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Editorial

On the occasion of Andro Bitsadze's 100th birthday anniversary (May 22, 1916–September 6, 1994)

This issue is dedicated to the 100 birthday anniversary of the outstanding Georgian mathematician, Corresponding member of the USSR Academy of Sciences, academician of the Georgian National Academy of Sciences, professor Andro Bitsadze.

His scientific activity is so diverse that it is impossible to describe it in a full measure. It seems reasonable to divide it into several stages keeping within the chronology. We will dwell only on those results that produced a broadest resonance in the mathematical society and determined development of many fields of mathematics in the works of subsequent generations.

Elliptic equations and systems together with the problems posed for them take an important place in A. Bitsadze's activity. The fact that the condition of uniform ellipticity of a linear equation or a system ensures fredholmity of the boundary value problem in the given domain, in particular, of the first boundary value problem, was assumed formerly indisputable. Irregularity of that fact has been illustrated by A. Bitsadze by a simple and clear for everyone example of elliptic system called later on Bitsadze's system.

As it turned out, the homogeneous Dirichlet problem for Bitsadze's system in an arbitrary circle of arbitrarily small radius has an infinite set of linearly independent solutions, thus the formula of their representation, containing an arbitrary analytic function has been obtained.

This fact seemed at that time unexpected and almost unbelievable, became a subject of discussions for many mathematicians trying to explain this phenomenon. At his known seminar, I. Gelfand made an attempt to explain this fact by multiplicity of characteristic roots of the system. In reply, A. Bitsadze has constructed an elliptic system with simple characteristic roots for which the Dirichlet problem in an arbitrary circle of arbitrarily small radius had likewise an infinite set of linearly independent solutions. On the basis of those simple and refined examples, the theory of boundary value problems for elliptic systems has acquired a great deal of new trends. The widely known theory of nonfredholm boundary value problems is one of those trends.

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Afterwards, there arose the natural question to single out the classes of elliptic systems with solvable, in a certain sense, boundary value problems, in particular, in the Fredholm, Noether, or Hausdorff sense. In this direction, it should be noted that for elliptic systems constructed by A. Bitsadze, the Dirichlet problem is normally Hausdorff solvable, if and only if the function, mapping conformally the given domain onto the circle, is rational. A. Bitsadze has singled out a class of elliptic second order systems that are at present called weakly connected in Bitsadze's sense for which the Dirichlet problem is always Fredholmian one.

It was assumed that the solvability of the boundary value problems is determined only by the principal part of the system. A. Bitsadze expressed somewhat different opinion that coefficients of the system with lower order derivatives may significantly affect the solvability of the problem. Indeed, as it became clear, the normally solvable in one or another sense boundary value problems for elliptic systems with Bitsadze's operator in the principal part may turn out to be unsolvable normally on adding lower order terms, or vice versa. Taking the above-mentioned facts into account, he introduced the notion of strongly connected elliptic systems whose particular cases are constructed by him for systems for which the same facts dealing with the normal solvability of the problem as for these particular cases of elliptic systems, are valid.

The above-mentioned fundamental effects were discovered by A. Bitsadze by using the apparatus of the theory of functions of a complex variable. In the development of boundary value problems for the second order elliptic systems the leading part belongs to the formula obtained by him for representation of a general solution of those systems.

The instrument of the theory of analytic functions and of one-dimensional singular integral equations allows one to investigate many boundary value problems with two independent variables. If there are more than two variables, there arise considerable difficulties due to the absence of a complete theory of multidimensional singular integral equations. Using the multidimensional analogue of the Sokhotski–Plemelj's theorem, A. Bitsadze has studied the first boundary value problem for the well-known Moisel–Theodorescu's system, reduced it to a multidimensional system of singular integral equations with a special matrix kernel and constructed the formula of its inversion which is called in literature "Bitsadze's inversion formula".

Among the problems formulated for multidimensional elliptic equations and systems, the problem with an oblique derivative is regarded as one of the basic ones. As far back as in G. Giraud's works, it has been shown that if the direction of the oblique derivative does not meet the tangent plane, then the problem will be solvable in Fredholm's sense. Otherwise, the situation changes insomuch that many researches were inclined to regard this problem atypical for elliptic equations. Considering just these nonstandard cases, A. Bitsadze has shown this problem not at all exceeds the bounds of typical problems and proved the theorems on a number and on the existence of solutions. As it became clear, the problem with an oblique derivative might turn out to be simultaneously underdetermined and overdetermined. To make the problem well-posed, it is necessary in some cases (proceeding from the structure of interconnections between the vector fields of an oblique derivative and the domain) to require that the supplementary boundary conditions at some points of the boundary be fulfilled. Those results were found interesting and earned great attention of specialists, as a consequence there appeared many important research works carried out by A. Bitsadze's disciples.

The objects of A. Bitsadze's investigations are always not ordinary. He studied the problems which are, as a rule, not subjected to standard conditions ensuring the existence and uniqueness of solutions. Such problems may be related to those suggested by A. Bitsadze for elliptic equations with parabolic degeneration with weighted conditions on the boundary. These problems were dictated owing to their practical necessity, and the condition of uniform ellipticity violates in such problems. They degenerate parabolically on the whole boundary, or on its certain part only.

Academician M. Keldysh restricted himself to the boundedness of solutions on the above-mentioned sets, releasing them from any kind of boundary conditions. A. Bitsadze replaced the requirement of boundedness of solutions by the weighted boundary conditions on the above-mentioned sets, taking thus into account the unbounded solutions, as well. These problems have brought to light new practical and theoretical validity of weighted functional spaces that before and after formulation of those problems have become the subject of a great number of research works.

The hyperbolic equations and systems are not less rich with the effects connected with parabolic degeneration. Many factors affect the solvability of the problems formulated here; they include an order of parabolic degeneration, orientation of a set of degeneration points with respect to the characteristic manifolds, etc. As distinct from a separately taken equation, hyperbolic systems show a lot of unexpected properties even without parabolic degeneration. Thus, for example, the well-known Goursat problem for a scalar equation is perfectly well-posed.

The hyperbolic system constructed by A. Bitsadze has shown that the corresponding Goursat homogeneous problem may have an infinite set of linearly independent solutions, and what is more, he has shown that the lower order terms of the system may significantly affect the well-posedness of that problem. Those facts have given a great impetus to many important researches and stimulated development of a series of scientific trends.

In the middle of the past century, mathematics has found new significant applications that should, seemingly, be explained by an unprecedented rate of technical progress. Achievement of transonic and supersonic velocities posed many problems including those of mixed type equations in which M. Lavrent'ev has shown special interest and awoken it in A. Bitsadze. Combining the methods of the theory of analytic functions, partial differential equations and singular integral equations, A. Bitsadze created a powerful and, at the same time, elegant apparatus, convenient for solving the problems formulated for the mixed type equations. Efficiency of the suggested method has been tested on the boundary value problems for Lavrent'ev–Bitsadze's equation, being the model of the well-known Tricomi's equations for which A. Bitsadze posed a great number of actual problems and stated significant facts known as "Bitsadzian facts". Here we will mention only the Bitsadze's extremum principle. For the Tricomi's equation, along with the Tricomi's problem, researchers studied also the Dirichlet problem expecting its solvability. This was needed, mainly, for a practical, concrete purpose.

A. Bitsadze has shown that this problem was not always well-posed, and for its solvability, it is necessary to release some part of the boundary of hyperbolic subdomain from the conditions. To formulate the problem responding practical purposes in which the whole boundary is occupied with the conditions, A. Bitsadze suggested several versions. One of the versions links the solution values at different points of the boundary by the functional low. This nonlocal problem was proved to be well-posed and prompted the ways of its natural generalization to a multidimensional case.

To every well-posed plane problem one can assign several spatial versions of which it is necessary to single out more rational ones. For example, the well-known Tricomi's problem has several versions of spatial generalization, where the structure of a set of type variation points becomes obvious. This set may turn out to be a surface oriented to the space or time. This moment determines two essentially different trends in the theory of boundary value problems for multidimensional mixed type equations. This subject-matter was not set aside. The problems posed here became the point of investigations for many specialists in different fields of mathematics, for example, in spectral theory.

All equations refer to different types, depending on their characteristic roots. If the equation, along with its real characteristic roots, has also complex ones, then it belongs to the composite type equations. Such equations include, for example, the Laplace differentiated equation. If instead of the Laplace equation is differentiated Tricomi's operator, we obtain the mixed-composite type operator. For the equation of such a complicated nature, A. Bitsadze has formulated a great deal of actual problems and obtained important results.

We have mentioned above the nonlocal problem in which the values of an unknown solution at different boundary points are interconnected. Of practical and theoretical interest are the problems in which the boundary values of solutions are connected by the specific law with their values on some set of interior points of the domain. These problems cannot be in the literal sense considered as the boundary value problems. Their investigation comes across practical difficulties very often. Among the problems of such a kind the Bitsadze–Samarski's nonlocal problem takes central place, and a great number of works are devoted to its general modifications.

A. Bitsadze constructed exact solutions of wide classes of nonlinear partial differential equations and systems covering Einstein's gravitation field equations, Heisenberg's equations of ferromagnetic theory, various models of Lorentz-covariant equations that can be found in the well-known monographs and reference books for exact solutions of the above-mentioned equations.

Analysing A. Bitsadze's scientific works, we can say with confidence that he did not keep to the beaten tracks, but paved new ways in mathematical science, determining its progress for many years ahead. The ideas and methods elaborated by A. Bitsadze serve at present as an inspiration source for numerous researches of his pupils and followers. A large number of A. Bitsadze's creative achievements, including those mentioned above, have become long ago a corner stone on which scientific trends in the modern theory of partial differential equations are constructed.

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Original article

Dirichlet problem for Laplace–Beltrami equation on hypersurfaces—FEM approximation☆

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Abstract

We consider Dirichlet boundary value problem for Laplace–Beltrami Equation On Hypersurface $\mathscr{S}$, when the Laplace–Beltrami operator on the surface is described explicitly in terms of Günter's differential operators. Using the calculus of Günter's tangential differential operators on hypersurfaces we establish Finite Element Method for the considered boundary value problem and obtain approximate solution in explicit form.

MSC: 53A05; 76M10

Keywords: Hypersurface; Günter's derivatives; Laplace–Beltrami equation; Finite Element Method

Let $\mathscr{S}$ be a C^2 smooth orientable surface in $\mathbb{R}^3$ with Lipschitz boundary $\partial\mathscr{S}$ given by an immersion

$$\zeta : \omega \to \mathscr{S}, \quad \omega \subset \mathbb{R}^2 \tag{1}$$

where ω is open simple connected domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\omega$ and let $\theta : \mathscr{S} \to \omega$ be the inverse mapping

$$\zeta \circ \theta = Id : \mathscr{S} \to \mathscr{S}, \quad \theta \circ \zeta = Id : \omega \to \omega.$$

Denote by $\nu(y),\ y \in \mathscr{S}$ the unit normal on $\mathscr{S}$ with the chosen orientation.

Günter's tangential derivatives $\mathscr{D}_j$ on $\mathscr{S}$ are defined by identities

$$\mathscr{D}_j := \partial_j - \nu_j(y)\partial_\nu,\ \ j = 1, 2, 3, \tag{2}$$

where $\partial_\nu = \sum_{k=1}^{3} \nu_k \partial_k$ denotes the normal derivative.

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Tangential derivatives can be applied to the definition of Sobolev spaces $\mathbb{W}^\ell_p(\mathscr{S}) = \mathbb{H}^\ell(\mathscr{S})$, $\ell \in \mathbb{N}^0$, $1 \le p < \infty$ on an ℓ-smooth surface $\mathscr{S}$ (see [1,2])

$$\mathbb{H}^\ell(\mathscr{S}) = \mathbb{W}^\ell_p(\mathscr{S}) := \left\{\varphi \in D'(\mathscr{S}) \ : \ \nabla^\alpha_{\mathscr{S}}\varphi \in \mathbb{L}_p(\mathscr{S}) \,, \ \forall\, \alpha \in \mathbb{N}^n_0 \,, \ |\alpha| \le \ell\right\}. \tag{3}$$

Equivalently, $\mathbb{W}^\ell_p(\mathscr{S})$ is the closure of the space $C^\ell(\mathscr{S})$ with respect to the norm

$$\|\varphi \,\big|\, \mathbb{W}^\ell_p(\mathscr{S})\,\| := \left[\sum_{|\alpha|\le\ell} \|\mathscr{D}_\alpha\varphi \,\big|\, \mathbb{L}_p(\mathscr{S})\|_p\right]^{1/p}.$$

The space $\mathbb{W}^\ell_p(\mathscr{S})$ can also be understood in distributional sense: derivative $\mathscr{D}_j\varphi \in \mathbb{L}_2(\mathscr{S})$ means that there exists a function in $\mathbb{L}_2(\mathscr{S})$ denoted by $\mathscr{D}_j\varphi$ such that

$$(\mathscr{D}_j\varphi, \psi) := (\varphi, \mathscr{D}^*_j\psi) := \int_{\mathscr{S}} \varphi(y)\overline{\mathscr{D}^*_j\psi(y)}\, d\sigma \qquad \forall\, \psi \in \mathbb{L}_2(\mathscr{S}).$$

Space $\mathbb{W}^\ell_2(\mathscr{S})$ is a Hilbert space with the scalar product

$$(\varphi, \ v)^{(\ell)}_{\mathscr{S}} := \sum_{|\alpha|\le\ell} \int_{\mathscr{S}} \mathscr{D}^\alpha_j \varphi(y)\overline{\mathscr{D}^\alpha_j \ v(y)}d\sigma. \tag{4}$$

Under the space $\mathbb{W}^{-\ell}_2(\mathscr{S})$ with a negative order $-\ell$, $\ell \in \mathbb{N}$, is understood, as usual, the dual space of distributions to the Sobolev space $\mathbb{W}^\ell_2(\mathscr{S})$.

Denote by $\boldsymbol{\Delta}_{\mathscr{S}}$ the Laplace–Beltrami operator on $\mathscr{S}$

$$\boldsymbol{\Delta}_{\mathscr{S}}\, \varphi = \sum_{j=1}^3 \mathscr{D}^2_j\varphi \qquad \forall\, \varphi \in C^2(\mathscr{S}). \tag{5}$$

Note, that if $\varphi \in C^2_0(\mathscr{S})$, $\quad \psi \in C^1(\mathscr{S})$, then due to Kelvin—Stokes theorem

$$(-\boldsymbol{\Delta}_{\mathscr{S}}\varphi, \psi)_{\mathscr{S}} = \sum_{j=1}^3 (\mathscr{D}\varphi_j, \mathscr{D}\psi_j)_{\mathscr{S}}. \tag{6}$$

From (6) immediately follows

Theorem 1. *If $\mathscr{S}$ is a C^1 smooth surface in $\mathbb{R}^3$, then Laplace–Beltrami operator*

$$-\boldsymbol{\Delta}_{\mathscr{S}} \ : \ \mathbb{W}^1_2(\mathscr{S}) \to \mathbb{W}^{-1}_2(\mathscr{S}) \tag{7}$$

is positive definite (see [3]*)*

$$(-\boldsymbol{\Delta}_{\mathscr{S}}\varphi, \varphi)_{\mathscr{S}} = \sum_{k=1}^3 (\mathscr{D}_k\varphi, \mathscr{D}_k\varphi)_{\mathscr{S}} = \|\nabla_{\mathscr{S}}\varphi\big|L_2(\mathscr{S})\|^2 > 0$$
$$\textit{for} \quad \forall \varphi \in \mathbb{W}^1_2(\mathscr{S}), \ \varphi \ne 0. \tag{8}$$

We consider the following Dirichlet boundary value problem for the Laplace–Beltrami equation

$$\begin{cases} \Delta_{\mathscr{S}} u(y) = f(y), & y \in \mathscr{S}, \\ u^+(y) = 0, & y \in \partial\mathscr{S}, \end{cases} \tag{9}$$

where $f \in \mathbb{L}_2(\mathscr{S})$.

From (6) follows variational formulation of (9):

Find a vector $\varphi \in \mathbb{H}^1_0(\mathscr{S})$ that

$$\sum_{k=1}^{3}(\mathscr{D}_k\varphi, \mathscr{D}_k\psi)_{\mathscr{S}} = -(f, \psi)_{\mathscr{S}} \qquad \forall\, \psi \in \mathbb{H}^{1/2}(\mathscr{S})^3. \tag{10}$$

Due to Theorem 1 and Poincaré inequality the sesquilinear form

$$a(\varphi, \psi) := \sum_{k=1}^{3}(\mathscr{D}_k\varphi, \mathscr{D}_k\psi)_{\mathscr{S}} \tag{11}$$

is bounded and coercive in $\mathbb{H}^1_0(\mathscr{S})$

$$M_1 \left\|\varphi\middle|\mathbb{H}^1(\mathscr{S})\right\|^2 \geq a(\varphi, \varphi) \geq M\left\|\varphi\middle|\mathbb{H}^1(\mathscr{S})\right\|^2, \qquad \forall\, \varphi \in \mathbb{H}^1_0(\mathscr{S}), \tag{12}$$

for some $M > 0,\ M_1 > 0$, therefore problem (10) possesses a unique solution by Lax–Milgram Theorem (see [4]).

Now we describe the discrete counterpart of the problem (cf. [5]).

Let X_h be a family of finite dimensional subspaces approximating $\mathbb{H}^1(\mathscr{S})$, i.e., such that $\bigcup_h X_h$ is dense in $\mathbb{H}^1(\mathscr{S})$.

Consider Eq. (10) in the finite-dimensional space X_h

$$a(\varphi_h, \psi_h) = \widetilde{f}(\psi_h) \quad \forall\psi \in X_h, \tag{13}$$

where $\tilde{f}(\psi_h) = -(f, \psi_h)_{\mathscr{S}}$.

Theorem 2. *Eq.* (13) *has the unique solution* $\varphi_h \in X_h$ *for all* $h > 0$. *This solution converges in* $\mathbb{H}^1(\mathscr{S})$ *to the solution* φ *of* (10) *as* $h \to 0$.

Proof. From the coercivity of sesquilinear form a immediately follows

$$\begin{aligned} c_1 \left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\|^2 &\leq a(\varphi_h, \varphi_h) = |\tilde{f}(\varphi_h)| \\ &\leq c_2 \left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\| \text{ for all } h. \end{aligned} \tag{14}$$

Let φ_h be the unique solution of the homogeneous equation:

$$a(\varphi_h, \psi_h) = 0 \text{ for all } \psi_h \in X_h. \tag{15}$$

Then (14) implies $\left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\| = 0$ and consequently, $\varphi_h = 0$. Therefore Eq. (13) has a unique solution. From (14) it follows also that

$$\left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\|^2 \leq \frac{c_2}{c_1}\left\|\varphi_h\middle|\mathbb{H}^1(\mathscr{S})\right\|.$$

Hence sequence $\{\|\varphi_h|\mathbb{H}^1(\mathscr{S})\|\}$ is bounded and we can extract a subsequence $\{\varphi_{h_k}\}$ which converges weakly to some $\varphi \in \mathbb{H}^1(\mathscr{S})$.

Let us take an arbitrary $\psi \in \mathbb{H}^1(\mathscr{S})$ and for each $h > 0$ choose $\psi_h \in X_h$ such, that $\psi_h \to \psi$ in $\mathbb{H}^1(\mathscr{S})$. Then from (13) we have

$$a(\varphi, \psi) = \tilde{f}(\psi), \quad \forall\psi \in \mathbb{H}^1(\mathscr{S}).$$

Hence, φ solves (10). Note, that since (10) is uniquely solvable, each subsequence $\{\varphi_{h_k}\}$ converges weakly to the same solution φ, and consequently the whole sequence $\{\varphi_h\}$ also converges weakly to φ. Now let us prove that it converges in the space $\mathbb{H}^1(\mathscr{S})$.

Indeed, due to (14) we have

$$\begin{aligned} c_1\|\varphi_h - \varphi\|^2 &\leq |a(\varphi_h - \varphi, \varphi_h - \varphi)| \leq |a(\varphi_h, \varphi_h - \varphi) - a(\varphi, \varphi_h - \varphi)| \\ &= c_1|\tilde{f}(\varphi_h) - a(\varphi_h, \varphi) - \tilde{f}(\varphi_h - \varphi)| \to c_1|\tilde{f}(\varphi) - a(\varphi, \varphi)| = 0, \end{aligned}$$

which completes the proof. □

We can choose spaces X_h in different ways. As an example let us describe the discretization method based on the representation of the surface $\mathscr{S}$ as a network of the triangle-shaped elements.

Let $(U_\alpha, \varkappa_\alpha)$ be a parametrization of $\mathscr{S}$. Here $\varkappa_\alpha : U_\alpha \subset \mathbb{R}^2 \to \mathscr{S}$ are injective differentiable mappings (diffeomorphisms) of open sets U_α of $\mathbb{R}^2$ into $\mathscr{S}$ such that $\bigcup_\alpha \varkappa_\alpha(U_\alpha) = \mathscr{S}$.

Let $h > 0$. We call $\mathfrak{S}_h$ a triangulation of $\mathscr{S}$ if $\mathscr{S}$ is represented as $\mathscr{S} = \bigcup_{\mathscr{T}_\gamma \in \mathfrak{S}_h} \mathscr{T}_\alpha$, where the sets $\mathscr{T}_\gamma$ possess the following properties:

1. Each $\mathscr{T}_\gamma$ is a subset of some $\varkappa_{\alpha_\gamma}(U_{\alpha_\gamma})$ and $T_\gamma := \varkappa_{\alpha_\gamma}^{-1}(\mathscr{T}_\gamma) \subset U_{\alpha_\gamma}$ is a triangle.
2. If $T_\gamma = \varkappa_\alpha^{-1}(\mathscr{T}_\gamma)$ and $T_\delta = \varkappa_\alpha^{-1}(\mathscr{T}_\delta)$ are subsets of the same U_α, then their intersection can be only a common vertex or a side.
3. Sides of the triangles $\varkappa_{\alpha_\gamma}^{-1}(\mathscr{T}_\gamma)$ do not exceed h.

Denote by $\mathscr{N}_{\mathfrak{S}}$ the set of nodes of the triangulation $\mathfrak{S}$, i.e. the set of all points $\varkappa_\alpha(P_\beta) \in \mathscr{S}$, where P_β are vertices of the triangles T_γ. Let $\zeta : \mathscr{N}_{\mathfrak{S}} \to \mathbb{R}$ be any mapping of $\mathscr{N}_{\mathfrak{S}}$ into $\mathbb{R}$, then it can be easily proved that there exists function $v_\zeta \in \mathbb{H}^1(\mathscr{S})$ such that:

1. $r_{\mathscr{N}_{\mathfrak{S}}} v_\zeta = \zeta$.
2. The restriction of $v_\zeta \circ \varkappa_\alpha$ on T_α is an affine function: $v_\zeta \circ \varkappa_\alpha(x_1, x_2) = a_1x_1 + a_2x_2 + a_3$.

Denote by X_h the set of all such functions, corresponding to the triangulation $\mathfrak{S}_h$. The set X_h consists of the piecewise-linear functions and therefore $\bigcup_h X_h$ is dense in $\mathbb{H}^{1/2}(\mathscr{S})^3$.

We can replace the triangle-shaped elements in the above-described network by quadrilateral, hexagonal or other type polygonal elements.

In particular, consider a case, when $\omega = U_\alpha$ in the above parametrization is a square part of $\mathbb{R}^2$

$$\omega = \{(x_1, x_2) : \ 0 < x_1 < 1, \ 0 < x_2 < 1\}, \quad \zeta(\omega) = \mathscr{S}.$$

Allocate N^2 nodes $P_{ij} = (i/(N+1), \ j/(N+1)), \ i, j = 1, \dots, N$ on ω.

Let $\alpha_k, \ k = 1, \dots, N$ be piecewise linear functions defined on segment $[0, \ 1]$ as follows:

$$\alpha_k(x) = \begin{cases} 0, & 0 \le x \le \dfrac{k-1}{N+1}, \\ (N+1)\left(x - \dfrac{k-1}{N+1}\right), & \dfrac{k-1}{N+1} < x \le \dfrac{k}{N+1}, \\ (N+1)\left(\dfrac{k+1}{N+1} - x\right), & \dfrac{k}{N+1} < x \le \dfrac{k+1}{N+1}, \\ 0, & \dfrac{k+1}{N+1} < x \le 1, \\ j = k, \dots, N, & \end{cases} \tag{16}$$

and denote by $\tilde{\varphi}_{ij}, \ i, j = 1, \dots, N$ functions

$$\tilde{\varphi}_{ij}(x_1, \ x_2) = \alpha_i(x_1)\alpha_j(x_2), \ i, j = 1, \dots, N, \quad (x_1, \ x_2) \in \omega. \tag{17}$$

Evidently, $\tilde{\varphi}_{ij}$ are continuous functions, which take their maximal value $\varphi_{ij}(P_{ij}) = 1$ at point P_{ij} and vanish outside the set

$$\omega_{ij} = \omega \cap \left\{(x_1, x_2) : \ 0 \le \left|x_1 - \frac{i}{N+1}\right| \le 1, \ 0 \le \left|x_2 - \frac{j}{N+1}\right| \le 1\right\}, \tag{18}$$

consequently, they belong to $\mathbb{H}^1(\omega)$ and are linearly independent (see [6]).

Denote by X_N the linear span of the functions $\varphi_{ij} \circ \vartheta, \quad i, j = 1, \dots, N$. The space X_N is N^2-dimensional space contained into $\mathbb{H}^1(\mathscr{S})$.

Consider Eq. (13) in the space X_N.

$$a(\varphi, \psi) = \tilde{f}(\psi) \quad \forall \psi \in X_N. \tag{19}$$

We sought the solution $\varphi \in X_N$ of Eq. (19) in the form

$$\varphi = \sum_{i,\,j=1}^{N} C_{ij}\,\varphi_{ij}, \tag{20}$$

where C_{ij} are unknown coefficients. Substituting φ in (19) and replacing ψ successively by $\varphi_{ij},\ i, j = 1, \dots, N$, we get the equivalent system of N^2 linear algebraic equations

$$\sum_{i,\,j=1}^{N} A_{ijkl}\,C_{ij} = f_{kl}, \quad k, l = 1, \dots, N, \tag{21}$$

where

$$A_{ijkl} = a\Big(\varphi_{ij},\, \varphi_{kl}\Big), \quad f_{kl} = \tilde{f}(\varphi_{kl}). \tag{22}$$

Matrix $A = A_{(ij)(kl)}$ is the Gram's matrix of the positive semidefinite bilinear form a, therefore it is a nonsingular matrix and Eq. (21) has a unique solution.

$$\varphi = \sum_{i,j,k,\,l=1}^{N} \big(A\big)^{-1}_{(ij)(kl)}\,\varphi_{ij}\,f_{kl}. \tag{23}$$

To calculate explicitly $A_{(ij)(kl)}$ and f_{kl} note, that

$$\begin{aligned} \mathscr{D}_m\varphi_{ij}(y) &= \partial_{y_m}\varphi_{ij}(y) + \nu_m\partial_\nu\varphi_{ij}(y) \\ &= \sum_{p=1}^{2} \partial_p\hat{\varphi}_{ij}(\vartheta(y)) \left(\partial_m\vartheta_p(y) + \nu_m \sum_{l=1}^{3} \nu_l\partial_l\vartheta_p(y)\right) \\ &= \sum_{p=1}^{2} \partial_p\hat{\varphi}_{ij}(\vartheta(y))\mathscr{D}_m\vartheta_p(y), \end{aligned} \tag{24}$$

$$\begin{aligned} A_{ijkl} &= a\Big(\varphi_{ij},\, \varphi_{kl}\Big) \\ &= \sum_{p,q=1}^{2} \int_{\mathscr{S}} \big(\partial_p\hat{\varphi}_{ij}(\vartheta(y))\big)\big(\partial_q\hat{\varphi}_{kl}(\vartheta(y))\big) \sum_{m=1}^{3} \mathscr{D}_m\vartheta_p(y)\mathscr{D}_m\vartheta_q(y)d\sigma, \end{aligned} \tag{25}$$

$$f_{kl} = -(f,\, \hat{\varphi}_{kl})_{\mathscr{S}} = -\int_{\mathscr{S}} f(y)\,\hat{\varphi}_{kl}(y)\,d\sigma. \tag{26}$$

Changing variables $y = \zeta(x), \quad x = \vartheta(y)$ on right side of (25) and taking into account, that supp $\big(\partial_p\hat{\varphi}_{ij}(x)\big) = \omega_{ij}$ we get

$$A_{ijkl} = \sum_{p,q=1}^{2} \int_{\omega_{ij}\cap\omega_{kl}} \big(\partial_p\hat{\varphi}_{ij}(x)\big)\big(\partial_q\hat{\varphi}_{kl}(x)\big) \sum_{m=1}^{3} \mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx, \tag{27}$$

$$f_{kl} = -\int_{\omega_{kl}} f(\zeta(x))\,\hat{\varphi}_{kl}(\zeta(x))|\sigma'(x)|dx \tag{28}$$

where $|\sigma'(x)|$ is a surface element of $\mathscr{S}$

$$|\sigma'(x)| = |\partial_1\vartheta(x) \times \partial_2\vartheta(x)|.$$

From (16)–(19), (27)–(28) we obtain explicit expressions of A_{ijkl} and f_{kl}, $1 \le i,\ j,\ k,\ l, \le N$

$$A_{ijkl} = 0, \ \text{ if } \ i < k-1 \ \text{ or } \ k < i-1 \ \text{ or } \ j < l-1 \ \text{ or } \ l < j-1,$$

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \Big[\delta_{p1}\Big(x_2 - \frac{j+1}{N+1}\Big) + \delta_{p2}\Big(x_1 - \frac{i+1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(x_2 - \frac{j}{N+1}\Big)$$
$$+ \delta_{q2}\Big(x_1 - \frac{i}{N+1}\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

if $k = i+1, \quad l = j+1$,

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}} \Big[\delta_{p1}\Big(\frac{j-1}{N+1} - x_2\Big) + \delta_{p2}\Big(\frac{i+1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(x_2 - \frac{j-1}{N+1}\Big)$$
$$+ \delta_{q2}\Big(x_1 - \frac{i}{N+1}\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$
$$+ (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \Big[\delta_{p1}\Big(\frac{j+1}{N+1} - x_2\Big) + \delta_{p2}\Big(\frac{i+1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j+1}{N+1} - x_2\Big)$$
$$+ \delta_{q2}\Big(\frac{i}{N+1} - x_1\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

if $k = i+1, \quad l = j$,

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}} \Big[\delta_{p1}\Big(\frac{j-1}{N+1} - x_2\Big) + \delta_{p2}\Big(\frac{i+1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j}{N+1} - x_2\Big)$$
$$+ \delta_{q2}\Big(x_1 - \frac{i}{N+1}\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

if $k = i+1, \quad l = j-1$,

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \Big[\delta_{p1}\Big(\frac{j+1}{N+1} - x_2\Big) + \delta_{p2}\Big(\frac{i-1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j}{N+1} - x_2\Big)$$
$$+ \delta_{q2}\Big(\frac{i}{N+1} - x_1\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$

if $k = i-1, \quad l = j+1$,

$$A_{ijkl} = (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}} \int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}} \Big[\delta_{p1}\Big(x_2 - \frac{j-1}{N+1}\Big) + \delta_{p2}\Big(x_1 - \frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(x_2 - \frac{j-1}{N+1}\Big)$$
$$+ \delta_{q2}\Big(x_1 - \frac{i-1}{N+1}\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$
$$+ (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \Big[\delta_{p1}\Big(\frac{j+1}{N+1} - x_2\Big) + \delta_{p2}\Big(x_1 - \frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(\frac{j+1}{N+1} - x_2\Big)$$
$$+ \delta_{q2}\Big(\frac{i-1}{N+1} - x_1\Big)\Big] \sum_{m=1}^{3} \mathscr{D}_m \vartheta_p(\zeta(x)) \mathscr{D}_m \vartheta_q(\zeta(x)) |\sigma'(x)| dx,$$
$$+ (N+1)^4 \sum_{p,q=1}^{2} \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}} \Big[\delta_{p1}\Big(x_2 - \frac{j-1}{N+1}\Big) + \delta_{p2}\Big(\frac{i+1}{N+1} - x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j-1}{N+1} - x_2\Big)$$

$$+\delta_{q2}\Big(\frac{i+1}{N+1}-x_1\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

$$+(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}}\int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}}\Big[\delta_{p1}\Big(\frac{j+1}{N+1}-x_2\Big)+\delta_{p2}\Big(\frac{i+1}{N+1}-x_1\Big)\Big]\Big[\delta_{q1}\Big(\frac{j+1}{N+1}-x_2\Big)$$

$$+\delta_{q2}\Big(\frac{i+1}{N+1}-x_1\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

if $k=i,\ \ l=j,$

$$A_{ijkl}=(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}}\int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}}\Big[\delta_{p1}\Big(x_2-\frac{j-1}{N+1}\Big)+\delta_{p2}\Big(x_1-\frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(\frac{j}{N+1}-x_2\Big)$$

$$+\delta_{q2}\Big(\frac{i-1}{N+1}-x_1\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

$$+(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}}\int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}}\Big[\delta_{p1}\Big(\frac{j-1}{N+1}-x_2\Big)+\delta_{p2}\Big(\frac{i+1}{N+1}-x_1\Big)\Big]\Big[\delta_{q1}\Big(x_2-\frac{j}{N+1}\Big)$$

$$+\delta_{q2}\Big(x_1-\frac{i+1}{N+1}\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

if $k=i,\ \ l=j-1,$

$$A_{ijkl}=(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}}\int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}}\Big[\delta_{p1}\Big(x_2-\frac{j-1}{N+1}\Big)+\delta_{p2}\Big(x_1-\frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(\frac{j-1}{N+1}-x_2\Big)$$

$$+\delta_{q2}\Big(\frac{i}{N+1}-x_1\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

$$+(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}}\int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}}\Big[\delta_{p1}\Big(x_2-\frac{j-1}{N+1}\Big)+\delta_{p2}\Big(x_1-\frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(x_2-\frac{j+1}{N+1}\Big)$$

$$+\delta_{q2}\Big(x_1-\frac{i}{N+1}\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

if $k=i-1,\ \ l=j,$

$$A_{ijkl}=(N+1)^4\sum_{p,q=1}^{2}\int_{\frac{i-1}{N+1}}^{\frac{i}{N+1}}\int_{\frac{j-1}{N+1}}^{\frac{j}{N+1}}\Big[\delta_{p1}\Big(x_2-\frac{j-1}{N+1}\Big)+\delta_{p2}\Big(x_1-\frac{i-1}{N+1}\Big)\Big]\Big[\delta_{q1}\Big(x_2-\frac{j}{N+1}\Big)$$

$$+\delta_{q2}\Big(x_1-\frac{i}{N+1}\Big)\Big]\sum_{m=1}^{3}\mathscr{D}_m\vartheta_p(\zeta(x))\mathscr{D}_m\vartheta_q(\zeta(x))|\sigma'(x)|dx,$$

if $k=i-1,\ \ l=j-1,$

$$f_{kl}=-(N+1)^2\int_{\frac{k-1}{N+1}}^{\frac{k}{N+1}}\int_{\frac{l-1}{N+1}}^{\frac{l}{N+1}}\Big(x_1-\frac{k-1}{N+1}\Big)\Big(x_2-\frac{l-1}{N+1}\Big)f(\zeta(x))|\sigma'(x)|dx$$

$$-(N+1)^2\int_{\frac{k}{N+1}}^{\frac{k+1}{N+1}}\int_{\frac{l-1}{N+1}}^{\frac{l}{N+1}}\Big(\frac{k+1}{N+1}-x_1\Big)\Big(x_2-\frac{l-1}{N+1}\Big)f(\zeta(x))|\sigma'(x)|dx$$

$$- (N+1)^2 \int_{\frac{k-1}{N+1}}^{\frac{k}{N+1}} \int_{\frac{l}{N+1}}^{\frac{l+1}{N+1}} \Big(x_1 - \frac{k-1}{N+1}\Big)\Big(\frac{l+1}{N+1} - x_2\Big) f(\zeta(x))|\sigma'(x)|dx$$

$$- (N+1)^2 \int_{\frac{k}{N+1}}^{\frac{k+1}{N+1}} \int_{\frac{l}{N+1}}^{\frac{l+1}{N+1}} \Big(\frac{k+1}{N+1} - x_1\Big)\Big(\frac{l+1}{N+1} - x_2\Big) f(\zeta(x))|\sigma'(x)|dx.$$

As an application of the aforementioned boundary value problem we can consider stationary state of heat conduction by an isotropic media, governed by the Laplace equations and constrained by classical Dirichlet–Neumann mixed boundary conditions for the Laplace equation in the layer domain $\Omega^\varepsilon := \mathscr{S} \times (-\varepsilon, \varepsilon)$ of thickness 2ε, where $\mathscr{S} \subset \mathscr{C}$ is a smooth subsurface of a closed hypersurface $\mathscr{C}$ with smooth nonempty boundary $\partial\mathscr{S}$

$$\Delta_{\Omega^\varepsilon}\varphi(y,t) = f(y,t), \quad (y,t) \in \mathscr{S} \times (-\varepsilon, \varepsilon), \tag{29}$$

$$\varphi^+(y,t) = g(y,t), \quad (y,t) \in \partial\mathscr{S} \times (-\varepsilon, \varepsilon), \tag{30}$$

$$\pm(\partial_t\varphi)^+(y, \pm\varepsilon) = q^\pm(y), \quad y \in \mathscr{S}, \tag{31}$$

where

$$\Delta_{\Omega^\varepsilon}\varphi = \sum_{j=1}^{4} \mathscr{D}_j^2\varphi + \mathscr{H}_{\mathscr{S}}^0\partial_\nu\varphi = \Delta_{\mathscr{S}}\varphi + \partial_t^2\varphi + \mathscr{H}_{\mathscr{S}}^0\partial_\nu\varphi$$

and $\mathscr{H}_{\mathscr{S}}^0$ is a Weingarten matrix

$$\mathscr{H}_{\mathscr{S}}^0(\mathscr{X}) := \big[\mathscr{D}_j\nu_k(\mathscr{X})\big]_{n\times n}, \qquad \mathscr{X} \in \mathscr{S}. \tag{32}$$

It can be proved that when the thickness 2ε of the layer domain Ω^ε with the mid-surface $\mathscr{C}$, tends to zero, this boundary-value problem "converges" in the sense of Γ convergence to the following Dirichlet boundary value problem for Laplace–Beltrami equation on $\mathscr{S}$

$$\begin{aligned} &\Delta_{\mathscr{S}}\varphi(y) = f_0(y) && y \in \mathscr{S}, \\ &\varphi^+(\tau) = g(\tau, 0), && \tau \in \partial\mathscr{S}, \end{aligned} \tag{33}$$

where

$$f_0(y) := f(y,0) - (\partial_t^2 G)(y,0) - \frac{1}{2}[q^+(y) + q^-(y)]\frac{1}{2}\big[(\partial_t G)(y,0) - (\partial_t G)(y,0)\big] \tag{34}$$

and $G(x,t)$ is a continuation of the boundary data $g(y,t)$, $(y,t) \in \partial\mathscr{S} \times (-\varepsilon, \varepsilon)$, from the boundary into the domain $(x,t) \in \Omega^\varepsilon$.

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Original article

Mixed boundary value problems of pseudo-oscillations of generalized thermo-electro-magneto-elasticity theory for solids with interior cracks

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Abstract

We investigate the mixed boundary value problems of the generalized thermo-electro-magneto-elasticity theory for homogeneous anisotropic solids with interior cracks. Using the potential methods and theory of pseudodifferential equations on manifolds with boundary we prove the existence and uniqueness of solutions. We analyse the asymptotic behaviour and singularities of the mechanical, electric, magnetic, and thermal fields near the crack edges and near the curves, where different types of boundary conditions collide. In particular, for some important classes of anisotropic media we derive explicit expressions for the corresponding stress singularity exponents and demonstrate their dependence on the material parameters. The questions related to the so called oscillating singularities are treated in detail as well.

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Keywords: Thermo-electro-magneto-elasticity; Green–Lindsay's model; Mixed boundary value problems; Cracks; Potential method; Pseudodifferential equations; Asymptotic behaviour of solutions

1. Introduction

The paper deals with three-dimensional boundary value problems (BVP) arising in the generalized thermo-electro-magneto-elasticity (GTEME) theory for homogeneous anisotropic solids with interior cracks.

The theory under consideration is associated with Green–Lindsay's model of thermo-electro-magneto-elasticity which describes full coupling of elastic, electric, magnetic, and thermal fields. Another feature of this model is that

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in contrast to the conventional theory of heat transfer, the heat propagation in Green–Lindsay's theory occurs with a finite speed (see [1,2]).

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields. For example, piezoelectric materials (electro-elastic coupling) have been used as ultrasonic transducers and micro-actuators; pyroelectric materials (thermal–electric coupling) have been applied in thermal imaging devices; and magnetoelastic coupling effects are used in modern signal detection systems and instrumentation (see [3–9] and the references therein).

Theories of thermoelasticity consistent with a finite speed propagation of heat recently are attracting increasing attention. In contrast to the conventional thermoelasticity theory, these nonclassical theories involve a hyperbolic-type heat transport equation, and are motivated by experiments exhibiting the actual occurrence of wave-type heat transport (second sound). Several authors have formulated these theories on different grounds, and a wide variety of problems revealing characteristic features of the theories has been investigated.

As it is well known from the classical mathematical physics and the classical elasticity theory, in general, solutions to crack type and mixed boundary value problems have singularities near the crack edges and near the lines where the types of boundary conditions change, regardless of the smoothness of given boundary data. Throughout the paper we shall refer to such lines as *exceptional curves*. The same effect can be observed also in the GTEME theory. In this paper, our main goal is a detailed theoretical investigation of regularity and asymptotic properties of thermo-mechanical and electro-magnetic fields near the exceptional curves. By explicit calculations we show that the stress singularity exponents essentially depend on the material parameters, in general.

We draw a special attention to the problem of oscillating singularities which is very important in engineering applications. Such singularities usually lead to some mechanical contradictions, e.g., overlapping of materials (see, e.g., [10] and the references therein). It turned out that there are classes of anisotropic media for which the oscillating singularities near the exceptional curves do not occur. In particular, *calcium phosphate based bioceramics*, such as *hydroxyapatite*, possess the above property. These materials are extensively used in medicine and dentistry [11,12].

Our main tools are the potential methods and the theory of pseudodifferential equations, which proved to be very efficient in deriving the asymptotic formulas. They allow us to calculate effectively the field singularity exponents by means of the characteristics related to the symbol matrices of the corresponding pseudodifferential operators. In our analysis we essentially apply the results obtained in the references [13–16,18,19].

To demonstrate the dependence of the singularity exponents on the material parameters let us compare behaviour of solutions to the crack type mixed boundary value problems near the exceptional curves for the Laplace equation (Zaremba type problem), for equations of the classical elasticity (e.g., the Lamé equations for an isotropic solid) and for the equations to generalized thermo-electro-magneto-elasticity equations for transversely-isotropic media.

Near the crack edge the asymptotic formulae for solutions of all the above three problems have the same form, namely,

$$a_0\, r^{1/2} + a_1\, r^{3/2} + \cdots , \tag{1.1}$$

where r is the distance from the reference point x to the crack edge [20,21].

We have quite a different situation near the exceptional curve, where the different types of boundary conditions (for example, the Dirichlet and Neumann type conditions) collide. Unlike the asymptotic expansion (1.1) of solutions to the Laplace equation the asymptotic expansion of the solutions to Lamé equations has the form

$$b_0\, r^{1/2} + b_1\, r^{1/2+i\,\delta} + b_2\, r^{1/2-i\,\delta} + \mathcal{O}(r^{3/2-\varepsilon}),$$

where ε is an arbitrary positive number, while the asymptotic expansion of a solution to the generalized thermo-electro-magneto-elasticity equations for transversely-isotropic case reads as

$$c_0\, r^{\gamma_1} + c_1\, r^{1/2+i\,\widetilde{\delta}} + c_2\, r^{1/2-i\,\widetilde{\delta}} + c_3\, r^{1/2} \ln r + c_4\, r^{1/2} + \mathcal{O}(r^{\gamma_2}),$$

where $\gamma_1 \in (0,\ 1/2)$, $\gamma_2 > 1/2$, and δ and $\widetilde{\delta}$ are real numbers. Note that $\gamma_1 - 1$ represents the dominant stress singularity exponent. The parameter γ_1 in general depends on the material constants and the geometry of the curve and may take an arbitrary value from the interval $(0,\ 1/2)$ (for details see Section 6). Thus, the stress singularity exponent essentially depends on the material constants and is less than $-1/2$, in general. Consequently, in the classical elasticity, we have oscillating stress singularities, while in the generalized thermo-electro-magneto-elasticity theory we have no oscillating stress singularities for the transversely isotropic case due to the inequality $\gamma_1 < 1/2$.

2. Formulation of the problem

2.1. Field equations

In this subsection, we collect the field equations of the generalized thermo-electro-magneto-elasticity (GTEME) for a general anisotropic case and introduce the corresponding matrix partial differential operators

Throughout the paper $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, σ_{ij} are the components of the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ are the components of the mechanical strain tensor, $E = (E_1, E_2, E_3)^\top$ and $H = (H_1, H_2, H_3)^\top$ are electric and magnetic fields respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials and

$$E = -\mathrm{grad}\,\varphi\,, \qquad H = -\mathrm{grad}\,\psi\,,$$

ϑ is the temperature change to a reference temperature T_0, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density.

We employ also the notation $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$; the superscript $(\cdot)^\top$ denotes transposition operation. In what follows the summation over the repeated indices is meant from 1 to 3, unless stated otherwise. Throughout the paper the over bar, applied to numbers and functions, denotes complex conjugation and the central dot denotes the scalar product of two vectors in the complex vector space $\mathbb{C}^N$, i.e., $a \cdot b \equiv (a, b) := \sum_{j=1}^N a_j \overline{b_j}$ for $a, b \in \mathbb{C}^N$. Over bar, applied to a subset $\mathcal{M}$ of Euclidean space $\mathbb{R}^N$, denotes the closure of $\mathcal{M}$, i.e. $\overline{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$, where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$.

In the GTEME theory we have the following governing equations:

The constitutive relations:

$$\sigma_{rj} = \sigma_{jr} = c_{rjkl}\,\varepsilon_{kl} - e_{lrj} E_l - q_{lrj} H_l - \lambda_{rj}(\vartheta + \nu_0\,\partial_t\vartheta), \qquad r, j = 1, 2, 3, \tag{2.1}$$

$$D_j = e_{jkl}\,\varepsilon_{kl} + \varkappa_{jl}\,E_l + a_{jl}\,H_l + p_j(\vartheta + \nu_0\,\partial_t\vartheta), \qquad j = 1, 2, 3, \tag{2.2}$$

$$B_j = q_{jkl}\,\varepsilon_{kl} + a_{jl}\,E_l + \mu_{jl} H_l + m_j(\vartheta + \nu_0\,\partial_t\vartheta), \qquad j = 1, 2, 3, \tag{2.3}$$

$$\varrho\,\mathcal{S} = \lambda_{kl}\,\varepsilon_{kl} + p_l\,E_l + m_l\,H_l + a_0 + d_0\,\vartheta + h_0\,\partial_t\vartheta. \tag{2.4}$$

The equations of motion:

$$\partial_j\sigma_{rj} + \varrho\,F_r = \varrho\,\partial_t^2 u_r, \qquad r = 1, 2, 3. \tag{2.5}$$

The quasi-static equations for electric and magnetic fields:

$$\partial_j D_j = \varrho_e\,, \qquad \partial_j B_j = \varrho_c\,. \tag{2.6}$$

The linearized energy equations:

$$\varrho\,T_0\,\partial_t\mathcal{S} = -\partial_j q_j + \varrho\,Q, \qquad q_j = -T_0\,\eta_{jl}\partial_l\vartheta. \tag{2.7}$$

Here the following notation is used: ϱ—the mass density, ϱ_e—the electric charge density, ϱ_c—the electric current density, $F = (F_1, F_2, F_3)^\top$—the mass force density, Q—the heat source intensity, c_{rjkl}—the elastic constants, e_{jkl}—the piezoelectric constants, q_{jkl}—the piezomagnetic constants, $\varkappa_{jk}$—the dielectric (permittivity) constants, μ_{jk}—the magnetic permeability constants, a_{jk}—the electromagnetic coupling coefficients, p_j, m_j, and λ_{rj}—coupling coefficients connecting dissimilar fields, η_{jk}—the heat conductivity coefficients, T_0—the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, ν_0 and h_0—two relaxation times, a_0 and d_0—constitutive coefficients.

The constants involved in the above equations satisfy the symmetry conditions:

$$\begin{gathered} c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \quad q_{klj} = q_{kjl}, \\ \varkappa_{kj} = \varkappa_{jk}, \quad \lambda_{kj} = \lambda_{jk}, \quad \mu_{kj} = \mu_{jk}, \quad a_{kj} = a_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad r, j, k, l = 1, 2, 3. \end{gathered} \tag{2.8}$$

From physical considerations it follows that (see, e.g., [22,23,2,1,24]):

$$\begin{gathered}c_{rjkl}\,\xi_{rj}\,\xi_{kl} \ge \delta_0\,\xi_{kl}\,\xi_{kl},\ \ \varkappa_{kj}\,\xi_k\,\xi_j \ge \delta_1\,|\xi|^2,\ \ \mu_{kj}\,\xi_k\,\xi_j \ge \delta_2\,|\xi|^2,\ \ \eta_{kj}\,\xi_k\,\xi_j \ge \delta_3\,|\xi|^2,\\ \text{for all}\ \ \xi_{kj} = \xi_{jk} \in \mathbb{R} \text{ and for all } \ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,\end{gathered} \tag{2.9}$$

$$\nu_0 > 0, \quad h_0 > 0, \quad d_0\nu_0 - h_0 > 0, \tag{2.10}$$

where δ_0, δ_1, δ_2, and δ_3 are positive constants depending on material parameters.

Due to the symmetry conditions (2.8), with the help of (2.9) we easily derive

$$\begin{gathered}c_{rjkl}\,\zeta_{rj}\,\overline{\zeta_{kl}} \ge \delta_0\,\zeta_{kl}\,\overline{\zeta_{kl}},\ \ \varkappa_{kj}\,\zeta_k\,\overline{\zeta_j} \ge \delta_1\,|\zeta|^2,\ \ \mu_{kj}\,\zeta_k\,\overline{\zeta_j} \ge \delta_2\,|\zeta|^2,\ \ \eta_{kj}\,\zeta_k\,\overline{\zeta_j} \ge \delta_3\,|\zeta|^2,\\ \text{for all}\ \ \zeta_{kj} = \zeta_{jk} \in \mathbb{C} \text{ and for all } \ \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3.\end{gathered} \tag{2.11}$$

More careful analysis related to the positive definiteness of the potential energy and the thermodynamical laws insure that the following 8×8 matrix

$$M = [M_{kj}]_{8\times 8} := \begin{bmatrix} [\varkappa_{jl}]_{3\times 3} & [a_{jl}]_{3\times 3} & [p_j]_{3\times 1} & [\nu_0 p_j]_{3\times 1} \\ [a_{jl}]_{3\times 3} & [\mu_{jl}]_{3\times 3} & [m_j]_{3\times 1} & [\nu_0 m_j]_{3\times 1} \\ [p_j]_{1\times 3} & [m_j]_{1\times 3} & d_0 & h_0 \\ [\nu_0 p_j]_{1\times 3} & [\nu_0 m_j]_{1\times 3} & h_0 & \nu_0 h_0 \end{bmatrix}_{8\times 8} \tag{2.12}$$

is positive definite. Note that the positive definiteness of M remains valid if the parameters p_j and m_j in (2.12) are replaced by the opposite ones, $-p_j$ and $-m_j$. Moreover, it follows that the matrices

$$\Lambda^{(1)} := \begin{bmatrix} [\varkappa_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} \end{bmatrix}_{6\times 6}, \qquad \Lambda^{(2)} := \begin{bmatrix} d_0 & h_0 \\ h_0 & \nu_0 h_0 \end{bmatrix}_{2\times 2} \tag{2.13}$$

are positive definite as well, i.e.,

$$\varkappa_{kj}\,\zeta'_k\,\overline{\zeta'_j} + a_{kj}\,(\zeta'_k\,\overline{\zeta''_j} + \overline{\zeta'_k}\,\zeta''_j) + \mu_{kj}\,\zeta''_k\,\overline{\zeta''_j} \ge \kappa_1\,(|\zeta'|^2 + |\zeta''|^2) \qquad \forall \zeta', \zeta'' \in \mathbb{C}^3, \tag{2.14}$$

$$d_0\,|z_1|^2 + h_0\,(z_1\,\overline{z_2} + \overline{z_1}\,z_2) + \nu_0 h_0\,|z_2|^2 \ge \kappa_2\,(|z_1|^2 + |z_2|^2) \qquad \forall z_1, z_2 \in \mathbb{C}, \tag{2.15}$$

with some positive constants κ_1 and κ_2 depending on the material parameters involved in (2.13).

With the help of the symmetry conditions (2.9) we can rewrite the constitutive relations (2.1)–(2.4) as follows

$$\sigma_{rj} = c_{rjkl}\,\partial_l u_k + e_{lrj}\,\partial_l\varphi + q_{lrj}\,\partial_l\psi - \lambda_{rj}\,(\vartheta + \nu_0\,\partial_t\vartheta), \quad r, j = 1, 2, 3, \tag{2.16}$$

$$D_j = e_{jkl}\,\partial_l u_k - \varkappa_{jl}\,\partial_l\varphi - a_{jl}\,\partial_l\psi + p_j\,(\vartheta + \nu_0\,\partial_t\vartheta), \quad j = 1, 2, 3, \tag{2.17}$$

$$B_j = q_{jkl}\,\partial_l u_k - a_{jl}\,\partial_l\varphi - \mu_{jl}\,\partial_l\psi + m_j\,(\vartheta + \nu_0\,\partial_t\vartheta), \quad j = 1, 2, 3, \tag{2.18}$$

$$\mathcal{S} = \lambda_{kl}\,\partial_l u_k - p_l\,\partial_l\varphi - m_l\,\partial_l\psi + a_0 + d_0\,\vartheta + h_0\,\partial_t\vartheta. \tag{2.19}$$

In the theory of generalized thermo-electro-magneto-elasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n = (n_1, n_2, n_3)$ have the form

$$\sigma_{rj}\,n_j = c_{rjkl}\,n_j\,\partial_l u_k + e_{lrj}\,n_j\,\partial_l\varphi + q_{lrj}\,n_j\,\partial_l\psi - \lambda_{rj}\,n_j\,(\vartheta + \nu_0\,\partial_t\vartheta)\,, \quad r = 1, 2, 3, \tag{2.20}$$

while the normal components of the electric displacement vector, magnetic induction vector and heat flux vector read as

$$D_j\,n_j = e_{jkl}\,n_j\,\partial_l u_k - \varkappa_{jl}\,n_j\,\partial_l\varphi - a_{jl}\,n_j\,\partial_l\psi + p_j\,n_j\,(\vartheta + \nu_0\,\partial_t\vartheta), \tag{2.21}$$

$$B_j\,n_j = q_{jkl}\,n_j\,\partial_l u_k - a_{jl}\,n_j\,\partial_l\varphi - \mu_{jl}\,n_j\,\partial_l\psi + m_j\,n_j\,(\vartheta + \nu_0\,\partial_t\vartheta), \tag{2.22}$$

$$q_j\,n_j = -T_0\,\eta_{jl}\,n_j\,\partial_l\vartheta.$$

For convenience we introduce the following matrix differential operator

$$\mathcal{T} = \mathcal{T}(\partial_x, n, \partial_t) = \big[\, \mathcal{T}_{pq}(\partial_x, n, \partial_t) \,\big]_{6\times 6}$$
$$:= \begin{bmatrix} [c_{rjkl}\, n_j\, \partial_l]_{3\times 3} & [e_{lrj}\, n_j\, \partial_l]_{3\times 1} & [q_{lrj}\, n_j\, \partial_l]_{3\times 1} & [-\lambda_{rj}\, n_j (1+\nu_0\,\partial_t)]_{3\times 1} \\ [-e_{jkl}\, n_j\, \partial_l]_{1\times 3} & \varkappa_{jl}\, n_j\, \partial_l & a_{jl}\, n_j\, \partial_l & -p_j\, n_j (1+\nu_0\,\partial_t) \\ [-q_{jkl}\, n_j\, \partial_l]_{1\times 3} & a_{jl}\, n_j\, \partial_l & \mu_{jl}\, n_j\, \partial_l & -m_j\, n_j (1+\nu_0\,\partial_t) \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\, n_j\, \partial_l \end{bmatrix}_{6\times 6}. \tag{2.23}$$

Evidently, for a smooth six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial_x, n, \partial_t)\, U = (\,\sigma_{1j}\, n_j,\ \sigma_{2j}\, n_j,\ \sigma_{3j}\, n_j,\ -D_j\, n_j,\ -B_j\, n_j,\ -T_0^{-1} q_j\, n_j)^\top. \tag{2.24}$$

Due to the constitutive equations, the components of the vector $\mathcal{T}\,U$ given by (2.24) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the fourth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector respectively with opposite sign, and finally the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector.

