End}(n, C)$. The matrix A is decomposed into blocks of submatrices and the system is viewed as Fuchsian over CP^1 with regular singular points at $x = t_1, \dots, t_p, \infty$. By special gauge transformation, $Y = PZ$, it is possible to determine all systems which are irreducible and free from accessory parameters, therefore there exists a new class of extensions of the Gauss hypergeometric function.

Let

$$(tI - B) \frac{dx}{dt} = Ax \quad (8)$$

is (7) type system. The following conditions for the coefficients of the system we assume:

(i) the matrix B is diagonal with eigenvalues $\lambda_1, \dots, \lambda_p$ that have multiplicities n_1, \dots, n_p satisfying the inequalities $n_1 \geq \dots \geq n_p$;

(ii) each diagonal block A_{ii} (in the same partition of A as B) is a diagonal matrix with distinct eigenvalues;

(iii) the matrix A is diagonalizable and it has eigenvalues ρ_1, \dots, ρ_q that have multiplicities m_1, \dots, m_q satisfying the relations $m_1 \geq \dots \geq m_q$.

It is known, that the condition $n_1 + m_1 \leq d$ (d is the rank of the system) is necessary for the system to be irreducible (a system is called irreducible if there is no transformation $P(t) \in GL(d, C(t))$ reducing the system to a system with a block triangular coefficient matrix). A classification of the above systems which are free of accessory parameters and satisfy the inequality $n_1 + m_1 \leq d$ is presented [6].

Consider particular case of the system (8), where $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, A is diagonalizable and has nonresonant nonnegative eigenvalues $\rho_1 = \rho_2, \rho_3, \rho_4$. Moreover, A has a block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11}, A_{22} are diagonalizable 2×2 matrices with nonresonant eigenvalues.

This equation is Fuchsian, has three singular points $\lambda_1, \lambda_2, \infty$ and its exponents at these points are as follows: $(a_{11}, a_{22}, 0, 0)$, $(0, 0, a_{33}, a_{44})$, $(\rho_1, \rho_2, \rho_3, \rho_4)$.

Proposition 4. The system (8) in assumptions above is accessory parameter free.

From this proposition follows, that it is possible to calculate its monodromy group in terms of the exponents (up to diagonal transformations) and obtain necessary and sufficient conditions of irreducibility for the monodromy group in terms of the exponents.

Theorem 2. The $d = 4$ dimensional system

$$dF = \left(I \otimes \begin{pmatrix} 0 & 0 \\ -\alpha\beta & -\gamma \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 0 & 1 \\ 0 & \gamma - (\alpha + \beta) \end{pmatrix} \otimes I \frac{dz}{z-1} \right) F.$$

is free from accessory parameters and have the form (8).

Final remark. The notion of a local system on CP^1 -{finite set of points} was introduced by Riemann in order to study the classical Gauss hypergeometric function, which he did by studying rank two local systems on CP^1 -{three points}. His investigation was a stunning success, in large part because any such local system is rigid. For example, in [10] proved that a bundle of rank 2 is rigid (that is, $\dim H^1(CP^1, \text{End}(E)) = 0$) if and only if it admits a connection which is holomorphic everywhere except at 3 points with at most logarithmic singularities. From the theorem 2 follows, that the notations of the paper [11] it is possible to extend in the four dimensional case also.

References

- [1] Haraoka, Yoshishige. Monodromy representations of systems of differential equations free from accessory parameters. SIAM J. Math. Anal. 25 (1994), no. 6, 15951621.
- [2] Shima, Kazuhisa. On the monodromy representation of a reducible hypergeometric differential equation. Japan. J. Math. (N.S.) 18 (1992), no. 2, 403438.
- [3] P. Deligne, Equations différentielles à points singuliers réguliers, Lecture Notes in Math., 163, Springer, Berlin, 1970;
- [4] Bolibrukh, A. A. Construction of a Fuchsian equation from a monodromy representation. Mat. Zametki 48 (1990), no. 5, 2234, 158;
- [5] Haraoka, Yoshishige. Canonical forms of differential equations free from accessory parameters. SIAM J. Math. Anal. 25 (1994), no. 4, 12031226.
- [6] Yokoyama, Toshiaki. On an irreducibility condition for hypergeometric systems. Funkcial. Ekvac. 38 (1995), no. 1, 1119.
- [7] Sasai, Takao; Tsuchiya, Susumu. On a fourth order Fuchsian differential equation of Okubo type. Funkcial. Ekvac. 34 (1991), no. 2, 211221.
- [8] Haraoka, Yoshishige. Monodromy of an Okubo system with non-semisimple exponents. Funkcial. Ekvac. 40 (1997), no. 3, 435457.

- [9] Katz, Nicholas M. Rigid local systems. *Annals of Mathematics Studies*, 139. Princeton University Press, Princeton, NJ, 1996.
- [10] Giorgadze, G. Density problem of monodromy representation of Fuchsian systems. *Further progress in analysis*, 347355, World Sci. Publ., Hackensack, NJ, 2009.
- [11] Giorgadze, G. Holomorphic structures and connections on lowdimensional vector bundles on the Riemann sphere. *Bull. Soc. Sci. Lett. odz Ser. Rech. Deform.* 38 (2002), 93100.

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On G-compactness of the classes of first and second order elliptic systems

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1 First order elliptic systems

1.1 G-convergence.

We denote by $\mathcal{B}(k_0; Q)$ the set of elliptic systems of two first-order equations expressed in the complex form as a single equation:

$$Au \equiv \partial_{\bar{z}} u + \mu \partial_z u + \nu \overline{\partial_z u}, \quad (1)$$

where

$$2\partial_{\bar{z}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}, \quad 2\partial_z = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y};$$

\bar{v} is the complex conjugate function to v ; μ and ν are measurable functions in Q satisfying the following condition:

$$\text{vrai sup}_{x \in Q} (|\mu(x)| + |\nu(x)|) \leq k_0 < 1, \quad (2)$$

where k_0 is a positive quantity (the ellipticity constant); $Q \subset \mathbb{R}^2$ is a bounded simply connected $C_{\nu_1, \dots, \nu_m}^{1+\alpha}$ -domain (which means that its boundary ∂Q is a piecewise smooth closed curve consisting of finitely many $C^{1+\alpha}$ arcs, $0 < \alpha < 1$; the interior angles at the corner points are $\nu_1\pi, \dots, \nu_m\pi$, $0 < \nu_j \leq 2$, $j = 1, \dots, m$).

We point out that each uniformly (homogeneous) elliptic system of two equations with real coefficients in $L_\infty(Q)$ can be represented in the form (1), (2) (see [1, Ch. 2, §7 and Ch. 3, §17]). Equation (1) is a generalization of Beltrami equation

$$A_0 u \equiv \partial_{\bar{z}} u + \mu \partial_z u = f, \quad \|\mu\|_{L_\infty(Q)} \leq k_0 < 1. \quad (3)$$

We denote by $\mathcal{B}_0(k_0; Q)$ the subclass of $\mathcal{B}(k_0; Q)$ consisting of Beltrami equations (3).

