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ON THE STABILITY OF SOLUTIONS OF LINEAR BOUNDARY VALUE PROBLEMS FOR THE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

M.ASHORDIA

ABSTRACT. Linear boundary value problem for the system of ordinary differential equations is considered. The question on the stability of the solution with respect to small perturbations of coefficients and boundary values is investigated.

რეზიუმე. ჩვეულებრივ წრფივ დიფერენციალურ განტოლებათა სისტემისათვის შესწავლილია საკითხი წრფივი სასაზღვრო ამოცანის ამონახსნის მდგრადობის შესახებ სისტემის კოეფიციენტებისა და სასაზღვრო მონაცემების მცირე შეშფოთების მიმართ.

Let $\mathcal{P}_0 : [a,b] \to \mathbb{R}^{n\times n}$ and $q_0 : [a.b] \to \mathbb{R}^n$ be integrable matrix- and vector-functions, respectively, $c_0 \in \mathbb{R}^n$ and let $l_0 : C([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ be a linear continuous operator such that the boundary value problem

$$\frac{dx}{dt} = \mathcal{P}_0(t)x + q_0(t), \tag{1}$$

$$l_0(x) = c_0 \tag{2}$$

has the unique solution x_0. Consider the sequences of integrable matrix- and vector-functions, $\mathcal{P}_k : [a,b] \to \mathbb{R}^{n\times n}$ $(k = 1,2,\dots)$ and $q_k : [a,b] \to \mathbb{R}^n$ $(k = 1,2,\dots)$, respectively, the sequence of constant vectors $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ and the sequence of linear continuous operators $l_k : C([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ $(k = 1,2,\dots)$. In [1,2] the sufficient conditions are given for the problem

$$\frac{dx}{dt} = \mathcal{P}_k(t)x + q_k(t), \tag{3}$$

$$l_k(x) = c_k \tag{4}$$

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to have the unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty} x_k(t) = x_0(t) \quad \text{uniformly on } [a,b]. \tag{5}$$

In the present paper the necessary and sufficient conditions are established for the sequence of boundary value problems of the form (3),(4) to have the above-mentioned property.

Throughout the paper the following notations and definitions will be used:

$\mathbb{R} =]-\infty, +\infty[$;

$\mathbb{R}^n$ is a space of real column n-vectors $x = (x_i)_{i=1}^n$ with the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$\mathbb{R}^{n\times n}$ is a space of real $n \times n$ matrices $X = (x_{ij})_{i,j=1}^n$ with the norm

$$\|X\| = \sum_{i,j=1}^n |x_{ij}|;$$

if $X = (x_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$, then $diag\, X$ is a diagonal matrix with diagonal components $x_{11}, \dots, x_{nn}$; X^{-1} is an inverse matrix to X; E is an identity $n \times n$ matrix;

$C([a,b];\mathbb{R}^n)$ is a space of continuous vector-functions $x : [a,b] \to \mathbb{R}^n$ with the norm

$$\|x\|_c = \max\{\|x(t)\| : a \le t \le b\};$$

$\widetilde{C}([a,b];\mathbb{R}^n)$ and $\widetilde{C}([a,b];\mathbb{R}^{n\times n})$ are the sets of absolutely continuous vector- and matrix- functions, respectively;

$L([a,b];\mathbb{R}^n)$ and $L([a,b];\mathbb{R}^{n\times n})$ are the sets of vector- and matrix-functions $x : [a,b] \to \mathbb{R}^n$ and $X : [a,b] \to \mathbb{R}^{n\times n}$, respectively, whose components are Lebesgue-integrable;

$\|l\|$ is the norm of the linear continuous operator $l : C([a,b];\mathbb{R}^n) \to \mathbb{R}^n$.

The vector-function $x : [a,b] \to \mathbb{R}^n$ is said to be a solution of the problem (1),(2) if it belongs to $\widetilde{C}([a,b];\mathbb{R}^n)$ and satisfies the condition (2) and the system (1) a.e. on $[a,b]$.

Definition 1. We shall say that the sequence $(\mathcal{P}_k, q_k, l_k)$ $(k = 1,2,\dots)$ belongs to $S(\mathcal{P}_0, q_0, l_0)$ if for every $c_0 \in \mathbb{R}^n$ and $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ satisfying the condition

$$\lim_{k\to+\infty} c_k = c_0 \tag{6}$$

the problem (3),(4) has the unique solution x_k for any sufficiently large k and (5) holds.

Along with (1),(2) and (3),(4) we shall consider the corresponding homogeneous problems

$$\frac{dx}{dt} = \mathcal{P}_0(t)x, \tag{1_0}$$

$$l_0(x) = 0 \tag{2_0}$$

and

$$\frac{dx}{dt} = \mathcal{P}_k(t)x, \tag{3_0}$$

$$l_k(x) = 0. \tag{4_0}$$

Theorem 1. *Let*

$$\lim_{k\to+\infty} l_k(y) = l_0(y) \quad \text{for } y \in \widetilde{C}([a,b];\mathbb{R}^n) \tag{7}$$

and

$$\lim_{k\to+\infty} \sup \|l_k\| < +\infty. \tag{8}$$

Then

$$\Big((\mathcal{P}_k, q_k, l_k)\Big)_{k=1}^{+\infty} \in S(\mathcal{P}_0, q_0, l_0) \tag{9}$$

if and only if there exist sequences of matrix- and vector-functions, $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ *and* $\varphi_k \in \widetilde{C}([a,b];\mathbb{R}^n)$ $(k = 1,2,\dots)$, *respectively, such that*

$$\lim_{k\to+\infty} \sup \int_a^b \|\mathcal{P}_k^*(\tau)\|\, d\tau < +\infty \tag{10}$$

and

$$\lim_{k\to+\infty} \Phi_k(t) = 0, \tag{11}$$

$$\lim_{k\to+\infty} \varphi_k(t) = 0, \tag{12}$$

$$\lim_{k\to+\infty} \int_a^t \mathcal{P}_k^*(\tau)\, d\tau = \int_a^t \mathcal{P}_0(\tau)\, d\tau, \tag{13}$$

$$\lim_{k\to+\infty} \int_a^t q_k^*(\tau)\, d\tau = \int_a^t q_0(\tau)\, d\tau \tag{14}$$

uniformly on $[a,b]$, *where*

$$\mathcal{P}_k^*(t) \equiv [E - \Phi_k(t)] \cdot \mathcal{P}_k(t) - \Phi_k'(t), \tag{15}$$

$$q_k^*(t) \equiv [E - \Phi_k(t)][\mathcal{P}_k(t)\varphi_k(t) + q_k(t) - \varphi_k'(t)]. \tag{16}$$

Theorem 1′. *Let* (7)*and* (8) *be satisfied. The* (9) *holds if and only if there exist sequences of matrix- and vector-functions,* $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ *and* $\varphi_k \in \widetilde{C}([a,b];\mathbb{R}^n)$ $(k = 1,2,\dots)$, *respectively, such that*

$$\lim_{k\to+\infty} \sup \int_a^b \|\mathcal{P}_k^*(\tau) - diag\,\mathcal{P}_k^*(\tau)\|\,d\tau < +\infty \tag{17}$$

and the conditions (11)–(13) *and*

$$\lim_{k\to+\infty} \int_a^t \exp\Big(-\int_a^\tau diag\,\mathcal{P}_k^*(s)ds\Big)\cdot q_k^*(\tau)\,d\tau =$$
$$= \int_a^t \exp\Big(-\int_a^\tau diag\,\mathcal{P}_0(s)ds\Big)\cdot q_0(\tau)\,d\tau \tag{18}$$

are fulfilled uniformly on $[a,b]$, *where*

$$\mathcal{P}_k^*(t) \equiv [\mathcal{P}_k(t) - \Phi_k(t)\mathcal{P}_k(t) - \Phi_k'(t)]\cdot[E - \Phi_k(t)]^{-1} \tag{19}$$

and $q_k^*(t)$ *is the vector-function defined by* (16).

Before proving this theorems, we shall give a theorem from [1] and some of its generalizations.

Theorem 2_0. *Let the conditions* (6)–(8),

$$\lim_{k\to+\infty} \sup \int_a^b \|\mathcal{P}_k(\tau)\|\,d\tau < +\infty \tag{20}$$

hold and let the following conditions

$$\lim_{k\to+\infty} \int_a^t \mathcal{P}_k(\tau)\,d\tau = \int_a^t \mathcal{P}_0(\tau)\,d\tau, \tag{21}$$

$$\lim_{k\to+\infty} \int_a^t q_k(\tau)\,d\tau = \int_a^t q_0(\tau)\,d\tau \tag{22}$$

hold uniformly on $[a,b]$. *Then* (9) *is satisfied*[1].

Theorem 2. *Let there exist sequences of matrix- and vector-functions,* $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ *and* $\varphi_k \in \widetilde{C}([a,b];\mathbb{R}^n)$ $(k = 1,2,\dots)$, *respectively, such that the conditions* (10),

$$\lim_{k\to+\infty} [c_k - l_k(\varphi_k)] = c_0 \tag{23}$$

hold and let the conditions (11),(13),(14) *be fulfilled uniformly on* $[a,b]$, *where* $\mathcal{P}_k^*(t)$ *and* $q_k^*(t)$ *are the matrix- and vector-functions defined by* (15) *and* (16), *respectively. Let, moreover, conditions* (7),(8) *hold.*

[1]See [1], Theorem 1.2.

Then for any sufficiently large k the problem (3),(4) *has the unique solution x_k and*

$$\lim_{k\to+\infty} \|x_k - \varphi_k - x_0\|_c = 0.$$

Proof. The transformation $z = x - \varphi_k$ reduces the problem (3),(4) to

$$\frac{dz}{dt} = \mathcal{P}_k(t)z + r_k(t), \tag{24}$$

$$l_k(z) = c_{k1}, \tag{25}$$

where $r_k(t) \equiv \mathcal{P}_k(t)\varphi_k(t) + q_k(t) - \varphi_k'(t)$, $c_{k1} = c_k - l_k(\varphi_k)$ $(k = 1, 2, \dots)$.

Let us show that for any sufficiently large k the homogeneous problem (3_0),(4_0) has only trivial solution.

Suppose this proposition is invalid. It can be assumed without loss of generality that for every natural k the problem (3_0),(4_0) has the solution x_k for which

$$\|x_k\|_c = 1. \tag{26}$$

Moreover, it is evident that the vector-function x_k is the solution of the system

$$\frac{dx}{dt} = \mathcal{P}_k^*(t)x + [\Phi_k(t) \cdot x_k(t)]'. \tag{27}$$

According to (11) and (26)

$$\lim_{k\to+\infty} [\Phi_k(t) \cdot x_k(t)] = 0$$

uniformly on $[a, b]$. Therefore the conditions of Theorem 2_0 are fulfilled for the sequence of problems (27),(4_0). Hence

$$\lim_{k\to+\infty} \|x_k\|_c = 0,$$

which contradicts (26). This proves that the problem (3_0),(4_0) has only trivial solution.

From this fact it follows that for any sufficiently large k the problem (24),(25) has the unique solution z_k.

It can easily be shown that the vector-function z_k satisfies the system

$$\frac{dz}{dt} = \mathcal{P}_k^*(t)z + r_k^*(t), \tag{28}$$

where $r_k^*(t) = [\Phi_k(t) \cdot z_k(t)]' + q_k^*(t)$.

Show that

$$\lim_{k\to+\infty} \sup \|z_k\|_c < +\infty. \tag{29}$$

Let this proposal be invalid. Assume without loss of generality that

$$\lim_{k\to+\infty} \|z_k\|_c = +\infty. \tag{30}$$

Put

$$u_k(t) = \|z_k\|_c^{-1} \cdot z_k(t) \quad \text{for } t \in [a,b] \ (k = 1, 2, \dots).$$

Then in view of (25) and (28), for every natural k the vector-function $u_k(t)$ will be the solution of the boundary value problem

$$\frac{du}{dt} = \mathcal{P}_k^*(t)u + s_k(t),$$
$$l_k(u) = \|z_k\|_c^{-1} \cdot c_{k1},$$

where $s_k(t) = \|z_k\|_c^{-1} \cdot r_k^*(t)$. (11),(14),(23) and (30) imply

$$\lim_{k\to+\infty} [\|z_k\|_c^{-1} \cdot c_{k1}] = 0$$

and

$$\lim_{k\to+\infty} \int_a^t s_k(\tau)\, d\tau = 0$$

uniformly on $[a,b]$. Hence, according to (10) and (13) the conditions of Theorem 2_0 are fulfilled for the sequence of the last boundary value problems. Therefore

$$\lim_{k\to+\infty} \|u_k\|_c = 0.$$

This equality contradicts the conditions $\|u_k\|_c = 1$ ($k = 1, 2, \dots$). The inequality (29) is proved.

In view of (11),(14) and (29)

$$\lim_{k\to+\infty} \int_a^t r_k^*(\tau)\, d\tau = \int_a^t q_0(\tau)\, d\tau$$

uniformly on $[a,b]$.

Applying Theorem 2_0 to the sequence of the problems (28),(25), we again convince ourselves that

$$\lim_{k\to+\infty} \|z_k - x_0\|_c = 0. \ \blacksquare$$

Corollary 1. *Let* (6)–(8),

$$\lim_{k\to+\infty} \sup \int_a^b \|\mathcal{P}_k(\tau) - \Phi_k(\tau)\mathcal{P}_k(\tau) - \Phi_k'(\tau)\|\, d\tau < +\infty$$

hold and let the conditions (11),(21),(22),

$$\lim_{k\to+\infty} \int_a^t \Phi_k(\tau)\mathcal{P}_k(\tau)\, d\tau = \int_a^t \mathcal{P}^*(\tau)\, d\tau$$

and

$$\lim_{k\to+\infty} \int_a^t \Phi_k(\tau) q_k(\tau)\, d\tau = \int_a^t q^*(\tau)\, d\tau$$

be fulfilled uniformly on $[a,b]$, *where* $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $\mathcal{P}^* \in L([a,b];\mathbb{R}^{n\times n})$, $q^* \in L([a,b];\mathbb{R}^n)$. *Let, moreover, the system*

$$\frac{dx}{dt} = \mathcal{P}_0^*(t)x + q_0^*(t),$$

where $\mathcal{P}_0^*(t) \equiv \mathcal{P}_0(t) - \mathcal{P}^*(t)$, $q_0^*(t) \equiv q_0(t) - q^*(t)$ *have unique solution satisfying the condition* (2). *Then*

$$\Big((\mathcal{P}_k, q_k, l_k)\Big)_{k=1}^{+\infty} \in S(\mathcal{P}_0^*, q_0^*, l_0).$$

Proof. It suffices to assume in Theorem 2 that $\varphi_k(t) \equiv 0$ and to notice that

$$\lim_{k\to+\infty} \int_a^t [E - \Phi_k(\tau)] \cdot \mathcal{P}_k(\tau)\, d\tau = \int_a^t \mathcal{P}_0^*(\tau)\, d\tau$$

and

$$\lim_{k\to+\infty} \int_a^t [E - \Phi_k(\tau)] \cdot q_k(\tau)\, d\tau = \int_a^t q_0^*(\tau)\, d\tau$$

uniformly on $[a,b]$. ■

Corollary 2. *Let* (6)–(8) *hold and let there exist a natural number* m *and matrix-functions* $\mathcal{P}_{0j} \in L([a,b],\mathbb{R}^{n\times n})$ $(j = 1,\dots,m)$ *such that*

$$\lim_{k\to+\infty} [\mathcal{P}_{km}(t) - \mathcal{P}_k(t)] = 0,$$

$$\lim_{k\to+\infty} \int_a^t [E + \mathcal{P}_{km}(\tau) - \mathcal{P}_k(\tau)] \cdot \mathcal{P}_k(\tau)\, d\tau = \int_a^t \mathcal{P}_0(\tau)\, d\tau,$$

$$\lim_{k\to+\infty} \int_a^t [E + \mathcal{P}_{km}(\tau) - \mathcal{P}_k(\tau)] \cdot q_k(\tau)\, d\tau = \int_a^t q_0(\tau)\, d\tau$$

uniformly on $[a,b]$ *where*

$$\mathcal{P}_{k1}(t) \equiv \mathcal{P}_k(t), \quad \mathcal{P}_{k\,j+1}(t) \equiv \mathcal{P}_{kj}(t) - \int_a^t [\mathcal{P}_{kj}(\tau) - \mathcal{P}_{0j}(\tau)]\, d\tau$$
$$(j = 1,\dots,m).$$

Let, moreover,

$$\lim_{k\to+\infty} \sup \int_a^b \|[E + \mathcal{P}_{km}(\tau) - \mathcal{P}_k(\tau)] \cdot \mathcal{P}_k(\tau) + $$
$$+[\mathcal{P}_{km}(\tau) - \mathcal{P}_k(t)]'\|d\tau < +\infty.$$

Then (9) *holds.*

Theorem 2′. *Let there exist sequences of matrix- and vector-functions,* $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ *and* $\varphi_k \in \widetilde{C}([a,b];\mathbb{R}^n)$ $(k = 1,2,\dots)$, *respectively such that the conditions* (17),(23) *hold and let the conditions* (11),(13) *and* (18) *be fulfilled uniformly on* $[a,b]$. *Here* $\mathcal{P}_k^*(t)$ *and* $q_k^*(t)$ *are the matrix- and vector-functions defined by* (19) *and* (16), *respectively. Then the conclusion of Theorem* 2 *is true.*

Proof. In view of (14), we may assume without loss of generality that for every natural k the matrix $E - \Phi_k(t)$ is invertible for $t \in [a,b]$.

For every $k \in \{0,1,\dots\}$ and $t \in [a,b]$ assume

$$\mathcal{P}_0^*(t) = \mathcal{P}_0(t), \ \ q_0^*(t) = q_0(t), \ \ \Phi_0(t) = 0, \ \ \varphi_0(t) = 0,$$
$$c_{k1} = c_k - l_k(\varphi_k), \ Q_k(t) = H_k(t) \cdot [\mathcal{P}_k^*(t) - diag\, \mathcal{P}_k^*(t)] \cdot H_k^{-1}(t),$$
$$r_k(t) = H_k(t) \cdot q_k^*(t),$$

where

$$H_k(t) = \exp\Big(- \int_a^t diag\, \mathcal{P}_k^*(\tau)\, d\tau\Big).$$

Moreover, assume

$$l_k^*(z) = l_k(x) \quad \text{for } \ z \in C([a,b];\mathbb{R}^n),$$

where $x(t) = [E - \Phi_k(t)]^{-1} \cdot H_k^{-1}(t) \cdot z(t)$.

From (13) it follows that $l_k^* : C([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ $(k = 0,1,\dots)$ is a sequence of linear continuous operators for which conditions (7) and (8) are satisfied.

For every $k \in \{0,1,\dots\}$ the transformation

$$z(t) = H_k(t) \cdot [E - \Phi_k(t)] \cdot [x(t) - \varphi_k(t)] \ \text{ for } t \in [a,b] \tag{31}$$

reduces the problem (3),(4) to

$$\frac{dz}{dt} = Q_k(t)z + r_k(t), \tag{32}$$
$$l_k^*(z) = c_{k1} \tag{33}$$

and the problem (1),(2) to

$$\frac{dz}{dt} = Q_0(t)z + r_0(t), \tag{34}$$

$$l_0^*(z) = c_0. \tag{35}$$

In view of (13) and (17) from Lemma 1.1 ([1], p.9) it follows that

$$\lim_{k\to+\infty} \int_a^t Q_k(\tau)\, d\tau = \int_a^t Q_0(\tau)\, d\tau$$

uniformly on $[a, b]$. According to Theorem 2_0 from the above and from (7),(8),(17),(18),(23) it follows that the problem (32),(33) has the unique solution z_k for any sufficiently large k, and

$$\lim_{k\to+\infty} \|z_k - z_0\|_c = 0,$$

where z_0 is the unique solution of the problem (34),(35). Therefore (11),(13) and (31) show that the statement of the theorem is true. ■

Corollary 3. *Let the conditions* (6)–(8),

$$\lim_{k\to+\infty} \sup \int_a^b \|\mathcal{P}_k(\tau) - diag\,\mathcal{P}_k(\tau)\|\, d\tau < +\infty$$

hold and let (21) *and*

$$\lim_{k\to+\infty} \int_a^t \exp\Big(-\int_a^\tau diag\,\mathcal{P}_k(s)ds\Big)\cdot q_k(\tau)\, d\tau =$$
$$= \int_a^t \exp\Big(-\int_a^\tau diag\,\mathcal{P}_0(s)ds\Big)\cdot q_0(\tau)\, d\tau$$

be fulfilled uniformly on $[a, b]$. *Then* (9) *holds.*

Remark. As compared with Theorem 2_0 and the results of [2], it is not assumed in Theorems 2 and 2′ that the equalities (21) and (22) hold uniformly on $[a, b]$. Below we will give an example of a sequence of boundary value problems for linear systems for which (9) holds but (21) is not fulfilled uniformly on $[a, b]$.

Example. Let $a = 0$, $b = 2\pi$, $n = 2$ and for every natural k and $t \in [0, 2\pi]$

$$\mathcal{P}_k(t) = \begin{pmatrix} 0 & p_{k1}(t) \\ 0 & p_{k2}(t) \end{pmatrix}, \quad \mathcal{P}_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\varphi_k(t) = q_k(t) = q_0(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

$$p_{k1}(t) = \begin{cases} (\sqrt{k} + \sqrt[4]{k}) \sin kt & \text{for } t \in I_k, \\ \sqrt{k} \sin kt & \text{for } t \in [0, 2\pi] \setminus I_k; \end{cases}$$

$$p_{k2}(t) = \begin{cases} -\alpha_k'(t) \cdot [1 - \alpha_k(t)]^{-1} & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0, 2\pi] \setminus I_k; \end{cases}$$

$$\beta_k(t) = \int_0^t [1 - \alpha_k(\tau)] \cdot p_{k1}(\tau)\, d\tau;$$

$$\alpha_k(t) = \begin{cases} 4\pi^{-1}(\sqrt[4]{k} + 1)^{-1} \sin kt & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0, 2\pi] \setminus I_k \end{cases}$$

where $I_k = \cup_{m=0}^{k-1}]2mk^{-1}\pi, (2m+1)k^{-1}\pi[$. Let, moreover, for every $k \in \{0, 1, \dots\}$ $Y_k(t)$ be the fundamental matrix of the system (3_0) satisfying

$$Y_k(a) = E.$$

It can easily be shown that for every natural k we have

$$Y_0(t) = E, \quad Y_k(t) = \begin{pmatrix} 1 & \beta_k(t) \\ 0 & 1 - \alpha_k(t) \end{pmatrix} \quad \text{for } t \in [0, 2\pi]$$

and

$$\lim_{k \to +\infty} Y_k(t) = Y_0(t)$$

uniformly on $[0, 2\pi]$, since

$$\lim_{k \to +\infty} \|\alpha_k\|_c = \lim_{k \to +\infty} \|\beta_k\|_c = 0.$$

Note that

$$\lim_{k \to +\infty} \int_0^{2\pi} p_{k1}(t)\, dt = 2 \lim_{k \to +\infty} \sqrt[4]{k} = +\infty.$$

Therefore neither the conditions of Theorem 2_0 nor the results of [2] are fulfilled.

On the other hand, if we assume that

$$\Phi_k(t) = E - Y_k^{-1}(t) \quad \text{for } t \in [0, 2\pi] \ (k = 1, 2, \dots),$$

then the conditions of Theorems 2 and 2′ will be fulfilled, and if we put

$$\Phi_k(t) = \begin{pmatrix} \alpha_k(t) & \beta_k(t) \\ 0 & 0 \end{pmatrix} \quad \text{for } t \in [0, 2\pi] \ (k = 1, 2, \dots),$$

then in this case only the conditions of Theorem 2 will be fulfilled, since

$$\limsup_{k \to +\infty} \int_0^{2\pi} |p_{k2}(t)| \, dt = +\infty.$$

Proof of Theorem 1. The sufficiency follows from Theorem 2, since in view of (6),(8) and (12), condition (23) holds.

Let us show the necessity. Let $c_k \in \mathbb{R}^n$ $(k = 0, 1, \dots)$ be an arbitrary sequence satisfying (6) and let $e_j = (\delta_{ij})_{i=1}^n$ $(j = 1, \dots, n)$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

In view of (9), we may assume without loss of generality that for every natural k the problem (3),(4) has the unique solution x_k.

For any $k \in \{0, 1, \dots\}$ and $j \in \{1, \dots, m\}$ assume

$$y_{kj}(t) = x_k(t) - x_{kj}(t) \quad (t \in [a, b]),$$

where x_{oj} and x_{kj} $(k = 1, 2, \dots)$ are the unique solutions of (1) and (3) satisfying

$$l_0(x) = c_0 - e_j \quad \text{and} \quad l_k(x) = c_k - e_j,$$

respectively. Moreover, for every $k \in \{0, 1, \dots\}$ by $Y_k(t)$ denote the matrix-function whose columns are $y_{k1}(t), \dots, y_{kn}(t)$.

It can easily be shown that y_{oj} and y_{kj} satisfy (1_0) and (3_0), respectively, and

$$l_k(y_{kj}) = e_j \quad (j = 1, \dots, n; \ k = 0, 1, \dots). \tag{36}$$

If for some k and $\alpha_j \in \mathbb{R}$ $(j = 1, \dots, n)$

$$\sum_{j=1}^n \alpha_j y_{kj}(t) = 0 \quad (t \in [a, b]),$$

then using (36)

$$\sum_{j=1}^n \alpha_j e_j = 0$$

and therefore

$$\alpha_1 = \dots = \alpha_n = 0,$$

i.e. Y_0 and Y_k are the fundamental matrices of the systems (1_0) and (3_0), respectively. Hence, (5) implies

$$\lim_{k\to+\infty} Y_k^{-1}(t) = Y_0^{-1}(t) \quad \text{uniformly on } [a,b]. \tag{37}$$

Let for every natural k and $t \in [a,b]$

$$\Phi_k(t) = E - Y_0(t)Y_k^{-1}(t), \tag{38}$$

$$\varphi_k(t) = x_k(t) - x_0(t). \tag{39}$$

Let us show (10)–(14). (11) and (12) are evident. Moreover, using equality

$$[Y_k^{-1}(t)]' = -Y_k^{-1}(t)\mathcal{P}_k(t) \quad \text{for } t \in [a,b]\ (k = 1,2,\dots),$$

it can be easily shown that

$$\mathcal{P}_k^*(t) = \mathcal{P}_0(t)Y_0(t)Y_k^{-1}(t) \quad \text{for } t \in [a,b]\ (k = 1,2,\dots)$$

and

$$\int_a^t q_k^*(\tau)\,d\tau = Y_0(t)Y_k^{-1}(t)x_0(t) - Y_0(a)Y_k^{-1}(a)x_0(a) -$$

$$- \int_a^t \mathcal{P}_0(\tau)Y_0(\tau)Y_k^{-1}(\tau)x_0(\tau)\,d\tau \quad \text{for } t \in [a,b]\ (k = 1,2,\dots).$$

Therefore, according to (37) the conditions (10),(13) and (14) are fulfilled uniformly on $[a,b]$. This completes the proof. ■

The proof of Theorem 1′ is analogous. We note only that Φ_k and φ_k are defined as in above.

The problem about the behaviour at $k \to +\infty$ of the solution of the Cauchy problem ($l_k(x) = x(t_0)$, $t_0 \in [a,b]$) and of the Cauchy-Nicoletti problem ($l_k(x) = (x_i(t_i))_{i=1}^n$, $t_i \in [a,b]$) are considered in [3-5]. Besides, in [6] the necessary conditions for the stability of the Cauchy problem are investigated.

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ON THE UNIQUENESS THEOREMS FOR THE EXTERNAL PROBLEMS OF THE COUPLE-STRESS THEORY OF ELASTICITY

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Abstract. A formula is obtained for the asymptotic representation of solutions of the basic equations of the couple-stress theory of elasticity. The formula is used in proving the uniqueness theorems of the external boundary value problems.

რეზიუმე. მიღებულია დრეკადობის მომენტური თეორიის ერთგვაროვან განტოლებათა სისტემის ამონახსნის ასიმპტოტური წარმოდგენის ფორმულა უსასრულოდ შორეული წერტილის მიდამოში. ამ ფორმულის გამოყენებით დამტკიცებულია გარე სასაზღვრო ამოცანების ამონახსნთა ერთადერთობის ახალი თეორემები.

0. Let Ω^+ be a bounded domain in the three-dimensional Euclidean space $\mathbb{R}^3$, and Ω^- a complement of Ω^+ to the entire space $\mathbb{R}^3$:$\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}^-$. The boundary value problems formulated for the domain Ω^- are called external. The uniqueness theorems for the external boundary value problems are valid only under some restrictions of the class of solutions at infinity ([1], [2]). These restrictions arose naturally from the Green formulas and consist in the requirement that both the solution and its derivatives vanish at infinity. The weakening of the restrictions is important from both theoretical and practical standpoints (for example, in constructing effective solutions). This question is discussed in the monograph [1] specially devoted to uniqueness theorems of the elasticity theory.

In recent years new results have been obtained for the external static problems of the classical elasticity theory [3]-[7]. In these works the authors have succeeded in weakening essentially the restrictions at infinity imposed on the class of solutions in which the uniqueness theorems are proved. The results were obtained by two different methods: in [4], [5] the proof was based on Korn's inequality, whereas in [3], [6], [7] use was made of the asymptotic representation of solutions in the neighbourhood of an isolated singular point (in particular, in the neighbourhood of the point at infinity). However, both methods were applied to the system of equations containing only derivatives of higher (second) order. The system of static equations of the classical elasticity theory is also such a system.

In this paper we show that the method of asymptotic representations of solutions in the neighbourhood of an isolated singular point (see [3], [6], [7]) can be as well applied to systems of equations containing derivatives of both higher and lower orders. This is exemplified by the system of static equations of the couple-stress theory of elasticity for a homogeneous anisotropic medium containing derivatives of second order, as well as derivatives of first and zero orders. Here we have derived the asymptotic representation of the solution of the said system in the neighbourhood of the point at infinity, which has enabled us to prove new uniqueness theorems for the external boundary value problems of the couple-stress theory of elasticity.

The derivation of asymptotic representations largely rests on the behaviour of the fundamental solution of the considered system at infinity.

1. A homogeneous system of the couple-stress theory of a homogeneous anisotropic micropolar elastic medium is written in the form [2], [9]

$$\begin{gathered} c_{ijlk}\frac{\partial^2 u_k}{\partial x_j \partial x_l} - c_{jilm}\varepsilon_{klm}\frac{\partial \omega_k}{\partial x_j} = 0, \\ c_{jmlk}\varepsilon_{ijm}\frac{\partial u_k}{\partial x_l} + c'_{jilk}\frac{\partial^2 \omega_k}{\partial x_j \partial x_l} - c_{jmlp}\varepsilon_{ijm}\varepsilon_{klp}\omega_k = 0, \quad i = 1,2,3, \end{gathered} \tag{1}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector, $\omega = (\omega_1, \omega_2, \omega_3)$ is the rotation vector, ε_{ijk} is the Levy-Civita symbol, c_{jilk}, c'_{jilk} $(i,j,l,k = 1,2,3)$ are the elastic constants. Here and in what follows the repetition of the index in the product means summation over this index.

It is assumed that the elastic coefficients c_{ijkl} and c'_{ijkl} satisfy the symmetry conditions

$$c_{ijkl} = c_{klij}, \quad c'_{ijkl} = c'_{klij} \tag{2}$$

and the energetic form is positive-definite

$$c_{ijkl}\xi_{ij}\xi_{kl} + c'_{ijkl}\eta_{ij}\eta_{kl} > 0 \quad \text{for } \xi_{ij}\xi_{ij} + \eta_{ij}\eta_{ij} \neq 0. \tag{3}$$

Let

$$
\begin{aligned}
&A(\partial_x) \equiv \|A_{ik}(\partial_x)\|_{6\times 6},\\
&A_{ik}(\partial_x) \equiv c_{jilk}\frac{\partial^2}{\partial x_j \partial x_l}, \quad i,k=1,2,3;\\
&A_{i,k+3}(\partial_x) \equiv -c_{jilm}\varepsilon_{klm}\frac{\partial}{\partial x_j}, \quad i,k=1,2,3;\\
&A_{i+3,k}(\partial_x) \equiv c_{jmlk}\varepsilon_{ijm}\frac{\partial}{\partial x_l}, \quad i,k=1,2,3;\\
&A_{i+3,k+3}(\partial_x) \equiv c'_{jilk}\frac{\partial^2}{\partial x_j \partial x_l} - c_{jmlp}\varepsilon_{ijm}\varepsilon_{klp}, \quad i,k=1,2,3.
\end{aligned}
$$

Denote by $U=(U_1,\dots,U_6)$ the six-component vector $U_i=u_i$ and $U_{i+3}=\omega_i$ $(i=1,2,3)$. Then the system (1) is written in the matrix form

$$
A(\partial_x)U=0 \quad \big(A_{ik}(\partial_x)U_k=0\big). \tag{4}
$$

2. Let us establish the properties of the fundamental matrix $\Phi = \|\Phi_{ij}\|_{6\times 6}$ of the operator $A(\partial_x)$ in the neighbourhood of the point at infinity. By virtue of the definition of the fundamental matrix we have

$$
A_{ik}(\partial_x)\Phi_{kj}(x)=\delta_{ij}\delta(x), \quad i,j=1,\dots,6,
$$

where δ_{ij} is the Kronecker symbol, δ is the Dirac function. Using the Fourier transform

$$
\hat{\varphi}(\xi) \equiv F[\varphi](\xi) = \int_{\mathbb{R}^3} e^{ix\cdot\xi}\varphi(x)\,dx,
$$

from this equality we obtain

$$
\begin{gathered}
-A_{ik}(\xi)\hat{\Phi}_{kj}(\xi)=\delta_{ij},\\
i,j=1,\dots 6,
\end{gathered}
$$

where

$$
\begin{gathered}
A(\xi) \equiv \|A_{ik}(\xi)\|_{6\times 6},\\
A_{ik}(\xi)=c_{jilk}\xi_j\xi_l, \quad A_{i,k+3}(\xi)=ic_{jilm}\varepsilon_{klm}\xi_j,\\
A_{i+3,k}(\xi)=-ic_{jmlk}\varepsilon_{ijm}\xi_l,\\
A_{i+3,k+3}(\xi)=c'_{jilk}\xi_j\xi_l+c_{jmlp}\varepsilon_{ijm}\varepsilon_{klp},\\
i,k=1,2,3.
\end{gathered} \tag{5}
$$

The matrix $A(\xi)$ is the invertible one if $|\xi| \equiv (\xi_i\xi_i)^{1/2} \neq 0$. Indeed, if $|\xi| \neq 0$ and $\eta_i \equiv \xi_i/|\xi|$, then

$$\begin{aligned} &\det A(\xi) = |\xi|^6 \det B(\eta, |\xi|), \\ &B(\eta,\rho) \equiv \|B_{ik}(\eta,\rho)\|_{6\times 6}, \\ &B_{ik}(\eta,\rho) = A_{ik}(\eta), \quad i \leq 3 \text{ or } k \leq 3; \\ &B_{i+3,k+3}(\eta,\rho) = \rho^2 c'_{jilk}\eta_j\eta_l + c_{jmlp}\varepsilon_{ijm}\varepsilon_{klp}, \quad i \leq 3, \quad k \leq 3. \end{aligned} \tag{6}$$

Now we will prove that $\det B(\eta,\rho) \neq 0$ for $\eta \neq 0$. Consider the expression

$$B_{ik}(\eta,\rho)U_iU_k = c_{jilk}\eta_j\eta_l u_i u_k + \rho^2 c'_{jilk}\eta_j\eta_l\omega_i\omega_k + c_{jmlp}\varepsilon_{ijm}\varepsilon_{klp}\omega_i\omega_k.$$

By virtue of (3) we have the estimates

$$\begin{aligned} &c_{jilk}\eta_j\eta_l u_i u_k \geq c_0(\eta_j u_i)(\eta_j u_i) = |\eta|^2 c_o u_i u_i, \\ &c'_{jilk}\eta_j\eta_l\omega_i\omega_k \geq c_0(\eta_j\omega_i)(\eta_j\omega_i) = |\eta|^2 c_o\omega_i\omega_i, \\ &c_{jmlp}\varepsilon_{ijm}\varepsilon_{klp}\omega_i\omega_k \geq c_0(\varepsilon_{ijm}\omega_i)(\varepsilon_{kjm}\omega_k) = 2c_o\omega_i\omega_i \end{aligned}$$

for some positive number c_0. Therefore

$$B_{ik}(\eta,\rho)U_iU_k \geq |\eta^2|c_0 u_i u_i + |\eta|^2\rho^2 c_0\omega_i\omega_i + 2c_0\omega_i\omega_i > 0$$

if $U \neq 0$ and $\eta \neq 0$. Therefore $\det B(\eta,\rho) \neq 0$ for $\eta \neq 0$.