Note that the following pairs are called like fields:

(i) $\{u = (u_1, u_2, u_3)^\top,\ (\sigma_{1j}\, n_j,\ \sigma_{2j}\, n_j,\ \sigma_{3j}\, n_j)^\top\}$—pair of mechanical fields,

(ii) $\{\varphi,\ -D_j\, n_j\}$—pair of electric fields,

(iii) $\{\psi,\ -B_j\, n_j\}$—pair of magnetic fields,

(iv) $\{\vartheta,\ -T_0^{-1} q_j\, n_j\}$—pair of thermal fields.

As we see all the thermo-mechanical and electro-magnetic characteristics can be determined by the six functions: three displacement components u_j, $j = 1, 2, 3$, temperature distribution ϑ, and the electric and magnetic potentials φ and ψ. Therefore, all the above field relations and the corresponding boundary value problems we reformulate in terms of these six functions.

First of all, from Eqs. (2.5)–(2.7) with the help of the constitutive relations (2.1)–(2.4) we derive the basic linear system of dynamics of the generalized thermo-electro-magneto-elasticity theory of homogeneous solids

$$\begin{aligned} & c_{rjkl}\, \partial_j \partial_l\, u_k(x,t) + e_{lrj}\, \partial_j \partial_l \varphi(x,t) + q_{lrj}\, \partial_j \partial_l \psi(x,t) - \lambda_{rj}\, \partial_j \vartheta(x,t) - \nu_0 \lambda_{rj}\, \partial_j \partial_t \vartheta(x,t) \\ & \quad - \varrho\, \partial_t^2\, u_r(x,t) = -\varrho F_r(x,t), \quad r = 1, 2, 3, \\ & -e_{jkl}\, \partial_j \partial_l u_k(x,t) + \varkappa_{jl}\, \partial_j \partial_l \varphi(x,t) + a_{jl}\, \partial_j \partial_l \psi(x,t) - p_j\, \partial_j \vartheta(x,t) - \nu_0\, p_j\, \partial_j \partial_t \vartheta(x,t) = -\varrho_e(x,t), \\ & -q_{jkl}\, \partial_j \partial_l u_k(x,t) + a_{jl}\, \partial_j \partial_l \varphi(x,t) + \mu_{jl}\, \partial_j \partial_l \psi(x,t) - m_j\, \partial_j \vartheta(x,t) - \nu_0\, m_j\, \partial_j \partial_t \vartheta(x,t) = -\varrho_c(x,t), \\ & -\lambda_{kl}\, \partial_t \partial_l u_k(x,t) + p_l\, \partial_l \partial_t \varphi(x,t) + m_l\, \partial_l \partial_t \psi(x,t) + \eta_{jl}\, \partial_j \partial_l \vartheta(x,t) - d_0\, \partial_t \vartheta(x,t) \\ & \quad - h_0\, \partial_t^2 \vartheta(x,t) = -T_0^{-1} \varrho\, Q(x,t). \end{aligned} \tag{2.25}$$

Let us introduce the matrix differential operator generated by the left hand side expressions in Eqs. (2.25),

$$A(\partial_x, \partial_t) = \big[A_{pq}(\partial_x, \partial_t)\big]_{6\times 6}$$
$$:= \begin{bmatrix} [c_{rjkl}\, \partial_j \partial_l - \varrho\, \delta_{rk}\, \partial_t^2\,]_{3\times 3} & [e_{lrj}\, \partial_j \partial_l]_{3\times 1} & [q_{lrj}\, \partial_j \partial_l]_{3\times 1} & [-\lambda_{rj}\, \partial_j (1+\nu_0 \partial_t)]_{3\times 1} \\ [-e_{jkl}\, \partial_j \partial_l]_{1\times 3} & \varkappa_{jl}\, \partial_j \partial_l & a_{jl}\, \partial_j \partial_l & -p_j\, \partial_j (1+\nu_0 \partial_t) \\ [-q_{jkl}\, \partial_j \partial_l]_{1\times 3} & a_{jl}\, \partial_j \partial_l & \mu_{jl}\, \partial_j \partial_l & -m_j\, \partial_j (1+\nu_0 \partial_t) \\ [-\lambda_{kl}\, \partial_l \partial_t]_{1\times 3} & p_l\, \partial_l \partial_t & m_l\, \partial_l \partial_t & \eta_{jl}\, \partial_j \partial_l - d_0\, \partial_t - h_0\, \partial_t^2 \end{bmatrix}_{6\times 6}. \tag{2.26}$$

Then Eqs. (2.25) can be rewritten in matrix form

$$A(\partial_x, \partial_t)\, U(x,t) = \Phi(x,t)\,, \tag{2.27}$$

where

$$U = (u_1, u_2, u_3, u_4, u_5, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top$$

is the sought for vector function and

$$\Phi = (\Phi_1, \ldots, \Phi_6)^\top := (-\varrho F_1, -\varrho F_2, -\varrho F_3, -\varrho_e, -\varrho_c, -\varrho T_0^{-1} Q)^\top \tag{2.28}$$

is the given vector function.

If all the functions involved in these equations are harmonic time dependent, that is they can be represented as the product of a function of the spatial variables (x_1, x_2, x_3) and the multiplier $\exp\{\tau\, t\}$, where $\tau = \sigma + i\omega$ is a complex parameter, we have the *pseudo-oscillation equations* of the generalized thermo-electro-magneto-elasticity theory. Note that the pseudo-oscillation equations can be obtained from the corresponding dynamical equations by the Laplace transform. If $\tau = i\omega$ is a pure imaginary number, with the so called frequency parameter $\omega \in \mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau = 0$, i.e., the functions involved in Eqs. (2.25) are independent of t, we get the *equations of statics*.

Recall that for a smooth function $v(t)$ which is exponentially bounded,

$$e^{-\sigma_0 t}\big[\, |v(t)| + |\partial v(t)| + |\partial_t^2 v(t)|\,\big] = \mathcal{O}(1) \ \text{ as } \ t \to +\infty, \quad \sigma_0 \geqslant 0, \tag{2.29}$$

the corresponding Laplace transform

$$\widehat{v}(\tau) \equiv L_{t\to\tau}[v(t)] := \int_0^{+\infty} e^{-\tau t}\, v(t)\, dt, \quad \tau = \sigma + i\omega, \quad \sigma > \sigma_0, \tag{2.30}$$

possesses the following properties

$$L_{t\to\tau}[\partial_t v(t)] := \int_0^{+\infty} e^{-\tau t}\, \partial_t v(t)\, dt = -v(0) + \tau\, \widehat{v}(\tau), \tag{2.31}$$

$$L_{t\to\tau}[\partial_t^2 v(t)] := \int_0^{+\infty} e^{-\tau t}\, \partial_t^2 v(t)\, dt = -\partial_t v(0) - \tau v(0) + \tau^2\, \widehat{v}(\tau). \tag{2.32}$$

Provided that all the functions involved in the dynamical equations (2.25) are exponentially bounded and applying the Laplace transform to the system (2.25), we obtain the following pseudo-oscillation equations:

$$\begin{aligned}
& c_{rjkl}\, \partial_j \partial_l\, \widehat{u}_k(x,\tau) - \varrho\, \tau^2\, \widehat{u}_r(x,\tau) + e_{lrj}\, \partial_j \partial_l \widehat{\varphi}(x,\tau) + q_{lrj}\, \partial_j \partial_l \widehat{\psi}(x,\tau) \\
& \quad - (1+\nu_0\tau)\lambda_{rj}\, \partial_j \widehat{\vartheta}(x,\tau) = -\varrho \widehat{F}_r(x,\tau) + \Psi_r^{(0)}(x,\tau), \quad r = 1,2,3, \\
& -e_{jkl}\, \partial_j \partial_l \widehat{u}_k(x,\tau) + \varkappa_{jl}\, \partial_j\, \partial_l \widehat{\varphi}(x,\tau) + a_{jl}\, \partial_j\, \partial_l \widehat{\psi}(x,\tau) - (1+\nu_0\tau) p_j\, \partial_j \widehat{\vartheta}(x,\tau) \\
& \quad = -\widehat{\varrho}_e(x,\tau) + \Psi_4^{(0)}(x,\tau), \\
& -q_{jkl}\, \partial_j \partial_l \widehat{u}_k(x,\tau) + a_{jl}\, \partial_j\, \partial_l \widehat{\varphi}(x,\tau) + \mu_{jl}\, \partial_j\, \partial_l \widehat{\psi}(x,\tau) - (1+\nu_0\tau) m_j\, \partial_j \widehat{\vartheta}(x,\tau) \\
& \quad = -\widehat{\varrho}_c(x,\tau) + \Psi_5^{(0)}(x,\tau), \\
& -\tau\, \lambda_{kl}\, \partial_l \widehat{u}_k(x,\tau) + \tau\, p_l\, \partial_l \widehat{\varphi}(x,\tau) + \tau\, m_l\, \partial_l \widehat{\psi}(x,\tau) + \eta_{jl}\, \partial_j \partial_l \widehat{\vartheta}(x,\tau) \\
& \quad - (\tau d_0 + \tau^2 h_0) \widehat{\vartheta}(x,\tau) = -T_0^{-1} \varrho\, \widehat{Q}(x,\tau) + \Psi_6^{(0)}(x,\tau),
\end{aligned} \tag{2.33}$$

where the overset "hat" denotes the Laplace transform of the corresponding function with respect to t (see (2.30)) and

$$\Psi^{(0)}(x,\tau) \;=\; \big(\Psi_1^{(0)}(x,\tau), \;\ldots\;, \Psi_6^{(0)}(x,\tau)\big)^\top$$
$$:= \begin{bmatrix}
-\varrho\, \tau\, u_1(x,0) - \varrho\, \partial_t u_1(x,0) - \nu_0\, \lambda_{1j}\, \partial_j \vartheta(x,0) \\
-\varrho\, \tau\, u_2(x,0) - \varrho\, \partial_t u_2(x,0) - \nu_0\, \lambda_{2j}\, \partial_j \vartheta(x,0) \\
-\varrho\, \tau\, u_3(x,0) - \varrho\, \partial_t u_3(x,0) - \nu_0\, \lambda_{3j}\, \partial_j \vartheta(x,0) \\
\nu_0\, p_j\, \partial_j \vartheta(x,0) \\
\nu_0\, m_j\, \partial_j \vartheta(x,0) \\
-\lambda_{kl}\, \partial_l u_k(x,0) + p_j\, \partial_l \varphi(x,0) + m_j\, \partial_l \psi(x,0) - (d_0 + \tau\, h_0)\vartheta(x,0) - h_0 \partial_t \vartheta(x,0)
\end{bmatrix}. \tag{2.34}$$

Note that the relations (2.30)–(2.32) can be extended to the spaces of generalized functions (see e.g., [25]).

In matrix form these pseudo-oscillation equations can be rewritten as

$$A(\partial_x,\tau)\,\widehat{U}(x,\tau)=\Psi(x,\tau)\,,$$

where

$$\widehat{U}=(\widehat{u}_1,\widehat{u}_2,\widehat{u}_3,\widehat{u}_4,\widehat{u}_5,\widehat{u}_6)^\top:=(\widehat{u},\widehat{\varphi},\widehat{\psi},\widehat{\vartheta})^\top$$

is the sought for vector function,

$$\Psi(x,\tau)=\big(\Psi_1(x,\tau),\ldots,\Psi_6(x,\tau)\big)^\top=\widehat{\Phi}(x,\tau)+\Psi^{(0)}(x,\tau)$$

with $\widehat{\Phi}(x,\tau)$ being the Laplace transform of the vector function $\Phi(x,t)$ defined in (2.28) and $\Psi^{(0)}(x,\tau)$ given by (2.34), and $A(\partial_x,\tau)$ is the pseudo-oscillation matrix differential operator generated by the left hand side expressions in Eq. (2.33),

$$A(\partial_x,\tau)\;=\;\big[A_{pq}(\partial_x,\tau)\big]_{6\times6}$$
$$:=\begin{bmatrix}[c_{rjkl}\partial_j\partial_l-\varrho\tau^2\delta_{rk}]_{3\times3} & [e_{lrj}\partial_j\partial_l]_{3\times1} & [q_{lrj}\partial_j\partial_l]_{3\times1} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3\times1}\\ [-e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -(1+\nu_0\tau)\,p_j\partial_j\\ [-q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\,\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -(1+\nu_0\tau)m_j\partial_j\\ [-\tau\lambda_{kl}\partial_l]_{1\times3} & \tau p_l\partial_l & \tau m_l\partial_l & \eta_{jl}\partial_j\partial_l-\tau^2h_0-\tau d_0\end{bmatrix}_{6\times6}. \tag{2.35}$$

It is evident that the operator

$$A^{(0)}(\partial_x):=\begin{bmatrix}[c_{rjkl}\partial_j\partial_l]_{3\times3} & [e_{lrj}\partial_j\partial_l]_{3\times1} & [q_{lrj}\partial_j\partial_l]_{3\times1} & [0]_{3\times1}\\ [-e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & 0\\ [-q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & 0\\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\partial_j\partial_l\end{bmatrix}_{6\times6}$$

is the principal part of the operators $A(\partial_x,\tau)$. Clearly, the symbol matrix $A^{(0)}(-i\,\xi)$, $\xi\in\mathbb{R}^3$, of the operator $A^{(0)}(\partial_x)$ is the principal homogeneous symbol matrix of the operator $A(\partial_x,\tau)$ for all $\tau\in\mathbb{C}$,

$$A^{(0)}(-i\,\xi):=\begin{bmatrix}[-c_{rjkl}\,\xi_j\,\xi_l]_{3\times3} & [-e_{lrj}\,\xi_j\xi_l]_{3\times1} & [-q_{lrj}\,\xi_j\xi_l]_{3\times1} & [0]_{3\times1}\\ [e_{jkl}\,\xi_j\xi_l]_{1\times3} & -\varkappa_{jl}\,\xi_j\xi_l & -a_{jl}\,\xi_j\xi_l & 0\\ [q_{jkl}\,\xi_j\xi_l]_{1\times3} & -a_{jl}\,\xi_j\xi_l & -\mu_{jl}\,\xi_j\xi_l & 0\\ [0]_{1\times3} & 0 & 0 & -\eta_{jl}\,\xi_j\,\xi_l\end{bmatrix}_{6\times6}.$$

From the symmetry conditions (2.8), inequalities (2.9) and positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.13) it follows that there is a positive constant C_0 depending only on the material parameters, such that

$$\mathrm{Re}\,\big(-A^{(0)}(-i\,\xi)\zeta\cdot\zeta\big)=\mathrm{Re}\,\Bigg(-\sum_{k,j=1}^{6}A^{(0)}_{kj}(-i\,\xi)\,\zeta_j\,\overline{\zeta_k}\Bigg)\geq\;C_0\,|\xi|^2\,|\zeta|^2$$
$$\text{for all}\;\;\xi\in\mathbb{R}^3\;\;\text{and}\;\;\text{for all}\;\;\zeta\in\mathbb{C}^6.$$

Therefore, $-A(\partial_x,\tau)$ is a non-selfadjoint strongly elliptic differential operator. We recall that the over bar denotes complex conjugation and the central dot denotes the scalar product in the respective complex vector space. By $A^*(\partial_x,\tau):=[\overline{A(-\partial_x,\tau)}]^\top=A^\top(-\partial_x,\overline{\tau})$ we denote the operator formally adjoint to $A(\partial_x,\tau)$,

$$A^*(\partial_x,\tau)\;=\;\Big[A^*_{pq}(\partial_x,\tau)\Big]_{6\times6}$$
$$:=\begin{bmatrix}[c_{rjkl}\partial_j\partial_l-\varrho\,\overline{\tau}^2\,\delta_{rk}]_{3\times3} & [-e_{lrj}\partial_j\partial_l]_{3\times1} & [-q_{lrj}\partial_j\partial_l]_{3\times1} & [\overline{\tau}\lambda_{kl}\partial_l]_{3\times1}\\ [e_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -\overline{\tau}\,p_l\,\partial_l\\ [q_{jkl}\partial_j\partial_l]_{1\times3} & a_{jl}\,\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -\overline{\tau}\,m_l\,\partial_l\\ [(1+\nu_0\overline{\tau})\lambda_{rj}\partial_j]_{1\times3} & (1+\nu_0\overline{\tau})\,p_j\partial_j & (1+\nu_0\overline{\tau})m_j\partial_j & \eta_{jl}\partial_j\partial_l-\overline{\tau}^2\,h_0-\overline{\tau}\,d_0\end{bmatrix}_{6\times6}. \tag{2.36}$$

Applying the Laplace transform to the dynamical constitutive relations (2.1)–(2.3) and (2.7) we get

$$\begin{aligned}
\widehat{\sigma_{rj}}(x,\tau) &= c_{rjkl}\,\widehat{\varepsilon_{kl}}(x,\tau) + e_{lrj}\partial_l\widehat{\varphi}(x,\tau) + q_{lrj}\partial_l\widehat{\psi}(x,\tau) - (1+\nu_0\tau)\lambda_{rj}\,\widehat{\vartheta}(x,\tau)\\
&\quad + \nu_0\lambda_{rj}\vartheta(x,0), \qquad r,j=1,2,3,\\
\widehat{D_j}(x,\tau) &= e_{jkl}\,\widehat{\varepsilon_{kl}}(x,\tau) - \varkappa_{jl}\,\partial_l\widehat{\varphi}(x,\tau) - a_{jl}\,\partial_l\widehat{\psi}(x,\tau) + (1+\nu_0\tau)p_j\widehat{\vartheta}(x,\tau)\\
&\quad - \nu_0 p_j\vartheta(x,0), \qquad j=1,2,3,\\
\widehat{B_j}(x,\tau) &= q_{jkl}\,\widehat{\varepsilon_{kl}}(x,\tau) - a_{jl}\,\partial_l\widehat{\varphi}(x,\tau) - \mu_{jl}\partial_l\widehat{\psi}(x,\tau) + (1+\nu_0\tau)m_j\widehat{\vartheta}(x,\tau)\\
&\quad - \nu_0 m_j\vartheta(x,0), \qquad j=1,2,3,\\
\widehat{q_j}(x,\tau) &= -\,T_0\,\eta_{jl}\partial_l\widehat{\vartheta}(x,\tau).
\end{aligned}$$

With the help of these equalities, the Laplace transform of the stress vector $\mathcal{T}(\partial_x,n,\partial_t)\,U(x,t)$ defined in (2.24) can be represented as follows

$$L_{t\to\tau}[\mathcal{T}(\partial_x,n,\partial_t)\,U(x,t)] = \mathcal{T}(\partial_x,n,\tau)\,\widehat{U}(x,\tau) + F^{(0)}(x),$$

where

$$\mathcal{T}(\partial_x,n,\tau)\,\widehat{U}(x,\tau) = \big(\widehat{\sigma_{1j}}n_j, \widehat{\sigma_{2j}}n_j, \widehat{\sigma_{3j}}n_j, -\widehat{D_j}n_j, -\widehat{B_j}n_j, -T_0^{-1}\widehat{q_j}n_j\big) - F^{(0)}(x),$$

$$F^{(0)}(x) := \begin{bmatrix} \nu_0\lambda_{1j}\,n_j\vartheta(x,0)\\ \nu_0\lambda_{2j}\,n_j\vartheta(x,0)\\ \nu_0\lambda_{3j}\,n_j\vartheta(x,0)\\ \nu_0 p_j\,n_j\vartheta(x,0)\\ \nu_0 m_j\,n_j\vartheta(x,0)\\ 0\end{bmatrix},$$

and the boundary operator $\mathcal{T}(\partial_x,n,\tau)$ reads as (cf. (2.23))

$$\begin{aligned}
\mathcal{T}(\partial_x,n,\tau) &= \big[\,\mathcal{T}_{pq}(\partial_x,n,\tau)\,\big]_{6\times6}\\
&:= \begin{bmatrix} [c_{rjkl}\,n_j\partial_l]_{3\times3} & [e_{lrj}\,n_j\partial_l]_{3\times1} & [q_{lrj}\,n_j\partial_l]_{3\times1} & [-(1+\nu_0\tau)\lambda_{rj}n_j]_{3\times1}\\ [-e_{jkl}\,n_j\partial_l]_{1\times3} & \varkappa_{jl}\,n_j\partial_l & a_{jl}\,n_j\partial_l & -(1+\nu_0\tau)p_jn_j\\ [-q_{jkl}\,n_j\partial_l]_{1\times3} & a_{jl}\,n_j\partial_l & \mu_{jl}\,n_j\partial_l & -(1+\nu_0\tau)m_jn_j\\ [0]_{1\times3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times6}.
\end{aligned} \tag{2.37}$$

Below, in Green's formulas there appears also the boundary operator $\mathcal{P}(\partial_x,n,\tau)$ associated with the adjoint differential operator $A^*(\partial_x,\tau)$,

$$\begin{aligned}
\mathcal{P}(\partial_x,n,\tau) &= \big[\,\mathcal{P}_{pq}(\partial_x,n,\tau)\,\big]_{6\times6}\\
&= \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times3} & [-e_{lrj}\,n_j\,\partial_l]_{3\times1} & [-q_{lrj}\,n_j\,\partial_l]_{3\times1} & [\overline{\tau}\,\lambda_{rj}\,n_j]_{3\times1}\\ [e_{jkl}\,n_j\,\partial_l]_{1\times3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l & -\overline{\tau}\,p_j\,n_j\\ [q_{jkl}\,n_j\,\partial_l]_{1\times3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l & -\overline{\tau}\,m_j\,n_j\\ [0]_{1\times3} & 0 & 0 & \eta_{jl}\,n_j\,\partial_l \end{bmatrix}_{6\times6}.
\end{aligned} \tag{2.38}$$

2.2. Green's formulas for the pseudo-oscillation model

Let $\Omega=\Omega^+$ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $S=\partial\Omega^+$ and $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega^+}$, $\overline{\Omega^+}=\Omega^+\cup S$. By $C^k(\overline{\Omega})$ we denote the subspace of functions from $C^k(\Omega)$ whose derivatives up to the order k are continuously extendable to $S=\partial\Omega$ from Ω.

The symbols $\{\,\cdot\,\}_S^+$ and $\{\,\cdot\,\}_S^-$ denote one sided limits on S from Ω^+ and Ω^- respectively. We drop the subscript in $\{\,\cdot\,\}_S^\pm$ if it does not lead to misunderstanding.

By L_p, $L_{p,loc}$, $L_{p,comp}$, W_p^r, $W_{p,loc}^r$, $W_{p,comp}^r$, H_p^s, and $B_{p,q}^s$ (with $r\geq0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\leq q\leq\infty$) we denote the well-known Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [26,27]). Recall that $H_2^r=W_2^r=B_{2,2}^r$, $H_2^s=B_{2,2}^s$, $W_p^t=B_{p,p}^t$, and $H_p^k=W_p^k$, for any $r\geq0$, for any

$s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. In our analysis we essentially employ also the spaces:

$$\widetilde{H}^s_p(\mathcal{M}) := \{f \ : \ f \in H^s_p(\mathcal{M}_0), \ \text{supp}\, f \subset \overline{\mathcal{M}}\},$$
$$\widetilde{B}^s_{p,q}(\mathcal{M}) := \{f \ : \ f \in B^s_{p,q}(\mathcal{M}_0), \ \text{supp}\, f \subset \overline{\mathcal{M}}\},$$
$$H^s_p(\mathcal{M}) := \{r_{\mathcal{M}} f \ : \ f \in H^s_p(\mathcal{M}_0)\},$$
$$B^s_{p,q}(\mathcal{M}) := \{r_{\mathcal{M}} f \ : \ f \in B^s_{p,q}(\mathcal{M}_0)\},$$

where $\mathcal{M}_0$ is a closed manifold without boundary and $\mathcal{M}$ is an open proper submanifold of $\mathcal{M}_0$ with nonempty boundary $\partial\mathcal{M} \neq \varnothing$; $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$. Below, sometimes we use also the abbreviations $H^s_2 = H^s$ and $W^s_2 = W^s$.

If a function $f \in B^s_{p,q}(\mathcal{M})$, where $\mathcal{M}$ is a proper part of a closed surface $\mathcal{M}_0$, can be extended by zero to the whole $\mathcal{M}_0$ preserving the space, we write $f \in \widetilde{B}^s_{p,q}(\mathcal{M})$ instead of $f \in r_{\mathcal{M}}\widetilde{B}^s_{p,q}(\mathcal{M})$.

For arbitrary vector functions

$$U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top \in \big[C^2(\overline{\Omega})\big]^6 \text{ and } U' = (u_1', u_2', u_3', \varphi', \psi', \vartheta')^\top \in \big[C^2(\overline{\Omega})\big]^6$$

we can derive the following Green's identities with the help of the Gauss integration by parts formula:

$$\int_\Omega \Big[A(\partial_x, \tau)\, U \cdot U' + \mathcal{E}_\tau(U, \overline{U'})\Big] dx = \int_S \{\mathcal{T}(\partial_x, n, \tau)U\}^+ \cdot \{U'\}^+ dS\,, \tag{2.39}$$

$$\int_\Omega \Big[U \cdot A^*(\partial_x, \tau)\, U' + \mathcal{E}_\tau(U, \overline{U'})\Big] dx = \int_S \{U\}^+ \cdot \{\mathcal{P}(\partial_x, n, \tau)U'\}^+ dS\,, \tag{2.40}$$

$$\int_\Omega \Big[A(\partial_x, \tau)\, U \cdot U' - U \cdot A^*(\partial_x, \tau)\, U'\Big] dx$$
$$= \int_S \Big[\{\mathcal{T}(\partial_x, n, \tau)U\}^+ \cdot \{U'\}^+ - \{U\}^+ \cdot \{\mathcal{P}(\partial_x, n, \tau)\, U'\}^+\Big] dS, \tag{2.41}$$

where the operators $A(\partial_x, \tau)$, $\mathcal{T}(\partial_x, n, \tau)$, $A^*(\partial_x, \tau)$ and $\mathcal{P}(\partial_x, n, \tau)$ are given in (2.35), (2.37), (2.36), and (2.38) respectively,

$$\begin{aligned}
\mathcal{E}_\tau(U, \overline{U'}) := {} & c_{rjkl}\, \partial_l u_k\, \overline{\partial_j u_r'} + \varrho\, \tau^2\, u_r\, \overline{u_r'} + e_{lrj}\, (\partial_l \varphi\, \overline{\partial_j u_r'} - \partial_j u_r\, \overline{\partial_l \varphi'}) \\
& + q_{lrj}\, (\partial_l \psi\, \overline{\partial_j u_r'} - \partial_j u_r\, \overline{\partial_l \psi'}) + \varkappa_{jl}\, \partial_l \varphi\, \overline{\partial_j \varphi'} + a_{jl}\, (\partial_l \varphi\, \overline{\partial_j \psi'} + \partial_j \psi\, \overline{\partial_l \varphi'}) \\
& + \mu_{jl}\, \partial_l \psi\, \overline{\partial_j \psi'} + \lambda_{kj}\, [\tau\, \overline{\vartheta'}\, \partial_j u_k - (1 + \nu_0 \tau) \vartheta\, \overline{\partial_j u_k'}] - p_l\, [\tau\, \overline{\vartheta'}\, \partial_l \varphi + (1 + \nu_0 \tau) \vartheta\, \overline{\partial_l \varphi'}] \\
& - m_l\, [\tau\, \overline{\vartheta'}\, \partial_l \psi + (1 + \nu_0 \tau) \vartheta\, \overline{\partial_l \psi'}] + \eta_{jl}\, \partial_l \vartheta\, \overline{\partial_j \vartheta'} + \tau\, (h_0 \tau + d_0)\, \vartheta\, \overline{\vartheta'}.
\end{aligned} \tag{2.42}$$

Note that the above Green's formula (2.39) by standard limiting procedure can be generalized to Lipschitz domains and to vector functions $U \in [W^1_p(\Omega)]^6$ and $U' \in [W^1_{p'}(\Omega)]^6$ with

$$A(\partial_x, \tau) U \in \big[L_p(\Omega)\big]^6, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

With the help of Green's formula (2.39) we can correctly determine a *generalized trace vector* $\{\mathcal{T}(\partial_x, n, \tau)U\}^+ \in [B^{-\frac{1}{p}}_{p,p}(S)]^6$ for a function $U \in [W^1_p(\Omega)]^6$ with $A(\partial_x, \tau)U \in [L_p(\Omega)]^6$ by the following relation (cf. [28–30])

$$\big\langle \{\mathcal{T}(\partial_x, n, \tau)U\}^+\,,\ \{U'\}^+ \big\rangle_S := \int_\Omega [\, A(\partial_x, \tau)\, U \cdot U' + \mathcal{E}_\tau(U, \overline{U'})\,]\, dx, \tag{2.43}$$

where $U' \in [W^1_{p'}(\Omega)]^6$ is an arbitrary vector function. Here the symbol $\langle \cdot\,, \cdot \rangle_S$ denotes the duality between the function spaces $[B^{-\frac{1}{p}}_{p,p}(S)]^6$ and $[B^{\frac{1}{p}}_{p',p'}(S)]^6$ which extends the usual L_2 inner product for complex valued vector functions,

$$\langle f, g\rangle_S = \int_S \sum_{j=1}^{6} f_j(x)\,\overline{g_j(x)}\,dS \quad \text{for} \quad f, g \in [L_2(S)]^6.$$

Evidently we have the following estimate

$$\|\{\mathcal{T}(\partial_x, n, \tau)U\}^+\|_{[B_{p,p}^{-1/p}(S)]^6} \le c_0 \left\{\|A(\partial_x, \tau)\,U\|_{[L_p(\Omega)]^6} + (1+|\tau|^2)\|U\|_{[W_p^1(\Omega)]^6}\right\},$$

where c_0 does not depend on U; in general c_0 depends on the material parameters and on the geometrical characteristics of the domain Ω.

Let us introduce a sesquilinear form on $[H_2^1(\Omega)]^6 \times [H_2^1(\Omega)]^6$

$$\mathcal{B}(U, V) := \int_\Omega \mathcal{E}_\tau(U, \overline{V})\,dx,$$

where $\mathcal{E}_\tau(U, \overline{V})$ is defined by (2.42). With the help of the relations (2.9) and (2.42), positive definiteness of the matrix (2.13) and the well known Korn's inequality we deduce the following estimate

$$\operatorname{Re}\mathcal{B}(U, U) \ge c_1\,\|U\|^2_{[H_2^1(\Omega)]^6} - c_2\,\|U\|^2_{[H_2^0(\Omega)]^6} \tag{2.44}$$

with some positive constants c_1 and c_2 depending on the material parameters (cf. [17,29]), which shows that the sesquilinear form defined in (2.44) is coercive.

From the Green formulas (2.39)–(2.41) by standard limiting procedure we derive similar formulas for the exterior domain Ω^- provided the vector functions $U,\ U' \in \mathbf{Z}(\Omega^-)$, where the class $\mathbf{Z}(\Omega^-)$ is defined as a set of functions U possessing the following asymptotic properties at infinity as $|x| \to \infty$:

$$\begin{aligned}
&u_k(x) = \mathcal{O}(|x|^{-2}), && \partial_j u_k(x) = \mathcal{O}(|x|^{-2}),\\
&\varphi(x) = \mathcal{O}(|x|^{-1}), && \partial_j \varphi(x) = \mathcal{O}(|x|^{-2}),\\
&\psi(x) = \mathcal{O}(|x|^{-1}), && \partial_j \psi(x) = \mathcal{O}(|x|^{-2}),\\
&\vartheta(x) = \mathcal{O}(|x|^{-2}), && \partial_j \vartheta(x) = \mathcal{O}(|x|^{-2}),\\
&k, j = 1, 2, 3.
\end{aligned} \tag{2.45}$$

Assume that $A^*(\partial_x, \tau)\,U'$ is compactly supported as well and U' satisfies the conditions of type (2.45). Then the following Green formulas hold for the exterior domain Ω^-:

$$\int_{\Omega^-}\Big[A(\partial_x, \tau)\,U \cdot U' + \mathcal{E}_\tau(U, \overline{U'})\Big]dx = -\int_S \{\mathcal{T}(\partial_x, n, \tau)U\}^- \cdot \{U'\}^- dS\,,$$

$$\int_{\Omega^-}\Big[U \cdot A^*(\partial_x, \tau)\,U' + \mathcal{E}_\tau(U, \overline{U'})\Big]dx = -\int_S \{U\}^- \cdot \{\mathcal{P}(\partial_x, n, \tau)U'\}^- dS\,,$$

$$\begin{aligned}\int_{\Omega^-}\Big[A(\partial_x, \tau)\,U \cdot U' - U \cdot A^*(\partial_x, \tau)\,U'\Big]dx \;=\; -\int_S\Big[&\{\mathcal{T}(\partial_x, n, \tau)U\}^- \cdot \{U'\}^-\\
&-\{U\}^- \cdot \{\mathcal{P}(\partial_x, n, \tau)\,U'\}^-\Big]dS,\end{aligned}$$

where $\mathcal{E}_\tau$ is defined by (2.42). We recall that the direction of the unit normal vector to $S = \partial\Omega^-$ is outward with respect to the domain $\Omega = \Omega^+$.

As we shall see below the fundamental matrix of the operator $A(\partial_x, \tau)$ with $\tau = \sigma + i\,\omega$, $\sigma > \sigma_0$, possesses the decay properties (2.45)

2.3. Boundary value problems for pseudo-oscillation equations

Throughout the paper we assume that the origin of the co-ordinate system belongs to Ω. Assume that the domain $\overline{\Omega}$ is occupied by an anisotropic homogeneous material with the above described generalized thermo-electro-magneto-elastic properties (henceforth such type of materials will be referred to as GTEME materials).

Further, we assume that $\partial\Omega$ is divided into two disjoint parts $S_D \neq \varnothing$ and S_N: $\partial\Omega = S = \overline{S}_D \cup \overline{S}_N$, $\overline{S}_D \cap \overline{S}_N = \varnothing$. Set $\partial S_D = \partial S_N =: \ell_m$. In what follows, for simplicity we assume that S, S_D, S_N, ℓ_m are C^∞-smooth.

Here we preserve the notation introduced in the previous subsections and formulate the boundary value problems for the pseudo-oscillation equations of the GTEME theory. The operators $A(\partial_x, \tau)$ and $\mathcal{T}(\partial_x, n, \tau)$ involved in the formulations below are determined by the relations (2.35) and (2.37) respectively. In what follows we always assume that

$$\tau = \sigma + i\omega, \quad \sigma > \sigma_0 \geqslant 0, \quad \omega \in \mathbb{R},$$

if not otherwise stated.

The Dirichlet pseudo-oscillation problem (D) $_\tau^+$: *Find a solution*

$$U = (u, \varphi, \psi, \vartheta)^\top \in [W_p^1(\Omega)]^6, \quad 1 < p < \infty$$

to the pseudo-oscillation equations of the GTEME theory,

$$A(\partial_x, \tau)\, U(x) = \Phi(x)\,, \qquad x \in \Omega, \tag{2.46}$$

satisfying the Dirichlet type boundary condition

$$\{U(x)\}^+ = f(x), \quad x \in S, \tag{2.47}$$

i.e.

$$\{u_r(x)\}^+ = f_r(x), \quad x \in S, \quad r = 1, 2, 3, \tag{2.48}$$

$$\{\varphi(x)\}^+ = f_4(x), \quad x \in S, \tag{2.49}$$

$$\{\psi(x)\}^+ = f_5(x), \quad x \in S, \tag{2.50}$$

$$\{\vartheta(x)\}^+ = f_6(x), \quad x \in S, \tag{2.51}$$

where $\Phi = (\Phi_1, \dots, \Phi_6)^\top \in [L_p(\Omega)]^6$, *and* $f = (f_1, \dots, f_6)^\top \in [B_{p,p}^{1-1/p}(S)]^6$, $1 < p < \infty$ *are given functions from the appropriate spaces*.

In the case when U satisfies the homogeneous equation

$$A(\partial_x, \tau)\, U(x) = 0, \qquad x \in \Omega, \tag{2.52}$$

we denote the corresponding problem by $(\mathrm{D})_{\tau,0}^+$.

The Neumann pseudo-oscillation problem $(N)_\tau^+$: *Find a regular solution*

$$U = (u, \varphi, \psi, \vartheta)^\top \in [W_p^1(\Omega)]^6, \quad 1 < p < \infty$$

to the pseudo-oscillation equations of the GTEME theory (2.46) *satisfying the Neumann type boundary condition*

$$\{\mathcal{T}(\partial_x, n, \tau) U(x)\}^+ = F(x), \qquad x \in S, \tag{2.53}$$

i.e.

$$\{[\mathcal{T}(\partial_x, n, \tau) U(x)]_r\}^+ \equiv \{\sigma_{rj}\, n_j\}^+ = F_r(x), \quad x \in S, \quad r = 1, 2, 3, \tag{2.54}$$

$$\{[\mathcal{T}(\partial_x, n, \tau) U(x)]_4\}^+ \equiv \{-D_j\, n_j\}^+ = F_4(x), \quad x \in S, \tag{2.55}$$

$$\{[\mathcal{T}(\partial_x, n, \tau) U(x)]_5\}^+ \equiv \{-B_j\, n_j\}^+ = F_5(x), \quad x \in S, \tag{2.56}$$

$$\{[\mathcal{T}(\partial_x, n, \tau) U(x)]_6\}^+ \equiv \{-T_0^{-1} q_j\, n_j\}^+ = F_6(x), \quad x \in S, \tag{2.57}$$

where $F = (F_1, \dots, F_6) \in [B_{p,p}^{-1/p}(S)]^6$, $1 < p < \infty$ *is a given vector function*.

In the case when U satisfies the homogeneous equation (2.52) we denote the corresponding problem by $(\mathrm{N})_{\tau,0}^+$.

Mixed boundary value problems for solids with interior cracks. Let us assume that a GTEME type solid occupying the simply connected domain $\overline{\Omega}$ contains an interior crack. We identify the crack surface as a two-dimensional, two-sided manifold Σ with the crack edge $\ell_c := \partial\Sigma$. We assume that Σ is a submanifold of a closed surface $S_0 \subset \Omega$ surrounding a domain $\overline{\Omega}_0 \subset \Omega$ and that Σ, S_0, and ℓ_c are C^∞-smooth. Denote $\Omega_\Sigma := \Omega \setminus \overline{\Sigma}$.

We write $v \in W_p^1(\Omega_\Sigma)$ if $v \in W_p^1(\Omega_0)$, $v \in W_p^1(\Omega \setminus \overline{\Omega_0})$, and $r_{S_0 \setminus \overline{\Sigma}}\{v\}^+ = r_{S_0 \setminus \overline{\Sigma}}\{v\}^-$.

Recall that throughout the paper $n = (n_1, n_2, n_3)$ stands for the exterior unit normal vector to $\partial\Omega$ and $S_0 = \partial\Omega_0$. This agreement defines the positive direction of the normal vector on the crack surface Σ.

We will consider the following problem $(\mathbf{MC})_\tau$:

(i) the magneto-piezoelectric elastic solid under consideration is mechanically fixed along the subsurface S_D, and at the same time there are given the temperature and the electric and magnetic potential functions (i.e., on S_D there are given the components of the vector $\{U\}^+$-Dirichlet conditions);

(ii) on the subsurface S_N there are prescribed the mechanical stress vector and the normal components of the heat flux, the electric displacement and magnetic induction vectors (i.e., on S_N there are given the components of the vector $\{\mathcal{T}U\}^+$-Neumann conditions);

(iii) the crack surface Σ is mechanically traction free and we assume that the temperature, electric and magnetic potentials, and the normal components of the heat flux, the electric displacement and magnetic induction vectors are continuous across the crack surface.

Reducing the nonhomogeneous differential equations (2.46) to the corresponding homogeneous ones, we can formulate the above problem mathematically as follows:

Find a vector $U = (u, \varphi, \psi, \theta)^\top = (u_1, u_2, u_3, u_4, u_5, u_6)^\top \in \left[W_p^1(\Omega_\Sigma)\right]^6$ *with* $1 < p < \infty$, *satisfying the homogeneous pseudo-oscillation differential equation of the GTEME theory*

$$A(\partial_x, \tau)\, U = 0 \ \text{ in } \ \Omega_\Sigma, \quad \tau = \sigma + i\,\omega, \quad \sigma > 0, \tag{2.58}$$

the crack conditions on Σ,

$$\{[\mathcal{T}U]_j\}^+ = F_j^+ \quad \text{on } \Sigma, \qquad j = \overline{1,3}, \tag{2.59}$$

$$\{[\mathcal{T}U]_j\}^- = F_j^- \quad \text{on } \Sigma, \qquad j = \overline{1,3}, \tag{2.60}$$

$$\{u_4\}^+ - \{u_4\}^- = f_4 \quad \text{on } \Sigma, \tag{2.61}$$

$$\{[\mathcal{T}U]_4\}^+ - \{[\mathcal{T}U]_4\}^- = F_4 \quad \text{on } \Sigma, \tag{2.62}$$

$$\{u_5\}^+ - \{u_5\}^- = f_5 \quad \text{on } \Sigma, \tag{2.63}$$

$$\{[\mathcal{T}U]_5\}^+ - \{[\mathcal{T}U]_5\}^- = F_5 \quad \text{on } \Sigma, \tag{2.64}$$

$$\{u_6\}^+ - \{u_6\}^- = f_6 \quad \text{on } \Sigma, \tag{2.65}$$

$$\{[\mathcal{T}U]_6\}^+ - \{[\mathcal{T}U]_6\}^- = F_6 \quad \text{on } \Sigma, \tag{2.66}$$

and the mixed boundary conditions on $S = \overline{S}_D \cup \overline{S}_N$,

$$\{U\}^+ = g^{(D)} \quad \text{on } S_D, \tag{2.67}$$

$$\{\mathcal{T}U\}^+ = g^{(N)} \quad \text{on } S_N. \tag{2.68}$$

We require that the boundary data possess the natural smoothness properties associated with the trace theorems,

$$\begin{aligned} &F_j^+, F_j^- \in B_{p,p}^{-\frac{1}{p}}(\Sigma), \quad j = 1,2,3; \quad f_4,\ f_5,\ f_6 \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma), \\ &F_4,\ F_5,\ F_6 \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \quad g^{(D)} \in \left[B_{p,p}^{1-\frac{1}{p}}(S_D)\right]^6, \quad g^{(N)} \in \left[B_{p,p}^{-\frac{1}{p}}(S_N)\right]^6, \\ &1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1. \end{aligned} \tag{2.69}$$

Moreover, the following compatibility conditions

$$F_j^+ - F_j^- \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma), \quad j = 1,2,3,$$

are to be satisfied.

The differential equation (2.58) is understood in the distributional sense, in general. We remark that if $U \in [W^1_p(\Omega_\Sigma)]^6$ solves the homogeneous differential equation then actually we have the inclusion $U \in [C^\infty(\Omega_\Sigma)]^6$ due to the ellipticity of the corresponding differential operators. In fact, U is a complex valued analytic vector function of spatial real variables (x_1, x_2, x_3) in Ω_Σ.

The Dirichlet-type conditions (2.61), (2.63), (2.65) and (2.67) are understood in the usual trace sense, while the Neumann-type conditions (2.59), (2.60), (2.62), (2.64), (2.66) and (2.68) involving boundary limiting values of the components of the vector $\mathcal{T}U$ are understood in the above described generalized functional sense related to Green's formula (2.43).

2.3.1. Uniqueness theorems for the pseudo-oscillation problems

We prove here the following uniqueness theorem for solutions to the pseudo-oscillation problems in the case of $p = 2$.

Theorem 2.1. *Let S be Lipschitz surface and $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$.*

(i) *The basic boundary value problem $(D)^+_\tau$ has at most one solution in the space $\left[W^1_2(\Omega)\right]^6$.*

(ii) *Solutions to the Neumann type boundary value problem $(N)^+_\tau$ in the space $\left[W^1_2(\Omega)\right]^6$ are defined modulo a vector of type $U^{(N)} = (0, 0, 0, b_1, b_2, 0)^\top$, where b_1 and b_2 are arbitrary constants.*

(iii) *Mixed type boundary value problem $(\mathbf{MC})_\tau$ has at most one solution in the space $\left[W^1_2(\Omega_\Sigma)\right]^6$.*

Proof. Due to the linearity of the boundary value problems in question it suffices to consider the corresponding homogeneous problems.

First we demonstrate the proof for the problems stated in the items (i) and (ii) of the theorem. Let $U = (u, \varphi, \psi, \vartheta)^\top \in \left[W^1_2(\Omega)\right]^6$ be a solution to the homogeneous problem $(\mathrm{D})^+_{\tau,0}$ or $(\mathrm{N})^+_{\tau,0}$. For arbitrary $U' = (u', \varphi', \psi', \vartheta')^\top \in \left[W^1_2(\Omega)\right]^6$ from Green's formula (2.43) we have

$$\int_\Omega \mathcal{E}_\tau(U, \overline{U'})\, dx = \big\langle \{\mathcal{T}(\partial_x, n, \tau)U\},\ \{U'\}^+ \big\rangle_{\partial\Omega}, \tag{2.70}$$

where $\mathcal{E}_\tau(U, \overline{U'})$ is given by (2.42).

If in (2.70) we substitute the vectors $(u_1, u_2, u_3, 0, 0, 0)^\top$, $(0, 0, 0, \varphi, 0, 0)^\top$, $(0, 0, 0, 0, \psi, 0)^\top$, and $(0, 0, 0, 0, 0, (1 + \nu_0\tau)[\overline{\tau}]^{-1}\vartheta)^\top$ for the vector U' successively and take into consideration the homogeneous boundary conditions, we get

$$\int_\Omega \big[c_{rjkl}\partial_l u_k \overline{\partial_j u_r} + \varrho\tau^2 u_r \overline{u_r} + e_{lrj}\, \partial_l\varphi\, \overline{\partial_j u_r} + q_{lrj}\, \partial_l\psi\, \overline{\partial_j u_r} - (1 + \nu_0\tau)\lambda_{kj}\vartheta\, \overline{\partial_j u_k}\,\big]dx = 0, \tag{2.71}$$

$$\int_\Omega \big[-e_{lrj}\, \partial_j u_r\, \overline{\partial_l\varphi} + \varkappa_{jl}\, \partial_l\varphi\, \overline{\partial_j\varphi} + a_{jl}\, \partial_j\psi\, \overline{\partial_l\varphi} - (1 + \nu_0\tau)p_l\, \vartheta\, \overline{\partial_l\varphi}\,\big]\, dx = 0, \tag{2.72}$$

$$\int_\Omega \big[-q_{lrj}\, \partial_j u_r\, \overline{\partial_l\psi} + a_{jl}\, \partial_l\varphi\, \overline{\partial_j\psi} + \mu_{jl}\, \partial_l\psi\, \overline{\partial_j\psi} - (1 + \nu_0\tau)m_l\, \vartheta\, \overline{\partial_l\psi}\,\big]\, dx = 0, \tag{2.73}$$

$$\int_\Omega \left\{(1 + \nu_0\overline{\tau})[\lambda_{kj}\overline{\vartheta}\partial_j u_k - p_l\overline{\vartheta}\partial_l\varphi - m_l\overline{\vartheta}\partial_l\psi + (h_0\tau + d_0)|\vartheta|^2] + \frac{1 + \nu_0\overline{\tau}}{\tau}\, \eta_{jl}\partial_l\vartheta\overline{\partial_j\vartheta}\right\}dx = 0. \tag{2.74}$$

Add to Eq. (2.71) the complex conjugate of Eqs. (2.72)–(2.74) and take into account the symmetry properties (2.8) to obtain

$$\begin{aligned}\int_\Omega \Big\{& c_{rjkl}\partial_l u_k\, \overline{\partial_j u_r} + \varrho\, \tau^2|u|^2 + \varkappa_{jl}\, \partial_l\varphi\, \overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\, \overline{\partial_j\varphi} + \partial_j\varphi\, \overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\, \overline{\partial_j\psi} \\ & - 2\,\mathrm{Re}\big[p_l(1 + \nu_0\tau)\vartheta\, \overline{\partial_l\varphi}\,\big] - 2\,\mathrm{Re}\big[m_l(1 + \nu_0\tau)\vartheta\, \overline{\partial_l\psi}\,\big] + (1 + \nu_0\tau)(h_0\overline{\tau} + d_0)\, |\vartheta|^2 \\ & + \frac{1 + \nu_0\tau}{\overline{\tau}}\, \eta_{jl}\, \partial_l\vartheta\, \overline{\partial_j\vartheta}\Big\}\, dx = 0. \end{aligned} \tag{2.75}$$

Due to the relations (2.11) and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.13), we find that

$$\begin{aligned}&c_{ijlk}\,\partial_i u_j\,\overline{\partial_l u_k} \geq 0, \quad \eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta} \geq 0,\\ &[\varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}\,(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi}] \geq \lambda_0(|\nabla\varphi|^2 + |\nabla\psi|^2),\end{aligned} \tag{2.76}$$

where λ_0 is a positive constant. Use the equalities

$$\begin{aligned}&\tau^2 = \sigma^2 - \omega^2 + 2i\sigma\omega, \quad \frac{1+\nu_0\tau}{\overline{\tau}} = \frac{\sigma + \nu_0(\sigma^2-\omega^2)}{|\tau|^2} + i\,\frac{\omega(1+2\sigma\nu_0)}{|\tau|^2},\\ &(1+\nu_0\tau)(h_0\overline{\tau} + d_0) = d_0 + \nu_0 h_0|\tau|^2 + (h_0+\nu_0 d_0)\sigma + i\,\omega(\nu_0 d_0 - h_0),\end{aligned}$$

and separate the imaginary part of (2.75) to deduce

$$\omega\int_\Omega\Big\{2\,\varrho\,\sigma\,|u|^2 + (\nu_0 d_0 - h_0)|\vartheta|^2 + \frac{1+2\sigma\nu_0}{|\tau|^2}\,\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta}\Big\}\,dx = 0.$$

By the inequalities in (2.10) and since $\sigma > \sigma_0 \geqslant 0$, we conclude $u = 0$ and $\vartheta = 0$ in Ω for $\omega \neq 0$. From (2.75) we then have

$$\int_\Omega\big[\varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}\,(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\,\partial_l\psi\,\overline{\partial_j\psi}\,\big]\,dx = 0.$$

Whence, in view of the last inequality in (2.76), we get $\partial_l\varphi = 0$, $\partial_l\psi = 0$, $l = 1, 2, 3$, in Ω. Thus, if $\omega \neq 0$,

$$u = 0, \quad \varphi = b_1 = \text{const}, \quad \psi = b_2 = \text{const}, \quad \vartheta = 0 \text{ in } \Omega. \tag{2.77}$$

If $\omega = 0$, then $\tau = \sigma > 0$ and (2.75) can be rewritten in the form

$$\begin{aligned}&\int_\Omega\Big\{c_{rjkl}\partial_l u_k\,\overline{\partial_j u_r} + \varrho\,\sigma^2|u|^2 + \frac{1+\nu_0\sigma}{\sigma}\,\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta}\Big\}\,dx\\ &\quad + \int_\Omega\Big\{\varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\,\overline{\partial_j\psi} - 2p_l(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta\,\overline{\partial_l\varphi}\,\big]\\ &\quad - 2m_l(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta\,\overline{\partial_l\psi}\,\big] + (1+\nu_0\sigma)(h_0\sigma + d_0)\,|\vartheta|^2\Big\}\,dx = 0.\end{aligned} \tag{2.78}$$

The integrand in the first integral is nonnegative. Let us show that the integrand in the second integral is also nonnegative. To this end, let us set

$$\zeta_j := \partial_j\varphi, \quad \zeta_{j+3} := \partial_j\psi, \quad \zeta_7 := -\vartheta, \quad \zeta_8 := -\sigma\vartheta, \quad j = 1, 2, 3,$$

and introduce the vector

$$\Theta := (\zeta_1, \zeta_2, \ldots, \zeta_8)^\top.$$

It can be easily checked that (summation over repeated indices is meant from 1 to 3)

$$\begin{aligned}&\varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\,\overline{\partial_j\psi} - 2p_l(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta\,\overline{\partial_l\varphi}\,\big]\\ &\qquad - 2m_l(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta\,\overline{\partial_l\psi}\,\big] + (1+\nu_0\sigma)(h_0\sigma + d_0)\,|\vartheta|^2\\ &\quad = [\varkappa_{jl}\,\partial_l\varphi + a_{jl}\,\partial_l\psi + p_j(-\vartheta) + \nu_0 p_j(-\sigma\vartheta)]\,\overline{\partial_j\varphi}\\ &\qquad + [a_{jl}\,\partial_l\varphi + \mu_{jl}\,\partial_l\psi + m_j(-\vartheta) + \nu_0 m_j(-\sigma\vartheta)]\,\overline{\partial_j\psi}\\ &\qquad + [p_l\partial_l\varphi + m_l\,\partial_l\psi + d_0(-\vartheta) + h_0(-\sigma\vartheta)](-\overline{\vartheta})\\ &\qquad + [\nu_0 p_l\partial_l\varphi + \nu_0 m_l\,\partial_l\psi + h_0(-\vartheta) + \nu_0 h_0(-\sigma\vartheta)](-\sigma\overline{\vartheta})\\ &\qquad + \sigma(d_0\nu_0 - h_0)\,|\vartheta|^2\\ &\quad = [\varkappa_{jl}\,\zeta_l + a_{jl}\,\zeta_{l+3} + p_j\zeta_7 + \nu_0 p_j\zeta_8]\,\overline{\zeta}_j\\ &\qquad + [a_{jl}\,\zeta_l + \mu_{jl}\,\zeta_{l+3} + m_j\zeta_7 + \nu_0 m_j\zeta_8]\,\overline{\zeta}_{j+3}\\ &\qquad + [p_l\zeta_l + m_l\,\zeta_{l+3} + d_0\zeta_7 + h_0\zeta_8]\,\overline{\zeta}_7\\ &\qquad + [\nu_0 p_l\zeta_l + \nu_0 m_l\,\zeta_{l+3} + h_0\zeta_7 + \nu_0 h_0\zeta_8]\,\overline{\zeta}_8\\ &\qquad + \sigma(d_0\nu_0 - h_0)\,|\vartheta|^2\\ &\quad = \sum_{p,q=1}^{8} M_{pq}\zeta_q\,\overline{\zeta}_p + \sigma(d_0\nu_0 - h_0)\,|\vartheta|^2 = M\,\Theta\cdot\Theta + \sigma(d_0\nu_0 - h_0)\,|\vartheta|^2 \geq C_0\,|\Theta|^2\end{aligned} \tag{2.79}$$

with some positive constant C_0 due to the positive definiteness of the matrix M defined by (2.12).

Therefore, from (2.78) we see that the relations (2.77) hold for $\omega = 0$ as well.

Thus the equalities (2.77) hold for arbitrary $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$, whence the items (i) and (ii) of the theorem follow immediately, since the homogeneous Dirichlet conditions for φ and ψ imply $b_1 = b_2 = 0$, while a vector $U^{(\mathcal{N})} = (0, 0, 0, b_1, b_2, 0)^\top$, where b_1 and b_2 are arbitrary constants, solves the homogeneous Neumann BVP $(N)^+_{\tau,0}$.

To prove the third item of the theorem we have to add together two Green's formulas of type (2.70) for the domains $\Omega \setminus \overline{\Omega_0}$ and Ω_0, where Ω_0 is the above introduced auxiliary domain $\Omega_0 \subset \Omega$. We recall that the crack surface Σ is a proper part of the boundary $S_0 = \partial\Omega_0 \subset \Omega$ and any solution to the homogeneous differential equation $A(\partial_x, \tau)\, U = 0$ of the class $\big[W^1_2(\Omega_\Sigma)\big]^6$ and its derivatives are continuous across the surface $S_0 \setminus \overline{\Sigma}$. If U is a solution to the homogeneous crack type BVP by the same approach as above, we arrive at the relation

$$\int_{\Omega_\Sigma} \Big\{ c_{rjkl}\partial_l u_k\, \overline{\partial_j u_r} + \varrho\,\tau^2 |u|^2 + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi} + a_{jl}(\partial_l\psi\,\overline{\partial_j\varphi} + \partial_j\varphi\,\overline{\partial_l\psi}) + \mu_{jl}\partial_l\psi\,\overline{\partial_j\psi}$$
$$- 2\,\mathrm{Re}\big[p_l(1+\nu_0\tau)\vartheta\,\overline{\partial_l\varphi}\,\big] - 2\,\mathrm{Re}\big[m_l(1+\nu_0\tau)\vartheta\,\overline{\partial_l\psi}\,\big] + (1+\nu_0\tau)(h_0\overline{\tau} + d_0)\,|\vartheta|^2$$
$$+ \frac{1+\nu_0\tau}{\overline{\tau}}\,\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta}\Big\}\, dx = 0. \tag{2.80}$$

The surface integrals vanish due to the homogeneous boundary and crack type conditions and the above mentioned continuity of solutions and its derivatives across the auxiliary surface $S_0 \setminus \overline{\Sigma}$. Therefore, the proof of item (iii) can be verbatim performed. □

3. Properties of potentials and boundary operators

The full symbol of the pseudo-oscillation differential operator $A(\partial_x, \tau)$ is elliptic provided $\mathrm{Re}\,\tau \neq 0$, i.e.,

$$\det A(-i\,\xi, \tau) \neq 0, \quad \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Moreover, the entries of the inverse matrix $A^{-1}(-i\,\xi, \tau)$ are locally integrable functions decaying at infinity as $\mathcal{O}(|\xi|^{-2})$. Therefore, we can construct the fundamental matrix $\Gamma(x, \tau) = [\Gamma_{kj}(x, \tau)]_{6\times6}$ of the operator $A(\partial_x, \tau)$ by means of the Fourier transform technique,

$$\Gamma(x, \tau) = \mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi, \tau)\big]. \tag{3.1}$$

The properties of the fundamental matrix $\Gamma(x, \tau)$ near the origin and at infinity, and the properties of corresponding layer potentials are studied in [31]. Here we collect some results which are necessary for our further analysis. Detailed proofs of the theorems below are similar to the proofs of their counterparts in [32,30,33–35].

