ONE BOUNDARY VALUE PROBLEM FOR THE PLATES

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Abstract. In this work we consider equations of equilibrium of the isotropic elastic shell. By means of Vekua's method, the system of differential equations for thin and shallow shells is obtained, when on upper and lower face surfaces displacements are assumed to be known. The general solution for approximations $N = 1$ is constructed. The concrete problem is solved.

1. Introduction

The refined theory of shells is constructed by reducing the three-dimensional problems of the theory of elasticity to the two-dimensional problems. I. Vekua had obtained the equations of shallow shells [1],[2]. It means that the interior geometry of the shell does not vary in thickness. This method for non-shallow shells in case of geometrical and physical nonlinear theory was generalized by T. Meunargia [3].

By means of Vekua's method, the system of differential equations for thin and shallow shells was obtained, when on upper and lower face surfaces displacements are assumed to be known [4].

The systems of equilibrium equations and stress-strain relations (Hooke's law) of the tow-dimensional shallow shells may be written in the following form [4]:

$$
\begin{cases}
\nabla_\alpha \overset{(m)}{\sigma}{}^{\alpha\beta} - b_\alpha^\beta \overset{(m)}{\sigma}{}^{\alpha 3} + \dfrac{2m+1}{h}\left(\overset{(m+1)}{\sigma}{}^{\beta 3} + \overset{(m+3)}{\sigma}{}^{\beta 3} + ... \right) + \overset{(m)}{\Phi}{}^{\beta} = 0, \\[3mm]
\nabla_\alpha \overset{(m)}{\sigma}{}^{\alpha 3} + b_\alpha^\beta \overset{(m)}{\sigma}{}^{\alpha}_{\beta} + \dfrac{2m+1}{h}\left(\overset{(m+1)}{\sigma}{}^{33} + \overset{(m+3)}{\sigma}{}^{33} + ... \right) + \overset{(m)}{\Phi}{}^{3} = 0,
\end{cases}
\tag{1}
$$

where

$$
\begin{cases}
\overset{(m)}{\sigma}{}^{\alpha\beta} = \lambda \left[\nabla_\gamma \overset{(m)}{u}{}^{\gamma} - 2H \overset{(m)}{u}{}^{3} - \dfrac{2m+1}{h}\left(\overset{(m-1)}{u}{}^{3} + \overset{(m-3)}{u}{}^{3} + \cdots \right) \right] a^{\alpha\beta} \\[3mm]
+\mu\left(\nabla^\beta \overset{(m)}{u}{}^{\alpha} + \nabla^\alpha \overset{(m)}{u}{}^{\beta} - 2b^{\alpha\beta} \overset{(m)}{u}{}^{3} \right) + \lambda \dfrac{2m+1}{h}\left(\overset{(+)}{u}{}_{3} - (-1)^m \overset{(-)}{u}{}_{3} \right) a^{\alpha\beta}, \\[3mm]
\overset{(m)}{\sigma}{}^{\alpha 3} = \mu\left[\nabla^\alpha \overset{(m)}{u}{}^{3} + b^\alpha_\beta \overset{(m)}{u}{}^{\beta} - \dfrac{2m+1}{h}\left(\overset{(m-1)}{u}{}^{\alpha} + \overset{(m-3)}{u}{}^{\alpha} + \cdots \right) \right] \\[3mm]
+\mu \dfrac{2m+1}{h}\left(\overset{(+)}{u}{}^{\alpha} - (-1)^m \overset{(-)}{u}{}^{\alpha} \right), \\[3mm]
\overset{(m)}{\sigma}{}^{33} = \lambda\left(\nabla_\gamma \overset{(m)}{u}{}^{\gamma} - 2H \overset{(m)}{u}{}^{3} \right) - (\lambda + 2\mu)\dfrac{2m+1}{h}\left(\overset{(m-1)}{u}{}^{3} + \overset{(m-3)}{u}{}^{3} + \cdots \right) \\[3mm]
+(\lambda + 2\mu)\dfrac{2m+1}{h}\left(\overset{(+)}{u}{}_{3} - (-1)^m \overset{(-)}{u}{}_{3} \right).
\end{cases}
\tag{2}
$$

Here λ and μ are Lame's constants, ∇_α are covariant derivatives on the midsurface, $a^{\alpha\beta}$ and $b^{\alpha\beta}$ are the contravariant components of the metric tensor and curvature tensor of the midsurface, H is middle curvature of the midsurface and

$$\left(\overset{(m)}{\sigma}{}^{ij}, \overset{(m)}{u}{}^i, \overset{(m)}{\Phi}{}^i \right) = \frac{2m+1}{2h} \int\limits_{-h}^{h} \left(\sigma^{ij}, u^i, \Phi^i \right) P_m \left(\frac{x_3}{h} \right) dx_3,$$

$$(m = 0, 1, 2, \cdots)$$

$$\overset{(\pm)}{u}{}^i = u^i(x^1, x^2, \pm h),$$

where σ^{ij} are contravariant components of the stress vectors, u^i are contravariant components of the displacement vector, Φ^i are contravariant components of the volume force, $P_m \left(\frac{x^3}{h} \right)$ are Legendre polynomials, x^1, x^2 are the Gaussian parameters of the midsurfaces, $x^3 = x_3$ is the thickness coordinate and h is the semi-thickness. So, we have the infinite system.

An infinite system of equations (1) has the advantage that it contains two independent variables - Gaussian coordinates x^1, x^2 of the midsurface. But the decrease in the number of independent variables is achieved by increasing the number of equations to infinity, which, naturally, has an obvious practical inconvenience. Therefore it is necessary to make the next step for a further simplification of the problem.

2. $N = 1$ approximation for plates

we consider $N = 1$ approximation for plates. In other words, in the previous equations it is assumed that

$$\overset{(m)}{\sigma}{}^{ij} = 0, \quad \overset{(m)}{u}{}^i = 0, \quad if\ m > 1.$$

As a result we obtain a finite system of equilibrium equations

$$\begin{cases} \partial_\alpha \overset{(0)}{\sigma}_{\alpha\beta} + \dfrac{1}{h} \overset{(1)}{\sigma}_{\beta 3} + \overset{(0)}{\Phi}_\beta = 0, \\[2mm] \partial_\alpha \overset{(0)}{\sigma}_{\alpha 3} + \dfrac{1}{h} \overset{(1)}{\sigma}_{33} + \overset{(0)}{\Phi}_3 = 0, \end{cases} \tag{3}$$

$$\begin{cases} \partial_\alpha \overset{(1)}{\sigma}_{\alpha\beta} + \overset{(1)}{\Phi}_\beta = 0, \\[2mm] \partial_\alpha \overset{(1)}{\sigma}_{\alpha 3} + \overset{(1)}{\Phi}_3 = 0, \end{cases} \tag{4}$$

where

$$\begin{cases} \overset{(0)}{\sigma}_{\alpha\beta} = \lambda \left(\partial_\gamma \overset{(0)}{u}_\gamma \right) \delta_{\alpha\beta} + \mu \left(\partial_\beta \overset{(0)}{u}_\alpha + \partial_\alpha \overset{(0)}{u}_\beta \right) + \dfrac{\lambda}{h} \left(\overset{(+)}{u}_3 - \overset{(-)}{u}_3 \right) \delta_{\alpha\beta}, \\[3mm] \overset{(0)}{\sigma}_{\alpha 3} = \mu \left(\partial_\alpha \overset{(0)}{u}_3 \right) + \dfrac{\mu}{h} \left(\overset{(+)}{u}_\alpha - \overset{(-)}{u}_\alpha \right), \\[3mm] \overset{(0)}{\sigma}_{33} = \lambda \left(\partial_\gamma \overset{(0)}{u}_\gamma \right) + \dfrac{\lambda + 2\mu}{h} \left(\overset{(+)}{u}_3 - \overset{(-)}{u}_3 \right), \end{cases} \tag{5}$$

$$
\begin{cases}
\overset{(1)}{\sigma}_{\alpha\beta} = \lambda \left(\partial_\gamma \overset{(1)}{u}_\gamma - \frac{3}{h} \overset{(0)}{u}_3 \right) \delta_{\alpha\beta} + \mu \left(\partial_\beta \overset{(1)}{u}_\alpha + \partial_\alpha \overset{(1)}{u}_\beta \right) + \frac{3\lambda}{h} \left(\overset{(+)}{u}_3 + \overset{(-)}{u}_3 \right) \delta_{\alpha\beta}, \\[2mm]
\overset{(1)}{\sigma}_{\alpha3} = \mu \left(\partial_\alpha \overset{(1)}{u}_3 - \frac{3}{h} \overset{(0)}{u}_\alpha \right) + \frac{3\mu}{h} \left(\overset{(+)}{u}_\alpha + \overset{(-)}{u}_\alpha \right), \\[2mm]
\overset{(1)}{\sigma}_{33} = \lambda \left(\partial_\gamma \overset{(1)}{u}_\gamma \right) - \frac{3(\lambda + 2\mu)}{h} \overset{(0)}{u}_3 + \frac{3(\lambda + 2\mu)}{h} \left(\overset{(+)}{u}_3 + \overset{(-)}{u}_3 \right).
\end{cases}
\tag{6}
$$

Substituting these expressions (5) and (6) into equation (3) and (4), we obtain the system of second-order partial differential equations:

$$
\begin{cases}
\mu \Delta \overset{(0)}{u}_1 + (\lambda + \mu) \partial_1 \overset{(0)}{\theta} + \frac{1}{h} \left(\mu \partial_1 \overset{(1)}{u}_3 - \frac{3\mu}{h} \overset{(0)}{u}_1 \right) = \overset{(0)}{\Psi}_1, \\[2mm]
\mu \Delta \overset{(0)}{u}_2 + (\lambda + \mu) \partial_2 \overset{(0)}{\theta} + \frac{1}{h} \left(\mu \partial_2 \overset{(1)}{u}_3 - \frac{3\mu}{h} \overset{(0)}{u}_2 \right) = \overset{(0)}{\Psi}_2, \\[2mm]
\mu \Delta \overset{(0)}{u}_3 + \frac{1}{h} \left(\lambda \overset{(1)}{\theta} - \frac{3(\lambda + 2\mu)}{h} \overset{(0)}{u}_3 \right) = \overset{(0)}{\Psi}_3,
\end{cases}
\tag{7}
$$

$$
\begin{cases}
\mu \Delta \overset{(1)}{u}_1 + (\lambda + \mu) \partial_1 \overset{(1)}{\theta} - \frac{3\lambda}{h} \partial_1 \overset{(0)}{u}_3 = \overset{(1)}{\Psi}_1, \\[2mm]
\mu \Delta \overset{(1)}{u}_2 + (\lambda + \mu) \partial_2 \overset{(1)}{\theta} - \frac{3\lambda}{h} \partial_2 \overset{(0)}{u}_3 = \overset{(1)}{\Psi}_2, \\[2mm]
\mu \Delta \overset{(1)}{u}_3 - \frac{3\mu}{h} \overset{(0)}{\theta} = \overset{(1)}{\Psi}_3,
\end{cases}
\tag{8}
$$

where $\overset{(m)}{\Psi}_i$ are the known values and

$$
\overset{(m)}{\theta} = \partial_1 \overset{(m)}{u}_1 + \partial_2 \overset{(m)}{u}_2, \quad m = 0, 1.
$$

Introducing the well-known differential operators

$$
\partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2),
$$

where $z = x_1 + ix_2$.

System (7) and (8) can be written in the complex form:
a) for the tension-pressure of plates

$$
\begin{cases}
\mu \Delta \overset{(0)}{u}_+ + 2(\lambda + \mu) \partial_{\bar{z}} \overset{(0)}{\theta} + \frac{1}{h} \left(2\mu \partial_{\bar{z}} \overset{(1)}{u}_3 - \frac{3\mu}{h} \overset{(0)}{u}_+ \right) = \overset{(0)}{\Psi}_+, \\[2mm]
\mu \Delta \overset{(1)}{u}_3 - \frac{3\mu}{h} \overset{(0)}{\theta} = \overset{(1)}{\Psi}_3,
\end{cases}
\tag{9}
$$

b) for the bending of plates

$$
\begin{cases}
\mu \Delta \overset{(1)}{u}_+ + 2(\lambda + \mu) \partial_{\bar{z}} \overset{(1)}{\theta} - \frac{6\lambda}{h} \partial_{\bar{z}} \overset{(0)}{u}_3 = \overset{(1)}{\Psi}_+, \\[2mm]
\mu \Delta \overset{(0)}{u}_3 + \frac{1}{h} \left(\lambda \overset{(1)}{\theta} - \frac{3(\lambda + 2\mu)}{h} \overset{(0)}{u}_3 \right) = \overset{(0)}{\Psi}_3,
\end{cases}
\tag{10}
$$

where $\Delta = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$ and

$$\overset{(m)}{u}_+ = \overset{(m)}{u}_1 + i\,\overset{(m)}{u}_2, \quad \overset{(m)}{\theta} = \partial_z\,\overset{(m)}{u}_z + \partial_{\bar{z}}\,\overset{(m)}{u}_+, \quad \overset{(m)}{\Psi}_+ = \overset{(m)}{\Psi}_1 + i\,\overset{(m)}{\Psi}_2.$$

The complex representation of the general solutions of the homogenous systems (9) and (10) are written in the following form [2, 5]:

$$
\begin{cases}
\overset{(0)}{u}_+ = f(z) + z\overline{f'(z)} + \dfrac{4(\lambda + 2\mu)h^2}{3\mu}\overline{f''(z)} + \overline{g'(z)} - \dfrac{ih}{3}\dfrac{\partial \omega(z,\bar{z})}{\partial \bar{z}}, \\[3mm]
\overset{(1)}{u}_3 = \dfrac{3}{2h}\left(\bar{z}f(z) + z\overline{f(z)}\right) + \dfrac{3}{2h}\left(g(z) + \overline{g(z)}\right),
\end{cases}
\tag{11}
$$

$$
\begin{cases}
\overset{(1)}{u}_+ = \dfrac{5\lambda + 6\mu}{3\lambda + 2\mu}\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + \dfrac{\lambda h}{2(\lambda + \mu)}\dfrac{\partial \chi(z,\bar{z})}{\partial \bar{z}}, \\[3mm]
\overset{(0)}{u}_3 = \chi(z,\bar{z}) + \dfrac{2\lambda h}{3(3\lambda + 2\mu)}\left(\varphi'(z) + \overline{\varphi'(z)}\right),
\end{cases}
\tag{12}
$$

where $f(z)$, $g(z)$, $\varphi(z)$ and $\psi(z)$ are any analytic functions of z, $\omega(z,\bar{z})$ and $\chi(z,\bar{z})$ are the general solutions of the following Helmholtz's equations, respectively:

$$\Delta\omega - \gamma^2\omega = 0, \quad \left(\gamma^2 = \dfrac{3}{h^2}\right),$$

$$\Delta\chi - \nu^2\chi = 0, \quad \left(\chi^2 = \dfrac{12(\lambda + \mu)h^2}{\lambda + 2\mu}\right).$$

From eqs. (5), (6) the following relations follow

$$
\begin{cases}
\overset{(0)}{\sigma}_{11} + \overset{(0)}{\sigma}_{22} = 2(\lambda + \mu)\overset{(0)}{\theta}, & \overset{(0)}{\sigma}_{11} - \overset{(0)}{\sigma}_{22} + 2i\,\overset{(0)}{\sigma}_{12} = 4\mu\partial_{\bar{z}}\,\overset{(0)}{u}_+, \\[3mm]
\overset{(1)}{\sigma}_{11} + \overset{(1)}{\sigma}_{22} = 2(\lambda + \mu)\overset{(1)}{\theta} - \dfrac{6\lambda}{h}\overset{(0)}{u}_3, & \overset{(1)}{\sigma}_{11} - \overset{(1)}{\sigma}_{22} + 2i\,\overset{(1)}{\sigma}_{12} = 4\mu\partial_{\bar{z}}\,\overset{(1)}{u}_+, \\[3mm]
\overset{(0)}{\sigma}_{13} + i\,\overset{(0)}{\sigma}_{23} = 2\mu\partial_{\bar{z}}\,\overset{(0)}{u}_3, & \overset{(1)}{\sigma}_{13} + i\,\overset{(1)}{\sigma}_{23} = 2\mu\partial_{\bar{z}}\,\overset{(1)}{u}_3 - \dfrac{3\mu}{h}\overset{(0)}{u}_+.
\end{cases}
\tag{13}
$$

3. The solution of the boundary problem for the circle

Let us solve the problem when the midsurface of the body is the circle with the radius R.

The boundary problem (in stresses) takes the form [3]:

$$
\begin{cases}
\overset{(m)}{\sigma}_{rr} + i\,\overset{(m)}{\sigma}_{r\alpha} = \dfrac{1}{2}\left[\overset{(m)}{\sigma}_{11} + \overset{(m)}{\sigma}_{22} - \left(\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + 2i\,\overset{(m)}{\sigma}_{12}\right)\left(\dfrac{d\bar{z}}{ds}\right)^2\right] = \overset{(m)}{F}_+, \\[3mm]
\overset{(m)}{\sigma}_{rn} = -\mathrm{Im}\left(\overset{(m)}{\sigma}_{+3}\dfrac{d\bar{z}}{ds}\right) = \overset{(m)}{F}_3, \quad \left(\overset{(m)}{\sigma}_{+3} = \overset{(m)}{\sigma}_{13} + i\,\overset{(m)}{\sigma}_{23}\right).
\end{cases}
\tag{14}
$$

Using eqs. (12) and (13) the boundary conditions are written as

$$
\begin{cases}
(\lambda + \mu)(f'(z) + \overline{f'(z)}) + \left(2\mu z \overline{f''(z)} + \dfrac{8(\lambda + 2\mu)}{3}\overline{f'''(z)}\right. \\[2mm]
\left. +2\mu \overline{g''(z)} - \dfrac{2\mu i h}{3}\dfrac{\partial^2 \omega(z,\bar{z})}{\partial \bar{z}^2}\right)e^{-2i\alpha} = \displaystyle\sum_{-\infty}^{+\infty} A_{n1}e^{in\alpha}, \quad r = R, \\[4mm]
\dfrac{\mu}{2h}\left(ih\dfrac{\partial \omega}{\partial \bar{z}} - \dfrac{4(\lambda + 2\mu)h^2}{3\mu}\overline{f''(z)}\right)e^{-i\alpha} \\[3mm]
-\dfrac{\mu}{2h}\left(ih\dfrac{\partial \omega(z,\bar{z})}{\partial \bar{z}} + \dfrac{4(\lambda + 2\mu)h^2}{3\mu}\overline{f''(z)}\right)e^{i\alpha} = \displaystyle\sum_{-\infty}^{+\infty} B_{n1}e^{in\alpha}, \quad r = R,
\end{cases}
\tag{15}
$$

$$
\begin{cases}
2\mu(\varphi'(z) + \overline{\varphi'(z)}) - \dfrac{3\lambda\mu}{(\lambda + 2\mu)h}\chi(z,\bar{z}) \\[3mm]
+2\mu\left(\dfrac{\lambda h}{2(\lambda + \mu)}\dfrac{\partial^2 \chi(z,\bar{z})}{\partial \bar{z}^2} - z\overline{\varphi''(z)} - \overline{\psi'(z)}\right)e^{-2i\alpha} = \displaystyle\sum_{-\infty}^{+\infty} A_{n2}e^{in\alpha}, \quad r = R, \\[4mm]
\left(\mu\dfrac{\partial \chi(z,\bar{z})}{\partial \bar{z}} + \dfrac{2\lambda\mu h}{3(3\lambda + 2\mu)}\overline{\varphi''(z)}\right)e^{-i\alpha} \\[3mm]
+\left(\mu\dfrac{\partial \chi(z,\bar{z})}{\partial z} + \dfrac{2\lambda\mu h}{3(3\lambda + 2\mu)}\varphi''(z)\right)e^{i\alpha} = \displaystyle\sum_{-\infty}^{+\infty} B_{n2}e^{in\alpha}, \quad r = R.
\end{cases}
\tag{16}
$$

Inside the domain the analytic functions $f(z)$, $g(z)$, $\varphi(z)$ and $\psi(z)$ will have the following form:

$$
f(z) = \sum_{n=1}^{+\infty} a_n e^{in\alpha}, \quad g(z) = \sum_{n=0}^{+\infty} b_n e^{in\alpha},
\tag{17}
$$

$$
\varphi(z) = \sum_{n=1}^{+\infty} c_n e^{in\alpha}, \quad \psi(z) = \sum_{n=1}^{+\infty} d_n e^{in\alpha}.
\tag{18}
$$

Solutions of the Helmholtz equations $\omega(z,\bar{z})$ and $\chi(z,\bar{z})$ inside of the domain are represented as follows

$$
\omega(z,\bar{z}) = \sum_{-\infty}^{+\infty} \alpha_n I_n(\gamma r)e^{in\alpha},
\tag{19}
$$

$$
\chi(z,\bar{z}) = \sum_{-\infty}^{+\infty} \beta_n I_n(\nu r)e^{in\alpha},
\tag{20}
$$

where $I_n(\cdot)$ are Bessel's modified functions.

