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ABSTRACTS

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Non-Instantaneous Impulsive Differential Equations with Finite State Dependent Delay and Ulam-Type Stability

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Abstract

We consider an initial value problem (IVP) for a nonlinear system of non-instantaneous impulsive differential equations with state dependent delay (NIDDE) and Ulam-type stability is studied.

1 Statement of the problem

Let the points $t_i, s_i \in [0,T]$: $s_0 = 0$, $t_{k+1} = T$, $0 < t_i < s_i < t_{i+1}$, $i = 1, 2, \dots, k$ be given. Consider the space $PC_0 = C([-r,0], E)$ endowed with the norm $\|y\|_{PC_0} = \sup\limits_{t\in[-r,0]} \{\|y(t)\|_E : y \in PC_0\}$; here E is a Banach space.

The intervals (s_i, t_{i+1}), $i = 0, 1, 2, \dots, k$ will be the intervals on which the fractional differential equation will be given and the intervals (t_i, s_i), $i = 1, 2, \dots, k$ will be called impulsive intervals and on these intervals impulsive conditions are given.

Consider the IVP for the NIDDE

$$\begin{aligned}
&x'(t) = f(t, x_{\rho(t,x_t)}) \ \text{ for } \ t \in (s_i, t_{i+1}], \ \ i = 0, 1, 2, \dots, k,\\
&x(t) = g_i(t, x(t_i)), \ \ t \in (t_i, s_i], \ \ i = 1, 2, \dots, k,\\
&x(t) = \phi(t) \ \text{ for } \ t \in [-r, 0],
\end{aligned} \tag{1.1}$$

where the functions $f : [0,T] \times PC_0 \to E$; $\rho : [0,T] \times PC_0 \to [0,T]$, $\phi : [-r,0] \to E$; $g_i : [t_i, s_i] \times E \to E$, $i = 1, 2, \dots, k$. Here for any $t \in [0,T]$ the notation $x_t(s) = x(t+s)$, $s \in [-r,0]$ is used, i.e. x_t represents the history of the state $x(t)$ from time $t-r$ up to the present time t. Note that for any $t \in [0,T]$ we let $y_{\rho(t,x_t)}(s) = x(\rho(t, x(t+s)) + s)$, $s \in [-r,0]$, i.e. the function ρ determines the state-dependent delay.

Remark 1.1. Note in the special case $\rho(t,x) \equiv t$ problem (1.1) reduces to an IVP for a delay non-instantaneous impulsive differential equation.

Let $\mathcal{PC}$ be the Banach space of all functions $y : [-r,T] \to E$ which are continuous on $[0,T]$ except for the points $t_i \in (0,T)$ at which $y(t_i+) = \lim\limits_{t \downarrow t_i} y(t)$ and $y(t_i-) = y(t_i) = \lim\limits_{t \uparrow t_i} y(t)$ exist and it is endowed with the norm $\|y\|_{\mathcal{PC}} = \sup\limits_{t\in[-r,T]} \{\|y(t)\|_E : y \in \mathcal{PC}\}$.

We consider the assumptions:

A1. The function $f \in C\big(\bigcup\limits_{i=0}^{k} [s_i, t_{i+1}] \times E, E\big)$.

A2. The function $\phi \in PC_0$.

A3. The function $\rho \in C\big(\bigcup\limits_{i=0}^{k} [s_i, t_{i+1}] \times E, [0,T]\big)$ is such that for any $t \in \bigcup\limits_{i=0}^{k} [s_i, t_{i+1}]$ and any function $u \in PC_0$ the inequality $\rho(t,u) \le t$ holds.

A4. The functions $g_i \in C([t_i, s_i] \times E, E)$, $i = 1, 2, \dots, k$.

Definition 1.1. The function $x \in \mathcal{PC}$ is a solution of the IVP (1.1) iff it satisfies the following integral-algebraic equation

$$x(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \phi(0) + \displaystyle\int_0^t f(s, x_{\rho(s,x_s)})\,ds, & t \in (0, t_1], \\ g_i(t, x(t_i)), & t \in (t_i, s_i], \quad i = 1,2,\dots,k, \\ g_i(s_i, x(t_i)) + \displaystyle\int_{s_i}^t f(s, x_{\rho(s,x_s)})\,ds, & t \in (s_i, t_{i+1}], \quad i = 1,2,\dots,k. \end{cases} \tag{1.2}$$

2 Ulam types stability

Let $\varepsilon > 0$, $\Psi \ge 0$ and $\Phi \in C\big(\bigcup\limits_{i=1}^{k} [s_i, t_{i+1}], [0,\infty)\big)$ be nondecreasing. We consider the following inequalities:

$$\begin{aligned} \big\|y'(t) - f(t, y_{\rho(t,y_t)})\big\|_E &\le \varepsilon \ \text{ for } \ t \in (s_i, t_{i+1}], \quad i = 0,1,2,\dots,k, \\ \big\|y(t) - g_i(t, y(t_i))\big\|_E &\le \varepsilon, \ \ t \in (t_i, s_i], \quad i = 1,2,\dots,k, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \big\|y'(t) - f(t, y_{\rho(t,y_t)})\big\|_E &\le \Phi(t) \ \text{ for } \ t \in (s_i, t_{i+1}], \quad i = 0,1,2,\dots,k, \\ \big\|y(t) - g_i(t, y(t_i))\big\|_E &\le \Psi, \ \ t \in (t_i, s_i], \quad i = 1,2,\dots,k, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \big\|y'(t) - f(t, y_{\rho(t,y_t)})\big\|_E &\le \varepsilon\Phi(t) \ \text{ for } \ t \in (s_i, t_{i+1}], \quad i = 0,1,2,\dots,k, \\ \big\|y(t) - g_i(t, y(t_i))\big\|_E &\le \varepsilon\Psi, \ \ t \in (t_i, s_i], \quad i = 1,2,\dots,k. \end{aligned} \tag{2.3}$$

The inequalities (2.1)–(2.3) have connections with the definitions of Ulam–Hyers stability, Ulam–Hyers–Rassias stability with respect to Φ, Ψ and generalized Ulam–Hyers–Rassias stability, respectively (for detailed definitions see, for example [2]).

Lemma 2.1. *Let assumptions* A1, A3, A4 *be satisfied.*

- *If $y \in \mathcal{PC}$ is a solution of inequalities* (2.1), *then it satisfies the following integral-algebraic inequalities*

$$\begin{cases} \Big\| y(t) - \phi(0) - \int\limits_0^t f(s, y_{\rho(s,y_s)})\,ds \Big\|_E \le \varepsilon t, & t \in (0, t_1], \\ \big\| y(t) - g_i(t, y(t_i)) \big\|_E \le \varepsilon, & t \in (t_i, s_i], \ \ i = 1, 2, \dots, k, \\ \Big\| y(t) - g_i(s_i, y(t_i)) - \int\limits_{s_i}^t f(s, y_{\rho(s,y_s)})\,ds \Big\|_E \le \varepsilon + \varepsilon(t - s_i), & t \in (s_i, t_{i+1}], \ \ k = 1, 2, \dots, k. \end{cases}$$

- *If $y \in \mathcal{PC}$ is a solution of inequalities* (2.2), *then it satisfies the following integral-algebraic inequalities*

$$\begin{cases} \Big\| y(t) - \phi(0) - \int\limits_0^t f(s, y_{\rho(s,y_s)})\,ds \Big\|_E \le \int\limits_0^t \Phi(s)\,ds, & t \in (0, t_1], \\ \big\| y(t) - g_i(t, y(t_i)) \big\|_E \le \Psi, & t \in (t_i, s_i], \ \ i = 1, 2, \dots, k, \\ \Big\| y(t) - g_i(s_i, y(t_i)) - \int\limits_{s_i}^t |f(s, y_{\rho(s,y_s)})|\,ds \Big\|_E \le \Psi + \int\limits_{s_i}^t \Phi(s)\,ds, & t \in (s_i, t_{i+1}], \ \ i = 1, 2, \dots, k. \end{cases}$$

Remark 2.1. We have a similar result for the inequality (2.3).

Next we discuss the existence of the solution of (1.1), given by Definition 1.1, using the Banach contraction principle.

Theorem 2.1 (*Existence result*)**.** *Let the following conditions be satisfied:*

1. *Assumption* A1 *is satisfied and there exists a constant $L_f > 0$ such that for any $t \in \bigcup\limits_{i=1}^k [s_i, t_{i+1}]$ and any functions $u, v \in \mathcal{PC}$ the inequality*

$$\big\| f(t, u_{\rho(t,u_t)}) - f(t, v_{\rho(t,v_t)}) \big\|_E \le L_f \| u_{\rho(t,u_t)} - v_{\rho(t,v_t)} \|_{PC_0}$$

holds.

2. *Assumption* A4 *is satisfied and there exist constants $L_{g_i} > 0$, $i = 1, 2, \dots, k$, such that*

$$\| g_i(t, x) - g_i(t, y) \|_E \le L_{g_i} \| x - y \|_E, \ \ t \in [t_i, s_i], \ \ x, y \in E, \ \ i = 1, 2, \dots, k.$$

3. *Assumptions* A2, A3 *are satisfied.*

4. *The inequality $\gamma = \max\limits_{i=1,2,\dots,k} L_{g_i} + \eta L_f < 1$ holds, where $\eta = \max\{t_{i+1} - s_i, \ i = 0, 1, \dots, k\}$.*

Then the initial value problem (1.1) *has a unique solution $x \in \mathcal{PC}$ as defined in Definition* 1.1.

Theorem 2.2 (*Stability results*)**.** *Let the conditions of Theorem* 2.1 *be satisfied.*

(i) *Assume for any* $\varepsilon > 0$ *inequality* (2.1) *has at least one solution* $y_\varepsilon \in \mathcal{PC}$*. Then problem* (1.1) *is Ulam–Hyers stable, i.e.*

$$\|x(t) - y_\varepsilon(t)\|_E < c_{f,g_i}\varepsilon, \ \ t \in [0,T]$$

with

$$c_{f,g_i} = 1 + (1+\eta)\sum_{j=1}^{k-1}\Big(\prod_{m=0}^{j-1} L_{g_{k-m}}\Big)e^{jL_f\eta} + \Big(\prod_{j=1}^{k} L_{g_j}\Big)\eta e^{(k+1)L_f\eta},$$

where x *is the solution of* (1.1).

(ii) *Suppose there exist constants* $\Psi \geq 0$, $\Lambda_\Phi > 0$ *and a function* $\Phi \in C\big(\bigcup\limits_{i=1}^{k}[s_i,t_{i+1}],[0,\infty)\big)$ *such that for any* $t \in [s_i,t_{i+1}]$, $i = 0,1,2,\dots,k$ *inequality* $\int\limits_{s_i}^{t}\Phi(s)\,ds \leq \Lambda_\Phi\Phi(t)$ *holds and for any* $\varepsilon > 0$ *inequality* (2.3) *has at least one solution* $y_\varepsilon(t) \in \mathcal{PC}$*. Then problem* (1.1) *is generalized Ulam–Hyers–Rassias stable with respect to* Φ, Ψ.

(iii) *Assume there exist constants* $\Psi \geq 0$, $\Lambda_\Phi > 0$ *and a function* $\Phi \in C\big(\bigcup\limits_{i=1}^{k}[s_i,t_{i+1}],[0,\infty)\big)$ *such that for any* $t \in [s_i,t_{i+1}]$, $i = 0,1,2,\dots,k$ *inequality* $\int\limits_{s_i}^{t}\Phi(s)\,ds \leq \Lambda_\Phi\Phi(t)$ *holds and inequality* (2.2) *has at least one solution* $y \in \mathcal{PC}$*. Then problem* (1.1) *is Ulam–Hyers–Rassias stable with respect to* Φ, Ψ, *i.e.* $\|x(t) - y(t)\|_E < c_{f,g_i}(\Psi + \Phi(t))$, $t \in [0,T]$ *with*

$$C = \max\{1,\Lambda_\Phi\}, \quad c_{f,g_i} = Ce^{L_f\eta}\Big(1 + \sum_{i=1}^{k}\prod_{m=0}^{i-1}(L_{g_{k-m}}e^{L_f\eta})\Big),$$

where x *is the solution of* (1.1).

Remark 2.2. Ulam stability properties of ordinary differential equations were studied in [2], for impulsive differential equations without any type of delays see [3] and for impulsive differential equations with variable delays see [4].

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On the Well-Posedness Question of the Modified Cauchy Problem for Linear Systems of Impulsive Equations with Singularities

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Let $I \subset \mathbb{R}$ be an interval non-degenerate in the point, $b_0 = \sup I$ and

$$I_0 = I \setminus \{b_0\}.$$

Consider the linear system of impulsive equations with fixed points of impulses actions

$$\frac{dx}{dt} = P(t)x + q(t) \ \text{ for a.a. } \ t \in I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = G_l x(\tau_l) + g_l \ \ (l = 1, 2, \dots), \tag{2}$$

where $P \in L_{loc}(I_0, \mathbb{R}^{n\times n})$, $q \in L_{loc}(I_0, \mathbb{R}^n)$, $G_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$, $g_l \in \mathbb{R}^n$ $(l = 1, 2, \dots)$, $\tau_l \in I_0$ $(l = 1, 2, \dots)$, $\tau_i \neq \tau_j$ if $i \neq j$, and $\lim\limits_{l\to+\infty} \tau_l = b_0$.

Let $H = \operatorname{diag}(h_1, \dots, h_n) : I_0 \to \mathbb{R}^{n\times n}$ be a diagonal matrix-functions with continuous diagonal elements $h_k : I_0 \to]0, +\infty[$ $(k = 1, \dots, n)$.

We consider the problem of the well-posedness of solution $x : I_0 \to \mathbb{R}^n$ of the system $(1), (2)$, satisfying the modified Cauchy condition

$$\lim_{t\to b_0} (H^{-1}(t)x(t)) = 0. \tag{3}$$

Along with the system $(1), (2)$ consider the perturbed singular system

$$\frac{dx}{dt} = \widetilde{P}(t)x + \widetilde{q}(t) \ \text{ for a.a. } \ t \in I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}, \tag{4}$$

$$x(\tau_l+) - x(\tau_l-) = \widetilde{G}_l x(\tau_l) + \widetilde{g}_l \ \ (l = 1, 2, \dots), \tag{5}$$

where $\widetilde{P} \in L_{loc}(I_0, \mathbb{R}^{n\times n})$, $\widetilde{q} \in L_{loc}(I_0, \mathbb{R}^n)$, $\widetilde{G}_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$, $\widetilde{g}_l \in \mathbb{R}^n$ $(l = 1, 2, \dots)$.

In the paper, we investigate the question when the unique solvability of the problem $(1), (2); (3)$ guarantees the unique solvability of the problem $(4), (5); (3)$ and also nearness of its solutions in the definite sense if matrix-functions P and $\widetilde{P}$, G_l and $\widetilde{G}_l$ $(l = 1, 2, \dots)$, and vector-functions q and $\widetilde{q}$ and g_l and $\widetilde{g}_l$ $(l = 1, 2, \dots)$ are accordingly close to each other.

The analogous problem for systems (1) of ordinary differential equations with singularities are investigated in [2–4].

The singularity of system (1) is considered in the sense that the matrix P and vector q functions, in general, are not integrable at the point b. In general, the solution of the problem $(1), (2); (3)$

is not continuous at the point b and, therefore, it is not a solution in the classical sense. But its restriction on every interval from I_0 is a solution of the system (1), (2). In connection with this we give the example from [4].

Let $\alpha > 0$ and $\varepsilon \in]0, \alpha[$. Then the problem

$$\frac{dx}{dt} = -\frac{\alpha\, x}{t} + \varepsilon |t|^{\varepsilon-1-\alpha}, \quad \lim_{t\to 0}(t^\alpha x(t)) = 0$$

has the unique solution $x(t) = |t|^{\varepsilon-\alpha} \operatorname{sgn} t$. This function is not solution of the equation on the set $I = \mathbb{R}$, but its restrictions on $]-\infty, 0[$ and $]0, +\infty[$ are solutions of that.

We give sufficient conditions guaranteeing the well-posedness of the problem (1), (2); (3). The analogous results belong to I. Kiguradze [3, 4] for the modified Cauchy problem for systems of ordinary differential equations with singularities.

Some boundary value problems for linear impulsive systems with singularities are investigated in [1] (see, also references therein).

In the paper, the use will be made of the following notation and definitions.

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} :\ x_{ij} \ge 0\ (i = 1, \dots, n;\ j = 1, \dots, m)\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$ at the point t.

$\widetilde{C}([a, b], D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a, b] \to D$.

$\widetilde{C}_{loc}(I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}, D)$ is the set of all matrix-functions $X : I_{t_0} \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from $I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}$ belong to $\widetilde{C}([a, b], D)$.

$L([a, b]; D)$ is the set of all integrable matrix-functions $X : [a, b] \to D$.

$L_{loc}(I_0; D)$ is the set of all matrix-functions $X : I_0 \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from I_0 belong to $L([a, b], D)$.

A vector-function $x \in \widetilde{C}_{loc}(I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}, \mathbb{R}^n)$ is said to be a solution of the system (1), (2) if

$$x'(t) = P(t)x(t) + q(t) \ \text{ for a.a. } \ t \in I_{t_0} \setminus \{\tau_l\}_{l=1}^{+\infty}$$

and there exist one-sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1, 2, \dots)$ such that the equalities (2) hold.

We assume that

$$\det(I_n + G_l) \neq 0 \ \ (l = 1, 2, \dots).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e. for the case when $P \in L_{loc}(I, \mathbb{R}^{n\times n})$ and $q \in L_{loc}(I, \mathbb{R}^n)$.

Let $\mathcal{N}_{t0} = \{l \in \mathbb{N} :\ t \le \tau_l < b\}$ and $I_0(\delta) = [b_0 - \delta, b_0[\cap I_0$ for every $\delta > 0$.

Definition. The problem (1), (2); (3) is said to be H-well-posed if it has the unique solution x and for every $\varepsilon > 0$ there exists $\eta > 0$ such that the problem (4), (5); (3) has the unique solution $\widetilde{x}$ and the estimate

$$\big\|H(t)(x(t) - \widetilde{x}(t))\big\| < \varepsilon \ \text{ for } \ t \in I$$

holds for every $\widetilde{P} \in L_{loc}(I_0, \mathbb{R}^{n\times n})$, $\widetilde{q} \in L_{loc}(I_0, \mathbb{R}^n)$, $\widetilde{G}_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$, $\widetilde{g}_l \in \mathbb{R}^n$ $(l = 1, 2, \dots)$ such that $\det(I_n + \widetilde{G}_l) \neq 0$ $(l = 1, 2, \dots)$,

$$\bigg\|\int\limits_t^{b-} H^{-1}(s)|\widetilde{P}(s) - P(s)|H(s)\,ds\bigg\| + \Big\|\sum_{l\in\mathcal{N}_{t0}} H^{-1}(\tau_l)|\widetilde{G}_l - G_l|H(\tau_l)\Big\| < \eta \ \text{ for } \ t \in I_0(\delta)$$

and

$$\bigg\|\int\limits_t^{b-} H^{-1}(s)|\widetilde{q}(s) - q(s)|\,ds\bigg\| + \Big\|\sum_{l\in\mathcal{N}_{t0}} H^{-1}(\tau_l)|\widetilde{g}_l - g_l|\,\Big\| < \eta \ \text{ for } \ t \in I_0(\delta).$$

Let $P_0 \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ and $G_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$. Then a matrix-function $C_0 : I_0 \times I_0 \to \mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous impulsive system

$$\frac{dx}{dt} = P_0(t)x, \tag{6}$$

$$x(\tau_l+) - x(\tau_l-) = G_{0l}x(\tau_l) \ \ (l = 1, 2, \dots), \tag{7}$$

if, for every interval $J \subset I_0$ and $\tau \in J$, the restriction of $C_0(\,\cdot\,, \tau) : I_0 \to \mathbb{R}^{n\times n}$ on J is the fundamental matrix of the system (6), (7) satisfying the condition $C_0(\tau, \tau) = I_n$. Therefore, C_0 is the Cauchy matrix of (6), (7) if and only if the restriction of C_0 on $J \times J$, for every interval $J \subset I_0$, is the Cauchy matrix of the system in the sense of definition given in [5].

Theorem. *Let there exist a matrix-function $P_0 \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ and constant matrices $G_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$ and $B_0, B \in \mathbb{R}_+^{n\times n}$ such that*

$$\det(I_n + G_{0l}) \neq 0 \ \ (l = 1, 2, \dots),$$
$$r(B) < 1,$$

and the estimates

$$|C_0(t, \tau)| \le H(t)\,B_0\,H^{-1}(\tau) \ \ \textit{for} \ \ b - \delta \le t \le \tau < b, \ \ \tau \neq \tau_l \ \ (l = 1, 2, \dots),$$
$$\big|C_0(t, \tau_l)G_{0l}(I_n + G_{0l})^{-1}\big| \le H(t)B_0H^{-1}(\tau_l) \ \ \textit{for} \ \ b - \delta \le t \le \tau_l < b \ \ (l = 1, 2, \dots)$$

and

$$\int\limits_t^{b-} \big|C_0(t, \tau)(P(\tau) - P_0(\tau))\big|H(\tau)\,d\tau + \sum_{l\in\mathcal{N}_{t0}} \big|C_0(t, \tau_l)G_{0l}(I_n + G_{0l})^{-1}\big|\,|G_l - G_{0l}|H(\tau_l) \le H(t)B \ \ \textit{for} \ \ t \in I_0(\delta)$$

hold for some $\delta > 0$, where C_0 is the Cauchy matrix of the system (5), (6)*. Let, moreover,*

$$\lim_{t\to b}\bigg(\bigg\|\int\limits_{t_0}^{t} H^{-1}(\tau)|C_0(t, \tau)|\,|q(\tau)|\,d\tau\bigg\| + \Big\|\sum_{l\in\mathcal{N}_{t0}} H^{-1}(\tau_l)\big|C_0(t, \tau_l)G_{0l}(I_n + G_{0l})^{-1}\big|\,|g_l|\,\Big\|\bigg) = 0.$$

Then the problem (1), (2); (3) *is H-well-posed.*

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On Nonpower-Law Behavior of Blow-up Solutions to Emden–Fowler Type Higher-Order Differential Equations

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1 Introduction

For the equation

$$y^{(n)} = p_0|y|^k \operatorname{sgn} y, \ \ n \ge 2, \ \ k > 1, \ \ p_0 > 0, \tag{1.1}$$

we study blow-up solutions, i.e. those with $\lim\limits_{x \to x^*-0} y(x) = \infty$.

The origin of the considered problem is described in [8, problem 16.4], and [6]. It was earlier proved for sufficiently large n (see [9]), for $n = 12$ (see [7]), for $n = 13, 14$ (see [4]), and for $n = 15$ (see [11]), that there exists $k = k(n) > 1$ such that equation (1.1) has a solution with nonpower-law behavior, namely,

$$y(x) = (x^* - x)^{-\alpha} h(\log(x^* - x)), \ \ x \to x^* - 0, \tag{1.2}$$

where h is a positive periodic non-constant function on $\mathbb{R}$. Now we prove this result for arbitrary $n \ge 12$.

Note that it was also proved for $n = 2$ (see [8]) and for $n = 3, 4$ [1], that all blow-up solutions have power-law asymptotic behavior:

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)), \ \ x \to x^* - 0, \tag{1.3}$$

with

$$\alpha = \frac{n}{k-1}\,, \ \ C = \Big(\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{p_0}\Big)^{\frac{1}{k-1}}. \tag{1.4}$$

Existence of a solution satisfying (1.3) was proved for arbitrary $n \ge 2$. For $2 \le n \le 11$ an $(n-1)$-parametric family of such solutions to equation (1.1) was proved to exist (see [1,2], [3, Ch. I(5.1)]). It was proved that for slightly superlinear equations of arbitrary order $n \ge 5$ all blow-up solutions have power-law asymptotic behavior (see [5]).

2 The main result

In this section, a result on existence of solutions with non-power behavior is formulated for equation (1.1) with $n \ge 12$.

Theorem 2.1. *For $n \ge 12$ there exists $k > 1$ such that equation* (1.1) *has a solution $y(x)$ with*

$$y^{(j)}(x) = (x^* - x)^{-\alpha-j} h_j(\log(x^* - x)), \ \ j = 0, 1, \dots, n-1,$$

where α is defined by (1.4) *and h_j are periodic positive non-constant functions on $\mathbb{R}$.*

3 Proof of the main result

To prove the main result we transform equation (1.1) into the dynamical system and use a version of the Hopf Bifurcation theorem (see [10]).

3.1 Transformation of equation (1.1)

Equation (1.1) can be transformed into a dynamical system (see [1] or [3, Ch. I(5.1)]), by using the substitution

$$x^* - x = e^{-t}, \quad y = (C + v)\, e^{\alpha t}, \tag{3.1}$$

where C and α are defined by (1.4). The derivatives $y^{(j)}$, $j = 0, 1, \dots, n-1$, become

$$e^{(\alpha+j)t} \cdot L_j(v, v', \dots, v^{(j)}),$$

where $v^{(j)} = \frac{d^j v}{dt^j}$, and L_j is a linear function with

$$L_j(0, 0, \dots, 0) = C\alpha(\alpha+1)\cdots(\alpha+j-1) \neq 0$$

and the coefficient of $v^{(j)}$ is equal to 1.

Thus (1.1) is transformed into

$$e^{(\alpha+n)t} \cdot L_n(v, v', \dots, v^{(n)}) = p_0(C+v)^k e^{\alpha k t}, \tag{3.2}$$

$$v^{(n)} = p_0(C+v)^k - p_0 C^k - \sum_{j=0}^{n-1} a_j v^{(j)}, \tag{3.3}$$

where a_j, $j = 1, \dots, n$, are the coefficients of $v^{(j)}$ in the linear function L_n, and are $(n-j)$-degree polynomial functions in α. Equation (3.3) can be written as

$$v^{(n)} = kC^{k-1} p_0 v - \sum_{j=0}^{n-1} a_j v^{(j)} + f(v), \tag{3.4}$$

where

$$f(v) = p_0\big((C+v)^k - C^k - kC^{k-1} v\big) = O(v^2),$$
$$f'(v) = O(v) \ \text{ as } \ v \to 0,$$

Suppose $V = (V_0, \dots, V_{n-1})$ is the vector with coordinates $V_j = v^{(j)}$, $j = 0, \dots, n-1$. Then equation (3.4) can be written as

$$\frac{dV}{dt} = AV + F(V), \tag{3.5}$$

where A is a constant $n \times n$ matrix, namely,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -\widetilde{a}_0 & -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} \end{pmatrix}$$

with

$$\widetilde{a}_0 = a_0 - kc^{k-1}p_0 = a_0 - k\alpha(\alpha+1)\cdots(\alpha+n-1) = a_0 - (\alpha+1)\cdots(\alpha+n-1)(\alpha+n) \tag{3.6}$$

and eigenvalues satisfying the equation

$$\begin{aligned} 0 = \det(A-\lambda E) &= (-1)^{n+1}(-\widetilde{a}_0 - a_1\lambda - \cdots - a_{n-1}\lambda^{n-1} - \lambda^n) \\ &= (-1)^{n+1}\big((\alpha+1)(\alpha+2)\cdots(\alpha+n) - (\lambda+\alpha)\cdots(\lambda+\alpha+n-1)\big), \end{aligned} \tag{3.7}$$

which is equivalent to

$$\prod_{j=0}^{n-1}(\lambda+\alpha+j) = \prod_{j=0}^{n-1}(1+\alpha+j). \tag{3.8}$$

F in (3.5) is the vector function $F(V) = (0,\dots,0,F_{n-1}(V))$ and $F_{n-1}(V) = f(V_0)$.

3.2 Preliminary results

Theorem 3.1 (Modification of the Hopf Theorem [10])**.** *Consider an α-parameterized dynamical system $\dot{x} = f(x,\alpha)$ where $f : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ is a C^r-function ($r \geq 3$) such that $f(0,\alpha) = 0$ for all $\alpha \in \mathbb{R}$. Suppose the Jacobian matrix $D_x f(0,\widetilde{\alpha}) \equiv A(\widetilde{\alpha})$ has $\pm i\beta$ as simple eigenvalues for some $\widetilde{\alpha} \in \mathbb{R}$. Let v and w be eigenvectors such that $Av = \beta iv$, $A^*w = \beta iw$, where A^* denotes the transpose conjugate matrix of the matrix A. Put*

$$\varphi \equiv \operatorname{Re}(e^{it}v), \quad \psi \equiv \operatorname{Re}(e^{it}w), \quad \Theta_j = \frac{1}{j!}\int_0^{2\pi} \Big(\frac{\partial^j(f_x)}{\partial\alpha^j}(0,\widetilde{\alpha})\varphi, \psi\Big)\,dt.$$

If $\Theta_c \neq 0$ for some odd number c, then $(0,\widetilde{\alpha})$ is a bifurcation point of periodic solutions of $\dot{x} = f(x,\alpha)$. More precisely, there exist continuous mappings $\varepsilon \mapsto \alpha(\varepsilon) \in \mathbf{R}$, $\varepsilon \mapsto T(\varepsilon) \in \mathbf{R}$, and $\varepsilon \mapsto b(\varepsilon) \in \mathbf{R}^n$ defined in a neighborhood of 0 and such that $\alpha(0) = \widetilde{\alpha}$, $T(0) = \frac{2\pi}{q}$, $b(0) = 0$, $b(\varepsilon) \neq 0$ for $\varepsilon \neq 0$, and the solutions to the problems $\dot{x} = f(x,\alpha(\varepsilon))$, $x(0) = b(\varepsilon)$ are $T(\varepsilon)$-periodic and non-constant.

To apply the Hopf Bifurcation theorem, we study equation (3.5) and the roots of the algebraic equation (3.8).

Lemma 3.1 ([4])**.** *For any integer $n \geq 12$ there exist $\alpha > 0$ and $q > 0$ such that*

$$\prod_{j=0}^{n-1}(qi+\alpha+j) = \prod_{j=0}^{n-1}(1+\alpha+j) \tag{3.9}$$

with $i^2 = -1$.

Lemma 3.2 ([4])**.** *For any $\alpha > 0$ and any integer $n > 1$ all roots $\lambda \in \mathbb{C}$ to equation (3.8) are simple.*

3.3 Proof of Theorem 2.1

We can obtain some useful formulas

$$\widetilde{a}_0 = \alpha(\alpha+1)\dots(\alpha+n-1) - (\alpha+1)\cdots(\alpha+n) = -n(\alpha+1)\dots(\alpha+n-1), \tag{3.10}$$

$$\frac{d^{n-1}(-\widetilde{a}_0)}{d\alpha^{n-1}} = n!, \quad \frac{d^{n-1}(-a_1)}{d\alpha^{n-1}} = -n!, \tag{3.11}$$

$$\frac{d^{n-2}(-\widetilde{a}_0)}{d\alpha^{n-2}} = n\Big((n-1)!\alpha + (n-2)!\,\frac{n(n-1)}{2}\Big) = \frac{(2\alpha+1)n!}{2}\,, \tag{3.12}$$

$$\frac{d^{n-1}(-a_2)}{d\alpha^{n-1}} = 0, \quad \frac{d^{n-2}(-a_2)}{d\alpha^{n-2}} = -(n-2)!\,\frac{n(n-1)}{2} = -\frac{n!}{2}\,. \tag{3.13}$$

By using (3.7), we can prove for n, α, q from Lemma 3.1 that the vector

$$v = (1, qi, -q^2, -q^3 i, q^4, \dots)$$

is an eigenvector of the matrix A corresponding to the eigenvalue qi. Consider also an eigenvector w of the matrix A^* corresponding to the eigenvalue qi, assuming its last coordinate to equal 1: $w = (\dots\dots, 1)$. Then

$$\varphi = \operatorname{Re}(e^{it}v) = \big(\cos t, -q\sin t, -q^2\cos t, q^3\sin t, q^4\cos t, \dots\big), \quad \psi = \operatorname{Re}(e^{it}w) = (\dots\dots, \cos t).$$

Using formulas (3.11)–(3.13), we obtain

$$\Theta_{n-1} = \frac{1}{(n-1)!}\int\limits_0^{2\pi} \left(\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ n! & -n! & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ -q\sin t \\ \vdots \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \cos t \end{pmatrix} \right) dt$$

$$= \frac{1}{(n-1)!}\int\limits_0^{2\pi} n!\,(\cos^2 t + q\sin t\cos t)\,dt = \pi n \neq 0,$$

$$\Theta_{n-2} = \frac{1}{(n-2)!}\int\limits_0^{2\pi} \left(\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \frac{(2\alpha+1)n!}{2} & \frac{d^{n-2}(-a_1)}{d\alpha^{n-2}} & -\frac{n!}{2} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ -q\sin t \\ -q^2\cos t \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \cos t \end{pmatrix} \right) dt$$

$$= \frac{\pi}{(n-2)!}\Big(\frac{(2\alpha+1)n!}{2} + \frac{q^2 n!}{2}\Big) = \frac{\pi n(n-1)}{2}(2\alpha + 1 + q^2) > 0.$$

So, if $n \geq 12$, then $\Theta_{n-1} > 0$, $\Theta_{n-2} > 0$ (since $\alpha > 0$), and either $n-1$ or $n-2$ is odd. Consequently, due to the above lemmas, all the conditions of Theorem 3.1 are fulfilled. Therefore, for any $n \geq 12$ there exists a family $\alpha_\varepsilon > 0$ such that equation (3.8) with $\alpha = \alpha_0$ has the imaginary roots $\lambda = \pm qi$ with q from Lemma 3.1 and, for sufficiently small ε, system (3.5) with $\alpha = \alpha_\varepsilon$ has an arbitrary small non-zero periodic solution $V_\varepsilon(t)$. In particular, the coordinate $V_{\varepsilon,0}(t) = v(t)$ of the vector $V_\varepsilon(t)$ is also a small periodic function with the same period. This function is non-zero, too. Otherwise, all $v^{(j)}$ and therefore $V_\varepsilon(t)$ itself should be zero. Then, taking into account (3.1), we obtain

$$y(x) = \big(C + v(-\ln(x^* - x))\big)(x^* - x)^{-\alpha}.$$

Put $h(s) = C + v(-s)$, which is a non-constant continuous periodic and positive for sufficiently small ε function, and obtain the required equality

$$y(x) = (x^* - x)^{-\alpha} h(\ln(x^* - x)).$$

In the similar way we obtain the related expressions for $y^{(j)}(x)$, $j = 0, 1, \dots, n-1$.

Theorem 2.1 is proved.

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On Dimensions of Subspaces Defined by Lyapunov Exponents of Families of Linear Differential Systems

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1 Introduction

Let M be a metric space. For a given positive integer n consider a family of linear differential systems depending on the parameter $\mu \in M$:

$$\dot{x} = A(t,\mu)x, \quad x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+ \equiv [0,+\infty), \tag{1.1}$$

such that the matrix function $A(\,\cdot\,,\mu)\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is continuous and bounded for each fixed $\mu \in M$ (generally speaking, the bound being dependent on μ). Therefore, fixing a value of the parameter $\mu \in M$ in the family (1.1), we obtain a linear differential system with continuous coefficients bounded on the semiaxis. The Lyapunov exponents of this system are denoted by $\lambda_1(\mu;A) \leqslant \cdots \leqslant \lambda_n(\mu;A)$. Thus for each $k = \overline{1,n}$ we get the function $\lambda_k(\,\cdot\,;A)\colon M \to \mathbb{R}$, which is called the k-th Lyapunov exponent of the family (1.1), and the vector function $\Lambda(\,\cdot\,;A)\colon M \to \mathbb{R}^n$ defined by $\Lambda(\mu;A) = (\lambda_1(\mu;A),\dots,\lambda_n(\mu;A))^\top$.

In the theory of Lyapunov exponents, a family of matrix functions $A(\,\cdot\,,\mu)$, $\mu \in M$ (as stated, all functions are continuous and bounded on the semiaxis), is considered under one of the following two natural assumptions: that the family is continuous either **a)** in the compact-open topology, or **b)** in the uniform topology. The condition **a)** is equivalent to the fact that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ of points from M converges to a point μ_0, then the sequence of functions $A(t,\mu_k)$ of the variable $t \geqslant 0$ converges to the function $A(t,\mu_0)$ as $k \to +\infty$ uniformly on each segment $[0,T] \subset \mathbb{R}_+$, while the condition **b)** is equivalent to the fact that this convergence is uniform over the whole semiaxis $\mathbb{R}_+$. Denote the class of families (1.1) that are continuous in the compact-open topology by $\mathcal{C}^n(M)$ and the class of those that are continuous in the uniform topology by $\mathcal{U}^n(M)$. It is clear that $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$. In what follows, we shall identify families (1.1) with the matrix-functions $A(\,\cdot\,,\,\cdot\,)$ defining them, and therefore write $A \in \mathcal{C}^n(M)$ or $A \in \mathcal{U}^n(M)$.

For families (1.1) V. M. Millionshchikov stated [9] the problem of description of their Lyapunov exponents as functions of a parameter. In other words, this problem is formulated as follows: for each $n \in \mathbb{N}$, $k = \overline{1,n}$, and metric space M describe the following classes of functions:

$$\Lambda_k(M;n,\mathcal{C}) = \big\{\lambda_k(\,\cdot\,;A):\ A \in \mathcal{C}^n(M)\big\} \ \text{ and } \ \Lambda_k(M;n,\mathcal{U}) = \big\{\lambda_k(\,\cdot\,;A):\ A \in \mathcal{U}^n(M)\big\}. \tag{1.2}$$

V. M. Millionshchikov proved that for any metric space M and family $A \in \mathcal{C}^n(M)$ each of the Lyapunov exponents $\lambda_k(\,\cdot\,;A)$ can be represented as the limit of a decreasing sequence of functions of the first Baire class. In particular, this implies that $\lambda_k(\,\cdot\,;A)$ is a function of the second Baire class on this space (this assertion followed from the essentially more general Millionshchikov theorem obtained by him in [8]). M. I. Rakhimberdiev proved [10] that the number of Baire class in the description above cannot be reduced even in the case of Lyapunov exponents of families from $\mathcal{U}^n(M)$. However, the problem of a complete description of the classes (1.2) until recently remained unsolved, the solution have been obtained in [6] and [4].

The description of the classes (1.2) is a special case of a more general problem – to describe for each $n \in \mathbb{N}$ and metric space M the following classes of vector functions:

$$\Lambda(M;n,\mathcal{C}) = \big\{\Lambda(\,\cdot\,;A):\ A \in \mathcal{C}^n(M)\big\} \ \text{ and } \ \Lambda(M;n,\mathcal{U}) = \big\{\Lambda(\,\cdot\,;A):\ A \in \mathcal{U}^n(M)\big\}. \tag{1.3}$$

For further discourse note that in the case $n = 1$, the description of the second of the classes (1.3) (i.e., of the class $\Lambda(M;1,\mathcal{U}) = \Lambda_1(M;1,\mathcal{U})$) is obvious: it consists of all continuous functions $M \to \mathbb{R}$.

Before presenting the main results on the description of the classes (1.2) and (1.3), recall the necessary definitions of the descriptive set theory [5, p. 267]. Let $\mathfrak{M}$ and $\mathfrak{N}$ be sets consisting of subsets of the space M. A function $f\colon M \to \mathbb{R}$ belongs to the class $(\mathfrak{M},{}^*)$ if for any $r \in \mathbb{R}$ the preimage $f^{-1}((r,+\infty))$ of the interval $(r,+\infty)$ belongs to $\mathfrak{M}$. A function $f\colon M \to \mathbb{R}$ belongs to the class $({}^*,\mathfrak{N})$ if for any $r \in \mathbb{R}$ the preimage $f^{-1}([r,+\infty))$ of the half-interval $[r,+\infty)$ belongs to $\mathfrak{N}$. Finally, a function f belongs to the class $(\mathfrak{M},\mathfrak{N})$ if it belongs to both classes $(\mathfrak{M},{}^*)$ and $({}^*,\mathfrak{N})$.

For any $n \in \mathbb{N}$, $k = \overline{1,n}$, and metric space M, the classes $\Lambda_k(M;n,\mathcal{C})$ are described in [6] – a function $f\colon M \to \mathbb{R}$ belongs to the class $\Lambda_k(M;n,\mathcal{C})$ if and only if it: 1) belongs to the class $({}^*,G_\delta)$ and 2) has an upper semi-continuous minorant. For any $n \geqslant 2$, $k = \overline{1,n}$, and metric space M, the description of the classes $\Lambda_k(M;n,\mathcal{U})$ is obtained in [4]: a function $f\colon M \to \mathbb{R}$ belongs to the class $\Lambda_k(M;n,\mathcal{U})$ if and only if it satisfies the condition 1) and the condition 2′) it has continuous minorant and majorant. As can be seen from the formulations above, the descriptions of the classes $\Lambda_k(M;n,\mathcal{C})$ and $\Lambda_k(M;n,\mathcal{U})$ are similar, however, their proofs differ quite significantly. For any $n \in \mathbb{N}$, $k = \overline{1,n}$, and metric space M, the class $\Lambda(M;n,\mathcal{C})$ is described in [6], and the description of the class $\Lambda(M;n,\mathcal{U})$ was announced in [1] (the full proof is given in [2]). Moreover, the description of both classes (1.3) is obtained by adding to the conditions 1) and 2) (respectively, to 1) and 2′)), which are necessary since $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$, the inequalities $f_1(\mu) \leqslant \cdots \leqslant f_n(\mu)$ for all $\mu \in M$. The latter inequalities obviously follow from the definition of the vector function $\Lambda(\,\cdot\,;A)$.

Let us emphasize that the description of the class $\Lambda(M;n,\mathcal{U})$ required for its proof an approach different from those used in [4, 6]. As noted above, the key part in the description of the class $\Lambda(M;n,\mathcal{U})$ is a (constructive) proof of the sufficiency of the conditions. Let us formulate this description [1, 2], since the results given below are closely related to it.

Theorem. *Let M be a metric space, an integer $n \geqslant 2$, and all components of a vector function $(f_1,\dots,f_n)^\top\colon M \to \mathbb{R}^n$ belong to the class $({}^*,G_\delta)$, have continuous minorant and majorant and satisfy the inequalities $f_1(\mu) \leqslant \cdots \leqslant f_n(\mu)$ for all $\mu \in M$. Then there exists a family $A \in \mathcal{U}^n(M)$ such that $\Lambda(\,\cdot\,;A) = (f_1,\dots,f_n)^\top$.*

If the given vector function is bounded:

$$\sup\Big\{\big\|(f_1(\mu),\dots,f_n(\mu))^\top\big\|:\ \mu \in M\Big\} < +\infty,$$

then the statement of the above theorem can be significantly strengthened. Denote by $\mathcal{Q}^n(M)$ the class of families (1.1) of the form $A(t,\mu) = B(t) + Q(t,\mu)$, $t \in \mathbb{R}_+$, $\mu \in M$, where $B(t)$ is a bounded $n \times n$ matrix, and $Q(t,\mu)$ is a bounded $n \times n$ matrix vanishing as $t \to +\infty$ uniformly with respect to μ.

The proof of the preceding theorem implies the following

Corollary 1. *For any metric space M, integer $n \geqslant 2$, and vector function $(f_1, \dots, f_n)^\top : M \to \mathbb{R}^n$ whose components belong to the class $({}^*, G_\delta)$, are bounded and satisfy the inequalities $f_1(\mu) \leqslant \dots \leqslant f_n(\mu)$ for all $\mu \in M$, there exists a family $A \in \mathcal{Q}^n(M)$ such that $\Lambda(\,\cdot\,; A) = (f_1, \dots, f_n)^\top$.*

Let us give some more corollaries of the theorem presented here, which answer a number of open questions.

V. M. Millionshchikov proved [8] that if M is a complete metric space, then for a family $A \in \mathcal{C}^n(M)$ the set $US_i(A)$ of upper semicontinuity points of the function $\lambda_i(\,\cdot\,; A)$ contains a dense G_δ-set for each $i = \overline{1,n}$. In other words, the upper semicontinuity of these functions is Baire typical in the space M. This statement is not true for the lower semicontinuity: in [11] for each $n \geqslant 1$ there is constructed a family $A \in \mathcal{C}^n([0,1])$ such that the set $LS_i(A)$ of lower semicontinuity points of the function $\lambda_i(\,\cdot\,; A)$, $i = \overline{1,n}$, is empty. A complete description of the n-tuples $(LS_1(A), \dots, LS_n(A))$ for any metric space M and a complete description of the n-tuples $(US_1(A), \dots, US_n(A))$ for any complete metric space M are obtained in [7] for the families $A \in \mathcal{C}^n(M)$. A family $A \in \mathcal{U}^n([0,1])$ for which the set $LS_i(A)$ is empty is constructed in [13] for any $n \geqslant 2$ and $i = \overline{1,n}$. Later, using the ideas of that paper and the results of [7], a complete description of the sets $LS_i(A)$, $i = \overline{1,n}$, for any metric space M and a complete description of the sets $US_i(A)$, $i = \overline{1,n}$, for any complete metric space M were obtained in [3] for the families $A \in \mathcal{U}^n(M)$.

Using the main theorem we can give a complete description of the n-tuples $(LS_1(A), \dots, LS_n(A))$ for any metric space M and a complete description of the n-tuples $(US_1(A), \dots, US_n(A))$ for any complete metric space M for the families $A \in \mathcal{U}^n(M)$ thus giving an answer to the problem stated in [3].

Corollary 2. *For any integer $n \geqslant 2$ and metric space M, an n-tuple $(M_1, \dots, M_n)$ of subsets of M is the n-tuple of the lower semicontinuity sets of the Lyapunov exponents of some family $A \in \mathcal{U}^n(M)$ (i.e., $M_i = LS_i(A)$, $i = \overline{1,n}$) if and only if each set M_i, $i = \overline{1,n}$, is $F_{\sigma\delta}$ and contains all isolated points of M. Moreover, in cases where such a family exists, it can be chosen from the class $\mathcal{Q}^n(M)$.*

Corollary 3. *For any integer $n \geqslant 2$ and complete metric space M, an n-tuple $(M_1, \dots, M_n)$ of subsets of M is the n-tuple of the upper semicontinuity sets of the Lyapunov exponents of some family $A \in \mathcal{U}^n(M)$ (i.e., $M_i = US_i(A)$, $i = \overline{1,n}$) if and only if each set M_i, $i = \overline{1,n}$, is a dense G_δ-set in M. Moreover, in cases where such a family exists, it can be chosen from the class $\mathcal{Q}^n(M)$.*

For each $\mu \in M$ denote by $S(\mu; A)$ the vector space of solutions of the system (1.1). As is well known, the sets $L_\alpha(\mu; A) \stackrel{\text{def}}{=} \{x \in S(\mu; A) :\ \lambda[x] < \alpha\}$ and $N_\alpha(\mu; A) \stackrel{\text{def}}{=} \{x \in S(\mu; A) :\ \lambda[x] \le \alpha\}$ are vector subspaces of the space $S(\mu; A)$ for any $\alpha \in \mathbb{R}$. Denote their dimensions by $d_\alpha(\mu; A)$ and $D_\alpha(\mu; A)$ respectively. Next we consider the natural question: what are the functions $\mu \mapsto d_\alpha(\mu; A)$ and $\mu \mapsto D_\alpha(\mu; A)$? A. N. Vetokhin proved [12] that if M is the space of all linear n-dimensional systems endowed with either of the topologies: compact-open or uniform, and the family (1.1) is defined by the equality $A(t, \mu) = \mu(t)$, $\mu \in M$, $t \in \mathbb{R}_+$, then the first function belongs exactly to the second Baire class, and the second one belongs exactly to the third Baire class.

The following statements contain a complete description of the classes $\{d_\alpha(\mu; A) :\ A \in \mathcal{C}^n(M)\}$, $\{d_\alpha(\mu; A) :\ A \in \mathcal{U}^n(M)\}$, $\{D_\alpha(\mu; A) :\ A \in \mathcal{C}^n(M)\}$, and $\{D_\alpha(\mu; A) :\ A \in \mathcal{U}^n(M)\}$ for any metric space M and numbers $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$.

Corollary 4. *Let an arbitrary metric space M and numbers $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and a function $f\colon M \to \{0, \dots, n\}$ be given. Then the equality $f = d_\alpha(\,\cdot\,; A)$ $(f = D_\alpha(\,\cdot\,; A))$ holds for some family $A \in \mathcal{C}^n(M)$ if and only if f belongs to the class (F_σ, F_σ) (respectively, to the class $(F_{\sigma\delta}, F_{\sigma\delta})$).*

Corollary 5. *Let an arbitrary metric space M and numbers $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and a function $f\colon M \to \{0,\dots,n\}$ be given. Then the equality $f = d_\alpha(\,\cdot\,;A)$ ($f = D_\alpha(\,\cdot\,;A)$) holds for some family $A \in \mathcal{U}^n(M)$ if and only if*

1) *in the case $n \geqslant 2$, the function f belongs to the class (F_σ, F_σ) (respectively, $(F_{\sigma\delta}, F_{\sigma\delta})$);*

2) *in the case $n = 1$, the function f is lower semicontinuous (respectively, upper semicontinuous).*

Moreover, for $n \geqslant 2$, if such a family exists, then it can be chosen from the class $\mathcal{Q}^n(M)$.

Corollaries 4 and 5 allow us to describe the sets of semicontinuity of functions $d_\alpha(\,\cdot\,;A)$ and $D_\alpha(\,\cdot\,;A)$ for families $A \in \mathcal{C}^n(M)$ and $A \in \mathcal{U}^n(M)$.

Corollary 6. *Let an arbitrary metric space M and numbers $\alpha \in \mathbb{R}$, and $n \geqslant 2$ ($n \geqslant 1$) be given. Then a set $S \subset M$ is the set of lower semicontinuity points of the function $d_\alpha(\,\cdot\,;A)$ for some family $A \in \mathcal{U}^n(M)$ ($A \in \mathcal{C}^n(M)$) if and only if S is a dense G_δ-subset. A set $S \subset M$ is the set of upper semicontinuity points of the function $d_\alpha(\,\cdot\,;A)$ for some family $A \in \mathcal{U}^n(M)$ ($A \in \mathcal{C}^n(M)$) if and only if S is a dense F_σ-subset. Moreover, for $n \geqslant 2$, if the mentioned family exists, then it can be chosen from the class $\mathcal{Q}^n(M)$.*

Corollary 7. *Let an arbitrary metric space M and numbers $\alpha \in \mathbb{R}$, and $n \geqslant 2$ ($n \geqslant 1$) be given. Then a set $S \subset M$ is the set of lower semicontinuity points of the function $D_\alpha(\,\cdot\,;A)$ for some family $A \in \mathcal{U}^n(M)$ ($A \in \mathcal{C}^n(M)$) if and only if S is a dense $F_{\sigma\delta}$-subset. A set $S \subset M$ is the set of upper semicontinuity points of the function $D_\alpha(\,\cdot\,;A)$ for some family $A \in \mathcal{U}^n(M)$ ($A \in \mathcal{C}^n(M)$) if and only if S is a dense $G_{\delta\sigma}$-subset. Moreover, for $n \geqslant 2$, if the mentioned family exists, it can be chosen from the class $\mathcal{Q}^n(M)$.*

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Analogue of the Erugin Theorem on the Absence of Strongly Irregular Periodic Solutions of Two-dimensional Linear Discrete Periodic System

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Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of natural, integer and real numbers, respectively, $z = (z_n) = (z(n))$ $(n \in \mathbb{N}) - l$-dimensional vector function (sequence), defined on $\mathbb{N}$ with values in $\mathbb{R}^l$, i.e. $z : \mathbb{N} \to \mathbb{R}^l$. The set of such sequences is denoted by S^l. Following [1, p. 69] we introduce the definition.

Definition 1. A sequence $z \in S^l$ is called periodic with a period $\omega \in \mathbb{N}$ (ω-periodic) if for any $n \in \mathbb{N}$ the equality $z_{n+\omega} = z_n$ holds.

Naturally, if the number ω is the period of the sequence z, then its multiples will also be the periods of this sequence, i.e. for any $n \in \mathbb{N}$, $m \in \mathbb{N}$, we have $z(n + m\omega) = z(n)$. Therefore, in the future, under the period of the sequence, as a rule, we will understand the smallest of the periods. In this case, in particular, any constant scalar sequence will be 1-periodic. The set of l-dimensional ω-periodic sequences is denoted by PS^l_ω.

Periodic sequences under certain conditions can be solutions of discrete (difference) systems. The problem of the existence and construction of periodic solutions of discrete equations and systems is considered in a sufficiently large number of papers [1, 4, 6] etc. In these papers solutions are mainly studied, the period of which coincides with the period of the equation. The results obtained in this direction are in many respects similar to the corresponding results for ordinary differential equations. However, in some cases there are significant differences. Note one of them.

As it is known [8], a nonlinear scalar periodic ordinary differential equation does not have non-constant periodic solutions such that the periods of the solution and equation are incommensurable. Moreover, N. P. Erugin proved in [5] that such solutions are absent in the linear nonstationary periodic system of two equations. It is interesting to investigate such questions for discrete equations and systems. For this purpose, we consider the system

$$x_{n+1} = X(x_n, y_n, n), \quad y_{n+1} = Y(x_n, y_n, n), \;\; n \in \mathbb{N}, \;\; \operatorname{col}(x, y) \in S^2, \tag{1}$$

the right side of which is ω-periodic, i.e. there exists the smallest $\omega \in \mathbb{N}$ such that for any fixed $n_0 \in \mathbb{N}$ equalities $X(x_{n_0}, y_{n_0}, n + \omega) = X(x_{n_0}, y_{n_0}, n)$, $Y(x_{n_0}, y_{n_0}, n + \omega) = Y(x_{n_0}, y_{n_0}, n)$ hold for all $n \in \mathbb{N}$. Further, the period of the system of the form (1) is understood as the period of its right side.

Analogous to [2], we introduce the following

Definition 2. A periodic solution with a period of the system (1) such that the numbers ω and Ω are coprime, we will call strongly irregular.

We note that the paper [7] shows the following: under certain conditions, the scalar discrete equation can admit a strongly irregular periodic solution. Indeed, let σ be an arbitrary odd number and $(h_n) \in PS^1_\sigma$. Take the discrete equation

$$x_{n+1} = -x_n - (1 - x_n^2)h_n. \tag{2}$$

The equation (2) has a solution

$$x_n = (-1)^n \tag{3}$$

with period $\Omega = 2$. As the numbers σ and Ω coprime, by Definition 2, the periodic solution (3) of the equation (2) is strongly irregular.

Thus, Massera's theorem [8] on the absence of strongly irregular periodic solutions for a scalar ordinary equation for difference equations, generally speaking, has no complete analog for discrete equations. An analogue of Massera's theorem for linear difference equations was obtained in [3]. In particular, it is shown that the scalar linear homogeneous periodic nonstationary discrete equation of the first order has not strongly irregular periodic solutions different from the constants.

It is quite natural to raise the question for the two-dimensional case: is there an analogue of the above theorem by N. P. Erugin on the two-dimensional linear system (1)

$$x_{n+1} = a_n x_n + b_n y_n, \quad y_{n+1} = c_n x_n + d_n y_n, \ \ n \in \mathbb{N}, \ \ x \in S^1, \ \ y \in S^1, \tag{4}$$

where the coefficient matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is ω-periodic, i.e. $A(n+\omega) = A(n)$ for all $n \in \mathbb{N}$ and at least one of its elements is different from the constant? As the following example shows, the answer to this question is generally negative. Indeed, take a linear discrete system

$$x_{n+1} = -x_n + b_n y_n, \quad y_{n+1} = d_n y_n, \ \ n \in \mathbb{N}, \ \ (b_n) \in PS^1_\omega, \ \ (d_n) \in PS^1_\omega, \tag{5}$$

where at least one of the coefficients (b_n), (d_n) is different from the constant, i.e. $\omega \geq 2$, and the greatest common divisor of numbers 2 and ω is 1. The system (5) has a periodic solution

$$x_n = (-1)^n, \quad y_n = 0, \ \ n \in \mathbb{N}. \tag{6}$$

The period of the solution (6) is coprime with the period of the system (5).

Our goal is to distinguish a class of linear two-dimensional discrete systems that have not strongly irregular periodic solutions.

Further, we say that the columns $H^{(1)}(n), \dots, H^{(k)}(n)$ of some matrix $H(n)$, $n \in \mathbb{N}$ are linearly independent if the identity

$$\alpha_1 H^{(1)}(n) + \cdots + \alpha_k H^{(k)}(n) \equiv 0, \ \ n \in \mathbb{N}, \ \ \alpha_1, \dots, \alpha_k \in \mathbb{R}$$

holds if and only if $\alpha_1 = \cdots = \alpha_k = 0$. Through $\operatorname{rank}_{\mathrm{col}} H$ denote the column rank of the matrix $H(n)$, $n \in \mathbb{N}$, i.e. the largest number of its linearly independent columns.

Suppose that the system (4) has a strongly irregular Ω-periodic solution

$$x_n = \varphi_n, \quad y_n = \psi_n, \quad \varphi(n+\Omega) = \varphi(n), \quad \psi(n+\Omega) = \psi(n), \ \ n \in \mathbb{N}, \tag{7}$$

where ω and Ω are coprime and $\Omega \geq 2$. This means that

$$\varphi_{n+1} \equiv a_n \varphi_n + b_n \psi_n, \quad \psi_{n+1} \equiv c_n \varphi_n + d_n \psi_n, \ \ n \in \mathbb{N}. \tag{8}$$

As the identities (8) are true for all $n \in \mathbb{N}$, there are also true

$$\varphi_{n+1+\Omega} \equiv a_{n+\Omega} \varphi_{n+\Omega} + b_{n+\Omega} \psi_{n+\Omega}, \quad \psi_{n+1} \equiv c_{n+\Omega} \varphi_{n+\Omega} + d_{n+\Omega} \psi_{n+\Omega}, \ \ n \in \mathbb{N}. \tag{9}$$

By virtue of the Ω-periodicity of functions φ_n, ψ_n, the identities (9) take the following form

$$\varphi_{n+1} \equiv a_{n+\Omega} \varphi_n + b_{n+\Omega} \psi_n, \quad \psi_{n+1} \equiv c_{n+\Omega} \varphi_n + d_{n+\Omega} \psi_n, \ \ n \in \mathbb{N}. \tag{10}$$

The identities (8), (10) implies the following

$$\begin{aligned} &(a_{n+\Omega} - a_n)\varphi_n + (b_{n+\Omega} - b_n)\psi \equiv p^{(11)}(n)\varphi_n + p^{(12)}(n)\psi_n \equiv 0, \\ &(c_{n+\Omega} - c_n)\varphi_n + (d_{n+\Omega} - d_n)\psi \equiv p^{(21)}(n)\varphi_n + p^{(22)}(n)\psi_n \equiv 0, \end{aligned} \quad n \in \mathbb{N}. \tag{11}$$

We form a matrix

$$P(n) = \begin{bmatrix} p^{(11)}(n) & p^{(12)}(n) \\ p^{(21)}(n) & p^{(22)}(n) \end{bmatrix}, \ \ n \in \mathbb{N}.$$

We denote by $P^{(j)}(n)$, $n \in \mathbb{N}$, $j = 1, 2$ the columns of this matrix. As $P(n) = A(n+\Omega) - A(n)$ and $A(n+\omega) \equiv A(n)$, $n \in \mathbb{N}$, the matrix function P is ω-periodic.

We show that the columns $P^{(1)}(n)$ and $P^{(2)}(n)$ are linearly dependent, i.e. there are exist such $\alpha_0, \beta_0 \in \mathbb{R}$, $\alpha_0^2 + \beta_0^2 \neq 0$, that $\alpha_0 P^{(1)}(n) + \beta_0 P^{(2)}(n) \equiv 0$, $n \in \mathbb{N}$. According to the assumption, at least one of the functions $x = \varphi$, $y = \psi$ is nonstationary. Therefore, there exists $n_0 \in \mathbb{N}$ for which the inequality $\varphi_{n_0}^2 + \psi_{n_0}^2 \neq 0$ holds. The identities (11) imply the justice of equalities

$$\varphi_{n_0+m\Omega} P^{(1)}(n_0 + m\Omega) + \psi_{n_0+m\Omega} P^{(2)}(n_0 + m\Omega) = 0, \ \ m \in \mathbb{N},$$

from which, on the basis of the Ω-periodicity of functions φ, ψ, we obtain the equality

$$\varphi_{n_0} P^{(1)}(n_0 + m\Omega) + \psi_{n_0} P^{(2)}(n_0 + m\Omega) = 0, \ \ m \in \mathbb{N}. \tag{12}$$

As the matrix P has a period ω, the equality (12) can be written as

$$\varphi_{n_0} P^{(1)}(n_0 + m\Omega + k\omega) + \psi_{n_0} P^{(2)}(n_0 + m\Omega + k\omega) = 0, \ \ k, m \in \mathbb{N}. \tag{13}$$

Since k, m are an arbitrary natural numbers and least common multiple of ω and Ω is 1, for any $n \in \mathbb{N}$ there exist such k, m that the equation $n = n_0 + m\Omega + k\omega$ holds. Therefore, $P^{(j)}(n_0 + m\Omega + k\omega) = P^{(j)}(n)$, $n \in \mathbb{N}$, $j = 1, 2$ for $k, m \in \mathbb{N}$. Hence, from the equations (13) we obtain

$$\varphi_{n_0} P^{(1)}(n) + \psi_{n_0} P^{(2)}(n) = 0, \ \ n \in \mathbb{N}. \tag{14}$$

By virtue of the fact that $\varphi_{n_0}^2 + \psi_{n_0}^2 \neq 0$, the identity (14) means that the columns of the matrix $P(n)$, $n \in \mathbb{N}$ are linearly dependent.

So, we have proved the following

Theorem. *If the system* (4) *has a nonstationary periodic solution such that the solution period is coprime with the system's period, then the columns of the matrix are linearly dependent.*

Corollary. *If the matrix* $P(n)$, $n \in \mathbb{N}$ *has a complete column rank, i.e.* $\operatorname{rank}_{\rm col} P = 2$, *the system* (4) *has not nonstationary strongly irregular periodic solutions.*

Remark 1. As shown above, the discrete periodic system (5) has a strongly irregular 2-periodic solution (6). The matrix $P(n)$, $n \in \mathbb{N}$ for this system has the form

$$P(n) = \begin{bmatrix} 0 & b(n+2) - b(n) \\ 0 & d(n+2) - d(n) \end{bmatrix}, \quad n \in \mathbb{N}. \tag{15}$$

The columns of this matrix are linearly dependent and its column rank in generall case is one.

Remark 2. In general, the linear dependence of the columns and rows of a discrete matrix is not equivalent. This is particularly confirmed by the example (15), where the matrix rows can be linearly dependent only if

$$b(n+2) - b(n) \equiv l(d(n+2) - d(n)), \quad l \in \mathbb{R}.$$

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On the Choice of Additional Initial Condition for Some Three-Level Difference Schemes

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Abstract. In this paper we study an initial boundary-value problem for the Regularized Long Wave (RLW) equation. A three-level conservative difference scheme is constructed and investigated. For each new level the obtained algebraic equations are linear with respect to the values of unknown function.

1 Introduction

We consider one-dimensional RLW equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\lambda u\,\frac{\partial u}{\partial x}-\mu\,\frac{\partial^3 u}{\partial x^2\partial t}=0, \tag{1.1}$$

with the physical boundary conditions $u\to 0$ as $x\to\pm\infty$. Here $u(x,t)$ represents the wave's amplitude, and λ and μ are positive parameters.

This equation describes phenomena with weak nonlinearity and dispersion waves, including, for example, ion-acoustic and magnetohidrodynamic waves in plasma.

The main difficulties of numerical solution of (1.1) consist in physical domain boundless and nonlinearity of the equation, therefore, it is expedient to restrict the computational domain to a finite one. Suppose that the initial data $u_0(x)$ is compactly supported in a finite domain $(a,b)\subset\mathbb{R}$ which contains the compact support of $u(x,t)$.

We consider RLW equation (1.1) with the homogeneous boundary conditions

$$u(a,t)=0,\quad u(b,t)=0,\ \ 0<t\le T,$$

and the initial condition

$$u(x,0)=u_0(x),\ \ a\le x\le b.$$

2 Construction of difference scheme

The domain $[a,b]\times[0,T]$ is divided into rectangle grids by

$$x_i=a+ih,\ \ t_j=j\tau,\ \ i=1,2,\dots,n,\ \ j=0,1,2,\dots,J,$$

where $h=(b-a)/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively. For discrete functions defined on the mesh we use notation $U_i^j=U(x_i,t_j)$, $U_i^j\sim u(x_i,t_j)$.