Let us introduce the single and double layer potentials:

$$V(h)(x) = \int_S \Gamma(x-y, \tau)\, h(y)\, d_yS,$$
$$W(h)(x) = \int_S \Big[\mathcal{P}(\partial_y, n(y), \overline{\tau})\big[\Gamma(x-y, \tau)\big]^\top\Big]^\top h(y)\, d_yS,$$

where $h = (h_1, h_2, \ldots, h_6)^\top$ is a density vector function.

Theorem 3.1. *Let $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. Then the single and double layer potentials can be extended to the following continuous operators*

$$V : \big[B^s_{p,q}(S)\big]^6 \to \big[B^{s+1+\frac{1}{p}}_{p,q}(\Omega)\big]^6, \qquad W : \big[B^s_{p,q}(S)\big]^6 \to \big[B^{s+\frac{1}{p}}_{p,q}(\Omega)\big]^6,$$
$$: \big[B^s_{p,q}(S)\big]^6 \to \big[B^{s+1+\frac{1}{p}}_{p,q,loc}(\Omega^-)\big]^6, \qquad : \big[B^s_{p,q}(S)\big]^6 \to \big[B^{s+\frac{1}{p}}_{p,q,loc}(\Omega^-)\big]^6,$$
$$: \big[B^s_{p,p}(S)\big]^6 \to \big[H^{s+1+\frac{1}{p}}_p(\Omega)\big]^6, \qquad : \big[B^s_{p,p}(S)\big]^6 \to \big[H^{s+\frac{1}{p}}_p(\Omega)\big]^6,$$
$$: \big[B^s_{p,p}(S)\big]^6 \to \big[H^{s+1+\frac{1}{p}}_{p,loc}(\Omega^-)\big]^6, \qquad : \big[B^s_{p,p}(S)\big]^6 \to \big[H^{s+\frac{1}{p}}_{p,loc}(\Omega^-)\big]^6.$$

Theorem 3.2. *Let*

$$h^{(1)} \in \big[B^{-1+s}_{p,q}(S)\big]^6,\ h^{(2)} \in \big[B^{s}_{p,q}(S)\big]^6,\ 1 < p < \infty,\ 1 \le q \le \infty,\ s > 0.$$

Then

$$\big\{V(h^{(1)})(z)\big\}^{\pm} = \int_S \Gamma(z-y,\tau)h^{(1)}(y)\,d_yS \quad on \ \ S,$$

$$\big\{W(h^{(2)})(z)\big\}^{\pm} = \pm\,\frac{1}{2}\,h^{(2)}(z) + \int_S \Big[\mathcal{P}(\partial_y, n(y), \overline{\tau})\big[\Gamma(z-y,\tau)\big]^{\top}\Big]^{\top} h^{(2)}(y)\,d_yS \quad on \ \ S.$$

The equalities are understood in the sense of the space $[B^{s}_{p,q}(S)\big]^6$.

Theorem 3.3. *Let* $h^{(1)} \in \big[B^{-\frac{1}{p}}_{p,q}(S)\big]^6$, $h^{(2)} \in \big[B^{1-\frac{1}{p}}_{p,q}(S)\big]^6$, $1 < p < \infty$, $1 \le q \le \infty$. *Then*

$$\big\{\mathcal{T}V(h^{(1)})(z)\big\}^{\pm} = \mp\,\frac{1}{2}\,h^{(1)}(z) + \int_S \mathcal{T}(\partial_z, n(z), \tau)\Gamma(z-y,\tau)\,h^{(1)}(y)\,d_yS \quad on \ \ S,$$

$$\big\{\mathcal{T}W(h^{(2)})(z)\big\}^{+} = \big\{\mathcal{T}W(h^{(2)})(z)\big\}^{-} \quad on \ \ S,$$

where the equalities are understood in the sense of the space $[B^{-\frac{1}{p}}_{p,q}(S)\big]^6$.

We introduce the following notation for the boundary operators generated by the single and double layer potentials:

$$\mathcal{H}(h)(z) = \int_S \Gamma(z-y,\tau)\,h(y)\,d_yS, \quad z \in S, \tag{3.2}$$

$$\mathcal{K}(h)(z) = \int_S \mathcal{T}(\partial_z, n(z), \tau)\Gamma(z-y,\tau)\,h(y)\,d_yS, \quad z \in S, \tag{3.3}$$

$$\mathcal{N}(h)(z) = \int_S \Big[\mathcal{P}(\partial_y, n(y), \overline{\tau})\big[\Gamma(z-y,\tau)\big]^{\top}\Big]^{\top} h(y)\,d_yS, \quad z \in S,$$

$$\mathcal{L}(h)(z) = \big\{\mathcal{T}W(h)(z)\big\}^{+} = \big\{\mathcal{T}W(h)(z)\big\}^{-}, \quad z \in S.$$

Actually, $\mathcal{H}$ is a weakly singular integral operator (pseudodifferential operator of order -1), $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators (pseudodifferential operator of order 0), and $\mathcal{L}$ is a singular integro-differential operator (pseudodifferential operator of order 1). These operators possess the following mapping and Fredholm properties (see [31]).

Theorem 3.4. *Let* $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. *Then the operators*

$$\begin{aligned}
\mathcal{H} : \big[B^{s}_{p,q}(S)\big]^6 &\ \to\ \big[B^{s+1}_{p,q}(S)\big]^6,\\
: \big[H^{s}_{p}(S)\big]^6 &\ \to\ \big[H^{s+1}_{p}(S)\big]^6,\\
\mathcal{K},\ \mathcal{N} : \big[B^{s}_{p,q}(S)\big]^6 &\ \to\ \big[B^{s}_{p,q}(S)\big]^6,\\
: \big[H^{s}_{p}(S)\big]^6 &\ \to\ \big[H^{s}_{p}(S)\big]^6,\\
\mathcal{L} : \big[B^{s}_{p,q}(S)\big]^6 &\ \to\ \big[B^{s-1}_{p,q}(S)\big]^6,\\
: \big[H^{s}_{p}(S)\big]^6 &\ \to\ \big[H^{s-1}_{p}(S)\big]^6
\end{aligned}$$

are continuous.

The operators $\mathcal{H}$ *and* $\mathcal{L}$ *are strongly elliptic pseudodifferential operators, while the operators* $\pm\frac{1}{2}\ I_6 + \mathcal{K}$ *and* $\pm\frac{1}{2}\ I_6 + \mathcal{N}$ *are elliptic, where* I_6 *stands for the* 6×6 *unit matrix.*

Moreover, the operators $\mathcal{H}$, $\frac{1}{2}\ I_6 + \mathcal{N}$ *and* $\frac{1}{2}\ I_6 + \mathcal{K}$ *are invertible, whereas the operators* $-\frac{1}{2}\ I_6 + \mathcal{K}$, $-\frac{1}{2}\ I_6 + \mathcal{N}$ *and* $\mathcal{L}$ *are Fredholm operators with zero index.*

There hold the following operator equalities

$$\mathcal{L}\,\mathcal{H} = -\frac{1}{4}\,I_6 + \mathcal{K}^2, \quad \mathcal{H}\,\mathcal{L} = -\frac{1}{4}\,I_6 + \mathcal{N}^2\,. \tag{3.4}$$

4. Existence and regularity of solutions to mixed BVP for solids with interior crack

If not otherwise stated, throughout this section we assume that

$$1 < p < \infty, \quad q \geq 1, \quad s \in \mathbb{R}.$$

Before we start analysis of the mixed problem we present here existence results for the basic Dirichlet and Neumann boundary value problems. Using Theorem 3.4 and the fact that the null spaces of strongly elliptic pseudodifferential operators acting in Bessel potential $H^s_p(S)$ and Besov $B^s_{p,q}(S)$ spaces actually do not depend on the parameters s, p, and q, we arrive at the following existence results (for details see [31]).

Theorem 4.1. *Let* $1 < p < \infty$ *and* $f \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$. *Then the pseudodifferential operator*

$$2^{-1}I_6 + \mathcal{N} \;:\; \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6 \to \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$$

is continuously invertible, the interior Dirichlet BVP (2.52), (2.47)–(2.51) *is uniquely solvable in the space* $[W^1_p(\Omega)]^6$ *and the solution is representable in the form of double layer potential* $U = W(h)$ *with the density vector function* $h \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$ *being a unique solution of the equation*

$$[\,2^{-1}I_6 + \mathcal{N}\,]\,h = f \;\; on \;\; S.$$

Theorem 4.2. (i) *Let a vector function* $U \in [W^1_p(\Omega)]^6$, $1 < p < \infty$ *solves the homogeneous differential equation* $A(\partial,\tau)U = 0$ *in* Ω. *Then it is uniquely representable in the form*

$$U(x) = V\big(\mathcal{H}^{-1}\{U\}^+\big)(x), \quad x \in \Omega,$$

where $\{U\}^+$ *is the trace of* U *on* S *from* Ω *and belongs to the space* $\big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$.

(ii) *Let a vector function* $U \in [W^1_{p,loc}(\Omega^-)]^6$, $1 < p < \infty$ *satisfy the decay conditions* (2.45), *and solve the homogeneous differential equation* $A(\partial,\tau)U = 0$ *in* Ω^-. *Then it is uniquely representable in the form*

$$U(x) = V\big(\mathcal{H}^{-1}\{U\}^-\big)(x), \quad x \in \Omega^-,$$

where $\{U\}^-$ *is the trace of* U *on* S *from* Ω^- *and belongs to the space* $\big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^6$.

Theorem 4.3. *Let* $1 < p < \infty$ *and* $F = (F_1,\dots,F_6)^\top \in \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^6$.

(i) *The operator*

$$-2^{-1}I_6 + \mathcal{K} \;:\; \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^6 \to \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^6 \tag{4.1}$$

is an elliptic pseudodifferential operator with zero index and has a two-dimensional null space $\Lambda(S) := \ker(-2^{-1}I_6 + \mathcal{K}) \subset [C^\infty(S)]^6$, *which represents a linear span of the vector functions*

$$h^{(1)},\; h^{(2)} \in \Lambda(S),$$

such that

$$V(h^{(1)}) = \Psi^{(1)} := (0,0,0,1,0,0)^\top \; and \; V(h^{(2)}) = \Psi^{(2)} := (0,0,0,0,1,0)^\top \; in \quad \Omega. \tag{4.2}$$

(ii) *The null space of the operator adjoint to* (4.1),

$$-2^{-1}I_6 + \mathcal{K}^* \;:\; \big[B^{\frac{1}{p}}_{p',p'}(S)\big]^6 \to \big[B^{\frac{1}{p}}_{p',p'}(S)\big]^6, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

is a linear span of the vectors $(0,0,0,1,0,0)^\top$ *and* $(0,0,0,0,1,0)^\top$.

(iii) *The equation*

$$[\,-2^{-1}I_6 + \mathcal{K}\,]\,h = F \; on \;\; S, \tag{4.3}$$

is solvable if and only if

$$\int_S F_4(x)\,dS = \int_S F_5(x)\,dS = 0. \tag{4.4}$$

(iv) *If the conditions* (4.4) *hold, then solutions to Eq.* (4.3) *are defined modulo a linear combination of the vector functions* $h^{(1)}$ *and* $h^{(2)}$.

(v) *If the conditions* (4.4) *hold, then the interior Neumann type boundary value problem* (2.52), (2.53)–(2.57) *is solvable in the space* $[W^1_p(\Omega)]^6$, $1 < p < \infty$ *and its solution is representable in the form of single layer potential* $U = V(h)$, *where the density vector function* $h \in \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^6$ *is defined by Eq.* (4.3). *A solution to the interior Neumann BVP in* Ω *is defined modulo a linear combination of the constant vector functions* $\Psi^{(1)}$ *and* $\Psi^{(2)}$ *given by* (4.2).

Remark 4.4. If boundary data of Dirichlet and Neumann boundary value problems $(D)^+_{\tau,0}$ and $(N)^+_{\tau,0}$ are sufficiently smooth, then the problems have regular solutions (see [31]).

Now we start investigation of the mixed boundary value problems for solids with interior cracks.

First let us note that the boundary conditions on the crack faces Σ, (2.59) and (2.60), can be transformed equivalently as

$$\big\{[\mathcal{T}U]_j\big\}^+ - \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ - F_j^- \in \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma), \qquad j=\overline{1,3},$$

$$\big\{[\mathcal{T}U]_j\big\}^+ + \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ + F_j^- \in B^{-\frac{1}{p}}_{p,p}(\Sigma), \qquad j=\overline{1,3}.$$

Thus, the boundary conditions (2.59)–(2.68) of the problem under consideration can be rewritten as

$$\big\{\mathcal{T}\,U\big\}^+ = g^{(N)} \qquad \text{on } S_N, \tag{4.5}$$

$$\big\{U\big\}^+ = g^{(D)} \qquad \text{on } S_D, \tag{4.6}$$

$$\big\{[\mathcal{T}U]_j\big\}^+ + \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ + F_j^- \qquad \text{on } \Sigma, \qquad j=\overline{1,3}, \tag{4.7}$$

$$\big\{u_4\big\}^+ - \big\{u_4\big\}^- = f_4 \qquad \text{on } \Sigma, \tag{4.8}$$

$$\big\{u_5\big\}^+ - \big\{u_5\big\}^- = f_5 \qquad \text{on } \Sigma, \tag{4.9}$$

$$\big\{u_6\big\}^+ - \big\{u_6\big\}^- = f_6 \qquad \text{on } \Sigma, \tag{4.10}$$

$$\big\{[\mathcal{T}U]_j\big\}^+ - \big\{[\mathcal{T}U]_j\big\}^- = F_j^+ - F_j^- \qquad \text{on } \Sigma, \qquad j=\overline{1,3}, \tag{4.11}$$

$$\big\{[\,\mathcal{T}\,U\,]_4\big\}^+ - \big\{[\,\mathcal{T}\,U\,]_4\big\}^- = F_4 \qquad \text{on } \Sigma, \tag{4.12}$$

$$\big\{[\,\mathcal{T}\,U\,]_5\big\}^+ - \big\{[\,\mathcal{T}\,U\,]_5\big\}^- = F_5 \qquad \text{on } \Sigma, \tag{4.13}$$

$$\big\{[\,\mathcal{T}\,U\,]_6\big\}^+ - \big\{[\,\mathcal{T}\,U\,]_6\big\}^- = F_6 \qquad \text{on } \Sigma. \tag{4.14}$$

We look for a solution of the boundary value problem (2.58)–(2.68) in the following form:

$$U = V(\mathcal{H}^{-1}\,h) + W_c(h^{(2)}) + V_c(h^{(1)}) \quad \text{in} \quad \Omega_\Sigma, \tag{4.15}$$

where $\mathcal{H}^{-1}$ is the operator inverse to the integral operator $\mathcal{H}$ defined by (3.2),

$$V_c(h^{(1)})(x) := \int_\Sigma \Gamma(x-y,\tau)\,h^{(1)}(y)\,d_yS,$$

$$W_c(h^{(2)})(x) := \int_\Sigma \big[\mathcal{P}(\partial_y, n(y), \overline{\tau})[\Gamma(x-y,\tau)]^\top\big]^\top\, h^{(2)}(y)\,d_yS,$$

$$V(\mathcal{H}^{-1}\,h)(x) := \int_S \Gamma(x-y,\tau)\;(\mathcal{H}^{-1}\,h)(y)\,d_yS,$$

$h^{(i)} = (h_1^{(i)}, \dots, h_6^{(i)})^\top$, $i = 1, 2$, and $h = (h_1, \dots, h_6)^\top$ are unknown densities,

$$h^{(1)} \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Sigma)\big]^6, \quad h^{(2)} \in \big[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(\Sigma)\big]^6, \quad h \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6. \tag{4.16}$$

Due to the above inclusions, clearly, in V_c and W_c we can take the closed surface S_0 as an integration manifold instead of the crack surface Σ. Recall that Σ is assumed to be a proper part of $S_0 = \partial\Omega_0 \subset \Omega$ (see Section 2.3).

The boundary and transmission conditions (4.5)–(4.14) lead to the equations:

$$r_{S_N}\mathcal{A}h + r_{S_N}\big[\mathcal{T}W_c(h^{(2)})\big] + r_{S_N}\big[\mathcal{T}V_c(h^{(1)})\big] = g^{(N)} \quad \text{on } S_N, \tag{4.17}$$

$$r_{S_D}h + r_{S_D}\big[W_c(h^{(2)})\big] + r_{S_D}V_c(h^{(1)}) = g^{(D)} \quad \text{on } S_D, \tag{4.18}$$

$$r_\Sigma\big[\mathcal{T}V\big(\mathcal{H}^{-1}h\big)\big]_j + r_\Sigma\big[\mathcal{L}_c h^{(2)}\big]_j + r_\Sigma\big[\mathcal{K}_c(h^{(1)})\big]_j = 2^{-1}\big(F_j^+ + F_j^-\big), \quad j = 1, 2, 3, \quad \text{on } \Sigma, \tag{4.19}$$

$$h_4^{(2)} = f_4 \quad \text{on } \Sigma, \tag{4.20}$$

$$h_5^{(2)} = f_5 \quad \text{on } \Sigma, \tag{4.21}$$

$$h_6^{(2)} = f_6 \quad \text{on } \Sigma, \tag{4.22}$$

$$h_j^{(1)} = F_j^- - F_j^+, \quad j = 1, 2, 3, \quad \text{on } \Sigma, \tag{4.23}$$

$$h_4^{(1)} = -F_4 \quad \text{on } \Sigma, \tag{4.24}$$

$$h_5^{(1)} = -F_5 \quad \text{on } \Sigma, \tag{4.25}$$

$$h_6^{(1)} = -F_6 \quad \text{on } \Sigma, \tag{4.26}$$

where $\mathcal{A} := \big(-2^{-1}\, I_5 + \mathcal{K}\big)\,\mathcal{H}^{-1}$ is the Steklov–Poincaré type operator on S, and

$$\mathcal{L}_c(h^{(2)})(z) := \big\{\mathcal{T}W_c(h^{(2)})(z)\big\}^+ = \big\{\mathcal{T}W_c(h^{(2)})(z)\big\}^- \quad \text{on } \Sigma,$$

$$\mathcal{K}_c(h^{(1)})(z) := \int_\Sigma \mathcal{T}(\partial_z, n(z), \tau)\Gamma(z-y, \tau)\, h^{(1)}(y)\, d_yS \quad \text{on } \Sigma.$$

As we see the sought for density $h^{(1)}$ and the last two components of the vector $h^{(2)}$ are determined explicitly by the data of the problem. Hence, it remains to find the density h and the first three components of the vector $h^{(2)}$.

The operator generated by the left hand side expressions of the above simultaneous equations, acting upon the unknown vector $(h, h^{(2)}, h^{(1)})$ reads as

$$\mathcal{Q} := \begin{bmatrix} r_{S_N}\mathcal{A} & r_{S_N}\mathcal{T}W_c & r_{S_N}\mathcal{T}V_c \\ r_{S_D} I_6 & r_{S_D} W_c & r_{S_D} V_c \\ r_\Sigma\,[\mathcal{T}V\big(\mathcal{H}^{-1}\big)]_{3\times 6} & r_\Sigma\,[\mathcal{L}_c]_{3\times 6} & r_\Sigma\,[\mathcal{K}_c]_{3\times 6} \\ [\,0\,]_{3\times 6} & r_\Sigma\, I^*_{3\times 6} & [\,0\,]_{3\times 6} \\ [\,0\,]_{6\times 6} & [\,0\,]_{6\times 6} & r_\Sigma\, I_6 \end{bmatrix}_{24\times 18},$$

where $[\,M\,]_{3\times 6}$ denotes the first three rows of a 6×6 matrix M, $[\,0\,]_{m\times n}$ stands for the corresponding zero matrix,

$$I^*_{3\times 6} := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times 6}.$$

This operator possesses the following mapping properties

$$\begin{aligned} \mathcal{Q}\ &: [H_p^s(S)]^6 \times [\widetilde{H}_p^s(\Sigma)]^6 \times [\widetilde{H}_p^{s-1}(\Sigma)]^6 \\ &\to [H_p^{s-1}(S_N)]^6 \times [H_p^s(S_D)]^6 \times [H_p^{s-1}(\Sigma)]^3 \times [\widetilde{H}_p^s(\Sigma)]^3 \times [\widetilde{H}_p^{s-1}(\Sigma)]^6, \\ \mathcal{Q}\ &: [B_{p,q}^s(S)]^6 \times [\widetilde{B}_{p,q}^s(\Sigma)]^6 \times [\widetilde{B}_{p,q}^{s-1}(\Sigma)]^6 \\ &\to [B_{p,q}^{s-1}(S_N)]^6 \times [B_{p,q}^s(S_D)]^6 \times [B_{p,q}^{s-1}(\Sigma)]^3 \times [\widetilde{B}_{p,q}^s(\Sigma)]^3 \times [\widetilde{B}_{p,q}^{s-1}(\Sigma)]^6, \\ &1 < p < \infty, \quad 1 \le q \le \infty, \quad s \in \mathbb{R}. \end{aligned} \tag{4.27}$$

Our main goal is to establish invertibility of the operators (4.27).

To this end, by introducing a new additional unknown vector we extend Eq. (4.18) from S_D onto the whole of S. We will do this in the following way. Denote by $g_0^{(D)}$ some fixed extension of $g^{(D)}$ from S_D onto the whole of S preserving the space and introduce a new unknown vector ϕ on S

$$\phi = h + r_S\left[W_c(h^{(2)})\right] + r_S\, V_c(h^{(1)}) - g_0^{(D)}. \tag{4.28}$$

It is evident that $\phi \in \left[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)\right]^6$ in accordance with (4.18), (4.16), (2.69), and the embedding $g_0^{(D)} \in \left[B_{p,p}^{1-\frac{1}{p}}(S)\right]^6$. Moreover, the restriction of Eq. (4.28) on S_D coincide with Eq. (4.18). Therefore, we can replace Eq. (4.18) in the system (4.17)–(4.26) by Eq. (4.28). Finally, we arrive at the following simultaneous equations with respect to unknowns h, ϕ, $h^{(2)}$ and $h^{(1)}$:

$$r_{S_N}\,\mathcal{A}\,h + r_{S_N}\left[\mathcal{T} W_c(h^{(2)})\right] + r_{S_N}\left[\mathcal{T}\,V_c(h^{(1)})\right] = g^{(N)} \quad \text{on } S_N, \tag{4.29}$$
$$h - \phi + r_S\left[W_c(h^{(2)})\right] + r_S\,V_c(h^{(1)}) = g_0^{(D)} \quad \text{on } S, \tag{4.30}$$
$$r_\Sigma\left[\mathcal{T} V\big(\mathcal{H}^{-1}h\big)\right]_j + r_\Sigma\left[\mathcal{L}_c\,h^{(2)}\right]_j + r_\Sigma\left[\mathcal{K}_c(h^{(1)})\right]_j = 2^{-1}\big(F_j^+ + F_j^-\big), \quad j=1,2,3, \quad \text{on } \Sigma, \tag{4.31}$$
$$h_4^{(2)} = f_4 \quad \text{on } \Sigma, \tag{4.32}$$
$$h_5^{(2)} = f_5 \quad \text{on } \Sigma, \tag{4.33}$$
$$h_6^{(2)} = f_6 \quad \text{on } \Sigma, \tag{4.34}$$
$$h_j^{(1)} = F_j^- - F_j^+, \quad j=1,2,3, \quad \text{on } \Sigma, \tag{4.35}$$
$$h_4^{(1)} = -F_4 \quad \text{on } \Sigma, \tag{4.36}$$
$$h_5^{(1)} = -F_5 \quad \text{on } \Sigma, \tag{4.37}$$
$$h_6^{(1)} = -F_6 \quad \text{on } \Sigma. \tag{4.38}$$

In what follows, for the zero vector $g^{(D)} = 0$ on S_D we always choose the fixed extension vector $g_0^{(D)} = 0$ on S.

Rewrite the system (4.29)–(4.38) in the equivalent form

$$\begin{aligned} &r_{S_N}\,\mathcal{A}\,\phi + r_{S_N}\,\mathcal{T} W_c(h^{(2)}) - r_{S_N}\,\mathcal{A}\,[r_S\,W_c(h^{(2)})] + r_{S_N}\,\mathcal{T}\,V_c(h^{(1)}) \\ &\qquad - r_{S_N}\,\mathcal{A}\,[r_S\,V_c(h^{(1)})] = g^{(N)} - r_{S_N}\,\mathcal{A}\,g_0^{(D)} \quad \text{on } S_N, \end{aligned} \tag{4.39}$$
$$-\phi + h + r_S\left[W_c(h^{(2)})\right] + r_{\partial\Omega}\,V_c(h^{(1)}) = g_0^{(D)} \quad \text{on } S, \tag{4.40}$$
$$r_\Sigma\left[\mathcal{T} V\big(\mathcal{H}^{-1}h\big)\right]_j + r_\Sigma\left[\mathcal{L}_c\,h^{(2)}\right]_j + r_\Sigma\left[\mathcal{K}_c(h^{(1)})\right]_j = 2^{-1}\big(F_j^+ + F_j^-\big), \quad j=1,2,3, \quad \text{on } \Sigma, \tag{4.41}$$
$$h_4^{(2)} = f_4 \quad \text{on } \Sigma, \tag{4.42}$$
$$h_5^{(2)} = f_5 \quad \text{on } \Sigma, \tag{4.43}$$
$$h_6^{(2)} = f_6 \quad \text{on } \Sigma, \tag{4.44}$$
$$h_j^{(1)} = F_j^- - F_j^+, \quad j=1,2,3, \quad \text{on } \Sigma, \tag{4.45}$$
$$h_4^{(1)} = -F_4 \quad \text{on } \Sigma, \tag{4.46}$$
$$h_5^{(1)} = -F_5 \quad \text{on } \Sigma, \tag{4.47}$$
$$h_6^{(1)} = -F_6 \quad \text{on } \Sigma. \tag{4.48}$$

Remark 4.5. The systems (4.17)–(4.26) and (4.39)–(4.48) are equivalent in the following sense:

(i) if $(h, h^{(2)}, h^{(1)})^\top$ solves the system (4.17)–(4.26), then $(\phi, h, h^{(2)}, h^{(1)})^\top$ with ϕ given by (4.28) where $g_0^{(D)}$ is some fixed extension of the vector $g^{(D)}$ from S_D onto the whole of S involved in the right hand side of Eq. (4.40), solves the system (4.39)–(4.48);

(ii) if $(\phi, h, h^{(2)}, h^{(1)})^\top$ solves the system (4.39)–(4.48), then $(h, h^{(2)}, h^{(1)})^\top$ solves the system (4.17)–(4.26).

The operator generated by the left hand sides of system (4.39)–(4.48) reads as

$$\mathcal{Q}_1 := \begin{bmatrix} r_{S_N}\,\mathcal{A} & [\,0\,]_{6\times6} & r_{S_N}\,\mathcal{R}_2 & r_{S_N}\,\mathcal{R}_1 \\ -r_S\,I_6 & r_S\,I_6 & r_S\,W_c & r_S\,V_c \\ [\,0\,]_{3\times6} & r_\Sigma\,[\mathcal{T}V(\mathcal{H}^{-1})]_{3\times6} & r_\Sigma\,[\mathcal{L}_c]_{3\times6} & r_\Sigma\,[\mathcal{K}_c]_{3\times6} \\ [\,0\,]_{3\times6} & [\,0\,]_{3\times6} & r_\Sigma\,I^*_{3\times6} & [\,0\,]_{3\times6} \\ [\,0\,]_{6\times6} & [\,0\,]_{6\times6} & [\,0\,]_{6\times6} & r_\Sigma\,I_6 \end{bmatrix}_{24\times24}, \tag{4.49}$$

where

$$\mathcal{R}_1 = \mathcal{T}\,V_c - \mathcal{A}\,[r_S\,V_c], \qquad \mathcal{R}_2 = \mathcal{T}\,W_c - \mathcal{A}\,[r_S\,W_c].$$

Here and in what follows $[\,M\,]_{6\times k}$ with $k<6$ denotes the first k columns of a 6×6 matrix M, while $[\,M\,]_{k\times6}$ denotes the first k rows of the same matrix, and $[\,M\,]_{k\times k}$ stands for the upper left $k\times k$ block of M.

This operator possesses the following mapping properties

$$\begin{aligned} \mathcal{Q}_1\ &: [\widetilde{H}^s_p(S_N)]^6 \times [H^s_p(S)]^6 \times [\widetilde{H}^s_p(\Sigma)]^6 \times [\widetilde{H}^{s-1}_p(\Sigma)]^6 \\ &\to [H^{s-1}_p(S_N)]^6 \times [H^s_p(S)]^6 \times [H^{s-1}_p(\Sigma)]^3 \times [\widetilde{H}^s_p(\Sigma)]^3 \times [\widetilde{H}^{s-1}_p(\Sigma)]^6, \\ \mathcal{Q}_1\ &: [\widetilde{B}^s_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [\widetilde{B}^s_{p,q}(\Sigma)]^6 \times [\widetilde{B}^{s-1}_{p,q}(\Sigma)]^6 \\ &\to [B^{s-1}_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [B^{s-1}_{p,q}(\Sigma)]^3 \times [\widetilde{B}^s_{p,q}(\Sigma)]^3 \times [\widetilde{B}^{s-1}_{p,q}(\Sigma)]^6, \\ &1<p<\infty, \quad 1\le q\le\infty, \quad s\in\mathbb{R}. \end{aligned} \tag{4.50}$$

Due to the above agreement about the extension of the zero vector we see that if the right hand side functions of the system (4.17)–(4.26) vanish then the same holds for the system (4.39)–(4.48) and vice versa.

The uniqueness Theorem 2.1 and properties of the single and double layer potentials imply the following assertion.

Lemma 4.6. *The null spaces of the operators $\mathcal{Q}$ and $\mathcal{Q}_1$ are trivial for $s=1/2$ and $p=2$.*

Now we start to analyse Fredholm properties of the operator $\mathcal{Q}_1$.

From the structure of the operator $\mathcal{Q}_1$ it is evident that we need only to study Fredholm properties of the operator generated by the upper left 15×15 block of the matrix operator (4.49),

$$\mathcal{M} := \begin{bmatrix} r_{S_N}\,\mathcal{A} & [\,0\,]_{6\times6} & r_{S_N}\,[\,\mathcal{R}_2]_{6\times3} \\ -r_S\,I_6 & r_S\,I_6 & r_S\,[\,W_c\,]_{6\times3} \\ [\,0\,]_{3\times6} & r_\Sigma\,[\mathcal{T}V(\mathcal{H}^{-1})]_{3\times6} & r_\Sigma\,[\,\mathcal{L}_c\,]_{3\times3} \end{bmatrix}_{15\times15}.$$

This operator has the following mapping properties:

$$\begin{aligned} \mathcal{M}\ &: [\widetilde{H}^s_p(S_N)]^6 \times [H^s_p(S)]^6 \times [\widetilde{H}^s_p(\Sigma)]^3 \\ &\to [H^{s-1}_p(S_N)]^6 \times [H^s_p(S)]^6 \times [H^{s-1}_p(\Sigma)]^3, \\ \mathcal{M}\ &: [\widetilde{B}^s_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [\widetilde{B}^s_{p,q}(\Sigma)]^3 \\ &\to [B^{s-1}_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [B^{s-1}_{p,q}(\Sigma)]^3, \\ &1<p<\infty, \quad 1\le q\le\infty, \quad s\in\mathbb{R}. \end{aligned} \tag{4.51}$$

For the principal part $\mathcal{M}_0$ of the operator $\mathcal{M}$ we have

$$\mathcal{M}_0 := \begin{bmatrix} r_{S_N}\,\mathcal{A} & [\,0\,]_{6\times6} & [\,0\,]_{6\times3} \\ -r_S\,I_6 & r_S\,I_6 & [\,0\,]_{6\times3} \\ [\,0\,]_{3\times6} & [\,0\,]_{3\times6} & r_\Sigma\,\mathcal{L}^{(1)} \end{bmatrix}_{15\times15}, \tag{4.52}$$

where

$$\mathcal{L}^{(1)} := \|[\mathcal{L}_c]_{kj}\|_{3\times3}, \quad \mathcal{L}_c = \|[\mathcal{L}_c]_{kl}\|_{6\times6}. \tag{4.53}$$

Clearly, the operator $\mathcal{M}_0$ has the same mapping properties as $\mathcal{M}$ and the difference $\mathcal{M} - \mathcal{M}_0$ is compact. Actually, $\mathcal{M} - \mathcal{M}_0$ is an infinitely smoothing operator.

The operators $\mathcal{L}_c$ and $\mathcal{A}$ are strongly elliptic pseudodifferential operators of order 1 (see [31]). From (4.53) we get then that $\mathcal{L}^{(1)}$ is a strongly elliptic pseudodifferential operator as well. Moreover, we have the following invertibility results.

Theorem 4.7. *Let $1 < p < \infty$, $1 \le q \le \infty$, $1/p - 1/2 < s < 1/p + 1/2$. Then the operators*

$$r_\Sigma\, \mathcal{L}^{(1)} \,:\, \big[\widetilde{H}_p^s(\Sigma)\big]^3 \to \big[H_p^{s-1}(\Sigma)\big]^3, \quad r_\Sigma\, \mathcal{L}^{(1)} \,:\, \big[\widetilde{B}_{p,q}^s(\Sigma)\big]^3 \to \big[B_{p,q}^{s-1}(\Sigma)\big]^3 \tag{4.54}$$

are invertible.

Proof. With the help of the first equality in (3.4) we derive that the principal homogeneous symbol matrix of the strongly elliptic pseudodifferential operator $\mathcal{L}_c$ reads as

$$\begin{aligned}\mathfrak{S}(\mathcal{L}_c; x, \xi) = \mathfrak{S}(\mathcal{L}_{S_0}; x, \xi) &:= \Big(-\frac{1}{4}\, I_6 + \mathfrak{S}^2(\mathcal{K}_{S_0}; x, \xi)\Big)\, [\, \mathfrak{S}(\mathcal{H}_{S_0}; x, \xi)\,]^{-1} \\ &= \Big(-\frac{1}{4}\, I_6 + \mathfrak{S}^2(\mathcal{K}_c; x, \xi)\Big)\, [\, \mathfrak{S}(\mathcal{H}_c; x, \xi)\,]^{-1}, \quad x \in \overline{\Sigma}, \ \ \xi \in \mathbb{R}^2 \setminus \{0\},\end{aligned}$$

where $\mathcal{H}_{S_0}$ and $\mathcal{K}_{S_0}$ are integral operators given by (3.2) and (3.3) with S_0 for S.

One can show that the principal homogeneous symbol matrix of the operator $\mathcal{K}_c$ is an odd matrix function in ξ, whereas the principal homogeneous symbol matrix of the operator $\mathcal{H}_c$ is an even matrix function in ξ. Consequently, the matrix $\mathfrak{S}(\mathcal{L}_c; x, \xi)$ is even in ξ (for details see [31]).

From these results it follows that $\mathcal{L}^{(1)}$ is a strongly elliptic pseudodifferential operator with even principal homogeneous symbol. Therefore the matrix

$$[\, \mathfrak{S}(\mathcal{L}^{(1)}; x, 0, +1)\,]^{-1}\, \mathfrak{S}(\mathcal{L}^{(1)}; x, 0, -1)$$

is the unit matrix and the corresponding eigenvalues equal to 1 (see Appendix A). Now, from Theorem A in Appendix A it follows that the operators (4.54) are Fredholm with zero index for $1 < p < \infty$, $1 \le q \le \infty$ and $1/p - 1/2 < s < 1/p + 1/2$. It remains to show that the corresponding null spaces are trivial. In turn, due to the same Theorem A (see Appendix A), it suffices to establish that the operator

$$r_\Sigma\, \mathcal{L}^{(1)} \,:\, \big[\widetilde{H}_2^{\frac12}(\Sigma)\big]^3 \to \big[H_2^{-\frac12}(\Sigma)\big]^3$$

is injective, i.e, we have to prove that the homogeneous equation

$$r_\Sigma\, \mathcal{L}^{(1)}\, \chi = 0 \quad \text{on} \quad \Sigma \tag{4.55}$$

possesses only the trivial solution in the space $\big[\widetilde{H}_2^{\frac12}(\Sigma)\big]^3$.

Let $\chi \in \big[\widetilde{H}_2^{\frac12}(\Sigma)\big]^3$ solve Eq. (4.55) and construct the double layer potential

$$U = (u_1, \ldots, u_6)^\top = W_c(\widetilde{\chi}), \quad \widetilde{\chi} = (\chi, 0, 0, 0)^\top\, .$$

In view of properties of the double layer potential and Eq. (4.55), it can easily be verified that the vector $U \in \big[W_2^1(\mathbb{R}^3 \setminus \overline{\Sigma}\,)\big]^6$ is a solution to the following crack type boundary transmission problem:

$$\begin{aligned} A(\partial_x, \tau)\, U &= 0 && \text{in } \ \mathbb{R}^3 \setminus \overline{\Sigma}, \\ \big\{[\,\mathcal{T}\, U\,]_j\big\}^+ = \big\{[\,\mathcal{T}\, U\,]_j\big\}^- &= 0, \quad j = 1, 2, 3, && \text{on } \ \Sigma, \\ \big\{u_k\big\}^+ - \big\{u_k\big\}^- &= 0, \quad k = 4, 5, 6, && \text{on } \ \Sigma, \\ \big\{[\,\mathcal{T}\, U\,]_k\big\}^+ - \big\{\,[\,\mathcal{T}\, U\,]_k\big\}^- &= 0, \quad k = 4, 5, 6, && \text{on } \ \Sigma \end{aligned}$$

and satisfy the decay conditions (2.45) at infinity, i.e., $U \in \mathbf{Z}(\mathbb{R}^3 \setminus \overline{\Sigma})$.

Applying Green's identity (2.70) by standard arguments we arrive at the equality $U = 0$ in $\mathbb{R}^3 \setminus \overline{\Sigma}$. Whence $\chi = (\chi_1, \chi_2, \chi_3)^\top = 0$ on Σ follows due to the equalities $\big\{u_j\big\}^+ - \big\{u_j\big\}^- = \chi_j$ on Σ, $j = 1, 2, 3$. This completes the proof. $\square$

Let λ_k, $k = \overline{1,6}$, be the eigenvalues of the matrix

$$a_0(x) := [\,\mathfrak{S}(\mathcal{A}; x, 0, +1)\,]^{-1}\, \mathfrak{S}(\mathcal{A}; x, 0, -1), \quad x \in \ell_m,$$

where $\mathfrak{S}(\mathcal{A}; x, \xi)$ with $x \in \overline{S}_N$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ is the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A}$.

We can show that $\lambda = 1$ is an eigenvalue of the matrix $a_0(x)$. It follows from the following technical lemma.

Lemma 4.8. *Let* $\mathbf{Q}$ *be the set of all non-singular* $k \times k$ *square matrices with complex-valued entries and having the structure*

$$\begin{bmatrix} [Q_{lj}]_{(k-1)\times(k-1)} & \{0\}_{(k-1)\times 1} \\ \{0\}_{1\times(k-1)} & Q_{kk} \end{bmatrix}_{k\times k}, \qquad k \in \mathbb{N}.$$

If $X, Y \in \mathbf{Q}$*, then* $XY \in \mathbf{Q}$ *and* $X^{-1} \in \mathbf{Q}$*. Moreover, if in addition* $X = [X_{jl}]_{k\times k}$ *and* $Y = [Y_{jl}]_{k\times k}$ *are strongly elliptic, i.e.*

$$\operatorname{Re}(X\zeta\dot{\zeta}) > 0, \quad \operatorname{Re}(Y\zeta\dot{\zeta}) > 0 \quad \text{for all} \quad \zeta \in \mathbb{C}^k \setminus \{0\},$$

and X_{kk} *and* Y_{kk} *are real numbers, then* $\lambda = X_{kk}Y_{kk} > 0$ *is an eigenvalue of the matrix* XY*.*

In particular if $X_{kk} = Y_{kk}^{-1}$*, then* $\lambda = 1$ *is an eigenvalue of the matrix* XY*.*

Let us introduce the notation

$$\delta' = \inf_{\substack{1\le j\le 6\\ x\in\ell_m}} \frac{1}{2\pi} \arg\lambda_j(x), \quad \delta'' = \sup_{\substack{1\le j\le 6\\ x\in\ell_m}} \frac{1}{2\pi} \arg\lambda_j(x). \tag{4.56}$$

Due to strong ellipticity of the operator $\mathcal{A}$ and since one eigenvalue, say λ_6 equals 1, we easily derive that

$$-\frac{1}{2} < \delta^{\,\prime} \le 0 \le \delta^{\,\prime\prime} < \frac{1}{2}\,.$$

Applying again Theorem A in Appendix A, we get (see [31], Lemma 5.20).

Theorem 4.9. *Let* $1 < p < \infty$*,* $1 \le q \le \infty$*,* $1/p - 1/2 + \delta'' < s < 1/p + 1/2 + \delta'$ *with* δ' *and* δ'' *given by* (4.56)*. Then the Steklov–Poincaré operators*

$$r_{S_N}\mathcal{A} : \big[\widetilde{H}^s_p(S_N)\big]^6 \to \big[H^{s-1}_p(S_N)\big]^6,$$
$$r_{S_N}\mathcal{A} : \big[\widetilde{B}^s_{p,q}(S_N)\big]^6 \to \big[B^{s-1}_{p,q}(S_N)\big]^6,$$

are invertible.

These assertions imply

Theorem 4.10. *Let*

$$1 < p < \infty, \quad 1 \le q \le \infty, \quad 1/p - 1/2 + \delta'' < s < 1/p + 1/2 + \delta' \tag{4.57}$$

with δ' *and* δ'' *given by* (4.56)*. Then the operators* (4.51) *are Fredholm with index 0.*

Proof. From Theorems 4.7 and 4.9 we conclude that for arbitrary p, q and s satisfying the conditions (4.57), the operators

$$\mathcal{M}_0 : [\widetilde{H}^s_p(S_N)]^6 \times [H^s_p(S)]^6 \times [\widetilde{H}^s_p(\Sigma)]^3 \\ \to [H^{s-1}_p(S_N)]^6 \times [H^s_p(S)]^6 \times [H^{s-1}_p(\Sigma)]^3\,,$$
$$\mathcal{M}_0 : [\widetilde{B}^s_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [\widetilde{B}^s_{p,q}(\Sigma)]^3 \\ \to [B^{s-1}_{p,q}(S_N)]^6 \times [B^s_{p,q}(S)]^6 \times [B^{s-1}_{p,q}(\Sigma)]^3\,,$$

with $\mathcal{M}_0$ defined in (4.52) are invertible. Therefore the operators (4.51) are Fredholm operators with index 0. $\square$

Now we are in a position to prove the invertibility of the operator $\mathcal{Q}_1$.

Theorem 4.11. *Let conditions* (4.57) *be satisfied. Then the operators* (4.50) *are invertible.*

Proof. From Theorem 4.10 it follows that the operator $\mathcal{Q}_1$ is Fredholm with index zero if (4.57) holds. By Lemma 4.6 we conclude then that for $s = 1/2$ and $p = 2$ it is invertible. The null-spaces and indices of the operators (4.50) are the same for all values of the parameter $q \in [1, +\infty]$, provided p and s satisfy the inequalities (4.57) (see [36, Ch. 3., Proposition 10.6]). Therefore, for these values of the parameters p and s they are invertible. In particular, the nonhomogeneous system (4.39)–(4.48) is uniquely solvable in the corresponding spaces. Moreover, it can be easily shown that the solution vectors $h, h^{(2)}, h^{(1)}$ do not depend on the extension of the vector $g^{(D)}$, while ϕ does. However, the sum $\phi + g_0^{(D)}$ is defined uniquely. □

Due to Remark 4.5 we conclude that the operators (4.27) are invertible if p, q and s satisfy the conditions (4.57).

With the help of this theorem we arrive at the following existence result for the original mixed BVP.

Theorem 4.12. *Let*

$$\frac{4}{3-2\delta''} < p < \frac{4}{1-2\delta'} \,. \tag{4.58}$$

with δ' and δ'' given by (4.56). *Then the BVP* (2.58)–(2.68) *has a unique solution U in the space $\big[W^1_p(\Omega_\Sigma)\big]^6$, which can be represented as $U = V(\mathcal{H}^{-1} h) + W_c(h^{(2)}) + V_c(h^{(1)})$ in Ω_Σ, where h, $h^{(2)}$ and $h^{(1)}$ are defined by the system* (4.17)–(4.25).

Proof. The condition (4.58) follows from the inequality (4.57) with $s = 1 - 1/p$. Now existence of a solution $U \in [W^1_p(\Omega_\Sigma)]^6$ with p satisfying (4.58) follows from Theorem 4.6. Due to the inequalities $-\frac{1}{2} < \delta' \le \delta'' < \frac{1}{2}$ we have $p = 2 \in \Big(\frac{4}{3-2\delta''}\,,\ \frac{4}{1-2\delta'}\Big)$. Therefore the unique solvability for $p = 2$ is a consequence of Theorem 2.1.

To show the uniqueness result for all other values of p from the interval (4.58) we proceed as follows. Let a vector $U \in [W^1_p(\Omega_\Sigma)]^6$ with p satisfying (4.58) be a solution to the homogeneous boundary value problem (2.58)–(2.68). Then, it is evident that

$$\begin{aligned}
\big\{U\big\}^+_S &\in [B^{1-\frac{1}{p}}_{p,p}(S)]^6, & \big\{\mathcal{T}U\big\}^+_S &\in [B^{-\frac{1}{p}}_{p,p}(S)]^6,\\
\big\{U\big\}^\pm_\Sigma &\in [B^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6, & \big\{\mathcal{T}U\big\}^\pm_\Sigma &\in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^6,\\
\big\{U\big\}^+_\Sigma - \big\{U\big\}^-_\Sigma &\in [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^6, & \big\{\mathcal{T}U\big\}^+_\Sigma - \big\{\mathcal{T}U\big\}^-_\Sigma &= 0 \ \text{ on } \ \Sigma.
\end{aligned}$$

By the general integral representation formula the vector U can be represented in Ω_Σ as

$$U = W_c(\{U\}^+_\Sigma - \{U\}^-_\Sigma) - V_c(\{\mathcal{T}U\}^+_\Sigma - \{\mathcal{T}U\}^-_\Sigma) + W(\{U\}^+_S) - V(\{\mathcal{T}U\}^+_S)\,,$$

i.e.,

$$U = U^* + W_c(h^{(2)}) + V_c(h^{(1)}) \quad \text{in} \quad \Omega_\Sigma, \tag{4.59}$$

where

$$\begin{aligned}
&h^{(1)} = \{\mathcal{T}U\}^+_\Sigma - \{\mathcal{T}U\}^-_\Sigma, \quad h^{(2)} := \{U\}^+_\Sigma - \{U\}^-_\Sigma \quad \text{on} \quad \Sigma,\\
&U^* := W(\{U\}^+_S) - V(\{\mathcal{T}U\}^+_S) \in \Big[W^1_p(\Omega)\Big]^6.
\end{aligned}$$

Note that U^* solves the homogeneous equation

$$A(\partial,\tau)U^* = 0 \ \text{ in } \ \Omega.$$

Denote $h := \{U^*\}^+_S$. Clearly, $h \in \big[B^{1-1/p}_{p,p}(S)\big]^6$. Since the Dirichlet problem possesses a unique solution in the space $[W^1_p(\Omega)]^6$ for arbitrary $p \in [1, +\infty)$, we can represent U^* uniquely in the form of a single layer potential, $U^* = V(\mathcal{H}^{-1} h)$ in Ω (for details see [31, Ch. 5, Section 5.6]). Therefore from (4.59) we get

$$U = V(\mathcal{H}^{-1} h) + W_c(h^{(2)}) + V_c(h^{(1)}) \quad \text{in} \quad \Omega_\Sigma.$$

Now, the homogeneous boundary and transmission conditions for U lead to the homogeneous system (cf. (4.17)–(4.25)) $\mathcal{Q}\Psi = 0$, where $\Psi = (h, h^{(2)}, h^{(1)})^\top$. Whence, $\Psi = 0$ follows immediately due to invertibility of $\mathcal{Q}$ (see Theorem 4.11). Consequently, $U = 0$ in Ω_Σ. □

Let us now present some regularity results for solutions of the mixed boundary value problem (2.58)–(2.68).

Theorem 4.13. *Let* $1 < t < \infty$, $1 \le q \le \infty$,

$$\frac{4}{3-2\delta''} < p < \frac{4}{1-2\delta'}, \quad \frac{1}{t} - \frac{1}{2} + \delta'' < s < \frac{1}{t} + \frac{1}{2} + \delta',$$

with δ' *and* δ'' *given by* (4.56)*, and let* $U \in [W^1_p(\Omega_\Sigma)]^6$ *be the solution of the boundary value problem* (2.58)–(2.68)*. Then the following regularity results hold:*

(i) *If*

$$F_j^+, F_j^- \in B^{s-1}_{t,t}(\Sigma), \quad F_j^+ - F_j^- \in \widetilde{B}^{s-1}_{t,t}(\Sigma), \quad j = 1,2,3, \quad F_k \in \widetilde{B}^{s-1}_{t,t}(\Sigma),$$
$$f_k \in \widetilde{B}^{s}_{t,t}(\Sigma), \quad k = 4,5,6, \quad g^{(D)} \in \left[B^{s}_{t,t}(S_D)\right]^6, \quad g^{(N)} \in \left[B^{s-1}_{t,t}(S_N)\right]^6,$$

then $U \in \left[H_t^{s+\frac{1}{t}}(\Omega_\Sigma)\right]^6$;

(ii) *If*

$$F_j^+, F_j^- \in B^{s-1}_{t,q}(\Sigma), \quad F_j^+ - F_j^- \in \widetilde{B}^{s-1}_{t,q}(\Sigma), \quad j = 1,2,3, \quad F_k \in \widetilde{B}^{s-1}_{t,q}(\Sigma),$$
$$f_k \in \widetilde{B}^{s}_{t,q}(\Sigma), \quad k = 4,5,6, \quad g^{(D)} \in \left[B^{s}_{t,q}(S_D)\right]^6, \quad g^{(N)} \in \left[B^{s-1}_{t,q}(S_N)\right]^6,$$

then $U \in \left[B_{t,q}^{s+\frac{1}{t}}(\Omega_\Sigma)\right]^6$;

(iii) *If* $\alpha > 0$ *and*

$$F_j^+, F_j^- \in B^{\alpha-1}_{\infty,\infty}(\Sigma), \quad F_j^+ - F_j^- \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \quad j = 1,2,3,$$
$$F_k \in \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Sigma), \quad f_k \in C^\alpha(\overline{\Sigma}), \quad r_{\ell_c} f_k = 0, \quad k = 4,5,6,$$
$$g^{(D)} \in \left[C^\alpha(\overline{S}_D)\right]^6, \quad g^{(N)} \in \left[B^{\alpha-1}_{\infty,\infty}(S_N)\right]^6,$$

then

$$U \in \bigcap_{\alpha'<\gamma} C^{\alpha'}(\overline{\Omega}_j), \quad j = 0,1,$$

where $\gamma = \min\{\alpha,\ 1/2 + \delta'\}$, $-1/2 < \delta' \le 0$ *and* Ω_0 *is an arbitrary proper subdomain of* Ω *such that* $\Sigma \subset \partial\Omega_0 = S_0 \in C^\infty$ *and* $\Omega_1 = \Omega \setminus \overline{\Omega}_0$.

Moreover, in one-sided interior and exterior neighbourhoods of the surface S_0 *the vector* U *has* $C^{\gamma'-\varepsilon}$*-smoothness with* $\gamma' = \min\{\alpha,\ 1/2\}$*, while in a one-sided interior neighbourhood of the surface* S *the vector* U *possesses* $C^{\gamma''-\varepsilon}$*-smoothness with* $\gamma'' = \min\{\alpha,\ 1/2 + \delta'\}$*; here* ε *is an arbitrarily small positive number.*

Proof. The proof is exactly the same as that of Theorem 5.22 in [31]. □

Remark 4.14. Theorem 4.13 describes global smoothness properties of solutions. Below, in Section 6.1, with the help of the asymptotic analysis, we will show that actually in a neighbourhood of the crack edge ℓ_c the functions u, φ and ψ have $C^{1/2}$ regularity while the temperature function ϑ possesses $C^{3/2}$ smoothness.

5. Asymptotic expansion of solutions

Here we investigate the asymptotic behaviour of solutions to the problem (2.58)–(2.68) near the exceptional curves ℓ_c and ℓ_m. For simplicity of description of the method applied below, we assume that the boundary data of the problem are infinitely smooth,

$$F_j^+,\ F_j^- \in C^\infty(\overline{\Sigma}), \quad F_j^+ - F_j^- \in C_0^\infty(\overline{\Sigma}), \ j = 1,2,3, \quad f_k,\ F_k \in C_0^\infty(\overline{\Sigma}), \ k = 4,5,6,$$
$$g^{(D)} \in \left[C^\infty(\overline{S}_D)\right]^6, \quad g^{(N)} \in \left[C^\infty(\overline{S}_N)\right]^6,$$

where $C_0^\infty(\overline{\Sigma})$ denotes a space of functions vanishing along with all tangential (to Σ) derivatives on $\ell_c = \partial\Sigma$.

In Section 4, we have shown that the boundary value problem (2.58)–(2.68) is uniquely solvable and the solution U can be represented by (4.15), where the densities are defined by Eqs. (4.17)–(4.26) or by the equivalent system (4.39)–(4.48).

Let $\Phi := (\phi, h, h^{(2)}, h^{(1)})^\top$ be a solution of the system (4.39)–(4.48):

$$\mathcal{Q}_1\,\Phi = G,$$

where G is the vector constructed by the right hand sides of the system,

$$G \in [C^\infty(\overline{S}_N)]^6 \times [C^\infty(S)]^6 \times [C^\infty(\overline{\Sigma})]^3 \times [C_0^\infty(\overline{\Sigma})]^9.$$

To establish the asymptotic behaviour of the vector U near the curves ℓ_c and ℓ_m, we rewrite (4.15) as follows

$$U = V\left(\mathcal{H}^{-1}\,\phi\right) + W_c\,(\widetilde{\chi}) + \mathcal{R}, \tag{5.1}$$

where

$$\mathcal{R} := -V\Big(\mathcal{H}^{-1}\Big[r_S\,W_c(h^{(2)}) + r_S\,V_c(h^{(1)}) - g_0^{(D)}\Big]\Big) + W_c(f_0) + V_c(h^{(1)}),$$

with $f_0 = (0, 0, 0, f_4, f_5, f_6)^\top$. Note that $r_{\overline{\Omega}_j}\,\mathcal{R} \in \big[C^\infty(\overline{\Omega}_j)\big]^6$, where Ω_j, $j = 0, 1$, are as in Theorem 4.13, item (iii), since

$$r_S\,W_c(h^{(2)}) + r_S\,V_c(h^{(1)}) - g_0^{(D)} \in [C^\infty(S)]^6,$$
$$h^{(1)} = \big(F_1^- - F_1^+, F_2^- - F_2^+, F_3^- - F_3^+, -F_4, -F_5, -F_6\big) \in [C_0^\infty(\overline{\Sigma})]^6,$$
$$h_4^{(2)} = f_4 \in C_0^\infty(\overline{\Sigma}), \quad h_5^{(2)} = f_5 \in C_0^\infty(\overline{\Sigma}), \quad h_6^{(2)} = f_6 \in C_0^\infty(\overline{\Sigma}).$$

Further, the vector $\widetilde{\chi}$ involved in (5.1) is defined as follows: $\widetilde{\chi} = (\chi, 0, 0, 0)^\top$, where $\chi = (\chi_1, \chi_2, \chi_3)^\top \equiv (h_1^{(2)}, h_2^{(2)}, h_3^{(2)})^\top$, and χ solves the pseudodifferential equation

$$r_\Sigma\mathcal{L}^{(1)}\,\chi = \Psi^{(1)} \quad \text{on} \quad \Sigma \tag{5.2}$$

with $\Psi^{(1)} = (\Psi_1^{(1)}, \Psi_2^{(1)}, \Psi_3^{(1)})^\top$. Evidently,

$$\Psi_j^{(1)} = 2^{-1}\,(F_j^+ + F_j^-) - r_\Sigma\,[T\,V\big(\mathcal{H}^{-1}\,h\big)]_j - r_\Sigma\,[\mathcal{K}_c(h^{(1)})]_j \in C^\infty(\overline{\Sigma}), \quad j = 1, 2, 3.$$

Finally, the vector ϕ involved in (5.1) solves the pseudodifferential equation

$$r_{S_N}\,\mathcal{A}\,\phi = \Psi^{(2)} \quad \text{on} \quad S_N, \tag{5.3}$$

where

$$\Psi^{(2)} = g^{(N)} - r_{S_N}\,\mathcal{A}\,g_0^{(D)} - r_{S_N}\,T\,W_c(h^{(2)}) + r_{S_N}\,\mathcal{A}\,[r_S\,W_c(h^{(2)})]$$
$$- r_{S_N}\,T\,V_c(h^{(1)}) + r_{S_N}\,\mathcal{A}\,[r_S\,V_c(h^{(1)})] \in [C^\infty(\overline{S}_N)]^6\,.$$

The principal homogeneous symbol $\mathfrak{S}(\mathcal{L}^{(1)}; x, \xi)$, $x \in \overline{\Sigma}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ of the pseudodifferential operator $\mathcal{L}^{(1)}$ is even with respect to the variable ξ and, therefore, the matrix

$$\big[\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, +1)\big]^{-1}\,\mathfrak{S}(\mathcal{L}^{(1)}; x, 0, -1), \quad x \in \ell_c,$$

is the unit matrix I_3. Consequently, all eigenvalues of this matrix equal to one,

$$\lambda_j(x) = 1, \quad j = \overline{1, 3}, \quad x \in \ell_c.$$

Applying a partition of unity, natural local co-ordinate systems and local diffeomorphisms, we can rectify ℓ_c and Σ locally in a standard way. For simplicity, let us denote the local rectified images of ℓ_c and Σ under this diffeomorphisms by the same symbols. Then we identify a one-sided neighbourhood (in Σ) of an arbitrary point $\widetilde{x} \in \ell_c$ as a part of the half-plane $x_2 > 0$. Thus, we assume that $(x_1, 0) \in \ell_c$ and $(x_1, x_{2,+}) \in \Sigma$ for $0 < x_{2,+} < \varepsilon$;. Clearly, $x_{2,+} = \mathrm{dist}(x, \ell_c)$.