It is obvious that A is linear over the field \mathbb{R} , while A_0 is linear over \mathbb{C} . Consider the Riemann-Hilbert (RH) boundary value problem

$$\begin{cases} Au = f \in L_2(Q; \mathbb{C}), \\ u \in \mathcal{W}_0(Q; \mathbb{C}). \end{cases} \quad (4)$$

where $\mathcal{W}_0(Q; \mathbb{C})$ is the subspace of $W_2^1(Q; \mathbb{C})$ space whose elements satisfy the following relations:

$$\operatorname{Re} u \in \mathring{W}_2^1(Q), \quad \int_Q \operatorname{Im} u \, dx = 0;$$

$W_2^1(Q; \mathbb{C})$ is the Sobolev space of complex-valued functions over the field of real numbers \mathbb{R} .

We have the following result.

Theorem 1 *The RH problem (4) is uniquely solvable for each right-hand side $f \in L_2(Q; \mathbb{C})$. Moreover, the following a priori estimates hold:*

$$\nu_0 \|\partial_{\bar{z}} u\|_{L_2(Q_1; \mathbb{C})} \leq \|Au\|_{L_2(Q_1; \mathbb{C})}, \quad (5)$$

$$\nu_0 \|\partial_{\bar{z}} u\|_{L_2(Q_1; \mathbb{C})}^2 \leq \operatorname{Re} \int_{Q_1} Au \overline{\partial_{\bar{z}} u} \, dx, \quad (6)$$

$$\operatorname{Re} \int_{Q_1} Au \overline{\partial_{\bar{z}} v} \, dx \leq \nu_1 \left(\operatorname{Re} \int_{Q_1} Au \overline{\partial_{\bar{z}} u} \, dx \right)^{\frac{1}{2}} \|\partial_{\bar{z}} v\|_{L_2(Q_1; \mathbb{C})}, \quad (7)$$

for all $u, v \in \mathcal{W}_0(Q_1; \mathbb{C})$,

where $\nu_0 = 1 - k_0$, $\nu_1 = \sqrt{(1 + k_0)(1 - k_0)^{-1}}$ and $Q_1 \subseteq Q$ is an arbitrary simply connected subdomain (of class $C_{\nu_1, \dots, \nu_m}^{1+\alpha}$).

The estimates (5) – (7) hold also for multiply connected domains.

We point out that the quantity

$$\|u\|_{\mathcal{W}_0(Q; \mathbb{C})} = \|\partial_{\bar{z}} u\|_{L_2(Q; \mathbb{C})} \quad \left(= \|\partial_z u\|_{L_2(Q; \mathbb{C})} \right)$$

defines a norm in the subspace $\mathcal{W}_0(Q; \mathbb{C}) \subset W_2^1(Q; \mathbb{C})$, which is equivalent to the norm of the space $W_2^1(Q; \mathbb{C})$.

Definition 1 We say that a sequence $\{A_k\} \in \mathcal{B}(k_0; Q)$ G -converges in the domain Q to an elliptic system $A \in \mathcal{B}(k_0; Q)$ (and we write $A_k \xrightarrow{G} A$, $G\text{-}\lim A_k = A$) if the sequence $\{\mathcal{A}_k^{-1}\}$ converges weakly to $\{\mathcal{A}^{-1}\}$, where \mathcal{A}_k and \mathcal{A} are the operators of the RH boundary value problems $A_k u_k = f \in L_2(Q; \mathbb{C})$, $Au = f \in L_2(Q; \mathbb{C})$, $u_k, u \in \mathcal{W}_0(Q; \mathbb{C})$.

In other words, $G\text{-}\lim A_k = A$ in the domain Q if for each $f \in L_2(Q; \mathbb{C})$ the sequence $\{u_k\}$ of solutions of the RH problem converges weakly in $\mathcal{W}_0(Q; \mathbb{C})$ to the corresponding solution of the G -limiting problem.

The G -limit is defined in the naturally. The following result holds.

Theorem 2 (i) The class $\mathcal{B}(k_0; Q)$ of generalized Beltrami operators is G -compact. In other words, each sequence $\{A_k\} \subset \mathcal{B}(k_0; Q)$ contains a subsequence that is G -convergent in the sense of Definition 1.

(ii) The class $\mathcal{B}_0(k_0; Q)$ of Beltrami operators is G -compact.

(iii) Assume that $A_k \xrightarrow{G} A$ in the domain Q and let $Q_1 \subseteq Q$. Then $A_k \xrightarrow{G} A$ also in domain Q_1 .

(iv) Let $A_k u_k = f_k$, $f_k \rightarrow f$ in $L_2(Q; \mathbb{C})$, $u_k \rightarrow u$ weakly in $W_2^1(Q; \mathbb{C})$ and let $G\text{-}\lim A_k = A$. Then $Au = f$.

1.2 Homogenization.

Assume that the coefficients μ and ν are almost-periodic (a.-p.) functions (in Bohr sense) satisfying condition (2) in \mathbb{R}^2 . Consider the family of operators $\{A_\varepsilon\}_{0 < \varepsilon \leq 1}$ acting from $\mathcal{W}_0(Q; \mathbb{C})$ into $\mathcal{L}_2(Q; \mathbb{C})$ and defined by the formula

$$A_\varepsilon u \equiv \partial_{\bar{z}} u + \mu^\varepsilon \partial_z u + \nu^\varepsilon \overline{\partial_z u}, \quad u \in \mathcal{W}_0(Q; \mathbb{C}), \quad (8)$$

where $0 < \varepsilon \leq 1$, $\mu^\varepsilon = \mu(\varepsilon^{-1}x)$, $\nu^\varepsilon = \nu(\varepsilon^{-1}x)$. For each ε , $0 < \varepsilon \leq 1$, the operator A_ε clearly belongs to the class $\mathcal{B}(k_0; Q)$.

Definition 2 We say that homogenization occurs for the family $\{A_\varepsilon\}_{0 < \varepsilon \leq 1}$ if there exists an operator $A_0 \in \mathcal{B}(k_0; Q)$ such that $A_\varepsilon \xrightarrow{G} A_0$ in the domain Q as $\varepsilon \rightarrow 0$. In this case A_0 is called the homogenized operator (and the corresponding equation is called the homogenized equation).

Let $\text{Trig}(\mathbb{R}^2)$ be set of trigonometric polynomials (i.e. finite sums of the form $u(x) = \sum e^{i\lambda \cdot x}$, $\lambda, x \in \mathbb{R}^2$) and B_2 is the Bezikovich space of a.-p. functions. We say that the element $p \in B_2$ satisfies the equation $A^*p = 0$ if the equation $\text{Re} \langle \bar{p}(\partial_{\bar{z}}u + \mu \partial_z u + \nu \overline{\partial_z u}) \rangle = 0$ holds for any element $u \in \text{Trig}(\mathbb{R}^2)$, where $\langle f \rangle$ is mean value of the element $f \in B_2$. We have the following result.