In some cases, for simplicity and not implying vagueness, we omit some indices of the discrete function. We introduce fictitious values U_{-1}^j, U_{n+1}^j which correspond to the abscissaes $x_{-1}=a-h$, $x_{n+1}=b+h$ and are defined by the equalities:

$$U_{-1}^j=0,\;\; U_{n+1}^j=0,\;\; j=0,1,2,\dots\,.$$

Let

$$Z_h^0=\big\{v=(v_i)\mid\; v_{-1}=v_0=v_n=v_{n+1}=0\big\}.$$

Define

$$\begin{gathered}
(U_i^j)_x=\frac{U_{i+1}^j-U_i^j}{h}\,,\quad (U_i^j)_{\overline{x}}=\frac{U_i^j-U_{i-1}^j}{h}\,,\\
(U_i^j)_{\dot{x}}=\frac{1}{2h}\,(U_{i+1}^j-U_{i-1}^j),\quad (U_i^j)_{\ddot{x}}=\frac{1}{4h}\,(U_{i+2}^j-U_{i-2}^j),\\
\overline{U}_i^{\,0}=\frac{U_i^1+U_i^0}{2}\,,\quad \overline{U}_i^{\,j}=\frac{U_i^{j+1}+U_i^{j-1}}{2}\;\text{ for }\; j\geq 1,\\
(U_i^j)_t=\frac{U_i^{j+1}-U_i^j}{\tau}\,,\quad (U_i^j)_{\overset{\circ}{t}}=\frac{1}{2\tau}\,(U_i^{j+1}-U_i^{j-1}).
\end{gathered}$$

Define the following averaging operators

$$\begin{gathered}
\dot{\mathcal{P}}u=\frac{1}{h^2}\int\limits_{x-h}^{x+h}\big(h-|x-\xi|\big)u(\xi,t)\,d\xi,\quad \ddot{\mathcal{P}}u=\frac{1}{4h^2}\int\limits_{x-2h}^{x+2h}\big(2h-|x-\xi|\big)u(\xi,t)\,d\xi,\\
\overset{\circ}{\mathcal{S}}u=\frac{1}{2\tau}\int\limits_{t-\tau}^{t+\tau}u(x,\zeta)\,d\zeta,\quad \widehat{\mathcal{S}}u=\frac{1}{\tau}\int\limits_{t}^{t+\tau}u(x,\zeta)\,d\zeta.
\end{gathered}$$

Let us consider some equalities connected with these operators

$$\dot{\mathcal{P}}\,\frac{\partial^2u}{\partial x^2}=u_{\overline{x}x},\quad \ddot{\mathcal{P}}\,\frac{\partial^2u}{\partial x^2}=u_{\dot{x}\dot{x}},\quad \overset{\circ}{\mathcal{S}}\,\frac{\partial u}{\partial t}=u_{\overset{\circ}{t}}.$$

It is easy to verify that

$$\dot{\mathcal{P}}u=u+\frac{h^2}{12}\,\frac{\partial^2u}{\partial x^2}+O(h^4),\quad \ddot{\mathcal{P}}u=u+\frac{4h^2}{12}\,\frac{\partial^2u}{\partial x^2}+O(h^4),$$

whence

$$(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})u=3u+O(h^4).$$

Let us act on (1.1) with the operator

$$\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}.$$

Notice that

$$\begin{gathered}
\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}\,\frac{\partial u}{\partial t}=\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})u_{\overset{\circ}{t}}=u_{\overset{\circ}{t}}+O(h^4),\\
\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}\,\frac{\partial^3u}{\partial x^2\partial t}=\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\Big(\frac{\partial^2u}{\partial x^2}\Big)_{\overset{\circ}{t}}=\frac{1}{3}\,(4u_{\overline{x}x\overset{\circ}{t}}-u_{\dot{x}\dot{x}\overset{\circ}{t}}).
\end{gathered}$$

Further,

$$\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}\,\frac{\partial u}{\partial x}=\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\,\frac{\partial\overline{u}}{\partial x}=\frac{1}{3}\,(4\overline{u}_{\dot{x}}-\overline{u}_{\ddot{x}})+O(\tau^2+h^4).$$

Finally, after some transformations we have

$$\begin{aligned}(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}\Big(u\,\frac{\partial u}{\partial x}\Big)&=(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\Big(u\,\frac{\partial\overline{u}}{\partial x}\Big)+O(\tau^2)=3u\frac{\partial\overline{u}}{\partial x}+O(\tau^2+h^4)\\&=\frac{4}{3}\,\big[\overline{u}_{\dot{x}}u+(\overline{u}u)_{\dot{x}}\big]-\frac{1}{3}\,\big[\overline{u}_{\ddot{x}}u+(\overline{u}u)_{\ddot{x}}\big]+O(\tau^2+h^4).\end{aligned}$$

Thus, we have the difference scheme

$$\begin{aligned}(U_i^j)_{\overset{\circ}{t}}+\Big(\frac{4}{3}\,(\overline{U}_i^j)_{\dot{x}}-\frac{1}{3}\,(\overline{U}_i^j)_{\ddot{x}}\Big)+\frac{4\lambda}{9}\,\kappa_1(\overline{U}_i^j,U_i^j)-\frac{\lambda}{9}\,\kappa_2(\overline{U}_i^j,U_i^j)\\-\mu\Big(\frac{4}{3}\,(U_i^j)_{\overline{x}x\overset{\circ}{t}}-\frac{1}{3}\,(U_i^j)_{\dot{x}\dot{x}\overset{\circ}{t}}\Big)=0,\ \ i=1,2,\dots,n-1;\ \ j=1,2,\dots,J-1,\ \ U\in Z_h^0,\end{aligned}\tag{2.1}$$

where

$$\kappa_1(U,V)=U_{\dot{x}}V+(UV)_{\dot{x}},\quad \kappa_2(U,V)=U_{\ddot{x}}V+(UV)_{\ddot{x}}.$$

The additional initial conditions (the values of unknown function on the first level) is found with two-level linear scheme:

$$\begin{aligned}(U_i^0)_t+\Big(\frac{4}{3}\,(\overline{U}_i^0)_{\dot{x}}-\frac{1}{3}\,(\overline{U}_i^0)_{\ddot{x}}\Big)+\frac{4\lambda}{9}\,\kappa_1(\overline{U}_i^0,U_i^0)\\-\frac{\lambda}{9}\,\kappa_2(\overline{U}_i^0,U_i^0)-\mu\Big(\frac{4}{3}\,(U_i^0)_{\overline{x}xt}-\frac{1}{3}\,(U_i^0)_{\dot{x}\dot{x}t}\Big)=0,\ \ i=1,2,\dots,n-1.\end{aligned}\tag{2.2}$$

It is proved that the difference scheme (2.1), (2.2) is uniquely solvable, conservative, absolutely stable and converges with rate $O(\tau^2+h^4)$.

Equations (2.2) are especially notable. Some authors suggest that this is the approximation of the differential equation using initial conditions and attempt to receive an approximation with the same order truncation error as for the differential equation. We think that (2.2) is an approximation of the initial conditions for the first level using the differential equation. It must be required an appropriate order of approximation of initial data. This is confirmed in our papers (see, e.g. [1–3]).

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Kneser Solutions to Second Order Nonlinear Equations with Indefinite Weight

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1 Introduction

Consider the nonlinear differential equation

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \ \ t \in [1, \infty), \tag{1.1}$$

where

$$\Phi(u) := |u|^\alpha \operatorname{sgn} u, \ \ \alpha > 0.$$

We study the problem of the existence of *Kneser solutions*, that is solutions x such that

$$x(t) > 0, \quad x'(t) < 0 \ \text{ for } \ t \in [1, \infty), \tag{1.2}$$

satisfying the boundary conditions

$$x(1) = c > 0, \quad \lim_{t \to \infty} x(t) = 0. \tag{1.3}$$

We assume that the functions a, b are continuous functions on $[1, \infty)$, $a(t) > 0$, and

$$J_a = \int_1^\infty \Psi\Big(\frac{1}{a(t)}\Big)\, dt < \infty,$$

where Ψ is the inverse function of Φ, that is $\Psi(u) := |u|^{1/\alpha} \operatorname{sgn} u$. The weight function b is bounded from above and is allowed to change sign (in)finite many times. The nonlinearity F is a continuous function on $[0, \infty)$ such that $F(u) > 0$ for $u > 0$ and

$$\limsup_{u \to 0+} \frac{F(u)}{\Phi(u)} < \infty. \tag{1.4}$$

This problem is motivated by [3] where some asymptotic BVPs are studied for (1.1) in case $F(u) = |u|^\beta \operatorname{sgn} u$, $\beta > 0$ and $b(t) \le 0$ for $t \ge 1$. There are few contributions to the solvability of the boundary value problems when the function b is allowed to change its sign. For example, the boundary value problem on the compact interval with the indefinite weight has been considered in [1].

In [4], our method used here is based on a fixed point theorem for operators defined in a Fréchet space stated in [2]. This approach does not require the explicit form of the fixed point operator but only good *a-priori* bounds. These bounds are obtained using the principal solutions of an associated linear or half-linear differential equations.

Our proofs are based on the following fixed point theorem.

Theorem 1 ([2]). *Consider the BVP on $[1,\infty)$,*

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \quad x \in S, \tag{1.5}$$

where S is a nonempty subset of the Fréchet space $C[1,\infty)$ of the continuous functions defined in $[1,\infty)$ endowed with the topology of uniform convergence on compact subsets of $[1,\infty)$.

Let G be a continuous function on $[0,\infty)\times[0,\infty)$ such that $F(d) = G(d,d)$ for any $d \in [0,\infty)$. Assume that there exist a nonempty, closed, convex and bounded subset $\Omega \subseteq C[1,\infty)$ and a bounded closed subset $S_1 \subseteq S \cap \Omega$ such that for any $u \in \Omega$ the BVP on $[1,\infty)$

$$(a(t)\Phi(x'))' + b(t)G(u(t),x(t)) = 0, \quad x \in S_1$$

admits a unique solution. Then the BVP (1.5) has at least a solution.

In the sequel, we introduce the notion of principal solution and disconjugacy for the half-linear equation

$$(a(t)\Phi(y'))' + \beta(t)\Phi(y) = 0, \tag{1.6}$$

where β is a continuous function for $t \geq 1$. When (1.6) is nonoscillatory, the notion of principal solution of (1.6) has been introduced in [7] by following the Riccati approach, see, also [6, Sections 2.2, 4.2]. Among all eventually different from zero solutions of the associated Riccati equation

$$w' + \beta(t) + R(t,w) = 0, \tag{1.7}$$

where

$$R(t,w) = \alpha|w|\Psi\Big(\frac{|w|}{a(t)}\Big),$$

there exists one, say w_x, which is continuable to infinity and is minimal in the sense that any other solution w of (1.7), which is continuable to infinity, satisfies $w_x(t) < w(t)$ as $t \to \infty$. This concept extends to the half-linear case the well-known notion of principal solution that was introduced in 1936 by W. Leighton and M. Morse for the linear case.

We recall that (1.6) is said to be *disconjugate* on an interval $I \subset [T,\infty)$ if any nontrivial solution of (1.6) has at most one zero on I. Equation (1.6) is disconjugate on $[T,\infty)$ if and only if it has the principal solution without zeros on (T,∞).

An important role in our considerations is played by a comparison theorem for the principal solutions of Sturm majorant and minorant half-linear equations established in [5]. It is worth to note that if $\alpha = 1$, the half-linear equation reduces to linear one and its principal solution can be characterized by the condition

$$\int_1^\infty \frac{1}{a(t)x^2(t)}\,dt = \infty. \tag{1.8}$$

However, the integral characterization of the principal solution of half-linear equations remains an open problem. Hence, in the half-linear case a different approach has been used.

2 Existence and uniqueness theorem: case $\alpha = 1$

Consider nonlinear equation with the Sturm–Liouville operator

$$(a(t)x')' + b(t)F(x) = 0. \tag{2.1}$$

In addition to assumptions stated in Introduction, we also assume here that F is differentiable on $[0,\infty)$ with bounded nonnegative derivative, that is

$$0 \le \frac{dF(u)}{du} \le K \ \text{ for } \ u \ge 0, \tag{2.2}$$

and satisfies

$$\lim_{u\to 0^+} \frac{F(u)}{u} = k_0, \quad \lim_{u\to\infty} \frac{F(u)}{u} = k_\infty, \tag{2.3}$$

where $0 \le k_0 \ne k_\infty$.

The following result has been stated in [4, Theorem 3], see also Remark 5.

Theorem 2. *Let $B > 0$ be such that*

$$b(t) \le B \ \text{ on } \ [1,\infty)$$

and assume that the linear differential equation

$$v'' + \frac{BK}{a(t)} v = 0 \tag{2.4}$$

is disconjugate on $[1,\infty)$. Then, for any $c > 0$, equation (2.1) has a unique solution x satisfying (1.2) and (1.3). Moreover, such solution x satisfies (1.8).

Example. Consider the equation

$$(t^2x')' + \frac{1}{4}\cos\Big(\frac{\pi t}{2}\Big) F(x) = 0 \ \ (t \ge 1), \tag{2.5}$$

where

$$F(u) = \frac{u}{1+\sqrt{u}}\,.$$

Then F satisfies (2.2), (2.3), $K = 1$ and $b(t) \le 1/4$ for $t \ge 1$. Hence equation (2.4) becomes the Euler equation

$$v'' + \frac{1}{4t^2}\, v = 0 \ \ (t \ge 1),$$

which has a principal solution $v = \sqrt{t}$ and thus it is disconjugate on $[1,\infty)$. By Theorem 2, for any $c > 0$, equation (2.5) has a unique Kneser solution satisfying (1.2), (1.3) and (1.8).

3 Existence theorem in the general case

Denote by b_+, b_-, respectively, the positive and the negative part of b, i.e., $b_+(t) = \max\{b(t),0\}$, $b_-(t) = -\min\{b(t),0\}$. Thus $b(t) = b_+(t) - b_-(t)$.

Denote by $\widetilde{F}$ the function

$$\widetilde{F}(v) = \frac{F(v)}{\Phi(v)} \ \text{ on } \ (0,\infty). \tag{3.1}$$

In view of (1.4), the function $\widetilde{F}$ is bounded in the neighbourhood of zero.

Using Theorem 1 and asymptotic properties of the half-linear equations, we obtain from [5, Theorem 1] the following result.

Theorem 3. *Let $c>0$ be fixed and M_c be such that*

$$\widetilde{F}(v)\le M_c \ \text{ on } \ [0,c].$$

Assume that the half-linear differential equation

$$(a_1(t)\Phi(y'))'+\beta_1(t)\Phi(y)=0, \tag{3.2}$$

where

$$a_1(t)\le a(t), \quad \beta_1(t)\ge M_c b_+(t) \ \text{ on } \ t\ge 1, \tag{3.3}$$

has a principal solution which is positive decreasing on $[1,\infty)$.

Then, the BVP (1.1), (1.3) has at least one solution x if any of the following conditions holds:

(i$_1$)

$$\lim_{T\to\infty}\int\limits_1^T |b(t)|\,\Phi\bigg(\int\limits_t^\infty \Psi\Big(\frac{1}{a(s)}\Big)\,ds\bigg)\,dt<\infty; \tag{3.4}$$

(i$_2$) *There exists $\overline{t}\ge 1$ such that $b_+(t)=0$ for any $t\ge\overline{t}$.*

Moreover, if (i$_1$) *holds, such solution x satisfies*

$$\lim_{t\to\infty}\frac{x(t)}{\int\limits_t^\infty \Psi(a^{-1}(s))\,ds}=\ell, \ \ 0<\ell<\infty. \tag{3.5}$$

Remark. A typical nonlinearity satisfying (1.4) is $F(u)=u^\beta$. A prototype of an half-linear equation (3.2) is the Euler type equation

$$(t^{1+\alpha}\Phi(y'))'+\Big(\frac{1}{1+\alpha}\Big)^{1+\alpha}\Phi(y)=0. \tag{3.6}$$

From [6, Theorem 4.2.4], the function

$$y_0(t)=\Big(\frac{1}{1+\alpha}\Big)^{1/\alpha}t^{-1/(1+\alpha)}$$

is the principal solution of (3.6). Moreover, y_0 is positive decreasing on the interval $[1,\infty)$ and so (3.6) is disconjugate on the same interval. Other examples can be found in [5].

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Resonance Case of Full Separation of Countable Linear Homogeneous Differential System with Coefficients of Oscillating Type

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Let

$$G(\varepsilon_0)=\{t,\varepsilon:\ t\in\mathbf{R},\ \varepsilon\in[0,\varepsilon_0],\ \varepsilon_0\in\mathbf{R}^+\}.$$

Definition 1. We say that the function $p(t,\varepsilon)$ belongs to the class $S(m;\varepsilon_0)$ ($m\in\mathbf{N}\cup\{0\}$) if

1) $p:G(\varepsilon_0)\to\mathbf{C}$;
2) $p(t,\varepsilon)\in C^m(G(\varepsilon_0))$ with respect to t;
3)
$$\frac{d^kp(t,\varepsilon)}{dt^k}=\varepsilon^kp_k^*(t,\varepsilon)\ \ (0\le k\le m)$$
and
$$\|p\|_{S(m,\varepsilon_0)}\stackrel{def}{=}\sum_{k=0}^m\sup_{G(\varepsilon_0)}|p_k^*(t,\varepsilon)|<+\infty.$$

Definition 2. We say that the function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$ ($m\in\mathbf{N}\cup\{0\}$) if

$$f(t,\varepsilon,\theta(t,\varepsilon))=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon)),$$

and

1) $f_n(t,\varepsilon)\in S(m,\varepsilon_0)$ ($n\in\mathbf{Z}$);
2)
$$\|f\|_{F(m;\varepsilon_0,\theta)}\stackrel{def}{=}\sum_{n=-\infty}^{\infty}\|f_n\|_{S(m;\varepsilon_0)}<+\infty;$$
3) $\theta(t,\varepsilon)=\int\limits_0^t\varphi(\tau,\varepsilon)\,d\tau$, $\varphi\in\mathbf{R}^+$, $\varphi\in S(m,\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\varphi(t,\varepsilon)=\varphi_0>0$.

Definition 3. We say that the infinite dimensional $x(t,\varepsilon)=\operatorname{col}(x_1(t,\varepsilon),x_2(t,\varepsilon),\dots)$ belongs to the class $S_1(m;\varepsilon_0)$ if $x_j\in S(m;\varepsilon_0)$ ($j=1,2,\dots$), and

$$\|x\|_{S_1(m;\varepsilon_0)}\stackrel{def}{=}\sup_j\|x_j\|_{S(m;\varepsilon_0)}<+\infty.$$

Definition 4. We say that the infinite dimensional matrix $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=1,2,\dots}$ belongs to the class $S_2(m;\varepsilon_0)$ if $a_{jk}\in S(m;\varepsilon_0)$, and

$$\|A\|_{S_2(m;\varepsilon_0)}\overset{def}{=}\sup_j\sum_{k=1}^{\infty}\|a_{jk}\|_{S(m;\varepsilon_0)}<+\infty.$$

Definition 5. We say that the infinite dimensional vector $x(t,\varepsilon,\theta)=\operatorname{col}(x_1(t,\varepsilon,\theta),x_2(t,\varepsilon,\theta),\dots)$ belongs to the class $F_1(m;\varepsilon_0,\theta)$ if $x_j\in F(m;\varepsilon_0;\theta)$ $(j=1,2,\dots)$, and

$$\|x\|_{F_1(m;\varepsilon_0,\theta)}\overset{def}{=}\sup_j\|x_j\|_{F(m;\varepsilon_0,\theta)}<+\infty.$$

Definition 6. We say that the infinite dimensional matrix $A(t,\varepsilon,\theta)=(a_{jk}(t,\varepsilon,\theta))_{j,k=1,2,\dots}$ belongs to the class $F_2(m;\varepsilon_0,\theta)$ if $a_{jk}\in F(m;\varepsilon_0,\theta)$, and

$$\|A\|_{F_2(m;\varepsilon_0,\theta)}\overset{def}{=}\sup_j\sum_{k=1}^{\infty}\|a_{jk}\|_{F(m;\varepsilon_0,\theta)}<+\infty.$$

Obviously, if $A\in F_2(m;\varepsilon_0;\theta)$, $x\in F_1(m;\varepsilon_0;\theta)$, then $Ax\in F_1(m;\varepsilon_0;\theta)$, and

$$\|Ax\|_{F_1(m;\varepsilon_0;\theta)}\le 2^m\|A\|_{F_2(m;\varepsilon_0;\theta)}\cdot\|x\|_{F_1(m;\varepsilon_0;\theta)}.$$

The condition $\|A\|_{F_2(m;\varepsilon_0;\theta)}<1$ guarantees the existence of a matrix

$$(E+A)^{-1}=E+\sum_{k=1}^{\infty}(-1)^kA^k,$$

where $E=\operatorname{diag}(1,1,\dots)$.

For any vector $x(t,\varepsilon,\theta)\in F_1(m;\varepsilon_0;\theta)$ we denote

$$\Gamma_n[x]=\frac{1}{2\pi}\int\limits_0^{2\pi}x(t,\varepsilon,\theta)\exp(-in\theta)\,d\theta,\quad n\in\mathbf{Z}.$$

For infinite dimensional vectors $x=\operatorname{colon}(x_1,x_2,\dots)$, $y=\operatorname{colon}(y_1,y_2,\dots)$ we denote $[x,y]=\operatorname{colon}(x_1y_1,x_2y_2,\dots)$.

We consider the following countable system of differential equations

$$\frac{dx}{dt}=\Lambda(t,\varepsilon)x+\mu B^{(0)}(t,\varepsilon,\theta)x+\mu^2B(t,\varepsilon,\theta)x, \tag{1}$$

where

$$\begin{gathered}
t,\varepsilon\in G(\varepsilon_0),\quad x=\operatorname{colon}(x_1,x_2,\dots),\\
\Lambda(t,\varepsilon)=\operatorname{diag}\big[\lambda_1(t,\varepsilon),\lambda_2(t,\varepsilon),\dots\big]\in S_2(m;\varepsilon_0),\\
B^{(0)}(t,\varepsilon,\theta)=\operatorname{diag}\big[b_1(t,\varepsilon,\theta),b_2(t,\varepsilon,\theta),\dots\big]\in F_2(m;\varepsilon_0;\theta),\\
B(t,\varepsilon,\theta)=(b_{jk}(t,\varepsilon,\theta))_{j,k=1,2,\dots}\in F_2(m;\varepsilon_0;\theta),\\
b_{jj}(t,\varepsilon,\theta)\equiv 0\ \ (j=1,2,\dots),\quad \mu\in(0,\mu_0)\subset\mathbf{R}^+.
\end{gathered}$$

We suppose

$$\lambda_j(t,\varepsilon)-\lambda_k(t,\varepsilon)=in_{jk}\varphi(t,\varepsilon), \tag{2}$$

$n_{jk} \in \mathbf{Z}$ $(j,k=1,2,\dots)$, $\varphi(t,\varepsilon)$ – function in Definition 2. In this sense we say that we have a resonance case.

We study the problem on the existence of the transformation of kind

$$x = (E + Q(t,\varepsilon,\theta,\mu))y, \tag{3}$$

$y = \operatorname{colon}(y_1, y_2, \dots)$, $Q(t,\varepsilon,\theta,\mu) = (q_{jk}(t,\varepsilon,\theta,\mu))_{j,k=1,2,\dots} \in F_2(m_1;\varepsilon_2;\theta)$ $(m_1 \le m, \varepsilon_1 \le \varepsilon_0)$, $q_{jj}(t,\varepsilon,\theta,\mu) \equiv 0$, which leads the system (4) to kind:

$$\frac{dy}{dt} = D(t,\varepsilon,\theta,\mu)y, \tag{4}$$
$$D(t,\varepsilon,\theta,\mu) = \operatorname{diag}\big[d_1(t,\varepsilon,\theta,\mu), d_2(t,\varepsilon,\theta,\mu), \dots\big] \in F_2(m_1,\varepsilon_1;\theta).$$

We consider the auxiliary countable system of differential equations

$$\frac{dz}{dt} = i\varphi(t,\varepsilon)\Lambda_1 z + \mu U(t,\varepsilon,\theta)z + g(t,\varepsilon,\theta) + \mu^2 C(t,\varepsilon,\theta)z + \mu^4[z, R(t,\varepsilon,\theta)z], \tag{5}$$

where

$$t,\varepsilon \in G(\varepsilon_0), \quad z = \operatorname{colon}(z_1, z_2, \dots), \quad \Lambda_1 = \operatorname{diag}[n_1, n_2, \dots], \quad n_j \in \mathbf{Z} \ \ (j=1,2,\dots),$$
$$U = \operatorname{diag}\big[u_1(t,\varepsilon,\theta), u_2(t,\varepsilon,\theta), \dots\big] \in F_2(m;\varepsilon_0;\theta),$$
$$g = \operatorname{colon}\big(g_1(t,\varepsilon,\theta), g_2(t,\varepsilon,\theta), \dots\big) \in F_1(m;\varepsilon_0;\theta),$$
$$C = (c_{jk}(t,\varepsilon,\theta))_{j,k=1,2,\dots} \in F_2(m;\varepsilon_0;\theta), \quad c_{jj} \equiv 0 \ \ (j=1,2,\dots),$$
$$R \in F_2(m;\varepsilon_0;\theta), \quad \mu \in (0,\mu_0) \subset \mathbf{R}^+.$$

Lemma 1. *Let the system* (5) *satisfy the next conditions:*

1) $\forall\, t,\varepsilon \in G(\varepsilon_0)$:
$$\int\limits_0^{2\pi} g_j(t,\varepsilon,\theta)\exp(-in_j\theta)\,d\theta = 0, \quad j=1,2,\dots;$$

2)
$$\inf_{G(\varepsilon_0)} \bigg| \int\limits_0^{2\pi} u_j(t,\varepsilon,\theta)\,d\theta \bigg| \ge \gamma > 0, \quad j=1,2,\dots.$$

Then there exists $\mu_1 \in (0,\mu_0)$ such that $\forall\,\mu \in (0,\mu_1)$ and $\forall\, q \in \mathbf{N}$ there exists the transformation of kind

$$z = \sum_{s=0}^{2q-1} \xi^{(s)}(t,\varepsilon,\theta)\mu^s + \Phi(t,\varepsilon,\theta,\mu)z^{(1)}, \tag{6}$$

$\xi^{(s)} \in F_1(m;\varepsilon_0;\theta)$, $\Phi \in F_2(m;\varepsilon_0;\theta)$, *which leads the system* (6) *to kind:*

$$\begin{aligned}\frac{dz^{(1)}}{dt} = \Big(\sum_{l=1}^{q} K^{(l)}(t,\varepsilon)\mu^l\Big)z^{(1)} + \varepsilon h^{(11)}(t,\varepsilon,\theta,\mu) + \mu^{2q}h^{(12)}(t,\varepsilon,\theta,\mu)\\ + \varepsilon V^{(1)}(t,\varepsilon,\theta,\mu)z^{(1)} + \mu^{q+1}P^{(1)}(t,\varepsilon,\theta,\mu)z^{(1)}\\ + \mu\big[R^{(11)}(t,\varepsilon,\theta,\mu)z^{(1)}, R^{(12)}(t,\varepsilon,\theta,\mu)z^{(1)}\big],\end{aligned}$$

where $K^{(l)} \in S_2(m;\varepsilon_0)$, and $\forall\,\mu \in (0,\mu_1)$; $h^{(11)}, h^{(12)} \in F_1(m-1;\varepsilon_0;\theta)$, $V^{(1)}, P^{(1)}, R^{(11)}, R^{(12)} \in F_2(m-1;\varepsilon_0;\theta)$.

We consider the countable linear homogeneous system of differential equations:

$$\frac{dx^{(0)}}{dt} = A(t,\varepsilon)x^{(0)}, \tag{7}$$

where $A(t,\varepsilon) \in S_2(m;\varepsilon_0)$.

Definition 7. The Green-matrix of the system (7) is the matrix $G(t,\tau,\varepsilon) = (g_{jk}(t,\tau,\varepsilon))_{j,k=1,2,\dots}$, such that

1) if $t \neq \tau$:
$$\frac{\partial G(t,\tau,\varepsilon)}{\partial t} = A(t,\varepsilon)G(t,\tau,\varepsilon), \quad \frac{\partial G(t,\tau,\varepsilon)}{\partial \tau} = -G(t,\tau,\varepsilon)A(\tau,\varepsilon);$$

2)
$$G(\tau+0,\tau,\varepsilon) - G(\tau-0,\tau,\varepsilon) = E, \quad G(t,t+0,\varepsilon) - G(t,t-0,\varepsilon) = -E.$$

If $t = \tau$, then Green-matrix is not defined.

Along with the system (7) consider the countable linear inhomogeneous system:

$$\frac{dx}{dt} = A(t,\varepsilon)x + f(t,\varepsilon,\theta), \tag{8}$$

where $f \in F_1(m;\varepsilon_0;\theta)$, matrix $A(t,\varepsilon)$ is the same as in the system (7).

Lemma 2. *Let the system* (7) *have the Green-matrix* $G(t,\tau,\varepsilon) = (g_{jk}(t,\tau,\varepsilon))_{j,k=1,2,\dots}$ *such that*

$$|g_{jk}(t,\tau,\varepsilon)| \leq M_0 \exp\big(-\gamma_0|t-\tau|\big),$$

where $M_0,\gamma_0 \in (0,+\infty)$*, and* M_0,γ_0 *do not depend on* t, τ, ε*. Then the system* (8) *has a unique particular solution* $x(t,\varepsilon,\theta) \in F_1(m;\varepsilon_0;\theta)$*, and there exists* $K_0 \in (0,+\infty)$ *such that*

$$\|x(t,\varepsilon,\theta)\|_{F_1(m;\varepsilon_0;\theta)} \leq \frac{K_0}{\gamma_0}\,\|f(t,\varepsilon,\theta)\|_{F_1(m;\varepsilon_0;\theta)}.$$

Lemma 3. *Let the system* (5) *be such that*

1) *the conditions of Lemma* 1 *hold;*

2) *for the linear homogeneous system*

$$\frac{dx}{dt} = \Big(\sum_{l=1}^{q} K^{(l)}(t,\varepsilon)\mu^l\Big)x,$$

where matrices $K^{(l)}(t,\varepsilon)$ *are defined by Lemma* 1*, there exists the Green-matrix* $G(t,\tau,\varepsilon,\mu) = (g_{jk}(t,\tau,\varepsilon,\mu))_{j,k=1,2,\dots}$ *such that*

$$|g_{jk}(t,\tau,\varepsilon,\mu)| \leq M_1 \exp\big(-\gamma_1\mu^{q_0}|t-\tau|\big),$$

$q_0 \in [1,q]$, $M_1,\gamma_1 \in (0,+\infty)$ *and do not depend on* t, τ, ε, μ.

Then there exist $\mu_2 \in (0,\mu_0)$, $\varepsilon_2(\mu) \in (0,\varepsilon_0)$ *such that* $\forall\,\mu \in (0,\mu_2)$, $\varepsilon \in (0,\varepsilon_2(\mu))$ *the system* (5) *has a particular solution, belonging to the class* $F_1(m-1;\varepsilon_2(\mu);\theta)$.

Now we return to the system (1) and make in it substitution (3). Taking into account the condition of diagonality of transformed system (4) and condition (2), we obtain the next countable system of differential equations for the elements q_{jk} $(j \neq k)$ of matrix Q:

$$\frac{dq_{jk}}{dt} = in_{jk}\varphi(t,\varepsilon)q_{jk} + \mu\big(b_j(t,\varepsilon,\theta) - b_k(t,\varepsilon,\theta)\big)q_{jk} + \mu^2 b_{jk}(t,\varepsilon,\theta) + \mu^2 \sum_{\substack{s=1\\(s\neq j,\, s\neq k)}}^{\infty} b_{js}(t,\varepsilon,\theta)q_{sk} - \mu^2 q_{jk} \sum_{\substack{s=1\\(s\neq k)}}^{\infty} b_{ks}(t,\varepsilon,\theta)q_{sk}, \quad j,k = 1,2,\dots; \ \ j \neq k. \tag{9}$$

The elements of the diagonal matrix D in system (4) are defined by formulas:

$$d_j(t,\varepsilon,\theta,\mu) = \lambda_j(t,\varepsilon) + \mu b_j(t,\varepsilon,\theta) + \mu \sum_{\substack{s=1\\(s\neq j)}}^{\infty} b_{js}(t,\varepsilon,\theta)q_{sj}(t,\varepsilon,\theta,\mu). \tag{10}$$

The substitution

$$q_{jk} = \mu^2 \widetilde{q}_{jk}, \quad j,k = 1,2,\dots; \ \ j \neq k$$

leads the system (9) to kind:

$$\frac{d\widetilde{q}_{jk}}{dt} = in_{jk}\varphi(t,\varepsilon)\widetilde{q}_{jk} + \mu\big(b_j(t,\varepsilon,\theta) - b_k(t,\varepsilon,\theta)\big)\widetilde{q}_{jk} + b_{jk}(t,\varepsilon,\theta) + \mu^2 \sum_{\substack{s=1\\(s\neq j,\, s\neq k)}}^{\infty} b_{js}(t,\varepsilon,\theta)\widetilde{q}_{sk} - \mu^4 \widetilde{q}_{jk} \sum_{\substack{s=1\\(s\neq k)}}^{\infty} b_{ks}(t,\varepsilon,\theta)\widetilde{q}_{sk}, \quad j,k = 1,2,\dots; \ \ j \neq k. \tag{11}$$

In the system (11) index k is fixed, then for any $k = 1,2,\dots$ system (11) is the separate countable system of the differential equations for $\widetilde{q}_{1k}, \widetilde{q}_{2k}, \dots, \widetilde{q}_{k-1,k}, \widetilde{q}_{k+1,k}, \dots$. It is not difficult to see that vector-form of such system has a kind (5). Then we can prove the validity of the next theorem.

Theorem. *Let for the system* (1) *hold* (2), *and for all* $k = 1,2,\dots$ *the system* (11) *satisfy all the conditions of Lemma* 3. *Then there exist* $\mu_3 \in (0,\mu_0)$, $\varepsilon_3(\mu) \in (0,\varepsilon_0)$ *such that* $\forall\, \mu \in (0,\mu_3)$, $\varepsilon \in (0,\varepsilon_3(\mu))$ *there exists the transformation of kind* (3), *where* $Q(t,\varepsilon,\theta,\mu) \in F_2(m-1;\varepsilon_3(\mu);\theta)$, *which leads the system* (1) *to kind* (4), *where the elements of diagonal matrix* $D(t,\varepsilon,\theta,\mu) \in F_2(m-1;\varepsilon_3(\mu);\theta)$ *are defined by formulas* (10).

Asymptotic Behaviour of Solutions of Third-Order Differential Equations with Rapid Varying Nonlinearities

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We consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0,+\infty[$ is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ for } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{or} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 equals either zero or $\pm\infty$, Δ_{Y_0} – some one-sided neighborhood of Y_0.

From identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} + 1 \ \text{ for } \ y \in \Delta_{Y_0}$$

and conditions (2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)}, \ \ y \to Y_0 \ \ (y \in \Delta_{Y_0}), \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

It means that in the considered equation the continuous function φ and its first order derivatives are [5, Chapter 3, Section 3.4, Lemmas 3.2, 3.3, pp. 91–92] rapidly change at $y \to Y_0$.

For two-term differential equations of the second order with nonlinearities satisfying condition (2), the asymptotic properties of solutions were studied in the works of M. Maric [5], V. M. Evtukhov and his students N. G. Drik, V. M. Kharkov, A. G. Chernikova [1–3].

In the works of V. M. Evtukhov, A. G. Chernikova [1] for the differential equation (1) of the second order in the case when φ satisfies condition (2), the asymptotic properties of so-called $P_\omega(Y_0,\lambda_0)$-solutions were studied with $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$. In this work, we propose the distribution of [1] results to third-order differential equations.

Solution y of the differential equation (1) specified on the interval $[t_0,\omega[\subset[a,\omega[$ calls $P_\omega(Y_0,\lambda_0)$-solution, if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y(t) = Y_0, \quad \lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} k = 1,2, \quad \lim_{t\uparrow\omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The goal of this work is to establish the necessary and sufficient conditions for the existence for the equation (1) $P_\omega(Y_0,\lambda_0)$-solutions in the non-singular case, when $\lambda_0 \in \mathbb{R}\setminus\{0,1,\frac{1}{2}\}$, as well as asymptotic for $t\uparrow\omega$ representations for such solutions and their derivatives up to the second order inclusively.

Without loss of generality, we will further assume that

$$\Delta_{Y_0}=\begin{cases}[y_0,Y_0[\,, & \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ ]Y_0,y_0], & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0,\end{cases} \tag{3}$$

where $y_0\in\mathbb{R}$ is such that $|y_0|<1$, when $Y_0=0$ and $y_0>1$ ($y_0<-1$), when $Y_0=+\infty$ (when $Y_0=-\infty$).

A function $\varphi:\Delta_{Y_0}\to\mathbb{R}\setminus\{0\}$, satisfying condition (2), belongs to the class $\Gamma_{Y_0}(Z_0)$, that was introduced in the work [1], which extends the class of function Γ, introduced by L. Khan (see, for example, [4, Chapter 3, Section 3.10, p. 175]). Using properties from this class, main results below are obtained.

We introduce the necessary auxiliary notation. We assume that the domain of the function $\varphi\in\Gamma_{Y_0}(Z_0)$ is determined by formula (3). Next, we set

$$\mu_0=\operatorname{sgn}\varphi'(y),\quad \nu_0=\operatorname{sgn}y_0,\quad \nu_1=\begin{cases}1, & \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1, & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the following functions

$$\pi_\omega(t)=\begin{cases}t, & \text{if } \omega=+\infty,\\ t-\omega, & \text{if } \omega<+\infty,\end{cases}\quad J(t)=\int_A^t\pi_\omega^2(\tau)p(\tau)\,d\tau,\quad \Phi(y)=\int_B^y\frac{ds}{\varphi(s)}\,,$$

where

$$A=\begin{cases}\omega, & \text{if } \displaystyle\int_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=const,\\ a, & \text{if } \displaystyle\int_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=\pm\infty,\end{cases}\qquad B=\begin{cases}Y_0, & \text{if } \displaystyle\int_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=const,\\ y_0, & \text{if } \displaystyle\int_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=\pm\infty.\end{cases}$$

Considering the definition of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1), we note that the numbers ν_0, ν_1 determine the signs of any $P_\omega(Y_0,\lambda_0)$-solution and of its first derivatives in some left neighborhood of ω. It is clear that the condition

$$\nu_0\nu_1<0 \ \text{ if } \ Y_0=0,\quad \nu_0\nu_1>0 \ \text{ if } \ Y_0=\pm\infty,$$

is necessary for the existence of such solutions.

Now we turn our attention to some properties of the function Φ. It retains a sign on the interval Δ_{Y_0}, tends either to zero or to $\pm\infty$, when $y\to Y_0$ and increasing by Δ_{Y_0}, because on this interval $\Phi'(y)=\frac{1}{\varphi(y)}>0$. Therefore, for it there is an inverse function $\Phi^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where due to the second of conditions (2) and the monotone increase of Φ^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases}\text{or} & 0,\\ \text{or} & +\infty,\end{cases}\qquad \Delta_{Z_0}=\begin{cases}[z_0,Z_0[\,, & \text{or } \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0], & \text{or } \Delta_{Y_0}=]Y_0,y_0],\end{cases}\qquad z_0=\varphi(y_0).$$

For $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac12\}$ with using Φ^{-1} we also introduce the auxiliary functions

$$q(t)=\frac{\alpha_0(\lambda_0-1)^2\pi_\omega^3(t)p(t)\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}(\lambda_0-1)J(t))\big)}{\lambda_0\Phi^{-1}\big(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t)\big)}\,,$$

$$H(t)=\frac{\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\varphi'\big(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\big)}{\varphi(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t)))}\,.$$

For equation (1) the following assertions are valid.

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0;1;\frac{1}{2}\}$. Then for the existence for the differential equation* (1), *$P_\omega(Y_0,\lambda_0)$-solutions, it is necessary to comply with the conditions*

$$\alpha_0\nu_1\lambda_0 > 0,$$
$$\nu_0\nu_1(2\lambda_0-1)(\lambda_0)\pi_\omega(t) > 0, \quad \alpha_0\mu_0\lambda_0 J(t) < 0 \ \text{ for } \ t \in (a,\omega),$$
$$\frac{\alpha_0}{\lambda_0}\lim_{t\uparrow\omega} J(t) = Z_0, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'(t)}{J(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} q(t) = \frac{2\lambda_0-1}{\lambda_0-1}.$$

Moreover, for each such solution, the asymptotic representations

$$y(t) = \Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big] \ \text{ for } \ t\uparrow\omega,$$
$$y'(t) = \frac{(2\lambda_0-1)}{(\lambda_0-1)}\,\frac{\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))}{\pi_\omega(t)}\,[1+o(1)] \ \text{ for } \ t\uparrow\omega,$$
$$y''(t) = \frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2}\,\frac{\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))}{\pi_\omega^2(t)}\,[1+o(1)] \ \text{ for } \ t\uparrow\omega$$

take place.

Theorem 2. *Let $\lambda_0 \in \mathbb{R}\setminus\{0;1;\frac{1}{2}\}$, there exist a finite or equal to $\pm\infty$ limit*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2}\,\sqrt[3]{\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^2}$$

and

$$\lim_{t\uparrow\omega}\Big[\frac{2\lambda_0-1}{\lambda_0-1} - q(t)\Big]|H(t)|^{\frac{2}{3}} = 0.$$

Then, the differential equation (1) *has at least one $P_\omega(Y_0,\lambda_0)$-solution which allows for $t\uparrow\omega$ the asymptotic representation*

$$y(t) = \Phi^{-1}\Big(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big],$$
$$y'(t) = \frac{2\lambda_0-1}{(\lambda_0-1)\pi_\omega(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\big[1+o(1)H^{-\frac{2}{3}}\big],$$
$$y''(t) = \frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)\pi_\omega^2(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\big[1+o(1)H^{-\frac{1}{3}}\big].$$

Moreover, in the case when $\mu_0\lambda_0(2\lambda_0-1)(\lambda_0-1) < 0$ there exists one-parameter family, but in the case $\mu_0\lambda_0(2\lambda_0-1)(\lambda_0-1) > 0$ there exists a two-parameter family.

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On the Problem on Minimization of the Functional Generated by a Sturm–Liouville Problem

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1 Introduction

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \ \ x \in (0,1), \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all real-valued locally integrable on $(0,1)$ functions with non-negative values such that the following integral condition holds

$$\int\limits_0^1 x^{\alpha}(1-x)^{\beta} Q^{\gamma}(x)\,dx = 1, \ \ \alpha,\beta,\gamma \in \mathbb{R}, \ \ \gamma \neq 0. \tag{1.3}$$

A function y is a *solution to problem* $(1.1),(1.2)$ if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0<\rho<\frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

For any function $Q \in T_{\alpha,\beta,\gamma}$ by H_Q we denote the closure of the set $C_0^{\infty}(0,1)$ with respect to the norm

$$\|y\|_{H_Q} = \bigg(\int\limits_0^1 {y'}^2 dx + \int\limits_0^1 Q(x) y^2\,dx \bigg)^{\frac{1}{2}}.$$

We consider the functional generated by problem $(1.1),(1.2)$

$$R[Q,y] = \frac{\int_0^1 {y'}^2\,dx - \int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}\,.$$

We give estimates for

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \ \inf_{y \in H_Q \setminus \{0\}} R[Q,y], \quad M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \ \inf_{y \in H_Q \setminus \{0\}} R[Q,y].$$

Remark 1.1. This work is the continuation of the study of estimates for the first eigenvalue of Sturm–Liouville problems with integral conditions on the potential, which was initiated by Y. V. Egorov and V. A. Kondratiev [1]. The history of the research can be found in [2].

2 Main results

2.1 On precise estimates for $M_{\alpha,\beta,\gamma}$ as $\gamma < -1$, $\alpha, \beta > 2\gamma - 1$

It is proved [3] that $M_{\alpha,\beta,\gamma} \leqslant \pi^2$ for all α, β, γ, $\gamma \neq 0$, and $M_{\alpha,\beta,\gamma} < \pi^2$ as $\gamma < 0$, $\alpha, \beta > 3\gamma - 1$.

In case of $\gamma < 0$, using the Hölder inequality for any functions $Q \in T_{\alpha,\beta,\gamma}$ and $y \in H_Q$, we obtain

$$\int\limits_0^1 Q(x)y^2\,dx \geqslant \Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\Big)^{\frac{\gamma-1}{\gamma}}$$

and

$$\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant \inf_{y\in H_Q\setminus\{0\}} G[y],$$

where

$$G[y] = \frac{\int\limits_0^1 y'^2\,dx - \big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\big)^{\frac{\gamma-1}{\gamma}}}{\int\limits_0^1 y^2\,dx}\,.$$

Consider the functional G in $H_0^1(0,1)$. It is proved [4] that for $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$ the functional G is bounded below in $H_0^1(0,1)$ and there exists

$$m = \inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y].$$

Similarly to [4] we prove that for $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$ any minimizing sequence of the functional G in $H_0^1(0,1)$ converges to some function $u \in H_0^1(0,1)$ and

$$\inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y] = G[u] = m.$$

As in the case of $\alpha = \beta = 0$ [2] we prove that function u is positive on $(0,1)$.

For $0 < \varepsilon < \frac{1}{3}$, we consider the function

$$v(x) = \begin{cases} 0, & x \in [0,\varepsilon] \cup [1-\varepsilon, 1], \\ u, & x \in (\varepsilon, 1-\varepsilon) \end{cases}$$

and its averaging v_ρ with $\rho = \frac{\varepsilon}{2}$ (see, for example, [5, I, § 1]). Then for any function $Q \in T_{\alpha,\beta,\gamma}$, we obtain

$$\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant \inf_{y\in H_Q\setminus\{0\}} G[y] \leqslant \lim_{\rho\to 0} G[v_\rho] = G[u] = m$$

and

$$M_{\alpha,\beta,\gamma} = \sup_{Q\in T_{\alpha,\beta,\gamma}} \inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant m.$$

On $(0,1)$ we consider the function $Q_*(x) = x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2}{\gamma-1}}$ which satisfies the integral condition (1.3) and $u \in H_{Q_*}$. Since the function u is the first eigenfunction for problem (1.1)–(1.3) for $Q = Q_*$ and the first eigenvalue $\lambda_1(Q_*) = m$, then

$$\inf_{y\in H_{Q*}\setminus\{0\}} R[Q_*,y] = R[Q_*,u] = m.$$

Therefore, $M_{\alpha,\beta,\gamma} \geqslant m$. Hence, the following theorem holds.

Theorem 2.1. *If $\gamma < -1$, $\alpha, \beta > 2\gamma - 1$ and $m = \inf\limits_{y \in H_0^1(0,1)\setminus\{0\}} G[y]$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_{Q_*}$, $u > 0$ on $(0,1)$, such that $M_{\alpha,\beta,\gamma} = R[Q_*, u]$, moreover, u satisfies the equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{\gamma+1}{\gamma-1}} \tag{2.1}$$

and the integral condition

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{2\gamma}{\gamma-1}}\, dx = 1. \tag{2.2}$$

2.2 On estimates for $M_{\alpha,\beta,\gamma}$ as $\gamma > 0$

Theorem 2.2.

- *If $\gamma > 1$, then $M_{\alpha,\beta,\gamma} = \pi^2$.*
- *If $0 < \gamma \leqslant 1$, $\alpha \leqslant 2\gamma - 1$, $-\infty < \beta < +\infty$ or $\beta \leqslant 2\gamma - 1$, $-\infty < \alpha < +\infty$, then $M_{\alpha,\beta,\gamma} = \pi^2$.*
- *If $0 < \gamma < 1$, $\alpha, \beta > 3\gamma - 1$, then $M_{\alpha,\beta,\gamma} < \pi^2$.*
- *If $0 < \gamma < 1/2$, $\alpha, \beta \geqslant 0$, then $M_{\alpha,\beta,\gamma} < \pi^2$.*
- *If $1/2 \leqslant \gamma < 1$, $2\gamma - 1 < \alpha, \beta \leqslant 3\gamma - 1$, then $M_{\alpha,\beta,\gamma} < \pi^2$.*

Remark 2.1. The result $M_{0,0,\gamma} < \pi^2$ as $0 < \gamma < 1/2$ was obtained in [6].

Remark 2.2. We can give some lower bounds for $M_{\alpha,\beta,\gamma}$ in cases of $\gamma < 0$ or $0 < \gamma < 1$:

- If $\gamma < 0$, $\alpha, \beta \geqslant 0$, then $M_{\alpha,\beta,\gamma} \geqslant \pi^2 - 1$.
- If $\gamma < 0$, $2\gamma - 1 < \alpha < 0 \leqslant \beta$, then $M_{\alpha,\beta,\gamma} \geqslant (1 - 4(\alpha - 2\gamma + 1)^{\frac{1}{\gamma}})\pi^2$.
- If $\gamma < 0$, $2\gamma - 1 < \beta < 0 \leqslant \alpha$, then $M_{\alpha,\beta,\gamma} \geqslant (1 - 4(\beta - 2\gamma + 1)^{\frac{1}{\gamma}})\pi^2$.
- If $\gamma < 0$, $2\gamma - 1 < \alpha, \beta < 0$, then $M_{\alpha,\beta,\gamma} \geqslant (1 + \theta^{\frac{1}{\gamma}} \cdot 2^{\frac{\theta + 4\gamma - 2}{\gamma}})\pi^2$, $\theta = \min\{\alpha, \beta\} - 2\gamma + 1$.

2.3 Some estimates for $m_{\alpha,\beta,\gamma}$ below

Theorem 2.3.

- *If $\gamma < 0$ or $0 < \gamma < 1$, then $m_{\alpha,\beta,\gamma} = -\infty$.*
- *If $\gamma = 1$ and $\alpha, \beta \leqslant 0$, then $m_{\alpha,\beta,\gamma} \geqslant \frac{3}{4}\,\pi^2$.*
- *If $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$ or $\alpha \leqslant 0 < \beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*
- *If $\gamma = 1$, $0 < \alpha, \beta \leqslant 1$, then $-\pi^2 \leqslant m_{\alpha,\beta,\gamma} \leqslant \pi^2$.*
- *If $\gamma > 1$ and $0 < \alpha$, $\beta \leqslant 2\gamma - 1$, then*

$$m_{\alpha,\beta,\gamma} \geqslant \Big(1 - 2^{\frac{3\gamma-2}{\gamma}} \Big(\frac{2\gamma - 1}{\gamma}\Big)^{\frac{2\gamma-1}{\gamma}}\Big)\pi^2.$$

- *If* $\gamma > 1$ *and* $\beta \leqslant 0 < \alpha \leqslant 2\gamma - 1$ *or* $\alpha \leqslant 0 < \beta \leqslant 2\gamma - 1$*, then*

$$m_{\alpha,\beta,\gamma} \geqslant \Big(1 - \Big(\frac{2\gamma - 1}{\gamma}\Big)^{\frac{2\gamma-1}{\gamma}}\Big)\pi^2.$$

- *If* $\gamma > 1$ *and* $\alpha, \beta \leqslant 0$*, then* $m_{\alpha,\beta,\gamma} \geqslant 0$*.*

Theorem 2.4. *If* $\gamma > 1$ *and* $\alpha, \beta < 2\gamma - 1$*, then there exist functions* $Q_* \in T_{\alpha,\beta,\gamma}$ *and* $u \in H_{Q_*}$*,* $u > 0$ *on* $(0, 1)$*, such that* $m_{\alpha,\beta,\gamma} = R[Q_*, u] = m$*, moreover,* u *satisfies equation* (2.1) *and the integral condition* (2.2)*.*

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Stability Analysis of Invariant Tori of Nonlinear Extensions of Dynamical Systems on Torus Using Quadratic Forms

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1 Introduction and preliminaries

A set of fundamental results of the mathematical theory of multifrequency oscillations have been developed by A. M. Samoilenko and summarized in [11]. In particular, these studies include the problems of the existence and stability of invariant manifolds of dynamical systems defined in the direct product of m-dimensional torus $\mathcal{T}_m$ and n-dimensional Euclidean space $\mathbb{R}^n$. In [5], the stability properties of invariant tori have been studied in terms of sign-definite quadratic forms. In this paper, we establish less restrictive (compared to [5]) conditions for exponential stability and instability of the trivial invariant torus of nonlinear extension of dynamical system on torus which are formulated in terms of quadratic forms that are sign-definite in nonwandering set Ω of dynamical system on torus and allowed to be sign-indefinite in $\mathcal{T}_m \setminus \Omega$. For further details we refer a reader to the extended version of this contribution [2]. The corresponding results for linear extensions of dynamical systems on torus have been obtained in [1, 3, 7–10].

We consider the following system defined in $\mathcal{T}_m \times \mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi, x)x, \tag{1.1}$$

where $\varphi = (\varphi_1, \dots, \varphi_m)^\top \in \mathcal{T}_m$, $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$, function P is continuous in $\mathcal{T}_m \times \mathbb{R}^n$ and for every $x \in \mathbb{R}^n$ $P(\,\cdot\,, x), a(\,\cdot\,) \in C(\mathcal{T}_m)$; $C(\mathcal{T}_m)$ is a space of continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ functions defined on $\mathcal{T}_m$. We assume that the following conditions hold:

$$\exists\, M > 0 \ \text{ such that } \ \forall\, (\varphi, x) \in \mathcal{T}_m \times \mathbb{R}^n \quad \|P(\varphi, x)\| \le M; \tag{1.2}$$

$$\forall\, r > 0 \ \ \exists\, L = L(r) > 0 \ \text{ such that } \ \forall\, x', x'', \ \|x'\| \le r, \ \|x''\| \le r, \ \forall\, \varphi \in \mathcal{T}_m$$
$$\big\|P(\varphi, x'') - P(\varphi, x')\big\| \le L\|x'' - x'\|; \tag{1.3}$$

$$\exists\, A > 0 \ \ \forall\, \varphi', \varphi'' \in \mathcal{T}_m \ \ \|a(\varphi'') - a(\varphi')\| \le A\|\varphi'' - \varphi'\|. \tag{1.4}$$

Condition (1.4) guarantees that the system

$$\frac{d\varphi}{dt} = a(\varphi) \tag{1.5}$$

generates a dynamical system on $\mathcal{T}_m$, which will be denoted by $\varphi_t(\varphi)$.

Definition 1.1 ([6])**.** A point $\varphi \in \mathcal{T}_m$ is called a nonwandering point of dynamical system (1.5) if there exist a neighbourhood $U(\varphi)$ and a moment of time $T = T(\varphi) > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \varnothing \ \ \forall\, t \geq T.$$

Let us denote by Ω a set of all nonwandering points of (1.5). Since $\mathcal{T}_m$ is a compact set, the set Ω is nonempty, invariant, and compact subset of $\mathcal{T}_m$ [11]. Additionally, the following holds:

Lemma 1.1 ([6])**.** *For any $\varepsilon > 0$ there exist $T(\varepsilon) > 0$ and $N(\varepsilon) > 0$ such that for any $\varphi \notin \Omega$ the corresponding trajectory $\varphi_t(\varphi)$ spends only a finite time that is bounded by $T(\varepsilon)$ outside the ε-neighbourhood of the set Ω, and leaves this set not more than $N(\varepsilon)$ times.*

Definition 1.2 ([11])**.** Trivial invariant torus $x = 0$, $\varphi \in \mathcal{T}_m$ of the system (1.1) is called exponentially stable if there exist constants $K > 0$, $\gamma > 0$, and $\delta > 0$ such that for all $\varphi \in \mathcal{T}_m$ and for all $x^0 \in \mathbb{R}^n$, $\|x^0\| \leq \delta$ it holds that

$$\forall\, t \geq 0 \ \ \|x(t,\varphi,x^0)\| \leq K\|x^0\|e^{-\gamma t},$$

where $x(t,\varphi,x^0)$ is a solution to the Cauchy problem

$$\frac{dx}{dt} = P(\varphi_t(\varphi),x)x, \quad x(0) = x^0.$$

In [4], the conditions for the exponential stability of the trivial invariant torus of the system (1.1) have been established in terms of the properties of function $\varphi \mapsto P(\varphi,0)$ in the nonwandering set Ω of dynamical system (1.5):

Lemma 1.2 ([4])**.** *Let*

$$\forall\, \varphi \in \Omega \ \ \lambda(\varphi,0) < 0, \tag{1.6}$$

where $\lambda(\varphi,x)$ is the largest eigenvalue of the matrix $\widehat{P}(\varphi,x) = \frac{1}{2}\,(P(\varphi,x) + P^T(\varphi,x))$. Then the trivial invariant torus of system (1.1) *is exponentially stable.*

The following example demonstrates the case when the trivial invariant torus is exponentially stable (this will be proven in Theorem 2.1), however the condition (1.6) does not hold.

Example 1.1. Consider a system defined in $\mathcal{T}_1 \times \mathbb{R}^2$

$$\frac{d\varphi}{dt} = -\sin^2\Big(\frac{\varphi}{2}\Big), \quad \begin{pmatrix} \dfrac{dx_1}{dt} \\[2mm] \dfrac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} \sin(\varphi + x_1 + x_2)x_1 & -x_2 \\ x_1 & -\sin(x_1 - x_2 - \varphi)x_2 \end{pmatrix}. \tag{1.7}$$

Dynamical system on torus $\mathcal{T}_1$ that are generated by (1.7) has a nonwandering set $\Omega = \{\varphi = 0\}$. However, the matrix $\widehat{P}(0,\overline{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does not satisfy condition (1.6).

In the following section, we prove new sufficient conditions that allow concluding exponential stability of trivial invariant torus of system (1.7).

2 Main results

For any $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$ let us denote

$$\widehat{S}(\varphi,x) = \frac{\partial S(\varphi,x)}{\partial \varphi}\, a(\varphi) + \frac{\partial S(\varphi,x)}{\partial x}\,(P(\varphi,x)x) + S(\varphi,x)P(\varphi,x) + P^T(\varphi,x)S(\varphi,x), \qquad (2.1)$$

where $S = S(\varphi,x)$ is a symmetric matrix of a class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$.

Theorem 2.1. *Let there exist a symmetric matrix $S = S(\varphi,x) \in C^1(\mathcal{T}_m \times \mathbb{R}^n)$ such that*

$$\forall\, \varphi \in \Omega \ \ S(\varphi,0) > 0, \ \ \widehat{S}(\varphi,0) < 0.$$

Then the trivial torus of system (1.1) *is exponentially stable.*

Example 2.1 (revisited)**.** Let us illustrate the usage of Theorem 2.1 for system (1.7). Let $S = S(\varphi,x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} > 0$. Then, $\widehat{S}(0,\overline{0}) = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} < 0$ which guarantees the exponential stability of the trivial invariant torus.

The following theorem provides sufficient conditions for instability of the trivial torus of system (1.1) in terms of sign-definite on the set Ω quadratic forms.

Theorem 2.2. *Let there exist a symmetric matrix $S = S(\varphi,x)$ of the class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$ such that for the matrix* (2.1) *and for the quadratic form $V(\varphi,x) = (S(\varphi,x)x,x)$ the following conditions hold:*

$$\forall\, \varphi \in \Omega \ \ \widehat{S}(\varphi,0) > 0,$$
$$\forall\, \delta > 0 \ \ \exists\, x_0 \in \mathbb{R}^n, \ \ \|x_0\| < \delta, \ \ \exists\, \varphi_0 \in \Omega \ \ \text{such that} \ \ V(\varphi_0,x_0) > 0.$$

Then the trivial torus of system (1.1) *is unstable.*

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Dulac–Cherkas Method for Detecting Exact Number of Limit Cycles for Planar Autonomous Systems

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We consider the autonomous system of differential equations on the real plane

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \ \ (x,y) \in \Omega \subset R^2, \ \ P(x,y), Q(x,y) \in C^1(\Omega). \tag{1}$$

The Dulac criterion [1, p. 226], [8,9] is one of the ways to obtain nonlocal solution of the problem of counting and localizing the limit cycles [7] of system (1). However, there is no regular methods for finding a connected domain Ω of localization of the limit cycles and for constructing the Dulac function in this domain. Therefore, this criterion was predominantly used for proving the absence of limit cycles in a simply-connected domain Ω or the existence of at most one limit cycle in a doubly connected domain Ω. L. A. Cherkas [2] suggested to develop the Dulac criterion and to construct a special Dulac function in a connected domain Ω where the number and localization of limit cycles can be determined by using transversal curves that correspond to such function. This criterion is referred to as the Dulac–Cherkas criterion and allows one to derive an upper bound for the number of limit cycles for many classes of systems (1) [3,4]. Additional research is needed to produce an exact estimate for the number of limit cycles but it is possible only in separate cases. Thus, our aim here is to present approaches developed by us to obtaining an exact nonlocal estimate for the number of limit cycles that surround one equilibrium point of system (1) and localizing these cycles. The Dulac or Dulac–Cherkas methods are applied sequentially two times to find closed transversal curves that divide the domain Ω in doubly connected subdomains surrounding the equilibrium point such that the system (1) has exactly one limit cycle in each of them.

The Dulac–Cherkas method as a generalization of the Dulac criterion consists in finding the Dulac–Cherkas function $\Psi(x,y)$ [3, p. 199].

Definition 1. A function $\Psi \in C^1(\Omega, R)$ is called as the Dulac–Cherkas function of system (1) in a domain Ω if there exists such a real number $k \neq 0$ that the following condition holds

$$\Phi(x,y) = k\Psi \operatorname{div} X + \frac{\partial \Psi}{\partial x}\, P + \frac{\partial \Psi}{\partial y}\, Q \geqslant 0 \ \ (\leqslant 0), \ \ \forall\,(x,y) \in \Omega \subset R^2, \tag{2}$$

where X is a vector field defined by system (1).

Remark 1. In inequality (2), it is usually assumed [1, p. 226], [2,8,9], [3, p. 68] that the function Φ can be zero on a set of the zero measure in the domain Ω, with no closed curve in this set being a limit cycle of system (1). However, Cherkas et al. [3, p. 312] showed that this requirement can be relaxed and replaced with the condition that the curve defined by the equation $\Phi(x,y) = 0$ is transversal.

Remark 2. If Ψ is a Dulac–Cherkas function of system (1) in the domain Ω, then $B = |\Psi|^{\frac{1}{k}}$ is a Dulac function in each subdomain Ω_i, where $\Psi > 0 (< 0)$, while any limit cycle Γ of system (1) that exists in Ω is rough and stable (unstable) under the condition that $k\Phi\Psi < 0 (> 0)$ on Γ.

To localize the limit cycles in the domain Ω, we introduce a set $W = \{(x,y) \in \Omega : \ \Psi(x,y) = 0\}$, that is transversal for the vector field X under condition (2) and is not intersected by the limit cycles of system (1).

The following assertion was proved in the monograph [3, p. 205].

Theorem 1 (the Dulac–Cherkas criterion)**.** *Suppose that in a connected domain Ω system* (1) *has the unique anti-saddle point of rest O, while Ψ is the Dulac–Cherkas function of system* (1) *with $k < 0$ in the domain Ω, where the set W consists of s mutually embedded ovals ω_i surrounding the point O. Then, system* (1) *has exactly one limit cycle in each of the $s-1$ ring-shaped subdomains Ω_i that are bounded by neighboring ovals ω_i and ω_{i+1} and can have at most s limit cycles in the domain Ω in total.*

The monograph [3] contains different ways for constructing the Dulac–Cherkas function which allows to estimate the upper bound for the number of limit cycles by using Theorem 1.

In cases where this approach is difficult to be implemented, it was suggested in [3, p. 334] to construct the Dulac function in the form of the product

$$B = |\Psi(x,y)|^{\frac{1}{k}} |\widetilde{\Psi}(x,y)|^{\frac{1}{\widetilde{k}}}, \quad k, \widetilde{k} \in R, \quad k\widetilde{k} \neq 0, \quad \Psi, \widetilde{\Psi} \in C^1(\Omega). \tag{3}$$

Theorem 2. *A function B of the form* (3) *is the Dulac function of system* (1) *in the domain Ω if the following condition is satisfied:*

$$\widetilde{\Phi} \equiv k\widetilde{k}\Psi\widetilde{\Psi} \operatorname{div} X + k\Psi \frac{d\widetilde{\Psi}}{dt} + \widetilde{k}\widetilde{\Psi} \frac{d\Psi}{dt} > 0 \ \ (< 0). \tag{4}$$

Let $W_0 = W \cup \widetilde{W}$, where

$$W = \big\{(x,y) \in \Omega : \ \Psi(x,y) = 0\big\}, \ \ \widetilde{W} = \big\{(x,y) \in \Omega : \ \widetilde{\Psi}(x,y) = 0\big\},$$

then the following assertions hold in the domain Ω: the set W_0 contains no equilibrium points of system (1)*; any trajectory of system* (1) *that encounters the set W_0 intersects it transversally; the set W_0 defines a curve with disjoint branches; and the limit cycles of system* (1) *that belong entirely to the domain Ω do not intersect the set W_0.*

Since the curves of the set W_0 divide the domain Ω in subdomains Ω_i in each of which B is a Dulac function in the classical sense, we find [3, p. 336] that the following assertion applies when evaluating the number of cycles of system (1) and localizing these cycles.