Applying the results obtained in Refs. [14] and [37] we can derive the following asymptotic expansion for the solution χ of the strongly elliptic pseudodifferential equation (5.2),

$$\chi(x_1, x_{2,+}) = c_0(x_1)\, x_{2,+}^{\frac{1}{2}} + \sum_{k=1}^{M} c_k(x_1)\, x_{2,+}^{\frac{1}{2}+k} + \chi_{M+1}(x_1, x_{2,+})\,, \tag{5.4}$$

where M is an arbitrary natural number, $c_k \in \left[C^\infty(\ell_c)\right]^3$, $k = 0, 1, \dots, M$, and the remainder term satisfies the inclusion

$$\chi_{M+1} \in \left[C^{M+1}(\ell^+_{c,\varepsilon})\right]^3, \quad \ell^+_{c,\varepsilon} = \ell_c \times [0, \varepsilon].$$

Note that, according to [37], the terms in the expansion (5.4) do not contain logarithms, since the principal homogeneous symbol $\mathfrak{S}(\mathcal{L}^{(1)}; x, \xi)$ of the pseudodifferential operator $\mathcal{L}^{(1)}$ is even in ξ.

To derive analogous asymptotic expansion for the solution vector ϕ of Eq. (5.3), we apply the same local technique as above to a one-sided neighbourhood (in S_N) of the curve ℓ_m and preserve the same notation for the local co-ordinates.

Consider a 6×6 matrix $a_0(x_1)$ constructed by the principal homogeneous symbol of the Steklov–Poincaré operator $\mathcal{A}$,

$$a_0(x_1) := \left[\mathfrak{S}(\mathcal{A}; x_1, 0, +1)\right]^{-1} \mathfrak{S}(\mathcal{A}; x_1, 0, -1), \quad (x_1, 0) \in \ell_m. \tag{5.5}$$

Note that unlike to the above considered case, now (5.5) is not the unit matrix and therefore we proceed as follows.

Denote by $\lambda_1(x_1), \dots, \lambda_6(x_1)$ the eigenvalues of the matrix a_0. Denote by μ_j, $j = 1, \dots, l$, $1 \le l \le 6$, the distinct eigenvalues and by m_j their algebraic multiplicities: $m_1 + \cdots + m_l = 6$. It is well known that the matrix $a_0(x_1)$ admits the following decomposition (see, e.g., [38], Chapter 7, Section 7)

$$a_0(x_1) = \mathcal{D}(x_1)\, \mathcal{J}_{a_0}(x_1)\, \mathcal{D}^{-1}(x_1), \quad (x_1, 0) \in \ell_m,$$

where $\mathcal{D}$ is 6×6 nondegenerate matrix with infinitely differentiable entries and $\mathcal{J}_{a_0}$ has a block diagonal structure

$$\mathcal{J}_{a_0}(x_1) := \mathrm{diag}\,\{\mu_1(x_1) B^{(m_1)}(1)\,, \dots,\ \mu_l(x_1) B^{(m_l)}(1)\}.$$

Here $B^{(\nu)}(t)$, $\nu \in \{m_1, \dots, m_l\}$, are upper triangular matrices:

$$B^{(\nu)}(t) = \|b^{(\nu)}_{jk}(t)\|_{\nu\times\nu}, \qquad b^{(\nu)}_{jk}(t) = \begin{cases} \dfrac{t^{k-j}}{(k-j)!}\,, & j < k, \\ 1, & j = k, \\ 0, & j > k, \end{cases}$$

i.e.,

$$B^{(\nu)}(t) = \begin{pmatrix} 1 & t & \dfrac{t^2}{2!} & \cdots & \dfrac{t^{\nu-2}}{(\nu-2)!} & \dfrac{t^{\nu-1}}{(\nu-1)!} \\ 0 & 1 & t & \cdots & \dfrac{t^{\nu-3}}{(\nu-3)!} & \dfrac{t^{\nu-2}}{(\nu-2)!} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{\nu\times\nu}.$$

Denote

$$B_0(t) := \mathrm{diag}\,\{B^{(m_1)}(t),\ \dots,\ B^{(m_l)}(t)\}.$$

Again, applying the results from Ref. [14] we derive the following asymptotic expansion for the solution ϕ of the strongly elliptic pseudodifferential equation (5.3)

$$\phi(x_1,x_{2,+}) = \mathcal{D}(x_1)\, x_{2,+}^{\frac{1}{2}+\Delta(x_1)}\, B_0\left(-\frac{1}{2\pi i}\log x_{2,+}\right)\mathcal{D}^{-1}(x_1)\, b_0(x_1)$$
$$+\sum_{k=1}^{M}\mathcal{D}(x_1)\, x_{2,+}^{\frac{1}{2}+\Delta(x_1)+k}\, B_k\big(x_1,\log x_{2,+}\big)+\phi_{M+1}(x_1,x_{2,+}), \tag{5.6}$$

where $b_0\in\big[C^\infty(\ell_m)\big]^6$, $\phi_{M+1}\in\big[C^{M+1}(\ell^+_{m,\varepsilon})\big]^6$, $\ell^+_{m,\varepsilon}=\ell_m\times[0,\varepsilon]$, and

$$B_k(x_1,t)=B_0\left(-\frac{t}{2\pi i}\right)\sum_{j=1}^{k(2m_0-1)} t^j\, d_{kj}(x_1).$$

Here $m_0=\max\{m_1,\dots,m_6\}$, the coefficients $d_{kj}\in\big[C^\infty(\ell_m)\big]^6$ and

$$\Delta := (\Delta_1,\dots,\Delta_6),$$
$$\Delta_j(x_1)=\frac{1}{2\pi\, i}\log\lambda_j(x_1)=\frac{1}{2\pi}\arg\lambda_j(x_1)+\frac{1}{2\pi\, i}\log|\lambda_j(x_1)|,$$
$$-\pi<\arg\lambda_j(x_1)<\pi,\quad (x_1,0)\in\ell_m,\quad j=\overline{1,6}.$$

Furthermore,

$$x_{2,+}^{\frac{1}{2}+\Delta(x_1)} := \operatorname{diag}\left\{x_{2,+}^{\frac{1}{2}+\Delta_1(x_1)},\dots,x_{2,+}^{\frac{1}{2}+\Delta_6(x_1)}\right\}.$$

Now, having at hand the formulae (5.4) and (5.6) with the help of the asymptotic expansion of potential-type functions obtained in [15] we can write the following spatial asymptotic expansions for the solution vector U of the boundary value problem (2.58)–(2.68) near the crack edge ℓ_c and near the collision curve ℓ_m.

(a) *Asymptotic expansion near the crack edge* ℓ_c:

$$U(x)=\sum_{\mu=\pm1}\Big[\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1} x_3^j\, z_{s,\mu}^{\frac{1}{2}-j}\, d^{(c)}_{sj}(x_1,\mu)$$
$$+\sum_{\substack{k,l=0\\ k+l+j+p\ge1}}^{M+2}\ \sum_{j+p=0}^{M+2-l} x_2^l\, x_3^j\, z_{s,\mu}^{\frac{1}{2}+p+k}\, d^{(c)}_{slkjp}(x_1,\mu)\Big]+U^{(c)}_{M+1}(x) \tag{5.7}$$

with the coefficients

$$d^{(c)}_{sj}(\,\cdot\,,\mu),\ d^{(c)}_{slkjp}(\,\cdot\,,\mu)\in\big[C^\infty(\ell_c)\big]^6 \quad\text{and}\quad U^{(c)}_{M+1}\in\big[C^{M+1}(\overline{\Omega}_j)\big]^6,\quad j=0,1.$$

Here Ω_j, $j=0,1$, are as in Theorem 4.13(iii), and

$$z_{s,+1}=-(x_2+x_3\zeta_{s,+1}),\qquad z_{s,-1}=x_2-x_3\zeta_{s,-1},$$
$$-\pi<\arg z_{s,\pm1}<\pi,\qquad \zeta_{s,\pm1}\in C^\infty(\ell_c), \tag{5.8}$$

where $\{\zeta_{s,\pm1}\}_{s=1}^{l_0}$ are the different roots in ζ of multiplicity n_s, $s=1,\dots,l_0$, of the polynomial $\det A^{(0)}\Big(\big[J_\varkappa^\top(x_1,0,0)\big]^{-1}\eta_\pm\Big)$ with $\eta_\pm=(0,\pm1,\zeta)^\top$, satisfying the condition $\operatorname{Re}\zeta_{s,\pm1}<0$. The matrix $J_\varkappa$ stands for the Jacobian matrix corresponding to the canonical diffeomorphism $\varkappa$ related to the local co-ordinate system. Under this diffeomorphism ℓ_c and Σ are locally rectified and we assume that $(x_1,0,0)\in\ell_c$, $x_2=\operatorname{dist}(x^{(\Sigma)},\ell_c)$, $x_3=\operatorname{dist}(x,\Sigma)$, where $x^{(\Sigma)}$ is the projection of the reference point $x\in\Omega_\Sigma$ onto the plane corresponding to the image of Σ under the diffeomorphism $\varkappa$.

Note that the coefficients $d^{(c)}_{sj}(\,\cdot\,,\mu)$ can be expressed by the first coefficient c_0 in the asymptotic expansion (5.4) (for details see [15, Theorem 2.3]).

(b) *Asymptotic expansion near the collision curve ℓ_m*:

$$U(x)=\sum_{\mu=\pm1}\left\{\sum_{s=1}^{l_0}\sum_{j=0}^{n_s-1}x_3^j\left[d_{sj}^{(m)}(x_1,\mu)\,z_{s,\mu}^{\frac12+\Delta(x_1)-j}\,B_0\left(-\frac{1}{2\pi i}\log z_{s,\mu}\right)\right]\widetilde{c}_j(x_1)\right.$$
$$\left.+\sum_{\substack{k,l=0\\k+l+j+p\ge1}}^{M+2}\sum_{j+p=0}^{M+2-l}x_2^l\,x_3^j\,d_{sljp}^{(m)}(x_1,\mu)\,z_{s,\mu}^{\frac12+\Delta(x_1)+p+k}\,B_{skjp}\big(x_1,\log z_{s,\mu}\big)\right\}+U_{M+1}^{(m)}(x), \tag{5.9}$$

where $d_{sj}^{(m)}(\cdot,\mu)$ and $d_{sljp}^{(m)}(\cdot,\mu)$ are matrices with entries belonging to the space $C^\infty(\ell_m)$, $\widetilde{c}_j\in\big[C^\infty(\ell_m)\big]^6$, $U_{M+1}^{(m)}\in\big[C^{M+1}(\overline{\Omega}_1)\big]^6$ and

$$z_{s,\mu}^{\kappa+\Delta(x_1)}:=\operatorname{diag}\big\{z_{s,\mu}^{\kappa+\Delta_1(x_1)},\ \dots,\ z_{s,\mu}^{\kappa+\Delta_6(x_1)}\big\},\ \kappa\in\mathbb{R},\quad \mu=\pm1,\quad x_1\in\ell_m;$$

$B_{skjp}(x_1,t)$ are polynomials with respect to the variable t with vector coefficients which depend on the variable x_1 and have the order $\nu_{kjp}=k(2m_0-1)+m_0-1+p+j$, in general, where $m_0=\max\{m_1,\dots,m_l\}$ and $m_1+\cdots+m_l=6$.

Note that the coefficients $d_{sj}^{(m)}(\cdot,\mu)$ can be calculated explicitly, whereas the coefficients $\widetilde{c}_j$ can be expressed by means of the first coefficient b_0 in the asymptotic expansion (5.6) (for details see [15, Theorem 2.3]).

Remark 5.1. Note that the above asymptotic expansions hold also true for finitely smooth data. In this case the asymptotic expansions can be obtained as in Ref. [16,14], and [15] with the help of the theory of anisotropic weighted Sobolev and Bessel potential spaces.

6. Analysis of singularities of solutions

Let $x'\in\ell_c$ and $\Pi_{x'}^{(c)}$ be the plane passing through the point x' and orthogonal to the curve ℓ_c. We introduce the polar coordinates (r,α), $r\ge0$, $-\pi\le\alpha\le\pi$, in the plane $\Pi_{x'}^{(c)}$ with pole at the point x'. Denote by $\Sigma^\pm$ the two different faces of the crack surface Σ. It is clear that $(r,\pm\pi)\in\Sigma^\pm$.

Denote the similar orthogonal plane to the curve ℓ_m by $\Pi_{x'}^{(m)}$ at the point $x'\in\ell_m$ and introduce there the polar coordinates (r,α), with pole at the point x'. The intersection of the plane $\Pi_{x'}^{(m)}$ and Ω_Σ can be identified with the half-plane $r\ge0$ and $0\le\alpha\le\pi$.

In these coordinate systems, the functions $z_{s,\pm1}$ given by (5.8) read as follows

$$z_{s,+1}=-r(\cos\alpha+\zeta_{s,+1}(x')\sin\alpha),$$
$$z_{s,-1}=r(\cos\alpha-\zeta_{s,-1}(x')\sin\alpha),$$

where $x'\in\ell_c\cup\ell_m$, $s=1,\dots,l_0$. We can rewrite asymptotic expansions (5.7) and (5.9) in more convenient forms, in terms of the variables r and α. Moreover, we establish more refined asymptotic properties.

6.1. Asymptotic analysis of solutions near the crack edge ℓ_c

The asymptotic expansion (5.7) yields

$$U=(u,\varphi,\psi,\vartheta)^\top=a_0(x',\alpha)\,r^{1/2}+a_1(x',\alpha)\,r^{3/2}+\cdots,$$

where r is the distance from the reference point $x\in\Pi_{x'}^{(c)}$ to the curve ℓ_c, and $a_j=(a_{j1},\dots,a_{j6})^\top$, $j=0,1,\dots,$ are smooth vector functions of $x'\in\ell_c$.

From this representation it follows that in one-sided interior and exterior neighbourhoods of the surface $S_0=\partial\Omega_0$ the vector $U=(u,\varphi,\psi,\vartheta)^\top$ has $C^{\frac12}$-smoothness.

More detailed analysis shows that $a_{06}=0$ and therefore for the temperature function we have the following asymptotic expansion

$$\vartheta=a_{16}(x',\alpha)\,r^{3/2}+a_{26}(x',\alpha)\,r^{5/2}+\cdots.$$

Indeed, we can see that $u_6 = \vartheta$ solves the segregated mixed transmission problem:

$$\eta_{ij}\partial_i\partial_j u_6 = Q^* \text{ in } \Omega \setminus S_0, \tag{6.1}$$

$$\{u_6\}^+ - \{u_6\}^- = \widetilde{f}_6 \text{ on } S_0, \tag{6.2}$$

$$\{[TU]_6\}^+ - \{[TU]_6\}^- = \widetilde{F}_6 \text{ on } S_0, \tag{6.3}$$

$$\{u_6\}^+ = g_6^{(D)} \text{ on } S_D, \tag{6.4}$$

$$\{[TU]_6\}^+ = g_6^{(N)} \text{ on } S_N \tag{6.5}$$

with

$$Q^* = \tau\, T_0\, \lambda_{il}\, \partial_l u_i - \tau\, T_0\, p_i\, \partial_i \varphi - \tau\, T_0\, m_i\, \partial_i \psi + \tau\, \alpha_0\, \vartheta, \quad [TU]_6 = \eta_{il}\, n_i\, \partial_l \vartheta,$$
$$\widetilde{f}_6 \in C^\infty(S_0), \quad \widetilde{F}_6 \in C^\infty(S_0), \quad g_6^{(D)} \in C^\infty(\overline{S}_D), \quad g_6^{(N)} \in C^\infty(\overline{S}_N),$$

where $\widetilde{f}_6$ and $\widetilde{F}_6$ are extensions of the functions f_6 and F_6 from Σ onto the whole of S_0 by zero, and $g_6^{(D)}$ and $g_6^{(N)}$ are the sixth components of the vectors $g^{(D)}$ and $g^{(N)}$, respectively.

The problem (6.1)–(6.5) is a classical transmission problem where transmission conditions are given on the closed interface surface S_0. Regularity of solutions to this problem near the line ℓ_c depends on smoothness of the right hand side function Q^*, since all the other data possess C^∞ smoothness on S_0 (cf. [31], Section 8.2.1).

Let $1 < t < \infty$, $1/t - 1/2 + \delta'' < s < 1/t + 1/2 + \delta'$. Then due to Theorem 4.13(i) we deduce

$$U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top \in \big[H_t^{s+1/t}(\Omega_\Sigma)\big]^6.$$

Whence $Q^* \in H_t^{s-1+\frac{1}{t}}(\Omega_\Sigma)$ follows. Using the mapping properties of the volume potential (see [39], Theorem 3.8) we conclude that $u_6 = \vartheta$ belongs to the space $H_t^{s+1+\frac{1}{t}}$ in one-sided neighbourhoods of S_0.

From the embedding theorem (see [26], Theorem 4.6.1) it then follows that for sufficiently large t there holds the inclusion $\vartheta \in C^{1+\varepsilon}$ in a neighbourhood of S_0 with some positive ε. Due to this regularity result, from the expansion

$$\vartheta = a_{06}(x', \alpha)\, r^{1/2} + a_{16}(x', \alpha)\, r^{3/2} + \cdots$$

it follows that $a_{06} = 0$, i.e., actually for ϑ we have

$$\vartheta = a_{16}(x', \alpha)\, r^{3/2} + a_{26}(x', \alpha)\, r^{5/2} + \cdots$$

and, consequently, ϑ possesses $C^{3/2}$-regularity in one-sided closed neighbourhoods of S_0.

6.2. *Asymptotic analysis of solutions near the curve ℓ_m*

The asymptotic expansion (5.9) yields

$$U(x) = \sum_{\mu=\pm 1} \sum_{s=1}^{l_0} \sum_{j=0}^{n_s - 1} c_{sj\mu}(x', \alpha)\, r^{\gamma + i\,\delta}\, B_0\Big(-\frac{1}{2\pi i}\log r\Big)\, \widetilde{c}_{sj\mu}(x', \alpha) + \cdots, \tag{6.6}$$

where

$$r^{\gamma + i\,\delta} := \operatorname{diag}\big\{r^{\gamma_1 + i\,\delta_1}, \ldots, r^{\gamma_6 + i\,\delta_6}\big\},$$
$$\gamma_j = \frac{1}{2} + \frac{1}{2\pi}\arg\lambda_j(x'), \quad \delta_j = \frac{1}{2\pi}\log|\lambda_j(x')|, \quad x' \in \ell_m, \;\; j = \overline{1, 6}, \tag{6.7}$$

and λ_j, $j = \overline{1, 6}$, are eigenvalues of the matrix

$$a_0(x') = \big[\mathfrak{S}(\mathcal{A}; x', 0, +1)\big]^{-1}\, \mathfrak{S}(\mathcal{A}; x', 0, -1), \quad x' \in \ell_m. \tag{6.8}$$

Here $\mathfrak{S}(\mathcal{A}; x', \xi)$ is the principal homogeneous symbol of the Steklov–Poincaré operator

$$\mathcal{A} = \big(-2^{-1} I_6 + \mathcal{K}\big)\, \mathcal{H}^{-1}.$$

Moreover, the eigenvalues λ_j, $j=\overline{1,6}$, can be expressed in terms of the eigenvalues β_j, $j=\overline{1,6}$, of the matrix $\mathfrak{S}(\mathcal{K};x',0,+1)$, where $\mathfrak{S}(\mathcal{K};x',\xi)$ is the principal homogeneous symbol matrix of the singular integral operator $\mathcal{K}$. Indeed, we have the following assertion (see [31, Lemma C.1]).

Lemma 6.1. *The principal homogeneous symbol* $\mathfrak{S}(\mathcal{K};x',\xi)$ $x'\in S$, $\xi=(\xi_1,\xi_2)$, *is an odd matrix-function with respect to* ξ *and*

$$\mathfrak{S}(\mathcal{K};x',\xi)=iR(x',\xi),$$

where the entries of the matrix $R(x',\xi)$ *are real-valued functions.*

Proof. Assume, that to every point $x_0\in\Sigma$ there corresponds some orthogonal local coordinate system such that a part of Σ located inside a sphere with a centre at x_0 admits the representation of the form

$$x_3=\gamma(x'),\quad x'=(x_1,x_2),\quad x=(x',\gamma(x'))\in\Sigma, \tag{6.9}$$

where $\gamma\in C^\infty$, $\gamma(0)=\frac{\partial\gamma(0)}{\partial x_1}=\frac{\partial\gamma(0)}{\partial x_2}=0$. The principal homogeneous symbol of the pseudodifferential operator $-\frac{1}{2}I_6+\mathcal{K}$ in the chosen local coordinate system has the form

$$\begin{aligned}
&\mathfrak{S}(-2^{-1}I_6+\mathcal{K};x',\xi)=\|\mathfrak{S}_{pq}(-2^{-1}I_6+\mathcal{K};x',\xi)\|_{6\times6},\quad p,q=1,\dots,6,\\
&\mathfrak{S}_{pq}(-2^{-1}I_6+\mathcal{K};x',\xi)=\frac{1}{2\pi}\int_{l^-}\frac{\mathcal{T}_{pk}(x',\alpha^\top(x')(i\xi,i\zeta))\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))}{\Delta(\alpha^\top(x')(i\xi,i\zeta))}d\zeta,\\
&\Delta(\alpha^\top(x')(i\xi,i\zeta))=\det\|A_{kq}\Big(\alpha^\top(x')(i\xi,i\zeta)\Big)\|_{6\times6},\\
&\alpha(x')=\begin{bmatrix}1&0&0\\0&1&0\\ \dfrac{\partial\gamma(x')}{\partial x_1}&\dfrac{\partial\gamma(x')}{\partial x_2}&-1\end{bmatrix},
\end{aligned}\tag{6.10}$$

where $\|A_{kq}(\alpha^\top(x')(i\xi,i\zeta))\|_{6\times6}$ and $\|\mathcal{T}_{pk}(x',\alpha^\top(x')(i\xi,i\zeta))\|_{6\times6}$ are the principal homogeneous symbol matrices of the operators $A(\partial_x,\tau)$ and $\mathcal{T}(\partial_x,n,\tau)$ respectively, written in the local coordinate system (6.9). $\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))$, $q,k=\overline{1,6}$, is a cofactor of $A_{kq}(\alpha^\top(x')(i\xi,i\zeta))$.

Represent the symbols $A(\alpha^\top(x')(i\xi,i\zeta))$ and $\mathcal{T}(x',\alpha^\top(x')(i\xi',i\zeta))$ as

$$\begin{aligned}
&A(\alpha^\top(x')(i\xi,i\zeta))=A^{(2)}(x',i\xi)+A^{(1)}(x',i\xi)(i\zeta)+A^{(0)}(x')(i\zeta)^2,\\
&\mathcal{T}(x',\alpha^\top(x')(i\xi,i\zeta))=\mathcal{T}^{(1)}(x',i\xi)+\mathcal{T}^{(0)}(x')(i\zeta),
\end{aligned}$$

where $A^{(j)}(x',i\xi)=\|A^{(j)}_{kq}(x',i\xi)\|_{6\times6}$, $j=0,1,2$, $\mathcal{T}^{(j)}(x',i\xi)=\|\mathcal{T}^{(j)}_{pk}(x',i\xi)\|_{6\times6}$, $j=0,1$ are homogeneous polynomials in ξ of degree j.

Taking into account (6.10) we get

$$\begin{aligned}
\mathfrak{S}_{pq}(-2^{-1}I_6+\mathcal{K};x',\xi)=&\frac{1}{2\pi}\mathcal{T}^{(0)}_{pk}(x')\int_{l^-}\frac{i\zeta\,\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))}{\Delta(\alpha^\top(x')(i\xi,i\zeta))}d\zeta\\
&+\frac{1}{2\pi}\mathcal{T}^{(1)}_{pk}(x',i\xi)\int_{l^-}\frac{\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))}{\Delta(\alpha^\top(x')(i\xi,i\zeta))}d\zeta.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{l^-}\frac{i\zeta\,\Delta_{qk}(\alpha^\top(x')(i\xi,i\zeta))}{\Delta(\alpha^\top(x')(i\xi,i\zeta))}d\zeta\\
&=-\frac{i}{12\pi}\frac{\widehat{A}^{(0)}_{qk}(x')}{\det[A^{(0)}(x')]}\int_{l^-}\frac{\partial_\zeta\Delta(\alpha^\top(x')(\xi,\zeta))}{\Delta(\alpha^\top(x')(\xi,\zeta))}d\zeta+\frac{i}{2\pi}\int_{l^-}\frac{\widetilde{\Delta}_{qk}(x',\xi,\zeta)}{\Delta(\alpha^\top(x')(\xi,\zeta))}d\zeta\\
&=\frac{1}{2}\frac{\widehat{A}^{(0)}_{qk}(x')}{\det[A^{(0)}(x')]}+\frac{i}{2\pi}\int_{l^-}\frac{\widetilde{\Delta}_{qk}(x',\xi,\zeta)}{\Delta(\alpha^\top(x')(\xi,\zeta))}d\zeta,
\end{aligned}$$

where $\widehat{A}^{(0)}_{qk}$ is the cofactor of $A^{(0)}_{kq}$ and $\widetilde{\Delta}_{qk}(x', \xi, \zeta)$ is a polynomial of degree 10 in ζ of the form

$$\widetilde{\Delta}_{qk}(x', \xi, \zeta) = \frac{\widehat{A}^{(0)}_{qk}(x')}{6\det[A^{(0)}(x')]}\partial_\zeta \Delta(\alpha^\top(x')(\xi, \zeta)) - \zeta\, \Delta_{kq}(\alpha^\top(x')(\xi, \zeta)).$$

Therefore

$$\begin{aligned}\mathfrak{S}_{pq}(\mathcal{K}; x', \xi) = {} & \frac{i}{2\pi}\mathcal{T}^{(0)}_{pk}(x') \int_{l^-} \frac{\widetilde{\Delta}_{qk}(x', \xi, \zeta)}{\Delta(\alpha^\top(x')(\xi, \zeta))} d\zeta \\ & - \frac{i}{2\pi}\mathcal{T}^{(1)}_{pk}(x', \xi) \int_{l^-} \frac{\Delta_{qk}(\alpha^\top(x')(\xi, \zeta))}{\Delta(\alpha^\top(x')(\xi, \zeta))} d\zeta, \quad p, q = \overline{1, 6}.\end{aligned} \tag{6.11}$$

Since $\widetilde{\Delta}_{qk}(x', \xi, \zeta)$ and $\Delta_{qk}(\alpha^\top(x'), \xi, \zeta)$ are polynomials of degree 10 in ζ, from (6.11) we can easily see that

$$\mathfrak{S}_{pq}(\mathcal{K}; x', -\xi) = -\mathfrak{S}_{pq}(\mathcal{K}; x', \xi)$$

and

$$\mathfrak{S}_{pq}(\mathcal{K}; x', \xi) = i\, R_{pq}(x', \xi), \quad p, q = \overline{1, 6},$$

where $R_{pq}(x', \xi), \quad p, q = \overline{1, 6}$, are real functions. □

Remark 6.2. It is not difficult to check that the principal homogeneous symbol $\mathfrak{S}(\mathcal{H}; x', \xi)$ of the pseudodifferential operator $\mathcal{H}$ is a real even matrix-function with respect to ξ (see Lemma C.2 in [31]).

Theorem 6.3. *Let $\lambda_j,\ j = \overline{1, 6}$, be the eigenvalues of the matrix* (6.8). *Then*

$$\lambda_j = \frac{1 + 2\beta_j}{1 - 2\beta_j}, \quad j = \overline{1, 6},$$

where $\beta_j,\ j = \overline{1, 6}$, are the eigenvalues of the matrix $\mathfrak{S}(\mathcal{K}; x', 0, +1)$.

Proof. The characteristic equation of the matrix a_0 given by (6.8) has the form

$$\det\Big\{\Big[(-2^{-1}I_6 + \sigma^+_{\mathcal{K}})\,[\sigma^+_{\mathcal{H}}]^{-1}\Big]^{-1}\Big[(-2^{-1}I_6 + \sigma^-_{\mathcal{K}})\,[\sigma^-_{\mathcal{H}}]^{-1}\Big] - \lambda I_6\Big\} = 0, \tag{6.12}$$

where

$$\sigma^\pm_{\mathcal{K}} = \mathfrak{S}(\mathcal{K}; x', 0, \pm 1), \quad \sigma^\pm_{\mathcal{H}} = \mathfrak{S}(\mathcal{H}; x', 0, \pm 1). \tag{6.13}$$

Since the matrix $\mathfrak{S}(\mathcal{K}; x', \xi)$ is odd and the matrix $\mathfrak{S}(\mathcal{H}; x', \xi)$ is even in ξ (see Lemma 6.1), we have $\sigma^-_{\mathcal{K}} = -\sigma^+_{\mathcal{K}}$ and $\sigma^-_{\mathcal{H}} = \sigma^+_{\mathcal{H}}$. Then the characteristic equation (6.12) can be rewritten as

$$\det\Big\{\sigma^+_{\mathcal{H}}\,[2^{-1}I_6 - \sigma^+_{\mathcal{K}}]^{-1}[2^{-1}I_6 + \sigma^+_{\mathcal{K}}]\,[\sigma^+_{\mathcal{H}}]^{-1} - \lambda I_6\Big\} = 0.$$

Since the matrices $\sigma^+_{\mathcal{H}}$ and $2^{-1}I_6 \pm \sigma^+_{\mathcal{K}}$ are non-singular, from the previous equality we derive

$$\det\Big\{\,[2^{-1}I_6 + \sigma^+_{\mathcal{K}}] - \lambda[2^{-1}I_6 - \sigma^+_{\mathcal{K}}]\Big\} = 0.$$

Consequently,

$$\det\Big[\sigma^+_{\mathcal{K}} + \frac{1}{2}\Big(\frac{1-\lambda}{1+\lambda}\Big)I_6\Big] = 0. \tag{6.14}$$

Let $\beta_j, \quad j = \overline{1, 6}$, be the eigenvalues of the matrix $\sigma^+_{\mathcal{K}}$. Then it follows from (6.14) that the eigenvalues λ_j of the matrix a_0 and the eigenvalues β_j of $\sigma^+_{\mathcal{K}}$ are related by the equation

$$\frac{\lambda_j - 1}{\lambda_j + 1} = 2\,\beta_j\,, \quad j = \overline{1, 6},$$

which completes the proof. □

It can be shown that $\lambda_6 = 1$, i.e., $\beta_6 = 0$ (for details see [31, Section 5.7]). Therefore, $\gamma_6 = 1/2$ and $\delta_6 = 0$ in accordance with (6.7). This implies that one could not expect better smoothness for solutions than $C^{1/2}$, in general.

More detailed analysis leads to the following refined asymptotic behaviour for the temperature function.

Theorem 6.4. *Near the line ℓ_m the function ϑ possesses the following asymptotic:*

$$\vartheta = b_0 r^{1/2} + \mathcal{R}, \tag{6.15}$$

where $\mathcal{R} \in C^{1+\delta'-\varepsilon}$ in a neighbourhood of ℓ_m and $1 + \delta' - \varepsilon > 1/2$ for sufficiently small $\varepsilon > 0$.

Proof. Indeed, $u_6 = \vartheta$ is a solution of the problem (6.1)–(6.5). Since the matrix $[\eta_{ij}]_{3\times 3}$ is positive definite, this problem can be reduced to a system of integral equations, where the principal part is described by the scalar positive-definite Steklov–Poincaré type operator $\mathcal{A} = (-\frac{1}{2} I + \mathcal{K}_{scalar})[H_{scalar}]^{-1}$ on S_N, where $\mathcal{K}_{scalar}$ is compact. This operator possesses an even principal homogeneous symbol $\mathfrak{S}(\mathcal{A}; x, \xi) = -\frac{1}{2}\mathfrak{S}([H_{scalar}]^{-1}; x, \xi) = -\frac{1}{2}[\mathfrak{S}(H_{scalar}; x, \xi)]^{-1}$ which is positive and even in ξ. Hence we can establish refined explicit asymptotic (6.15) for the temperature function $u_6 = \vartheta$ in a neighbourhood of ℓ_m. □

From (6.15) it follows that:

(i) The leading exponent for $u_6 = \vartheta$ in a neighbourhood of line ℓ_m equals $1/2$;
(ii) Logarithmic factors are absent in the first term of the asymptotic expansion of ϑ;
(iii) The temperature function ϑ does not oscillate in a neighbourhood of the collision curve ℓ_m and for the heat flux vector we have no oscillating singularities.

In what follows, we will consider particular type GTEME materials and analyse the exponents $\gamma_j + i\delta_j$ which determine the behaviour of $u = (u_1, u_2, u_3)$, φ, and ψ near the line ℓ_m. Non-zero parameters δ_j lead to the so called oscillating singularities for the first order derivatives of u, φ, and ψ, in general. In turn, this yields oscillating stress singularities which sometimes lead to mechanical contradictions, for example, to overlapping of materials. So, from the practical point of view, it is important to single out classes of solids for which the oscillating effects do not occur.

To this end, we will consider a special class of bodies belonging to the **422** (Tetragonal) or **622** (Hexagonal) class of crystals for which the corresponding system of differential equations reads as follows (see, e.g., [40])

$$\begin{aligned}
&(c_{11}\,\partial_1^2 + c_{66}\,\partial_2^2 + c_{44}\,\partial_3^2)u_1 + (c_{12} + c_{66})\,\partial_1\partial_2 u_2 + (c_{13} + c_{44})\,\partial_1\partial_3 u_3 \\
&\quad - e_{14}\,\partial_2\partial_3\varphi - q_{15}\,\partial_2\partial_3\psi - \widetilde{\gamma}_1\,\partial_1\vartheta - \varrho\,\tau^2 u_1 = F_1, \\
&(c_{12} + c_{66})\,\partial_2\partial_1 u_1 + (c_{66}\,\partial_1^2 + c_{11}\,\partial_2^2 + c_{44}\,\partial_3^2)\,u_2 + (c_{13} + c_{44})\,\partial_2\partial_3 u_3 \\
&\quad + e_{14}\partial_1\partial_3\varphi - q_{15}\partial_1\partial_3 u_2 - \widetilde{\gamma}_1\partial_2\vartheta - \varrho\,\tau^2 u_2 = F_2, \\
&(c_{13} + c_{44})\,\partial_3\partial_1 u_1 + (c_{13} + c_{44})\,\partial_3\partial_2 u_2 + (c_{44}\,\partial_1^2 + c_{44}\,\partial_2^2 + c_{33}\,\partial_3^2)\,u_3 \\
&\quad - \widetilde{\gamma}_3\,\partial_3\vartheta - \varrho\,\tau^2 u_3 = F_3, \\
&e_{14}\partial_2\partial_3 u_1 - e_{14}\partial_1\partial_3 u_2 + (\varkappa_{11}\,\partial_1^2 + \varkappa_{11}\,\partial_2^2 + \varkappa_{33}\,\partial_3^2)\,\varphi - (1 + \nu_0\tau)p_3\,\partial_3\vartheta = F_4, \\
&q_{15}\partial_2\partial_3 u_1 - q_{15}\partial_1\partial_3 u_2 + (\mu_{11}\,\partial_1^2 + \mu_{11}\,\partial_2^2 + \mu_{33}\,\partial_3^2)\,\psi - (1 + \nu_0\tau)m_3\,\partial_3\vartheta = F_5, \\
&-\tau\,T_0\,(\widetilde{\gamma}_1\,\partial_1 u_1 + \widetilde{\gamma}_1\,\partial_2 u_2 + \widetilde{\gamma}_3\,\partial_3 u_3) + \tau\,T_0\,p_3\,\partial_3\varphi + \tau\,T_0\,m_3\,\partial_3\psi \\
&\quad + (\eta_{11}\,\partial_1^2 + \eta_{11}\,\partial_2^2 + \eta_{33}\,\partial_3^2)\,\vartheta - (\tau\,d_0 + \tau^2 h_0)\,\vartheta = F_6,
\end{aligned} \tag{6.16}$$

where c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, c_{66}, are the elastic constants, e_{14} is the piezoelectric constant, q_{15} is the piezomagnetic constant, $\varkappa_{11}$ and $\varkappa_{33}$ are the dielectric constants, μ_{11} and μ_{33} are the magnetic permeability constants, $\widetilde{\gamma}_1 = (1+\nu_0\tau)\lambda_{11} = (1+\nu_0\tau)\lambda_{21}$ and $\widetilde{\gamma}_3 = (1+\nu_0\tau)\lambda_{31}$ are the thermal strain constants, η_{11} and η_{33} are the thermal conductivity constants, p_3 is the pyroelectric constant and m_3 is the pyromagnetic constant. Note that in the case of the Hexagonal crystals (**622** class), we have $c_{66} = (c_{11} - c_{12})/2$.

Note that some important polymers and bio-materials are modelled by the above partial differential equations, for example, the *collagen–hydroxyapatite* is one example of such a material. This material is widely used in biology and medicine (see [12]). The other important example is TeO_2 [40].

In this model the thermoelectromechanical stress operator is defined as

$$T(\partial_x, n) = \left\| T_{jk}(\partial_x, n) \right\|_{6\times 6}$$

with

$$\begin{aligned}
&T_{11}(\partial_x,n) = c_{11}n_1\partial_1 + c_{66}n_2\partial_2 + c_{44}n_3\partial_3, && T_{12}(\partial_x,n) = c_{12}n_1\partial_2 + c_{66}n_2\partial_1,\\
&T_{13}(\partial_x,n) = c_{13}n_1\partial_3 + c_{44}n_3\partial_1, && T_{14}(\partial_x,n) = -e_{14}n_3\partial_2,\\
&T_{15}(\partial_x,n) = -q_{15}n_3\partial_2, && T_{16}(\partial_x,n) = -\widetilde{\gamma}_1\, n_1,\\
&T_{21}(\partial_x,n) = c_{66}n_1\partial_2 + c_{12}n_2\partial_1, && T_{22}(\partial_x,n) = c_{66}n_1\partial_1 + c_{11}n_2\partial_2 + c_{44}n_3\partial_3,\\
&T_{23}(\partial_x,n) = c_{13}n_2\partial_3 + c_{44}n_3\partial_2, && T_{24}(\partial_x,n) = e_{14}n_3\partial_1,\\
&T_{25}(\partial_x,n) = q_{15}n_3\partial_1, && T_{26}(\partial_x,n) = -\widetilde{\gamma}_1\, n_2,\\
&T_{31}(\partial_x,n) = c_{44}n_1\partial_3 + c_{13}n_3\partial_1, && T_{32}(\partial_x,n) = c_{44}n_2\,\partial_3 + c_{13}n_3\partial_2,\\
&T_{33}(\partial_x,n) = c_{44}n_1\partial_1 + c_{44}n_2\partial_2 + c_{33}n_3\partial_3, && T_{34}(\partial_x,n) = 0,\\
&T_{35}(\partial_x,n) = 0, && T_{36}(\partial_x,n) = -\widetilde{\gamma}_3\, n_3,
\end{aligned}$$

$$\begin{aligned}
&T_{41}(\partial_x,n) = e_{14}n_2\partial_3, && T_{42}(\partial_x,n) = -e_{14}n_1\partial_3,\\
&T_{43}(\partial_x,n) = e_{14}(n_2\partial_1 - n_1\partial_2), && T_{44}(\partial_x,n) = \varkappa_{11}(n_1\partial_1 + n_2\partial_2) + \varkappa_{33}n_3\partial_3,\\
&T_{45}(\partial_x,n) = 0, && T_{46}(\partial_x,n) = -p_3n_3,\\
&T_{51}(\partial_x,n) = q_{15}n_2\partial_3, && T_{52}(\partial_x,n) = -q_{15}n_1\partial_3,\\
&T_{53}(\partial_x,n) = q_{15}(n_2\partial_1 - n_1\partial_2), && T_{54}(\partial_x,n) = 0,\\
&T_{55}(\partial_x,n) = \mu_{11}(n_1\partial_1 + n_2\partial_2) + \mu_{33}n_3\partial_3, && T_{56}(\partial_x,n) = -m_3n_3,\\
&T_{6j}(\partial_x,n) = 0, \text{ for } j = 1,2,3,4,5, && T_{66}(\partial_x,n) = \eta_{11}(n_1\partial_1 + n_2\partial_2) + \eta_{33}n_3\partial_3.
\end{aligned}$$

The material constants satisfy the following inequalities which follow from positive definiteness of the internal energy form (see (2.9)–(2.10))

$$\begin{aligned}
&c_{11} > |c_{12}|, \quad c_{44} > 0, \quad c_{66} > 0, \quad c_{33}(c_{11} + c_{12}) > 2c_{13}^2,\\
&\varkappa_{11} > 0, \quad \varkappa_{33} > 0, \quad \eta_{11} > 0, \quad \eta_{33} > 0, \quad \mu_{11} > 0, \quad \mu_{33} > 0.
\end{aligned}$$

From (2.11), (2.14), (2.15), it follows also that

$$\varkappa_{33} > \frac{p_3^2\, T_0}{d_0}, \quad \mu_{33} > \frac{m_3^2\, T_0}{d_0}.$$

Under these conditions the corresponding mixed boundary value problem in question is uniquely solvable.

Furthermore, we assume that mechanical and electric fields are coupled, i.e. $e_{14} \neq 0$, that $\frac{\mu_{11}}{\varkappa_{11}} = \frac{\mu_{33}}{\varkappa_{33}} = \alpha$ and the surface S is parallel to the plane of isotropy (i.e., to the plane $x_3 = 0$) in some neighbourhood of ℓ_m.

We will show that under these conditions we can find the exponents involved in the asymptotic expansions of solutions explicitly in terms of the material constants.

In this case the symbol matrix $\sigma_{\mathcal{K}}^+ = \mathfrak{S}(\mathcal{K}; x', 0, +1)$ is calculated explicitly and has the form (see Appendix B):

$$\sigma_{\mathcal{K}}^+ = \begin{bmatrix} 0 & 0 & 0 & A_{14} & A_{15} & 0\\ 0 & 0 & A_{23} & 0 & 0 & 0\\ 0 & A_{32} & 0 & 0 & 0 & 0\\ A_{41} & 0 & 0 & 0 & 0 & 0\\ A_{51} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$A_{14} = -i\,\frac{e_{14}\,c_{66}\,(b_2-b_1)}{2\,b_1\,b_2\sqrt{B}} - i\,\frac{e_{14}q_{15}^2}{\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1b_2+\varkappa_{11})}{\sqrt{B}}\right], \tag{6.17}$$

$$A_{41} = -i\,\frac{e_{14}\,\varkappa_{33}\,(b_2-b_1)}{2\sqrt{B}}, \tag{6.18}$$

$$A_{15} = -i\,\frac{q_{15}\,c_{66}\,(b_2-b_1)}{2\alpha\,b_1\,b_2\sqrt{B}} - i\,\frac{q_{15}e_{14}^2}{\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1b_2+\varkappa_{11})}{\sqrt{B}}\right], \tag{6.19}$$

$$A_{51} = -i\,\frac{q_{15}\,\varkappa_{33}\,(b_2-b_1)}{2\sqrt{B}}, \tag{6.20}$$

$$b_1 = \sqrt{\frac{A-\sqrt{B}}{2\,c_{44}\,\varkappa_{33}}}, \qquad b_2 = \sqrt{\frac{A+\sqrt{B}}{2\,c_{44}\,\varkappa_{33}}},$$

$$\widetilde{e}_{14} = \left(e_{14}^2 + \alpha^{-1}q_{15}^2\right)^{1/2}, \quad \alpha = \frac{\mu_{11}}{\varkappa_{11}} = \frac{\mu_{33}}{\varkappa_{33}} > 0,$$

$$A = \widetilde{e}_{14}^2 + c_{44}\,\varkappa_{11} + c_{66}\,\varkappa_{33} > 0, \qquad B = A^2 - 4\,c_{44}\,c_{66}\,\varkappa_{11}\,\varkappa_{33} > 0, \quad A > \sqrt{B}.$$

It can be proved that $A_{14}A_{41} + A_{15}A_{51} < 0$ (see Appendix B).

To calculate the entries A_{23} and A_{32}, we have to consider two cases. We set

$$C := c_{11}\,c_{33} - c_{13}^2 - 2\,c_{13}\,c_{44}, \quad D := C^2 - 4\,c_{44}^2\,c_{33}c_{11}.$$

First, let $D > 0$. Then it follows from the positive definiteness of the internal energy that $C > \sqrt{D}$ and we have

$$A_{23} = i\,\frac{c_{44}\,(d_2-d_1)\,(c_{11}-c_{13}\,d_1d_2)}{2\,d_1\,d_2\sqrt{D}}, \tag{6.21}$$

$$A_{32} = -i\,\frac{c_{44}\,(d_2-d_1)\,(c_{33}\,d_1\,d_2-c_{13})}{2\,d_1\,d_2\sqrt{D}}, \tag{6.22}$$

$$d_1 = \sqrt{\frac{C-\sqrt{D}}{2\,c_{44}\,c_{33}}}, \qquad d_2 = \sqrt{\frac{C+\sqrt{D}}{2\,c_{44}\,c_{33}}}.$$

These equalities imply $A_{23}\,A_{32} > 0$.

Now, let $D < 0$. We get

$$A_{23} = i\,\frac{a\,c_{44}(\sqrt{c_{11}\,c_{33}}-c_{13})}{\sqrt{-D}}, \quad A_{32} = -i\,\frac{a\,c_{44}(\sqrt{c_{11}\,c_{33}}-c_{13})}{\sqrt{-D}}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}},$$

where

$$a = \frac{1}{2}\sqrt{\frac{-C+2c_{44}\sqrt{c_{11}c_{33}}}{c_{44}c_{33}}} > 0.$$

One can easily check that again

$$A_{23}A_{32} = \frac{c_{44}^2\,a^2(\sqrt{c_{11}\,c_{33}}-c_{13})^2}{-D}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}} > 0.$$

The characteristic polynomial of the matrix $\sigma_{\mathcal{K}}^{+}$ can be represented as

$$\det(\sigma_{\mathcal{K}}^{+} - \beta\, I) = \det \begin{bmatrix} -\beta & A_{14} & 0 & 0 & A_{15} & 0 \\ A_{41} & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & A_{23} & 0 & 0 \\ 0 & 0 & A_{32} & -\beta & 0 & 0 \\ A_{51} & 0 & 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta \end{bmatrix}$$
$$= \beta^2 \left(\beta^2 - A_{23}A_{32}\right)\left(\beta^2 - (A_{14}A_{41} + A_{15}A_{51})\right).$$

Therefore, we have the following expressions for the eigenvalues of the matrix $\sigma_{\mathcal{K}}^{+}$:

$$\beta_{1,2} = \mp i\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}, \quad \beta_{3,4} = \mp\sqrt{A_{23}A_{32}}\,, \quad \beta_5 = \beta_6 = 0.$$

Then by Theorem 6.3

$$\lambda_1 = \frac{1 - 2i\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}}{1 + 2i\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}},\ \lambda_2 = \frac{1}{\lambda_1},\ \lambda_3 = \frac{1 - 2\sqrt{A_{23}A_{32}}}{1 + 2\sqrt{A_{23}A_{32}}},\ \lambda_4 = \frac{1}{\lambda_3},\ \lambda_5 = \lambda_6 = 1.$$

Note that $|\lambda_1| = |\lambda_2| = 1$. Moreover, since λ_3 and λ_4 are real, they are positive (see Appendix A).

Applying the above results we can explicitly write the exponents of the first terms of the asymptotic expansions of solutions (see (6.7)):

$$\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\arctan 2\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}, \quad \delta_1 = 0,$$
$$\gamma_2 = \frac{1}{2} + \frac{1}{\pi}\arctan 2\sqrt{-(A_{14}A_{41} + A_{15}A_{51})}, \quad \delta_2 = 0,$$
$$\gamma_3 = \gamma_4 = \frac{1}{2}, \qquad \delta_3 = -\delta_4 = \widetilde{\delta} = \frac{1}{2\pi}\log\frac{1 - 2\sqrt{A_{23}A_{32}}}{1 + 2\sqrt{A_{23}A_{32}}},$$
$$\gamma_5 = \gamma_6 = \frac{1}{2}, \qquad \delta_5 = \delta_6 = 0.$$

Note that $B_0(t)$ has the following form

$$B_0(t) = \begin{bmatrix} I_4 & [0]_{4\times 2} \\ [0]_{4\times 2} & B^{(2)}(t) \end{bmatrix}, \qquad \text{where} \quad B^{(2)}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Now we can draw the following conclusions:

1. The solutions of the problem possess the following asymptotic behaviour near the line ℓ_m:

$$(u,\ \varphi,\ \psi)^\top = c_0\, r^{\gamma_1} + c_1\, r^{\frac{1}{2}+i\,\widetilde{\delta}} + c_2\, r^{\frac{1}{2}-i\,\widetilde{\delta}} + c_3 r^{\frac{1}{2}}\ln r + c_4 r^{\frac{1}{2}} + c_5 r^{\gamma_2} + \cdots$$
$$\vartheta = b_3\, r^{1/2} + b_4\, r^{\gamma_2} + \cdots .$$

As we see, the exponent γ_1 characterizing the behaviour of u, φ, and ψ near the line ℓ_m depends on the elastic, piezoelectric, piezomagnetic, dielectric, and permeability constants, and does not depend on the thermal constants. Moreover, γ_1 takes values from the interval $(0,\ 1/2)$.

For the general anisotropic case these exponents also depend on the geometry of the line ℓ_m, in general.

2. Since $\gamma_1 < 1/2$, we have not oscillating singularities for physical fields in some neighbourhood of the curve ℓ_m.

Note that in the classical elasticity theory (for both isotropic and anisotropic solids) for mixed BVPs the dominant exponents are $1/2$, $1/2 \pm i\,\widetilde{\delta}$ with $\widetilde{\delta} \neq 0$ and consequently we have oscillating stress singularities at the collision curve ℓ_m.

The following questions arise naturally:

(a) does there exist a class of GTEME type media for which the real part of the principal exponent defining the dominant stress singularity near the line ℓ_m does not depend on the material constants?

(b) does there exist a class of GTEME type media for which the real part of the principal exponent equals 1/2?

As we will see below, both question have positive answers.

Indeed, let us consider the class of GTEME type media with cubic anisotropy. Note that such materials as $Bi_{12}GeO_{20}$ and GaAs belong to this class (see, e.g., [40]). The latter material is widely used in the electronic industry.

The corresponding system of differential equations in this case reads as:

$$\begin{aligned}
&(c_{11}\,\partial_1^2 + c_{44}\,\partial_2^2 + c_{44}\,\partial_3^2)u_1 + (\,c_{12} + c_{44}\,)\,\partial_1\partial_2 u_2 + (\,c_{12} + c_{44}\,)\,\partial_1\partial_3 u_3\\
&\quad + 2e_{14}\,\partial_2\partial_3\varphi + 2q_{15}\,\partial_2\partial_3\psi - \widetilde{\gamma}_1\,\partial_1\vartheta - \varrho\,\tau^2\,u_1 = F_1,\\
&(\,c_{12} + c_{44}\,)\,\partial_2\partial_1 u_1 + (\,c_{44}\,\partial_1^2 + c_{11}\,\partial_2^2 + c_{44}\,\partial_3^2\,)\,u_2 + (\,c_{12} + c_{44}\,)\,\partial_2\partial_3 u_3\\
&\quad + 2e_{14}\partial_1\partial_3\varphi + 2q_{15}\partial_1\partial_3\psi - \widetilde{\gamma}_1\partial_2\vartheta - \varrho\,\tau^2\,u_2 = F_2,\\
&(\,c_{12} + c_{44}\,)\,\partial_3\partial_1 u_1 + (\,c_{12} + c_{44}\,)\,\partial_3\partial_2 u_2 + (\,c_{44}\,\partial_1^2 + c_{44}\,\partial_2^2 + c_{11}\,\partial_3^2\,)\,u_3\\
&\quad + 2e_{14}\partial_1\partial_2\varphi + 2q_{15}\partial_1\partial_2\psi - \widetilde{\gamma}_3\,\partial_3\vartheta - \varrho\,\tau^2\,u_3 = F_3,\\
&-2e_{14}\partial_2\partial_3 u_1 - 2e_{14}\partial_1\partial_3 u_2 - 2e_{14}\partial_1\partial_2 u_3 + (\varkappa_{11}\partial_1^2 + \varkappa_{11}\partial_2^2 + \varkappa_{11}\partial_3^2)\varphi - p_3\partial_3\vartheta = F_4,\\
&-2q_{15}\partial_2\partial_3 u_1 - 2q_{15}\partial_1\partial_3 u_2 - 2q_{15}\partial_1\partial_2 u_3 + (\mu_{11}\partial_1^2 + \mu_{11}\partial_2^2 + \mu_{11}\partial_3^2)\psi - m_3\,\partial_3\vartheta = F_5,\\
&-\tau T_0(\widetilde{\gamma}_1\partial_1 u_1 + \widetilde{\gamma}_1\partial_2 u_2 + \widetilde{\gamma}_3\,\partial_3 u_3) + \tau T_0 p_3\partial_3\varphi + \tau T_0 m_3\partial_3\psi\\
&\quad + (\,\eta_{11}\,\partial_1^2 + \eta_{11}\,\partial_2^2 + \eta_{33}\,\partial_3^2\,)\,\vartheta - \tau\,d_0\,\vartheta = F_6.
\end{aligned} \tag{6.23}$$

The elastic, dielectric, permeability and thermal constants involved in the governing equations satisfy the following conditions:

$$\begin{aligned}
&c_{11} > 0, \quad c_{44} > 0, \quad -1/2 < c_{12}/c_{11} < 1, \quad \varkappa_{11} > 0, \quad \mu_{11} > 0,\\
&\varkappa_{33} > \frac{p_3^2\,T_0}{d_0}, \quad \mu_{33} > \frac{m_3^2\,T_0}{d_0}, \quad \eta_{11} > 0, \quad \eta_{33} > 0.
\end{aligned} \tag{6.24}$$

Introduce the notation,

$$D := C^2 - 4\,c_{11}^2\,c_{44}^2, \quad C := c_{11}^2 - c_{12}^2 - 2\,c_{12}\,c_{44}, \quad a := \frac{1}{2}\sqrt{\frac{-C + 2c_{44}\sqrt{c_{11}}}{c_{44}c_{11}}} > 0.$$

In the case under consideration, the matrix $\sigma_{\mathcal{K}}^{+}$ is self-adjoint and reads as:

$$\sigma_{\mathcal{K}}^{+} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & A_{23} & 0 & 0 & 0\\ 0 & A_{32} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.25}$$

where

$$\begin{aligned}
&A_{23} = \overline{A_{32}} = i\,\frac{c_{44}\,(d_2 - d_1)(c_{11} - c_{12})}{2\sqrt{D}} \quad \text{for} \quad D > 0,\\
&d_1 = \sqrt{\frac{C - \sqrt{D}}{2c_{44}c_{11}}}, \quad d_2 = \sqrt{\frac{C + \sqrt{D}}{2c_{44}c_{11}}},\\
&A_{23} = \overline{A_{32}} = i\,\frac{c_{44}\,a\,(c_{11} - c_{12})}{2\sqrt{-D}} \quad \text{for} \quad D < 0.
\end{aligned}$$

The corresponding eigenvalues read as (see Theorem 6.3)

$$\begin{aligned}
&\beta_j = 0, \;\; j = 1, 2, 5, 6, \quad \beta_{3,4} = \pm|A_{23}|,\\
&\lambda_j = 1, \;\; j = 1, 2, 5, 6, \quad \lambda_3 = \frac{1 + 2|A_{23}|}{1 - 2|A_{23}|} > 0, \quad \lambda_4 = \frac{1}{\lambda_3}\,,
\end{aligned}$$

and

$$\gamma_j = \frac{1}{2},\ \ j = \overline{1,6},\ \ \delta_j = 0,\ \ j = 1,2,5,6,\ \ \delta_3 = -\delta_4 = \widetilde{\delta} = \frac{1}{2\pi}\log\frac{1+2|A_{23}|}{1-2|A_{23}|}.$$

From Lemma 6.1, Remark 6.2 and equalities (6.25) and (6.8) we derive

$$a_0 = \Big[(-2^{-1}I_6 + \sigma^+_{\mathcal{K}})\,[\sigma^+_{\mathcal{H}}]^{-1}\Big]^{-1}\Big[(-2^{-1}I_6 + \sigma^-_{\mathcal{K}})\,[\sigma^-_{\mathcal{H}}]^{-1}\Big] = \sigma^+_{\mathcal{H}}\,\widetilde{a}_0\,[\sigma^+_{\mathcal{H}}]^{-1},$$

where

$$\widetilde{a}_0 = [2^{-1}I_6 - \sigma^+_{\mathcal{K}}]^{-1}\,[2^{-1}I_6 + \sigma^+_{\mathcal{K}}].$$

This matrix is self-adjoint due to the equality (6.25) and it is similar to a diagonal matrix, i.e., there is a unitary matrix $\mathcal{D}$ such that $\mathcal{D}\,\widetilde{a}_0\,[\mathcal{D}]^{-1}$ is diagonal. Therefore the matrix a_0 can be reduced to a diagonal matrix by the non-degenerate matrix $\sigma^+_{\mathcal{H}}\mathcal{D}^{-1}$. In turn, this implies that $B_0(t) = I$ and the leading terms of the asymptotic expansion (6.6) near the curve ℓ_m do not contain logarithmic factors.