In the boundary conditions (15) we substitute the corresponding expressions (17), (19) and compare the coefficients at identical degrees. We obtain the following system of equations

$$
\begin{cases}
(\lambda + \mu)(n + 1)R^n a_{n+1} - \dfrac{\mu i}{2h}I_{n+2}(\gamma R)\alpha_n = A_{n1}, \\[3mm]
\dfrac{i\mu\gamma}{4}\left(I_{n+1}(\gamma R) - I_{n-1}(\gamma R)\right)\alpha_n - 2(\lambda + 2\mu)n(n + 1)R^{n-1}a_{n+1} = B_{n1} \\[3mm]
\left[(\lambda + \mu)R^n + 2\mu nR^{n-1} + \dfrac{8(\lambda + 2\mu)h^2}{3}(n - 1)nR^{n-2}\right](n + 1)a_{n+1} \\[3mm]
+2\mu(n - 1)nb_n = \bar{A}_{-n1}.
\end{cases}
\tag{21}
$$

The solutions of the system (21) have the following forms:

$$\text{Re}a_1 = \frac{\text{Re}A_{01}}{2(\lambda + \mu)}, \quad \alpha_0 = -\frac{2h\text{Im}A_{01}}{\mu I_2(\gamma R)},$$

$$a_{n+1} = \frac{2I_{n+2}(\gamma R)B_{n1} + (I_{n+1}(\gamma R) - I_{n-1}(\gamma R))\gamma h A_{n1}}{(n+1)R^{n-1}((\lambda + \mu)(I_{n+1}(\gamma R) - I_{n-1}(\gamma R))\gamma h R - 4(\lambda + 2\mu)nI_{n+2}(\gamma R))},$$

$$\alpha_n = \frac{2h[(n+1)(\lambda + \mu)R^n a_{n+1} - A_{n1}]}{\mu i I_{n+2}(\gamma R)},$$

$$b_n = \frac{\bar{A}_{-n1}}{2\mu n(n-1)} - \left[\frac{(\lambda + \mu)R^n}{2\mu n(n-1)} + \frac{R^{n-1}}{n-1} + \frac{4(\lambda + 2\mu)h^2 R^{n-2}}{3}\right](n+1)a_{n+1}.$$

Now by substituting (18), (20) into (16) we obtain the system of algebraic equations:

$$\begin{cases} \dfrac{3\lambda\mu}{(\lambda + 2\mu)h}\Big(I_{n+2}(\nu R) - I_n(\nu R)\Big)\beta_n + 2\mu(n+1)R^n c_{n+1} = A_{n2}, \\[2mm] \dfrac{\mu\nu}{2}\Big(I_{n+1}(\nu R) + I_{n-1}(\nu R)\Big)\beta_n + \dfrac{2\lambda\mu h}{3(3\lambda + 2\mu)}n(n+1)c_{n+1} = B_{n2}, \\[2mm] \dfrac{3\lambda\mu}{(\lambda + 2\mu)h}\Big(I_{n-2}(\nu R) - I_n(\nu R)\Big)\beta_n + 2\mu(n+1)(1-n)R^n c_{n+1} \\[2mm] -2\mu(n-1)R^{n-2}d_{n-1} = \bar{A}_{-n2}. \end{cases} \quad (22)$$

For coefficients c_n, d_n and β_n we have:

$$c_{n+1} = \frac{(3\lambda + 2\mu)[6\lambda I_n'(\nu R)B_{n2} - (\lambda + 2\mu)h^2 I_n''(\nu R)A_{n2}]}{4\lambda^2\mu h n(n+1)R^{n-1}I_n'(\nu R) - 2(\lambda + 2\mu)(3\lambda + 2\mu)\mu h^2(n+1)R^n I_n''(\nu R)},$$

$$\beta_n = \frac{(\lambda + 2\mu)h(A_{n2} - 2\mu(n+1)R^n c_{n+1})}{3\lambda\mu I_n'(\nu R)},$$

$$d_{n-1} = \frac{3\lambda\Big(I_{n-2}(\nu R) - I_n(\nu R)\Big)}{2(\lambda + 2\mu)h(n-1)R^{n-2}}\beta_n - (n+1)R^2 c_{n+1} - \frac{\bar{A}_{-n2}}{2\mu(n-1)R^{n-2}},$$

$$\beta_0 = \frac{B_{02}}{\mu\nu I_1(\nu R)}, \quad \text{Re}c_1 = \frac{\text{Re}A_{02}}{4\mu} - \frac{3\lambda I_0'(\nu R)B_{02}}{(\lambda + 2\mu)\nu h I_1(\nu R)},$$

where

$$I_n'(\nu R) = I_{n+2}(\nu R) - I_n(\nu R), \quad I_n''(\nu R) = I_{n+1}(\nu R) + I_{n-1}(\nu R).$$

It is easy to prove that the absolute and uniform convergence of the series obtained in the circle (including the contours) when the functions set on the boundaries have sufficient smoothness.

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ABOUT ONE METHOD OF CONSTRUCTION OF APPROXIMATE SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS

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Abstract. A simple algorithm for construction of the approximate solution of some classical and nonlocal boundary value problems of the mathematical physics is considered. The efficiency of the offered algorithm for construction of the approximate solutions of problems is shown on the examples of two-dimensional classical and nonlocal boundary value problems of the theory of elasticity and for two-dimensional equations of Laplace and Helmholtz.

Keywords and phrases: Boundary value problems, approximate solution, nonlocal problems.

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1. Introduction

In this work a simple algorithm for construction of the approximate solution of some boundary value problems of the mathematical physics is considered. The mentioned algorithm has been offered in [1]. We may call a considered method a semi-analytical method. From the approximate methods known in the literature it is the closest to a method of fundamental solutions [2-4] and a boundary elements method [5-9].

In the work the main relations of the offered method for the problems of the two-dimensional equations of Laplace and Helmholtz and for problems of the plane theory of thermoelasticity are obtained. By means of this method the approximate solutions for several classical boundary value problems and nonlocal problems of Bitsadze-Samarskii type [10-21] are constructed and exact solutions of these problems are known in advance. The relevant exact and approximate solutions are compared with each other and appropriate conclusions are drawn.

2. Problems for the Laplaces two dimensional equation

Let Oxy be a rectangular cartesian coordinate system on the plane. We consider the Laplace equation

$$\Delta u = 0, \tag{1}$$

where $\Delta(\cdot) = (\cdot)_{,xx} + (\cdot)_{,yy}$ is a two-dimensional laplacian, $(\cdot)_{,x} \equiv \dfrac{\partial(\cdot)}{\partial x}$, $(\cdot)_{,y} \equiv \dfrac{\partial(\cdot)}{\partial y}$; $u(x,y)$ is a scalar function.

First we consider the simply connected domain Ω with a sufficiently smooth boundary L . The domain Ω covers the origin of coordinates. On a contour L the $2N+1$ points with coordinates of $(x_1, y_1), (x_2, y_2), ..., (x_{2N+1}, y_{2N+1})$ are more or less evenly distributed (Fig. 1).

The approximate solution is sought in the form of

$$\bar{u} = a_0 + \sum_{n=1}^{N} r^n(x,y)[a_n \cos(n\theta(x,y)) + b_n \sin(n\theta(x,y))], \qquad (2)$$

where $a_0, a_1, \cdots, a_n, b_1, \cdots, b_n$ are sought-for real coefficients; $r(x,y) = \sqrt{x^2 + y^2}$,

$$\theta(x,y) = \begin{cases} \arctan \dfrac{y}{x}, & x > 0, \\[2mm] \arctan \dfrac{y}{x} + \pi, & x < 0, \ y \geq 0, \\[2mm] \arctan \dfrac{y}{x} - \pi, & x < 0, \ y < 0, \\[2mm] \dfrac{\pi}{2}, & x = 0, \ y > 0, \\[2mm] -\dfrac{\pi}{2}, & x = 0, \ y < 0. \end{cases}$$

The partial derivatives of $\bar{u}(x,y)$ are expressed by the formulas

$$\bar{u}_{,x} = \sum_{n=1}^{N} nr^{n-1}(x,y)[a_n \cos((n-1)\theta(x,y)) + b_n \sin((n-1)\theta(x,y))],$$

$$\bar{u}_{,y} = \sum_{n=1}^{N} nr^{n-1}(x,y)[-a_n \sin((n-1)\theta(x,y)) + b_n \cos((n-1)\theta(x,y))]. \qquad (3)$$
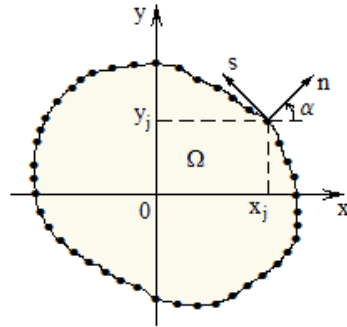
Fig. 1. The simply connected domain Ω

The algorithm of construction of the approximate solution is stated on the example of the classical mixed boundary value problem. The contour L is divided into two contours L_1 and L_2 so that by $L_1 \bigcap L_2 = \varnothing$ and $\overline{L}_1 \bigcap \overline{L}_2 = L$ (Fig. 1). Let us assume that the contour L_1 includes points of $(x_1, y_1), (x_2, y_2), \cdots, (x_{N_1}, y_{N_1})$ and the contour L_2 includes points of $(x_{N_1+1}, y_{N_1+1}), (x_{N_1+2}, y_{N_1+2}), \cdots, (x_{2N+1}, y_{2N+1})$. On the contour L_1 the value of the sought-for function is set, and on the contour L_2 - of the value of its normal derivative

$$\begin{cases} u|_{L_1} = f_1(x,y), & (x,y) \in L_1 \\ u_{,n}|_{L_2} = f_2(x,y), & (x,y) \in L_2, \end{cases} \qquad (4)$$

where $f_1(x, y)$ and $f_2(x, y)$ are the functions defined on the boundary; $(\cdot)_{,n}$ derivative in the direction $\vec{n} = (\cos \alpha, \sin \alpha)$, i. e.

$$u_{,n} = u_{,x} \cos \alpha + u_{,y} \sin \alpha. \tag{5}$$

External unit normal in a point (x_j, y_j) on the boundary is designated through $(\cos \alpha_j, \sin \alpha_j)$.

When $j = 1, 2, \cdots, N_1$ in the formula (2) x and y are replaced through x_j and y_j respectively. The expressions obtained $f_1(x_j, y_j)$ are equated to the corresponding values of the boundary conditions (4). Similarly, when $j = N_1 + 1, N_1 + 2, \cdots, 2N + 1$ in the formula (3) x and y are replaced through x_j and y_j. The expressions received are substituted in (4), where instead of α value α_j is substituted. The resulting expressions are equated to the corresponding values $f_2(x_j, y_j)$ of the boundary conditions (4).

Thus, we obtain the system of the linear algebraic $2N + 1$ equations with $2N + 1$ unknown $a_0, a_1, ..., a_N, b_1, ..., b_N$

$$\begin{cases} a_0 + \sum_{n=1}^{N} (A_{1nj} a_n + A_{2nj} b_n) = f_1(x_j, y_j), & j = 1, 2, \cdots, N_1, \\ \sum_{n=1}^{N} (B_{1nj} a_n + B_{2nj} b_n) = f_2(x_j, y_j), & j = N_1 + 1, N_1 + 2, \cdots, 2N + 1, \end{cases} \tag{6}$$

where

$$A_{1nj} = r^n(x_j, y_j) \cos(n\theta(x_j, y_j)),$$
$$A_{2nj} = r^n(x_j, y_j) \sin(n\theta(x_j, y_j)),$$
$$B_{1nj} = nr^{n-1}(x_j, y_j)[\cos((n-1)\theta(x_j, y_j)) \cos \alpha_j + \sin((n-1)\theta(x_j, y_j)) \sin \alpha_j],$$
$$B_{2nj} = nr^{n-1}(x_j, y_j)[-\sin((n-1)\theta(x_j, y_j)) \cos \alpha_j + \cos((n-1)\theta(x_j, y_j)) \sin \alpha_j].$$

After solving the system (6), its solution $(a_0, a_1, ..., a_N, b_1, ..., b_N)$ is substituted in the formula (2) and thus we've got the approximate solution of a boundary value problem (1), (4).

Example 1. As an example we consider a classical problem of Dirichlet in elliptic domain $V = \{(x, y) | \ x^2 + 4y^2 < 1\}$. The boundary of domain V is the ellipse of S, which is set parametrically $x = \cos t, \ y = 0.5 \sin t, \ 0 \le t < 2\pi$. Thus, the following problem is considered

$$\Delta u = 0 \quad in \ \ V,$$
$$u|_S = 0.5(x^2 + y^2)|_{(x,y) \in S}.$$

The exact solution of this problem is the following function

$$u = 0.2 + 0.3(x^2 - y^2).$$

On the boundary S the points $\left(\cos \dfrac{\pi}{36}(j-1), 0.5 \sin \dfrac{\pi}{36}(j-1) \right), j = 1, 2, ..., 71$ are marked (Fig. 2). The approximate solution is sought in the form (2), where $N = 35$. Meeting the boundary conditions in the marked points, we've got the system of the algebraic 71 equations with 71 unknown.
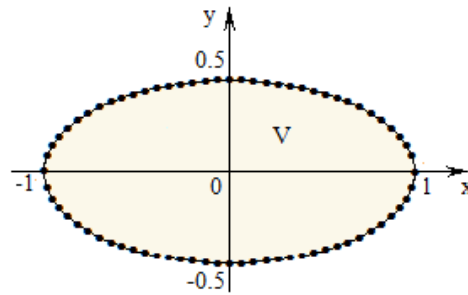
Fig. 2. The domain V with the points marked on the boundary

After solving this system, the resulting solution is substituted in (2) ($N = 35$) and we've got the approximate solution.

The appropriate program is made in the Maple12. Numerical results are specified in Table 1.

Tab. 1. Numerical results for the problem 1

(x, y)	$\bar{u}(x, y)$	$u(x, y)$	$\lvert \bar{u}(x, y) - u(x, y) \rvert$
$(0.01, 0)$	0.2000300000	0.20003	0
$(0.1, 0)$	0.2030000000	0.20300	0
$(0.5, 0)$	0.2750000000	0.27500	0
$(0.9, 0)$	0.4429999995	0.44300	$5.0 \cdot 10^{-10}$
$(0.2, -0.2)$	0.2000000000	0.20000	0
$(0, 0.3)$	0.1730000000	0.17300	0
$(0.8, 0.1)$	0.3890000001	0.38900	10^{-10}

As Table 1 shows the constructed approximate solution may be called the exact solution of the problem of Dirichlet.

The approximate solutions for multi-connected domains are constructed analogously. For simplicity the doubly connected domain Ω, bounded by the simple closed contours L_1 and L_2 is considered from which the last one embraces the latter and the previous embraces the origin of coordinates. On these contours the points $2(2N + 1)$ with the coordinates $(x_1, y_1), (x_2, y_2), ..., (x_{2(2N+1)}, y_{2(2N+1)})$ are more or less evenly distributed (Fig. 3).

Fig. 3. The doubly connected domain Ω

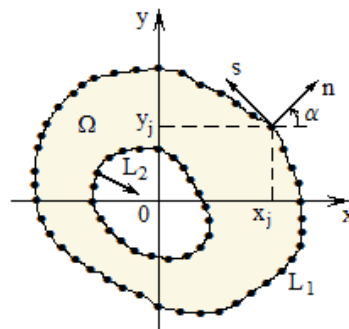The approximate solution is sought in the following form

$$\bar{u} = a\ln r(x,y) + a_0 + \sum_{n=1}^{N} r^{-n}(x,y)[a_n \cos(n\theta(x,y)) + b_n \sin(n\theta(x,y))] \\ + r^n(x,y)[c_n \cos(n\theta(x,y) + d_n \sin(n\theta(x,y))]. \tag{7}$$

Partial derivatives of $\bar{u}(x,y)$ function are expressed by means of the formulas

$$\bar{u}_{,x} = \frac{ax}{r^2(x,y)} + \sum_{n=1}^{N} -nr^{-n-1}(x,y)[a_n \cos((n+1)\theta(x,y)) \\ + b_n \sin((n+1)\theta(x,y))] + nr^{n-1}(x,y)[c_n \cos((n-1)\theta(x,y)) \\ + d_n \sin((n-1)\theta(x,y))], \tag{8}$$

$$\bar{u}_{,y} = \frac{ay}{r^2(x,y)} + \sum_{n=1}^{N} -nr^{-n-1}(x,y)[a_n \sin((n+1)\theta(x,y)) \\ - b_n \cos((n+1)\theta(x,y))] + nr^{n-1}(x,y)[-c_n \sin((n-1)\theta(x,y)) \\ + d_n \cos((n-1)\theta(x,y))], \tag{9}$$

Using the formulas (7)-(9), (5) of the simply connected domain considered above, the boundary conditions are satisfied point-wise in the points selected on the boundary. As a result the we've got a system of the linear algebraic $4N+2$ equations with $4N+2$ unknowns $a, a_0, a_1, ..., a_N, b_1, ..., b_N, c_1, ..., c_N, d_1, ..., d_N$.

The considered way can be applied to construct the approximate solution of rather a wide class of tasks for harmonic functions. The example of construction of the approximate solution of nonlocal problem of Bitsadze-Samarskii for the doubly connected domain bounded by the rectangular contours is given below.

Example 2. Let the domain V represent the doubly connected domain $V = V_1\backslash \overline{V}_2$, where $V_1 = \{-2 < x < 3, -2 < y < 2\}$, $V_2 = \{-1 < x < 1, -1 < y < 1\}$ (Fig. 4). We consider below the nonlocal problem of Bitsadze-Samarskii

$$\Delta u = 0 \quad in \ \ V,$$

$$u(-2,y) = -\frac{2}{4+y^2} - y^2 + 20, \quad -2 \le y \le 2,$$

$$u(x,\pm 2) = \frac{x}{x^2+4} + x^2 - 5x + 2, \quad -2 < x \le 3,$$

$$u(3,y) - u(2,y) = \frac{3}{9+y^2} - \frac{2}{4+y^2}, \quad -2 < y < 2,$$

$$u_{,x}(-1,y) = \frac{y^2-1}{(y^2+1)^2} - 7, \quad -1 \le y < 1,$$

$$u_{,y}(x,\pm 1) = \mp\Big(\frac{2x}{(x^2+1)^2} + 2\Big), \quad -1 \le x < 1,$$

$$u_{,x}(1, y) = \frac{y^2 - 1}{(y^2 + 1)^2} - 3, \quad -1 < y \leq 1.$$

The exact solution of this problem is as follows

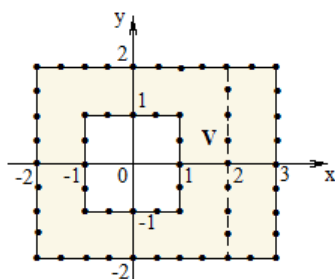$$u(x, y) = \frac{x}{x^2 + y^2} + x^2 - 5x - y^2 + 6.$$

Fig. 4. Doubly connected domain V, in which nonlocal problem is solved

On an external contour beginning from the point $(3, 0)$, with a step 0.5 points 36 are marked. Analogously, on an internal contour beginning from a point $(1, 0)$, with the same frequency 16 more points are marked. On an internal contour two more points with coordinates $(0.75, -1.0)$ and $(-0.75, 1.0)$ are marked. In fig. 4 also 7 points are marked on the segment inside the body where nonlocal conditions are set. The approximate solution sought in the form (7), where $N = 13$. Boundary and nonlocal conditions are satisfied in the marked points.