Theorem 3. *Suppose that in a connected domain Ω system* (1) *has the unique anti-saddle equilibrium point O and possesses a function B of the form* (3) *that satisfies condition* (4) *for $k < 0$, $\widetilde{k} < 0$. If sets W and $\widetilde{W}$ in the domain Ω consist of, respectively, s and $\widetilde{s}$ mutually embedded ovals that surround O, then in each of the $s + \widetilde{s} - 1$ ring-shaped subdomains Ω_i that are bounded by neighboring ovals ω_i and ω_{i+1} of the set W_0, system* (1) *has exactly one limit cycle, which is stable* (*unstable*) *for $\widetilde{\Phi}/(k\widetilde{k}\Psi\widetilde{\Psi}) < 0$ (> 0). System* (1) *can have at most $s + \widetilde{s}$ limit cycles in the domain Ω in total.*

However, none of the above theorems provides an exact estimate for the number of limit cycles of the considered systems (1), since to establish the existence or absence of a limit cycle in the external doubly connected subdomain Ω_s or $\Omega_{s+\widetilde{s}}$, one needs to conduct additional research and examine the influence of the other equilibrium points of rest or construct an additional transversal closed curve that embraces an external oval that corresponds to the function $B = |\Psi|^{\frac{1}{k}}$ or a function B of the form (3).

Now we will present our approaches to establishing the exact number of limit cycles of system (1) in the domain Ω, the approaches being based on constructing a closed transversal curve that surrounds the external oval of the function B in a doubly connected subdomain Ω_s with the use of an additional application of the Dulac or Dulac–Cherkas criterion. The gist of the first approach is expressed by the following assertion.

Theorem 4. *Suppose that the assumptions of Theorem* 1 *are valid, and system* (1) *has a second Dulac–Cherkas function* $\widetilde{\Psi}(x,y)$ *for* $\widetilde{k} < 0$ *in the domain* Ω *such that the set* $\widetilde{W}$ *consists of* $s+1$ *ovals in* Ω *that surround the point* O*. Then system* (1) *has exactly* s *limit cycles in the domain* Ω*.*

Proof. By virtue of Theorem 1, the existence of a Dulac–Cherkas function $\Psi(x,y)$ that defines s ovals in the domain Ω implies the existence of $s-1$ limit cycles of system (1) in the ring-shaped domains Ω_i, $i = 1, \dots, s-1$, bounded by neighboring ovals ω_i and ω_{i+1} and admits the existence of one limit cycle in the doubly connected subdomain Ω_s. By virtue of Theorem 1, the existence of the second Dulac–Cherkas function $\widetilde{\Psi}(x,y)$, that defines $s+1$ ovals in the domain Ω implies the existence of s limit cycles of system (1) in the ring-shaped domains $\widetilde{\Omega}_i$, $i = 1, \dots, s$, bounded by neighboring ovals of the set $\widetilde{W}$ and admits the existence of one limit cycle in the doubly connected subdomain $\widetilde{\Omega}_{s+1}$, that lies in between the external oval of the set $\widetilde{W}$ and the boundary $\partial\Omega$ of the domain Ω. The simultaneous existence of the functions Ψ and $\widetilde{\Psi}$ guarantees the existence of one limit cycle in the subdomain $\Omega_s \setminus \widetilde{\Omega}_{s+1}$ and rules out the existence of a limit cycle in the subdomain $\widetilde{\Omega}_{s+1}$. Hence it follows that system (1) has exactly s limit cycles in the domain Ω. It completes the proof of the theorem. $\square$

A second approach can be described as follows.

Theorem 5. *Suppose that the assumptions of Theorem* 1 *hold, and, in addition, that in the domain* Ω *system* (1) *has a Dulac function* B *of the form* (3) *that satisfies the assumptions of Theorem* 3*, with the set* $\widetilde{W}$ *consisting of a single oval that is situated in the doubly connected domain* Ω_s *and surrounds all the ovals of the set* W*. Then system* (1) *has exactly* s *limit cycles in the domain* Ω*.*

Proof. The existence of $s-1$ limit cycles of system (1) in the case where the Dulac–Cherkas function $\Psi(x,y)$ exists can be proved similarly to Theorem 4. The existence of one more limit cycle in the doubly connected subdomain $\widetilde{\Omega}_s \subset \Omega_s$ in between the external oval of the set W and the single oval of the set $\widetilde{W}$ follows from Theorem 3. The simultaneous existence of the function Ψ and a function B of the form (3) guarantees that system (1) has exactly s limit cycles in the domain Ω. The proof of the theorem is complete. $\square$

If the usage of Theorem 5 does not enable the construction of a function $\widetilde{\Psi}$, that satisfies inequality (3), one can relinquish the sign-definiteness of the function $\widetilde{\Phi}$ and use the condition of transversality of the set

$$V = \big\{(x,y) \in \Omega : \ \widetilde{\Phi} = 0\big\}$$

with respect to the vector field X of system (1). This constitutes the essence of the third approach.

Theorem 6. *Suppose that the assumptions of Theorem* 1 *are valid and there exists such a function* $\widetilde{\Psi}(x,y) \in C^1(\Omega)$ *with* $\widetilde{k} < 0$ *that in the domain* Ω *the set* $\widetilde{W}$ *intersects neither the set* V *nor the set* W*. Then the set* $\widetilde{W}$ *is transversal to the vector field* X *and is disjoint with the limit cycles of system* (1) *that belong entirely to the domain* Ω*.*

Proof. We consider the set $\widetilde{W}$. Since $\widetilde{W}$ and V are disjoint sets, it follows that the condition $\widetilde{\Phi} > 0(< 0)$ is satisfied on the set $\widetilde{W}$. By virtue of inequality (4), the condition $k\Psi\frac{d\widetilde{\Psi}}{dt} > 0$ (< 0) is satisfied on the curve $\widetilde{\Psi} = 0$ along any solution of system (1). Since the set $\widetilde{W}$ does not intersect the set W, it follows from the above inequality that the condition $\frac{d\widetilde{\Psi}}{dt} > 0$ (< 0) is satisfied. Consequently, any trajectory of system (1) intersects the curve $\widetilde{\Psi} = 0$ transversally.

Without loss of generality, we consider the case $\frac{d\widetilde{\Psi}}{dt} > 0$. Let us show that the limit cycles cannot intersect the curve $\widetilde{\Psi} = 0$. Suppose the contrary is true, then a point on the limit cycle can get with time onto the curve $\widetilde{\Psi} = 0$ only from a set in which $\widetilde{\Psi} < 0$ and should necessarily leave into a set in which $\widetilde{\Psi} > 0$. However, when moving along the limit cycle, the point should return into the original position in the domain $\widetilde{\Psi} = 0$, which is impossible in view of the inequality? $\frac{d\widetilde{\Psi}}{dt} > 0$. The obtained contradiction implies that the limit cycles of system (1) cannot intersect the curve $\widetilde{\Psi} = 0$ and it completes the proof. □

Remark 3. Theorems 4–6 persist if the function $\widetilde{\Psi}$ is found not in the entire domain Ω but only in the domain Ω_s or in its doubly connected subdomain $G_s \subset \Omega_s$ surrounding the equilibrium point O.

Theorem 7. *Suppose that the assumptions of Theorem* 1 *are valid and system* (1) *has a closed transversal curve that lies in a doubly connected subdomain* Ω_s *that surrounds the external oval of the set* W*, two of them forming the boundary of a ring-shaped domain* $\widetilde{\Omega}_s \subset \Omega_s$*. Then, if the trajectories of system* (1) *enter, as* t *increases, the interior of the domain* $\widetilde{\Omega}_s$ *from outside* (*or vice versa*) *through the boundary* $\partial\widetilde{\Omega}_s$*, then there exists the unique stable* (*or unstable*) *limit cycle of system* (1) *in the subdomain* $\widetilde{\Omega}_s$ *and system* (1) *has exactly* s *limit cycles in the domain* Ω *in total.*

Proof. According to Theorem 4, the existence of a Dulac–Cherkas function $\Psi(x, y)$ ensures the existence of $s - 1$ limit cycles of system (1) encircled by the external oval ω_s of the set W. In accordance with the Dulac criterion, system (1) can have no more than one limit cycle in the doubly connected subdomain Ω_s. On the other hand, if the trajectories of system (1) enter, as t increases, the interior of the subdomain $\widetilde{\Omega}_s$ from outside (or vice versa) through the boundary $\partial\widetilde{\Omega}_s$, then, according to the Poincare theorem [3, p. 64], there exists at least one stable (or unstable) limit cycle in the subdomain $\widetilde{\Omega}_s$. Thus, we establish the uniqueness of the limit cycle in $\widetilde{\Omega}_s$. Consequently, system (1) has exactly s limit cycles in the domain Ω. The proof is complete. □

A detailed presentation of the approaches developed by us and their application to some classes of systems (1) are contained in our paper [5]. Our paper [6] also shows that these approaches can be effectively implemented to establish the exact number of limit cycles surrounding several equilibrium points of systems (1), the total Poincaré index of which is +1.

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Theorems on Functional Differential Inequalities

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Consider the system of functional differential inequalities

$$\mathcal{D}(\sigma(t))\big[u'(t)-\ell(u)(t)\big]\geq 0 \ \text{ for a.e. } \ t\in[a,b], \tag{1}$$

$$\varphi(u)\geq 0, \tag{2}$$

where $\ell : C([a,b];\mathbb{R}^n)\to L([a,b];\mathbb{R}^n)$ is a linear bounded operator, $\varphi : C([a,b];\mathbb{R}^n)\to\mathbb{R}^n$ is a linear bounded functional, $\sigma=(\sigma_i)_{i=1}^n$, $\sigma_i:[a,b]\to\{-1,1\}$ are functions of bounded variation, and $\mathcal{D}(\sigma(t))=\operatorname{diag}(\sigma_1(t),\dots,\sigma_n(t))$. In the present contribution, we establish conditions guaranteeing that every absloutely continuous vector-valued function u satisfying (1) and (2) admits also the inequality $u(t)\geq 0$ for $t\in[a,b]$. For this purpose we will need the following notation and definitions.

$\mathbb{R}$ is a set of all real numbers, $\mathbb{R}_+=[0,+\infty[$, $\mathbb{R}^n$ is a space of n-dimensional column vectors $x=(x_i)_{i=1}^n$ with elements $x_i\in\mathbb{R}$ $(i=1,\dots,n)$, $\mathbb{R}^{n\times n}$ is a space of $n\times n$-matrices $X=(x_{ij})_{i,j=1}^n$ with elements $x_{ij}\in\mathbb{R}$ $(i,j=1,\dots,n)$, $\mathbb{R}^n_+$ and $\mathbb{R}^{n\times n}_+$ are sets of non-negative column vectors and matrices, respectively. The inequalities between vectors and matrices are understood componentwise. If 0 and 1 are used as vectors, then 0 is a zero column vector and 1 is a column vector with all components equal to one; δ_{ik} is the Kronecker's symbol; X^{-1} is the inverse matrix to X; $r(X)$ is the spectral radius of the matrix X; Θ is a zero matrix.

$C([a,b];\mathbb{R}^n)$ is a Banach space of continuous vector-valued functions $x=(x_i)_{i=1}^n:[a,b]\to\mathbb{R}^n$ endowed with the norm

$$\|x\|_C=\max\bigg\{\sum_{i=1}^n|x_i(t)| :\ t\in[a,b]\bigg\}.$$

$AC([a,b];\mathbb{R}^n)$ is a set of absolutely continuous vector-valued functions $x:[a,b]\to\mathbb{R}^n$.

$L([a,b];\mathbb{R}^n)$ is a Banach space of Lebesgue integrable vector-valued functions $p=(p_i)_{i=1}^n$: $[a,b]\to\mathbb{R}^n$ endowed with the norm

$$\|p\|_L=\int\limits_a^b\sum_{i=1}^n|p_i(s)|\,ds.$$

$\mathcal{L}^n_{ab}$ is a set of linear bounded operators $\ell:C([a,b];\mathbb{R}^n)\to L([a,b];\mathbb{R}^n)$.

$\mathcal{C}^{n,*}_{ab}$ is a set of linear bounded functionals $\varphi:C([a,b];\mathbb{R}^n)\to\mathbb{R}^n$.

For any $\ell\in\mathcal{L}^n_{ab}$, the operators $\ell_i:C([a,b];\mathbb{R}^n)\to L([a,b];\mathbb{R})$ and $\ell_{ik}:C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ $(i,k=1,\dots,n)$ are defined as follows:

- for any $v\in C([a,b];\mathbb{R}^n)$, $\ell_i(v)$ is the i-th component of the vector-valued function $\ell(v)$;
- for any $z\in C([a,b];\mathbb{R})$ we put $\ell_{ik}(z)=\ell_i(\widehat{z})$, where $\widehat{z}=(\delta_{ik}z)_{i=1}^n$.

For any functional $\varphi\in\mathcal{C}^{n,*}_{ab}$ we define the functionals φ_i : $C([a,b];\mathbb{R}^n)\to\mathbb{R}$ and φ_{ik} : $C([a,b];\mathbb{R})\to\mathbb{R}$ in a similar way. Moreover, we put $\Phi=\big(\varphi_{ik}(1)\big)_{i,k=1}^n$.

Definition 1. An operator $\ell \in \mathcal{L}_{ab}^n$ is said to be σ-*positive* if the relation

$$\mathcal{D}(\sigma(t))\ell(u)(t) \geq 0 \text{ for a.e. } t \in [a,b] \tag{3}$$

is fulfilled whenever $u \in C([a,b];\mathbb{R}^n)$ is such that

$$u(t) \geq 0 \text{ for } t \in [a,b] \tag{4}$$

holds. A set of σ-positive operators is denoted by $\mathcal{P}_{ab}^n(\sigma)$.

Definition 2. We will say that an operator $\ell \in \mathcal{L}_{ab}^n$ belongs to the set $\mathcal{P}_{ab}^{n,+}(\sigma)$ if the relation (3) is fulfilled whenever $u \in AC([a,b];\mathbb{R}^n)$ is such that (4) and

$$\mathcal{D}(\sigma(t))u'(t) \geq 0 \text{ for a.e. } t \in [a,b] \tag{5}$$

hold.

Remark 1. Obviously, $\mathcal{P}_{ab}^n(\sigma) \subsetneq \mathcal{P}_{ab}^{n,+}(\sigma)$.

Definition 3. We will say that a pair of operators $(\ell,\varphi) \in \mathcal{L}_{ab}^n \times \mathcal{C}_{ab}^{n,*}$ belongs to the set $\mathcal{S}_{ab}^n(\sigma)$ if every function $u \in AC([a,b];\mathbb{R}^n)$ satisfying (1), (2) admits also (4).

Remark 2. Obviously, if $(\ell,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$, then the problem

$$u'(t) = \ell(u)(t) + q(t) \text{ for a.e. } t \in [a,b], \quad \varphi(u) = c$$

has a unique solution $u \in AC([a,b];\mathbb{R}^n)$ for every $q \in L([a,b];\mathbb{R}^n)$ and $c \in \mathbb{R}^n$, and this solution is non-negative if $\mathcal{D}(\sigma(t))q(t) \geq 0$ for a. e. $t \in [a,b]$ and $c \geq 0$.

In the formulation of the main results, the inclusion $(0,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$ is used. Therefore, we present here some basic implication of this inclusion.

Proposition 1. *Let* $(0,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$. *Then*

(i) $\det \Phi \neq 0$,

(ii) $\Phi^{-1} \geq \Theta$.

Proposition 2. *Let* $(0,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$ *and let* $u \in AC([a,b];\mathbb{R}^n)$ *satisfy* (5). *Then*

$$u(t) \geq \Phi^{-1}\varphi(u) \;\; \textit{for} \;\; t \in [a,b].$$

Main results

Theorem 1. *Let* $\ell \in \mathcal{P}_{ab}^n(\sigma)$, $(0,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$. *Then* $(\ell,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$ *iff there exists* $\gamma \in AC([a,b];\mathbb{R}^n)$ *such that*

$$\mathcal{D}(\sigma(t))\big[\gamma'(t) - \ell(\gamma)(t)\big] \geq 0 \;\; \textit{for a.e.} \;\; t \in [a,b],$$
$$\gamma(t) > 0 \;\; \textit{for} \;\; t \in [a,b], \quad \Phi^{-1}\varphi(\gamma) > 0.$$

Proof. Necessity: If $(\ell,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$, then according to Remark 2 the problem

$$u'(t) = \ell(u)(t) + \ell(1)(t) \text{ for a.e. } t \in [a,b], \quad \varphi(u) = 0$$

is uniquely solvable. Moreover, $u(t) \geq 0$ for $t \in [a,b]$. Put $\gamma(t) = u(t) + 1$ for $t \in [a,b]$. Then

$$\mathcal{D}(\sigma(t))\big[\gamma'(t) - \ell(\gamma)(t)\big] = 0 \ \text{ for a.e. } \ t \in [a,b],$$
$$\gamma(t) > 0 \ \text{ for } \ t \in [a,b], \quad \Phi^{-1}\varphi(\gamma) = \Phi^{-1}(\varphi(u) + \Phi \cdot 1) > 0.$$

Sufficiency: Let u satisfy $(1), (2)$ with $u_j(t_j) < 0$ for some $j \in \{1, \dots, n\}$ and $t_j \in [a,b]$. Put

$$\lambda_i = \max\Big\{ -\frac{u_i(t)}{\gamma_i(t)} : \ t \in [a,b]\Big\} \ \ (i = 1, \dots, n)$$

and let

$$\lambda = \max\big\{\lambda_1, \dots, \lambda_n\big\} > 0.$$

Then $w(t) \overset{def}{=} \lambda\gamma(t) + u(t) \geq 0$ for $t \in [a,b]$, and there exist $i_0 \in \{1, \dots, n\}$ and $t_0 \in [a,b]$ such that $w_{i_0}(t_0) = \lambda\gamma_{i_0}(t_0) + u_{i_0}(t_0) = 0$. Consequently,

$$\mathcal{D}(\sigma(t))w'(t) \geq \mathcal{D}(\sigma(t))\ell(w)(t) \geq 0 \ \text{ for a.e. } \ t \in [a,b].$$

According to Proposition 2,

$$w(t) \geq \Phi^{-1}\varphi(w) = \Phi^{-1}(\lambda\varphi(\gamma) + \varphi(u)) > 0,$$

a contradiction. □

Theorem 2. *Let ℓ admit the representation $\ell = \ell^+ - \ell^-$ where $\ell^+, \ell^- \in \mathcal{P}^n_{ab}(\sigma)$. Let, moreover,*

$$\ell \in \mathcal{P}^{n,+}_{ab}(\sigma), \ \ (\ell^+, \varphi) \in \mathcal{S}^n_{ab}(\sigma), \ \ (0, \varphi) \in \mathcal{S}^n_{ab}(\sigma).$$

Then $\ell \in \mathcal{S}^n_{ab}(\sigma)$.

Proof. Let u satisfy $(1), (2)$. According to Remark 2 there exists a unique solution x to the problem

$$x'(t) = \mathcal{D}(\sigma(t))\big[\mathcal{D}(\sigma(t))u'(t)\big]_- \ \text{ for a.e. } \ t \in [a,b], \quad \varphi(x) = 0.$$

Moreover, we have $x(t) \geq 0$ for $t \in [a,b]$. Put $w(t) = u(t) + x(t)$ for $t \in [a,b]$. Then $w(t) \geq u(t)$ for $t \in [a,b]$,

$$\mathcal{D}(\sigma(t))w'(t) = \big[\mathcal{D}(\sigma(t))u'(t)\big]_+ \geq 0 \ \text{ for a.e. } \ t \in [a,b], \quad \varphi(w) \geq 0.$$

Thus, $w(t) \geq 0$ for $t \in [a,b]$. Let $A_i = \{t \in [a,b] : w_i'(t) = u_i'(t)\}$ and put

$$q(t) \overset{def}{=} \mathcal{D}(\sigma(t))\big[u'(t) - \ell(u)(t)\big] \ \text{ for a.e. } \ t \in [a,b].$$

Then, for every $i \in \{1, \dots, n\}$, we have

$$\sigma_i(t)w_i'(t) = \begin{cases} \sigma_i(t)u_i'(t) = \sigma_i(t)\displaystyle\sum_{k=1}^n \big[\ell^+_{ik}(u_k)(t) - \ell^-_{ik}(u_k)(t)\big] + q_i(t) & \\ \qquad\qquad \leq \sigma_i(t)\displaystyle\sum_{k=1}^n \big[\ell^+_{ik}(w_k)(t) - \ell^-_{ik}(u_k)(t)\big] + q_i(t) \ \text{ for } \ t \in A_i, & \\ 0 \qquad\qquad \text{for a.e. } \ t \in [a,b] \setminus A_i. & \end{cases}$$

On the other hand,

$$\mathcal{D}(\sigma(t))\big[\ell^+(w)(t) - \ell^-(u)(t)\big] + q(t) \geq \mathcal{D}(\sigma(t))\ell(w)(t) + q(t) \geq 0 \ \text{ for a.e. } \ t \in [a,b].$$

Consequently,

$$\mathcal{D}(\sigma(t))\big[w'(t)-\ell^{+}(w)(t)\big]\le -\mathcal{D}(\sigma(t))\ell^{-}(u)(t)+q(t) \ \text{ for a.e. } \ t\in[a,b].$$

Put $z(t)=u(t)-w(t)$ for $t\in[a,b]$. Then

$$\mathcal{D}(\sigma(t))\big[z'(t)-\ell^{+}(z)(t)\big]\ge 0 \ \text{ for a.e. } \ t\in[a,b], \quad \varphi(z)=0,$$

and so $z(t)\ge 0$ for $t\in[a,b]$, i.e. $u(t)\ge w(t)\ge 0$ for $t\in[a,b]$. □

As a consequences of the main results we formulate corollaries in the case when σ is a constant function. Therefore, in what follows we assume that $\sigma(t)=(\sigma_i)_{i=1}^n$ for $t\in[a,b]$ with $\sigma_i\in\{-1,1\}$. First consider the system with deviating arguments

$$\sigma_i\Big[u_i'(t)-\sum_{k=1}^{n}\big(p_{ik}(t)u_k(\tau_{ik}(t))-g_{ik}(t)u_k(\mu_{ik}(t))\big)\Big]\ge 0 \ \text{ for a.e. } \ t\in[a,b], \tag{6}$$

$$u_i(a)\ge 0 \ \text{ if } \ \sigma_i=1, \quad u_i(b)\ge 0 \ \text{ if } \ \sigma_i=-1, \tag{7}$$

where $\sigma_i p_{ik},\sigma_i g_{ik}\in L([a,b];\mathbb{R}_+)$, $\tau_{ik},\mu_{ik}:[a,b]\to[a,b]$ are measurable functions.

Corollary 1. *Let*

$$\sigma_i\big(p_{ik}(t)-g_{ik}(t)\big)\ge 0, \quad \sigma_i\sigma_k g_{ik}(t)\big(\tau_{ik}(t)-\mu_{ik}(t)\big)\ge 0 \ \textit{ for a.e. } \ t\in[a,b].$$

Let, moreover, there exist $A=\big(a_{ik}\big)_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ *such that* $r(A)<1$ *and*

$$\int_a^b\bigg(\sigma_i\big(p_{ik}(t)-g_{ik}(t)\big)+\sigma_i g_{ik}(t)\int_{\mu_{ik}(t)}^{\tau_{ik}(t)}\sum_{j=1}^{n}p_{kj}(s)\,ds\bigg)\,dt\le a_{ik}.$$

Then every $u\in AC([a,b];\mathbb{R}^n)$ *that satisfies* (6), (7) *is non-negative.*

The next corollary deals with the second-order differential inequality with deviations together with mixed boundary value conditions

$$u''(t)\le -p(t)u(\tau(t))+g(t)u(\mu(t)) \ \text{ for a.e. } \ t\in[a,b], \ \ u(a)\ge 0, \ \ u'(b)\ge 0. \tag{8}$$

Here $p,g\in L([a,b];\mathbb{R}_+)$ and $\tau,\mu:[a,b]\to[a,b]$ are measurable functions.

Corollary 2. *Let*

$$\tau(t)\le t, \ \ p(t)\ge g(t), \ \ g(t)(\tau(t)-\mu(t))\ge 0 \ \textit{ for a.e. } \ t\in[a,b].$$

Let, moreover, there exists $\lambda_1,\lambda_2\in\mathbb{R}_+$ *such that*

$$\int_0^{+\infty}\frac{ds}{\lambda_1+\lambda_2 s+s^2}\ge b-a,$$

$$p(t)-g(t)+g(t)(\tau(t)-\mu(t))\int_{\tau(t)}^{t}p(s)\,ds+g(t)\int_{\mu(t)}^{\tau(t)}(s-\mu(t))p(s)\,ds\le\lambda_1 \ \textit{ for a.e. } \ t\in[a,b],$$

$$g(t)(\tau(t)-\mu(t))\le\lambda_2 \ \textit{ for a.e. } \ t\in[a,b],$$

and at least one of the last three inequalities is strict. Then every $u\in AC^1([a,b];\mathbb{R})$ *that satisfies* (8) *is non-negative and nondecreasing.*

Baer's Classification of Characteristic Exponents in the Full Perron's Effect of Their Value Change

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We consider the linear differential system

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^2, \;\; t \geq 0, \tag{1}$$

with a bounded continuously differentiable matrix of coefficients $A(t)$ and with negative characteristic exponents $\lambda_1(A) \leq \lambda_2(A) < 0$. This system is a linear approximation for the nonlinear system

$$\dot{y} = A(t)y + f(t,y), \;\; y = (y_1, y_2) \in \mathbb{R}^2, \;\; t \geq 0. \tag{2}$$

In addition, the so-called m-perturbation of $f(t,y)$ is continuously differentiable in its arguments $t \geq 0$ and $y_1, y_2 \in \mathbb{R}$ and has the order $m > 1$ of smallness in some neighborhood of the origin and admissible growth outside of it:

$$\|f(t,y)\| \leq C_f \|y\|^m, \;\; m > 1, \;\; y \in \mathbb{R}^2, \;\; t \geq 0, \tag{3}$$

where C_f is a positive constant.

Perron's effect [28], [27, pp. 50, 51] of sign and value change in characteristic exponents claims the existence of such system (1) with the negative Lyapunov exponents and 2-perturbation (3) that all nontrivial solutions of the perturbed system (2) turn out to be infinitely extendable and have finite Lyapunov exponents equal to:

1) the negative higher exponent λ_2 of the initial system (1) for the solutions starting at the initial moment on the axis $y_1 = 0$ (that allows one to consider Perron's effect incomplete);

2) any one positive value for all the rest solutions (calculated in [10, pp. 13–15]).

In our works [3–8, 11–24], we obtained various versions of the full Perron's effect when all nontrivial solutions of the nonlinear system (2) with m-perturbation (3) are infinitely extendable (this is not so in a general case) and have finite positive Lyapunov exponents for negative exponents of the system of linear approximation (1). These versions correspond to: different types of the set $\lambda(A,f) \subset (0, +\infty)$ of characteristic Lyapunov exponents of all nontrivial solutions of the perturbed system (2), distribution of those solutions with respect to the exponents from the set $\lambda(A,f)$ and, finally, an arbitrary order of systems (1) and (2). In particular, in our last works [14, 15], we obtained a continual version of the full Perron's effect with an arbitrarily given segment, a set $\lambda(A,f) \subset (0, +\infty)$ of characteristic exponents of the perturbed system (2).

In the full Perron's effect, the question dealing, in particular, with a most general type of the set $\lambda(A,f)$ of characteristic exponents (of all nontrivial solutions) of the perturbed system (2), i.e., the question on a full description of that set, remains still open. The aim of the present work is to establish that in the full Perron's effect of value change in characteristic exponents their set $\lambda(A,f)$ is the Suslin's one [2, pp. 97, 98, 192], realizing thus the first stage of the above description. Towards this end, it will be proved that within the framework of the effect under consideration the characteristic exponent

$$\lambda[y(\cdot,y_0)] \equiv \overline{\lim_{t\to+\infty}}\, \frac{1}{t}\, \ln \|y(t,y_0)\|$$

of every nontrivial solution $y(t,y_0)$ of system (2), being the function of the initial vector $y_0 = y(0,y_0) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, is the function of the second Bare's class [2, p. 248]. Thus its set of values

$$\Lambda(A,f) \equiv \big\{\lambda[y(\,\cdot\,,y_0)] : y_0 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}\big\}$$

belongs to the class of Suslin's sets [2, pp. 97,98, 192].

The perturbed differential system (2) realizing the full Perron's effect of values change, whose all nontrivial solutions take their origin in some neighbourhood of its zero solution and have, by the definition, positive exponents, may be called exponentially nonstable. In an opposite case, in no way connected with the Perron's effect, when the exponentially stable system (1) is such that any system (2) with m-perturbation f is likewise exponentially stable, we studied the set [9] $\Lambda_0(A,f) = \bigcap\limits_{\rho>0} \Lambda_\rho(A,f)$, where $\Lambda_\rho(A,f)$ is a set of Lyapunov's exponents of nontrivial solutions of system (2), emanating for $t=0$ from the ρ-neighbourhood of zero. For the set $\Lambda_0(A,f) \subset (-\infty,0)$, we obtained the following results. In [9], for an arbitrary segment $[\alpha,\beta] \subset (-\infty,0)$, we constructed the system (2) for which $\Lambda_0(A,f) = [\alpha,\beta]$. In [29], these constructions were extended to the sets $\Lambda_0(A,f) \subset (-\infty,0)$ consisting of a countable number of connectedness components. Finally, in [1], the family of sets $\Lambda_0(A,f)$ is described completely; it consists of bounded Suslin's sets of the negative semi-axis whose exact upper bound is negative.

The essentials of the Baer's classification of Lyapunov exponents and other asymptotic characteristics of solutions of parametric differential systems, as the functions of a parameter, were laid by V. M. Millionshchikov. Its subsequent development is connected with the works of M. I. Rakhimberdiev, I. N. Sergeyev, E. A. Barabanov, A. N. Vetokhin, V. V. Bykov and their pupils.

We will consider a more general, as compared with (2), the n-dimwnsional differential system

$$\dot{y} = F(t,y), \quad y \in \mathbb{R}^n, \quad t \ge 0, \tag{4}$$

with a continuously differentiable in its arguments $t>0$ and $y_1,\dots,y_n \in \mathbb{R}$ right-hand side $F(t,y)$ satisfying the condition $F(t,\mathbf{0}) \equiv \mathbf{0}$, $t \ge 0$.

The following theorem is valid.

Theorem. *Let all nontrivial solutions $y(t,y_0)$ of system* (4) *be infinitely extendable and have finite characteristic exponents. Then the characteristic exponent $\lambda[y(\,\cdot\,,y_0)]$ of those solutions is the function of the 2nd Baer's class of their initial vectors $y_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.*

Getting back to the full Perron's effect of value change in negative characteristic exponents of the system of linear approximation (1), for the whole set $\Lambda(A,f)$ of positive Lyapunov exponents of all nontrivial solutions of the perturbed system (2), we obtain the following

Corollary. *Let all nontrivial solutions $y(t,y_0)$, $y_0 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ of system* (2) *be infinitely extendable and have finite positive Lyapunov exponents. Then the characteristic exponent $\lambda[y(\,\cdot\,,y_0)]$ of those solutions is the function of the 2nd Baer' class of their initial values $y_0 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, whereas the whole set $\Lambda(A,f)$ of exponents of nontrivial solutions is Suslin's one.*

Remark 1. The above corollary is likewise valid for the n-dimensional analogue of the full Perron's effect.

Remark 2. In addition to the monograph by G. A. Leonov [27] the works due to V. V. Kozlov [25, 26] had a stimulating influence on our investigations of Perron's effect of sign and value change in characteristic exponents.

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On Additive Averaged Semi-Discrete Scheme for One Nonlinear Multi-Dimensional Integro-Differential Equation

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The present note is devoted to the nonlinear multi-dimensional integro-differential equation of parabolic type. The well-posedness of the initial-boundary value problem with first kind boundary condition and convergence of additive averaged semi-discrete scheme with respect to time variable are studied. The investigated equation is kind of natural generalization, on the one hand, of equations describing applied problems of mathematical physics and, on the other hand, of nonlinear parabolic equations considered, for example, in [14] and [18]. The studied equation is based on well-known Maxwell's system arising in mathematical simulation of electromagnetic field penetration into a substance [11].

Maxwell's system is complex and its investigation and numerical resolution still yield for special cases (see, for example, [9] and the references therein). In [3] the mentioned system was proposed in the integro-differential form. The literature on the questions of existence, uniqueness, and regularity of solutions to Maxwell's system and models of such integro-differential types is very rich. In [1–8, 12, 13], as well as in a number of other works the solvability of the initial-boundary value problems for this type integro-differential models in scalar cases are studied. The well-posedness of those problems in [1–8] are proved using a modified version of Galerkin's method and compactness arguments that are used in [14, 18] for investigation nonlinear elliptic and parabolic equations.

Let us note that the unique solvability and large time behavior of initial-boundary value problems for investigated in this note multi-dimensional integro-differential type equations at first are given in [4].

These questions and numerical resolution of initial-boundary value problems are discussed in many works as well (see, for example, [1–9, 12, 13, 16, 17] and the references therein).

Many authors study Rothe's type semi-discrete scheme with respect to time variable, semi-discrete schemes with spatial variable, finite element and finite difference approximations for a integro-differential models (see, for example, [5–10, 14, 16, 17] and the references therein).

It is very important to study decomposition analogs for the above-mentioned multi-dimensional integro-differential equation and systems too. At present there are some effective economic algorithms for solving the multi-dimensional problems (see, for example, [14, 15] and the references therein).

In this paper the existence and uniqueness of solutions of initial-boundary value problems is given. Main attention is paid to investigation of Rothe's type semi-discrete additive averaged scheme.

Let us formulate the studied problem. Let Ω be bounded domain in the n-dimensional Euclidean space R^n with sufficiently smooth boundary $\partial\Omega$. In the domain $Q=\Omega\times(0,T)$ of the variables $(x,t)=(x_1,x_2,\dots,x_n,t)$, where T is a positive constant, let us consider the following equation:

$$\frac{\partial U}{\partial t}-\sum_{i=1}^{n}\left\{\frac{\partial}{\partial x_i}\left[1+\int_0^t\left|\frac{\partial U}{\partial x_i}\right|^q d\tau\right]^p\left|\frac{\partial U}{\partial x_i}\right|^{q-2}\frac{\partial U}{\partial x_i}\right\}=f(x,t),\quad (x,t)\in Q, \tag{1}$$

with the homogeneous boundary and initial conditions:

$$U(x,t)=0,\ \ (x,t)\in\partial\Omega\times[0,T], \tag{2}$$

$$U(x,0)=0,\ \ x\in\overline{\Omega}, \tag{3}$$

where p, q are constants and f is a given function.

Principal characteristic peculiarity of the equation (1) is connected with the appearance of the higher order nonlinear terms depended on the time integral in the coefficients with high order derivatives. These circumstances requires different discussions than it is usually necessary for the solution of local differential problems.

The problem (1)–(3) is similar to problems considered in [2, 4, 7, 12]. Unique solvability and discrete analogs of initial-boundary value problem for one-dimensional case of equation (1) are studied in [5]. Using modified version of Galerkin's method and compactness arguments as in [14,18] the following statement is obtained.

Theorem 1. *If $0<p\le 1$, $q\ge 2$, $f\in W_2^1(Q)$, $f(x,0)=0$, then there exists the unique solution U of problem (1)–(3) satisfying the following properties:*

$$U\in L_{pq+q}(0,T;\overset{\circ}{W}{}^1_{pq+q}(\Omega)),\quad \frac{\partial U}{\partial t}\in L_2(Q),$$

$$\sqrt{\psi}\,\frac{\partial U}{\partial x_j}\bigg(\Big|\frac{\partial U}{\partial x_i}\Big|^{\frac{q-2}{2}}\frac{\partial U}{\partial x_i}\bigg)\in L_2(Q),\quad \sqrt{T-t}\,\frac{\partial U}{\partial t}\bigg(\Big|\frac{\partial U}{\partial x_i}\Big|^{\frac{q-2}{2}}\frac{\partial U}{\partial x_i}\bigg)\in L_2(Q),\ \ i,j=1,\dots,n,$$

where $\psi\in C^\infty(\overline{\Omega})$, $\psi(x)>0$ for $x\in\Omega$; $\frac{\partial\psi}{\partial\nu}=0$ for $x\in\partial\Omega$ and ν is the outer normal of $\partial\Omega$.

Here we used usual L_p and W_p^k, $\overset{\circ}{W}{}_p^k$ Sobolev spaces.

Using the scheme of investigation as in [4] it is not difficult to get the results of exponential asymptotic behavior of solution as $t\to\infty$ of the initial-boundary value problems for the equation (1) with nonhomogeneous initial condition.

On $[0,T]$, let us introduce a net with mesh points denoted by $t_j=j\tau$, $j=0,1,\dots,J$, with $\tau=T/J$.

Coming back to problem (1)–(3), let us construct the following additive averaged Rothe's type scheme:

$$\eta_i\,\frac{u_i^{j+1}-u^j}{\tau}=\frac{\partial}{\partial x_i}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big|\frac{\partial u_i^k}{\partial x_j}\Big|^q\bigg)^p\,\frac{\partial u_i^{j+1}}{\partial x_i}\bigg]+f_i^{j+1}, \tag{4}$$

with the homogeneous boundary and initial $u_i^0=u^0=0$ conditions, where $u_i^j(x)$, $i=1,\dots,n$, $j=0,1,\dots,J-1$ are solutions of the problems (4). The notations in (4) are as follows:

$$u^j(x)=\sum_{i=1}^n\eta_i u_i^j(x),\quad \sum_{i=1}^n\eta_i=1,\ \ \eta_i>0,\quad \sum_{i=1}^n f_i^{j+1}(x)=f^{j+1}(x)=f(x,t_{j+1}),$$

where u^j denotes approximation of an exact solution U of the problem (1)–(3) at t_j. We use usual norm $\|\cdot\|$ of the space $L_2(\Omega)$.

Theorem 2. *If problem (1)–(3) has sufficiently smooth solution, then the solution of the problem (4) with homogeneous initial and boundary conditions converges to the solution of the problem (1)–(3) and the following estimate is true*

$$\|U^j-u^j\|=O(\tau^{1/2}),\ \ j=1,\dots,J.$$

Let us note that the results analogous to Theorem 2 for the following integro-differential models are obtained in the works [6–8]:

$$\frac{\partial U}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\left(1+\int_0^t\Big|\frac{\partial U}{\partial x_i}\Big|^2\,dx\,d\tau\right)\frac{\partial U}{\partial x_i}\right]=f(x,t),$$

and

$$\frac{\partial U}{\partial t}-\sum_{i=1}^{n}\left(1+\int_\Omega\int_0^t\Big|\frac{\partial U}{\partial x_i}\Big|^2\,dx\,d\tau\right)\frac{\partial^2 U}{\partial x_i^2}=f(x,t).$$

It was mentioned in [7] that it is very important to construct and investigate (4) type semi-discrete additive schemes for more general type nonlinearities. The purpose of this work was to expand the previously studied cases. Thus, in this note we studied more wide class of nonlinearity.

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Productivity of Riccati Differential Equations

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1 Introduction

Consider the second order half-linear differential equation

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{E}$$

where α is a positive constant, $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$, $a \geq 0$, and $\varphi_\alpha(u) = |u|^\alpha \operatorname{sgn} u$, $u \in \mathbf{R}$.

We assume that equation (E) is nonoscillatory. Given a solution $x(t)$ of (E) we call the function $p(t)\varphi_\alpha(x'(t))$ the *quasi-derivative* of $x(t)$ and denote it by $Dx(t)$. If $u(t)$ and $v(t)$ are defined by

$$u(t) = \frac{Dx(t)}{\varphi_\alpha(x(t))} \quad \text{and} \quad v(t) = \frac{x(t)}{\varphi_{1/\alpha}(Dx(t))}\,,$$

then they satisfy the first order nonlinear differential equations

$$u' = -q(t) - \alpha p(t)^{-\frac{1}{\alpha}}|u|^{1+\frac{1}{\alpha}}, \tag{R1}$$

$$v' = p(t)^{-\frac{1}{\alpha}} + \frac{1}{\alpha}\,q(t)|v|^{1+\alpha}, \tag{R2}$$

for all large t. Equations (R1) and (R2) are referred to as the first and the second Riccati equations associated with (E). Note that (R2) has recently been discovered by Mirzov [3]. Conversely, suppose that (R1) and (R2) have solutions $u(t)$ and $v(t)$ defined for all large t, say on $[T,\infty)$. Such solutions $u(t)$ and $v(t)$ are termed *global solutions* of (R1) and (R2), respectively. Form the function $x(t)$ on $[T,\infty)$ by one of the following formulas which are collectively called the *reproducing formulas*

$$x(t) = \exp\bigg(\int\limits_T^t p(s)^{-\frac{1}{\alpha}}\varphi_{\frac{1}{\alpha}}(u(s))\,ds\bigg) \ \text{ or } \ x(t) = \exp\bigg(-\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\varphi_{\frac{1}{\alpha}}(u(s))\,ds\bigg), \tag{1.1}$$

$$x(t)=\frac{1}{\varphi_{\frac{1}{\alpha}}(u(t))}\,\exp\bigg(-\frac{1}{\alpha}\int_T^t\frac{q(s)}{u(s)}\,ds\bigg)\ \text{ or }\ x(t)=\frac{1}{\varphi_{\frac{1}{\alpha}}(u(t))}\,\exp\bigg(\frac{1}{\alpha}\int_t^\infty\frac{q(s)}{u(s)}\,ds\bigg),$$

$$x(t)=\exp\bigg(\int_T^t\frac{ds}{p(s)^{\frac{1}{\alpha}}v(s)}\bigg)\ \text{ or }\ x(t)=\exp\bigg(-\int_t^\infty\frac{ds}{p(s)^{\frac{1}{\alpha}}v(s)}\bigg),$$

$$x(t)=v(t)\exp\bigg(-\frac{1}{\alpha}\int_T^t q(s)\varphi_\alpha(v(s))\,ds\bigg)\ \text{ or }\ x(t)=v(t)\exp\bigg(\frac{1}{\alpha}\int_t^\infty q(s)\varphi_\alpha(v(s))\,ds\bigg).\tag{1.2}$$

Then, $x(t)$ gives a nonoscillatory solution of equation (E) on $[T,\infty)$. This shows that equation (E) is nonoscillatory if and only if the Riccati equation (R1) (or (R2)) has a global solution.

We expect that the Riccati equations will be more productive in the sense that all nonoscillatory solutions of equation (E) can be reproduced from the global solutions of (R1) and/or (R2). As a result of our efforts made in [2] it has turned out that a majority of solutions of (E) can really be reproduced by solving (R1) and (R2) by means of fixed point techniques. Worthy of note is that both (R1) and (R2) are indispensable in the reproduction processes.

2 Main results

We need the following notations:

$$I_p=\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt,\quad I_q=\int_a^\infty q(t)\,dt,$$

$$P_\alpha(t)=\int_a^t p(s)^{-\frac{1}{\alpha}}\,ds\ \text{ if }\ I_p=\infty,\quad \pi_\alpha(t)=\int_t^\infty p(s)^{-\frac{1}{\alpha}}\,ds\ \text{ if }\ I_p<\infty,$$

$$Q(t)=\int_a^t q(s)\,ds\ \text{ if }\ I_q=\infty,\quad \rho(t)=\int_t^\infty q(s)\,ds\ \text{ if }\ I_q<\infty.$$

Of crucial importance is the following classification of nonoscillatory solutions of (E). Let $x(t)$ be a solution of (E) such that $x(t)Dx(t)\neq 0$ on $[T,\infty)$. Both $x(t)$ and $Dx(t)$ are monotone and have the limits $x(\infty)=\lim\limits_{t\to\infty}x(t)$ and $Dx(\infty)=\lim\limits_{t\to\infty}Dx(t)$ in the extended real number system. The pair $(x(\infty),Dx(\infty))$, referred to as the terminal state of $x(t)$, is a decisive indicator of the asymptotic behavior at infinity of a solution $x(t)$ of (E). All possible types of terminal states of solutions $x(t)$ of (E) can be enumerated as follows.

(I) (The case where $I_p=\infty\wedge I_q<\infty$) (All solutions satisfy $x(t)Dx(t)>0$)

- I(i): $|x(\infty)|=\infty$, $0<|Dx(\infty)|<\infty$,
- I(ii): $|x(\infty)|=\infty$, $Dx(\infty)=0$,
- I(iii): $0<|x(\infty)|<\infty$, $Dx(\infty)=0$.

(II) (The case where $I_p<\infty\wedge I_q=\infty$) (All solutions satisfy $x(t)Dx(t)<0$)

- II(i): $0<|x(\infty)|<\infty$, $|Dx(\infty)|=\infty$,
- II(ii): $x(\infty)=0$, $|Dx(\infty)|=\infty$,

II(iii): $x(\infty) = 0$, $0 < |Dx(\infty)| < \infty$.

(III) (The case where $I_p < \infty \wedge I_q < \infty$)

$$\begin{aligned} \mathrm{III(i)}\ &= \mathrm{I(iii)}\ (x(t)Dx(t) > 0), \\ \mathrm{III(ii)}\ &= \mathrm{II(iii)}\ (x(t)Dx(t) < 0), \end{aligned}$$

III(iii): $0 < |x(\infty)| < \infty$, $0 < |Dx(\infty)| < \infty$) ($x(t)Dx(t) > 0$ or $x(t)Dx(t) < 0$).

The existence of solutions of the types I(i), I(iii), II(i) and II(iii) can be completely characterized.

Theorem 2.1. *Assume that $I_p = \infty \wedge I_q < \infty$.*

(i) (E) *has a solution of type* I(i) *if and only if* $\int\limits_a^\infty q(t)P_\alpha(t)^\alpha\,dt < \infty$.

(ii) (E) *has a solution of type* I(iii) *if and only if* $\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}\rho(t)^{\frac{1}{\alpha}}\,dt < \infty$.

Theorem 2.2. *Assume that $I_p < \infty \wedge I_q = \infty$.*

(i) (E) *has a solution of type* II(i) *if and only if* $\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}Q(t)^{\frac{1}{\alpha}}\,dt < \infty$.

(ii) (E) *has a solution of type* II(iii) *if and only if* $\int\limits_a^\infty q(t)\pi_\alpha(t)^\alpha\,dt < \infty$.

Only the proofs of the "if" parts of Theorem 2.1 are outlined.

Proof of the "if" part of Theorem 2.1-(i). Choose $T > a$ so that $\int\limits_T^\infty q(s)P_\alpha(s)^\alpha\,ds \le \alpha/(\alpha+1)2^{\alpha+1}$, define the set

$$\mathcal{V} = \big\{v \in C_{P_\alpha}[T,\infty) : \ P_\alpha(t) \le v(t) \le 2P_\alpha(t),\ t \ge T\big\},$$

where $C_{P_\alpha}[T,\infty)$ denotes the Banach space of all continuous functions $w(t)$ on $[T,\infty)$ such that $|w(t)|/P_\alpha(t)$ is bounded with the norm $\|w\|_{P_\alpha} = \sup\{|w(t)|/P_\alpha(t) : \ t \ge T\}$, and show that the integral operator given by

$$Gv(t) = P_\alpha(t) + \frac{1}{\alpha}\int\limits_T^t q(s)|v(s)|^{\alpha+1}\,ds, \ \ t \ge T,$$

is a contraction such that $\|Gv_1 - Gv_2\|_{P_\alpha} \le \frac{1}{2}\|v_1 - v_2\|_{P_\alpha}$ for any $v_1, v_2 \in \mathcal{V}$. Therefore, G has a unique fixed point $v \in \mathcal{V}$ which gives a solution $v(t)$ of (R2) on $[T,\infty)$ such that $v(t) \sim P_\alpha(t)$ as $t \to \infty$. With this $v(t)$ define $x(t)$ by the second formula in (1.2). Then, it is a solution of (E) satisfying $x(t) \sim P_\alpha(t)$ and $Dx(t) \sim 1$ as $t \to \infty$.

Proof of the "if" part of Theorem 2.1-(ii). Choose $T > a$ so that $\int\limits_T^\infty p(s)^{-\frac{1}{\alpha}}\rho(s)^{\frac{1}{\alpha}}\,ds \le 1/(\alpha+1)2^{1+\frac{1}{\alpha}}$ and consider the set

$$\mathcal{U} = \big\{v \in C_0[T,\infty) : \ \rho(t) \le u(t) \le 2\rho(t),\ t \ge T\big\},$$

where $C_0[T,\infty)$ denotes the set of all continuous functions $w(t)$ on $[T,\infty)$ tending to zero as $t \to \infty$. It is a Banach space with the sup-norm $\|w\|_0 = \sup\{|w(t)| : t \ge T\}$. Show that the integral operator given by

$$Fu(t) = \rho(t) + \alpha \int_t^\infty p(s)^{-\frac{1}{\alpha}} |u(s)|^{1+\frac{1}{\alpha}}\, ds, \ \ t \ge T,$$

is a contraction such that $\|Fu_1 - Fu_2\|_0 \le \frac{1}{2}\|u_1 - u_2\|_0$ for any $u_1, u_2 \in \mathcal{U}$. Let $u \in \mathcal{U}$ be a unique fixed point of F. Then, it is a solution $u(t)$ of (R1) on $[T,\infty)$ such that $u(t) \sim \rho(t)$ as $t \to \infty$. Using this $u(t)$ define $x(t)$ according to the second reproducing formula of (1.1). Then, it is a positive solution of (E) satisfying $x(t) \sim 1$ and $Dx(t) \sim \rho(t)$ as $t \to \infty$.

Note that any solution of the type III(iii) of (E) in the case $I_p < \infty \wedge I_q < \infty$ can also be reproduced from a suitable solution of (R1) or (R2).

As for solutions of the types I(ii) and II(ii) of (E), often referred to as *intermediate solutions*, very little is known about their existence and asymptotic behavior at infinity. In [2] we have indicated several nontrivial cases of (E) whose intermediate solutions can actually be reproduced with the aid of (R1) and (R2).

Theorem 2.3.

(i) *Assume that $I_p = \infty \wedge I_q < \infty$. Equation* (E) *has an intermediate solution of the type* I(ii) *if*

$$\int_a^\infty p(t)^{-\frac{1}{\alpha}} \rho(t)^{\frac{1}{\alpha}}\, dt = \infty, \quad \int_a^\infty q(t) P_\alpha(t)^\alpha\, dt < \infty.$$

(ii) *Assume that $I_p < \infty \wedge I_q = \infty$. Equation* (E) *has an intermediate solution of the type* II(ii) *if*

$$\int_a^\infty q(t)\pi_\alpha(t)^\alpha\, dt = \infty, \quad \int_a^\infty p(t)^{-\frac{1}{\alpha}} Q(t)^{\frac{1}{\alpha}}\, dt < \infty.$$

Outline of proof of (i). Let any constant $A > 1$ be given. Put $r(t) = \int_t^\infty q(s)P_\alpha(s)^\alpha\, ds$ and choose $T > a$ so that $r(T) \le (A-1)^\alpha A^{-\alpha-1}$. Define the integral operator F by

$$Fu(t) = \rho(t) + \alpha \int_t^\infty p(s)^{-\frac{1}{\alpha}} |u(s)|^{1+\frac{1}{\alpha}}\, ds, \ \ t \ge T,$$

and let it act on the set $\mathcal{U}$ defined by

$$\mathcal{U} = \big\{u \in C[T,\infty) : \ \rho(t) \le u(t) \le Ar(t)P(t)^{-\alpha}, \ t \ge T\big\},$$

which is a closed convex subset of the locally convex space $C[T,\infty)$.

Then, it can be shown that F is a continuous self-map of $\mathcal{U}$ sending $\mathcal{U}$ into a relatively compact subset of $C[T,\infty)$. Therefore, by the Schauder–Tychonoff fixed point theorem there exists a u in $\mathcal{U}$ such that $u = Fu$, which means that $u(t)$ is a global solution of (R1). With this $u(t)$ apply the first reproducing formula of (1.1) to construct a positive solution $x(t)$ of (E) on $[T,\infty)$. This is an intermediate solution of the type I(ii) since it is easily verified that $x(\infty) = \infty$ and $Dx(\infty) = 0$.

Remark. Some of our results are already known; see e.g., [1]. However, our approach based on the Riccati equations makes the asymptotic analysis of equation (E) much easier and clearer.

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On One Mixed Problem for One Class of Second Order Nonlinear Hyperbolic Systems with the Dirichlet and Poincare Boundary Conditions

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In the domain $D_T:\ 0<x<l,\ 0<t<T$ consider the following mixed problem

$$u_{tt}-u_{xx}+Au_x+Bu_t+Cu+f(x,t,u)=F(x,t),\ \ (x,t)\in D_T, \tag{1}$$
$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\ \ 0\le x\le l, \tag{2}$$
$$(Mu_x+Nu_t+Su)(0,t)=0,\quad u(l,t)=0,\ \ 0\le t\le T, \tag{3}$$

where A, B, C, M, N, S are given n-th order quadratic real matrix-functions; $f=(f_1,\dots,f_n)$, $F=(F_1,\dots,F_n)$, $\varphi=(\varphi_1,\dots,\varphi_n)$ and $\psi=(\psi_1,\dots,\psi_n)$ are given and $u=(u_1,\dots,u_n)$ is an unknown real vector-functions, $n\ge 2$.

Below we consider the problem (1)–(3) in a classical statement, when its regular solution is searched in the class $C^2(\overline{D}_T)$ and it is supposed that the problem data have corresponding smoothness and in the points $(0,0)$ and $(l,0)$ satisfy second order agreement conditions.

Divide the domain D_l, being a quadrat with the center in $O_1(\frac{l}{2},\frac{l}{2})$, into four triangles:

$$D_l^1:=OO_1O_2,\ \ D_l^2:=OO_1O_3,\ \ D_l^3:=O_2O_1O_4,\ \ D_l^4:=O_3O_1O_4,$$

where

$$O=(0,0),\ \ O_2=(l,0),\ \ O_3=(0,l),\ \ O_4=(l,l).$$

Assuming that

$$\det(M-N)(0,t)\neq 0,\ \ 0\le t\le l,$$

the problem (1)–(3) can be equivalently reduced to the Volterra type nonlinear integro-differential equation with respect to variable t

$$u(x,t)=(Tu)(x,t),\ \ (x,t)\in D_l,$$

where

$$\begin{aligned}(Tu)(x,t)&=\chi_1^1(x,t)\varphi(x-t)+\chi_2^1(x,t)\varphi(x+t)\\&\quad+\int\limits_{P_1^1P_2^1}\big[\Lambda_1^1(x,t;\xi)\varphi(\xi)+\Lambda_2^1(x,t;\xi)\psi(\xi)\big]\,d\xi\\&\quad+\int\limits_{D_{x,t}^1}K_1(x,t;\xi,\eta)\big[F(\xi,\eta)-f(\xi,\eta,u)\big]\,d\xi\,d\eta,\ \ P^1(x,t)\in D_l^1,\end{aligned} \tag{4}$$

where $P_1^1=(x-t,0)$, $P_2^1=(x+t,0)$, $D_{x,t}^1$ is a triangle $P_1^1P^1P_2^1$, while χ_i^1, Λ_i^1 and K_1 well-defined matrices;

$$\begin{aligned}(Tu)(x,t)=\chi_1^2(x,t)\varphi(0)+\chi_2^2(x,t)\varphi(t-x)+\chi_3^2\varphi(t+x)\\ +\int\limits_{OP_3^2}\big[\Lambda_1^2(x,t;\xi)\varphi(\xi)+\Lambda_2^2(x,t;\xi)\psi(\xi)\big]\,d\xi\\ +\int\limits_{D_{x,t}^2}K_2(x,t;\xi,\eta)\big[F(\xi,\eta)-f(\xi,\eta,u)\big]\,d\xi\,d\eta,\ \ P^2(x,t)\in D_l^2,\end{aligned}\tag{5}$$

where $P_1^2=(0,t-x)$, $P_2^2=(t-x,0)$, $P_3^2=(t+x,0)$, $D_{x,t}^2$ is a quadrangle $OP_1^2P^2P_3^2$, while χ_i^2, Λ_i^2 and K_2 are well-defined matrices;

$$\begin{aligned}(Tu)(x,t)=\chi_1^3(x,t)\varphi(x-t)+\chi_2^3(x,t)\varphi(x+t-l)\\ +\int\limits_{P_1^3O_1}\big[\Lambda_1^3(x,t;\xi)\varphi(\xi)+\Lambda_2^3(x,t;\xi)\psi(\xi)\big]\,d\xi\\ +\int\limits_{D_{x,t}^3}K_3(x,t;\xi,\eta)\big[F(\xi,\eta)-f(\xi,\eta,u)\big]\,d\xi\,d\eta,\ \ P^3(x,t)\in D_l^3,\end{aligned}\tag{6}$$

where $P_1^3=(x-t,0)$, $P_2^3=(x+t-l,0)$, $P_3^3=(l,x+t-l)$, $D_{x,t}^3$ is a quadrangle $P_3P_1^3O_1P_3^3$, while χ_i^3, Λ_i^3 and K_3 are well-defined matrices;

$$\begin{aligned}(Tu)(x,t)=\chi_1^4(x,t)\varphi(0)+\chi_2^4(x,t)\varphi(t-x)+\chi_3^4(x,t)\varphi(2l-x-l)\\ +\int\limits_{OO_1}\big[\Lambda_1^4(x,t;\xi)\varphi(\xi)+\Lambda_2^4(x,t;\xi)\psi(\xi)\big]\,d\xi\\ +\int\limits_{D_{x,t}^4}K_4(x,t;\xi,\eta)\big[F(\xi,\eta)-f(\xi,\eta,u)\big]\,d\xi\,d\eta,\ \ P^4(x,t)\in D_l^4,\end{aligned}\tag{7}$$

where $P_1^4=(0,t-x)$, $P_2^4=(t-x,0)$, $P_3^4=(2l-x-t,0)$, $P_4^4=(l,x+t-l)$, $D_{x,t}^4$ is a quadrangle $P^4P_1^4OO_1P_4^4$, while χ_i^4, Λ_i^4 and K_4 are well-defined matrices.

For $f=0$ the formulas (4)–(7) give the solution of the posed linear problem in quadratures.

Notice, on supposition that $f\in C(D_\infty\times\mathbb{R})$ the problem (1)–(3) is locally always solvable, i.e. there exists a number $T_0=T_0(F,\varphi,\psi)>0$ such that for $T<T_0$ the problem is solvable in domain D_T. Besides, without additional requirements on the increment of nonlinearity of vector-function f and its structure, the problem (1)–(3) may not have a solution.

Relationships Between Different Kinds of Stochastic Stability for Functional Differential Equations

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1 Notation and preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis (see, e.g. [5]), where Ω is a set of elementary probability events, $\mathcal{F}$ is a σ-algebra of all events on Ω, $(\mathcal{F}_t)_{t\geq 0}$ is a right continuous family of σ-subalgebras of $\mathcal{F}$, P is a probability measure on $\mathcal{F}$; all the above σ-algebras are assumed to be complete with respect to (w.r.t. in what follows) the measure P, i.e. they contain all subsets of zero measure; the symbol E stands for the expectation related to the probability measure P.

In the sequel, we use an arbitrary yet fixed norm $|\cdot|$ in R^n, the real-valued index p satisfying the assumption $0\leq p\leq\infty$ and a continuous positive function $\gamma(t)$ defined for all $t\geq 0$.

By $Z=(z_1,\dots,z_m)^T$ we denote an m-dimensional semimartingale (see, e.g. [5]). A most popular particular case of Z is the standard Brownian motion (the Wiener process) $\mathcal{B}=(\mathcal{B}_1,\dots,\mathcal{B}_m)^T$.

The general linear stochastic functional differential equation is defined as follows (see, e.g. [2]):

$$dx(t)=(Vx)(t)\,dZ(t)\ \ (t\geq 0), \tag{1.1}$$

and the initial condition reads in this case as

$$x(0)=x_0\in R^n. \tag{1.2}$$

Here V is a k-linear Volterra operator (see below), which is defined in certain linear spaces of vector-valued stochastic processes.

By the k-linearity of the operator V we mean the property

$$V(\alpha_1x_1+\alpha_2x_2)=\alpha_1Vx_1+\alpha_2Vx_2,$$

which holds for all $\mathcal{F}_0$-measurable, bounded and scalar random values α_1, α_2 and all stochastic processes x_1, x_2 belonging to the domain of the operator V.

According to the paper [3] the following classes of linear stochastic equations can be rewritten in the form (1.2):

(a) *Systems of linear ordinary* (*i.e. non-delay*) *stochastic differential equations driven by an arbitrary semimartingale* (*in particular, systems of ordinary Itô equations*);

(b) *Systems of linear stochastic differential equations with discrete delays driven by a semimartingale* (*in particular, systems of Itô equations with discrete delays*);

(c) *Systems of linear stochastic differential equations with distributed delays driven by a semimartingale* (*in particular, systems of Itô equations with distributed delays*);

(d) *Systems of linear stochastic integro-differential equations driven by a semimartingale* (*in particular, systems of Itô integro-differential equations*);

(e) *Systems of linear stochastic functional difference equations driven by a semimartingale* (*in particular, systems of Itô functional difference equations*).

2 Lyapunov stability and M-stability

In this section we study different kinds of stochastic Lyapunov stability of the zero solution of the linear equation (1.1) with respect to the initial data (1.2). Let us start with the precise definitions.

Definition 2.1. The zero solution of the equation (1.1) is called

1. *weakly stable in probability* if for any $\varepsilon > 0$, $\delta > 0$ there is $\eta(\varepsilon, \delta) > 0$ such that $P\{\omega \in \Omega: |x(t, x_0)| > \varepsilon\} < \delta$ for all $|x_0| < \eta$ and $t \geq 0$;

2. *asymptotically weakly stable in probability* if it is weakly stable in probability and if, in addition, for any $\varepsilon > 0$ and all $x_0 \in R^n$ one has
$$P\big\{\omega \in \Omega: \ |x(t, x_0)| > \ \varepsilon\big\} \longrightarrow 0 \ \text{ as } \ t \to +\infty;$$

3. *stable in probability* if for any $\varepsilon, \delta > 0$ there is $\eta(\varepsilon, \delta) > 0$ such that
$$P\Big\{\omega \in \Omega: \ \sup_{t \geq 0} |x(t, x_0)| > \varepsilon\Big\} < \delta \ \text{ for all } \ |x_0| < \eta;$$

4. *asymptotically stable in probability* if it is stable in probability and if, in addition, for any $\varepsilon > 0$ and all $x_0 \in R^n$ one has $P\{\omega \in \Omega: \ |x(t, x_0)| > \varepsilon\} \to 0$ as $t \to +\infty$;

5. p-stable if for any $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that $|x_0| < \eta$ implies $E|x(t, x_0)|^p \leq \varepsilon$ for all $t \geq 0$;

6. asymptotically p-stable if it is p-stable and, in addition, $\lim\limits_{t \to +\infty} E|x(t, x_0)|^p = 0$ for all $x_0 \in R^n$;

7. exponentially p-stable if there exist positive constants K, β such that the inequality
$$E|x(t, x_0)|^p \leq K|x_0|^p \exp\{-\beta t\}$$
holds true for all $t \geq 0$ and all $x_0 \in R^n$;

8. *stable with probability* 1 if $\sup\limits_{t \geq 0} |x(t, x_\nu)| \to 0$ with probability 1 whenever $|x_\nu| \to 0$ as $\nu \to +\infty$;

9. *asymptotically stable with probability* 1 if it is stable with probability 1 and if, in addition, $|x(t, x_0)| \to 0$ as $t \to +\infty$ for all $x_0 \in R^n$;

10. *strongly stable with probability* 1 if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that
$$P\Big\{\omega \in \Omega : \sup_{t\geq 0} |x(t,x_0)| \leq \varepsilon\Big\} = 1$$
whenever $|x_0| < \eta$;

11. *strongly asymptotically stable with probability* 1 if it is strongly stable with probability 1 and if, in addition, for any $\varepsilon > 0$, $x(t,x_0)$ tends to 0 with probability 1 as $t \to +\infty$ for all $x_0 \in R^n$.

Remark 2.2. The initial condition x_0 can also be random. In this case the norm of x_0 should be adjusted accordingly.

For brevity, we will also write "the equation (1.1) is stable" in a certain sense instead of "the zero solution of the equation (1.1) is stable" in this sense.

In the sequel the following linear spaces of stochastic processes will be used:

- $L^n(Z)$ consists of all predictable $n \times m$-matrix stochastic processes on $[0,+\infty)$, the rows of which are locally integrable w.r.t. the semimartingale Z (see, e.g. [5]);
- D^n consists of all n-dimensional stochastic processes on $[0,+\infty)$, which can be represented as
$$x(t) = x(0) + \int_0^t H(s)\,dZ(s),$$
where $x(0) \in R^n$, $H \in L^n(Z)$.

In addition to Lyapunov stability, one can consider the so-called "M-stability".

Definition 2.3. Let $x(\,\cdot\,,x_0)$ be the solution of the initial value problem (1.1)–(1.2) defined on $[0,\infty)$ and M be a certain subspace of the space D^n. We say that the equation (1.1) is M-stable if $x(\,\cdot\,,x_0) \in M$ for any $x_0 \in R^n$.

The spaces below ("M-spaces") are crucial for studying the stochastic Lyapunov stabilities listed above.