Theorem 3 *The kernel of the operator A^* in B_2 is a two-dimensional subspace. And there exists a basis $\{p_1, p_2\}$ in this kernel that satisfies the following conditions: $\langle p_1 \rangle = 1$, $\langle p_2 \rangle = i$.*

Moreover, in the case of a Beltrami operator the basis vectors can be selected so as to satisfy $p_2 = ip_1$.

We now state a result on homogenization.

Theorem 4 *Homogenization occurs for the family (8), and the homogenized operator*

$$A_0 u \equiv \partial_{\bar{z}} u + \mu^0 \partial_z u + \nu^0 \overline{\partial_z u}, \quad u \in \mathcal{W}_0(Q; \mathbb{C}),$$

has the constant coefficients defined by the equalities

$$\mu^0 = \langle \mu \mathcal{Q} + \bar{\nu} \mathcal{P} \rangle, \quad \nu^0 = \langle \bar{\mu} \mathcal{P} + \nu \mathcal{Q} \rangle,$$

where

$$\mathcal{P} = 2^{-1}(p_1 + ip_2), \quad \mathcal{Q} = 2^{-1}(\bar{p}_1 + i\bar{p}_2),$$

and p_1, p_2 are the basis vectors from Theorem 3.

We point out that in the case of the Beltrami operator the homogenized operator is Beltrami operator too.

2 Second order elliptic systems

We denote by $\mathcal{B}_2(k_0; Q)$ the set of elliptic systems of two second-order equations expressed in the complex form as a single equation:

$$Au \equiv \partial_{z\bar{z}}^2 u + \mu \partial_{z\bar{z}}^2 u + \nu \partial_{\bar{z}\bar{z}}^2 \bar{u},$$

where μ and ν are measurable functions in Q , $C_{\nu_1, \dots, \nu_m}^{1+\alpha}$, satisfying the condition (2).

Consider the Poincare boundary value problem

$$\begin{cases} Au = f \in L_2(Q; \mathbb{C}), \\ u \in \mathcal{W}_0(Q; \mathbb{C}). \end{cases} \quad (9)$$

where $\mathcal{W}_0(Q; \mathbb{C})$ is the subspace of $W_2^2(Q; \mathbb{C})$ space whose elements satisfy the following relations:

$$\operatorname{Re} u|_{\partial Q} = 0, \quad \operatorname{Re} \partial_z u|_{\partial Q} = 0, \quad \int_Q \operatorname{Im} u \, dx = 0, \quad \int_Q \operatorname{Im} \partial_z u \, dx = 0.$$

We have the following result.

Theorem 5 *The Poincare problem (9) is uniquely solvable for each right-hand side $f \in L_2(Q; \mathbb{C})$. Moreover, a priori estimates (5)–(7), where instead of the Cauchy – Riemann operator $\partial_{\bar{z}}$ we have the Laplacian $\partial_{z\bar{z}}^2$, hold.*

On the basis of the theorem 5 one can receive precisely the same results of G-convergence and homogenization as well as in case of first order systems. They are formulated similar to the theorems 2 – 4.

References

- [1] I. N. Vekua, *Generalized analytic functions*, 2nd ed., Nauka, Moscow 1988; English transl. of 1st ed., Pergamon, London-Paris-Frankfurt; Addison-Wesley, Reading, MA 1962.

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Dirichlet problem for generalized Cauchy–Riemann systems with a singular point

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Let G be a given domain in the z -plane and $A(z), B(z)$ — two given functions defined on G . A complex-valued function $w = w(z)$ is called *generalized analytic function* if it satisfies the equation

$$\partial_{\bar{z}}w + A(z)w + B(z)\bar{w} = 0, \quad (1)$$

where $\partial_{\bar{z}}w = \frac{1}{2}(w_x + i \cdot w_y)$, $z \in G$. Remark that the derivative $\partial_{\bar{z}}w$ may be understood in Sobolev's sense. The book ([1]) of I. N. Vekua contains a general theory of the differential equation (1) under assumption that the coefficients $A(z), B(z)$ belong to $L_p(G)$, $p > 2$. The Vekua theory is closely related to the theory of holomorphic functions by so-called similarity principle.

It is known (see e.g. [2]–[4]), that the Vekua theory for the equation (1) breaks down if coefficients $A(z), B(z)$ do not belong to the space $L_p(G)$, $p > 2$. Therefore, it is necessary to make a separate investigation for equations with such coefficients as $A(z) = \frac{1}{z}$, $B(z) = \frac{1}{\bar{z}}$ and others. A large variety of results for such equations (1) with singular coefficients and boundary value problems is obtained in the works [2]–[9]. In this direction of generalized analytic functions the most important results were obtained by Mikhailov L., Vinogradov V., Usmanov Z., Blied N., Tungatarov A., Begehr H., D, Q. Dai, Reissig M., Saks R., Makatsaria G., Meziani A. and others.

*This paper is the last publication of Alexei Timofeev, who tragically passed when this volume was in press of preparation.

1 Dirichlet problem for model Cauchy-Riemann systems

In [5] the following boundary value problem

$$\partial_{\bar{z}}v - \frac{\lambda + \delta|z|^s}{2\bar{z}}\bar{v}(z) = 0, z \in G \setminus \{0\} = \{z : 0 < |z| < 1\}, \quad (2)$$

$$\Re v(z)|_{\Gamma} = f(t), |t| = 1, f \in C(\Gamma), \quad (3)$$

was studied, where $\lambda, \delta \in \mathbb{C} \setminus \{0\}$, $s \in \mathbb{N}$. R. Saks proved the following

Theorem 1 *Let $\lambda > 0$, $\delta > 0$, $s \in \mathbb{N}$, $f \in C^{1,\alpha}(\Gamma)$, $0 \leq \alpha \leq 1$. Then the solution $v(z)$ of the problem (2)–(3) is uniquely determined. This solution belongs to the class $C(\bar{G}) \cap C^1(G \setminus \{0\})$.*

By using scheme of [5] and Rouché's theorem we can prove the following

Theorem 2 *Let $s \in \mathbb{N}$, $f \in C^{1,\alpha}(\Gamma)$, $0 \leq \alpha \leq 1$. Then for every $\lambda \neq 0$, $\arg \lambda \neq \pi$, there exists positive constant $R = R(\lambda)$ such that for any $\delta \in U_R \setminus \{0\}$ with given fixed λ there exists a uniquely determined solution $v(z) \in C(\bar{G}) \cap C^1(G \setminus \{0\})$ of the problem (2)–(3).*

2 Dirichlet problem for generalized Cauchy–Riemann systems in spaces described by modulus continuity

A function $p(t)$ is said to belong to P ([7]) if

1. $p(t)$ is a positive function on the interval $(0, t_p]$;
2. $p(t)$ is an increasing function on the interval $(0, t_p]$;
3. there exists right-sided limit $\lim_{t \rightarrow +0} p(t) = 0$;
4. the integral $\int_0^{t_p} \frac{dt}{p(t)}$ is finite.

We can continue $p(t)$ onto $[t_p, 1]$ by $p(t_p)$ for $t \in [t_p, 1]$. In what follows we will call such the functions *weights*.