Let us represent $\det B(\eta, r)$ as follows:

$$\det B(\eta,\rho) = \sum_{k=0}^{6} a_k(\eta)\rho^k,$$

where $a_k(\eta)$ are homogeneous polynomials of η of order $k+6$. In particular,

$$a_0(\eta) = \det B(\eta, 0).$$

As proved above,

$$a_0(\eta) \neq 0, \quad \sum_{k=0}^{6} a_k(\eta)\rho^k \neq 0 \quad \text{for } \eta \neq 0. \tag{7}$$

Write $\det A(\xi)$ in the form

$$\det A(\xi) = |\xi|^6 \sum_{k=0}^{6} a_k(\eta)|\xi|^k. \tag{8}$$

By virtue of (7) $A(\xi)$ is the invertible matrix for $|\xi| \neq 0$. Therefore

$$\widehat{\Phi}_{ik}(\xi) = -A_{ik}^{-1}(\xi), \quad i,k = 1,\ldots,6.$$

Let us now estimate the elements of the matrix $\widehat{\Phi}(\xi)$. First we will prove the validity of the representation

$$\widehat{\Phi}_{ik}(\xi) = \widehat{\Phi}^{(1)}_{ik}(\xi) + \widehat{\Phi}^{(2)}_{ik}(\xi), \qquad i,k = 1,\dots,6, \tag{9}$$

where $\widehat{\Phi}^{(1)}_{ik}(\xi)$ are homogeneous functions of order -2 for $i,k = 1,2,3$, of order -1 for $i = 1,2,3$ and $k = 4,5,6$, and for $i = 4,5,6$ and $k = 1,2,3$; of order 0 for $i,k = 4,5,6$; $\widehat{\Phi}^{(2)}_{ik}(\xi)$ admits the estimates

$$\begin{aligned} &|\partial^\alpha \widehat{\Phi}^{(2)}_{ik}(\xi)| \le c_\alpha |\xi|^{-|\alpha|-1}, \quad i \le 3,\ k \le 3; \\ &|\partial^\alpha \widehat{\Phi}^{(2)}_{ik}(\xi)| \le c_\alpha |\xi|^{-|\alpha|}, \quad i \le 3,\ k \ge 4 \text{ or } i \ge 4,\ k \le 3; \\ &|\partial^\alpha \widehat{\Phi}^{(2)}_{ik}(\xi)| \le c_\alpha |\xi|^{1-|\alpha|}, \quad i \ge 4,\ k \ge 4, \end{aligned} \tag{10}$$

$|\xi| \ne 0$, and α is an arbitrary multiindex, $c_\alpha = \text{const}$.

Let $F \equiv (F_1,\dots,F_6)$ be some vector and $V_i \equiv A^{-1}_{ik} F_k$. Repeating the above arguments for the matrix $B(\eta, r)$, we can readily prove

$$A_{ik}(\xi) V_i V_k \ge c_0 |\xi|^2 V_i V_i + 2c_0 \sum_{i=1}^{4} V_i^2. \tag{11}$$

Let us fix the index p. If $F_k = \delta_{kp}$, $k = 1,\dots,6$, then $V_i = A^{-1}_{ik}(\xi)\delta_{kp} = A^{-1}_{ip}(\xi)$ $(i = 1,\dots,6)$. The substitution of the obtained value of V_i in (11) leads to

$$A^{-1}_{pp}(\xi) \ge c_0 |\xi|^2 \sum_{k=1}^{6} \left(A^{-1}_{kp}(\xi)\right)^2 + 2c_0 \sum_{k=4}^{6} \left(A^{-1}_{kp}(\xi)\right)^2. \tag{12}$$

Hence

$$\begin{gathered} \sum_{k,p=1}^{6} \left(A^{-1}_{kp}(\xi)\right)^2 \le \frac{1}{c_0 |\xi|^2} \sum_{p=1}^{6} A^{-1}_{pp}(\xi) \le \frac{c_1}{|\xi|^2} \left(\sum_{k,p=1}^{6} \left(A^{-1}_{kp}(\xi)\right)^2 \right)^{1/2}, \\ |A^{-1}_{kp}(\xi)| \le c_1 |\xi|^{-2}, \quad c_1 = \text{const}, \quad k,p = 1,\dots,6. \end{gathered} \tag{13}$$

From (12) and (13) yield

$$\begin{gathered} \sum_{k=4}^{6} \left(A^{-1}_{kp}(\xi)\right)^2 \le \frac{1}{2c_0} A^{-1}_{pp}(\xi) \le \frac{c_1}{2c_0} |\xi|^{-2}, \\ |A^{-1}_{kp}(\xi)| \le c_2 |\xi|^{-1}, \quad k = 4,5,6; \ p = 1,\dots,6. \end{gathered} \tag{14}$$

Since $A_{ik}(\xi) = A_{ki}(-\xi)$ $(i,k = 1,\dots,6)$, from (14) we have

$$|A^{-1}_{kp}(\xi)| \le c_2 |\xi|^{-1}, \quad c_1 = \text{const}, \quad k = 1,\dots,6; \ p = 4,5,6. \tag{14'}$$

Considering (12) for $p \geq 4$, we obtain

$$\sum_{k,p=4}^{6} \left(A_{kp}^{-1}(\xi)\right)^2 \leq \frac{1}{2c_0} \sum_{p=4}^{6} A_{pp}^{-1}(\xi) \leq c_1 \Big(\sum_{k,p=4}^{6} \left(A_{kp}^{-1}(\xi)\right)^2 \Big)^{1/2}.$$

Therefore

$$|A_{kp}^{-1}(\xi)| \leq c_1, \quad k,p = 4,5,6. \tag{15}$$

Now we will prove the representation (9). Let $i \leq 3$ and $k \leq 3$. Write $\widehat{\Phi}_{ik}(\xi)$ in the form

$$\widehat{\Phi}_{ik}(\xi) = -A_{ik}^{-1}(i\xi) = -\frac{M_{ik}(\xi)}{\det A(\xi)},$$

where $M_{ik}(\xi)$ is the cofactor of the element $A_{ik}(\xi)$ in the matrix $A(\xi)$. Therefore $M_{ik}(\xi)$ is the polynomial of ξ. Since $\det A(\xi) = |\xi|^6 \det B(\eta, |\xi|)$ and $|\widehat{\Phi}_{ik}(\xi)| \leq c|\xi|^{-2}$, it is obvious that $M_{ik}(\xi)$ is represented in the form

$$M_{ik}(\xi) = |\xi|^4 \sum_{j=0}^{6} b_j^{ik}(\eta)|\xi|^j,$$

where $b_j(\eta)$ is the homogeneous polynomial of η of order $j+4$ ($j = 1,\dots,6$). Thus

$$\widehat{\Phi}_{ik}(\xi) = -\frac{1}{|\xi|^2} \frac{\sum_{j=0}^{6} b_j^{ik}(\eta)|\xi|^j}{\sum_{j=0}^{6} a_j(\eta)|\xi|^j}.$$

Setting

$$\begin{aligned} \widehat{\Phi}_{ik}^{(1)}(\xi) &= -\frac{1}{|\xi|^2} \cdot \frac{b_0^{ik}(\eta)}{a_0(\eta)} = -\frac{1}{|\xi|^2} \frac{b_0^{ik}\left(\frac{\xi}{|\xi|}\right)}{a_0\left(\frac{\xi}{|\xi|}\right)}, \\ \widehat{\Phi}_{ik}^{(2)}(\xi) &= -\frac{1}{|\xi|} \frac{\sum_{j=1}^{6}(a_0(\eta)b_j^{ik}(\eta) - b_0^{ik}(\eta)a_j(\eta))|\xi|^{j-1}}{a_0(\eta)\sum_{j=0}^{6} a_j(\eta)|\xi|^j}, \end{aligned} \tag{16}$$

we obtain the required representation (9), since $\widehat{\Phi}_{ik}^{(1)}(\xi)$ is a homogeneous function of ξ of order -2 and $\widehat{\Phi}_{ik}^{(2)}(\xi)$ satisfies the condition (10) for $i,k = 1,2,3$.

In a similar manner one can prove the validity of the representation (9) for the rest of i and k.

Let us now estimate the matrix $\Phi(x)$. From the equality (9) we have

$$\Phi_{ik}(x) = \Phi_{ik}^{(1)}(x) + \Phi_{ik}^{(2)}(x), \quad i,k = 1,\dots,6. \tag{17}$$

The first term in (17) is the inverse Fourier transform of the homogeneous function $\hat{\Phi}^{(1)}_{ik}(\xi)$, and therefore $\Phi^{(1)}_{ik}(\xi)$ is a homogeneous function of order $-3-q$, where q is the order of the homogeneous function $\hat{\Phi}^{(1)}_{ik}(\xi)$. Thus for $\Phi^{(1)}_{ik}(\xi)$ we have the estimates

$$\begin{aligned} &|\partial^\alpha \Phi^{(1)}_{ik}(x)| \leq c|x|^{-|\alpha|-1}, \\ &\qquad i \leq 3, \quad k \leq 3; \\ &|\partial^\alpha \Phi^{(1)}_{ik}(x)| \leq c|x|^{-|\alpha|-2}, \\ &\qquad i \leq 3, \quad k \geq 4 \ \text{ or } \ i \geq 3, \quad k \leq 4; \\ &|\partial^\alpha \Phi^{(1)}_{ik}(x)| \leq c|x|^{-|\alpha|-3}, \\ &\qquad i, \ k \geq 4, \quad c = \text{const}. \end{aligned} \tag{18}$$

Next we will estimate the second term in (17). It will be shown that in the neighbourhood of the point at infinity

$$\begin{aligned} &\partial^\alpha \Phi^{(2)}_{ik}(x) = o(|x|^{-|\alpha|-1}), \\ &\qquad i \leq 3, \quad k \leq 3; \\ &\partial^\alpha \Phi^{(2)}_{ik}(x) = o(|x|^{-|\alpha|-2}), \\ &\qquad i \leq 3, \quad k \geq 4 \ \text{ or } \ i \geq 3, \quad k \leq 4; \\ &\partial^\alpha \Phi^{(2)}_{ik}(x) = o(|x|^{-|\alpha|-3}), \\ &\qquad i \geq 4, \quad k \geq 4. \end{aligned} \tag{19}$$

We introduce the functions ω_0 and ω_1, where $\omega_1 = 1 - \omega_0$ and ω_0 possesses the following properties:

$$\omega_0 \in C^\infty(\mathbb{R}^3), \quad \operatorname{supp} \omega_0 \subset B(0,1), \quad \omega_0(x) = 1 \ \text{ if } \ |x| \leq \frac{1}{2}.$$

Here $B(0,1)$ is the ball with centre 0 and radius 1 in $\mathbb{R}^3$. Obviously,

$$\hat{\Phi}^{(2)}_{ik}(\xi) = \hat{\Phi}^{(2)}_{ik}(\xi)\omega_0(\xi) + \hat{\Phi}^{(2)}_{ik}(\xi)\omega_1(\xi),$$

$$\Phi^{(2)}_{ik}(x) = \overset{0}{\Phi}{}^{(2)}_{ik}(x) + \overset{1}{\Phi}{}^{(2)}_{ik}(x),$$

where

$$\overset{0}{\Phi}{}^{(2)}_{ik}(x) = F^{-1}[\hat{\Phi}^{(2)}_{ik}\omega_0](x), \quad \overset{1}{\Phi}{}^{(2)}_{ik}(x) = F^{-1}[\hat{\Phi}^{(2)}_{ik}\omega_1](x).$$

F^{-1} is the inverse Fourier transform operator.

Let $i \leq 3$, $k \leq 3$ and $|\beta| < \alpha + 2$. Then by virtue of (10) the function $\partial^\beta(\xi^\alpha \hat{\Phi}^{(2)}_{ik}(\xi)\omega_0(\xi))$ is absolutely integrable on $\mathbb{R}^3$ and therefore the inverse Fourier transform of this function tends to zero at infinity, but

$$F^{-1}[\partial^{\beta}(\xi^{\alpha}\hat{\Phi}^{(2)}_{ik}(\xi)\omega_0(\xi))](x) = (-1)^{|\alpha|} i^{|\alpha|+|\beta|} x^{\beta}\partial^{\alpha} \overset{0}{\Phi}{}^{(2)}_{ik}(x).$$

Thus, if $|\beta| = |\alpha| + 1$, then

$$\lim_{|x|\to\infty} x^{\beta}\partial^{\alpha} \overset{0}{\Phi}{}^{(2)}_{ik}(x) = 0$$

and therefore

$$\partial^{\alpha} \overset{0}{\Phi}{}^{(2)}_{ik}(x) = o(|x|^{-|\alpha|-1}). \tag{20}$$

Let us estimate $\overset{1}{\Phi}{}^{(2)}_{ik}(x)$. If $n \geq |\alpha| + 2$, then

$$\Delta^{n}(\xi^{\alpha}\hat{\Phi}^{(2)}_{ik}(\xi)\omega_1(\xi)) \in L_1(\mathbb{R}^3)$$

and the Fourier transform of this expression tends to zero at infinity:

$$(-1)^{n}|x|^{2n}\partial^{\alpha} \overset{1}{\Phi}{}^{(2)}_{ik}(x) = o(1).$$

Therefore for any $n \geq |\alpha| + 2$

$$\partial^{\alpha} \overset{1}{\Phi}{}^{(2)}_{ik}(x) = o(|x|^{-2n}). \tag{21}$$

(20) and (21) imply the first estimate in (19). The rest of the estimates are proved in the same manner.

3. The derivation of the asymptotic representation formula for the solution of the system (1) in the neighbourhood of the point at infinity is based on the Green formulas. We will give these formulas.

Let Ω be a bounded domain in $\mathbb{R}^3$ with a piecewise-smooth boundary $\partial\Omega$, $U = (U_1,\dots,U_6)$, $V = (V_1,\dots,V_6)$, $U \in C^2(\bar{\Omega})$ and $V \in C^2(\bar{\Omega})$. Then

$$\int_{\Omega}(V_i(x)A_{ik}U_k(x) - U_k(x)A_{ki}(\partial_x)V_i(x))dx =$$
$$= \int_{\partial\Omega}(V_i(y)T_{ik}(\partial_y,\nu)U_k(y) - U_k(y)T_{ki}(\partial_y,\nu)V_i(y))d_yS, \tag{22}$$

where $T(\partial_y, \nu) \equiv \|T_{ik}(\partial_y)\|_{6\times 6}$ is the boundary stress operator defined on $\partial\Omega$ by the relations

$$\begin{aligned} T_{ik}(\partial_y, \nu) &= c_{jilk}\nu_j \frac{\partial}{\partial y_l}, \\ T_{i,k+3}(\partial_y, \nu) &= -c_{jilm}\nu_j \varepsilon_{klm}, \\ T_{i+3,k}(\partial_y, \nu) &= 0, \\ T_{i+3,k+3}(\partial_y, \nu) &= c'_{jilk}\nu_j \frac{\partial}{\partial y_l}, \quad i,k = 1,2,3. \end{aligned} \tag{23}$$

Here $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit normal to $\partial\Omega$ at the point y, external with respect to Ω.

If $U = (U_1, \dots, U_6)$ is the solution of the system (1) in the domain Ω, belonging to the class $C^2(\Omega) \cap C^1(\bar{\Omega})$, then $\forall x \in \Omega$:

$$\begin{aligned} u_j(x) = \int_{\partial\Omega} (U_i(y) T_{ik}(\partial_y, \nu)\Phi_{kj}(y - x) - \\ -\Phi_{kj}(y - x) T_{ki}(\partial_y, \nu) U_i(y)) dy\, S. \end{aligned} \tag{24}$$

The formulas (22) and (24) are proved by the standard techniques [2], [6], [8]

4. Let us formulate the theorem of the asymptotic representation of a solution of the system (1) in the neighbourhood of the point at infinity.

Theorem 1. *Let Ω be a domain from $\mathbb{R}^3$ containing the neighbourhood of the point at infinity, U be a solution of the system (1) in Ω of the class $C^2(\Omega)$, and*

$$U_i(z) = o(|z|^{p+1}), \quad i = 1, \dots, 6 \tag{25}$$

in the neighbourhood of the point at infinity, where p is a nonnegative integer number. Then the representation

$$\begin{aligned} U_j(x) = \sum_{|\alpha| \le p} c_j^{(\alpha)} x^\alpha + \sum_{|\beta| \le q} d_k^{(\beta)} \partial^\beta \Phi_{jk}(x) + \psi_j(x) \\ j = 1, \dots, 6 \end{aligned} \tag{26}$$

holds in the neigbourhood of the point at infinity. Here $c_j^{(\alpha)}$ and $d_k^{(\beta)}$ are the constants, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are the multiindexes, q is an arbitrary nonnegative integer number, and the function ψ_j admits the estimate

$$\partial^\gamma \psi_j(x) = O(|x|^{-2-|\gamma|-q}), \quad j = 1, \dots, 6 \tag{27}$$

in the neighbourhood of $|x| = \infty$, *where* $\gamma \equiv (\gamma_1, \gamma_2, \gamma_3)$ *is an arbitrary multiindex.*

Moreover, each of three terms on the right-hand side of (26) *is the solution of the system* (1) *in the neighbourhood of* $|x| = \infty$.

Proof. Let $x \in \Omega$ and a positive number r be chosen such that $x \in B(0, r/8)$ and $\mathbb{R}^3 \setminus B(0, r/8) \subset \Omega$. Write the formula (24) for the domain $\Omega_r \equiv B(0,r) \cap \Omega$. We will have

$$\begin{aligned} U_j(x) = \int_{\partial\Omega} \Big(U_i(y) T_{ik}(\partial_y, \nu) \Phi_{kj}(y-x) - \\ -\Phi_{kj}(y-x) T_{ki}(\partial_y, \nu) U_i(y)\Big) d_y S + \\ + \int_{\partial B(0,r)} \Big(U_i(y) T_{ik}(\partial_y, \nu) \Phi_{kj}(y-x) - \\ -\Phi_{kj}(y-x) T_{ki}(\partial_y, \nu) U_i(y)\Big) d_y S. \end{aligned} \tag{28}$$

Represent $\Phi_{kj}(y-x)$, in the neighbourhood of the point y, by the Taylor's formula

$$\begin{aligned} \Phi_{kj}(y-x) &= \sum_{|\alpha| \le p+1} \frac{(-1)^{|\alpha|} x^\alpha}{\alpha!} \partial^\alpha \Phi_{kj}(y) + R_{kj}(x,y), \\ R_{kj}(x,y) &= \sum_{|\alpha| \le p+2} \frac{(-1)^{|\alpha|} x^\alpha}{\alpha!} \partial^\alpha \Phi_{kj}(y - \theta x), \quad 0 < \theta < 1. \end{aligned} \tag{29}$$

By virtue of (18) and (19) we readily ascertain that the estimates

$$\begin{aligned} &|\partial_y^\beta R_{kj}(x,y)| \le a^{\beta,p}(r) |y|^{-p-|\beta|-3}, \\ &\qquad k,j \le 3; \\ &|\partial_y^\beta R_{kj}(x,y)| \le a^{\beta,p}(r) |y|^{-p-|\beta|-4}, \\ &\qquad k \le 3,\ j \ge 4 \ \text{ or } \ k \ge 4,\ j \le 3; \\ &|\partial_y^\beta R_{kj}(x,y)| \le a^{\beta,p}(r) |y|^{-p-|\beta|-4}, \\ &\qquad k,j \ge 4 \end{aligned} \tag{30}$$

are fulfilled for any x and y satisfying the conditions $|x| < r/8$ and $r/4 \le |y| \le r$. Taking into account (29), from (28) we obtain

$$U_j(x) = U_j^{(0)}(x) + \sum_{|\alpha| \le p+1} \frac{(-1)^{|\alpha|} c_j^{(\alpha)}(r)}{\alpha!} x^\alpha + I_j(p, r, x), \quad j = 1, \dots, 6, \tag{31}$$

where

$$U_j^{(0)}(x) \equiv \int_{\partial\Omega} \Big(U_i(y)T_{ik}(\partial_y,\nu)\Phi_{kj}(y-x) - $$
$$- \Phi_{kj}(y-x)T_{ki}(\partial_y,\nu)U_i(y)\Big)d_yS, \tag{32}$$

$$c_j^{(\alpha)}(r) \equiv \int_{\partial\Omega} \Big(U_i(y)T_{ik}(\partial_y,\nu)\partial^\alpha\Phi_{kj}(y) - $$
$$- \partial^\alpha\Phi_{kj}(y)T_{ki}(\partial_y,\nu)U_i(y)\Big)d_yS, \tag{33}$$

$$I_j(p,r,x) \equiv \int_{\partial B(0,r)} \Big(U_i(y)T_{ik}(\partial_y,\nu)R_{kj}(x,y) - $$
$$- R_{kj}(x,y)T_{ki}(\partial_y,\nu)U_i(y)\Big)d_yS. \tag{34}$$

It is not difficult to prove (cf. [6]) that $c_j^{(\alpha)}(r)$ does not depend on r, and, introducing the notation

$$c_j^{(\alpha)} \equiv \frac{(-1)^{|\alpha|}}{\alpha!}c_j^{(\alpha)}(r),$$

we obtain the equality

$$U_j(x) = U_j^{(0)}(x) + \sum_{|\alpha|\le p+1} c_j^{(\alpha)}x^\alpha + I_j(p,r,x),$$

from which we conclude that $I_j(p,r,x)$ does not depend on r either. Thus, if we prove

$$\lim_{r\to\infty} I_j(p,r,x) = 0,$$

we will obtain

$$U_j(x) = U_j^{(0)}(x) + \sum_{|\alpha|\le p+1} c_j^{(\alpha)}x^\alpha. \tag{35}$$

Let the function $\omega:\mathbb{R}^3\to\mathbb{R}$, $\omega\in C_0^\infty(\mathbb{R}^3)$, $\operatorname{supp}\omega\subset B(0,3)\backslash B(0,1/3)$, $\omega(y)=1$ for $1/2<|y|<2$. Then the estimate

$$|\partial^\alpha\omega^{(r)}(y)| \le b^{(\alpha)}r^{-|\alpha|} \tag{36}$$

holds for the function $\omega^{(r)}(y)\equiv\omega(y/r)$.

Rewriting the formula (22) for the domain $B(0,r)\backslash B(0,r/4)$, in which V is replaced by the function

$$V \equiv \Big(R_{1j}^{(r)}(x,\cdot),\dots,R_{6j}^{(r)}(x,\cdot)\Big),$$
$$R_{kj}^{(r)}(x,y) \equiv \omega^{(r)}(y)R_{kj}(x,y),$$

we obtain

$$I_j(p,r,x) = \int_{B(0,r)\setminus B(0,r/4)} U_i(z) A_{ik}(\partial_z) R_{kj}^{(r)}(x,z)\, dz, \qquad (37)$$
$$j = 1,\dots,6.$$

On account of (30) we have the estimates

$$|A_{ik}(\partial_z) R_{kj}(x,z)| \le a(x)|z|^{-p-5}, \quad i \le 3,\ j \le 3;$$
$$|A_{ik}(\partial_z) R_{kj}(x,z)| \le a(x)|z|^{-p-6}, \quad i \le 3,\ j \ge 4;$$
$$|A_{ik}(\partial_z) R_{kj}(x,z)| \le a(x)|z|^{-p-4}, \quad i \ge 4,\ j \le 3;$$
$$|A_{ik}(\partial_z) R_{kj}(x,z)| \le a(x)|z|^{-p-5}, \quad i \ge 4,\ j \ge 4.$$

Taking these estimates and restrictions (25) into account, we obtain

$$\lim_{r\to\infty} I_j(p,r,x) = 0.$$

The representation (35) is thus derived. Note that, due to (25), in the formula (35) the constants $c_j^{(\alpha)} = 0$ if $\alpha = p+1$, and therefore we have the representation

$$U_j(x) = U_j^{(0)}(x) + \sum_{|\alpha|\le p} c_j^{(\alpha)} x^\alpha.$$

Let us transform this representation in the form (26). To this effect, in the formula (32) we will represent $\Phi_{kj}(y-x)$ by the Taylor's formula. Since $A_{kj}(\xi) = A_{jk}(-\xi)$, we have $A_{kj}^{-1}(\xi) = A_{jk}^{-1}(-\xi)$, and therefore $\Phi_{kj}(y-x) = \Phi_{jk}(x-y)$. Choose a positive number r_0 such that $\mathbb{R}^3\setminus B(0,r_0) \subset \Omega$. Then, if $y \in \partial\Omega$ and $x \in \mathbb{R}^3\setminus B(0,2r_0)$, we will have the expansion

$$\Phi_{kj}(y-x) = \Phi_{jk}(x-y) =$$
$$= \sum_{|\alpha|\le q} \frac{(-1)^{|\alpha|} y^\alpha}{\alpha!} (\partial^\alpha \Phi_{jk})(x) + \psi_{jk}(x,y),$$
$$\psi_{kj}(x,y) = \sum_{|\alpha|=q+1} \frac{(-1)^{q+1} y^\alpha}{\alpha!} (\partial^\alpha \Phi_{jk})(x-\theta y), \qquad (38)$$
$$0 < \theta < 1.$$

Applying the estimates (18), (19), we show

$$
\begin{aligned}
&|\partial_x^\beta \psi_{jk}(x,y)| \le c_{jk}^{(\beta)}(y)|x|^{-q-|\beta|-2}, \\
&\qquad j \le 3, \quad k \le 3; \\
&|\partial_x^\beta \psi_{jk}(x,y)| \le c_{jk}^{(\beta)}(y)|x|^{-q-|\beta|-3}, \\
&\qquad j \le 3, \quad k \ge 4 \ \text{ or } \ j \ge 4, \quad k \le 3; \\
&|\partial_x^\beta \psi_{jk}(x,y)| \le c_{jk}^{(\beta)}(y)|x|^{-q-\beta-4}, \\
&\qquad j \ge 4, \quad k \ge 4.
\end{aligned}
\tag{39}
$$

The substitution of (38) in (32) gives

$$
U_j^{(0)}(x) = \sum_{|\alpha| \le q} d_k^{(\alpha)} \partial^\alpha \Phi_{jk}(x) + \psi_j(x), \tag{40}
$$

$$
\psi_j(x) = (-1)^q \sum_{|\alpha|=q} \int_{\partial\Omega} U_i(y)\frac{y^\alpha}{\alpha!} T_{ik}(\partial x, \nu)(\partial^\alpha \Phi_{jk})(x) d_y S -
$$

$$
- \int_{\partial\Omega} (U_i(y) T_{ik}(\partial x, \nu)\psi_{jk}(x,y) - \psi_{jk}(x,y) T_{ki}(\partial y, \nu) U_i(y)) d_y S.
$$

Now, due to (18), (19) and (39), we obtain

$$
|\partial^\gamma \psi_j(x)| \le c_j^{(\gamma)} |x|^{-|\gamma|-2-q}, \quad j = 1, \dots, 6. \ \blacksquare
$$

Remark. Theorem 1 can also be proved when the condition (25) is replaced by the conditions of Theorem 2 from [6].

5. Theorem 1 can be used, in particular, to prove uniqueness theorems for the external boundary value problems of the couple-stress theory of elasticity, and to weaken the restrictions imposed on the class of solutions. As an example, let us consider the first external problem:

In the domain Ω^- with the piecewise-smooth boundary $\partial\Omega$, find a solution U of the system (1) of the class $C^1(\bar{\Omega}) \cap C^2(\Omega)$, satisfying the boundary condition

$$
\forall y \in \partial\Omega : \lim_{\Omega^- \ni x \to y} U(x) = \varphi(y)
$$

and the condition at infinity

$$
\lim_{|x| \to \infty} U(x) = 0.
$$

Theorem 2. *The first external problem of the couple-stress theory of elasticity has one solution at most.*

Proof. Let U be a solution of the first external problem. Then the expansion (26) holds for U. Setting $p = 0$, $q = 0$ in (26), we obtain the equality

$$U_j(x) = c_j^{(0)} + d_k^{(0)}\Phi_{jk}(x) + \psi_j(x), \quad j = 1, \dots, 6.$$

All terms on the right-hand side of this equality, except $c_j^{(0)}$, tend to zero as $|x| \to \infty$. Therefore $c_j^{(0)} = 0$, $j = 1, \dots, 6$. Now we conclude from (18), (19), (27) that

$$\partial^\alpha U_j(x) = O(|x|^{-|\alpha|-1}), \quad j = 1, 2, 3;$$
$$\partial^\alpha U_j(x) = O(|x|^{-|\alpha|-2}), \quad j = 4, 5, 6.$$

Now repeating the arguments, say, from [2], we readily obtain the proof of Theorem 2. ■

The uniqueness theorems for the other external boundary value problems of the couple-stress elasticity are proved in a quite similar manner.

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ON THE CORRECT FORMULATION OF ONE MULTIDIMENSIONAL PROBLEM FOR STRICTLY HYPERBOLIC EQUATIONS OF HIGHER ORDER

S. KHARIBEGASHVILI

ABSTRACT. A theorem of the unique solvability of the first boundary value problem in the Sobolev weighted spaces is proved for higher order strictly hyperbolic systems in the conic domain with special orientation.

რეზიუმე. მაღალი რიგის მკაცრად ჰიპერბოლური განტოლებისათვის გარკვეული ორიენტაციის მქონე კონუსურ არეში დამტკიცებულია პირველი სასაზღვრო ამოცანის ცალსახად ამოხსნადობის თეორემა სობოლევის წონით სივრცეებში.

In the space R^n, $n > 2$, let us consider a strictly hyperbolic equation of the form

$$p(x,\partial)u(x) = f(x), \tag{1}$$

where $\partial = (\partial_1, \dots, \partial_n)$, $\partial_j = \partial/\partial x_j$, $p(x,\xi)$ is a real polynomial of order $2m$, $m > 1$, with respect to $\xi = (\xi_1, \dots, \xi_n)$, f is the known and u is the unknown function. It is assumed that in the equation (1) the coefficients at higher derivatives are constant and the other coefficients are finite and infinitely differentiable in R^n.

Let D be a conic domain in R^n, i.e. D together with a point $x \in D$ contains the entire beam tx, $0 < t < \infty$. Denote by Γ the cone ∂D. It is assumed that D is homeomorphic onto the conic domain $x_1^2 + \cdots + x_{n-1}^2 - x_n^2 < 0$, $x_n > 0$ and $\Gamma' = \Gamma \backslash O$ is a connected $(n-1)$-dimensional manifold of the class C^∞, where O is the vertex of the cone Γ.

Consider the problem: Find in the domain D the solution $u(x)$ of

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the equation (1) by the boundary conditions

$$\left.\frac{\partial^i u}{\partial \nu^i}\right|_{\Gamma'} = g_i, \quad i = 0, \dots, m-1, \tag{2}$$

where $\nu = \nu(x)$ is the outward normal to Γ' at a point $x \in \Gamma'$, g_i, $i = 0, \dots, m-1$, are the known real functions.

Note that the problem (1), (2) is considered in [1–6] for one hyperbolic-type equation of second order when Γ is a characteristic conoid. In [7] this problem is considered for a wave equation when the conic surface Γ is not characteristic at any point and has a time-type orientaton. A multidimensional analogue of the problem is treated in [8–10] for the case when one part of the cone Γ is characteristic and the other part is a time-type hyperplane. Other multidimensional analogues of the Goursat problem for hyperbolic systems of first and second order are investigated in [11–15].

In this paper we consider the question whether the problem (1), (2) can be correctly formulated in special weighted spaces $W^k_\alpha(D)$ when the cone Γ is assumed not to be characteristic but having a quite definite orientation.

Denote by $p_0(\xi)$ the characteristic polynomial of the equation (1), i.e. the higher homogeneous part of the polynomial $p(x, \xi)$. The strict hyperbolicity of the equation (1) implies the existence of a vector $\zeta \in R^n$ such that the straight line $\xi = \lambda\zeta + \eta$, where $\eta \in R^n$ is an arbitrarily chosen vector not parallel to ζ and λ is the real parameter, intersects the cone of normals $K : p_0(\xi) = 0$ of the equation (1) at $2m$ different real points. In other words, the equation $p_0(\lambda\zeta + \eta) = 0$ with respect to λ has $2m$ different real roots. The vector ζ is called a spatial-type normal. As is well-known, a set of all spatial-type normals form two connected centrally-symmetric convex conic domains whose boundaries K_1 and K_{2m} give the internal cavity of the cone of normals K [3]. The surface $S \subset R^n$ is called characteristic at a point $x \in S$ if the normal to S at the point x belongs to the cone K.

Let the vector ζ be a spatial-type normal and the vector $\eta \neq 0$ change in the plane orthogonal to ζ. Then for λ the roots of the characteristic polynomial $p_0(\lambda\zeta + \eta)$ can renumerated so that $\lambda_{2m}(\eta) < \lambda_{2m-1}(\eta) < \cdots < \lambda_1(\eta)$. It is obvious that the vectors $\lambda_i(\eta)\zeta + \eta$ cover the cavities K_i of K when the η changes on the plane othogonal to ζ. Since $\lambda_{m-j}(\eta) = -\lambda_{m+j+1}(-\eta)$, $0 \leq j \leq m-1$, the cones K_{m-j} and K_{m+j+1} are centrally symmetric with respect to the point $(0, \dots, 0)$. As is well-known, by the bicharacteristics of the equation (1) we understand straight beams whose orthogonal planes are tangential planes to one of the cavities K_i at the point different from the vertex.

Assume that there exists a plane π_0 such that $\pi_0 \cap K_m = \{(0,\dots,0)\}$. This means that the cones $K_1, \dots, K_m$ are located on one side of π_0 and the cones $K_{m+1}, \dots, K_{2m}$ on the other. Set $K_i^* = \cap_{\eta \in K_i}\{\xi \in R^n : \xi \cdot \eta < 0\}$, where $\xi \cdot \eta$ is the scalar product of ξ and η. Since $\pi_0 \cap K_m = \{(0,\dots,0)\}$, K_i^* is a conic domain and $K_m^* \subset K_{m-1}^* \subset \dots \subset K_1^*$, $K_{m+1}^* \subset K_{m+2}^* \subset \dots \subset K_{2m}^*$. It is easy to verify that $\partial(K_i^*)$ is a convex cone whose generatrices are bicharacteristics; note that in this case none of the bicharacteristics of the equation (1) comes from the point $(0,\dots,0)$ into the cone $\partial(K_m^*)$ or $\partial(K_{m+1}^*)$ [3].