- $$M_0^\gamma = \Big\{x:\ x \in D^n \text{ such that for any } \delta > 0 \text{ there is } K > 0, \text{ for which } \sup_{t\geq 0} P\big\{\omega:\ \omega \in \Omega,\ |\gamma(t)x(t)| > K\big\} < \delta\Big\};$$
- $$\widehat{M}_0^\gamma = \Big\{x:\ x \in D^n \text{ such that for any } \delta > 0 \text{ there is } K > 0, \text{ for which } P\big\{\omega:\ \omega \in \Omega,\ \sup_{t\geq 0} |\gamma(t)x(t)| > K\big\} < \delta\Big\};$$
- $$M_p^\gamma = \big\{x:\ x \in D^n,\ \sup_{t\geq 0} E|\gamma(t)x(t)|^p < \infty\big\}\ (0 < p < \infty);$$
- $$\widehat{M}_p^\gamma = \big\{x:\ x \in D^n,\ E\sup_{t\geq 0} |\gamma(t)x(t)|^p < \infty\big\}\ (0 < p < \infty);$$
- $$M_\infty^\gamma = \widehat{M}_\infty^\gamma = \big\{x:\ x \in D^n,\ \operatorname*{ess\,sup}_{(t,\omega)\in[0,+\infty[\,\times\Omega} |\gamma(t)x(t)| < \infty\big\};$$

For $\gamma(t) = 1$ $(t \geq 0)$ we also put

- $M_p^1 = M_p$ and $\widehat{M}_p^1 = \widehat{M}_p$ $(0 \le p \le \infty)$.

Let B be a linear subspace of the space $L^n(Z)$ equipped with some norm $\|\cdot\|_B$. For a given positive and continuous function $\gamma(t)$ $(t \in [0,\infty))$ we define $B^\gamma = \{f : \ f \in B, \ \gamma f \in B\}$. The latter space becomes a linear normed space if we put $\|f\|_{B^\gamma} := \|\gamma f\|_B$. By this, the linear spaces M_p^γ, $\widehat{M}_p^\gamma$ become normed spaces if $1 \le p \le \infty$.

Remark 2.4. The above spaces can also be described as follows. Let $L_\infty(X)$ be the space consisting of all essentially bounded functions $g : [0,\infty) \to X$, while $\mathcal{L}_p(Y)$ be the space of measurable $(p = 0)$, p-integrable $(0 < p < \infty)$, essentially bounded $(p = \infty)$ functions $h : \Omega \to Y$, where X and Y are arbitrary separable Banach spaces. Then it is easy to see that $M_p^\gamma = L_\infty(\mathcal{L}_p(R^n))$ and $\widehat{M}_p^\gamma = \mathcal{L}_p(L_\infty(R^n))$ for all $0 \le p \le \infty$ and an arbitrary positive and continuous function $\gamma : [0,\infty) \to R$. This means that the above list of the M-spaces covers all possible combinations of Lebesgue spaces with respect to the variable $\omega \in \Omega$ and spaces of essentially bounded functions with respect to the variable $t \in [0,\infty)$. As we will see, this list covers also all types of stochastic Lyapunov stability described in Definition 2.1.

Below we use the following assumptions on a continuous positive function $\gamma(t)$, $t \in [0,\infty)$:

Property $\gamma 1$: the function γ satisfies the conditions $\gamma(t) \ge \sigma$ $(t \in [0,+\infty))$, $\sigma > 0$ and $\lim\limits_{t\to+\infty} \gamma(t) = +\infty$.

Property $\gamma 2$: $\gamma(t) = \exp\{\beta t\}$ for some $\beta > 0$.

The theorem below describes relationships between the different kinds of the stochastic Lyapunov stability and the associated M-stabilities.

Theorem 2.5. *The following statements are valid for the equation* (1.1)*:*

1. *weak stability in probability is equivalent to the M_0-stability;*
2. *weak asymptotic stability in probability is equivalent to the M_0^γ-stability for some γ satisfying Property $\gamma 1$;*
3. *stability in probability is equivalent to the $\widehat{M}_0$-stability;*
4. *if $0 < p < \infty$, then p-stability is equivalent to the M_p-stability;*
5. *if $0 < p < \infty$, then asymptotic p-stability is equivalent to the M_p^γ-stability for some γ satisfying Property $\gamma 1$;*
6. *if $0 < p < \infty$, then exponential p-stability is equivalent to the M_p^γ-stability for some γ satisfying Property $\gamma 2$;*
7. *stability with probability* 1 *is equivalent to the $\widehat{M}_0$-stability;*
8. *strong stability with probability* 1 *is equivalent to the M_∞-stability;*
9. *strong asymptotic stability with probability* 1 *is equivalent to the M_∞^γ-stability for some γ satisfying Property $\gamma 1$.*

Using these results we can study relationships between different kinds of stochastic Lyapunov stability and M-stability.

Corollary 2.6. *Let $p \in [0,\infty]$. Then the following are valid for the stochastic functional differential equation* (1.1)*:*

1. $\widehat{M}_p$*-stability implies stability with probability* 1*;*
2. $\widehat{M}_p^\gamma$*-stability with* γ *satisfying Property* $\gamma 1$ *implies asymptotic stability with probability* 1*.*
3. $\widehat{M}_\infty^\gamma$*-stability with* γ *satisfying Property* $\gamma 1$ *implies strong asymptotic stability with probability* 1*.*

Corollary 2.7. *For the equation* (1.1) *we have:*

1. *if* $0 < q < p < \infty$*, then* p*-stability* (*resp. asymptotic, exponential* p*-stability*) *implies* q*-stability* (*resp. asymptotic, exponential* q*-stability*)*;*
2. *if* $0 < p < \infty$*, then* p*-stability* (*resp. asymptotic* p*-stability*) *implies weak stability in probability* (*resp. weak asymptotic stability in probability*)*;*
3. *stability in probability* (*resp. asymptotic stability in probability*) *implies weak stability with probability* 1 (*resp. weak asymptotic stability with probability* 1)*.*
4. *stability in probability is equivalent to stability with probability* 1*.*

The proof of the theorem and the corollaries as well as some applications can be found in [4].

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On the Solvability of the Boundary value Problem for One Class of Higher-Order Semilinear Partial Differential Equations

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In the cylindrical domain $D_T := \Omega \times (0,T)$, where Ω is a Lipschitz domain in $\mathbb{R}^n$, consider a boundary value problem on finding a solution $u = u(x,t)$ to the equation

$$L_f := \frac{\partial^{4k} u}{\partial t^{4k}} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \Big(a_{ij} \frac{\partial u}{\partial x_i} \Big) + f(u) = F \tag{1}$$

by the boundary conditions

$$u\big|_{\Gamma} = 0, \tag{2}$$

$$\frac{\partial^i u}{\partial t^i}\bigg|_{\Omega_0 \cup \Omega_T} = 0, \ \ i = 0, \dots, 2k-1, \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function, $a_{ij} = a_{ji} = a_{ij}(x) \in C^1(\overline{\Omega})$, $i,j = 1, \dots, n$, $F = F(x,t)$ are the given, and $u = u(x,t)$ is an unknown real functions, k is a natural number, $n \geq 2$. Here $\Gamma := \partial\Omega \times (0,T)$ is the lateral face of the cylinder D_T, $\Omega_0 : \ x \in \Omega, \ t = 0$ and $\Omega_T : \ x \in \Omega, \ t = T$ are upper and lower bases of this cylinder, respectively.

Below, we assume that operator $K := \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_j} \big(a_{ij}(x) \frac{\partial u}{\partial x_i} \big)$ is evenly elliptical in $\overline{\Omega}$, i.e.

$$k_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq k_1 |\xi|^2 \ \ \forall\, x \in \overline{\Omega}, \ \ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n, \tag{4}$$

where $k_0, k_1 = const > 0$, $|\xi|^2 = \sum\limits_{i=1}^{n} \xi_i^2$. Note that (4) implies the hypoellipticity of the linear part of operator L_f from (1), i.e. L_0 is hypoelliptic for each $x = x_0 \in \overline{\Omega}$.

Denote by $C^{2,4k}(\overline{D}_T, \partial D_T)$ the space of functions u continuous in $\overline{D}_T$, having continuous partial derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $\frac{\partial^l u}{\partial t^l}$, $i,j = 1, \dots, n$; $l = 1, \dots, 4k$, in $\overline{D}_T$. Assume

$$C_0^{2,4k}(\overline{D}_T, \partial D_T) := \bigg\{ u \in C^{2,4k}(\overline{D}_T) : \ u\big|_{\Gamma} = 0, \ \frac{\partial^i u}{\partial t^i}\bigg|_{\Omega_0 \cup \Omega_T} = 0, \ i = 0, \dots, 2k-1 \bigg\}.$$

Introduce the Hilbert space $W_0^{1,2k}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)} = \int\limits_{D_T} \bigg[u^2 + \sum_{i=1}^{2k} \Big(\frac{\partial^i u}{\partial t^i} \Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u}{\partial x_i} \Big)^2 \bigg] \, dx \, dt$$

of the classical space $C_0^{2,4k}(\overline{D}_T, \partial D_T)$.

Remark 1. From definition of the space $W_0^{1,2k}(D_T)$ it follows that if $u \in W_0^{1,2k}(D_T)$, then $u \in \overset{\circ}{W}{}_2^1(D_T)$ and $\frac{\partial^i u}{\partial t^i} \in L_2(D_T)$, $i = 2, \dots, 2k$. Here $W_2^1(D_T)$ is the well-known Sobolev space consisting of the elements of $L_2(D_T)$, having the first order generalized derivatives from $L_2(D_T)$, and $\overset{\circ}{W}{}_2^1(D_T) = \{u \in W_2^1(D_T) : \ u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the sense of the trace theory.

Below, on the function $f = f(u)$ we impose the following requirements

$$f \in C(\mathbb{R}), \ \ |f(u)| \le M_1 + M_2 |u|^\alpha, \ \ u \in \mathbb{R}, \tag{5}$$

where $M_i = const \ge 0$, $i = 1, 2$, and

$$0 \le \alpha = const < \frac{n+1}{n-1}\,. \tag{6}$$

Remark 2. The embedding operator $I : W_2^1(\overline{D}_T) \to L_q(D_T)$ represents a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$. At the same time the Nemitski operator $N : L_q(D_T) \to L_2(D_T)$, acting by the formula $Nu = -f(u)$, due to (5) is continuous and bounded if $q \ge 2\alpha$. Thus, since due to (6) we have $2\alpha < \frac{2(n+1)}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Therefore, in this case the operator

$$N_0 = NI : \overset{\circ}{W}{}_2^1(D_T) \to L_2(D_T)$$

will be continuous and compact. Besides, from $u \in W_0^{1,2k}(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u_m \to u$ in the space $W_0^{1,2k}(D_T)$, then $f(u_m) \to f(u)$ in the space $L_2(D_T)$.

Definition 1. Let function f satisfy the conditions (5) and (6), $F \in L_2(D_T)$. The function $u \in W_0^{1,2k}(D_T)$ is said to be a weak generalized solution of the problem (1)–(3), if for any $\varphi \in W_0^{1,2k}(D_T)$ the integral equality

$$\int\limits_{D_T} \bigg[\frac{\partial^{2k} u}{\partial t^{2k}} \cdot \frac{\partial^{2k} \varphi}{\partial t^{2k}} + \sum_{i,j=1}^{n} a_{ij}(x)\, \frac{\partial u}{\partial x_i}\, \frac{\partial \varphi}{\partial x_j} \bigg]\, dx\, dt + \int\limits_{D_T} f(u)\varphi\, dx\, dt \\ = \int\limits_{D_T} F\varphi\, dx\, dt \ \ \forall\, \varphi \in C_0^{2,4k}(\overline{D}_T, \partial D_T)$$

is valid.

It is not difficult to verify that if the solution of the problem (1)–(3) in the sense of Definition 1 belongs to the class $C_0^{2,4k}(D_T, \partial D_T)$, then it will also be a classical solution of this problem.

Theorem. *Let the conditions* (5), (6) *and*

$$\lim_{|u| \to \infty} \inf \frac{f(u)}{u} \ge 0 \tag{7}$$

be fulfilled. Then for any $F \in L_2(D_T)$ the problem (1)–(3) *has at least one weak generalized solution $u \in W_0^{1,2k}(D_T)$.*

Remark 3. Let us note that if along with the conditions (5)–(7) imposed on function f to demand that it is monotonous, then the solution $u \in W_0^{1,2k}(D_T)$ of the problem (1)–(3), the existence of which is stated in the theorem, is unique. As show the examples, when the conditions imposed on nonlinear function f are violated, then the problem (1)–(3) may not have a solution.

Existence of Optimal Controls for Functional-Differential Systems on Semi Axis

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We study functional-differential equations on the semi-axis which are nonlinear with respect to the phase variables and linear with respect to the control. Sufficient conditions for existence of optimal control in terms of the right-hand side and the quality criterion are obtained. Connection between the solutions of the problems on infinite and finite intervals is studied and results about these connections are proven.

Let $h > 0$ be a constant, describing the delay. By $|\cdot|$ we denote a vector norm in R^d, and by $\|\cdot\|$ the norm of $d \times m$-matrices, which agrees with the vector norm. We introduce the necessary functional spaces which we use in this paper. Let $C = C([-h,0];R^d)$ be the Banach space of continuous functions from $[-h,0]$ into R^d with the uniform norm $\|\varphi\|_C = \max\limits_{\theta\in[-h,0]} |\varphi(\theta)|$, and let $L_p = L_p([-h,0];R^m)$, $p > 1$ be the Banach space of p-integrable m-dimensional vector-valued functions with the norm

$$\|\varphi\|_{L_p} = \bigg(\int\limits_{-h}^{0} |\varphi(s)|^p\,ds\bigg)^{1/p}.$$

Let x be continuous function on $[0,\infty)$ and let $\varphi \in C$. If $x(0) = \varphi(0)$, then the function

$$x(t,\varphi) = \begin{cases} \varphi(t), & t \in [-h,0], \\ x(t), & t \geq 0 \end{cases}$$

is continuous for $t \geq 0$. In the standard way for each $t \geq 0$ we can introduce an element $x_t(\varphi) \in C$ by the expression $x_t(\varphi) = x(t+\theta,\varphi)$, $\theta \in [-h,0]$. Further, instead of $x_t(\varphi)$ we write x_t.

Let $t \in [0,\infty)$, and D be a domain in $[-h,\infty) \times C$ with boundary ∂D.

In this paper, we study optimal control problems for systems of functional-differential equations ($\dot{x} = dx(t)/dt$)

$$\dot{x}(t) = f_1(t,x_t) + \int\limits_{-h}^{0} f_2(t,x_t,y)u(t,y)\,dy, \;\; t \in [0,\tau], \;\; x(t) = \varphi_0(t), \;\; t \in [-h,0], \tag{1}$$

with one of the next cost criterion

$$J[u] = \int\limits_{0}^{\tau} \big(e^{-\gamma t}A(t,x_t) + B(t,u(t,\cdot))\big)\,dt \longrightarrow \inf, \tag{2}$$

$$J[u] = \int\limits_{0}^{\tau} \bigg(e^{-\gamma t}A(t,x_t) + \int\limits_{-h}^{0} |u(t,y)|^2\,dy\bigg) \longrightarrow \inf. \tag{3}$$

These problems are considered on the infinite horison $t \geq 0$. Here $\varphi_0 \in C$ is a fixed element such that $(0, \varphi_0) \in D$, $x(t)$ is the phase vector in R^d, and x_t is the corresponding phase vector in C, τ is the moment when (t, x_t) reaches the boundary ∂D for the first time or $\tau = \infty$ otherwise. Also, $f_1 : D \to R^d$, $f_2 : D \times [-h, 0] \to M^{d\times m}$ – $d \times m$-dimensional matrix such that for each $(t, \varphi) \in D$ $f_2(t, \varphi, \cdot\,)$ belongs to the space $L_q([-h, 0]; M^{d\times m})$ with the norm

$$\|f_2(t,\varphi)\|_{L_q} = \bigg(\int\limits_{-h}^{0} \|f_2(t,\varphi,y)\|^q \, dy\bigg)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

$A : D \to R^+$, $B : [0, \infty) \times L_p \to R^+$ are given mappings.

The control parameter $u \in L_p([0, \infty) \times [-h, 0])$ is m-dimensional vector function such that for almost all $(t, y), u(t, y) \in W$, $0 \in W$, where W is a convex and closed set in R^m.

For each control function, we define corresponding solution (trajectory) of (1). A continuous function $x(t)$ is a solution of (1) on the interval $[-h, T]$, if it satisfies the following conditions: $x(t) = \varphi_0(t)$, $t \in [-h, 0]$; $(t, x_t) \in D$ for $t \in [0, T]$; for $t \in [0, T]$ $x(t)$ satisfies the integral equation

$$x(t) = \varphi_0(0) + \int\limits_{0}^{t} \bigg[f_1(s, x_s) + \int\limits_{-h}^{0} f_2(s, x_s, y)u(s, y)\, dy\bigg]\, ds.$$

The control function $u(t, \cdot\,)$ is considered admissible for the problems (1), (2) and (1), (3), if: $u(t, y) \in L_p([0, \infty) \times [-h, 0]$; $u(t, y) \in W$ for almost all $t \geq 0$, $y \in [-h, 0]$; the solution $x(t)$ corresponding to the control $u(t, \cdot\,)$ exists on the interval $[-h, \tau]$, $\tau > 0$; $|J[u]| < \infty$.

Let $V(\varphi_0)$ denote the Bellman function for the problem on the infinite horison and let $V_T(\varphi_0)$ be the Bellman function for the corresponding problem on some finite interval $[0, T]$.

In [4] it was shown that system (1) includes as particular cases the usual optimal control problem for functional-differential equations

$$\dot{x}(t) = f(t, x_t) + g(t, x_t)u(t), \quad u \in L_p([0, \infty); R^m), \tag{4}$$

for equations with maximum, and for system of ordinary differential equations.

The choice of the control $u(t, \cdot\,) \in L_p([0, \infty); [-h, 0])$ for each t as an element of the function space is justified (determined) by two reasons:

1) the given problem to be similar to the general functional-operator form of an optimal control problem where $u(t) \in W$ and W is a topological space (see, for example, [1]).

2) the given class of problems includes some problems with applications to economics (see [2,3]).

The goal of this work is to generalize the results obtained in [4] to the infinite horison $[0, \infty)$ and to clarify the relation between problems on finite and infinite intervals. It turns out that by the means of optimal control for finite interval, it is possible to construct easily minimizers for the problem on infinite horison.

Let D be a domain in $[-h, \infty) \times C$, and ∂D be its boundary. We introduce the notations $D_t = \{\varphi \in C, (t, \varphi) \in D\}$, $D_c = \bigcup\limits_{t\geq 0} D_t$, where D_c is bounded in C.

Assumption 1. *The admissible controls are m-dimensional vector functions $u(t, y) \in L_p([0, \infty) \times [-h, 0]; R^m)$ such that for almost all $t \geq 0$ and $y \in [-h, 0]$ we have $u(t, y) \in W$, where W is a convex closed set in R^m and $0 \in W$ and there exists $J[u]$.*

The set of admissible controls we denote as $\mathcal{U}$.

Assumption 2. *The mappings $f_1(t,\varphi):D\to R^d$ and $f_2(t,\varphi,y):D\times[-h,0]\to M^{d\times m}$ are defined and measurable with respect to all arguments in the domain D and $D_1=\{(t,\varphi)\in D, y\in[-h,0]\}$, respectively. Moreover, these functions satisfy in D and D_1, with respect to φ the condition for linear growth and the Lipchitz condition, i.e., there exists constant $K>0$ such that*

$$|f_1(t,\varphi)|+\|f_2(t,\varphi,y)\|\le K(1+\|\varphi\|_C), \tag{5}$$

for $(t,\varphi)\in D$, $y\in[-h,0]$,

$$\big|f_1(t,\varphi_1)-f_1(t,\varphi_2)\big|+\big\|f_2(t,\varphi_1,y)-f_2(t,\varphi_2,y)\big\|\le K\|\varphi_1-\varphi_2\|_C, \tag{6}$$

for $(t,\varphi_1),(t,\varphi_2)\in D$.

Assumption 3.

1) *The mapping $A:D\to R$, $A(t,\varphi)\ge 0$ for $(t,\varphi)\in D$ is defined and continuous in D and for $(t,\varphi)\in D$ there is a constant $K_A>0$ such that $A(t,\varphi)\le K_A(1+\|\varphi\|_C)$;*

2) *the mapping $B:[0,\infty)\times L_p\to R$ is measurable with respect to all its arguments and there are constants $a>0$, $a_1>0$ such that $a_1\|z\|^p_{L_p}\ge B(t,z)\ge a\|z\|^p_{L_p}$ if $t\ge 0$;*

3) *for each $t\ge 0$, $B(t,z)$ is strongly differentiable with respect to z and for $t\ge 0$ and $z\in L_p$ the Frechet derivative $\frac{\partial B}{\partial z}$ satisfies the estimate*

$$\Big\|\frac{\partial B}{\partial z}\Big\|_{\mathcal{L}(L_p;R^1)}\le a_2\|z\|^{p-1}_{L_p}$$

for some constant $a_2>0$, independently of t and z. Here $\|\cdot\|_{\mathcal{L}(L_p;R^1)}$ is the uniform operator norm in the space of linear continuous functionals over L_p.

The main results of this work are given by the following theorems.

Theorem 1. *Suppose that Assumptions* 1–3 *are satisfied. Then there exists a solution (x^*,u^*) of the problems* $(1),(2)$ *and* $(1),(3)$.

Let $T>0$ be fixed. By (x^*_T,u^*_T) we denote the solution of the problems $(1),(2)$ or $(1),(3)$ on $[0,T]$.

For the problem on infinite horison, we define

$$u_{T,\infty}(t,\,\cdot\,)=\begin{cases} u^*_T(t,\,\cdot\,), & t\in[0,T],\\ 0, & t>T\end{cases} \tag{7}$$

and $x^{T,\infty}(t)$ is corresponding trajectory.

It is obvious that the given control is admissible for the original problem. Again, $(u^*(t,\,\cdot\,),x^*(t))$ is an optimal pair for the problem $(1),(2)$, τ – the time at which the solution x^*_t reaches the boundary ∂D.

Theorem 2. *Suppose that Assumptions* 1–3 *are satisfied, then we have:*

1)
$$V_T(\varphi_0)\to V(\varphi_0),\ \ T\to\infty;$$

2) *there is a sequence $T_n\to\infty$, $n\to\infty$, such that the sequence $\{u_{T_n,\infty}\}$ is minimizer for the problem* $(1),(2)$, *i.e.*

$$J[u_{T_n,\infty}]\longrightarrow V,\ \ n\to\infty; \tag{8}$$

3) *there is a sequence $T_n \to \infty$, $n \to \infty$, such that*

$$u_{T_n,\infty} \xrightarrow{w} u^*, \quad n \to \infty \tag{9}$$

weekly in $L_p([0,\infty) \times [-h,0]; R^m)$;

4) *pointwise on $[0,\tau^*]$, uniformly on each finite interval*

$$x^{T_n,\infty}(t) \longrightarrow x^*(t), \quad n \to \infty.$$

If the problem $(1),(2)$ has unique solution, then the convergence in (8), (9) occurs for all $T \to \infty$.

Remark. In the conditions of Theorem 2 for the functional (3) all statements of Theorem 2 are valid, if the weak convergence of optimal controls (9) is replaced with strong convergence in $L_2([0,\infty) \times [-h,0]; R^m)$.

The next theorem is about the case when the domain D_c in the statement of the problem is unbounded. As it is shown in [4], the solution of the original problem cannot go to infinity in finite time. However, it can increase without bound in such a way that the integrals in (2) and (6) become divergent for all admissible controls. Now we give a theorem which guarantees existence of optimal control in this case. So, we assume that it is possible that D is unbounded domain in $[-h,\infty) \times C$ but the set of control values W is bounded in R^m. Without loss of generality, we can assume that W is a ball with radius r.

Theorem 3. *If the conditions of Theorem 1 are satisfied and $\gamma < (hr+1)K$, then the problems $(1),(2)$ and $(1),(3)$ have solutions.*

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The Dirichlet Problem for Second Order Essentially Singular Ordinary Differential Equations

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On a finite open interval $]a,b[$, we consider the differential equation

$$u'' = f(t,u) \tag{1}$$

with the Dirichlet boundary conditions

$$u(a+) = 0, \quad u(b-) = 0, \tag{2}$$

where $f :]a,b[\times \mathbb{R} \to \mathbb{R}$ is a continuous function, $u(a+)$ and $u(b-)$ are, respectively, the right and the left limits of the function u at the points a and b.

We are interested in the case where the function f has a nonintegrable singularity in the time variable at the points a and b.

In the earlier known theorems of the existence and uniqueness of a solution of the singular boundary value problem (1), (2) it was assumed that

$$\int\limits_a^b (t-a)(b-t)|f(t,0)|\,dt < +\infty$$

(see, e.g., [1–9] and the references therein). Unlike them, the results below cover the case when for arbitrary $x \in \mathbb{R}$ and $\ell > 0$ the condition

$$\int\limits_a^b (t-a)^{\ell}(b-t)^{\ell}|f(t,x)|\,dt = +\infty \tag{3}$$

is fulfilled. The results are new also for the linear differential equation

$$u'' = p(t)u + q(t), \tag{4}$$

where p and $q :]a,b[\to \mathbb{R}$ are continuous functions with singularities at the points a and b.

We use the following notation.

$\mathbb{R}$ is the set of real numbers, $[x]_- = \frac{|x|-x}{2}$.

Definition 1. The linear homogeneous differential equation

$$u'' = p(t)u \tag{4_0}$$

with continuous coefficients $p :]a,b[\to \mathbb{R}$ is said to be ***nonoscillatory in the interval*** $[a,b]$ if every its nontrivial solution, satisfying the initial condition

$$u(a+) = 0,$$

satisfies also the inequalities

$$u(t) \neq 0 \ \text{ for } \ a < t < b, \quad \liminf_{t\to b} |u(t)| > 0.$$

Definition 2. The function $G :]a,b[\times]a,b[\to\mathbb{R}$ is said to be ***Green's function of problem*** (4_0)**,** (2) if for every $s\in]a,b[$ the function $u(t)=G(t,s)$ is continuous in the interval $]a,b[$ and satisfies the boundary conditions (2), while the restrictions of u to $]a,s[$ and $]s,b[$ are the solutions of equation (4_0) and

$$u'(s+)-u'(s-)=1.$$

If G is Green's function of problem $(4_0),(2)$, we put

$$H(p)(s)=\sup\big\{|G(t,s)|:\ a<t<b\big\}\ \text{ for }\ a<s<b.$$

Proposition 1. *If*

$$\int\limits_a^b (t-a)(b-t)[p(t)]_-\,dt<+\infty \tag{5}$$

and the homogeneous problem $(4_0),(2)$ *has only the trivial solution, then there exists a unique Green's function of that problem, and*

$$\sup\Big\{\frac{H(p)(s)}{(s-a)(b-s)}\ :\ a<s<b\Big\}<+\infty.$$

Theorem 1. *If the homogeneous problem* $(4_0),(2)$ *has only the trivial solution and along with* (5) *the condition*

$$\int\limits_a^b H(p)(t)|q(t)|\,dt<+\infty \tag{6}$$

is fulfilled, then problem $(4),(2)$ *is uniquely solvable and its solution admits the representation*

$$u(t)=\int\limits_a^b G(t,s)q(s)\,ds\ \ \textit{for}\ \ a<t<b, \tag{7}$$

where G *is Green's function of problem* $(4_0),(2)$.

Corollary 1. *Let there exist a nondecreasing in some right neighbourhood of the point* a *and a nonincreasing in some left neighbourhood of the point* b *continuously differentiable function* $\delta :]a,b[\to]0,+\infty[$ *such that*

$$\delta(a+)=\delta'(a+)=0,\quad \delta(b-)=\delta'(b-)=0,$$
$$\liminf_{t\to a}(\delta^2(t)p(t))>0,\quad \liminf_{t\to b}(\delta^2(t)p(t))>0.$$

If, moreover,

$$\int\limits_a^b (t-a)(b-t)[p(t)]_-\,dt\le b-a,\qquad \int\limits_a^b \delta(t)|q(t)|\,dt<+\infty,$$

then problem $(4),(2)$ *is uniquely solvable and its solution admits representation* (7).

Remark 1. Green's formula (7) has been derived earlier only in the case, where

$$\int\limits_a^b (t-a)(b-t)|p(t)|\,dt<+\infty,\qquad \int\limits_a^b (t-a)(b-t)|q(t)|\,dt<+\infty$$

(see [6, Theorem 1.1]), but Theorem 1 covers the case in which these functions have at the points a and b singularities of infinite order. Indeed, if

$$\delta(t) \equiv \exp\Big(-\frac{1}{t-a}-\frac{1}{b-t}\Big),$$
$$p(t) \equiv p_0(t)\delta^{-2}(t), \quad q(t) \equiv q_0(t)\delta^{-1}(t),$$

where $p_0 :]a,b[\, \to]1,+\infty[\,$, $q_0 : [a,b] \to [1,+\infty[$ are arbitrary continuous functions, then for any $\ell > 0$, the equalities

$$\int\limits_a^b (t-a)^\ell (b-t)^\ell p(t)\,dt = +\infty, \quad \int\limits_a^b (t-a)^\ell (b-t)^\ell q(t)\,dt = +\infty$$

are fulfilled. Nevertheless, according to Corollary 1, problem (4), (2) is uniquely solvable and its solution admits representation (7).

Theorem 2. *Let on the set $]a,b[\,\times\mathbb{R}$ the inequality*

$$f(t,x)\operatorname{sgn}(x) \ge p(t)|x| + q(t) \tag{8}$$

be fulfilled, where $p :]a,b[\, \to \mathbb{R}$ and $q :]a,b[\, \to]-\infty,0]$ are continuous functions. If, moreover, the homogeneous equation (4_0) is nonoscillatory and conditions (5) *and* (6) *hold, then problem* (1), (2) *has at least one solution.*

Corollary 2. *Let on the set $]a,b[\,\times\mathbb{R}$ inequality* (8) *be fulfilled, where $p :]a,b[\, \to \mathbb{R}$ and $q :]a,b[\, \to [0,+\infty[$ are continuous functions and, in addition, p is continuously differentiable and nonincreasing (nondecreasing) in some right neighbourhood of the point a (in some left neighbourhood of the point b). If, moreover,*

$$p(a+) = +\infty, \ \lim_{t\to a}\big(p^{-3/2}(t)p'(t)\big) = 0, \quad p(b-) = +\infty, \ \lim_{t\to b}\big(p^{-3/2}(t)p'(t)\big) = 0,$$
$$\int\limits_a^b (t-a)(b-t)[p(t)]_-\,dt \le b-a, \quad \int\limits_a^b \frac{|q(t)|}{\sqrt{1+|p(t)|}}\,dt < +\infty,$$

then problem (1), (2) *has at least one solution.*

Theorem 3. *Let on the set $]a,b[\,\times\mathbb{R}$ the condition*

$$(f(t,x)-f(t,y))\operatorname{sgn}(x-y) \ge p(t)|x-y| \tag{9}$$

be fulfilled, where $p :]a,b[\, \to \mathbb{R}$ is a continuous function satisfying condition (5). *If, moreover, the homogeneous equation (4_0) is nonoscillatory and*

$$\int\limits_a^b H(p)(t)|f(t,0)|\,dt < +\infty,$$

then problem (1), (2) *has one and only one solution.*

Corollary 3. *Let on the set $]a,b[\,\times\mathbb{R}$ condition* (9) *be fulfilled, where $p :]a,b[\, \to \mathbb{R}$ is a function satisfying the conditions of Corollary* 2. *If, moreover,*

$$\int\limits_a^b \frac{|f(t,0)|}{\sqrt{1+|p(t)|}}\,dt < +\infty,$$

then problem (1), (2) *has one and only one solution.*

Example 1. Let

$$f(t,x)=\sum_{k=1}^{n}p_k(t)|x|^{\lambda_k}\operatorname{sgn}x+p_0(t)\exp\Big(\frac{2}{t-a}+\frac{2}{b-t}\Big)u+q_0(t)\exp\Big(\frac{1}{t-a}+\frac{1}{b-t}\Big),$$

where $p_k:]a,b[\to[0,+\infty[$ $(k=1,\dots,n)$, $p_0:]a,b[\to[1,+\infty[$, $q_0:[a,b]\to[1,+\infty[$ are continuous function, $\lambda_k=const>0$ $(k=1,\dots,n)$. Then for arbitrary $x\in\mathbb{R}$ and $\ell>0$ condition (3) is fulfilled. On the other hand, according to Corollary 3, problem (1), (2) has one and only one solution.

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Periodic Solutions of Higher Order Nonlinear Hyperbolic Equations

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Let $m_1,\dots,m_n$ be positive integers. Consider the periodic problem

$$u^{(\mathbf{m})}=f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big), \tag{1}$$

$$u(\mathbf{x}+\boldsymbol{\omega}_i)=u(\mathbf{x})\ \ (i=1,\dots,n). \tag{2}$$

Here $\mathbf{x}=(x_1,\dots,x_n)$, $\boldsymbol{\omega}=(\omega_1,\dots,\omega_n)$, $\boldsymbol{\omega}_{\mathbf{i}}=(0,\dots,\omega_i,\dots,0)$, $\mathbf{m}=(m_1,\dots,m_n)$ is a multi-index,

$$u^{(\mathbf{m})}(\mathbf{x})=\frac{\partial^{m_1+\cdots+m_n}u(\mathbf{x})}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}\,,$$

$\mathcal{D}^{\mathbf{m}}[u]=(u^{(\boldsymbol{\alpha})})_{\boldsymbol{\alpha}\le\mathbf{m}}$, $\widetilde{\mathcal{D}}^{\mathbf{m}}[u]=(u^{(\boldsymbol{\alpha})})_{\boldsymbol{\alpha}<\mathbf{m}}$, $f\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}+\mathbf{1}})$ and $C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}+\mathbf{1}})$ is the space of continuous functions $v(\mathbf{x},\mathbf{Z})$ that are $\boldsymbol{\omega}$-periodic with respect to the variable $\mathbf{x}$, i.e.

$$v(\mathbf{x}+\boldsymbol{\omega}_i,\mathbf{Z})=v(\mathbf{x},\mathbf{Z})\ \ (i=1,\dots,n).$$

By a solution of problem (1),(2) we understand a classical solution, i.e., a function $u\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ satisfying equation (1) everywhere in $\mathbb{R}^n$.

Problems on doubly periodic solutions for hyperbolic equations of the second and fourth orders were studied in [1–3]. Problem (1), (2) for the case $n>2$ remained virtually unstudied until recently. The linear case of problem (1), (2) was investigated in [4].

Throughout the paper the following notations will be used:

$\mathbf{m}=(m_1,\dots,m_n)$, $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)$.

$\mathbb{R}^{\boldsymbol{\alpha}}=\mathbb{R}^{\alpha_1\times\cdots\times\alpha_n}$.

$\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)<\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)\Longleftrightarrow\alpha_i\le\beta_i$ $(i=1,\dots,n)$ and $\boldsymbol{\alpha}\ne\boldsymbol{\beta}$.

$\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\le\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)\Longleftrightarrow\boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.

$\mathbf{0}=(0,\dots,0)$, $\mathbf{1}=(1,\dots,1)$, $\mathbf{1}_i=(0,\dots,0,1,0,\dots,0)$.

$\operatorname{supp}\boldsymbol{\alpha}=\{i\ \ \alpha_i>0\}$, $\|\boldsymbol{\alpha}\|=|\alpha_1|+\cdots+|\alpha_n|$.

$\boldsymbol{\Upsilon}_{\mathbf{m}}=\{\boldsymbol{\alpha}<\mathbf{m}:\ \alpha_i=m_i$ for some $i\in\{1,\dots,n\}\}$.

$\boldsymbol{\omega}=(\omega_1,\dots,\omega_n)$, $\boldsymbol{\omega}_{\mathbf{i}}=(0,\dots,\omega_i,\dots,0)$.

$\Omega=[0,\omega_1]\times\cdots\times[0,\omega_n]$.

$\mathbf{x}_{\boldsymbol{\alpha}}=(\chi(\alpha_1)x_1,\dots,\chi(\alpha_n)x_n)$, where $\chi(\alpha)=0$ if $\alpha=0$, and $\chi(\alpha)=1$ if $\alpha>0$. $\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(x_{i_1},\dots,x_{i_l})$, where $\{i_1,\dots,i_l\}=\operatorname{supp}\boldsymbol{\alpha}$.

$\mathbf{Z}=(z_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}<\mathbf{m}}$; $f_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z})=\frac{\partial f(\mathbf{x},\mathbf{Z})}{\partial z_{\boldsymbol{\alpha}}}$.

The variables $z_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha}\in\boldsymbol{\Upsilon}_{\mathbf{m}})$ are called ***principal phase variables*** of the function $f(\mathbf{x},\mathbf{Z})$.

$C^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u:\Omega\to\mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}\le\mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)}=\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

$C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the Banach space of $\boldsymbol{\omega}$-periodic continuous functions, i.e. functions that are ω_i-periodic with respect to the variable x_i $(i = 1, \dots, n)$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} \le \mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

$\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the Banach space of $\boldsymbol{\omega}$-periodic continuous functions, i.e. functions that are ω_i-periodic with respect to the variable x_i $(i = 1, \dots, n)$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} \le \mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

If $z_0 \in \widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ and $r > 0$, then

$$\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(z_0; r) = \Big\{ z \in \widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n) : \ \|z - z_0\|_{\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}} \le r \Big\}.$$

$C^{\mathbf{m},k}_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\boldsymbol{\beta}})$ the space of continuous functions $v(\boldsymbol{x}, \mathbf{Z})$ such that $v(\,\cdot\,, \mathbf{Z}) \in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ for every $\mathbf{Z} \in \mathbb{R}^{\boldsymbol{\beta}}$ and $v(\mathbf{x}, \,\cdot\,) \in C^k(\mathbb{R}^{\boldsymbol{\beta}})$ for every $\mathbf{x} \in \mathbb{R}^n$.

Let $p_{0\,\boldsymbol{\alpha}} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \boldsymbol{m})$ and let $z \in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ be an arbitrary function. Along with the equation (1) consider the following equations

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\lambda\,\boldsymbol{\alpha}}[z](\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{3}$$

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\lambda\,\boldsymbol{\alpha}}[z](\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{4}$$

and

$$u^{(\mathbf{m})} = (1-\lambda) \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{0\,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + \lambda f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big), \tag{5}$$

where $\lambda \in [0, 1]$, $p_{\boldsymbol{\alpha}}[z](\mathbf{x}) = f_{\boldsymbol{\alpha}}(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[z](\mathbf{x}))$, and

$$p_{\lambda\,\boldsymbol{\alpha}}[z](\mathbf{x}) = (1-\lambda) p_{0\,\boldsymbol{\alpha}}(\mathbf{x}) + \lambda p_{\boldsymbol{\alpha}}[z](\mathbf{x}).$$

Definition 1. Let the function $f(\mathbf{x}, \mathbf{Z})$ be continuously differentiable with respect to the phase variables $\mathbf{v}$. We say that problem (1), (2) to is *strongly* (u_0, r)*-well-posed*, if:

(I) it has a solution $u_0(\mathbf{x})$;

(II) in the neighborhood $\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0; r)$ u_0 is the unique solution;

(III) there exists $\varepsilon_0 > 0$, $\delta_0 > 0$ and $M_0 > 0$ such that for any $\delta \in (0, \delta_0)$, and $\widetilde{f}(\mathbf{x}, \mathbf{Z})$ satisfying the inequalities

$$\sum_{\boldsymbol{\alpha}<\mathbf{m}} \big| f_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) - \widetilde{f}_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) \big| < \varepsilon_0, \tag{6}$$

$$\big| f(\mathbf{x}, \mathbf{Z}) - \widetilde{f}(\mathbf{x}, \mathbf{Z}) \big| < \delta \tag{7}$$

in the neighborhood $\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0;r)$ the problem

$$u^{(\mathbf{m})}=\widetilde{f}(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]),$$
$$u(\mathbf{x}+\boldsymbol{\omega}_i)=u(\mathbf{x})\ \ (i=1,\dots,n)$$

has a unique solution $\widetilde{u}$ and

$$\|u-\widetilde{u}\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}<M_0\delta.$$

Definition 2. Problem (1), (2) is called *strongly well-posed* if it is strongly (u_0,r)-well-posed for every $r>0$.

Theorem 1. *Let the function $f(\mathbf{x},Z)$ be continuously differentiable with respect to the phase variables, and let there exist a positive number M_0 such that*

$$|f_{\boldsymbol{\alpha}}(\mathbf{x},Z)|\le M_0 \ \text{ for } \ (\mathbf{x},Z)\in\mathbb{R}^n\times\mathbb{R}^{\mathbf{m+1}}.$$

Furthermore, let for arbitrary $z\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ and $\lambda\in[0,1)$ problem (3), (2) be well–posed and its solution u_λ admit the estimate

$$\|u_\lambda\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}\le M\|q\|_{C_{\boldsymbol{\omega}}},$$

where M is a positive number independent of λ, z and q. Then problem (1), (2) is strongly well–posed.

Consider the "perturbed" equation

$$u^{(\mathbf{m})}=f(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u])+q(\mathbf{x},\mathcal{D}^{\mathbf{m\text{-}1}}[u]). \tag{8}$$

Theorem 2. *Let the function f satisfy all of the conditions of Theorem 1, and let $q\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}})$ be such that*

$$\lim_{\|\mathbf{Z}\|\to+\infty}\frac{|q(\mathbf{x},\mathbf{Z})|}{\|\mathbf{Z}\|}=0 \tag{9}$$

uniformly on $\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}}$. Then problem (8), (2) has at least one solution

Theorem 3. *Let the function $f(\mathbf{x},Z)$ be continuously differentiable with respect to the phase variables, and let there exist a positive number M and a nondecreasing continuous function $\eta:[0,+\infty)\to[0,+\infty)$, $\eta(0)=0$ such that:*

(i) *for every $\lambda\in[0,1)$ an arbitrary solution u_λ of problem (5), (2) admits the estimates*

$$u_\lambda\in\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(0;M),\ \ \|w_{\lambda\,\delta}\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}\le\eta(|\delta|),$$

where $w_{\lambda\,\delta}(\mathbf{x})=u_\lambda(\mathbf{x}+\delta)-u_\lambda(\mathbf{x})$;

(ii) *problem (4), (2) is well–posed for every $\lambda\in[0,1)$ and $z\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$, $\|z\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}\le M$;*

(iii) *problem (4), (2) has only the trivial solution for $\lambda=1$ and arbitrary $z\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$, $\|z\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}\le M$.*

Then problem (1), (2) has a solution $u_0\in\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(0;M)$, and it is strongly strongly *(u_0,r) well-posed for some $r>0$.*

Consider the equations of even and odd orders:

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\leq \mathbf{m}} \Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x}, \mathcal{D}^{\boldsymbol{\alpha}}[u]\big)u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m-1}}[u]), \tag{10}$$

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\leq \mathbf{m}} \Big(p_{\boldsymbol{\alpha}}\big(\mathbf{x}, \mathcal{D}^{\boldsymbol{\alpha}}[u]\big)\ u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\alpha})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m-1}}[u]) \tag{11}$$

and

$$u^{(2\mathbf{m}+\mathbf{1}_n)} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\leq \mathbf{m}} (p_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}_n}(\mathbf{x}, \mathcal{D}^{\boldsymbol{\alpha}+\mathbf{1}_n}[u]))^{(\boldsymbol{\beta})} + \sum_{\boldsymbol{\alpha}\leq \mathbf{m}} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})u^{(2\boldsymbol{\alpha})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m-1}}[u]). \tag{12}$$

Theorem 4. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\boldsymbol{\alpha}+\mathbf{1}})$ $(\boldsymbol{\alpha}, \boldsymbol{\beta} \leq \boldsymbol{m})$*, and let* $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}})$ *satisfy equality* (9) *uniformly on* $\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}}$*. Furthermore, let there exist* $\delta > 0$ *such that*

$$\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\leq \mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}, \mathbf{Z})v_{\boldsymbol{\alpha}}v_{\boldsymbol{\beta}} \geq \delta \sum_{\boldsymbol{\alpha}\leq \mathbf{m}} v^2_{\boldsymbol{\alpha}} \ \ \textit{for}\ \ (\mathbf{x}, \mathbf{Z}) \in \mathbb{R}^n \times \mathbb{R}^{\mathbf{2}\boldsymbol{m}+\mathbf{1}}.$$

Then problem (10), (2) *has at least one solution.*

Corollary 1. *Let* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{\alpha},\|\boldsymbol{\alpha}\|}_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\boldsymbol{\alpha}+\mathbf{1}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$*, and let* $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}})$ *satisfy equality* (9) *uniformly on* $\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}}$*. Furthermore, let there exist* $\delta > 0$ *such that*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) \geq \delta \ \ \textit{for}\ \ (\mathbf{x}, \mathbf{Z}) \in \mathbb{R}^n \times \mathbb{R}^{\mathbf{2}\boldsymbol{m}+\mathbf{1}} \ \ (\boldsymbol{\alpha} \leq \boldsymbol{m}).$$

Then problem (11), (2) *has at least one solution.*

Theorem 5. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\boldsymbol{\alpha}+\mathbf{1}})$ $(\boldsymbol{\alpha}, \boldsymbol{\beta} \leq \boldsymbol{m})$*, and let* $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}})$ *satisfy equality* (14) *uniformly on* $\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}}$*. Furthermore, let there exist* $\delta > 0$ *such that*

$$\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\leq \mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}_n}(\mathbf{x}, \mathbf{Z})z_{\boldsymbol{\alpha}}\, z_{\boldsymbol{\beta}} \geq \delta \sum_{\boldsymbol{\alpha}\leq \mathbf{m}} z^2_{\boldsymbol{\alpha}} \ \ \textit{for}\ \ (\mathbf{x}, \mathbf{Z}) \in \mathbb{R}^n \times \mathbb{R}^{\mathbf{2}\boldsymbol{m}+\mathbf{1}}$$

and

$$(-1)^{\|\boldsymbol{\alpha}\|}\sigma p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) \geq \delta \ \ \textit{for}\ \ \mathbf{x} \in \mathbb{R}^n \ \ (\boldsymbol{\alpha} \leq \boldsymbol{m}).$$

Then problem (12), (2) *has at least one solution.*

Remark 1. In Theorems 1–3 continuous differentiability of the function $f(\mathbf{x}, \mathbf{Z})$ with respect to the phase variables $\mathbf{Z}$ can be replaced by Lipschitz continuity, although that will make the formulation of the theorems more cumbersome. However, Lipschitz continuity of the function $f(\mathbf{x}, \mathbf{Z})$ with respect to the *principal* phase variables $z_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}})$ is essential and cannot be replaced by Hölder continuity with the exponent $\gamma \in (0, 1)$.

As an example consider the two–dimensional problem

$$u^{(2,2)} = u^{(2,0)} + u^{(0,2)} - \delta^{1-\gamma}|u^{(0,2)} - u|^{\gamma}\operatorname{sgn}(u^{(0,2)} - u) - u - \sin x_2, \tag{13}$$

$$u(x_1 + 2\pi, x_2) = u(x_1 + 2\pi, x_2), \quad u(x_1, x_2 + 2\pi) = u(x_1, x_2) \tag{14}$$

where $\delta \geq 0$ and $\gamma \in (0, 1)$.

Let u be a solution of problem (10), (11). Set:

$$v(x_1, x_2) = u^{(0,2)}(x_1, x_2) - u(x_1, x_2). \tag{15}$$

Then v is a solution of the problem

$$v^{(2,0)} = v - \delta^{1-\gamma}|v|^\gamma \operatorname{sgn}(v) - \sin x_2, \tag{16}$$

$$v(x_1 + 2\pi, x_2) = v(x_1, x_2). \tag{17}$$

If $\delta = 0$, then it is clear that problem (16), (17) is a uniquely solvable linear periodic problem with the solution

$$v(x_1, x_2) \equiv \sin x_2,$$

and problem (10), (11) is a well–posed linear problem with the solution

$$u(x_1, x_2) \equiv u(x_2) = \int\limits_{x_2-2\pi}^{x_2} \frac{\cosh(x_2 - t - \pi)}{2\sinh(\pi)} \sin t \, dt.$$

Let us show that problem (10), (11) has no classical solutions for sufficiently small $\delta > 0$. For that it is sufficient to show that for sufficiently small $\delta > 0$ problem (16), (17) has no solution that is continuous with respect to x_2.

Problem (16), (17) is a periodic problem for an ordinary differential equation depending on the parameter x_2. It has a solution $v(x_1, x_2) \equiv v^*(x_2)$, where, for every x_2, $v^*(x_2)$ is the root of the equation

$$v - \delta^{1-\gamma}|v|^\gamma \operatorname{sgn}(v) - \sin x_2 = 0. \tag{18}$$

One can easily show that problem (16), (17) is solvable for every x_2 if $\delta \in (0, 1)$. Moreover, if $\delta \in (0, 2^{\frac{1}{\gamma-1}})$, then problem (16), (17) is uniquely solvable for $x_2 = \frac{\pi}{2}$, and its solution is positive. The latter fact implies that $v^*(\frac{\pi}{2}) > \delta$.

Let $\delta \in \big(0, 2^{\frac{1}{\gamma-1}}\big)$, and let $v(x_1, x_2)$ be a solution of problem (16), (17) that is a continuous function of x_2. Then $v(x_1, \frac{\pi}{2}) = v^*(\frac{\pi}{2}) > \delta$. Due to continuity there exists $\varepsilon > 0$ such that

$$v(x_1, x_2) \geq \delta \ \text{ for } \ x_2 \in \Big[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\Big] \subset (0, \pi). \tag{19}$$

But then problem (16), (17) is uniquely solvable for $x_2 \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. Indeed, let $v_1(x_1) \geq \delta$ and $v_2(x_1) \geq \delta$ be arbitrary solutions of problem (16),(17) for some $x_2 \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. Then $v(x_1) = v_2(x_1) - v_1(x_1)$ is a solution of the problem

$$v'' = (1 - \theta(x_1, x_2))v, \quad v(x_1 + 2\pi) = v(x_1), \tag{20}$$

where

$$\theta(x_1, x_2) = \gamma \int\limits_0^1 \frac{\delta^{1-\gamma}}{(v_1(x_1, x_2) + (1-t)(v_2(x_1, x_2) - v_1(x_2, x_1)))^{1-\gamma}} \, dt \leq \gamma < 1. \tag{21}$$

The latter inequality implies that problem (20) has only the trivial solution, i.e. problem (16), (17) is uniquely solvable. Consequently, $v(x_1, x_2) = v^*(x_2)$ for $x_2 \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. However, it is easy to see that a positive root of equation (18) is strictly bigger than δ for $x_2 \in (0, \pi)$. Hence

$$v(x_1, x_2) = v^*(x_2) > \delta \ \text{ for } \ x_2 \in \Big[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\Big] \subset (0, \pi). \tag{22}$$

From (19)–(22) one can easily deduce that

$$v(x_1, x_2) = v^*(x_2) > \delta \ \text{ for } \ x_2 \in (0, \pi). \tag{23}$$

Similarly one can show that

$$v(x_1, x_2) = v^*(x_2) < -\delta \ \text{ for } \ x_2 \in (-\pi, 0). \tag{24}$$

(23) and (24) imply that $v^*(0+) = \delta$ and $v^*(0-) = -\delta$. Thus $v(x_1, x_2) \equiv v^*(x_2)$ is discontinuous at 0. Consequently, problem (13), (14) has no classical solutions for sufficiently small $\delta \in (0, 2^{\frac{1}{\gamma-1}})$.

This is the result of the fact that the righthand side of equation (13) is not Lipschitz continuous with respect to the principal phase variables, but instead is a Hölder continuous function with the exponent $\gamma \in (0, 1)$.

Remark 2. The aforementioned example also demonstrates that:

(A) Condition (6) in Definition 1 is optimal and cannot be relaxed;

(B) Only inequality (7), without inequality (6) does not guarantee even solvability of a perturbed problem.

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On One System of Nonlinear Partial Integro-Differential Equations with Source Terms

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Let us consider the following system of nonlinear integro-differential equations:

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\Big[a(S)\,\frac{\partial U}{\partial x}\Big]+f(U)=0,\quad \frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\Big[a(S)\,\frac{\partial V}{\partial x}\Big]+f(V)=0, \tag{1}$$

where

$$S(x,t)=1+\int\limits_0^t\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau$$

and $a=a(S)$, $f=f(U)$ and $f=f(V)$ are given functions, constraints on which will be specified later.

The above-mentioned system with source terms is based on the well-known system of Maxwell's equations [12] by reducing it to the following integro-differential model [4]

$$\frac{\partial H}{\partial t}=-\operatorname{rot}\Big[a\Big(\int\limits_0^t|\operatorname{rot}H|^2\,d\tau\Big)\operatorname{rot}H\Big], \tag{2}$$

where $H=(H_1,H_2,H_3)$ is a vector of the magnetic field.

In the rectangle $[0,1]\times[0,\infty]$ let us consider the following initial-boundary value problem with mixed boundary conditions:

$$U(0,t)=\frac{\partial U(x,t)}{\partial x}\Big|_{x=1}=V(0,t)=\frac{\partial V(x,t)}{\partial x}\Big|_{x=1}=0,\ \ t\ge 0, \tag{3}$$

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x),\ \ x\in[0,1], \tag{4}$$

where U_0 and V_0 are given functions.

Study of the models of type (2) have begun in [4]. In that work, in particular, based on Galerkin's modified method and compactness arguments as in [14, 18] for nonlinear parabolic equations the theorems of existence of a solution of the initial-boundary value problem with first kind boundary conditions for scalar and one-dimensional space case when $a(S)=1+S$ and uniqueness for more general cases are proven. One-dimensional scalar variant for the case $a(S)=(1+S)^p$, $0<p\le 1$ is studied in [2]. Asymptotic behavior as $t\to\infty$ of solutions of initial-boundary value problems for (2) type models are studied in [3, 6, 7, 9, 13, 16] and in a number of other works as well. In those works main attention is paid to one-dimensional cases. Finite element analogues and Galerkin's method algorithm as well as construction and investigation of semi-discrete and finite difference schemes for (2) type one-dimensional integro-differential models are studied in [1, 5, 7–11, 13, 15–17] and in other works as well for the linear case of diffusion coefficient.

The following statement is true [5, 8].

Theorem 1. *If $a = a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$, f is positively defined and monotonically increased function, $U_0, V_0 \in H^1(0,1)$, $U_0(0) = \frac{dU_0(x)}{dx}\big|_{x=1} = V_0(0) = \frac{dV_0(x)}{dx}\big|_{x=1} = 0$, and problem* (1), (3), (4) *has a solution, then it is unique and exponential stabilization of solution as $t \to \infty$ takes place.*

On $[0,1]\times[0,T]$, where T is a positive constant, let us introduce a net with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, \dots, M$; $j = 0, 1, \dots, N$ with $h = 1/M$, $\tau = T/N$ and let us consider the finite discrete scheme for problem (1), (3), (4):

$$\begin{gathered}
\frac{u_i^{j+1} - u_i^j}{\tau} - \bigg\{ a\Big(\tau \sum_{k=1}^{j+1} \big[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2\big]\Big) u_{\bar{x},i}^{j+1} \bigg\}_x + f(u_i^{j+1}) = 0, \\
\frac{v_i^{j+1} - v_i^j}{\tau} - \bigg\{ a\Big(\tau \sum_{k=1}^{j+1} \big[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2\big]\Big) v_{\bar{x},i}^{j+1} \bigg\}_x + f(v_i^{j+1}) = 0, \\
i = 1, 2, \dots, M-1; \;\; j = 0, 1, \dots, N-1, \\
u_0^j = u_{\bar{x},M}^j = v_0^j = v_{\bar{x},M}^j = 0, \;\; j = 0, 1, \dots, N, \\
u_i(0) = U_{0,i}, \quad v_i(0) = V_{0,i}, \;\; i = 0, 1, \dots, M,
\end{gathered} \tag{5}$$

where the well-known notations of forward and backward derivatives are used.

Applying the u_i^{j+1} and v_i^{j+1} multiplicators for the first and second equations of system (5) respectively, it is not difficult to get the inequalities:

$$\|u^n\|^2 + \tau h \sum_{j=1}^{n} \sum_{i=1}^{M} (u_{i,\bar{x}}^j)^2 < C, \quad \|v^n\|^2 + \tau h \sum_{j=1}^{n} \sum_{i=1}^{M} (v_{i,\bar{x}}^j)^2 < C, \;\; n = 1, 2, \dots, N. \tag{6}$$

Here and in what follows C is a positive constant independent of τ and h.

The a priori estimates (6) guarantee the global solvability of problem (5).

The following statement is true.

Theorem 2. *If $a = a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$, f is positively defined and monotonically increased function and problem* (1), (3), (4) *has a sufficiently smooth solution, then the solution of problem* (5) *tends to the solution of the continuous problem* (1), (3), (4) *as $h \to 0$, $\tau \to 0$ and the following estimates are true:*

$$\|u^j - U^j\| \leq C(\tau + h), \quad \|v^j - V^j\| \leq C(\tau + h).$$

We have carried out numerous numerical experiments for problem (1), (3), (4) with different kinds of right hand sides and initial-boundary conditions. The obtained numerical results are in accordance to the theoretical findings.

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Asymptotic Behaviour of $P_\omega(Y_0, 0)$-Solutions of Second-Order Nonlinear Differential Equations with Regularly and Rapidly Varying Nonlinearities

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Consider the differential equation

$$y'' = \sum_{i=1}^{m} \alpha_i p_i(t)\varphi_i(y), \tag{1}$$

where $\alpha_i \in \{-1, 1\}$ $(i = \overline{1,m})$, $p_i : [a, \omega[\to]0, +\infty[$ $(i = \overline{1,m})$ are continuous functions, $-\infty < a < \omega \le +\infty$; $\varphi_i : \Delta_{Y_0} \to]0, +\infty[$ $(i = \overline{1,m})$, where Δ_{Y_0} is a one-sided neighborhood of Y_0, Y_0 is equal either to zero or $\pm\infty$, are continuous functions for $i = \overline{1,l}$ and twice continuously differentiable for $i = \overline{l+1,m}$, and for each $i \in \{1, \dots, l\}$ for some $\sigma_i \in \mathbb{R}$

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_i(\lambda y)}{\varphi_i(y)} = \lambda^{\sigma_i} \ \text{ for any } \ \lambda > 0, \tag{2}$$

and for each $i \in \{l+1, \dots, m\}$ –

$$\varphi_i'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_i(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_i''(y)\varphi_i(y)}{\varphi_i'^2(y)} = 1. \tag{3}$$

It follows from the conditions (2) and (3) that φ_i $(i = \overline{1,l})$ are regularly varying functions, as $y \to Y_0$, of orders σ_i and φ_i $(i = \overline{l+1,m})$ are rapidly varying functions, as $y \to Y_0$ (see [4, Introduction, pp. 2, 4]).

Definition. A solution y of the differential equation (1) is called $P_\omega(Y_0, \lambda_0)$ – solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on some interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the following conditions

$$\lim_{t\uparrow\omega} y(t) = Y_0, \quad \lim_{t\uparrow\omega} y'(t) = \begin{cases} \text{either} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad \lim_{t\uparrow\omega} \frac{y'^2(t)}{y''(t)y(t)} = \lambda_0.$$

By its asymptotic properties, the class of $P_\omega(Y_0, \lambda_0)$ – solutions is split into 4 non-intersecting subsets that correspond to the next value of the parameter λ_0

$$\lambda_0 \in \mathbb{R} \setminus \{0, 1\}, \quad \lambda_0 = 1, \quad \lambda_0 = 0, \quad \lambda_0 = \pm\infty.$$

The existence conditions of $P_\omega(Y_0, \lambda_0)$ – solutions of the differential equation (1) and asymptotic representations, as $t \uparrow \omega$, of such solutions and their first-order derivatives, are established for each of these cases in the case where, for some $s \in \{1, \dots, m\}$

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} = 0 \ \text{ for all } \ i \in \{1, \dots, m\} \setminus \{s\}, \tag{4}$$

i.e., where the right-hand side of Eq. (1) for each such solution y is equivalent for $t \uparrow \omega$ to one term with regularly or rapidly varying nonlinearity (see [1–3]).

In this paper, we formulate the main results obtained for the case $\lambda_0 = 0$.

Let

$$\Delta_{Y_0} = \Delta_{Y_0}(b), \ \text{ where } \ \Delta_{Y_0}(b) = \begin{cases} [b, Y_0[\,, & \text{if } \ \Delta_{Y_0} \ \text{ is a left neighborhood of } \ Y_0, \\ ]Y_0, b], & \text{if } \ \Delta_{Y_0} \ \text{ is a right neighborhood of } \ Y_0, \end{cases}$$

and the number b satisfy the inequalities

$$|b| < 1 \ \text{ as } \ Y_0 = 0 \ \text{ and } \ b > 1 \ \ (b < -1) \ \text{ as } \ Y_0 = +\infty \ \ (Y_0 = -\infty).$$

We set

$$\nu_0 = \operatorname{sign} b, \quad \nu_1 = \begin{cases} 1, & \text{if } \ \Delta_{Y_0}(b) = [b, Y_0[\,, \\ -1, & \text{if } \ \Delta_{Y_0}(b) = \,]Y_0, b], \end{cases} \quad \pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t - \omega, & \text{if } \ \omega < +\infty, \end{cases}$$

$$J_{1s}(t) = \int_{A_{1s}}^{t} p_s(\tau)\,d\tau, \quad J_{2s}(t) = \int_{A_{2s}}^{t} J_{1s}(\tau)\,d\tau, \quad J_{3s}(t) = \int_{A_{3s}}^{t} \pi_\omega(\tau) p_{0s}(\tau)\,d\tau,$$

$$H_s(y) = \int_{B_s}^{y} \frac{du}{\varphi_s(u)}\,, \quad B_s = \begin{cases} b, & \text{if } \displaystyle\int_b^{Y_0} \frac{dy}{\varphi_s(y)} = \pm\infty, \\ Y_0, & \text{if } \displaystyle\int_b^{Y_0} \frac{dy}{\varphi_s(y)} = const, \end{cases} \quad Z_s = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}(b)}} H_s(y),$$

$$J_{\varphi_s}(t) = \int_{A_{\varphi_s}}^{t} p_{0s}(\tau)\varphi_s\big(H_s^{-1}(-\alpha_s J_{3s}(\tau))\big)\,d\tau, \quad E_s(t) = \alpha_s \pi_\omega^2(t) p_{0s}(t) \varphi_s'\big(H_s^{-1}(-\alpha_s J_{3s}(t))\big),$$

$$G_s(t) = \frac{y\varphi_s'(y)}{\varphi_s(y)}\bigg|_{y = H_s^{-1}(-\alpha_s J_{3s}(t))}, \quad \Phi_s(t) = \frac{y\big(\frac{\varphi_s'(y)}{\varphi_s(y)}\big)'}{\frac{\varphi_s'(y)}{\varphi_s(y)}}\Bigg|_{y = H_s^{-1}(-\alpha_s J_{3s}(t))},$$

$$\mu_s = \operatorname{sign} \varphi_s'(y), \quad \gamma_s = \lim_{t \uparrow \omega} \frac{E_s(t)\Phi_s(t)}{G_s(t)}\,, \quad \psi_s(t) = \int_{t_0}^{t} \frac{|E_s(\tau)|^{\frac{1}{2}}}{\pi_\omega(\tau)}\,d\tau,$$

where $s \in \{1, \dots, m\}$, $p_{0s} : [a, \omega[\, \to]0, +\infty[$ are continuous functions so that $p_{0s}(t) \sim p_s(t)$ as $t \uparrow \omega$, every limit of integration A_{1s}, A_{2s}, A_{3s}, A_{φ_s} is equal to either a or ω and is chosen so that the corresponding integral tends either to $\pm\infty$, or to zero with $t \uparrow \omega$, t_0 is some number of $[a, \omega[\,$.