As a result we obtain the asymptotic expansion leading to the positive answers to the questions (a) and (b) stated above,

$$\begin{aligned}(u,\varphi,\psi)^\top &= c_0\,r^{1/2} + c_1\,r^{1/2+i\,\widetilde{\delta}} + c_2\,r^{1/2-i\,\widetilde{\delta}} + \mathcal{O}(r^{3/2-\varepsilon}),\\ \vartheta &= b_0\,r^{1/2} + \mathcal{O}(r^{3/2-\varepsilon}),\end{aligned}$$

where ε is an arbitrary positive number. Consequently, u, φ, ψ, and ϑ possess $C^{1/2}$-regularity in a neighbourhood of the collision curve ℓ_m.

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Appendix A. Some results for pseudodifferential equations on manifolds with boundary

Here we collect some results describing the Fredholm properties of strongly elliptic pseudodifferential operators on a compact manifold with boundary. They can be found in [36,16,41,19].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, nonselfintersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathcal{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}},\ \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \ldots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\mathfrak{S}(\mathcal{A}; x, 0, \ldots, 0, +1)\,]^{-1}\ \mathfrak{S}(\mathcal{A}; x, 0, \ldots, 0, -1)\,, \quad x \in \partial\overline{\mathcal{M}},$$

and let

$$\delta_j(x) = \mathrm{Re}\,\big[\,(2\pi\,i)^{-1}\ln\lambda_j(x)\,\big],\ \ j = 1, \ldots, N.$$

Here $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty, 0]$. Due to the strong ellipticity of $\mathcal{A}$ we have the strict inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$. The numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. In particular, if the eigenvalue λ_j is real, then λ_j is positive.

Note that when $\mathfrak{S}(\mathcal{A}, x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ or when it is an even matrix in ξ we have $\delta_j(x) = 0$ for $j = 1, \ldots, N$, since all the eigenvalues $\lambda_j(x)$ ($j = \overline{1,N}$) are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators are characterized by the following theorem.

Theorem A. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant c_0 such that*

$$\operatorname{Re}\Big(\sigma_{\mathcal{A}}(x,\xi)\zeta\cdot\zeta\Big) \ge c_0\,|\zeta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\zeta \in \mathbb{C}^N$. Then

$$\mathcal{A} : \widetilde{H}^s_p(\mathcal{M}) \to H^{s-\nu}_p(\mathcal{M}), \qquad \mathcal{A} : \widetilde{B}^s_{p,q}(\mathcal{M}) \to B^{s-\nu}_{p,q}(\mathcal{M}), \tag{A.1}$$

are Fredholm operators with index zero if

$$\frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\,1\le j\le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\,1\le j\le N} \delta_j(x). \tag{A.2}$$

Moreover, the null-spaces and indices of the operators (A.1) are the same (for all values of the parameter $q \in [1, +\infty]$) provided p and s satisfy the inequality (A.2).

We essentially use this theorem in Section 4 to prove the existence and regularity results of solutions to mixed boundary value problems for solids with interior cracks.

Appendix B. Calculation of the symbolic matrices

In this section we calculate the principal homogeneous symbol matrix $\sigma^+_{\mathcal{K}} = \mathfrak{S}(\mathcal{K}; x_1, 0, +1)$ corresponding to the system (6.16) (**422** and **622** classes). To this end we write the fundamental matrix (2.1) in the form (see [31, Section 3])

$$\begin{aligned}\Gamma(x,\tau) &= \mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi,\tau)\big]\\ &= \mathcal{F}^{-1}_{\xi'\to x'}\Big[\pm\frac{1}{2\pi}\int_{l^\pm}\big[A(-i\xi',-i\zeta,\tau)\big]^{-1} e^{-i\zeta x_3}\,d\zeta\Big],\end{aligned} \tag{B.1}$$

where the sign "−" corresponds to the case $x_3 > 0$ and the sign "+" to the case $x_3 < 0$. Here $x' = (x_1, x_2)$, $\xi' = (\xi_1, \xi_2)$, $\xi = (\xi', \xi_3)$, $l^+(l^-)$ is a closed contour with positive counterclockwise orientation enveloping all the roots of the polynomial $\det A(-i\xi', -i\zeta, \tau)$ with respect to the variable ζ in the half-plane $Im\,\zeta > 0$ ($Im\,\zeta < 0$).

First, we write the principal homogeneous symbols $A^{(0)}$ and $\mathcal{T}^{(0)}$ of the operators $A(\partial_x, \tau)$ and $\mathcal{T}(\partial, n)$ at a point $\widetilde{\zeta} = (0, 1, \zeta)$. Choosing a local coordinate system appropriately, we can assume that the exterior unit normal vector at this point reads as $n = (0, 0, 1)$. Then we have

$$A^{(0)}(\widetilde{\zeta}) = -\begin{bmatrix} A^{(0)}_{11} & 0 & 0 & A^{(0)}_{14} & A^{(0)}_{15} & 0\\ 0 & A^{(0)}_{22} & A^{(0)}_{23} & 0 & 0 & 0\\ 0 & A^{(0)}_{23} & A^{(0)}_{33} & 0 & 0 & 0\\ -A^{(0)}_{14} & 0 & 0 & A^{(0)}_{44} & 0 & 0\\ -A^{(0)}_{15} & 0 & 0 & 0 & A^{(0)}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & A^{(0)}_{66}\end{bmatrix}, \tag{B.2}$$

$$\mathcal{T}^{(0)}(\widetilde{\zeta}, n) = -\begin{bmatrix} ic_{44}\zeta & 0 & 0 & -ie_{14} & -iq_{14} & 0\\ 0 & ic_{44}\zeta & ic_{44} & 0 & 0 & 0\\ 0 & ic_{13} & ic_{33}\zeta & 0 & 0 & 0\\ 0 & 0 & 0 & i\varkappa_{33}\zeta & 0 & 0\\ 0 & 0 & 0 & 0 & i\mu_{33}\zeta & 0\\ 0 & 0 & 0 & 0 & 0 & i\eta_{33}\zeta\end{bmatrix}, \tag{B.3}$$

where

$$\begin{aligned}
&A^{(0)}_{11} = c_{44}\zeta^2 + c_{66}, \quad A^{(0)}_{14} = -e_{14}, \qquad A^{(0)}_{15} = -q_{15},\\
&A^{(0)}_{22} = c_{44}\zeta^2 + c_{11}, \quad A^{(0)}_{23} = (c_{13} + c_{44})\zeta, \quad A^{(0)}_{33} = c_{33}\zeta^2 + c_{44},\\
&A^{(0)}_{44} = \varkappa_{33}\zeta^2 + \varkappa_{11}, \quad A^{(0)}_{55} = \mu_{33}\zeta^2 + \mu_{11}, \quad A^{(0)}_{66} = \eta_{33}\zeta^2 + \eta_{11}.
\end{aligned}$$

Recall, that we assume $\frac{\mu_{11}}{\varkappa_{11}} = \frac{\mu_{33}}{\varkappa_{33}} = \alpha$.

From (3.3), (B.1)–(B.3), (6.13), and Theorem 3.3 it follows that

$$\begin{aligned}
-\frac{1}{2}I + \sigma^{+}_{\mathcal{K}} &= \lim_{x_3\to 0}\frac{1}{2\pi}\int_{l^+}\mathcal{T}^{(0)}(\widetilde{\zeta}, n)\big[A^{(0)}(\widetilde{\zeta})\big]^{-1}e^{-i\zeta x_3}d\zeta\\
&= \frac{1}{2\pi}\int_{l^+}\mathcal{T}^{(0)}(\widetilde{\zeta}, n)\big[A^{(0)}(\widetilde{\zeta})\big]^{-1}d\zeta = \|A_{kj}\|_{6\times 6},
\end{aligned} \tag{B.4}$$

where

$$\begin{aligned}
&A_{11} = \frac{i}{2\pi}\int_{l^+}\frac{c_{44}\varkappa_{33}\zeta^3 + (c_{44}\varkappa_{11} + \tilde{e}^2_{14})\zeta}{P_1(\zeta)}d\zeta,\\
&A_{14} = -\frac{i}{2\pi}\int_{l^+}\frac{c_{66}e_{14}}{P_1(\zeta)}d\zeta - \frac{i}{2\pi}\int_{l^+}\frac{e_{14}q^2_{15}\zeta^2}{Q(\zeta)}d\zeta\\
&A_{15} = -\frac{i}{2\pi}\int_{l^+}\frac{c_{66}q_{15}}{\alpha P_1(\zeta)}d\zeta - \frac{i}{2\pi}\int_{l^+}\frac{e^2_{14}q_{15}\zeta^2}{Q(\zeta)}d\zeta\\
&A_{1j} = 0, \quad j = 2, 3, 6, \quad A_{22} = \frac{i}{2\pi}\int_{l^+}\frac{c_{33}c_{44}\zeta^3 - c_{13}c_{44}\zeta}{P_2(\zeta)}d\zeta,\\
&A_{23} = -\frac{i}{2\pi}\int_{l^+}\frac{c_{13}c_{44}\zeta^2 - c_{11}c_{44}}{P_2(\zeta)}d\zeta, \quad A_{2j} = 0, \quad j = 1, 4, 5, 6,\\
&A_{3j} = 0, \quad j = 1, 4, 5, 6, \quad A_{32} = -\frac{i}{2\pi}\int_{l^+}\frac{c_{33}c_{44}\zeta^2 - c_{13}c_{44}}{P_2(\zeta)}d\zeta,\\
&A_{33} = \frac{i}{2\pi}\int_{l^+}\frac{c_{33}c_{44}\zeta^3 + (c_{11}c_{33} - c_{13}c_{44} - c^2_{13})\zeta}{P_2(\zeta)}d\zeta,\\
&A_{41} = -\frac{i}{2\pi}\int_{l^+}\frac{e_{14}\varkappa_{33}\zeta^2}{P_1(\zeta)}d\zeta, \quad A_{4j} = 0, \quad j = 2, 3, 5, 6,\\
&A_{44} = \frac{i}{2\pi}\int_{l^+}\frac{c_{44}\varkappa_{33}\zeta^3 + c_{66}\varkappa_{33}\zeta}{P_1(\zeta)}d\zeta + \frac{i}{2\pi}\int_{l^+}\frac{\varkappa_{33}q^2_{15}\zeta^3}{Q(\zeta)}d\zeta,\\
&A_{51} = -\frac{i}{2\pi}\int_{l^+}\frac{q_{15}\varkappa_{33}\zeta^2}{P_1(\zeta)}d\zeta, \quad A_{5j} = 0, \quad j = 2, 3, 4, 6\\
&A_{55} = \frac{i}{2\pi}\int_{l^+}\frac{c_{44}\varkappa_{33}\zeta^3 + c_{66}\varkappa_{33}\zeta}{P_1(\zeta)}d\zeta + \frac{i}{2\pi}\int_{l^+}\frac{\alpha\varkappa_{33}e^2_{14}\zeta^3}{Q(\zeta)}d\zeta,\\
&A_{66} = \frac{i}{2\pi}\int_{l^+}\frac{\eta_{33}\zeta}{\eta_{33}\zeta^2 + \eta_{11}}d\zeta, \quad A_{6j} = 0, \quad j = \overline{1, 5},\\
&P_1(\zeta) = c_{44}\varkappa_{33}\zeta^4 + (c_{44}\varkappa_{11} + c_{66}\varkappa_{33} + \widetilde{e}^2_{14})\zeta^2 + c_{66}\varkappa_{11},\\
&P_2(\zeta) = c_{33}c_{44}\zeta^4 + (c_{11}c_{33} - 2c_{13}c_{44} - c^2_{13})\zeta^2 + c_{11}c_{44},\\
&Q(\zeta) = \alpha(\varkappa_{33}\zeta^2 + \varkappa_{11})P_1(\zeta), \quad \widetilde{e}_{14} = \left(e^2_{14} + \alpha^{-1}q^2_{15}\right)^{1/2}.
\end{aligned}$$

Denote by $\zeta^{(j)}_1$, $\zeta^{(j)}_2$, $j = 1, 2$, the roots of the polynomials P_j with positive imaginary part. Evidently, $\zeta^{(1)}_1$, $\zeta^{(1)}_2$ and $\zeta^{(3)} = i\sqrt{\eta_{11}/\eta_{33}}$ are then the roots of $Q(\zeta)$ with positive imaginary parts.

We have the following explicit formulas,

$$\zeta_1^{(1)} = ib_1 = i\sqrt{\frac{A-\sqrt{B}}{2c_{44}\varkappa_{33}}}, \qquad \zeta_2^{(1)} = ib_2 = i\sqrt{\frac{A+\sqrt{B}}{2c_{44}\varkappa_{33}}},$$
$$\zeta_1^{(2)} = id_1 = i\sqrt{\frac{C-\sqrt{D}}{2c_{44}c_{33}}}, \qquad \zeta_2^{(2)} = id_2 = i\sqrt{\frac{C+\sqrt{D}}{2c_{44}c_{33}}},$$
$$A = \widetilde{e}_{14}^2 + c_{44}\varkappa_{11} + c_{66}\varkappa_{33} > 0, \qquad B = A^2 - 4c_{44}c_{66}\varkappa_{11}\varkappa_{33} > 0,$$
$$C = c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}, \qquad D = C^2 - 4c_{44}^2 c_{33}c_{11}.$$

Note that, if $D > 0$, then the roots $\zeta_1^{(2)}$ and $\zeta_2^{(2)}$ are purely imaginary. For $D < 0$ the roots are complex numbers with opposite real parts and equal imaginary parts:

$$\zeta_1^{(2)} = a + i\,b, \qquad \zeta_2^{(2)} = -a + i\,b, \qquad a > 0, \qquad b > 0.$$

Curvilinear integrals participating in (B.4) can be calculated explicitly by applying theory of residues and Cauchy's theorem

$$\int_{l^+} \frac{d\zeta}{P_1(\zeta)} = \frac{i\,\pi\,(\zeta_2^{(1)} - \zeta_1^{(1)})}{\zeta_1^{(1)}\zeta_2^{(1)}\sqrt{B}}, \qquad \int_{l^+} \frac{d\zeta}{P_2(\zeta)} = \frac{i\,\pi\,(\zeta_2^{(2)} - \zeta_1^{(2)})}{\zeta_1^{(2)}\zeta_2^{(2)}\sqrt{D}},$$
$$\int_{l^+} \frac{\zeta}{P_1(\zeta)}d\zeta = 0, \qquad \int_{l^+} \frac{\zeta}{P_2(\zeta)}d\zeta = 0,$$
$$\int_{l^+} \frac{\zeta^2}{P_1(\zeta)}d\zeta = -\frac{i\,\pi}{\sqrt{B}}(\zeta_2^{(1)} - \zeta_1^{(1)}), \qquad \int_{l^+} \frac{\zeta^2}{P_2(\zeta)}d\zeta = -\frac{i\,\pi}{\sqrt{D}}(\zeta_2^{(2)} - \zeta_1^{(2)}),$$
$$\int_{l^+} \frac{\zeta^3}{P_1(\zeta)}d\zeta = \frac{i\,\pi}{c_{44}\varkappa_{33}}, \qquad \int_{l^+} \frac{\zeta^3}{P_2(\zeta)}d\zeta = \frac{i\,\pi}{c_{44}c_{33}},$$
$$\int_{l^+} \frac{\zeta^2}{Q(\zeta)}d\zeta = \frac{\pi}{\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1b_2 + \varkappa_{11})}{\sqrt{B}}\right],$$

$$\int_{l^+} \frac{\zeta^3}{Q(\zeta)}d\zeta = 2\pi i \sum_{k=1}^{3} \frac{\zeta^3}{Q'(\zeta)}\bigg|_{\zeta=\zeta_k}$$
$$= -\frac{\pi i}{\alpha}\Bigg(\frac{b_1^2}{(\varkappa_{11} - \varkappa_{33}b_1^2)(c_{44}\varkappa_{11} + c_{66}\varkappa_{33} + \widetilde{e}_{14}^2 - 2c_{44}\varkappa_{33}b_1^2)}$$
$$+ \frac{b_2^2}{(\varkappa_{11} - \varkappa_{33}b_2^2)(c_{44}\varkappa_{11} + c_{66}\varkappa_{33} + \widetilde{e}_{14}^2 - 2c_{44}\varkappa_{33}b_2^2)} + \frac{b_3^2}{\varkappa_{33}P_1(ib_3)}\Bigg).$$

Note, that the last equality implies that the integrals

$$\frac{i}{2\pi}\int_{l^+} \frac{\varkappa_{33}q_{15}^2\zeta^3}{Q(\zeta)}d\zeta \quad \text{and} \quad \frac{i}{2\pi}\int_{l^+} \frac{\alpha\varkappa_{33}e_{14}^2\zeta^3}{Q(\zeta)}d\zeta$$

which are involved in A_{44} and A_{55} are real, therefore due to Lemma 6.1 they must be zero.

As a result we obtain

$$A_{jj} = -\frac{1}{2}, \quad j = \overline{1,6}, \quad A_{1j} = 0, \quad j = 2,3,6,$$
$$A_{14} = \frac{e_{14}c_{66}(\zeta_2^{(1)} - \zeta_1^{(1)})}{2\zeta_1^{(1)}\zeta_2^{(1)}\sqrt{B}} - \frac{e_{14}q_{15}^2}{2\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[i\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(\zeta_2-\zeta_1)(-\varkappa_{33}\,\zeta_1\zeta_2 + \varkappa_{11})}{\sqrt{B}}\right],$$
$$A_{15} = \frac{q_{15}c_{66}(\zeta_2^{(1)} - \zeta_1^{(1)})}{2\alpha\zeta_1^{(1)}\zeta_2^{(1)}\sqrt{B}} - \frac{e_{14}^2q_{15}}{2\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[i\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(\zeta_2-\zeta_1)(-\varkappa_{33}\,\zeta_1\zeta_2 + \varkappa_{11})}{\sqrt{B}}\right],$$
$$A_{2j} = 0, \quad j = 1,4,5,6, \quad A_{23} = -\frac{c_{44}(\zeta_2^{(2)} - \zeta_1^{(2)})(c_{11} + c_{13}\zeta_1^{(2)}\zeta_2^{(2)})}{2\zeta_1^{(2)}\zeta_2^{(2)}\sqrt{D}},$$

$$A_{3j} = 0, \quad j = 1, 4, 5, 6, \quad A_{32} = -\frac{c_{44}(\zeta_2^{(2)} - \zeta_1^{(2)})(c_{33}\zeta_1^{(2)}\zeta_2^{(2)} + c_{13})}{2\zeta_1^{(2)}\zeta_2^{(2)}\sqrt{D}},$$
$$A_{41} = -\frac{e_{14}\varkappa_{33}(\zeta_2^{(1)} - \zeta_1^{(1)})}{2\sqrt{B}}, \quad A_{4j} = 0, \quad j = 2, 3, 5, 6,$$
$$A_{51} = -\frac{q_{15}\varkappa_{33}(\zeta_2^{(1)} - \zeta_1^{(1)})}{2\sqrt{B}}, \quad A_{5j} = 0, \quad j = 2, 3, 4, 6,$$
$$A_{6j} = 0, \quad j = \overline{1, 5}.$$

Now, taking into account that

$$\zeta_j^{(1)} = i\,b_j, \quad b_j > 0, \quad j = 1, 2,$$
$$\zeta_j^{(2)} = i\,d_j, \quad d_j > 0, \quad j = 1, 2, \quad \text{if } D > 0,$$
$$\zeta_1^{(2)} = a + i\,b, \quad \zeta_2^{(2)} = -a + i\,b, \quad a > 0, \quad b > 0, \quad \zeta_1^{(2)}\zeta_2^{(2)} = -\sqrt{\frac{c_{11}}{c_{33}}}, \quad \text{if } D < 0,$$

we obtain (6.17)–(6.22).

One can calculate the homogeneous symbol matrix $\sigma_{\mathcal{K}}^+ = \sigma_{\mathcal{K}}(x_1, 0, +1)$ corresponding to the system (6.23) quite similarly.

Now we prove that

$$A_{14}A_{41} + A_{15}A_{51} < 0. \tag{B.5}$$

In view of the inequalities (6.24) and the relation

$$A_{14}A_{41} + A_{15}A_{51} = -\frac{c_{66}\varkappa_{33}(b_2 - b_1)^2\widetilde{e}_{14}^2}{4Bb_1b_2} - \frac{1}{\alpha}\frac{e_{14}^2q_{15}^2\varkappa_{33}(b_2 - b_1)}{2\sqrt{B}\varkappa_{11}\widetilde{e}_{14}^2}\Big[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2 - b_1)(\varkappa_{33}b_1b_2 + \varkappa_{11})}{\sqrt{B}}\Big],$$

and since

$$b_2 - b_1 > 0, \quad b_1 > 0, \quad B > 0,$$

it is sufficient to show that

$$\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2 - b_1)(\varkappa_{33}b_1b_2 + \varkappa_{11})}{\sqrt{B}} > 0. \tag{B.6}$$

Rewrite this inequality as

$$\varkappa_{11}B > c_{44}^2\varkappa_{33}(b_2 - b_1)^2(\varkappa_{33}b_1b_2 + \varkappa_{11})^2. \tag{B.7}$$

Taking into account the equalities

$$(b_2 - b_1)^2 = \tfrac{A}{c_{44}\varkappa_{33}} - 2\sqrt{\tfrac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}}, \quad b_1b_2 = \sqrt{\tfrac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}}, \quad B = A^2 - 4c_{66}c_{44}\varkappa_{11}\varkappa_{33},$$

we find that (B.7) is equivalent to the relation

$$\varkappa_{11}\Big(A^2 - 4c_{44}c_{66}\varkappa_{11}\varkappa_{33}\Big) > c_{44}\Big(A - 2\sqrt{c_{44}c_{66}\varkappa_{11}\varkappa_{33}}\Big)\left(\varkappa_{33}\sqrt{\frac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}} + \varkappa_{11}\right)^2.$$

In turn the last inequality is equivalent to the following one

$$\varkappa_{11}\Big(A + 2\sqrt{c_{44}c_{66}\varkappa_{11}\varkappa_{33}}\Big) > c_{44}\left(\varkappa_{33}\sqrt{\frac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}} + \varkappa_{11}\right)^2.$$

At last, substituting here $A = \widetilde{e}_{14}^2 + c_{44}\varkappa_{11} + c_{66}\varkappa_{33}$ we arrive at the evident inequality

$$\widetilde{e}_{14}^2 + \left(\sqrt{c_{44}\varkappa_{11}} + \sqrt{c_{44}\varkappa_{11}}\right)^2 > \left(\sqrt{c_{44}\varkappa_{11}} + \sqrt{c_{44}\varkappa_{11}}\right)^2.$$

Thus (B.6) is valid and consequently (B.5) holds as well.

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Original article

A Novel similarity measure based on eigenvalue distribution

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Abstract

Due to the rapidly increasing interests of effective and efficient data processing, the developments of similarity measure have been significantly expanded. This paper defines the eigenvalue distribution as a criterion of measuring similarity in a multivariate system. The primary evaluations are conducted by simulations with the assistances and comparisons of several empirical statistical tests. Furthermore, the proposed measure is conducted in simultaneous real case scenario by adopting the bootstrap re-sampling technique. It also overcomes the difficulty of different series lengths in the multivariate system. Moreover, it does not have pre-assumptions on distributions, and it can be easily employed and efficiently computed.

Keywords: Similarity measure; Eigenvalue distribution; Singular value decomposition; Multivariate

1. Introduction

The studies of similarity have been overwhelmingly explored and applied in various disciplines on many different formats, for example, numerical values [1,2], images [3,4], genes [5–7], chemical subjects [8–10], words [11,12] and so on. According to [13], the similarity measure is the most essential core element of time series classification and clustering. Therefore, the development of better similarity measure can significantly assist the improvement of data analysis efficiency. According to [14], the similarity measure is closely related to the distance measure, as the distance is defined as a quantitative degree of how far apart two objects are. Consequently, studies of distance and similarity are significantly connected and crucial in terms of solving many pattern recognition related problems, such as clustering technique [15,16], Taxonomy [17,18], image registration [19,20], etc.

As one of the crucial difficulties in similarity measure is that the different types of features are not comparable, this paper proposes the novel similarity measure based on the eigenvalue distribution, which is inspired by the dynamical approach and embedding theorem where a one dimensional time series will be transferred to multidimensional time series in a Hankel matrix. Hankel matrix has many features as a square matrix, where gives a sequence of the

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one dimensional time series, also defines the dynamical state-space. This paper is the initial attempt of adopting eigenvalue distribution into formulating a similarity measure in the multivariate system. Time series under evaluation are embedded into multidimensional matrices and combined either vertically or horizontally to be transformed into a Hankel matrix, where the eigenvalues can be extracted by Singular Value Decomposition (SVD) technique accordingly. As Aristotle claimed in [21], the Formal Cause is "the account of what-it-is-to-be", or "what makes a thing one thing rather than many things". Based on the "formal cause" claimed by Aristotle, here in this paper, we define the corresponding distribution of extracted eigenvalues as the "formal" criterion for developing a novel similarity measure. The successful implementation of this novel similarity measure can overcome the limitations of nonlinear dynamic, complex fluctuations and the possibility of distinguishing similarity for particular or selected features.

In order to evaluate the reliability of eigenvalue distribution as the similarity measure, three empirical statistical tests together with the real case scenario are overwhelmingly considered. Possible circumstances during the formulation process of the new measure are comprehensively evaluated with brief introductions and comparisons in following sections.

In general, this paper is structured as follows: Section 2 briefly introduce the techniques for obtaining the corresponding eigenvalue distribution. The review of some empirical methods and the formulation of proposed novel similarity measure are listed in Section 3. Section 4 provides the empirical results and evaluations by simulations, whilst the real case scenario results are stated in Section 5. Finally, the discussion and conclusion are summarized in Sections 6 and 7 respectively.

2. Eigenvalue distribution

To overcome the difficulty of existing diverse and incomparable features, the novel similarity measure extracts the corresponding eigenvalue distributions as the formal criterion by considering the elements of time series as a whole without removing any nonlinear or complex features. Note that as the structures of constructing Hankel matrix containing multiple variables differ, including both horizontal and vertical forms.

Consider M time series with different series length N_i $Y_{N_i}^{(i)} = (y_1^{(i)}, \dots, y_{N_i}^{(i)})(i = 1, \dots, M)$. In this case, the standard univariate form can be acquired by setting $M = 1$. Firstly, we transfer a one-dimensional time series $Y_{N_i}^{(i)}$ in to a multidimensional matrix $[X_1^{(i)}, \dots, X_{K_i}^{(i)}]$ with vectors $X_j^{(i)}$ that equals to $(y_j^{(i)}, \dots, y_{j+L_i-1}^{(i)})^T \in \mathbf{R}^{L_i}$, where $L_i(2 \le L_i \le N_i/2)$ is the window length for each series with length N_i and $K_i = N_i - L_i + 1$. We can then get the trajectory matrix $\mathbf{X}^{(i)} = [X_1^{(i)}, \dots, X_{K_i}^{(i)}] = (x_{mn})_{m,n=1}^{L_i,K_i}$ after this step. The above procedure for each series separately provides M different $L_i \times K_i$ trajectory matrices $\mathbf{X}^{(i)}(i = 1, \dots, M)$.

To construct a block Hankel matrix in the vertical form we need to have $K_1 = \cdots = K_M = K$. Accordingly, this version enables us to have various window length L_i and different series length N_i, but similar K_i for all series. The result of this step is the following block Hankel trajectory matrix:

$$\mathbf{X}_V = \begin{bmatrix} \mathbf{X}^{(1)} \\ \vdots \\ \mathbf{X}^{(M)} \end{bmatrix}.$$

Note that $\mathbf{X}_V$ indicates that the output of the first step is a block Hankel trajectory matrix formed in a vertical form.

Then, the SVD of $\mathbf{X}_V$ is performed in the following step. Note that the SVD technique is closely related to the Singular Spectrum Analysis technique and its multivariate extension, which have been widely applied in a range of different fields and a multitude of fairly precise results proved it as a powerful and applicable technique [22,29,23–28, 30–35]. Denote $\lambda_{V_1}, \dots, \lambda_{V_{L_{sum}}}$ as the eigenvalues of $\mathbf{X}_V\mathbf{X}_V^T$, arranged in decreasing order $(\lambda_{V_1} \ge \cdots \lambda_{V_{L_{sum}}} \ge 0)$ and $U_{V_1}, \dots, U_{V_{L_{sum}}}$, the corresponding eigenvectors, where $L_{sum} = \sum_{i=1}^{M} L_i$. Note also that the structure of the matrix $\mathbf{X}_V\mathbf{X}_V^T$ is as follows:

$$\mathbf{X}_V\mathbf{X}_V^T = \begin{bmatrix} \mathbf{X}^{(1)}\mathbf{X}^{(1)T} & \mathbf{X}^{(1)}\mathbf{X}^{(2)T} & \cdots & \mathbf{X}^{(1)}\mathbf{X}^{(M)T} \\ \mathbf{X}^{(2)}\mathbf{X}^{(1)T} & \mathbf{X}^{(2)}\mathbf{X}^{(2)T} & \cdots & \mathbf{X}^{(2)}\mathbf{X}^{(M)T} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}^{(M)}\mathbf{X}^{(1)T} & \mathbf{X}^{(M)}\mathbf{X}^{(2)T} & \cdots & \mathbf{X}^{(M)}\mathbf{X}^{(M)T} \end{bmatrix}.$$

The structure of the matrix $\mathbf{X}_V\mathbf{X}_V^T$ is similar to the variance–covariance matrix in the classical multivariate statistical analysis literature. The matrix $\mathbf{X}^{(i)}\mathbf{X}^{(i)T}$ for the series $Y_{N_j}^{(j)}$, appears along the main diagonal and the products of two Hankel matrices $\mathbf{X}^{(i)}\mathbf{X}^{(j)T}$ $(i \neq j)$, which are related to the series $Y_{N_i}^{(i)}$ and $Y_{N_j}^{(i)}$, appears in the off-diagonal. The SVD of $\mathbf{X}_V$ can be written as $\mathbf{X}_V = \mathbf{X}_{V_1} + \cdots + \mathbf{X}_{V_{Lsum}}$, where $\mathbf{X}_{V_i} = \sqrt{\lambda_{V_i}} U_{V_i} V_{V_i}^T$ and $V_{V_i} = \mathbf{X}_V^T U_{V_i} / \sqrt{\lambda_{V_i}}$ ($\mathbf{X}_{V_i} = 0$ if $\lambda_{V_i} = 0$).

Moreover, the horizontal form decomposition is proved to produce more reliable and consistent eigenvalue distributions. Note that the eigenvalue distributions by vertically and horizontally formed techniques are both carefully considered and compared (detailed results are available upon request from authors). Hence, all tests in the following sections are based on eigenvalues conducted by decomposition stage of the horizontal form.

3. Similarity measures

The distributions of eigenvalues of the trajectory matrices are here considered as the "formal" criterion of measuring the similarity between two series. The explorations of the significance of the Hankel matrix and its corresponding eigenvalues can be found in many different areas (for example [29,36–39]). In addition, more details about the empirical distribution of the eigenvalues of the Hankel matrix divided by its trace can be found in [40,42,41].

In order to evaluate whether the extracted eigenvalues are similar or not to conclude the similarity between two tested series, three empirical statistical tests (Chi-squared Test, Log-likelihood Goodness of Fit Test and Kolmogorov–Smirnov Test) are adopted. Various distance and similarity measures are comprehensively reviewed and categorized in [14], therefore, we do not reproduce here. Since the proposed similarity measure is expected to have no assumption or limitation on measuring tested series with only the empirical distributions, some tests that are commonly used to evaluate the consistency with the empirical distributions cannot be properly suitable here (i.e. Shapiro–Wilk Test [43], Hellinger Distance [44], Kullback Leibler Divergence [45], Anderson–Darling Test [46]). Therefore, only brief introductions of the suitable empirical statistical tests are provided respectively as follows.

3.1. *Similarity measures*

In general, coordinates and the cumulative distribution function (CDF) are the most generally accepted concepts to represent the examined subject. We briefly summarize several important and dominant measurements that are referred for formulating the novel similarity measure due to the special feature of eigenvalue distribution.

3.1.1. *Chi-squared test*

As an improved distance measure comparing to Euclidean distance, the Chi-squared statistic can be simply considered as the summation of squared Euclidean distances of two vectors (by considering them in a n dimensional space domain, where n is the number of observations for both vectors) over the corresponding "coordinates" of the domain vector. The Chi-squared distribution (also known as Helmertian distribution) [47] is one of the most significantly applied probability distributions, and it is most commonly accepted for measuring the distance or similarity level between two probability distributions. Pearson [48] adopted the Chi-squared distribution in the goodness of fit domain and conducted the Chi-squared test, which statistically evaluates the observed data about its goodness of fit level and consistency with an expected distribution. Here in this paper, it is adopted for comparing the eigenvalue distributions of two series (or one examined series with the benchmark population) as evidence of similarity. The Chi-squared statistic formula is:

$$\chi^2(C, E) = \sum_{i=1}^{Z} \frac{(C_i - E_i)^2}{E_i}, \tag{1}$$

where Z is the number of levels of categories; C is the observed frequency and E is the expected count.

Therefore, in terms of Chi-squared test between two tested variables, assume Z_A and Z_B are the number of levels of categorized variables A and B, so the degree of freedom can be calculated by $df = (Z_A - 1) \times (Z_B - 1)$. The expected counts/frequencies is computed by

$$E_{Z_{A,B}} = (C_{Z_A} \times C_{Z_B})/n, \tag{2}$$

where C_Z refers to observed counts at specific level of category and n indicates the total observation number. Consequently, the corresponding Chi-squared statistics is:

$$\chi^2(A, B) = \sum \frac{(C_{Z_{A,B}} - E_{Z_{A,B}})^2}{E_{Z_{A,B}}}. \tag{3}$$

3.1.2. Log-likelihood goodness of fit test

The Log-likelihood Goodness of Fit Test is actually based on the commonly used Chi-squared test statistics in [48]. According to [49], the Log-likelihood statistic formula is:

$$G = 2\sum_i f_i \cdot ln\left(\frac{f_i}{q_i}\right), \tag{4}$$

where the f_i refers to the observed frequency, whilst q_i indicates the expected frequency. More specifically, the test is adopted for evaluating whether the eigenvalue distribution of the examined series fit well to the eigenvalue distribution of the benchmark series.

3.1.3. Kolmogorov–Smirnov test

The Kolmogorov–Smirnov Test (K–S Test) was firstly proposed in [50]. As a non-parametric statistical test, it quantifies the distance based on the CDF with no assumption about the distribution of data. It can be adopted to examine the similarity level of one distribution to empirical distribution, more importantly, K–S test is also applicable for evaluating the similarity of distributions of two random samples. The K–S test statistic is defined as below, which we mainly follows [51]:

$$D_n = sup_x |F_n(x) - F(x)|, \tag{5}$$

where F refers to the theoretical cumulative distribution function, F_n represents the cumulative distribution up to n observations, sup_x indicates the supremum of the set of distances, and D_n refers to the supremum distance reached up to n observations. In terms of the two-sample case of K–S Test, the corresponding test statistic formula is:

$$D_{n,n'i} = sup_x |F_{1,n}(x) - F_{2,n'}(x)|, \tag{6}$$

note that $F_{1,n}$ and $F_{2,n'}$ are the corresponding distribution function for two tested samples respectively.

Specifically for the proposed similarity measure method based on eigenvalue distribution, two-sample K–S Test is adopted to determine whether the "benchmark" populations created by the dominate series has consistent eigenvalue distribution as the other series.

3.2. Novel similarity measure using eigenvalue distribution

By setting the eigenvalue distribution as our criterion and adopting the empirical methods listed above, the hypotheses of the novel similarity measure are stated as below:

Null hypothesis (H_0): there is no significant difference between the eigenvalue distributions of matrices by two tested series.

Alternative hypothesis (H_a): there is a significant difference between the eigenvalue distributions of matrices by two tested series.

The null hypothesis is rejected when the p-value is less than the 5% significance level, and therefore we conclude that the set of eigenvalues are not similar and consequently two test series are different. While if the p-value is very close to or equal to 1, we conclude that the two tested series are similar as they share very similar or even identical eigenvalue distributions.

As the proposing method of measuring similarity based on eigenvalue distribution is considering a possible implementation of detecting "Formal Cause", different benchmarks of comparison will lead to different results. Consider two random variables X and Y, "how similar is X to Y" and "how similar is Y to X" are two different

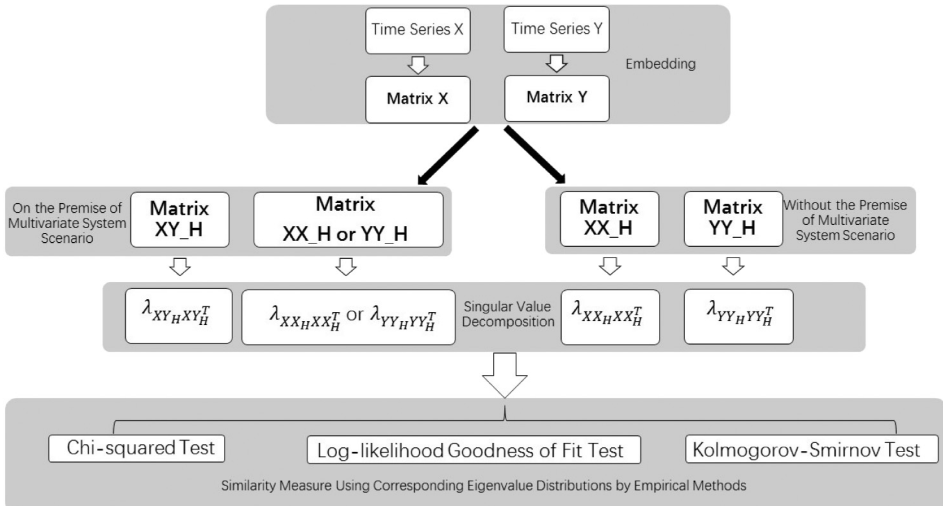

Fig. 1. The flowchart of the novel similarity measure using eigenvalue distribution.

questions, especially when the distribution of eigenvalues is the criterion. We should not expect exactly "same" results when we compare X to Y and Y to X, while the expected final outcomes that define "similar" or "different" should not vary. For instance, if the principle is to answer the question of how similar is Y to X, the eigenvalue distribution by corresponding matrix ($\mathbf{XY}_H$ or $\mathbf{XX}_H$ determined by with or without the premise of multivariate system) will be considered as the "benchmark" for further evaluation. Hence, if the other eigenvalue distribution by $\mathbf{XX}_H$ or $\mathbf{YY}_H$ (determined by with or without the premise of multivariate system respectively) is statistically similar with the "benchmark" eigenvalue distribution, Y will then concluded as similar with X.

Moreover, in order to ensure the consistency and comparability, the default window length is set as about 1/10 of the time series length. This will be fairly number to include almost all significant eigenvalues without containing too much unimportant ones. With a relatively larger window length, the information will be split either flatly or partly flatly by more eigenvalues, and the differences will be split to be less significant to be identified; in contrast, a smaller window length will result in the fewer amount of eigenvalues with more significant differences for all or some of the eigenvalues. Without considering the consistency to be comparable, the most proper window length will be selected heavily depends on the feature of the series being analyzed with the principle of relatively maximizing the significant information with possibly small number of eigenvalues.

A flowchart is provided in Fig. 1 that briefly summarizes the formulation and evaluation process of this proposing similarity measure. Note that in terms of simulation, corresponding process is repeated 1000 times respectively, and the population of tested series are generated by involving random white noises that being maintained at about 10% of the range of tested series.

The similarity measure is firstly built on the premise of multivariate system with the benchmark series as the dominant role. Therefore, we evaluate the similarity of multivariate system formed by X and Y by comparing it to the benchmark multivariate system formed by X and X or Y and Y respectively (determined by which series is considered as the benchmark series). This will be considered as the scenario of on the premise of multivariate system.

Another question raise here is that we can only compare the system of X and X with Y and Y. This refers to the scenario of without the premise of multivariate system. Note that all evaluations will be performed on the corresponding eigenvalue distributions generated by the systems formed respectively. The detailed test results of simulations with and without the premise scenarios will be separately presented in the following sections.

4. Empirical results

Three statistical tests are adopted for evaluating this novel similarity measure and examining the similarity measure criterion of eigenvalue distribution, which are briefly introduced previously: Chi-squared Test, Log-likelihood Goodness of Fit Test, Kolmogorov–Smirnov Test. In order to evaluate the performance of the proposed method, various types of simulated series are tested by being separated into two groups of circumstances: the similar group and the different group, additionally the different choices of "benchmark" are also considered in each group. The test results are summarized in Tables 1 and 2 by each empirical statistical method. Thus, the robustness of accepting eigenvalue distribution as similarity measure criterion are preliminarily examined, followed by the tests under simultaneous real case scenario by employing bootstrap re-sampling technique. We have managed to obtain consistently promising results as simulative expectations, which convincingly prove the consistent, robust performances of this novel similarity measure on several different types of simulated series. The initials of various types of generated series are listed below for the sake of simplifying the expressions:

1.	WN	White noise.
2.	UD[0, 1]	Uniform distribution series [0, 1].
3.	UD[−1, 1]	Uniform distribution series [−1, 1].
4.	EP[1]	Exponential distribution series rate 1.
5.	SINE[−1, 1]	Sine wave series [−1, 1].

4.1. *On the premise of multivariate system scenario*

Regarding the scenario of on the premise of multivariate system, we evaluate the similarity of eigenvalue distributions extracted from the matrices $\mathbf{XY}_H$ and $\mathbf{XX}_H$ (or $\mathbf{YY}_H$ determined by which series is considered as the benchmark series), respectively. Note that $\mathbf{XY}_H$ is created from two time series X_N and Y_N simultaneously, and $\mathbf{XX}_H$ (or $\mathbf{YY}_H$) is formed by X_N (or Y_N) with itself respectively. The corresponding test results of eigenvalue distributions as novel similarity measure by three different empirical methods are summarized in Table 1. Note that the bold number indicates the best performance option in corresponding comparable level.

The Chi-squared test results show positive outcomes as expected for the "similar" group on both numbers of observations scenarios, whilst in terms of the "different" group, the tests can perform better for longer series. However, there are still significantly unexpected results (p-value is close to 1) for the UD[0,1] & EP[1] and UD[−1,1] & SINE[−1,1] combinations, especially the results vary greatly for the UD[0,1] & SINE[−1,1] and EP[1] & SINE[−1,1] cases. As mentioned earlier, the population for comparison is created by the "benchmark" series, therefore differences are expected when switching the "benchmark" series, however, opposite results for the same pair of series are not robust as expected, and it is even worse than the cases of indicating "similar" for the groups that are expected to be "different".

In terms of the log-likelihood goodness of fit test results, expected results for the "similar" group are confirmed in accordance with the simulation results. P-values are equal to 1, which indicate that it is almost 100% sure to accept the null hypothesis, therefor very similar or identical eigenvalue distributions prove the expected conclusion of "similar". Regarding the expected to be "different" group, both long and short series length, 1000 and 100 observations, show generally consistent significant results, except the UD[0,1] & EP[1] combination. Since UD[0,1] and EP[1] indeed show similar eigenvalue distributions and the differences are between the tails, the log-likelihood goodness of fit test is not sensitive for detecting differences of distributions with flat tails. However, the advantage of this test can be noticed in the shorter length of observation scenario; the results are almost stable and consistent with the expected results of highly significant statistics.

Test results of K–S test show positive results as expected for the "similar" group on both numbers of observations scenarios. In terms of $N = 100$ case for "different" group of combinations, only the UD[0,1] & UD[−1,1] combination can be detected with 10% of significance level, however, the differences between switching dominant series to create "benchmark" populations are not significant. Comparing to the results of previous tests, the inconsistency is worse than less sensitivity of accurate detection, it has to be noticed that the two sample K–S test shows great performance on consistency and stability, even in the quite unstable and greatly varied scenarios that other tests cannot even provide uniformed results. In addition, for the "different" group with $N = 1000$ case, almost all results are as expected to be significant (majority is under 5%, only a few are under 10%). Note that the EP[1] & SINE[−1,1] combination is the only one that K–S test could not detect significantly, and this is mostly because that K–S test is not that much sensitive to the differences at tail, also the natural character of eigenvalue distribution

Table 1
Similarity measure evaluation by three different tests on simulated groups of series on the premise of multivariate system scenario.

			Chi-squared test				Log-likelihood GOF test				K–S test			
			$N = 100$ $L = 10$		$N = 1000$ $L = 100$		$N = 100$ $L = 10$		$N = 1000$ $L = 100$		$N = 100$ $L = 10$		$N = 1000$ $L = 100$	
			p-value		p-value		p-value		p-value		p-value		p-value	
	X	Y	Y→X	X→Y	Y→X	X→Y	Y→X	X→Y	Y→X	X→Y	Y→X	X→Y	Y→X	X→Y
Similar	UD[0,1]	UD[0,1]	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
	UD[−1,1]	UD[−1,1]	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
	EP[1]	EP[1]	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
	SINE[−1,1]	SINE[−1,1]	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
Different	UD[0,1]	UD[−1,1]	0.14	**0.00**	0.00	0.00	**0.04**	0.04	0.00	0.00	0.07	0.05	**0.00**	**0.00**
	UD[0,1]	EP[1]	0.76	0.98	0.88	1.00	0.81	1.00	0.99	1.00	**0.77**	**0.65**	**0.01**	**0.01**
	UD[0,1]	SINE[−1,1]	0.98	0.35	1.00	0.00	**0.00**	**0.00**	**0.00**	**0.00**	0.76	0.89	0.02	0.01
	UD[−1,1]	EP[1]	**0.01**	0.88	0.00	0.01	0.05	**0.35**	0.00	0.00	0.49	0.65	**0.00**	**0.00**
	UD[−1,1]	SINE[−1,1]	1.00	1.00	1.00	0.99	**0.00**	**0.00**	**0.00**	**0.00**	0.11	0.41	0.10	0.10
	EP[1]	SINE[−1,1]	1.00	0.20	1.00	0.00	**0.00**	**0.00**	**0.00**	**0.00**	0.45	0.60	0.56	0.53

Table 2
Similarity measure evaluation by three different tests on simulated groups of series without the premise of multivariate system scenario.

			Chi-squared test		Log-likelihood GOF test		K–S test	
			$N = 100$ $L = 10$	$N = 1000$ $L = 100$	$N = 100$ $L = 10$	$N = 1000$ $L = 100$	$N = 100$ $L = 10$	$N = 1000$ $L = 100$
	X	Y	p-value	p-value	p-value	p-value	p-value	p-value
Similar	UD[0,1]	UD[0,1]	0.99	1.00	0.99	1.00	0.99	0.99
	UD[−1,1]	UD[−1,1]	0.99	1.00	0.99	1.00	0.99	0.98
	EP[1]	EP[1]	0.99	1.00	0.99	1.00	0.98	0.93
	SINE[−1,1]	SINE[−1,1]	0.99	1.00	0.99	1.00	0.99	0.99
Different	UD[0,1]	UD[−1,1]	0.01	0.00	**0.01**	0.00	0.03	**0.00**
	UD[0,1]	EP[1]	0.72	0.73	0.72	0.64	**0.61**	**0.00**
	UD[0,1]	SINE[−1,1]	**0.52**	0.45	0.54	0.49	0.76	**0.01**
	UD[−1,1]	EP[1]	0.22	0.00	**0.18**	0.00	0.23	**0.00**
	UD[−1,1]	SINE[−1,1]	**0.96**	0.46	0.96	0.50	0.98	**0.03**
	EP[1]	SINE[−1,1]	**0.58**	**0.49**	0.59	0.50	0.99	0.57

for both types of series vary at the tail part with increasing differences when the window length of structuring matrix increases.

4.2. Without the premise of multivariate system scenario

In terms of the scenario without the premise of multivariate system, the similarity measure is performed on the eigenvalue distributions extracted from the matrices $\mathbf{XX_H}$ and $\mathbf{YY_H}$ respectively. To be consistent with the previous evaluation process, we consider both similar and different groups of series and evaluate the performance of similarity measure by 1000 time simulations. Note that this time there is no premise of a multivariate system, therefore, the evaluation by simulated series will have no assumption on benchmark series. Hence, for each pair of series, there is only one test statistic conducted. The default number of observation is 1000 and default window length is 100. All statistical tests results are listed in Table 2. Note that the bold number indicates the best performance option in corresponding comparable level.

It is worth to be noted that due to the algorithm of applying Chi-square test and Log-likelihood goodness of fit test for two sample test, it is necessary to define one of the tested series as dominant series and re-scale the assumption of distribution in the first place for the further tests. Consequently, for the scenario of without the premise of multivariate system, the simulations of 1000 times are equally shared by both series in one pair of tested series. Therefore, both series have same quantity of chances to be the dominant series to re-scale the assumption distribution. K–S test do not have assumptions on any distribution, hence simulations for two sample test of K–S test here do not have significant difference comparing to the corresponding process of previous scenario on the premise of multivariate system.

According to Table 2, all statistical tests provide consistent results on both short and long series for the similar group, which all show p-value nearly equal or identical to 1. Hence it indicates significantly similar eigenvalue distributions consequently the similarity between tested series. However, in terms of the different group, both Chi-squared test and Log-likelihood goodness of fit test could not detect most of the differences properly except the UD[0,1] & UD[−1,1] and UD[−1,1] & EP[1] combinations. It is mostly because of the variation and instability caused by switching dominant series for re-scale distribution assumption. Even for the longer series case, most of the results get smaller p-values (which indicates different eigenvalue distributions), they are still not significant enough as we expected for the generated different group. K–S test is proved to outperform the other two tests for the long series case, also it can accurately detect the similarity or differences for both simulated groups. Even for the short series case, the results of K–S test are fairly close to the results of the other tests. Unlike the previous test results of log-likelihood goodness of fit test, it does not show good performance on short series this time. In general, by considering the scenario without the premise of a multivariate system, the K–S test is confirmed again as the most proper statistics to be adopted for the new similarity measure based on eigenvalue distribution.

5. Similarity measure in simultaneous real case scenario by bootstrap re-sampling

Based on previous evaluations of eigenvalue distribution as similarity measure criterion by simulations, it can be summarized that the eigenvalue distribution can be considered as a proper criterion of measure similarity by adopting proper statistical test; K–S test outperforms others in the large data size domain with consistent results as simultaneously expected.

Considering the real case scenario, data can be assumed to be formed by signal and noise. Therefore, we cannot simulate noises to form and produce the population of dominate series as the benchmark to measure similarity. Consequently, we adopt bootstrap re-sampling technique [52] to conduct the population of dominate series with specific confidence level and evaluate how similar the other tested variable is to the benchmark population under the specific confidence level circumstance. Note that the newly proposed method can certainly be performed without any re-sampling process if there are already clear information of its population. The corresponding population will only be generated by re-sampling for obtaining the information of its population. Due to the nature of similarity we mentioned previously, the similarity level of X to Y and Y to X are two different questions regarding the differences of the benchmark. Consequently, the re-sampling process will consider two different cases by choosing different original series to create the population.

A flowchart is provided in Fig. 2 that briefly summarizes the formulation process under the simultaneous real case scenario by bootstrap re-sampling. For instance, when the principle is to obtain the population of benchmark series X, thus, the population of $\lambda_{XY_H XY_H^T}$ or $\lambda X X_H X X_H^T$ (determined by with or without the premise of multivariate system) are conducted, which are formed by eigenvalues distributions within specific confidence interval of K–S statistics. Therefore, if the confidence level is fixed as 95%, we can conduct the population of eigenvalue distributions that indicate significantly 95% similarity level with benchmark series. To this end, we can evaluate the other series by comparing its corresponding K–S statistics with the range of K–S statistics by the population. Hence, we can identify the similarity level respectively with necessary adjustment of confidence level in the bootstrap re-sampling stage. The results by representative simultaneous groups of series are provided in Table 3. In terms of the similar group, the similar group shows consistent results for both short and long series, in which, 95% significant level indicates tested series share at least 95% of similarity based on the eigenvalue distributions from the corresponding matrices. According to the previous evaluations of K–S test on short and long series for different group, we here only consider to evaluate the performance on long series in accordance to its previous promising results in simulations (symbol \ for short series in Table 3). The 5% significant level refers to that the test statistics does not fit even when the confidence level of bootstrap re-sampling is set as 5%. This significantly indicates that tested series can be considered different as they are not similar even for 5% significant level.

6. Discussion

Although as a novel similarity measure based on eigenvalue distribution with proven robustness and consistent performances, it is also certain that it is still the beginning of developing this new measure. The types of series in simulations are relatively limited, and there are still numerous choices of more complex series or combinations of

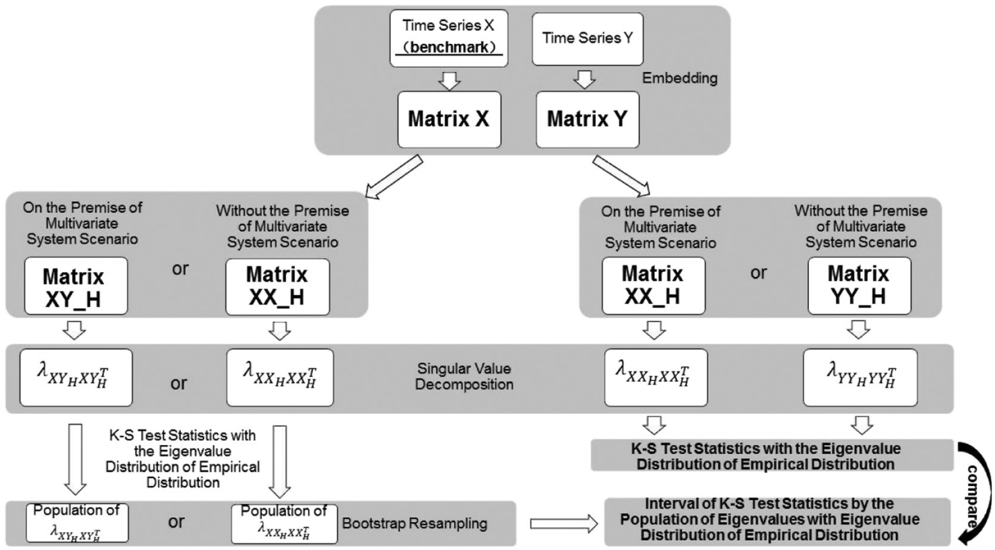

Fig. 2. The flowchart of the simultaneous real case scenario by bootstrap re-sampling.