Let us consider

Condition 1. The surface Γ' is characteristic at none of its points and each generatrix of the cone Γ has the direction of a spatial-type normal; moreover, $\Gamma \subset K_m^* \cup 0$ or $\Gamma \subset K_{m+1}^* \cup 0$.

Denote by $W_\alpha^k(D)$, $k \geq 2m$, $-\infty < \alpha < \infty$, the functional space with the norm [16]

$$\|u\|^2_{W_\alpha^k(D)} = \sum_{i=0}^{k} \int_D r^{-2\alpha-2(k-i)} \left\| \frac{\partial^i u}{\partial x^i} \right\|^2 dx,$$

where

$$r = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}, \quad \frac{\partial^i u}{\partial x^i} = \frac{\partial^i u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \quad i = i_1 + \dots + i_n.$$

The space $W_\alpha^k(\Gamma)$ is defined in a similar manner.

Consider the space

$$V = W_{\alpha-1}^{k+1-2m}(D) \times \prod_{i=0}^{m-1} W_{\alpha-\frac{1}{2}}^{k-i}(\Gamma).$$

Assume that to the problem (1), (2) there corresponsponds the unbounded operator

$$T : W_\alpha^k(D) \to V$$

with the definition domain $\Omega_T = W_{\alpha-1}^{k+1}(D) \subset W_\alpha^k(D)$, acting by the formula

$$Tu = \left(p(x,\partial)u, u\Big|_{\Gamma'}, \dots, \frac{\partial^i u}{\partial \nu^i}\Big|_{\Gamma'}, \dots, \frac{\partial^{m-1} u}{\partial \nu^{m-1}}\Big|_{\Gamma'} \right), \quad u \in \Omega_T.$$

It is obvious that the operator T admits the closure $\overline{T}$.

The function u is called a strong solution of the problem (1), (2) of the class $W_\alpha^k(D)$ if $u \in \Omega_{\overline{T}}$, $\overline{T}u = (f, g_0, \dots, g_{m-1}) \in V$, which is equivalent to the existence of a sequence $u_i \in \Omega_T = W_{\alpha-1}^{k+1}(D)$

such that $u_i \to u$ in $W^k_\alpha(D)$ and $\left(p(x,\partial)u_i, u_i|_{\Gamma'}, \dots, \frac{\partial^{m-1}u_i}{\partial\nu^{m-1}}\Big|_{\Gamma'}\right) \to (f, g_0, \dots, g_{m-1})$ in V.

Below, by a solution of the problem (1), (2) of the class $W^k_\alpha(D)$ we will mean a strong solution of this problem in the sense as indicated above.

We will prove

Theorem. *Let condition 1 be fulfilled. Then there exists a real number $\alpha_0 = \alpha_0(k) > 0$ such that for $\alpha \geq \alpha_0$ the problem (1), (2) is uniquely solvable in the class $W^k_\alpha(D)$ for any $f \in W^{k+1-2m}_{\alpha-1}(D)$, $g_i \in W^{k-i}_{\alpha-\frac{1}{2}}(\Gamma)$, $i = 0, \dots, m-1$, and to obtain the solution u we have the estimate*

$$\|u\|_{W^k_\alpha(D)} \leq c\Big(\sum_{i=1}^{m-1} \|g_i\|_{W^{k-i}_{\alpha-\frac{1}{2}}(\Gamma)} + \|f\|_{W^{k+1-2m}_{\alpha-1}(D)}\Big), \tag{3}$$

where c is a positive constant not depending on f, g_i, $i = 0, \dots, m-1$.

Proof. First we will show that the corollaries of condition 1 are the conditions as follows: Take any point $P \in \Gamma'$ and choose a Cartesian system $x_1^0, \dots, x_n^0$ connected with this point and having vertex at P such that the x_n^0-axis be directed along the generatrix of Γ passing through P and the x_{n-1}^0-axis be directed along the inward normal to Γ at this point.

Condition 2. The surface Γ' is characteristic at none of its points. Each generatrix of the cone Γ has the direction of a spatial-type normal, and exactly m characteristic planes of the equation (1) pass through the $(n-2)$-dimensional plane $x_n^0 = x_{n-1}^0 = 0$ connected with an arbitrary point $P \in \Gamma'$ into the angle $x_n^0 > 0$, $x_{n-1}^0 > 0$.

Denote by $\widetilde{p_0}(\xi)$ the characteristic polynomial of the equation (1) written in terms of the coordinate system $x_1^0, \dots, x_n^0$, connected with an arbitrarily chosen point $P \in \Gamma'$.

Condition 3. The surface Γ' is characteristic at none of its point. Each generatrix of the cone Γ has the direction of a spatial-type normal and for $Re\, s > 0$ the number of roots $\lambda_j(\xi_1, \dots, \xi_{n-2}, s)$, if we take into account the multiplicity of the polynomial $\widetilde{p_0}(i\xi_1, \dots, i\xi_{n-2}, \lambda, s)$ with $Re\, \lambda_j < 0$, is equal to m, $i = \sqrt{-1}$.

When condition 3 is fulfilled, the polynomial $\widetilde{p_0}(i\xi_1, \dots, i\xi_{n-2}, \lambda, s)$ can be written as the product $\Delta_-(\lambda)\Delta_+(\lambda)$, where for $Re\, s > 0$ the roots of the polynomials $\Delta_-(\lambda)$ and $\Delta_+(\lambda)$ lie, respectively, to the left and to the right of the imaginary axis, while the coefficients are continuous for s, $Re\, s \geq 0$, $(\xi_1, \dots, \xi_{n-2}) \in R^{n-2}$, $\xi_1^2 + \cdots + \xi_{n-2}^2 + |s|^2 = 1$ [17]. On the left side of the boundary conditions (2) to the differential operator $b_j(x, \partial)$, $0 \leq j \leq m-1$, written in terms of the coordinate system $x_1^0, \dots, x_n^0$ connected with the point $P \in \Gamma'$, there corresponds the characteristic polynomial $b_j(\xi) = \xi_{n-1}^j$. Therefore, since the degree of the polynomial $\Delta_-(\lambda)$ is equal to m, the following condition will be fulfilled:

Condition 4. For any point $P \in \Gamma'$ and any s, $Re\, s \geq 0$, and $(\xi_1, \dots, \xi_{n-2}) \in R^{n-2}$ such that $\xi_1^2 + \cdots + \xi_{n-2}^2 + |s|^2 = 1$, the polynomials $b_j(i\xi_1, \dots, i\xi_{n-2}, \lambda, s) = \lambda^j$, $j = 0, \dots, m-1$, are linearly independent like the polynomials of λ modulo $\Delta_-(\lambda)$.

We will now show that condition 1 implies condition 2, while the latter implies condition 3. Let us consider the case $\Gamma \subset K_{m+1}^* \cup O$. The second case $\Gamma \subset K_m^* \cup O$ is treated similarly.

Let $P \in \Gamma'$ and $x_1^0, \dots, x_n^0$ be the coordinate system connected with this point. Since the generatrix γ of the cone Γ passing through this point is a spatial-type normal, the plane $x_n^0 = 0$ passing through the point P is a spatial-type plane. Denote by $K_j^\wedge$ the boundary of the convex shell of the set K_j and by $K_j^\perp$ the set which is the union of all bicharacteristics corresponding to the cone K_j and coming out of the point O along the outward normal to K_j, $1 \leq j \leq 2m$. It is obvious that $(K_j^\wedge)^* = K_j^*$, $\partial(K_j^*) = (K_j^\wedge)^\perp$. We will show that the plane π_1, parallel to the plane $x_n^0 = 0$ and passing through the point $(0, \dots, 0)$, is the plane of support to the cone $K_m^\wedge$ at the point $(0, \dots, 0)$. Indeed, it is obvioius that the plane $N \cdot \xi = 0$, $N \in R^n \backslash (0, \dots, 0)$, $\xi \in R^n$ is the plane of support to $K_m^\wedge$ at the point $(0, \dots, 0)$ iff the normal vector N to this plane taken with the sign $+$ or $-$ belongs to the conic domain closure $(K_m^\wedge)^* = K_m^*$. Now it remains for us to note that the conic domains K_m^* and K_{m+1}^* are centrally symmetric with respect to the point $(0, \dots, 0)$, and the generatrix Γ passing through the point P is perpendicular to the plane π_1 and, by the condition, belongs to the set $K_{m+1}^* \cup O$. Since $x_n^0 = 0$ is a spatial-type plane, the two-dimensional plane $\sigma : x_1^0 = \cdots = x_{n-2}^0 = 0$ passing through the generatrix γ which is directed along the spatial-type normal intersects the cone of normals K_p of the equation (1) with vertex at the point P by $2m$ different real straight lines [3]. The planes orthogonal to these straight lines and passing through $(n-2)$-dimensional plane $x_n^0 = x_{n-1}^0 = 0$, give all $2m$ characteric planes passing through the $(n-2)$-dimensional plane $x_n^0 = x_{n-1}^0 = 0$. The straight lines $x_n^0 = 0$ and $x_{n-1}^0 = 0$ divide the two-dimensional plane σ into four right angles

$$\sigma_1 : x^0_{n-1} > 0,\ x^0_n > 0;\quad \sigma_2 : x^0_{n-1} < 0,\ x^0_n > 0;$$
$$\sigma_3 : x^0_{n-1} < 0,\ x^0_n < 0;\quad \sigma_4 : x^0_{n-1} > 0,\ x^0_n < 0.$$

One can readily see that exactly m characteristic planes of the equation (1) pass through the $(n-2)$-dimensional plane $x^0_n = x^0_{n-1} = 0$ into the angle $x^0_n > 0$, x^0_{n-1} iff exactly m straight lines from the intersection of σ_4 with the two-dimensional plane σ pass into the angle K_P. The latter fact really occurs, since: 1) the plane $x^0_n = 0$ is the plane of support to $K^\wedge_m$ and therefore to all $K_1, \dots, K_{2m}$; 2) the planes $x^0_n = 0$, $x^0_{n-1} = 0$ are not characteristic because the generatrices of Γ have a spatial-type direction and Γ is not characteristic at the point P.

Now it will be shown that condition 2 implies condition 3. By virtue of condition 2 the plane $x^0_{n-1} = 0$ is not characteristic and therefore for λ the polynomial $\widetilde{p_0}(i\xi_1, \dots, i\xi_{n-2}, \lambda, s)$ has exactly $2m$ roots. In this case, if $Re\, s > 0$, the number of roots $\lambda_j(\xi_1, \dots, \xi_{n-2}, s)$, with the multiplicity of the polynomial $\widetilde{p_0}(i\xi_1, \dots, i\xi_{n-2}, \lambda, s)$ taken into account, will be equal to m provided that $Re\, \lambda_j < 0$. Indeed, recalling that the equation (1) is hyperbolic, the equation $\widetilde{p_0}(i\xi_1, \dots, i\xi_{n-2}, \lambda, s) = 0$ has no purely imaginary roots with respect to λ. Since the roots λ_j are continuous functions of s, we can determine the number of roots λ_j with $Re\, \lambda_j < 0$ by passing to the limits as $Re\, s \to +\infty$. Since the equality

$$\widetilde{p}_0(i\xi_1, \dots, i\xi_{n-2}\lambda, s) = s^{2m}\widetilde{p}_0\Big(i\frac{\xi_1}{s}, \dots, i\frac{\xi_{n-2}}{s}, \frac{\lambda}{s}, 1\Big)$$

holds, it is clear that the ratios λ_j/s , where λ_j are the roots of the equation $\widetilde{p_0}(i\xi_1, \dots, i\xi_{n-2}, \lambda, s) = 0$, tend to the roots μ_j of the equation $\widetilde{p_0}(0, \dots 0, \mu, 1) = 0$ as $Re\, s \to +\infty$. The latter roots are real and different because the equation (1) is hyperbolic. If s taken positive and sufficiently large, then for $\mu_j \neq 0$ we have $\lambda_j = s\mu_j + o(s)$. But $\mu_j \neq 0$, since the plane $x^0_n = 0$ is not characteristic. Therefore the number of roots λ_j with $Re\, \lambda_j < 0$ coincides with the number of roots μ_j with $\mu_j < 0$. Since the characteristic planes of the equation (1), passing through the $(n-2)$-dimensional plane $x^0_n = x^0_{n-1} = 0$, are determined by the equalities $\mu_j x^0_{n-1} + x^0_n = 0$, $j = 1, \dots, 2m$, condition 2 implies that for $Re\, \lambda_j < 0$ the number of roots λ_j is equal to m.

We will another equivalent description of the space $W^k_\alpha(D)$. On the unit sphere $S^{n-1} : x_1^2 + \cdots + x_n^2 = 1$ choose a coordinate system

$(\omega_1, \dots, \omega_{n-1})$ such that in the domain D the transformation

$$I : \tau = \log r, \ \ \omega_j = \omega_j(x_1, \dots, x_n), \ \ j = 1, \dots, n-1,$$

be one-to-one, nondegenerate and infinitely differentiable. Since the cone $\Gamma = \partial D$ is strictly convex at the point $O(0, \dots, 0)$, such coordinates evidently exist. As a result of the above transformation, the domain D will become the infinite cylinder G bounded by the infinitely differentiable surface $\partial G = I(\Gamma')$.

Introduce the functional space $H^k_\gamma(G)$, $-\infty < \gamma < \infty$, with the norm

$$\|v\|^2_{H^k_\gamma(G)} = \sum_{i_1+j=0}^{k} \int_G e^{-2\gamma\tau} \left\| \frac{\partial^{i_1+j} v}{\partial \tau^{i_1} \partial \omega^j} \right\|^2 d\omega\, d\tau$$

where

$$\frac{\partial^{i_1+j} v}{\partial \tau^{i_1} \partial \omega^j} = \frac{\partial^{i_1+j} v}{\partial \tau^{i_1} \partial \omega_1^{j_1} \cdots \partial \omega_{n-1}^{j_{n-1}}}, \ \ j = j_1 + \dots + j_{n-1}.$$

As shown in [16], a function $u(x) \in W^k_\alpha(D)$ iff $\tilde{u} = u(I^{-1}(\tau, \omega)) \in H^k_{(\alpha+k)-\frac{n}{2}}(G)$, and the estimates

$$c_1 \|\tilde{u}\|_{H^k_{(\alpha+k)-\frac{n}{2}}(G)} \le \|u\|_{W^k_\alpha(D)} \le c_2 \|\tilde{u}\|_{H^k_{(\alpha+k)-\frac{n}{2}}(G)}$$

hold, where I^{-1} is the inverse transformation of I and the positive constants c_1 and c_2 do not depend on u.

It can be easily verified that the condition $v \in H^k_\gamma(G)$ is equivalent to the condition $e^{-\gamma\tau} v \in W^k(G)$, where $W^k(G)$ is the Sobolev space. Denote by $H^k_\gamma(\partial G)$ a set of ψ such that $e^{-\gamma\tau}\psi \in W^k(\partial G)$, and by $W^k_{\alpha-\frac{1}{2}}(\Gamma)$ a set of all φ for which $\tilde{\varphi} = \varphi(I^{-1}(\tau, \omega)) \in H^k_{(\alpha+k)-\frac{n}{2}}(\partial G)$. Assume that

$$\|\varphi\|_{W^k_{\alpha-\frac{1}{2}}(\Gamma)} = \|\tilde{\varphi}\| H^k_{(\alpha+k)-\frac{n}{2}}(\partial G).$$

Spaces $W^k_\alpha(D)$ possess the following simple properties:

1) if $u \in W^k_\alpha(D)$, then $\frac{\partial^i u}{\partial x^i} \in W^{k-i}_\alpha(D)$, $0 \le i \le k$;

2) $W^{k+1}_{\alpha-1}(D) \subset W^k_\alpha(D)$;

3) if $u \in W^{k+1}_{\alpha-1}(D)$, then by the well-known embedding theorems we have $u\big|_\Gamma \in W^k_{\alpha-\frac{1}{2}}(\Gamma)$, $\frac{\partial^i u}{\partial \nu^i}\big|_{\Gamma'} \in W^{k-i}_{\alpha-\frac{1}{2}}(\Gamma)$, $i = 1, \dots, m-1$;

4) if $u \in W^{k+1}_{\alpha-1}(D)$, then $f = p(x, \partial)u \in W^{k+1-2m}_{\alpha-1}(D)$.

In what follows we will need, in spaces $W^k_\alpha(D)$, $W^k_{\alpha-\frac{1}{2}}(\Gamma)$ other norms depending on the parameter $\gamma = (\alpha + k) - \frac{n}{2}$ and equivalent to the original norms.

Set

$$R^n_{\omega,\tau} = \{-\infty < \tau < \infty,\ -\infty < \omega_i < \infty,\ i = 1, \dots, n-1\},$$
$$R^n_{\omega,\tau,+} = \{(\omega,\tau) \in R^n_{\omega,\tau} : \omega_{n-1} > 0\},\ \omega' = (\omega_1, \dots, \omega_{n-2}),$$
$$R^{n-1}_{\omega',\tau} = \{-\infty < \tau < \infty,\ -\infty < \omega_i < \infty,\ i = 1, \dots, n-2\}.$$

Denote by $\tilde{v}(\xi_1, \dots, \xi_{n-2}, \xi_{n-1}, \xi_n - i\gamma)$ the Fourier transform of the function $e^{-\gamma\tau} v(\omega, \tau)$, i.e.,

$$\tilde{v}(\xi_1, \dots, \xi_{n-1}, \xi_n - i\gamma) = (2\pi)^{-\frac{n}{2}} \int v(\omega,\tau) e^{-i\omega\xi' - i\tau\xi_n - \gamma\tau} d\omega\, d\tau,$$
$$i = \sqrt{-1},\quad \xi' = (\xi_1, \dots, \xi_{n-1}),$$

and by $\hat{v}(\xi, \dots, \xi_{n-2}, \omega_{n-1}, \xi_n - i\gamma)$ the partial Fourier transform of the function $e^{-\gamma\tau} v(\omega,\tau)$ with respect to ω', τ.

We can introduce the following equivalent norms

$$|||v|||^2_{R^n,k,\gamma} = \int_{R^n} (\gamma^2 + |\xi|^2)^k \|\tilde{v}(\xi_1, \dots, \xi_{n-1}, \xi_n - i\gamma)\|^2 d\xi,$$
$$|||v|||^2_{R^n_+,k,\gamma} = \int_0^\infty \int_{R^{n-1}} \sum_{j=0}^k (\gamma^2 + |\xi'|^2)^{k-j} \times$$
$$\times \left\| \frac{\partial^j}{\partial \omega^j_{n-1}} \hat{v}(\xi_1, \dots, \xi_{n-2}, \omega_{n-1}, \xi_n - i\gamma) \right\|^2 d\xi' d\omega_{n-1},$$

in the above-considered spaces $H^k_\gamma(R^n_{\omega,\tau})$ and $H^k_\gamma(R^n_{\omega,\tau,+})$.

Let $\varphi_1, \dots, \varphi_N$ be the partitioning of unity into $G' = G \cap \{\tau = 0\}$, where $G = I(D)$, i.e., $\sum_{j=1}^N \varphi_j(\omega) \equiv 1$ in $G', \varphi_j \in C^\infty(\overline{G}')$, the supports of functions $\varphi_1, \dots, \varphi_{N-1}$ lie in the boundary half-neighbourhoods, while the support of function φ_N inside G'. Then for $\gamma = (\alpha + k) - \frac{n}{2}$ the equalities

$$|||u|||^2_{G,k,\gamma} = \sum_{j=1}^{N-1} |||\varphi_j u|||^2_{R^n_{+,k,\gamma}} + |||\varphi_N u|||^2_{R^n,k,\gamma},$$
$$|||u|||^2_{\partial G,k,\gamma} = \sum_{j=1}^{N-1} |||\varphi_j u|||^2_{R^{n-1}_{\omega',\tau,k,\gamma}} \tag{4}$$

define equivalent norms in the spaces $W^k_\alpha(D)$ and $W^k_{\alpha-\frac{1}{2}}(\Gamma)$, where the norms on the right sides of these equalities are taken in the terms of local coordinates [17].

First we assume that the equation (1) contains only higher terms, i.e. $p(x,\xi) \equiv p_0(\xi)$. The equation (1) and the boundary conditions

(2) written in terms of the coordinates ω, τ have the form

$$e^{-2m\tau}A(\omega,\partial)u = f,$$

$$e^{-i\tau}B_i(\omega,\partial)u\Big|_{\partial G} = g_i, \quad i = 0,\dots,m-1,$$

or

$$A(\omega,\partial)u = \tilde{f}, \tag{5}$$

$$B_i(\omega,\partial)u\Big|_{\partial G} = \tilde{g}_i, \quad i = 0,\dots,m-1, \tag{6}$$

where $A(\omega,\partial)$ and $B_i(\omega,\partial)$ are respectively the differential operators of orders $2m$ and i, with infinitely differentiable coefficients depending only on ω, while $\tilde{f} = e^{2m\tau}f$ and $\tilde{g}_i = e^{i\tau}g_i$, $i = 0,1,\dots,m-1$.

Thus, for the transformation $I : D \to G$, the unbounded operator T of the problem (1), (2) transforms to the unbounded operator

$$\tilde{T} : H^k_\gamma(G) \to H^{k+1-2m}_\gamma(G) \times \prod_{i=0}^{m-1} H^{k-i}_\gamma(\partial G)$$

with the definition domain $H^{k+1}_\gamma(G)$, acting by the formula

$$\tilde{T}u = \left(A(\omega,\partial)u, B_0(\omega,\partial)u\Big|_{\partial G},\dots,B_{m-1}(\omega,\partial)u\Big|_{\partial G}\right)$$

where $\gamma = (\alpha + k) - \frac{n}{2}$. Note that written in terms of the coordinates ω, τ the functions $f = (\omega,\tau) \in H^{k+1-2m}_{\gamma-2m}(G)$, $g_i(\omega,\tau) \in H^{k-i}_{\gamma-i}(\partial G)$, $i = 0,\dots,m-1$, if $f(x) \in W^{k+1-2m}_{\alpha-1}(D)$, $g_i(x) \in W^{k-i}_{\alpha-\frac{1}{2}}(\Gamma)$, $i = 0,\dots,m-1$. Therefore the functions $\tilde{f} = e^{2m\tau}f \in H^{k+1-2m}_\gamma(G)$, $\tilde{g}_i = e^{i\tau}g_i \in H^{k-i}_\gamma(\partial G)$, $i = 0,\dots,m-1$.

Since by condition 1 each generatrix of the cone Γ has the direction of a spatial-type normal, due to the convexity of K_m each beam coming from the vertex O into the conic domain D also has the direction of a spatial-type normal. Therefore the equation (4) is strictly hyperbolic with respect the τ-axis. It was asown above that the fulfilment of condition 1 implies the fulfilment of condition 4. Therefore, according to the results of [17], for $\gamma \geq \gamma_0$, where γ_0 is a sufficiently large number, the operator $\overline{\tilde{T}}$ has the bounded right inverse operator $\overline{\tilde{T}}^{-1}$. Thus for any $\tilde{f} \in H^{k+1-2m}_\gamma(G)$, $\tilde{g}_i \in H^{k-i}_\gamma(\partial G)$, $i = 0,\dots,m-1$, when $\gamma \geq \gamma_0$, the problem (5), (6) is uniquely solvable in the space $H^k_\gamma(G)$

and for the solution u we have the estimate

$$|||u|||^2_{G,k,\gamma} \leq C\Big(\sum_{i=0}^{m-1} |||\tilde{g}_i|||_{\partial G,k-i,\gamma} + \frac{1}{\gamma}|||\tilde{f}|||_{G,k+1-2m,\gamma}\Big) \qquad (7)$$

with the positive constant C not depending on γ, f and $\tilde{g}_i$, $i = 0,\dots,m-1$.

Hence it immediately follows that the theorem and the estimate (3) are valid in the case $p(x,\xi) \equiv p_0(\xi)$. ■

Remark. The estimate (7) with the coefficient $\frac{1}{\gamma}$ at $|||\tilde{f}|||_{G,k+1-2m,\gamma}$, obtained in the appropriately chosen norms (4), enables one to prove the theorem also when the equation (1) contains lower terms, since the latter give arbitrarily small perturbations for sufficiently large γ.

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ON COHOMOTOPY-TYPE FUNCTORS

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Abstract. The paper deals with Chogoshvili cohomotopy functors which are defined by extending a cohomology functor given on some special auxiliary subcategories of the category of topological spaces. The question of choosing these subcategories is discussed. In particular, it is shown that in the singular case to define absolute groups it is sufficient that auxiliary subcategories should have as objects only spheres S^n, Moore spaces $P^n(t) = S^{n-1} \cup_t e^n$ and one-point unions of these spaces.

რეზიუმე. განხილულია ჭოღოშვილის კოჰომოტოპიური ფუნქტორები, რომლებიც განიმარტება ტოპოლოგიურ სივრცეთა კატეგორიის სპეციალურ დამხმარე ქვეკატეგორიებიდან კოჰომოლოგიის ფუნქტორის გავრცელებით მთელ კატეგორიამდე. შესწავლილია ამ ქვეკატეგორიების შერჩევის საკითხი. კერძოდ, ნაჩვენებია, რომ სინგულარულ შემთხვევაში აბსოლუტური ჯგუფების განმარტებისათვის დამხმარე ქვეკატეგორიების ობიექტებად საკმარისია სფეროები S^n, მურის სივრცეები $P^n(t) = S^{n-1} \cup_t e^n$ და ამ სივრცეთა თაიგულები.

In [2, 3] for any cohomology theory $H = \{H^n\}$ given on some category K of pairs of topological spaces the sequence

$$\Pi = \{\Pi^n\}, \quad n = 0, 1, 2, \ldots$$

of contravariant functors Π^n is constructed from K into the category of abelian groups with the coboundary operator $\delta^\#$ which commutes with the induced homomorphisms $\varphi^\#$, $\varphi \in K$. Functors Π^n possess the properties of semi-exactness and homotopy and are connected with H by the natural transformations

$$d : H^n \to \Pi^n$$

which are the natural equivalences on a certain subcategory K_n of K. Constructing of such functors is reduced to the problem of extending the functor given on an auxiliary subcategory K_n to the whole category

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K. The problem is solved by means of the theory of inverse systems of groups with sets of homomorphisms of Hurewiz, Dugundji and Dowker [4].

Functors Π^n are dual to the homotopy functors associated in the sense of Bauer [1] with a given homological structure. It should be noted that functors Π^n have some of the basic properties of the Borsuk cohomotopy, but they differ from the latter.

In [5-7] functors Π^n were investigated under the assumption that K is the category of pairs of topological spaces with a base point and base point preserving maps, and H is the singular integral theory of cohomology. We will adhere to the same assumption throughout this paper (base points are not indicated here). To define functors Π^n we need auxiliary subcategories K_n. We have to consider a problem of choosing these subcategories.

For convenience we recall the definition of a limit of the inverse system of groups with sets of homomorphisms (see [4]). Let ω be a partially ordered set, and $\{G_\sigma\}$ a system of abelian groups indexed by the elements of ω. Besides, let for each pair $\rho < \sigma$ sets $H_{\sigma\rho} \subset Hom\,(G_\sigma, G_\rho)$ be given such that if $\rho < \sigma < \tau$ and $\varphi_1 \in H_{\sigma\rho}$, $\varphi_2 \in H_{\tau\sigma}$, then the composition $\varphi_1\varphi_2 \in H_{\tau\rho}$. Then, by definition, $\varprojlim G_\sigma$ is a subgroup of the group ΠG_σ and its elements are elements $g = \{g_\sigma\} \in \Pi G_\sigma$ such that for each pair $\rho < \sigma$ and $\varphi \in H_{\sigma\rho}$ we have $\varphi(g_\sigma) = g_\rho$.

It should be noted that this theory of [4] essentialy is Kan's extension theory on it's early stage, however quite sufficicient for our purpose.

The results of this paper earlier were announced in [6,7]

1. Preliminaries

In this section we will give the definitions of subcategories K_n and functors Π^n and discuss some of their properties.

Let e^m be the unit m-cell of the m-dimensional euclidean space $\mathbb{R}^m$. By e^0 we denote some fixed point (base point). Let $\widetilde{K}$ be the small full subcategory of K whose objects are all finite CW-complexes X for which $X^0 = e^0$ and X^k is the adjunction space obtained by adjoining a finite number of e^k to X^{k-1}, $k > 0$. We denote by $\widetilde{\widetilde{K}}$ the small full subcategory of K whose objects are all CW-pairs (X, X') for which X and X' are the objects of $\widetilde{K}$.

Now we shall define auxiliary subcategories K_n, $n > 3$, by the following two conditions (cf. [2, 5]):

1) K_n is an arbitrary small full subcategory of K; each object of K_n is a pair (X, X') of linearly- and simply-connected spaces, satisfing the conditions that $\pi_2(X, X') = 1$, the homology modules $H_*(X)$ and $H_*(X')$ are of finite type, $H^i(X, X') = 0$, $i < n$, and $H^i(X) = H^i(X') = 0$, $0 < i < n-1$;

2) K_n contains all possible objects of $\widetilde{\widetilde{K}}$.

We denote by F_n^r an auxiliary subcategory of objects only of $\widetilde{\widetilde{K}}$.

If $n = 3$ we assume that K_3 is an arbitrary (containing all possible objects of $\widetilde{K}$) small full subcategory of K whose all objects are linearly- and simply-connected spaces X for which $H_*(X)$ is a module of finite type and $H^2(X) = 0$.

Let (R, R') be an object of K. Consider a set of indices $\omega(R, R'; n)$ of all pairs $\alpha = (X, X'; f)$, where (X, X') is an object of K_n and

$$f : (X, X') \to (R, R')$$

is a continuous map of K. Let $\omega(R, R'; n)$ be ordered as follows: $\alpha < \beta$, where $\beta = (Y, Y'; g)$ if there is a map

$$\varphi : (X, X') \to (Y, Y')$$

of K_n such that

$$g\varphi = f. \tag{1}$$

Assume that to every $\alpha \in \omega(R, R'; n)$ there corresponds the n-dimensional cohomology group $H_\alpha = H^n(X, X')$ and to every ordered pair $\alpha < \beta$ there corresponds the set of homomorphisms $\{\varphi^*\}$, where

$$\varphi^* : H^n(Y, Y') \to H^n(X, X')$$

are the induced homomorphisms in the H theory.

We have obtained the inverse system of the group H_α with sets of homomorphisms. Cohomotopic groups of Chogoshvili are determined by the formula

$$\Pi^n(R, R'; K_n) = \varprojlim H_\alpha.$$

We denote by $\Pi^n(R; K_n)$ the absolute group $\Pi^n(R, *; K_n)$, where $*$ is a base point, and by p_α the α-coordinate of an element $p \in \Pi^n(R, R'; K_n)$. Note that for $n = 3$ we have determined the absolute groups only.

Let

$$\alpha = (X, X'; f), \quad \beta = (X, X'; g),$$
$$\alpha, \beta \in \omega(R, R'; n), \quad p \in \Pi^n(R, R'; K_n)$$

and let the maps f and g be homotopic, i.e., $f \sim g$. Let I be the unit segment.

Lemma 1.1. *If the subcategory K_n contains, alongside with (X, X'), the pair $(X \times I, X' \times I)$, then $p_\alpha = p_\beta$.*

Proof. See [5], p. 83. ■

Let

$$\alpha = (X, X'; f), \quad \beta = (Y, Y'; g),$$
$$\alpha, \beta \in \omega(R, R'; n), \quad p \in \Pi^n(R, R'; K_n).$$

Moreover, let us have a map

$$\varphi : (X, X') \to (Y, Y')$$

from K_n such that the maps $g\varphi$ and f are homotopic, i.e.,

$$g\varphi \sim f. \tag{2}$$

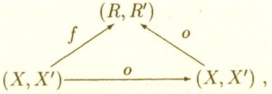
Lemma 1.2. *If the subcategory K_n contains, alongside with (X, X'), the pair $(X \times I, X' \times I)$, then $\varphi^*(p_\beta) = p_\alpha$.*

Proof. Consider the index $(X, X'; g\varphi) = \alpha_1 < \beta$ and apply Lemma 1.1 to α_1 and α. ■

Let $\alpha = (X, X'; f)$, where f is null-homotopic, and $p \in \Pi^n(R, R'; K_n)$.

Corollary 1.3. *If the subcategory K_n contains, alongside with (X, X'), the pair $(X \times I, X' \times I)$, then $p_\alpha = 0$.*

Proof. Apply Lemma 1.2 to the homotopy commutative diagram



where o denotes the constant map. ■

2. Main Results

Let K'_n be a subcategory of K''_n, where K'_n and K''_n are two auxiliary subcategories, and let $(R, R') \in K$, $p \in \Pi^n(R, R'; K''_n)$,

$$\alpha \in {}'\omega(R, R'; n) \subset {}''\omega(R, R'; n).$$

Then, as one can easily verify, the formula $[\lambda(p)]_\alpha = p_\alpha$ defines the restriction homomorphism

$$\lambda : \Pi^n(R, R'; K''_n) \to \Pi^n(R, R'; K'_n).$$

In Section 3 we will prove

Theorem 2.1. *The homomorphism λ defines the natural equivalence of the functors $\Pi^n(-,-; K''_n)$ and $\Pi^n(-,-; K'_n)$, $n > 3$. In particular, all functors $\Pi^n(-,-; K_n)$ are naturally equivalent to the functor $\Pi^n(-,-; F^r_n)$.*

Remark 2.2. Theorem 2.1 shows that in choosing a subcategory K_n we can restrict ourselves only to the finite CW-pairs. On the other hand, from Theorem 2.1 it follows that for the convenience of construction and proof we can regard an arbitrary admissible pair as an object of K_n.

Remark 2.3. One can easily show that Theorem 2.1 holds for the absolute groups when $n > 2$. Moreover, in defining the absolute groups, to choose the subcategory K_n we can restrict ourselves to the absolute pairs $(X, *)$, i.e., $X' = *$ (see the definition of K_n and [5]).

In the remainder of this section we will consider the absolute groups only. Therefore to define the functors Π^n, $n > 2$, we can use auxiliary subcategories F^a_n consisting of finite CW-complexes. More exactly, F^a_n is a full subcategory $\widetilde{K}$ whose objects are all spaces X for which $\pi_1(X) = 1$ and $H^2(X) = \cdots = H^{n-1}(X) = 0$.

We intend here to study the problem dealing with a possibility of further reducing subcategories K_n provided that groups $\Pi^n(R; K_n)$ and the results from [5-7] remain unchanged. To this effect, relying on Lemma 1.2, in the definition of $\Pi^n(R; K_n)$ we replace condition (1) by condition (2) (see Section 1). We will stick to this definition in the sequel.

Let S^n denote the n-dimensional unit sphere of the euclidean space $\mathbb{R}$ and e^n the unit disk. Denote by $P^n(t)$, $t > 1$, $n > 2$ the Moore space $S^{n-1} \cup_t e^n$. Also assume that $P^n(1) = S^n$.

Consider now the full subcategory F_n^b of F_n^a whose objects are all finite one-point unions of spaces $P^n(t)$, $t \geq 1$. The subcategory F_n^b will be regarded as an auxiliary subcategory.

The following theorem will be proved in Section 4.

Theorem 2.4. *The restriction homomorphism defines the natural equivalence of the functors* $\Pi^n(-;F_n^a)$ *and* $\Pi^n(-;F_n^b)$, $n > 2$.

We introduce the notations:

1) $P_j^n(t) = P^n(t)$, where j is a positive integer, $n > 2$, $t \geq 1$;

2) $X_k^n = \vee_{t=1}^k \left(\vee_{j=1}^k P_j^n(t) \right)$;

3) $Q^n = \varinjlim X_k$ (by inclusion maps $X_k^n \to X_{k+1}^n$).