Examples of weights:

1. $p(t) = t^\alpha, 0 < \alpha < 1$;
2. $p(t) = t \cdot \ln^\beta \frac{1}{t}, \beta > 1$;
3. $p(t) = t \cdot \ln \frac{1}{t} \cdot \ln \ln \frac{1}{t} \cdot \dots \cdot \underbrace{(\ln \dots \ln \frac{1}{t})}_m^\beta, \beta > 1$.

Let $G = \{z : |z| < 1\}$. For every $p(t) \in P$ we can define the following spaces of functions

$$s_p(G) = \left\{ B(z) \in L_{\infty,loc}(G \setminus \{0\}) : \sup_{G \setminus \{0\}} (|B(z)| \cdot p(|z|)) < \infty \right\},$$

$$S_p(G) = \bigcup_{p \in P} s_p(G).$$

The linear space $S_p(G)$ is contained in $L_2(G)$. In [7] the following is proved.

Theorem 3 *Let us consider the Dirichlet problem*

$$\partial_{\bar{z}} w(z) + b(z) \bar{w}(z) = 0, z \in G = \{z \in \mathbb{C} : |z| < 1\}, \quad (4)$$

$$\Re w|_{\partial G} = g(z), \Im w|_{z_0=1} = h, \quad (5)$$

where $b \in S_p(G)$, $g \in C^{\lambda_0}(\partial G)$, $\lambda_0 \in (0, 1)$ and $h \in \mathbb{R}$. Then there exists a uniquely determined solution $w = w(z)$ which belongs to $C(\bar{G}) \cap C^{\lambda_0}(\bar{G} \setminus \{0\})$.

Recall that a *modulus of continuity* is a function $\mu(t)$, defined on the interval $(0, \delta)$ and satisfying the following conditions:

1. $\mu(t) \geq 0, t > 0$;
2. $\lim_{t \rightarrow +0} \mu(t) = 0$;
3. $\mu(t)$ does not decrease for $t > 0$;
4. for any $t_1, t_2 \in (0, \delta)$ $\mu(t_1 + t_2) \leq \mu(t_1) + \mu(t_2)$.

The condition 4 is known to hold if we assume that $\frac{\mu(t)}{t}$ does not increase for $t > 0$. Let $\mu_{1,0}(t) := t$, $\mu_{1,k}(t) := t \cdot \left(\ln \frac{1}{t}\right)^k$, $0 < t < \frac{1}{e}$, $k \geq 1$. With a given closed bounded subset $K \subset \mathbb{C}$ and modulus of continuity $\mu(t)$ we can denote the class of continuous functions $C_\mu(K)$ obeying the condition

$$\|f\|_\mu := \max \left\{ \sup_{z \in K} |f(z)|, \sup_{z_1 \neq z_2} \frac{|f(z_1) - f(z_2)|}{\mu(|z_1 - z_2|)} \right\} < \infty.$$

Using the results of boundary behaviour of holomorphic functions ([10]–[11]) and the scheme of [7] it can be proved the following

Theorem 4 *Let us consider the Dirichlet problem (4)–(5) where $b \in S_p(G)$, $g \in C_{\mu_{1,0}}(\partial G)$, $z_0 \in \partial G$ is a fixed point. Then there exists a uniquely determined solution $w = w(z)$ which belongs to $C(\overline{G}) \cap C_{\mu_{1,5}}(\overline{G} \setminus \{0\})$.*

Remark. It is interesting to prove the analogue of theorem 4 for general case of modulus of continuity.

References

- [1] Vekua I. N., *Generalized Analytic Functions*, Oxford: Reading, 1962.
- [2] Mikhailov L. G., *A New Class of Singular Integral Equations and its Application to Differential Equations with Singular Coefficients*, Berlin: Academie-Verlag, 1970.
- [3] Usmanov Z. D., *Generalized Cauchy-Riemann systems with a singular point*, Longman, Harlow, 1997.
- [4] Blied N. K., *Generalized analytic functions in fractional spaces*, Pitman Monographs and Surveys and Applied Mathematics, Longman, Harlow, 1997.
- [5] Saks R., *Riemann-Hilbert Problem for New Class of Model Vekua Equations with Singular Degeneration*, *Applicable analysis/* 1999. Vol. 73 (1–2). P. 201–211.
- [6] Reissig M., Timofeev A., *Special Vekua Equations with Singular Coefficients*, *Applicable Analysis/* 1999. Vol. 73 (1–2). P. 187–199.

- [7] Reissig M., Timofeev A., *Dirichlet problems for generalized Cauchy-Riemann systems with singular coefficients*, Complex variables/ 2005. Vol. 73 (1–2). P. 653–672.
- [8] Makatsaria G., *Correct boundary value problems for some classes of singular elliptic differential equations on a plane*, Mem. Differential equations. Math. Phys./ 2005. Vol. 34. P. 115–134.
- [9] Akhalaia G., Makatsaria G., Manjavidze N. *Some problems for elliptic systems on the plane*, Proceedings of 6-th International ISAAC Congress/ August 13–18, 2007. Ankara. P. 303–310.
- [10] Ilchukov A. S., Timofeev A. Y., *The Dirichlet problem for holomorphic functions in spaces with modulus of continuity (in Russian)*, Vestnik Udmurtskogo Universiteta/ 2010. (1). P. 58–65.
- [11] Napalkov V. V., Timofeev A. Y. *The Dirichlet problem for holomorphic functions in generalized Hölder spaces*, Doklady Mathematics/ 2010. Vol. 81 (3). P. 1–3.

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The solution of the Vekua generalized boundary value problem of the membrane theory of thin shells

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Introduction

In the view of membrane theory of convex shells [1] the problem of the realization moment less intense balance of the elastic shell, median surface S of which, is an interior part of the ovaloid S_0 of strictly positive Gaussian curvature regular of class $W^{3,p}$, $p > 2$, with piecewise-smooth boundary $L = \bigcup_{j=1}^n L_j$ consisting with finitely many arcs L_j ($j = 1, \dots, n$) regular of class $C^{1,\varepsilon}$ where $0 < \varepsilon < 1$ is studied. It is assumed that at every point of arc L_j the projection $u(s)$ of the stress vector to the direction of the vector $\mathbf{r}(s) = \{\alpha(s), \beta(s)\}$, belonging to the surface S , with tangential and normal components α, β respectively is given, where s — is the natural parameter, $\alpha^2 + \beta^2 = 1$, Holder continuous $\alpha(s), \beta(s), u(s)$, on each of the arcs L_j , $\beta(s)$ is the function of fixed sign on L , a vector field \mathbf{r} as a vector functions $\mathbf{r}(c)$ of the boundary point c having discontinuities of the first kind at the corner points c_j ($j = 1, \dots, n$). Further, such vector field \mathbf{r} we will call *admissible*. A problem T for median surface S with smooth boundary under condition of continuity on L of the vector field \mathbf{r} is set by I. N. Vekua [1]. In the same paper its simplest and special cases ($\alpha \equiv 0$ or $\beta \equiv 0$ L) are studied.