Theorem 1. *Let $\sigma_s \neq 1$ for some $s \in \{1, \dots, l\}$ and there exist finite or equal to infinity limit*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_{1s}'(t)}{J_{1s}(t)}\,.$$

For existence of $P_\omega(Y_0, 0)$ – solutions of equation (1), satisfied the limit relations (4), it is necessary that the inequalities

$$\alpha_s \nu_0 (1 - \sigma_s) J_{2s}(t) > 0, \ \ \alpha_s \nu_1 \pi_\omega(t) < 0 \ \text{ as } \ t \in \,]a, \omega[\tag{5}$$

and conditions

$$\alpha_s \lim_{t\uparrow\omega} J_{2s}(t) = Z_s, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J'_{1s}(t)}{J_{1s}(t)} = -1, \quad \lim_{t\uparrow\omega} \frac{J^2_{1s}(t)}{p_s(t)J_{2s}(t)} = 0, \tag{6}$$

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s J_{2s}(t)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s J_{2s}(t)))} = 0 \ \text{ for all } \ i \in \{1,\dots,l\} \setminus \{s\}, \tag{7}$$

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s J_{2s}(t)(1+\delta_i)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s J_{2s}(t)))} = 0 \ \text{ for all } \ i \in \{l+1,\dots,m\}$$

hold, where δ_i are arbitrary numbers of a one-sided neighborhood of zero. Moreover, for each of such solutions the following asymptotic representations hold

$$y(t) = H_s^{-1}(\alpha_s J_{2s}(t))[1+o(1)] \ \text{ at } \ t \uparrow \omega, \tag{8}$$

$$y'(t) = \frac{J_{1s}(t)H_s^{-1}(\alpha_s J_{2s}(t))}{(1-\sigma_s)J_{2s}(t)}\,[1+o(1)] \ \text{ at } \ t\uparrow\omega. \tag{9}$$

Theorem 2. *Let $\sigma_s \neq 1$ for some $s \in \{1,\dots,l\}$, conditions* (5)–(7) *hold and*

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s J_{2s}(t)(1+u)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s J_{2s}(t)))} = 0 \ \text{ for all } \ i \in \{l+1,\dots,m\}$$

uniformly with respect to $u \in [-\delta,\delta]$ for any $0 < \delta < 1$. Then the differential equation (1) *has $P_\omega(Y_0,0)$ – solutions that admit the asymptotic representations* (8) *and* (9). *Moreover, if $\alpha_s\nu_0(1-\sigma_s)\pi_\omega(t) < 0$ as $t \in]a,\omega[$, there is a one-parameter family of such solutions in case $\omega = +\infty$ and two-parameter family in case $\omega < +\infty$.*

Theorem 3. *Let for some $s \in \{l+1,\dots,m\}$ the function p_s might be representable in the form*

$$p_s(t) = p_{0s}(t)[1+r_s(t)], \ \text{ where } \ \lim_{t\uparrow\omega} r_s(t) = 0, \tag{10}$$

$p_{0s} : [a,\omega[\to]0,+\infty[$ is a continuously differentiable function, $r_s : [a,\omega[\to]-1,+\infty[$ is a continuous function, and let the conditions

$$\frac{\varphi_s(y)\varphi'_i(y)}{\varphi'_s(y)\varphi_i(y)} = O(1) \ \ (i = \overline{l+1,m}) \ \text{ for } \ y \to Y_0 \tag{11}$$

hold. Then, for the existence of $P_\omega(Y_0,0)$ – solutions of the differential equation (1) *satisfying conditions* (4), *it is necessary that, there exist finite or equal to infinity limit*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J'_{\varphi_s}(t)}{J_{\varphi_s}(t)}\,,$$

and the conditions

$$\alpha_s\nu_1\pi_\omega(t) < 0, \quad \alpha_s\mu_s J_{3s}(t) > 0 \ \text{ as } \ t \in]a,\omega[\,, \tag{12}$$

$$-\alpha_s \lim_{t\uparrow\omega} J_{3s}(t) = Z_s, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J'_{\varphi_s}(t)}{J_{\varphi_s}(t)} = -1, \quad \lim_{t\uparrow\omega} \frac{\pi^2_\omega(t)p_{0s}(t)\varphi_s(H_s^{-1}(-\alpha_s J_{3s}(t)))}{H_s^{-1}(-\alpha_s J_{3s}(t))} = 0, \tag{13}$$

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(-\alpha_s J_{3s}(t)))}{p_s(t)\varphi_s(H_s^{-1}(-\alpha_s J_{3s}(t)))} = 0 \ \text{ for all } \ i \in \{1,\dots,m\} \setminus \{s\} \tag{14}$$

be satisfied. Moreover, each such solutions has the asymptotic representations

$$y(t) = H_s^{-1}(-\alpha_s J_{3s}(t))\Big[1 + \frac{o(1)}{G_s(t)}\Big] \ \text{ at } \ t\uparrow\omega, \tag{15}$$

$$y'(t) = -\alpha_s\pi_\omega(t)p_{0s}(t)\varphi_s\big(H_s^{-1}(-\alpha_s J_{3s}(t))\big)[1+o(1)] \ \text{ at } \ t\uparrow\omega. \tag{16}$$

Theorem 4. *Let for some $s \in \{l+1,\dots,m\}$ the conditions* (10), (11), (12)–(14) *be satisfied and*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) J_{3s}'(t)}{J_{3s}(t)} = \eta_s, \quad \text{where} \;\; \eta_s \in \mathbb{R}.$$

Then:

1) *if $\eta_s > 0$ or $\eta_s = 0$ and $\alpha_s\mu_s = 1$, the differential equation* (1) *has a one-parameter family of $P_\omega(Y_0,0)$ – solutions with the asymptotic representations* (15) *and* (16)*;*

2) *if $\eta_s < 0$ or $\eta_s = 0$ and $\alpha_s\mu_s = -1$, there is a two-parameter family of $P_\omega(Y_0,0)$ – solutions which admit the asymptotic representations* (15), (16) *in case $\omega < +\infty$ and there is at least one such solution in case $\omega = +\infty$.*

Theorem 5. *Let for some $s \in \{l+1,\dots,m\}$ the function p_s be representable in the form* (10), *let conditions* (11), (12)–(14) *hold, and let the limits* (*which are finite or equal to $\pm\infty$*)

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) J_{\varphi_s}''(t)}{J_{\varphi_s}'(t)}\,, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}(b)}} \frac{\big(\frac{\varphi_s'(y)}{\varphi_s(y)}\big)'}{\big(\frac{\varphi_s'(y)}{\varphi_s(y)}\big)^2} \cdot \sqrt{\Big|\frac{y\varphi_s'(y)}{\varphi_s(y)}\Big|}\,, \qquad \gamma_s = \lim_{t\uparrow\omega} \frac{E_s(t)\Phi_s(t)}{G_s(t)}\,, \qquad \lim_{t\uparrow\omega} \frac{\psi_s''(t)\psi_s(t)}{\psi_s'^2(t)}$$

exist. Then:

1) *if $\alpha_s\mu_s = 1$, the differential equation* (1) *has a one-parameter family of $P_\omega(Y_0,0)$ – solutions which admit the asymptotic representations* (15) *and* (16) *and are such that their derivatives satisfy the asymptotic relation*

$$y'(t) = -\alpha_s \pi_\omega(t) p_{0s}(t) \varphi_s\big(H_s^{-1}(-\alpha_s J_{3s}(t))\big)\big[1 + |E_s(t)|^{-\frac{1}{2}} o(1)\big] \;\; \text{at} \;\; t\uparrow\omega;$$

2) *if $\alpha_s\mu_s = -1$ and*

$$\gamma_s \neq \frac{1}{3}\,; \quad \lim_{t\uparrow\omega} \psi_s(t) r_s(t) = 0, \quad \lim_{t\uparrow\omega} \psi_s^2(t)\Big[r_s(t) + 2 + \frac{\pi_\omega(t) J_{\varphi_s}''(t)}{J_{\varphi_s}'(t)}\Big] = 0,$$
$$\lim_{t\uparrow\omega} \frac{\psi_s(t)}{E_s(t)} = 0 \;\; \text{at} \;\; \gamma_s = 0, \quad \lim_{t\uparrow\omega} \psi_s^2(t) \sum_{\substack{i=1\\ i\neq s}}^{m} \frac{p_i(t)\varphi_i(H_s^{-1}(-\alpha_s J_{3s}(t)))}{p_s(t)\varphi_s(H_s^{-1}(-\alpha_s J_{3s}(t)))} = 0,$$

the differential equation (1) *has a $P_\omega(Y_0,0)$ – solution with asymptotic representations*

$$y(t) = H_s^{-1}(-\alpha_s J_{3s}(t))\Big[1 + \frac{o(1)}{G_s(t)\psi_s(t)}\Big] \;\; \text{at} \;\; t\uparrow\omega,$$
$$y'(t) = -\alpha_s \pi_\omega(t) p_{0s}(t) \varphi_s\big(H_s^{-1}(-\alpha_s J_{3s}(t))\big)\big[1 + |E_s(t)|^{-\frac{1}{2}} \psi_s^{-1}(t) o(1)\big] \;\; \text{at} \;\; t\uparrow\omega.$$

Moreover, there exists a two-parameter family of such solutions in case when $\gamma_s \in (0,1/3)$ or $\gamma_s = 0$ and $\alpha_s\nu_1 = 1$.

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On Asymptotic Behavior of Solutions to Second-Order Differential Equations with General Power-Law Nonlinearities

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1 Introduction

Consider the second-order nonlinear differential equation

$$y'' = p(x, y, y')|y|^{k_0}|y'|^{k_1} \operatorname{sgn}(yy'), \quad k_0 > 0, \quad k_1 > 0, \quad k_0,\, k_1 \in \mathbb{R} \tag{1.1}$$

with positive continuous in x and Lipschitz continuous in u, v function $p(x, u, v)$ satisfying the inequalities

$$0 < m \le p(x, u, v) \le M < +\infty. \tag{1.2}$$

The results on the behavior of solutions depending on the nonlinearity exponents k_0, k_1 and qualitative properties of solutions was studied in [11].

The asymptoptic behavior of solutions to (1.1) in the case $k_1 = 0$ is described in [5, 6]. In the case $p = p(x)$ asymptotic behavior of solutions to (1.1) is obtained by V. M. Evtukhov [7]. Using methods described in [1, 2, 4] by I. V. Astashova, the behavior of solutions to (1.1) near domain boundaries is considered with respect to the values k_0 and k_1.

The following definitions are used further.

Definition 1.1 ([4])**.** A solution $y : (a, b) \to \mathbb{R}$, $-\infty \le a < b \le +\infty$ to an ordinary differential equation is called a *μ-solution* if

(1) the equation has no other solutions equal to y on some subinterval (a, b) and not equal to y at some point in (a, b);

(2) the equation either has no solution equal to y on (a, b) and defined on another interval containing (a, b) or has at least two such solutions which differ from each other at points arbitrary close to the boundary of (a, b).

Definition 1.2 ([8])**.** A solution satisfying at some finite point x^* the conditions $\lim\limits_{x\to x^*} |y'(x)| = \infty$, $\lim\limits_{x\to x^*} |y(x)| < \infty$ is called a *black hole* solution.

Definition 1.3 ([9])**.** A μ-solution satisfying at finite point (its domain boundary) $\widetilde{x}$ the conditions $\lim\limits_{x\to\widetilde{x}} y'(x) = 0$ and $\lim\limits_{x\to\widetilde{x}} y(x) \ne 0$ is called a *white hole* solution.

Definition 1.4 ([10])**.** A solution to equation (1.1) is called a *Kneser solution at decreasing argument* on the interval $(-\infty; x_0)$ if $y(x) > 0$, $y'(x) > 0$ for any $x < x_0$.

Definition 1.5 ([10])**.** A solution to equation (1.1) is called a *negative Kneser solution* on the interval $(x_0; +\infty)$ if $y(x) < 0$, $y'(x) > 0$ for any $x > x_0$.

Definition 1.6 ([10])**.** A μ-solution $y(x)$ to equation (1.1) is called a *singular of the type II at a point* $a \in \mathbb{R}$ if $\lim\limits_{x\to a} y(x) = \lim\limits_{x\to a} y'(x) = 0$.

2 Main results

Lemma 2.1. *Let the function $p(x,u,v)$ be continuous in x, Lipschitz continuous in u, v and satisfying inequalities* (1.2). *Then all μ-solutions to equation* (1.1) *are monotonous.*

Denote
$$\alpha = \frac{2-k_1}{k_0+k_1-1}\,, \quad C = \Big(\frac{|\alpha|^{1-k_1}|\alpha+1|}{p_0}\Big)^{\frac{1}{k_0+k_1-1}}.$$

Theorem 2.1. *Suppose $k_0+k_1<1$. Let the function $p(x,u,v)$ be continuous in x, Lipschitz continuous in u, v and satisfying inequalities* (1.2). *Let there also exist the following limits of $p(x,u,v)$:*

(1) p_+ *as* $x\to+\infty$, $u\to+\infty$, $v\to+\infty$;

(2) p_- *as* $x\to-\infty$, $u\to-\infty$, $v\to+\infty$.

Denote $p_a=p(a,0,0)$ for any $a\in\mathbb{R}$. Then $\alpha<-1$ and all increasing μ-solutions to equation (1.1) *according to their asymptotic behavior can be divided into three types:*

1. *Increasing solutions defined on the whole axis with zero at some point x_0:*
$$y(x)=C(p_-)(x_0-x)^{-\alpha}(1+o(1)), \quad x\to-\infty,$$
$$y(x)=C(p_+)(x-x_0)^{-\alpha}(1+o(1)), \quad x\to+\infty.$$

2. *Positive singular solutions defined on semi-axis $(a,+\infty)$:*
$$y(x)=C(p_a)(x-a)^{-\alpha}(1+o(1)), \quad x\to a+0,$$
$$y(x)=C(p_+)(x-a)^{-\alpha}(1+o(1)), \quad x\to+\infty.$$

3. *Negative singular solutions defined on semi-axis $(-\infty,b)$:*
$$y(x)=C(p_-)(b-x)^{-\alpha}(1+o(1)), \quad x\to-\infty,$$
$$y(x)=C(p_b)(b-x)^{-\alpha}(1+o(1)), \quad x\to b-0.$$

Theorem 2.2. *Suppose $k_0+k_1>1$, $k_1<2$. Let the function $p(x,u,v)$ be continuous in x, Lipschitz continuous in u, v and satisfying inequalities* (1.2). *Let there also exist the following limits of $p(x,u,v)$:*

(1) P^a *as* $x\to a-0$, $u\to+\infty$, $v\to+\infty$, *for every* $a\in\mathbb{R}$;

(2) P_a *as* $x\to a+0$, $u\to-\infty$, $v\to+\infty$, *for every* $a\in\mathbb{R}$;

(3) P_+ *as* $x\to+\infty$, $u\to0$, $v\to0$;

(4) P_- *as* $x\to-\infty$, $u\to0$, $v\to0$.

Then $\alpha>0$ and all maximally extended increasing solutions to (1.1) *according to their asymptotic behavior can be divided into three types:*

1. *Increasing solutions with two vertical asymptotes $x=x_*$ and $x=x^*$, $x_*<x^*$:*
$$y=C(P^{x^*})(x^*-x)^{-\alpha}(1+o(1)), \quad x\to x^*-0,$$
$$y=-C(P_{x_*})(x-x_*)^{-\alpha}(1+o(1)), \quad x\to x_*+0.$$

2. *Kneser solution at decreasing argument defined on semi-axis* $(-\infty, x^*)$*:*

$$y = C(P_-)|x|^{-\alpha}(1+o(1)), \quad x \to -\infty,$$
$$y = C(P^{x^*})(x^* - x)^{-\alpha}(1+o(1)), \quad x \to x^* - 0.$$

3. *Negative Kneser solutions defined on semi-axis* $(x_*, +\infty)$*:*

$$y = -C(P_{x_*})(x - x_*)^{-\alpha}(1+o(1)), \quad x \to x_* + 0,$$
$$y = -C(P_+)x^{-\alpha}(1+o(1)), \quad x \to +\infty.$$

Theorem 2.3. *Suppose* $0 < k_1 < 1$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Then any maximally extended increasing solution* $y(x)$ *to* (1.1) *is a black hole solution defined on the interval* (x_*, x^*)*, and the limit* $\lim\limits_{x \to x^* - 0} y(x) = y^*$ *satisfies the following inequalities:*

$$\Big(\frac{k_0+1}{M(k_1-2)}\Big)^{\frac{1}{k_0+1}} (y'(x_0))^{-\frac{k_1-2}{k_0+1}} \le |y^*| \le \Big(\frac{k_0+1}{m(k_1-2)}\Big)^{\frac{1}{k_0+1}} (y'(x_0))^{-\frac{k_1-2}{k_0+1}}.$$

The same inequalities hold for the limit $y_* = \lim\limits_{x \to x_* + 0} y(x)$*.*

Theorem 2.4. *Suppose* $k_1 > 2$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Let there also exist limits* p^+ *as* $x \to +\infty$*,* $u \to -\infty$*,* $v \to 0$ *and* p^- *as* $x \to -\infty$*,* $u \to -\infty$*,* $v \to 0$*. Then* $-1 < \alpha < 0$ *and any increasing solution to* (1.1) *has a zero at some point* x_0 *and has the following asymptotic behavior:*

$$y(x) = -C(p^+)(x - x_0)^{-\alpha}(1+o(1)), \quad x \to +\infty,$$
$$y(x) = C(p^-)(x_0 - x)^{-\alpha}(1+o(1)), \quad x \to -\infty.$$

Theorem 2.5. *Suppose* $k_0 > 0$*,* $1 \le k_1 < 2$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Then any decreasing solution* $y(x)$ *to equation* (1.1) *is defined on the whole axis, has a zero at some point* x_0 *and has two horizontal asymptotes* $y = y_+ < 0$ *at* $x \to +\infty$ *and* $y = y_- > 0$ *at* $x \to -\infty$*. Moreover,*

$$\frac{k_0+1}{M(2-k_1)}\,|y'(x_0)|^{2-k_1} \le |y_\pm|^{k_0+1} \le \frac{k_0+1}{m(2-k_1)}\,|y'(x_0)|^{2-k_1}.$$

Theorem 2.6. *Suppose* $k_0 > 0$*,* $0 < k_1 < 1$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Then any decreasing* μ*-solution* $y(x)$ *to equation* (1.1) *is defined on a finite interval* (x_-, x_+)*, has a zero at some point* x_0 *and the limits* $y_+ = \lim\limits_{x \to x_+ - 0} y(x)$ *and* $y_- = \lim\limits_{x \to x_- + 0}$ *satisfy the estimate from Theorem 2.5.*

Corollary 2.1. *Suppose* $k_0 > 0$*,* $0 < k_1 < 2$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Then any decreasing solution* $y(x)$ *to equation* (1.1) *is defined on the whole axis and the limits* $y_\pm = \lim\limits_{x \to \pm\infty} y(x)$ *satisfy the following inequalities:*

$$\Big(\frac{m}{M}\Big)^{\frac{1}{k_0+1}} \le \Big|\frac{y_+}{y_-}\Big| \le \Big(\frac{M}{m}\Big)^{\frac{1}{k_0+1}}.$$

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Asymptotic Representations of One Class Solutions of Second-Order Differential Equations

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Consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')[1+\psi(t,y,y')], \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\, \to]0,+\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ $(i=0,1)$ are continuous and regular varying as $y^{(i)} \to Y_i$ $(i=0,1)$ functions of orders σ_i $(i=0,1)$, Δ_{Y_i} $(i \in \{0,1\})$ is a one-side neighborhood of Y_i and $Y_i \in \{0;\pm\infty\}$ $(i \in \{0,1\})$, $\psi : [a,\omega[\, \times \Delta_{Y_0} \times \Delta_{Y_1} \to \mathbf{R}$ is a continuous function such that the condition

$$\lim_{\substack{t\uparrow\omega \\ (y,z)\to(Y_0,Y_1) \\ (y,z)\in\Delta_{Y_0}\times\Delta_{Y_1}}} \psi(t,y,z) = 0$$

holds. We assume that the numbers μ_i $(i=0,1)$ given by the formula

$$\mu_i = \begin{cases} 1, & \text{if either } \ Y_i = +\infty, \ \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is a right neighborhood of the point } 0, \\ -1, & \text{if either } \ Y_i = -\infty, \ \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is a left neighborhood of the point } 0, \end{cases}$$

satisfy the relations

$$\mu_0\mu_1 > 0 \ \text{ for } \ Y_0 = \pm\infty \quad \text{and} \quad \mu_0\mu_1 < 0 \ \text{ for } \ Y_0 = 0. \tag{2}$$

Conditions (2) are necessary for the existence of solutions of Eq. (1) defined in the left neighborhood of ω and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \ \text{ for } \ t \in [t_0,\omega[\,, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1). \tag{3}$$

We study Eq. (1) on class $P_\omega(Y_0,Y_1,\lambda_0)$-solutions, that is defined as follows.

Definition. A solution y of Eq. (1) on the interval $[t_0,\omega[\, \subset [a,\omega[$ is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if, in addition to (3), it satisfies the condition

$$\lim_{t\uparrow\omega} \frac{[y'(t)]^2}{y(t)y''(t)} = \lambda_0.$$

Depending on λ_0 these solutions have different asymptotic properties. For $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$ in [1] such ratios

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0-1}\,, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{1}{\lambda_0-1}\,,$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t-\omega & \text{if } \ \omega < +\infty \end{cases}$$

are established.

By the definition of a regularly varying function [5, Chapter 1, Section 1.1, 9–10 of the Russian translation], each of the functions φ_i $(i \in \{0,1\})$ admits a representation of the form

$$\varphi_i(z) = |z|^{\sigma_i} L_i(z),$$

where $L_i : \Delta_{Y_i} \to]0,+\infty[$ is a continuous function slowly varying as $y \to Y_i$. Moreover, there exist continuously differentiable functions (see [5, Chapter 1, Section 1.1, 10–15 of the Russian translation]) $L_{ii} : \Delta_{Y_i} \to]0,+\infty[$ slowly varying as $y \to Y_i$ $(i = 0,1)$ and satisfying the conditions

$$\lim_{\substack{z\to Y_i\\ z\in \Delta_{Y_i}}} c \frac{L_i(z)}{L_{ii}(z)} = 1, \quad \lim_{\substack{z\to Y_i\\ z\in \Delta_{Y_i}}} \frac{zL'_{ii}(z)}{L_{ii}(z)} = 0 \ \ (i = 0,1).$$

Asymptotic representations and conditions of the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions in case $\sigma_0 + \sigma_1 \neq 1$ are obtained in [4]. Here we study the behavior of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions in case $\sigma_0 + \sigma_1 = 1$ and $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$, when it becomes close in some sense to the linear, which is studied in detail in the monograph [3]. The theorem is a generalization of the result of work [2] for Eq. (1).

We choose a number $b \in \Delta_{Y_0}$ such that the inequality

$$|b| < 1 \ \text{ for } \ Y_0 = 0, \quad b > 1 \ \ (b < -1) \ \text{ for } \ Y_0 = +\infty \ \ (Y_0 = -\infty)$$

is respected and put

$$\begin{aligned} &\Delta_{Y_0}(b) = [b, Y_0[\ \text{ if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ &\Delta_{Y_0}(b) =]Y_0, b] \ \text{ if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0. \end{aligned}$$

Now we introduce auxiliary functions and notation as follows:

$$\Phi : \Delta_{Y_0}(b) \to \mathbb{R}, \quad \Phi(y) = \int_B^y \frac{ds}{sL_0(s)}\,, \quad B = \begin{cases} b & \text{if } \displaystyle\int_b^{Y_0} \frac{ds}{sL_0(s)} = \pm\infty,\\ Y_0 & \text{if } \displaystyle\int_b^{Y_0} \frac{ds}{sL_0(s)} = const, \end{cases}$$

$$Z = \lim_{y\to Y_0} \Phi(y) = \begin{cases} 0 & \text{if } B = Y_0,\\ +\infty & \text{if } B = b, \ \mu_0\mu_1 > 0,\\ -\infty & \text{if } B = b, \ \mu_0\mu_1 < 0, \end{cases} \quad \mu_2 = \begin{cases} 1 & \text{if } B = b,\\ -1 & \text{if } B = Y_0, \end{cases}$$

$$I_0(t) = \int_{A_0}^t p(\tau)|\pi_\omega(\tau)|^{-\sigma_1} L_1\big(\mu_1|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}\big)\,d\tau, \quad I_1(t) = \int_{A_1}^t p(\tau)|\pi_\omega(\tau)|^{\sigma_0} L_1\big(\mu_1|\pi_\omega(\tau)|^{\frac{1}{\lambda_0-1}}\big)\,d\tau,$$

where the integration limits $A_i \in \{a;\omega\}$ $(i = 0,1)$ are chosen so as to ensure that the integrals I_i $(i = 0,1)$ tend either to zero or to $\pm\infty$ as $t \uparrow \omega$.

Theorem. *Let $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ and let the function $L_0(\Phi^{-1}(z))$ is regular varying of γ-th order as $z \to Z$, moreover, let the orders σ_i $(i = 0,1)$ of the functions φ_i $(i = 0,1)$ regularly varying as $y^{(i)} \to Y_i$ $(i = 0,1)$ satisfy the condition $\sigma_0 + \sigma_1 = 1$. Then, for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the differential equation* (1)*, it is necessary and, if the condition*

$$(1 + \lambda_0)(1 + \lambda_0 + \lambda_0\gamma) \neq 0$$

is satisfied, sufficient that

$$\lim_{t\uparrow\omega}\frac{|\pi_\omega(t)|^{\sigma_0}p(t)L_1(\mu_1|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}})}{I_0(t)}=-\beta,\quad \lim_{t\uparrow\omega}\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)=Z,$$

$$\lim_{t\uparrow\omega}p(t)|\pi_\omega(t)|^{1+\sigma_0}L_1\big(\mu_1|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}\big)L_0\Big(\Phi^{-1}\big(\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)\big)\Big)=\frac{|\lambda_0|^{\sigma_0}}{|\lambda_0-1|^{1+\sigma_0}},$$

and the sign conditions

$$\mu_2\pi_\omega(t)I_1(t)>0,\quad \mu_0\mu_1\lambda_0(\lambda_0-1)\pi_\omega(t)>0\ \textit{ for }\ t\in\,]a,\omega[$$

hold. Moreover, each solution of this kind admits the asymptotic representations

$$\begin{aligned}\Phi(y(t))&=\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)[1+o(1)],\\ \frac{y'(t)}{y(t)}&=\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}p(t)|\pi_\omega(t)|^{\sigma_0}\\ &\qquad\times L_1\big(\mu_1|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}\big)L_0\Big(\Phi^{-1}\big(\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)\big)\Big)\ \textit{ as }\ t\uparrow\omega,\end{aligned}$$

and such solutions form a one-parameter family if

$$(\lambda_0-1)(1+\lambda_0+\gamma\lambda_0)I_1(t)<0\ \textit{ for }\ t\in\,]a,\omega[\,,$$

and two-parameter family if

$$(\lambda_0-1)(1+\lambda_0+\gamma\lambda_0)I_1(t)>0$$

and

$$(\lambda_0^2-1)\pi_\omega(t)>0\ \textit{ for }\ t\in\,]a,\omega[\,.$$

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The Asymptotic Behaviour of Solutions of Systems of Differential Equations Partially Solved Relatively to the Derivatives with Non-Square Matrices

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One of the methods of investigation of systems of differential equations which are not resolved relatively to the derivatives in the real-valued domain was suggested by R. Grabovskaya and J. Diblic [1]. It was developed in the complex domain in the articles by G. Samkova, N. Sharay, E. Michalenko, D. Limanska [2–6] and others. The current article is a continuation of the researching of systems of differential equations that are not resolved relatively to the derivatives in the complex domain.

Let us consider the system of ordinary differential equations

$$A(z)Y' = B(z)Y + f(z, Y, Y'), \tag{1}$$

where matrices $A, B : D_1 \to \mathbb{C}^{m\times p}$, $D_1 = \{z : \ |z| < R_1, \ R_1 > 0\} \subset \mathbb{C}$, matrices $A(z)$, $B(z)$ are analytic in the domain D_{10}, $D_{10} = D_1 \setminus \{0\}$, the pencil of matrices $A(z)\lambda - B(z)$ is singular on the condition that $z \to 0$, function $f : D_1 \times G_1 \times G_2 \to \mathbb{C}^m$, where domains $G_k \subset \mathbb{C}^p$, $0 \in G_k$, $k = 1, 2$, function $f(z, Y, Y')$ is analytic in $D_{10} \times G_{10} \times G_{20}$, $G_{k0} = G_k \setminus \{0\}$, $k = 1, 2$.

Let us study the system of ordinary differential equations (1) on the conditions that $m > p$ and $rang A(z) = p$ on condition that $z \in D_1$.

Without loss of the generality, let's assume that matrices $A(z)$, $B(z)$ and vector-function $f(z, Y, Y')$ take the forms

$$A(z) = \begin{pmatrix} A_1(z) \\ A_2(z) \end{pmatrix}, \quad B(z) = \begin{pmatrix} B_1(z) \\ B_2(z) \end{pmatrix}, \quad f(z, Y, Y') = \begin{pmatrix} f_1(z, Y, Y') \\ f_2(z, Y, Y') \end{pmatrix},$$

$A_1 : D_1 \to \mathbb{C}^{p\times p}$, $A_2 : D_1 \to \mathbb{C}^{(m-p)\times p}$, $B_1 : D_1 \to \mathbb{C}^{p\times p}$, $B_2 : D_1 \to \mathbb{C}^{(m-p)\times p}$, $\det A_1(z) \neq 0$ on the condition that $z \in D_1$, $f_1 : D_1 \times G_1 \times G_2 \to C^p$, $f_2 : D_1 \times G_1 \times G_2 \to C^{m-p}$.

In this view the system (1) may be written as:

$$\begin{cases} Y' = A_1^{-1}(z)B_1(z)Y + A_1^{-1}(z)f_1(z, Y, Y'), & (2.1) \\ A_2(z)Y' = B_2(z)Y + f_2(z, Y, Y'), & (2.2) \end{cases} \tag{2}$$

where $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10}, $A_1^{-1}(z)f_1(z, Y, Y')$ is analytic vector-function in the domain $D_{10} \times G_{10} \times G_{20}$. Then vector-function $A_1^{-1}(z)f_1(z, Y, Y')$ has an isolated singularity in the point $(0, 0, 0)$. Thus, according to the theorem about an isolated singularity for a function of several complex variables, point $(0, 0, 0)$ is a removable singularity of the function $A_1^{-1}(z)f_1(z, Y, Y')$.

Let us complete definition of vector-function $A_1^{-1}(z)f_1(z, Y, Y')$ in the point $(0, 0, 0)$ thus it became analytic function in the domain $D_1 \times G_1 \times G_2$ and, without loss of the generality, let's assume that $A_1^{-1}(0)f_1(0, 0, 0) = 0$.

Let us consider two cases:

1. $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10} and has a removable singularity in the point $z=0$;
2. $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10} and has a pole of order r in the point $z=0$.

For the first case let us introduce the following notations

$$A_1^{-1}(z)B_1(z)=P^{(1)}(z),A_1^{-1}f_1(z,Y,Y')=F(z,Y,Y').$$

Then the system (2.1) may be written as

$$Y'=P^{(1)}(z)Y+F(z,Y,Y'), \tag{3}$$

where $P^{(1)}:D_1\to\mathbb{C}^{p\times p}, P^{(1)}(z)$ is analytic matrix in the domain D_1, $F(z,Y,Y')$ is analytic vector-function in the domain $D_1\times G_1\times G_2$.

For the second case let us introduce the following notations

$$A_1^{-1}(z)B_1(z)=z^{-r}P^{(2)}(z),A_1^{-1}f_1(z,Y,Y')=F(z,Y,Y').$$

Then the system (2.1) may be written as

$$Y'=z^{-r}P^{(2)}(z)Y+F(z,Y,Y'), \tag{4}$$

where $P^{(2)}:D_1\to\mathbb{C}^{p\times p}$, $P^{(2)}(z)$ is analytic matrix in the domain D_1.

We study the questions of the analytic solutions existence of the system (2) for both cases that satisfy the initial condition

$$Y(z)\to 0 \ \text{ on the condition that } \ z\to 0, \ \ z\in D_{10}, \tag{5}$$

and additional condition

$$Y'(z)\to 0 \ \text{ on the condition that } \ z\to 0, \ \ z\in D_{10}, \tag{6}$$

are considered.

The sufficient conditions of the existence of analytical solutions for the systems of differential equations (3) and (4), partially solved relatively to the derivatives, in the presence of a removable singularity or a pole z=0, were found. It was found an estimate for these solutions in the domain with the zero-point on a border.

The theorems on the existence of at least one analytic solution in the complex domain of the Cauchy problem (1)–(5) with the additional condition (6) are established for both cases. Moreover, the asymptotic behavior of these solutions in this domain is studied.

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On Instability of Millionshchikov Linear Systems with a Parameter

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We consider one-parameter family of linear differential systems

$$\dot{x} = A_\mu(t)x, \quad x \in \mathbb{R}^2, \quad t \geq 0 \tag{1_μ}$$

with the coefficient matrix $A_\mu(t) := d_k(\mu)\operatorname{diag}[1, -1]$, $2k-1 \leq t < 2k$, $A_\mu(t) := (\mu + b_k)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $2k-2 \leq t < 2k-1$, where $\mu, b_k \in \mathbb{R}$, $d_k(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{N}$.

In [4] we established the positivity of senior Lyapunov characteristic exponent of system (1_μ) for parameter values of positive Lebesgue measure, assumed that $d_k(\,\cdot\,)$ is independent on μ and the condition $d_k(\mu) \equiv d_k \geq d > 0$, $k \in \mathbb{N}$, holds. The proof of the result above substantially uses special complex matrices.

For all $\alpha_n \in \mathbb{R}$, $n \in \mathbb{N}$, let

$$b_{2^n} := b_{2^{n-1}} + \alpha_n, \quad b_{2^n+k} := b_k, \quad k = \overline{1, 2^n - 1}, \quad d_k(\mu) \equiv d_0(\mu) > 2^{20}, \quad k \in \mathbb{N}. \tag{2}$$

Systems of this type give rise to various one-parameter families with a wide range of asymptotic properties. For example, V. M. Millionshchikov used them in works [5, 6] (see, as well [3]) to prove an existence of irregular under Lyapunov linear differential systems with limit-periodic and quasi-periodic coefficients.

Method of these papers essentially use the estimations for eigenvalues and eigenvectors of system (1_μ) Cauchy matrix. Another way for investigation was initiated by the criterium due E. A. Barabanov of linear system regularity, that consist in the application of Cauchy matrix singular form (see the equality (5_n)).

In this paper we prove an existence of parameter value $\mu \in \mathbb{R}$ such that the corresponding system (1_μ) is unstable under condition (2) and if the function $d_0(\,\cdot\,)$ is continuous.

Let us denote the sequences $\{\psi_k(\mu)\}_{k=1}^{+\infty} \subset \mathbb{R}$ and $\{\eta_k(\mu)\}_{k=1}^{+\infty} \subset \mathbb{R}$ by the equalities $\psi_1(\mu) := \mu$, $\eta_1(\mu) = d_0(\mu)$, $\psi_{k+1} = \psi_k + \varphi_k/2$,

$$(\operatorname{ch}\eta_{k+1})\sin\varphi_k = \sin\xi_k, \quad k \in \mathbb{N}, \tag{3}$$

where $\xi_k := 2\psi_k + \zeta_k$, $\zeta_k := \sum\limits_{j=1}^{k} \alpha_j$, $\varphi_k \in (-2^{-1}\pi, 2^{-1}\pi]$ are defined by the formula

$$\operatorname{ctg}\varphi_k = (\operatorname{ch} 2\eta_k)\operatorname{ctg}\xi_k. \tag{4}$$

Let $X_{A_\mu}(t, s)$, $t, s \geq 0$, is the Cauchy matrix for system (1_μ).

Lemma 1. *Foe all $n \in \mathbb{N}$, $\mu \in \mathbb{R}$ under conditions* (2) *and* (3) *the next equalities hold*

$$X_{A_\mu}(2^n, 0) = U(\xi_n - \psi_n)\begin{pmatrix} \eta_n & 0 \\ 0 & \eta_n^{-1} \end{pmatrix} U(\psi_n), \tag{5_n}$$

$$\operatorname{sh}\eta_{k+1} = (\operatorname{sh} 2\eta_k)\cos\xi_k. \tag{6}$$

Lemma 2. *For every continuous function $f(\,\cdot\,):[a,b]\to\mathbb{R}$, $a,b\in\mathbb{R}$, such that $f(a)\le c<d\le f(b)$, the closed interval $[p,q]\subset[a,b]$ exists with the property $f([p,q])=[c,d]$.*

Theorem. *For all $\alpha_n\in[-\pi/2,\pi/2]$, $n\in\mathbb{N}$, b_k and $d_k(\,\cdot\,)$, chosen accordinaly (2), the senior characteristic exponent of system (1_μ) is positive for some $\mu\in\mathbb{R}$, whereas the function $d_0(\,\cdot\,)$ is continuous.*

Proof. Let us denote

$$V_\varepsilon(\alpha):=\big\{\varkappa\in[-2^{-1}\pi,2^{-1}\pi]:\ |\sin(\varkappa-\alpha)|<\sin\varepsilon\big\}.$$

For every $k\in\mathbb{N}$ let

$$W_{k+1}:=[-2^{-1}\pi,2^{-1}\pi]\setminus\Big(\bigcup_{j=1}^{k}V_{2^{-j}-2^{-k-1}}(\zeta_j-2^{-1}\pi)\Big),\quad W_1:=(-\pi,\pi].$$

For all $j\in\{1,\dots,k\}$ a unic $\beta_{2j}(k),\beta_{2j+1}(k)\in(-2^{-1}\pi,2^{-1}\pi]$ exist such that

$$\sin(\beta_{2j+\delta}(k)-\zeta_j+2^{-1}\pi)=(-1)^\delta\sin(2^{-j}-2^{-k-1}),\ \ \delta\in\{0,1\}.$$

A substitution $j(\,\cdot\,):\{1,\dots,2k\}\to\{1,\dots,2k\}$ exist with the facility that the sequence $\{\beta_{j(i)}(k)\}_{i=1}^{2k}\subset(-2^{-1}\pi,2^{-1}\pi)$ do not decrease.

Let $\beta_{j(0)}:=-2^{-1}\pi$, $\beta_{j(2k+1)}:=2^{-1}\pi$.

The bound ∂W_{k+1} of the set W_{k+1} satisfies the inclusions

$$\partial W_{k+1}\subset\{-2^{-1}\pi,2^{-1}\pi\}\cup\Big(\bigcup_{j=1}^{k}\partial V_{2^{-j}-2^{-k-1}}(\zeta_j-2^{-1}\pi)\Big)\subset\{\beta_j(k)\}_{j=0}^{2k+1}. \tag{7}$$

We shall build the set $I_k\subset\{0,\dots,2k\}$ by the next way. Because of (7) for all $i\in\{0,\dots,2k\}$ or the relation $L_{i,k+1}:=[\beta_{j(i)},\beta_{j(i+1)}]\in W_{k+1}$ holds, in this case we set $I_k\ni i$, or, otherwise, the inclusion $L_{i,k+1}\in[-2^{-1}\pi,2^{-1}\pi]\setminus W_{k+1}$ is true. In the last case let $I_k\not\ni i$.

For every $i\in I_k$ let

$$b_i:=2^{-1}(\beta_{j(i)}+\beta_{j(i+1)})\in[-2^{-1}\pi,2^{-1}\pi],\quad c_i:=2^{-1}(\beta_{j(i+1)}-\beta_{j(i)})\in[-2^{-1}\pi,2^{-1}\pi].$$

Next equalities hold

$$L_{i,k+1}=\big\{\varphi\in[-2^{-1}\pi,2^{-1}\pi]:\ |\sin(\varphi-b_i)|\le\sin c_i\big\},\quad W_k=\bigcup_{i\in I_k}L_{i,k+1}.$$

If $k=0$, we set $I_0=1$, $L_{1,1}=[-2^{-1}\pi,2^{-1}\pi]$.

Assume the first that $\mu_{2j-1},\mu_{2j}\in\mathbb{R}$, $j\in I_{k-1}$, exist for some $k\in\mathbb{N}$ such that the equality holds

$$\sin\xi_k(M_{i,k})=\sin L_{i,k},\quad M_{i,k}:=[\mu_{2i-1},\mu_{2i}],\ \ i\in I_{k-1} \tag{8_k}$$

and, the second, that in the case $k>1$ we have the inclusion

$$M_k:=\bigcup_{j\in I_{k-1}}M_{j,k}\subset M_{k-1}. \tag{9_k}$$

Let us denote

$$s_k:=\sum_{j=1}^{k-1}2^{-j}j,\ \ s_1:=0.$$

Assume that the next inequality holds

$$\operatorname{sh} \ln \eta_k(\mu) \geq 2^{(9-s_k)2^k}. \tag{10_k}$$

Due to (8_k) for all $\mu \in M_k$ the inclusion $\xi_k(\mu) \in \mathbb{R} \setminus V_{2^{-k-1}}(\zeta_k - 2^{-1}\pi))$ is true, that imply the inequalities

$$|\cos \xi_k(\mu)| \geq \sin 2^{-k-1} \geq 2^{-k-2}. \tag{11}$$

For all $\mu \in M_k$ the formulas (6), (10_k) and (11) give the estimation

$$\operatorname{sh} \ln \eta_{k+1}(\mu) \overset{(6)}{=} \operatorname{sh} \ln \eta_k^2 \cos \xi_k(\mu) \overset{(11)}{\geq} 2^{-k-2} \operatorname{sh} \ln \eta_k^2(\mu) \overset{(10_k)}{\geq} 2^{(9-s_k)2^{k+1}-2k} \geq 2^{(9-s_{k+1})2^{k+1}}.$$

Hence we have the relation (10_{k+1}).

We set

$$S_k(\alpha) := \sum_{j=1}^{k} \alpha^j j.$$

For all $\alpha \in (-1, 1)$ we obtain the equalities

$$S_{+\infty}(\alpha) = \Big(\sum_{j=1}^{+\infty} \alpha^j\Big)'_\alpha = \big((1-\alpha)^{-1}\big)'_\alpha = 2(1-\alpha)^{-2}.$$

Since that the next relations hold

$$s_k \leq s_{+\infty} = \sum_{j=1}^{+\infty} 2^{-j} j = S_{+\infty}(2^{-1}) = 8.$$

Hence, in view of (10_k), we have the estimate

$$\operatorname{sh} \ln \eta_k(\mu) \geq 2^{2^k}. \tag{12_k}$$

For all $i \in I_k$ the inclusion $V_{2^{-k-1}}(L_{i,k+1}) \subset W_k$ is true. Since that, because of $L_{i,k+1}$ is the closed interval, there exists $j_i \in I_{k-1}$ such that the relation $V_{2^{-k-1}}(L_{i,k+1}) \subset L_{j_i,k}$ holds.

Due to (4), (11) and (12_k), we have the estimates

$$\begin{aligned} &|\varphi_k(\mu) \leq 2|\sin \varphi_k(\mu)| \\ &\qquad \leq 2|\operatorname{tg} \varphi_k(\mu)| \overset{(4)}{=} 2(\operatorname{ch} 2\eta_k(\mu))^{-1} \operatorname{tg} \xi_k(\mu) \leq 4e^{-2\eta_k(\mu)} |\cos \xi_k(\mu)|^{-1} \overset{(11),\,(12_k)}{\leq} 2^{-k-1}. \end{aligned} \tag{13}$$

Hence the next inclusion holds

$$\psi_{k+1}(\mu_{2j-\delta}) \overset{(12)}{\in} V_{2^{-k-1}}(\psi_k(\mu_{2j-\delta})), \ \ \delta = \overline{0,1}. \tag{14}$$

Let us denote the function $f(\,\cdot\,) : \mathbb{R} \to [-1, 1]$ by the formula $f(\mu) := \sin \xi_{k+1}(\mu)$.

Because of (14) and due to (8_k), we have the inequality

$$|f(\mu_{2j-\delta})| \geq \sin(c_{j,k} - 2^{-k-1}) =: \varkappa. \tag{15}$$

Let us denote $s := \operatorname{sgn}(f(\mu_{2j}) - f(\mu_{2j-1}))$, $g(\mu) := sf(\mu)$.

The relation (15) implies the estimates

$$g(\mu_{2j-1,k}) \leq -\varkappa < 0 < \varkappa \leq g(\mu_{2j,k}). \tag{16}$$

Because of continuity of the function $\eta_1(\,\cdot\,)$, $\varphi_{k+1}(\,\cdot\,)$ is also continuous, hence such is the function $g(\,\cdot\,)$. Since that and in view of (16) the function $g(\,\cdot\,)$ satisfies conditions of Lemma 2, in which one have denote $[a,b]:=[\mu_{2j-1,k},\mu_{2j,k}]$, $[c,d]:=[-\varkappa,\varkappa]$.

Hence, because of this lemma, there exists a closed interval $M_{i,k+1}:=[\mu_{2i-1,k+1},\mu_{2i,k+1}]\subset M_{j,k}$ such that $g(M_{i,k+1})=[-\varkappa,\varkappa]=\sin L_{i,k+1}$, that is (8_{k+1}) holds. Beside of that we have the inclusion (9_{k+1}).

Note that in the case $k=1$ the equalities $I_0=1$, $L_1=[-2^{-1}\pi,2^{-1}\pi]$ are true, since that, if denote $M_1:=M_{1,1}=[\mu_{1,1},\mu_{1,2}]:=[-2^{-1}\pi,2^{-1}\pi]$, we obtain the relation $\sin\xi_1(M_{1,1})=\sin([-\pi+a_1,\pi+a_1])=[-1,1]=\sin L_1$, that is, the equality (8_1) holds.

Due to (2), we have the inequalities

$$\operatorname{sh}\ln\eta_1(\mu)=2^{-1}(\eta_1(\mu)-\eta_1^{-1}(\mu))\overset{(2)}{\geq}2^{-1}(2^{20}-2^{-20})\geq 2^{18}=2^{(9-s_1)2^1},$$

that implies the estimates (10_1).

Under induction, we obtain the relations (8_n), (9_n) and (10_n) for every $1<n\in\mathbb{N}$.

Due to (8_k), the positivity of Lebesgue measure for the set W_k implies the inequality $M_k\neq\varnothing$. Hence, in view of (8_n), $n\in\mathbb{N}$, we have the existence of $\mu_{+\infty}\in M_{+\infty}:=\operatorname*{Lim}\limits_{k\to+\infty}M_k$.

Because of (5_n) and (12_n), in view of the Lyapunov formula for the senior characteristic exponent of system (1_μ) [2], the next estimates hold

$$\lambda_{\max}(A_{\mu_{+\infty}})=\varlimsup_{t\to+\infty}t^{-1}\ln\|X_{A_{\mu_{+\infty}}}(t,0)\|\geq\varlimsup_{n\to+\infty}2^{-n}\ln\|X_{A_{\mu_{+\infty}}}(2^n,0)\|\overset{(5_n),(12_n)}{\geq}1.$$

They theorem is proved. □

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Global Components of Positive Bounded Variation Solutions of a One-Dimensional Capillarity Problem

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In this paper we study the topological structure of the set of positive bounded variation solutions of the quasilinear Neumann problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+u'^2}}\Big)' = \lambda a(x) f(u) \quad \text{in } (0,1), \\ u'(0)=0, \;\; u'(1)=0, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter, $a \in L^\infty(0,1)$ changes sign, $f \in C^1(\mathbb{R})$ satisfies $f(s)$, $s > 0$ for all $s \neq 0$ and $f'(0) = 1$. Problem (1) is a particular version of

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = g(x,u) \quad \text{in } \Omega, \\ -\dfrac{\nabla u \cdot \nu}{\sqrt{1+|\nabla u|^2}} = \sigma \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. This model plays a central role in the mathematical analysis of a number of geometrical and physical issues, such as prescribed mean curvature problems for cartesian surfaces in the Euclidean space [11, 19, 22–25, 30, 45, 46], capillarity phenomena for incompressible fluids [16, 20, 21, 27, 28], and reaction-diffusion processes where the flux features saturation at high regimes [12, 29, 44].

Although there is a large amount of literature devoted to the existence of positive solutions for semilinear elliptic problems with indefinite nonlinearities [1–3, 7, 8, 26, 33, 37], no results were available for the problem (2), even in the one-dimensional case (1), before [35, 36], where we began the analysis of the effects of spatial heterogeneities in the simplest prototype problem (1). Even if part of our discussion in this paper has been influenced by some results in the context of semilinear equations, it must be stressed that the specific structure of the mean curvature operator, $u \mapsto -\operatorname{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big)$, makes the analysis in this paper much more delicate and sophisticated, as (1) may determine spatial patterns which exhibit sharp transitions between adjacent profiles, up to the formation of discontinuities [9, 10, 12, 17, 18, 29, 40, 42]. This special feature explains why the existence intervals of regular positive solutions of [14, 15, 39] are smaller than those given in the former references when dealing with bounded variation solutions. It is a well-agreed fact that the space of bounded variation functions is the most appropriate setting for discussing these topics. The precise notion of bounded variation solution of (1) used in this paper has been basically introduced in [5, 6] and it has been extensively used and discussed later (see, e.g., [35, 38, 40–43]).

Definition 1 (Bounded variation solution)**.** A bounded variation solution of problem (1) is a function $u \in BV(0,1)$ such that

$$\int_0^1 \frac{Du^a D\phi^a}{\sqrt{1+(Du^a)^2}}\,dx + \int_0^1 \frac{Du^s}{|Du^s|}\,D^s\phi = \int_0^1 \lambda a f(u)\phi\,dx \tag{3}$$

for all $\phi \in BV(0,1)$ such that $|D\phi^s|$ is absolutely continuous with respect to $|Du^s|$.

In Definition 1 the following notations are used for every $v \in BV(0,1)$ (we refer to, e.g., [4,13] for any required additional detail):

- $Dv = Dv^a dx + Dv^s$ is the Lebesgue–Nikodym decomposition of the Radon measure Dv in its absolutely continuous part $Dv^a dx$, with density function Dv^a, and its singular part Dv^s, with respect to the Lebesgue measure dx in $\mathbb{R}$.
- $|Dv|$, $|Dv^a|$ and $|Dv^s|$ stand for the absolute variations of the measures Dv, Dv^a and Dv^s, respectively; thus, the Lebesgue–Nikodym decomposition of $|Dv|$ is given by
$$|Dv| = |Dv|^a\,dx + |Dv|^s = |Dv^a|\,dx + |Dv^s|.$$
- $\frac{Dv}{|Dv|}$ and $\frac{Dv^s}{|Dv^s|}$ denote the density functions of Dv and Dv^s, respectively, with respect to their absolute variations $|Dv|$ and $|Dv^s|$.

In [35], we discussed the existence and the multiplicity of positive bounded variation solutions of (1) under various representative configurations of the behavior at zero and at infinity of the function f. The solutions of [35] can be singular, for as they may exhibit jump discontinuities at the nodal points of the weight function a, while they are regular, at least of class C^1, on each open interval where the weight function a has a constant sign. Instead, in [36] we investigated the existence and the non-existence of positive regular solutions. Some of the most intriguing findings of [35,36] can be synthesized by saying that the solutions of (1) obtained in [35] are regular as long as they are small, in a sense to be precised later, whereas they develop singularities as they become sufficiently large. This is in complete agreement with the peculiar structure of the mean curvature operator, which combines the regularizing features of the 2-laplacian, when ∇u is sufficiently small, with the severe sharpening effects of the 1-laplacian, when ∇u becomes larger.

A natural question arising at the light of these novelties is the problem of ascertaining whether or not these regular and singular solutions can be obtained, simultaneously, by establishing the existence of connected components of bounded variation solutions bifurcating from $(l,u)=(l,0)$, which stem regular from $(l,0)$ and develop singularities as their sizes increase; thus establishing the coexistence along the same component of both regular and singular solutions, as synoptically illustrated by the two bifurcation diagrams in Figure 1. Although this phenomenology has been already documented by the special example of [36, Section 8], by means of a rather sophisticated phase plane analysis, solving this problem in our general setting still was a challenge.

The main aim of this work is establishing the existence of two connected components, $\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^+$, of the closure of the set of positive bounded variation solutions of problem (1),

$$\mathcal{S}^> = \big\{(\lambda,u) \in [0,+\infty) \times BV(0,1):\ u>0 \text{ is a solution of (1)}\big\} \cup \big\{(0,0),(\lambda_0,0)\big\},$$

emanating from the line $\{(l,0):\ l \in \mathbb{R}\}$ of the trivial solutions, at the two principal eigenvalues $l=0$ and $l=l_0$ of the linearization of (1) at $u=0$,

$$\begin{cases} -u'' = \lambda a(x) u & \text{in } (0,1),\\ u'(0)=u'(1)=0. \end{cases} \tag{4}$$

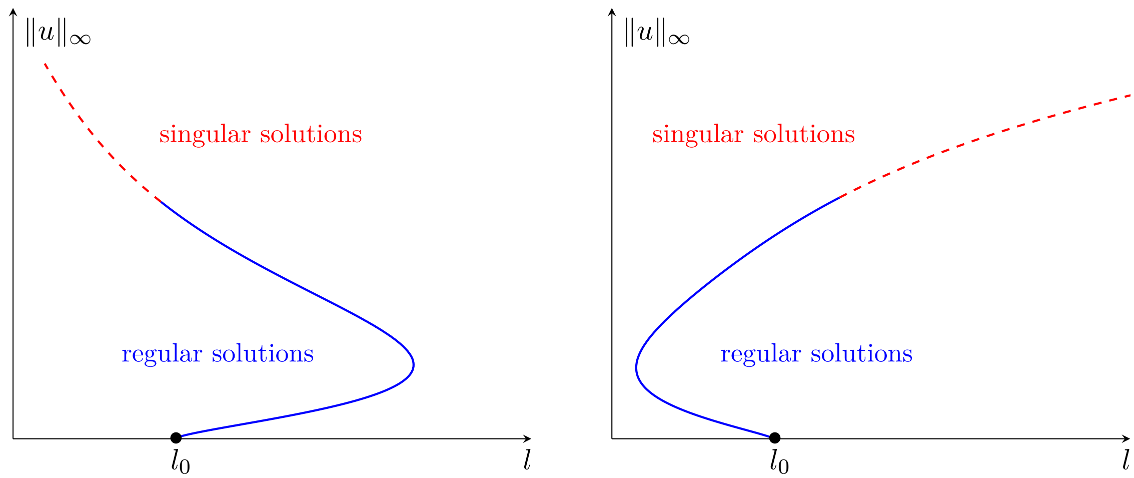

Figure 1. Global bifurcation diagrams emanating from the positive principal eigenvalue l_0, according to the nature of the potential $\int_0^s f(t)\,dt$ of f: superlinear at infinity (on the left), or sublinear at infinity (on the right).

Precisely, our main global bifurcation theorem (see [34] for the proof) can be stated as follows.

Theorem 1. *Assume that $f \in C^1(\mathbb{R})$ satisfies $f(s)s > 0$ for all $s \neq 0$, $f'(0) = 1$, and, for some constants $\kappa > 0$ and $p > 2$, $|f'(s)| \le \kappa\,(|s|^{p-2} + 1)$ for all $s \in \mathbb{R}$. Moreover, suppose that a satisfies $\int_0^1 a(x)\,dx < 0$ and there is $z \in (0,1)$ such that $a(x) > 0$ a.e. in $(0,z)$ and $a(x) < 0$ a.e. in $(z,1)$. Then there exist two subsets of $\mathcal{S}^>$, $\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^>$ such that*

- *$\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^>$ are maximal in $\mathcal{S}^>$ with respect to the inclusion, are connected with respect to the topology of the strict convergence in $BV(0,1)$*[1]*, and are unbounded in $\mathbb{R} \times L^p(0,1)$;*
- *$(0,0) \in \mathcal{C}_0^>$ and $(\lambda_0, 0) \in \mathcal{C}_{\lambda_0}^>$;*
- *$\{(0,r) : r \in [0,+\infty)\} \subseteq \mathcal{C}_0^>$;*
- *if $(\lambda, u) \in \mathcal{C}_0^> \cup \mathcal{C}_{\lambda_0}^>$ and $u \neq 0$, then $\operatorname{ess\,inf} u > 0$;*
- *if $(\lambda, 0) \in \mathcal{C}_0^> \cup \mathcal{C}_{\lambda_0}^>$ for some $\lambda > 0$, then $\lambda = \lambda_0$;*
- *either $\mathcal{C}_0^> \cap \mathcal{C}_{\lambda_0}^> = \varnothing$, or $(\lambda_0, 0) \in \mathcal{C}_0^+$ and $(0,0) \in \mathcal{C}_{\lambda_0}^>$ and, in such case, $\mathcal{C}_0^> = \mathcal{C}_{\lambda_0}^>$;*
- *there exists a neighborhood U of $(0,0)$ in $\mathbb{R} \times L^p(0,1)$ such that $\mathcal{C}_0^> \cap U$ consists of regular solutions of (1);*
- *there exists a neighborhood V of $(\lambda_0, 0)$ in $\mathbb{R} \times L^p(0,1)$ such that $\mathcal{C}_{\lambda_0}^> \cap V$ consists of regular solutions of (1).*

Theorem 1 appears to be the first global bifurcation result for a quasilinear elliptic problem driven by the mean curvature operator in the setting of bounded variation functions. The absence in the existing literature of any previous result in this direction might be attributable to the fact that mean curvature problems are fraught with a number of serious technical difficulties which do not

[1] See [4, Definition 3.14]

arise when dealing with other non-degenerate quasilinear problems. As a consequence, our proof of Theorem 1 is extremely delicate, even though the problem (1) is one-dimensional. The main technical difficulties coming from the eventual lack of regularity of solutions of (1) as they grow, which does not allow us to work neither in spaces of differentiable functions, nor in Sobolev spaces. Instead, this lack of regularity forces us to work in the frame of the Lebesgue spaces L^p, where the cone of positive functions has empty interior and most of the global path-following techniques in bifurcation theory fail. Thus, to get most of the conclusions of Theorem 1, a number of highly non-trivial technical issues must be previously overcome. Among them count the reformulation of (1) as a suitable fixed point equation, the proof of the differentiability of the associated underlying operator, the search for the most appropriate global bifurcation setting, as well as solving the tricky problem of the preservation of the positivity of the solutions along both components, for as in the L^p context a positive solution, a priori, could be approximated by changing sign solutions. Naturally, none of these rather pathological situations cannot arise when dealing with classical regular problems, like those considered in [32].

For simplicity, here we have restricted ourselves to deal with the simplest situation when the function a possesses a single interior node z, and thus the positive solutions of (1) are monotone. As our proof relies, on a pivotal basis, on this special feature, getting a proof of this theorem in the general case when a has an intricate nodal behavior might be a real challenge plenty of technical difficulties. The validity of Theorem 1 in more general settings remains therefore an open problem.

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On Adaptive Sequences to Evaluate Izobov Exponential Exponents

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Consider a linear system
$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$
with piecewise continuous and bounded coefficient matrix A and with the Cauchy matrix X_A. Together with the system (1) consider a perturbed system
$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$
with piecewise continuous and bounded perturbation matrix Q. Denote the higher exponent of (2) by $\lambda_n(A+Q)$.

One of the basic problem of Lyapunov exponents theory is to describe the influence of perturbations of coefficients from various classes on asymptotic properties of system (2). Usually these perturbations are considered as small in some sense. For example, the value $\Lambda(\mathfrak{M}, A) := \sup\{\lambda_n(A+Q) : \ Q \in \mathfrak{M}\}$ is known as attainable bound of upward mobility of higher exponent of (2) with perturbations from $\mathfrak{M}$, see [4, p. 157], [8], [11, p. 39], [10, p. 46], [17]. The following classes are commonly used to calculate $\Lambda(\mathfrak{M}, A)$:

Infinitesimal perturbations [18]
$$Q(t) \to 0, \ \ t \to +\infty, \tag{3}$$

exponentially small perturbations [9]
$$\|Q(t)\| \leq C(Q)\exp(-\sigma(Q)t), \ \ C(Q) > 0, \ \ \sigma(Q) > 0; \tag{4}$$

σ-perturbations [7]:
$$\|Q(t)\| \leq C(Q)\exp(-\sigma t), \ \ C(Q) > 0, \ \ \sigma > 0; \tag{5}$$

power perturbations
$$\|Q(t)\| \leq C(Q)t^{-\gamma}, \ \ C(Q) > 0, \ \ \gamma > 0; \tag{6}$$

generalized power perturbations [1, 2]
$$\|Q(t)\| \leq C(Q)\exp(-\sigma\theta(t)), \ \ C(Q) > 0, \ \ \sigma > 0, \tag{7}$$
$$\|Q(t)\| \leq C(Q)\exp(-\sigma(Q)\theta(t)), \quad C(Q) > 0, \quad \sigma(Q) > 0, \tag{8}$$
where θ is a positive function satisfying some additional conditions;

infinitesimal average [18] and integrable perturbations [3]
$$\lim_{t\to+\infty} \frac{1}{t}\int\limits_0^t \|Q(t)\|\,dt = 0, \qquad \int\limits_0^{+\infty} \|Q(t)\|\,dt < +\infty, \tag{9}$$

and their modifications with some positive weights φ and powers $p \geq 1$, see [4, p. 309], [5, 12, 13, 15, 16],

$$\lim_{t\to+\infty} \frac{1}{t} \int\limits_0^t \varphi(\tau)\|Q(\tau)\|^p\,d\tau = 0, \qquad \int\limits_0^{+\infty} \varphi(\tau)\|Q(t)\|^p\,dt < +\infty. \tag{10}$$

Sometimes [1–3, 12, 13, 15, 16] to calculate $\Lambda(\mathfrak{M})$ we can construct an algorithm analogous to a famous Izobov algorithm for σ-exponent [7]

$$\nabla_\sigma(A) = \overline{\lim_{k\to\infty}} \frac{\xi_k(\sigma)}{k}\,, \tag{11}$$
$$\xi_k(\sigma) = \max_{i\leq k} \big\{ \ln\|X_A(k,i)\| + \xi_i(\sigma) - \sigma i \big\}, \ \ \xi_0 = 0, \ \ k \in \mathbb{N} \cup \{0\}.$$

For classes (5)–(7), (10), and the first of (9) we can write it in the general form

$$\Lambda(\mathfrak{M}, A) = \overline{\lim_{k\to\infty}} \frac{\ln \eta_k}{k}\,, \tag{12}$$
$$\eta_k = \max_{i\leq k} \big\{ \|X_A(k,i)\|\beta(i)\eta_i \big\}, \ \ \eta_0 = 1, \ \ k \in \mathbb{N} \cup \{0\},$$

where $\beta(k)$, $\beta(0) > 0$ is some nonegative function depending on $\mathfrak{M}$, e.g. $\beta(i) = e^{-\sigma i}$ for σ-perturbations. We shall consider β as a functional parameter of the algorithm.

The quantity η_k is always positive, because the maximum in (12) can not be reached at some $i \in \mathbb{N}$ if $\beta(i)$ is zero. We shall refer to this property of the algorithm (12) as adaptivity.

Alternatively, in some other cases [1, 2, 9, 18] we have formulas like the following Millionshcikov formula [4, p. 99], [8], [10, p. 48], [17]

$$\Omega(A) = \lim_{T\to+\infty} \overline{\lim_{k\to\infty}} \frac{1}{mT} \sum_{k=1}^{m} \ln\|X_A(kT, kT-T)\|, \tag{13}$$

for the central exponent. One of such classes is the class of exponential perturbations, see formula (4). For exponential exponent $\nabla_0(A)$ corresponding to them [9] we have

$$\nabla_0(A) = \lim_{\theta\to1+0} \overline{\lim_{m\to\infty}} \frac{1}{\theta^m} \sum_{k=1}^{m} \ln\|X_A(\theta^k, \theta^{k-1})\|, \tag{14}$$

Also classes (3), (4), (8), and the second of (9) have the analogous expression for $\Lambda(\mathfrak{M}, A)$. The smallness classes $\mathfrak{M}$ for which $\Lambda(\mathfrak{M})$ has the representation of the form similar to (13), are called limit classes [1, 2].

One of the most important differences between representations (13) or (14) and algorithm (12) is that the sequence to calculate $\nabla_\sigma(A)$ is determined by system (1) itself, and $\nabla_0(A)$ or $\Omega(A)$ are calculated using strictly prescribed sequences. This rigidity does not allow us to construct analogues of formulas (13) and (14) for the perturbation classes with degenerations as it was done for algorithms of the type (12) in [14].

Let $\mathbb{T}$ be the set of all sequences $t_k \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, monotonically increasing to $+\infty$. For each $\tau \in \mathbb{T}$ put

$$\Omega(A, \tau) = \overline{\lim_{k\to\infty}} \frac{1}{t_{k+1}} \sum_{i=0}^{k} \ln\|X_A(t_{i+1}, t_i)\|.$$

We say that some family of sequences depending on a functional parameter β is adaptive if β is not zero at any element of each of these sequences.

We say that a one-parametric family $\mathbb{S}_\alpha$ of sequences is admissible for a class $\mathfrak{M}$ if for some α_0 the equality

$$\Lambda(\mathfrak{M},A)=\lim_{\alpha\to\alpha_0}\sup_{\tau\in\mathbb{S}_\alpha}\Omega(A,\tau)$$

holds.

For any $\theta>1$ $\mathbb{T}_\theta$ by $\mathbb{T}$ let us denote the set of all sequences from $\mathbb{T}$ satisfying the condition $\lim\limits_{k\to+\infty} t_k^{-1}t_{k+1}\geq\theta$.

Lemma. *The equality*

$$\nabla_0(A)=\lim_{\theta\to1+0}\sup_{\tau\in\mathbb{T}_\theta}\Omega(A,\tau)$$

holds.

Together with the property A established in [7] for the families of finite sequences implementing the σ-exponent $\nabla_\sigma(A)$, the above lemma allow us to give an algorithm for adaptive construction of sequences implementing the exponential exponent $\nabla_0(A)$. We can prove analogous lemmas for some other limit classes of perturbations.