Let Q_n be the full subcategory of K consisting of one object Q^n, $n > 2$. The subcategory will also be regarded as an auxiliary subcategory. Note that the module $H_*(Q^n)$ is not obviously of the finite type. Therefore none of the above-defined auxiliary subcategories contains Q^n.

The following theorem will be proved in Section 5.

Theorem 2.5. *The functors* $\Pi^n(-;F_n^b)$ *and* $\Pi^n(-;Q_n)$ *are naturally equivalent,* $n > 2$.

3. Proof of Theorem 2.1

Let us prove that λ is natural. Assume

$$f : (S, S') \to (R, R')$$

to be an arbitrary map of K. Consider the diagram

$$\begin{array}{ccc} \Pi^n(R, R'; K_n'') & \xrightarrow{\lambda} & \Pi^n(R, R'; K_n') \\ \downarrow f^{\#} & & \downarrow f^{\#} \\ \Pi^n(S, S'; K_n'') & \xrightarrow{\lambda} & \Pi^n(S, S'; K_n') \end{array}$$

Let

$$\alpha = (X, X'; g) \in {}'\omega(S, S', n) \subset {}''\omega(S, S'; n),$$
$$\beta = f(\alpha) = (X, X'; fg) \in {}'\omega(R, R'; n) \subset {}''\omega(R, R'; n)$$

and let $p \in \Pi^n(R, R'; K_n'')$. We have

$$\left[\lambda(f^{\#}(p))\right]_\alpha = \left[f^{\#}(p)\right]_\alpha = p_\beta,$$
$$\left[f^{\#}(\lambda(p))\right]_\alpha = \left[\lambda(p)\right]_\beta = p_\beta,$$

which proves that λ is natural.

Let (X, X') be an arbitrary object of some auxiliary subcategory K_n. Using the standard technique of the homotopy theory, we can construct a CW-pair (A, B) from the subcategory K_n and a map

$$\varphi : (A, B) \to (X, X')$$

such that homomorphisms φ^* induced by φ in the H theory will be isomorphisms up to any pregiven dimension. Let now $p \in \Pi^n(R, R'; K_n)$ and

$$\alpha = (X, X'; f) \in \omega(R, R'; n).$$

Consider the index

$$\beta = (A, B; f\varphi) \in \omega(R, R'; n).$$

Then $\beta < \alpha$ and therefore $p_\beta = \varphi^*(p_\alpha)$. Hence $p_\alpha = \varphi^{*-1}(p_\beta)$. From the above reasoning and the definition of auxiliary subcategories it now follows that if $p \in \Pi^n(R, R'; K_n'')$ and $\lambda(p) = 0$, then $p = 0$. Thus λ is a monomorphism.

Let L_n, $n > 3$, be a full subcategory of the category K whose objects are all pairs (X, X') for which X and X' are linearly- and simply-connected spaces, $\pi_2(X, X') = 1$, the homology modules $H_*(X)$ and $H_*(X')$ are of the finite type, $H^i(X, X') = 0$, $i < n$, and $H^i(X) = H^i(X') = 0$, $0 < i < n - 1$.

Consider some full subcategories of L_n:

1) $L_n^{(1)}$ are CW-pairs;

2) $L_n^{(2)}$ are CW-pairs with a finite number of cells in all demensions;

3) $L_n^{(3)}$ are finite CW-pairs;

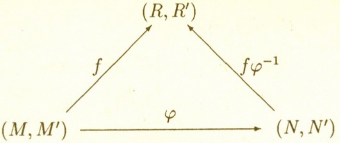
4) $L_n^{(4)} = \widetilde{\widetilde{K}} \cap \mathrm{L}_n^{(3)} = F_n^\tau$.

We will gradually extend the thread defining element $p \in \Pi^n(R, R'; F_n^\tau)$ from the category $L_n^{(4)}$ onto $L_n^{(3)}$, then onto $L_n^{(2)}$, $L_n^{(1)}$ and, finally, onto L_n. Such an extension already implies that λ is epimorphic.

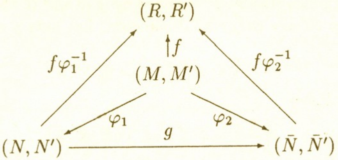
Let (M, M') be an arbitrary object of $L_n^{(3)}$; also let

$$f : (M, M') \to (R, R')$$

be an arbitrary map. Consider the diagram



where $(N, N') \in L_n^{(4)}$ and φ is a homeomorphism. Let $\alpha = (M, M'; f)$ and $\beta = (N, N'; f\varphi^{-1})$. It is assumed that $p_\alpha = \varphi^*(p_\beta)$. We will show that p_α does not depend on a choice of the homeomorphism φ. Consider the diagram



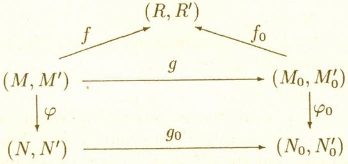
where φ_1 and φ_2 are two different homeomorphisms and $g = \varphi_2\varphi_1^{-1}$. The indices β_1 and β_2 will be defined similarly to β. We have

$$(f\varphi_2^{-1})g = f\varphi_2^{-1}\varphi_2\varphi_1^{-1} = f\varphi_1^{-1}.$$

Therefore $\beta_1 < \beta_2$. Then

$$\varphi_2^*(p_{\beta_2}) = (g\varphi_1)^*(p_{\beta_2}) = \varphi_1^*(g^*(p_{\beta_2})) = \varphi_1^*(p_{\beta_1}).$$

Consider the commutative diagram



where φ and φ_0 are homeomorphisms, $g_0 = \varphi_0 g \varphi^{-1}$.

Let

$$\alpha = (M, M'; f), \quad \alpha_0 = (M_0, M_0'; f_0),$$
$$\beta = (N, N'; f\varphi^{-1}), \quad \beta_0 = (N_0, N_0'; f_0\varphi_0^{-1}).$$

We have

$$(f_0\varphi_0^{-1})g_0 = f_0\varphi_0^{-1}\varphi_0 g\varphi^{-1} = f_0 g\varphi^{-1} = f\varphi^{-1}.$$

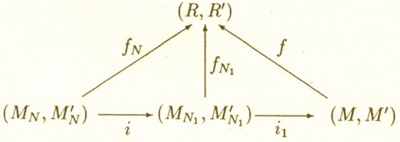
Therefore $\beta < \beta_0$. Then

$$g^*(p_{\alpha_0}) = g^*(\varphi_0^*(p_{\beta_0})) = (\varphi_0 g)^*(p_{\beta_0}) = (g_0\varphi)^*(p_{\beta_0}) = \\ = \varphi^*(g_0^*(p_{\beta_0})) = \varphi^*(p_\beta) = p_\alpha.$$

We have thus extended the thread of the element p onto the category $L_n^{(3)}$.

Let, now, $(M, M') \in L_n^{(2)}$ and $f : (M, M') \to (R, R')$ be an arbitrary map. By $i : X_k \to X$ we denote here the standard embedding, where X_k is the k-skeleton of the CW-complex X. Let $N_1 > N > n + 1$ be arbitrary integers.

Consider the commutative diagram



where $f_{N_1} = fi_1$ and $f_N = f_{N_1}i$. Also consider the indices

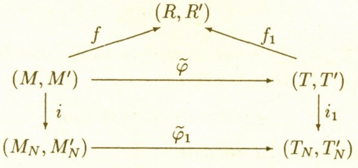
$$\alpha = (M_N, M'_N; f_N), \quad \beta = (M_{N_1}, M'_{N_1}; f_{N_1}), \quad \gamma = (M, M'; f).$$

Assume $p_\gamma = i_1^{*-1}(p_\beta)$. We have

$$(i_1 i)^{*-1}(p_\alpha) = i_1^{*-1}(i^{*-1}(p_\alpha)) = i_1^{*-1}(p_\beta),$$

where the last equality evidently follows from the fact that $\alpha < \beta$. Therefore p_γ does not depend on a choice of the number N_1.

Now consider the diagram



where $\varphi : (M, M') \to (T, T')$ is a map such that $f_1\varphi = f$, $\tilde{\varphi}$ is a cellular approximation of φ and $\widetilde{\varphi_1} = \widetilde{\varphi}|(M_N, M'_N)$. We have $f \sim f_1\widetilde{\varphi}$ and

$$(f_1 i_1)\widetilde{\varphi_1} = (f_1\widetilde{\varphi})i \sim fi.$$

Also consider the indices

$$\alpha = (M_N, M'_N; fi), \quad \beta = (T_N, T'_N; f_1 i_1).$$

Applying Lemma 1.2, we have

$$\varphi^*(i_1^{*-1}(p_\beta)) = (\widetilde{\varphi^* i_1^{*-1}})(p_\beta) = i^{*-1}(\widetilde{\varphi_1}^*(p_\beta)) = i^{*-1}(p_\alpha).$$

We have thus extended the thread of the element p onto the category $L_n^{(2)}$.

Let now $(\bar{M}, \bar{M}')$ be an arbitrary object of $L_n^{(1)}$ and let

$$\bar{g} : (\bar{M}, \bar{M}') \to (R, R')$$

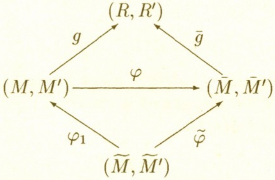
be an arbitrary map. Using the standard technique of the homotopy theory, we can, under our assumptions, construct a map

$$\varphi : (M, M') \to (\bar{M}, \bar{M}')$$

such that $(M, M') \in L_n^{(2)}$ and φ is a homotopy equivalence. Let $g = \bar{g}\varphi$. Assume

$$p_\alpha = \varphi^{*-1}(p_\beta),$$

where $\alpha = (\bar{M}, \bar{M}'; \bar{g})$, $\beta = (M, M'; g)$. We will show that p_α does not depend on a choice of φ. Consider the diagram



where $(\widetilde{M}, \widetilde{M}') \in L_n^{(2)}$, $\widetilde{\varphi}$ and φ_1 are homotopy equivalences and $\varphi \sim \widetilde{\varphi}\varphi_1$. Let $\widetilde{\beta} = (\widetilde{M}, \widetilde{M}'; \bar{g}\widetilde{\varphi})$. Then

$$g = \bar{g}\varphi \sim \bar{g}\widetilde{\varphi}\varphi_1 = (\bar{g}\widetilde{\varphi})\varphi_1.$$

Applying Lemma 1.2, we have

$$\widetilde{\varphi}^{*-1}(p_{\widetilde{\beta}}) = \widetilde{\varphi}^{*-1}(\varphi_1^{*-1}(p_\beta)) = \varphi^{*-1}(p_\beta).$$

Consider now an arbitrary map

$$\varphi_0 : (\bar{M}, \bar{M}') \to (\bar{T}, \bar{T}')$$

from the category $L_n^{(1)}$ and the diagram

$$\begin{array}{ccc}
 & (R,R') & \\
 {}^{\bar{g}}\nearrow & & \nwarrow^{\bar{g}_1} \\
(\bar{M},\bar{M}') & \xrightarrow{\quad \varphi_0 \quad} & (\bar{T},\bar{T}') \\
\downarrow \varphi & & \varphi_1 \uparrow \;\; \downarrow \tilde{\varphi}_1 \\
(M,M') & \xrightarrow{\quad \tilde{\varphi}_0 \quad} & (T,T')
\end{array}$$

where $\bar{g} = \bar{g}_1\varphi_0$, φ and φ_1 are homotopy eqivalences, $\tilde{\varphi}_1$ is the homotopy inverse of φ_1, $\widetilde{\varphi_0} = \widetilde{\varphi_1}\varphi_0\varphi$ and $(M,M'),(T,T') \in L_n^{(2)}$. We have

$$(\bar{g}_1\varphi_1)\widetilde{\varphi_0} = \bar{g}_1\varphi_1\widetilde{\varphi_1}\varphi_0\varphi \sim \bar{g}_1\varphi_0\varphi = \bar{g}\varphi.$$

Also consider the indices

$$\beta = (M, M'; \bar{g}\varphi), \quad \beta_1 = (T, T'; \bar{g}_1\varphi_1).$$

Then, applying Lemma 1.2, we have

$$\varphi_0^*(\varphi_1^{*-1}(p_{\beta_1})) = (\varphi_0^*\varphi_1^{*-1})(p_{\beta_1}) = (\varphi^{*-1}\widetilde{\varphi_0}^*)(p_{\beta_1}) = \varphi^{*-1}(p_\beta).$$

Let, finally, (X, X') be an arbitrary object of L_n and let

$$w_X : (S(X), S(X')) \to (X, X')$$

be the natural projection of the singular complex $S(X)$ onto X. Let

$$f : (X, X') \to (R, R')$$

be an arbitrary map. Also consider the indices

$$\alpha = (X, X'; f), \quad \beta = (S(X), S(X'); f w_X).$$

Assume

$$p_\alpha = w_X^{*-1}(p_\beta)$$

and consider the commutative diagram

$$\begin{array}{ccc} & (R,R') & \\ & {\scriptstyle f}\nearrow \qquad \nwarrow{\scriptstyle g} & \\ (X,X') & \xrightarrow{\quad\varphi\quad} & (Y,Y') \\ \downarrow{\scriptstyle \omega_X} & & \downarrow{\scriptstyle \omega_Y} \\ (S(X),S(X')) & \xrightarrow{\quad\bar{\varphi}\quad} & (S(Y),S(Y')) \end{array}$$

where $g\varphi = f$ and $\bar{\varphi}$ is the cellular map induced by φ. We have

$$(g w_Y)\overline{\varphi} = g\varphi w_X = f w_X.$$

Also consider the indices

$$\beta_X = (S(X), S(X'); f\omega_X), \quad \beta_Y = (S(Y), S(Y'); g\omega_Y).$$

Then

$$\varphi^*\left(\omega_Y^{*-1}(p_{\beta_Y})\right) = \omega_X^{*-1}\left(\bar{\varphi}^*(p_{\beta_Y})\right) = \omega_X^{*-1}(p_{\beta_X}).$$

This completes the proof of Theorem 2.1. ■

4. Proof of Theorem 2.4

Consider some full subcategories of F_n^a (see Section 2):

0) $K_n^{(0)} = F_n^a$;

1) $K_n^{(1)}$-objects are CW-complexes with one vertex and without cells of dimensions $1, 2, \ldots, n-2$;

2) $K_n^{(2)}$-objects are CW-complexes with one vertex and cells of dimensions $n-1$, n and $n+1$ only;

3) $K_n^{(3)}$-objects are CW-complexes with one vertex and cells of dimensions $n-1$ and n only;

4) $K_n^{(4)} = F_n^b$.

Let $R = (R, *)$ be an arbitrary space from K. All subcategories $K_n^{(i)}$, $0 \leq i \leq 4$, will be regarded as auxiliary ones.

Let

$$\lambda_i : \Pi^n(R; K_n^{(i)}) \to \Pi^n(R; K_n^{(i+1)}), \quad 0 \leq i \leq 3,$$

be the natural restriction homomorphisms.

Let L_n' and L_n'' be two small full subcategories of the category K consisting of the spaces $(X, *)$, and $L_n' \subset L_n''$. It is assumed that the following condition is satisfied: for each $X \in L_n''$ there is $Y \in L_n'$ such that Y has the same homotopy type as X. Consider L_n' and L_n'' as auxiliary subcategories. Let

$$\bar{\lambda} : \Pi^n(R; L_n'') \to \Pi^n(R; L_n')$$

be the natural restriction homomorphism.

Lemma 4.1. $\bar{\lambda}$ *is a natural isomorphism.*

Proof. We prove first that the homomorphism $\bar{\lambda}$ is a monomorphism. Let $p \in \Pi^n(R; L''_n)$, $\bar{\lambda}(p) = 0$ and

$$\alpha = (X; f) \in {}''\omega(R; n)$$

be an arbitrary index. Let $Y \in L'_n$ and the map $\varphi : Y \to X$ be a homotopy equivalence. Consider the index

$$\beta = (Y; f\varphi) \in {}'\omega(R; n) \subset {}''\omega(R; n).$$

We have $\beta < \alpha$. Then

$$\varphi^*(p_\alpha) = p_\beta = [\bar{\lambda}(p)]_\beta = 0.$$

Therefore $p_\alpha = 0$ and $p = 0$.

Let us now prove that the homomorphism $\bar{\lambda}$ is an epimorphism. Let $q \in \Pi^n(R; L'_n)$ and

$$\alpha = (X; f) \in {}''\omega(R; n)$$

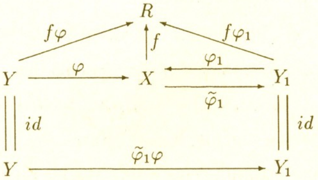
be an arbitrary index. Let $Y \in L'_n$ and the map $\varphi : Y \to X$ be a homotopy equivalence. Consider the index

$$\beta = (Y; f\varphi) \in {}'\omega(R; n)$$

and assume

$$p_\alpha = \varphi^{*-1}(q_\beta).$$

We will show that p_α does not depend on a choice of φ. Consider the diagram



where $Y, Y_1 \in L'_n$, φ and φ_1 are homotopy equivalences, $\tilde{\varphi}_1$ is the homotopy inverse of φ_1. Consider the index

$$\gamma = (Y_1; f\varphi_1) \in {}'\omega(R; n).$$

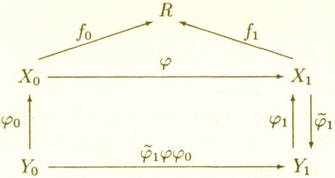
Then $(f\varphi_1)(\tilde{\varphi}_1\varphi) \sim f\varphi$ and therefore $\beta < \gamma$. In this case

$$\varphi_1^*\big(\varphi^{*-1}(q_\beta)\big) = \varphi_1^*\Big(\varphi^{*-1}\big((\tilde{\varphi}_1\varphi)^*(q_\gamma)\big)\Big) =$$
$$= (\varphi_1^*\varphi^{*-1}\varphi^*\tilde{\varphi}_1^*)(q_\gamma) = (\varphi_1^*\tilde{\varphi}_1^*)(q_\gamma) = q_\gamma.$$

Hence

$$\varphi^{*-1}(q_\beta) = \varphi_1^{*-1}(q_\gamma).$$

We will show that the set $\{p_\alpha\}$ defines an element of the group $\Pi^n(R; L_n'')$. Consider the diagram



where $f_1\varphi = f_0$, φ_0 and φ_1 are homotopy equivalences, $\tilde{\varphi}_1$ is the homotopy inverse of φ_1, $X_0, X_1 \in L_n''$ and $Y_0, Y_1 \in L_n'$. Then

$$(f_1\varphi_1)(\tilde{\varphi}_1\varphi\varphi_0) \sim f_1\varphi\varphi_0 = f_0\varphi_0.$$

Also consider the indices

$$\alpha = (X_0; f_0), \quad \beta = (X_1; f_1), \quad \alpha, \beta \in {}''\omega(R; n),$$
$$\alpha_1 = (Y_0; f_0\varphi_0), \quad \beta_1 = (Y_1; f_1\varphi_1), \quad \alpha_1, \beta_1 \in {}'\omega(R; n) \subset {}''\omega(R; n).$$

Then $\alpha < \beta$, $\alpha_1 < \beta_1$ and we have

$$\varphi^*(p_\beta) = \varphi^*\big(\varphi_1^{*-1}(q_{\beta_1})\big) = \varphi_0^{*-1}\big((\tilde{\varphi}_1\varphi\varphi_0)^*(q_{\beta_1})\big) = \varphi_0^{*-1}(q_{\alpha_1}) = p_\alpha.$$

Finally, let us prove that $\lambda(p) = q$. Assume that $X \in L_n' \subset L_n''$ and

$$\alpha = (X; f) \in {}'\omega(R; n) \subset {}''\omega(R; n).$$

Define p_α by taking $\varphi = id : X \to X$. then

$$[\lambda(p)]_\alpha = p_\alpha = id^{*-1}(q_\alpha) = q_\alpha.$$

This completes the proof of Lemma 4.1. ■

As a consequence of the foregoing lemma we have

Proposition 4.2. λ_0 *is a natural isomorphism.*

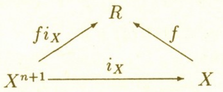
Proof. Every CW-complex from $K_n^{(0)}$ is homotopically equivalent to some CW-complex from $K_n^{(1)}$. ■

Proposition 4.3. λ_1 *is a natural isomorphism.*

Proof. We will prove in the first place that λ_1 is a monomorphism. Let $p \in \Pi^n(R; K_n^{(1)})$ and $\lambda_1(p) = 0$. Consider the index

$$\alpha = (X; f) \in {}^{(1)}\omega(R; n), \quad X \in K_n^{(1)},$$

and the diagram



(3)

where X^{n+1} is the $(n+1)$-skeleton of X and i_X is the standard embedding. Let

$$\beta = (X^{n+1}; f\, i_X) \in {}^{(2)}\omega(R; n) \subset {}^{(1)}\omega(R; n).$$

Then $\beta < \alpha$ and we have

$$i_X^*(p_\alpha) = p_\beta = [\lambda_1(p)]_\beta = 0.$$

Therefore $p = 0$.

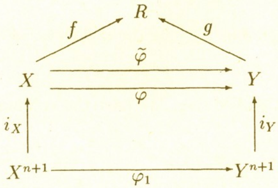
Assume now that $p \in \Pi^n(R; K_n^{(2)})$ and

$$\alpha = (X; f) \in {}^{(1)}\omega(R; n).$$

Consider the diagram (3), the index β and assume that

$$q_\alpha = i_X^{*-1}(p_\beta).$$

We will show that the set $\{q_\alpha\}$ defines an element of $\Pi^n(R; K_n^{(1)})$. Consider the diagram



where $g\varphi \sim f$, $\tilde{\varphi}$ is a cellular approximation of φ and $\varphi_1 = \tilde{\varphi}|X^{n+1}$.

Then

$$(gi_Y)\varphi_1 = g\tilde{\varphi}i_X \sim fi_X.$$

Also consider the indices

$$\alpha_1 = (Y; g) \in {}^{(1)}\omega(R; n),$$
$$\beta_1 = (Y^{n+1}; gi_Y) \in {}^{(2)}\omega(R; n).$$

We have $\beta < \beta_1$ and

$$\varphi^*(q_{\alpha_1}) = \tilde{\varphi}^*(q_{\alpha_1}) = \tilde{\varphi}^*\left(i_Y^{*-1}(q_{\beta_1})\right) = i_X^{*-1}\left(\varphi_1^*(q_{\beta_1})\right) = i_X^{*-1}(q_\beta) = q_\alpha.$$

Finally, let us prove that $\lambda_1(q) = p$.

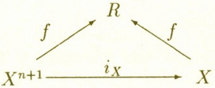
Consider the index

$$\alpha = (X; f) \in {}^{(2)}\omega(R; n),$$

where

$$X = X^{n+1} \in K_n^{(2)} \subset K_n^{(1)}$$

and the diagram



where $i_X = id$. Then

$$[\lambda_1(q)]_\alpha = i_X^{*-1}(p_\alpha) = p_\alpha.$$

This completes the proof of Proposition 4.3. ■

Proposition 4.4. *λ_2 is a natural isomorphism.*

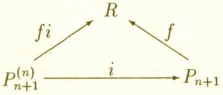
Proof. Let $q \in \Pi^n(R; K_n^{(2)})$ and $\lambda_2(q) = 0$. Consider the index

$$\alpha = (P_{n+1}; f) \in {}^{(2)}\omega(R; n),$$

where $P_{n+1} \in K_n^{(2)}$. Let $P_{n+1}^{(n)}$ be the n-skeleton of P_{n+1} and

$$i : P_{n+1}^{(n)} \to P_{n+1}$$

be the standanrd embedding. We have the commutative diagram



(4)

Also consider the index

$$\beta = (P_{n+1}^{(n)}; fi) \in {}^{(3)}\omega(R;n) \subset {}^{(2)}\omega(R;n).$$

We have $\beta < \alpha$. Let

$$\to H^n(P_{n+1}, P_{n+1}^{(n)}) \to H^n(P_{n+1}) \xrightarrow{i^*} H^n(P_{n+1}^{(n)}) \to$$

be a part of the cohomological exact sequence for the pair $(P_{n+1}, P_{n+1}^{(n)})$. Since $H^n(P_{n+1}, P_{n+1}^{(n)}) = 0$, i^* is a monomorphism. Then

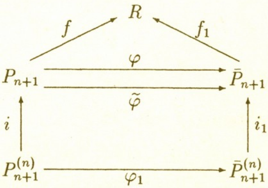
$$i^*(q_\alpha) = q_\beta = [\lambda_2(q)]_\beta = 0.$$

Therefore $q_\alpha = 0$ and $q = 0$. Hence λ_2 is a monomorphism.

Let now $q \in \Pi^n(R; K_n^{(3)})$. Consider again the commutative diagram (4) and the corresponding indices α and β. We will prove below that $q_\beta \in Im\, i^*$. Therefore

$$p_\alpha = i^{*-1}(q_\beta)$$

is the correct definition. Let us show that the set $\{p_\alpha\}$ defines an element $p \in \Pi^n(R; K_n^{(2)})$. Consider the diagram



where $f_1\varphi \sim f$, $\tilde{\varphi}$ is a cellular approximation of φ and $\varphi_1 = \tilde{\varphi}|p_{n+1}^{(n)}$. Consider the indices

$$\alpha_1 = (\bar{p}_{n+1}; f_1) \in {}^{(2)}\omega(R;n), \quad \beta_1 = (\bar{p}_{n+1}^{(n)}; f_1 i_1) \in {}^{(3)}\omega(R;n).$$

Since

$$(f_1 i_1)\varphi_1 = f_1\tilde{\varphi} i \sim fi,$$

we obtain $\beta < \beta_1$ and therefore $\varphi_1^*(q_{\beta_1}) = q_\beta$. We have

$$i^{*-1}(\varphi_1^*(q_{\beta_1})) = \tilde{\varphi}^*(i_1^{*-1}(q_{\beta_1}))$$

and

$$\varphi^*(p_{\alpha_1}) = \tilde{\varphi}^*(p_{\alpha_1}) = \tilde{\varphi}^*(i_1^{*-1}(q_{\beta_1})) = i^{*-1}(\varphi_1^*(q_{\beta_1})) = i^{*-1}(q_\beta) = p_\alpha.$$

Finally, we will prove that $\lambda_2(p) = q$. Let

$$\alpha = (P_{n+1}; f) \in {}^{(3)}\omega(R;n) \subset {}^{(2)}\omega(R;n)$$

be an arbitrary index, $P_{n+1} \in K_n^{(3)}$. Thus $P_{n+1}^{(n)} = P_{n+1}$. Then the map

$$i : P_{n+1}^{(n)} \to P_{n+1}$$

is the identity map: $i = id$. Therefore for $\beta = (P_{n+1}^{(n)}; fi)$ we have $\beta = \alpha$. In this case

$$[\lambda_2(p)]_\alpha = p_\alpha = i^{*-1}(q_\beta) = id^{*-1}(q_\alpha) = q_\alpha.$$

It remains to prove that $q_\beta \in Im\, i^*$. Consider the characteristic map of the CW-complex P_{n+1}

$$\Phi : \big(C(\vee S^n), \vee S^n\big) \to \big(P_{n+1}, P_{n+1}^{(n)}\big),$$

where $C(\vee S^n)$ denotes the cone over $\vee S^n$ and $\vee$ denotes the finite one-point union of spaces. Let $\varphi = \Phi|(\vee S^n)$. Consider the commutative diagram

$$\begin{array}{ccccc} & & R & & \\ & \overset{fi\varphi}{\nearrow} & \uparrow{\scriptstyle fi} & \overset{f}{\nwarrow} & \\ \vee S^n & \xrightarrow{\ \varphi\ } & P_{n+1}^{(n)} & \xrightarrow{\ i\ } & P_{n+1} \end{array}$$

Since $i\varphi \sim 0$, we obtain $fi\varphi \sim 0$. Let

$$\gamma = (\vee S^n; fi\varphi) \in {}^{(3)}\omega(R;n).$$

We have $\gamma < \beta$ and $q_\gamma = 0$ (see the proof of Corollary 1.3). In this case

$$\varphi^*(q_\beta) = q_\gamma = 0.$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n(P_{n+1}) & \xrightarrow{\ i^*\ } & H^n(P_{n+1}^{(n)}) & \xrightarrow{\ \delta\ } & H^{n+1}(P_{n+1}, P_{n+1}^{(n)}) & & \\ & & & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \Phi^*} & & \\ & & 0 & \longrightarrow & H^n(\vee S^n) & \xrightarrow{\ \delta\ } & H^{n+1}(C(\vee S^n), \vee S^n) & \longrightarrow & 0 \end{array}$$

where Φ^* is an isomorphism. Then, since $\varphi^*(q_\beta) = 0$, we have $\delta(q_\beta) = 0$. Therefore $q_\beta \in Im\, i^*$. This completes the proof of Proposition 4.4. ■

Lemma 4.5. *For each CW-complex P from the category $K_n^{(3)}$ there is a CW-complex $\bar{P}$ from the category $K_n^{(4)}$ such that $\bar{P}$ has the homotopy type of P.*

Proof. By the condition $H_i(P) = 0, i \neq 0, n-1, n$; $H_{n-1}(P) \approx \pi_{n-1}(P)$ are finite abelian groups and $H_n(P)$ is a finitely generated free abelian group. Thus the group $\pi_{n-1}(P)$ can be represented in the form

$$\pi_{n-1}(P) \approx Z_{r_1} \oplus Z_{r_2} \oplus \cdots \oplus Z_{r_t},$$

where Z_{r_i}, $i = 1, 2, \ldots, t$, are cyclic groups of order r_i. Consider the corresponding system

$$\xi_i : S^{n-1} \to P, \quad 1 \leq i \leq t,$$

of generators in the group $\pi_{n-1}(P)$ and define, by means of ξ_i, the map

$$f : \vee_{i=1}^{t} P^n(r_i) \to P.$$

Then f induces isomorphisms in homotopy and homology in dimensions $\leq n-1$. Now consider a system $h_1, h_2, \ldots, h_s$ of generators in the group $H_n(P)$. The Hurewicz homomorphism $\pi_n(P) \to H_n(P)$ for the space P is an epimorphism. In this case we can consider maps

$$\varphi_k : S^n \to P, \quad k = 1, 2, \ldots, s,$$

such that $\varphi_{k*}(1) = h_k$, where $1 \in H_n(S^n)$. Assume

$$\bar{P} = \left(\vee_{i=1}^{t} P^n(r_i)\right) \bigvee \left(\vee_{k=1}^{s} P_k^n(1)\right),$$

where $P_k^n(1) = P^n(1) = S^n$, and define by means of the maps f and φ_k the map

$$\varphi = f \vee (\vee \varphi_k) : \bar{P} \to P.$$

Then φ induces isomorphisms of all homology groups. Therefore, under our assumptions, the map φ is a homotopy equivalence. This proves Lemma 4.5. ■

Lemmas 4.1 and 4.5 imply

Proposition 4.6. λ_3 *is a natural isomorphism.*

Propositions 4.2 - 4.4 and 4.6 imply Theorem 2.4.

5. Proof of Theorem 2.5

Let $h \in H^n(Q^n)$ and

$$i_{j,t} : P^n(t) \to Q^n$$

be standard embeddings. Assume

$$\varepsilon(h) = \left\{i^*_{j,t}(h)\right\} \in \prod_{j,t} H^n\left(P^n_j(t)\right).$$

Obviously, we have

Lemma 5.1. *The correspondence*

$$\varepsilon : H^n(Q^n) \to \prod_{j,t} H^n\left(P^n_j(t)\right)$$

is an isomorphism.

In the sequel, for convenience, the subcategories F^b_n and Q_n will be denoted by $K^{(4)}_n$ and $K^{(5)}_n$. Let $R = (R, *)$ be an arbitrary space from K. Let $q \in \Pi^n(R; K^{(4)}_n)$ and

$$\alpha = (Q^n; f) \in {}^{(5)}\omega(R; n)$$

be an arbitrary index. Assume $f_{j,t} = f i_{j,t}$ and consider the indices

$$\alpha_{j,t} = (P^n(t); f_{j,t}) \in {}^{(4)}\omega(R; n).$$

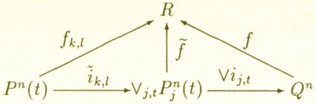
Let

$$p_\alpha = [\lambda_4(q)]_\alpha = \varepsilon^{-1}\left(\left\{q_{\alpha_{j,t}}\right\}\right).$$

We will show that the set $\{p_\alpha\}$ defines an element of the group $\Pi^n(R; K^{(5)}_n)$ and the natural isomorphism

$$\lambda_4 : \Pi^n(R; K^{(4)}_n) \to \Pi^n(R; K^{(5)}_n).$$

By $\vee$ we will denote the symbol of finite one-point union of spaces. Consider the commutative diagram



where $\tilde{i}_{k,l}$ and $\vee i_{j,t}$ are standard embeddings, and the indices

$$\alpha = (Q^n; f) \in {}^{(5)}\omega(R; n),$$
$$\beta = (\vee_{j,t} P_j^n(t); \tilde{f}) \in {}^{(4)}\omega(R; n),$$
$$\alpha_{k,l} = (P^n(t); f_{k,l}) \in {}^{(4)}\omega(R; n).$$

We have

$$(\vee i_{j,t})\tilde{i}_{k,l} = i_{k,l}.$$

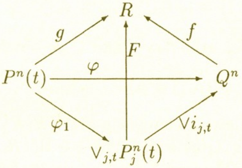
Therefore $\alpha_{k,l} < \beta$. Then

$$\tilde{i}^*_{k,l}\Big((\vee i_{j,t})^*(p_\alpha)\Big) = i^*_{k,l}(p_\alpha) = q_{\alpha_{k,l}}.$$

Since this is true for arbitrary k and l, we have

$$(\vee i_{j,t})^*(p_\alpha) = q_\beta. \tag{5}$$

Now consider the diagram



where $f\varphi \sim g$, $F = f(\vee i_{j,t})$ and $(\vee i_{j,t})\varphi_1 = \varphi$ (since $P^n(t)$ is a compact space and φ is a contunuous map, it follows that there exists a map φ_1). Then

$$F\varphi_1 = f\Big(\vee_{j,t}\, i_{j,t}\Big)\varphi_1 = f\varphi \sim g.$$

Consider the indices

$$\beta = (P^n(t), g) \in {}^{(4)}\omega(R; n),$$
$$\beta_1 = \Big(\vee_{j,t}\, P_j^n(t); F\Big) \in {}^{(4)}\omega(R; n).$$

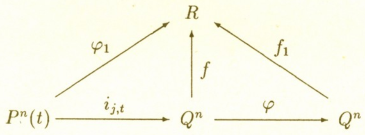
Then $\beta < \beta_1$, and by (5) we have

$$\varphi^*(p_\alpha) = \varphi_1^*\big((\vee_{j,t})^*(p_\alpha)\big) = \varphi_1^*(q_{\beta_1}) = q_\beta.$$

Thus

$$\varphi^*(p_\alpha) = q_\beta. \tag{6}$$

Finally, consider the diagram



where $f_1\varphi \sim f$ and $fi_{j,t} = \varphi_1$. Consider the indices

$$\alpha_1 = (Q^n; f_1) \in {}^{(5)}\omega(R; n),$$
$$\alpha_{j,t} = (P^n(t); \varphi_1) \in {}^{(4)}\omega(R; n).$$

Thus $\alpha < \alpha_1$. Therefore by (6) we have

$$i_{j,t}^*\big(\varphi^*(p_{\alpha_1})\big) = (\varphi i_{j,t})^*(p_{\alpha_1}) = q_{\alpha_{j,t}}.$$

Since this equality is true for arbitrary j and t, we obtain

$$\varphi^*(p_{\alpha_1}) = p_\alpha.$$

Therefore the set $\{p_\alpha\}$ defines an element $p \in \Pi^n(R; K_n^{(5)})$. The map λ_4 is now defined by setting $\lambda_4(q) = p$, where $q \in \Pi^n(R; K_n^{(4)})$.