The solution of the nonlocal problem is tabulated to solution of the problem of system of the linear algebraic 54 equations with 54 unknown. After solving this system, the resulting solution is substituted in (7) ($N = 13$) and we've got the approximate solution.

The appropriate program is made in the Maple 12. Numerical results are presented in Table 2.

Tab. 2. Numerical results for a problem 2

(x, y)	$\bar{u}(x, y)$	$u(x, y)$	$|\bar{u}(x, y) - u(x, y)|$
$(2.0, 0)$	0.5000066859	0.5	$6.66859 \cdot 10^{-6}$
$(1.6, 1.8)$	-2.404136519	-2.404137931	$1.412 \cdot 10^{-6}$
$(0.4, 1.74)$	1.257888466	1.257886259	$2.203 \cdot 10^{-6}$
$(-1.43, -2.25)$	9.931201879	9.931201249	$6.3 \cdot 10^{-7}$
$(0.7, -1.23)$	1.826596858	1.826593235	$3.623 \cdot 10^{-6}$
$(-1.5, 1.5)$	13.16666710	13.16666667	$4.3 \cdot 10^{-7}$
$(3.0, -2.0)$	-3.769230771	-3.769230769	$2.0 \cdot 10^{-9}$

3. Problems of the plane theory of thermoelasticity

Let consider the plane deformation parallel to the plane Oxy for the homogeneous transversely isotropic thermoelastic body.

If the plane of an isotropie is parallel to the Oxy plane then the homogenous system of the equations of thermoelastic equilibrium in displacements has the form [22, 23, 25]

$$\begin{cases} \mu\Delta u + \dfrac{1}{2}\dfrac{E_1 E_2}{(1-\nu_1)E_2 - 2\nu_2^2 E_1}(u_{,x}+v_{,y})_{,x} - \beta T_{,x} = 0, \\[3mm] \mu\Delta v + \dfrac{1}{2}\dfrac{E_1 E_2}{(1-\nu_1)E_2 - 2\nu_2^2 E_1}(u_{,x}+v_{,y})_{,y} - \beta T_{,y} = 0, \end{cases} \tag{10}$$

where μ are shear modulus $\mu = \dfrac{E_1}{2(1+\nu_1)}$; ν_1, ν_2 and E_1, E_2 Poisson's coefficients and Young's modulus in the Oxy and in the direction of perpendicular thereto, respectively. u and v are components of the displacement vector along axes x and y, respectively; β constant depending on the thermal properties of material $\beta = \dfrac{E_1 E_2(\alpha_1 + \nu_2\alpha_2)}{(1-\nu_1)E_2 - 2\nu_2^2 E_1}$; α_1, α_2 are the coefficients of the linear thermal expansion; T is the temperature changes in the elastic body satisfying the Laplace equation

$$\Delta T = 0. \tag{11}$$

Duhamel-Neumann relations has the form

$$\sigma_{xx} = \frac{2\mu}{(1-\nu_1)E_2 - 2\nu_2^2 E_1}[(E_2 - \nu_2^2 E_1)u_{,x} + (\nu_1 E_2 + \nu_2^2 E_1)v_{,y}] - \beta T,$$

$$\sigma_{yy} = \frac{2\mu}{(1-\nu_1)E_2 - 2\nu_2^2 E_1}[(\nu_1 E_2 + \nu_2^2 E_1)u_{,x} + (E_2 - \nu_2^2 E_1)v_{,y}] - \beta T,$$

$$\sigma_{xy} = \sigma_{yx} = \mu(u_{,y} + v_{,x}), \tag{12}$$

$$\sigma_{zz} = \frac{\nu_2 E_1 E_2}{(1-\nu_1)E_2 - 2\nu_2^2 E_1}(u_{,x} + v_{,y}) - \beta T,$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{zz}$ are components of the stresses tensor. Other components of a tensor of stresses in case of plane deformation equal to zero.

Next, we construct the general representation of the system of equations (10) by means of harmonic functions (Kolosov-Muskhelishvilis formula).

The first equation of the system (10) is differentiated by x, the second - by y and are added up. Given the fact, that we've got the T harmonic function

$$\Delta[(c+\mu)(u_{,x} + v_{,y})] = 0, \tag{13}$$

where denotation is entered

$$c := \frac{1}{2}\frac{E_1 E_2}{(1-\nu_1)E_2 - 2\nu_2^2 E_1}.$$

If the second equation of the system (10) is differentiated by x , and the first equation is differentiated by y and to consider their difference, we'll obtain

$$\Delta[\mu(v_{,x} - u_{,y})] = 0. \tag{14}$$

The notation is introduced

$$\theta := (c + \mu)(u_{,x} + v_{,y}), \quad \omega := \mu(v_{,x} - u_{,y}). \tag{15}$$

Thus, according to (13) and (14), θ and ω are harmonic functions

$$\Delta\theta = 0, \quad \Delta\omega = 0. \tag{16}$$

According to notation (15)

$$\begin{cases} u_{,x} + v_{,y} = \dfrac{\theta}{c + \mu} \\ v_{,x} - u_{,y} = \dfrac{\omega}{\mu} \end{cases} \tag{17}$$

From (17) there follows

$$\Delta u = \frac{\theta_{,x}}{c + \mu} - \frac{\omega_{,y}}{\mu}, \quad \Delta v = \frac{\theta_{,y}}{c + \mu} + \frac{\omega_{,x}}{\mu}. \tag{18}$$

Formulas (18) are substituted in the system (10) and the notation introduced in this section are accounted

$$\begin{cases} (\theta - \beta T)_{,x} - \omega_{,y} = 0, \\ (\theta - \beta T)_{,y} + \omega_{,x} = 0. \end{cases} \tag{19}$$

As θ and ω are harmonic functions, from (19) we have

$$\theta = a\varphi + \beta T = 0.5[(a\varphi^* + \beta T^*)_{,x} + (a\tilde{\varphi} + \beta\tilde{T})_{,y}], \tag{20}$$

$$\omega = 0.5a(-\varphi^*_{,y} + \tilde{\varphi}_{,x}), \tag{21}$$

where a is any real constant other than zero; $\varphi^*, \tilde{\varphi}$ and $T^*, \tilde{T}$ are the mutually conjugate harmonic functions

$$\varphi^*_{,x} = \tilde{\varphi}_{,y} = \varphi, \quad \varphi^*_{,y} = -\tilde{\varphi}_{,x},$$

$$T^*_{,x} = \tilde{T}_{,y} = T, \quad T^*_{,y} = -\tilde{T}_{,x},$$

Relations (20) and (21) are substituted in system (17)

$$\begin{cases} \left(u - \dfrac{a}{2(c + \mu)}\varphi^* - \dfrac{\beta}{\beta(c + \mu)}T^*\right)_{,x} - \left(v - \dfrac{a}{2(c + \mu)}\tilde{\varphi} - \dfrac{\beta}{2(c + \mu)}\tilde{T}\right)_{,y} = 0, \\ v_{,x} - u_{,y} = \dfrac{a}{2\mu}(-\varphi^*_{,y} + \tilde{\phi}_{,x}) = 0. \end{cases} \tag{22}$$

The first equation of system (22) is identically satisfied, if

$$u = \Phi_{,y} + \frac{a}{2(c + \mu)}\varphi^* + \frac{\beta}{2(c + \mu)}T^*, \quad v = -\Phi_{,x} + \frac{a}{2(c + \mu)}\tilde{\varphi} + \frac{\beta}{2(c + \mu)}\tilde{T}. \tag{23}$$

The equalities (23) are substituted in the second equation (22) as a result of which we've got the equation relating to the function Φ

$$\Delta\Phi = \frac{ca}{2\mu(c+\mu)}(\varphi^*_{,y} - \tilde{\varphi}_{,x}) + \frac{\beta}{2(c+\mu)}(-T^*_{,y} + \tilde{T}_{,x}). \tag{24}$$

The general solution of equation (24) is presented in the form

$$\Phi = \frac{ca}{4\mu(c+\mu)}(y\varphi^* - x\tilde{\varphi}) + b\psi + \frac{\beta}{4(c+\mu)}(-yT^* + x\tilde{T}). \tag{25}$$

where ψ is an arbitrary harmonic function, b is any real constant other than zero.

Constants a and b may be represented as follows

$$a = \frac{c+\mu}{c}, \qquad b = \frac{1}{2\mu},$$

and the formula (25) is substituted in the ratio (23)

$$2\mu u = \frac{c+2\mu}{2c}\varphi^* + 0.5(y\varphi^*_{,y} - x\tilde{\varphi}_{,y}) + \psi_{,y} + \frac{\mu\beta}{2(c+\mu)}(T^* - yT^*_{,y} + x\tilde{T}_{,y}), \tag{26}$$

$$2\mu v = \frac{c+2\mu}{2c}\tilde{\varphi} + 0.5(x\tilde{\varphi}_{,x} - y\varphi^*_{,x}) - \psi_{,x} + \frac{\mu\beta}{2(c+\mu)}(\tilde{T} - x\tilde{T}_{,x} + yT^*_{,x}). \tag{27}$$

By substituting (26) and (27) in the formulas (12) we've obtained the following expressions for stress tensor components

$$\sigma_{xx} = \varphi + 0.5(y\varphi^*_{,xy} - x\tilde{\varphi}_{,xy}) + \psi_{,xy} - \frac{\beta\mu}{2(c+\mu)}(2T + yT^*_{,xy} - x\tilde{T}_{,xy}),$$

$$\sigma_{yy} = \varphi - 0.5(y\varphi^*_{,xy} - x\tilde{\varphi}_{,xy}) - \psi_{,xy} - \frac{\beta\mu}{2(c+\mu)}(2T - yT^*_{,xy} + x\tilde{T}_{,xy}),$$

$$\sigma_{xy} = 0.5(y\varphi^*_{,yy} + x\tilde{\varphi}_{,xx}) + \psi_{,yy} - \frac{\beta\mu}{2(c+\mu)}(yT^*_{,yy} + x\tilde{T}_{,xx}), \tag{28}$$

$$\sigma_{zz} = 2\nu_2\varphi - \frac{(1-2\nu_2)c+\mu}{c+\mu}\beta T.$$

For simplification of representations (26) - (28) the following notation is introduced

$$\phi = \varphi - \frac{\mu\beta}{c+\mu}T, \quad \phi^* = \varphi^* - \frac{\mu\beta}{c+\mu}T^*, \quad \tilde{\phi} = \tilde{\varphi} - \frac{\mu\beta}{c+\mu}\tilde{T}. \tag{29}$$

ϕ is a harmonic function, and ϕ^* and $\tilde{\phi}$ are the mutually conjugate harmonic functions

$$\phi^*_{,x} = \tilde{\phi}_{,y} = \phi, \quad \phi^*_{,y} = -\tilde{\phi}_{,x}.$$

From (29) functions $\varphi, \varphi^*, \tilde{\varphi}$ are defined and are substituted in the formulas (26) - (28). As a result we obtain displacement representations

$$2\mu u = \frac{c+2\mu}{2c}\phi^* + 0.5(y\phi^*_{,y} - x\tilde{\phi}_{,y}) + \psi_{,y} + \frac{\beta\mu}{c}T^*, \tag{30}$$

$$2\mu v = \frac{c+2\mu}{2c}\tilde{\phi} + 0.5(x\tilde{\phi}_{,x} - y\phi^*_{,x}) - \psi_{,x} + \frac{\beta\mu}{c}\tilde{T}. \tag{31}$$

The following representations are fair for stresses

$$\sigma_{xx} = \phi + 0.5(y\phi_{,y} - x\phi_{,x}) + \psi_{,xy},$$

$$\sigma_{yy} = \phi - 0.5(y\phi_{,y} - x\varphi_{,x}) - \psi_{,xy},$$

$$\tag{32}$$

$$\sigma_{xy} = -0.5(y\phi_{,x} + x\phi_{,x}) + \psi_{,yy},$$

$$\sigma_{zz} = 2\nu_2\phi - (1 - 2\nu_2)\beta T.$$

The analogs of formulas of Kolosov-Muskhelishvili [24] of (30)-(32) plane theories of thermoelasticity for transversely isotropic bodies may be used both for construction of exact solutions of boundary value problems and for construction of approximate solutions of a wide class of problems.

In case of finite simply connected domain the harmonic functions $\phi^*, \tilde{\phi}, \phi$ are represented by the following finite series

$$\phi^* = a_0 + \sum_{n=1}^{N} r^n(x,y)[a_n\cos(n\theta(x,y)) + b_n\sin(n\theta(x,y))],$$

$$\tilde{\phi} = b_0 + \sum_{n=1}^{N} r^n(x,y)[a_n\sin(n\theta(x,y)) - b_n\cos(n\theta(x,y))], \tag{33}$$

$$\phi = \sum_{n=1}^{N} nr^{n-1}(x,y)[a_n\cos((n-1)\theta(x,y)) + b_n\sin((n-1)\theta(x,y))].$$

As the formulas (30), (31) show the constants a_0, b_0 correspond to rigid displacement of a body, therefore they are equal to zero $a_0 = b_0 = 0$. The harmonic function ψ is represented as

$$\psi = \sum_{n=1}^{N} r^n(x,y)[c_n\cos(n\theta(x,y)) + d_n\sin(n\theta(x,y))]. \tag{34}$$

Analogously, the harmonic functions $T^*, \tilde{T}, T$ are also represented as

$$T^* = t_0 + \sum_{n=1}^{N_T} r^n(x,y)[t_n\cos(n\theta(x,y)) + \tau_n\sin(n\theta(x,y))],$$

$$\tilde{T} = \tau_0 + \sum_{n=1}^{N_T} r^n(x,y)[t_n\sin(n\theta(x,y)) - \tau_n\cos(n\theta(x,y))], \tag{35}$$

$$T = \sum_{n=1}^{N_T} nr^{n-1}(x,y)[t_n\cos((n-1)\theta(x,y)) + \tau_n\sin((n-1)\theta(x,y))].$$

For construction of the approximate solution of problems, the representations (33)-(35) are substituted in the formulas (30)-(32), if necessary formulas of transformation

of components of a vector and a tensor of the second rank are used and the conditions set are satisfied point-wise. The problem is tabulated to the solution of square system of the linear algebraic equations for required expansion coefficients (33)-(35).

The example of a nonlocal problem of Bitsadze-Samarskii in case of the plane theory of elasticity for rectangular domain is given below.

Example 3. We consider the domain $V = \{-2.5 < x < 2.5, -2 < y < 2\}$ (Fig. 5). In the domain V it is required to find such solution of system (10) (where $c = 3, \mu = 1, T = 0$ is accepted), which satisfies the following conditions (see [1])

$$u = -5.5y^2 + 14.375, \quad x = -2.5, -2 \le y \le 2,$$

$$v = 7.0y, \quad x = -2.5, -2 \le y \le 2,$$

$$\sigma_{yy}|_{y=2} = -2.0x - 1, \quad -2.5 < x < 2.5,$$

$$\sigma_{yx}|_{y=2} - \sigma_{yx}|_{y=1} = -14.0, \quad -2.5 < x < 2.5,$$

$$u = -5.5y^2 + 16.875, \quad x = 2.5, -2 \le y \le 2,$$

$$v = -8.0y, \quad x = 2.5, -2 \le y \le 2,$$

$$u = 2.5x^2 + 0.5x - 22.0, \quad y = -2, -2.5 < x < 2.5,$$

$$v = 6.0x + 1.0, \quad y = -2, -2.5 < x < 2.5.$$

The exact solution of this problem is as follows

$$u = 2.5x^2 - 5.5y^2 + 0.5x,$$

$$v = -0.3xy - 0.5y.$$

The boundary counter of the considered domain is divided by points into 72 equal segments. 19 points are also distributed evenly on a segment inside the domain where nonlocal conditions are set. The approximate solutions are sought as follows

$$\bar{u} = 0.5 \sum_{n=1}^{36} r^{n-1} \left\{ \left[\frac{5}{6} r \cos(n\theta) - \frac{n}{2} y \sin((n-1)\theta) - \frac{n}{2} x \cos((n-1)\theta) \right] a_n \right.$$

$$+ \left[\frac{5}{6} r \sin(n\theta) + \frac{n}{2} y \cos((n-1)\theta) - \frac{n}{2} x \sin((n-1)\theta) \right] b_n$$

$$\left. - n \sin((n-1)\theta) c_n + n \cos((n-1)\theta) d_n \right\},$$

$$\bar{v} = 0.5 \sum_{n=1}^{36} r^{n-1} \left\{ \left[\frac{5}{6} r \sin(n\theta) + \frac{n}{2} y \sin((n-1)\theta) - \frac{n}{2} x \cos((n-1)\theta) \right] a_n \right.$$

$$- \left[\frac{5}{6} r \cos(n\theta) + \frac{n}{2} y \cos((n-1)\theta) + \frac{n}{2} x \sin((n-1)\theta) \right] b_n$$

$$\left. - n \cos((n-1)\theta) c_n - n \sin((n-1)\theta) d_n \right\}.$$

The components of the stress tensor σ_{yy} and σ_{yx} appearing in the reference condition are presented in the form of the following finite rows

$$\sigma_{yy} = \sum_{n=1}^{36} nr^{n-2}\left\{\left[r\cos((n-1)\theta) + \frac{n-1}{2}y\sin((n-2)\theta) + \frac{n-1}{2}x\cos((n-2)\theta)\right]a_n\right.$$

$$+\left[r\sin((n-1)\theta) - \frac{n-1}{2}y\cos((n-2)\theta) + \frac{n-1}{2}x\sin((n-2)\theta)\right]b_n$$

$$\left.+(n-1)\sin((n-2)\theta)c_n - (n-1)\cos((n-2)\theta)d_n\right\},$$

$$\sigma_{yx} = \sum_{n=1}^{36} \frac{n(n-1)}{2}r^{n-2}\left\{\left[-y\cos((n-2)\theta) + x\sin((n-2)\theta)\right]a_n\right.$$

$$-\left[y\sin((n-2)\theta) + x\cos((n-2)\theta)\right]b_n$$

$$\left.-2\cos((n-2)\theta)c_n - 2\sin((n-2)\theta)d_n\right\}.$$

In the last four formulas the coordinates of points marked on the boundary and inside the domain are substituted and the corresponding boundary and nonlocal conditions are satisfied on them. As a result we obtained the system consisting of the 144-linear algebraic equations and containing 144 unknowns $(a_1, ..., a_{36}, b_1, ..., b_{36}, c_1, ..., c_{36}, d_1, ..., d_{36})$. After solving this system by means of the formulas given above one can easily find components of a vector of displacement and a tensor of stresses.
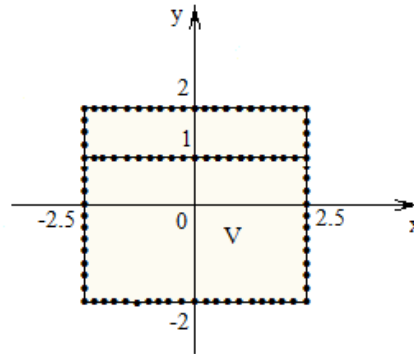

Fig. 5. The domain V in which the nonlocal problem of the plane theory of elasticity is solved

The appropriate program is made in the Maple 12. Numerical results are presented in Table 3, where $\bar{u}$ and $\bar{v}$ denote the approximate values of components of the displacement vector.