Table 3
Simultaneous real case similarity measure results by bootstrap re-sampling.

			$N = 100$ $L = 10$				$N = 1000$ $L = 100$			
			Y to X		X to Y		Y to X		X to Y	
	X	Y	Y/N	Sig level	Y/N	Sig level	Y/N	Sig level	Y/N	Sig level
Similar	UD[0,1]	UD[0,1]	✓	95%	✓	95%	✓	95%	✓	95%
	UD[−1,1]	UD[−1,1]	✓	95%	✓	95%	✓	95%	✓	95%
	EP[1]	EP[1]	✓	95%	✓	95%	✓	95%	✓	95%
Different	UD[0,1]	UD[−1,1]	\	\	\	\	✓	5%	✓	5%
	UD[0,1]	EP[1]	\	\	\	\	✓	5%	✓	5%
	UD[−1,1]	EP[1]	\	\	\	\	✓	5%	✓	5%

Note: ✓indicates the result is correctly proved by the measure.

series haven not been explored. The bootstrap re-sampling by K–S statistics for some real data (especially large size of data that is much longer than the default 1000 observations in simulation) may take a longer time of calculation, which makes it crucial to find a more straight forward process to identify the population information as the benchmark. Also, the performance in short series is not as good as its effort on long series. However, there are also numerous possibilities to improve this novel measure further: more representative data patterns, more types of noises with different levels of variations and more options of window lengths are planned to be explored as the second stage of improving this new measure; in terms of time series with different frequencies, it can also provide possible solution by adopting SSA technique with specific modification accordingly; one significant implementation area of similarity measure is time series classification, therefore, the evaluations of its performances on classifications of some empirical data are in process.

7. Conclusion

In general, we overcome the difficulties and develop a novel similarity measure based on eigenvalue distribution by combining the SVD technique. It is the initial attempt of adopting this technique in terms of the similarity measure. The evaluation results are promising and robust as we have considered many possible circumstances in the formulation process. We have examined the robustness of adopting eigenvalue distribution as proper criterion of measuring similarity; additionally, we have found that K–S test outperforms others in the large data size domain with consistent results as simultaneously expected. Furthermore, the simultaneous real case scenario is evaluated by adopting the bootstrap re-sampling technique to prevent the possible impacts during the process of creating benchmark population. Consistent results are achieved in the simultaneous real case scenario indicating the robust performance of distinguishing various "similar" or "different" groups of series.

This novel similarity measure can work properly on long series, and it does not require any assumption of distributions during the measuring process. The computation is reasonably efficient and can be easily employed by modifying currently available R packages. By considering eigenvalue distribution as the criterion of similarity measure, the amount of computation is significantly reduced for large data set. More importantly, this novel similarity measure can work with time series with different lengths and still identify the significant features for evaluations. Furthermore, the signal and noise of time series are considered as a whole without one fixed model. In brief, this novel similarity measure contributes to providing a measurement that has no limitations of series length, series with nonlinear features or complex fluctuations, series sharing both signal and noises as similarities, etc. It is absolutely worth looking forward to its developments and performance of implementations on various disciplines in the close future.

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Original article

Unilateral contact problems with a friction

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Abstract

The boundary contact problem for a micropolar homogeneous elastic hemitropic medium with a friction is investigated. Here, on a part of the elastic medium surface with a friction, instead of a normal component of force stress there is prescribed the normal component of the displacement vector. We give their mathematical formulation of the Problem in the form of spatial variational inequalities. We consider two cases, the so-called coercive case (when elastic medium is fixed along some part of the boundary) and semi-coercive case (the boundary is not fixed). Based on our variational inequality approach, we prove the existence and uniqueness theorems and show that solutions continuously depend on the data of the original problem. In the semi-coercive case, the necessary condition of solvability of the corresponding contact problem is written out explicitly. This condition under certain restrictions is sufficient, as well.

Keywords: Elasticity theory; Hemitropic solids; Contact problem with a friction; Variational inequality

1. Introduction

In the present paper we investigate the one-sided contact problem for a homogeneous hemitropic elastic medium with a friction. In the considered model of the theory of elasticity, as distinct from the classical theory, every elementary medium particle undergoes both displacement and rotation. In this case, all mechanical values are expressed by means of the displacement and rotation vectors.

In their works [1] and [2], E. and F. Cosserats created and presented the model of solid medium in which every material point has six degrees of freedom, three of which are defined by displacement components and the other three by the components of rotation (for the history of the model of elasticity see [3–9] and references therein).

A micropolar medium, not isotropic with respect to the inversion, is called a hemitropic or noncentrosymmetric medium.

Improved mathematical models describing hemitropic properties of elastic materials have been obtained and considered in [10] and [11]. The main equations of that model are interconnected and generate a matrix second

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order differential operator of dimension 6×6. Particular problems for solid media of hemitropic theory of elasticity have been considered in [12,13,8] and [9]. The basic boundary value problems and also the transmission problems of the hemitropic theory of elasticity with the use of the potential method for smooth and nonsmooth Lipschitz domains were studied in [12], the one-sided contact problems of statics of the hemitropic theory of elasticity free from friction were investigated in [14–18], and the contact problems of statics and dynamics with a friction were considered in [19–29]. Analogous one-sided problems of classical linear theory of elasticity have been considered in many works and monographs (see [30–35] and the references therein).

In the present work, we present the basic equations of statics of the theory of elasticity for homogeneous hemitropic media in a vector–matrix form, introduce the generalized stress operator and quadratic form of potential energy. Then we describe mathematical model of boundary conditions which show the contact between a hemitropic medium and a solid body with regard for the friction effect. We will consider the case, where some part of the elastic medium boundary is fixed mechanically. The problem is reduced equivalently to the variational inequality, the question on the existence and uniqueness of a weak solution of the initial problem is treated, and a continuous Lipschitz dependence of the solution on the data of the problem is investigated. Further, we will investigate more complicated cases, where friction is considered on the whole medium boundary. In such cases, the corresponding mathematical problem is, in general, unsolvable. The necessary conditions of solvability are established and the sufficient conditions for the existence of a solution are formulated explicitly.

2. Basic equations and Green's formulas

2.1. Basic equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a C^∞-smooth boundary $S = \partial\Omega$, $\overline{\Omega} = \Omega \cup S$. The domain Ω is assumed to be filled with a homogeneous hemitropic material.

The basic equilibrium equations in the hemitropic theory of elasticity written in components of the displacement and rotation vectors are of the form

$$
\begin{aligned}
(\mu+\alpha)\ \ &\Delta u(x) + (\lambda+\mu-\alpha)\,\mathrm{graddiv}\,u(x) + (\varkappa+\nu)\Delta\omega(x) \\
&+ (\delta+\varkappa-\nu)\,\mathrm{graddiv}\,\omega(x) + 2\alpha\,\mathrm{curl}\,\omega(x) + \rho F(x) = 0, \\
(\varkappa+\nu)\ \ &\Delta u(x) + (\delta+\varkappa-\nu)\,\mathrm{graddiv}\,u(x) + 2\alpha\,\mathrm{curl}\,u(x) + (\gamma+\varepsilon)\Delta\omega(x) \\
&+ (\beta+\gamma-\varepsilon)\,\mathrm{graddiv}\,\omega(x) + 4\nu\,\mathrm{curl}\,\omega(x) - 4\alpha\omega(x) + \rho\Psi(x) = 0,
\end{aligned}
\tag{2.1}
$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator, $\partial_j = \partial/\partial x_j$, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, $\omega = (\omega_1, \omega_2, \omega_3)^\top$ is the vector of rotation, $F = (F_1, F_2, F_3)^\top$ and $\Psi = (\Psi_1, \Psi_2, \Psi_3)^\top$ are the mass force and mass moment calculated per unit of mass, ρ is density of the elastic medium, and α, β, γ, δ, λ, μ, ∂, $\varkappa$ and ε are elastic constants (see [11,13]). Here and in what follows, the symbol $(\cdot)^\top$ denotes transposition.

We introduce a matrix differential operator corresponding to the left-hand side of system (2.1):

$$
\begin{aligned}
&L(\partial) = \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) \end{bmatrix}_{6\times 6}, \\
&L^{(1)}(\partial) := (\mu+\alpha)\Delta I_3 + (\lambda+\mu-\alpha)Q(\partial), \\
&L^{(2)}(\partial) = L^{(3)}(\partial) := (\varkappa+\nu)\Delta I_3 + (\delta+\varkappa-\nu)Q(\partial) + 2\alpha R(\partial), \\
&L^{(4)}(\partial) := \big[(\gamma+\varepsilon)\Delta - 4\alpha\big]I_3 + (\beta+\gamma-\varepsilon)Q(\partial) + 4\nu R(\partial),
\end{aligned}
$$

where I_k is the unit $k \times k$-matrix and

$$
Q(\partial) = \big[\partial_k\,\partial_j\big]_{3\times 3}, \quad R(\partial) = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}.
$$

The system of Eqs. (2.1) can be rewritten in the matrix form

$$
L(\partial)U(x) + \mathcal{G}(x) = 0, \quad x \in \Omega,
$$

where $U = (u, \omega)^\top$ and $\mathcal{G} = (\rho F, \rho\Psi)^\top$.

By $T(\partial, n)$ we denote the generalized stress operator of dimension 6×6 (see [13]):

$$T(\partial,n)=\begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) \end{bmatrix}, \quad T^{(j)}=\big[T^{(j)}_{pq}\big]_{3\times 3}, \quad j=\overline{1,4},$$

where

$$T^{(1)}_{pq}(\partial,n) :=(\mu+\alpha)\delta_{pq}\partial_n+(\mu-\alpha)n_q\partial_p+\lambda n_p\partial_q,$$

$$T^{(2)}_{pq}(\partial,n) :=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,$$

$$T^{(3)}_{pq}(\partial,n) :=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q,$$

$$T^{(4)}_{pq}(\partial,n) :=(\gamma+\varepsilon)\delta_{pq}\partial_n+(\gamma-\varepsilon)n_q\partial_p+\delta n_p\partial_q-2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.$$

Here, $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outward (with respect to Ω) unit normal vector at the point $x \in S$, and $\partial_n = \partial/\partial n$ is the normal derivative in the direction of the vector n. The six-component generalized stress vector has the form

$$T(\partial,n)U=(\mathcal{T}U,\mathcal{M}U)^\top,$$

where $\mathcal{T}U := T^{(1)}u + T^{(2)}\omega$ is the force stress vector and $\mathcal{M}U := T^{(3)}u + T^{(4)}\omega$ is the moment stress vector.

2.2. Green's formulas

For the real-valued vector functions $U = (u, \omega)^\top$ and $U' = (u', \omega')^\top$ of the class $[C^2(\overline{\Omega})]^6$ the following Green's formula [13]

$$\int_\Omega \big[L(\partial)U\cdot U'+E(U,U')\big]dx=\int_S\big\{T(\partial,n)U\big\}^+\cdot\big\{U'\big\}^+dS, \tag{2.2}$$

is valid, where $\{\cdot\}^+$ denotes the trace operator on S from Ω, and $E(\cdot\,,\cdot)$ is a bilinear form defined by the equality

$$\begin{aligned} E(U,U') = E(U',U) = \sum_{p,q=1}^{3}\Big\{&(\mu+\alpha)u'_{pq}u_{pq}+(\mu-\alpha)u'_{pq}u_{qp} \\ &+(\varkappa+\nu)\big(u'_{pq}\omega_{pq}+\omega'_{pq}u_{pq}\big)+(\varkappa-\nu)\big(u'_{pq}\omega_{qp}+\omega'_{pq}u_{qp}\big)+(\gamma+\varepsilon)\omega'_{pq}\omega_{pq} \\ &+(\gamma-\varepsilon)\omega'_{pq}\omega_{qp}+\delta\big(u'_{pp}\omega_{qq}+\omega'_{qq}u_{pp}\big)+\lambda u'_{pp}u_{qq}+\beta\omega'_{pp}\omega_{qq}\Big\}, \end{aligned}$$

where u_{pq} and ω_{pq} are the so-called tensors of deformation and torsion-bending for hemitropic media,

$$u_{pq}=u_{pq}(U)=\partial_p u_q-\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k, \quad \omega_{pq}=\omega_{pq}(U)=\partial_p\omega_q, \quad p,q=1,2,3. \tag{2.3}$$

Here and in the sequel, by $a \cdot b$ we denote the scalar product of two vectors $a, b \in \mathbb{R}^m : a\cdot b=\sum_{j=1}^m a_j b_j$.

Under certain assumptions on elastic constants (see [11,17,24]), specific energy of deformation $E(U, U)$ is a positive definite quadratic form with respect to $u_{pq}(U)$ and $\omega_{pq}(U)$, i.e., there exists a positive number $C_0 > 0$, depending only on the elastic constants, such that

$$E(U,U)\ge C_0\sum_{p,q=1}^{3}\big[u^2_{pq}+\omega^2_{pq}\big].$$

The following assertion describes the null space of the energy quadratic form $E(U, U)$ (see [13]).

Lemma 2.1. *Let $U = (u, \omega)^\top \in [C^1(\overline{\Omega})]^6$ and $E(U, U) = 0$ in Ω. Then*

$$u(x) = [a \times x] + b, \quad \omega(x) = a, \quad x \in \Omega,$$

where a and b are arbitrary three-dimensional constant vectors and $[\cdot \times \cdot]$ denotes the cross product of two vectors.

Vectors of the type $([a \times x] + b, a)$ are called generalized rigid vectors. We observe that a generalized rigid displacement vector vanishes, i.e., $a = b = 0$ if it is zero at a single point.

Throughout the paper, $L_p(\Omega)$ $(1 \le p \le \infty)$, $L_2(\Omega) = H^0(\Omega)$ and $H^s(\Omega) = H^s_2(\Omega)$, $s \in \mathbb{R}$, denote, respectively, the Lebesgue and Bessel potential spaces (see e.g., [36,37]). The corresponding norms we denote by the symbols $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$. By $\mathcal{D}(\Omega)$ we denote the class of $C^\infty(\Omega)$ functions with support in the domain Ω. If M is an open proper part of the manifold $\partial\Omega$, i.e., $M \subset \partial\Omega$, $M \ne \partial\Omega$, then by $H^s(M)$ we denote the restriction of the space $H^s(\partial\Omega)$ on M,

$$H^s(M) := \{r_M \varphi : \varphi \in H^s(\partial\Omega)\},$$

where r_M stands for the restriction operator on the set M. Further, let

$$\widetilde{H}^s(M) := \{\varphi \in H^s(\partial\Omega) : \operatorname{supp} \varphi \subset \overline{M}\}.$$

From the positive definiteness of the energy form $E(\cdot, \cdot)$ with respect to the variables (2.3) it follows that

$$B(U, U) := \int_\Omega E(U, U) dx \ge 0. \tag{2.4}$$

Moreover, there exist positive constants C_1 and C_2, depending only on the material parameters, such that the following Korn's type inequality (see (Part I, Section 12, [32]))

$$B(U, U) \ge C_1 \|U\|^2_{[H^1(\Omega)]^6} - C_2 \|U\|^2_{[H^0(\Omega)]^6} \tag{2.5}$$

holds for an arbitrary real-valued vector function $U \in [H^1(\Omega)]^6$.

Remark 2.2. If $U \in [H^1(\Omega)]^6$ and on some part $S^* \subset \partial\Omega$ the trace $\{U\}^+$ vanishes, i.e., $r_{S^*}\{U\}^+ = 0$, we have the strict Korn's inequality

$$B(U, U) \ge C \|U\|^2_{[H^1(\Omega)]^6} \tag{2.6}$$

with some positive constant $C > 0$ which does not depend on the vector U. This follows from (2.5) and the fact that in this case $B(U, U) > 0$ for $U \ne 0$ (see, e.g., [38], [32, Ch. 2, Exercise 2.17]).

Remark 2.3. By the standard limiting arguments, Green's formula (2.2) can be extended to the Lipschitz domains and to the vector function $U \in [H^1(\Omega)]^6$ with $L(\partial)U \in [L_2(\Omega)]^6$ and $U' \in [H^1(\Omega)]^6$ (see [38,36]),

$$\int_\Omega \big[L(\partial)U \cdot U' + E(U, U')\big] dx = \big\langle \{T(\partial, n)U\}^+, \{U'\}^+ \big\rangle_{\partial\Omega}, \tag{2.7}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes duality between the spaces $[H^{-1/2}(\partial\Omega)]^6$ and $[H^{1/2}(\partial\Omega)]^6$ which generalizes the usual inner product in the space $[L_2(\partial\Omega)]^6$. By virtue of this relation, the generalized trace of the stress operator $\{T(\partial, n)U\}^+ \in [H^{-1/2}(\partial\Omega)]^6$ is correctly determined.

3. Contact problems with a friction

3.1. Pointwise and variational formulation of the contact problem

Let the boundary S of the domain Ω be divided into two open, connected and nonoverlapping parts S_1 and S_2 of positive measure, $S = \overline{S}_1 \cup \overline{S}_2$, $S_1 \cap S_2 = \varnothing$. Assume that the hemitropic elastic body occupying the domain Ω is in contact with another rigid body along the subsurface S_2.

Definition 1. A vector function $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ is said to be a weak solution of the equation

$$L(\partial)U + \mathcal{G} = 0, \quad \mathcal{G} \in [L_2(\Omega)]^6 \tag{3.1}$$

in the domain Ω if

$$B(U, \Phi) = \int_\Omega \mathcal{G} \cdot \Phi \, dx \quad \forall\, \Phi \in [\mathcal{D}(\Omega)]^6,$$

where the bilinear form $B(\cdot, \cdot)$ is given by formula (2.4).

For the normal and tangential components of the force stress vector we will use, respectively, the following notation:

$$(\mathcal{T}U)_n := \mathcal{T}U \cdot n, \quad (\mathcal{T}U)_s := \mathcal{T}U - n(\mathcal{T}U)_n.$$

Further, let

$$\mathcal{G} = \big(\rho F, \rho \Psi\big)^\top \in \big[L_2(\Omega)\big]^6, \quad \varphi \in \big[H^{-1/2}(S_2)\big]^3 \quad \text{and} \quad g \in L_\infty(S_2), \quad g \geq 0.$$

Consider the following contact problem of statics with a friction.

Problem A. Find a vector function $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ which is a weak solution of Eq. (3.1) and satisfies the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$ and the following conditions:

$$r_{S_1}\{U\}^+ = 0 \quad \text{on} \quad S_1, \tag{3.2}$$

$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \quad \text{on} \quad S_2, \tag{3.3}$$

$$r_{S_2}\{u_n\}^+ = 0 \quad \text{on} \quad S_2, \tag{3.4}$$

$$\text{if} \quad \big|r_{S_2}\{(\mathcal{T}U)_s\}^+\big| < g, \quad \text{then} \quad r_{S_2}\{u_s\}^+ = 0, \tag{3.5}$$

if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and

$$\lambda_1\, r_{S_2}\{u_s\}^+ = -\lambda_2\, r_{S_2}\{(\mathcal{T}U)_s\}^+. \tag{3.6}$$

The conditions (3.2) and (3.4) are understood in the usual trace sense, whereas (3.3) is understood in the generalized functional sense described in Remark 2.3.

This problem can be reformulated as a variational inequality. To this end, let us introduce on the space $[H^1(\Omega)]^3$ the following continuous convex functional

$$j(v) = \int_{S_2} g\big|\{v_s\}^+\big| dS, \quad v \in [H^1(\Omega)]^3. \tag{3.7}$$

Next, we define the closed convex subset of $[H^1(\Omega)]^6$,

$$K(\Omega) := \big\{V = (v, \omega)^\top \in [H^1(\Omega)]^6 : r_{S_1}\{V\}^+ = 0,\ r_{S_2}\{v_n\}^+ = 0\big\}.$$

Consider the following variational inequality: Find $U = (u, \omega)^\top \in K(\Omega)$ such that the variational inequality

$$B(U, V - U) + j(v) - j(u) \geq \int_\Omega \mathcal{G} \cdot (V - U) dx + \big\langle \varphi, r_{S_2}\{w - \omega\}^+ \big\rangle_{S_2} \tag{3.8}$$

holds for all $V = (v, w)^\top \in K(\Omega)$.

Here and in what follows, the symbol $\langle \cdot\ ,\ \cdot \rangle_M$ denotes the duality relation between the corresponding dual pairs $X(M)$ and $X^*(M)$. In particular, the brackets $\langle \cdot\ ,\ \cdot \rangle_{S_2}$ in (3.8) denote the duality relation between the spaces $[H^{-1/2}(S_2)]^3$ and $[\widetilde{H}^{1/2}(S_2)]^3$.

3.2. Equivalence

Here we prove the following equivalence result.

Theorem 3.1. *If a vector function $U \in K(\Omega)$ solves the variational inequality* (3.8)*, then it is a solution of Problem* A*, and vice versa.*

Proof. Let $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ be a solution of Problem A. With the help of Green's formula (2.7), we get

$$\big\langle \{T(\partial, n)U\}^+, \{V-U\}^+\big\rangle_S - B(U, V-U) + \int_\Omega \mathcal{G}\cdot(V-U)dx = 0$$

for all $V = (v, w)^\top \in K(\Omega)$.

Since $r_{S_1}\{V-U\}^+ = 0$ and $r_{S_2}\{v_n - u_n\}^+ = 0$, these equations can be rewritten as

$$B(U, V-U) + j(v) - j(u) = \int_\Omega \mathcal{G}\cdot(V-U)dx + \langle \varphi, r_{S_2}\{w-\omega\}^+\rangle_{S_2} + \int_{S_2}\Big[\big\{(\mathcal{T}U)_s\big\}^+\cdot\{v_s-u_s\}^+ + g\big(|\{v_s\}^+| - |\{u_s\}^+|\big)\Big]dS. \tag{3.9}$$

It is easy to see that if the conditions (3.5) and (3.6) hold, then

$$r_{S_2}\big\{(\mathcal{T}U)_s\big\}^+\cdot r_{S_2}\{v_s-u_s\}^+ + g\Big(\big|r_{S_2}\{v_s\}^+\big| - \big|r_{S_2}\{u_s\}^+\big|\Big) \geq 0 \quad \text{on} \quad S_2.$$

Using this inequality, from (3.9) we obtain

$$B(U, V-U) + j(v) - j(u) \geq \int_\Omega \mathcal{G}\cdot(V-U)dx + \big\langle \varphi, r_{S_2}\{w-\omega\}^+\big\rangle_{S_2} \quad \forall\, V = (v, w)^\top \in K(\Omega).$$

Thus, $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ is a solution of the variational inequality (3.8).

Let now $U \in K(\Omega)$ be a solution of the variational inequality (3.8). Substituting $U \pm \Phi$ for V in (3.8) with an arbitrary $\Phi \in [\mathcal{D}(\Omega)]^6$, we obtain

$$B(U, \Phi) = \int_\Omega \mathcal{G}\cdot\Phi\, dx \quad \forall\, \Phi \in [\mathcal{D}(\Omega)]^6$$

which implies that U is a weak solution of Eq. (3.1).

By virtue of the interior regularity theorems (see [32]), we have $U \in [H^2(\Omega')]^6$ for every $\overline{\Omega'} \subset \Omega$. Hence, the following equation holds in the domain Ω

$$L(\partial)U + \mathcal{G} = 0.$$

Using again Green's formula, we have

$$B(U, V-U) - \big\langle r_{S_2}\big\{(\mathcal{T}U)_s\big\}^+, r_{S_2}\{v_s-u_s\}^+\big\rangle_{S_2} - \big\langle r_{S_2}\big\{\mathcal{M}U\big\}^+, r_{S_2}\{w-\omega\}^+\big\rangle_{S_2} = \int_\Omega \mathcal{G}\cdot(V-U)dx \quad \forall\, V = (v, w)^\top \in K(\Omega). \tag{3.10}$$

We subtract (3.10) from inequality (3.8) and get

$$\big\langle r_{S_2}\big\{(\mathcal{T}U)_s\big\}^+, r_{S_2}\{v_s-u_s\}^+\big\rangle_{S_2} + \int_{S_2} g\Big(\big|\{v_s\}^+\big| - \big|\{u_s\}^+\big|\Big)dS + \big\langle r_{S_2}\big\{\mathcal{M}U\big\}^+ - \varphi, r_{S_2}\{w-\omega\}^+\big\rangle_{S_2} \geq 0 \tag{3.11}$$

for all $V = (v, w)^\top \in K(\Omega)$.

Choose $V = (v, w)^\top \in K(\Omega)$ such that

$$r_{S_2}\{v\}^+ = r_{S_2}\{u\}^+ \quad \text{and} \quad r_{S_2}\{w\}^+ = r_{S_2}\{\omega\}^+ \pm r_{S_2}\psi,$$

where $\psi \in [\widetilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector function. Then (3.11) yields

$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \quad \text{on} \quad S_2,$$

i.e., (3.3) holds. The conditions (3.2) and (3.4) are satisfied automatically, since $U \in K(\Omega)$. Therefore the relation (3.11) can be rewritten as

$$\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{v_s - u_s\}^+\rangle_{S_2} + \int_{S_2} g\Big(\big|\{v_s\}^+\big| - \big|\{u_s\}^+\big|\Big)dS \ge 0$$

for all $V = (v, w)^\top \in K(\Omega)$.

Let $\psi \in [\widetilde{H}^{1/2}(S_2)]^3$. Since $\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s\rangle_{S_2} = \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi\rangle_{S_2}$ and $|r_{S_2}\psi_s| \le |r_{S_2}\psi|$, therefore taking $r_{S_2}\psi_s$ in the place of $r_{S_2}\{v_s\}^+$, we obtain

$$\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s\rangle_{S_2} + \int_{S_2} g|\psi|dS - \Big\{\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s\rangle_{S_2} + \int_{S_2} g\big|\{u_s\}^+\big|dS\Big\} \ge 0 \quad \forall\, \psi \in \big[\widetilde{H}^{1/2}(S_2)\big]^3. \tag{3.12}$$

Further, let $t \ge 0$ be an arbitrary number and take $\pm t\psi$ for ψ in (3.12),

$$t\Big\{\pm\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s\rangle_{S_2} + \int_{S_2} g|\psi|dS\Big\} - \Big\{\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi_s\rangle_{S_2} + \int_{S_2} g\big|\{u_s\}^+\big|dS\Big\} \ge 0 \quad \forall\, \psi \in \big[\widetilde{H}^{1/2}(S_2)\big]^3$$

whence by sending t first to $+\infty$ and then to 0, we easily derive

$$\Big|\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\varphi\rangle_{S_2}\Big| \le \int_{S_2} g|\psi|dS \quad \forall\, \psi \in \big[\widetilde{H}^{1/2}(S_2)\big]^3, \tag{3.13}$$

$$\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\{u_s\}^+\rangle_{S_2} + \int_{S_2} g\big|\{u_s\}^+\big|dS \le 0. \tag{3.14}$$

Now we prove that $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$. Towards this end, we consider on the space $[\widetilde{H}^{1/2}(S_2)]^3$ the linear functional

$$\Phi(\psi) = \langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi\rangle_{S_2} \quad \forall\, \psi \in \big[\widetilde{H}^{1/2}(S_2)\big]^3.$$

Inequality (3.13) shows that the functional Φ is continuous on the space $r_{S_2}[\widetilde{H}^{1/2}(S_2)]^3$ with respect to the topology, induced by the space $[L_1(S_2)]^3$. Since the space $r_{S_2}[\widetilde{H}^{1/2}(S_2)]^3$ is dense in $[L_1(S_2)]^3$, the functional Φ can be continuously extended to the space $[L_1(S_2)]^3$ preserving the norm. Therefore, by the Riesz theorem, there is the function $\Phi^* \in [L_\infty(S_2)]^3$ such that

$$\Phi(\psi) = \int_{S_2} \Phi^* \cdot \psi \, dS \quad \forall\, \psi \in [L_1(S_2)]^3.$$

Thus,

$$\langle r_{S_2}\{(\mathcal{T}U)_s\}^+, r_{S_2}\psi\rangle_{S_2} = \int_{S_2} \Phi^* \cdot \psi \, dS \quad \forall\, \psi \in \big[\widetilde{H}^{1/2}(S_2)\big]^3,$$

that is,

$$\langle r_{S_2}\{(\mathcal{T}U)_s\}^+ - \Phi^*, r_{S_2}\psi\rangle_{S_2} = 0 \quad \text{for all} \quad \psi \in \big[\widetilde{H}^{1/2}(S_2)\big]^3,$$

which implies that

$$r_{S_2}\{(\mathcal{T}U)_s\}^+ = \Phi^* \in \big[L_\infty(S_2)\big]^3.$$

It is well known that for an arbitrary function $\widetilde{\psi} \in L_\infty(S_2)$ there is a sequence $\widetilde{\psi}_e \in C^\infty(S_2)$ with $\operatorname{supp}\widetilde{\varphi}_e \subset S_2$ such that (see, e.g., [39, Lemma 1.4.2])

$$\lim_{\ell\to\infty} \widetilde{\varphi}_\ell(x) = \widetilde{\psi}(x) \quad \text{for almost all} \quad x \in S_2$$

and

$$\left|\widetilde{\varphi}_\ell(x)\right| \le \operatorname*{ess\,sup}_{y\in S_2} \left|\widetilde{\psi}(y)\right| \quad \text{for almost all} \quad x \in S_2.$$

Therefore, from inequality (3.13), by the Lebesgue dominated convergence theorem, it follows that

$$\int_{S_2} \Big[\pm\big\{(\mathcal{T}U)_s\big\}^+ \cdot \psi - g|\psi|\Big] dS \le 0 \quad \forall\, \psi \in \big[L_\infty(S_2)\big]^3.$$

In the place of ψ we can put $\chi\,\psi$, where $\psi \in \big[L_\infty(S_2)\big]^3$ and χ is the characteristic function of an arbitrary measurable subset $\Gamma \subset S_2$. As a result, we arrive at the inequality $\pm\{(\mathcal{T}U)_s\}^+ \cdot \psi - g|\psi| \le 0$ on S_2 for all $\psi \in \big[L_\infty(S_2)\big]^3$ and, consequently, by choosing $\psi = \{(\mathcal{T}U)_s\}^+$, we finally get

$$\left|r_{S_2}\big\{(\mathcal{T}U)_s\big\}^+\right| \le g \quad \text{on} \quad S_2. \tag{3.15}$$

In view of (3.14) and (3.15), we obtain

$$r_{S_2}\big\{(\mathcal{T}U)_s\big\}^+ \cdot r_{S_2}\{u_s\}^+ + g\left|r_{S_2}\{u_s\}^+\right| = 0 \quad \text{on} \quad S_2. \tag{3.16}$$

Now, it is evident that if $\left|r_{S_2}\big\{(\mathcal{T}U)_s\big\}^+\right| < g$, then (3.16) implies $r_{S_2}\{u_s\}^+ = 0$. Also, if $\left|r_{S_2}\big\{(\mathcal{T}U)_s\big\}^+\right| = g$, then (3.16) can be rewritten as follows:

$$g\left|r_{S_2}\{u_s\}^+\right|(\cos\alpha + 1) = 0 \quad \text{on} \quad S_2,$$

where α is the angle lying between the vectors $r_{S_2}\{(\mathcal{T}U)_s\}^+$ and $r_{S_2}\{u_s\}^+$ at the point $x \in S_2$. Therefore there exist the functions $\lambda_1(x) \ge 0$ and $\lambda_2(x) \ge 0$ such that $\lambda_1(x) + \lambda_2(x) > 0$ and

$$\lambda_1(x) r_{S_2}\big\{u_s(x)\big\}^+ = -\lambda_2(x) r_{S_2}\big\{(\mathcal{T}U)_s\big\}^+ \quad \text{on} \quad S_2.$$

Moreover, we may assume that λ_1 belongs to the same class as $\{(\mathcal{T}U)_s\}^+$, and λ_2 belongs to the same class as $\{u_s\}^+$.

Thus, the conditions (3.5) and (3.6) of Problem A hold as well, and the proof is complete. □

4. The existence and uniqueness theorems

Here we investigate the so-called coercive case, where the measure of the Dirichlet part of the boundary is positive, i.e., meas $S_1 > 0$.

Theorem 4.1. *The variational inequality* (3.8) *has at most one solution.*

Proof. Let $U = (u, \omega)^\top \in K(\Omega)$ and $U' = (u', \omega')^\top \in K(\Omega)$ be two solutions of the variational inequality (3.8). Then

$$B(U, U' - U) + j(u') - j(u) \ge (\mathcal{G}, U' - U) + \left\langle \varphi, r_{S_2}\{\omega' - \omega\}^+ \right\rangle_{S_2}$$

and

$$B(U', U - U') + j(u) - j(u') \ge (\mathcal{G}, U - U') + \left\langle \varphi, r_{S_2}\{\omega - \omega'\}^+ \right\rangle_{S_2}.$$

Summing these inequalities and applying the property (2.4), we easily derive that $B(U - U', U - U') = 0$. Therefore $U - U' = ([a \times x] + b, a)$ in Ω, where $a, b \in \mathbb{R}^3$ are arbitrary constant vectors (see Lemma 2.1). Since $r_{S_1}\{U - U'\}^+ = 0$, we conclude that $a = b = 0$, i.e., $U = U'$ in Ω. □

To prove the existence result, we introduce on the set $K(\Omega)$ the following functional:

$$J(V) = \frac{1}{2} B(V, V) + j(v) - \int_{\Omega} \mathcal{G} \cdot V \, dx - \langle \varphi, r_{S_2}\{w\}^{+} \rangle_{S_2} \quad \forall\, V = (v, w)^{\top} \in K(\Omega). \tag{4.1}$$

Due to the symmetry property of the form $B(U, V)$, it is not difficult to show that the variational inequality (3.8) is equivalent to the minimization problem for the functional (4.1) on the closed convex set $K(\Omega)$, i.e., the variational inequality (3.8) is equivalent to the following minimizing problem:

Find $U_0 \in K(\Omega)$ such that

$$J(U_0) = \inf_{V \in K(\Omega)} J(V).$$

In turn, in accordance with the general theory of variational inequalities (see, e.g., [30,40]), the solvability of the minimization problem immediately follows from the coercivity of the functional $\mathcal{J}$, i.e., from the property

$$J(V) \to +\infty, \quad \text{when} \quad \|V\|_{[H^1(\Omega)]^6} \to \infty, \quad V \in K(\Omega). \tag{4.2}$$

Since B is positive and bounded below on $K(\Omega)$, due to (2.6) and the inequality $j(v) \geq 0$, it is easy to see by the trace theorem that

$$J(V) \geq C_1 \|V\|^2_{[H^1(\Omega)]^6} - C_2 \|V\|_{[H^1(\Omega)]^6},$$

where C_1 and C_2 are some positive constants, independent of V. This inequality shows that the functional (4.1) is coercive on the set $K(\Omega)$. Therefore we have the following existence result for Problem A.

Theorem 4.2. *Let $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S_2)]^3$, $g \in L_\infty(S_2)$ and $g \geq 0$. Then Problem* A *has a unique solution in $[H^1(\Omega)]^6$, depending continuously on the data $\mathcal{G}$, φ and g of the problem.*

Proof. The unique solvability follows from the equivalence Theorem 3.1, uniqueness Theorem 4.1 and the coercivity property (4.2) (see Theorem 2.1 in [40]).

Further, we establish the continuous dependence of solutions on the data of Problem A.

Let $U = (u, \omega)^{\top} \in [H^1(\Omega)]^6$ and $\widetilde{U} = (\widetilde{u}, \widetilde{\omega})^{\top} \in [H^1(\Omega)]^6$ be two solutions of Problem A, corresponding to the data $\mathcal{G}$, φ, g and $\widetilde{\mathcal{G}}$, $\widetilde{\varphi}$, $\widetilde{g}$, respectively. Thus we have two variational inequalities of type (3.8): the first inequality for U and the second one for $\widetilde{U}$. Substituting $V = \widetilde{U}$ into the first inequality and $V = U$ into the second one and taking their sum, we obtain

$$-B(U - \widetilde{U}, U - \widetilde{U}) - \int_{S_2} (g - \widetilde{g})\Big(\big|\{u_s\}^{+}\big| - \big|\{\widetilde{u}_s\}^{+}\big|\Big) dS$$
$$- \int_{\Omega} (\mathcal{G} - \widetilde{\mathcal{G}}) \cdot (U - \widetilde{U}) dx - \langle \varphi - \widetilde{\varphi}, r_{S_2}\{\omega - \widetilde{\omega}\}^{+} \rangle_{S_2}.$$

Taking into account the last inequality, the inclusion $U, \widetilde{U} \in K(\Omega)$ and the strong Korn's inequality (2.6) (see Remark 2.2), we obtain

$$\|U - \widetilde{U}\|_{[H^1(\Omega)]^6} \leq C\Big(\|g - \widetilde{g}\|_{L_2(S_2)} + \|\mathcal{G} - \widetilde{\mathcal{G}}\|_{[L_2(\Omega)]^6} + \|\varphi - \widetilde{\varphi}\|_{[H^{-1/2}(S_2)]^3}\Big)$$

with some positive constant C, not depending on U and $\widetilde{U}$ and on the data of the problem under consideration. This estimate shows the desired Lipschitz dependence of the solution on the data of the problem. □

5. The semicoercive case

Let $S_1 = \varnothing$, $S_2 = S$, $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S)]^3$, $g \in L_\infty(S)$ and $g \geq 0$. Consider the boundary contact problem.

Problem B. Find a vector function $U = (u, \omega)^{\top} \in [H^1(\Omega)]^6$ which is a weak solution of Eq. (3.1), satisfying the inclusion $\{(\mathcal{T}U)_s\}^{+} \in [L_\infty(S)]^3$ and on the surface S the following boundary conditions:

$$\{\mathcal{M}U\}^{+} = \varphi, \quad \{u_n\}^{+} = 0,$$
$$\text{if} \quad \big|\{(\mathcal{T}U)_s\}^{+}\big| < g, \quad \text{then } \{u_s\}^{+} = 0,$$

if $\big|\{(\mathcal{T}U)_s\}^+\big| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and

$$\lambda_1\{u_s\}^+ = -\lambda_2\big\{(\mathcal{T}U)_s\big\}^+.$$

Analogously to the previous coercive case (see Theorem 3.1), we can show that Problem B is equivalent to the following variational inequality:

$$B(U, V-U) + j(v) - j(u) \geq \int_\Omega \mathcal{G}\cdot(V-U)dx + \big\langle\varphi, \{w-\omega\}^+\big\rangle_S \tag{5.1}$$

which holds for all $V = (v, w)^\top \in [H^1(\Omega)]^6$. Here

$$j(v) = \int_S g\big|\{v_s\}^+\big|dS.$$

Let $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ be a solution of the variational inequality (5.1).
Substituting first $V = 0$ and then $V = 2U$ into inequality (5.1), we obtain

$$B(U, U) + j(u) = \int_\Omega \mathcal{G}\cdot U\,dx + \big\langle\varphi, \{\omega\}^+\big\rangle_S. \tag{5.2}$$

By virtue of (5.2), from (5.1) we derive

$$B(U, V) + j(v) \geq \int_\Omega \mathcal{G}\cdot V\,dx + \big\langle\varphi, \{w\}^+\big\rangle_S. \tag{5.3}$$

Thus inequality (5.1) is equivalent to the simultaneous relations (5.2) and (5.3).
Substituting $-V$ in the place of V in (5.3), we get

$$\left|\int_\Omega \mathcal{G}\cdot V\,dx + \big\langle\varphi, \{w\}^+\big\rangle_S - B(U, V)\right| \leq j(v) \tag{5.4}$$

for all $V = (v, \omega)^\top \in [H^1(\Omega)]^6$.
By $\mathcal{R}$ we denote the set of solutions of the equation $B(U, U) = 0$ in the space $[H^1(\Omega)]^6$ (see Lemma 2.1),

$$\mathcal{R} := \Big\{\xi = (\rho, a)^\top \in \big[H^1(\Omega)\big]^6;\ \rho = [a\times x] + b,\ a, b \in \mathbb{R}^3\Big\}.$$

By the substitution of an arbitrary $\xi = (\rho, a)^\top \in \mathcal{R}$ in the place of V in (5.4), we derive the necessary condition of solvability of the variational inequality (5.1),

$$\left|\int_\Omega \mathcal{G}\cdot\xi\,dx + \big\langle\varphi, a\big\rangle_S\right| \leq \int_S g\big|\{\rho_s\}^+\big|dS \tag{5.5}$$

for all $\xi = (\rho, a)^\top \in \mathcal{R}$.
Let in (5.5) we have the strict inequality. Then taking into account the fact that the space $\mathcal{R}$ has finite dimension, $\dim\mathcal{R} = 6$, it is easy to see that (5.5) is equivalent to the relation

$$\int_S g\big|\{\rho_s\}^+\big|dS - \left|\int_\Omega \mathcal{G}\cdot\xi\,dx + \big\langle\varphi, a\big\rangle_S\right| \geq C\|\xi\|_{[L_2(\Omega)]^6} \tag{5.6}$$

with some positive constant C, and for all $\xi \in \mathcal{R}\setminus\{0\}$.
Let $\mathcal{P}_\mathcal{R}$ be an orthogonal projection operator of the space $[H^1(\Omega)]^6$ on $\mathcal{R}$, in the sense of the space $[L_2(\Omega)]^6$, i.e., $\forall\,V \in [H^1(\Omega)]^6 : V = W + \xi$, where $\xi = (\rho, a)^\top = \mathcal{P}_\mathcal{R}V \in \mathcal{R}$, and

$$W = (\eta, \zeta)^\top \in \mathcal{R}^\perp := \left\{U \in [H^1(\Omega)]^6 : \int_\Omega U\cdot\xi\,dx = 0,\ \forall\,\xi\in\mathcal{R}\right\}.$$

Due to inequality (2.5) and Lemma 5.1 in [19], the bilinear form B is semicoercive, i.e., there is a positive constant C_0 such that

$$B(V,V) \geq C_0 \| V - \mathcal{P}_{\mathcal{R}} V \|^2_{[H^1(\Omega)]^6} = C_0 \| W \|^2_{[H^1(\Omega)]^6} \quad \forall\, V \in \big[H^1(\Omega)\big]^6. \tag{5.7}$$

Therefore, for all $V \in [H^1(\Omega)]^6$, due to (5.6) and (5.7), we have

$$\begin{aligned} J(V) = J(W+\xi) &= \frac{1}{2} B(W,W) + j(\eta+\rho) - j(\rho) \\ &\quad - \int_\Omega \mathcal{G} \cdot W\, dx - \langle \varphi, \{\zeta\}^+ \rangle_S - \int_\Omega \mathcal{G} \cdot \xi\, dx - \langle \varphi, a \rangle_S + j(\rho) \\ &\geq C_0 \| W \|^2_{[H^1(\Omega)]^6} + C \| \xi \|_{[L_2(\Omega)]^6} - C_1 \| W \|_{[H^1(\Omega)]^6} + j(\eta+\rho) - j(\rho), \end{aligned}$$

for some positive constants C, C_0, C_1.

Let us now estimate $j(\eta+\rho) - j(\rho)$. We have

$$j(\eta+\rho) - j(\rho) = \int_S g\big(\big|\{(\eta+\rho)_s\}^+\big| - \big|\{\rho_s\}^+\big|\big) dS \geq - \int_S g \big|\{\eta_s\}^+\big| dS \geq -C_2 \| W \|_{[H^1(\Omega)]^6},$$

where the positive constant C_2 is independent of η and ρ.

Taking into account this inequality, we finally have

$$J(V) \geq C_0 \| W \|^2_{[H^1(\Omega)]^6} + C \| \xi \|_{[L_2(\Omega)]^6} - C_3 \| W \|_{[H^1(\Omega)]^6}$$

with some positive constants, whence it follows that

$$J(V) \to \infty, \quad \text{when } \| V \|_{[H^1(\Omega)]} \to \infty, \quad V \in \big[H^1(\Omega)\big]^6$$

i.e., the functional is coercive and the minimization problem for this functional is solvable. Consequently, the corresponding variational inequality (5.1) is solvable, as well (see [30,33]). Further, just as in Theorem 4.1, for the two possible solutions U and U^* to the variational inequality (5.1) of the class $[H^1(\Omega)]^6$, we easily derive $B(U - U^*, U - U^*) = 0$, which implies

$$U - U^* = \big([a \times x] + b, a\big), \quad a, b \in \mathbb{R}^3.$$

Thus we have the following existence and uniqueness

Theorem 5.1. *Let $S_1 = \varnothing$, $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S)]^3$, $g \in L_\infty(S)$, $g \geq 0$ and the condition* (5.6) *be fulfilled. Then the variational inequality* (5.1) *is solvable in the space* $[H^1(\Omega)]^6$*. Moreover, the solutions are defined modulo generalized rigid displacement vectors.*

Remark 5.2. Analogously to the noncoercive case, we can study the problem, when on a part of the boundary S_1 instead of the Dirichlet condition (3.2) there is assigned the following fractional boundary condition

$$r_{S_1} \big\{ T(\partial, n) U \big\}^+ = Q,$$

where $Q \in [\widetilde{H}^{-1/2}(S_1)]^6$. Moreover, we assume that the vector φ appearing in the condition (3.3) belongs to the space $[\widetilde{H}^{-1/2}(S_2)]^3$.

In this case, instead of (3.8) we have the following variational inequality: Find $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ such that $\forall\, V = (v, w)^\top \in [H^1(\Omega)]^6$

$$B(U, V-U) + j(v) - j(u) \geq \int_\Omega \mathcal{G} \cdot (V-U) dx + \langle r_{S_1} Q, r_{S_1} \{V - U\}^+ \rangle_{S_1} + \langle r_{S_2} \varphi, r_{S_2} \{w - \omega\}^+ \rangle_{S_2}. \tag{5.8}$$

The necessary condition for the solvability of the variational inequality (5.8) reads now as

$$\left| \int_\Omega \mathcal{G} \cdot \xi\, dx + \langle r_{S_1} Q, r_{S_1} \{\xi\}^+ \rangle_{S_1} + \langle r_{S_2} \varphi, a \rangle_{S_2} \right| \leq \int_{S_2} g \big|\{\rho_s\}^+\big| dS,$$

where $\xi = (\rho, a)^\top \in \mathcal{R}$ is an arbitrary generalized rigid displacement vector.

Let us assume that in the necessary condition we have the strict inequality. Since $\mathcal{R}$ is finite-dimensional, we can show that the strict inequality is equivalent to the condition: there is a positive constant C such that the inequality

$$\int_{S_2} g\big|\{\rho_s\}^+\big|dS - \left|\int_{\Omega} \mathcal{G}\cdot\xi\,dx + \langle r_{S_1}Q, r_{S_1}\{\xi\}^+\rangle_{S_1} + \langle r_{S_2}\varphi, a\rangle_{S_2}\right| \geq C\|\xi\|_{[L_2(\Omega)]^6} \tag{5.9}$$

holds for all $\xi \in \mathcal{R}\setminus\{0\}$. This condition is sufficient for the variational inequality (5.8) to be solvable.

Thus, we have the following existence result.

Theorem 5.3. *Let* mes $S_1 > 0$, $\mathcal{G} \in [L_2(\Omega)]^6$, $Q \in [\widetilde{H}^{-1/2}(S_1)]^6$, $\varphi \in [\widetilde{H}^{-1/2}(S_2)]^3$, $g \in L_\infty(S_2)$, $g \geq 0$ *and the condition* (5.9) *be fulfilled. Then the variational inequality* (5.8) *is solvable and the solution minimizes the functional*

$$J(V) = \frac{1}{2}B(V,V) + j(V) - \int_{\Omega}\mathcal{G}\cdot V\,dx - \langle r_{S_1}Q, r_{S_1}\{V\}^+\rangle_{S_1} - \langle r_{S_2}\varphi, r_{S_2}\{w\}^+\rangle_{S_2},$$
$$V = (v,w)^\top \in [H^1(\Omega)]^6$$

on the space $[H^1(\Omega)]^6$. *Solutions of the variational inequality* (5.8) *are defined modulo generalized rigid displacement vector.*

Remark 5.4. Let the boundary $S = \partial\Omega$ fall into three mutually disjoint portions S_1, S_T and S_2 such that $\overline{S}_1 \cup \overline{S}_T \cup \overline{S}_2 = S$, $\overline{S}_1 \cap \overline{S}_2 = \varnothing$. Analogously to the coercive case, we can study the problem, when on a part of the boundary S_T there is assigned the traction boundary condition

$$r_{S_T}\big\{T(\partial,n)U\big\}^+ = Q,$$

where $Q \in [H^{-1/2}(S_T)]^6$. The conditions on the boundaries S_1 and S_2 in this case remain the same as in Problem A.

In this case we have the following variational inequality:

Find $U = (u,\omega)^\top \in K(\Omega)$ such that $\forall\, V = (v,w)^\top \in K(\Omega)$,

$$B(U, V-U) + j(v) - j(u) \geq (\mathcal{G}, V-U) + \langle Q, r_{S_T}\{V-U\}^+\rangle_{S_T} + \langle\varphi, r_{S_2}\{w-\omega\}^+\rangle_{S_2},$$

where the functional j is defined by formula (3.7).

The proof of the existence and uniqueness theorems for this case can be carried out by repeating word for word the above arguments.

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Original article

On micropolar elastic cusped prismatic shells

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Abstract

A huge literature is devoted to the study of cusped prismatic shells on the basis of the classical theory of elasticity. It was stimulated by the works of I. Vekua. I. Vekua considered very important to carry out investigations of boundary value and initial boundary value problems for such bodies, since they are connected with degenerate partial differential equations and, therefore, are not classical, in general. The present paper is devoted to cusped prismatic shells on the basis of the theory of micropolar elasticity. Namely, on the basis of the $N = 0$ approximation of hierarchical models for micropolar elastic cusped prismatic shells constructed by the I. Vekua dimension reduction method.

Keywords: Micropolar elasticity; Cusped prismatic shells; Degenerate partial differential equations

1. Introduction

A huge literature is devoted to the study of cusped prismatic shells on the basis of the classical theory of elasticity. It was stimulated by the work [1] of I. Vekua (see also [2]). I. Vekua considered very important to carry out investigations of boundary value (BVP) and initial boundary value (IBVP) problems for such bodies, since they are connected with degenerate partial differential equations (PDE) and, therefore, are not classical, in general. A survey of results obtained in this direction one can find in [3] (see also the references therein). The present paper is devoted to cusped prismatic shells on the basis of the theory of micropolar elasticity (see, e.g. [4,5]). Namely, on the basis of the $N = 0$ approximation of hierarchical models for micropolar elastic cusped prismatic shells constructed by the I. Vekua dimension reduction method.

The paper is organized as follows. In Section 2 we give an exposition of the governing equations of the $N = 0$ approximation of hierarchical models of micropolar elastic prismatic shells and briefly sketch the Nth approximation. Section 3 contains an analysis of the system of PDEs constructed in Section 2. Peculiarities of well-posedness of boundary conditions (BCs) for micropolar elastic cusped prismatic shells are revealed.

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2. $N = 0$ approximation

Let $Ox_1x_2x_3$ be an anticlockwise-oriented rectangular Cartesian frame of origin O. We conditionally assume the x_3-axis vertical. The elastic body Ω is called a prismatic shell [1–3] if it is bounded from above and below by, respectively, the surfaces (so called face surfaces)

$$x_3 = \overset{(+)}{h}(x_1, x_2) \quad \text{and} \quad x_3 = \overset{(-)}{h}(x_1, x_2), \qquad (x_1, x_2) \in \omega,$$

laterally by a cylindrical surface Γ of generatrix parallel to the x_3-axis and its vertical dimension is sufficiently small compared with other dimensions of the body. $\overline{\omega} := \omega \cup \partial\omega$ is the so-called projection of the prismatic shell on $x_3 = 0$.

Let the thickness of the prismatic shell be

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) \quad \begin{cases} > 0 & \text{for} \quad (x_1, x_2) \in \omega, \\ \geq 0 & \text{for} \quad (x_1, x_2) \in \partial\omega \end{cases}$$

and

$$2\widetilde{h}(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2).$$

If the thickness of the prismatic shell vanishes on some subset of $\partial\omega$, it is called cusped one.

Let us note that the lateral boundary of the standard shell is orthogonal to the "middle surface" of the shell, while the lateral boundary of the prismatic shell is orthogonal to the prismatic shell's projection on $x_3 = 0$.

Let $t \in T := [0, +\infty[$ be time, $T_+ :=]0, \infty[$, $\overline{\Omega} \times T$ denote the Cartesian product, $u_i \in C^2(\Omega \times T_+)$, $i = 1, 2, 3$, be displacements, $\omega_i \in C^2(\Omega \times T_+)$, $i = 1, 2, 3$, be microrotations, $e_{ij} \in C^1(\Omega \times T_+)$ be the asymmetric strain tensor, $u_{ji} \in C^1(\Omega \times T_+)$ be the asymmetric microstrain (torsion-flexure) tensor, $X_{ji} \in C^1(\Omega \times T_+)$ be the asymmetric force–stress tensor, $\chi_{ji} \in C^1(\Omega \times T_+)$ be the asymmetric couple stress tensor, $\Phi_i \in C(\Omega \times T_+)$ and $\Psi_i \in C(\Omega \times T_+)$ be the fields of volume forces and volume couples, respectively, ρ be the density, $\mathcal{I}$ be the rotational inertia of the medium, λ, μ, $\widetilde{\alpha}$, $\widetilde{\beta}$, ν and ε be the elasticity constants of the medium, $\mu > 0$, $3\lambda + 2\mu > 0$, $\widetilde{\alpha} > 0$, $\widetilde{\beta} > 0$, $\nu > 0$, $3\varepsilon + 2\nu > 0$, $\in_{ijk}$ be the Levi-Civita symbol. Here C^2 and C^1 are classes of twice and once, correspondingly, continuously differentiable functions in the domain under consideration; C is a class of continuous functions on the sets under consideration. Throughout the paper Einstein's rule of summation is used for Latin indexes from 1 to 3, and for Greek indexes from 1 to 2.

In order to construct governing equations of the $N = 0$ approximation of hierarchical models, using Vekua's dimension reduction method, we integrate within the limits $\overset{(-)}{h}$, $\overset{(+)}{h}$ with respect to the thickness variable x_3 the following governing equations of the micropolar theory of elasticity (see [4,5] and the references therein):

Motion equations

$$X_{ji,j} + \Phi_i = \rho \ddot{u}_i, \qquad i = 1, 2, 3, \tag{1}$$

$$\chi_{ji,j} + \in_{ijk} X_{jk} + \Psi_i = \mathcal{I}\ddot{\omega}_i, \qquad i = 1, 2, 3; \tag{2}$$

Kinematic equations

$$u_{ji} = u_{i,j} - \in_{kji} \omega_k = e_{ji} + \in_{kji} (\theta_k - \omega_k), \quad i, j = 1, 2, 3, \tag{3}$$

$$\omega_{ji} = \omega_{i,j}, \quad i, j = 1, 2, 3; \tag{4}$$

Constitutive equations

$$\begin{aligned} X_{ij} &= \lambda\delta_{ij}u_{kk} + (\mu + \tilde{\alpha})u_{ij} + (\mu - \tilde{\alpha})u_{ji} = \lambda u_{k,k}\delta_{ij} \\ &\quad + (\mu + \tilde{\alpha})u_{j,i} - (\mu + \tilde{\alpha}) \in_{kij} \omega_k + (\mu - \tilde{\alpha})u_{i,j} - (\mu - \tilde{\alpha}) \in_{kji} \omega_k \\ &= \lambda u_{k,k}\delta_{ij} + (\mu + \tilde{\alpha})u_{j,i} + (\mu - \tilde{\alpha})u_{i,j} - 2\tilde{\alpha} \in_{kij} \omega_k, \quad i, j = 1, 2, 3, \end{aligned} \tag{5}$$

$$\begin{aligned} \chi_{ij} &= \varepsilon\delta_{ij}\omega_{kk} + (\nu + \tilde{\beta})\omega_{ij} + (\nu - \tilde{\beta})\omega_{ji} \\ &= \varepsilon\omega_{k,k}\delta_{ij} + (\nu + \tilde{\beta})\omega_{j,i} + (\nu - \tilde{\beta})\omega_{i,j}, \quad i, j = 1, 2, 3, \end{aligned} \tag{6}$$

considered in the domain

$$\Omega := \{x := (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ (x_1, x_2) \in \omega, \ \overset{(-)}{h}(x_1, x_2) < x_3 < \overset{(+)}{h}(x_1, x_2)\}$$

occupied by the prismatic shell with the projection ω on the plane $x_3 = 0$.

By the corresponding calculations as values of tractions and couple stress vectors on the face surfaces we take their prescribed values, while as values of displacements and microrotations we take their approximate values calculated from their Fourier–Legendre expansions (see Remark 1) on the face surfaces corresponding to $N = 0$ approximation.