1. A Reimann-Hilbert Problem

Further, a surface S is assumed connected surface. Let us introduce the following notations: \mathfrak{J} is the mapping of the surface S to the complex plane

$\zeta = u^1 + iu^2$, defined by the choice of a conjugate isometric parametrization (u^1, u^2) on the surface S , $D = \mathfrak{J}(S)$ is a bounded domain in the plane ζ with boundary $\Gamma = \bigcup_{j=1}^n \mathfrak{J}(L_j)$ containing the corner points $\zeta_j = \mathfrak{J}(c_j)$. According to [1], the problem T is accumulated to searching for (in the plane D) a complex-valued solution $w(\zeta)$ of the equation

$$w_{\bar{\zeta}} + B(\zeta)\bar{w}(\zeta) = F(\zeta), \quad (1)$$

where $i^2 = -1$, $\partial_{\bar{\zeta}} = \frac{1}{2} \left(\frac{\partial}{\partial u^1} + i \frac{\partial}{\partial u^2} \right)$ is the operator of complex differentiation, $w(\zeta)$ is the *complex stress function* expressed with a contravariant of *stress tensor* components and coefficients in the metric form on the surface, $B(\zeta)$ is a given by the surface S function of class $L_p(\bar{D})$, $p > 2$. $F(\zeta)$ is *the complex function of the shell external load* according to the defined boundary condition

$$\operatorname{Re} \{ \lambda(\zeta) w(\zeta) \} = g(\sigma, K, k_s, \tau_g, X), \quad \zeta \in \Gamma, \quad (2)$$

in which

$$\lambda(\zeta) = [s^1(\zeta) + is^2(\zeta)] [\beta(\zeta)(t^1(\zeta) + it^2(\zeta)) - \alpha(\zeta)(s^1(\zeta) + is^2(\zeta))], \quad (3)$$

s^i ($i = 1, 2$) are the coordinates of the tangent to Γ unitary vector, t^i ($i = 1, 2$) are the coordinates of unitary vector \vec{t} of \mathbf{t} direction on the surface, being the \mathfrak{J} -image of $\boldsymbol{\tau}$ direction on the surface S^0 orthogonal to the curvature L direction, where function values $\alpha(\zeta)$, $\beta(\zeta)$ coincide with function values $\alpha(c)$, $\beta(c)$ in the point $c = \mathfrak{J}^{-1}(\zeta)$, g is the determined function of self-argument. K , k_s , τ_g are the Gaussian curvature, the normal curvature and geodesic torsion of the surface, respectively, of the surface in the direction of the boundary $c = \mathfrak{J}^{-1}(\zeta)$, X is *the normal component of the surface and volume forces per unit area*. Let us mention that the right member of equality (2) as the argument function ζ Holder continuous on each of the arcs Γ_j , and undergoing discontinuities of the first kind at the points ζ_j ($j = 1, \dots, n$). Let us call problem (1), (2) problem R and discontinuity points ζ_j of the function λ — *a node* of boundary condition (2), (3).

2. Results

Let $\nu_j\pi$ and $\gamma_j\pi$ ($0 < \nu_j < 2$; $j = 1, \dots, n$) be the values of interior angle at corner points c_i of the boundary L and $\zeta_j = \mathfrak{J}(c_j)$ respectively. We will assume that the surface S and its boundary L are oriented so as while going

around L the surface remains on the left and the direction of the tangent to $\sigma(c)$ in the point c of the vector coincides with the positive direction of the curve traversal L . Let us introduce the following notations: $\sigma_j^{(1)}$ and $\sigma_j^{(2)}$ ($\tau_j^{(1)}$ and $\tau_j^{(2)}$, $r_j^{(1)}$ and $r_j^{(2)}$) are limit values of the vector field $\sigma(c)$ ($\tau(c)$, $r(c)$) in the point c_j on the left and right respectively while going around the curve L in the positive direction. Here the pair of vectors $\nu_j^{(1)} = -\sigma_j^{(1)}$, $\nu_j^{(2)} = \sigma_j^{(2)}$ sets the interior angle in the point c_j . The corner point c_j together with the ordered set of vectors $(\nu_j^{(1)}, \nu_j^{(2)}, r_j^{(1)}, r_j^{(2)})$ we will call *the node* $c_j(R)$ of the problem R. Let us indicate by $\theta_j^{(k)}$ the angle between vectors $\nu_j^{(k)}$ and $r_j^{(k)}$ ($k = 1, 2; 0 \leq \theta_j^{(k)} \leq \pi$).

Statement. *The node* $\zeta_i(R)$ corresponding to the node $c_j(R)$ is a singular node (Muskhelishvili) and is valid if and only if when there are such integrals ℓ , m and s ($m = 1, 2$) as

$$\begin{aligned} \Omega(\theta_i^{(1)}, \theta_i^{(2)}, \nu_i) &\equiv (-1)^m \arccos \omega \left(\theta_i^{(1)} + \theta_i^{(2)}, \theta_i^{(1)} - \theta_i^{(2)} \right) + \\ &+ (-1)^s \arcsin \left(\sqrt{K / (k_i^{(1)} k_i^{(2)})} \sin \nu_i \pi \right) + \pi \ell = 0, \end{aligned}$$

where $\omega(u, v) = \left[\frac{1}{2}(T + S) \cos u + \frac{1}{2}(T - S) \cos v + \frac{1}{2}(M_{12} - M_{21}) \sin u + \frac{1}{2}(M_{12} + M_{21}) \sin v \right] \left[(1 - M_{11} \sin(u+v))(1 - M_{22} \sin(u-v)) \right]^{-1/2}$, $T = \mathbf{t}_i^{(1)} \cdot \mathbf{t}_i^{(2)}$, $S = \mathbf{s}_i^{(1)} \cdot \mathbf{s}_i^{(2)}$, $M_{11} = \mathbf{s}_i^{(1)} \cdot \mathbf{t}_i^{(1)}$, $M_{12} = \mathbf{s}_i^{(1)} \cdot \mathbf{t}_i^{(2)}$, $M_{21} = \mathbf{s}_i^{(2)} \cdot \mathbf{t}_i^{(1)}$, $M_{22} = \mathbf{s}_i^{(2)} \cdot \mathbf{t}_i^{(2)}$, $k_i^{(s)}$ ($s = 1, 2$) are main curvatures in the point c_i .

This statement results from the definition of *nonsingular node* [2], expression (3) and known properties [1] of mapping \mathfrak{J} .

Let us introduce the *subsidiary* classification of the corner points c_j of the boundary L : we will call corner point c_j a point of type k ($k = 1, \dots, 4$), if $\frac{k-1}{2}\pi < \gamma_i \leq \frac{k}{2}\pi$ in the corresponding point ζ_j . The point c_j of type k and the set of *admissible* pairs of vectors $\mathbf{r}_j^{(1)}, \mathbf{r}_j^{(2)}$ ($\alpha^2 + \beta^2 = 1, \beta \geq 0$) set many *nodes* $c_j(R)$ of the problem R. Then each k set of *nonsingular* nodes $c_j(R)$ can be divided into three non-crossing classes $T_k^{(i)}$ ($i = 1, 2, 3; 1 \leq k \leq 4$), besides the belonging of the point $c_j(R)$ to one of the classes are defined with the value ℓ, m, s ($-2 \leq \ell \leq 2$) the sign $\Omega(\theta_j^{(1)}, \theta_j^{(2)}, \nu_j)$.