Tab. 3. Numerical results for a problem 3

(x, y)	$\bar{u}(x, y)$	$u(x, y)$	$|\bar{u}(x, y) - u(x, y)|$
$(0, 0)$	$4.166195286 \cdot 10^{-8}$	0	$4.166195286 \cdot 10^{-9}$
$(-1.0, 1.0)$	-3.499999843	-3.500	$1.57 \cdot 10^{-7}$
$(1.5, -1.5)$	-5.999999978	-6.000	$2.2 \cdot 10^{-8}$
$(1.2, -0.8)$	0.6800000324	0.680	$3.24 \cdot 10^{-8}$
$(-1.7, 1.5)$	-5.999999651	-6.000	$3.49 \cdot 10^{-7}$
$(2.2, -1.4)$	2.420000034	2.420	$3.4 \cdot 10^{-8}$
$(1.25, 1.75)$	-12.31250013	-12.31250	$1.3 \cdot 10^{-7}$

(x, y)	$\bar{v}(x, y)$	$v(x, y)$	$|\bar{v}(x, y) - v(x, y)|$
$(0, 0)$	$-1.190721638 \cdot 10^{-7}$	0	$1.190721638 \cdot 10^{-7}$
$(-1.0, 1.0)$	2.499999817	2.500	$1.83 \cdot 10^{-7}$
$(1.5, -1.5)$	7.499999968	7.500	$3.2 \cdot 10^{-8}$
$(1.2, -0.8)$	3.279999938	3.280	$6.2 \cdot 10^{-8}$
$(-1.7, 1.5)$	6.899999694	6.900	$3.06 \cdot 10^{-7}$
$(2.2, -1.4)$	9.939999959	9.940	$4.1 \cdot 10^{-8}$
$(1.25, 1.75)$	-7.437500134	-7.43750	$1.34 \cdot 10^{-7}$

As numerical results show the considered method gives the good approximate solution for nonlocal mixed boundary value problem of the plane theory of elasticity.

4. Problems for the Helmholtzs two dimensional equation

Let on the plane Oxy there be a domain Ω (shown in Fig. 3). In this domain the following equation of Helmholtz is considered

$$\Delta\omega - \zeta^2\omega = 0 \quad in \quad \Omega, \tag{36}$$

where ζ is any real constant other than zero.

The approximate solution is sought as follows

$$
\begin{aligned}
\bar{\omega} = {} & a_0 I_0(\zeta r(x, y)) + b_0 K_0(\zeta r(x, y)) \\
& + \sum_{n=1}^{N} \{ I_n(\zeta r(x, y))[a_n \cos(n\theta(x, y)) + b_n \sin(n\theta(x, y))] \\
& + K_n(\zeta r(x, y))[c_n \cos(n\theta(x, y)) + d_n \sin(n\theta(x, y))] \},
\end{aligned}
\tag{37}
$$

where $I_n(\zeta r)$ and $K_n(\zeta r)$ are modified Bessel functions of n order according to [26].

Partial derivatives of functions $\bar{\omega}(x, y)$ are expressed by means of the formulas

$$
\begin{aligned}
\bar{\omega}_{,x} = {} & \frac{\zeta x}{r}(a_0 I_1(\zeta r) - b_0 K_1(\zeta r) \\
& + \frac{\zeta x}{2r} \sum_{n=1}^{N} \{ (I_{n-1}(\zeta r) + I_{n+1}(\zeta r))[a_n \cos((n-1)\theta(x, y)) + b_n \sin((n-1)\theta(x, y))] \\
& - (K_{n-1}(\zeta r) + K_{n+1}(\zeta r))[c_n \cos((n-1)\theta(x, y)) + d_n \sin((n-1)\theta(x, y))] \},
\end{aligned}
\tag{38}
$$

$$\bar{\omega}_{,y} = \frac{\zeta y}{r}(a_0 I_1(\zeta r) - b_0 K_1(\zeta r))$$

$$+\frac{\zeta y}{2r}\sum_{n=1}^{N}\{(I_{n-1}(\zeta r) + I_{n+1}(\zeta r))[-a_n \sin((n-1)\theta(x,y))$$

$$+b_n \cos((n-1)\theta(x,y))] + (K_{n-1}(\zeta r) + K_{n+1}(\zeta r))[c_n \sin((n-1)\theta(x,y))$$

$$-d_n \cos((n-1)\theta(x,y))]\}. \tag{39}$$

By means of the formulas (37)-(39) one can construct the approximate solutions of various boundary value problems or boundary value contact problems for Helmholtz's equation (36).

An example of nonlocal problem of Bitsadze-Samarskii for the Helmholtz's equation is given below.

Example 4. The Helmholtz equation in a rectangle $V = \{-3 < x < 3, -2 < y < 2\}$(Fig. 6) is given as an example to find such a function ω satisfying the following conditions

$$\Delta\omega - \frac{\pi^2}{12}\omega = 0 \quad in \quad V, \tag{40}$$

$$\omega(-3,y) - \sqrt{2}\omega(-1.5,y) + \omega(0,y) = 0, \quad -2 < y < 2,$$

$$\omega(x,\pm 2) = e^{\pm\frac{2\pi}{3}}\sin\frac{\pi x}{6}, \quad -3 \le x \le 3,$$

$$\omega(3,y) = e^{\frac{\pi y}{3}}, \quad -2 < y < 2.$$

It is easy to verify that the exact solution of the problem set is as follows

$$\omega(x,y) = e^{\frac{\pi y}{3}}\sin\frac{\pi x}{6}.$$

The approximate solution of the considered nonlocal problem is sought in the form of the sum

$$\bar{\omega} = a_0 I_0\Big(\frac{\sqrt{3}\pi}{6}r(x,y)\Big) + \sum_{n=1}^{39}\Big\{I_n\Big(\frac{\sqrt{3}\pi}{6}r(x,y)\Big)[a_n \cos(n\theta(x,y))$$

$$+b_n \sin(n\theta(x,y))]\Big\}. \tag{41}$$

Beginning from a point (-3, 0) on the boundary of the considered rectangle with a step 0.25, 79 points are evenly distributed. 15 points are evenly distributed on each piece inside the domain where nonlocal conditions are set. After satisfying the given boundary conditions and nonlocal conditions we've obtained the system of the linear algebraic 79 equations with 79 unknowns. The solution of this system $(a_0, a_1, ..., a_{39}, b_1, ..., b_{39})$ is substituted in formula (41) representing the approximate solution of the stated problem. The constructed approximate solution satisfies the Helmholtz equation in the domain V and satisfies the boundary conditions and nonlocal conditions in the respective points marked in advance.
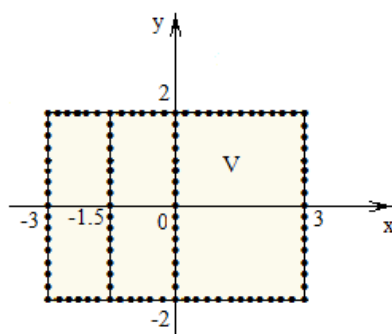
Fig. 6. Domain V, in which the nonlocal problem for Helmholtz's equation is considered

The appropriate program is made in the Maple 12. Numerical results are presented in the table 4.

Tab. 4. Numerical results for the problem 4.

(x, y)	$\bar{\omega}(x, y)$	$\omega(x, y)$	$\lvert\bar{\omega}(x, y) - \omega(x, y)\rvert$
$(-3.0, -1.5)$	-0.2078795840	-0.2078795765	$7.5 \cdot 10^{-9}$
$(-1.75, 1.75)$	-4.958529035	-4.958529038	$3.0 \cdot 10^{-9}$
$(0, -1.5)$	$-3.564486401 \cdot 10^{-10}$	0	$3.564486401 \cdot 10^{-10}$
$(0.5, -2.0)$	0.03187219544	0.03187219654	$1.1 \cdot 10^{-9}$
$(1.0, 1.5)$	2.405238691	2.405238689	$2.0 \cdot 10^{-9}$
$(1.5, 1.25)$	2.618033198	2.618032200	$2.0 \cdot 10^{-9}$
$(3.0, 1.5)$	4.810477384	4.810477377	$7.0 \cdot 10^{-9}$

As the table shows the constructed approximate solution of the nonlocal problem is a good approximation to the exact solution of this problem.

5. Conclusion. In the work we propose the simple method of the approximate solution of boundary value problems of mathematical physics. The approximate solutions of such two-dimensional classical and nonlocal boundary value problems for Laplace's and Helmholtz's equations and the theory of elasticity, the exact solutions of which are known in advance, are constructed by the proposed method.

We believe that by means of the considered algorithm it is possible to receive quite good approximate solutions of some boundary value problems of mathematical physics.

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ONE PROBLEM OF THE BENDING OF A PLATE FOR A CURVILINEAR QUADRANGULAR DOMAIN WITH A RECTILINEAR CUT

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Abstract. In the present paper we consider the problem of bending of a plate for a curvilinear quadrangular domain with a rectilinear cut. It is assumed that the external boundary of the domain composed of segments (parallel to the abscissa axis) and arcs of one and the same circumference. The internal boundary is the rectilinear cut (parallel to the Ox-axis). The plate is bent by normal moments applied to rectilinear segments of the boundary, the arcs of the boundary are free from external forces, while the cut edges are simply supported. The problem is solved by the methods of conformal mappings and boundary value problems of analytic functions. The sought complex potentials which determine the bending of the midsurface of the plate are constructed effectively (in the analytical form). Estimates are given of the behavior of these potentials in the neighborhood of the corner points.

Keywords and phrases: The bending of a plate, conformal mapping, Riemann-Hilbert problem for circular ring.

AMS subject classification (2010): 74B05.

1. Statement of the problem

Let a homogeneous Isotropic plate on a plane $z = x + iy$ of a complex variable occupy the doubly-connected domain S, the external boundary of the domain is composed of segments (parallel to the abscissa axis) and arcs of one and the same circumference. The internal boundary is the rectilinear cut (parallel to the Ox-axis).

We will assume that normal bending moments M_n act on each rectilinear sections $L_0^{(1)} = A_1A_1$, $L_0^{(2)} = A_3A_4$ of the external boundary, the arcs $L_0^{(3)} = A_2A_3$, $L_0^{(k)} = A_4A_1$ of the boundary are free from external forces, while the cut $L_1 = B_1B_2$ edges are simply supported and for better clearness, we consider the symmetric case. We denote by $\alpha^0\pi$ the value of internal (with respect to the domain S) vertex angles $A_k (k = 1, ..., 4)$ (we mean the angles between the segments $L_0^{(1)}$, $L_0^{(2)}$ and the tangent arcs $L_0^{(3)}$ and $L_0^{(4)}$) and we will choose as the positive direction on the boundary $L = L_0 \cup L_1$ ($L_0 = \overset{4}{\underset{k=1}{\cup}} L_0^{(k)}$, $L_1 = \overset{2}{\underset{m=1}{\cup}} L_1^{(m)}$, $L_1^{(1)} = B_1B_2$, $L_1^{(2)} = B_2B_1$) which leaves the region S on the left. Let $\alpha(t)$ and $\beta(t)$ be the angles lying between the Ox-axis and the outer normals to the contours L_0 and L_1 at the point $t \in L$, where

$$\alpha(t) = \begin{cases} \dfrac{\pi}{2}(2k-1), & t \in L_0^{(k)}, \ k = 1,2, \\ \arg t, & t \in L_0^{(k)}, \ k = 3,4, \end{cases} \qquad \beta(t) = \begin{cases} \dfrac{\pi}{2}, & t \in L_1^{(1)}, \\ -\dfrac{\pi}{2}, & t \in L_1^{(2)}. \end{cases}$$
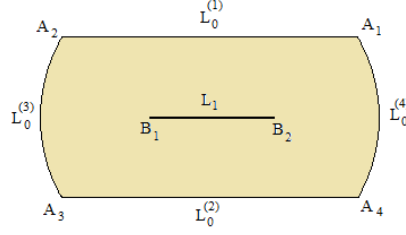
Fig. 1

The problem consists in defining the bending deflection of the middle surface of the plate and establishing the situations of the concentration of stresses near the angular points which in turn depend on the behavior of Kolosov-Muskhelishvili potentials at these points.

Analogous problems of plane elasticity and plate bending for finite doubly-connected domains bounded by polygons are considered in [1, 4].

2. Solution of the problem

Let us recall some results concerning the conformal mapping of a doubly-connected domain $S^{(0)}$ onto the circular ring $D_0\{1 < |\zeta| < R_0\}$. The derivative of the function $\omega(\varsigma)$ is the solution of the Riemann-Hilbert problems for the circular ring [5]

$$\text{Re}[i\sigma e^{-i\nu_0(\sigma)}\omega'(\sigma)] = 0, \quad \sigma \in l, \tag{1}$$

where $l = l_0 \cup l_1$, $l_0 = \{|\sigma| = R\}$, $l_1 = \{|\sigma| = 1\}$, $\nu_0(\sigma) = \alpha[\omega(\sigma)] = \alpha_0(\sigma)$, $\sigma \in l_0$, $\nu_0(\sigma) = \beta[\omega(\sigma)] = \beta_0(\sigma)$, $\sigma \in l_1$.

To solve the problem (1) (with respect to the function $\omega'(\zeta)$) of the class $h(b_1, b_2)$ [6] (the index of the given class problem (1) is equal to zero), it is necessary and sufficient that the condition

$$\prod_{k=1}^{4} R^2 a_k^{\alpha_k^0 - 1} \cdot \prod_{m=1}^{2} b_m = 1, \quad (a_k = \omega^{-1}(A_k), \; b_m = \omega^{-1}(B_m)), \tag{2}$$

be fulfilled, and a solution itself is given by the formula

$$\omega'(\zeta) = K^0 e^{\gamma(\zeta)} B(\zeta), \tag{3}$$

where

$$\gamma(\zeta) = \frac{1}{2\pi i} \sum_{j=-\infty}^{\infty} \int_{l_0} \frac{\ln(R^2 \sigma^{-2} e^{2i\alpha_0(\sigma)})}{\sigma - R^{2j}\zeta} d\sigma, \quad B(\zeta) = \prod_{j=-\infty}^{\infty} \prod_{m=1}^{2} (R^{2j}\zeta - b_m), \tag{4}$$

with k^0 as an arbitrary real constant.

Based on the results given in [6, §78], we conclude that the function $e^{\gamma(\varsigma)}$ near the points $a_k(k = \overline{1,4})$ can be written in the form

$$e^{\gamma(\varsigma)} = \prod_{k=1}^{4} (\zeta - a_k)^{\alpha_k^0 - 1} \Omega^0(\zeta), \tag{5}$$

where Ω^0 is the function holomorphic near the point a_k and tending to definite nonzero limits as $\varsigma \to a_k$.

Thus, for a conformally mapping function bounded at the points a_k from (4) we obtain the formula

$$\omega'(\zeta) = K^0 \prod_{k=1}^{4} (\zeta - a_k)^{\alpha_k^0 - 1} \Omega^0(\zeta) B(\varsigma). \tag{6}$$

Let us now return to the considered problem. According to the approximate theory of the bending of a plate, the bending deflection $w(x, y)$ of the midsurface of the plate in the case considered satisfies the biharmonic equation

$$\Delta^2 w(x, y) = 0, \quad z = x + iy \in S$$

and the boundary conditions

$$M_n(t) = f(t), \quad \frac{\partial w}{\partial s} = 0, \quad t \in L_0^{(1)} \cup L_0^{(2)},$$

$$M_n(t) = 0, \quad \frac{\partial w}{\partial n} = 0, \quad t \in L_0^{(3)} \cup L_0^{(4)}, \tag{7}$$

$$w(t) = 0, \quad M_n(t) = 0, \quad t \in L_1, \quad N(t) = 0, \quad t \in L_0 \cup L_1,$$

where $M_n(t)$ is the normal bending moments, $N(t)$ is the shearing force.

Using the well-known formulae [6-8] we have

$$\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s} = e^{-i\nu(t)}[\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}],$$

$$2D_0(\sigma - 1)d[\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}] = \left[M_n(t) + i\int_0^s N(t)ds \right] dt, \tag{8}$$

$$\nu(t) = \alpha(t), \quad t \in L_0, \quad \nu(t) = \beta(t), \quad t \in L_1, \quad \varkappa = (\sigma + 3)(1 - \sigma)^{-1},$$

where σ is Poisson ratio, D_0 is the cylindrical stiffness of the plate.

By virtue of condition (7) and formula (8) with respect to the required functions $\varphi(z)$ and $\psi(z)$ we obtain the boundary problems

$$\mathrm{Re}\left[ie^{-i\nu(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}) \right] = 0,$$

$$\mathrm{Re}\left[ie^{-i\nu(t)}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}) \right] = F_0^{(1)}(t), \quad t \in L_0^{(1)} \cup L_0^{(2)}, \tag{9}$$

$$\mathrm{Re}\left[ie^{-i(\nu(t)+\frac{\pi}{2})}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}) \right] = 0,$$

$$\mathrm{Re}\left[ie^{-i(\nu(t)+\frac{\pi}{2})}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}) \right] = F_0^{(2)}(t), \quad t \in L_0^{(3)} \cup L_0^{(4)}, \tag{10}$$

$$\mathrm{Re}\left[ie^{-i\nu(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}) \right] = 0,$$

$$\mathrm{Re}\left[ie^{-i\nu(t)}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}) \right] = c^{(1)}(t), \quad t \in L_1, \tag{11}$$

where

$$F_0^{(1)}(t) = -[2D_0(\sigma - 1)]^{-1} \int_0^s M_n(t)ds + c^{(0)}(t), \quad t \in L_0^{(1)} \cup L_0^{(2)},$$

$$F_0^{(2)}(t) = \mathrm{Re}[2D_0(\sigma - 1)]^{-1}t^{-1}c^{(0)}(t), \quad t \in L_0^{(3)} \cup L_0^{(4)},$$

$$c^{(0)}(t) = c_k^{(0)} = const, \ \ t \in L_0^{(k)} \ \ (k = \overline{1,4}),$$

$$c^{(1)}(t) = c_k^{(1)} = const, \ \ t \in L_1^{(k)} \ \ (k = 1,2).$$

The constant $c_k^{(j)}$ $(j = 0,1)$ are unknown in advance and must be determined when solving the problem in such a away that the function $\varphi(z)$ and $\bar{z}\varphi'(z) + \psi(z)$ extend continuously into to domain $S \cup L$.

These boundary problems are in turn divided into two problems

$$\mathrm{Re}\left[ie^{-i\Delta(t)}\varphi(t)\right] = F(t), \ \ t \in L_0 \cup L_1, \tag{12}$$

$$\mathrm{Re}\left[ie^{-i\Delta(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})\right] = 0, \ \ t \in L_0^{(1)} \cup L_0^{(2)}, \tag{13}$$

where $\Delta(t) = \alpha(t)$, $t \in L_0^{(1)} \cup L_1^{(2)}$; $\Delta(t) = \dfrac{\pi}{2} + \arg t$, $t \in L_0^{(3)} \cup L_1^{(4)}$; $\Delta(t) = \beta(t)$, $t \in L_1$; $F(t) = F_0^{(1)}$, $t \in L_0^{(1)} \cup L_1^{(2)}$; $F(t) = F_0^{(2)}$, $t \in L_0^{(3)} \cup L_1^{(4)}$; $F(t) = c^{(1)}(t)$, $t \in L_1$.

Let us consider problem (12). After the conformal mapping of the domain S onto the circular ring D, this problem for the function $\chi(\zeta) = \zeta^{-1}\varphi_0(\zeta)$ $(\varphi_0(\zeta) = \varphi[\omega(\zeta)])$ reduces to the Riemann-Hilbert problem for a circular ring

$$\mathrm{Re}\left[i\sigma e^{-i\Delta_0(\sigma)}\chi(\sigma)\right] = F_0(\sigma), \ \ \sigma \in l, \tag{14}$$

where $\Delta_0(\sigma) = \Delta[\omega(\sigma)]$, $F_0(\sigma) = F[\omega(\sigma)]$, $\sigma \in l$.