Let $q_1, q_2 \in \Pi^n(R; K_n^{(4)})$. Then we have

$$i_{j,t}^*\big([\lambda_4(q_1 + q_2)]_\alpha\big) = (q_1 + q_2)_{\alpha_{j,t}} = (q_1)_{\alpha_{j,t}} + (q_2)_{\alpha_{j,t}} = i_{j,t}^*\big([\lambda_4(q_1)]_\alpha\big) +$$
$$+ i_{j,t}^*\big([\lambda_4(q_2)]_\alpha\big) = i_{j,t}^*\big([\lambda_4(q_1)]_\alpha + [\lambda_4(q_2)]_\alpha\big) = i_{j,t}^*\big([\lambda_4(q_1) + \lambda_4(q_2)]_\alpha\big).$$

Since this equality holds for arbitrary j and t, we have

$$\lambda_4(q_1 + q_2) = \lambda_4(q_1) + \lambda_4(q_2).$$

Let now $\varphi : S \to R$ be an arbitrary map. Consider the diagram

$$\begin{array}{ccc} \Pi^n(R;K_n^{(4)}) & \xrightarrow{\lambda_4} & \Pi^n(R;K_n^{(5)}) \\ \downarrow \varphi^{\#} & & \downarrow \varphi^{\#} \\ \Pi^n(S;K_n^{(4)}) & \xrightarrow{\lambda_4} & \Pi^n(S;K_n^{(5)}) \end{array} ,$$

element $q \in \Pi^n(R;K_n^{(4)})$, indices $\alpha, \alpha_{j,t}$ and

$$\beta = \varphi(\alpha) = (Q^n; g) \in {}^{(5)}\omega(R;n),$$

$$\beta_{j,t} = (P^n(t); g_{j,t}) \in {}^{(4)}\omega(R;n),$$

where $g = \varphi f$, $g_{j,t} = g i_{j,t}$. Then $\beta_{j,t} = \varphi(\alpha_{j,t})$ and we have

$$\left[\varphi^{\#}(\lambda_4(q))\right]_\alpha = [\lambda_4(q)]_\beta = \varepsilon^{-1}\left(\left\{q_{\beta_{j,t}}\right\}\right).$$

On the other hand, we have

$$\left[\lambda_4(\varphi^{\#}(q))\right]_\alpha = \varepsilon^{-1}\left(\left\{[\varphi^{\#}(q)]_{\alpha_{j,t}}\right\}\right) = \varepsilon^{-1}\left(\left\{q_{\varphi(\alpha_{j,t})}\right\}\right) = \varepsilon^{-1}\left(\left\{q_{\beta_{j,t}}\right\}\right).$$

Thus λ_4 is a natural homomorphism.

We will prove that λ_4 is a monomorphism. Let $q \in \Pi^n(R;K_n^{(4)})$ and $\lambda_4(q) = 0$. Consider an arbitrary index

$$\beta = \left(\vee_{j,t} P_j^n(t); g\right) \in {}^{(4)}\omega(R;n)$$

and define the map $f : Q^n \to R$ by taking

$$f\left((\vee i_{j,t})(x)\right) = g(x), \quad x \in \vee_{j,t} P_j^n(t),$$

$$f\left(Q^n - (\vee i_{j,t})(\vee_{j,t} P_j^n(t))\right) = *.$$

Consider the index

$$\alpha = (Q^n; f) \in {}^{(5)}\omega(R;n)$$

and the commutative diagram

$$\begin{array}{ccc} & R & \\ {}^{g}\nearrow & & \nwarrow^{f} \\ \vee_{j,t} P_j^n(t) & \xrightarrow{\vee i_{j,t}} & Q^n \end{array}$$

then by (5) we have

$$q_\beta = \left(\vee_{j,t}\, i_{j,t}\right)^*\left([\lambda_4(q)]_\alpha\right) = \left(\vee_{j,t}\, i_{j,t}\right)^*(0) = 0.$$

Therefore $q = 0$ and λ_4 is a monomorphism.

Further, we will prove that λ_4 is an epimorphism. Let $p \in \Pi^n(R; K_n^{(5)})$ and

$$\alpha = (X; f) \in {}^{(4)}\omega(R; n)$$

be an arbitrary index, $X \in K_n^{(4)}$. Therefore the space X can be represented in the form

$$X = \vee_{j,t} P_j^n(t)$$

where j is the index indicating a certain arrangement of the identical subspaces of X. Then we have natural embedding $i : X \to Q^n$. Define the map $l : Q^n \to X$ by taking

$$l(i(x)) = x, \quad x \in X;$$
$$l(Q^n - i(X)) = *.$$

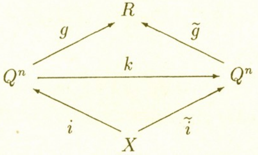
Let $g = fl$. We have $gi = f$. Let

$$\beta = (Q^n; g) \in {}^{(5)}\omega(R; n)$$

Assume that

$$q_\alpha = i^*(p_\beta). \tag{7}$$

Consider a different arrangement of subspaces of X. Let the map $\tilde{i}$ and the index $\tilde{\beta} = (Q^n; \tilde{g})$ be defined in the same way as i and β, respectively. Consider the commutative diagram

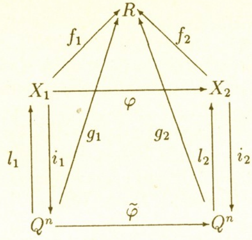


where the map k can be defined by a certain permutation of subspaces of Q^n. Then $\beta < \tilde{\beta}$ and we have

$$\tilde{i}^*(p_{\tilde{\beta}}) = i^*(k^*(p_{\tilde{\beta}})) = i^*(p_\beta).$$

Thus definition (7) is correct.

Consider the diagram



where $X_1, X_2 \in K_n^{(4)}$, $f_2\varphi \sim f_1$, $\tilde{\varphi} = i_2\varphi l_1$, $f_1 = g_1 i_1$, $f_2 = g_2 i_2$. Then

$$\tilde{\varphi} i_1 = i_2 \varphi l_1 i_1 = i_2 \varphi,$$
$$g_2\tilde{\varphi} = f_2 l_2 i_2 \varphi l_1 = f_2 \varphi l_1 \sim f_1 l_1 = g_1.$$

Consider the indices

$$\alpha_t = (X_t; f_t) \in {}^{(4)}\omega(R; n), \quad t = 1, 2;$$
$$\beta_t = (Q^n; g_t) \in {}^{(5)}\omega(R; n), \quad t = 1, 2.$$

Then $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$ and we have

$$\varphi^*(q_{\alpha_2}) = \varphi^*\big(i_2^*(p_{\beta_2})\big) = i_1^*\big(\tilde{\varphi}^*(p_{\beta_2})\big) = i_1^*(p_{\beta_1}) = q_{\alpha_1}.$$

Therefore the set $\{q_\alpha\}$ defines an element $q \in \Pi^n(R; K_n^{(4)})$.

Finally, let us prove that $\lambda_4(q) = p$. Consider an arbitrary index

$$\alpha = (Q^n; f) \in {}^{(5)}\omega(R; n)$$

and the index

$$\alpha_{j,t} = (P^n(t); f i_{j,t}) \in {}^{(4)}\omega(R; n).$$

Then $i_{j,t} = i$ and we have

$$i_{j,t}^*\big([\lambda_4(q)]_\alpha\big) = q_{\alpha_{j,t}} = i_{j,t}^*(p_\alpha).$$

Since this equality is true for arbitrary j and t, we have $[\lambda_4(q)]_\alpha = p_\alpha$. Therefore $\lambda_4(q) = p$. This completes the proof of Theorem 2.5. ■

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A SELFADJOINT "SIMULTANEOUS CROSSING OF THE AXIS"

KURT KREITH

ABSTRACT. By constructing the corresponding Green's function in a trapezoidal domain, we establish the existence of selfadjoint realizations of $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$ incorporating boundary conditions of the form $u(s,0) = u(s,T) = 0$. Such operators correspond to the historically important concept of a "simultaneous crossing of the axis" for vibrating strings.

რეზიუმე. ტრაპეციის ფორმის არისათვის გრინის ფუნქციის აგების საშუალებით დადგენილია $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$ ოპერატორის თვითშეუღლებულობა $u(s,0) = u(s,T) = 0$ სასაზღვრო პირობებში. ასეთი ოპერატორები შეესაბამება „ღერძის ერთობლივი გადაკვეთის„ ამოცანას რხევითი სიმებისათვის

1. Introduction. Given Poisson's equation

$$\begin{aligned} \Delta u \equiv u_{xx} + u_{yy} = f(x,y) \quad &\text{in} \quad R, \\ u = 0 \quad &\text{on} \quad \partial R \end{aligned} \tag{1.1}$$

in a rectangle

$$R = \{(x,y) : 0 \le x \le h,\ 0 \le y \le k\},$$

separation of variables leads to Fourier series solution

$$u(x,y) = \sum a_{mn} \sin\frac{m\pi x}{h} \sin\frac{n\pi y}{k}$$

with

$$a_{mn} = \frac{-4}{hk\pi^2(\frac{m^2}{h^2} + \frac{n^2}{k^2})} \iint_R f(x,y) \sin\frac{m\pi x}{h} \sin\frac{n\pi y}{k} dx\, dy. \tag{1.2}$$

The validity of this formal solution is related to the fact that Δ has a selfadjoint realization in $L^2(R)$ corresponding to the boundary condition $u = 0$ on ∂R.

1991 *Mathematics Subject Classification.* 35L20.

If one naively attempts the same approach to

$$\begin{aligned} Au \equiv u_{xx} - u_{yy} &= f(x,y) \quad \text{in} \quad R, \\ u &= 0 \quad \text{on} \quad \partial R, \end{aligned} \tag{1.3}$$

separation of variables again provides the same form of series solution. However, now

$$a_{mn} = \frac{-4}{hk\pi^2(\frac{m^2}{h^2} - \frac{n^2}{k^2})} \iint_R f(x,y) \sin\frac{m\pi x}{h} \sin\frac{n\pi y}{k} dx\, dy, \tag{1.4}$$

and the fact that the Dirichlet problem is not well posed for (1.3) is reflected by the sign change in going from (1.2) to (1.4). While the series solution for $u(x,y)$ makes formal sense for irrational values of h/k, its instability precludes the existence of a Green's function $G(x,y;\xi,\eta)$ and a corresponding representation of this "solution" in the form

$$u(x,y) = \iint_R G(x,y;\xi,\eta) f(\xi,\eta) d\xi\, d\eta.$$

One therefore would not expect to find a selfadjoint realization of A in $L^2(R)$ corresponding to $u = 0$ on ∂R.

The fact remains, however, that problems such as (1.3) have a certain appeal. Replacing x by a spatial variable s and y by a temporal variable t, it is natural, in the theory of vibrating strings, to consider

$$\begin{aligned} Au \equiv u_{tt} - u_{ss} &= f(s,t), \\ u(s,0) = u(s,T) &= 0. \end{aligned} \tag{1.5}$$

Here the boundary conditions can be interpreted as calling for a "simultaneous crossing of the axis" at $t = 0$ and $t = T$. As described by Cannon and Dostrovski [1], the physical concept played an important role in early attempts by both Brook Taylor and Johann Bernoulli to model vibrating strings.

Also, if one interprets the classical Sturmian theory for $(py')' + qy = 0$ in terms of the motion of a mass $p(t)$ subject to a linear restoring force $-q(t)y$, then it becomes very attractive to consider (1.5) as part of an effort to generalize Sturmian theory to hyperbolic PDEs (see for example [4]). While one would not expect to find a selfadjoint realization of A corresponding to (1.3), there do exist both historical and mathematical reasons for seeking selfadjoint realizations which incorporate (1.5).

The purpose of this paper is to show that, by considering (1.5) on a trapezoid

$$R = \{(s,t) : t \leq s \leq L - t;\ 0 \leq t \leq T\} \tag{1.6}$$

with $L \geq 2T > 0$, it becomes possible to establish selfadjoint realizations of (1.5) in terms of additional boundary conditions on the characteristics

$$s - t = 0 \quad \text{and} \quad s + t = L \quad (0 \leq t \leq T).$$

In case $L = 2T$, these results are related to ones obtained by Kalmenov [2], albeit by very different techniques.

2. Fundamental Singularities. If one seeks a representation of solutions of $y'' = f(x)$ in the form

$$y = \int_a^{x^-} \Gamma(x,\xi)f(\xi)d\xi + \int_{x^+}^{b} \Gamma(x,\xi)f(\xi)d\xi,$$

two applications of Leibniz's rule readily lead to the conditions

$$\Gamma_{xx} = 0 \quad \text{for} \quad x \neq \xi, \quad \Gamma(x,x^-) - \Gamma(x,x^+) = 0,$$

and

$$\Gamma_x(x,x^-) - \Gamma_x(x,x^+) = 1$$

as characterizing a fundamental singularity for $\frac{d^2}{dx^2}$. In [3] this familiar idea is extended to obtain a characterization of a fundamental singularity for $\frac{\partial^2}{\partial x \partial y}$. Representing a solution of $u_{xy} = f(x,y)$ in the form

$$u(x,y) = \int_c^{y^-} \left[\int_a^{x^-} \Gamma(x,y;\xi,\eta)f(\xi,\eta)d\xi + \int_{x^+}^{b} \Gamma(x,y;\xi,\eta)f(\xi,\eta)d\xi \right] d\eta +$$
$$+ \int_{y^+}^{d} \left[\int_a^{x^-} \Gamma(x,y;\xi,\eta)f(\xi,\eta)d\xi + \int_{x^+}^{b} \Gamma(x,y;\xi,\eta)f(\xi,\eta)d\xi \right] d\eta,$$

repeated applications of Leibniz's rule lead to the following characterization of a fundamental singularity for $\frac{\partial^2}{\partial x \partial y}$:

(i) $\Gamma_{xy} = 0$ for $x \neq \xi$ and $y \neq \eta$,

(ii) $\Gamma_x(x,y;\xi,y^-) = \Gamma_x(x,y;\eta,y^+)$,

$\Gamma_y(x,y;x^-,\eta) = \Gamma_y(x,y;x^+,\eta)$,

(iii) $\Gamma(x,y;x^-,y^-) - \Gamma(x,y;x^-,y^+) - \Gamma_x(x,y;x^+,y^-) +$

$+ \Gamma(x,y;x^+,y^+) = 1.$

Transforming such singularities into the (s,t)-plane by $t = y + x$, $s = y - x$ (and taking note of the fact that $|J(\begin{smallmatrix} x, & y \\ s, & t \end{smallmatrix})| = \frac{1}{2}$), one obtains, as a special case, the following

Lemma 2.1. *If $G(s,t;\sigma,\tau)$ satisfies*

$$G(s,t;\sigma,\tau)=\begin{cases}\frac{1}{4} & \text{for } |t-\tau|>|s-\sigma| \\ 0 & \text{for } |s-\sigma|>|t-\tau|\end{cases} \tag{2.1}$$

or

$$G(s,t;\sigma,\tau)=\begin{cases}-\frac{1}{4} & \text{for } |s-\sigma|>|t-\tau| \\ 0 & \text{for } |t-\tau|>|s-\sigma|,\end{cases} \tag{2.2}$$

then G is a fundamental singularity for $A=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial s^2}$. Furthermore, G is symmetric in the sense that $G(s,t;\sigma,\tau)=G(\sigma,\tau,s,t)$.

Proof. Transforming back into the (ξ,η)-plane, the above values for G yield

$$\Gamma(x,y;x^-,y^-)-\Gamma(x,y;x^-,y^+)-\Gamma(x,y;x^+,y^-)+\Gamma(x,y;x^+,y^+)=1. \quad ■$$

Lemma 2.2. *If γ is a real constant and $G(s,t;\sigma,\tau)$ satisfies*

$$G(s,t;\sigma,\tau)=\begin{cases}0 & \text{for } |t-\tau|>|s-\sigma| \\ \gamma & \text{for } s-\sigma>|t-\tau| \\ -\gamma & \text{for } |\sigma-s|>|t-\tau|,\end{cases}$$

then G is "nonsingular" in the sense that

$$A\iint_R G(s,t;\sigma,\tau)f(\sigma,\tau)d\sigma\,d\tau=0$$

for R a neighborhood of (s,t) in the (σ,τ)-plane.

Proof. Transforming back into the (ξ,η)-plane, the above values for G yield

$$\Gamma(x,y;x^-,y^-)-\Gamma(x,y;x^-,y^+)-\Gamma(x,y;x^+,y^-)+\Gamma(x,y;x^+,y^+)=0.$$

in place of (iii).

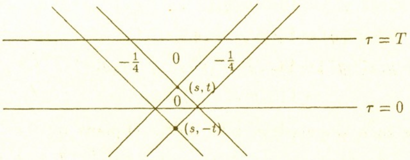

Figure 1

3. Construction of Green's Functions. To construct symmetric Green's functions corresponding to (1.5), we shall apply the method of images to symmetric fundamental singularities of the form (2.1). In order to satisfy $u(s,0) = 0$, we consider a pair of fundamental singularities to form

$$H(s,t;\sigma,\tau) = G(s,t;\sigma,\tau) - G(s,-t;\sigma,\tau).$$

Restricting $H(s,t;\sigma,\tau)$ to the strip $0 \leq t \leq T$, we have

$$H(s,t;\sigma,\tau) = \begin{cases} 0 & \text{for } |t-\tau| > |s-\sigma| \\ -\frac{1}{4} & \text{for } |\tau + t| > |\sigma - s| > |t-\tau| \end{cases}$$

(see Figure 1). Since H is composed of symmetric singularities, it is again symmetric.

Noting that $H(s,T;\sigma,\tau) \neq 0$, we now reflect about $t = T$ to consider $H(s,t;\sigma,\tau) - H(s,2T-t;\sigma,\tau)$ as depicted in Figure 2.

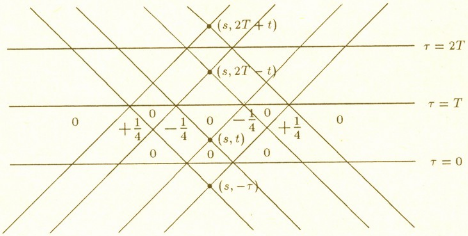

Figure 2

As confirmed by Lemmas 1.1 and 1.2, this again yields a symmetric fundamental singularity for A, one which corresponds to $u(s,0) = 0$. While this construction does not, in general, correspond to $u(s,T) = 0$, continued reflections about $\tau = 0$ and $\tau = T$ do eventually achieve this condition for bounded domains. This fact is evident from the fundamental singularity depicted in Figure 3, which vanishes except in rectangular regions defined by characteristics emanating from (s,t) and reflected by $\tau = 0$ and $\tau = T$.

Since all these rectangular regions approach the empty set as $t \to 0$ or $t \to T$, we see that

$$u(s,t) = A^{-1}f \equiv \iint_R H(s,t;\sigma,\tau)f(\sigma,\tau)d\sigma\, d\tau$$

does satisfy $u(s,0) = u(s,T) = 0$. Since $H(s,t;\sigma,\tau) = H(\sigma,\tau;s,t)$, A^{-1} is a completely continuous selfadjoint operator in $L^2(R)$ whose range manifests "a simultaneous crossing of the axis".

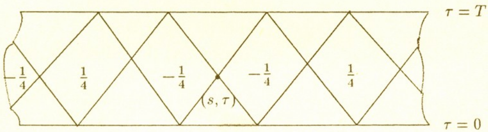

Figure 3

4. The Space like Boundary. Given a domain R contained in the rectangle $0 \le s \le S$, $0 \le t \le T$ the construction of §3 yields a selfadjoint realization of $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$ whose domain satisfies $u(s,0) = u(s,T) = 0$. What this construction fails in general to do is to characterize the domain of A in terms of boundary conditions for which

$$\iint_R (vAu - uAv)ds\, dt = 0.$$

It is here that the geometry of the problem enters in an essential way.
We consider a trapezoidal region

$$R(k,\theta) = \{(s,t) : 0 \le t \le s \le 2kT + \theta - t;\ 0 \le t \le T\},$$

where k is a positive integer and $0 \le \theta < 2T$. In this case it will be possible to determine boundary conditions on the characteristics

$$t - s = 0 \quad \text{and} \quad t + s = 2kT + \theta$$

which, together with $u(s,0) = u(s,T) = 0$, characterize selfadjoint realizations of A.

In case $\theta = 0$, the broken characteristic connecting (T,T) with $((2k-1)T,T)$ divides R into $2k-1$ congruent triangles (see Figure 4 for $k = 2$).

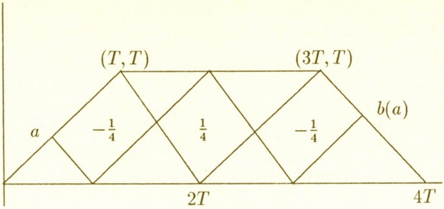

Figure 4

Corresponding to $a = (s, s)$ on the characteristic $t - s = 0$ there is a point $b(a) = ((2k-1)T + s, T - s)$ on the characteristic $t + s = 2kT$. The pairs of parallel lines

$$t - s = \nu T \quad \text{and} \quad t - s = \nu T + 2s,$$
$$t + s = \nu T + 2s \quad \text{and} \quad t + s = 2(\nu + 1)T$$

define a sequence of $2k - 1$ rectangles in R in which the fundamental singularity $H(s, t; \sigma, \tau)$ of §3 assumes the values $\frac{(-1)^\nu}{4}$ for $\nu = 1, \ldots, 2k-1$. However, this is the same function $H(s, t; \sigma, \tau)$ obtained by locating the fundamental singularity at $b(a)$. These observations establish the following.

Theorem 4.1. *Given a trapezoidal domain*

$$R(k, 0) = \{(s, t) : 0 \leq t \leq s \leq 2kT - t;\ 0 \leq t \leq T\},$$

the boundary conditions

$$u(s, 0) = 0 \quad \textit{for} \quad 0 \leq s \leq 2kT,$$
$$u(s, T) = 0 \quad \textit{for} \quad T \leq s \leq (2k-1)T,$$
$$u(s, s) = u((2k-1)T + s, T - s) \quad \textit{for} \quad 0 \leq s \leq T,$$

correspond to a selfadjoint realization of $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$ *in* $L^2(R)$.

Remark. For $k = 1$, the trapezoid becomes a characteristic triangle and we obtain the selfadjoint operator studied by Kalmenov [2].

There remains the problem of a general trapezoid with $0 < \theta < 2T$ for which selfadjoint boundary conditions will involve four points:

$$a = (s, s) \quad \text{for} \quad 0 < s < T,$$

$$b(a) = \begin{cases} ((2k-1)T + \frac{\theta}{2} + s,\ T + \frac{\theta}{2} - s) & \text{for} \quad s > \frac{\theta}{2} \\ ((2kT + \frac{\theta}{2} + s,\ \frac{\theta}{2} - s) & \text{for} \quad s < \frac{\theta}{2} \end{cases}$$

$$c = \left(\frac{\theta}{2}, \frac{\theta}{2}\right), d = \left(2kT + \frac{\theta}{2}, \frac{\theta}{2}\right).$$

Here a and $b(a)$ can be connected by broken characteristics reflected by the lines $t = 0$ and $t = T$. There is a similar relationship between c and $((2k-1)T + \theta, T)$ and between d and (T, T).

Our principal result is the following

Theorem 4.2. *Given a trapezoidal domain*

$$R(k, \theta) = \{(s, t) : 0 \le t \le s \le 2kT - t + \theta;\ 0 \le t \le T\},$$

the boundary conditions

$$u(s, 0) = 0 \quad \textit{for} \quad 0 \le s \le 2kT + \theta,$$
$$u(s, T) = 0 \quad \textit{for} \quad T \le s \le (2k-1)T + \theta,$$
$$u(a) + \frac{1}{2}u(c) = u(b) + \frac{1}{2}u(d),$$

correspond to a selfadjoint realization of $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$ *in* $L^2(R)$.

Proof. The proof consists of applying the construction of §3 to $(s, t) = a, b, c$ and d and noting the rectangles in which this construction assigns the values $\pm\frac{1}{4}$. In the figures below (for $k = 2$) we denote the value $\pm\frac{1}{4}$ resulting from point a by $\pm a$, etc.

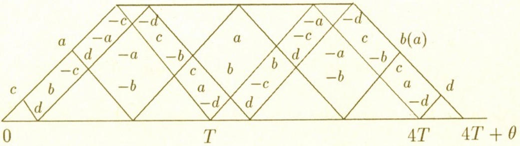

Figure 5

A rather tedious calculation shows that this construction *always* decomposes R into rectangles in which H assumes one of the following values

$$\pm(a+b),$$
$$\pm(a+c-d),$$
$$\pm(b-c+d).$$

Since all of these expressions are made to vanish by choosing

$$a=\frac{1}{4},\quad b=-\frac{1}{4},\quad c=-\frac{1}{8},\quad d=\frac{1}{8}$$

it follows that the composite singularity corresponding to

$$u(a)-u(b)-\frac{1}{2}u(c)+\frac{1}{2}u(d)$$

vanishes identically in $R\times R$. Therefore all functions in the range of

$$A^{-1}f=\iint_R H(s,t;\sigma,\tau)f(\sigma,\tau)\,d\sigma\,d\tau$$

will satisfy

$$u(a)-u(b)+\frac{u(d)-u(c)}{2}=0.$$

Remarks.

1. Other selfadjoint boundary conditions can be obtained by choosing $a=\frac{1}{4}$, $b=-\frac{1}{4}$, c arbitrary, and $d=c+\frac{1}{4}$.

2. By way of physical interpretation of these results, it seems that a simultaneous crossing of the axis is an unreasonable requirement for a driven string tied down at both end points. If, however, one is willing to shorten the string by moving in from both ends at the speed of propagation, then it is possible to manipulate these ends so as to achieve a simultaneous crossing of the axis for arbitrary $T>0$.

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STABILIZATION OF FUNCTIONS AND ITS APPLICATION

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ABSTRACT. The concepts of polynomial stabilization, strong polynomial stabilization and strong stabilization are introduced for a fundamental system of solutions of linear differential equations. Some criteria of such kind stabilizations and applications to the theory of existence and uniqueness of solutions of ordinary differential equations are given. An abstract scheme of the obtained results is presented for Banach spaces.

რეზიუმე. შემოღებულია პოლინომური სტაბილიზაციის, მკაცრი პოლინომური სტაბილიზაციისა და წრფივი დიფერენციალური განტოლების ამონახსნთა ფუნდამენტური სისტემის მიმართ მკაცრი სტაბილიზაციის ცნებები. დადგენილია ასეთი სტაბილიზაციების კრიტერიუმები. ეს შედეგები გამოყენებულია ჩვეულებრივი დიფერენციალური განტოლებებისათვის სასაზღვრო ამოცანების თეორიაში და მოცემულია ბანახის სივრცეებში მათი გადატანის აბსტრაქტული სქემა.

1. Polynomial and Strong Polynomial Stabilization

Let us introduce concepts of stabilization and strong stabilization of functions. We begin by considering stabilization as $t \to +\infty$ of a function to a polynomial

$$P(t) = \sum_{m=0}^{n-1} c_m t^m \tag{1}$$

of degree at most $n-1$, where n is a fixed natural number.

Definition 1. An $n-1$ times differentiable function $x(t)$ on the infinite half-interval $[t_0, +\infty)$, $t_0 \in \mathbb{R}$ ($\mathbb{R}$ is the real line) is said to stabilize as $t \to +\infty$ to the polynomial (1) if

$$\lim_{t\to+\infty} \left(x(t) - P(t)\right)^{(j)} = 0, \quad j = 0, 1, \ldots, n-1. \tag{2}$$

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Let us write in this case $x(t) \sim P(t)$.

If such a polynomial exists (for a given function), then it is unique.

We introduce the notation

$$(I_m x)(t) = \int_t^{+\infty} dt_1 \int_{t_1}^{+\infty} dt_2 \cdots \int_{t_{m-1}}^{+\infty} x(t_m)\, dt_m, \quad m \in \mathbb{N}.$$

It is possible to obtain a good enough description of functions stabilizing to polynomials in the class of functions having n and not $n-1$ derivatives as assumed by Definition 1.

Theorem 1. *A function $x(t)$ having a locally integrable derivative of order n on the half-interval $[t_0, +\infty)$ stabilizes as $t \to +\infty$ to a polynomial of degree at most $n-1$ iff the integral*

$$(I_n x^{(n)})(t_0) = \int_{t_0}^{+\infty} dt_1 \int_{t_1}^{+\infty} dt_2 \cdots \int_{t_{n-1}}^{+\infty} x^{(n)}(t_n)\, dt_n \qquad (3)$$

converges.

Theorem 2. *If the integral* (3) *converges, then the function $x(t)$ stabilizes as $t \to +\infty$) to the the given polynomial* (1) *iff*

$$x(t) = P(t) + (-1)^n (I_n x^{(n)})(t). \qquad (4)$$

The property (2) of the polynomial $P(t)$ is analogous to that of the Taylor polynomial of a function for the finite point t_0 when $t \to t_0$. However, in contrast to the latter polynomial, the polynomial with the property (2) for a given function $x(t)$ exists only for one number n.

The conditions of Theorem 1 are fulfilled if

$$\int_{t_0}^{+\infty} t^{m-1} |x^{(n)}(t)|^p\, dt < +\infty, \quad 1 \le p < +\infty, \quad m \in \mathbb{N}, \quad m > pn.$$

This case was considered by S.L. Sobolev [1]. V.N. Sedov [2] and the author [3] obtained some generalizations when the integral $\int_{t_0}^{+\infty} \varphi(t) \times |x^{(n)}(t)|^p\, dt$ is finite for a nonnegative function φ. The general case, i.e. Theorems 1 and 2, is treated in [4, 5].

The coefficients of the polynomial (1) to which the function $x(t)$ stabilizes as $t \to +\infty$, can be calculated (see [4]) by the formula

$$c_{n-m} = \frac{1}{(n-m)!} \left[\sum_{j=1}^{m} \frac{(-1)^{m-j}}{(m-j)!} x^{(n-j)}(t_0) t_0^{m-j} + (-1)^{m-1} \sum_{k=0}^{m-1} \frac{t_0^k}{k!} (I_{m-k} x^{(n)})(t_0) \right], \quad m = 1, 2, \ldots, n.$$

If the function $x(t)$ is $n-1$ times differentiable on $[t_0, +\infty)$, then there exists its one and only one representation

$$x(t) = \sum_{m=0}^{n-1} y_{x,m}(t) t^m \tag{5}$$

such that

$$x^{(k)}(t) = \sum_{m=k}^{n-1} \frac{m!}{(m-k)!} y_{x,m}(t) t^{m-k}, \quad k = 0, 1, \ldots, n-1, \quad t \geq t_0, \tag{6}$$

i.e., the behaviour of the coefficients $y_{x,m}(t)$ is that as if they were constants by $n-1$ times differentiation of the expression occurring on the right-hand side of the equality (5). Such representations of functions will be called polynomial Lagrange representations. Representations of this kind emerge when we use Lagrange's method of variation of constants for solving linear nonhomogeneous ordinary differential equations.

Definition 2. An $n-1$ times differentiable function $x(t)$ on the half-interval $[t_0, +\infty)$ is said to strongly stabilize as $t \to +\infty$ to the polynomial (1) if

$$\lim_{t \to +\infty} y_{x,m}(t) = c_m, \qquad m = 0, 1, \ldots, n-1, \tag{7}$$

where $y_{x,m}(t)$ are the coefficients of the polynomial Lagrange representation of $x(t)$.

In this case let us write $x(t) \approx P(t)$.

If the function $x(t)$ has n derivatives, then for derivatives of the coefficients of its polynomial Lagrange representation we have the formula

$$y'_{x,n-k}(t) = \frac{(-1)^{k-1}}{(n-k)!(k-1)!} x^{(n)}(t) t^{k-1}, \quad k = 1, 2, \ldots, n.$$

Theorem 3. *A function $x(t)$ having a locally integrable derivative of order n on the half-interval $[t_0, +\infty)$ strongly stabilizes as $t \to +\infty$ to the polynomial* (1) *iff the integral*

$$\int_{t_0}^{+\infty} t^{n-1} x^{(n)}(t)\, dt \tag{8}$$

converges.

We observe that if the inegral (8) converges, then so does the integral (3), but some examples show that the converse statement is wrong [4]. Thus if some function strongly stabilizes as $t \to +\infty$ to a polynomial, then it stabilizes to a polynomial too, but not conversely. More exactly, the following theorem is valid.

Theorem 4. *If the integral (8) converges, then a function $x(t)$ stabilizes as $t \to +\infty$ to the polynomial (1) iff this function strongly stabilizes as $t \to +\infty$ to the same polynomial.*

We would like to indicate a special role of polynomials

$$Q_r(t) = \sum_{j=0}^{r} \frac{(-1)^j}{j!} x^j, \quad r = 0, 1, \dots,$$

for the polynomial Lagrange representation of functions.

Theorem 5. *The coefficients $y_{x,m}(t)$ of the polynomial Lagrange representation of an $n-1$ times differentiable function $x(t)$ on $[t_0, +\infty)$ can be calculated by the formula*

$$y_{x,m}(t) = \frac{1}{m!} Q_{n-m-1}\left(t \frac{d^m}{dt^m}\right) x(t), \quad m = 0, 1, \dots, n-1.$$

The following criterion plays an important role for strong stabilization to polynomials.

Theorem 6. *If the integral (8) converges, then a function $x(t)$, having a locally integrable derivative of order n on $[t_0, +\infty)$, strongly stabilizes to the polynomial (1) iff the identity*

$$x(t) = P(t) + \sum_{m=0}^{n-1} \frac{(-1)^{n-m}}{m!(n-m-1)!} t^m \int_t^{+\infty} s^{n-m-1} x^{(n)}(s)\, ds \tag{9}$$

holds.

We introduce some Banach spaces for stabilized and strongly stabilized functions.

Let $\widetilde{X}_t$ be a set of all $n-1$ times continuously differentiable functions on the half-interval $[t, +\infty)m$ $t \geq t_0$ which stabilize as $t \to +\infty$ to polynomilas of degree at most $n-1$, i.e.,

$$\widetilde{X}_t = \left\{ C^{n-1}[t, +\infty) : \exists P \sim x \right\}.$$

We shall use the notation

$$P_x(t) = \sum_{m=0}^{n-1} c_{x,m} t^m \tag{10}$$

for the polynomial to which the given function $x(t)$ stabilizes as $t \to +\infty$ and assume that

$$\mathbf{c}_x = (c_{x,0}, c_{x,1}, \dots, c_{x,n-1}). \tag{11}$$

Theorem 7. *The set $\widetilde{X}_t$ is a Banach space with the norm*

$$\|x\|_t = \|x - P_x\|_{C^{n-1}[t,+\infty)} + |\mathbf{c}_x|. \tag{12}$$

The polynomial stabilization is continuous with respect to the norm (12).

Let $\widetilde{\widetilde{X}}_t$ be a set of all $n-1$ times continuously differentiable on the half-interval $[t, +\infty)$, $t \geq t_0$, functions strongly stabilized as $t \to +\infty$ to polynomials of degree at most $n-1$, i.e.,

$$\widetilde{\widetilde{X}} = \left\{x \in C^{n-1}[t, +_\infty) \,:\, \exists P \approx x\right\}$$

and let

$$\mathbf{y}_x = (y_{x,0}, y_{x,1}, \dots, y_{x,n-1}),$$

where $y_{x,m} = y_{x,m}(t)$ are the coefficients of the polynomial Lagrange representation of the given function $x(t)$. Thus, if $x(t) \approx P(t)$, then

$$\lim_{t \to +\infty} \mathbf{y}_x(t) = \mathbf{c}_x. \tag{13}$$

Theorem 8. *The set $\widetilde{\widetilde{X}}_t$ is a Banach space with the norm*

$$|||x|||_t = \|y_x\|_{C_n[t,+\infty)},$$

where $C_n[t, +\infty)$ is a Banach space of all continuous and bounded on the half-interval $[t, +\infty)$ n-dimensional vector functions with the uniform norm.