From (1), (2), (3), (4) we get

$$X_{\beta i0,\beta} + X_i^0 = \rho \ddot{u}_{i0}, \quad i = 1, 2, 3; \tag{7}$$

$$\chi_{\beta i0,\beta} + \in_{ijk} X_{jk0} + \chi_j^0 = \mathcal{I} \ddot{\omega}_{j0}, \quad i = 1, 2, 3, \tag{8}$$

in ω, where

$$X_i^0 := Q_{\overset{(+)}{\nu} i} \cdot \sqrt{\left(\overset{(+)}{h}_{,1}\right)^2 + \left(\overset{(+)}{h}_{,2}\right)^2 + 1} + Q_{\overset{(-)}{\nu} i} \cdot \sqrt{\left(\overset{(-)}{h}_{,1}\right)^2 + \left(\overset{(-)}{h}_{,2}\right)^2 + 1} + \Phi_{i0},$$

$$\chi_j^0 := \Theta_{\overset{(+)}{\nu} i} \cdot \sqrt{\left(\overset{(+)}{h}_{,1}\right)^2 + \left(\overset{(+)}{h}_{,2}\right)^2 + 1} + \Theta_{\overset{(-)}{\nu} i} \cdot \sqrt{\left(\overset{(-)}{h}_{,1}\right)^2 + \left(\overset{(-)}{h}_{,2}\right)^2 + 1} + \Psi_{i0},$$

$Q_{\overset{(\pm)}{\nu}}$ and $\Theta_{\overset{(\pm)}{\nu}}$ are tractions and couple stress vectors prescribed on face surfaces (in what follows superscripts $(+)$ and $(-)$ mean values on upper and lower face surfaces, correspondingly),

$$u_{\beta i0} = u_{i0,\beta} - \overset{(+)}{u}_i \overset{(+)}{h}_{,\beta} + \overset{(-)}{u}_i \overset{(-)}{h}_{,\beta} - \in_{k\beta i} \omega_{k0}, \quad \beta = 1, 2; \ i = 1, 2, 3, \tag{9}$$

$$u_{3i0} = \overset{(+)}{u}_i - \overset{(-)}{u}_i - \in_{k3i} \omega_{k0} = \overset{(+)}{u}_i - \overset{(-)}{u}_i - \in_{\gamma 3i} \omega_{\gamma 0}, \quad i = 1, 2, 3, \tag{10}$$

$$\omega_{\beta i0} = \omega_{i0,\beta} - \overset{(+)}{\omega}_i \overset{(+)}{h}_{,\beta} + \overset{(-)}{\omega}_i \overset{(-)}{h}_{,\beta}, \quad \beta = 1, 2, \ i = 1, 2, 3; \tag{11}$$

$$\omega_{3i0} = \overset{(+)}{\omega}_i - \overset{(-)}{\omega}_i, \quad i = 1, 2, 3. \tag{12}$$

In indices 0 means integrated values of the corresponding quantities which are called the zero order moments.

In the $N = 0$ approximation we assume

$$u_i(x_1, x_2, x_3, t) = \frac{1}{2} v_{i0}(x_1, x_2, t) = \frac{1}{2} \frac{u_{i0}(x_1, x_2, t)}{h(x_1, x_2)}, \tag{13}$$

$$\omega_i(x_1, x_2, x_3, t) = \frac{1}{2} \eta_{i0}(x_1, x_2, t) = \frac{1}{2} \frac{\omega_{i0}(x_1, x_2, t)}{h(x_1, x_2)}. \tag{14}$$

Evidently, by virtue of (9), (13), (14),

$$u_{\beta i0} = h v_{i0,\beta} - \in_{k\beta i} h \eta_{k0}, \tag{15}$$

in view of (10), (14),

$$u_{3i0} = - \in_{\gamma 3i} h \eta_{\gamma 0}, \tag{16}$$

whence,

$$u_{330} = 0, \qquad u_{320} = - \in_{123} \omega_{10} = \omega_{10}, \qquad u_{310} = - \in_{231} \omega_{20} = -\omega_{20}. \tag{17}$$

By virtue of (11), (14),

$$\omega_{\beta i0} = h \eta_{i0,\beta}, \quad \beta = 1, 2, \ i = 1, 2, 3. \tag{18}$$

In view of (12), (14)

$$\omega_{3i0} = 0, \quad i = 1, 2, 3. \tag{19}$$

From (5) and (6), we obtain

$$X_{ij0} = \lambda\delta_{ij}u_{kk0} + (\mu+\tilde{\alpha})u_{ij0} + (\mu-\tilde{\alpha})u_{ji0}, \quad i, j = 1, 2, 3, \tag{20}$$

and

$$\chi_{ij0} = \varepsilon\delta_{ij}\omega_{kk0} + (\nu+\tilde{\beta})\omega_{ij0} + (\nu-\tilde{\beta})\omega_{ji0}, \quad i, j = 1, 2, 3, \tag{21}$$

respectively.

Remark 1. The governing equations for the Nth approximation can be constructed in the similar way.

From (17), (15) it follows

$$u_{kk0} = u_{\gamma\gamma0} = hv_{\gamma0,\gamma}. \tag{22}$$

From (19), (18) we obtain

$$\omega_{kk0} = \omega_{\gamma\gamma0} = h\eta_{\gamma0,\gamma}. \tag{23}$$

Since

$$\in_{k\alpha\beta}\eta_{k0} = \in_{3\alpha\beta}\eta_{30}, \tag{24}$$

we have

$$\in_{312}\eta_{30} = \eta_{30}, \qquad \in_{321}\eta_{30} = -\eta_{30}.$$

From (20), taking into account (22), (15), (24), we find

$$\begin{aligned} X_{\beta\alpha0} &= \lambda hv_{\gamma0,\gamma}\delta_{\beta\alpha} + (\mu+\tilde{\alpha})h(v_{\alpha0,\beta} - \in_{k\beta\alpha}\eta_{k0}) + (\mu-\tilde{\alpha})h(v_{\beta0,\alpha} - \in_{k\alpha\beta}\eta_{k0}) \\ &= \lambda hv_{\gamma0,\gamma}\delta_{\beta\alpha} + (\mu+\tilde{\alpha})h(v_{\alpha0,\beta} - \in_{3\beta\alpha}\eta_{30}) + (\mu-\tilde{\alpha})h(v_{\beta0,\alpha} - \in_{3\alpha\beta}\eta_{30}) \\ &= \lambda hv_{\gamma0,\gamma}\delta_{\beta\alpha} + (\mu+\tilde{\alpha})hv_{\alpha0,\beta} + (\mu-\tilde{\alpha})hv_{\beta0,\alpha} + 2\tilde{\alpha}h\in_{3\alpha\beta}\eta_{30}, \quad \alpha, \beta = 1, 2; \end{aligned} \tag{25}$$

From (20), taking into account (15), (16), we find

$$\begin{aligned} X_{3\beta0} &= (\mu+\tilde{\alpha})h(-\in_{\gamma3\beta}\eta_{\gamma0}) + (\mu-\tilde{\alpha})h(v_{30,\beta} - \in_{k\beta3}\eta_{k0}) \\ &= -(\mu+\tilde{\alpha})h\in_{\gamma3\beta}\eta_{\gamma0} + (\mu-\tilde{\alpha})h(v_{30,\beta} - \in_{\gamma\beta3}\eta_{\gamma0}) \\ &= (\mu-\tilde{\alpha})hv_{30,\beta} + 2\tilde{\alpha}h\in_{\gamma\beta3}\eta_{\gamma0}. \end{aligned} \tag{26}$$

From (20), by virtue of (22), (16) we get

$$X_{330} = \lambda hv_{\gamma0,\gamma}.$$

From (20), in view of (16), (15), we obtain

$$\begin{aligned} X_{\beta30} &= (\mu+\tilde{\alpha})h(v_{30,\beta} - \in_{\gamma\beta3}\eta_{\gamma0}) + (\mu-\tilde{\alpha})h(-\in_{\gamma3\beta}\eta_{\gamma0}) \\ &= (\mu+\tilde{\alpha})hv_{30,\beta} - 2\tilde{\alpha}h\in_{\gamma\beta3}\eta_{\gamma0}. \end{aligned} \tag{27}$$

From (21), taking into account (23), (18), we have

$$\chi_{\beta\alpha0} = \varepsilon h\eta_{\gamma0,\gamma}\delta_{\beta\alpha} + (\nu+\tilde{\beta})h\eta_{\alpha0,\beta} + (\nu-\tilde{\beta})h\eta_{\beta0,\alpha}. \tag{28}$$

From (21), by virtue of (19), (18), we get

$$\chi_{3\beta0} = (\nu-\tilde{\beta})h\eta_{30,\beta}.$$

From (21), in view of (23), (19), we obtain

$$\chi_{330} = \varepsilon h\eta_{\gamma0,\gamma}.$$

From (21), according to (18), (19), we have

$$\chi_{\beta30} = (\nu+\tilde{\beta})h\eta_{30,\beta}. \tag{29}$$

Substituting (25) into (7), we arrive at the equation

$$\lambda\delta_{\beta\alpha}(hv_{\gamma 0,\gamma})_{,\beta} + (\mu+\tilde{\alpha})(hv_{\alpha 0,\beta})_{,\beta} + (\mu-\tilde{\alpha})(hv_{\beta 0,\alpha})_{,\beta} + 2\tilde{\alpha} \in_{3\alpha\beta}(h\eta_{30})_{,\beta} + X^0_\alpha = \rho h\ddot{v}_{\alpha 0}, \quad \alpha = 1, 2.$$

Hence,

$$(\mu+\tilde{\alpha})(hv_{\alpha 0,\beta})_{,\beta} + \lambda(hv_{\gamma 0,\gamma})_{,\alpha} + (\mu-\tilde{\alpha})(hv_{\beta 0,\alpha})_{,\beta} + 2\tilde{\alpha} \in_{\alpha\beta 3}(h\eta_{30})_{,\beta} + X^0_\alpha = \rho h\ddot{v}_{\alpha 0}, \quad \alpha = 1, 2. \tag{30}$$

Substituting (27) into (7), we have

$$(\mu+\tilde{\alpha})(hv_{30,\beta})_{,\beta} - 2\tilde{\alpha} \in_{\gamma\beta 3}(h\eta_{\gamma 0})_{,\beta} + X^0_3 = \rho h\ddot{v}_{30}.$$

Therefore,

$$(\mu+\tilde{\alpha})(hv_{30,\beta})_{,\beta} - 2\tilde{\alpha}[(h\eta_{10})_{,2} - (h\eta_{20})_{,1}] + X^0_3 = \rho h\ddot{v}_{30}. \tag{31}$$

Substituting (28) into (8), we get

$$\varepsilon(h\eta_{\gamma 0,\gamma})_{,\alpha} + (\nu+\tilde{\beta})(h\eta_{\alpha 0,\beta})_{,\beta} + (\nu-\tilde{\beta})(h\eta_{\beta 0,\alpha})_{,\beta} + \in_{\alpha jk} X_{jk0} + \chi^0_\alpha = \mathcal{I}h\ddot{\eta}_{\alpha 0}, \quad \alpha = 1, 2; \tag{32}$$

Substituting (29) into (8), we obtain

$$(\nu+\tilde{\beta})(h\eta_{30,\beta})_{,\beta} + \in_{3jk} X_{jk0} + \chi^0_3 = \mathcal{I}h\ddot{\eta}_{30}. \tag{33}$$

Since, by virtue of (25)–(27),

$$\begin{aligned}
\in_{1jk} X_{jk0} &= \in_{123} X_{230} + \in_{132} X_{320} = X_{230} - X_{320} = 2\tilde{\alpha}hv_{30,2} - 4\tilde{\alpha}h\eta_{10},\\
\in_{2jk} X_{jk0} &= \in_{213} X_{130} + \in_{231} X_{310} = -X_{130} + X_{310}\\
&= -2\tilde{\alpha}hv_{30,1} - 4\tilde{\alpha}h\eta_{20},\\
\in_{3jk} X_{jk0} &= \in_{312} X_{120} + \in_{321} X_{210} = X_{120} - X_{210}\\
&= (\mu+\tilde{\alpha})hv_{20,1} + (\mu-\tilde{\alpha})hv_{10,2} + 2\tilde{\alpha}h \in_{321} \eta_{30}\\
&\quad - (\mu+\tilde{\alpha})hv_{10,2} - (\mu-\tilde{\alpha})hv_{20,1} - 2\tilde{\alpha}h \in_{312} \eta_{30}\\
&= 2\tilde{\alpha}h(v_{20,1} - v_{10,2}) - 4\tilde{\alpha}h\eta_{30},
\end{aligned}$$

from (32), (33) we get

$$(\nu+\tilde{\beta})(h\eta_{10,\beta})_{,\beta} + \varepsilon(h\eta_{\gamma 0,\gamma})_{,1} + (\nu-\tilde{\beta})(h\eta_{\beta 0,1})_{,\beta} + 2\tilde{\alpha}hv_{30,2} - 4\tilde{\alpha}h\eta_{10} + \chi^0_1 = \mathcal{I}h\ddot{\eta}_{10}, \tag{34}$$

$$(\nu+\tilde{\beta})(h\eta_{20,\beta})_{,\beta} + \varepsilon(h\eta_{\gamma 0,\gamma})_{,2} + (\nu-\tilde{\beta})(h\eta_{\beta 0,2})_{,\beta} - 2\tilde{\alpha}hv_{30,1} - 4\tilde{\alpha}h\eta_{20} + \chi^0_2 = \mathcal{I}h\ddot{\eta}_{20}, \tag{35}$$

$$(\nu+\tilde{\beta})(h\eta_{30,\beta})_{,\beta} + 2\tilde{\alpha}h(v_{20,1} - v_{10,2}) - 4\tilde{\alpha}h\eta_{30} + \chi^0_3 = \mathcal{I}h\ddot{\eta}_{30}. \tag{36}$$

3. Analysis of the constructed system

System (30), (31), (34)–(36) splits into two independent systems (30), (36) and (31), (34), (35) with respect to $v_{\alpha 0}$, $\alpha = 1, 2$, η_{30} and v_{30}, $\eta_{\alpha 0}$, $\alpha = 1, 2$, correspondingly.

If $h(x_1, x_2) > 0$ on $\overline{\omega}$, in the static case the system (30), (31), (34)–(36) becomes elliptic and for existence and uniqueness theorems of the Dirichlet problem general results (see e.g. [6,7,4] and the references therein) can be applied.

Let us consider prismatic shells with a cusped edge $\omega_0 \subseteq \partial\omega$, where the thickness $2h(x_1, x_2)$ vanishes:

$$\omega_0 := \{(x_1, x_2) \in \partial\omega : 2h(x_1, x_2) = 0\}.$$

Evidently, ω_0 is a closed set.

Dirichlet Problem. Find a solution $w_{i0} \in C^2(\omega) \cap C(\overline{\omega})$, $i = \overline{1, 6}$

$$(w_{10}, w_{20}, w_{30}, w_{40}, w_{50}, w_{60}) := (v_{10}, v_{20}, v_{30}, \eta_{10}, \eta_{20}, \eta_{30}),$$

of the governing system (30), (31), (34)–(36) in ω, satisfying the BCs

$$w_{i0}(x_1, x_2) = \varphi_i(x_1, x_2), \quad (x_1, x_2) \in \partial\omega, \ i = \overline{1, 6},$$

where φ_i, $i = \overline{1, 6}$, are given continuous on $\partial\omega$ functions.

Keldysh Problem. Find a bounded solution $w_{i0} \in C^2(\omega) \cap C(\overline{\omega} \backslash \omega_0)$, $i = \overline{1,6}$, of the governing system (30), (31), (34)–(36) in ω, satisfying the BCs

$$w_{i0}(x_1, x_2) = \varphi_i(x_1, x_2), \quad (x_1, x_2) \in \partial\omega \backslash \omega_0, \ i = \overline{1,6},$$

where φ_i, $i = \overline{1,6}$, are given continuous on $(\partial\omega) \backslash \omega_0$ functions.

The following theorem is true [8] (compare with [9], where $m_1 = 0$).

Theorem 2. *If the coefficients a_α, $\alpha = 1, 2$, and c of the equation*

$$x_2^{m_\alpha} u_{,\alpha\alpha} + a_\alpha(x_1, x_2) u_{,\alpha} + c(x_1, x_2) u = 0, \quad c \leq 0, \ m_\alpha = const \geq 0, \ \alpha = 1, 2,$$

are analytic in $\overline{\omega}$ bounded by a sufficiently smooth arc $(\partial\omega \backslash \omega_0)$ lying in the half-plane $x_2 \geq 0$ and by a segment ω_0 of the x_1-axis, then

(i) *if either $m_2 < 1$, or $m_2 \geq 1$, $a_2(x_1, x_2) < x_2^{m_2-1}$ in $\bar{I}_\delta$ for some $\delta = const > 0$, where*

$$I_\delta := \{(x_1, x_2) \in \omega \,:\, 0 < x_2 < \delta\},$$

the Dirichlet problem is correct;

(ii) *if $m_2 \geq 1$, $a_2(x_1, x_2) \geq x_2^{m_2-1}$ in I_δ and $a_1(x_1, x_2) = O(x_2^{m_1})$, $x_2 \to 0_+$ (O is the Landau symbol), the Keldysh problem is correct.*

Remark 3. If $1 < m_2 < 2$, $a_2(x, 0) \leq 0$, the Dirichlet problem is correct.

Remark 4. Using the method applied in [10] (see pages 58, 68–74), it is not difficult to verify that the theorem is also true for Hölder continuous c and a_α, $\alpha = 1, 2$, on $\overline{\omega}$, provided:

(i) $\lim_{x_2 \to 0_+} x_2^{1-m_2} a_2(x_1, x_2) = a_0 = const < 1$ for $(x_1, 0) \in \omega_0$ when $0 \leq m_2 < 1$;

(ii) if $a_2(x_1^0, 0) = 0$ for a fixed $(x_1^0, 0) \in \omega_0$ when $1 < m_2 < 2$, then there exists such a $\delta = const > 0$ that $a_2(x_1^0, x_2) = \tilde{a}_2(x_1^0, x_2) x_2$ with bounded $\tilde{a}_2(x_1^0, x_2)$ for $0 \leq x_2 < \delta$.

For the sake of transparency of revealing the peculiarities of well-posedness of non-classical, in general, BCs let us consider the case

$$\begin{aligned} & h = h(x_2); \ \eta_{10} = \eta_{10}(x_2, t), \qquad \eta_{20} = \eta_{20}(x_2, t), \qquad v_{30} = v_{30}(x_2, t), \\ & x_2 \in [0, L], \quad L = const. \end{aligned} \tag{37}$$

From (31) we have

$$(\mu + \tilde{\alpha})(h v_{30,2})_{,2} - 2\tilde{\alpha}(h\eta_{10})_{,2} + X_3^0 = \rho h \ddot{v}_{30}(x_2, t). \tag{38}$$

From (34) we get

$$(\nu + \tilde{\beta})(h\eta_{10,2})_{,2} + 2\tilde{\alpha} h v_{30,2} - 4\tilde{\alpha} h \eta_{10} + \chi_1^0 = \mathcal{I} h \ddot{\eta}_{10}. \tag{39}$$

From (35) we obtain

$$(\nu + \tilde{\beta})(h\eta_{20,2})_{,2} + \varepsilon(h\eta_{20,2})_{,2} + (\nu - \tilde{\beta})(h\eta_{20,2})_{,2} - 4\tilde{\alpha} h \eta_{20} + \chi_2^0 = \mathcal{I} h \ddot{\eta}_{20},$$

whence,

$$(2\nu + \varepsilon)(h\eta_{20,2})_{,2} - 4\tilde{\alpha} h \eta_{20} + \chi_2^0 = \mathcal{I} h \ddot{\eta}_{20}. \tag{40}$$

In the static case, determining $h v_{30,2}$ from (39) and substituting into (38), we get

$$\begin{aligned} & -\frac{(\mu + \tilde{\alpha})(\nu + \tilde{\beta})}{2\tilde{\alpha}} (h\eta_{10,2})_{,22} + 2(\mu + \tilde{\alpha})(h\eta_{10})_{,2} \\ & -2\tilde{\alpha}(h\eta_{10})_{,2} - \frac{\mu + \tilde{\alpha}}{2\tilde{\alpha}} \chi_{1,2}^0 + X_3^0 = 0, \end{aligned}$$

i.e.,

$$-\frac{(\mu+\tilde{\alpha})(\nu+\tilde{\beta})}{2\tilde{\alpha}}(h\eta_{10,2})_{,22}+2\mu(h\eta_{10})_{,2}-\frac{\mu+\tilde{\alpha}}{2\tilde{\alpha}}\chi^0_{1,2}+X^0_3=0. \tag{41}$$

From (41) after integration we find

$$(h\eta_{10,2})_{,2}-\frac{4\tilde{\alpha}\mu}{(\mu+\tilde{\alpha})(\nu+\tilde{\beta})}h\eta_{10} +\frac{1}{\nu+\tilde{\beta}}\chi^0_1-\frac{2\tilde{\alpha}}{(\mu+\tilde{\alpha})(\nu+\tilde{\beta})}\int_{x_2^0}^{x_2}X^0_3(\tau)d\tau+c_1=0,\quad c_1=const. \tag{42}$$

Whence we derive

$$\eta_{10,22}+\frac{h_{,2}}{h}\eta_{10,2}-\frac{4\tilde{\alpha}\mu}{(\mu+\tilde{\alpha})(\nu+\tilde{\beta})}\eta_{10} +\frac{1}{\nu+\tilde{\beta}}h^{-1}\chi^0_1-\frac{2\tilde{\alpha}}{(\mu+\tilde{\alpha})(\nu+\tilde{\beta})}h^{-1}\int_{x_2^0}^{x_2}X^0_3(\tau)d\tau+c_1h^{-1}=0. \tag{43}$$

From (40) we have

$$(2\nu+\varepsilon)(h\eta_{20,2})_{,2}-4\tilde{\alpha}h\eta_{20}+\chi^0_2=0. \tag{44}$$

So, under assumptions (37) for $\eta_{10},\ \eta_{20},\ v_{30}$, according to Remark 4, from (43) (i.e., (42)), (44) and (38) it follows that if

$$h(x_2)=h_0x_2^{\kappa},\ x_2\in[0,L],\ h_0,\kappa=const>0,\ L=const, \tag{45}$$

the Dirichlet type problem is well-posed when $\kappa<1$, *i.e.,* $\int_0^L h^{-1}(t)dt<+\infty$, *the Keldysh type problem is well-posed when* $\kappa\geq 1$, *i.e.,* $\int_0^L h^{-1}(t)dt=+\infty$.

Let now

$$h=h(x_2),\qquad v_{10}=v_{10}(x_2,t),\qquad v_{20}=v_{20}(x_2,t),\qquad \eta_{30}=\eta_{30}(x_2,t), \quad x_2\in[0,L],\quad L=const. \tag{46}$$

For $\alpha=1$ from (30) we have

$$(\mu+\tilde{\alpha})(hv_{10,2})_{,2}+2\tilde{\alpha}(h\eta_{30})_{,2}+X^0_1=\rho h\ddot{v}_{10}. \tag{47}$$

For $\alpha=2$ from (30) we have

$$(\mu+\tilde{\alpha})(hv_{20,2})_{,2}+\lambda(hv_{20,2})_{,2}+(\mu-\tilde{\alpha})(hv_{20,2})_{,2}+X^0_2=\rho h\ddot{v}_{20},$$

i.e.,

$$(\lambda+2\mu)(hv_{20,2})_{,2}+X^0_2=\rho h\ddot{v}_{20}. \tag{48}$$

From (36) we have

$$(\nu+\tilde{\beta})(h\eta_{30,2})_{,2}-2\tilde{\alpha}hv_{10,2}-4\tilde{\alpha}h\eta_{30}+\chi^0_3=\mathcal{I}h\ddot{\eta}_{30}. \tag{49}$$

In the static case from (49) we get

$$hv_{10,2}=\frac{\nu+\tilde{\beta}}{2\tilde{\alpha}}(h\eta_{30,2})_{,2}-2h\eta_{30}+\frac{1}{2\tilde{\alpha}}\chi^0_3. \tag{50}$$

Substituting (50) into (47) for the static case, we obtain

$$\frac{(\mu+\tilde{\alpha})(\nu+\tilde{\beta})}{2\tilde{\alpha}}(h\eta_{30,2})_{,22}-2\mu(h\eta_{30})_{,2}+\frac{(\mu+\tilde{\alpha})}{2\tilde{\alpha}}\chi^0_{3,2}+X^0_1=0.$$

Therefore, after integration we find

$$(h\eta_{30,2})_{,2} - \frac{4\mu\tilde{\alpha}}{(\mu+\tilde{\alpha})(\nu+\tilde{\beta})}h\eta_{30} + \frac{1}{\nu+\tilde{\beta}}\chi_3^0 + \frac{2\tilde{\alpha}}{(\mu+\tilde{\alpha})(\nu+\tilde{\beta})}\int_{x_2^0}^{x_2} X_1^0(\tau)d\tau + c_2 = 0, \quad x_2^0 \in \,]0, L[. \tag{51}$$

So, in the case (46), according to Remark 4, from (47), (48), (51) we arrive at the same (see the previous case (37)) result for $v_{\alpha0}$, $\alpha = 1, 2$, η_{30} in the sense of setting BCs.

Note, that cusped edge does not affect on setting initial conditions (ICs), and they remain classical, since on the line $t = 0$ Eqs. (30), (31), (34)–(36) do not degenerate.

Let us consider a particular static case, when

$$v_{i0} \equiv 0, \qquad \eta_{i0} \not\equiv 0, \quad i = 1, 2, 3.$$

It means that points of the shell under consideration do not displace, while they possess microrotations. As it follows from (30), (31), such a state is realizable only if

$$X_\alpha^0 = -2\widetilde{\alpha} \in_{3\alpha\beta}\Big(h\eta_{30}\Big)_{,\beta}, \quad \alpha = 1, 2,$$

$$X_3^0 = 2\widetilde{\alpha}\Big[(h\eta_{10})_{,2} - (h\eta_{20})_{,1}\Big],$$

(i.e., applied surface forces and volume forces should be chosen appropriately). In this case, from (34), (35), (36) it follows that the governing equations have the following form

$$(\nu+\widetilde{\beta})\Big(h\eta_{10,\beta}\Big)_{,\beta} + \varepsilon\Big(h\eta_{\gamma0,\gamma}\Big)_{,1} + (\nu-\widetilde{\beta})\Big(h\eta_{\beta0,1}\Big)_{,\beta} - 4\widetilde{\alpha}h\eta_{10} + \chi_1^0 = 0, \tag{52}$$

$$(\nu+\widetilde{\beta})\Big(h\eta_{20,\beta}\Big)_{,\beta} + \varepsilon\Big(h\eta_{\gamma0,\gamma}\Big)_{,2} + (\nu-\widetilde{\beta})\Big(h\eta_{\beta0,2}\Big)_{,\beta} - 4\widetilde{\alpha}h\eta_{20} + \chi_2^0 = 0, \tag{53}$$

$$(\nu+\widetilde{\beta})\Big(h\eta_{30,\beta}\Big)_{,\beta} - 4\widetilde{\alpha}h\eta_{30} + \chi_3^0 = 0. \tag{54}$$

If the thickness $2h$ has the form (45) and ω is adjacent to the x_1-axis, then the cusped edge ω_0 is a segment of the x_1-axis, where Eq. (54) has the order degeneration and Theorem 2 can be applied. Namely, according to Remark 4 *for* $\kappa < 1$ *the Dirichlet and for* $\kappa \geq 1$ *the Keldysh problems are well-posed*.

If at the boundary couple stress vector should be prescribed, then at the cusped edge, by virtue of (28), (29), it leads to the weighted (with the weight h) Neuman type BC with respect to the microrotations.

For investigation of the system (52), (53) the approach developed in [11,12] can be used. The results concerning peculiarities of setting BCs will be similar to the results for Eq. (54).

4. Conclusion

1. Investigation of BVPs (static problems) and IBVPs (dynamical problems) for micropolar elastic cusped (tapered) prismatic shells leads to study of BVPs and IBVPs for the corresponding elliptic and hyperbolic PDEs and systems of PDEs with the order degeneration on a part corresponding to the cusped edge. Therefore, problems under consideration are non-classical, in general.

2. If at boundary displacements and microrotations should be prescribed, then in certain cases the cusped edge should be freed from BCs at all.

3. If at boundary the traction and couple stress vector should be prescribed, then at the cusped edge it leads to the weighted Neuman problems with respect to the displacements and microrotations. The displacements and microrotations are unbounded, in general, in a neighbourhood of the cusped edge.

4. The peculiarities of setting BCs at cusped edges for the displacements and microrotations are exactly the same.

5. Presence of cusped edges does not affect on setting ICs; ICs remain classical.

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Original article

The second Darboux problem for the wave equation with integral nonlinearity

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Abstract

For a one-dimensional wave equation with integral nonlinearity, the second Darboux problem is considered for which the questions on the existence and uniqueness of a global solution are investigated.

Keywords: Darboux problem; Integral nonlinearity; Wave equation; Global solvability

1. Statement of the problem

In a plane of independent variables x and t we consider the wave equation with integral nonlinearity of the type

$$L_\lambda u := u_{tt} - u_{xx} + \lambda g\left(x, t, u, \int_{\alpha(t)}^{\beta(t)} u(x,t)dx\right) = f(x,t), \tag{1.1}$$

where $\lambda \neq 0$ is the given real constant; g, α, β and f are the given and u is an unknown real functions of their arguments.

By $D_T := \{(x,t) \in \mathbb{R}^2 : -\widetilde{k}_2 t < x < \widetilde{k}_1 t, 0 < t < T; 0 < \widetilde{k}_i := \text{const} < 1, i = 1, 2\}$ we denote a triangular domain lying inside of a characteristic angle $\Lambda := \{(x,t) \in \mathbb{R}^2 : t > |x|\}$ and bounded by the segments $\widetilde{\gamma}_{1,T} : x = \widetilde{k}_1 t, 0 \le t \le T$, $\widetilde{\gamma}_{2,T} : x = -\widetilde{k}_2 t, 0 \le t \le T$ and $\widetilde{\gamma}_{3,T} : t = T, -\widetilde{k}_2 T \le x \le \widetilde{k}_1 T$. For $T = +\infty$, $D_\infty := \{(x,t) \in \mathbb{R}^2 : -\widetilde{k}_2 t < x < \widetilde{k}_1 t, 0 < t < +\infty\}$ (Fig. 1.1).

For Eq. (1.1), let us consider the second Darboux problem on finding in the domain D_T a solution $u(x,t)$ of the above equation by the boundary conditions (see e.g., [1, p. 107]; [2, p. 228])

$$u|_{\widetilde{\gamma}_{i,T}} = 0, \quad i = 1, 2. \tag{1.2}$$

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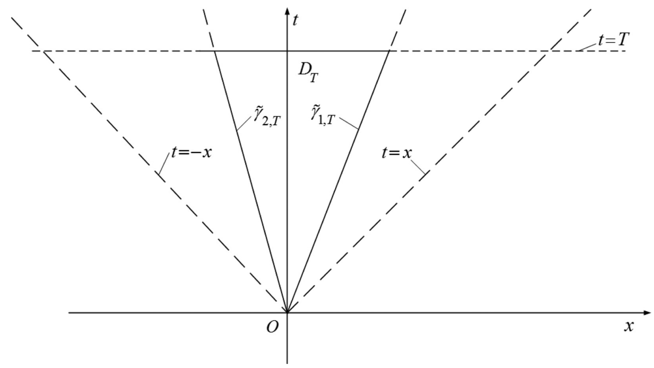

Fig. 1.1.

Below, when investigating problem (1.1), (1.2) it will be assumed that

$$-\widetilde{k}_2\, t \le \alpha(t) < \beta(t) \le \widetilde{k}_1\, t, \quad 0 < t < \infty. \tag{1.3}$$

For linear hyperbolic equations of second order with one spatial variable, a great number of works were devoted to the questions of the well-posedness of the Darboux problem (see, e.g., [2,3] and references therein). As it turned out, the presence of a weak nonlinearity in the equation affects the correctness of formulation even in the case of the first Darboux problem (see, e.g., [4–10]). Note that hyperbolic equations with nonlocal nonlinearities of type (1.1) have been considered in many works (see, e.g., [11–14] and references therein). In the present work it is shown that under definite conditions on the growth of nonlinear function $g = g(x, t, s_1, s_2)$ with respect to the variables s_1, s_2 the second Darboux problem (1.1), (1.2) is globally solvable.

Definition 1.1. Let $\alpha, \beta \in C([0, T])$, $g \in C(\overline{D}_T \times \mathbb{R}^2)$, $f \in C(\overline{D}_T)$. The function u is said to be a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T if $u \in C(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ such that $u_n \to u$ and $L_\lambda u_n \to f$ in the space $C(\overline{D}_T)$, as $n \to \infty$, where $\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T) := \{v \in C^2(\overline{D}_T) : v|_{\Gamma_T} = 0\}$, $\Gamma_T := \widetilde{\gamma}_{1,T} \cup \widetilde{\gamma}_{2,T}$.

Remark 1.1. Note that two different approximations with given properties define the same function in Definition 1.1. Obviously, the classical solution of problem (1.1), (1.2) from the space $\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ is a strong generalized solution of that problem of the class C in the domain D_T. In its turn, if a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T belongs to the space $C^2(\overline{D}_T)$, then it will be a classical solution of that problem, as well.

Definition 1.2. Let $\alpha, \beta \in C([0, \infty))$, $g \in C(\overline{D}_\infty \times \mathbb{R}^2)$, $f \in C(\overline{D}_\infty)$. We say that problem (1.1), (1.2) is globally solvable in the class C, if for any finite $T > 0$, this problem has a strong generalized solution of the class C in the domain D_T.

2. An a priori estimate of solution of problem (1.1), (1.2)

Let us consider the following condition imposed on the function g:

$$\big|g(x, t, s_1, s_2)\big| \le a + b|s_1| + c|s_2|, \quad (x, t, s_1, s_2) \in \overline{D}_T \times \mathbb{R}^2, \tag{2.1}$$

where $a, b, c = \text{const} \ge 0$.

Lemma 2.1. *Let the condition* (2.1) *be fulfilled. Then for a strong generalized solution of problem* (1.1), (1.2) *of the class C in the domain D_T the following a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \le c_1 \|f\|_{C(\overline{D}_T)} + c_2 \tag{2.2}$$

with nonnegative constants c_i, $i = 1, 2$, independent of u and f, is valid.

Proof. Let u be a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T. Then by virtue of Definition 1.1, there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|L_\lambda u_n - f\|_{C(\overline{D}_T)} = 0. \tag{2.3}$$

Denote

$$f_n := L_\lambda u_n. \tag{2.4}$$

Multiplying both parts of equality (2.4) by u_{nt} and integrating with respect to the domain $D_\tau := \{(x,t) \in D_T : t < \tau\}$, $0 < \tau \le T$, we obtain

$$\frac{1}{2}\int_{D_\tau} (u_{nt}^2)_t\, dx\, dt - \int_{D_\tau} u_{nxx} u_{nt}\, dx\, dt + \lambda \int_{D_\tau} g\Big(x,t,u_n, \int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big) u_{nt}\, dx\, dt = \int_{D_\tau} f_n u_{nt}\, dx\, dt.$$

Assume $\omega_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \le T$. Then taking into account that $u_n|_{\Gamma_T} = 0$, the integration by parts of the left-hand side of the last equality yields

$$\begin{aligned} 2\int_{D_\tau} f_n u_{nt}\, dx\, dt &= \int_{\Gamma_\tau} \frac{1}{\nu_t}\big[(u_{nx}\nu_t - u_{nt}\nu_x)^2 + u_{nt}^2(\nu_t^2 - \nu_x^2)\big]ds \\ &\quad + \int_{\omega_\tau} (u_{nx}^2 + u_{nt}^2)dx + 2\lambda \int_{D_\tau} g\Big(x,t,u_n, \int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big) u_{nt}\, dx\, dt, \end{aligned} \tag{2.5}$$

where $\nu := (\nu_x, \nu_t)$ is the unit vector of the outer normal to ∂D_τ, and $\Gamma_\tau := \Gamma_T \cap \{t \le \tau\}$.

Taking into account that $\nu_t \frac{\partial}{\partial x} - \nu_x \frac{\partial}{\partial t}$ is the inner differential operator on Γ_T and $u_n|_{\Gamma_T} = 0$, we have

$$(u_{nx}\nu_t - u_{nt}\nu_x)\big|_{\Gamma_\tau} = 0. \tag{2.6}$$

Since $D_\tau : -\widetilde{k}_2 t < x < \widetilde{k}_1 t$, $t < \tau$, it is easy to see that

$$(\nu_t^2 - \nu_x^2)\big|_{\Gamma_\tau} < 0, \quad \nu_t\big|_{\Gamma_\tau} < 0. \tag{2.7}$$

Bearing in mind (2.6) and (2.7), from (2.5) we obtain

$$w_n(\tau) := \int_{\omega_\tau} (u_{nx}^2 + u_{nt}^2)dx \le 2\int_{D_\tau} f_n u_{nt} dx\, dt - 2\lambda \int_{D_\tau} g\Big(x,t,u_n, \int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big) u_{nt}\, dx\, dt. \tag{2.8}$$

In view of (2.1), we have

$$\begin{aligned} \Big| g\Big(x,t,u_n, \int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big) u_{nt}\Big| &\le \Big(a + b|u_n| + c\Big|\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big|\Big)|u_{nt}| \\ &\le \frac{1}{2}\Big(a + b|u_n| + c\Big|\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big|\Big)^2 + \frac{1}{2}u_{nt}^2 \\ &\le \frac{3}{2}a^2 + \frac{3}{2}b^2u_n^2 + \frac{3}{2}c^2\Big(\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big)^2 + \frac{1}{2}u_{nt}^2. \end{aligned} \tag{2.9}$$

If $(x,t) \in \overline{D}_T$, then owing to (1.3), $u_n|_{\Gamma_T} = 0$ and Schwartz inequality, we have

$$\begin{aligned} |u_n(x,t)| &= \Big|u_n(-\widetilde{k}_2 t, t) + \int_{-\widetilde{k}_2 t}^{x} u_{nx}(s,t)ds\Big| = \Big|\int_{-\widetilde{k}_2 t}^{x} u_{nx}(s,t)ds\Big| \\ &\le \Big(\int_{-\widetilde{k}_2 t}^{x} 1^2 ds\Big)^{\frac{1}{2}} \Big(\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds\Big)^{\frac{1}{2}} \le \sqrt{2t}\Big(\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds\Big)^{\frac{1}{2}}, \end{aligned} \tag{2.10}$$

$$\Big(\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big)^2 \le \int_{\alpha(t)}^{\beta(t)} 1^2 dx \int_{\alpha(t)}^{\beta(t)} u_n^2(x,t)dx \le 2t \int_{\alpha(t)}^{\beta(t)} u_n^2(x,t)dx. \tag{2.11}$$

It follows from (2.8), (2.10) and (2.11) that

$$\Big|\int_{\alpha(t)}^{\beta(t)} u_n^2(x,t)dx\Big| \le 2t\int_{\alpha(t)}^{\beta(t)}\Big[2t\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds\Big]dx$$
$$\le (2t)^2\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t}dx\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t} u_{nx}^2(s,t)ds = 4t^3(\widetilde{k}_1+\widetilde{k}_2)\int_{\omega_t} u_{nx}^2dx \le 8\,t^3\int_{\omega_t}(u_{nx}^2+u_{nt}^2)dx = 8\,t^3 w_n(t),$$

whence we get

$$\int_{D_\tau}\Big|\int_{\alpha(t)}^{\beta(t)} u_n(\xi,t)d\xi\Big|^2 dx\,dt = \int_0^\tau dt\int_{\omega_t}\Big|\int_{\alpha(t)}^{\beta(t)} u_n(\xi,t)d\xi\Big|^2 dx \le \int_0^\tau dt\int_{\omega_t} 8\,t^3 w_n(t)dx$$
$$= \int_0^\tau 8\,t^3 w_n(t)\,\mathrm{mes}\,\omega_t dt \le 16\,\tau^4\int_0^\tau w_n(t)dt. \tag{2.12}$$

From (2.9) and (2.12), we now obtain

$$\int_{D_\tau} g\Big(x,t,u_n,\int_{\alpha(t)}^{\beta(t)} u_n(x,t)dx\Big)u_{nt}dx\,dt \le \frac{3}{2}a^2\,\mathrm{mes}\,D_\tau + \frac{3}{2}b^2\int_{D_\tau} u_n^2dx\,dt$$
$$+\Big(24\,c^2\tau^4+\frac{1}{2}\Big)\int_0^\tau w_n(t)dt. \tag{2.13}$$

Further, in view of (2.10), we have

$$\int_{D_\tau} u_n^2dx\,dt = \int_0^\tau dt\int_{\omega_t} u_n^2(x,t)dx \le \int_0^\tau dt\int_{\omega_t}\Big(2\,t\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds\Big)dx$$
$$\le \int_0^\tau dt\int_{\omega_t}\Big(2\,t\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t} u_{nx}^2(s,t)ds\Big)dx \le \int_0^\tau \mathrm{mes}\,\omega_t\Big(2\,t\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t} u_{nx}^2(s,t)ds\Big)dt$$
$$\le 4\,\tau^2\int_0^\tau dt\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t} u_{nx}^2(s,t)ds = 4\,\tau^2\int_0^\tau dt\int_{\omega_t} u_{nx}^2dx$$
$$\le 4\,\tau^2\int_0^\tau dt\int_{\omega_t}(u_{nx}^2+u_{nt}^2)dx = 4\,\tau^2\int_0^\tau w_n(t)dt. \tag{2.14}$$

Taking into account (2.13), (2.14) and the fact that $\mathrm{mes}\,D_\tau \le \tau^2 \le T^2$, $2f_nu_{nt} \le u_{nt}^2+f_n^2$, as well as

$$\int_{D_\tau} u_{nt}^2dx\,dt \le \int_0^\tau w_n(t)dt,$$

from (2.8) we get

$$w_n(\tau) \le |\lambda|\Big(3\,a^2T^2 + 12\,b^2T^2\int_0^\tau w_n(t)dt + 48\,c^2T^4\int_0^\tau w_n(t)dt + \int_0^\tau w_n(t)dt\Big) + \int_0^\tau w_n(t)dt$$
$$+\|f_n\|^2_{L_2(D_T)} \le \Big[|\lambda|\big(12\,b^2T^2+48\,c^2T^4+1\big)+1\Big]\int_0^\tau w_n(t)dt + 3|\lambda|a^2T^2$$
$$+\|f_n\|^2_{L_2(D_T)},\quad 0<\tau\le T.$$

Hence according to the Gronwall's lemma, it follows that

$$w_n(\tau) \le \Big(3|\lambda|a^2T^2+\|f_n\|^2_{L_2(D_T)}\Big)\exp\Big(T\Big[|\lambda|\big(12\,b^2T^2+48\,c^2T^4+1\big)+1\Big]\Big),\quad 0<\tau\le T. \tag{2.15}$$

If $(x,t)\in\overline{D}_T$, then owing to (2.8), (2.10) and (2.15), we have

$$\big|u_n(x,t)\big|^2 \le 2\,t\int_{-\widetilde{k}_2 t}^{x} u_{nx}^2(s,t)ds \le 2\,T\int_{-\widetilde{k}_2 t}^{\widetilde{k}_1 t}\big(u_{nx}^2+u_{nt}^2\big)dx = 2\,T\,w_n(t)$$
$$\le 2\,T\big(3|\lambda|a^2T^2+\|f_n\|^2_{L_2(D_T)}\big)\exp\Big(T\Big[|\lambda|\big(12\,b^2T^2+48\,c^2T^4+1\big)+1\Big]\Big).$$

This implies that

$$\|u_n\|_{C(\overline{D}_T)} \le c_1 \|f_n\|_{C(\overline{D}_T)} + c_2, \tag{2.16}$$

where

$$\begin{aligned} c_1 &= \sqrt{2T}\, \exp\Big(\frac{T}{2}\Big[|\lambda|\big(12\, b^2T^2 + 48\, c^2T^4 + 1\big) + 1\Big]\Big), \\ c_2 &= a\, T\sqrt{6T|\lambda|}\, \exp\Big(\frac{T}{2}\Big[|\lambda|\big(12\, b^2T^2 + 48\, c^2T^4 + 1\big) + 1\Big]\Big). \end{aligned} \tag{2.17}$$

By virtue of (2.3), passing in inequality (2.16) to the limit, as $n \to \infty$, we obtain the estimate (2.2) which proves Lemma 2.1. □

Remark 2.1. If in inequality (2.1) the number $a = 0$, then in the a priori estimate (2.2) the value $c_2 = 0$. In this case estimate (2.2) takes the form

$$\|u\|_{C(\overline{D}_T)} \le c_1 \|f\|_{C(\overline{D}_T)},$$

hence from $f = 0$ it follows that $u = 0$, which in a linear case implies the uniqueness of a solution of problem (1.1), (1.2).

3. Equivalent reduction of problem (1.1), (1.2) to a nonlinear integral equation of Volterra type

In new independent variables $\xi = \frac{1}{2}(t + x)$, $\eta = \frac{1}{2}(t - x)$ the domain D_T will go over to a triangular domain G_T with vertices at the points $O(0, 0)$, $Q_1\big(\frac{1+\widetilde{k}_1}{2}\, T, \frac{1-\widetilde{k}_1}{2}\, T\big)$, $Q_2\big(\frac{1-\widetilde{k}_2}{2}\, T, \frac{1+\widetilde{k}_2}{2}\, T\big)$ of the plane of variables ξ, η, and problem (1.1), (1.2) will go over to the problem

$$\widetilde{L}_\lambda v := v_{\xi\eta} + \lambda\, K\, v = \widetilde{f}(\xi, \eta), \quad (\xi, \eta) \in G_T, \tag{3.1_λ}$$

$$v\big|_{\gamma_{i,T}} = 0, \quad \gamma_{i,T} := O\, Q_i, \quad i = 1, 2, \tag{3.2_λ}$$

with respect to a new unknown function $v(\xi, \eta) := u(\xi - \eta, \xi + \eta)$; $\widetilde{f}(\xi, \eta) := f(\xi - \eta, \xi + \eta)$.

Here, the operator K acts by the formula

$$(K\, v)(\xi, \eta) = g\bigg(\xi - \eta, \xi + \eta, v, \int_{\alpha(\xi+\eta)}^{\beta(\xi+\eta)} v(\xi, \eta) d\xi - v(\xi, \eta) d\eta\bigg), \tag{3.3}$$

$$\begin{aligned} \gamma_{1,T} &: \eta = k_1\xi, \quad 0 \le \xi \le \xi_0 := 2^{-1}(1 + \widetilde{k}_1)T, \\ \gamma_{2,T} &: \xi = k_2\eta, \quad 0 \le \eta \le \eta_0 := 2^{-1}(1 + \widetilde{k}_2)T, \end{aligned} \tag{3.4}$$

$$0 < k_i := \frac{1 - \widetilde{k}_i}{1 + \widetilde{k}_i} < 1, \quad i = 1, 2. \tag{3.5}$$

Analogously to Definition 1.1, we introduce the notion of a strong generalized solution v of problem (3.1_λ), (3.2_λ) of the class C in the domain G_T.

If $P_0(\xi, \eta) \in G_T$, we denote by $P_1M_0P_0N_0$ a rectangle, characteristic with respect to Eq. (3.1_λ) whose vertices N_0 and M_0 lie, respectively, on the segments $\gamma_{1,T}$ and $\gamma_{2,T}$, that is, by virtue of (3.4): $N_0 := (\xi, k_1\xi)$, $M_0 := (k_2\eta, \eta)$, $P_1 := (k_2\eta, k_1\xi)$. Since $P_1 \in G_T$, we construct analogously the characteristic rectangle $P_2M_1P_1N_1$ whose vertices N_1 and M_1 lie, respectively, on the segments $\gamma_{1,T}$ and $\gamma_{2,T}$. Continuing this process, we obtain the characteristic rectangle $P_{i+1}M_iP_iN_i$ for which $N_i \in \gamma_{1,T}$, $M_i \in \gamma_{2,T}$, and $N_i := (\xi_i, k_1\xi_i)$, $M_i := (k_2\eta_i, \eta_i)$, $P_{i+1} := (k_2\eta_i, k_1\xi_i)$ if $P_i := (\xi_i, \eta_i)$, $i > 0$ (Fig. 3.1).

It is not difficult to see that

$$\begin{aligned} P_{2n} &= \big((k_1k_2)^n\xi, (k_1k_2)^n\eta\big), & P_{2n+1} &= \big((k_1k_2)^n k_2\eta, (k_1k_2)^n k_1\xi\big), \quad n = 0, 1, 2, \ldots, \\ M_{2n} &= \big((k_1k_2)^n k_2\eta, (k_1k_2)^n\eta\big), & M_{2n+1} &= \big((k_1k_2)^{n+1}\xi, (k_1k_2)^n k_1\xi\big), \quad n = 0, 1, 2, \ldots, \\ N_{2n} &= \big((k_1k_2)^n\xi, (k_1k_2)^n k_1\xi\big), & N_{2n+1} &= \big((k_1k_2)^n k_2\eta, (k_1k_2)^{n+1}\eta\big), \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{3.6}$$

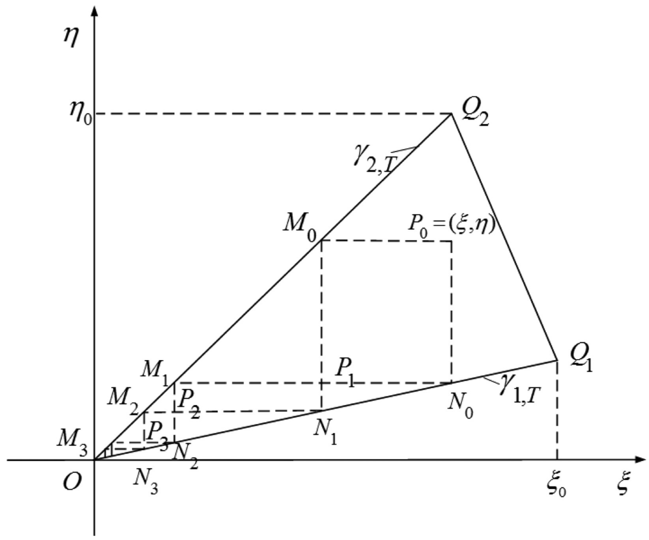

Fig. 3.1.

Consider first a linear case, i.e., when in problem (3.1_λ), (3.2_λ) the parameter $\lambda = 0$. If v is a strong generalized solution of problem (3.1_0), (3.2_0) of the class C in the domain G_T, then considering the function v as a solution of the Goursat problem for equation (3.1_0), in the rectangle $P_{i+1}M_iP_iN_i$ with data on characteristic segments $P_{i+1}N_i$ and $P_{i+1}M_i$, we have (see, e.g., [15, p. 173]),

$$v(P_i) = v(M_i) + v(N_i) - v(P_{i+1}) + \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1, \quad i = 0, 1, \ldots.$$

Thus, by virtue of equality (3.2_0), it follows that

$$\begin{aligned} v(\xi,\eta) &= v(P_0) = v(M_0) + v(N_0) - v(P_1) + \int_{P_1M_0P_0N_0} \widetilde{f}\, d\xi_1\, d\eta_1 \\ &= -v(P_1) + \int_{P_1M_0P_0N_0} \widetilde{f}\, d\xi_1\, d\eta_1 \\ &= -v(M_1) - v(N_1) + v(P_2) - \int_{P_2M_1P_1N_1} \widetilde{f}\, d\xi_1\, d\eta_1 + \int_{P_1M_0P_0N_0} \widetilde{f}\, d\xi_1\, d\eta_1 \\ &= v(P_2) - \int_{P_2M_1P_1N_1} \widetilde{f}\, d\xi_1\, d\eta_1 \\ &\quad + \int_{P_1M_0P_0N_0} \widetilde{f}\, d\xi_1\, d\eta_1 = \cdots = (-1)^n v(P_n) \\ &\quad + \sum_{i=0}^{n-1} (-1)^i \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1, \quad (\xi,\eta) \in G_T. \end{aligned} \tag{3.7}$$

Since the point P_n from (3.7) tends to the point $O(0,0)$, as $n \to \infty$, by (3.2_0), we have $\lim_{n\to\infty} v(P_n) = 0$. Hence, passing in equality (3.7) to the limit, as $n \to \infty$, for a strong generalized solution v of problem (3.1_0), (3.2_0) of the class C in the domain G_T, we obtain the following integral representation:

$$v(\xi,\eta) = \sum_{i=0}^{\infty} (-1)^i \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1, \quad (\xi,\eta) \in G_T. \tag{3.8}$$

Remark 3.1. Since $\widetilde{f} \in C(\overline{G}_T)$ and there take place inequalities (3.5), and moreover, owing to (3.6),

$$\text{mes}\, P_{i+1}M_iP_iN_i = (k_1k_2)^i(\xi - k_2\eta)(\eta - k_1\xi), \tag{3.9}$$

the series in the right-hand side of equality (3.8) is uniformly and absolutely convergent.

Remark 3.2. From the above reasoning it follows that for any $\widetilde{f} \in C(\overline{G}_T)$, linear problem (3.1_0), (3.2_0) has a unique strong generalized solution v of the class C in the domain G_T which is representable in the form of uniformly and absolutely converging series (3.8).

Introduce into consideration the operator $\widetilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ acting by the formula

$$(\widetilde{L}_0^{-1}\widetilde{f})(\xi, \eta) := \sum_{i=0}^{\infty}(-1)^i \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1, \quad (\xi, \eta) \in G_T. \tag{3.10}$$

Remark 3.3. According to (3.10) and Remark 3.2, a unique strong generalized solution v of problem (3.1_0), (3.2_0) of the class C in the domain G_T is representable in the form $v = \widetilde{L}_0^{-1}\widetilde{f}$, and owing to (3.5), (3.9), we have the estimate

$$\begin{aligned} \left|v(\xi, \eta)\right| &\le \sum_{i=0}^{\infty} \int_{P_{i+1}M_iP_iN_i} |\widetilde{f}|\, d\xi_1\, d\eta_1 \le (\xi + \eta)^2 \|\widetilde{f}\|_{C(\overline{G}_T)} \sum_{i=0}^{\infty}(k_1k_2)^i \\ &\le \frac{2(\xi^2 + \eta^2)}{1 - k_1k_2} \|\widetilde{f}\|_{C(\overline{G}_T)} \le \frac{1 + \widetilde{k}^2}{1 - k_1k_2} T^2 \|\widetilde{f}\|_{C(\overline{G}_T)}, \quad \widetilde{k} := \max\{\widetilde{k}_1, \widetilde{k}_2\}, \end{aligned}$$

whence in its turn it follows that

$$\|\widetilde{L}_0^{-1}\|_{C(\overline{G}_T)\to C(\overline{G}_T)} \le \frac{1 + \widetilde{k}^2}{1 - k_1k_2} T^2. \tag{3.11}$$

Lemma 3.1. *The function $v \in C(\overline{G}_T)$ is a strong generalized solution of problem* (3.1_λ), (3.2_λ) *of the class C in the domain G_T, if and only if this function is a continuous solution of the following nonlinear Volterra type integral equation*

$$v(\xi, \eta) + \lambda(\widetilde{L}_0^{-1}K\, v)(\xi, \eta) = (\widetilde{L}_0^{-1}\widetilde{f})(\xi, \eta), \quad (\xi, \eta) \in G_T. \tag{3.12}$$

Proof. Indeed, let $v \in C(\overline{G}_T)$ be a solution of Eq. (3.12). Since $\widetilde{f} \in C(\overline{G}_T)$, and the space $C^2(\overline{G}_T)$ is dense in $C(\overline{G}_T)$ (see, e.g., [16, p. 37]), there exists a sequence of functions $\widetilde{f}_n \in C^2(\overline{G}_T)$ such that $\widetilde{f}_n \to \widetilde{f}$ in the space $C(\overline{G}_T)$, as $n \to \infty$. Analogously, since $v \in C(\overline{G}_T)$, there exists a sequence of functions $w_n \in C^2(\overline{G}_T)$ such that $w_n \to v$ in the space $C(\overline{G}_T)$, as $n \to \infty$. Assume $v_n := -\lambda\widetilde{L}_0^{-1}Kw_n + \widetilde{L}_0^{-1}\widetilde{f}_n$, $n = 1, 2, \ldots$. Taking into account (3.5), (3.6), (3.9) and (3.10), it is easy to see that $v_n \in C^2(\overline{G}_T)$, and $v_n|_{\gamma_{i,T}} = 0$, $i = 1, 2$. On the one hand, by virtue of estimate (3.1_λ) and equality (3.12), we have $v_n \to -\lambda\widetilde{L}_0^{-1}Kv + \widetilde{L}_0^{-1}\widetilde{f} = v$ in the space $C(\overline{G}_T)$, as $n \to \infty$, i.e., $v_n \to v$ in $C(\overline{G}_T)$, as $n \to \infty$. On the other hand, $\widetilde{L}_0v_n = -\lambda Kw_n + \widetilde{f}_n$, but since $\lim_{n\to\infty} \|v_n - v\|_{C(\overline{G}_T)} = 0$, $\lim_{n\to\infty} \|w_n - v\|_{C(\overline{G}_T)} = 0$ and $\lim_{n\to\infty} \|\widetilde{f}_n - \widetilde{f}\|_{C(\overline{G}_T)} = 0$, in view of (2.3) we have $\widetilde{L}_\lambda v_n = \widetilde{L}_0v_n + \lambda Kv_n = -\lambda Kw_n + \widetilde{f}_n + \lambda Kv_n = -\lambda(Kw_n - Kv) + \lambda(Kv_n - Kv) + \widetilde{f}_n \to \widetilde{f}$ in the space $C(\overline{G}_T)$, as $n \to \infty$. Thus, the function $v \in C(\overline{G}_T)$ is a strong generalized solution of problem (3.1_λ), (3.2_λ) of the class C in the domain G_T. The converse is obvious. □

4. The case of global solvability of problem (1.1), (1.2) in the class of continuous functions

Lemma 4.1. *The operator $\widetilde{L}_0^{-1}$ defined by formula* (3.10) *is the linear continuous operator acting from the space $C(\overline{G}_T)$ to the space $C^1(\overline{G}_T)$.*

Proof. To prove the lemma, we first show that for $\widetilde{f} \in C(\overline{G}_T)$, the series in the right-hand side of (3.10) differentiated formally with respect to ξ and to η converges uniformly on the set $\overline{G}_T$. Indeed, as it can be easily verified, we have

$$\frac{\partial}{\partial \xi}\left[\sum_{i=0}^{\infty}(-1)^i \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1\right] = \sum_{n=0}^{\infty}\left[(k_1k_2)^n \int_{N_{2n}P_{2n}} \widetilde{f}\, d\eta_1 + (k_1k_2)^{n+1}\int_{P_{2n+2}M_{2n+1}} \widetilde{f}\, d\eta_1 - (k_1k_2)^n k_1 \int_{M_{2n+1}N_{2n}} \widetilde{f}\, d\xi_1\right], \tag{4.1}$$

$$\frac{\partial}{\partial \eta}\left[\sum_{i=0}^{\infty}(-1)^i \int_{P_{i+1}M_iP_iN_i} \widetilde{f}\, d\xi_1\, d\eta_1\right] = \sum_{n=0}^{\infty}\left[(k_1k_2)^n \int_{M_{2n}P_{2n}} \widetilde{f}\, d\xi_1 + (k_1k_2)^{n+1}\int_{P_{2n+2}N_{2n+1}} \widetilde{f}\, d\xi_1 - (k_1k_2)^n k_2 \int_{N_{2n+1}M_{2n}} \widetilde{f}\, d\eta_1\right]. \tag{4.2}$$

By virtue of (3.6), the equalities

$$|N_{2n}P_{2n}| = (k_1k_2)^n(\eta - k_1\xi), \quad |P_{2n+2}M_{2n+1}| = (k_1k_2)^n k_1(\xi - k_2\eta), \quad |M_{2n+1}N_{2n}| = (k_1k_2)^n(1-k_1k_2)\xi,$$
$$|M_{2n}P_{2n}| = (k_1k_2)^n(\xi - k_2\eta), \quad |P_{2n+2}N_{2n+1}| = (k_1k_2)^n k_2(\eta - k_1\xi), \quad |N_{2n+1}M_{2n}| = (k_1k_2)^n(1-k_1k_2)\eta,$$

hold, hence with regard for (3.5), it follows that the series (4.1) and (4.2) converge uniformly and absolutely, and we have the estimate

$$\max\left\{\left\|\frac{\partial}{\partial \xi}\,(\widetilde{L}_0^{-1}\widetilde{f})\right\|_{C(\overline{G}_T)}, \left\|\frac{\partial}{\partial \eta}\,(\widetilde{L}_0^{-1}\widetilde{f})\right\|_{C(\overline{G}_T)}\right\} \leq \frac{3}{1-(k_1k_2)^2}\, T\,\|\widetilde{f}\|_{C(\overline{G}_T)}.$$

Thus by virtue of 3.1 and the fact that $\|v\|_{C^1} := \max\{\|v\|_C,\ \|v_\xi\|_C,\ \|v_\eta\|_C\}$, we obtain the assertion of Lemma 4.1. □

Remark 4.1. Since the space $C^1(\overline{G}_T)$ is compactly embedded into $C(\overline{G}_T)$ (see, e.g., [17, p. 135]), the operator $\widetilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ in view of (3.1_λ) and Lemma 4.1 is linear and compact one.