Let us consider the homogeneous problem corresponding to problem (14)

$$\mathrm{Re}\left[i\sigma e^{-i\Delta_0(\sigma)}\chi(\sigma)\right] = 0, \ \ \sigma \in l, \tag{15}$$

Although problem (15) is different from problem (1) we can use it [5] and its solution is given by the formula

$$\chi(\zeta) = \omega'(\zeta)T(\zeta), \tag{16}$$

where $T(\zeta) = \prod\limits_{j=-\infty}^{\infty}\prod\limits_{k=1}^{4}(R^{2j}\zeta - a_k)^{-\frac{1}{2}}$, $\omega'(\zeta)$ is defined by formula (6).

Thus we have obtained the factorization coefficient of problem (15) in the form

$$e^{2i\Delta_0(\sigma)}\frac{\bar{\sigma}}{\sigma} = \frac{\omega'(\sigma)T(\sigma)}{\overline{\omega'(\sigma)} \cdot \overline{T(\sigma)}}, \ \ \sigma \in l.$$

With the obtained results taken into account, from the boundary conditions (14) for the function

$$\Omega(\zeta) = i\varphi_0(\zeta)[\zeta\omega'(\zeta)T(\zeta)]^{-1} \tag{17}$$

we obtain the Dirichlet problem for a circular ring

$$\mathrm{Re}\left[\Omega(\sigma)\right] = F_0(\sigma)e^{i\Delta_0(\sigma)}[\sigma\omega'(\sigma)T(\sigma)]^{-1}, \ \ \sigma \in l. \tag{18}$$

A solvability condition of problem (18) has the form

$$\int\limits_{l} \frac{F_0(\sigma)e^{i\Delta_0(\sigma)}}{\sigma^2\omega'(\sigma)T(\sigma)}d\sigma = 0, \tag{19}$$

and its solution is given by the formula

$$\Omega(\zeta) = \frac{1}{\pi i} \sum_{j=-\infty}^{\infty} \int_l \frac{F_0(\sigma)e^{i\Delta_0(\sigma)}d\sigma}{(\sigma - R^{2j}\zeta)\sigma\omega'(\sigma)T(\sigma)} + ic_0^*, \tag{20}$$

where c_0^* is an arbitrary real constant.

Thus, using (17) and (20), for the function $\varphi_0(\zeta)$ we obtain the formula

$$\varphi_0(\zeta) = \omega'(\zeta)T(\zeta)M(\zeta), \tag{21}$$

where

$$M(\zeta) = -\frac{\zeta}{\pi} \left[\sum_{j=-\infty}^{\infty} \int_l \frac{F_0(\sigma)e^{i\Delta_0(\sigma)}d\sigma}{(\sigma - R^{2j}\zeta)\sigma\omega'(\sigma)T(\sigma)} + c_0^* \right]. \tag{22}$$

Since the function $\omega'(\zeta)T(\zeta)$ at the points a_k $(k = \overline{1,4})$ has singularities of the form $|\zeta - a_k|^{\alpha_k^0 - \frac{3}{2}}$, for the function $\varphi_0(\zeta)$ to be continuously extendable into the domain $D \cup l$ it is necessary and sufficient for the following conditions to be satisfied

$$M(a_k) == 0, \quad k = \overline{1,4}. \tag{23}$$

Since $\varphi'(z) = \frac{\varphi_0'(\zeta)}{\omega'(\zeta)}$, from (21) we have

$$\varphi'(z) = \frac{\omega''(\zeta)}{\omega'(\zeta)}T(\zeta)M(\zeta) + [T(\zeta)M(\zeta)]'. \tag{24}$$

Bearing in mind both the behavior of the Cauchy type integral in the neighborhood on the points density discontinuity [6] and that of the conformally mapping fuction in the neighborhood of angular points [9], we conclude that near the points b_k $(k = 1, 2)$

$$\omega(\zeta) = B + (\zeta - b)^2[N_0 + N_1(\zeta - b) + \cdots],$$
$$\frac{\omega''(\zeta)}{\omega'(\zeta)} = \frac{1}{\zeta - b} + E_1 + E_2(\zeta - b) + \cdots, \tag{25}$$
$$T(\zeta)M(\zeta) = \frac{k_0}{\zeta - b} + k_1 + k_2(\zeta - b) + \cdots,$$

where b is one of the points b_k, B is the preimage of the point b, $N_0,...,k_2,...$ are some constants.

Thus, using (24) and (25), near a point B we have the estimates

$$\left|\varphi'(z)\right| < M_1 |z - B|^{-\frac{1}{2}}, \quad \left|\varphi''(z)\right| < M_2 |z - B|^{-\frac{3}{2}}, \quad M_1, M_2 = const.$$

By a similar reasoning to the above, it is proved that $\varphi'(z)$ is almost bounded (i.e. has singularities of logarithmic type $\ln(z - A)$) near the points A_k $(k = \overline{1,4})$.

After finding the function $\varphi(z)$, the definition of the function $\psi(z)$ by (13) reduces to the following problem which is analogous to problem (12)

$$\mathrm{Re}\left[ie^{i\Delta(t)}R(t)\right] = \Gamma(t), \quad t \in L, \tag{26}$$

where

$$R(z) = \psi(z) + P(z)\varphi'(z),$$

$$\Gamma(t) = F(t) + \operatorname{Re}\left[ie^{i\Delta(t)}(P(t) - \bar{t})\varphi'(t)\right], \ \ t \in L,$$

and $P(z)$ is an interpolation polynomial satisfying the condition $P(B_k) = \bar{B}_k$ $(k = 1, 2)$, $\bar{B}_k$ is a number conjugate to B_k.

The use of the polynomial $P(z)$ makes bounded the right-hand part of the boundary condition (26) so that the solution of this problem can be constructed in an analogous manner as above (see problem (12)), while the solvability condition (with the assumption that the function $\psi(z)$ is continuous up to the boundary) will be analogous to conditions (19) and (23).

All these conditions are represented as an inhomogeneous system with real coefficient with respect to 8 constants $c_k^{(0)}$ $(k = \overline{1, 4})$, $c_m^{(1)}$ $(m = 1, 2)$, c_0^*, c_0^{**} (c_0^{**} is a real constant which occurs when solving problem (26)). For the definition of these constant we have 8 equations. It is proved that the obtained system is uniquely solvable and therefore the problem posed has a unique solution.

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HIGHER ORDER DIFFERENCE EQUATIONS WITH PROPERTIES **A** AND **B**

Khachidze N.

Abstract. The following higher order difference equation

$$\Delta^{(n)}u(k) + p(k)|u(\sigma(k))|^\lambda \text{sign}(u(\sigma(k))) = 0$$

is considered, where $n \geq 2$, $0 < \lambda < 1$, $p : N \to R$, $\sigma : N \to N$, $\sigma(k) \geq k + 1$.

Necessary conditions are obtained for the above equation to have monotone solutions. The obtained results are also new for the oscillation of solutions.

Keywords and phrases: Property **A**, Property **B**, oscillation.

AMS subject classification (2010): 34K11.

1. Introduction

Consider the higher order difference equation

$$\Delta^{(n)}u(k) + p(k)|u(\sigma(k))|^\lambda \text{sign}(u(\sigma(k))) = 0, \tag{1.1}$$

where $n \geq 2$, $0 < \lambda < 1$, $p : N \to R$, $\sigma : N \to N$, $\sigma(k) \geq k + 1$.

Here

$$\Delta^{(0)}u(k) = u(k), \ \ \Delta^{(1)}u(k) = u(k+1) - u(k), \ \ \Delta^{(i)}u(k) = \Delta^{(1)} \circ \Delta^{(i-1)}u(k)$$
$$(i = 2, \ldots, n).$$

It will always be assumed that either the condition

$$p(k) \geq 0 \quad \text{for} \quad k \in N, \tag{1.2}$$

or

$$p(k) \leq 0 \quad \text{for} \quad k \in N \tag{1.3}$$

holds.

For each $k \in N$ denote $N_k = \{k, k + 1, \ldots\}$.

Definition 1.1. Let $k_0 \in N$. A function $u : N_{k_0} \to R$ is said to be a proper solution of equation (1.1), if it satisfies (1.1) on N_{k_0} and

$$\sup\{|u(k)| : k \geq s\} > 0 \quad \text{for any} \quad s > k_0.$$

Definition 1.2. Let $k_0 \in N$. A proper solution $u : N_{k_0} \to R$ of equation (1.1) is said to be oscillatory if for any $k \in N_{k_0}$ there are $k_1, k_2 \in N_{k_0}$ such that $u(k_1)u(k_2) < 0$. Otherwise the solution is called nonoscillatory.

Definition 1.3. We say that equation (1.1) has Property **A** if any its proper solutions either is oscillatory or satisfies

$$|\Delta^{(i)}u(k)| \downarrow 0 \quad \text{for} \quad k \uparrow +\infty \quad (i = 0, 1, \ldots, n-1), \tag{1.4}$$

when n is odd.

Definition 1.4. We say that equation (1.1) has Property **B** if any of its proper solutions is oscillatory or satisfies either (1.4) or

$$|\Delta^{(i)}u(k)| \uparrow +\infty \quad \text{for} \quad k \uparrow +\infty \quad (i = 0, 1, \dots, n-1), \tag{1.5}$$

when n is even, either is oscillatory or satisfies (1.5) when n is odd.

For a functional differential equation, similar problems were considered in [1–4] (see also the references therein). Oscillatory properties for first and second order difference equations are studied in [5–9].

In the present paper we give sufficient conditions for equation (1.1) to have properties **A** and **B**.

2. Necessary condition of the existence of monotone solutions

For any $k_0 \in N$ denote by $U_{k_0,l}$ the set of solutions $u : N_{k_0} \to R$ of equation (1.1) which satisfies the condition:

$$\Delta^{(i)}u(k) > 0 \quad \text{for} \quad k \geq k_0 \quad i = 0, \dots, l-1,$$
$$(-1)^i \Delta^{(i)}u(k) \geq 0 \quad \text{for} \quad k \geq k_0 \quad i = l, \dots, n.$$

Theorem 2.1. *Let $0 < \lambda < 1$, $k_0 \in N$, condition (1.3) ((1.4)) be fulfilled, $l \in \{1, 2, \dots, n-1\}$, $l+n$ be odd ($l+n$ be even) and $U_{k_0,l} \neq \varnothing$.*

Moreover, if

$$\sum_{k=1}^{+\infty} k^{n-l}(\sigma(k))^{\lambda(l-1)}|p(k)| = +\infty \tag{2.1}$$

then for any $\delta \in [0; \lambda]$ and $i \in N$ we have

$$\sum_{k=1}^{+\infty} k^{n-l-1+\lambda-\delta}(\sigma(k))^{\lambda(l-1)}[\rho_{l,i}(\sigma(k))]^{\delta}|p(k)| < +\infty,$$

where

$$\rho_{l,1}(k) = \left(\frac{1-\lambda}{l!(n-l)!} \sum_{i=1}^{k-1} \sum_{j=i}^{+\infty} j^{n-l-1}(\sigma(j))^{\lambda(l-1)}|p(j)| \right)^{\frac{1}{1-\lambda}}, \tag{2.2}$$

$$\rho_{l,s}(k) = \frac{1-\lambda}{l!(n-l)!} \sum_{i=1}^{k} \sum_{j=i}^{+\infty} j^{n-l-1}(\sigma(j))^{\lambda(l-1)}|p(j)|(\rho_{l,s-1}(\sigma(j)))^{\lambda} \quad (s = 2, 3, \dots).$$
$$\tag{2.3}$$

3. Sufficient conditions of nonexistence of monotone solutions

Theorem 3.1 *Let conditions* (1.2) ((1.3)) (2.1) *be fulfilled, $l \in \{1, \ldots, n-1\}$, let $l + n$ be odd ($l + n$ be even) and for any $\delta \in [0, \lambda]$ and $i \in N$*

$$\sum_{k=i}^{+\infty} k^{n-l-1+\lambda-\delta} (\sigma(k))^{\lambda(l-1)} (\rho_{l,i}(\sigma(k)))^{\delta} |p(k)| = +\infty \qquad (3.1)$$

then for any $k_0 \in N$, $U_{l,k_0} = \varnothing$, where $\rho_{l,i}$ is defined by (2.2) *and* (2.3).

Theorem 3.2. *Let conditions* (1.2) ((1.3)) (2.1), *for any $\gamma \in (0; 1)$*

$$\liminf_{k \to +\infty} k^{\gamma} \sum_{j=k}^{+\infty} j^{n-l-1} (\sigma(j))^{\lambda(l-1)} |p(j)| > 0$$

be fulfilled, $l \in \{1, \ldots, n-1\}$, let $l + n$ be odd ($l + n$ be even) and for any $\alpha \in (1; +\infty)$

$$\liminf_{k \to +\infty} \frac{\sigma(k)}{k^{\alpha}} > 0.$$

Moreover, if either

$$\alpha\lambda \geq 1,$$

or

$$\alpha\lambda < 1 \quad and \quad \sum_{k=1}^{+\infty} k^{n-l-1+\frac{\alpha\lambda(1-\lambda)}{1-\alpha\lambda}-\varepsilon} (\sigma(k))^{\lambda(l-1)} |p(k)| = +\infty$$

is fulfilled. Then for any $k_0 \in N$, $U_{l,k_0} = \varnothing$.

4. Difference equations with property A

Theorem 4.1. *Let conditions* (1.2) (2.1) *be fulfilled, $l \in \{1, \ldots, n-1\}$, let $l + n$ be odd and for any $\delta \in [0, \lambda]$ and let $k \in N$ (3.1) be fulfilled. Moreover, if*

$$\sum_{k=1}^{n} k^{n-1} p(k) = +\infty, \qquad (4.1)$$

when n is odd, then Equation (1.1) *has Property* **A**.

Theorem 4.2. *Let conditions* (1.2) *and*

$$\liminf_{k \to +\infty} \frac{(\sigma(k))^{\lambda}}{k} > 0$$

be fulfilled. Then for the equation (1.1) *to have Property* **A**, *it is sufficient that*

$$\sum_{k=1}^{+\infty} k^{n-2+\lambda} p(k) = +\infty.$$

Theorem 4.3. *Let conditions* (1.2) *and*

$$\limsup_{k\to+\infty} \frac{(\sigma(k))^\lambda}{k} < +\infty$$

be fulfilled. Then for equation (1.1) *to have Property* **A**, *it is sufficient that conditions* (4.1) *and*

$$\sum_{k=1}^{+\infty} k^\lambda (\sigma(k))^{\lambda(n-2)} p(k) = +\infty$$

be fulfilled.

5. Difference equations with property B

Theorem 5.1. *Let conditions* (1.3), (2.1) *be fulfilled, $l \in \{1, \ldots, n-1\}$, $l+n$ is even and for any $\delta \in [0, \lambda]$ and let $k \in N$ (3.1) be fulfilled. Moreover, if*

$$\sum_{k=1}^{+\infty} k^{n-1} |p(k)| = +\infty, \tag{5.1}$$

when n is even, then equation (1.1) *has Property* **B**.

Theorem 5.2. *Let conditions* (1.3) *and*

$$\liminf_{k\to+\infty} \frac{(\sigma(k))^\lambda}{k} > 0$$

be fulfilled. Then for equation (1.1) *to have Property* **B**, *it is sufficient that condition*

$$\sum_{k=1}^{+\infty} k^{n-2+\lambda} |p(k)| = +\infty$$

be fulfilled.

Theorem 5.3. *Let conditions* (1.3) *and*

$$\limsup_{k\to+\infty} \frac{(\sigma(k))^\lambda}{k} < +\infty$$

be fulfilled. Then for equation (1.1) *to have Property* **B**, *it is sufficient that conditions* (5.1),

$$\sum_{k=1}^{+\infty} k^{\lambda+1} (\sigma(k))^{\lambda(n-3)} |p(k)| = +\infty$$

and

$$\sum_{k=1}^{+\infty} (\sigma(k))^{\lambda(n-1)} |p(k)| = +\infty$$

be fulfilled.

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ON OSCILLATORY PROPERTIES OF SOLUTIONS OF ALMOST LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Koplatadze R.

Abstract. The differential equation

$$u^{(n)}(t) + F(u)(t) = 0$$

is considered, where $F : C(R_+; R) \to L_{\text{loc}}(R_+; R)$ is a continuous mapping. In the case operator F has almost linear minorant, sufficient conditions are established for equation to have Properties **A** and **B**.

Keywords and phrases: Property **A**, Property **B**, oscillation.

AMS subject classification (2010): 34K11.

1. Introduction

This work deals with the investigation of oscillatory properties of solutions of a functional differential equation

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1.1}$$

where $F : C(R_+; R) \to L_{\text{loc}}(R_+; R)$ is a continuous mapping.

Let $\tau \in C(R_+; R_+)$, $\lim_{t \to +\infty} \tau(t) = +\infty$. Denote by $V(\tau)$ the set of continuous mappings F satisfying the condition $F(x)(t) = F(y)(t)$ holds for any $t \in R_+$ and $x, y \in C(R_+; R)$ provided that $x(s) = y(s)$ for $s \geq \tau(t)$. For any $t_0 \in R_+$, we denote by $H_{t_0,\tau}$ the set of all functions $u \in C(R_+; R)$ satisfying $u(t) \neq 0$ for $t \geq t_1$, where $t_1 = \min\{t_0, \tau_*(t_0)\}$, $\tau_*(t) = \inf\{\tau(s) : s \geq t\}$.

It will always be assumed that either the condition

$$F(u)(t)\, u(t) \geq 0 \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau} \tag{1.2}$$

or the condition

$$F(u)(t)\, u(t) \leq 0 \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau} \tag{1.3}$$

is fulfilled.

A function $u : [t_0, +\infty) \to R$ is said to be a proper solution of equation (1.1), if it is locally absolutely continuous along with its derivatives up to the order $n-1$ inclusive, $\sup\{|u(s)| : s \in [t, +\infty)\} > 0$ for $t \geq t_0$ and there exists a function $u_* \in C(R_+; R)$ such that $u_*(t) \equiv u(t)$ on $[t_0, +\infty)$ and the equality

$$u_*^{(n)}(t) + F(u_*)(t) = 0$$

holds for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise, the solution u is said to be nonoscillatory.

Definition 1.1. We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory, when n is even and either is oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.4}$$

when n is odd.

Definition 1.2. We say that equation (1.1) has Property **B** if any of its proper solution either is oscillatory, or satisfies either (1.4) or

$$\left|u^{(i)}(t)\right| \uparrow +\infty \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.5}$$

when n is even, and either is oscillatory or satisfies (1.5) when n is odd.

A. Kneser was the first who showed the condition

$$\liminf_{t \to +\infty} t^{n/2} p(t) > 0$$

is sufficient for the equation

$$u^{(n)}(t) + p(t)\, u(t) = 0 \tag{1.6}$$

to have Property **A** [1]. This theorem for Property **A** (for Property **B**) was essentially generalized by Kondrat'ev [2] (by Chanturia [3]). Their methods was based on a comparison theorem which enables one to obtain optimal results for establishing oscillatory properties of solutions of equation (1.6). Koplatadze [4,5] proved integral comparison theorems of two types for differential equations with deviated arguments. The theorems of the first type enables one not only to generalize the above mentioned results for equations with deviated arguments, but to improve Chanturia's result concerning Property **B** even in the case of equation (1.6).