Theorem. *For each of classes* (5)–(7)*, there exist a one-parametric family of admissible sequences.*

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On Unreachable Values of Boundary Functionals for Overdetermined Boundary Value Problems with Constraints

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1 Introduction

The classical formulation of the general linear boundary value problem (BVP) for linear ordinary differential system

$$(\mathcal{L}x)(t) \equiv \dot{x}(t) + A(t)x(t) = f(t), \quad t \in [0,T], \tag{1.1}$$

where $A(t)$ is a $n \times n$-matrix with elements summable on $[0,T]$, supposes that we are interested in the study of the question about the existence of solutions to (1.1) that satisfy the boundary conditions

$$\ell x = \beta \tag{1.2}$$

with linear bounded vector-functional $\ell = col(\ell_1, \dots, \ell_n)$ defined on the space of absolutely continuous functions $x : [0,T] \to R^n$ (see below more in detail). The key point in (1.1), (1.2) is that the number of linearly independent components ℓ_i in (1.2) equals the dimension of (1.1). In such a case, the unique solvability of BVP (1.1), (1.2) for $f = 0$, $\beta = 0$ implies the everywhere and unique solvability. If this is not the case, we have very specific situation with either the underdetermined BVP or the overdetermined BVP [11].

Linear BVP's for differential equations with ordinary derivatives, that lack the everywhere and unique solvability, are met with in various applications. Among these applications are some problems in Economic Dynamics [10, 12]. Results on the solvability and solutions representation for these BVP's are widely used as an instrument of investigating weakly nonlinear BVP's [6]. General results concerning linear BVP's for an abstract functional differential equation (AFDE) are given in [5]. As for linear overdetermined BVP's for AFDE in general, the principal results by L. F. Rakhmatullina are given in detail in [2, 3, 5].

In this paper, we consider the case that the number of linearly independent boundary conditions is greater than the dimension of the null-space of the corresponding homogeneous equation and study the BVP for FDE in an essentially different statement. Namely, the question we discuss is as follows: does there exist at least one free term f in the given linear FDE such that (1.2) holds for a fixed β, taking into account some given pointwise constraints with respect to $f(t)$ on $[0,T]$. Next we give a description for the set of unreachable β's, i.e. those for which f does not exist.

2 A class of boundary value problems

In this section, we consider a system of functional differential equations with aftereffect that, formally speaking, is a concrete realization of the AFDE, and, on the other hand, it covers many kinds of dynamic models with aftereffect (integro-differential, delayed differential, differential difference) [9, 12].

Let us introduce the functional spaces where operators and equations are considered. Fix a segment $[0,T] \subset R$. By $L_2^n = L_2^n[0,T]$ we denote the Hilbert space of square summable functions $v : [0,T] \to R^n$ endowed with the inner product $(u,v) = \int\limits_0^T u'(t)v(t)\,dt$ ($\cdot'$ is the symbol of transposition). The space $AC_2^n = AC_2^n[0,T]$ is the space of absolutely continuous functions $x : [0,T] \to R^n$ such that $\dot{x} \in L_2^n$ with the norm $\|x\|_{AC_2^n} = |x(0)| + \sqrt{(\dot{x},\dot{x})}$, where $|\cdot|$ stands for the norm of R^n.

Consider the functional differential equation

$$\mathcal{L}x \equiv \dot{x} - \mathcal{K}\dot{x} - A(\cdot)x(0) = f, \tag{2.1}$$

where the linear bounded operator $\mathcal{K} : L_2^n \to L_2^n$ is defined by

$$(\mathcal{K}z)(t) = \int\limits_0^t K(t,s)z(s)\,ds, \quad t \in [0,T],$$

the elements $k_{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $0 \le s \le t \le T$ and such that $|k_{ij}(t,s)| \le u(t)v(s)$, $i,j = 1,\dots,n$, $u,v \in L_2^1[0,T]$, $(n \times n)$-matrix A has elements that are square summable on $[0,T]$.

In what follows we will use some results from [1, 3, 8, 9] concerning (2.1). The homogeneous equation (2.1) ($f(t) = 0$, $t \in [0,T]$) has the fundamental $(n \times n)$-matrix $X(t)$:

$$X(t) = E_n + V(t),$$

where E_n is the identity $(n \times n)$-matrix, each column $v_i(t)$ of the $(n \times n)$-matrix $V(t)$ is a unique solution to the Cauchy problem

$$\dot{v}(t) = \int\limits_0^t K(t,s)\dot{v}(s)\,ds + a_i(t), \quad v(0) = 0, \quad t \in [0,T],$$

where $a_i(t)$ is the i-th column of A.

The solution to (2.1) with the initial condition $x(0) = 0$ has the representation

$$x(t) = (Cf)(t) = \int\limits_0^t C(t,s)f(s)\,ds,$$

where $C(t,s)$ is the Cauchy matrix [8] of the operator $\mathcal{L}$. This matrix can be defined (and constructed) as the solution to

$$\frac{\partial}{\partial t}\,C(t,s) = \int\limits_s^t K(t,\tau)\,\frac{\partial}{\partial \tau}\,C(\tau,s)\,d\tau + K(t,s), \quad 0 \le s \le t \le T,$$

under the condition $C(s,s) = E_n$. The properties of the Cauchy matrix used below are studied in detail in [9].

The matrix $C(t,s)$ is expressed in terms of the resolvent kernel $R(t,s)$ of the kernel $K(t,s)$. Namely,

$$C(t,s) = E_n + \int\limits_s^t R(\tau,s)\,d\tau.$$

The general solution to (2.1) has the form

$$x(t)=X(t)\alpha+\int_0^t C(t,s)f(s)\,ds,$$

with an arbitrary $\alpha\in R^n$.

The general linear BVP is the system (2.1) supplemented by the linear boundary conditions

$$\ell x=\beta,\ \ \beta\in R^N, \tag{2.2}$$

where $\ell: AC_2^n\to R^N$ is a linear bounded vector functional. Let us recall the representation of ℓ:

$$\ell x=\int_0^T \Phi(s)\dot{x}(s)\,ds+\Psi x(0). \tag{2.3}$$

Here Ψ is a constant $(N\times n)$-matrix, Φ is $(N\times n)$-matrix with elements that are square summable on $[0,T]$. We assume that the components $\ell_i: AC_2^n\to R$, $i=1,\dots,N$, of ℓ are linearly independent.

BVP (2.1), (2.2) is well-posed if $N=n$. In such a situation, the BVP is uniquely solvable for any $f\in L_2^n[0,T]$ and $\beta\in R^n$ if and only if the matrix

$$\ell X=(\ell X^1,\dots,\ell X^n),$$

where X^j is the j-th column of X, is nonsingular, i.e. $\det\ell X\neq 0$.

In the sequel we assume that $N>n$ and the system $\ell^i: AC_2^n\to R$, $i=1,\dots,N$, can be splitted into two subsystems $\ell^1: AC_2^n\to R^n$ and $\ell^2: AC_2^n\to R^{N-n}$ such that the BVP

$$\mathcal{L}x=f,\ \ \ell^1x=\beta^1 \tag{2.4}$$

is uniquely solvable. Without loss of generality we will consider the case that ℓ^1 is defined by $\ell^1x\equiv x(0)$, formed by the first n components of ℓ, and the elements of $\beta^1=0$ in (2.4) are the corresponding components of β. Thus ℓ^2 will stand for the final $(N-n)$ components of ℓ, and elements of $\beta^2\in R^{N-n}$ are defined as the final $(N-n)$ components of β. Let us write ℓ_1 in the form

$$\ell^1x=\int_0^T \Phi_1(s)\dot{x}(s)\,ds+\Psi_1x(0),$$

where $\Phi_1(s)=0$ and $\Psi_1=E_n$ are the corresponding rows of $\Phi(s)$ and Ψ, respectively, in (2.3). Similarly,

$$\ell_2x=\int_0^T \Phi_2(s)\dot{x}(s)\,ds+\Psi_2x(0).$$

Put

$$\Theta_i(s)=\Phi_i(s)+\int_s^T \Phi_i(\tau)C'_\tau(\tau,s)\,d\tau,\ \ i=1,2.$$

In the case that f is not constrained, it is shown in [11] that under the condition of nonsingularity of the matrix

$$W=\int_0^T \Theta_2(s)\Theta_2'(s)\,ds \tag{2.5}$$

BVP $(2.1),(2.2)$ is solvable for all $\beta^2 \in R^{N-n}$ if

$$f(t) = f_0(t) + \varphi(t),$$

where

$$f_0(t) = \Theta_2'(t)[W^{-1}\beta^2]$$

and $\varphi(\,\cdot\,) \in L_2^n$ is an arbitrary function that is orthogonal to each column of $\Theta_2'(\,\cdot\,)$:

$$\int_0^T \Theta_2(s)\varphi(s)\,ds = 0.$$

Here we consider the case of the pointwise constraints

$$c_i \le f_i(t) \le d_i, \;\; t \in [0,T], \;\; c_i \le d_i, \;\; i = 1,\dots,n, \tag{2.6}$$

with respect to components $f_i(t)$ of the column $f(t) = col(f_1(t),\dots,f_n(t))$. Denote $\mathcal{V} = [c_1,d_1] \times \cdots \times [c_n,d_n]$.

In the sequel it is assumed that the elements of $\Phi_2(t)$ are piecewise continuous on $[0,T]$.

To formulate the main theorem, let us introduce some notation. For any $\lambda \in R^{N-n}$ and $t \in [0,T]$, we define $z(t,\lambda)$ by the equality

$$z(t,\lambda) = \max\big(\lambda'\Theta_2(t)v : \; v \in \mathcal{V}\big).$$

Define $v(t,\lambda)$ as the centroid of the collection of the unite mass points belonging to $\mathcal{V}$ and bringing the value $z(t,\lambda)$ to the functional $v \to \lambda' \cdot \Theta_2(t) \cdot v$.

Theorem. *Let a collection* $\{\lambda_i \in R^{N-n}, \; i = 1,\dots,m\}$ *be fixed, and a collection* $\{q_i \in R, \; i = 1,\dots,m\}$ *be such that the inequalities*

$$\lambda_i' \int_0^T \Theta(t) \cdot v(t,\lambda_i)\,dt \le q_i, \;\; i = 1,\dots,m,$$

hold. Define $\mathcal{P}$ *as the set of all* $\rho \in R^{N-n}$ *such that the inequalities*

$$\lambda_i' \cdot \rho \le q_i, \;\; i = 1,\dots,m,$$

are fulfilled. Then all $\beta^2 \in R^{N-n}$ *outside the polyhedron* $\mathcal{P}$ *are unreachable for BVP* $(2.1),(2.2)$ *under constraints* (2.6).

The proof of the theorem is based on [7, Theorem 7.1].

Example. Let us consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-1) + f_1(t), \\ \dot{x}_2(t) &= -x_2(t) + f_2(t), \end{aligned} \quad t \in [0,3],$$

where $x_2(s) = 0$ if $s < 0$, with the initial conditions

$$x_1(0) = 0, \;\; x_2(0) = 0,$$

and additional conditions as follows:

$$x_1(3) - x_2(2) = \beta_1, \;\; x_2(3) + x_1(2) = \beta_2,$$

under the constraints

$$0 \le f_i(t) \le 2, \quad i = 1, 2.$$

Here we have

$$C(t,s) = \begin{pmatrix} 1 & \int\limits_s^t \chi_{[1,3]}(\tau)\chi_{[0,\tau-1]}(s)\exp(1-\tau+s)\,d\tau \\ 0 & \exp(s-t) \end{pmatrix},$$
$$\ell^2 x = col\big(x_1(3) - x_2(2), x_2(3) + x_1(2)\big),$$
$$\Theta_2(s) = \begin{pmatrix} C_{1,1}(3,s) - \chi_{[0,2]}(s)C_{2,1}(2,s) & C_{1,2}(3,s) - \chi_{[0,2]}(s)C_{2,2}(2,s) \\ C_{2,1}(3,s) + \chi_{[0,2]}(s)C_{1,1}(2,s) & C_{2,2}(3,s) + \chi_{[0,2]}(s)C_{1,2}(2,s) \end{pmatrix},$$

where $C_{j,k}(t,s)$, $j,k = 1,2$ are the components of $C(t,s)$. It should be noted that for W defined by (2.5) the inequality $\det W > 5$ holds.

By application of theorem for the case $\lambda_i = col(\sin(i\pi/4), \cos(i\pi/4))$, $i = 1, \dots, 8$, we obtain that all points (β_1, β_2) outside the intersection of the quadrangle with corners $\{(-1.35, 1.10), (1.02, -1.30), (5.40, 7.90), (7.90, 5.50)\}$ and the quadrangle with corners $\{(-0.60, 0), (-0.60, 6.55), (7.05, 0), (7.05, 6.55)\}$ are unreachable for the problem under consideration.

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The Periodic Problem for the Second Order Integro-Differential Equations with Distributed Deviation

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On the interval $I=[0,\omega]$, consider the second order linear integro-differential equation

$$u''(t)=p_0(t)u(t)+\int\limits_0^\omega p(t,\,s)u(\tau(t,s))\,ds+q(t), \tag{0.1}$$

and the nonlinear functional differential equation

$$u''(t)=F(u)(t)+q(t), \tag{0.2}$$

with the periodic two-point boundary conditions

$$u^{(i-1)}(\omega)-u^{(i-1)}(0)=c_i \ \ (i=1,2), \tag{0.3}$$

where $c_1,c_2\in R$, $p_0,f,q\in L_\infty(I,R)$, $p\in L_\infty(I^2,R)$, $\tau:I^2\to I$ is a measurable function, and $F:C'(I,R)\to L_\infty(I,R)$ is a continuous operator.

By a solution of problem $(0.2),(0.3)$ we understand a function $u:I\to R$, which is absolutely continuous together with its first derivative, satisfies equation (0.2) almost everywhere on I and satisfies conditions (0.3).

Our work is motivated by some original results for the functional differential equations with argument deviation (see [1, 2, 4]), and the results of Nieto [5] and Kuo-Shou Chiu [3].

Here we establish theorems which in some sense complete and generalize the results of the works cited above as well as some other known results. We first describe some classes of unique solvability for linear problems $(0.1),(0.3)$, and then on the basis of these results, we prove the existence theorems for nonlinear problem $(0.2),(0.3)$. The conditions we obtain take into account the effect of argument deviation, and in some sense are optimal.

Throughout the paper we use the following notation.

$R=\,]-\infty,+\infty[\,$, $R_+=[0,+\infty[\,$.

$C(I;R)$ is the Banach space of continuous functions $u:I\to R$ with the norm $\|u\|_C=\max\{|u(t)|:\ t\in I\}$.

$C'(I;R)$ is the Banach space of functions $u:I\to R$ which are continuous together with their first derivatives with the norm $\|u\|_{C'}=\max\{|u(t)|+|u'(t)|:\ t\in I\}$.

$L(I;R)$ is the Banach space of Lebesgue integrable functions $p:I\to R$ with the norm $\|p\|_L=\int\limits_0^\omega|p(s)|\,ds$.

$L_\infty(I,R)$ is the space of essentially bounded measurable functions $p : I \to R$ with the norm $\|p\|_\infty = \operatorname{ess\,sup}\{|p(t)| : t \in I\}$.

$L_\infty(I^2,R)$ is the set of such functions $p : I^2 \to R$, that for any fixed $t \in I$, $p(t,\cdot\,) \in L(I,R)$, and $\int\limits_0^\omega |p(\,\cdot\,,s)|\,ds \in L_\infty(I,R)$.

Also for arbitrary $p_0, p_1 \in L_\infty(I,R)$, $p \in L_\infty(I^2,R)$, and measurable $\tau : I^2 \to I$ we will use the notation:

$$\ell_0(p_0,p)(t) = |p_0(t)| + \int\limits_0^\omega |p(t,s)|\,ds,$$

$$\ell_1(p,\tau) = \frac{2\pi}{\omega}\bigg(\int\limits_0^\omega \bigg(\int\limits_0^\omega |p(\xi,s)|\,|\tau(\xi,s)-\xi|\,ds\bigg)\,d\xi\bigg)^{1/2}.$$

Definition 0.1. Let $\sigma \in \{-1,1\}$, and $\tau : I \to I$ be the measurable function. We say that the vector-function $(h_0,h) : I \to R^2$, where $h_0 \in L_\infty(I,R_+)$ and $h \in L_\infty(I^2,R_+)$, belongs to the set P^σ_τ, if for an arbitrary vector-function $(p_0,p) : I \to R^2$ with such measurable components, that

$$0 \le \sigma p_0(t) \le h_0(t), \quad 0 \le \sigma p(t,s) \le h(t,s) \ \text{ for } \ t,s \in I,$$
$$p_0(t) + \int\limits_0^\omega p(t,s)\,ds \not\equiv 0, \tag{0.4}$$

the homogeneous problem

$$v''(t) = p_0(t)v(t) + \int\limits_0^\omega p(t,s)v(\tau(t,s))\,ds,$$
$$v^{(i-1)}(\omega) - v^{(i-1)}(0) = 0 \ \ (i=1,2),$$

has no nontrivial solution.

1 Linear problem

Proposition 1.1. *Let $\sigma \in \{-1,1\}$,*

$$h_0 \in L_\infty(I,R_+), \ \ h \in L_\infty(I^2,R_+), \quad h_0(t) + \int\limits_0^\omega h(t,s)\,ds \not\equiv 0,$$

and for almost all $t \in I$ the inequality

$$\frac{1-\sigma}{2}\,\ell_0(h_0,h)(t) + \ell_1(h,\tau)\ell_0^{1/2}(h_0,h)(t) < \frac{4\pi^2}{\omega^2}$$

holds. Then

$$(h_0,h) \in P^\sigma_\tau. \tag{1.1}$$

Theorem 1.1. *Let $\sigma \in \{-1,1\}$, $\sigma p_0 \in L_\infty(I,R_+)$, $\sigma p \in L_\infty(I^2,R_+)$, and condition* (0.4) *be fulfilled. Moreover, let for almost all $t \in I$ the inequality*

$$\frac{1-\sigma}{2}\,\ell_0(p_0,p)(t) + \ell_1(p,\tau)\ell_0^{1/2}(p_0,p)(t) < \frac{4\pi^2}{\omega^2} \tag{1.2}$$

hold. Then problem (0.1), (0.3) *is uniquely solvable.*

Let $p_0 \equiv 0$, $\tau(t,s) \equiv t - \nu(t,s)$, and $0 \le \nu(t,s) \le t$ for $t, s \in I$. Then equation (0.1) transforms to the next equation

$$u''(t) = \int\limits_0^\omega p(t,s)u(t-\nu(t,s))\,ds + q(t), \tag{1.3}$$

and from Theorem 1.1 it follows

Corollary 1.1. *Let conditions* $p \in L_\infty(I^2, R_+)$, $\int\limits_0^\omega p(t,s)\,ds \not\equiv 0$, *and for almost all* $t \in I$ *the inequality*

$$\int\limits_0^\omega \int\limits_0^\omega p(\xi,s)\nu(\xi,s)\,ds\,d\xi \int\limits_0^\omega p(t,s)\,ds < \frac{4\pi^2}{\omega^2}$$

hold. Then problem (1.3), (0.3) *is uniquely solvable.*

Corollary 1.2. *Let* $n \ge 3$, *and the function* $p_1 \in L_\infty(I, R_+)$ *be such that for almost all* $t \in I$ *the inequality*

$$\int\limits_0^\omega \int\limits_0^t p_1(s)|\tau(s) - t|\,ds\,dt \int\limits_0^\omega p_1(s)\,ds \le \frac{4\pi^2[(n-3)!\,]^2}{\omega^{2(n-2)}}$$

holds. Then the problem

$$u^{(n)}(t) = p_1(t)u(\tau(t)) + q(t), \tag{1.4}$$

under the two-point boundary conditions

$$u^{(i-1)}(\omega) - u^{(i-1)}(0) = c_i, \;\; u^{(j-1)}(0) = c_j \;\; (i = 1, 2; \;\; j = 3, \dots, n),$$

where $c_k \in R$ $(k = 1, \dots, n)$, $p_1 \in L_\infty(I, R)$, *and* $\tau : I \to I$ *is a measurable function, is uniquely solvable.*

If $p_0 \equiv 0$ and $\tau(t,s) = \tau(t)$ for $t, s \in I$, then equation (0.1) transforms to the equation (1.4) with $n = 2$, $p_1(t) = \int\limits_0^\omega p(t,s)\,ds$, and then from Theorem 1.1 it follows

Corollary 1.3. *Let* $p_1 \in L_\infty(I, R_+)$ *be such that for almost all* $t \in I$ *the inequality*

$$p_1(t) \int\limits_0^\omega p_1(s)|\tau(s) - s|\,ds < \frac{4\pi^2}{\omega^2}$$

holds. Then problem (1.4), (0.3) *when* $n = 2$ *is uniquely solvable.*

2 Nonlinear problem

Definition 2.1. We say that the operator F belongs to the Carathéodory's local class and write $F \in K(C', L_\infty)$, if $F : C'(I, R) \to L_\infty(I, R)$ is the continuous operator, and for an arbitrary $r > 0$,

$$\sup\big\{|F(x)(t)| : \;\|x\|_{C'} \le r, \; x \in C'(I, R)\big\} \in L_\infty(I, R_+).$$

Definition 2.2. Let $\sigma \in \{-1,1\}$, inclusion (1.1) hold and the operators $V_0 : C'(I,R) \to L_\infty(I,R)$, $V : C'(I,R) \to L_\infty(I^2,R)$ be continuous. Then we say that $(V_0,V) \in E(h_0,h,P_\tau^\sigma)$, if for all $x \in C'(I,R)$ the conditions

$$0 \le \sigma V_0(x)(t) \le h_0(t), \ \ 0 \le \sigma V(x)(t,s) \le h(t,s) \ \text{ for } \ t,s \in I$$

hold, and

$$\inf\big\{\|L(x,1)\|_L : \ x \in C'(I,R)\big\} > 0,$$

where

$$L(x,y)(t) = V_0(x)(t)y(t) + \int_0^\omega V(x)(t,s)y(\tau(t,s))\,ds.$$

Also here it is assumed that the function sgn is defined by the equality

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x \ge 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then the next theorem is true.

Theorem 2.1. *Let $\sigma \in \{-1,\,1\}$, and*

$$(V_0 + \widetilde{V}_0, V) \in E(h_0,h,P_\tau^\sigma),$$

where the operators σV_0, $\sigma\widetilde{V}_0$ are nonnegative.

Moreover, let the constant $r_0 > 0$, the operator $F \in K(C',L_\infty)$, and the function $g_0 \in L(I,R_+)$, be such that the conditions

$$g_0(t) \le \sigma\big(F(x)(t) - L(x,x)(t)\big)\operatorname{sgn} x(t) \le \big|\widetilde{V}_0(x)(t)x(t)\big| + \eta\big(t,\|x\|_{C'}\big) \ \ \textit{for} \ \ t \in I, \ \ \|x\|_{C'} \ge r_0,$$

and

$$|c_2| \le \int_0^\omega g_0(s)\,ds - \bigg|\int_0^\omega q(s)\,ds\bigg|$$

hold, where the function $\eta : I \times R_+ \to R_+$ is summable in the first argument, nondecreasing in the second one, and admits the condition

$$\lim_{\rho\to+\infty} \frac{1}{\rho} \int_0^\omega \eta(s,\rho)\,ds = 0.$$

Then problem (0.2), (0.3) *has at least one solution.*

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Conditions for Unique Solvability of the Two-Point Neumann Problem

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On a finite interval $[a, b]$, we consider the differential equation

$$u'' = f(t, u) \tag{1}$$

with the Neumann two-point boundary conditions

$$u'(a) = c_1, \quad u'(b) = c_2, \tag{2}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions, while c_1 and c_2 are real constants.

A number of interesting and unimprovable in a certain sense results concerning the existence and uniqueness of a solution of problem (1), (2) are known (see, e.g., [1–8] and the references therein). Jointly with I. Kiguradze [9] we have proved a general theorem on the existence and uniqueness of a solution of that problem which is a nonlinear analogue of the first Fredholm theorem. Below we give this theorem and its corollaries containing unimprovable sufficient conditions, different from the above mentioned results, for the unique solvability of problem (1), (2).

We use the following notation.

$\mathbb{R}$ is the set of real numbers; $\mathbb{R}_- =]-\infty, 0]$;

$$[x]_- = \frac{|x| - x}{2};$$

$L([a, b])$ is the space of Lebesgue integrable on $[a, b]$ real functions.

Definition 1. Let $p_i \in L([a, b])$ $(i = 1, 2)$ and

$$p_1(t) \le p_2(t) \ \text{ for almost all } \ t \in [a, b]. \tag{3}$$

We say that the vector function (p_1, p_2) belongs to the set $\boldsymbol{\mathcal{N}\mathrm{eum}([a, b])}$ if for any measurable function $p : [a, b] \to \mathbb{R}$, satisfying the inequality

$$p_1(t) \le p(t) \le p_2(t) \ \text{ for almost all } \ t \in [a, b],$$

the homogeneous Neumann problem

$$u'' = p(t)u, \tag{4}$$

$$u'(a) = 0, \quad u'(b) = 0 \tag{5}$$

has only the trivial solution.

Theorem 1. *Let on the set $[a, b] \times \mathbb{R}$ the inequality*

$$p_1(t)|x - y| \le (f(t, x) - f(t, y)) \operatorname{sgn}(x - y) \le p_2(t)|x - y| \tag{6}$$

be satisfed, where $(p_1, p_2) \in \boldsymbol{\mathcal{N}\mathrm{eum}([a, b])}$. Then problem (1), (2) has one and only one solution.

Corollary 1. *Let on the set* $[a,b]\times\mathbb{R}$ *condition* (6) *hold, where* $p_i\in L([a,b])$ $(i=1,2)$ *are the functions satisfying inequalities* (3)*. Let, moreover,*

$$\int\limits_a^b p_2(t)\,dt\le 0,\quad \operatorname{mes}\{[t\in[a,b]:\ p_2(t)<0\}>0, \tag{7}$$

and there exist a number $\lambda\ge 1$ *such that*

$$\int\limits_a^b [p_1(t)]_-^\lambda\,dt\le \frac{4(b-a)}{\pi^2}\Big(\frac{\pi}{b-a}\Big)^{2\lambda}. \tag{8}$$

Then problem (1), (2) *has one and only one solution.*

Corollary 2. *Let on the set* $[a,b]\times\mathbb{R}$ *inequality* (6) *hold, where* $p_1:[a,b]\to\mathbb{R}_-$ *and* $p_2:[a,b]\to\mathbb{R}$ *are integrable functions satisfying inequalities* (3) *and* (7)*. Let, moreover, there exist* $t_0\in\,]a,b[$ *such that the function* p_2 *is non-increasing and non-decreasing in the intervals* $]a,t_0[$ *and* $]t_0,b[$*, respectively, and*

$$\int\limits_a^{t_0}\sqrt{|p_1(t)|}\,dt\le\frac{\pi}{2}\,,\quad \int\limits_{t_0}^b\sqrt{|p_1(t)|}\,dt\le\frac{\pi}{2}\,,\quad \int\limits_a^b\sqrt{|p_1(t)|}\,dt<\pi. \tag{9}$$

Then problem (1), (2) *has one and only one solution.*

The following two corollaries concern the linear differential equation

$$u''=p(t)u+q(t), \tag{10}$$

where p and $q\in L([a,b])$.

Corollary 3. *Let*

$$\int\limits_a^b p(t)\,dt\le 0,\quad \operatorname{mes}\{t\in[a,b]:\ p(t)<0\}>0, \tag{11}$$

and let there exist a number $\lambda\ge 1$ *such that*

$$\int\limits_a^b [p(t)]_-^\lambda\,dt\le \frac{4(b-a)}{\pi^2}\Big(\frac{\pi}{b-a}\Big)^{2\lambda}. \tag{12}$$

Then problem (10), (2) *has one and only one solution.*

Corollary 4. *Let there exist a number* $t_0\in\,]a,b[$ *such that the function* p *along with* (11) *satisfies the conditions*

$$p_0(t)=\operatorname{ess\,sup}\big\{[p(s)]_-:\ a<s<t\big\}<+\infty\quad\text{for } a<t<t_0, \tag{13}$$

$$p_0(t)=\operatorname{ess\,sup}\big\{[p(s)]_-:\ t<s<b\big\}<+\infty\quad\text{for } t_0<t<b, \tag{14}$$

$$\int\limits_a^{t_0}\sqrt{p_0(t)}\,dt\le\frac{\pi}{2}\,,\quad \int\limits_{t_0}^b\sqrt{p_0(t)}\,dt\le\frac{\pi}{2}\,,\quad \int\limits_a^b\sqrt{p_0(t)}\,dt<\pi. \tag{15}$$

Then problem (10), (2) *has one and only one solution.*

Remark 1. In the case, where instead of (11) the more hard condition

$$p(t) \le 0 \ \text{ for } \ a < t < b, \quad \operatorname{mes}\{t \in [a,b] : \ p(t) < 0\} > 0 \tag{16}$$

is satisfied, the results analogous to Corollary 3 previously were obtained in [4,5,8]. More precisely, in [8] it is required that along with (16) the inequalities

$$\int\limits_a^b |p(t)|\,dt \le \frac{4}{b-a}\,, \quad \operatorname*{ess\,sup}\{|p(t)| : \ a \le t \le b\} < +\infty$$

be satisfied (see [8, Theorem 3]), while in [4] and [5] it is assumed, respectively, that

$$\int\limits_a^b |p(t)|\,dt \le \frac{4}{b-a}$$

(see [4, Corollary 1.2]), and

$$\int\limits_a^b |p(t)|^\lambda\,dt \le \frac{4(b-a)}{\pi^2}\Big(\frac{\pi}{b-a}\Big)^{2\lambda},$$

where $\lambda \equiv const \ge 1$ (see [5, Corollary 1.3]).

Example 1. Suppose

$$p(t) \equiv -\Big(\frac{\pi}{b-a}\Big)^2,$$

ε is arbitrarily small positive number, while λ is so large that

$$\Big(1+\frac{\varepsilon}{\pi}\Big)^\lambda > \frac{\pi}{2}\,.$$

Then instead of (12) the inequality

$$\int\limits_a^b [p(t)]_-^\lambda\,dt < \frac{4(b-a)}{\pi^2}\Big(\frac{\pi+\varepsilon}{b-a}\Big)^{2\lambda} \tag{17}$$

is satisfied. On the other hand, the homogeneous problem (4), (5) has a nontrivial solution $u_0(t) = \cos\frac{\pi(t-a)}{b-a}$, and the nonhomogeneous problem (10), (2) has no solution if only

$$c_1 + c_2 + \int\limits_a^b u_0(t)q(t)\,dt \ne 0.$$

Consequently, condition (12) in Corollary 3 is unimprovable and it cannot be replaced by condition (17).

The above example shows also that condition (8) in Corollary 1 is unimprovable in the sense that it cannot be replaced by the condition

$$\int\limits_a^b [p_1(t)]_-^\lambda\,dt < \frac{4(b-a)}{\pi^2}\Big(\frac{\pi+\varepsilon}{b-a}\Big)^{2\lambda},$$

where ε is a positive constant independent of λ.

Note that condition (8) in Corollary 1 is unimprovable also in the case where $\lambda = 1$, and it cannot be replaced by the condition

$$\int_a^b [p_1(t)]_- \, dt < \frac{4+\varepsilon}{b-a}$$

no matter how small $\varepsilon > 0$ would be (see [4, p. 357, Remark 1.1]).

Example 2. Suppose $t_0 \in]a,b[$ and

$$p(t) = \begin{cases} -\dfrac{\pi^2}{4(t_0-a)^2} & \text{for } \ a \le t \le t_0, \\ -\dfrac{\pi^2}{4(b-t_0)^2} & \text{for } \ t_0 < t \le b. \end{cases}$$

Then inequalities (13), (14) hold, and instead of (15) we have

$$\int_a^{t_0} \sqrt{p_0(t)} \, dt = \frac{\pi}{2}\,, \quad \int_{t_0}^b \sqrt{p_0(t)} \, dt = \frac{\pi}{2}\,.$$

On the other hand, the homogeneous problem (4), (5) has a nontrivial solution

$$u_0(t) = \begin{cases} (t_0-a)\cos\dfrac{\pi(t-a)}{2(t_0-a)} & \text{for } \ a \le t \le t_0, \\ (t_0-b)\cos\dfrac{\pi(b-t)}{2(b-t_0)} & \text{for } \ t_0 < t \le b, \end{cases}$$

while the nonhomogeneous problem (10), (2) has no solution if only

$$(t_0-a)c_1 + (b-t_0)c_2 + \int_a^b u_0(t)q(t) \, dt \neq 0.$$

Consequently, condition (15) in Corollary 4 is unimprovable in the sense that it cannot be replaced by the condition

$$\int_a^{t_0} \sqrt{p_0(t)} \, dt \le \frac{\pi}{2}\,, \quad \int_{t_0}^b \sqrt{p_0(t)} \, dt \le \frac{\pi}{2}\,.$$

From the above said it is also clear that condition (9) in Corollary 2 is unimprovable and it cannot be replaced by the condition

$$\int_a^{t_0} \sqrt{|p_1(t)|} \, dt \le \frac{\pi}{2}\,, \quad \int_{t_0}^b \sqrt{|p_1(t)|} \, dt \le \frac{\pi}{2}\,.$$

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Stability Properties of Uniform Attractors for Parabolic Impulsive Systems

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An important problem in the theory of impulsive systems of differential equations [13] is a qualitative study of discontinuous (or impulsive) dynamical systems. In the case of an infinite-dimensional phase space, one of the most effective tools for studying the qualitative behavior of solutions is the theory of global attractors [4, 7]. The transfer of basic concepts and results of the theory of attractors to impulsive dynamical systems has a fundamental problem – the absence of continuous dependence of solutions on the initial data. Using the notion of a uniform attractor [4, 12], in [8], we were able to prove the existence of a minimal compact uniformly attracting set for a class of weakly nonlinear impulsive parabolic equations. Later in the works [5, 6, 9] this approach was extended to other classes of impulsive systems. It turned out that in the case when the trajectories of an impulsive dynamical system reach the impulsive set infinitely many times, the uniform attractor can have a non-empty intersection with the impulsive set and be neither invariant nor stable with respect to the impulsive semi-flow. The invariance of the non-impulsive part of a uniform attractor for different classes of impulsive systems was proved in [3, 5]. In [10], for the first time conditions for the impulsive semi-flow, which guarantee the stability of the non-impulsive part of the uniform attractors, were proposed. In this paper, we refine these conditions and apply them to study the stability of a uniform attractor of a weakly nonlinear two-dimensional impulsive-perturbed parabolic system.

Let us consider the impulsive dynamical system (further the impulsive DS) $G = G(V, M, I)$, which is defined on the normalized space X. It means that we consider the mapping $G : R_+ \times X \to X$, which is constructed from the continuous semigroup $V : R_+ \times X \to X$, the impulsive set $M \subset X$ and the impulsive map $I : M \to X$ using the following rule [11]: if for $x \in X$ for every $t > 0$ $V(t, x) \notin M$, then $G(t, x) = V(t, x)$; otherwise

$$G(t,x) = \begin{cases} V(t-t_n), & t \in [t_n, t_{n+1}), \\ x^+_{n+1}, & t = t_{n+1}, \end{cases} \tag{1}$$

where $t_0 = 0$, $t_{n+1} = \sum\limits_{k=0}^{n} s_k$, $x^+_{n+1} = IV(s_n, x^+_n)$, $x^+_0 = x$, s_n are moments of impulsive perturbation, characterized by a condition $V(s_n, x^+_n) \in M$. Under conditions

$$\begin{gathered} M \ \text{ is closed}, \ \ M \cap IM = \varnothing, \\ \forall\, x \in M \ \ \exists\, \tau = \tau(x) > 0, \ \ \forall\, t \in (0, \tau) \ \ V(t, x) \notin M, \\ \forall\, x \in X \ \ t \to G(t, x) \ \text{ is defined on } \ [0, +\infty) \end{gathered} \tag{2}$$

formula (1) defines a semigroup $G : R_+ \times X \to X$ [2, 8].

Remark 1. From the condition (2) and the continuity of V follows [2, 5] that for every $x \in X$ either there is moments of time $s := s(x) > 0$ such that $\forall\, t \in (0, s)$ $V(t, x) \notin M$, $V(s, x) \in M$, or $\forall\, t > 0$ $V(t, x) \cap M = \varnothing$ (and in this case we set $s(x) = \infty$).

Definition 1 ([8])**.** A compact set $\Theta \subset X$ is called a uniform attractor of the impulsive DS G, if

1) Θ is uniformly attracting set, i.e.,

$$\forall\, B \in \beta(X) \quad \operatorname{dist}(G(t,B),\Theta) \longrightarrow 0, \ \ t \to \infty;$$

2) Θ is minimal closed set which satisfies 1).

Remark 2. A uniform attractor can be not invariant with respect to G. In that case the equality

$$\forall\, t \geq 0 \ \ \Theta = G(t,\Theta)$$

will not be fulfilled [8].

Theorem 1 ([5])**.** *Let impulsive* DS G *be dissipative, that is*

$$\exists\, B_0 \in \beta(X) \ \ \forall\, B \in \beta(X), \ \ \exists\, T = T(B) \ \ \forall\, t \geq T \ \ G(t,B) \subset B_0. \tag{3}$$

Then G has a uniform attractor Θ if and only if G is asymptotically compact, i.e. $\forall\,\{x_n\} \in \beta(X)$ $\forall\,\{t_n \nearrow \infty\}$ *the sequence* $\{G(t_n,x_n)\}$ *is precompact. Herewith,*

$$\Theta = \omega(B_0) := \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} G(t,B_0)}.$$

Definition 2 ([1])**.** The set $A \subset X$ is called a stable with respect to the semi-flow G, if

$$A = D^{+}(A) := \bigcup_{x\in A} \big\{y \mid \ y = \lim G(t_n,x_n), \ x_n \to x, \ t_n \geq 0\big\}. \tag{4}$$

In [10] it was shown that the uniform attractor of an impulsive DS may not satisfy the property (4), however, using additional assumptions about the nature of the behavior of the trajectories in the neighborhood of the impulsive set, we manage to obtain the following result which clarifies the statement of Theorem 1, 2 from [10].

Theorem 2. *Let impulsive* DS $G = (V,M,I)$ *satisfy conditions* (2), (3) *and have the uniform attractor* Θ*. Let impulsive mapping* $I : M \to X$ *and semi-group* $V : R_{+} \times X \to X$ *be continuous and in addition, the conditions met:*

- *for an arbitrary sequence* $x_n \to x \in \Theta \setminus M$

$$\begin{cases} s(x) = \infty, & \textit{if } \ s(x_n) = \infty \ \textit{ for infinitely many } n, \\ s(x_n) \to s(x), & \textit{otherwise}; \end{cases}$$

- *for an arbitrary sequence* $x_n \to x \in \Theta \cap M$

$$\textit{either } \ s(x_n) = \infty \ \textit{ for infinitely many } \ n, \quad \textit{or } \ s(x_n) \to 0.$$

Then the following equality is fulfilled:

$$\Theta = \overline{\Theta \setminus M}. \tag{5}$$

Moreover, Θ *is invariant in the sense that*

$$\forall\, t \geq 0 \ \ G(t,\Theta \setminus M) = \Theta \setminus M, \tag{6}$$

and stable in the sense that

$$D^{+}(\Theta \setminus M) \subset \overline{\Theta \setminus M}. \tag{7}$$

Let $\Omega \subset R^n$, $n \ge 1$ is a bounded domain. Using the unknown functions $u(t,x)$, $v(t,x)$ in $(0,+\infty) \times \Omega$ we consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + \varepsilon f_1(u,v), \\ \dfrac{\partial v}{\partial t} = a\Delta v + 2b\Delta u + \varepsilon f_2(u,v), \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \tag{8}$$

where $\varepsilon > 0$ is a small parameter,

$$a > 0, \quad |b| < a. \tag{9}$$

Nonlinear perturbation $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C^1(R^2)$ satisfies the conditions:

$$\exists\, C > 0 \ \ \forall\, u, v \in R \quad |f_1(u,v)| + |f_2(u,v)| \le C, \ \ Df(u,v) \ge -C, \tag{10}$$

which guarantee the single-valued global solvability of the problem (8) in a phase space $X = L^2(\Omega) \times L^2(\Omega)$ with the norm $\|z\|_X = \sqrt{\|u\|^2 + \|v\|^2}$, where here and further $\|\cdot\|$ and $(\cdot,\cdot)$ are the norm and the scalar product in $L^2(\Omega)$.

Let $\{\lambda_i\}_{i=1}^{\infty} \subset (0,+\infty)$, $\{\psi_i\}_{i=1}^{\infty} \subset H_0^1(\Omega)$ be solutions of the spectral problem $\Delta\psi = -\lambda\psi$, $\psi \in H_0^1(\Omega)$.

For fixed $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\mu > 0$ the following impulsive problem is considered on the solutions of (8):

when the phase point $z(t)$ meets the impulsive set

$$M = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X \ \Big| \ |(u,\psi_1)| \le \gamma, \ \alpha(u,\psi_1) + \beta(v,\psi_1) = 1 \right\}, \tag{11}$$

it is instantly translated by the impulsive map $I : M \to M'$ to the new position $Iz \in M'$, where

$$M' = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X \ \Big| \ |(u,\psi_1)| \le \gamma, \ \alpha(u,\psi_1) + \beta(v,\psi_1) = 1 + \mu \right\}. \tag{12}$$

We will consider the following class of impulsive mappings:

$$\text{for } \ z = \sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M, \quad I(z) = I_1 \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \psi_1 + \sum_{i=2}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M',$$

where $I_1 : R^2 \to R^2$ is specified continuous mapping.

In [9], it was proved that under the additional condition

$$2\beta\gamma \le 1$$

the problem (8)–(12) for sufficiently small ε generates an impulsive DS G_ε which has a uniform attractor Θ_ε.

The main result of this paper is the following theorem.

Theorem 3. *Let $f_1 \equiv 0$. Then for sufficiently small $\varepsilon > 0$ the uniform attractor Θ_ε of the impulsive* DS G_ε*, generated by the problem* (8)–(12)*, is invariant and stable in the sense* (5)–(7)*.*

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The Generalized Jacobi–Poisson Theorem of Building First Integrals for Hamiltonian Systems

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1 Introduction

Consider the canonical Hamiltonian system with n degrees of freedom

$$\frac{dq_i}{dt} = \partial_{p_i} H(t,q,p), \quad \frac{dp_i}{dt} = -\,\partial_{q_i} H(t,q,p), \ \ i = 1,\dots,n, \tag{1.1}$$

where $q = (q_1,\dots,q_n) \in \mathbb{R}^n$ and $p = (p_1,\dots,p_n) \in \mathbb{R}^n$ are the generalized coordinates and momenta, respectively, $t \in \mathbb{R}$, and the Hamiltonian $H : D \to \mathbb{R}$ is a twice continuously differentiable function on the domain $D = T \times G$, $T \subset \mathbb{R}$, $G \subset \mathbb{R}^{2n}$.

To avoid ambiguity, we give the following notation and definitions.

The *Poisson bracket* of functions $u, v \in C^1(D)$ is the function

$$[u,v] : (t,q,p) \longrightarrow \sum_{i=1}^{n} \Big(\partial_{q_i} u(t,q,p)\,\partial_{p_i} v(t,q,p) - \partial_{p_i} u(t,q,p)\,\partial_{q_i} v(t,q,p) \Big) \ \text{ for all } \ (t,q,p) \in D.$$

A function $\mathrm{g} \in C^1(D')$ is called a *first integral on the domain* $D' \subset D$ of the Hamiltonian system (1.1) if $\mathfrak{G}\,\mathrm{g}(t,q,p) = 0$ for all $(t,q,p) \in D'$, where the linear differential operator

$$\mathfrak{G}(t,q,p) = \partial_t + \sum_{i=1}^{n} \big(\partial_{p_i} H(t,q,p)\,\partial_{q_i} - \partial_{q_i} H(t,q,p)\,\partial_{p_i} \big) \ \text{ for all } \ (t,q,p) \in D.$$

A smooth manifold $\mathrm{g}(t,q,p) = 0$ is said to be an *integral manifold* of the Hamiltonian system (1.1) if the derivative of the function $\mathrm{g} \in C^1(D')$ by virtue of the Hamiltonian system (1.1) is the identically zero on the manifold $\mathrm{g}(t,q,p) = 0$, i.e.,

$$\mathfrak{C}\,\mathrm{g}(t,q,p) = \Phi(t,q,p), \quad \Phi(t,q,p)\big|_{\mathrm{g}(t,q,p)=0} = 0 \ \text{ for all } \ (t,q,p) \in D'.$$

By $\mathrm{I}_{D'}$ $(\mathrm{M}_{D'})$ denote the set of all first integrals (integral manifolds) on the domain D' of the Hamiltonian system (1.1). The phrase "the function g is an integral manifold with function Φ on the domain D' of the Hamiltonian system (1.1)" is denoted by $(\mathrm{g}, \Phi) \in \mathrm{M}_{D'}$. For the current state of the theory of integrability see the monographs [2, 4, 5, 7–9] and the references therein.

Among the general methods of building first integrals of the Hamiltonian system (1.1), the Jacobi–Poisson method is of particular importance. It gives the possibility to find the additional (third) first integral of the Hamiltonian system (1.1) by two known first integrals of the Hamiltonian system (1.1). And thus, in certain cases, to build an integral basis of the Hamiltonian system (1.1). Due to this property, the Jacobi-Poisson method is included in almost all monographs and textbooks on analytical mechanics (see, for example, [6, pp. 298–306], [1, p. 216], [3, pp. 85–86]) and formulated as the following statement.

Theorem 1.1 (the Jacobi–Poisson theorem)**.** *Suppose twice continuously differentiable functions* $\mathrm{g}_1 : D' \to \mathbb{R}$ *and* $\mathrm{g}_2 : D' \to \mathbb{R}$ *are first integrals on the domain* D' *of the Hamiltonian system* (1.1)*. Then the Poisson bracket*

$$\mathrm{g}_{12} : (t,q,p) \longrightarrow \big[\,\mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p)\,\big] \ \textit{ for all } \ (t,q,p) \in D', \ \ D' \subset D, \tag{1.2}$$

of the functions g_1 *and* g_2 *is also a first integral of the Hamiltonian system* (1.1)*.*

In his *Lectures on Dynamics* [7, pp. 298–306], C. G. J. Jacobi referred to Poisson's theorem as "one of the most remarkable theorems of the whole of integral calculus. In the particular case when $H = T - U$, it is the fundamental theorem of analytical mechanics. …After I discovered this theorem I communicated it to the Academies of Berlin and Paris as an entirely new discovery. But I noticed soon after that this theorem had already been discovered and forgotton for 30 years, because one did not appreciate its real meaning, but had only used it as a lemma in a entirely different problem".

Of course, the Jacobi–Poisson theorem does not always supply further first integrals. In some cases the result is trivial, the Poisson bracket being a constant. In other cases the first integral obtained is simply a function of the original integrals. If neither of these two possibilities occurs, however, then the Poisson bracket is a further first integral of the Hamiltonian system (1.1).

The aim of this paper is to develop the Jacobi–Poisson method for integral manifolds of the Hamiltonian system (1.1).

2 Main results

Theorem 2.1. *Suppose* $(\mathrm{g}_k, \Phi_k) \in \mathrm{M}_{D'}$ *and* $\mathrm{g}_k \in C^2(D')$, $k = 1, 2$*. Then the Poisson bracket* $[\mathrm{g}_1, \mathrm{g}_2] \in \mathrm{I}_{D'}$ *if and only if the following identity holds*

$$\big[\,\mathrm{g}_1(t,q,p), \Phi_2(t,q,p)\,\big] = \big[\,\mathrm{g}_2(t,q,p), \Phi_1(t,q,p)\,\big] \ \textit{ for all } \ (t,q,p) \in D'. \tag{2.1}$$

Proof. Since $(\mathrm{g}_k, \Phi_k) \in \mathrm{M}_{D'}$, $k = 1, 2$, we have

$$\mathfrak{G}\,\mathrm{g}_k(t,q,p) = \Phi_k(t,q,p) \ \text{ for all } \ (t,q,p) \in D', \ \ k = 1, 2.$$

From these identities it follows that

$$\partial_t\, \mathrm{g}_k(t,q,p) = \Phi_k(t,q,p) - \big[\,\mathrm{g}_k(t,q,p), H(t,q,p)\,\big] \ \text{ for all } \ (t,q,p) \in D', \ \ k = 1, 2.$$

Using these identities and the properties of Poisson brackets (time derivative, bilinearity, anticommutativity, and Jacobi identity), we obtain the derivative of the function (1.2) by virtue of the Hamiltonian system (1.1)

$$\begin{aligned}
\mathfrak{G}\big[\,\mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p)\,\big] &= \partial_t\big[\,\mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p)\,\big] + \Big[\big[\,\mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p)\,\big], H(t,q,p)\Big] \\
&= \big[\,\partial_t \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p)\,\big] + \big[\,\mathrm{g}_1(t,q,p), \partial_t \mathrm{g}_2(t,q,p)\,\big] + \Big[\big[\,\mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p)\,\big], H(t,q,p)\Big] \\
&= \Big[\Phi_1(t,q,p) - \big[\,\mathrm{g}_1(t,q,p), H(t,q,p)\,\big], \mathrm{g}_2(t,q,p)\Big] \\
&\quad + \Big[\mathrm{g}_1(t,q,p), \Phi_2(t,q,p) - \big[\,\mathrm{g}_2(t,q,p), H(t,q,p)\,\big]\Big] + \Big[\big[\,\mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p)\,\big], H(t,q,p)\Big] \\
&= \big[\,\Phi_1(t,q,p), \mathrm{g}_2(t,q,p)\,\big] - \Big[\big[\,\mathrm{g}_1(t,q,p), H(t,q,p)\,\big], \mathrm{g}_2(t,q,p)\Big] + \big[\,\mathrm{g}_1(t,q,p), \Phi_2(t,q,p)\,\big] \\
&\quad - \Big[\mathrm{g}_1(t,q,p), \big[\,\mathrm{g}_2(t,q,p), H(t,q,p)\,\big]\Big] + \Big[\big[\,\mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p)\,\big], H(t,q,p)\Big]
\end{aligned}$$

$$= \big[\,\mathrm{g}_1(t,q,p),\Phi_2(t,q,p)\,\big] - \big[\,\mathrm{g}_2(t,q,p),\Phi_1(t,q,p)\,\big] + \bigg(\Big[\big[\,H(t,q,p),\mathrm{g}_1(t,q,p)\,\big],\mathrm{g}_2(t,q,p)\Big]$$

$$+ \Big[\big[\,\mathrm{g}_2(t,q,p),H(t,q,p)\,\big],\mathrm{g}_1(t,q,p)\Big] + \Big[\big[\,\mathrm{g}_1(t,q,p),\mathrm{g}_2(t,q,p)\,\big],H(t,q,p)\Big]\bigg)$$

$$= \big[\,\mathrm{g}_1(t,q,p),\Phi_2(t,q,p)\,\big] - \big[\,\mathrm{g}_2(t,q,p),\Phi_1(t,q,p)\,\big] \ \text{ for all } \ (t,q,p)\in D'.$$

Therefore the Poisson bracket (1.2) of the integral manifolds g_1 and g_2 of system (1.1) is a first integral of the Hamiltonian system (1.1) if and only if the identity (2.1) is true. □

Remark. If the function

$$\Phi:(t,q,p)\longrightarrow \big[\,\mathrm{g}_1(t,q,p),\Phi_2(t,q,p)\,\big] - \big[\,\mathrm{g}_2(t,q,p),\Phi_1(t,q,p)\,\big] \ \text{ for all } \ (t,q,p)\in D'$$

such that the following identity holds

$$\Phi(t,q,p)\big|_{[\mathrm{g}_1(t,q,p),\mathrm{g}_2(t,q,p)]=0} = 0 \ \text{ for all } \ (t,q,p)\in D',$$

then the Poisson bracket (1.2) is an integral manifold of the Hamiltonian system (1.1).

As a consequence of Theorem 2.1, we obtain

Corollary 2.1. *Let* $\mathrm{g}_1\in \mathrm{I}_{D'}, (\mathrm{g}_2,\Phi_2)\in \mathrm{M}_{D'}$, $\mathrm{g}_k\in C^2(D')$, $k=1,2$. *Then the Poisson bracket* $[\mathrm{g}_1,\mathrm{g}_2]\in \mathrm{I}_{D'}$ *if and only if the functions* g_1 *and* Φ_2 *are in involution, i.e.,*

$$\big[\,\mathrm{g}_1(t,q,p),\Phi_2(t,q,p)\big] = 0 \ \textit{ for all } \ (t,q,p)\in D'.$$

If $\mathrm{g}_1,\mathrm{g}_2\in \mathrm{I}_{D'}$, then from Theorem 2.1 (or Corollary 2.1), we have the statement of the Jacobi–Poisson theorem (Theorem 1.1).

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On a Weighted Problem for Functional Differential Equations with Decreasing Non-Linearity

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We study the weighted boundary value problem

$$u'(t) = (gu)(t), \quad t \in (a, b], \tag{1}$$

$$\lim_{t \to a+} \varrho(t)u(t) \in \mathbb{R} \ \text{ exists}, \tag{2}$$

$$\int\limits_a^b \varrho(t)|u'(t)|\,dt < +\infty, \tag{3}$$

where $-\infty < a < b < \infty$, $\varrho : (a, b] \to (0, +\infty)$ is a non-decreasing absolutely continuous function such that $\lim\limits_{t \to a+} \varrho(t) = 0$. We assume that $g : C((a, b], \mathbb{R}) \to L_{1;\,\mathrm{loc}}((a, b], \mathbb{R})$ is non-increasing in the sense that $(gu_1)(t) \le (gu_0)(t)$ for a.e. $t \in (a, b]$ for arbitrary pairs of functions $\{u_0, u_1\} \subset C((a, b], \mathbb{R})$ such that $u_1(t) \ge u_0(t)$, $t \in (a, b]$. In particular, the case of neutral type equations is excluded from consideration.

By a solution of equation (1), we mean a locally absolutely continuous function $u : (a, b] \to \mathbb{R}$ satisfying (1) almost everywhere on the interval $(a, b]$. In particular, solutions of (1) may be unbounded in a neighbourhood of the point a.

The formulation has been motivated, in particular, by a relation to boundary value problems with conditions at infinity, integral boundary conditions on unbounded intervals [1, 3], and Kneser type solutions with possible blow-up [2, 4].

The following notation is used.

$C((a, b], \mathbb{R})$ is the set of continuous functions $u : (a, b] \to \mathbb{R}$.

$L_1([a, b], \mathbb{R})$ is the set of Lebesgue integrable functions $u : [a, b] \to \mathbb{R}$.

$L_{1;\,\mathrm{loc}}((a, b], \mathbb{R})$ is the set of functions $u : (a, b] \to \mathbb{R}$ such that $u|_{[a_0,b]} \in L_1([a_0, b], \mathbb{R})$ for any $a_0 \in (a, b)$.

$\widetilde{C}([a, b], \mathbb{R})$ is the set of absolutely continuous functions $u : [a, b] \to \mathbb{R}$.

$\widetilde{C}_{\mathrm{loc}}((a, b], \mathbb{R})$ is the set of all the locally absolutely continuous functions $u : (a, b] \to \mathbb{R}$ (i.e., $u|_{[a_0,b]} \in \widetilde{C}([a_0, b], \mathbb{R})$ for any $a_0 \in (a, b)$).

$\widetilde{C}_{\mathrm{loc};\,\varrho}((a, b], \mathbb{R})$ is the set of all $u \in \widetilde{C}_{\mathrm{loc}}((a, b], \mathbb{R})$ with $\varrho u' \in L_1((a, b], \mathbb{R})$ such that the limit $\lim\limits_{t \to a+} \varrho(t)u(t)$ exists and is finite.

Let ψ_0, ψ_1 be functions from $\widetilde{C}_{\mathrm{loc};\,\varrho}((a, b], \mathbb{R})$ such that

$$(-1)^i(\psi_1^{(i)}(t) - \psi_0^{(i)}(t)) \ge 0, \quad t \in (a, b], \quad i = 0, 1, \tag{4}$$

and

$$l_{\psi_0,\psi_1} := \inf\big\{\psi_1(t) - \psi_0(t) : \ t \in (a,b]\big\}. \tag{5}$$

The value l_{ψ_0,ψ_1} is positive if the graphs of ψ_0 and ψ_1 do not touch each other. For any pair ψ_0, ψ_1 with the above properties, the set of functions u such that

$$\psi_0(t) + (1-\theta) l_{\psi_0,\psi_1} \le u(t) \le \psi_1(t) - \theta\, l_{\psi_0,\psi_1}, \ \ t \in (a,b], \tag{6}$$

$$\psi_1'(t) \le u'(t) \le \psi_0'(t), \ \ t \in (a,b], \tag{7}$$

is non-empty for any $\theta \in [0,1]$. Introduce the set $S_\theta(\psi_0,\psi_1)$ by putting

$$S_\theta(\psi_0,\psi_1) := \big\{u \in \widetilde{C}_{\mathrm{loc};\,\varrho}((a,b],\mathbb{R}) : \ (6) \text{ and } (7) \text{ hold}\big\} \tag{8}$$

for $\theta \in [0,1]$.

For any $\theta \in [0,1]$, the set $S_\theta(\psi_0,\psi_1)$ describes the area obtained by shifting the graphs of ψ_0 and ψ_1, respectively, upwards and downwards, in the ratio $1-\theta : \theta$, until they touch each other. Clearly, this happens at the points of the set

$$\big\{t \in (a,b] : \ \psi_1(t) - \psi_0(t) = l_{\psi_0,\psi_1}\big\}. \tag{9}$$

The typical situation is that where $(-1)^i\psi_i$, $i = 0,1$, are non-decreasing and, hence, set (9) is a singleton consisting of the point b.

Theorem. *Let the mapping $g : C((a,b],\mathbb{R}) \to L_{1;\,\mathrm{loc}}((a,b],\mathbb{R})$ in* (1) *be non-increasing and, moreover,*

$$\varrho\, g\Big(\frac{\lambda}{\varrho}\Big) \in L_1((a,b],\mathbb{R}) \tag{10}$$

for any $\lambda \in \mathbb{R}$. Furthermore, let there exist certain functions ψ_0 and ψ_1 in $\widetilde{C}_{\mathrm{loc};\,\varrho}((a,b],\mathbb{R})$ with properties (4) *such that*

$$(-1)^k\big(\psi_k'(t) - (g\psi_k)(t)\big) \ge 0, \ \ t \in (a,b], \ \ k = 0,1. \tag{11}$$

Then for any $\theta \in [0,1]$ equation (1) *has a solution $u \in \widetilde{C}_{\mathrm{loc};\,\varrho}((a,b],\mathbb{R})$ such that $u \in S_\theta(\psi_0,\psi_1)$.*

Under the conditions assumed, one can guarantee the existence of solutions in the corresponding weighted space and specify certain bounds for u and u'. These bounds allow us to select solutions with different growth rates while we are still working in the same weighted space. Indeed, consider, e. g., the simple equation

$$u'(t) = \frac{\phi(u(1))}{t} - \frac{\psi(u(1))}{t^2}\,, \ \ t \in (0,1], \tag{12}$$

where $\phi(s) = 2\pi^{-1}\operatorname{arccot} s - 1/2$ and $\psi(s) = 2\pi^{-1}\arctan s + 1/2$ for all $s \in (-\infty,\infty)$. It is easy to see that any u satisfying (12) has the form

$$u_\lambda(t) = \lambda + \phi(\lambda)\ln t + \Big(\frac{1}{t} - 1\Big)\,\psi(\lambda), \ \ t \in (0,1], \tag{13}$$

where $\lambda \in \mathbb{R}$, and since $|\phi(\lambda)| + |\psi(\lambda)| > 0$, it follows that $u_\lambda(t)$ is unbounded as $t \to 0+$ for any λ. If $\lambda \ne -1$, then $\psi(\lambda) \ne 0$ and the growth of $|u_\lambda(t)|$ as $t \to 0+$ is of order $1/t$, whereas $u_{-1}(t) = -1 + \ln t$ has only logarithmic growth. Note that the corresponding operator g for (12) is non-increasing.

For equation (12), conditions (4), (11) are satisfied, in particular, with

$$\psi_0(t) = 0, \quad \psi_1(t) = \frac{1}{t} - 1,$$

and, hence, the theorem claims that (12) has solutions u with the properties $0 \le u(t) \le -1 + 1/t$, $-1/t^2 \le u'(t) \le 0$, $u(1) = 0$, which indeed hold, e.g., for $u_0(t) = (\ln t + t^{-1} - 1)/2$ (see (13)). On the other hand, by choosing

$$\psi_0(t) = -1 + \mu \ln t, \quad \psi_1(t) = -1$$

with $\mu > 1$, we get the bounds $-1 + \mu \ln t \le u(t) \le -1$, $0 \le u'(t) \le \mu t^{-1}$, $u(1) = -1$ that fit only the solution $u_{-1}(t) = -1 + \ln t$ and do not cover u_λ with $\lambda \ne -1$. Note that (10) is satisfied in this case for $\rho(t) = t^\alpha$ with $\alpha > 1$.

If g is a linear operator of the form

$$(gu)(t) = -p(t)u(\tau(t)) + q(t), \ \ t \in (a, b],$$

where p and q are locally integrable, $p \ge 0$, and $\tau : (a, b] \to (a, b]$ is a measurable function, condition (10) reduces to the relations

$$\int\limits_a^b p(t)\,\frac{\varrho(t)}{\varrho(\tau(t))}\,dt < \infty, \quad \int\limits_a^b \varrho(t)|q(t)|\,dt < \infty, \tag{14}$$

which determine the corresponding class of equations for which the theorem can be applied. As an example, consider the linear equation with advanced argument

$$u'(t) = -\frac{u(t^\gamma)}{t} + q(t), \ \ t \in (0, 1], \tag{15}$$

where q is locally integrable and $\gamma \in (0, 1)$. The function $p(t) = 1/t$ satisfies (14) with $\varrho(t) = t^\alpha$, $t \in (0, 1]$, $\alpha > 1$. Then, for arbitrary $\mu > 0$, $\theta \in [0, 1]$, and q satisfying the estimate

$$|q(t)| \le \mu h(t), \ \ t \in (0, 1],$$

where $h(t) = t^{-2} - t^{-\gamma-1}$, $t \in (0, 1]$, the corresponding problem (15), (2), (3) has a solution u with the terminal value $u(1) = (1 - 2\theta)\mu$ such that

$$-\frac{\mu}{t} + 2(1-\theta)\mu \le u(t) \le \frac{\mu}{t} - 2\theta\mu, \quad -\frac{\mu}{t^2} \le u'(t) \le \frac{\mu}{t^2}\,,$$

respectively, for all and almost all $t \in (0, 1]$. This follows from the theorem applied with $\psi_i(t) = (-\mu)^{i+1} t^{-1}$, $i = 0, 1$. Furthermore, if

$$-\mu h(t) \le \sigma q(t) \le \frac{\mu_0}{t}\,, \ \ t \in (0, 1],$$

for some $\sigma \in \{-1, 1\}$, $0 < \mu_0 \le \mu$, then for any $\theta \in [0, 1]$ there is a monotone solution with $u(1) = (\frac{1}{2}(\sigma+1) - \theta)\mu + (\frac{1}{2}(\sigma-1) + \theta)\mu_0$ such that

$$\mu \le \sigma u(t) + \Big(\sigma\theta + \frac{1-\sigma}{2}\Big)(\mu - \mu_0) \le \frac{\mu}{t}\,, \ \ -\frac{\mu}{t^2} \le \sigma u'(t) \le 0.$$

In particular, for $q = -\sigma\mu h$, the problem in question admits the solution $u(t) = \sigma\mu t^{-1}$.

The conditions assumed do not exclude the possibility of existence of non-trivial solutions of homogeneous problems. For example, by taking $\psi_i(t) = (-1)^{i+1} \exp(2(t^{-2} - 1))$, $i = 0, 1$, we find that the equation

$$u'(t) = -\frac{4}{t^3}\, u\big(\sqrt{t}\,\big) + q(t), \ \ t \in (0, 1],$$

has a solution in the set $\widetilde{C}_{\mathrm{loc};\,\varrho}((0,1],\mathbb{R})$ for $\varrho(t)=\exp(-\alpha t^{-2})$, $\alpha>2$, if

$$|q(t)|\le \frac{4}{e^2t^3}\,e^{\frac{2}{t^2}}\big(1-e^{\frac{2(t-1)}{t^2}}\big),\ \ t\in(0,1].$$

One can verify by direct substitution that $u(t)=\frac{\lambda}{t^4}$ is a solution of the corresponding homogeneous problem for any λ.

The theorem ensures the existence of solutions lying between ψ_0 and ψ_1 with terminal values filling the corresponding interval. This does not exclude the possibility of existence of solutions which escape from the regions in question. For example, consider the functional differential equation

$$u'(t)=\frac{1}{t^2}\big(1-\exp(t)-u(\exp(-t))\big),\ \ t\in(0,1]. \tag{16}$$

Defining g according to the right-hand side of (16) and choosing the weight ϱ in the form $\varrho(t)=t^{\alpha}$, $\alpha>1$, we find that equation (16) satisfies conditions (10).