Let $c_{i_1}(R), \dots, c_{i_m}(R)$ be *nonsingular* nodes of the problem R. Following [3], the solution of the problem is to be found in class $h_{i_1, \dots, i_m}^{1, q}$, $2 < q < p$. Let

us denote by $N_k^{(i)}$ the number of *corner points of the problem R* belonging to the class $T_k^{(i)}$ ($1 \leq k \leq 4$, $1 \leq i \leq 3$, $\sum_{k=1}^4 \sum_{i=1}^3 N_k^{(i)} = n$).

Theorem 1 is valid.

Theorem 1. Let S be the set above simply connected surface regular of class $W^{3,p}$, $p > p_0$ where $p_0 = \max\{1, \nu_1/\pi, \dots, \nu_n/\pi\}$ and c_{i_1}, \dots, c_{i_m} are arbitrary *distinguished points* from the nodes $c_j(\mathbb{R})$, ℓ is the number of all singular points. If

$$N \equiv \sum_{j=1}^3 \left[\sum_{k=1}^4 (4 - (j+k)) N_k^{(j)} \right] \geq 3 + m + \ell - n,$$

then the problem R is uniquely solvable in class $h_{i_1, \dots, i_m}^{1,q}$, $2 < q < 2p/(2 + p(1 - 1/p_0))$. If $N < 3 + m + \ell - n$, then the problem R is uniquely solvable in the indicated class if and only if when the right member of equality (2) meets the finite number of the integral type tractability conditions.

When proving the technic [3] of the boundary condition index evaluation and the results of the paper [4] are used.

In the case when on each of the arcs L_j ($1 \leq j \leq n$) the condition $\alpha \cdot \beta = 0$ is met the result obtained allows a clarification (see [3]) and provides the solution of the *mixed* boundary value problem by Vekua [1]. We mention that in the case with the smooth boundary L and the continuity if the vector field \mathbf{r} the undoubted problem T solution is possible only for the manifold surfaces [1].

Now let c_j be the arbitrary distinguished points of the surface S_0 , $(\boldsymbol{\nu}_j^{(1)}, \boldsymbol{\nu}_j^{(2)})$ are the pairs of different directions on the surface at the point c_j ($j = 1, \dots, n$). We denote by S_ν , $\nu = (\nu_1, \dots, \nu_n)$ the class of all simply connected surfaces with the piecewise smooth boundary that is a part of ovaloid S_0 and which met the following conditions: any corner point of the boundary is the point from the number of points c_j ($j = 1, \dots, n$); $\nu_j\pi$ is the value if the interior angle at the corner point of the class surface S_ν ($0 < \nu_j < 2$); directions which meet at the point c_j of the boundary arcs are the directions $(\boldsymbol{\nu}_j^{(1)}, \boldsymbol{\nu}_j^{(2)})$. Let us mention that the set of points c_j and the set of pairs $(\boldsymbol{\nu}_j^{(1)}, \boldsymbol{\nu}_j^{(2)})$ are setting the finite number of classes S_ν .

Theorem 2. Let $\boldsymbol{\nu}_j^{(1)}$ be the arbitrary distinguished directions at the points c_j ($j = 1, \dots, n$) respectively, m is the arbitrary fixed integral $0 \leq m \leq$

$3n - 3$. Then at each point c_j we can show the direction $\nu_j^{(2)}$ and the class S_ν of the surfaces corresponding to the set of pairs $(\nu_j^{(1)}, \nu_j^{(2)})$ for which the problem R is unconditionally solvable in the class of unlimited solutions with any continuous admissible field \vec{r} and its solution definitely depends on m real parameters.

Remark. In the statement of the theorem the direction $\nu_j^{(2)}$ can be substituted by some connected class of directions $\nu_j^{(2)}(\varepsilon)$, continuously depending on the real parameter ε and including the direction $\nu_j^{(2)}$.

For some special cases of the surface S boundaries and *admissible* fields \vec{r} the exact results in the form of *geometric criterion of unconditional solvability*.

References

- [1] I. N. Vekua, *Generalized Analytic Functions* (Fizmatgiz, Moscow, 1959) [in Russian].
- [2] N. I. Muskhelishvili, *Singular Integral Equations* (Fizmatgiz, Moscow, 1968) [in Russian].
- [3] E.V. Tyurikov, *Geometric Analogue of the Vekua-Goldenveizer Problem*, Doklady Mathematics. 2009. Vol. 79, No. 1, pp. 83–86.
- [4] E. V. Tyurikov, *Boundary Value Problems of the Theory of Infinitesimal Bendings of the Surfaces with Piecewise-smooth boundary*. Mat. Sb., 1977, No. 3 (7), pp. 445–462.

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რეზიუმე

მეოცე საუკუნის ორმოცდაათიან წლებში ი.ვეკუამ და ლ. ბერსმა შენიშნეს, რომ ანალიზურ ფუნქციათა ძირითადი თეორები შესაძლებელია გავრცელებული იქნას ფუნქციათა უფრო ფართო კლასზე, ვიდრე ანალიზურ ფუნქციათა სივრცეა. ვეკუასა და ბერსის საგნისადმი განსხვავებულმა და ორიგინალურმა ხედვამ მოგვცა ახალი, საუკეთესო თვისებების მქონე ფუნქციონალური სივრცე, რომლის ელემენტები დღეს განზოგადებული ანალიზური ფუნქციების (И.Векуа, Обобщенные аналитические функции. Москва, 1959) ანუ ფსევდოანალიზური ფუნქციების (L.Bers, Theory of Pseudo-Analytic Functions, New York, 1953) სახელითაა ცნობილი.

ი.ვეკუას მიდგომა, რომელიც კომპლექსურ სიბრტყეზე პირველი რიგის წრფივი ელიფსური სისტემების თეორიას ემყარება, ნაყოფიერი გამოდგა არა მარტო ფუნქციათა თეორიისათვის, არამედ მეცნიერების მომიჯნავე დარგებში გამოყენების თვალსაზრისითაც. ვეკუასეულმა ელიფსური სისტემების კომპლექსური ფორმით ჩაწერამ და ამონახსნის ანალიზური სახით წარმოდგენამ გააფართოვა ფუნქციათა თეორიის სასაზღვრო ამოცანების გამოყენების არეალი, ნათელი გახადა თეორიის გავრცელების შესაძლებლობა ვექტორულ ფუნქციებზე, მოხერხდა, აგრეთვე, განზოგადებული ანალიზური ფუნქციების მეთოდებით ბელტრამის განტოლების სრული ანალიზი, რომელიც თავის მხრივ ორგანოზომილებიანი ანალიზური მრავალსახეობების კლასიფიკაციის, მათემატიკური ფიზიკის კლასიკური და თანამედროვე დარგების: გარსთა თეორიის, ველის კონფორმული და ტოპოლოგიური თეორიების, იანგ-მილსის თეორიის მნიშვნელოვანი ინსტრუმენტია.