2. Some Applications of Polynomial and Strong Polynomial Stabilizations to the Existence and Uniqueness of Solutions of Ordinary Differential Equations

Let us consider a differential equation

$$x^{(n)}(t) = f(t, x, x', \dots, x^{(n-1)}), \tag{14}$$

where $f : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

First we shall study the problem of stabilization of solutions of the equation (14) to the given polynomial (1):

$$x(t) \sim P(t). \tag{15}$$

Note that we do not obtain simpler problems by the change of the variables $t = 1/s$ or $x = x_1$, $x' = x_2, \dots, x^{(n-1)} = x_n$.

Theorem 9. *A solution $x(t)$ of the equation* (14) *stabilizes as $t \to +\infty$ to the polynomial* (1) *iff it is a solution of the integral equation*

$$x(t) = P(t) + (-1)^n \big(I_n f(\cdot, x, x', \dots, x^{(n-1)})\big)(t). \tag{16}$$

It is useful to note that in the case of strong stabilization of solutions of the equation (14) we naturally obtain another integral equation which is equivalent to the differential equation (14) (see Theorem 11 below).

Definition 3. Let g be a function such that $g : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ and let Y be a set of some functions $n-1$ times differentiable on the half-interval $[t_0, +\infty)$.

The integral $(I_n g(\cdot, x, x', \dots, x^{(n-1)}))(t_0)$ is called strongly uniformly convergent (on the n-dimensional half-interval $[t_0, +\infty)^n$) with respect to the set Y if for every $\varepsilon > 0$ there exists $t_\varepsilon \geq t_0$ such that for all functions $x(t) \in Y$ and for all $t > t_\varepsilon$ we have the inequalities

$$\left|\big(I_m g(\cdot, x, x', \dots, x^{(n-1)})\big)(t)\right| < \varepsilon, \quad m = 1, 2, \dots, n.$$

We introduce the notation

$$\widetilde{X}_T(0) = \{x \in \widetilde{X}_T : x \sim 0\}, \quad \widetilde{X}_T(P) = \widetilde{X}_T(0) + P,$$
$$\widetilde{Q}_T(P, r) = \{x : \widetilde{X}_T(P) : \|x - P\|_T \leq r\}, \quad T \geq t_0,$$
$$\mathbf{x} = (x, x', \dots, x^{(n-1)}), \text{ in particular, } \mathbf{P} = (P, P', \dots, P^{(n-1)}), \tag{17}$$
$$f(t, \mathbf{x}) = f(t, x, x', \dots, x^{(n-1)}).$$

Theorem 10. *If the polynomial* (1) *is given, if for every $r > 0$ the integral*

$$\big(I_n f(\cdot, x)\big)(t_0)$$

is strongly uniformly convergent with respect to the ball $\widetilde{Q}_{t_0}(P, r)$, if for every $r > 0$ there exists a function $\varphi_r : [t_0, +\infty) \to [0, +\infty)$ such that

$$(I_n \varphi_r)(t_0) < +\infty$$

and for all $t \geq t_0$, $\boldsymbol{\xi} \in \mathbb{R}^n$, $\boldsymbol{\eta} \in \mathbb{R}^n$, $|\boldsymbol{\xi}| \leq r$, $|\boldsymbol{\eta}| \leq r$ the inequality

$$|f(t, \mathbf{P} + \boldsymbol{\eta}) - f(t, \mathbf{P} + \boldsymbol{\xi})| \leq \varphi_r(t) |\boldsymbol{\eta} - \boldsymbol{\xi}| \tag{18}$$

holds, then there exists a $T \geq t_0$ *such that on the half-interval* $[T, +\infty)$ *there exists one and only one solution of the equation* (14) *which stabilizes as* $t \to +\infty$ *to the given polynomial* (1).

An equation of the type $x' = f(x)/(1+t^2)$, where for every $r > 0$ the function $f(x)$ is bounded on the set of all functions $x(t) \in \widetilde{X}_{t_0}$ belonging to the ball $\|x\|_{C[t_0,+\infty)} \leq r$, is an example of equations satisfying the conditions of Theorem 10.

Note that solutions of the equation (14), for which the conditions of Theorem 10 are fulfilled, can have no absolutely integrable derivatives. The simplest example of such an equation is $x' = \sin t/t$.

Let us now consider the case of strong stabilization of solutions as $t \to +\infty$ to a polynomial.

Theorem 11. *A solution* $x(t)$ *of the equation* (14) *strongly stabilizes as* $t \to +\infty$ *to the polynomial* (1) *iff it is a solution of the integral equation*

$$x(t) =$$

$$= P(t) + \sum_{m=0}^{n-1} \frac{(-1)^{n-m}}{m!(n-m-1)!} t^m \int_t^{+\infty} s^{n-m-1} f(s, x, x', \dots, x^{(n-1)})\, ds.$$

Setting now $\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^n$, $P(t, \boldsymbol{\xi}) = \sum_{m=0}^{n-1} \xi_m t^m$, $\widetilde{\widetilde{Q}}_T(r) = \{x \in \widetilde{\widetilde{X}} : |||x|||_T \leq r\}$ and using the notation (17), we obtain

$$\mathbf{P}(t, \xi) = (P(t, \xi), P'(t, \xi), \dots, P^{(n-1)}(t, \xi)).$$

Theorem 12. *If the polynomial* (1) *is given, if for every* $r > 0$ *the integral*

$$\int_{t_0}^{+\infty} t^{n-1} f(t, \mathrm{x})\, dt$$

uniformly converges with respect to the ball $\widetilde{\widetilde{Q}}_{t_0}(r)$, *if for every* $r > 0$ *there exists a function* $\psi_r : [t_0, +\infty) \to [0, +\infty)$ *such that*

$$\int_{t_0}^{+\infty} t^{n-1} \psi_r(t)\, dt < +\infty \tag{19}$$

and for all $t \geq t_0$, $\boldsymbol{\xi} \in \mathbb{R}^n$, $\boldsymbol{\eta} \in \mathbb{R}^n$, $|\boldsymbol{\xi}| \leq r$, $|\boldsymbol{\eta}| \leq r$, *we have the inequality*

$$|f(t, \mathbf{P}(t, \boldsymbol{\eta})) - f(t, \mathbf{P}(t, \boldsymbol{\xi}))| \leq \psi_r(t) |\boldsymbol{\eta} - \boldsymbol{\xi}|, \tag{20}$$

then there exists a $T \geq t_0$ *such that on the half-interval* $[T, +\infty)$ *there exists one and only one solution of* (14) *which strongly stabilizes as* $t \to +\infty$ *to the given polynomial* (1).

It is obvious that, in contrast to simple stabilization, in the case of strong stabilization we need some other generalization of the Lipschitz condition (compare the relations (18) and (20)).

The theorem about continuous dependence of solutions of the equation (14) on the stabilization data holds in the space $\overset{\approx}{X}_{t_0}$.

Theorem 13. *Let the polynomials* $P((t, \mathrm{a}_j)$, $\mathrm{a}_j \in \mathbb{R}^n$ $j = 1, 2$ *of degree at most* $n-1$ *be given. If the solution* $x_j(t)$ *of the equation* (14) *is defined on the half-interval* $[t_0, +\infty)$, *if it strongly stabilizes as* $t \to +\infty$ *to the polynomial* $P(t, \mathrm{a}_j)$, $j = 1, 2$, *and if for some* $r > \max_{j=1,2} |||x_j|||_{t_0}$ *there exists* $\psi_r(t)$ *a function satisfying the conditions* (19), (20), *then the inequality*

$$|||x_2 - x_1|||_{t_0} \leq |\mathrm{a}_2 - \mathrm{a}_1| n \exp n \int_{t_0}^{+\infty} t^{n-1} \psi_r(t)\, dt$$

is valid.

3. General Case of the Strong Stabilization Problem

Let the equation

$$Lx = f(t, x, x', \ldots, x^{(n-1)}), \tag{21}$$

be given, where

$$L = \frac{d^n}{dt^n} + \sum_{k=0}^{n-1} p_m(t) \frac{d^m}{dt^m} \tag{22}$$

$p_m(t)$ are continuous functions on the interval (a, b), $m = 0, 1, \ldots, n-1$, $-\infty \leq a < b \leq +\infty$.

In the case of polynomial stabilization we have $L = \frac{d^n}{dt^n}$, $a = t_0 \in \mathbb{R}$ and $b = +\infty$. For an arbitrarily chosen linear operator L it is possible to write the equation (14) in the form (21), and conversely. Of course the right-hand sides of the equations will be different. So the equation (21) is not an equation of a new type but only a new notation.

Assume that

$$v_1, v_2, \ldots, v_n \tag{23}$$

in some fundamental system of solutions of the equation

$$Lx = 0. \tag{24}$$

Let $x(t)$ be an $n-1$ times differentiable function on the interval (a,b) and

$$x(t)=\sum_{j=1}^{n} y_{x,j}(t)v_j(t), \quad a<t<b, \tag{25}$$

be its Lagrange representation with respect to the system (23), i.e. a representation such that

$$x^{(m)}(t)=\sum_{j=1}^{n} y_{x,j}(t)v_j^{(m)}(t), \quad m=0,1,\dots,n-1, \quad a<t<b. \tag{26}$$

For every $n-1$ times differentiable on the interval (a,b) function there exists its one and only one Lagrange representation (25), since the determinant of the system of linear equations (26) with respect to the variables $y_{x,j}$, $j=1,2,\dots,n$, is the Wronskian of the system (23).

Let now $v(t)\in \mathit{ker}\,L$; therefore

$$v(t)=\sum_{j=1}^{n} c_j v_j(t), \tag{27}$$

where c_j are some constants, $j=1,2,\dots,n$; also let k, l be some nonnegative integers, $1\le k+l\le n$.

Definition 4. An $n-1$ times differentiable function on the interval (a,b) is said to strongly (k,l)-stabilize to the function (27) if

$$\lim_{t\to a} y_{x,j}(t)=c_j, \quad j=1,2,\dots,k,$$
$$\lim_{t\to b} y_{x,j}(t)=c_j, \quad j=k+1,k+2,\dots,k+l.$$

In this case let us write

$$x(t)\underset{(k,l)}{\approx} v(t).$$

The general problem is to find solutions of the equation (21) which strongly (k,l)-stabilize to a given function $v(t)\in \mathit{ker}\,L$. Let us briefly discuss this problem. It includes the classical boundary problems on finite segments, the Cauchy problem and some new problems.

Indeed, if all coefficients $p_m(t)$, $m=0,1,\dots,n-1$, of the operator (22) are continuous on the half-interval $(a,b]$, $b\in\mathbb{R}$, then all functions (23) with all their derivatives up to order n inclusive are continuous at the point $t = b$. Therefore the system of identities (26) implies as $t \to b$ that

$$x^{(m)}(b) = \sum_{j=1}^{n} c_j v_j^{(m)}(b), \quad m = 0, 1, \dots, n-1.$$

From this we conclude that to give the function $v(t) \in ker\, L$, i.e., to give the coefficients $c_1, c_2, \dots, c_n$, is equivalent to give the Cauchy data $x(b)$, $x'(b), \dots, x^{(n-1)}(b)$. Therefore in this case the problem of strong $(0, n)$-stabilization of solutions of the equation (21) is equivalent to the Cauchy problem.

In the case when $k \geq 1$, $l \geq 1$ and $-\infty < a < b < +\infty$, it is possible to see (when the coefficients $p_m(t)$, $m = 0, 1, \dots, n-1$, are continuous at $t = a$ and $t = b$) that the problem of strong (k, l)-stabilization of solutions to a given $v \in ker\, L$ is equivalent to a classical boundary value problem of the type when $x(a)$, $x'(a)$, ... $x^{(k-1)}(a)$, $x(b)$, $x'(b)$, ..., $x^{(l-1)}(b)$ are given.

An example of the new problem is given in [6]. There the Euler equation

$$Lx + f = 0 \tag{28}$$

with L as a quadratic integral functional depending on a function and its derivatives up to order n inclusive is considered. Under some restrictions imposed on the coefficients of the integrand of the given functional the existence and uniqueness of a generalized solution on the half-interval $[t_0, +\infty)$ are proved when the following stabilization data are given:

1) the solution stabilizes as $t \to +\infty$ to some polynomial (1);

2) the values $x^{(i_1)}(t_0), \dots, x^{(i_k)}(t_0)$ and the coefficients $c_{j_1}, c_{j_2}, \dots, c_{j_l}$ of the polynomial (1) are given.

Some conditions are established for the indices

$$\{i_\mu\}_{\mu=1}^{\mu=k}, \quad \{j_\nu\}_{\nu=1}^{\nu=l} \tag{29}$$

when one and only one generalized solution exists.

Let the indices (29) be increasing sequences of integers belonging to the set $\{0, 1, \dots, n-1\}$:

$$0 \leq i_1 < i_2 < \dots < i_k \leq n-1, \quad 1 \leq k \leq n,$$
$$0 \leq j_1 < j_2 < \dots < j_l \leq n-1, \quad 1 \leq l \leq n.$$

If $k+l=n$, we introduce the notation $\{\bar{i}_\nu\}_{\nu=1}^{\nu=l}$ for the complement of the set of indices $\{i_\mu\}$ to the set $\{0,1,\dots,n-1\}$. Assume that

$$\bar{i}_1 < \bar{i}_2 < \cdots < \bar{i}_l.$$

If $k+l=n$, then the conditions

$$j_1 \le \bar{i}_1,\ j_2 \le \bar{i}_2,\ \dots,\ j_l \le \bar{i}_l$$

are called the Pólya ones. When one of the sets $\{i_\mu\}_{\mu=1}^{\mu=k}$ or $\{j_\nu\}_{\nu=1}^{\nu=l}$ is empty, the system (29) is also said to satisfy the Pólya conditions.

These conditions were introduced by Pólya. He proved (see [7]) that there exists one and only one polynomial (1) with the given values

$$P^{(i_\mu)}(0),\quad P^{(j_\nu)},\quad \mu=1,2\dots,k,\quad \nu=1,2,\dots,l,\quad k+l=n,$$

iff the system of indices (29) satisfies the Pólya conditions. It is evidently a purely algebraic problem.

In the general case, i.e., when $k+l\le 2n$, the system (29) is called complete if it contains some subsystem satisfying the Pólya conditions.

For the complete system (29) we evidently have $n\le k+l\le 2n$.

If the system (29) is complete, then there exists one and only one generalized solution $x(t)$ of the Euler equation (28) with the given values $x^{(i_\mu)}(t_0)$, c_{j_ν}, $\mu=1,2,\dots,k$, $\nu=1,2,\dots,l$ where c_{j_ν} are some coefficients of the polynomial (1) to which the solution $x(t)$ stabilizes as $t\to+\infty$. It is interesting to note that in contrast to the Pólya case it is a purely analytic problem. If the system of indices (29) is not complete, then there exist examples for which the problem under consideration has more than one solution.

One can prove that the n times continuously differentiable on the interval (a,b) function strongly $(k,n-k)$-stabilizes $(k=0,1,\dots,n-1)$ to the function $v(t)\in ker\,L$ iff the identity

$$x(t)=v(t)+\int_a^b G_k(t,s)Lx(s)ds \tag{30}$$

holds.

Here $G_k(t,s)$ is the generalized Green function. This function strongly $(k,n-k)$-stabilizes to zero (at the ends of the interval (a,b)), but it and its appropriate derivatives do not in general tend to zero as $t\to a$ and $t\to b$, as it should be if $G_k(t,s)$ were the ordinary Green function.

Let us assume now that the solution of the equation (21) strongly $(0,n)$-stabilizes to a function $v(t)\in ker\,L$. Then the standard change of variables

$$x=x_1,\ x'=x_2,\ \dots,\ x^{(n-1)}=x_n$$

is very advisable. We obtain a system of the type

$$Lx = f(t, x), \quad x = (x_1, x_2, \dots, x_n),$$
$$L = \frac{d}{dt} + A, \tag{31}$$

where A is a continuous matrix of order $n \times n$ on the interval (a, b), and $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}^n$.

Let

$$v_i = (v_{i1}, v_{i2}, \dots, v_{in}), \quad i = 1, 2, \dots, n, \tag{32}$$

be a fundamental system of the homogeneous equation $Lx = 0$,

$$V = (v_{ij}), \quad i, j = 1, 2, \dots, n, \tag{33}$$

be a fundamental matrix of the system (32), $x(t)$ be a differentiable function on the interval (a, b),

$$x(t) = \sum_{i=1}^{n} y_{x,i}(t) v_i(t) \tag{34}$$

(by analogy with scalar functions this representation of the vector function $x(t)$ will be called the Lagrange representation); also let

$$y_x(t) = (y_{x,1}(t), y_{x,2}(t), \dots, y_{x,n}(t)). \tag{35}$$

The vector function $x(t)$ is called strongly stabilized as $t \to b$ to the function

$$v(t) = \sum_{i=1}^{n} c_i v_i(t) \in ker L \tag{36}$$

if

$$\lim_{t \to b} y_x(t) = c, \quad c = (c_1, c_2, \dots, c_n). \tag{37}$$

The function $x(t)$ strongly stabilizes to the function (35) iff the identity

$$x(t) = Vc - V \int_t^b V^{-1}(s) Lx(s) ds$$

holds.

The problem of strongly $(0, n)$-stabilized solutions of the equation (21) is reduced to the problem of strongly stabilized solutions of the equation (31). The formula (34) implies $x = Vy$ and for the vector function y we obtain the equation (see [8])

$$y' = V^{-1} f(t, Vy)$$

thereby the condition (37) has to be fulfilled.

Like in the case of strong stabilization to polynomials, one can obtain the existence and uniqueness theorem for solutions of the system (31) in a neighbourhood of the point $t = b$ only if these solutions strongly stabilize as $t \to b$ to some function $\mathrm{v} \in ker\, L$ (see [9]).

4. Absract Scheme

The main idea of this paper is to establish (under some restrictions) that for every given solution $v(t)$ of the linear homogeneous equation $Lx = 0$ there exists one and only one solution $x(t)$ of the nonhomogeneous equation $Lx = f(t, x)$ if only this solution $x(t)$ stabilizes to the solution $v(t)$.

The first question in the case of abstract spaces is connected with defining the concept of stabilization, since in the function case this concept is based on the concept of the limit of functions. It is very advisable to use for this purpose the generalization of the representations of functions (4), (9) and (30).

Let X and Y be linear spaces, $L : X \to Y$, $F : X \to Y$, $S : Y \to X$, $Y = L(X)$, where L and S are linear operators, and F is in general a nonlinear operator; also let

$$LS = Id$$

where Id is the identity operator of the space Y onto itself. Then the following decomposition in the direct sum holds:

$$X = ker\, L \oplus S(Y).$$

Definition 5. An element $x \in X$ is called S-stabilized to an element $v \in ker\, L$ if

$$x = v + SLx.$$

In this case we can write $x \underset{S}{\sim} v$.

Under the above-given assumptions, for every element $v \in ker\, L$ the equations $Lx = Fx$ and $x = v + SFx$ are equivalent on the set of all elements of the space X which are S-stabilized to the given element $v \in ker\, L$.

If X, Y are Banach spaces and SF is a contracting operator, then for any $v \in ker\, L$ there exists one and only one solution $x \underset{S}{\sim} v$ in the space X.

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LIMIT DISTRIBUTION OF THE INTEGRATED SQUARED ERROR OF TRIGONOMETRIC SERIES REGRESSION ESTIMATOR

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ABSTRACT. Limit distribution is studied for the integrated squared error of the projection regression estimator (2) constructed on the basis of independent observations (1). By means of the obtained limit theorems the test is given for verifying the hypothesis about the regression and the power of this test is calculated in the case of Pitman alternatives.

რეზიუმე. შესწავლილია რეგრესიისა და (1) დამოუკიდებელ დაკვირვებათა საფუძველზე აგებული მისი (2) პროექციული შეფასების სხვაობის კვადრატის ინტეგრალის ზღვარითი განაწილება. მიღებული ზღვარითი თეორემების მეშვეობით მოცემულია რეგრესიის შესახებ ჰიპოთეზის შესამოწმებელი ტესტი და გამოთვლილია მისი სიმძლავრე პიტმენის ალტერნატივებისათვის.

Let observations $Y_1, Y_2, \dots, Y_n$ be represented as

$$Y_i = \mu(x_i) + \varepsilon_i, \quad i = \overline{1,n}, \tag{1}$$

where $\mu(x)$, $x \in [-\pi, \pi]$, is the unknown regression function to be estimated by observations Y_i; x_i, $i = \overline{1,n}$, are the known numbers, and $-\pi = x_0 < x_1 < \cdots < x_n \le \pi$, ε_i, $i = \overline{1,n}$, are independent equidistributed random variables; $E\varepsilon_1 = 0$, $E\varepsilon_1^2 = \sigma^2$ and $E\varepsilon_1^4 < \infty$.

The problem of nonparametric estimation of the regression function $\mu(x)$ for the model (1) has a recent history and has been treated only in few papers. In particular, a kernel estimator of the Rosenblatt–Parzen type for $\mu(x)$ was proposed for the first time in [1].

Assume that $\mu(x)$ is representable as a converging series in $L_2(-\pi, \pi)$ with respect to the orthonormal trigonometric system

$$\Big\{(2\pi)^{-1/2},\ \pi^{-1/2}\cos ix,\ \pi^{-1/2}\sin ix\Big\}_{i=1}^{\infty}.$$

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Consider the estimator of the function $\mu(x)$ constructed by the projection method of N.N. Chentsov [2]

$$\mu_{nN}(x) = \frac{a_{0n}}{2} + \sum_{i=1}^{N} a_{in}\cos ix + b_{in}\sin ix, \tag{2}$$

where $N = N(n) \to \infty$ for $n \to \infty$ and

$$a_{in} = \frac{1}{\pi}\sum_{j=1}^{n} Y_j\Delta_j \cos ix_j, \quad b_{in} = \frac{1}{\pi}\sum_{j=1}^{n} Y_j\Delta_j \sin ix_j,$$
$$\Delta_j = x_j - x_{j-1}, \quad j = \overline{1,n}, \quad i = \overline{0,N}.$$

The estimator (2) can be rewritten in a more compact way as

$$\mu_{nN} = \sum_{j=1}^{n} Y_j\Delta_j K_N(x - x_j),$$

where $K_N(u) = \frac{1}{2\pi}\sum_{|r|\le N} e^{iru}$ is the Dirichlet kernel.

In [3], p.347, N.V. Smirnov considered estimators of the type (2) for a specially chosen class of functions $\mu(x)$ in the case of equidistant points $x_j \in [-\pi,\pi]$ and of independent and normally distributed observation errors ε_i. In [4] an estimator of the type (2) is obtained, which is asymptotically equivalent to projection estimators which are optimal in the sense of some accuracy criterion. The asymptotics of the mean value of the integrated squared error of the estimator (2) is considered in [5].

It is of interest to investigate the limit distribution of the integrated squared error

$$\int_{-\pi}^{\pi} [\mu_{nN}(x) - \mu(x)]^2 dx,$$

which is the goal pursued in this paper. The metod used to prove the theorems below is based on the functional limit theorem for a sequence of semimartingales [6].

Denote

$$U_{nN}=\frac{n}{2\pi(2N+1)}\int_{-\pi}^{\pi}\left[\mu_{nN}(x)-E\mu_{nN}(x)\right]^2dx,$$

$$Q_{ir}=\Delta_i\Delta_rK_N(x_i-x_r),\quad \sigma_{nN}^2=\frac{n^2\sigma^4}{\pi^2(2N+1)^2}\sum_{r=2}^{n}\sum_{j=1}^{r-1}Q_{jr}^2,$$

$$\eta_{ik}=\frac{n}{\pi(2N+1)\sigma_{nN}}\varepsilon_i\varepsilon_kQ_{ik},$$

$$\xi_1=0,\quad \xi_k=\sum_{i=1}^{k-1}\eta_{ik},\quad k=\overline{2,n},\quad \xi_k=0,\quad k>n,$$

and assume that $\mathcal{F}_k$ is σ-algebra generated by random variables $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k$, $\mathcal{F}_0=(\phi,\Omega)$.

Lemma 1 ([7], p.179). *The stochastic sequence $(\xi_k,\mathcal{F}_k)_{k\ge1}$ is a martingale-difference.*

Lemma 2. *Let $p(x)$ be the known positive continuously differentiable distribution density on $[-\pi,\pi]$, and points x_i be chosen from the relation $\int_{-\pi}^{x_i}p(u)\,du=\frac{i}{n}$, $i=\overline{1,n}$.*

If $\frac{N\ln N}{n}\to0$ for $n\to\infty$, then

$$EU_{nN}=\theta_1+O\left(\frac{N\ln N}{n}\right),\quad \theta_1=\frac{\sigma^2}{(2\pi)^2}\int_{-\pi}^{\pi}p^{-1}(u)\,du,\qquad(3)$$

$$(2N+1)\sigma_{nN}^2\to\theta_2=\frac{\sigma^4}{4\pi^3}\int_{-\pi}^{\pi}p^{-2}(u)\,du.\qquad(4)$$

Proof. From the definition of x_i we easily obtain

$$\Delta_i=\frac{1}{np(x_i)}\left[1+O\left(\frac{1}{n}\right)\right],$$

where $O\left(\frac{1}{n}\right)$ is uniform with respect to $i=\overline{1,n}$.

Hence it follows that

$$Q_{ir}=\frac{1}{n^2p(x_i)p(x_r)}K_N(x_i-x_r)\left[1+O\left(\frac{1}{n}\right)\right].\qquad(5)$$

Taking into account the relation

$$\max_{-\pi\le u\le\pi}|K_N(u)|=O(N)\qquad(6)$$

and (5), we find

$$\sigma_{nN}^2=\frac{\sigma^4}{2\pi^2(2N+1)^2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}K_N^2(x_i-x_j)\frac{1}{[p(x_i)p(x_j)]^2}+O\left(\frac{1}{n}\right).\quad(7)$$

Let $F(x)$ be a distribution function with density $p(x)$ and $F_n(x)$ be an empirical distribution function of the "sample" $x_1, x_2, \dots, x_n$, i.e. $F_n(x) = n^{-1}\sum_{k=1}^{n} I_{(-\infty,x)}(x_k)$, where $I_A(\cdot)$ is the indicator of the set A. Then the right side of (7) can be written as the integral

$$\sigma_{nN}^2 = \frac{\sigma^4}{2\pi^2(2N+1)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N^2(t-s)\frac{dF_n(t)\,dF_n(s)}{[p(t)p(s)]^2} + O\Big(\frac{1}{n}\Big).$$

Further we have

$$\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N^2(t-s)\frac{dF_n(t)\,dF_n(s)}{[p(t)p(s)]^2} - \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N^2(t-s)\frac{dF(t)\,dF(s)}{[p(t)p(s)]^2}\right| \le$$

$$\le I_1 + I_2,$$

$$I_1 = \left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N^2(t-s)\frac{dF_n(s)}{[p(t)p(s)]^2}\big[dF_n(t) - dF(t)\big]\right|,$$

$$I_2 = \left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N^2(t-s)\frac{dF(t)}{[p(t)p(s)]^2}\big[dF_n(s) - dF(s)\big]\right|.$$

By integration by parts in the internal integral in I_1 we readily obtain

$$I_1 \le 2\int_{-\pi}^{\pi}\frac{dF_n(s)}{p^2(s)}\int_{-\pi}^{\pi}\big|dF_n(t) - dF(t)\big|\,\Big|\big(K_N'(t-s)p(t) - $$

$$-K_N(t-s)p'(t)\big)K_N(t-s)/p^3(t)\Big|\,dt. \tag{8}$$

Since $\sup_{-\pi\le x\le\pi}|F_n(x) - F(x)| = O\big(\frac{1}{n}\big)$ and the following relations [8][1]

$$\max_{-\pi\le u\le\pi}|K_N'(u)| = O(N^2), \quad \int_{-\pi}^{\pi} K_N^2(u)\,du = 2N+1,$$
$$\int_{-\pi}^{\pi}|K_N(u)|\,du = O(\ln N) \tag{9}$$

are fulfilled, from (8) we have the estimate

$$I_1 = O\left(\frac{N^2\ln N}{n}\right).$$

In the same manner we show that

$$I_2 = O\left(\frac{N^2\ln N}{n}\right).$$

Therefore

$$(2N+1)\sigma_{nN}^2 = \frac{\sigma^4}{4\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Phi_N(s-t)\frac{dt\,ds}{p(s)p(t)} + O\left(\frac{N\ln N}{n}\right), \tag{10}$$

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[1]See p. 115 in the Russian version of [8]: "Mir", Moscow, 1965.

where $\Phi_N(u) = \frac{2\pi}{2N+1} K_N^2(u)$ is the Fejér kernel.

We shall complete the definition of the function p^{-1} outside $[-\pi, \pi]$ as regards its periodicity and also note that $K_N(u)$ and $\Phi_N(u)$ are periodic functions with the period 2π. The continued function will be denoted by $g(x)$. Then

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_N(s-t)\frac{dt\,ds}{p(s)p(t)} = \int_{-\pi}^{\pi} p^{-2}(x)\,dx + \chi_n,$$

where

$$|\chi_n| \le \int_{-\pi}^{\pi} |\sigma_N(x) - g(x)|dx,$$

$$\sigma_N(x) = \int_{-\pi}^{\pi} \Phi_N(u)g(x-u)du.$$

Hence, on account of the theorem on convergence of the Fejér integral $\sigma_N(x)$ to $g(x)$ in the norm of the space $L_1(-\pi, \pi)$ (see [9], p.481), we have $\chi_n \to 0$ for $n \to \infty$.

Therefore

$$(2N+1)\sigma_{nN}^2 \to \frac{\sigma^4}{4\pi^3}\int_{-\pi}^{\pi} p^{-2}(x)\,dx.$$

Now we shall prove (3). We have

$$D\mu_{nN}(x) = \sigma^2 \sum_{j=1}^{n} \frac{1}{np^2(x_j)} K_N^2(x - x_j)\left[1 + O\left(\frac{1}{n}\right)\right].$$

Applying the same reasoning as in deriving (10), we find

$$D\mu_{nN}(x) = \frac{\sigma^2}{n}\int_{-\pi}^{\pi} K_N^2(x-s)\frac{ds}{p(s)} + O\left(\frac{N^2 \ln N}{n^2}\right). \qquad (11)$$

Therefore

$$EU_{nN} = \frac{\sigma^2}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_N(t-s)\frac{ds\,dt}{p(s)} + O\left(\frac{N \ln N}{n}\right) =$$

$$= \frac{\sigma^2}{(2\pi)^2}\int_{-\pi}^{\pi} p^{-1}(s)\,ds + O\left(\frac{N \ln N}{n}\right). \quad \blacksquare$$

Denote by the symbol $\xrightarrow{d}$ the convergence in distribution, and let ξ be a random variable having normal distribution with the zero mean and variance 1.

Theorem 1. *Let x_i $i = \overline{1,n}$ be the same as in Lemma 2 and $\frac{N^2 \ln N}{n} \to 0$ for $n \to \infty$. Then as n increases*

$$\sqrt{2N+1}(U_{nN} - \theta_1)\theta_2^{-1/2} \xrightarrow{d} \xi.$$

Proof. We have

$$\frac{U_{nN} - EU_{nN}}{\sigma_{nN}} = H_n^{(1)} + H_n^{(2)},$$

where

$$H_n^{(1)} = \sum_{j=1}^{n} \xi_j,$$

$$H_n^{(2)} = \frac{n}{2\pi(2N+1)\sigma_{nN}} \sum_{i=1}^{n} (\varepsilon_i^2 - E\varepsilon_i^2) Q_{ii}.$$

$H_n^{(2)}$ converges to zero in probability. Indeed,

$$DH_n^{(2)} \le \frac{n^2 E\varepsilon_1^4}{(2\pi)^2(2N+1)^2\sigma_{nN}^2} \sum_{i=1}^{n} Q_{ii}^2 =$$

$$= \frac{E\varepsilon_1^4}{(2\pi)^2(2N+1)^2\sigma_{nN}^2 \cdot n^2} \sum_{i=1}^{n} \frac{1}{(p(x_i))^4} K_N^2(0)\left(1 + O\left(\frac{1}{n}\right)\right) \le$$

$$\le C\frac{1}{n\sigma_{nN}^2} = O\left(\frac{N}{n}\right),$$

whence $H_n^{(2)} \xrightarrow{P} 0$. Here and in what follows C is the positive constant varying from one formula to another and the letter P above the arrow denotes convergence in probability.

We will prove now that $H_n^{(1)} \xrightarrow{d} \xi$. To this end we will verify the validity of Corollaries 2 and 6 of Theorem 2 from [6]. We have to show whether the the conditions contained in these statements are fulfilled for asymptotic normality of the square-integrable martingale-difference, which, by Lemma 1, is our sequence $\{\xi_k, \mathcal{F}_k\}_{k\ge1}$.

A direct calculation shows that $\sum_{k=1}^{n} E\xi_k^2 = 1$. Asymptotic normality will take place if for $n \to \infty$

$$\sum_{k=1}^{n} E\left[\xi_k^2 \cdot I(|\xi_k| \ge \varepsilon) \mid \mathcal{F}_{k-1}\right] \to 0 \tag{12}$$

and

$$\sum_{k=1}^{n} \xi_k^2 \xrightarrow{P} 1. \tag{13}$$

It is shown in [6] that the fulfillment of (13) and the condition $\sup_{1\le k\le n} |\xi_k| \xrightarrow{P} 0$ implies the validy of (12) as well.

Since for $\varepsilon > 0$

$$P\Big\{ \sup_{1\le k\le n} |\xi_k| \ge \varepsilon \Big\} \le \varepsilon^{-4} \sum_{k=1}^{n} E\xi_k^4,$$

to prove $H_n^{(1)} \xrightarrow{d} \xi$ we have to verify only (13) by the relation (15) to be given below.