The ordinary differential equation with deviating argument

$$u^{(n)}(t) + p(t)\left|u(\sigma(t))\right|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0 \tag{1.7}$$

is a particular case of equation (1.1) where $p \in L_{\text{loc}}(R_+; R)$, $\mu \in C(R_+; (0, +\infty))$, $\sigma \in C(R_+; R)$ and $\lim\limits_{t \to +\infty} \sigma(t) = +\infty$. In case $\lim\limits_{t \to +\infty} \mu(t) = 1$, we call differential equation (1.7) almost linear, while if $\liminf\limits_{t \to +\infty} \mu(t) \neq 1$, or $\limsup\limits_{t \to +\infty} \mu(t) \neq 1$, then we call equation (1.7) the essentially nonlinear generalised Emden-Fowler type differential equation.

In the present paper developing ideas of [6,7], the both cases of Properties **A** and **B** will be studied when operator F has almost linear minorant.

Investigation of almost linear differential equations, in our opinion for the first time was carried out [6–8].

2. Almost linear functional differential equation with property A

Theorem 2.1. *Let* $F \in V(\tau)$, *conditions* (1.2) *and*

$$\left|F(u)(t)\right| \geq \sum_{i=1}^{n} p_i(t) \int_{\alpha_i t}^{\beta_i t} s^{\gamma_i} \left|u(s)\right|^{1 + \frac{d_i}{\ln s}} ds \quad \text{for} \quad t \geq t_0 > 1, \; u \in H_{t_0, \tau} \tag{2.1}$$

be fulfilled, where

$$p_i \in L_{\text{loc}}(R_+; R_+), \quad 0 < \alpha_i < \beta_i, \quad \gamma_i \in (-1, +\infty), \quad d_i \in R \quad (i = 1, \ldots, m). \quad (2.2)$$

Then for the equation (1.1) to have Property **A** *it is sufficient that*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \frac{-\lambda(\lambda - 1) \cdots (\lambda - n + 1) e^{-\frac{\lambda d}{m}} \Big(\prod_{i=1}^m (1 + \gamma_i + \lambda) \Big)^{\frac{1}{m}}}{\Big(\prod_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda} \big) \Big)^{\frac{1}{m}}} : \lambda \in [0, n-1] \right\},$$

where

$$\gamma = \sum_{i=1}^m \gamma_i, \quad d = \sum_{i=1}^m d_i. \quad (2.3)$$

Theorem 2.2. *Let* $F \in V(\tau)$, *conditions (1.2), (2.1) and (2.2) be fulfilled, where* $\beta_i \le 1$, $d_i \in (-\infty, 0]$ $(i = 1, \ldots, m)$. *Then the condition*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \frac{-\lambda(\lambda - 1) \cdots (\lambda - n + 1) e^{-\frac{\lambda d}{m}} \Big(\prod_{i=1}^m (1 + \gamma_i + \lambda) \Big)^{\frac{1}{m}}}{\Big(\prod_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda} \big) \Big)^{\frac{1}{m}}} : \lambda \in [n-2, n-1] \right\}$$

is sufficient for equation (1.1) to have Property **A**, *where* d *and* γ *are given by (2.3).*

Theorem 2.3. *Let* $F \in V(\tau)$, *conditions (1.2), (2.1) and (2.2) be fulfilled, where* $\alpha_i \ge 1$, $d_i \in [0, +\infty)$ $(i = 1, \ldots, m)$. *Then for equation (1.1) to have Property* **A**, *it is sufficient that the condition*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \frac{-\lambda(\lambda - 1) \cdots (\lambda - n + 1) e^{-\frac{\lambda d}{m}} \Big(\prod_{i=1}^m (1 + \gamma_i + \lambda) \Big)^{\frac{1}{m}}}{\Big(\prod_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda} \big) \Big)^{\frac{1}{m}}} : \lambda \in [0, 1] \right\}$$

holds when n is even and the condition

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \Bigg\{ \frac{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\frac{\lambda d}{m}} \Big(\prod_{i=1}^m (1+\gamma_i+\lambda) \Big)^{\frac{1}{m}}}{\Big(\prod_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda} \big) \Big)^{\frac{1}{m}}}$$

$$: \lambda \in [1,2] \cup [n-2,n-1] \Bigg\}$$

holds when n is odd, where d and γ are given by (2.3).

Theorem 2.4. *Let $F \in V(\tau)$, conditions (1.2) and*

$$\big| F(u)(t) \big| \geq \sum_{i=1}^m p_i(t) \big| u(\alpha_i t) \big|^{1+\frac{d_i}{\ln t}} \quad for \quad t \geq t_0 > 1, \quad u \in H_{t_0,\tau} \qquad (2.4)$$

be fulfilled, where

$$p_i \in L_{\mathrm{loc}}(R_+; R_+), \quad \alpha_i > 0, \quad d_i \in R. \qquad (2.5)$$

Then for equation (1.1) to have Property **A***, it is sufficient that*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \Bigg\{ -\lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m \alpha_i e^{d_i} \Big)^{-\frac{\lambda}{m}} : \lambda \in [0,n-1] \Bigg\}.$$

Theorem 2.5. *Let $F \in V(\tau)$, conditions (1.2), (2.4) and (2.5) be fulfilled, where $\alpha_i \leq 1$ and $d_i \in (-\infty,0]$ $(i=1,\ldots,m)$. Then for equation (1.1) to have Property* **A***, it is sufficient that*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \Bigg\{ -\lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m \alpha_i e^{d_i} \Big)^{-\frac{\lambda}{m}} : \lambda \in [n-2,n-1] \Bigg\}.$$

Theorem 2.6. *Let $F \in V(\tau)$, conditions (1.2), (2.4) and (2.5) be fulfilled, where $\alpha_i > 1$ and $d_i \in [0,+\infty)$ $(i=1,\ldots,m)$. Then for equation (1.1) to have Property* **A***, it is sufficient that the condition*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \Bigg\{ -\lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m \alpha_i e^{d_i} \Big)^{-\frac{\lambda}{m}} : \lambda \in [0,1] \Bigg\}$$

holds when n is even and

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \Big\{ -\lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m \alpha_i e^{d_i} \Big)^{-\frac{\lambda}{m}} : \lambda \in [1,2] \cup [n-2, n-1] \Big\}$$

holds when n is odd.

3. Almost linear functional differential equation with property B

Theorem 3.1. *Let $F \in V(\tau)$, conditions (1.3), (2.1) and (2.2) be fulfilled. Then for the equation (1.1) to have Property* **B** *it is sufficient that*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \frac{\lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m (1+\gamma_i+\lambda) \Big)^{\frac{1}{m}} e^{-\frac{\lambda d}{m}}}{\Big(\prod_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda} \big) \Big)^{\frac{1}{m}}} : \lambda \in [0, n-2] \right\},$$

where γ and d are given by (2.3).

Theorem 3.2. *Let $F \in V(\tau)$, conditions (1.3), (2.1) and (2.2) be fulfilled, where $\beta_i \le 1$, $d_i \in (-\infty, 0]$ $(i = 1, \ldots, m)$. Then for equation (1.1) to have Property* **B**, *it is sufficient that the condition*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \frac{\lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m (1+\gamma_i+\lambda) \Big)^{\frac{1}{m}} e^{-\frac{\lambda d}{m}}}{\Big(\prod_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda} \big) \Big)^{\frac{1}{m}}} : \lambda \in [n-3, n-2] \right\}$$

holds when n is even and the condition

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \frac{\lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m (1+\gamma_i+\lambda) \Big)^{\frac{1}{m}} e^{-\frac{\lambda d}{m}}}{\Big(\prod_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda} \big) \Big)^{\frac{1}{m}}} \right.$$

$$\left. : \lambda \in [0,1] \cup [n-3, n-2] \right\}$$

holds when n is odd, where γ and d are given by (2.3).

Theorem 3.3. *Let $F \in V(\tau)$, conditions (1.3), (2.1) and (2.2) be fulfilled, where $\alpha_i \geq 1$, $d_i \in [0, +\infty)$ $(i = 1, \ldots, m)$. Then for equation (1.1) to have Property* **B***, it is sufficient that the condition*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\frac{\lambda d}{m}} \Big(\prod_{i=1}^m (1+\gamma_i+\lambda) \Big)^{\frac{1}{m}}}{\Big(\prod_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda} \big) \Big)^{\frac{1}{m}}} : \lambda \in [1,2] \right\}$$

holds when n is even and the condition

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\frac{\lambda d}{m}} \Big(\prod_{i=1}^m (1+\gamma_i+\lambda) \Big)^{\frac{1}{m}}}{\Big(\prod_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda} \big) \Big)^{\frac{1}{m}}} : \lambda \in [0,1] \right\}$$

holds when n is odd, where d and γ are given by (2.3).

Theorem 3.4. *Let $F \in V(\tau)$, conditions (1.3), (2.1) and (2.5) be fulfilled. Then for equation (1.1) to have Property* **B***, it is sufficient that*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m \alpha_i e^{d_i} \Big)^{-\frac{\lambda}{m}} : \lambda \in [0, n-2] \right\}.$$

Theorem 3.5. *Let $F \in V(\tau)$, conditions (1.3), (2.4) and (2.5) be fulfilled, where $\alpha_i \leq 1$ and $d_i \in (-\infty, 0]$ $(i = 1, \ldots, m)$. Then for equation (1.1) to have Property* **B***, it is sufficient that the condition*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \left\{ \lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m \alpha_i e^{d_i} \Big)^{-\frac{\lambda}{m}} : \lambda \in [n-3, n-2] \right\}$$

holds when n is even and the condition

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-2} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \Big\{ \lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m \alpha_i e^{d_i} \Big)^{-\frac{\lambda}{m}}$$

$$: \lambda \in [0,1] \cup [n-3, n-2] \Big\}$$

holds when n is odd.

Theorem 3.6. *Let* $F \in V(\tau)$*, conditions* (1.3)*,* (2.4) *and* (2.5) *be fulfilled, where* $\alpha_i > 1$ *and* $d_i \in [0, +\infty)$ $(i = 1, \dots, m)$*. Then for equation* (1.1) *to have Property* **B**, *it is sufficient that the condition*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-2} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \Big\{ \lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m \alpha_i e^{d_i} \Big)^{-\frac{\lambda}{m}} : \lambda \in [1,2] \Big\}$$

holds when n is even and the condition

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-2} \Big(\prod_{i=1}^m p_i(s) \Big)^{\frac{1}{m}} ds$$

$$> \frac{1}{m} \max \Big\{ \lambda(\lambda-1)\cdots(\lambda-n+1) \Big(\prod_{i=1}^m \alpha_i e^{d_i} \Big)^{-\frac{\lambda}{m}} : \lambda \in [0,1] \Big\}$$

holds when n is odd.

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THE ISOMETRIC SYSTEM OF COORDINATES AND THE COMPLEX FORM OF THE SYSTEM OF EQUATIONS FOR THE NON-SHALLOW AND NONLINEAR THEORY OF SHELLS

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Abstract. In this paper, the 3-D geometrically and physically nonlinear theories of non-shallow shells are considered. The isometrical system of coordinates is of special interest, since in this system we can obtain bases equations of the theory of shells in a complex form. This circumstance makes is possible to apply the methods developed by N. Muskhelishvili and his disciples by means of the theory of functions of a complex variable and integral equations

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1. Introduction

The refined theory of shells is constructed by reducing the three-dimensional problems of the theory of elasticity to the two-dimensional problems [1, 2]. I. Vekua constructed several versions of the refined linear theory of thin and shallow shells, containing, the regular processes by means of the method of reduction of 3-D problems of elasticity to 2-D ones [1].

By thin and shallow shells I.Vekua means 3-D shell type elastic bodies satisfying the following conditions [3]

$$a_\alpha^\beta - x^3 b_\alpha^\beta \cong \alpha_\alpha^\beta \quad -h \leq x^3 = x_3 \leq h, \quad \alpha, \beta = 1, 2, \qquad (*)$$

where a_α^β and b_α^β are mixed components of the metric and curvature tensors of the midsurface of the shell, x^3 is the thickness coordinate and h is the semi-thickness.

In the sequel, under non-shallow shells we wean elastic bodies free from the assumption of the type (*) or, more exactly, the bodies with the conditions

$$a_\alpha^\beta - x_3 b_\alpha^\beta \neq a_\alpha^\beta \Rightarrow |hb\beta_\alpha| \leq q < 1.$$

Such kind of shells are called shells with varying in thickness geometry, or non-shallow shells.

2. System of geometrically and physically nonlinear equations for non-shallow shells

We write the equation of equilibrium of an elastic shell-type body in a vector form which is convenient for reduction to the 2-D equations

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} \vec{\sigma}^i}{dx^i} + \vec{\Phi} = 0 \Rightarrow \hat{\nabla}_i \vec{\sigma}^i + \vec{\Phi} = 0, \qquad (1)$$

where g is the discriminant of the metric quadratic form of the 3-D domain Ω, $\hat{\nabla}_i$ are covariant derivatives with respect to the space coordinates x^i, $\vec{\Phi}$ is an external force, $\vec{\sigma}^i$ are the contravariant constituents of the stress vector $\vec{\sigma}_{\underset{(\vec{l})}{*}}$ acting in the area with the normal $\overset{*}{\vec{l}}$ and representable as the Cauchy formulas as follows

$$\vec{\sigma}_{\underset{(\vec{l})}{*}} = \vec{\sigma}^i \overset{*}{l}_i, \quad \overset{*}{l}_i = \overset{*}{\vec{l}}\vec{R}_i.$$

A material is said to be hyper-elastic if the stresses are obtained by means of the strain energy function

$$\sigma^{ij} = \frac{\partial \exists}{\partial e_{ij}},$$

where σ^{ij} are contravariant components of the stress tensor, $\exists$ is the strain energy function, and e_{ij} are covariant components of the strain tensor.

The theory of hyper-elasticity of the second order has the form [2, 3]

$$\begin{aligned}
\exists &= \frac{1}{2}E^{ijpq}e_{ij}e_{pq} + \frac{1}{3}E^{ijpqsk}e_{ij}e_{pq}e_{sk}, \\
e_{ij} &= \frac{1}{2}(\vec{R}_i\partial_j\vec{U} + \vec{R}_j\partial_i\vec{U} + \partial_i\vec{U}\partial_j\vec{U}) \\
\sigma^{ij} &= E^{ijpq}e_{pq} + E^{ijpqsk}e_{pq}e_{sk}, \quad \vec{\sigma}^i = \sigma^{ij}(\vec{R}_j + \partial_j\vec{U})
\end{aligned} \tag{2}$$

where E^{ijpq} and E^{ijpqsk} are coefficients of elasticity of the first and second order and $\vec{U}$ is the displacement vector.

Coefficients of elasticity of the first order for isotropic elastic bodies are expressed by the two Lamé coefficients

$$E^{ijpq} = \lambda g^{ij}g^{\mu q} + \mu(g^{ip}g^{jq} + g^{iq}g^{jp}), \quad (g^{ij} = \vec{R}^i\vec{R}^j) \tag{3}$$

and coefficients of elasticity of the second order are defined by the formula

$$E^{ijpqsk} = (E_1 + E_2)g^{ij}g^{pq}g^{sk} - E_2 g^{ij}g^{pk}g^{qs} + E_3 g^{ip}g^{jq}g^{sk} + E_4 g^{is}g^{pq}g^{jk}, \tag{4}$$

where E_1, E_2, E_3 and E_4 are modules of elasticity of the second order for isotropic elastic bodies.

Here $\vec{R}_i$ and $\vec{R}^i$ are covariant and contravariant base vectors of the space.

3. The coordinate system in a shell normally connected with a surface

Let Ω denote a shell and a domain of the space occupied by the shell. Inside the shell, we consider a smooth surface S with respect to which the shell Ω lies symmetrically. The surface S is called the midsurface of the shell Ω. To construct the theory of shells, we use more convenient coordinate system which is normally connected with the midsurface S. This means that the radius-vector $\vec{R}$ of any point of the domain Ω can be represented in the form

$$\vec{R}(x^1, x^2, x^3) = \vec{r}(x^1, x^2) + x^3\vec{n}(x^1, x^2),$$

where $\vec{R}$ and $\vec{n}$ are respectively the radius-vector and the unit vector of the normal of the surface $S(x^3 = 0)$ and (x^1, x^2) are the Gaussian parameters of the midsurfaces S.

The covariant and contravariant basis vectors $\vec{R}_i$ and $\vec{R}^i$ of the surfaces $\hat{S}(x^3 = \text{const})$, and the corresponding basis vectors $\vec{r}_i$ and $\vec{r}^i$ of the midsurface S $(x^3 = 0)$ are connected by the following relations:

$$\vec{R}_i = A_{i.}^{j} \vec{r}_j = A_{ij} \vec{r}^j, \quad \vec{R}^i = A_{.j}^{i} \vec{r}^j = A^{ij} \vec{r}_j, \quad (i, j = 1, 2, 3),$$

where

$$A_{i.}^{j} = \begin{cases} a_\alpha^\beta - x_3 b_\alpha^\beta, & i = \alpha, \ j = \beta, \\ \delta_i^3, & j = 3, \end{cases} \qquad \vec{r}_i, \vec{r}^i = \begin{cases} \vec{r}_\alpha, \vec{r}^\alpha, & i = \alpha, \\ \vec{n}, \vec{n}, & i = 3, \end{cases}$$

$$A_{.j}^{i} = \begin{cases} \dfrac{(1 - 2Hx_3)a_\beta^\alpha + x_3 b_\beta^\alpha}{1 - 2Hx_3 + Kx_3^2}, & i = \alpha, \ j = \beta, \\ \delta_i^3, & j = 3. \end{cases}$$

Here $(a_{\alpha\beta}, a^{\alpha\beta}, a_\alpha^\beta)$ and $(b_{\alpha,\beta}, b^{\alpha\beta}, b_\alpha^\beta)$ are the components (covariant, contravariant and mixed) of the metric and curvature tensors of the midsurface S. By H and K we denote a middle and Gaussian curvature of the surface S, where

$$2H = b_\alpha^\alpha = b_1^1 + b_2^2, \quad K = b_1^1 b_2^2 - b_2^1 b_1^2.$$

It should be noted that for the refined theory of non-shallow shells (Koiter, Naghdi, Lurie) these relations have the form

$$\vec{R}^\alpha \cong (a_\beta^\alpha + x_3 b_\beta^\alpha)\vec{r}^\beta, \quad \vec{R}_\alpha = (a_\alpha^\beta - x_3 b_\alpha^\beta)\vec{r}_\beta.$$

The main quadratic forms of the midsurface S $(x_3 = 0)$ have the forms

$$I = ds^2 = a_{\alpha\beta} dx^\alpha dx^\beta, \quad II = K_s ds^2 = b_{\alpha\beta} dx^\alpha dx^\beta,$$

where k_s is the normal courvative of the S and

$$a_{\alpha\beta} = \vec{r}_\alpha \vec{r}_\beta, \quad b_{\alpha\beta} = -\vec{n}_\alpha \vec{r}_\beta, \quad k_s = b_{\alpha\beta} s^\alpha s^\beta, \quad \vec{r}_\alpha = \partial_\alpha \vec{r}, \quad s^\alpha = \frac{dx^\alpha}{ds}.$$

It is necessary to rewrite the relation (1-4) in terms of the midsurface S of the shell Ω.