განზოგადებული ანალიზური ფუნქციების თეორიის და მისი მეთოდების დამუშავება და გადრმავება, აგრეთვე მისი გამოყენებით მათემატიკური ფიზიკის ამოცანების კვლევა, ამჟამად მსოფლიოს მრავალ წამყვან მათემატიკურ ცენტრში მიმდინარეობს. მათ შორისაა თბილისის სახელმწიფო უნივერსიტეტის ი.ვეკუას გამოყენებითი მათემატიკის ინსტიტუტი, სადაც სხვადასხვა პერიოდში დღემდე შესრულდა მნიშვნელოვანი გამოკვლევები თავად ი.ვეკუას მიერ დაარსებულ “კომპლექსური ანალიზისა და მისი გამოყენებების” განყოფილებაში, რომელსაც შემდგომ გ.მანჯავიძე ხელმძღვანელობდა. ამჟამად, ი.ვეკუას გამოყენებითი მათემატიკის ინსტიტუტის კომპლექსური ანალიზის ჯგუფის (გ.ახალაია, გ.გიორგაძე, ე.გორდაძე, ნ.მანჯავიძე, გ.მაქაცარია, ნ.ქალდანი, ვ.ჯიქია) ინტერესის სფეროა ირე-გულარული ელიფსური სისტემები რიმანის ზედაპირებზე და რიმან-

ჰილბერტის სასაზღვრო ამოცანა ასეთი სისტემებისათვის. კერძოდ, არარეგულარული ელიფსური დიფერენციალურ განტოლებათა სისტემების თვისობრივი კვლევა; სინგულარული ელიფსური სისტემების ამონახსნთა (ფსევდოანალიზური, პოლიანალიზური) სივრცის გამოკვლევა განსაკუთრებული წერტილის მიდამოში, მათ შორის იმ შემთხვევაშიც, როდესაც განტოლების სინგულარული წერტილი ამონახსნის განშტოების წერტილია; ელიფსური სისტემების კლასიფიკაცია განსაკუთრებული წერტილების მიხედვით; რიმან-ჰურვიცის ფორმულის ანალოგის მიღება კომპაქტური რიმანის ზედაპირისათვის და სხვა.

ჩვეულებრივ დიფერენციალურ განტოლებათა თეორიაში კარგად ცნობილი ფუქსის სისტემების შემთხვევაში განტოლებათა სისტემისაგან ინდუცირებული მერომორფულ ბმულობიანი ჰოლომორფული ფიბრაციის ინვარიანტები (სრული ჩერნის რიცხვი, გახლეჩვის ტიპი) ბუნებრივად უკავშირდება რიმან-ჰილბერტის სასაზღვრო ამოცანის რიცხვით (ინდექსი, კერძო ინდექსები) ინვარიანტებს. ამ კონტექსტში საინტერესო ამოცანაა პირველი რიგის პოლუსების მქონე ელიფსური სისტემების გამოკვლევა უბან-უბან მუდმივი მატრიცული ფუნქციის ფაქტორიზაციასთან ერთად სხვადასხვა სახის წონიან სივრცეებში. აგრეთვე, წრფივი შეუღლების სასაზღვრო ამოცანა ზოგად გახსნილ წირებზე ცვლადმაჩვენებლიან ლებეგის სივრცეში. აქვე აღვნიშნოთ, რომ კომის ტიპის ინტეგრალის სასაზღვრო მნიშვნელობა შეიძლება წირის განსხვავებულ უბნებში სხვადასხვა ბუნების იყოს, რასაც ცვლადი მაჩვენებელი უკეთ აფიქსირებს, ვიდრე მუდმივი. დღემდე არ შესწავლილა Q -ჰოლომორფული ვექტორ-ფუნქციებისათვის შეუღლების წრფივი ამოცანა, კერძოდ ის შემთხვევა, როდესაც Q მატრიცა აკმაყოფილებს ე.წ. კომუტაციურობის პირობას. საინტერესო და პერსპექტიული ამოცანაა მისგან ინდუცირებული ჰოლომორფული ფიბრაციის კვთების სივრცის, როგორც წრფივი შეუღლების ამოცანის ამონახსნთა სივრცის გამოკვლევა კონათა თეორიის აპარატის გამოყენებით.

კარგად ცნობილია, რომ მატრიცულ რიმან-ჰილბერტის სასაზღვრო (წრფივი შეუღლების) ამოცანასთან ბუნებრივად დაკავშირებული მატრიც-ფუნქციის ფაქტორიზაციის პრობლემა. ეს უკანასკნელი გვაძლევს გეომეტრიულ ობიექტს - ვექტორულ ფიბრაციას რიმანის სფეროზე. სასაზღვრო ამოცანის ამონახსნთა სივრცე ჰოლომორფული ფიბრაციის კვთების სივრცესთან იგივდება. მატრიც-ფუნქციის კერძო ინდექსები და ამოცანის ინდექსი, შესაბამისად, ვექტორული ფიბრაციის გახლეჩვის ტიპს და სრულ ჩერნის რიცხვს წარმოადგენს. ასეთი მიდგომა საშუალებას იძლევა რიმან-

ჰილბერტის ამოცანა დაისვას და ამოიხსნას კომპაქტურ რიმანის ზედაპირებზე, მატრიც-ფუნქცია შეიცვალოს მარყუჟით, ჰოლომორფული ფიბრაციის დეფორმაციის ინვარიანტების საშუალებით გამოისახოს კერძო ინდექსები ორ და სამ განზომილებიან შემთხვევებში.

რიმან-ჰილბერტის მონოდრომიული ამოცანა (ჰილბერტის 21-ე პრობლემა), რომელიც გულისხმობს სიბრტყეზე მონიშნული წერტილებისა და მათზე მიწერილი გადაუგვარებელი მატრიცების საშუალებით აივოს ფუქსის სისტემა, რომლისთვისაც მონიშნული წერტილები იქნება განსაკუთრებული წერტილები, ხოლო მონოდრომიის მატრიცები კი უკვე მოცემულ მატრიცებს დაემთხვევა, დასმული იყო რიმანის მიერ ერთ-ერთ უკანასკნელ ნაშრომში და ჰილბერტის მიერ ამოხსნილი იქნა ერთ-ერთ კერძო შემთხვევაში. ზოგადი ამოცანა მან თავის ცნობილ პრობლემათა ნუსხაში შეიტანა. მე-20 საუკუნის 90-იან წლებამდე ითვლებოდა, რომ ჰილბერტის 21-ე პრობლემა პლემელიმ ამოხსნა. მიუხედავად რამდენიმე ავტორის მითითებისა, რომ პლემელის დამტკიცება შეცდომას შეიცავდა, ითვლებოდა, რომ პლემელის ეს უზუსტობა გამოსწორებადია და საბოლოო შედეგი მართებულია. ა. ბოლიბრუხმა ააგო კონტრმაგალითი და აჩვენა, რომ ეს ასე არ არის და ჰილბერტის 21-ე პრობლემის ამოხსნადობის საკითხი მკაცრადაა დამოკიდებული მონოდრომიის წარმოდგენაზე და აგრეთვე, განსაკუთრებული წერტილების განლაგებაზე, ანუ წერტილების კონფიგურაციაზე რიმანის სფეროზე. ეს თავის მხრივ ნიშნავს, რომ პრობლემის ამოხსნადობა დამოკიდებულია მონიშნული წერტილების მქონე რიმანის ზედაპირის კომპლექსურ/კონფორმულ სტრუქტურაზე. რიმანის ზედაპირის კომპლექსური სტრუქტურა კი განისაზღვრება ბელტრამის განტოლებით. ამასთან დაკავშირებით საინტერესო ამოცანად მიგვაჩნია რიმან-ჰილბერტის მონოდრომიული ამოცანის ამოხსნადობის კრიტერიუმის მიღება ბელტრამის დიფერენციალის ტერმინებში.