It is easy to verify that problem (16), (2), (3) with this ϱ has a one-parametric family of solutions

$$u(t)=-\frac{1}{t}-\lambda\ln t. \tag{17}$$

For $\psi_0(t)=-t^{-1}+2\ln t$, $\psi_1(t)=-t^{-1}-2\ln t$, the application of the theorem would result in the existence of solutions u such that

$$2\ln t\le u(t)+\frac{1}{t}\le -2\ln t,\quad -\frac{2}{t}\le u'(t)-\frac{1}{t^2}\le\frac{2}{t}\,,\quad u(1)=-1, \tag{18}$$

and such solutions are indeed obtained from (17) for $|\lambda|\le 2$. However, if $|\lambda|>2$, then solution (17) has the same terminal value -1 but does not satisfy conditions (18) any more.

In the cases where $\psi_0=c_0$ or $\psi_1=c_1$, where $c_0\le\psi_1(b)$ and $c_1\ge\psi_0(b)$, the solutions dealt with in the theorem are obviously monotone, and their terminal values fill, respectively, the intervals $[c_0,\psi_1(b)]$, $[\psi_0(b),c_1]$. With non-constant bounding functions, the solution, generally speaking, need not be monotone.

Under the conditions assumed, the set of solutions of the weighted problem in question possesses the least and the greatest elements.

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Nondecreasing Solutions of Singular Differential Equations

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1 Introduction

We investigate solutions of the initial value problem (IVP)

$$\big(p(t)\phi(u'(t))\big)' + p(t)f(\phi(u(t))) = 0, \ \ t \in (0,\infty), \tag{1.1}$$

$$u(0) = u_0, \quad u'(0) = 0, \ \ u_0 \in [L_0, 0), \tag{1.2}$$

where

$$\phi \in C^1(\mathbb{R}), \ \ \phi'(x) > 0 \ \text{ for } \ x \in (\mathbb{R} \setminus \{0\}), \tag{1.3}$$

$$\phi(\mathbb{R}) = \mathbb{R}, \ \ \phi(0) = 0, \tag{1.4}$$

$$L_0 < 0 < L, \ \ f(\phi(L_0)) = f(0) = f(\phi(L)) = 0, \tag{1.5}$$

$$f \in \mathrm{Lip}[\phi(L_0), \phi(L)], \ \ xf(x) > 0 \ \text{ for } \ x \in \big((\phi(L_0), \phi(L)) \setminus \{0\}\big), \tag{1.6}$$

$$p \in C[0,\infty) \cap C^1(0,\infty), \ \ p'(t) > 0 \ \text{ for } \ t \in (0,\infty), \ \ p(0) = 0. \tag{1.7}$$

A function $u \in C^1[0,\infty)$ with $\phi(u') \in C^1(0,\infty)$ which satisfies equation (1.1) for every $t \in (0,\infty)$ is called a *solution* of equation (1.1). If moreover u satisfies the initial conditions (1.2), then u is called a *solution* of IVP (1.1), (1.2).

Equation (1.1) has the constant solutions $u(t) \equiv L$, $u(t) \equiv 0$ and $u(t) \equiv L_0$.

Consider a solution u of IVP (1.1), (1.2) with $u_0 \in [L_0, 0)$ and denote

$$u_{sup} = \sup\big\{u(t) : \ t \in [0,\infty)\big\}.$$

- If $u_{sup} < L$, then u is called a *damped solution* of IVP (1.1), (1.2).
- If $u_{sup} = L$ and u is nondecreasing (i.e. $\lim\limits_{t\to\infty} u(t) = L$), then u is called a *homoclinic solution* of IVP (1.1), (1.2).
- The homoclinic solution is called a *regular homoclinic solution*, if $u(t) < L$ for $t \in [0,\infty)$ and a *singular homoclinic solution*, if there exists $t_0 > 0$ such that $u(t) = L$ for $t \in [t_0, \infty)$.
- If $u_{sup} > L$, then u is called an *escape solution* of IVP (1.1), (1.2).

In particular, we find additional conditions for p, ϕ and f which guarantee for some $u_0 \in [L_0, 0)$ the existence of a nondecreasing solution of IVP (1.1), (1.2) converging to L for $t \to \infty$. Note that if we extend the function p in equation (1.1) from the half-line onto $\mathbb{R}$ as an even function and assume that ϕ is odd, then any solution u of IVP (1.1), (1.2) with $\lim\limits_{t\to\infty} u(t) = L$ fulfils $\lim\limits_{t\to-\infty} u(t) = L$, hence u is *a homoclinic solution*. This is a motivation for our above definition. Due to condition (1.7) the function $1/p(t)$ may not be integrable on $[0,1]$ and consequently equation (1.1) has a time singularity at $t = 0$. Problems of this type arise in hydrodynamics [4] or in the nonlinear field theory [3], where homoclinic solutions play an important role in the study of behaviour of corresponding differential models.

Our first attempts in this subject have been made for the equation without ϕ-Laplacian

$$((t)u'(t))' + q(t)f(u(t)) = 0, \ \ t \in (0,\infty),$$

with $p \equiv q$ in [6–8] and for $p \not\equiv q$ in [1,9].

2 Existence and asymptotic properties of solutions of IVP

Here we present an overview of results from [2] and [10] which we need to get a homoclinic solution of IVP (1.1), (1.2). Since values of any homoclinic solution belong to $[L_0, L]$, we can assume without loss of generality

$$f(x) = 0 \ \text{ for } \ x \le \phi(L_0), \ \ x \ge \phi(L). \tag{2.1}$$

Theorem 2.1 (Existence of solutions)**.** *Assume* (1.3)–(2.1)*. Then, for each starting value* $u_0 \in [L_0, 0)$*, there exists a solution of IVP* (1.1), (1.2)*.*

Theorem 2.2 (Damped solutions)**.** *Let* (1.3)–(2.1) *hold and let*

$$\exists\, \overline{B} \in (L_0, 0): \ F(\overline{B}) = F(L), \ \ \textit{where} \ \ F(x) = \int\limits_0^x f(\phi(s))\,\mathrm{d}s, \ \ x \in \mathbb{R}, \tag{2.2}$$

and

$$\lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{2.3}$$

Then every solution of IVP (1.1), (1.2) *with the starting value* $u_0 \in [\overline{B}, 0)$ *is damped.*

Assume in addition that

$$\lim_{x\to 0} |x|(\phi^{-1})'(x) < \infty, \tag{2.4}$$

and that u *is a damped solution of IVP* (1.1), (1.2) *with the starting value* $u_0 \in (L_0, 0)$*. Then* u *is a unique solution of this IVP.*

Theorem 2.3 (Escape solutions)**.** *Let* (1.3)–(2.3) *hold. Then there exist infinitely many escape solutions of IVP* (1.1), (1.2) *with starting values in* $[L_0, \overline{B})$*.*

Assume in addition that (2.4) *hold and that* u *is an escape solutions of IVP* (1.1), (1.2) *with the starting value* $u_0 \in (L_0, \overline{B})$*. Then* u *is a unique solution of this IVP.*

The next theorem describes asymptotic behaviour of damped, homoclinic and escape solutions starting at $u_0 \in (L_0, 0)$.

Theorem 2.4. *Let* (1.3)–(2.3) *hold and let* u *be a solution of IVP* (1.1), (1.2) *with the starting value* $u_0 \in (L_0, 0)$*. Then*

$$u(t) > L_0 \ \ \textit{and} \ \exists\, \widetilde{c} > 0 \ \ \textit{such that} \ \ |u'(t)| \le \widetilde{c} \ \ \textit{for} \ \ t \in (0,\infty). \tag{2.5}$$

The constant $\widetilde{c}$ *depends on* L_0*,* L_1*,* ϕ *and* f *and does not depend on* p *and* u*.*

1. *Assume that* $u_{sup} < L$*, i.e.* u *is a damped solution.*

   - *Let* $\theta > 0$ *be the first zero of* u*. Then there exists* $\theta < a < b$ *such that*

$$u(a) \in (0, L), \quad u'(t) > 0 \ \ \textit{on} \ (0, a), \quad u'(a) = 0, \quad u'(t) < 0 \ \ \textit{on} \ (a, b). \tag{2.6}$$

- *Let* $u < 0$ *on* $[0, \infty)$. *Then*

$$u'(t) > 0 \ \ \textit{for} \ \ t \in (0, \infty), \quad \lim_{t \to \infty} u(t) = 0, \quad \lim_{t \to \infty} u'(t) = 0. \tag{2.7}$$

2. *Assume that* $u_{sup} > L$, *i.e.* u *is an escape solution. Then*

$$u'(t) > 0 \ \ \textit{for} \ \ t \in (0, \infty). \tag{2.8}$$

3. *Assume that* $u_{sup} = L$. *Then there are two possibilities.*

- $u(t) < L$ *for* $t \in [0, \infty)$ *which yields*

$$u'(t) > 0 \ \ \textit{for} \ \ t \in (0, \infty), \quad \lim_{t \to \infty} u(t) = L, \quad \lim_{t \to \infty} u'(t) = 0, \tag{2.9}$$

and u *is a regular homoclinic solution.*

- *There exists* $t_0 > 0$ *such that* $u(t_0) = L$, $u'(t_0) = 0$ *which implies*

$$u'(t) > 0 \ \ \textit{for} \ \ t \in (0, t_0), \tag{2.10}$$

and there exists a singular homoclinic solution v, *where* $v = u$ *on* $[0, t_0]$ *and* $v = L$ *on* $[t_0, \infty)$.

Consider a solution $u \not\equiv L_0$ of IVP (1.1), (1.2) with $u_0 = L_0$. Since $L_0 < 0$, there exists $\varepsilon > 0$ such that $u(t) < 0$ for $t \in [0, \varepsilon]$, and by (2.1), $f(\phi(u(t))) \leq 0$ for $t \in [0, \varepsilon]$. Integrating (1.1) over $[0, t]$ we get

$$p(t)\phi(u'(t)) = -\int_0^t p(s) f(\phi(u(s)))\,\mathrm{d}s \geq 0, \quad t \in [0, \varepsilon].$$

Hence $u'(t) \geq 0$ and $u(t)$ is nondecreasing on $[0, \varepsilon]$. Consequently, since $u \not\equiv L_0$, there exists a maximal $a_0 \geq 0$ such that

$$u(t) = L_0 \ \text{ on } [0, a_0] \text{ and } u \text{ is increasing in a right neighbouhood of } a_0. \tag{2.11}$$

Therefore all assertions of Theorem 2.4 are valid also for $u_0 = L_0$ if we replace 0 by a_0.

3 Existence of homoclinic solutions

IVP (1.1), (1.2) can be transformed on the equivalent integral equation

$$u(t) = u_0 + \int_0^t \phi^{-1}\bigg(-\frac{1}{p(s)} \int_0^s p(\tau) f(\phi(u(\tau)))\,\mathrm{d}\tau \bigg)\,\mathrm{d}s, \quad t \in [0, \infty). \tag{3.1}$$

Assumption (1.3) implies that ϕ is locally Lipschitz continuous on $\mathbb{R}$, but if $\phi'(0) = 0$, then

$$\lim_{x \to 0} (\phi^{-1})'(x) = \infty,$$

and so ϕ^{-1} does not fulfil the Lipschitz condition on intervals containing 0. If values of u are between L_0 and L, we see that

$$\lim_{s \to 0+} \frac{1}{p(s)} \int_0^s p(\tau) f(\phi(u(\tau)))\,\mathrm{d}\tau = 0.$$

Therefore ϕ^{-1} in (3.1) is considered on an interval containing zero. Hence, in order to prove the uniqueness for IVP (1.1), (1.2) if $\phi'(0)=0$, we need to use some new condition for ϕ^{-1} instead of the Lipschitz one. For such condition see (2.4). Then we get the main result published in [5] and contained in the next theorem.

Theorem 3.1 (Homoclinic solutions)**.** *Let* (1.3)–(1.7) *and* (2.2)–(2.4) *hold. Further assume that*

$$\text{there exists a right neighbourhood of } \phi(L_0), \text{ where } f \text{ is decreasing.} \tag{3.2}$$

Then there exists $u_0^* \in [L_0, \overline{B})$ such that a solution u_h of IVP (1.1), (1.2) *with $u_0 = u_0^*$ is homoclinic.*

A typical model example of (1.1) is an equation with the α-Laplacian $\phi(x)=|x|^\alpha \operatorname{sgn} x$, $x\in\mathbb{R}$, where $\alpha\geq 1$. Then $\phi'(x)=\alpha|x|^{\alpha-1}$ and conditions (1.3) and (1.4) are fulfilled. If $\alpha>1$, then $\phi'(0)=0$, ϕ' is nonincreasing on $(-\infty,0)$ and nondecreasing on $(0,\infty)$. Further,

$$\phi^{-1}(x)=|x|^{\frac{1}{\alpha}}\operatorname{sgn} x, \quad (\phi^{-1})'(x)=\frac{1}{\alpha}\,|x|^{\frac{1}{\alpha}-1}, \quad \lim_{x\to 0}(\phi^{-1})'(x)=\infty,$$

which yields that ϕ^{-1} is not Lipschitz continuous at 0. Since

$$\lim_{x\to 0} x(\phi^{-1})'(x)=\frac{1}{\alpha}\lim_{x\to 0} x|x|^{\frac{1}{\alpha}-1}=0,$$

we see that the α-Laplacian $\phi(x)=|x|^\alpha\operatorname{sgn} x$ fulfils (2.4). If we take $p(t)=t^\beta$, $t\in[0,\infty)$, where $\beta>0$, then p fulfils (1.7). As an example of f satisfying conditions (1.5) and (1.6) we can choose

$$f(x)=x(x-\phi(L_0))(\phi(L)-x), \quad x\in\mathbb{R}.$$

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On Existence of Solutions with Prescribed Number of Zeros to High-Order Emden–Fowler Equations with Regular Nonlinearity and Variable Coefficient

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1 Introduction

The problem of existence of solutions with a countable number of zeros on a given domain to Emden–Fowler type equations is investigated. Consider the equation

$$y^{(n)}+p(t,y,y',\dots,y^{(n-1)})|y|^k\operatorname{sgn}y=0,\ \ 0<m\leqslant p(t,\xi_1,\dots,\xi_n)\leqslant M<+\infty,\quad t\in\mathbb{R},\qquad(1.1)$$

where $n\in\mathbb{N}$, $n\geqslant 2$, $k\in\mathbb{R}$, $k>1$, the function $p(t,\xi_1,\dots,\xi_n)$ is continuous, and Lipschitz continuous in $\xi_1,\dots,\xi_n$.

We prove that equation (1.1) has solutions with a countable set of zeros on every finite interval $[a,b)$. The existence of solutions with a given finite number of zeros was considered in the previous papers, and results from them will be used to prove the main result. Namely, [3] is devoted to the case of the third- and the fourth-order Emden–Fowler type equations with constant p, [4,6] deal with the third-order equation with a variable coefficient, and [5,8] expand the previous results to the higher-order case. They based on the result obtained in [1,2]. Some results of the papers [3–6,8] can be summarized as

Theorem 1.1. *For any integer $S\geq 2$ and any finite interval $[a,b]\subset\mathbb{R}$ equation (1.1) has a solution $y(t)$ defined on the interval, $y(t)$ has exactly S zeros on the interval and $y(a)=0$, $y(b)=0$.*

Now, this theorem is expanded to the new case.

2 The main result

Theorem 2.1 ([7])**.** *For any finite interval $[a,b)\subset\mathbb{R}$ equation (1.1) has a solution $y(t)$ defined on the interval, $y(t)$ a countable set of zeros on the interval and $y(a)=0$.*

3 Sketch of the proof

The idea of the proof is similar to that of the proof of the main result from [8]. Suppose that $y(t)$ is a maximally extended solution to (1.1) with initial data $y(a)=0,y'(a)=y_1>0,\dots,y^{(n-1)}(a)=y_{n-1}>0$. In [1] it is proved that $y(t)$ has the countable number of zeroes. By t_N we denote a position of the N-th zero of $y(t)$ after the point a. In [8] it was proved that t_N is a continuous function on $(y_1,\dots,y_{n-1})$. Lower and upper estimates of the continuous function $t_N(y_1,\dots,y_{n-1})$ show that the N-th zero of the solution can be located at any point on the axis after a, hence solution with exactly N zeros can be defined on any $[a,b]$, if we choose appropriate initial data.

Proof of Theorem 2.1 has the same idea with some minor modifications. We know (see, for example, [1, Ch. 7]) that t_N tends to some finite limit t_* as $N \to +\infty$, but the solution itself is not defined at the point t_*. It appears that $t_*(y_1, \dots, y_{n-1})$ is also a continuous function of the variables $(y_1, \dots, y_{n-1})$ – like $t_N(y_1, \dots, y_{n-1})$. In addition, we obtain upper and lower estimates of t_* with the help of [1, p. 193, Lemmas 7.1, 7.2, 7.3] and Theorem 1.1.

We prove the continuity of $t_*(y_1, \dots, y_{n-1})$ using the continuity of every $t_N(y_1, \dots, y_{n-1})$ and lemmas [1, p. 193, Lemmas 7.1, 7.2, 7.3], since they give some estimates on the distance between t_N and t_{N+1} in comparison with the distance between t_N and t_{N-1}. The proposition of discontinuity of $t_*(y_1, \dots, y_{n-1})$ contradicts with those estimates.

4 Future plans

Papers [4, 5] demonstrate that Theorem 1.1 still holds true when $k \in (0, 1)$, so in future I hope to expand Theorem 2.1 on this case as well.

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On Solution of Some Non-Linear Integral Boundary Value Problem

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We study the non-linear integral boundary value problem

$$\frac{dx(t)}{dt} = f\Big(t, x(t), \frac{dx(t)}{dt}\Big), \ \ t \in [a,b], \tag{1}$$

$$g\bigg(x(a), x(b), \int\limits_a^b h(s, x(s))\,ds\bigg) = d. \tag{2}$$

We suppose that $f : [a,b] \times D \times D_1 \to \mathbb{R}^n$ is continuous function defined on bounded sets $D \subset \mathbb{R}^n$, $D^1 \subset \mathbb{R}^n$ (domain $D := D_\rho$ will be concretized later, see (8), D^1 is given) and $d \in \mathbb{R}^n$ is a given vector. Moreover, $f, g : D \times D \times D_2 \to \mathbb{R}^n$ and $h : [a,b] \times D \to \mathbb{R}^n$ are Lipschitzian in the following form

$$\big|f(t,u,v) - f(t,\widetilde{u},\widetilde{v})\big| \le K_1|u - \widetilde{u}| + K_2|v - \widetilde{v}|, \tag{3}$$

$$\big|g(u,w,p) - g(\widetilde{u},\widetilde{w},\widetilde{p})\big| \le K_3|u - \widetilde{u}| + K_4|w - \widetilde{w}| + K_5|p - \widetilde{p}|, \tag{4}$$

$$\big|h(t,u) - h(t,\widetilde{u})\big| \le K_6|u - \widetilde{u}| \tag{5}$$

for any $t \in [a,b]$ fixed, all $\{u, \widetilde{u}\} \subset D$, $\{v, \widetilde{v}\} \subset D^1$, $\{w, \widetilde{w}\} \subset D$, $\{p, \widetilde{p}\} \subset D_2$, where $D_2 := \Big\{ \int\limits_a^b h(t, x(t))\,dt : \ t \in [a,b], \ x \in D \Big\}$ and $K_1 - K_6$ are non-negative square matrices of dimension n. The inequalities between vectors are understood componentwise. A similar convention is adopted for the operations "absolute value", "max", "min". The symbol I_n stands for the unit matrix of dimension n, $r(K)$ denotes a spectral radius of a square matrix K.

By the solution of the problem (1), (2) we understand a continuously differentiable function with property (2) satisfying (1) on $[a,b]$.

In the sequel, we will use an approach that was suggested in [1]. We fix certain bounded sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$ and focus on the solutions x of the given problem with property $x(a) \in D_a$ and $x(b) \in D_b$. Instead of the non-local boundary value problem (1), (2), we consider the parameterized family of two-point "model-type" problems with simple separated conditions

$$\frac{dx(t)}{dt} = f\Big(t, x(t), \frac{dx(t)}{dt}\Big), \ \ t \in [a,b], \tag{6}$$

$$x(a) = z, \quad x(b) = \eta, \tag{7}$$

where $z = (z_1, z_2, \dots, z_n)$, $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ are considered as parameters.

If $z \in \mathbb{R}^n$ and ρ is a vector with non-negative components, $B(z,\rho) := \{\xi \in \mathbb{R}^n : \ |\xi - z| \le \rho\}$ stands for the componentwise ρ neighbourhood of z. For given two bounded connected sets $D_a \subset$

$\mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set $D_{a,b} := (1-\theta)z + \theta\eta,\ z \in D_a,\ \eta \in D_b,\ \theta \in [0,1]$ and its componentwise ρ-neighbourhood by putting

$$D = D_\rho := B(D_{a,b}, \rho) := \bigcup_{\xi \in D_{a,b}} B(\xi, \rho). \tag{8}$$

We suppose that

$$r(K_2) < 1, \quad r(Q) < 1, \tag{9}$$

where

$$Q := \frac{3(b-a)}{10}\, K, \quad K = K_1 + K_2[I_n - K_2]^{-1}K_1. \tag{10}$$

On the base of function $f : [a,b] \times D \times D^1 \to \mathbb{R}^n$ we introduce the vector

$$\delta_{[a,b],D,D^1}(f) := \frac{1}{2}\Big[\max_{(t,x)\in[a,b]\times D\times D^1} f(t,x,y) - \min_{(t,x)\in[a,b]\times D\times D^1} f(t,x,y)\Big] \tag{11}$$

and suppose that the $\rho-$neighbourhood in (8) such that

$$\rho \geq \frac{b-a}{2}\, \delta_{[a,b],D,D^1}(f). \tag{12}$$

Investigation of solutions of parameterized problem (6) and (7) is connected with the properties of the following special sequence of functions well posed on the interval $t \in [a,b]$

$$x_0(t,z,\eta) := z + \frac{t-a}{b-a}\,[\eta - z] = \Big[1 - \frac{t-a}{b-a}\Big]z + \frac{t-a}{b-a}\,\eta, \quad t \in [a,b], \tag{13}$$

$$x_{m+1}(t,z,\eta) = z + \int_a^t f\Big(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\Big)\, ds$$

$$-\frac{t-a}{b-a}\int_a^b f\Big(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\Big)\, ds + \frac{t-a}{b-a}\,[\eta - z], \quad t \in [a,b], \quad m = 0,1,2,\dots, \tag{14}$$

Theorem 1. *Let assumptions* (3)–(5) *and* (9) *hold. Then, for all fixed* $(z,\eta) \in D_a \times D_b$*:*

1. *The functions of the sequence* (14) *are continuously differentiable functions on the interval* $t \in [a,b]$*, have values in the domain* $D = D_\rho$ *and satisfy the two-point separated boundary conditions* (7)*.*

2. *The sequence of functions* (14) *in* $t \in [a,b]$ *converges uniformly as* $m \to \infty$ *to the limit function*

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta), \tag{15}$$

satisfying the two-point separated boundary conditions (7)*.*

3. *The limit function* $x_\infty(t,z,\eta)$ *is a unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_a^t f\Big(s, x(s), \frac{dx(s)}{ds}\Big)\, ds - \frac{t-a}{b-a}\int_a^b f\Big(s, x(s), \frac{dx(s)}{ds}\Big)\, ds + \frac{t-a}{b-a}\,[\eta - z], \tag{16}$$

i.e. it is the solution of the Cauchy problem for the modified system of integro-differential equations:

$$\frac{dx}{dt} = f\Big(t, x, \frac{dx(t)}{dt}\Big) + \frac{1}{b-a}\,\Delta(z,\eta), \quad x(a) = z, \tag{17}$$

where $\Delta(z,\eta) : D_a \times D_b \to \mathbb{R}^n$ *is a mapping given by formula*

$$\Delta(z,\eta) := [\eta - z] - \int\limits_a^b f\Big(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\Big)\,ds. \tag{18}$$

4. *The following error estimate holds:*

$$\big|x_\infty(t,z,\eta) - x_m(t,z,\eta)\big| \leqslant \frac{10}{9}\,\alpha_1(t,a,b-a)Q^m(1_n - Q)^{-1}\delta_{[a,b],D,D^1}(f), \tag{19}$$

for any $t \in [a,b]$ *and* $m \geq 0$, *where* $\delta_{[a,b],D,D^1}(f)$ *is given in* (11) *and*

$$\alpha_1(t,a,b-a) = 2(t-a)\Big(1 - \frac{t-a}{b-a}\Big), \quad \alpha_1(t,a,b-a) \leq \frac{b-a}{2}\,. \tag{20}$$

Theorem 2. *Under the assumption of Theorem 1, the limit function* $x_\infty(t,z,\eta) : [a,b] \times D_a \times D_b \to \mathbb{R}^n$ *defined by* (15) *is a continuously differentiable solution of the original BVP* (1), (2) *if and only if the pair of vectors* (z,η) *satisfies the system of* $2n$ *determining algebraic equations*

$$\begin{cases} \Delta(z,\eta) = \eta - z - \displaystyle\int\limits_a^b f\Big(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\Big)\,ds = 0, \\ g\Big(x_\infty(a,z,\eta), x_\infty(b,z,\eta), \displaystyle\int\limits_a^b h(s, x_\infty(s,z,\eta))\,ds\Big) - d = 0. \end{cases} \tag{21}$$

Note that similarly as in [2] the solvability of the determining system (21) on the base of (3)–(5) and (9) can be established by studying its m-th approximate versions:

$$\begin{cases} \Delta_m(z,\eta) = \eta - z - \displaystyle\int\limits_a^b f(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\Big)\,ds = 0, \\ g\Big(x_m(a,z,\eta), x_m(b,z,\eta), \displaystyle\int\limits_a^b h(s, x_m(s,z,\eta))\,ds\Big) - d = 0, \end{cases} \tag{22}$$

where m is fixed.

Let us apply the approach described above to the system of differential equations

$$\begin{cases} \dfrac{dx_1(t)}{dt} = \dfrac{1}{2}\,x_2^2(t) - t\,\dfrac{dx_2(t)}{dt}\,x_1(t) + \dfrac{1}{32}\,t^3 - \dfrac{1}{32}\,t^2 + \dfrac{9}{40}\,t, \\ \dfrac{dx_2(t)}{dt} = \dfrac{1}{2}\,\dfrac{dx_1(t)}{dt}\,x_1(t) - t^2x_2(t) + \dfrac{15}{64}\,t^3 + \dfrac{1}{8}\,t + \dfrac{1}{4}\,, \end{cases} \quad t \in [a,b] = [0,1], \tag{23}$$

considered with non-linear two-point boundary conditions

$$\begin{gathered} x_1(0)x_2(1) + \Big[\int\limits_0^1 x_1(s)\,ds\Big]^2 = -\frac{311}{14400}\,, \\ x_1(1)x_2(0) - \int\limits_0^1 x_2(s)\,ds = -\frac{1}{8}\,. \end{gathered} \tag{24}$$

Introduce the vector of parameters $z = col(z_1, z_2)$, $\eta = col(\eta_1, \eta_2)$. Let us consider the following choice of the subsets D_a, D_b and D^1:

$$D_a = D_b = \big\{(x_1, x_2) : \ -0.1 \le x_1 \le 0.2, \ -0.2 \le x_2 \le 0.3\big\}, \tag{25}$$
$$D^1 = \Big\{\Big(\frac{dx_1}{dt}, \frac{dx_2}{dt}\Big) : \ -0.1 \le \frac{dx_1}{dt} \le 0.3, \ -0.1 \le \frac{dx_2}{dt} \le 0.3\Big\}.$$

In this case $D_{a,b} = D_a = D_b$. For $\rho = col(\rho_1, \rho_2)$ involved in (12), we choose the vector $\rho = col(0.4; 0.4)$. Then, in view of (25) the set (8) takes the form

$$D = D_\rho = \big\{(x_1, x_2) : \ -0.5 \le x_1 \le 0.6, \ -0.6 \le x_2 \le 0.7\big\}. \tag{26}$$

A direct computations show that the conditions (3), (9), (10) hold with

$$K_1 = \begin{bmatrix} 0.3 & 0.3 \\ 0.15 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0.2 \\ 0.1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0.3367346939 & 0.5102040816 \\ 0.1836734694 & 1.051020408 \end{bmatrix}$$

and, therefore,

$$Q = \begin{bmatrix} 0.1010204082 & 0.1530612245 \\ 0.05510204082 & 0.3153061224 \end{bmatrix}, \quad r(Q) = 0.349278 < 1.$$

Furthermore, in view of (11)

$$\delta_{[a,b],D,D^1}(f) := \frac{1}{2}\Big[\max_{(t,x)\in[a,b]\times D\times D^1} f(t,x,y) - \min_{(t,x)\in[a,b]\times D\times D^1} f(t,x,y)\Big] = \begin{bmatrix} 0.31 \\ 0.7325 \end{bmatrix},$$
$$\rho = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} \ge \frac{b-a}{2}\,\delta_{[a,b],D,D_1}(f) = \begin{bmatrix} 0.155 \\ 0.36625 \end{bmatrix}.$$

We thus see that all the conditions of Theorem 1 are fulfilled, and the sequence of functions (14) for this example is uniformly convergent.

Applying Maple 14, we carried out the calculations.

It is easy to check that

$$x_1^*(t) = \frac{t^2}{8} - \frac{1}{10}, \quad x_2^*(t) = \frac{t}{4} \tag{27}$$

is a continuously differentiable solution of the problem (1), (2). For a different number of approximations m, we obtain from (22) the following numerical values for the introduced parameters which are presented in Table 1:

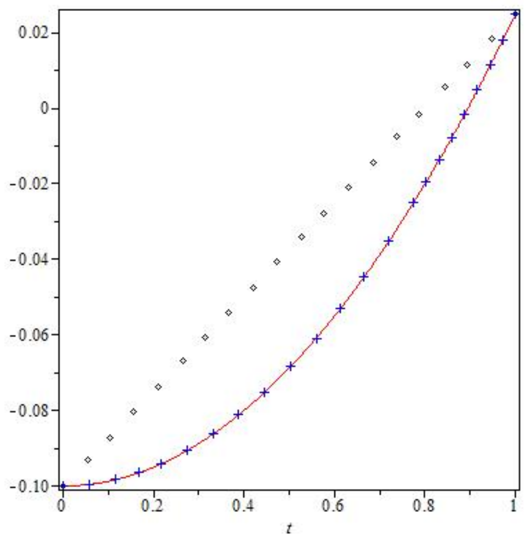
Table 1.

m	z_1	z_2	η_1	η_2
0	-0.089643967	-0.0002812586	0.03176891	0.25026338
1	-0.0994489263	0.00051937347	0.0255001973	0.2504687527
4	-0.0999998827	$7.744981 \cdot 10^{-8}$	0.02499999973	0.3535533902
6	-0.1000000004	$-2.263731 \cdot 10^{-10}$	0.0249999996	0.2499999996
Exact	-0.1	0	0.025	0.25

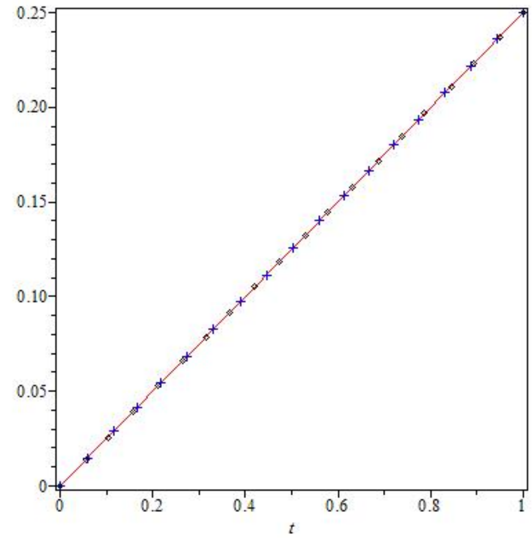
On the Figure 1 one can see the graphs of the exact solution (solid line) and its zero ($\Diamond$) and sixth approximation ($\times$) for the first and second coordinates.

The error of the sixth approximation ($m = 6$) for the first and second components:

$$\max_{t\in[0,1]} |x_1^*(t) - x_{61}(t)| \le 1 \cdot 10^{-9}, \quad \max_{t\in[0,1]} |x_2^*(t) - x_{62}(t)| \le 5 \cdot 10^{-9}.$$



a) first component



b) second component

Figure 1.

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Asymptotic Behavior of Solutions for One Class of Third Order Nonlinear Differential Equations

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Consider the differential equation

$$y''' = \alpha_0 p(t) y |\ln |y||^{\sigma}, \tag{1}$$

where $\alpha_0 \in \{-1;1\}$, $p : [a, \omega) \to (0, +\infty)$ is a continuous function, $\sigma \in \mathbb{R}$, $\infty < a < \omega \le +\infty$. It belongs to the equations class of the form

$$y''' = \alpha_0 p(t)\, L(y), \tag{2}$$

where $\alpha_0 \in \{-1;1\}$,$p : [a, \omega) \to (0, +\infty)$ is a continuous function, $\infty < a < \omega \le +\infty$, function L continuous and positive in a one-sided neighborhood Δ_{Y_0} points Y_0 (Y_0 equals either 0 or $\pm\infty$).

For equations of the form (2) in the works of A. Stekhun and V. Evtukhov [4, 9] there was investigated the question of the existence and asymptotic behavior when $t \to \omega$ of the endangered and unlimited solutions. The method of studying the equation of the form (2) assumed the presence of significant linearity of the function $L(y)$. In the equation (1) the function $L(y) = y\,|\ln |y||^{\sigma}$ is in some sense close to linear and requires improvements in research methods.

For second order equations of the form (1) in the works of V. Evtukhov and M. Jaber [1,3] there was investigated the question of the existence and asymptotic behavior, when $t \uparrow \omega$ all, so-called $P_\omega(\lambda_0)$-solution.

Solution y of the equation (1), specified on the interval $[t_y, w) \subset [a, \omega)$ is said to be $P_\omega(\lambda_0)$-solution, if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or } \ 0, \\ \text{or } \ \pm\infty, \end{cases} \qquad (k = 0, 1, 2), \quad \lim_{t\uparrow\omega} \frac{[y''(t)]^2}{y'''(t) y'(t)} = \lambda_0 \tag{3}$$

Earlier in the articles [6–8] were obtained the results in the case, when $\lambda_0 \in R \setminus \{0, -1, \frac{1}{2}\}$. The goal of the work to establish existence conditions for the equation (1) of $P_\omega(\pm\infty)$-solutions and also asymptotic representations, when $t \uparrow \omega$ such solutions and their derivative to the second order.

We introduce the necessary notation for further, assuming

$$q(t) = p(t)\pi_\omega^3(t)\,|\ln \pi_\omega^2(t)|^{\sigma}, \quad Q(t) = \int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,|\ln \pi_\omega^2(t)|^{\sigma}\, d\tau,$$

where

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \ w = +\infty, \\ t - \omega, & \text{if } \ w < +\infty. \end{cases}$$

Theorem 1. *For the existence of $P_\omega(\pm\infty)$-solutions of (1), it is necessary and sufficient the conditions*

$$\lim_{t\uparrow\omega} q(t)=0, \quad \lim_{t\uparrow\omega} Q(t)=\infty \tag{4}$$

to be satisfied. Moreover, for each such solution the following asymptotic representations, when $t\uparrow\omega$

$$\begin{gathered}\ln|y(t)|=\ln\pi_\omega^2(t)+\frac{\alpha_0}{2}\,Q(t)[1+o(1)],\\ \ln|y'(t)|=\ln|\pi_\omega(t)|+\frac{\alpha_0}{2}\,Q(t)[1+o(1)], \quad \ln|y''(t)|=\frac{\alpha_0}{2}\,Q(t)[1+o(1)]\end{gathered} \tag{5}$$

take place.

Indeed, if $y:[t_y,\omega[\,\to\mathbb{R}$ is a $P_\omega(\pm\infty)$-solution of the equation (1), then the conditions (3) are met and the following limit relations are true:

$$\lim_{t\uparrow\omega}\frac{y'''(t)\pi_\omega(t)}{y''(t)}=0, \quad \lim_{t\uparrow\omega}\frac{y''(t)\pi_\omega(t)}{y'(t)}=1, \tag{6}$$

$$\lim_{t\uparrow\omega}\frac{y''(t)\pi_\omega^2(t)}{y(t)}=2, \quad \lim_{t\uparrow\omega}\frac{y'(t)\pi_\omega(t)}{y(t)}=2. \tag{7}$$

Without loss of generality, we can assume that $y''(t)$, $y'(t)$, $\ln|y(t)|$ are non-zero when $t\in[t_y,\omega[\,$. Therefore, considering the limiting relations (7) and formulas

$$y(t)\sim\frac{1}{2}\,\pi_\omega^2(t)\,y''(t), \quad \ln|y(t)|\sim\ln\pi_\omega^2(t) \ \text{ when } \ t\uparrow\omega,$$

from equation (1) we get

$$y'''(t)=\alpha_0 p(t)\,\frac{\pi_\omega^2(t)}{2}\,|\ln\pi_\omega^2(t)|^\sigma y''(t)\,[1+o(1)]. \tag{8}$$

Hence, in view of the first of limiting relations (6), it follows that

$$p(t)\pi_\omega^3(t)\,|\ln\pi_\omega^2(t)|^\sigma\longrightarrow 0 \ \text{ when } \ t\uparrow\omega,$$

that is, the first of the conditions (4) of the theorem is satisfied. Dividing now (8) by $y''(t)$ and integrating obtained relation on the interval from t_y to t, come to a conclusion considering the first from conditions (4) that $\int\limits_{t_y}^{\omega} p(t)\pi_\omega^2(t)|\ln\pi_\omega^2(t)|^\sigma\,dt=\infty$ and when $t\uparrow w$ the asymptotic relation

$$\ln|y''(t)|=\frac{\alpha_0}{2}\int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,|\ln\pi_\omega^2(\tau)|^\sigma\,d\tau\,[1+o(1)]$$

take place, that is, the second of the theorem conditions (4) is met and the third of the asymptotic relations (5).

The validity of the first and second asymptotic representations (5) directly follows from the third, considering that $y(t)\sim\frac{1}{2}\,\pi_\omega^2(t)\,y''(t)$ and $y'(t)\sim\pi_\omega(t)\,y''(t)$ when $t\uparrow\omega$.

Assuming that conditions (4) are met, we reduce equation (1) using transformations

$$\begin{gathered}\ln|y(t)|=\ln\pi_\omega^2(\tau)[1+v_1(\tau)], \quad \frac{y'(t)}{y(t)}=\frac{2[1+v_2(\tau)]}{\pi_\omega(t)}\,,\\ \Big(\frac{y'(t)}{y(t)}\Big)'=\frac{-2\,[1+v_3(\tau)]}{\pi_\omega^2(t)}\,, \quad \tau=\beta\ln|\pi_w(t)|, \quad \beta=\begin{cases}1, & \text{when } \ w=+\infty,\\ -1, & \text{when } \ w<+\infty,\end{cases}\end{gathered} \tag{9}$$

to a system of differential equations

$$\begin{cases} v_1' = \dfrac{1}{\tau}\,[v_2 - v_1], \\ v_2' = \beta[v_2 - v_3], \\ v_3' = \beta\big[f(\tau) + \sigma f(\tau)v_1 + 6v_2 - 4v_3 + V(\tau, v_1, v_2, v_3)\big], \end{cases} \tag{10}$$

in which

$$f(\tau) = f(\tau(t)) = \alpha_0 q(t), \quad V(\tau, v_1, v_2, v_3) = 12v_2^2 + 4v_2^3 - 6v_2v_3 + f(\tau)\big[\,|1+v_1|^\sigma - 1 - \sigma v_1\big].$$

For the system (10) all the conditions of the Theorem 2.6 from the work [2] are satisfied. According to that theorem the system (10) has at least one solution $(v_1, v_2, v_3) : [\tau_1, +\infty) \to R^3(\tau_1 \geq \tau_0)$, converges to zero when $\tau \to +\infty$, to which, due to replacements (9), matches the solution $y(t)$ of the differential equation (1), allowing the asymptotic representations (5) when $t \uparrow \omega$.

Theorem 2. *Let the function $p : [a, \omega) \to (0, +\infty)$ be continuously differentiable and along with the conditions* (4) *the following conditions*

$$\int\limits_a^\omega |q'(t)|\,dt < +\infty, \quad \int\limits_a^\omega \frac{q^2(t)}{|\pi_\omega(t)|}\,dt < +\infty, \quad \int\limits_a^\omega \frac{q(t)|Q(t)|}{\pi_\omega(t)\ln|\pi_\omega(t)|}\,dt < +\infty$$

hold. Then for any $c \neq 0$ equation (1) *has $P_\omega(\pm\infty)$-solution. Furthermore, for every such solution the following asymptotic representations when $t \to w$*

$$y(t) = \pi_\omega^2(t)\,e^{\alpha_0 Q(t)}[c + o(1)],$$
$$y'(t) = \pi_\omega(t)\,e^{\alpha_0 Q(t)}[2c + o(1)], \quad y''(t) = e^{\alpha_0 Q(t)}[2c + o(1)]$$

take place.

Let present a corollary of these theorems, when $\sigma = 0$, i.e. for the following linear differential equation

$$y''' = \alpha_0 p(t) y, \tag{11}$$

where $\alpha_0 \in \{-1; 1\}$, $\sigma \in \mathbb{R}$, $p : [a, w) \to (0, +\infty)$ – continuous function; $a < w \leq +\infty$.

Corollary 1. *For the existence of $P_\omega(\pm\infty)$-solutions of* (11), *it is necessary and sufficient the conditions*

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^3(t) = 0, \quad \lim_{t\uparrow\omega} \int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau = \infty \tag{12}$$

to be fulfilled. Furthermore, for any such solution the following asymptotic representations, when $t \uparrow \omega$

$$\ln|y(t)| = \ln\pi_\omega^2(t) + \frac{\alpha_0}{2}\int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\,[1 + o(1)],$$

$$\ln|y'(t)| = \ln|\pi_\omega(t)| + \frac{\alpha_0}{2}\int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\,[1 + o(1)],$$

$$\ln|y''(t)| = \frac{\alpha_0}{2}\int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\,[1 + o(1)]$$

take place.

Corollary 2. *Let the function $p : [a,\omega) \to (0,+\infty)$ be continuously-differentiable and along with the conditions* (12) *the following conditions*

$$\int_a^\omega \big|(p(t)\pi_\omega^3(t))'\big|\,dt < +\infty, \quad \int_a^\omega p^2(t)|\pi_\omega^5(t)|\,dt < +\infty,$$

$$\int_a^\omega \frac{p(t)\pi_\omega^2(t)}{\ln|\pi_\omega(t)|} \int_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\,dt < +\infty$$

hold. Then for any $c \neq 0$ equation (11) *has $P_\omega(\pm\infty)$-solution. Furthermore, for any such solution the following asymptotic representations, when $t \to w$:*

$$y(t) = \pi_\omega^2(t)\exp\bigg(\alpha_0\int_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\bigg)[c + o(1)],$$

$$y'(t) = \pi_\omega(t)\exp\bigg(\alpha_0\int_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\bigg)[2c + o(1)],$$

$$y''(t) = \exp\bigg(\alpha_0\int_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\bigg)[2c + o(1)]$$

take place.

The obtained asymptotes are consistent with the already known results for linear differential equations (see [5, Chapter 1]).

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Necessary Conditions of Optimality for the Optimal Control Problem with Several Delays and the Continuous Initial Condition

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Let $O \subset \mathbb{R}^n$ be an open set and $U \subset \mathbb{R}^r$ be a convex compact set. Let $h_{i2} > h_{i1} > 0$, $i = \overline{1,s}$ and $\theta_k > \cdots > \theta_1 > 0$ be given numbers and n-dimensional function $f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k)$, $(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in I \times O^{1+s} \times U^{1+k}$ satisfies the following conditions: for almost all fixed $t \in I = [a, b]$ the function $f(t, \cdot\,) : I \times O^{1+s} \times U^{1+k} \to \mathbb{R}^n$ is continuous and continuously differentiable in $(x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in O^{1+s} \times U^{1+k}$; for each fixed $(x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in O^{1+s} \times U^{1+k}$, the function $f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k)$ and the matrices $f_x(t, \cdot\,)$, $f_{x_i}(t, \cdot\,)$, $i = \overline{1,s}$ and $f_u(t, \cdot\,)$, $f_{u_i}(t, \cdot\,)$, $i = \overline{1,k}$ are measurable on I; for any compact set $K \subset O$ there exists a function $m_K(t) \in L_1(I, [0, \infty))$ such that

$$\big|f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k)\big| \\ + |f_x(t, x, \cdot\,)| + \sum_{i=1}^{s} |f_{x_i}(t, x, \cdot\,)| + |f_u(t, x, \cdot\,)| + \sum_{i=1}^{k} |f_{u_i}(t, x, \cdot\,)| \le m_K(t)$$

for all $(x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in K^{1+s} \times U^{1+k}$ and for almost all $t \in I$.

Furthermore, let Φ be the set of continuous functions $\varphi(t) \in N$, $t \in I_1 = [\widehat{\tau}, b]$, where $\widehat{\tau} = a - \max\{h_{12}, \dots, h_{s2}\}$, $N \subset O$ is a convex compact set; Ω is the set of measurable functions $u(t) \in U$, $t \in I_2 = [a - \theta_k, b]$.

To each element $v = (t_0, t_1, \tau_1, \dots, \tau_s, \varphi, u) \in A = I \times I \times [h_{11}, h_{12}] \times \cdots \times [h_{s1}, h_{s2}] \times \Phi \times \Omega$ on the interval $[t_0, t_1]$ we assign the delay controlled functional differential equation

$$\dot{x}(t) = f\Big(t, x(t), x(t - \tau_1), \dots, x(t - \tau_s), u(t), u(t - \theta_1), \dots, u(t - \theta_k)\Big) \tag{1}$$

with the continuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0]. \tag{2}$$

The condition (2) is called continuous because always $x(t_0) = \varphi(t_0)$.

Definition 1. Let $\nu = (t_0, t_1, \tau_1, \dots, \tau_s, \varphi, u) \in A$. A function $x(t) = x(t; \nu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$ is called a solution of equation (1) with the continuous initial condition (2), or the solution corresponding to ν and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let the scalar-valued functions $q^i(t_0, t_1, \tau_1, \dots, \tau_s, x_0, x_1)$, $i = \overline{0,l}$ be continuously differentiable on $I^2 \times [h_{11}, h_{12}] \times \cdots \times [h_{s1}, h_{s2}] \times O^2$.

Definition 2. An element $\nu=(t_0,t_1,\tau_1,\dots,\tau_s,\varphi,u)\in A$ is said to be admissible if the corresponding solution $x(t)=x(t;\nu)$ satisfies the boundary conditions

$$q^i\big(t_0,t_1,\tau_1,\dots,\tau_s,\varphi(t_0),x(t_1)\big)=0,\ \ i=\overline{1,l}. \tag{3}$$

Denote by A_0 the set of admissible elements.

Definition 3. An element $\nu_0=(t_{00},t_{10},\tau_{10},\dots,\tau_{s0},\varphi_0,u_0)\in A_0$ is said to be optimal if for an arbitrary element $\nu\in A_0$ the inequality

$$q^0\big(t_{00},t_{10},\tau_{10},\dots,\tau_{s0},\varphi_0(t_{00}),x_0(t_{10})\big)\le q^0\big(t_0,t_1,\tau_1,\dots,\tau_s,\varphi(t_0),x(t_1)\big) \tag{4}$$

holds. Here $x_0(t)=x(t;\nu_0)$ and $x(t)=x(t;\nu)$.

The problem (1)–(4) is called the optimal control problem with the continuous initial condition.

Theorem 1. *Let ν_0 be an optimal element with $t_{00},t_{10}\in(a,b)$ and the following conditions hold:*

1) *the function $\varphi_0(t)$ is absolutely continuous and $\dot\varphi_0(t)$ is bounded;*

2) *the function*
$$f_0(w)=f\big(w,u_0(t),u_0(t-\theta_1),\dots,u_0(t-\theta_k)\big),$$
where $w=(t,x,x_1,\dots,x_s)\in I\times O^{1+s}$ is bounded on $I\times O^{1+s}$;

3) *there exists the finite limits*
$$\lim_{t\to t_{00}-}\dot\varphi_0(t)=\dot\varphi_0^-,\quad \lim_{w\to w_0}f_0(w)=f_0^-,\ \ w\in(a,t_{00}]\times O^{1+s},$$
where
$$w_0=\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0})\big);$$

4) *there exists the finite limit*
$$\lim_{w\to w_1}f_0(w)=f_1^-,\ \ w\in(t_{00},t_{10}]\times O^{1+s},$$
$$w_1=\big(t_{10},x_0(t_{10}),x_0(t_{10}-\tau_{10}),\dots,x_0(t_{10}-\tau_{s0})\big).$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\ne 0$, with $\pi_0\le 0$, and a solution $\psi(t)=(\psi_1(t),\dots,\psi_n(t))$ of the equation

$$\dot\psi(t)=-\psi(t)f_{0x}[t]-\sum_{i=1}^s\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}],\ \ t\in[t_{00},t_{10}],\ \ \psi(t)=0,\ \ t>t_{10}, \tag{5}$$

where
$$f_{0x}[t]=f_{0x}\big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0(t-\tau_{s0})\big),$$
such that the following conditions hold;

5) *the conditions for the moments t_{00} and t_{10}:*
$$\pi Q_{0t_0}+(\pi Q_{0x_0}+\psi(t_{00}))\dot\varphi_0^-\ge\psi(t_{00})f^-,\ \ \pi Q_{0t_1}\ge-\psi(t_{10})f_1^-,$$
where
$$Q_{0t_0}=\frac{\partial}{\partial t_0}\,Q\big(t_{00},t_{10},\tau_{10},\dots,\tau_{s0},\varphi_0(t_{00}),x_0(t_{10})\big),\ \ Q=(q^0,\dots,q^l)^T;$$

6) *the conditions for the delays* τ_{i0}, $i=\overline{1,s}$,

$$\pi Q_{0\tau_i}=\int\limits_{t_{00}}^{t_{10}}\psi(t)f_{0x_i}[t]\,\dot{x}_0(t-\tau_{i0})\,dt,\ \ i=\overline{1,s};$$

7) *the maximum principle for the initial function* $\varphi_0(t)$,

$$\begin{aligned}&[Q_{0x_0}+\psi(t_{00})]\varphi_0(t_{00})+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi_0(t)\,dt\\&\qquad=\max_{\varphi(t)\in\Phi}\bigg\{[Q_{0x_0}+\psi(t_{00})]\varphi(t_{00})+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi(t)\,dt\bigg\};\end{aligned}$$

8) *the linearized integral maximum principle for the control function* $u_0(t)$,

$$\begin{aligned}&\int\limits_{t_{00}}^{t_{10}}\psi(t)\Big[f_{0u}[t]u_0(t)+\sum_{i=1}^{k}f_{0u_i}[t]u_0(t-\theta_i)\Big]\,dt\\&\qquad=\max_{u(t)\in\Omega}\int\limits_{t_{00}}^{t_{10}}\psi(t)\Big[f_{0u}[t]u(t)+\sum_{i=1}^{k}f_{0u_i}[t]u(t-\theta_i)\Big]\,dt;\end{aligned}$$

9) *the condition for the function* $\psi(t)$

$$\psi(t_{10})=\pi Q_{0x_1}.$$

Theorem 2. *Let* ν_0 *be an optimal element with* $t_{00},t_{10}\in(a,b)$ *and the conditions* 1), 2) *of Theorem* 1 *hold. Moreover, there exists the finite limits*

$$\lim_{t\to t_{00}+}\dot{\varphi}_0(t)=\dot{\varphi}_0^+,\quad \lim_{w\to w_0}f_0(w)=f_0^+,\ \ w\in[t_{00},b)\times O^{1+s},$$
$$\lim_{w\to w_1}f_0(w)=f_1^+,\ \ w\in[t_{10},b)\times O^{1+s}.$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$, *with* $\pi_0\le 0$, *and a solution* $\psi=(\psi_1(t),\dots,\psi_n(t))$ *of the equation* (5) *such that the conditions* 6)–9) *hold. Moreover,*

$$\pi Q_{0t_0}+(\pi Q_{0x_0}+\psi(t_{00}))\dot{\varphi}_0^+\le\psi(t_{00})f_0^+,\ \ \pi Q_{0t_1}\le-\psi(t_{10})f_1^+,$$

Theorem 3. *Let* ν_0 *be an optimal element with* $t_{00},t_{10}\in(a,b)$ *and the following conditions hold: the function* $\varphi_0(t)$ *is continuously differentiable; the function* $f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_k)$ *is continuous; the function* $f(t,x,x_1,\dots,x_s,u_0(t),u_0(t-\theta_1),\dots,u_0(t-\theta_k))$ *is continuous at points* t_{00},t_{10}. *Then there exist a vector* $\pi=(\pi_0,\dots,\pi_l)\neq 0$, *with* $\pi_0\le 0$, *and a solution* $\psi=(\psi_1(t),\dots,\psi_n(t))$ *of the equation* (5) *such that the conditions* 6)–9) *hold. Moreover,*

$$\pi Q_{0t_0}+(\pi Q_{0x_0}+\psi(t_{00}))\varphi_0(t_{00})=\psi(t_{00})f_0[t_{00}],\ \ \pi Q_{0t_1}=-\psi(t_{10})f_0[t_{10}],$$

where

$$f_0[t]=f\Big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0(t-\tau_{s0}),u_0(t),u_0(t-\theta_1),\dots,u_0(t-\theta_k)\Big).$$

Theorem 3 is a corollary to Theorems 1 and 2. On the basis of variation formulas [2,3] Theorems 1, 2 are proved by the scheme given in [1,4].

Acknowledgment

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On a Fundamental Matrix of Linear Homogeneous Differential System with Coefficients of Oscillating Type

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Let
$$G(\varepsilon_0)=\big\{t,\varepsilon:\ 0<\varepsilon<\varepsilon_0,\ -L\varepsilon^{-1}\le t\le L\varepsilon^{-1},\ 0<L<+\infty\big\}.$$

Definition 1. We say that a function $p(t,\varepsilon)$ belongs to the class $S_0(m;\varepsilon_0)$ $(m\in\mathbf{N}\cup\{0\})$ if

1) $p:G(\varepsilon_0)\to\mathbf{C}$;
2) $p(t,\varepsilon)\in C^m(G(\varepsilon_0))$ with respect to t;
3)
$$\frac{d^kp(t,\varepsilon)}{dt^k}=\varepsilon^kp_k^*(t,\varepsilon)\ \ (0\le k\le m),$$
$$\|p\|_{S_0(m;\varepsilon_0)}\overset{def}{=}\sum_{k=0}^m\sup_{G(\varepsilon_0)}|p_k^*(t,\varepsilon)|<+\infty.$$

Under the slowly varying function we mean the function of the class $S_0(m;\varepsilon_0)$.

Definition 2. We say that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F_0(m;\varepsilon_0;\theta)$ $(m\in\mathbf{N}\cup\{0\})$ if this function can be represented as:
$$f(t,\varepsilon,\theta(t,\varepsilon))=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon)),$$
and

1) $f_n(t,\varepsilon)\in S_0(m;\varepsilon_0)$;
2)
$$\|f\|_{F_0(m;\varepsilon_0;\theta)}\overset{def}{=}\sum_{n=-\infty}^{\infty}\|f_n\|_{S_0(m;\varepsilon_0)}<+\infty;$$
3) $\theta(t,\varepsilon)=\int\limits_0^t\varphi(\tau,\varepsilon)\,d\tau$, $\varphi(t,\varepsilon)\in\mathbf{R}^+$, $\varphi(t,\varepsilon)\in S_0(m;\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\varphi(t,\varepsilon)=\varphi_0>0$.

Definition 3. We say that a vector-function $a(t,\varepsilon)=\operatorname{colon}(a_1(t,\varepsilon),\dots,a_N(t,\varepsilon))$ belongs to the class $S_1(m;\varepsilon_0)$ if $a_j(t,\varepsilon)\in S_0(m;\varepsilon_0)$ $(j=\overline{1,N})$. We say that a matrix-function $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$ belongs to the class $S_2(m;\varepsilon_0)$ if $a_{jk}(t,\varepsilon)\in S_0(m;\varepsilon_0)$ $(j,k=\overline{1,N})$.

We define the norms:
$$\|a(t,\varepsilon)\|_{S_1(m;\varepsilon_0)}=\max_{1\le j\le N}\|a_j(t,\varepsilon)\|_{S_0(m;\varepsilon_0)},$$
$$\|A(t,\varepsilon)\|_{S_2(m;\varepsilon_0)}=\max_{1\le j\le N}\sum_{k=1}^N\|a_{jk}(t,\varepsilon)\|_{S_0(m;\varepsilon_0)}.$$

Definition 4. We say that a vector-function $b(t,\varepsilon,\theta)=\operatorname{colon}(b_1(t,\varepsilon,\theta),\dots,b_N(t,\varepsilon,\theta))$ belongs to the class $F_1(m;\varepsilon_0;\theta)$ if $b_j(t,\varepsilon,\theta)\in F_0(m;\varepsilon_0;\theta)$ $(j=\overline{1,N})$. We say that a matrix-function $B(t,\varepsilon,\theta)=(b_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,N}}$ belongs to the class $F_2(m;\varepsilon_0;\theta)$ if $b_{jk}(t,\varepsilon,\theta)\in F_0(m;\varepsilon_0;\theta)$ $(j,k=\overline{1,N})$.

We define the norms:

$$\|b(t,\varepsilon,\theta)\|_{F_1(m;\varepsilon_0;\theta)}=\max_{1\le j\le N}\|b_j(t,\varepsilon,\theta)\|_{F_0(m;\varepsilon_0;\theta)},$$

$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)}=\max_{1\le j\le N}\sum_{k=1}^N\|b_{jk}(t,\varepsilon,\theta)\|_{F_0(m;\varepsilon_0;\theta)}.$$

Thus, the matrix $B(t,\varepsilon,\theta)$ has a kind

$$B(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}B_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon)),$$

where $B_n(t,\varepsilon)\in S_2(m;\varepsilon_0)$, and

$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)}\le\sum_{n=-\infty}^{\infty}\|B_n(t,\varepsilon)\|_{S_2(m;\varepsilon_0)}.$$

It is easy to obtain that if $A,B\in F_2(m;\varepsilon_0;\theta)$, then $AB\in F_2(m;\varepsilon;\theta)$, and

$$\|AB\|_{F_2(m;\varepsilon_0;\theta)}\le 2^m\|A\|_{F_2(m;\varepsilon_0;\theta)}\cdot\|B\|_{F_2(m;\varepsilon_0;\theta)}.$$

For $A(t,\varepsilon,\theta)\in F_2(m;\varepsilon_0;\theta)$ we denote

$$\Gamma_n[A]=\frac{1}{2\pi}\int_0^{2\pi}A(t,\varepsilon,\theta)\exp(-in\,\theta)\,d\theta\ \ (n\in\mathbf{Z}).$$

We consider the next system of differential equations

$$\frac{dx}{dt}=\big(\Lambda(t,\varepsilon)+\varepsilon P(t,\varepsilon,\theta)\big)x, \tag{1}$$

where $\varepsilon\in(0,\varepsilon_0)$, $\Lambda(t,\varepsilon)=\operatorname{diag}(\lambda_1(t,\varepsilon),\dots,\lambda_N(t,\varepsilon))\in S_2(m;\varepsilon_0)$, $P(t,\varepsilon,\theta)\in F_2(m;\varepsilon_0;\theta)$.

We study the problem about the structure of fundamental matrix of the system (1).

Consider the linear homogeneous system

$$\frac{dx}{dt}=\varepsilon A(t,\varepsilon)x, \tag{2}$$

where $\varepsilon\in(0,\varepsilon_0)$, $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}\in S_2(m;\varepsilon_0)$. Then there exists a matrizant $X(t,\varepsilon)$ of the system (2).

Lemma 1. *If $X(t,\varepsilon)$ is the matrizant of the system* (2)*, then $X(t,\varepsilon)$, $X^{-1}(t,\varepsilon)$ belongs to the class $S_2(m;\varepsilon_0)$.*

Lemma 2. *Let we have the matrix equation*

$$\frac{dX}{dt}=\varepsilon A(t,\varepsilon,\theta), \tag{3}$$

where $\varepsilon\in(0,\varepsilon_0)$, $A(t,\varepsilon,\theta)\in F_2(m;\varepsilon_0;\theta)$. Then there exists a solution $X(t,\varepsilon,\theta)$ of the equation (3) *which belongs to the class $F_2(m;\varepsilon_0;\theta)$, and there exists $K\in(0,+\infty)$ which does not depend on $A(t,\varepsilon,\theta)$ such that*

$$\|X(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)}\le K\|A(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)}.$$

Theorem 1. *Let the system* (1) *be such that*

$$\inf_{G(\varepsilon_0)} \big|\mathrm{Re}\,(\lambda_j(t,\varepsilon)-\lambda_k(t,\varepsilon)\big| \geq \gamma > 0 \ \ (j \neq k),$$

and $m \geq 1$. *Then there exists* $\varepsilon^* \in (0,\varepsilon_0)$ *such that for all* $\varepsilon \in (0,\varepsilon^*)$ *there exists a fundamental matrix* $X^{(1)}(t,\varepsilon,\theta)$ *of the system* (1) *which has a kind*

$$X^{(1)}(t,\varepsilon,\theta) = R^{(1)}(t,\varepsilon,\theta)\exp\bigg(\int\limits_0^t \Lambda^{(1)}(\tau,\varepsilon)\,d\tau\bigg),$$

where $R^{(1)}(t,\varepsilon,\theta) \in F_2(m-1;\varepsilon^*;\theta)$, $\Lambda^{(1)}(t,\varepsilon)$ *– the diagonal matrix, belonging to the class* $S(m-1;\varepsilon^*)$.

Theorem 2. *Let the system* (1) *be such that*

$$\Lambda(t,\varepsilon) = i\varphi(t,\varepsilon)J,$$

where $\varphi(t,\varepsilon)$ *is function in the Definition* 2, $J = \mathrm{diag}\,(n_1,\dots,n_N)$, $n_j \in \mathbf{Z}$ $(j=\overline{1,N})$, *and* $m \geq 1$. *Then there exists* $\varepsilon^{**} \in (0,\varepsilon_0)$ *such that for all* $\varepsilon \in (0,\varepsilon^{**})$ *there exists a fundamental matrix* $X^{(2)}(t,\varepsilon,\theta)$ *of the system* (1) *which has a kind:*

$$X^{(2)}(t,\varepsilon,\theta(t,\varepsilon)) = \exp(i\theta(t,\varepsilon)J)R^{(2)}(t,\varepsilon,\theta(t,\varepsilon)),$$

where $R^{(2)}(t,\varepsilon,\theta(t,\varepsilon)) \in F_2(m-1;\varepsilon^{**};\theta)$.

Proof. We make a substitution in the system (1)

$$x = \exp(i\theta(t,\varepsilon)J)y, \tag{4}$$

where y is a new unknown N-dimensional vector. We obtain

$$\frac{dy}{dt} = \varepsilon Q(t,\varepsilon,\theta)y, \tag{5}$$

where $Q(t,\varepsilon,\theta) = \exp(-i\theta(t,\varepsilon)J)P(t,\varepsilon,\theta)\exp(i\theta(t,\varepsilon)J)$ belongs to the class $F_2(m;\varepsilon_0;\theta)$.

Now in the system (5) we make the substitution

$$y = (E + \varepsilon\Phi(t,\varepsilon,\theta))z, \tag{6}$$

where the matrix Φ is defined from the equation

$$\varphi(t,\varepsilon)\,\frac{\partial\Phi}{\partial\theta} = Q(t,\varepsilon,\theta) - U(t,\varepsilon),$$

in which $U(t,\varepsilon) = \Gamma_0[Q(t,\varepsilon,\theta)]$. Then

$$\Phi(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty} \frac{\Gamma_n[Q(t,\varepsilon,\theta)]}{in\,\varphi(t,\varepsilon)}\,\exp(in\,\theta) \in F_2(m;\varepsilon_0;\theta).$$

As a result of the substitution (6) we obtain

$$\frac{dz}{dt} = \varepsilon\big(U(t,\varepsilon) + \varepsilon V(t,\varepsilon,\theta)\big)z, \tag{7}$$

where the matrix V is defined from the equation

$$(E+\varepsilon\Phi(t,\varepsilon,\theta))V = Q(t,\varepsilon,\theta)\Phi(t,\varepsilon,\theta) - \Phi(t,\varepsilon,\theta)U(t,\varepsilon) - \frac{1}{\varepsilon}\,\frac{\partial\Phi(t,\varepsilon,\theta)}{\partial t}\,. \tag{8}$$

The matrix $\frac{1}{\varepsilon}\frac{\partial\Phi}{\partial t}$ belongs to the class $F_2(m-1;\varepsilon_0;\theta)$, then there exists $\varepsilon_2 \in (0,\varepsilon_0)$ such that for all $\varepsilon \in (0,\varepsilon_2)$ the equation (8) is solved with respect to V, and $V(t,\varepsilon,\theta)$ belongs to the class $F_2(m-1;\varepsilon_2;\theta_0)$.