We will establish $\sum_{k=1}^{n} \xi_k^2 \xrightarrow{P} 1$. For this it suffices to make sure that $E(\sum_{k=1}^{n} \xi_k^2 - 1))^2 \to 0$ for $n \to \infty$, i.e. due to $\sum_{i=1}^{n} E\xi_i^2 = 1$

$$E\Big(\sum_{k=1}^{n} \xi_k^2\Big)^2 = \sum_{k=1}^{n} E\xi_k^4 + 2 \sum_{1\le k_1<k_2\le n} E\xi_{k_1}^2\xi_{k_2}^2 \to 1. \tag{14}$$

In the first place we find that $\sum_{k=1}^{n} E\xi_k^4 \to 0$ for $n \to \infty$. By virtue of the definitions of ξ_k and η_{ij} we write

$$\sum_{k=1}^{n} E\xi_k^4 = L_n^{(1)} + L_n^{(2)},$$

where

$$L_n^{(1)} = \frac{n^4}{\pi^4(2N+1)^4\sigma_{nN}^4} E\varepsilon_1^4(E\varepsilon_1^4 - 3\sigma^4) \sum_{k=2}^{n}\sum_{j=1}^{k-1} Q_{jk}^4,$$

$$L_n^{(2)} = \frac{3n^4\sigma^4 E\varepsilon_1^4}{(2N+1)^4\sigma_{nN}^4\pi^4} \sum_{k=2}^{n} \Big(\sum_{j=1}^{k-1} Q_{jk}^2\Big)^2.$$

From (5) and (6) we obtain

$$|L_n^{(1)}| = C\frac{1}{n^4N^4\sigma_{nN}^4} \sum_{k=2}^{n}\sum_{j=1}^{k-1} \frac{K_N'^4(x_j - x_k)}{[p(x_j)p(x_k)]^4}\Big[1 + O\Big(\frac{1}{n}\Big)\Big] \le$$

$$\le Cn^{-2}\sigma_{nN}^{-4} = O\Big(\Big(\frac{N}{n}\Big)^2\Big)$$

and also

$$|L_n^{(2)}| = C\frac{1}{n^2 N^4 \sigma_{nN}^4} \sum_{k=2}^{n} \left(\frac{1}{n} \sum_{j=1}^{k-1} \frac{K_N^2(x_j - x_k)}{p(x_j)p(x_k)} \left[1 + O\left(\frac{1}{n}\right)\right] \right)^2 \leq$$

$$\leq C\frac{1}{n^2 N^4 \sigma_{nN}^4} \sum_{k=1}^{n} \left(\frac{1}{n} \sum_{j=1}^{k-1} K_N^2(x_j - x_k) \right)^2 =$$

$$= C\frac{1}{n^2 N^4 \sigma_{nN}^4} \sum_{k=2}^{n} \left(\int_{-\pi}^{\pi} \frac{K_N^2(x_k - u)}{p^2(u)} dF_n(u) \right)^2 \leq$$

$$\leq C\frac{1}{n^2 N^4 \sigma_{nN}^4} \sum_{k=2}^{n} \left\{ \left[K_N^2(x_k - u) p^{-1}(u)\, du \right]^2 + \right.$$

$$\left. + \left[\int_{-\pi}^{\pi} K_N^2(x_k - u) p^{-2}(u)\, d\big(F_n(u) - F(u)\big) \right]^2 \right\}^2.$$

Hence, taking into account the relation (9) and the formula of integration by parts, we have

$$|L_n^{(2)}| = O\left(\frac{1}{n}\right).$$

Therefore

$$\sum_{k=1}^{n} E\xi_k^4 \to 0 \quad \text{for } n \to \infty. \qquad (15)$$

Let us now establish that $2\sum_{1 \leq k_1 < k_2 \leq n} E\xi_{k_1}^2 \xi_{k_2}^2 \to 1$ for $n \to \infty$. The definition of ξ_i implies

$$\xi_{k_1}^2 \xi_{k_2}^2 = \left(\sum_{i=1}^{k_1 - 1} \eta_{ik_1}^2 \right) \left(\sum_{i=1}^{k_2 - 1} \eta_{ik_2}^2 \right) + \left(\sum_{i=1}^{k_1 - 1} \eta_{ik_1}^2 \right) \left(\sum_{i \neq s=1}^{k_2 - 1} \eta_{ik_2} \eta_{sk_2} \right) +$$

$$+ \left(\sum_{i=1}^{k_2 - 1} \eta_{ik_2}^2 \right) \left(\sum_{s \neq t=1}^{k_1 - 1} \eta_{sk_1} \eta_{tk_1} \right) + \left(\sum_{s \neq t=1}^{k_1 - 1} \eta_{sk_1} \eta_{tk_1} \right) \left(\sum_{k \neq r=1}^{k_2 - 1} \eta_{kk_1} \eta_{rk_2} \right) =$$

$$= B_{k_1 k_2}^{(1)} + B_{k_1 k_2}^{(2)} + B_{k_1 k_2}^{(3)} + B_{k_1 k_2}^{(4)}.$$

Therefore

$$2 \sum_{1 \leq k_1 < k_2 \leq n} E\xi_{k_1}^2 \xi_{k_2}^2 = \sum_{i=1}^{4} A_n^{(i)},$$

where

$$A_n^{(i)} = 2 \sum_{1 \leq k_1 < k_2 \leq n} EB_{k_1 k_2}^{(i)}, \quad i = \overline{1,4}.$$

In the first place we consider $A_n^{(3)}$. By the definition of η_{ij} we obtain

$$E\eta_{ik_2}^2\eta_{sk_1}\eta_{tk_1} = 0, \quad s \neq t, \ k_1 < k_2.$$

Thus

$$A_n^{(3)} = 0. \tag{16}$$

Let us derive an estimate of $A_n^{(2)}$. Divide the sum $EB_{k_1k_2}^{(2)}$ into two parts:

$$EB_{k_1k_2}^{(2)} = \sum_{i=1}^{k_1-1}\sum_{r\neq s=1}^{k_1} E\eta_{ik_1}^2\eta_{rk_2}\eta_{sk_2} + \sum_{i=1}^{k_1-1}\sum_{r\neq s=k_1+1}^{k_2-1} E\eta_{ik_1}^2\eta_{rk_2}\eta_{sk_2}.$$

The second term is equal to zero, since i cannot coincide with r or with s and $r \neq s$; in this case $E\eta_{ik_1}^2\eta_{rk_2}\eta_{sk_2} = 0$, and $E\eta_{ik_1}^2\eta_{rk_2}\eta_{sk_2} = 0$ also in the first term each time except the case $s = k_1$ or $r = k_1$.

Thus

$$EB_{k_1k_2}^{(2)} = 2\sum_{i=1}^{k_1-1} E\left(\eta_{ik_1}^2\eta_{ik_2}\eta_{k_1k_2}\right).$$

Hence, using the definition of η_{ij} and the inequality $|Q_{ij}| \leq C\frac{N}{n^2}$ obtained from (5) and (6), we find

$$\left|EB_{k_1k_2}^{(2)}\right| \leq C\frac{1}{(2N+1)^2\sigma_{nN}^4}\sum_{i=1}^{k_1-1} Q_{ik_1}^2. \tag{17}$$

Next, taking into account statement (4) of Lemma 2 and the definition of σ_{nN}^2, from (17) we have

$$|A_n^{(2)}| \leq C\frac{n}{N^2\sigma_{nN}^4}\sum_{k_1=2}^{n}\sum_{i=1}^{k_1-1} Q_{ik_1}^2 \leq C\frac{1}{n\sigma_{nN}^2} = O\left(\frac{N}{n}\right). \tag{18}$$

Consider now $A_n^{(4)}$. By the definition of η_{ij} we obtain

$$A_n^{(4)} = \frac{8n^4}{\pi^4(2N+1)^4\sigma_{nN}^4}\sum_{s<t<k_1<k_2} Q_{sk_1}Q_{sk_2}Q_{tk_1}Q_{tk_2} \leq$$

$$\leq C\frac{n^4}{N^4\sigma_{nN}^4}\left[\left|\sum_{s,t,k_1,k_2} Q_{sk_1}Q_{sk_2}Q_{tk_1}Q_{tk_2}\right| + \right.$$

$$\left. + \left|\sum_{k_1,s,t} Q_{k_1s}^2 Q_{k_1t}^2\right| + \left|\sum_{k_1,s,t} Q_{k_1t}Q_{st}Q_{k_1s}Q_{ss}\right|\right] =$$

$$= C\frac{n^4}{N^4\sigma_{nN}^4}\left[|E_1| + |E_2| + |E_3|\right]. \tag{19}$$

According to (5) and (6) we write

$$E_1 = n^{-7} \sum_{s,t,k_1} K_N(x_s - x_{k_1}) K_N(x_t - x_{k_1}) \times$$
$$\times \int_{-\pi}^{\pi} K_N(x_s - u) K_N(x_t - u)\, dF_n(u) + O\left(\frac{N^2}{n}\right).$$

Hence, integrating by parts and taking into account (9), we obtain

$$E_1 = n^{-7} \int_{-\pi}^{\pi} \sum_{s,t,k_1} K_N(x_s - x_{k_1}) K_N(x_t - x_{k_1}) \times$$
$$K_N(x_s - u) K_N(x_t - u) p(u)\, du + O\left(\frac{N^4 \ln N}{n^5}\right). \tag{20}$$

Applying the same operations three times, we represent (20) in the form

$$E_1 = n^{-4} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_N(z-u) K_N(z-t) K_N(y-u) K_N(y-t) \times$$
$$\times p(y) p(u) p(z) p(t)\, du\, dt\, dz\, dy + O\left(\frac{N^4 \ln N}{n^5}\right) =$$
$$= O\left(\frac{N \ln^3 N}{n^4}\right) + O\left(\frac{N^4 \ln N}{n^5}\right).$$

Thus

$$\frac{n^4}{N^4 \sigma_{nN}^4} |E_1| = O\left(\frac{\ln^3 N}{N}\right) + O\left(\frac{N^2 \ln N}{n}\right). \tag{21}$$

Further, it is not difficult to show

$$\frac{n^4}{N^4 \sigma_{nN}^4} |E_2| = O\left(\frac{N^2}{n}\right),$$
$$\frac{n^4}{N^4 \sigma_{nN}^4} |E_3| = O\left(\frac{N^2}{n}\right). \tag{22}$$

Therefore (19), (21) and (22) imply

$$A_n^{(4)} = O\left(\frac{N^2 \ln N}{n}\right) + O\left(\frac{\ln^3 N}{N}\right). \tag{23}$$

Finally, we will show that $A_n^{(1)} \to 1$ for $n \to \infty$. For this represent $A_n^{(1)}$ in the form

$$A_n^{(1)} = Q_n^{(1)} + Q_n^{(2)},$$

where

$$Q_n^{(1)} = 2 \sum_{k_1<k_2} \Big(\sum_{i=1}^{k_1-1} E\eta_{ik_1}^2 \Big)\Big(\sum_{j=1}^{k_2-1} E\eta_{ik_2}^2 \Big),$$

$$Q_n^{(2)} = 2\Big(\sum_{k_1<k_2} EB_{k_1k_2}^{(1)} - \sum_{k_1<k_2} \Big(\sum_{i=1}^{k_1-1} E\eta_{ik_1}^2 \Big)\Big(\sum_{j=1}^{k_2-1} E\eta_{ik_2}^2 \Big)\Big).$$

From the definition of σ_{nN}^2 it follows that

$$Q_n^{(1)} = 1 - \sum_{k=2}^{n} \Big(\sum_{i=1}^{k-1} E\eta_{ik}^2 \Big)^2,$$

where

$$\sum_{k=2}^{n} \Big(\sum_{i=1}^{k-1} E\eta_{ik}^2 \Big)^2 \le C \frac{n^4}{N^4\sigma_{nN}^4} \sum_{k=2}^{n} \Big(\sum_{i=1}^{k-1} Q_{ik}^2 \Big)^2 \le$$

$$\le C \frac{1}{n\sigma_{nN}^4} = O\Big(\frac{N^2}{n}\Big).$$

Therefore

$$Q_n^{(1)} = 1 + O\big(N^2/n\big). \tag{24}$$

Let us now show that $Q_n^{(2)} \to 0$. $Q_n^{(2)}$ can be written as

$$Q_n^{(2)} = 2 \sum_{k_1<k_2} \Big[\sum_{i=1}^{k_1-1} \Big(\operatorname{cov}(\eta_{ik_1}^2, \eta_{ik_2}^2) + \operatorname{cov}(\eta_{ik_1}^2, \eta_{k_1k_2}^2) \Big) \Big].$$

But

$$E\eta_{ik_1}^2 \eta_{ik_2}^2 \le C \frac{n^4}{N^4\sigma_{nN}^4} Q_{ik_1}^2 \cdot Q_{ik_2}^2 \le$$

$$\le C \frac{1}{n^4N^4\sigma_{nN}^4} \Big(\max_{-\pi\le u\le\pi} |K_N(u)| \Big)^4 = O\Big(\frac{1}{n^4\sigma_{nN}^4}\Big).$$

Similarly,

$$E\eta_{ij}^2 = O\big(n^{-2}\sigma_{nN}^{-2}\big).$$

Therefore

$$\operatorname{cov}(\eta_{ik_1}^2, \eta_{ik_2}^2) = O\Big(\frac{1}{n^4\sigma_{nN}^4}\Big). \tag{25}$$

Further, since $\sum_{1\le k_1<k_2\le n}(k_1 - 1) = O(n^3)$, (25) implies

$$Q_n^{(2)} = O\Big(\frac{N^2}{n}\Big). \tag{26}$$

Thus, according to (24) and (26)

$$A_n^{(1)} = 1 + O\big(N^2/n\big). \qquad (27)$$

Combining the relations (16), (18), (23) and (27), we finally obtain

$$E\Big(\sum_{k=1}^{n} \xi_k^2 - 1\Big)^2 \to 0 \quad \text{for } n \to \infty.$$

Therefore

$$\frac{U_{nN} - EU_{nN}}{\sigma_{nN}} \xrightarrow{d} \xi.$$

Further, due to Lemma 2, $EU_{nN} = \theta_1 + O(\frac{N \ln N}{n})$ and $(2N+1)\sigma_{nN}^2 \to \theta_2$, and hence we obtain

$$(2N+1)^{1/2}(U_{nN} - \theta_1)\theta_2^{-1/2} \xrightarrow{d} \xi. \quad \blacksquare$$

Denote

$$T_{nN} = \frac{n}{2\pi(2N+1)} \int_{-\pi}^{\pi} \big[\mu_{nN}(x) - \mu(x)\big]^2 dx.$$

Theorem 2. *Let x_i, $i = \overline{1,n}$, be the same as in Lemma 2 and the function $\mu(x)$ with period 2π have bounded derivatives up to the second order. Moreover, if $N^2 \ln N/n \to 0$ and $n \ln^2 N/N^{9/2} \to 0$ for $n \to \infty$, then*

$$\sqrt{2N+1}(T_{nN} - \theta_1)\theta_2^{-1/2} \xrightarrow{d} \xi.$$

Before we proceed to proving the theorem, we have to show

$$\int_{-\pi}^{\pi} |K_N'(u)| du = O(N \ln N). \qquad (28)$$

Denote $\widetilde{D}_\nu(u) = \sum_{k=1}^{\nu} \sin ku$. Then by virtue of the Abel transformation we have

$$K_N'(u) = -\sum_{k=1}^{N} k \sin ku = \sum_{\nu=1}^{N-1} \widetilde{D}_\nu(u) + N\widetilde{D}_N.$$

It is well-known [8] that $\eta_\nu = (\ln \nu)^{-1} \int_{-\pi}^{\pi} |\widetilde{D}_\nu(u)| du \to 1$ for $\nu \to \infty$. Denote $b_N = \sum_{\nu=1}^{N-1} \ln \nu$. Then by Toeplitz lemma

$$R_N = \frac{1}{b_N} \sum_{\nu=1}^{N-1} \ln \nu \cdot \eta_\nu \to 1.$$

Therefore

$$\int_{-\pi}^{\pi} |K_N'(u)|du \le \sum_{\nu=1}^{N-1} \int_{-\pi}^{\pi} |\widetilde{D}_\nu(u)|du + N\int_{-\pi}^{\pi} |\widetilde{D}_N(u)|du =$$
$$= b_N \cdot R_N + N\int_{-\pi}^{\pi} |\widetilde{D}_N(u)|du = O(N\ln N).$$

Let us return to the proof of the theorem. We have

$$T_{nN} = U_{nN} + A_{1n} + A_{2n},$$
$$A_{1n} = \frac{n}{\pi(2N+1)}\int_{-\pi}^{\pi}\Big[\mu_{nN}(x) - E\mu_{nN}(x)\Big]\Big[E\mu_{nN}(x) - \mu(x)\Big]dx,$$
$$A_{2n} = \frac{n}{2\pi(2N+1)}\int_{-\pi}^{\pi}\Big[E\mu_{nN}(x) - \mu(x)\Big]^2 dx.$$

It is not difficult to find

$$\sqrt{2N+1}E|A_{1n}| \le \frac{n\sigma^2}{2\pi\sqrt{2N+1}}\Big(\sum_{j=1}^{n}\Delta_j^2\Big[\int_{-\pi}^{\pi} K_n(y-x_j)\times$$
$$\times\Big(E\mu_{nN}(y) - \mu(x)\Big)dy\Big]^2\Big)^{1/2}.$$

But

$$E\mu_{nN}(y) = \int_{-\pi}^{\pi}\mu(x)\frac{1}{p(x)}K_N(y-x)\,dF_n(x)\Big(1 + O\Big(\frac{1}{n}\Big)\Big)$$

and

$$\int_{-\pi}^{\pi}\mu(x)p^{-1}(x)K_N(y-x)\,dF_n(x) =$$
$$= \int_{-\pi}^{\pi}\mu(x)K_N(y-x) + O\Big(\frac{1}{n}\int_{-\pi}^{\pi}|K_N'(u)|du\Big).$$

It is well-known ([10], p.22) that

$$\int_{-\pi}^{\pi}\mu(x)K_N(y-x)\,dx = \mu(y) + O\Big(\frac{\ln N}{N^2}\Big)$$

uniformly in $y \in [-\pi,\pi]$. By virtue of (28) this gives us

$$E\mu_{nN}(x) = \mu(x) + O\Big(\frac{\ln N}{N^2}\Big) + O\Big(\frac{N\ln N}{n}\Big). \qquad (29)$$

Therefore

$$\sqrt{2N+1}E|A_{1n}| \le C\left[\left(\frac{n\ln^2 N}{N^{9/2}}\right)^{1/2}\frac{\ln N}{N^{1/4}} + \right.$$

$$\left. +\left(\frac{N^2\ln N}{n}\right)^{1/2}\frac{\ln^{3/2} N}{\sqrt{N}}\right] \to 0. \qquad (30)$$

Further, from (29) we have

$$\sqrt{2N+1}A_{2n} \le C\left(\frac{n\ln^2 N}{N^{9/2}} + \frac{N^2}{n}\frac{\ln^2 N}{\sqrt{N}}\right) \to 0. \qquad (31)$$

Finally, the statement of Theorem 2 directly follows from Theorem 1, (30) and (31).

Using Theorems 1 and 2, it is easy to solve the question concerning testing the hypothesis about $\mu(x)$. Given σ^2, it is required to verify the hypothesis $H_0 : \mu(x) = \mu_0(x)$. The critical region is defined approximately by the inequality

$$U_{nN} \ge d_n(\alpha) \quad \text{or} \quad T_{nN} \ge d_n(\alpha),$$

where

$$d_n(\alpha) = \sigma^2\big(L_1 + (2N+1)^{-1/2}L_2\big)\lambda_\alpha,$$

$$L_1 = ((2\pi)^{-2}\int_{-\pi}^{\pi} p^{-1}(x)\,dx, \quad L_2 = \left(\frac{1}{4\pi^3}\int_{-\pi}^{\pi} p^{-2}(x)\,dx\right)^{1/2},$$

and λ_α is the quantile of level α of standard normal distribution.

Let now σ^2 be unknown. We call an $\sqrt{N}$-consistent estimate of variance σ^2, for instance,

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\big(Y_i - \mu_{n\lambda}(x_i)\big)^2,$$

where $\lambda = \lambda(n) \to \infty$ is a sequence such that $\frac{\lambda}{N} \to 0$, $\frac{N\ln^2\lambda}{\lambda^4} \to 0$ and $\frac{N\lambda^4}{n} \to 0$ for $n \to \infty$.

Indeed, using the expressions (11) and (29), we easily find

$$\sqrt{N}(ES_n^2 - \sigma^2) = O\left(\left(\frac{N\lambda}{n}\right)^{1/2}\right) + O\left(\frac{N^{1/2}\ln\lambda}{\lambda^2}\right). \qquad (32)$$

Denote

$$Z_j = Y_j - R_j,$$

$$R_j = \sum_{k=1}^{n} Y_k\Delta_k K_\lambda(x_j - x_k).$$

Then

$$n^2 DS_n^2 = \sum_{j=1}^{n} DZ_j^2 + \sum_{i \neq i_1} \operatorname{cov}(Z_i^2, Z_{i_1}^2).$$

Simple calculations show that

$$\operatorname{cov}(Z_j^2, Z_{j_1}^2) = O\left(\frac{\lambda^4}{n}\right).$$

Therefore

$$DS_n^2 = O\left(\frac{\lambda^4}{n}\right).$$

This and (32) imply

$$\sqrt{N}(S_n^2 - \sigma^2) \xrightarrow{P} 0.$$

Corollary. *Let the conditions of Theorem 2 be fulfilled. Moreover, let* $\frac{\lambda}{n} \to 0$, $\frac{N\lambda^4}{n} \to 0$ *and* $\frac{N \ln^2 \lambda}{\lambda^4} \to 0$. *Then*

$$S_n^{-2} L_2^{-1} \sqrt{2N+1}(U_{nN} - S_n^2 L_1) \xrightarrow{d} \xi,$$

$$S_n^{-2} L_2^{-1} \sqrt{2N+1}(T_{nN} - S_n^2 L_1) \xrightarrow{d} \xi.$$

This corollary enables one to construct a test for verifying $H_0 : \mu(x) = \mu_0(x)$. The critical region is defined approximately by the inequality

$$U_{nN} \geq \tilde{d}_n(\alpha) \quad \text{or} \quad T_{nN} \geq \tilde{d}_n(\alpha),$$

where $\tilde{d}_n(\alpha)$ is obtained from $d_n(\alpha)$ by using S_n^2 instead of σ^2.

Consider now the local behaviour of the test power in the case when the critical region is of the form $\{x \in R^1,\ x \geq d_n(\alpha)\}$. More exactly, find a distribution of the quadratic functional U_{nN} under a sequence of alternatives close to the hypothesis $H_0 : \mu(x) = \mu_0(x)$. The sequence is written as

$$H_1 : \bar{\mu}(x) = \mu_0(x) + \gamma_n \varphi(x) + o(\gamma_n), \tag{33}$$

where $\gamma_n \to 0$ appropriately and $o(\gamma_n)$ is uniform in $x \in [-\pi, \pi]$.

Theorem 3. *Let* $\bar{\mu}_n(x)$ *satisfy the conditions of Theorem 2. If* $2N + 1 = n^\delta$, $\gamma_n = n^{-1/2+\delta/4}$, $\frac{2}{9} < \delta < \frac{1}{2}$, *then under the alternative* H_1 *the statistic* $(2N+1)^{1/2}(U_{nN} - \theta_1)$ *is distributed in the limit normally* $\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi^2(u)du, \sqrt{\theta_2}\right)$.

Proof. Let us represent U_{nN} as the sum

$$U_{nN} = \frac{n}{2\pi(2N+1)} \int_{-\pi}^{\pi} \left(\mu_{nN}(x) - E_1\mu_{nN}(x)\right)^2 dx + \\ + \frac{n}{\pi(2N+1)}\gamma_n \int_{-\pi}^{\pi} \left[\mu_{nN}(x) - E_1\mu_{nN}(x)\right]\tilde{\varphi}_n(x)\,dx + \\ + \frac{n}{2\pi(2N+1)}\gamma_n^2 \int_{-\pi}^{\pi} \tilde{\varphi}_n^2(x)\,dx = A_1(n) + A_2(n) + A_3(n),$$

where $E_1(\cdot)$ denotes the mathematical expectation under the hypothesis H_1,

$$\tilde{\varphi}_n(x) = \sum_{j=1}^{n} \varphi(x_j)\Delta_j K_n(x - x_j).$$

Due to Theorem 1 one can readily make sure that $\sqrt{2N+1}(A_1(n)-\theta_1)$ is distributed asymptotically normal $(0, \sqrt{\theta_2})$.

By analogy with the proof of Lemma 2 we find

$$\sqrt{2N+1}A_3(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\varphi(y)K_N(x-y)dy\right)^2 dx + O\left(\frac{N^2\ln N}{n}\right).$$

Hence, by virtue of theorem 2 from [9], p.474, we have

$$\sqrt{2N+1}A_3(n) \to \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi^2(u)du.$$

Further, for our choice of N and γ_n we can show by simple calculations that

$$\sqrt{2N+1}E|A_2(n)| \le C\left(\frac{\ln^2 n}{n^{\delta/4}} + \frac{\ln n}{n^{1-7\delta/4}}\right). \quad \blacksquare$$

Thus the local behaviou of the power $P_{H_1}(U_{nN} \ge d_n(\alpha))$ is

$$P_{H_1}\left(U_{nN} \ge d_n(\alpha)\right) \to 1 - \Phi\left(\lambda_\alpha - \theta_2^{-1/2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi^2(u)\,du\right). \quad (34)$$

Since $\int_{-\pi}^{\pi}\varphi^2(u)du > 0$ and is equal to zero iff $\varphi(x) = 0$, from (34) we conclude that the test for the hypothesis $H_0 : \mu(x) = \mu_0(x)$ against alternatives of the form (33) is asymptotically strictly unbiased.

Remark. Similar results can be obtained by the same method for the kernel estimator of Priestley and Chao [1].

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ON THE INTEGRAL BERNSTEIN OPERATORS IN SOME CLASSES OF MEASURABLE BIVARIATE FUNCTIONS*

ROMAN TABERSKI

Abstract. The main two theorems cencern the approximations of (complex-valued) functions on the real plane by the sums of Bernstein pseudoentire functions. They are formulated and proved in Section 4, after the prior determination of the suitable integral operators. Analogous results for pseudopolynomial approximations were obtained by Brudnyǐ, Gonska and Jetter ([2],[3]).

რეზიუმე. ნაშრომში შესწავლილია კომპლექსური მნიშვნელობის მქონე ორი ცვლადის ფუნქციათა ბერნშტეინის ფსევდომთელი ფუნქციებით მიახლოების საკითხები.

1. Preliminaries. Let $L_{loc}(R)$ [resp. $AC_{loc}(R)$] be the set of all univariate (complex-valued) functions Lebesgue-integrable (absolutely continuous) on every compact subinterval of $R := (-\infty, \infty)$. Denote by $L^p_{loc}(R^2)$, $1 \le p \le \infty$ the set of all measurable bivariate (complex-valued) functions Lebesgue-integrable with p-th power (essentially bounded when $p = \infty$) on every finite two-dimensional integral lying on the plane $R^2 = R \times R$; write $L_{loc}(R^2)$ instead of $L^1_{loc}(R^2)$. Denote by $C(R)$ [resp. $C(R^2)$] the set of all (complex-valued) functions continuous on $R\,[R^2]$.

Given any bivariate (complex-valued) function $f \equiv f(\cdot,\cdot)$ measurable on R^2, the quantity

$$\|f\|_p := \begin{cases} \left(\iint_{R^2} |f(u,v)|^p du\, dv\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{(u,v)\in R^2} |f(u,v)| & \text{if } p = \infty \end{cases} \tag{1}$$

is finite or infinite. In case $\|f\|_p < \infty$ the function f is said to be of class $L^p \equiv L^p(R^2)$, in symbols $f(\cdot,\cdot) \in L^p$. Analogous meanings have the notation: $f(\cdot,\cdot) \in L^p_{loc}(R^2)$, $g(\cdot, v) \in L_{loc}(R)$, etc.

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Let $f \equiv f(\cdot,\cdot)$ be a (complex-valued) function defined on R^2 and let $k, l \in N_0 \equiv \{0,1,2,\dots\}$. Determine the partial differences of f at a point $(x,y) \in R^2$, i.e.

$$^1\Delta_\lambda^k f(x,y) := \sum_{\mu=0}^{k}(-1)^{k-\mu}\binom{k}{\mu} f(x+\mu\lambda, y),$$

$$^1\Delta_\eta^l f(x,y) := \sum_{\nu=0}^{k}(-1)^{l-\nu}\binom{l}{\nu} f(x, y+\nu\eta),$$

with the real increments λ, η. Introduce also the mixed difference

$$\Delta_{\lambda,\eta}^{k,l} := {}^1\Delta_\lambda^k\left({}^2\Delta_\eta^l f(x,y)\right).$$

It is easy to see that

$$\Delta_{\lambda,\eta}^{k,l} = \sum_{\mu=0}^{k}\sum_{\nu=0}^{l}(-1)^{k+l-\mu-\nu}\binom{k}{\mu}\binom{l}{\nu} f(x+\mu\lambda, y+\nu\eta).$$

In particular, $\Delta_{\lambda,\eta}^{0,0} f(x,y) = f(x,y)$ and

$$\Delta_{\lambda,\eta}^{1,1} f(x,y) = f(x+\lambda, y+\eta) - f(x+\lambda, y) - f(x, y+\eta) + f(x,y).$$

Further,

$$\Delta_{\lambda,\eta}^{1,1}\left(\Delta_{\lambda,\eta}^{k,l} f(x,y)\right) = \Delta_{\lambda,\eta}^{k+1,l+1} f(x,y).$$

The first (weak) derivative of f at (x,y) is given by

$$f^{(1)}(x,y) := \lim_{\lambda\to 0}\frac{1}{\lambda^2}\Delta_{\lambda,\lambda}^{1,1} f(x,y)$$

whenever the right-hand side exists. The (weak) derivatives of f of higher orders are defined successively:

$$f^{(j)}(x,y) := \left(f^{(j-1)}\right)^{(1)}(x,y) \quad \text{for } j = 2,3,\dots.$$

Moreover, by convention, $f^{(0)}(x,y) \equiv f(x,y)$ and

$$f^{(m,n)}(x,y) \equiv \frac{\partial^m}{\partial x^m}\left(\frac{\partial^n}{\partial y^n} f(x,y)\right) \quad \text{for } m,n \in N_0.$$

Considering any (complex-valued) function f measurable on R^2, one can define its mixed L^p-modulus of smoothness:

$$\omega_{k,l}(\delta_1,\delta_2; f)_p := \sup\left\{\|\Delta_{\lambda,\eta}^{k,l} f\|_p : 0 \le \lambda \le \delta_1,\ 0 \le \eta \le \delta_2\right\}.$$

This quantity, with fixed $p \geq 1$ and $k, l \in N_0$, may be finite or infinite for positive numbers δ_1, δ_2. If there exist three non-negative numbers M, α, β such that

$$\omega_{k,l}(\delta_1, \delta_2; f)_p \leq M \delta_1^\alpha \delta_2^\beta \quad \text{for all } \delta_1, \delta_2 \in (0,1]$$

we say that f belongs to the Hölder class $H^{(k,l)}_{\alpha,\beta,p}$. More generally, if

$$\|f\|^{(k,l)}_{\varphi,\psi,p} \equiv \sup\left\{\frac{1}{\varphi(\lambda)\psi(\eta)}\|\Delta^{k,l}_{\lambda,\eta} f\|_p : 0 < \lambda, \eta \leq 1\right\} < \infty, \tag{2}$$

where φ, ψ mean positive non-decreasing functions on $(0,1]$ and $\varphi(1) = \psi(1) = 1$, the function f is said to be of class $H^{(k,l)}_{\varphi,\psi,p}$. In case $\varphi(\delta) = \delta^\alpha$, $\psi(\delta) = \delta^\beta$ for $\delta \in (0,1]$, the left-hand side of identity (2) will be signified by $\|f\|^{(k,l)}_{\alpha,\beta,p}$.

Denote by E_σ [resp. $E_{\sigma,\tau}$] the set of all univariate (bivariate) entire functions of exponential type of order σ $[(\sigma, \tau)]$ at most. Clearly, if $F(\cdot,\cdot) \in E_{\sigma,\tau}(\sigma, \tau \geq 0)$ then $F(\cdot, v) \in E_\sigma$ and $F(u, \cdot) \in E_\tau$ for all $u, v \in R$. Moreover, $F(\cdot,\cdot) \in C(R^2)$. In the case when $\Phi(\cdot, v) \in E_\sigma$ (resp. $\Psi(u, \cdot) \in E_\tau$) for almost every $v \in R$ $[u \in R]$, $\Phi(z, \cdot) \in L_{loc}(R)$ $[\Psi(\cdot, z) \in L_{loc}(R^2)]$ for every complex number z, and $\Phi(\cdot,\cdot) \in L_{loc}(R^2)$ $[\Psi(\cdot,\cdot) \in L_{loc}(R^2)]$, we call Φ $[\Psi]$ the pseudoentire function of class W^1_σ $[W^2_\tau]$.

The aim of this paper is to present the Jackson type theorems, in L^p-norms (1) and seminorms (2), for some (complex-valued) functions defined and measurable on R^2. We begin with auxiliary results about the mixed differences and Bernstein's singular integrals used in our approximation problems.

2. Estimates for the mixed differences and moduli of smoothness. Consider a (complex-valued) function $f \equiv f(\cdot,\cdot)$ defined and measurable on the plane R^2. Denote by k, l two non-negative integers. Take an arbitrary p satisfying the condition: $1 \leq p \leq \infty$

Proposition 1. *If $\lambda, \eta \in R$ and $n \in N \equiv \{1, 2, 3, \dots\}$ then*

$$\|\Delta^{k,l}_{m\lambda,n\eta} f\|_p \leq m^k n^l \|\Delta^{k,l}_{\lambda,\eta} f\|_p. \tag{3}$$

Proof. Given arbitrary $x, y \in R$, let

$$g(x,y) := {}^2\Delta^l_{n\eta} f(x,y).$$

By identity (5) of [5], p.116,

$$g(x,y) = \sum_{\nu_1=0}^{n-1} \cdots \sum_{\nu_l=0}^{n-1} {}^2\Delta^l_\eta f(x, y + \nu_1\eta + \cdots + \nu_l\eta)$$

and

$$^1\Delta^k_{m\lambda} g(x,y) = \sum_{\mu_1=0}^{m-1} \cdots \sum_{\mu_k=0}^{m-1} {}^1\Delta^k_{\lambda} g(x+\mu_1\lambda+\cdots+\mu_k\lambda, y).$$

Hence

$$^1\Delta^{k,l}_{m\lambda,n\eta} f(x,y) = \sum_{\mu_1=0}^{m-1} \cdots \sum_{\mu_k=0}^{m-1} \sum_{\nu_1=0}^{n-1} \cdots$$

$$\cdots \sum_{\nu_l=0}^{n-1} \Delta^{k,l}_{\lambda,\eta} f(x+\mu_1\lambda+\cdots+\mu_k\lambda, y+\nu_1\eta+\cdots+\nu_l\eta).$$

Applying Minkowski's inequality, we get at once estimate (3). ■

From (3) it follows that

$$\omega_{k,l}(m\delta_1, n\delta_2; f)_p \leq m^k n^l \omega_{k,l}(\delta_1,\delta_2; f)_p \quad (m,n \in N) \tag{4}$$

for all non-negative numbers δ_1, δ_2, whence , in case $a, b \geq 0$

$$\omega_{k,l}(a\delta_1, b\delta_2; f)_p \leq ([a]+1)^k([b]+1)^l \omega_{k,l}(\delta_1,\delta_2; f)_p. \tag{5}$$

Clearly, the estimates (3)-(5) are useful only with the finite right-hand sides.

Proposition 2. *Let f have the partial derivatives*

$$f^{(0,n)}(u,\cdot), \ldots, f^{(m-1,n)}(u,\cdot) \in L_{loc}(R) \quad (m,n \in N)$$

for every $u \in R$, and let $f^{(m-1,n)}(\cdot, v) \in AC_{loc}(R)$ for almost every $v \in R$. Further, given any $c > 0$, suppose the existence of positive number M_c such that

$$\operatorname*{ess\,sup}_{-c \leq u \leq c} |f^{(m,n)}(u,v)| \leq M_c$$

for almost every $v \in [-c,c]$. Then, in case $k \geq m$ and $\lambda, \eta \in R$,

$$\|\Delta^{k,l}_{\lambda,\eta} f\|_p \leq |\lambda|^m |\eta|^n \|\Delta^{k-m,l-n}_{\lambda,\eta} f^{(m,n)}\|_p. \tag{6}$$

Proof. By our assumption, the (Lebesgue) integrals

$$\int_0^\eta \cdots \int_0^\eta f^{(j,n)}(u, w+t_1+\cdots+t_n)dt_1\cdots dt_n \equiv F_j(u,w)$$

$$(j = 0,1,\ldots,m)$$

exist for all $u, w \in R$ if $j < m$ and for almost every u and every $w \in R$ if $j = m$.