Relation (1) can be written as follows:

$$\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a} \vartheta \vec{\sigma}^\alpha}{\partial x^\alpha} + \frac{\partial \vartheta \vec{\sigma}^3}{\partial x^3} + \vartheta \vec{\Phi} = 0, \quad (\vartheta = 1 - 2Hx_3 + Kx_3).$$

from (2), (3), (4) we obtain

$$\vec{\sigma}^i = \sigma^{ij}(\vec{R}_j + \partial_j \vec{U}) = (E^{ijpq} + E^{ijpqsk} e_{sk}) e_{pq}(\vec{R}_j + \partial_j \vec{U})$$

$$\Rightarrow \vec{\sigma}^i = \tfrac{1}{2} A_{i_1}^{i} [M^{i_1 j_1 p_1 q_1} + \tfrac{1}{2} M^{i_1 j_1 p_1 q_1 s_1 k_1}$$

$$\times (A_{k_1}^{k} \vec{r}_{s_1} \partial_k \vec{U} + A_{s_1}^{s} A_{k_1}^{k} \partial_s \vec{U} \partial_k \vec{U})]$$

$$\times (A_{p_1}^{p} \vec{r}_{q_1} \partial_p \vec{U} + A_{q_1}^{q} \vec{r}_{p_1} \partial_q \vec{U} + A_{p_1}^{p} A_{q_1}^{q} \partial_p \vec{U} \partial_q \vec{U})(\vec{r}_{j_1} + A_{j_1}^{j} \partial_j \vec{U}),$$

where

$$M^{i_1 j_1 p_1 q_1} = \lambda a^{i_1 j_1} a^{p_1 q_1} + \mu(a^{i_1 p_1} a^{j_1 q_1} + a^{i_1 q_1} a^{j_1 p_1})$$

$$M^{i_1 j_1 p_1 q_1 s_1 k_1} = (E_1 + E_2) a^{i_1 j_1} a^{p_1 q_1} - E_2 a^{i_1 j_1} a^{p_1 k_1} q^{q_1 s_1}$$

$$+ E_3 a^{i_1 p_1} a^{j_1 q_1} a^{s_1 k_1} + E_4 a^{i_1 s_1} a^{p_1 q_1} a^{j_1 k_1},$$

$$(a^{ij} = \vec{r}^i \vec{r}^j).$$

4. Isometric system of coordinates

The isometrical system of coordinates in the surface S is of special interest, since in this system we can obtain bases equations of the theory of shells in a complex form, which in turn, allows one for a rather wide class of problems to construct complex representation of general solutions by means of analytic functions of one variable $z = x' + ix^2$. This circumstance makes is possible to apply the methods developed by N. Muskhelishvili and his disciples by means of the theory of functions of a complex variable and integral equations [1].

The main quadratic forms in this of coordinates are of the type

$$I = ds^2 = \Lambda(x^1, x^2)[(dx^1)^2 + (dx^2)^2] = \Lambda(z, \bar{z}) dz d\bar{z}, \quad (\Lambda > 0)$$

$$II = b_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{2}[\bar{Q} dz^2 + 2H dz d\bar{z} + Q d\bar{z}^2],$$

where

$$Q = \frac{1}{2}(b_1^1 - b_2^2 + 2i b_2^1), \quad 2H = b_1^1 + b_2^2.$$

Introducing the well-known differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)$$

and the notation

$$\vec{\tau}^i = \sqrt{\frac{g}{a}}\vec{\sigma}^i, \quad \vec{F} = \sqrt{\frac{g}{a}}\vec{\Phi},$$

$$\sqrt{\frac{g}{a}} = \vartheta = 1 - 2H x_3 + K x_3^2,$$

we obtain the following complex writing both for the system of equations of equilibrium and for "Hooke's Law"

$$\frac{1}{\Lambda}\frac{\partial}{\partial z}[\Lambda(\tau_1^1 - \tau_2^2 + i\tau_2^1 + i\tau_1^2)] + \frac{\partial}{\partial \bar{z}}[\Lambda(\tau_1^1 + \tau_2^2 + i\tau_2^1 - i\tau_1^2)]$$

$$-\Lambda(H\tau_3^+ + Q\bar{\tau}_3^+) + \frac{\partial\tau_+^3}{\partial x^3} + F_+ = 0,$$

$$\frac{1}{\Lambda}\left(\frac{\partial\Lambda\tau_3^+}{\partial z} + \frac{\partial\Lambda\bar{\tau}_3^+}{\partial \bar{z}}\right) + H(\tau_1^1 + \tau_2^2)$$

$$+\mathrm{Re}[\bar{Q}(\tau_1^1 - \tau_2^2 + i\tau_2^1 - i\tau_1^2)] + \frac{\partial\tau_3^3}{\partial x^3} + F_3 = 0,$$

where

$$\tau_1^1 - \tau_2^2 + i(\tau_2^1 + \tau_1^2) = \vec{\tau}^+ \vec{r}_+ = \sqrt{\frac{g}{a}} \Big\{ [\lambda\Theta + \mu(\vec{R}^+ \partial_z \vec{U} + \bar{\vec{R}}^+ \partial_{\bar{z}} \vec{U}$$

$$+ 2\partial^z \vec{U} \partial^{\bar{z}} \vec{U})](\vec{R}^+ + 2\partial^{\bar{z}} \vec{U})\vec{r}_+ + \mu[2(\vec{R}_+ + \partial^{\bar{z}} \vec{U})\partial_{\bar{z}} \vec{U}(\bar{\vec{R}}^+ + 2\partial^{\bar{z}} \vec{U})\vec{r}_+$$

$$+ (\vec{R}_+ \partial_3 \vec{U} + 2\vec{n} \partial^{\bar{z}} \vec{U} + 2\partial^{\bar{z}} \vec{U} \partial_3 \vec{U})\partial_3 \vec{U}] \Big\},$$

$$\tau_1^1 + \tau_2^2 + i(\tau_2^1 - \tau_1^2) = \bar{\vec{\tau}}^+ \bar{\vec{r}}_+ = \sqrt{\frac{g}{a}} \Big\{ \lambda\Theta + \mu(\vec{R}_+ \partial^z \vec{U} + \bar{\vec{R}}^+ \partial_{\bar{z}} \vec{U}$$

$$+ 2\partial^z \vec{U} \partial_{\bar{z}} \vec{U})(\bar{\vec{R}}^+ + 2\partial^z \vec{U})\vec{r}_+ + \mu[2(\bar{\vec{R}}^+ \partial_{\bar{z}} \vec{U} + \partial_z \vec{U} \partial^{\bar{z}} \vec{U})$$

$$(\vec{R}^+ + 2\partial^{\bar{z}} \vec{U})\vec{r}_+ + (\vec{R}^+ \partial_3 \vec{U} + 2(\vec{n} + \partial^{\bar{z}} U)\partial^z U]\partial_3 \vec{U}_+] \Big\}$$

$$\tau_3^+ = (\vec{\tau}^1 + i\tau^2)\vec{n} = \sqrt{\frac{g}{a}} \Big\{ 2[\lambda\Theta + \mu(\vec{R}^+ \partial_z \vec{U} + \bar{\vec{R}}^+ \partial_{\bar{z}} \vec{U} + 2\partial^z \vec{U} \partial_{\bar{z}} \vec{U})$$

$$(\vec{n} \partial^{\bar{z}} \vec{U})] + \mu[2(\vec{R}^+ \partial_{\bar{z}} \vec{U} + \partial^{\bar{z}} \vec{U} \partial_z \vec{U})(\vec{n} \partial^z \vec{U}) +$$

$$(\vec{R}^+ \partial_3 \vec{U} + 2\vec{n} \partial^{\bar{z}} \vec{U} + 2\partial^{\bar{z}} \vec{U} \partial_3 \vec{U})(1 + \partial_3 U_3)] \Big\},$$

$$\tau_+^3 = \vec{\tau}^3 \vec{r}_+ = \sqrt{\frac{g}{a}} \Big\{ [\lambda\Theta + \mu(2\vec{n} \partial^3 \vec{U} + \partial_3 \vec{U} \partial^3 \vec{U}]\partial_3 \vec{U}_+$$

$$+ \mu(\vec{n} \partial^{\bar{z}} \vec{U} + \frac{1}{2}\bar{\vec{R}}^+ + \partial_3 \vec{U} \partial_3 \vec{U} \partial_z \vec{U})(\vec{R}_+ + 2\partial_z \vec{U})\vec{r}_+$$

$$+ (\vec{n} \partial^{\bar{z}} \vec{U} + \frac{1}{2}\vec{R}^+ \partial_z \vec{U} \partial_3 \vec{U} \partial^{\bar{z}} \vec{U})(\bar{\vec{R}}_+ + 2\partial_{\bar{z}} \vec{U})\vec{z}_+ \Big\}$$

$$\tau_3^3 = \vec{\tau}^3 \vec{n} = \sqrt{\frac{g}{a}} \Big\{ [\lambda\Theta + \mu(2\vec{n} \partial^3 \vec{U} + \partial_3 \vec{U} \partial^3 \vec{U}](1 + \partial_3 \vec{U})$$

$$+ 2\mu[(\vec{n} \partial^z \vec{U}_+ + \frac{1}{2}\vec{R}^+ \partial_3 \vec{U} + \partial^z \vec{U} \partial_3 \vec{U})(\vec{n} \partial_{\bar{z}} \vec{U})$$

$$+ (\vec{n} \partial^{\bar{z}} \vec{U} + \frac{1}{2}\vec{R}^+ \partial_3 \vec{U} + \partial_3 \vec{U} \partial_z \vec{U})\vec{n} \partial_z \vec{U}] \Big\}.$$

Then

$$\Theta = \vec{R}^+ \partial_z \vec{U} + \bar{\vec{R}}^+ \partial_{\bar{z}} \vec{U} + 2\partial_z \vec{U} \partial^{\bar{z}} \vec{U} + \partial_3 U_3 + \frac{1}{2}(\partial_3 \vec{U})^2,$$

$$\partial^z \vec{U} = \frac{1}{2}[(\vec{R}^+ \bar{\vec{R}}^+)\partial_z \vec{U}_+ + (\bar{\vec{R}} \vec{R}^+)_{\partial_z} \vec{U}],$$

$$\vec{R}^+ = \vec{R}^1 + i\vec{R}^2, \quad \vec{R}_+ = \vec{R}_1 + i\vec{R}_2,$$

$$\vec{R}^+ = \vartheta^{-1}[(1 - Hx_3)\vec{r}^+ + x_3 Q \bar{\vec{r}}_+],$$

$$\vec{r}^+ = \vec{r}^1 + i\vec{r}^2, \quad \vec{r}_+ = \vec{r}_1 + i\vec{r}_2,$$

$$\vec{R}^+ \vec{R}^+ = \frac{4x_3}{\Lambda}\frac{\lambda - Hx_3}{\vartheta^2}Q,$$

$$\vec{R}^+ \bar{\vec{R}}^+ = \frac{2}{\Lambda}\frac{(1 - Hx_3)^2 + x_3^2 Q\bar{Q}}{\vartheta^2} = \frac{2}{\Lambda}\frac{\vartheta + 2x_3^2 Q\bar{Q}}{\vartheta^2},$$

$$\vec{R}^+ \vec{r}_+ = \frac{2}{\vartheta} Q x_3, \quad \vec{\bar{R}}^+ \vec{r}_+ = \frac{2}{\vartheta}(1 - H x_3),$$

$$\vec{r}^+ \vec{r}^+ = 0, \quad \vec{r}^+ \vec{\bar{r}}^+ = \frac{2}{\Lambda}, \quad \vec{r}_+ \vec{\bar{r}}_+ = 2,$$

$$F_+ = F_1 + F_2, \quad U_+ = U + iU_2, \quad U^+ = U^1 + iU^2.$$

We have the formulas

$$\vec{r}^+ \partial_z \vec{U} = \frac{1}{\lambda} \partial_z U_+ - H U_3,$$

$$\vec{r}^+ \partial_{\bar{z}} \vec{U} = \partial_{\bar{z}} U^+ - Q U_3,$$

$$\vec{n} \partial_{\bar{z}} \vec{U} = \partial_{\bar{z}} U_3 + \frac{1}{2}(\bar{Q} U_+ + H \bar{U}_+).$$

The displacement vector $\vec{U}$, representable in the form

$$\vec{U} = U^\alpha \vec{r}_\alpha + U^3 \vec{n} = U_\alpha \vec{r}^\alpha + U_3 \vec{n} = U_{(e)} \vec{l} + U_{(s)} \vec{s} + U_3 \vec{n} \quad (U_3 = U^3)$$

can be rewritten as follows:

$$\vec{U} = \frac{1}{2}(U^+ \vec{\bar{r}}_+ + \bar{U}^+ \vec{r}_+) + U_3 \vec{n}$$

or

$$\vec{U} = \mathrm{Im}\left[\left(U_{(l)} + iU_{(s)}\right) \frac{dz}{ds} \vec{r}_+\right] + U_3 \vec{n}$$

where

$$U^+ = \vec{U} \vec{r}, \quad U_+ = \vec{U} \vec{r}_+, \quad \vec{U}_{(\vec{l})} = \vec{U} \vec{l}, \quad U_s = \vec{U} \vec{s}.$$

Here $\vec{s}$ and $\vec{l}$ are the unit tangent vector and tangential normal of the midsurface $S(x_3 = 0)$. The expression for the unit tangent vector $\hat{\vec{s}}$ and the tangential normal $\hat{\vec{l}}$ of the surface $\hat{S}(x_3 = const)$ have the forms

$$\hat{\vec{s}} = \frac{d\vec{R}}{d\hat{s}} = [(1 - x_s k_s)\vec{s} + x_s \tau_s \vec{l}]\frac{ds}{d\hat{s}},$$

$$\hat{\vec{l}} = \hat{\vec{s}} \times \vec{n} = [(1 - x_3 k_s)\vec{l} - x_3 \tau_s \vec{s}]\frac{ds}{d\hat{s}},$$

and

$$d\hat{s} = \sqrt{1 - 2x_3 k_s + (k_s^2 + l_s^2)x_3^2}\, ds,$$

$$\left(\hat{\vec{l}} \times \hat{\vec{s}} = \vec{n}\right)$$

where $d\hat{s}$ and ds are linear elements of the surfaces $\hat{S}$ and S, τ_s is the geodesic version of the surface S.

The formula

$$\hat{\vec{l}} \vec{R}_\alpha = (1 - 2H x_3 + K x_3^2)(\vec{l} \vec{r}_\alpha)\frac{ds}{d\hat{s}}.$$

which is necessary in writing the reduced basic boundary-value problems in stresses, is also valid.

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SOLUTION OF SOME BOUNDARY VALUE PROBLEMS OF STATICS OF THE
THEORY OF ELASTIC MIXTURE IN AN INFINITE DOMAIN WITH AN
ELLIPTICAL HOLE

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Abstract. For homogeneous equation of statics of the linear theory of elastic mixture in
the case of an outside the elliptical domain we consider the two boundary value problems
which are analogous to III and IV exterior boundary value problem of the classic theory of
elasticity. Applying the representation of the stress vector by the so-called mutually adjoint
vector functions we obtain effective solutions (Poisson type formulas) of the problems.

Keywords and phrases: Elastic mixture, singular integral equation with a Hilbert ker-
nel, general representation of the displacement and stress vectors, analogues of the general
Kolosov-Muskhelishvilis representations, adjoint vector-function.

AMS subject classification (2010): 74E35, 74E20, 74C05.

1. Introduction

The basic two-dimensional boundary value problems statics of the linear theory of
elastic mixtures are studied in [1], [3]-[7] and also by many other authors.

In the paper we consider two boundary value problems for homogeneous equation
of statics of the linear theory of elastic mixtures in an infinite domain with an elliptical
hole, which for the cases of simple connected finite and infinite domains has been
studied by M. Basheleishvili in [5].

To solve the problems we use the method described in [2, §28] and [4]. Applying the
representation of the stress vector by the so-called mutually adjoint vector-functions
the problems are reduced to the singular integral equations with Hilbert kernels, and
owing to the above result, the solution of the problems can be reduced to the first order
linear differential equations.

The solutions of the problems are represented in the form of Poisson type formulas.

2. Some auxiliary formulas and operators

The homogeneous equation of statics of the theory of elastic mixtures in a complex
form looks as follows [4]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0,$$

where $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

$U = (u_1+iu_2, u_3+iu_4)^T$, $u^{'} = (u_1, u_2)^T$ and $u^{''} = (u_3, u_4)^T$ are partial displacements,

$$K = -\frac{1}{2}lm^{-1}, \quad l = \begin{bmatrix} l_4 & l_5 \\ l_5 & l_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\Delta_0}\begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix}, \quad \Delta_0 = m_1 m_3 - m_2^2 > 0,$$

$$m_k = l_k + \frac{1}{2}l_{3+k}, \quad k = 1, 2, 3, \quad l_1 = \frac{a_2}{d_2}, \quad l_2 = -\frac{c}{d_2}, \quad l_3 = \frac{a_1}{d_2},$$

$$a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5, \quad d_2 = a_1 a_2 - c^2, \quad l_1 + l_4 = \frac{a_2 + b_2}{d_1}, \quad (2.2)$$

$$l_2 + l_5 = -\frac{c+d}{d_1}, \quad l_3 + l_6 = \frac{a_1 + b_1}{d_1}, \quad d_1 = (a_1 + b_1)(a_2 + b_2) - (c+d)^2,$$

$$b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\frac{\rho_2}{\rho}, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2\frac{\rho_1}{\rho}, \quad \rho = \rho_1 + \rho_2,$$

$$\alpha_2 = \lambda_3 - \lambda_4, \quad d = \mu_3 + \lambda_3 - \lambda_5 - \alpha_2\frac{\rho_1}{\rho} \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2\frac{\rho_2}{\rho}.$$

Here μ_1, μ_2, μ_3 and λ_p, $\quad p = \overline{1,5}$ are elastic modules characterizing mechanical properties of a mixture, ρ_1 and ρ_2 are its particular densities. The elastic constants $\mu_1, \mu_2, \mu_3, \quad \lambda_p, \quad p = \overline{1,5}$ and particular densities ρ_1 and ρ_2 will be assumed to satisfy the conditions of inequality [1].

In [4] M. Basheleishvili obtained the following representations:

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) + \frac{1}{2} \; lz\overline{\varphi'(z)} + \overline{\psi(z)}, \quad (2.3)$$

$$TU = \begin{pmatrix} (Tu)_2 - i(Tu)_1 \\ (Tu)_4 - i(Tu)_3 \end{pmatrix} = \frac{\partial}{\partial S(x)}\left(-2\varphi(z) + 2\mu U(x)\right), \quad (2.4)$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions, $(TU)_p$ $(p = \overline{1,4})$ are components of the stress vector [1],

$$\mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \quad m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad \det\mu = \Delta_1 > 0,$$

$\frac{\partial}{\partial S(x)} = n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1}$, $\quad n_1$ and n_2 are the projections of the unit vector of the normal onto the axes x_1 and x_2.

Formulas (2.3) and (2.4) are analogous to the Kolosov-Muskhelishvilis formulas for the linear theory of elastic mixture.

To investigate the problems we use the vector [4]

$$V = \begin{pmatrix} V_1 + iV_2 \\ V_3 + iV_4 \end{pmatrix} = i\left[-m\varphi(z) + \frac{1}{2}lz\overline{\varphi'(z)} + \overline{\psi(z)}\right]. \quad (2.5)$$

As is known (see [4]) V is a vector adjoint to U.

From (2.3) , (2.4) and (2.5) we obtain

$$TU = \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} = \frac{\partial}{\partial S(x)}\left[(2\mu - m^{-1})U - im^{-1}V\right]. \quad (2.6)$$

3. Statement of the posed boundary value problems and the uniqueness theorems

Let an infinite isotropic plane be weakened by an elliptic hole with the semi-axis a and b $(a > b)$. This unbound domain will be denoted by D^-. The symmetry axis of the ellipse is taken at the coordinate axis, and the major axis coincides with the real axis ox_1. By L we denote the elliptic curve $(a\cos\theta, b\sin\theta) \in L$.

We consider the following boundary value problems: Find in the domain D^- a vector $U = (u_1 + iu_2, u_3 + iu_4)^T$ which belongs to the class $C^2(D^-) \bigcap C^{1,\alpha}(D^- \bigcup L)$ is a solution of equation (2.1) and satisfies only one of the following conditions on the boundary L

$$(nU)^- = f^{(1)}, \quad (STU)^- = f^{(2)}, \tag{3.1}$$

$$(SU)^- = F^{(1)}, \quad (nTU)^- = F^{(2)}, \tag{3.2}$$

where $f^{(j)}$ and $F^{(j)}$, $\quad j = 1, 2$ are the given scalar complex functions on the boundary L, note that

$$(f^{(1)}, F^{(1)}) \in C^{1,\alpha}(L), \quad (f^{(2)}, F^{(2)}) \in \sigma^{0,\alpha}(L), \quad \alpha > 0.$$

In the vicinity of infinity the vector $U = (u_1 + iu_2, u_3 + iu_4)^T$ satisfies the following conditions:

$$u_k = 0(1), \quad |x|^2 \frac{\partial u_k}{\partial x_j} = 0(1), \quad j = 1, 2, \quad k = \overline{1,4}, \quad |x|^2 = x_1^2 + x_2^2.$$

It will be assumed that the stress and rotation components vanish at infinity; moreover, we suppose that the principal vector of external forces applied to the contour of the hole is equal to zero.