როგორც კარგადაა ცნობილი, კომპლექსური ფუნქციები მონიშნულ რიმანის სფეროზე შესაძლებელია აღიწერონ მრავალსახსრული მექანიზმის მოდულების სივრცის ტერმინებში. წარმოქმნილი კავშირი რიმან-ჰილბერტის მონოდრომიულ ამოცანასა და სახსრულების მოდულების სივრცეს შორის საინტერესო და პერსპექტიულ ამოცანად გვესახება.

რიმან-ჰილბერტის სასაზღვრო ამოცანა რიმანმა იგივე ნაშრომში დასვა როგორც მონოდრომიის ამოცანის ამოხსნის ერთ-ერთი ზერხი. პლემელი სწორედ ამ გზას გაჰყვა. ამოცანა უბან-უბან მუდმივი სასაზღვრო მატრიცით მან მიიყვანა ამოცანაზე უწყვეტი სასაზღვრო მატრიც-ფუნქციით,

რომლის ამოხსნის შემდეგ საძიებელი დიფერენციალური ფორმა ააგო. პლემელის შეცდომა სავარაუდოა გამოწვეული იყო დიფერენციალური და ინტეგრალური განტოლების მატრიცული ჩანაწერის და მათზე მანიპულირების არასაკმარისი ტექნიკით. მოგვიანებით ასეთი ტექნიკა ჰილბერტის 21-ე პრობლემისათვის განავითარა ი.ლაპო-დანილევსკიმ, ხოლო წრფივი შეუღლების ამოცანისათვის კი ნ.მუსხელიშვილმა და ნ. ვეკუამ ერთობლივ ნაშრომში. ამის შემდეგ ეს ორი ამოცანა შეისწავლებოდა დამოუკიდებლად.

მიუხედავად ასეთი ორგანული (შინაარსობრივი) კავშირისა ამ ორ ამოცანას შორის და მიუხედავად იმისა, რომ წრფივი შეუღლების ამოცანა შესწავლილია განზოგადებული ანალიზური ფუნქციებისათვის და კექტორებისათვის, სინგულარული ელიფსური სისტემების შესწავლაში მიღწეულია გარკვეული პროგრესი, კავშირი მის შესაბამის განსაკუთრებული წერტილების მქონე (არარეგულარულ) ელიფსურ სისტემებთან აქამდე მითითებული არ არის. რაც ჩვენი აზრით განზოგადებული ანალიზური ფუნქციათა თეორიის ერთ-ერთი მნიშვნელოვანი პრობლემაა.

როგორც ცნობილია, რიმან-ჰილბერტის მონოდრომიული ამოცანა წარმოქმნის ჰოლომორფულ ფიბრაციას მერომორფული ბმულობით. ეს უკანაგნელი კი იძლევა ჰილბერტის 21-ე პრობლემის ამოხსნადობის აუცილებელ და საკმარის პირობას. ოდნავ შესუსტებული სახით ანალოგიური შედეგი ვრცელდება რიმანის ზედაპირებზე და ფუნქციებზე მნიშვნელობით კომპლექსურ ლის ჯგუფებში. მონოდრომიული ამოცანის დასმა და შესწავლა ზოგადი ელიფსური სისტემებისათვის ან რომელიმე ქვეკლასისათვის, ჩვენი აზრით, ერთ-ერთი აქტუალური და მნიშვნელოვანი ამოცანაა არა მარტო ელიფსური სისტემების ამონახსნთა სივრცის შესასწავლად, არამედ L_p -კლასის ბმულობების ასაგებად და მოდულების სივრცის გამოსაკვლევად. მიღებული შედეგები საშუალებას მოგვცემს გავაფართოვოთ ელიფსური სისტემების გამოყენების არეალი მათემატიკურ ფიზიკასა და ტექნიკაში: დრეკადობის თეორიაში, კონფორმული, ყალიბური და ტოპოლოგიური ველის თეორიებში, კვანტურ გამოთვლებში.

ზემოთ დასმული პრობლემის განსჯის მიზნით ი.ვეკუას გამოყენებითი მათემატიკის ინსტიტუტის კომპლექსური ანალიზის ჯგუფის მიერ ორგანიზებულ საერთაშორისო კონფერენციაში მონაწილეობის მისაღებად მოწვეულნი იქნენ ამ დარგის მსოფლიოში წამყვანი მეცნიერები. ეს წიგნი სრულად ფარავს კონფერენციის თემატიკას. აღსანიშნავია, რომ აქ წარმოდგენილია ოთხი თაობის ავტორები, მათ შორის თბილისის სახელმწიფო უნივერსიტეტის სტუდენტები. კრებული დასაბეჭდად მზადდებოდა, როდესაც გავიგეთ

ჩვენი მეგობრისა და კოლეგის, ა.ტიმოფეევის ტრაგიკულად დაღუპვის ამბავი. კრებულში შეტანილია ა.ტიმოფეევის უკანასკნელი ნაშრომი მისი ხსოვნის უკდავსაყოფად.

განსაკუთრებით გვინდა აღვნიშნოთ განზოგადებულ ანალიზურ ფუნქციასთან თეორიაში კლასიკური შრომების ავტორის, პროფ. ბ.ბოიარსკის წვლილი კონფერენციის მხარდაჭერასა და სტიმულირებაში. წიგნში შეტანილია მისი ნაშრომი, რომელიც მტკიცებულებათა გამჭვირვალობის, დროში მეცნიერული აზრის დინების ისტორიის და დიდაქტიკის ერთ-ერთი საუკეთესო ნიმუშია.

მადლობა გვინდა გადავუხადოთ დარგის ცნობილ სპეციალისტებს, გამორჩეული ნაშრომების ავტორებს ვ.ადუკოვს, კ.ბეგერს, ვ.კრაზჩენკოს, ს.კრუშკალს, ვ.პალამოდოვს, ს.პლასას, ს.როგოზინს, რ.საქს, ვ.ტუჩკეს, ზ.უსმანოვს ყურადღებისა და მორალური მხარდაჭერისათვის.



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on
Generalized Analytic Functions and Their
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- **Boundary value problems for analytic and generalized analytic functions and vectors;**
- **Quasiconformal mappings;**
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