Together with the system (7) we consider the truncated system

$$\frac{dz^{(0)}}{dt} = \varepsilon U(t,\varepsilon)z^{(0)}. \tag{9}$$

Continuity of the matrix $U(t,\varepsilon)$ with respect to t for all $\varepsilon \in (0,\varepsilon_0)$ guarantees the existence of the matrizant $Z^{(0)}(t,\varepsilon)$ of the system (9), and by virtue the Lemma 1 $Z^{(0)}(t,\varepsilon)$, $(Z^{(0)}(t,\varepsilon))^{-1}$ belong to the class $S_2(m-1;\varepsilon_0)$.

We make in the system (7) the substitution

$$z = Z^{(0)}(t,\varepsilon)\xi, \tag{10}$$

where ξ – the new unknown vector. We obtain

$$\frac{d\xi}{dt} = \varepsilon^2 W(t,\varepsilon,\theta)\xi, \tag{11}$$

where $W = (Z^{(0)}(t,\varepsilon))^{-1}V(t,\varepsilon,\theta)Z^{(0)}(t,\varepsilon)) \in F_2(m-1;\varepsilon_2;\theta)$.

Now we show that there exists a substitution

$$\xi = (E+\varepsilon\Psi(t,\varepsilon,\theta))\eta, \tag{12}$$

where $\Psi \in F_2(m-1;\varepsilon_3;\theta)$ ($\varepsilon_3 \in (0,\varepsilon_2)$), which leads the system (11) to the system

$$\frac{d\eta}{dt} = O\eta, \tag{13}$$

where O – the null $(N\times N)$-matrix. Really, we define the matrix Ψ from the equation

$$\frac{d\Psi}{dt} = \varepsilon W(t,\varepsilon,\theta) + \varepsilon^2 W(t,\varepsilon,\theta)\Psi. \tag{14}$$

Consider the truncated equation

$$\frac{d\Psi^{(0)}}{dt} = \varepsilon W(t,\varepsilon,\theta).$$

By virtue of Lemma 2 this equation has a solution $\Psi^{(0)}(t,\varepsilon,\theta) \in F_2(m-1;\varepsilon_2;\theta)$.

We construct the process of successive approximations, used as initial approximation $\Psi^{(0)}(t,\varepsilon,\theta)$, and the subsequent approximations defining as solutions from the class $F_2(m-1;\varepsilon_2;\theta)$ of the matrix-equations

$$\frac{d\Psi^{(k+1)}}{dt} = \varepsilon W(t,\varepsilon,\theta) + \varepsilon^2 W(t,\varepsilon,\theta)\Psi^{(k)},\ \ k=0,1,2,\dots\,. \tag{15}$$

Each of these solutions exists by virtue of Lemma 2. Then we have

$$\frac{d(\Psi^{(k+1)}-\Psi^{(k)})}{dt} = \varepsilon^2 W(t,\varepsilon,\theta)(\Psi^{(k)}-\Psi^{(k-1)}),\ \ k=1,2,\dots\,.$$

By virtue of Lemma 2 and unequality (2) we obtain

$$\|\Psi^{(k+1)} - \Psi^{(k)}\|_{F_2(m-1;\varepsilon_2;\theta)} \leq \varepsilon 2^{m-1} K \|\Psi^{(k)} - \Psi^{(k-1)}\|_{F_2(m-1;\varepsilon_2;\theta)}, \quad k = 1, 2, \dots$$

(K is defined in the Lemma 2), therefore the convergence of the process (15) is guaranteed by the unequality $0 < \varepsilon < \varepsilon_3$, where $\varepsilon_3 2^{m-1} K < 1$. As a result of the process (15) we obtain the solution $\Psi(t, \varepsilon, \theta)$, belonging to the class $F_2(m-1; \varepsilon_3; \theta)$, of the equation (14).

The matrizant of the system (13) is E. Thus, by virtue of (4), (6), (10), (12) we obtain that the fundamental matrix of the system (1) has a kind:

$$X^{(2)}(t, \varepsilon, \theta) = \exp(i\theta(t, \varepsilon) J)(E + \varepsilon \Phi(t, \varepsilon, \theta)) Z^{(0)}(t, \varepsilon)(E + \varepsilon \Psi(t, \varepsilon, \theta)),$$

and the Theorem 2 is proved. $\square$

Remark. In the sense of the condition of Theorem 2 we say that we have a resonance case.

Differential Equations in Modelling Motion of Dislocations†

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Most of the technologically important materials are crystals, where atoms are arranged in a periodic lattice of a defined symmetry (cubic, hexagonal, orthorhombic, etc.). It is known that a plastic deformation of body-centred cubic metals is governed by the thermally activated motion of *screw dislocations*. Dislocations are line defects in crystals, that are caused by the finite rate of solidification because the atoms do not have sufficient time to take perfect lattice positions. Each dislocation is characterized by the so-called *Burgers vector* $\vec{b}$ and the tangential vector $\vec{u}$. We distinguish two basic types of dislocation segments: *edge* segment ($\vec{b} \perp \vec{u}$) and *screw* segment ($\vec{b} \parallel \vec{u}$), see Figure 1.

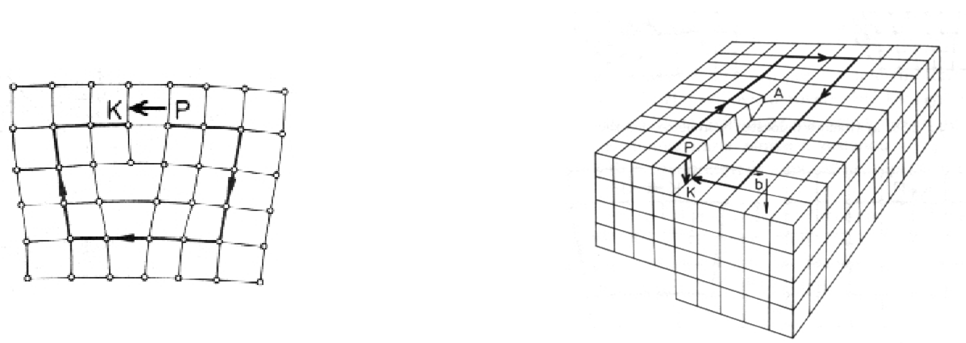

Figure 1. Edge and screw dislocations in a simple cubic lattice

If none of these conditions is satisfied, we speak about a *mixed* segment.

In this thesis, we consider the so-called $1/2\langle 111\rangle$ screw dislocation in a body-centred cubic lattice. In that case, the tangential vector $\vec{u}$ of the dislocation line has the direction of a body diagonal of the cubes. We choose a slip plane as shown in Figure 2 and introduce an appropriate coordinate system. The motion of screw dislocations in a slip plane is thermally activated – they move due to the applied load and this motion is aided by thermal fluctuations. The dislocation first moves by the applied shear stress τ as a straight line from $y = 0$ to $y = y_0$, where the value of y_0 is given by the relation $\Gamma'(y_0) = \tau b$ (see Figure 3).

Here Γ denotes the so-called Peierls barrier representing lattice friction that acts against moving the dislocation. From the straight *initiated shape*, the dislocation vibrates due to the finite thermal energy and reaches its *activated shape* (see Figure 3). This activated shape of the dislocation determines the the activation enthalpy for the motion of the dislocation under the applied stress τ.

In the paper [1], the following relation is derived for the enthalpy corresponding to the shape of the dislocation $y = y(x)$:

$$H_\tau(y) = \int\limits_{-\infty}^{+\infty} \Big[\Gamma(y(x))\sqrt{1 + [y'(x)]^2} - \Gamma(y_0) - \tau b(y(x) - y_0)\Big]\,\mathrm{d}x.$$

†The problem was suggested by Roman Gröger from the Institute of Physics of Materials of the Czech Academy of Sciences (e-mail: `groger@ipm.cz`).

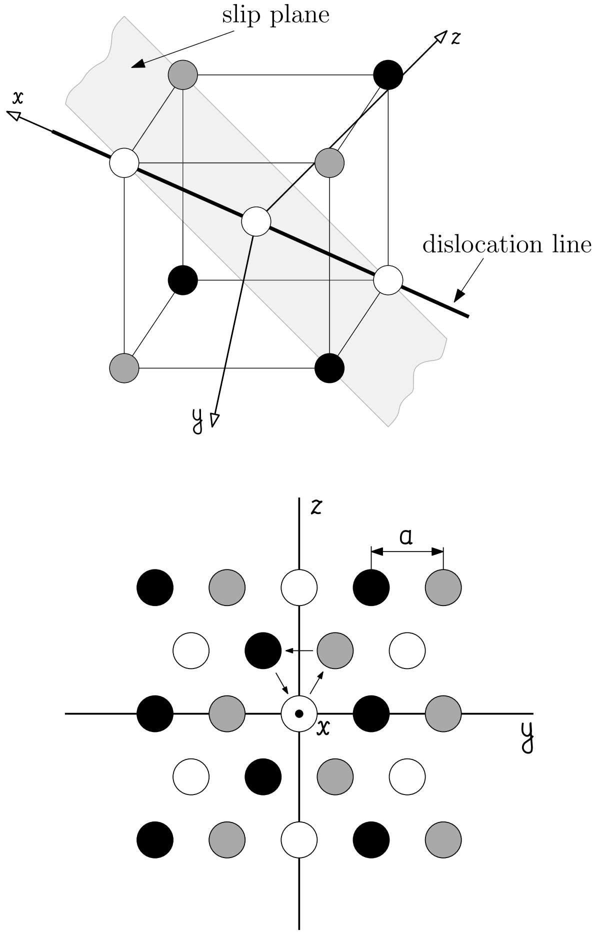

Figure 2. Coordinate system

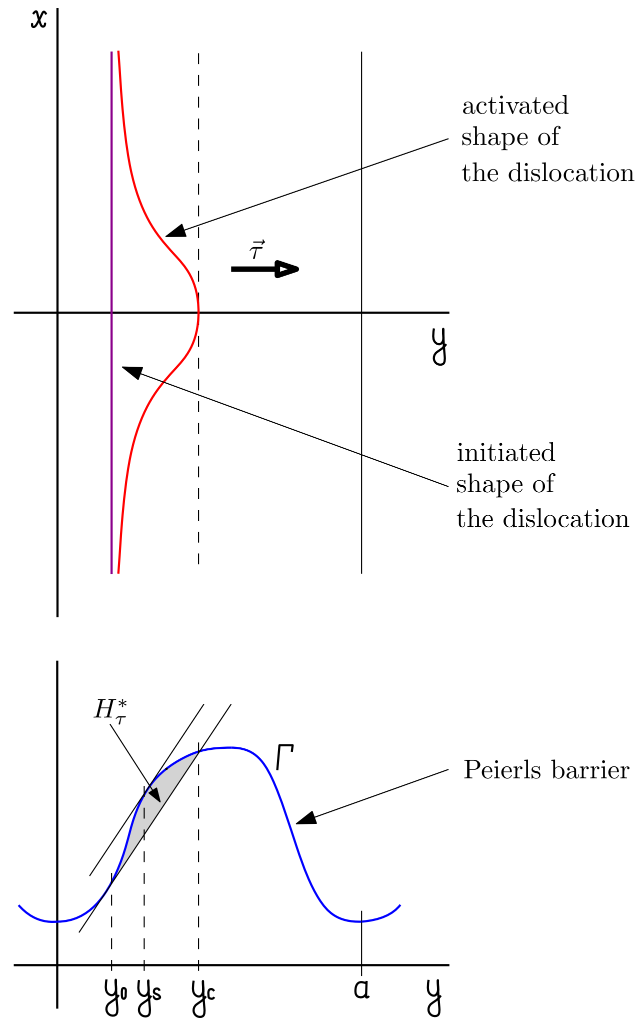

Figure 3. Peierls barrier

The first term under the integral sign corresponds to the energy of a curved dislocation, the second term deals with the energy of the straight dislocation, and the third term represents the work done by the stress τ on changing the shape from y_0 to y. We are looking for the shape of the dislocation $y = y(x)$ with fixed ends $y(\pm\infty) = y_0$, that corresponds to the minimum of the enthalpy H_τ. Such a shape of the dislocation is called *activated shape* and, as was mentioned above, it determines the value H^*_τ of the activation enthalpy for the motion of the dislocation under the given shear stress τ. Applying the Euler-Lagrange equation to the described variational problem leads to the boundary value problem

$$\frac{\Gamma(y)y''}{\sqrt{1+[y']^2}} = \Gamma'(y) - \tau b\sqrt{1+[y']^2}\,, \tag{1}$$

$$\lim_{x\to-\infty} y(x) = y_0, \quad \lim_{x\to-\infty} y(x) = y_0. \tag{2}$$

Hence, the *activated shape* of the dislocation can be mathematically described as a non-constant solution to the boundary value problem (1), (2). Recall that, in equation (1), τ is the share stress, b stands for the magnitude of the Burgers vector, and Γ denotes the Peierls barrier (see Figure 3).

Motivated by the shape of the Peierls barrier Γ discussed in [1], we introduce the assumption

$$\left.\begin{array}{c}\Gamma\in C^2(\mathbb{R};\,]0,+\infty[\,)\text{ is an }a\text{-periodic function,}\\ \text{there exists }\ 0<y_0<y_c<a\ \text{ such that}\\ \Gamma'(y_0)=\tau b,\quad \Gamma'(y_c)<\tau b,\\ \Gamma(y)>\Gamma(y_0)+\tau b(y-y_0)\ \text{ for }\ y\in[0,y_c[\,\setminus\{y_0\},\\ \Gamma(y)<\Gamma(y_0)+\tau b(y-y_0)\ \text{ for }\ y\in\,]y_c,a],\end{array}\right\}\qquad (A_1)$$

which allows one to prove the following theorem.

Theorem 1. *Let $a,b,\tau>0$ and the function Γ satisfy assumption (A_1). Then problem* (1), (2) *has a unique* (*up to a translation*) *non-constant solution.*

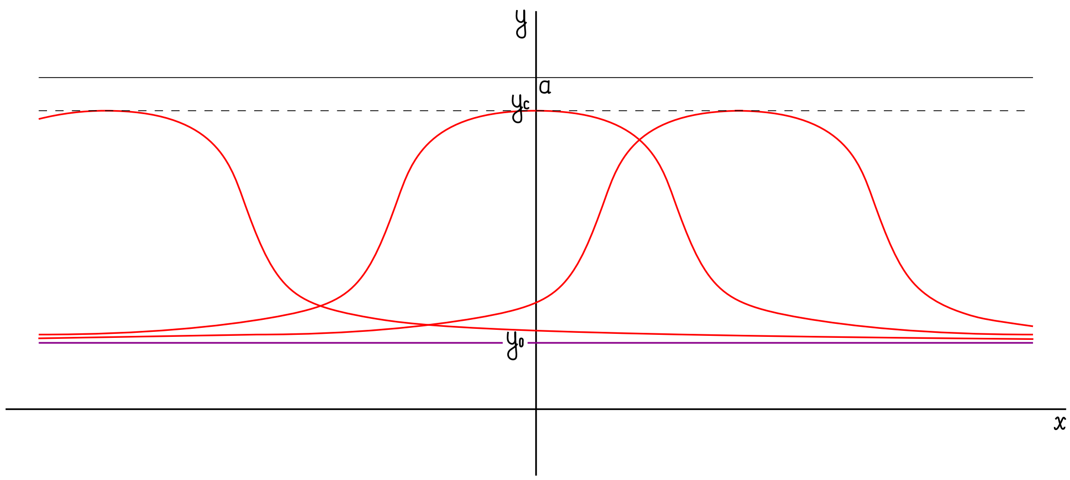

Figure 4. Solutions to problem (1), (2) – activated shape of the dislocation

Remark 2. It follows from the proof of Theorem 1 that each solution to problem (1), (2) is a solution to the Cauchy problem

$$\frac{\Gamma(y)y''}{\sqrt{1+[y']^2}}=\Gamma'(y)-\tau b\sqrt{1+[y']^2}\,;\quad y(0)=\alpha_1,\ \ y'(0)=\alpha_2$$

for some $\alpha_1\in\,]y_0,y_c]$, $k\in\{1,2\}$, and $\alpha_2=(-1)^k\sqrt{[\frac{\Gamma(\alpha_1)}{\Gamma(y_0)+\tau b(\alpha_1-y_0)}]^2-1}\,$.

From the mathematical point of view, it is interesting task to investigate a shape of each solution to equation (1). Assume that, in addition to (A_1), the Peierls barrier Γ satisfies the following condition

$$\text{there exists a unique }\ y_s\in\,]y_0,y_0+a[\ \text{ such that }\ \Gamma'(y_s)=\tau b. \qquad (A_2)$$

Then we can derive qualitative properties of all solutions to equation (1) and describe the phase portrait of (1) in detail, see Figures 5 and 6 on below.

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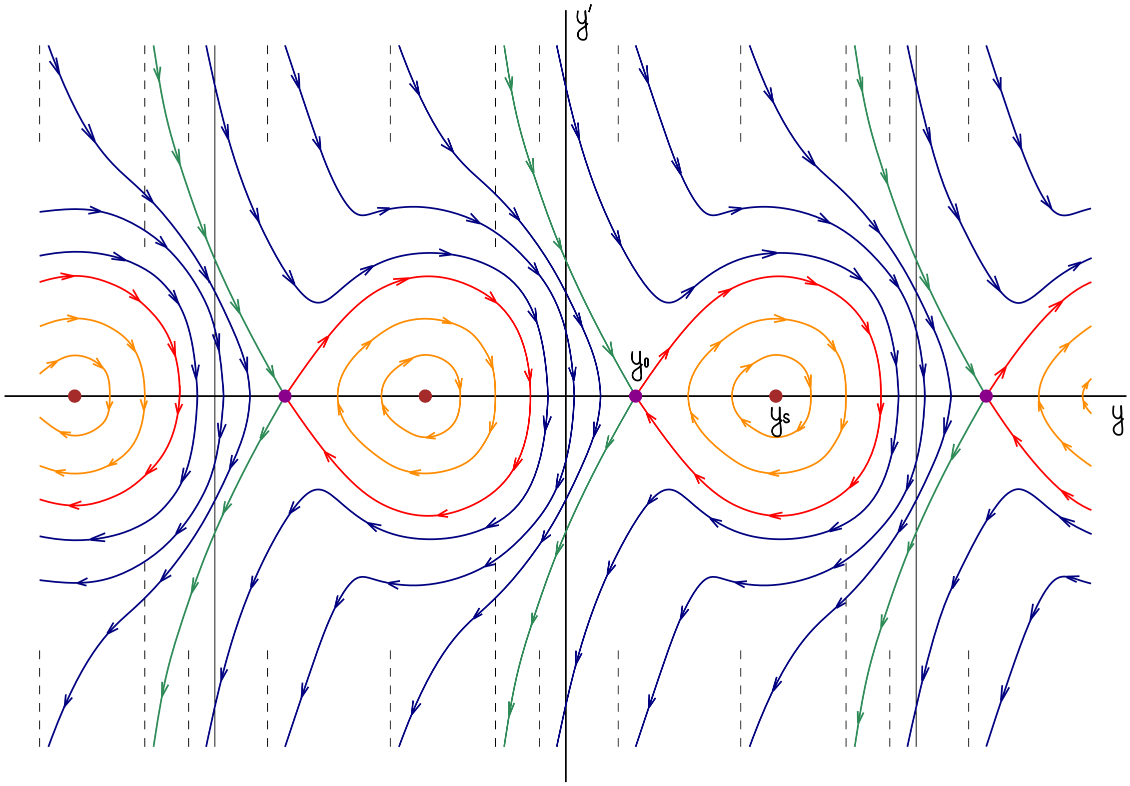

Figure 5. Phase portrait of equation (1)

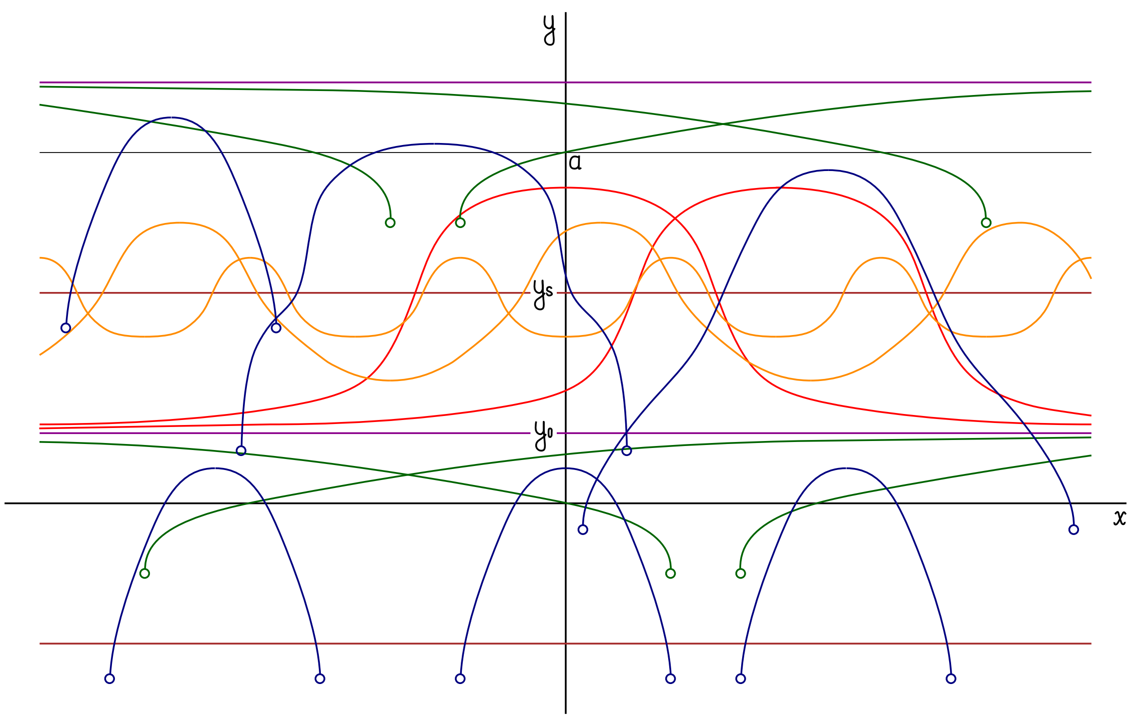

Figure 6. Graphs of various solutions to equation (1), colours of solutions correspond to colours of orbits in Fig. 5

On One Inverse Problem for the Linear Controlled Neutral Differential Equation

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Let $t_0 < t_1$ be fixed numbers and let $x_0 \in \mathbb{R}^n$ be a fixed vector. By Φ and Ω we denote, respectively, the sets of measurable initial functions $\varphi(t) = (\varphi^1(t), \dots, \varphi^n(t))^T$, $t \in [t_0 - \tau, t_0]$, $\varphi^i(t) \in [-1, 1]$, $i = \overline{1, n}$ and control functions $u(t) = (u^1(t), \dots, u^r(t))^T$, $t \in [t_0, t_1]$, $u^i(t) \in [-1, 1]$, $i = \overline{1, r}$.

To each element $w = (\varphi(t), g(t), u(t)) \in W = \Phi^2 \times \Omega$ we assign the linear neutral differential equation

$$\dot{x}(t) = Ax(t) + Bx(t - \tau) + C\dot{x}(t - \tau) + Du(t), \ \ t \in [t_0, t_1] \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), \quad \dot{x}(t) = g(t), \ \ t \in [t_0 - \tau, t_0), \ \ x(t_0) = x_0, \tag{2}$$

where A, B, C, D are given constant matrices with appropriate dimensions.

Definition 1. Let $w = (\varphi(t), g(t), u(t)) \in W$. A function $x(t) = x(t; w) \in \mathbb{R}^n, t \in [t_0 - \tau, t_1]$ is called a solution of differential equation (1) with the initial condition (2) if $x(t)$ satisfies the initial condition (2), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere.

The inverse problem: Let $y \in Y = \{y \in \mathbb{R}^n : \ \exists\, w \in W, \ x(t_1; w) = y\}$ be a given vector. Find element $w \in W$ such that the following condition holds $x(t_1; w) = y$. The vector y, as rule, by distinct error is beforehand given.Thus instead of the vector y we have $\widehat{y}$ (so called an observed vector) which is an approximation to the y and, in general, $\widehat{y} \notin Y$.Therefore it is natural to change posed inverse problem by the following approximate problem.

The approximate inverse problem: Find an element $w \in W$ such that the deviation

$$\frac{1}{2}\,|x(t_1; w) - \widehat{y}\,|^2 = \frac{1}{2} \sum_{i=1}^{n} \big[x^i(t_1; w) - \widehat{y}^{\,i}\big]^2$$

takes the minimal value.

It is clear that the approximate inverse problem is equivalent to the following optimization problem:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + C\dot{x}(t-\tau) + Du(t), \ \ t \in [t_0, t_1], \tag{3}$$

$$x(t) = \varphi(t), \ \ \dot{x}(t) = g(t), \ \ t \in [t_0 - \tau, t_0), \ \ x(t_0) = x_0, \tag{4}$$

$$J(w) = \frac{1}{2}\,|x(t_1; w) - \widehat{y}\,|^2 \longrightarrow \min, \ \ w \in W. \tag{5}$$

The problem (3)–(5) is called the optimal control problem corresponding to the inverse problem.

Theorem 1 ([4]). *There exists an optimal element $w_0 = (\varphi_0(t), g_0(t), u_0(t))$ for the problem* (3)–(5), *i.e.* $J(w_0) = \inf_{w\in W} J(w)$.

Regularization of the optimal control problem (3)–(5)**.** Now we consider the regularized optimal control problem

$$\dot{x}(t) = Ax + Bx(t-\tau) + C\dot{x}(t-\tau) + Du(t), \tag{6}$$

$$x(t) = \varphi(t), \ \ \dot{x}(t) = g(t), \ \ t \in [t_0 - \tau, t_0), \ \ x(t_0) = x_0, \tag{7}$$

$$\begin{aligned} J(w;\delta) = \frac{1}{2}\,|x(t_1; w) - \widehat{y}\,|^2 + \delta_1 \int_{t_0}^{t_1} \alpha(t)|\varphi(t-\tau)|^2\,dt \\ + \delta_2 \int_{t_0}^{t_1} \alpha(t)|g(t-\tau)|^2\,dt + \delta_3 \int_{t_0}^{t_1} |u(t)|^2\,dt \longrightarrow \min, \ \ w \in W, \end{aligned} \tag{8}$$

where $\delta = (\delta_1, \delta_2, \delta_3)$, $\delta_i > 0$, $i = 1, 2, 3$ and $\alpha(t)$ is the characteristic function of the interval $[t_0, t_0 + \tau]$.

Theorem 2. *For every δ the problem* (6)–(8) *has the unique optimal element $w_\delta = (\varphi_\delta(t), g_\delta(t), u_\delta(t))$ and*

$$\lim_{\delta\to 0} J(w_\delta; \delta) = J(w_0).$$

It is natural that for sufficiently small δ the element w_δ can be considered as an approximate optimal element of the problem (3)–(5) and consequently as an approximate solution of the approximate inverse problem.

Theorem 3. *For the optimality of an element w_δ it suffices to fulfill the conditions:*

$$\psi(t+\tau)B\varphi_\delta(t) - \delta_1|\varphi_\delta(t)|^2 = \max_{\varphi\in[-1,1]^n} \big[\psi(t+\tau)B\varphi - \delta_1|\varphi|^2\big], \ \ t \in [t_0 - \tau, t_0], \tag{9}$$

$$\psi(t+\tau)Cg_\delta(t) - \delta_2|g_\delta(t)|^2 = \max_{g\in[-1,1]^n} \big[\psi(t+\tau)Cg - \delta_2|g|^2\big], \ \ t \in [t_0 - \tau, t_0], \tag{10}$$

$$\psi(t)Du_\delta(t) - \delta_3|u_\delta(t)|^2 = \max_{u\in[-1,1]^r} \big[\psi(t)Du - \delta_3|u|^2\big], \ \ t \in [t_0, t_1]. \tag{11}$$

Here $\psi(t)$, in general, is discontinuous at points $t_1 - k\tau$, $k = 1, 2, \dots$ and $(\psi(t), \chi(t))$ is a solution of the system

$$\begin{cases} \dot{\chi}(t) = -\psi(t)A - \psi(t+\tau)B, \\ \psi(t) = \chi(t) + C\psi(t+\tau) \end{cases} \tag{12}$$

with the initial condition

$$\psi(t_1) = \chi(t_1) = \widehat{y} - x(t_1; w_\delta), \ \ \psi(t) = 0, \ \ t > t_1. \tag{13}$$

Let

$$\psi(t+\tau)B := (\varrho^1(t),\dots,\varrho^n(t)), \quad \psi(t+\tau)C := (\sigma^1(t),\dots,\sigma^n(t)),$$
$$\psi(t)D := (\gamma^1(t),\dots,\gamma^r(t)).$$

Using these notations, from (9)–(11), respectively, it follow

$$\begin{aligned}
\varrho^i(t)\varphi^i_\delta(t) - \delta_1(\varphi^i_\delta(t))^2 &= \max_{\varphi^i\in[-1,1]} \big[\varrho^i(t)\varphi^i - \delta_1(\varphi^i)^2\big], \ \ i=\overline{1,n},\\
\sigma^i(t)g^i_\delta(t) - \delta_2(g^i_\delta(t))^2 &= \max_{g^i\in[-1,1]} \big[\sigma^i(t)g^i - \delta_2(g^i)^2\big], \ \ i=\overline{1,n},\\
\gamma^i(t)u^i_\delta(t) - \delta_3(u^i_\delta(t))^2 &= \max_{u^i\in[-1,1]} \big[\gamma^i(t)u^i - \delta_3(u^i)^2\big], \ \ i=\overline{1,r}.
\end{aligned}$$

From the last relations we get

$$\varphi^i_\delta(t) = \begin{cases} -1 & \text{if } \dfrac{\varrho^i(t)}{2\delta_1} \le -1,\\[2mm] \dfrac{\varrho^i(t)}{2\delta_1} & \text{if } \dfrac{\varrho^i(t)}{2\delta_1} \in [-1,1],\\[2mm] 1 & \text{if } \dfrac{\varrho^i(t)}{2\delta_1} \ge 1, \end{cases} \quad g^i_\delta(t) = \begin{cases} -1 & \text{if } \dfrac{\sigma^i(t)}{2\delta_2} \le -1,\\[2mm] \dfrac{\sigma^i(t)}{2\delta_3} & \text{if } \dfrac{\sigma^i(t)}{2\delta_2} \in [-1,1],\\[2mm] 1 & \text{if } \dfrac{\sigma^i(t)}{2\delta_2} \ge 1, \end{cases}$$
$$u^i_\delta(t) = \begin{cases} -1 & \text{if } \dfrac{\gamma^i(t)}{2\delta_3} \le -1,\\[2mm] \dfrac{\gamma^i(t)}{2\delta_2} & \text{if } \dfrac{\gamma^i(t)}{2\delta_3} \in [-1,1],\\[2mm] 1 & \text{if } \dfrac{\gamma^i(t)}{2\delta_3} \ge 1. \end{cases}$$

Iterative process for the approximate solution of the regularization problem (6)–(8)**.** Let $\varphi_1(t)\in\Phi, g_1(t)\in\Phi$ and $u_1(t)\in\Omega$ be starting approximation of the initial functions and the control function. We construct the sequences $\{x_k(t)\}$, $\{\psi_k(t)\}$, $\{\varphi_k(t)\}$, $\{g_k(t)\}$, $\{u_k(t)\}$ by the following iteration process:

1) for given $\varphi_k(t), g_k(t)\in\Phi$ and $u_k(t)\in\Omega$ find $x_k(t)$: the solution of the differential equation

$$\dot x(t) = Ax(t) + Bx(t-\tau) + C\dot x(t-\tau) + Du_k(t), \ \ t\in[t_0,t_1]$$

with the initial condition

$$x(t)=\varphi_k(t), \dot x(t) = g_k(t), \ \ t\in[t-\tau,t_0), \ \ x(t_0)=x_0;$$

2) if a stopping criterion is satisfied stop, stopping criterion can be for example the value of $J(w_k;\delta)$ is less than before given number ε, where $w_k=(\varphi_k(t),g_k(t),u_k(t))$;

3) find $(\psi_k(t),\chi_k(t))$: the solution of the differential equation (12) with the initial condition

$$\psi(t_1)=\chi(t_1)=\widehat y - x(t_1;w_k)\psi(t)=0, \ \ t>t_1;$$

4) put $k := k + 1$ and find the next iterates $\varphi_{k+1}(t)$, $g_{k+1}(t)$ and $u_{k+1}(t)$

$$\varphi^i_{k+1}(t) = \begin{cases} -1 & \text{if } \dfrac{\varrho^i_k(t)}{2\delta_1} \le -1, \\ \dfrac{\varrho^i_k(t)}{2\delta_1} & \text{if } \dfrac{\varrho^i_k(t)}{2\delta_1} \in [-1,1], \\ 1 & \text{if } \dfrac{\varrho^i_k(t)}{2\delta_1} \ge 1, \end{cases} \quad g^i_{k+1}(t) = \begin{cases} -1 & \text{if } \dfrac{\sigma^i_k(t)}{2\delta_2} \le -1, \\ \dfrac{\sigma^i_k(t)}{2\delta_2} & \text{if } \dfrac{\sigma^i_k(t)}{2\delta_2} \in [-1,1], \\ 1 & \text{if } \dfrac{\sigma^i_k(t)}{2\delta_2} \ge 1, \end{cases}$$

$$u^i_{k+1}(t) = \begin{cases} -1 & \text{if } \dfrac{\gamma^i_k(t)}{2\delta_3} \le -1, \\ \dfrac{\gamma^i_k(t)}{2\delta_3} & \text{if } \dfrac{\gamma^i_k(t)}{2\delta_3} \in [-1,1], \\ 1 & \text{if } \dfrac{\gamma^i_k(t)}{2\delta_3} \ge 1. \end{cases}$$

Here

$$\psi_k(t+\tau)B := (\varrho^1_k(t), \dots, \varrho^n_k(t)), \quad \psi_k(t+\tau)C := (\sigma^1_k(t), \dots, \sigma^n_k(t)),$$
$$\psi_k(t)D := (\gamma^1_k(t), \dots, \gamma^r_k(t));$$

5) go to 1).

Theorem 4. *The following relations are valid:*

$$\lim_{k\to\infty} \chi_k(t) = \chi_\delta(t), \quad \lim_{k\to\infty} x_k(t) = x_\delta(t) \ \ \textit{uniformly for } \ t \in [t_0, t_1],$$
$$\lim_{k\to\infty} \sup_{t\in[t_0,t_1]} \psi_k(t) = \psi_\delta(t), \quad \lim_{k\to\infty} \varphi_k(t) = \varphi_\delta(t), \quad \lim_{k\to\infty} g_k(t) = g_\delta(t)$$

weekly in the space $L_1([t_0-\tau, t_0], \mathbb{R}^n)$, $\lim\limits_{k\to\infty} u_k(t) = u_\delta(t)$ *weekly in the space* $L_1([t_0, t_1], \mathbb{R}^r)$. *Moreover,* $w_\delta = (\varphi_\delta(t), g_\delta(t), u_\delta(t))$ *is the optimal element,* $x_\delta(t) = x(t; w_\delta)$, $(\psi_\delta(t), \chi_\delta(t))$ *is the solution of the equation* (12) *with the initial condition* (13).

Theorems 2–4 are proved on the basis of results obtained in [1–3].

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On Topological Classifications of Some Classes of Complex Differential Systems

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1 Covering foliations

The foliations theory began with works of H. Poincaré. It have began an independent scientific field and actually is consider as an efficient tool in the topological investigations. Here we consider foliations of a special type, referred to as covering foliations [5]. We will consider the problem of topological classification of covering foliations determinated by the complex linear differential systems and homogeneous projective matrix Riccati equations.

Definition 1.1. Let A and B be path connected smooth varieties of dimensions $\dim A = n$ and $\dim B = m$. Smooth foliation $\mathfrak{F}$ of dimension m on the variety $A \times B$, locally transversal to $A \times \{b\}$ for all $b \in B$, we will name ***a covering foliation***, if the projection $p\colon A \times B \to B$ on the second factor defines for each layer of it foliation covering of the variety B.

Definition 1.2. Let $\mathfrak{F}_c$ be a layer of the covering foliation $\mathfrak{F}$, containing the point $c \in A \times B$. ***The phase group*** $Ph(\mathfrak{F}, b_0)$, $b_0 \in B$, of the covering foliation $\mathfrak{F}$ we will name the group of the diffeomorphisms $\operatorname{Diff}(A, \pi_1(B, b_0))$ of the actions on the phase layer A by fundamental group $\pi_1(B, b_0)$ with noted point b_0, defined under formulae $\Phi^\gamma(a) = q \circ r \circ s$ for all $a \in A$, for all $\gamma \in \pi_1(B, b_0)$, where r is a lifting of one of ways $s(\tau) \subset B$ for all $\tau \in [0, 1]$, corresponding to the element γ of the group $\pi_1(B, b_0)$, on the layer $\mathfrak{F}_{(a,s(0))}$ of the covering foliation $\mathfrak{F}$ in the point $(a, s(0))$, and $q : A \times B \to A$ is a projection to the first factor.

It is easy to see that owing to path connectivity and smoothness of the variety B, then phase groups $Ph(\mathfrak{F}, b_1)$ and $Ph(\mathfrak{F}, b_2)$ are smoothly conjugated for any two points b_1 and b_2 of the base B. Therefore further we will speak simply about of ***the phase group*** $Ph(\mathfrak{F})$ of the covering foliation $\mathfrak{F}$, not connecting it with any point of the base B.

Definition 1.3. We will say that the covering foliation $\mathfrak{F}^1$ on the variety $A_1 \times B_1$ is ***topologically equivalent*** to the covering foliation $\mathfrak{F}^2$ on the variety $A_2 \times B_2$ if there exists the homeomorphism $h : A_1 \times B_1 \to A_2 \times B_2$ such that $q_2 \circ h(A_1 \times B_1) = A_2$, $h(\mathfrak{F}^1_{c_1}) = \mathfrak{F}^2_{h(c_1)}$ for all $c_1 \in A_1 \times B_1$, where q_2 is a projection to the first factor.

Definition 1.4. Let $\mathfrak{F}(\lambda)$ is a smooth family of covering foliations, $\mathfrak{F}(\lambda^0) = \mathfrak{F}$, $\lambda = (\lambda_1, \dots, \lambda_l)$. We will say that the covering foliations $\mathfrak{F}$ is ***structurally stable*** if for all enough small δ any covering foliation $\mathfrak{F}(\lambda)$ is topologically equivalent to it, where norm $\|\lambda - \lambda^0\| < \delta$.

Theorem 1.5. *For topological equivalence of the covering foliations $\mathfrak{F}^1$ and $\mathfrak{F}^2$ it is necessary and sufficient existence of the isomorphism μ of the fundamental groups $\pi_1(B_1)$ and $\pi_1(B_2)$, generated by the homeomorphism $g_\mu : B_1 \to B_2$ of the bases, and existence of the homeomorphism $f : A_1 \to A_2$ of phase layers such that $f \circ \Phi_1^{\gamma_1} = \Phi_2^{\mu(\gamma_1)} \circ f$ for all $\gamma_1 \in \pi_1(B_1)$, where $\Phi_\xi^{\gamma_\xi} \in Ph(\mathfrak{F}^\xi)$, $\gamma_\xi \in \pi_1(B_\xi)$, $\xi = 1, 2$.*

Concepts of smooth and real holomorphic equivalence of covering foliations are similarly introduced. Also corresponding analogues of Theorem 1.5 are similarly proved.

2 Complex nonautonomous linear differential systems

Consider the complex nonautonomous linear differential systems

$$dw = \sum_{j=1}^{m} A_j(z_1, \dots, z_m)w\,dz_j \tag{2.1}$$

and

$$dw = \sum_{j=1}^{m} B_j(z_1, \dots, z_m)w\,dz_j, \tag{2.2}$$

ordinary at $m = 1$ and completely solvable at $m > 1$, where $w = (w_1, \dots, w_n)$, square matrices $A_j(z_1, \dots, z_m) = \|a_{ikj}(z_1, \dots, z_m)\|$ and $B_j(z_1, \dots, z_m) = \|b_{ikj}(z_1, \dots, z_m)\|$ of the order n consist from holomorphic functions $a_{ikj} : A \to \mathbb{C}$ and $b_{ikj} : B \to \mathbb{C}$, $i = 1, \dots, n$, $k = 1, \dots, n$, $j = 1, \dots, m$, path connected holomorphic varieties A and B are holomorphically equivalent each other. The general solutions of systems (2.1) and (2.2) define covering foliations L^1 and L^2, accordingly, on the varieties $\mathbb{C}^n \times A$ and $\mathbb{C}^n \times B$. The phase group $Ph(L^1)$ of the covering foliation L^1 is generated by the forming nondegenerate linear transformations $P_r w$ for all $w \in \mathbb{C}^n$, $P_r \in GL(n, \mathbb{C})$, $r \in I$, and the phase group $Ph(L^2)$ of the covering foliation L^2 is generated by the forming nondegenerate linear transformations $Q_r w$ for all $w \in \mathbb{C}^n$, $Q_r \in GL(n, \mathbb{C})$, for all $r \in I$, where I is some set of indexes. Also the phase group $Ph(L^1)$ (the phase group $Ph(L^2)$) define the monodromy group of system (2.1) (system (2.2)). In the case $n = 1$, topological equivalence of the scalar equations (2.1) and (2.2) is studied in article [3]. Notice that it is a case integrated in quadratures. We will assume further $n > 1$.

Definition 2.1. A set $\{\lambda_1, \dots, \lambda_n\}$ of nonzero complex numbers we will name ***simple*** if $\lambda_k \setminus \lambda_l \notin s_{lk}^{\pm 1}$, $s_{lk} \in \mathbb{N}$, $l \neq k$, $k = 1, \dots, n$, $l = 1, \dots, n$, and a square matrix of the size $n > 1$ we will name ***simple*** if it has simple structure and simple collection of eigenvalues.

Theorem 2.2. *Let the matrices $P_r = S \operatorname{diag}\{p_{1r}, \dots, p_{nr}\}\, S^{-1}$, $Q_r = T \operatorname{diag}\{q_{1r}, \dots, q_{nr}\}\, T^{-1}$, and the matrixes $\ln P_r$ and $\ln Q_r$ be simple for all $r \in I$. Then for the topological equivalence of systems* (2.1) *and* (2.2) *it is necessary and sufficient existence of such permutations $\mu : I \to I$, $\varrho : (1, \dots, n) \to (1, \dots, n)$ and complex numbers α_k with $\operatorname{Re}\alpha_k > -1$, $k = 1, \dots, n$, that either $q_{\varrho(k)\mu(r)} = p_{kr}|p_{kr}|^{\alpha_k}$ for all $r \in I$, or $q_{\varrho(k)\mu(r)} = \overline{p}_{kr}|p_{kr}|^{\alpha_k}$ for all $r \in I$, $k = 1, \dots, n$.*

Theorem 2.3. *From a topological equivalence of systems* (2.1) *and* (2.2) *with the non-Abelian monodromy groups of general situation follows their real holomorphic equivalence.*

Theorem 2.4. *Systems* (2.1) *and* (2.2) *are smooth* (*real holomorphic*) *equivalent if and only if its monodromy groups are $\mathbb{R}$-linearly conjugated for some permutation $\mu : I \to I$.*

Theorem 2.5. *System* (2.1) *is structurally stable if and only if it monodromy group have one independent generator and the conditions of Theorem* 2.2 *are fulfilled for the matrix P_1.*

3 Complex nonautonomous homogeneous projective matrix Riccati equations

Consider the complex nonautonomous homogeneous projective matrix Riccati equations [5]

$$dv = \sum_{j=1}^{m} A_j(z_1, \dots, z_m)v\,dz_j \tag{3.1}$$

and
$$dv=\sum_{j=1}^{m}B_j(z_1,\dots,z_m)v\,dz_j. \tag{3.2}$$
ordinary at $m=1$ and completely solvable at $m>1$, where $v=(v_1,\dots,v_{n+1})$ are homogeneous coordinates, square matrices $A_j(z_1,\dots,z_m)=\|a_{ikj}(z_1,\dots,z_m)\|$ and $B_j(z_1,\dots,z_m)=\|b_{ikj}(z_1,\dots,z_m)\|$ of the order $n+1$ consist from holomorphic functions $a_{ikj}:A\to\mathbb{C}$ and $b_{ikj}:B\to\mathbb{C}$, $i=1,\dots,n+1$, $k=1,\dots,n+1$, $j=1,\dots,m$, path connected holomorphic varieties A and B are holomorphically equivalent each other. The general solutions of systems (3.1) and (3.2) define covering foliations PL^1 and PL^2, accordingly, on the varieties $\mathbb{C}P^n\times A$ and $\mathbb{C}P^n\times B$. The phase group $Ph(PL^1)$ of the covering foliation PL^1 is generated by the forming nondegenerate linear-fractional transformations P_rv for all $v\in\mathbb{C}P^n$, $P_r\in GL(n+1,\mathbb{C})$, $r\in I$, and the phase group $Ph(PL^2)$ of the covering foliation PL^2 is generated by the forming nondegenerate linear-fractional transformations Q_rv for all $v\in\mathbb{C}P^n$, $Q_r\in GL(n+1,\mathbb{C})$, for all $r\in I$, where I is some set of indexes. Also the phase group $Ph(L^1)$ (the phase group $Ph(L^2)$) define the holonomy group of system (3.1) (system (3.2)).

Theorem 3.1. *Let at $n=1$ the matrices $P_r=S\operatorname{diag}\{p_{1r},p_{2r}\}S^{-1}$ for all $r\in I$, $Q_r=T\operatorname{diag}\{q_{1r},q_{2r}\}T^{-1}$ for all $r\in I$. Then for the topological equivalence of systems* (3.1) *and* (3.2) *it is necessary and sufficient existence of such permutation $\mu:I\to I$ and complex number α with $\operatorname{Re}\alpha\neq-1$ that either*
$$\frac{q_{1r}}{q_{2r}}=\frac{p_{1r}}{p_{2r}}\Big|\frac{p_{1r}}{p_{2r}}\Big|^{\alpha}\ \text{ for all } r\in I,$$
or
$$\frac{q_{1r}}{q_{2r}}=\frac{\overline{p}_{1r}}{\overline{p}_{2r}}\Big|\frac{p_{1r}}{p_{2r}}\Big|^{\alpha}\ \text{ for all } r\in I.$$

Theorem 3.2. *Let the matrices $P_r=S\operatorname{diag}\{p_{1r},\dots,p_{n+1,r}\}S^{-1}$, $Q_r=T\operatorname{diag}\{q_{1r},\dots,q_{n+1,r}\}T^{-1}$, sets of numbers $\big\{\ln\frac{p_{1r}}{p_{n+1,r}},\dots,\ln\frac{p_{nr}}{p_{n+1,r}}\big\}$ and $\big\{\ln\frac{q_{1r}}{q_{n+1,r}},\dots,\ln\frac{q_{nr}}{q_{n+1,r}}\big\}$ are simple, for all $r\in I$. Then for the topological equivalence of systems* (3.1) *and* (3.2) *it is necessary and sufficient existence of such permutations $\mu:I\to I$, $\varrho:(1,\dots,n+1)\to(1,\dots,n+1)$ and complex number α with* $\operatorname{Re}\alpha>-1$, *that either*
$$\frac{q_{\varrho(k)\mu(r)}}{q_{\varrho(n+1)\mu(r)}}=\frac{p_{kr}}{p_{n+1,r}}\Big|\frac{p_{kr}}{p_{n+1,r}}\Big|^{\alpha}\ \text{ for all } r\in I,\ \ k=1,\dots,n,$$
or
$$\frac{q_{\varrho(k)\mu(r)}}{q_{\varrho(n+1)\mu(r)}}=\frac{\overline{p}_{kr}}{\overline{p}_{n+1,r}}\Big|\frac{p_{kr}}{p_{n+1,r}}\Big|^{\alpha}\ \text{ for all } r\in I,\ \ k=1,\dots,n.$$

Theorem 3.3. *From a topological equivalence of systems* (3.1) *and* (3.2) *with the non-Abelian holonomy groups of general situation follows their real holomorphic equivalence.*

Theorem 3.4. *Systems* (3.1) *and* (3.2) *are smooth* (*real holomorphic*) *equivalent if and only if its holonomy groups are conjugated either by linear-fractional transformation or by antiholomorphic linear-fractional transformation for some permutation $\mu:I\to I$.*

Theorem 3.5. *System* (3.1) *is structurally stable if and only if $n=1$, it holonomy group have one independent generator and $|p_{11}p_{21}^{-1}|\neq1$.*

4 Complex autonomous linear differential systems

At first we will consider complex completely solvable [2] (at $m > 1$) nondegenerate [4] linear discrete dynamic systems (L^1) and (L^2), defined by linear maps $A_j w$ for all $w \in \mathbb{C}^n$, $j = 1, \dots, m$, and $B_j w$ for all $w \in \mathbb{C}^n$, $j = 1, \dots, m$, accordingly, where $n > 1$, $1 < m < n-1$, $A_j \in GL(n, \mathbb{C})$ and $B_j \in GL(n, \mathbb{C})$, $j = 1, \dots, m$, origin O of space $\mathbb{C}^n$ is a unique fixed point of each of these systems.

Definition 4.1. Systems (L^1) and (L^2) we will name ***topologically equivalent*** if there exists the homeomorphism $h : \mathbb{C}^n \to \mathbb{C}^n$, translating the layers of the foliation, organised by basis of nondegenerate absolute invariants [4] of system (L^1), into the layers of the foliation, organised by basis of nondegenerate absolute invariants of system (L^2).

In article [6] the criterion of topological equivalence of systems (L^1) and (L^2) of general situation has been obtained. Completely solvable linear discrete dynamic system (L^1) is put in the flow

$$\exp\Big(\sum_{j=1}^{m} z_j \ln A_j\Big) w \ \text{ for all } \ w \in \mathbb{C}^n,$$

defined by the completely solvable autonomous linear differential system

$$dw = \sum_{j=1}^{m} \ln A_j w \, dz_j. \tag{4.1}$$

Therefore on the basis of results of article [6] it is possible to realize topological classification of the autonomous linear differential system (4.1) of general situation.

Notice that topological classification of ordinary system (4.1) (i.e. at $m = 1$) of general situation has been realize in articles [3] and [1].

5 Complex autonomous homogeneous projective matrix Riccati equations

At first we will consider complex completely solvable (at $m > 1$) nondegenerate linear-fractional discrete dynamic systems (PL^1) and (PL^2), defined by linear-fractional maps $A_j v$ for all $v \in \mathbb{C}P^n$, $j = 1, \dots, m$, and $B_j v$ for all $v \in \mathbb{C}P^n$, $j = 1, \dots, m$, accordingly, where $n > 1$, $1 < m < n-1$, $A_j \in GL(n+1, \mathbb{C})$ and $B_j \in GL(n+1, \mathbb{C})$, $j = 1, \dots, m$, each of these systems has exactly $n+1$ fixed points on $\mathbb{C}P^n$.

Definition 5.1. Systems (PL^1) and (PL^2) we will name ***topologically equivalent*** if there exists the homeomorphism $h : \mathbb{C}P^n \to \mathbb{C}P^n$, translating the layers of the foliation, organised by basis of nondegenerate absolute invariants of system (PL^1), into the layers of the foliation, organised by basis of nondegenerate absolute invariants of system (PL^2).

In article [6] the criterion of topological equivalence of systems (PL^1) and (PL^2) of general situation has been obtained. Completely solvable linear discrete dynamic system (PL^1) is put in the flow

$$\exp\Big(\sum_{j=1}^{m} z_j \ln A_j\Big) v \ \text{ for all } \ v \in \mathbb{C}P^n,$$

defined by the completely solvable autonomous homogeneous projective matrix Riccati equation

$$dv = \sum_{j=1}^{m} \ln A_j v \, dz_j. \tag{5.1}$$

Therefore on the basis of results of article [6] it is possible to realize topological classification of the autonomous linear differential system (5.1) of general situation.

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On Relations Between Perron and Grobman Regularity Coefficients of Parametric Linear Differential Equations

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For any $n \in \mathbb{N}$ we consider the linear system of differential equations

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1}$$

with a continuous coefficient $n \times n$ matrix uniformly bounded on the time half-line. Along with system (1), consider the adjoint system

$$\dot{y} = -A^{\mathrm{T}}(t)y, \quad y \in \mathbb{R}^n, \quad t \geq 0. \tag{2}$$

Obviously, the adjoint to system (2) is system (1); therefore, systems (1) and (2) are said to be mutually adjoint. Everywhere below, we identify system (1) with its coefficient matrix.

The so-called Perron and Lyapunov regularity coefficients $\sigma_P(A)$ and $\sigma_L(A)$, respectively, defined for each system (1) play an important role in the asymptotic theory of linear differential systems [3,4]. They essentially specify the response of system (1) to linear exponentially decreasing perturbations and nonlinear perturbations of a higher smallness order; in particular, the vanishing of at least one (and hence both) of them is equivalent to the Lyapunov regularity of system (1).

Let $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ be the Lyapunov exponents of system (1) arranged in nondescending order, and let $\mu_1(A) \geq \cdots \geq \mu_n(A)$ be the Lyapunov exponents of the adjoint system (2) arranged in nonascending order. By Sp we denote the trace of a matrix. Then, by definition,

$$\sigma_P(A) = \max_{1 \leq i \leq n} \big\{\lambda_i(A) + \mu_i(A)\big\} \ \text{ and } \ \sigma_L(A) = \sum_{i=1}^{n} \lambda_i(A) - \varliminf_{t \to +\infty} \frac{1}{t} \int_0^t \operatorname{Sp} A(\tau)\, d\tau.$$

It was shown in the monograph [2, § 1] that the regularity coefficients of any n-dimensional system (1) satisfy the inequalities

$$0 \leq \sigma_P(A) \leq \sigma_L(A) \leq n\sigma_P(A). \tag{3}$$

In the paper [5], it has been shown that inequalities (3) describe all possible relations between the regularity coefficients of differential systems. In other words, it was shown that for any positive integer n and ordered pair of numbers $(p;\ell)$ satisfying the inequalities $0 \leq p \leq \ell \leq np$, there exists a system A such that $\sigma_P(A) = p$ and $\sigma_L(A) = \ell$.

Let M be a metric space. Along with the individual system (1) we consider a family of linear differential systems

$$\dot{x} = A(t,\xi)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{4}$$

such that for every $\xi \in M$ the matrix-valued function $A(\,\cdot\,,\xi)\colon [0,+\infty) \to \mathbb{R}^{n\times n}$ is continuous and uniformly bounded on the time half-line, i.e. there exists $a_\xi \in \mathbb{R}$ such that $\sup\limits_{t\in[0,+\infty)} \|A(t,\xi)\| \leq a_\xi$.
Moreover, we suppose that the family of matrix-valued functions $A(\,\cdot\,,\xi)$, $\xi \in M$, is continuous in

compact-open topology, in other words, if a sequence $(\xi_k)_{k\in\mathbb{N}}$, $\xi_k \in M$, converges to ξ_0, then the sequence of functions $A(\,\cdot\,,\xi_k)$ converges to $A(\,\cdot\,,\xi_0)$ uniformly on every interval of $[0,+\infty)$. For a symbol $\varkappa \in \{P,L\}$ by $\sigma^A_\varkappa(\,\cdot\,)\colon M \to \mathbb{R}$ we denote a function acting by the rule $\xi \mapsto \sigma_\varkappa(A(\,\cdot\,,\xi))$. In a natural way a problem of complete description of pair $(\sigma^A_P(\,\cdot\,),\sigma^A_L(\,\cdot\,))$ arises. First we need introduce some notation to formulate a solution of this problem.

Let $f(\,\cdot\,)$ be a real-valued function defined on some set M. For a number $r \in \mathbb{R}$ and for the function $f(\,\cdot\,)$ the Lebesgue set $[f \geq r]$ is defined as the set $[f \geq r] \overset{\text{def}}{=} \{t \in M\colon\ f(t) \geq r\}$. If M is a topological space then G_δ stands for a system of subsets in M which can be represented as countable intersections of open sets. We say [1, pp. 223–224] that a function $f(\,\cdot\,)\colon M \to \mathbb{R}$ belongs to the class $({}^*,G_\delta)$, or $f(\,\cdot\,)$ is a function of the class $({}^*,G_\delta)$ if its Lebesgue set satisfies the condition $[f \geq r] \in G_\delta$ for any $r \in \mathbb{R}$.

Theorem. *For functions $p(\,\cdot\,),\ell(\,\cdot\,)\colon M \to \mathbb{R}$ there exists a parametric system* (4) *such that $\sigma^A_P(\,\cdot\,) \equiv p(\,\cdot\,)$ and $\sigma^A_L(\,\cdot\,) \equiv \ell(\,\cdot\,)$ if and only if $p(\,\cdot\,)$, $\ell(\,\cdot\,)$ are functions of the class $({}^*,G_\delta)$ and for every $\xi \in M$ the following inequalities*

$$0 \leq p(\xi) \leq \ell(\xi) \leq np(\xi)$$

hold.

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On the Sets of Lower Semicontinuity Points and Upper Semicontinuity Points of Topological Entropy with Continuous Dependence on a Parameter

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1 Statement of the problems

Let us present definitions needed in what follows. Let X be a compact metric space with the metric d. Take a continuous mapping $f : X \to X$. By $f^{\circ n}$ we denote the n-th iteration of f, i.e.,

$$f^{\circ n} = \underbrace{f \circ \cdots \circ f}_{n}, \quad n = 0, 1, 2, \dots;$$

$f^{\circ 0} \equiv \mathrm{id}$ by the definition. Along with the original metric d, we introduce a nondecreasing sequence $(d_n^f)_{n\in\mathbb{N}}$ of metrics on X defined by the equality

$$d_n^f(x, y) = \max_{0\le i\le n-1} d\big(f^{\circ i}(x), f^{\circ i}(y)\big), \quad n \in \mathbb{N}, \quad x, y \in X.$$

By $B_f(x, \varepsilon, n)$ we denote the open ball with the center x and radius ε in the metric d_n^f, i.e.,

$$B_f(x, \varepsilon, n) = \big\{y \in X : \ d_n^f(x, y) < \varepsilon\big\}.$$

A set $E \subset X$ is called an (f, ε, n)-cover if

$$X \subset \bigcup_{x\in E} B_f(x, \varepsilon, n).$$

For each (f, ε, n)-cover we find the number of its elements; let $S_d(f, \varepsilon, n)$ be the least of these numbers. The *topological entropy* of the dynamical system generated by a continuous mapping f is defined as follows [1]:

$$h_{\mathrm{top}}(f) = \lim_{\varepsilon\to 0} \varlimsup_{n\to\infty} \frac{1}{n} \ln S_d(f, \varepsilon, n). \tag{1.1}$$

Note that the topological entropy is independent of the choice of a metric generating the given topology on X and hence is well defined by (1.1).

Given a metric space $\mathcal{M}$ and a jointly continuous map

$$f : \mathcal{M} \times X \to X \tag{1.2}$$

we define the function

$$\mu \longmapsto h_{\mathrm{top}}(f_\mu(\,\cdot\,)). \tag{1.3}$$

It was proved in [3] that, in the case of $X = [0;1]$, the function (1.3) is lower semicontinuous. In the general case (for arbitrary X), the function (1.3) is not necessarily lower semicontinuous. For example, consider the family of maps $f_\mu : X_1 \to X_1$, where

$$X_1 = \big\{z \in \mathbf{C} :\ |z| \leqslant 1\big\}, \quad f_\mu(z) = \begin{cases} 0 & \text{if } z = 0, \\ \mu \, \dfrac{z^2}{|z|} & \text{if } z \neq 0, \end{cases} \qquad \mu \in [0;1].$$

Take a $\mu \in [0;1)$ and an $\varepsilon > 0$. There exists a positive integer $n(\mu,\varepsilon)$ such that

$$d\big(f_\mu^i(z), f_\mu^i(w)\big) \leqslant d\big(f_\mu^i(z), 0) + d(0, f_\mu^i(w)\big) \leqslant 2\mu^i < \varepsilon,$$

for any $i \geqslant n(\mu,\varepsilon)$ and any points $z,w \in X$; therefore, for any positive integer $n \geqslant n(\mu,\varepsilon)$ we have

$$d_n^{f_\mu}(z,w) = \max_{0 \leqslant i \leqslant n-1} d\big(f_\mu^i(z), f_\mu^i(w)\big) \leqslant \max\big\{d_{n(\mu,\varepsilon)}^{f_\mu}(z,w), \varepsilon\big\}.$$

Hence if $n \geqslant n(\mu,\varepsilon)$, then

$$S_d(f_\mu, \varepsilon, n) \leqslant S_d(f_\mu, \varepsilon, n(\mu,\varepsilon)).$$

It follows that

$$0 \leqslant h_{\mathrm{top}}(f_\mu) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \, \frac{1}{n} \, \ln S_d(f_\mu, \varepsilon, n) \leqslant \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \, \frac{1}{n} \, \ln S_d(f_\mu, \varepsilon, n(\mu,\varepsilon)) = 0.$$

Thus, for $\mu \in [0;1)$ we have $h_{\mathrm{top}}(f_\mu) = 0$.

For each positive integer $k \geqslant 4$, we set

$$\varepsilon_k = \sqrt{2\Big(1 - \cos\Big(\frac{2\pi}{2^k}\Big)\Big)}\,.$$

Given a positive integer $n \geqslant 4$, consider the set

$$\mathcal{Z} = \Big\{z_m = \exp\Big(\frac{2\pi m i}{2^{k+n}}\Big)\Big\}, \quad m = 0, \dots, 2^{k+n} - 1.$$

If the distance between two points z_p and z_q of $\mathcal{Z}$ satisfies the inequality $d(z_p, z_q) \geqslant \varepsilon_k$, then $d_n^{f_1}(z_p, z_q) \geqslant \varepsilon_k$, and if the distance between z_p and z_q satisfies the inequality $d(z_p, z_q) < \varepsilon_k$, then there exists an $l \leqslant n - 1$ such that

$$d_n^{f_1}(z_p, z_q) \geqslant d\big(f_1^l(z_p), f_1^l(z_q)\big) \geqslant \varepsilon_k.$$

Thus, for any two points of $\mathcal{Z}$ we have $d_n^{f_1}(z_p, z_q) \geqslant \varepsilon_k$. This implies

$$S_d(f_1, \varphi_0, \varepsilon_k, n) \geqslant 2^{k+n},$$

whence

$$h_{\mathrm{top}}(f_1) = \lim_{k \to \infty} \overline{\lim_{n \to \infty}} \, \frac{1}{n} \, \ln S_d(f_1, \varepsilon_k, n) \geqslant \ln 2.$$

Thus, the function $\mu \mapsto h_{\mathrm{top}}(f_\mu)$ is discontinuous at $\mu = 1$. Moreover, it is not lower semicontinuous at $\mu = 1$.

In the present paper we study the sets of upper semicontinuity and lower semicontinuity points of the function (1.3).

2 The typicality of the lower semicontinuity of topological entropy

Theorem 2.1. *If $\mathcal{M}$ is a complete metric space, then for any map* (1.2)*, the set of lower semicontinuity points of the function* (1.3) *is everywhere dense G_δ-set in the space $\mathcal{M}$.*

Consider the Baire space $\mathfrak{B}$

$$\mathfrak{B} = \big\{x = (x_1, x_2, \dots):\ x_k \in \{0,1\},\ k \in \mathbb{N}\big\}$$

of 0-1-sequences with the metric defined by the formula

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-\min\{k:\, x_k \neq y_k\}} & \text{if } x \neq y. \end{cases}$$

Then the metric space $\mathfrak{B}$ is compact.

Theorem 2.2. *Let $\mathcal{M} = X = \mathfrak{B}$, then for the map*

$$f\big((\mu_1, \mu_2, \dots), (x_1, x_2, \dots)\big) = (x_{1+\mu_1}, x_{2+\mu_2}, \dots)$$

the set of lower semicontinuity points of the function (1.3) *is not an F_σ-set in the space $\mathcal{M}$.*

Let $C(\mathfrak{B}, \mathfrak{B})$ be the space of continuous mappings of $\mathfrak{B}$ into $\mathfrak{B}$ with the metric

$$\varrho(f,g) = \max_{x \in \mathfrak{B}} d(f(x), g(x)).$$

Block [2] found that topological entropy is discontinuous at every point in space $C(\mathfrak{B}, \mathfrak{B})$.

Theorem 2.3. *The set of zeros of the function*

$$h_{\text{top}} : C(\mathfrak{B}, \mathfrak{B}) \to [0, +\infty) \tag{2.1}$$

coincides with the set of its lower semicontinuity points.

From Theorem 2.1 it follows that the set of zeros of the function (2.1) is an everywhere dense G_δ-set in the space $C(\mathfrak{B}, \mathfrak{B})$.

3 Emptiness of the set of upper semicontinuity points of topological entropy

Yomdin [5] and Newhouse [4] proved that the topological entropy of C^∞-diffeomorphisms on a compact Riemannian manifold is upper semicontinuous.

Theorem 3.1. *For any map* (1.2)*, the set of upper semicontinuity points of the function* (1.3) *is an $F_{\sigma\delta}$-set in the space $\mathcal{M}$.*

Theorem 3.2. *Let $\mathcal{M} = X = \mathfrak{B}$, then there exists a map* (1.2) *such that the set of upper semicontinuity points of the function* (1.3) *is empty.*

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