Let us denote by $(III_*)^-$ and $(IV_*)^-$ the problems (2.1), (3.1) and (2.1), (3.2) respectively.
The following assertion is true [5].

Theorem 3.1. *The problems $(III_*)^-$ and $(IV_*)^-$ are uniquely solvable.*

4. Solution of the $(III_*)^-$ and $(IV_*)^-$ problems

For the solution of the problems we use the method developed in [2]. Let us note that the solution of the first BVP of statics of the linear theory of elastic mixture for an infinite plane with an elliptic hole reads as ([7] or [3])

$$U(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - \tau_1\overline{\tau_1})F(\theta)d\theta}{1 - \tau_1 e^{i\theta} - \overline{\tau_1}e^{-i\theta} + \tau_1\overline{\tau_1}} - \frac{KA_0}{2\pi} \int_0^{2\pi} \frac{\overline{F(\theta)}\overline{\tau_1}e^{-i\theta}d\theta}{(1 - \overline{\tau_1}e^{-i\theta})^2}, \tag{4.1}$$

where $U^- = F \in C^{1,\alpha}(L)$, $\quad \alpha > 0, (a\cos\theta, \quad b\sin\theta) \in L$; $\quad K = -\frac{1}{2}lm^{-1}$ (see (2.2)),

$$A_0 = (1 - \eta_1\overline{\eta_1})\left(\overline{\eta_1}^{-1} - \eta_2\right)(\overline{\eta_1} - \overline{\eta_2})^{-1}, \quad \tau_1 = \eta_1^{-1}, \quad |\tau_1| < 1,$$

$$\eta_1 = \frac{z + \sqrt{z^2 - a^2 + b^2}}{a + b}, \quad \eta_2 = \frac{z - \sqrt{z^2 - a^2 + b^2}}{a + b}, \quad z = x_1 + ix_2.$$

If $x = (x_1 x_2)$ belong to the boundary of the ellipse then $x_1 = a \cos \theta_0, x_2 = b \sin \theta_0$, and $\tau_1 = e^{-i\theta}$, $\overline{\tau_1} = e^{i\theta_0}$ and $A_0 = 0$.

Further, note that the adjoint vector of (4.1) has the form

$$V(x) = \begin{pmatrix} V_1 + iV_2 \\ V_3 + iV_4 \end{pmatrix} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{(\tau_1 e^{i\theta} - \overline{\tau_1} e^{-i\theta})F(\theta)d\theta}{1 - \tau_1 e^{i\theta} - \overline{\tau_1} e^{-i\theta} + \tau_1 \overline{\tau_1}} +$$

$$(4.2)$$

$$+ \frac{KA_0}{2\pi i} \int_0^{2\pi} \frac{\overline{F(\theta)}\,\overline{\tau_1} e^{-i\theta} d\theta}{(1 - \overline{\tau_1} e^{-i\theta})^2}.$$

1^0. A solution of the problem $(III)^-$ is sought in the form (see 4.1.)

$$U(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - \tau_1 \overline{\tau_1})(nq + S\chi)d\theta}{1 - \tau_1 e^{i\theta} - \overline{\tau_1} e^{-i\theta} + \tau_1 \overline{\tau_1}} - \frac{KA_0}{2\pi} \int_0^{2\pi} \frac{\overline{\tau_1} e^{-i\theta}(n\overline{q} + S\overline{\chi})d\theta}{(1 - \overline{\tau_1} e^{-i\theta})^2}, \qquad (4.3)$$

where $(nU)^- = q = f^{(1)}$ is given by (3.1) and $(SU)^- = \chi$ is the unknown function

$$n = (n_1, n_2)^T = \frac{(b \cos \theta, a \sin \theta)^T}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}},$$

$$(4.4)$$

$$S = (-n_2, n_1)^T = \frac{(-a \sin \theta, b \cos \theta)^T}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}.$$

We remark also that, on $(a \cos \theta_0, b \sin \theta_0) \in L$

$$(U(\theta_0))^- = n(\theta_0)q(\theta_0) + S(\theta_0 \chi(\theta_0)), \qquad (4.5)$$

$$(V(\theta_0))^- = \int_0^{2\pi} ctg \frac{\theta - \theta_0}{2} \left[n(\theta)q(\theta) + S(\theta)\chi(\theta) \right] d\theta. \qquad (4.6)$$

Using now (2.6) and taking into account (4.5) and (4.6) for the boundary value of the stress vector we obtain

$$\sqrt{a^2 \sin^2 \theta_0 + b^2 cos^2 \theta_0}(TU)^- = (2\mu - m^{-1}) \left(\frac{dU}{d\theta_0} \right)^-$$

$$+ \frac{m^{-1}}{2\pi i} \int_0^{2\pi} ctg \frac{\theta - \theta_0}{2} \left(\frac{dU}{d\theta} \right)^- d\theta. \qquad (4.7)$$

If we take into account (4.4) and condition $(STU)^- = f^{(2)}$ (see(3.1)) then (4.7) can be rewritten in the form of one equation

$$\left[(2\mu - m^{-1}) \left(\frac{dU}{d\theta_0} \right)^- \right] \begin{pmatrix} -a \sin \theta_0 \\ b \cos \theta_0 \end{pmatrix} + \left[\frac{m^{-1}}{2\pi i} \int_0^{2\pi} ctg \frac{\theta - \theta_0}{2} \left(\frac{dU}{d\theta} \right)^- d\theta \right]$$

$$\times \begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix} = (a^2\sin^2\theta_0 + b^2\cos^2\theta_0)f^{(2)}(\theta_0). \tag{4.8}$$

Represent U^- in the form (see (3.1) and (4.4))

$$(U(\theta_0))^- = \begin{pmatrix} b\cos\theta_0 \\ a\sin\theta_0 \end{pmatrix} f(\theta_0) + \begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix} h(\theta_0), \tag{4.9}$$

where

$$f(\theta_0) = \frac{f^{(1)}(\theta_0)}{\sqrt{a^2\sin^2\theta_0 + b^2\cos^2\theta_0}}, \tag{4.10$_1$}$$

$$h(\theta_0) = \frac{(S(\theta_0)U(\theta_0))^-}{\sqrt{a^2\sin^2\theta_0 + b^2\cos^2\theta_0}} = \frac{\chi(\theta_0)}{\sqrt{a^2\sin^2\theta_0 + b^2\cos^2\theta_0}}. \tag{4.10$_2$}$$

Substituting (4.9) in (4.8) after obvious transformations we get

$$\left[(2\mu - m^{-1})H'(\theta_0)\right]\begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix} + \left[\frac{m^{-1}}{2\pi i}\int_0^{2\pi} ctg\frac{\theta-\theta_0}{2}H'(\theta)d\theta\right]\begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix}$$

$$= \Phi(\theta_0), \tag{4.11}$$

where

$$H(\theta) = \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} h(\theta), \tag{4.12}$$

$$\Phi(\theta_0) = (a^2\sin^2\theta_0 + b^2\cos^2\theta_0)f^{(2)}(\theta_0) - (2\mu - m^{-1})\left[\begin{pmatrix} b\cos\theta_0 \\ a\sin\theta_0 \end{pmatrix} f(\theta_0)\right]' \begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix}$$

$$- \frac{m^{-1}}{2\pi i}\int_0^{2\pi} ctg\frac{\theta-\theta_0}{2}\left[\begin{pmatrix} b\cos\theta \\ a\sin\theta \end{pmatrix} f(\theta)\right]' d\theta \begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix}. \tag{4.13}$$

Bearing in mind the formulas

$$ctg\frac{\theta-\theta_0}{2}\begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix} = \begin{pmatrix} a\cos\theta + a\cos\theta_0 \\ b\sin\theta + b\sin\theta_0 \end{pmatrix} + ctg\frac{\theta-\theta_0}{2}\begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix},$$

after some calculations we can rewrite (4.11) in the form

$$\left[(2m\mu - E)H'(\theta)\right]m^{-1}\begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix}$$

$$+ \frac{1}{2\pi i}\int_0^{2\pi} ctg\frac{\theta-\theta_0}{2}H'(\theta)m^{-1}\begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} d\theta - iM = \Phi(\theta_0). \tag{4.14}$$

where

$$M = \frac{1}{2\pi} \int_0^{2\pi} H(\theta) m^{-1} \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} d\theta. \tag{4.15}$$

Applying the formula of composition of integrals with Hilbert kernels (*see*[2], §28)

$$\frac{1}{4\pi^2} \int_0^{2\pi} ctg\frac{\theta_0 - \theta^*}{2} d\theta_0 \int_0^{2\pi} ctg\frac{\theta - \theta_0}{2} P(\theta) d\theta = -P(\theta^*) + \frac{1}{2\pi} \int_0^{2\pi} P(\theta) d\theta,$$

from (4.14) we find

$$H'(\theta_0) m^{-1} \begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix} + \frac{1}{2\pi i} \int_0^{2\pi} ctg\frac{\theta - \theta_0}{2} \left[(2m\mu - E)H'(\theta) \right] m^{-1} \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} d\theta$$

$$-N = \frac{1}{2\pi i} \int_0^{2\pi} ctg\frac{\theta - \theta_0}{2} \phi(\theta) d\theta, \tag{4.16}$$

where

$$N = \frac{1}{2\pi} \int_0^{2\pi} H'(\theta) m^{-1} \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} d\theta. \tag{4.17}$$

The equalities (4.14) and (4.16) result in

$$\left[(2m\mu - 2E)H'(\theta_0) \right] m^{-1} \begin{pmatrix} -a\sin\theta_0 \\ b\cos\theta_0 \end{pmatrix}$$

$$-\frac{1}{2\pi i} \int_0^{2\pi} ctg\frac{\theta_0 - \theta^*}{2} \left[(2m\mu - 2E)H'(\theta) \right] m^{-1} \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} d\theta + N - iM$$

$$= \phi(\theta_0) - \frac{1}{2\pi i}) \int_0^{2\pi} ctg\frac{\theta - \theta_0}{2} \phi(\theta) d\theta \tag{4.18}$$

Thus, for determining $\left[(2m\mu - 2E)H'(\theta) \right] m^{-1} \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix}$ we have obtained a singular integral equation (4.18) with the Hilbert kernel.

Taking into account the fact that, when $f^{(1)} = f^{(2)} = 0$, then $U(x) = 0, x \in D^-$, (see theorem 3.1), also $\phi = 0, h = 0, H = 0$ and $M = N = 0$, (see $(4.10)_1$, $(4.10)_2$, (4.15) and (4.17)) we can conclude that solution of the equation (4.18) is

$$\left[(2m\mu - 2E)H'(\theta) \right] m^{-1} \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} = \phi(\theta) - N + iM.$$

The last formula yields (see (4.12))

$$h'(\theta) + \frac{1}{2}\frac{r'(\theta)}{r(\theta)} h(\theta = \frac{\phi(\theta)}{r(\theta)} - \frac{N - iM}{r(\theta)}, \tag{4.19}$$

where

$$r(\theta) = 2\left[a^2\left(\mu_1 - \frac{m_3}{\Delta_0}\right)\sin^2\theta - ab\left(\mu_3 + \frac{m_2}{\Delta_0}\right)sin2\theta + b^2\left(\mu_2 - \frac{m_1}{\Delta_0}\right)\cos^2\theta\right] \neq 0,$$

$$0 \leq \theta \leq 2\pi. \tag{4.20}$$

Here (see [6])

$$\left(\mu_1 - \frac{m_3}{\Delta_0}\right)\left(\mu_2 - \frac{m_1}{\Delta_0}\right) - \left(\mu_3 + \frac{m_2}{\Delta_0}\right)^2 > 0, \quad \Delta_0 = m_1 m_3 - m_2^2 > 0. \tag{4.21}$$

From (4.19) by integration we obtain

$$h(\theta) = \frac{C}{\sqrt{r(\theta)}} + \frac{1}{\sqrt{r(\theta)}}\int_0^\theta \frac{\phi(\theta_0) - N + iM}{\sqrt{r(\theta_0)}}d\theta_0, \tag{4.22}$$

where C is an arbitrary constant

As it is known conditions $f^{(1)} = f^{(2)} = 0$ imply that $U(x) = 0, x \in D^-$ and $\phi = H = h = M = N = 0$. Therefore from (4.22) we obtain $C = 0$ and finally

$$h(\theta) = \frac{1}{\sqrt{r(\theta)}}\int_0^\theta \frac{\phi(\theta_0) - N + iM}{\sqrt{r(\theta_0)}}d\theta. \tag{4.23}$$

Now let us find $N - iM$. Since $h(\theta)$ is periodic with the period 2π, i.e. $h(\theta + 2\pi) = h(\theta)$ (see (4.9) $(4.10)_1$ and $(4.10)_2$ and $r(2\pi) = r(0) \neq 0$ (see (4.20) and (4.21)) therefore from (4.23) we obtain

$$N - iM = \frac{\int_0^{2\pi}\phi(\theta)(r(\theta))^{-\frac{1}{2}}\ d\theta}{\int_0^{2\pi}(r(\theta))^{-\frac{1}{2}}\ d\theta}.$$

Having found $h(\theta)$ by formula $(4.10)_2$ we obtain value of $S(\theta)\chi(\theta)$ and after by (4.3) we obtain the solution of the problem $(III_*)^-$ represented in the form of Poisson type formula.

Thus the $(III_*)^-$ boundary value problem is solved. The BVP $(IV_*)^-$ is solved quite analogously.

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ერთი სასაზღვრო ამოცანა ფირფიტებისათვის

ბ. გულუა

ნაშრომში განხილულია იზოტროპული ერთგვაროვანი დრეკადი გარსის წონასწორობის განტოლებათა სისტემა. ვეკუას მეთოდით მიღებულია ორგანზომილებიანი განტოლებათა სრული სისტემა, როცა სხეულის ზედა და ქვედა პირეულებზე მოცემულია გადააგილების ვექტორი. ფირფიტისათვის აგებულია ზოგადი ამონახსნი N=1 მიახლოებაში. ამოხსნილია ამოცანა წრისათვის, როცა საზღვარზე მოცემულია ძაბვის ვექტორი.

ზოგიერთი სასაზღვრო ამოცანის მიახლოებითი ამონახსნის აგების ერთი მეთოდის შესახებ

რ. ჯანჯღავა

ნაშრომში განხილულია მათემატიკური ფიზიკის ზოგიერთი კლასიკური და არალოკალური სასაზღვრო ამოცანის მიახლოებითი ამონახსნის აგების მარტივი ალგორითმი. განსახილველი ალგორითმის გამოყენებით, დრეკადობის ბრტყელი თეორიის შემთხვევაში, ასევე ლაპლასისა და ჰელმჰოლცის ორგანზომილებიანი განტოლებებისათვის, აგებულია ზოგიერთი სასაზღვრო ამოცანის მიახლოებითი ამონახსნი. აგებული მიახლოებითი ამონახსნები შედარებულია შესაბამის ზუსტ ამონახსნებთან.

ფირფიტის ღუნვის ერთი ამოცანის შესახებ მრუდწირული ოთხკუთხა არისათვის სწორხაზოვანი ჭრილით

გ. კაპანაძე, ბ. გულუა

განიხილება ფირფიტის ღუნვის ამოცანა მრუდწირული ოთხკუთხა არისათვის სწორხაზოვანი ჭრილით. ფირფიტის გარე საზღვარი შედგება აბსცისთა ღერძის პარალელური სწორხაზოვანი მონაკვეთებითა და ერთი და იგივე წრეწირის რკალებით, ხოლო შიგა საზღვარი წარმოადგენს აბსცისთა ღერძის პარალელურ ჭრილს. მეტი თვალსაჩინოებისათვის ვიხილავთ სიმეტრიულ შემთხვევას. ვთვლით, რომ გარე საზღვრის სწორხაზოვან მონაკვეთებზე მოქმედებენ თანაბრადგანაწილებული ნორმალური მღუნავი მომენტები, წრეწირის რკალები თავისუფალნი არიან გარეგანი დატვირთვებისაგან, ხოლო შიგა საზღვარი (ჭრილი) – სახსრულადაა დაყრდნობილი.

ამოცანის ამოსახსნელად გამოყენებულია კონფორმულ ასახვათა და ანალი- ზურ ფუნქციათა სასაზღვრო ამოცანების მეთოდები და საძიებელი კომპლექსური პოტენციალები, რომელთა საშუალებითაც გამოისახება ფირფიტის შუა ზედაპირის ჩაღუნვა, აგებულია ეფექტურად (ანალიზური ფორმით). გამოკვლეულია აღნიშნული პოტენციალების ყოფაქცევა კუთხის წვეროებისა და ჭრილის ბოლოების მახლობლობაში.

მაღალი რიგის სხვაობიანი განტოლებები A და B თვისებით

ნ. ხაჩიძე

მაღალი რიგის სხვაობიანი განტოლებებისათვის მიღებულია მონოტონური ამონახსნების არსებობის აუცილებელი პირობები. გარდა ამისა, დადგენილია ამონახსნების ოსცილაციურობის ახალი ტიპის კრიტერიუმები.

თითქმის წრფივი ფუნქციონალურ-დიფერენციალურ განტოლებათა ამონახსნების ოსცილაციური თვისებების შესახებ

რ. კოპლატაძე

მაღალი რიგის ფუნქციონალურ-დიფერენციალურ განტოლებებისათვის, როცა განტოლების მარჯვენა მხარეს გააჩნია თითქმის წრფივი მინორანტი, მიღებულია A ან B თვისების არსებობის საკმარისი პირობები.

იზომეტრიული კოორდინატთა სისტემა და არადამრეცი არაწრფივი გარსების განტოლებათა სისტემის კომპლექსური სახე

თ. მეუნარგია

იზომეტრიული კოორდინატთა სისტემის გამოყენებით სხეულის წონასწორობის განტოლებათა სისტემა და ჰუკის კანონი ჩაწერილია კომპლექსური სახით არადამრეცი და არაწრფივი გარსებისათვის.

დრეკად ნარევთა თეორიის სტატიკის ზოგიერთი სასაზღვრო ამოცანის ამოხსნა უსასრულო არეში ელიფსური ხვრელით

კ. სვანაძე

ნაშრომში დრეკად ნარევთა წრფივი თეორიის სტატიკის ერთგვაროვანი განტოლებისათვის, ელიფსის გარე არეს შემთხვევაში ეფექტურადაა ამოხსნილი იმ ტიპის ორი ამოცანა, რომლებიც კლასიკურ დრეკადობის თეორიაში ცნობილია III და IV ამოცანების სახით.

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ამოცანების ამონახსნები მოიცემა პუასონის ტიპის ინტეგრალური ფორმულების სახით.

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♦ ინფორმაცია ავტორის შესახებ: გვარი და სახელი, დაწესებულება და მისამართი, ელ-ფოსტა;

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♦ სტატია იქნება რეცენზირებული; სტატია, რომელიც არ გამოქვეყნდება ავტორს არ დაუბრუნდება;

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Hartman P. Ordinary differential equations, *John Wiley & Sons, New York-London-Sydney,* 1964.

Hartman P. A differential equation with non-unique solutions, *Amer. Math. Monthly*, **70** (1963), 255-259.

Vekua, I. N. A class of statically definable problems of the theory of shells. (Russian) *Sakharth. SSR Mecn. Akad. Moambe* **83,** 2 (1976), 273-276.

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