Given $u, w \in R$, one has

$$\left|\frac{1}{h}\int_0^h f^{(m,n)}(u+s, w+t_1+\cdots+t_n)ds\right| \leq M_c$$
$$\text{uniformly in } h \in [-1,1]$$

$(h \neq 0)$ for almost every point $(t_1, \cdots, t_n)$ of the n-dimensional interval $[-|\eta|, |\eta|]^n$ whenever $c \geq \max(|u|+1, |w|+n|\eta|)$. Moreover,

$$\lim_{h\to 0}\frac{1}{h}\int_0^h f^{(m,n)}(u+s, w+t_1+\cdots+t_n)ds =$$
$$= f^{(m,n)}(u, w+t_1+\cdots+t_n)$$

for almost every u. Hence

$$\lim_{h\to 0}\frac{1}{h}\Big\{F_{m-1}(u+h, w) - F_{m-1}(u,w)\Big\} =$$
$$= \int_0^\eta \cdots \int_0^\eta f^{(m,n)}(u, w+t_1+\cdots+t_n)dt_1\cdots dt_n,$$

i.e.

$$F_{m-1}^{(1,0)}(u,w) = F_m(u,w) \quad \text{for a.e. } u \in R \text{ and every } w \in R,$$

by the Lebesgue dominated convergence theorem.

Next, when $m \geq 2$, $(u,w) \in R^2$ and $0 < |h| \leq 1$,

$$\frac{1}{h}\Big\{F_{m-2}(u+h, w) - F_{m-2}(u,w)\Big\} =$$
$$= \int_0^\eta \cdots \int_0^\eta \left\{\frac{1}{h}\int_0^h f^{(m-1,n)}(u+s, w+t_1+\cdots+t_n)ds\right\}dt_1\cdots dt_n,$$

$$\left|\frac{1}{h} f^{(m-1,n)}(u+s, w+t_1+\cdots+t_n)ds\right| =$$
$$= \left|\frac{1}{h}\left\{\int_0^s f^{(m,n)}(u+z, w+t_1+\cdots+t_n)dz + \right.\right.$$
$$\left.\left. + f^{(m-1,n)}(u, w+t_1+\cdots+t_n)\right\}ds\right| \leq$$
$$\leq \left|\frac{1}{h}|f^{(m,n)}(u+z, w+t_1+\cdots+t_n)|dz\right| +$$
$$+ |f^{(m-1,n)}(u, w+t_1+\cdots+t_n)|$$

and

$$\lim_{h\to 0}\frac{1}{h}\int_0^h f^{(m-1,n)}(u+s, w+t_1+\cdots+t_n)ds =$$
$$= f^{(m-1,n)}(u, w+t_1+\cdots+t_n)$$

for almost all $(t_1,\dots,t_n) \in [-|\eta|,|\eta|]^n$. Therefore, as previously,

$$F_{m-2}^{(1,0)}(u,w) = F_{m-1}(u,w) \quad \text{for all} \quad (u,w) \in R^2$$

Analogously, when $m \geq 3$,

$$F_{m-3}^{(1,0)}(u,w) = F_{m-2}(u,w) \quad \text{for all} \quad (u,w) \in R^2, \quad \text{etc.}$$

Consequently, if $m \geq 1$,

$$F_0^{(m-1,0)}(u,w) = F_{m-1}(u,w) \quad \text{for all} \quad (u,w) \in R^2,$$

and

$$F_0^{(m,0)}(u,w) = F_m(u,w) \quad \text{for a.e.} \quad u \in R \quad \text{and every} \quad w \in R$$

Further, in case $-c \leq a \leq b \leq c$ and $|w| + n|\eta| \leq c$,

$$|F_{m-1}(b,w) - F_{m-1}(a,w)| =$$

$$= \left| \int_0^\eta \cdots \int_0^\eta \left\{ \int_a^b f^{(m,n)}(s, w + t_1 + \cdots + t_n) ds \right\} dt_1 \cdots dt_n \leq \right.$$

$$\leq M_c(b-a)|\eta|^n.$$

Hence

$$F_{m-1}(\cdot,w) \in AC_{loc}(R) \quad \text{for every} \quad w \in R.$$

By identity (4) of [5], p.116,

$$\Delta_{\lambda,\eta}^{m,n} f(x,y) = {}^1\Delta_\lambda^m \left({}^2\Delta_\eta^n f(x,y) \right) = {}^1\Delta_\lambda^m F_0(x,y) =$$

$$= \int_0^\lambda \cdots \int_0^\lambda F_0^{(m,0)}(x + s_1 + \cdots + s_m, y) ds_1 \cdots ds_m =$$

$$= \int_0^\lambda \cdots \int_0^\lambda F_m(x + s_1 + \cdots + s_m, y) ds_1 \cdots ds_m$$

for arbitrary $x, y, \lambda, \eta \in R$. Thus

$$\Delta_{\lambda,\eta}^{m,n} f(x,y) = \int_0^\lambda \cdots \int_0^\lambda \left\{ \int_0^\eta \cdots \right.$$

$$\left. \cdots \int_0^\eta f^{(m,n)}(x+s_1+\cdots+s_m, y+t_1+\cdots+t_n) dt_1 \dots dt_n \right\} ds_1 \dots ds_m.$$

Observing that

$$\Delta_{\lambda,\eta}^{k,l} f(x,y) = \Delta_{\lambda,\eta}^{k-m,l-n} \left(\Delta_{\lambda,\eta}^{m,n} f(x,y) \right) =$$

$$= \int_0^\lambda \cdots \int_0^\lambda \left\{ \int_0^\eta \cdots \int_0^\eta \Delta_{\lambda,\eta}^{k-m,l-n} \times \right.$$

$$\left. \times f^{(m,n)}(x + s_1 + \cdots + s_m, y + t_1 + \cdots + t_n) dt_1 \cdots dt_n \right\} ds_1 \cdots ds_m$$

and applying the generalized Minkowski's inequality, we obtain (6). ■

Estimate (6) immediately implies

$$\omega_{k,l}(\delta_1,\delta_2;f)_p \le \delta_1^m\delta_2^n\omega_{k-m,l-n}(\delta_1,\delta_2;f^{(m,n)})_p \quad \text{for } \delta_1,\delta_2 \ge 0. \tag{7}$$

Proposition 3. *Let* $(f_0, f_1, \dots, f_\rho)$ *be a system of (complex-valued) functions of two real variables, such that* $f_\rho \in L_{loc}(R^2)$ $(\rho \in N)$ *and, for* $j = 1, \dots, \rho$,

$$f_{\rho-j}(x,y) = f_{\rho-j}(x,0) + f_{\rho-j}(0,y) - f_{\rho-j}(0,0) + \\ + \int_0^x \int_0^y f_{\rho-j+1}(u,v)\,du\,dv \quad \text{if } x,y \ne 0,$$

$f_{\rho-j}(x,0)$, $f_{\rho-j}(0,y)$ *are defined for all real* x,y *and, when* $j < \rho$, $f_{\rho-j}(\cdot,0)$, $f_{\rho-j}(0,\cdot) \in L_{loc}(R)$. *Suppose that the integer* k *is greater than or equal to* ρ. *Then, for all* $\lambda,\eta \in R$

$$\|\Delta_{\lambda,\eta}^{k,k} f_0\|_p \le |\lambda\eta|^\rho \|\Delta_{\lambda,\eta}^{k-\rho,k-\rho} f_\rho\|_p. \tag{8}$$

Proof. Given arbitrary $x,y,\lambda,\eta \in R$, we have

$$\Delta_{\lambda,\eta}^{1,1} f_0(x,y) = \int_x^{x+y}\int_y^{y+\eta} f_1(s,t)\,ds\,dt = \\ = \int_0^\lambda \int_0^\eta f_1(x+s_1, y+t_1)\,ds_1dt_1,$$

$$\Delta_{\lambda,\eta}^{2,2} f_0(x,y) = \int_0^\lambda \int_0^\eta \Delta_{\lambda,\eta}^{1,1} f_1(x+s_1,y+t_1)\,ds_1dt_1 = \\ = \int_0^\lambda \int_0^\eta \left\{ \int_0^\lambda \int_0^\eta f_2(x+s_1+s_2, y+t_1+t_2)ds_2dt_2 \right\} ds_1dt_1, \quad \text{etc.}$$

Therefore

$$\Delta_{\lambda,\eta}^{k,k} f_0(x,y) = \Delta_{\lambda,\eta}^{k-\rho,k-\rho}\left(\Delta_{\lambda,\eta}^{\rho,\rho} f_0(x,y)\right) = \\ = \int_0^\lambda \cdots \int_0^\lambda \int_0^\eta \cdots \int_0^\eta \Delta_{\lambda,\eta}^{k-\rho,k-\rho} \times \\ \times f_\rho(x+s_1+\cdots+s_\rho, y+t_1+\cdots+t_\rho)ds_1\cdots ds_\rho dt_1\cdots dt_\rho.$$

This immediately implies estimate (8). ■

From (8) it follows that

$$\omega_{k,k}(\delta_1,\delta_2;f_0)_p \le (\delta_1\delta_2)^\rho \omega_{k-\rho,k-\rho}(\delta_1,\delta_2;f_\rho)_p \tag{9}$$

for all non-negative numbers δ_1,δ_2.

3. Basic properties of the Bernstein singular integrals. Consider the entire functions g_σ, $G_{\sigma,k}$ of exponential type of positive order σ, with positive integer parameters r, k given by

$$g_\sigma(z) := \left(\frac{1}{z}\sin\frac{\sigma z}{2r}\right)^{2r}, \quad G_{\sigma,k}(\zeta) := \sum_{\mu=1}^{k}(-1)^\mu \frac{1}{\mu}\binom{k}{\mu} g_\sigma\left(\frac{\zeta}{\mu}\right).$$

Write

$$\gamma_\sigma := \int_R g_\sigma(t)\,dt = 2\left(\frac{\sigma}{2r}\right)^{2r-1}\int_0^\infty \left(\frac{\sin v}{v}\right)^{2r} dv.$$

Suppose that $f \equiv f(\cdot,\cdot)$ is a (complex-valued) function defined and measurable on R^2, such that

$$I(f) \equiv \iint_{R^2} \frac{|f(u,v)|}{(1+u^{2r})(1+v^{2r})}\,du\,dv < \infty. \tag{10}$$

Take arbitrary $\tau > 0$, $l \in N$ and complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ $(x_m, y_m \in R)$. Introduce the singular integrals

$$J_{\sigma,\tau}[f](z_1,z_2) := (\gamma_\sigma\gamma_\tau)^{-1}\iint_{R^2} f(u,v)G_{\sigma,k}(z_1-u)\times$$
$$\times G_{\tau,l}(z_2-v)\,du\,dv, \tag{11}$$

$$J^1_\sigma[f](z_1,v) := \gamma_\sigma^{-1}\int_R f(u,v)G_{\sigma,k}(z_1-u))\,du \quad (v \in R), \tag{12}$$

$$J^2_\tau[f](u,z_2) := \gamma_\tau^{-1}\int_R f(u,v)G_{\tau,l}(z_2-v)\,dv \quad (u \in R), \tag{13}$$

which are due to S.Bernstein ([1], pp.421-432).

The double integral (11) exists in the Lebesgue sense for all complex z_1, z_2; the single (Lebesgue) integrals (12) and (13) exist for every complex z_1 [resp. z_2] and almost every real v [resp. u]. More precise assertions will be presented below.

Proposition 4. *The relation*

$$J_{\sigma,\tau}[f](z_1,z_2) = O\Big((1+x_1^{2r})(1+x_2^{2r})e^{\sigma|y_1|+\tau|y_2|}I(f)\Big) \tag{14}$$

holds uniformly in $x_1, y_1, x_2, y_2 \in R$; moreover, $J_{\sigma,\tau}[f]$ coincides with some bivariate entire function.

Proof. Putting

$$S(z_1,z_2) := \iint_{R^2} f(u,v)g_\sigma(z_1-u)g_\tau(z_2-v)\,du\,dv,$$

we can, formally, write

$$S(z_1,z_2) = \left(\int_{x_1-1}^{x_1+1}\int_{x_2-1}^{x_2+1} + \int_{x_1+1}^{\infty}\int_{x_2+1}^{\infty} + \right.$$
$$\left. + \int_{x_1+1}^{\infty}\int_{x_2-1}^{x_2+1} + \cdots\right) f(u,v) g_\sigma(z_1-u) g_\tau(z_2-v)\,du\,dv =$$
$$= T_0(z_1,z_2) + T_1(z_1,z_2) + T_2(z_1,z_2) + \cdots .$$

In view of (10), $f \in L_{loc})R^2)$. Hence the term $T_0(z_1,z_2)$ exists because $g_\sigma \in E_\sigma$, $g_\tau \in E_\tau$. Furthermore,

$$|T_0(z_1,z_2)| \le \int_{x_1-1}^{x_1+1}\int_{x_2-1}^{x_2+1} |f(u,v)| \left(\frac{\sigma}{2r}\right)^{2r} e^{\sigma|y_1|}\left(\frac{\tau}{2r}\right)^{2r} e^{\tau|y_2|}du\,dv \le$$
$$\le \left(\frac{\sigma\tau}{4r^2}\right)^{2r} e^{\sigma|y_1|+\tau|y_2|}\int_{x_1-1}^{x_1+1}\int_{x_2-1}^{x_2+1} \frac{|f(u,v)|}{(1+u^{2r})(1+v^{2r})}du\,dv\,\cdot$$
$$\cdot\left\{1+(|x_1|+1)^{2r}\right\}\left\{1+(|x_2|+1)^{2r}\right\} \le$$
$$\le 4\left(\frac{\sigma\tau}{4r^2}\right)^{2r} e^{\sigma|y_1|+\tau|y_2|} I(f)(|x_1|+1)^{2r}(|x_2|+1)^{2r}.$$

Next

$$|T_1(z_1,z_2)| \le \int_{x_1+1}^{\infty}\int_{x_2+1}^{\infty} |f(u,v)| e^{\sigma|y_1|}(x_1-u)^{-2r} e^{\tau|y_2|}(x_2-v)^{-2r}\,du\,dv \le$$
$$\le e^{\sigma|y_1|+\tau|y_2|}\int_{x_1+1}^{\infty}\int_{x_2+1}^{\infty}\frac{|f(u,v)|}{(1+u^{2r})(1+v^{2r})}du\,dv\,\cdot$$
$$\cdot\left\{1+(|x_1|+1)^{2r}\right\}\left\{1+(|x_2|+1)^{2r}\right\} \le$$
$$\le 4e^{\sigma|y_1|+\tau|y_2|} I(f)(|x_1|+1)^{2r}(|x_2|+1)^{2r},$$

$$|T_2(z_1,z_2)| \le \int_{x_1+1}^{\infty}\int_{x_2-1}^{\infty} |f(u,v)| e^{\sigma|y_1|}(x_1-u)^{-2r}\left(\frac{\tau}{2r}\right)^{2r} e^{\tau|y_2|}du\,dv \le$$
$$\le 4\left(\frac{\tau}{2r}\right)^{2r} e^{\sigma|y_1|+\tau|y_2|} I(f)(|x_1|+1)^{2r}(|x_2|+1)^{2r}, \text{ etc.}$$

Thus

$$S(z_1,z_2) = O\Big((1+x_1^{2r})(1+x_2^{2r})e^{\sigma|y_1|+\tau|y_2|} I(f)\Big) \tag{15}$$

uniformly in $x_1, y_1, x_2, y_2 \in R$.

Evidently, the left-hand side of (15) can be replaced by

$$S_{\mu,\nu}(z_1,z_2) := \iint_{R^2} f(u,v) g_\sigma\left(\frac{z_1-u}{\mu}\right) g_\tau\left(\frac{z_2-v}{\nu}\right) du\,dv \quad (\mu,\nu \in N).$$

Further,

$$J_{\sigma,\tau}[f](z_1, z_2) = \frac{1}{\gamma_\sigma \gamma_\tau} \sum_{\mu=1}^{k} \sum_{\nu=1}^{l} (-1)^{\mu+\nu} \binom{k}{m} \binom{l}{\nu} \frac{1}{\mu\nu} S_{\mu,\nu}(z_1, z_2).$$

Therefore, the uniform relation (14) is established. Applying the Lebesgue dominated convergence theorem, it can easily be proved that the function $J_{\sigma,\tau}[f]$ is continuous at every point belonging to the space of pairs of complex numbers.

The (Lebesgue) integrals

$$\int_{-n}^{n} \int_{-n}^{n} f(u,v) g_\sigma\left(\frac{z_1 - u}{\mu}\right) g_\tau\left(\frac{z_2 - v}{\nu}\right) du\, dv \quad (\mu, \nu, n \in N)$$

define some entire functions $F_{\mu,\nu,n}$ of two complex variables z_1, z_1, because they have the partial derivatives

$$\frac{\partial F_{\mu,\nu,n}}{\partial z_1}(z_1, z_2) = \frac{1}{\mu} \int_{-n}^{n} \int_{-n}^{n} f(u,v) g'_\sigma\left(\frac{z_1 - u}{\mu}\right) g_\tau\left(\frac{z_2 - v}{\nu}\right) du\, dv$$

and

$$\frac{\partial F_{\mu,\nu,n}}{\partial z_2}(z_1, z_2) = \frac{1}{\nu} \int_{-n}^{n} \int_{-n}^{n} f(u,v) g_\sigma\left(\frac{z_1 - u}{\mu}\right) g'_\tau\left(\frac{z_2 - v}{\nu}\right) du\, dv$$

An easy calculation shows that

$$\lim_{n\to\infty} F_{\mu,\nu,n}(z_1, z_2) = S_{\mu,\nu}(z_1, z_2)$$

uniformly in z_1, z_2 belonging to two arbitrary bounded sets of complex numbers. Hence the well-known Weierstrass theorem ensures that all $S_{\mu,\nu}$ are entire functions of two variables. Consequently, $J_{\sigma,\tau}[f]$ is a bivariate entire function for every fixed pair (σ, τ) of positive numbers. Obviously, in view of (14), $J_{\sigma,\tau}[f](\cdot,\cdot) \in E_{\sigma,\tau}$. ■

Proposition 5. *The singular integral* (12), *with a positive parameter* σ, *has the following basic properties:*

(i) $J^1_\sigma[f](\cdot, v) \in E_\sigma$ *for almost every* $v \in R$,

(ii) $J^1_\sigma[f](z_1, \cdot) \in L_{loc}(R)$ *for every complex* z_1,

(iii) $J^1_\sigma[f](\cdot,\cdot) \in L_{loc}(R^2)$.

Proof of (i). By Fubini's theorem, condition (10) implies

$$\left| \int_{-\infty}^{\infty} \frac{f(u,v)}{1+u^{2r}} du \right| < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{|f(u,v)|}{1+u^{2r}} du < \infty$$

for almost every $v \in R$. Hence, for these v and $z_1 = x_1 + iy_1$,

$$\varphi_\mu(z_1, v) \equiv \int_{-\infty}^{\infty} \left| f(u,v) g_\sigma\left(\frac{z_1 - u}{\mu}\right) \right| du < \infty \quad (\mu = 1, \ldots, k)$$

and

$$Y(z_1,v) \equiv \int_R f(u,v)G_{\sigma,k}(z_1-u)du =$$
$$= \sum_{\mu=1}^{k}(-1)^{\mu}\frac{1}{\mu}\binom{k}{\mu}\int_{-\infty}^{\infty} f(u,v)g_{\sigma}\left(\frac{z_1-u}{\mu}\right)du.$$

Writing

$$\varphi_{\mu}(z_1,v) = \left(\int_{x_1-1}^{x_1+1} + \int_{x_1+1}^{\infty} + \int_{-\infty}^{x_1-1}\right)\left|f(u,v)g_{\sigma}\left(\frac{z_1-u}{\mu}\right)\right|du$$

and proceeding as in the proof of Proposition 4, we obtain

$$\varphi_{\mu}(z_1,v) = O\Big((1+x_1^{2r})e^{(\sigma/\mu)|y_1|}\int_R |f(u,v)|(1+u^{2r})^{-1}du\Big) \quad (16)$$

uniformly in x_1, y_1, v. This immediately implies the O-relation for $Y(z_1,v)$, in which the right side is as in (16) with $\mu=1$.

Further, $Y(\cdot,v)$ are entire functions, by Weierstrass theorem. Consequently, $Y(\cdot,v)\in E_{\sigma}$ for almost every v. Thus the assertion (i) is obtained. ■

Proof of (ii). Given $\mu, n \in N$ and any complex number $z_1 = x_1 + iy_1$, we have

$$\int_{-\infty}^{\infty}\int_{-n}^{n}\left|f(u,v)g_{\sigma}\left(\frac{z_1-u}{\mu}\right)\right|du\,dv \leq$$
$$\leq \int_{x_1-1}^{x_1+1}\int_{-n}^{n}|f(u,v)|\left(\frac{\sigma}{2r}\right)^{2r}e^{\sigma|y_1|/\mu}du\,dv +$$
$$+\left(\int_{-\infty}^{x_1-1}\int_{-n}^{n} + \int_{x_1+1}^{\infty}\int_{-n}^{n}\right)|f(u,v)|\left(\frac{\mu}{x_1-u}\right)^{2r}e^{\sigma|y_1|/\mu}du\,dv < \infty.$$

Applying Fubini's theorem we conclude that $Y(z_1,\cdot)\in L_{loc}(R)$, which implies (ii). ■

Proof of (iii). Let $\mu, n \in N$, $x_1 \in R$. By Tonelli's theorem,

$$\int_{-\infty}^{\infty}\int_{-n}^{n}\int_{-n}^{n}\left|f(u,v)g_{\sigma}\left(\frac{x_1-u}{\mu}\right)\right|du\,dv\,dx_1 =$$
$$= \int_{-n}^{n}\left\{\int_{-\infty}^{\infty}\int_{-n}^{n}|f(u,v)|g_{\sigma}\left(\frac{x_1-u}{\mu}\right)du\,dv\right\}dx_1. \quad (17)$$

The inner double integral of the right-hand side of (17) does not exceed

$$\int_{x_1-1}^{x_1+1}\int_{-n}^{n}|f(u,v)|\left(\frac{\sigma}{2r}\right)^{2r}du\,dv+$$
$$+\left(\int_{-\infty}^{x_1-1}\int_{-n}^{n}+\int_{x_1+1}^{\infty}\int_{-n}^{n}\right)|f(u,v)|\left(\frac{\mu}{x_1-u}\right)^{2r}du\,dv\le$$
$$\le\left(\frac{\sigma}{2r}\right)^{2r}\int_{-n-1}^{n+1}\int_{-n}^{n}|f(u,v)|du\,dv+2\mu^{2r}(1+n)^{2r})(1+n^{2r})I(f).$$

Thus the left-hand side of (17) is finite.

Now, the Fubini theorem ensures that $Y(\cdot,\cdot)\in L_{loc}(R^2)$, and (iii) follows. ■

Analogous properties of the singular integral (13) can easily be formulated. They also will be used in the sequel.

4. Approximations by the sums of pseudoentire functions. Let f and the operators $J_{\sigma,\tau}$, J_σ^1, J_τ^2 be as in Section 3. Putting

$$\Phi(z_1,v):=-J_{\sigma,\tau}[f](z_1,v)-J_\sigma^1[f](z_1,v),$$
$$\Psi(u,z_2):=J_\tau^2[f](u,z_2),$$

where $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ ($x_j,y_j\in R$) and $u,v\in R$, it can easily be observed (see Propositions 4,5) that $\Phi(\cdot,\cdot)\in W_\sigma^1$, $\Psi(\cdot,\cdot)\in W_\tau^2$.

Introduce the approximant

$$Q_{\sigma,\tau}[f](x_1,x_2):=\Phi(x_1,x_2)+\Psi(x_1,x_2),$$

which is defined almost everywhere on R^2 and $Q_{\sigma,\tau}[f]\in L_{loc}(R^2)$. Assuming that $k,l\le 2r-2$ ($k,l,r\in N$) and $1\le p\le\infty$, we will present some Jackson type estimates.

In considerations the symbols $C_j(q,\dots)$ will mean positive constants depending on the indicated parameters $q,\dots$, only.

Theorem 1. *Under the restriction* $0<\sigma,\tau<\infty$,

$$\|f-Q_{\sigma,\tau}[f]\|_p\le 2^{k+l}C_1(r)\omega_{k,l}(1/\sigma,1/\tau;f)_p.$$

Proof. Take into account the real numbers x_1,x_2 for which $J_\sigma^1[f](\cdot,x_2)\in E_\sigma$, $J_\tau^2[f](x_1,\cdot)\in E_\tau$. In this case,

$$f(x_1,x_2)-Q_{\sigma,\tau}[f](x_1,x_2)=$$
$$=\frac{(-1)^{k+l}}{\gamma_\sigma\gamma_\tau}\iint_{R^2}\Delta_{s,t}^{k,l}f(x_1,x_2)g_\sigma(s)g_\tau(t)\,ds\,dt. \qquad (18)$$

Hence, by Minkowski's inequality and (5),

$$\gamma_\sigma\gamma_\tau\|f - Q_{\sigma,\tau}[f]\|_p \leq$$
$$\leq \omega_{k,l}(1/\sigma, 1/\tau; f)_p \iint_{R^2}(\sigma|s|+1)^k(\tau|t|+1)^l g_\sigma(s)g_\tau(t)\,ds\,dt.$$

Further,

$$\frac{1}{\gamma_\sigma}\int_R(\sigma|s|+1)^k g_\sigma(s)\,ds \leq \frac{2}{\gamma_\sigma}\Big\{2^k\int_0^{1/\sigma} g_\sigma(s)\,ds +$$
$$+2^k\sigma^k\int_{1/\sigma}^\infty s^k g_\sigma(s)\,ds\Big\} \leq 2^k\Big\{1+\frac{2}{\gamma_\sigma}\sigma^k\int_{1/\sigma}^\infty s^{k-2r}ds\Big\} \leq$$
$$\leq 2^k\Big\{1+\frac{2}{\gamma_\sigma}\sigma^{2r-1}\Big\}.$$

Observing that

$$\gamma_\sigma \geq 2\Big(\frac{\sigma}{2r}\Big)^{2r-1}\int_0^{\pi/2}\Big(\frac{2}{\pi}\Big)^{2r}dv = 2\Big(\frac{\sigma}{\pi r}\Big)^{2r-1},$$

we obtain

$$\frac{1}{\gamma_\sigma}\int_R(\sigma|s|+1)^k g_\sigma(s)\,ds \leq 2^k\Big\{1+(\pi r)^{2r-1}\Big\}.$$

Thus

$$\|f - Q_{\sigma,\tau}[f]\|_p \leq \omega_{k,l}(1/\sigma,1/\tau;f)_p\cdot 2^{k+l}\Big\{1+(\pi r)^{2r-1}\Big\}^2,$$

and the proof is complete. ■

Corollary 1. *Let f satisfy all conditions of Proposition 2, with some positive integers $m \leq k$, $n \leq l$. Then*

$$\|f - Q_{\sigma,\tau}[f]\|_p \leq 2^{k+l}C_1(r)\sigma^{-m}\tau^{-n}\omega_{k-m,l-n}(1/\sigma,1/\tau;f^{(m,n)})_p$$

This estimate is an immediate consequence of Theorem 1 and inequality (7).

Corollary 2. *Consider the bivariate functions $f_0, f_1, \ldots, f_\rho$ defined in Proposition 3, with $f_0(\cdot,0), f_0(0,\cdot) \in L_{loc}(R)$ and $f_{\rho-j}(\cdot,0), f_{\rho-j}(0,\cdot) \in C(R)$ when $1 \leq j \leq \rho-1$. Suppose that for some non-negative numbers $a, b < 2r-\rho-1$ and for $j = 1,\ldots,\rho$ the relations*

$$f_\rho(x,y) = O\big((1+|x|^a)(1+|y|^b)\big),$$
$$f_{\rho-j}(x,0) = O\big(1+|x|^{a+j}\big),\quad f_{\rho-j}(0,y) = O\big(1+|y|^{b+j}\big))$$

hold uniformly in $x, y \in R$. Then if $k = l \geq \rho$, the function $f = f_0$ and its approximant $Q_{\sigma,\tau}[f]$ have weak derivatives of order $\rho-1$ everywhere on R^2 and

$$\|f^{(\mu)} - Q^{(\mu)}_{\sigma,\tau}[f]\|_p \leq C_2(k,r)(\sigma\tau)^{\mu-\rho}\omega_{k-\rho+\mu,k-\rho+\mu}\Big(\frac{1}{\sigma},\frac{1}{\tau};f_\rho\Big)_p$$

for $\mu = 0,\dots,\rho-1$. Under the additional assumption $f_\rho \in C(R^2)$, also the derivatives $f^{(\rho)}$ and $Q^{(\rho)}_{\sigma,\tau}[f]$ exist on R^2 and the last inequality remains valid for $\mu = \rho$.

Indeed, an easy calculation shows that

$$f^{(\mu)}(x,y) = f_\mu(x,y) \quad \text{and} \quad Q^{(\mu)}_{\sigma,\tau}[f](x,y) = Q_{\sigma,\tau}[f_\mu](x,y)$$

for all $(x,y) \in R^2$, whenever $0 \leq \mu \leq \rho-1$ or $\mu = \rho$ and $f_\rho \in C(R^2)$. By Theorem 1 and the suitable estimate analogous to (9),

$$\|f_\mu - Q_{\sigma,\tau}[f]\|_p \leq C_2(k,r)\omega_{k,k}(1/\sigma,1/\tau;f_\mu)_p \leq$$
$$\leq C_2(k,r)(\sigma\tau)^{\mu-\rho}\omega_{k-\rho+\mu,k-\rho+\mu}(1/\sigma,1/\tau;f_\rho)_p \quad \text{if } 0 \leq \mu \leq \rho.$$

The conclusion is now evident.

Theorem 2. *Let $f \in H^{(k,l)}_{\varphi,\psi,p}$, where φ,ψ are as in Section 1. Denote by α and β two non-negative numbers such that $t^{-\alpha}\varphi(t)$, $t^{-\beta}\psi(t)$ are non-decreasing on $(0,1]$. Then, for $\sigma,\tau \geq 1$,*

$$\|f - Q_{\sigma,\tau}[f]\|^{(k,l)}_{\alpha,\beta,p} \leq C_3(k,l,r)\|f\|^{(k,l)}_{\varphi,\psi,p}\big\{\sigma^\alpha\varphi(1/\sigma) + \tau^\beta\psi(1/\tau)\big\}.$$

Proof. Write

$$D_{\sigma,\tau}(x,y) := f(x,y) - Q_{\sigma,\tau}[f](x,y) \quad (x,y \in R)$$

In view of (18),

$$\Delta^{k,l}_{\lambda,\eta}D_{\sigma,\tau}(x,y) = \frac{1}{\gamma_\sigma\gamma_\tau}\times$$
$$\times\iint_{R^2} g_\sigma(s)g_\tau(t)\sum_{\mu=0}^{k}\sum_{\nu=0}^{l}(-1)^{\mu+\nu}\binom{k}{\mu}\binom{l}{\nu}\Delta^{k,l}_{s,t}f(x+\mu\lambda, y+\nu\eta)\,ds\,dt$$

for all $\lambda,\eta \in R$ and almost all $(x,y) \in R^2$, whence, by Minkowski's inequality,

$$\|\Delta^{k,l}_{\lambda,\eta}D_{\sigma,\tau}\| \leq \frac{2^{k+l}}{\gamma_\sigma\gamma_\tau}\iint_{R^2} g_\sigma(s)g_\tau(t)\|\Delta^{k,l}_{s,t}f\|_p\,ds\,dt$$

Proceeding now as in the proof of Theorem 1, we obtain

$$\|\Delta^{k,l}_{\lambda,\eta}D_{\sigma,\tau}\|_p \leq 4^{k+l}C_1(r)\|f\|^{(k,l)}_{\varphi,\psi,p}\varphi(1/\sigma)\psi(1/\tau). \tag{19}$$

On the other hand, equality (18) leads to

$$\Delta^{k,l}_{\lambda,\eta} D_{\sigma,\tau}(x,y) = \frac{1}{\gamma_\sigma \gamma_\tau} \times$$

$$\times \iint_{R^2} g_\sigma(s) g_\tau(t) \sum_{\mu=0}^{k} \sum_{\nu=0}^{l} (-1)^{\mu+\nu} \binom{k}{\mu} \binom{l}{\nu} \Delta^{k,l}_{\lambda,\eta} f(x+\mu s, y+\nu t)\, ds\, dt$$

for almost all $(x,y) \in R^2$, which implies

$$\|\Delta^{k,l}_{\lambda,\eta} D_{\sigma,\tau}\|_p \le 2^{k+l} \|f\|^{(k,l)}_{\varphi,\psi,p} \varphi(\lambda)\psi(\eta) \ \text{ when } \lambda,\eta \in (0,1]. \qquad (20)$$

Next,

$$\|D_{\sigma,\tau}\|^{(k,l)}_{\alpha,\beta,p} = \sup\Big\{\Omega(\lambda,\eta) : 0 < \lambda,\eta \le 1\Big\},$$

where

$$\Omega(\lambda,\eta) = \lambda^{-\alpha}\eta^{-\beta} \|\Delta^{k,l}_{\lambda,\eta} D_{\sigma,\tau}\|_p.$$

If $1/\sigma \le \lambda \le 1$ and $1/\tau \le \eta \le 1$ then, by (19),

$$\Omega(\lambda,\eta) \le 4^{k+l} C_1(r) \|f\|^{(k,l)}_{\varphi,\psi,p} \sigma^\alpha \varphi(1/\sigma) \tau^\beta \psi(1/\tau).$$

From (20) it follows that

$$\Omega(\lambda,\eta) \le 2^{k+l} \|f\|^{(k,l)}_{\varphi,\psi,p} \times$$

$$\times \begin{cases} \sigma^\alpha \varphi(1/\sigma) \tau^\beta \psi(1/\tau) & \text{if } 0 < \lambda \le 1/\sigma,\ 0 < \eta \le 1/\tau, \\ \sigma^\alpha \varphi(1/\sigma) & \text{if } 0 < \lambda \le 1/\sigma,\ 1/\tau \le \eta \le 1, \\ \tau^\beta \psi(1/\tau) & \text{if } 1/\sigma \le \lambda \le 1,\ 0 < \eta \le 1/\tau. \end{cases}$$

Hence

$$\|D_{\sigma,\tau}\|^{(k,l)}_{\alpha,\beta,p} \le 2^{k+l} \|f\|^{(k,l)}_{\varphi,\psi,p} \Big\{ 2^{k+l} C_1(r) \sigma^\alpha \varphi(1/\sigma) \tau^\beta \psi(1/\tau) +$$
$$+ \sigma^\alpha \varphi(1/\sigma) \tau^\beta \psi(1/\tau) + \sigma^\alpha \varphi(1/\sigma) + \tau^\beta \psi(1/\tau) \Big\},$$

and the proof is completed (cf. [4], Th.2). ■

Corollary 3. *If*

$$\varphi(\delta) = \delta^{\alpha'}, \quad \psi(\delta) = \delta^{\beta'} \ \textit{for all } \delta \in (0,1]$$

then, in case $\alpha' \ge \alpha \ge 0$ *and* $\beta' \ge \beta \ge 0$,

$$\|f - Q_{\sigma,\tau}[f]\|^{(k,l)}_{\alpha,\beta,p} = O\Big(\sigma^{\alpha-\alpha'} + \tau^{\beta-\beta'}\Big) \ \textit{uniformly in } \sigma,\tau \ge 1.$$

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ავტორთა საყურადღებოდ

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