We rewrite Eq. (3.12) in the form

$$v = A\,v := \widetilde{L}_0^{-1}(-\lambda\, K\, v + \widetilde{f}), \tag{4.3}$$

where the operator $A : C(\overline{G}_T) \to C(\overline{G}_T)$ is continuous and compact, since the nonlinear operator $K : C(\overline{G}_T) \to C(\overline{G}_T)$, acting by formula (3.3), is bounded and continuous, whereas the linear operator $\widetilde{L}_0^{-1} : C(\overline{G}_T) \to C(\overline{G}_T)$ is, according to Remark 4.1, compact. At the same time, by Lemmas 2.1 and 3.1, and by equalities (2.17), for an arbitrary parameter $\tau \in [0,1]$ and for any solution $v \in C(\overline{G}_T)$ of equation $v = \tau A v$, the a priori estimate $\|v\|_{C(\overline{G}_T)} \leq c_1\|\widetilde{f}\|_{C(\overline{G}_T)} + c_2$ with the same nonnegative constants c_1 and c_2 as in (2.1), not depending on v, τ and $\widetilde{f}$, is valid. Therefore, by the Leray–Schauder's theorem (see, e.g., [18, p. 375]), Eq. (4.3) under the condition of Lemma 2.1 has at least one solution $v \in C(\overline{G}_T)$. Thus, owing to Lemma 3.1, we have proved the following.

Theorem 4.1. *Let $\alpha, \beta \in C([0,T])$, $g \in C(\overline{D}_T \times \mathbb{R}^2)$, $f \in C(\overline{D}_T)$ and condition* (2.1) *be fulfilled. Then problem* (1.1), (1.2) *has at least one strong generalized solution of the class C in the domain D_T in the sense of Definition* 1.1.

Corollary 4.1. *Let $\alpha, \beta \in C([0,\infty])$, $g \in C(\overline{D}_\infty \times \mathbb{R}^2)$, $f \in C(\overline{D}_\infty)$ and condition* (2.1) *for $(x,t) \in \overline{D}_\infty$ be fulfilled. Then problem* (1.1), (1.2) *is globally solvable in the class C in the sense of Definition* 1.2.

5. The smoothness and uniqueness of a solution of problem (1.1), (1.2). The existence of a global solution in D_∞

From equalities (3.12), (4.1), (4.2), by Lemmas 3.1 and 4.1 we immediately have

Lemma 5.1. *Let u be a strong generalized solution of problem* (1.1), (1.2) *of the class C in the domain D_T in the sense of Definition* 1.1. *Then if $\alpha, \beta \in C^1([0,T])$, $g \in C^1(\overline{D}_T \times \mathbb{R}^2)$ and $f \in C^1(\overline{D}_T)$, then $u \in C^2(\overline{D}_T)$.*

Lemma 5.2. *For $g \in C^1(\overline{D}_T \times \mathbb{R}^2)$, problem* (1.1), (1.2) *fails to have more than one strong generalized solution of the class C in the domain D_T.*

Proof. Indeed, assume that problem (1.1), (1.2) has two possible different strong generalized solutions u^1 and u^2 of the class C in the domain D_T. By Definition 1.1, there exists a sequence of functions $u_n^i \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, $i = 1, 2$, such that

$$\lim_{n\to\infty} \|u_n^i - u^i\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|L_\lambda u_n^i - f\|_{C(\overline{D}_T)} = 0, \quad i = 1, 2. \tag{5.1}$$

Assume $v_n := u_n^2 - u_n^1$. It can be easily seen that the function $v_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ is a classical solution of the problem

$$L_0 w_n + \lambda g_n^1 v_n + \lambda g_n^2 \int_{\alpha(t)}^{\beta(t)} v_n dx = f_n, \tag{5.2}$$

$$v_n\big|_{\Gamma_T} = 0. \tag{5.3}$$

Here,

$$\begin{aligned} g_n^1 &:= \int_0^1 g_{s_1}\left[x, t, u_n^1 + s(u_n^2 - u_n^1), \int_{\alpha(t)}^{\beta(t)} u_n^1 dx\right] ds, \\ g_n^2 &:= \int_0^1 g_{s_2}\left[x, t, u_n^2, \int_{\alpha(t)}^{\beta(t)} u_n^1 dx + s\int_{\alpha(t)}^{\beta(t)} (u_n^2 - u_n^1) dx\right] ds, \end{aligned} \tag{5.4}$$

$$f_n := L_\lambda u_n^2 - L_\lambda u_n^1, \tag{5.5}$$

where we have used the following obvious equality

$$\begin{aligned} \varphi(x_2, y_2) - \varphi(x_1, y_1) = {}& (x_2 - x_1) \int_0^1 \varphi_x\big[x_1 + s(x_2 - x_1), y_1\big] ds \\ & + (y_2 - y_1) \int_0^1 \varphi_y\big[x_2, y_1 + s(y_2 - y_1)\big] ds \end{aligned}$$

for the function $\varphi(x, y)$.

Assume

$$A := \left\{(x, t, s_1, s_2) \in \overline{D}_T \times \mathbb{R}^2 : (x, t) \in \overline{D}_T,\ \ |s_1| \le c_1 \|f\|_{C(\overline{D}_T)} + c_2,\ \ |s_2| \le 2\,T\,c_1\big(\|f\|_{C(\overline{D}_T)} + c_2\big)\right\}$$

and

$$B := \max\left\{\|g_{s_1}\|_{C(\overline{A})}, \|g_{s_2}\|_{C(\overline{A})}\right\}. \tag{5.6}$$

Taking into account the a priori estimate (2.2), for the functions u_n^1 and u_n^2, with regard for (5.4)–(5.6), we have

$$\left| g_n^1 v_n + g_n^2 \int_{\alpha(t)}^{\beta(t)} v_n dx \right| \le B\left(|v_n| + \left| \int_{\alpha(t)}^{\beta(t)} v_n dx \right|\right). \tag{5.7}$$

Now, by virtue of (5.7), Lemma 2.1 and Remark 2.1 applied to the case when in inequality (2.1) $a = 0$, $b = B$, $c = B$ for the solution v_n of problem (5.2), (5.3) we have the following estimate:

$$\|v_n\|_{C(\overline{D}_T)} \le \sqrt{2T}\, \exp\left(\frac{T}{2}\Big[|\lambda|\big(12\,B^2T^2 + 48\,B^2T^4 + 1\big) + 1\Big]\right) \|f_n\|_{C(\overline{D}_T)}. \tag{5.8}$$

Since owing to (5.1),

$$\|u_2 - u_1\| = \lim_{n\to\infty} \|v_n\|_{C(\overline{D}_T)}, \quad \lim_{n\to\infty} \|f_n\|_{C(\overline{D}_T)} = 0,$$

therefore passing in estimate (5.8) to the limit, as $n \to \infty$, we obtain

$$\|u_2 - u_1\|_{C(\overline{D}_T)} \leq 0,$$

i.e., $u_1 = u_2$, which contradicts our assumption. Thus Lemma 5.2 is proved. □

Theorem 5.1. *Let* $\alpha, \beta \in C^1([0, +\infty))$, $g \in C^1(\overline{D}_\infty \times \mathbb{R}^2)$ *and condition* (2.1) *be fulfilled. Then for any* $f \in C^1(\overline{D}_\infty)$, *problem* (1.1), (1.2) *has the unique global classical solution* $u \in \overset{\circ}{C}{}^2(\overline{D}_\infty, \Gamma_\infty)$ *in the domain* D_∞.

Proof. If $f \in C^1(\overline{D}_\infty)$ and condition (2.1) is fulfilled, then according to Theorem 4.1 and Lemmas 5.1 and 5.2, in the domain D_T for $T = n$ there exists the unique classical solution $u \in \overset{\circ}{C}{}^2(\overline{D}_n, \Gamma_n)$ of problem (1.1), (1.2). Since u_{n+1} is likewise a classical solution of problem (1.1), (1.2) in the domain D_n, by Lemma 5.2, we have $u_{n+1}|_{D_n} = u_n$. Therefore, the function u constructed in the domain D_∞ by the rule $u(x, t) = u_n(x, t)$ for $n = [t] + 1$, where $[t]$ is integer part of the number t, and $(x, t) \in D_\infty$, will be the unique classical solution of problem (1.1), (1.2) in the domain D_∞ of the class $\overset{\circ}{C}{}^2(\overline{D}_\infty, \Gamma_\infty)$. Thus Theorem 5.1 is proved. □

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Original article

Finite difference scheme for one nonlinear parabolic integro-differential equation

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Abstract

Initial–boundary value problem with mixed boundary conditions for one nonlinear parabolic integro-differential equation is considered. The model is based on Maxwell system describing the process of the penetration of a electromagnetic field into a substance. Unique solvability and asymptotic behavior of solution are fixed. Main attention is paid to the convergence of the finite difference scheme. More wide cases of nonlinearity that already were studied are investigated.

MSC: 65N06; 45K05; 35K55

Keywords: Nonlinear parabolic integro-differential equation; Initial–boundary value problem; Mixed boundary conditions; Finite difference scheme; Convergence

1. Introduction

Integro-differential equations of parabolic type arise in the study of various problems (see, for example, [1–5] and references therein). One such model is obtained at mathematical modeling of processes of electromagnetic field penetration in the substance. It is shown that in the quasi-stationary approximation the corresponding system of Maxwell equations [6] can be rewritten in the following form [7]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rot\,H|^2\,d\tau\right) rot\,H\right], \tag{1.1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that integro-differential models of (1.1) type are complex and still yields to the investigation only for special cases (see, for example, [3,7–20] and references therein).

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Study of the models of type (1.1) has begun in the work [7]. In particular, for the case $a(S) = 1 + S$ the theorems of existence of solution of the first boundary value problem for scalar and one-dimensional space case and uniqueness for more general cases are proved in this work. One-dimensional scalar variant for the case $a(S) = (1 + S)^p$, $0 < p \leq 1$ is studied in [9]. Investigations for multi-dimensional space cases at first are carried out in the work [10]. Multidimensional space cases are also discussed in the following works [14,18].

Asymptotic behavior as $t \to \infty$ of solutions of initial–boundary value problems for (1.1) type models is studied in the works [3,11,14–16] and in a number of other works as well. In these works main attentions, are paid to one-dimensional analogs.

Interest to above-mentioned integro-differential model is more and more arising and initial–boundary value problems with different kinds of boundary and initial conditions are considered. Particular attention should be paid to construction of numerical solutions and to their importance for integro-differential models. Finite element analogs and Galerkin method algorithm as well as settling of semi-discrete and finite difference schemes for (1.1) type one-dimensional integro-differential models are studied in [12,16,20–22] and in the other works as well (see [3] and references therein).

Our main aim is to study finite difference scheme for numerical solution of initial–boundary value problem with mixed boundary conditions for the one-component and one-dimensional analog of (1.1) system. Attention is paid to the investigation of more wide cases of nonlinearity than already were studied.

This article is organized as follows. In Section 2 the formulation of the problem and unique solvability and asymptotic behavior of solution are fixed. Main attention is paid to construction and investigation of finite difference scheme in Section 3. We conclude the paper with some discussions in Section 4.

2. Formulation of the problem. Unique solvability and asymptotic behavior of solution

If the magnetic field has the form $H = (0, 0, U)$, $U = U(x, t)$, then from (1.1) we obtain the following nonlinear integro-differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \tag{2.1}$$

where

$$S(x, t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau. \tag{2.2}$$

In the domain $[0, 1] \times [0, \infty)$ let us consider the following initial–boundary value problem for (2.1), (2.2):

$$U(0, t) = \left.\frac{\partial U(x, t)}{\partial x}\right|_{x=1} = 0, \tag{2.3}$$

$$U(x, 0) = U_0(x), \tag{2.4}$$

where U_0 is a given function.

The study of unique solvability and long-time behavior of solution of the problem (2.1)–(2.4) is actual.

The following statement [13] shows the exponential stabilization of the solution of problem (2.1)–(2.4) in the norm of the space $C^1(0, 1)$.

Theorem 2.1. *If* $a(S) = (1 + S)^p$, $0 < p \leq 1$ *and* $U_0 \in H^3(0, 1)$, $U_0(0) = \left.\frac{dU_0(x)}{dx}\right|_{x=1} = 0$, *then for the solution of problem* (2.1)–(2.4) *the following estimates hold as* $t \to \infty$:

$$\left|\frac{\partial U(x, t)}{\partial x}\right| \leq C \exp\left(-\frac{t}{2}\right), \quad \left|\frac{\partial U(x, t)}{\partial t}\right| \leq C \exp\left(-\frac{t}{2}\right),$$

uniformly in x on [0, 1].

Using the compactness method, a modified version of the Galerkin method [5,23] the unique solvability can be proven.

Let us note that same results are true for problem with first type homogeneous conditions on whole boundary (see, for example, [3,14] and references therein).

3. Finite difference scheme

In $[0, 1] \times [0, T]$ let us consider again problem (2.1)–(2.4) written in the following form:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[\left(1+\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)^p \frac{\partial U}{\partial x}\right], \tag{3.1}$$

$$U(0,t) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = 0, \tag{3.2}$$

$$U(x,0) = U_0(x), \tag{3.3}$$

where $0 < p \le 1$, T is positive number and U_0 is a given function.

On $[0, 1] \times [0, T]$ let us introduce a net with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, \ldots, M$; $j = 0, 1, \ldots, N$ with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i, t_j) is designed by u_i^j and the exact solution to problem (3.1)–(3.3) by U_i^j. We will use the following known notations [24]:

$$u_{x,i}^j = \frac{u_{i+1}^j - u_i^j}{h}, \quad u_{\bar{x},i}^j = \frac{u_i^j - u_{i-1}^j}{h}, \quad u_{t,i}^j = \frac{u_i^{j+1} - u_i^j}{\tau}.$$

Introduce inner product and norm:

$$(u^j, v^j) = h\sum_{i=1}^{M-1} u_i^j v_i^j, \quad \|u^j\| = (u^j, u^j)^{1/2}.$$

For problem (3.1)–(3.3) let us consider the following finite difference scheme:

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \left\{\left(1+\tau\sum_{k=1}^{j+1}(u_{\bar{x},i}^k)^2\right)^p u_{\bar{x},i}^{j+1}\right\}_x = f_i^j, \quad i = 1, 2, \ldots, M-1; \quad j = 0, 1, \ldots, N-1, \tag{3.4}$$

$$u_0^j = u_{\bar{x},M}^j = 0, \quad j = 0, 1, \ldots, N, \tag{3.5}$$

$$u_i^0 = U_{0,i}, \quad i = 0, 1, \ldots, M. \tag{3.6}$$

Multiplying Eq. (3.4) scalarly by u_i^{j+1}, it is not difficult to get the inequality

$$\|u^n\|^2 + \sum_{j=1}^{n} \|u_{\bar{x}}^j\|^2 \tau < C, \quad n = 1, 2, \ldots, N, \tag{3.7}$$

where here and below C is a positive constant independent from τ and h.

The a priori estimate (3.7) guarantee the stability of the scheme (3.4)–(3.6). Note, that it is easy to prove the uniqueness of the solution of the scheme (3.4)–(3.6) too.

The main statement of the present section can be stated as follows.

Theorem 3.1. *If problem* (3.1)–(3.3) *has a sufficiently smooth solution* $U(x,t)$, *then the solution* $u^j = (u_1^j, u_2^j, \ldots, u_M^j)$, $j = 1, 2, \ldots, N$ *of the difference scheme* (3.4)–(3.6) *tends to the solution of continuous problem* (3.1)–(3.3) $U^j = (U_1^j, U_2^j, \ldots, U_M^j)$, $j = 1, 2, \ldots, N$ *as* $\tau \to 0$, $h \to 0$ *and the following estimate is true*

$$\|u^j - U^j\| \le C(\tau + h). \tag{3.8}$$

Proof. To prove Theorem 3.1 let us introduce the difference $z_i^j = u_i^j - U_i^j$. We have:

$$z_{t,i}^{j+1} - \left\{ \left(1+\tau\sum_{k=1}^{j+1}(u_{\bar{x},i}^k)^2\right)^p u_{\bar{x},i}^{j+1} - \left(1+\tau\sum_{k=1}^{j+1}(U_{\bar{x},i}^k)^2\right)^p U_{\bar{x},i}^{j+1} \right\}_x = -\psi_i^j, \tag{3.9}$$
$$z_0^j = z_{\bar{x},M}^j = 0,$$
$$z_i^0 = 0,$$

where

$$\psi_i^j = O(\tau + h).$$

Multiplying Eq. (3.9) scalarly by $\tau z^{j+1} = \tau\left(z_1^{j+1}, z_2^{j+1}, \dots, z_{M-1}^{j+1}\right)$, using the discrete analog of the formula of integration by parts we get

$$\|z^{j+1}\|^2 - \left(z^{j+1}, z^j\right) + \tau h \sum_{i=1}^{M} \left\{ \left(1+\tau\sum_{k=1}^{j+1}(u_{\bar{x},i}^k)^2\right)^p u_{\bar{x},i}^{j+1} \right.$$
$$\left. - \left(1+\tau\sum_{k=1}^{j+1}(U_{\bar{x},i}^k)^2\right)^p U_{\bar{x},i}^{j+1} \right\} z_{\bar{x},i}^{j+1} = -\tau\left(\psi^j, z^{j+1}\right). \tag{3.10}$$

Note that,

$$\left\{ \left(1+\tau\sum_{k=1}^{j+1}\left(u_{\bar{x},i}^k\right)^2\right)^p u_{\bar{x},i}^{j+1} - \left(1+\tau\sum_{k=1}^{j+1}\left(U_{\bar{x},i}^k\right)^2\right)^p U_{\bar{x},i}^{j+1} \right\} \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)$$
$$= \int_0^1 \frac{d}{d\mu} \left\{ \left(1+\tau\sum_{k=1}^{j+1}\left[U_{\bar{x},i}^k + \mu\left(u_{\bar{x},i}^k - U_{\bar{x},i}^k\right)\right]^2\right)^p \left[U_{\bar{x},i}^{j+1} + \mu\left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)\right] \right\} d\mu$$
$$\times \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right) = 2p\int_0^1 \left(1+\tau\sum_{k=1}^{j+1}\left[U_{\bar{x},i}^k + \mu\left(u_{\bar{x},i}^k - U_{\bar{x},i}^k\right)\right]^2\right)^{p-1}$$
$$\times \tau\sum_{k=1}^{j+1}\left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]\left(u_{\bar{x},i}^k - U_{\bar{x},i}^k\right)$$
$$\times \left[U_{\bar{x},i}^{j+1} + \mu\left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)\right] d\mu \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)$$
$$+ \int_0^1 \left(1+\tau\sum_{k=1}^{j+1}\left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^p \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right) d\mu \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)$$
$$= 2p\int_0^1 \left(1+\tau\sum_{k=1}^{j+1}\left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^{p-1}$$
$$\times \tau\sum_{k=1}^{j+1}\left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]\left(u_{\bar{x},i}^k - U_{\bar{x},i}^k\right)$$
$$\times \left[U_{\bar{x},i}^{j+1} + \mu\left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)\right]\left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right) d\mu$$
$$+ \int_0^1 \left(1+\tau\sum_{k=1}^{j+1}\left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^p \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)^2 d\mu$$

$$= 2p \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^{p-1} \xi_i^{j+1}(\mu)\xi_{t,\,i}^j(\mu)\, d\mu$$
$$+ \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2\right)^{p} \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)^2 d\mu,$$

where

$$\xi_i^{j+1}(\mu) = \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu\left(u_{\bar{x},i}^k - U_{\bar{x},i}^k\right)\right]\left(u_{\bar{x},i}^k - U_{\bar{x},i}^k\right),$$
$$\xi_i^0(\mu) = 0,$$

and therefore,

$$\xi_{t,\,i}^j(\mu) = \left[U_{\bar{x},i}^{j+1} + \mu\left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)\right]\left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right).$$

Introducing the following notation

$$s_i^{j+1}(\mu) = \tau \sum_{k=1}^{j+1} \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2$$

from the previous equality we have

$$\left\{\left(1 + \tau \sum_{k=1}^{j+1} \left(u_{\bar{x},i}^k\right)^2\right)^p u_{\bar{x},i}^{j+1} - \left(1 + \tau \sum_{k=1}^{j+1} \left(U_{\bar{x},i}^k\right)^2\right)^p U_{\bar{x},i}^{j+1}\right\}\left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)$$
$$= 2p \int_0^1 \left(1 + s_i^{j+1}(\mu)\right)^{p-1} \xi_i^{j+1}\xi_{t,\,i}^j\, d\mu + \int_0^1 \left(1 + s_i^{j+1}(\mu)\right)^p \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)^2 d\mu.$$

After substituting this equality in (3.10) we get

$$\|z^{j+1}\|^2 - (z^{j+1}, z^j) + 2\tau hp \sum_{i=1}^M \int_0^1 \left(1 + s_i^{j+1}(\mu)\right)^{p-1} \xi_i^{j+1}\xi_{t,\,i}^j\, d\mu$$
$$+ \tau h \sum_{i=1}^M \int_0^1 \left(1 + s_i^{j+1}(\mu)\right)^p \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)^2 d\mu = -\tau(\psi^j, z^{j+1}). \tag{3.11}$$

Taking into account restriction $p > 0$ and relations

$$s_i^{j+1}(\mu) \geq 0,$$
$$(z^{j+1}, z^j) = \frac{1}{2}\|z^{j+1}\|^2 + \frac{1}{2}\|z^j\|^2 - \frac{1}{2}\|z^{j+1} - z^j\|^2,$$
$$\tau\, \xi_i^{j+1}\xi_{t,\,i}^j = \frac{1}{2}\left(\xi_i^{j+1}\right)^2 - \frac{1}{2}\left(\xi_i^j\right)^2 + \frac{\tau^2}{2}\left(\xi_{t,\,i}^j\right)^2$$

from (3.11) we have

$$\|z^{j+1}\|^2 - \frac{1}{2}\|z^{j+1}\|^2 - \frac{1}{2}\|z^j\|^2 + \frac{1}{2}\|z^{j+1} - z^j\|^2$$
$$+ hp \sum_{i=1}^M \int_0^1 \left(1 + s_i^{j+1}(\mu)\right)^{p-1} \left[\left(\xi_i^{j+1}\right)^2 - \left(\xi_i^j\right)^2\right] d\mu$$
$$+ \tau^2 hp \sum_{i=1}^M \int_0^1 \left(1 + s_i^{j+1}(\mu)\right)^{p-1} \left(\xi_{t,\,i}^j\right)^2 d\mu + \tau h \sum_{i=1}^M \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)^2 \leq -\tau(\psi^j, z^{j+1}). \tag{3.12}$$

From (3.12) we arrive at

$$\frac{1}{2}\|z^{j+1}\|^2-\frac{1}{2}\|z^j\|^2+\frac{\tau^2}{2}\|z_t^j\|^2+hp\sum_{i=1}^{M}\int_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu$$
$$+\tau\|z_{\bar{x}}^{j+1}\|^2\le\frac{\tau}{2}\|\psi^j\|^2+\frac{\tau}{2}\|z^{j+1}\|^2. \tag{3.13}$$

Using discrete analog of Poincaré inequality [24]

$$\|z^{j+1}\|^2\le\|z_{\bar{x}}^{j+1}\|^2$$

from (3.13) we get

$$\|z^{j+1}\|^2-\|z^j\|^2+\tau^2\|z_t^j\|^2+2hp\sum_{i=1}^{M}\int_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu$$
$$+\tau\|z_{\bar{x}}^{j+1}\|^2\le\tau\|\psi^j\|^2. \tag{3.14}$$

Summing (3.14) from $j=0$ to $j=n-1$ we arrive at

$$\|z^n\|^2+\tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2+2hp\sum_{j=0}^{n-1}\sum_{i=1}^{M}\int_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu$$
$$+\tau\sum_{j=0}^{n-1}\|z_{\bar{x}}^{j+1}\|^2\le\tau\sum_{j=0}^{n-1}\|\psi^j\|^2. \tag{3.15}$$

Note, that since $s_i^{j+1}(\mu)\ge s_i^j(\mu)$ and $p\le 1$, for the second line of last formula we have

$$\sum_{j=0}^{n-1}\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]$$
$$=\left(1+s_i^1(\mu)\right)^{p-1}\left(\xi_i^1\right)^2-\left(1+s_i^1(\mu)\right)^{p-1}\left(\xi_i^0\right)^2$$
$$+\left(1+s_i^2(\mu)\right)^{p-1}\left(\xi_i^2\right)^2-\left(1+s_i^2(\mu)\right)^{p-1}\left(\xi_i^1\right)^2$$
$$+\cdots+\left(1+s_i^n(\mu)\right)^{p-1}\left(\xi_i^n\right)^2-\left(1+s_i^n(\mu)\right)^{p-1}\left(\xi_i^{n-1}\right)^2$$
$$=\left(1+s_i^n(\mu)\right)^{p-1}\left(\xi_i^n\right)^2+\sum_{j=1}^{n-1}\left[\left(1+s_i^j(\mu)\right)^{p-1}-\left(1+s_i^{j+1}(\mu)\right)^{p-1}\right]\left(\xi_i^j\right)^2\ge 0.$$

Taking into account the last relation and (3.15) one can deduce

$$\|z^n\|^2+\tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2+\tau\sum_{j=0}^{n-1}\|z_{\bar{x}}^{j+1}\|^2\le\tau\sum_{j=0}^{n-1}\|\psi^j\|^2. \tag{3.16}$$

From (3.16) we get (3.8), and Theorem 3.1 thus is proved.

4. Conclusions

Nonlinear integro-differential parabolic equation associated with the penetration of an electromagnetic field in a substance is considered. Unique solvability and asymptotic behavior of solution of initial–boundary value problem (3.1)–(3.3) are fixed. The finite difference scheme (3.4)–(3.6) is constructed and investigated. One must note that convergence of the semi-discrete scheme for problem (3.1)–(3.3) for $0<p\le 1$ was proven in [13]. The fully discrete analogs for $p=1$ for this type of models and different kind of boundary conditions are studied in [12] and in a number of other works (see, for example, [3] and references therein). In [13] it was noted that it is important to construct and

investigate fully discrete finite difference schemes and finite element analogs studied in this note type models for more general type nonlinearities and for multi-dimensional cases as well. So, in the present work the finite difference scheme is investigated for the case of the nonlinearity $0 < p \leq 1$.

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Original article

Calculation of Lebesgue integrals by using uniformly distributed sequences

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Abstract

A certain modified version of Kolmogorov's strong law of large numbers is used for an extension of the result of C. Baxa and J. Schoißengeier (2002) to a maximal set of uniformly distributed sequences in (0, 1) which strictly contains the set of all sequences having the form $(\{\alpha n\})_{n\in\mathbf{N}}$ for some irrational number α and having the full ℓ_1^∞-measure, where ℓ_1^∞ denotes the infinite power of the linear Lebesgue measure ℓ_1 in (0, 1).

Keywords: Uniformly distributed sequence; Lebesgue integral; Strong law of large numbers

1. Introduction

A useful technique for numerical calculation of one-dimensional Riemann integral for a real-valued Riemann integrable function over [0, 1] in terms of uniformly distributed sequences firstly was given in 1916 by Hermann Weyl's celebrated theorem as follows:

Theorem A (*[1], Corollary 1.1, p. 3*). *The sequence of real numbers* $(x_n)_{n\in N} \in [0,1]^\infty$ *is uniformly distributed in* $[0,1]$ *if and only if for every real-valued Riemann integrable function* f *on* $[0,1]$ *we have*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} f(x_n)}{N} = \int_0^1 f(x)dx. \tag{1}$$

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Main corollaries of this theorem were used successfully in Diophantine approximations (see, for example, [2]) and have applications to Monte-Carlo integration (see, for example, [1,3,4]). During the last decades the methods of the theory of uniform distribution modulo one have been intensively used for calculation of improper Riemann integrals (see, for example, [5,6]).

Note that the set S of all uniformly distributed sequences in $[0, 1]$ viewed as a subset of $[0, 1]^\infty$ has full ℓ_1^∞-measure, where ℓ_1^∞ denotes the infinite power of the linear Lebesgue measure ℓ_1 in $[0, 1]$. So each element of the set S can be used for calculation of one-dimensional Riemann integral for an arbitrary Riemann integrable real-valued function in $[0, 1]$. For an arbitrary Lebesgue integrable function f in $[0, 1]$, there naturally arises the following question.

Question 1. *What is a maximal subset S_f of S each element of which can be used for calculation of the Lebesgue integral over* $[0, 1]$ *by the formula* (1) *and whether this subset has the full ℓ_1^∞-measure?*

In this note we consider two tasks:

The first task is an investigation of Question 1 by using Kolmogorov strong law of large numbers.

The second task is an improvement of the following result of C. Baxa and J. Schoißengeier.

Theorem B (*[6], Theorem 1, p. 271*)**.** *Let α be an irrational number,* $\mathbf{Q}$ *be a set of all rational numbers and $F \subseteq [0, 1] \cap \mathbf{Q}$ be finite. Let $f : [0, 1] \to R$ be an integrable, continuous almost everywhere and locally bounded on $[0, 1] \setminus F$. Assume further that for every $\beta \in F$ there is some neighborhood U of β such that f is either bounded or monotone in $[0, \beta) \cap U$ and in $(\beta, 1] \cap U$ as well. Then the following conditions are equivalent:*

1. $\lim_{n\to\infty} \frac{f(\{k\alpha\})}{n} = 0$*;*
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\{k\alpha\})$ *exists;*
3. $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\{k\alpha\}) = \int_{(0,1)} f(x)dx$*,*

where $\{\cdot\}$ denotes the fractional part of the real number.

More precisely, we plan to extend the result of Theorem B to a maximal set $D_f \subset S$ and $E_f \subseteq (0, 1)^\infty$ strictly containing the set S^* of all sequences of the form $(\{\alpha n\})_{n\in\mathbf{N}}$ where α is an irrational number and to calculate ℓ_1^∞ measures of D_f and E_f, respectively.

The paper is organized as follows.

In Section 2 we consider some auxiliary notions and facts from the theory of uniformly distributed sequences and probability theory. In Section 3 we present our main results. In Section 4 we discuss our main result.

2. Preliminary notes/materials and methods

Definition 1. A sequence $s_1, s_2, s_3, \dots$ of real numbers from the interval $[0, 1]$ is said to be uniformly distributed in the interval $[0, 1]$ if for any subinterval $[c, d]$ of the $[0, 1]$ we have

$$\lim_{n\to\infty} \frac{\#(\{s_1, s_2, s_3, \dots, s_n\} \cap [c, d])}{n} = d - c, \tag{2}$$

where # denotes the counting measure.

Example 1 (*[1], Exercise 1.12, p. 16*)**.** The sequence of all multiples of an irrational α

$$0, \{\alpha\}, \{2\alpha\}, \{3\alpha\} \cdots \tag{3}$$

is uniformly distributed in $(0, 1)$, where $\{\cdot\}$ denotes the fractional part of the real number.

Lemma 1 (*[1] Theorem 2.2, p. 183*)**.** *Let S be a set of all elements of* $[0, 1]^\infty$ *which are uniformly distributed in the interval* $[0, 1]$. *Then* $\ell_1^\infty(S) = 1$.

Lemma 2 (*Kolmogorov–Khinchin [7], Theorem 1, p. 371*)**.** *Let (X, S, μ) be a probability space and let $(\xi_n)_{n\in\mathbf{N}}$ be the sequence of independent random variables for which $\int_X \xi_n(x)d\mu(x) = 0$. If $\sum_{n=1}^{\infty} \int_X \xi_n^2(x)d\mu(x) < \infty$, then the series $\sum_{n=1}^{\infty} \xi_n$ converges with probability* 1.

Lemma 3 (*Toeplitz Lemma [7], Lemma 1, p. 377*)**.** *Let $(a_n)_{n\in\mathbf{N}}$ be a sequence of non-negative numbers, $b_n = \sum_{i=1}^{n} a_i$, $b_n > 0$ for each $n \geq 1$ and $b_n \uparrow \infty$, when $n \to \infty$. Let $(x_n)_{n\in\mathbf{N}}$ be a sequence of real numbers such that $\lim_{n\to\infty} x_n = x$. Then*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{n} a_j x_j = x. \tag{4}$$

In particular, if $a_n = 1$ for $n \in \mathbf{N}$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = x. \tag{5}$$

Lemma 4 (*Kronecker Lemma [7], Lemma 2, p. 378*)**.** *Let $(b_n)_{n\in\mathbf{N}}$ be an increasing sequence of positive numbers such that $b_n \uparrow \infty$, when $n \to \infty$, and let $(x_n)_{n\in\mathbf{N}}$ be a sequence of real numbers such that the series $\sum_{k\in\mathbf{N}} x_k$ converges. Then*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{n} b_j x_j = 0. \tag{6}$$

In particular, if $b_n = 0$, $x_n = \frac{y_n}{n}$ and the series $\sum_{n=1}^{\infty} \frac{y_n}{n}$ converges then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} y_k}{n} = 0. \tag{7}$$

Below we give the proof of a certain modification of the Kolmogorov Strong Law of Large Numbers (cf. [7], Theorem 3, p. 379).

Lemma 5. *Let $(X, \mathbf{F}, \mu)$ be a probability space and let $\mathbf{L}(X)$ be a class of all real-valued Lebesgue measurable functions on X. Let μ^{∞} be an infinite power of the probability measure μ. Then for $f \in \mathbf{L}(X)$ we have $\mu^{\infty}(A_f) = 1$, where A_f is defined by*

$$A_f = \left\{ (x_k)_{k\in\mathbf{N}} : (x_k)_{k\in\mathbf{N}} \in X^{\infty} \ \& \ \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_X f(x)dx \right\}. \tag{8}$$

Proof. Without loss of generality, we can assume that f is non-negative. We put $\xi_k((x_i)_{i\in\mathbf{N}}) = f(x_k)$ for $k \in \mathbf{N}$ and $(x_i)_{i\in\mathbf{N}} \in X^{\infty}$. We put also

$$\eta_k((x_i)_{i\in\mathbf{N}}) = \frac{1}{k}\left[\xi_k((x_i)_{i\in\mathbf{N}}) \chi_{\{\omega:\xi_k(\omega)<k\}}((x_i)_{i\in\mathbf{N}}) - \int_{X^{\infty}} \xi_k((z_i)_{i\in\mathbf{N}}) \chi_{\{\omega:\xi_k(\omega)<k\}}((z_i)_{i\in\mathbf{N}}) d\mu^{\infty}((z_i)_{i\in\mathbf{N}}) \right] \tag{9}$$

for $(x_i)_{i\in\mathbf{N}} \in X^{\infty}$.

Note that $(\eta_k)_{k\in\mathbf{N}}$ is the sequence of independent random variables for which $\int_{X^{\infty}} \eta_k d\mu^{\infty} = 0$.

We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_{X^{\infty}} \eta_n^2((x_i)_{i\in\mathbf{N}}) d\mu^{\infty}((x_i)_{i\in\mathbf{N}}) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{X^{\infty}} \xi_n^2((x_i)_{i\in\mathbf{N}}) \chi_{\{(y_i)_{i\in\mathbf{N}}:\xi_n((y_i)_{i\in\mathbf{N}})<n\}}((x_i)_{i\in\mathbf{N}}) d\mu^{\infty}((x_i)_{i\in\mathbf{N}}) \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\int_{X^{\infty}} \xi_n((x_i)_{i\in\mathbf{N}}) \chi_{\{(y_i)_{i\in\mathbf{N}}:\xi_n((y_i)_{i\in\mathbf{N}})<n\}}((x_i)_{i\in\mathbf{N}}) d\mu^{\infty}((x_i)_{i\in\mathbf{N}}) \right)^2
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{X^\infty} f(x_n)^2 \chi_{\{(y_i)_{i\in\mathbf{N}}: f(y_n)<n\}}((x_i)_{i\in\mathbf{N}}) d\mu^\infty((x_i)_{i\in\mathbf{N}})$$

$$- \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\int_{X^\infty} f(x_n) \chi_{\{(y_i)_{i\in\mathbf{N}}: f(y_n)<n\}}((x_i)_{i\in\mathbf{N}}) d\mu^\infty((x_i)_{i\in\mathbf{N}}) \right)^2$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_X f^2(x) \chi_{\{\omega: f(\omega)<n\}}(x) d\mu(x) - \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\int_X f(x) \chi_{\{\omega: f(\omega)<n\}}(x) d\mu(x) \right)^2$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^2} \int_X f^2(x) \chi_{\{\omega: f(\omega)<n\}}(x) d\mu(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \int_X f^2(x) \chi_{\{\omega: k-1\leq f(\omega)<k\}}(x) d\mu(x)$$

$$= \sum_{k=1}^{\infty} \int_X f^2(x) \chi_{\{\omega: k-1\leq f(\omega)<k\}}(x) d\mu((x)) \sum_{n=k}^{\infty} \frac{1}{n^2} \leq 2 \sum_{k=1}^{\infty} \frac{1}{k} \int_X f^2(x) \chi_{\{\omega: k-1\leq f(\omega)<k\}}(x) d\mu(x)$$

$$\leq 2 \sum_{k=1}^{\infty} \int_X f(x) \chi_{\{\omega: k-1\leq f(\omega)<k\}}(x) d\mu((x)) = 2 \int_X f(x) d\mu(x). \tag{10}$$

Since

$$\sum_{n=1}^{\infty} \int_X \eta_n^2((x_i)_{i\in\mathbf{N}}) d\mu((x_i)_{i\in\mathbf{N}}) < +\infty, \tag{11}$$

by using Lemma 2 we get

$$\mu\Bigg\{(x_i)_{i\in\mathbf{N}} : \sum_{k=1}^{\infty} \frac{1}{k} \Bigg[f(x_k) \chi_{\{(y_i)_{i\in\mathbf{N}}: f(y_k)<k\}}((x_i)_{i\in\mathbf{N}})$$

$$- \int_{X^\infty} \xi_k((z_i)_{i\in\mathbf{N}}) \chi_{\{(y_i)_{i\in\mathbf{N}}: f(y_k)<k\}}((z_i)_{i\in\mathbf{N}}) d\mu^\infty((z_i)_{i\in\mathbf{N}}) \Bigg] \text{ is convergent} \Bigg\} = 1. \tag{12}$$

Now by Lemma 4 we get that

$$\mu^\infty\Bigg\{(x_i)_{i\in\mathbf{N}} : \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \Bigg[f(x_k) \chi_{\{(y_i)_{i\in\mathbf{N}}: f(y_k)<k\}}((x_i)_{i\in\mathbf{N}})$$

$$- \int_{X^\infty} \xi_k((z_i)_{i\in\mathbf{N}}) \chi_{\{(y_i)_{i\in\mathbf{N}}: f(y_k)<k\}}((z_i)_{i\in\mathbf{N}}) d\mu^\infty((z_i)_{i\in\mathbf{N}}) \Bigg] = 0 \Bigg\} = 1. \tag{13}$$

Note that

$$\sum_{n=1}^{\infty} \mu^\infty(\{(x_i)_{i\in\mathbf{N}} : \xi_1((x_i)_{i\in\mathbf{N}}) \geq n\})$$

$$= \sum_{n=1}^{\infty} \sum_{k\geq n} \mu^\infty\{(x_i)_{i\in\mathbf{N}} : k \leq \xi_1((x_i)_{i\in\mathbf{N}}) < k+1\}$$

$$= \sum_{k=1}^{\infty} k\mu^\infty\{(x_i)_{i\in\mathbf{N}} : k \leq \xi_1((x_i)_{i\in\mathbf{N}}) < k+1\}$$

$$= \sum_{k=0}^{\infty} \int_{X^\infty} k \chi_{\{(y_j)_{j\in\mathbf{N}}: k\leq \xi_1((y_i)_{i\in\mathbf{N}})<k+1\}}((z_i)_{i\in\mathbf{N}}) d\mu^\infty((z_i)_{i\in\mathbf{N}})$$

$$\leq \sum_{k=0}^{\infty} \int_{X^\infty} \xi_1((z_i)_{i\in\mathbf{N}}) \chi_{\{(y_j)_{j\in\mathbf{N}}: k\leq \xi_1((y_j)_{j\in\mathbf{N}})<k+1\}}((z_i)_{i\in\mathbf{N}}) d\mu^\infty((z_i)_{i\in\mathbf{N}})$$

$$= \int_{X^\infty} \xi_1((z_i)_{i\in\mathbf{N}}) d\mu^\infty((z_i)_{i\in\mathbf{N}}) < +\infty. \tag{14}$$

Since $(\xi_k)_{k\in\mathbf{N}}$ is a sequence of equally distributed random variables on X^∞, we have

$$\sum_{n=1}^{\infty}\mu^\infty(\{(x_i)_{i\in\mathbf{N}} : \xi_k((x_i)_{i\in\mathbf{N}}) \geq n\}) \leq \int_{X^\infty}\xi_1((x_i)_{i\in\mathbf{N}})d\mu^\infty((x_i)_{i\in\mathbf{N}}) < +\infty, \tag{15}$$

which by the well-known Borel–Cantelli lemma implies that

$$\mu^\infty(\{(x_i)_{i\in\mathbf{N}} : \xi_n((x_i)_{i\in\mathbf{N}}) \geq n\}\ i.o.) = 0. \tag{16}$$

The last relation means that

$$\mu^\infty(\{(x_i)_{i\in\mathbf{N}} : (\exists N((x_i)_{i\in\mathbf{N}}))(\forall n \geq N((x_i)_{i\in\mathbf{N}})) \to \xi_n((x_i)_{i\in\mathbf{N}}) < n\}) = 1. \tag{17}$$

Thus, we have obtained the validity of the equality $\mu^\infty(A_f^*) = 1$, where

$$\begin{aligned} A_f^* = \Bigg\{(x_i)_{i\in\mathbf{N}} : \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\Bigg[& f(x_k)\chi_{\{(y_i)_{i\in\mathbf{N}}:f(y_k)<k\}}((x_i)_{i\in\mathbf{N}}) \\ & - \int_{X^\infty}\xi_k((z_i)_{i\in\mathbf{N}})\chi_{\{(y_i)_{i\in\mathbf{N}}:f(y_k)<k\}}((z_i)_{i\in\mathbf{N}})d\mu^\infty((z_i)_{i\in\mathbf{N}})\Bigg] = 0 \\ & \&(\exists N((x_i)_{i\in\mathbf{N}}))(\forall n > N((x_i)_{i\in\mathbf{N}}) \to \xi_n((x_i)_{i\in\mathbf{N}}) < n)\Bigg\}. \end{aligned} \tag{18}$$

Now it is obvious that for $(x_i)_{i\in\mathbf{N}} \in A_f^*$, we have

$$\begin{aligned} 0 &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\Bigg[f(x_k)\chi_{\{(y_i)_{i\in\mathbf{N}}:f(y_k)<k\}}((x_i)_{i\in\mathbf{N}}) \\ &\quad - \int_{X^\infty}\xi_k((z_i)_{i\in\mathbf{N}})\chi_{\{(y_i)_{i\in\mathbf{N}}:f(y_k)<k\}}((z_i)_{i\in\mathbf{N}})d\mu^\infty((z_i)_{i\in\mathbf{N}})\Bigg] \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=N((x_i)_{i\in\mathbf{N}})}^{N}\Bigg[f(x_k)\chi_{\{(y_i)_{i\in\mathbf{N}}:f(y_k)<k\}}((x_i)_{i\in\mathbf{N}}) \\ &\quad - \int_{X^\infty}\xi_k((z_i)_{i\in\mathbf{N}})\chi_{\{(y_i)_{i\in\mathbf{N}}:f(y_k)<k\}}((z_i)_{i\in\mathbf{N}})d\mu^\infty((z_i)_{i\in\mathbf{N}})\Bigg] \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=N((x_i)_{i\in\mathbf{N}})}^{N}\Bigg[f(x_k) - \int_X f(x)\chi_{\{y:f(y)<k\}}(x)d\mu(x)\Bigg] \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\Bigg[f(x_k) - \int_X f(x)\chi_{\{y:f(y)<k\}}(x)d\mu(x)\Bigg]. \end{aligned} \tag{19}$$

Since

$$\lim_{k\to\infty}\int_X f(x)\chi_{\{y:f(y)<k\}}(x)d\mu(x) = \int_X f(x)d\mu(x), \tag{20}$$

by Lemma 3 we get

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\int_X f(x)\chi_{\{y:f(y)<k\}}(x)d\mu(x) = \int_X f(x)d\mu(x) \tag{21}$$

which implies that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}f(x_k) = \int_X f(x)d\mu(x) \tag{22}$$

for each $(x_i)_{i\in\mathbf{N}} \in A_f^*$.

The validity of the inclusion $A_f^* \subseteq A_f$ ends the proof of Lemma 5. □

Remark 1. Formulation of Lemma 2.4 (cf. [8], p. 285) needs a certain specification. More precisely, it should be formulated for sequences $(x_k)_{k\in N} \in S \cap A^*_f$, where S comes from Lemma 1 and, A^*_f comes from the proof of Lemma 5 when $(X, \mathbf{F}, \mu) = ((0, 1), \mathbf{B}(0, 1), \ell_1)$. Since $\ell_1^\infty(S \cap A^*_f) = 1$, such reformulated Lemma 2.4 can be used for the proof of Corollary 4.2 (cf. [8], p. 296).

3. Results and discussion

By using Lemmas 1 and 5, we get

Theorem C. *Let f be a Lebesgue integrable real-valued function on* $(0, 1)$. *Then we have*

$$\ell_1^\infty\Bigg(\Bigg\{(x_k)_{k\in\mathbf{N}} : (x_k)_{k\in\mathbf{N}} \in [0,1]^\infty \ \&\ (x_k)_{k\in\mathbf{N}} \text{ is uniformly distributed in } (0,1) \ \&\ \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(x_k) = \int_0^1 f(x)dx\Bigg\}\Bigg) = 1. \tag{23}$$

Proof. Note that

$$\Bigg\{(x_k)_{k\in\mathbf{N}} : (x_k)_{k\in\mathbf{N}} \in [0,1]^\infty \ \&(x_k)_{k\in\mathbf{N}} \text{ is uniformly distributed in } (0,1) \ \&\ \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(x_k) = \int_0^1 f(x)dx\Bigg\} = S \cap A_f, \tag{24}$$

where S comes from Lemma 1 and A_f comes from Lemma 5 when $(X, \mathbf{F}, \mu) = ((0, 1), \mathbf{B}(0, 1), \ell_1)$. □

Note that the answer to Question 1 is contained in the following statement.

Theorem D. *The set $S_f = A_f \cap S$ is a maximal subset of S each element of which can be used for calculation of the Lebesgue integral over* $[0, 1]$ *by the formula* (1) *and* $\ell_1^\infty(S_f) = 1$.

Observation 1. *Let $f : (0, 1) \to \mathbf{R}$ be a Lebesgue integrable function. Then we have $A_f \subseteq B_f$, where*

$$B_f = \Bigg\{(x_k)_{k\in\mathbf{N}} : (x_k)_{k\in\mathbf{N}} \in (0,1)^\infty \ \&\ \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(x_k) \text{ exists}\Bigg\}. \tag{25}$$

Observation 2. *Let $f : (0, 1) \to \mathbf{R}$ be a Lebesgue integrable function. Then we have $B_f \subseteq C_f$, where*

$$C_f = \Bigg\{(x_k)_{k\in\mathbf{N}} : (x_k)_{k\in\mathbf{N}} \in (0,1)^\infty \ \&\ \lim_{N\to\infty}\frac{f(x_N)}{N} = 0\Bigg\}. \tag{26}$$

Proof. Let $(x_k)_{k\in\mathbf{N}} \in B_f$. Then we get

$$\begin{aligned}
\lim_{N\to\infty}\frac{f(x_N)}{N} &= \lim_{N\to\infty}\frac{1}{N}\left(\sum_{k=1}^{N} f(x_k) - \sum_{k=1}^{N-1} f(x_k)\right) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(x_k) - \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N-1} f(x_k) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(x_k) - \lim_{N-1\to\infty}\frac{N-1}{N}\left(\frac{1}{N-1}\sum_{k=1}^{N-1} f(x_k)\right) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(x_k) - \lim_{N-1\to\infty}\frac{1}{N-1}\sum_{k=1}^{N-1} f(x_k) = 0. \quad \square
\end{aligned} \tag{27}$$

Remark 2. Note that for each Lebesgue integrable function f in $(0,1)$, the following inclusion $S \cap A_f \subseteq S \cap C_f$ holds true, but the converse inclusion is not always valid. Indeed, let $(x_k)_{k\in N}$ be an arbitrary sequence of uniformly distributed numbers in $(0,1)$. Then the function $f : (0,1) \to \mathbf{R}$, defined by $f(x) = \chi_{(0,1)\setminus\{x_k:k\in\mathbf{N}\}}(x)$ for $x \in (0,1)$(here $\chi_{(0,1)\setminus\{x_k:k\in\mathbf{N}\}}(x)$ denotes an indicator function of the set $(0,1) \setminus \{x_k : k \in \mathbf{N}\}$ in $(0,1)$) is Lebesgue integrable, $(x_k)_{k\in N} \in C_f \cap S$ but $(x_k)_{k\in N} \notin A_f \cap S$ because

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = 0 \neq 1 = \int_{(0,1)} f(x)dx. \tag{28}$$

Theorem E. *Let $f : (0,1) \to \mathbf{R}$ be a Lebesgue integrable function. Then the set D_f of all uniformly distributed sequences in $(0,1)$ for which the following conditions*

1. $\lim_{n\to\infty} \frac{f(x_n)}{n} = 0$;
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k)$ *exists;*
3. $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_{(0,1)} f(x)dx$;

are equivalent, has full ℓ_1^∞-measure and

$$D_f = (A_f \cap S) \cup (S \setminus C_f),$$

where S, A_f and C_f come from Lemma 1, Lemma 5 (when $(X, \mathbf{F}, \mu) = ((0,1), \mathbf{B}(0,1), \ell_1)$) and Observation 2*, respectively.*

Proof. By Lemma 1 we know that $\ell_1^\infty(S) = 1$. By Lemma 5 when $(X, \mathbf{F}, \mu) = ((0,1), \mathbf{B}((0,1)), \ell_1)$ we know that $\ell_1^\infty(A_f) = 1$. Following Observations 1 and 2 we have $A_f \subseteq B_f \subseteq C_f$. Since $S_f = A_f \cap B_f \cap C_f \cap S = A_f \cap S$, we get

$$\ell_1^\infty(S_f) = \ell_1^\infty(A_f \cap S) = 1. \tag{29}$$

Since $S_f \subseteq D_f$ we end the proof of theorem. □

Corollary 1. *Let $\mathbf{Q}$ be a set of all rational numbers of $[0,1]$ and $F \subseteq [0,1] \cap \mathbf{Q}$ be finite. Let $f : [0,1] \to R$ be Lebesgue integrable, ℓ_1-almost everywhere continuous and locally bounded on $[0,1]\setminus F$. Assume that for every $\beta \in F$ there is some neighborhood U_β of β such that f is either bounded or monotone in $[0,\beta) \cap U_\beta$ and in $(\beta,1] \cap U_\beta$ as well. Let S, A_f and C_f come from Lemma 1, Lemma 5 (when $(X, \mathbf{F}, \mu) = ((0,1), \mathbf{B}(0,1), \ell_1)$) and* Observation 2*, respectively. We put*

$$D_f = (A_f \cap S) \cup (S \setminus C_f).$$

Then for $(x_k)_{k\in\mathbf{N}} \in D_f$ the following conditions are equivalent:

1. $\lim_{n\to\infty} \frac{f(x_n)}{n} = 0$;
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k)$ *exists;*
3. $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_{(0,1)} f(x)dx$.

4. Conclusion

Note that D_f is maximal subset of the set S for which conditions 1–3 participated in the formulation of Corollary 1 are equivalent, provided that for each $(x_k)_{k\in N} \in D_f$ the sentences 1–3 are true or false simultaneously, and for each $(x_k)_{k\in N} \in S \setminus D_f$ the sentences 1–3 are not true or false simultaneously. This extends the main result of Baxa and Schoißengeier [6] because, the class S^* of all sequences of the form $(\{n\alpha\})_{n\in N}$ is in D_f for each irrational number α, and no every element of D_f can be presented in the same form. For example,

$$(\{(n + 1/2(1 - \chi_{\{k:k\geq 2\}}(n)))\pi^{\chi_{\{k:k\geq 2\}}(n)}\})_{n\in\mathbf{N}} \in D_f \setminus S^*, \tag{30}$$

where $\{\cdot\}$ denotes the fractional part of the real number and $\chi_{\{k:k\geq 2\}}$ denotes the indicator function of the set $\{k : k \geq 2\}$.

Similarly, setting

$$E_f = A_f \cup \big(((0,1)^\infty \setminus A_f) \cap ((0,1)^\infty \setminus B_f) \cap ((0,1)^\infty \setminus C_f)\big) = A_f \cup \big((0,1)^\infty \setminus C_f\big), \tag{31}$$

we get a maximal subset of $(0,1)^\infty$ for which conditions 1–3 participated in the formulation of Corollary 1 are equivalent, provided that for each $(x_k)_{k\in N} \in E_f$ the sentences 1–3 are true or false simultaneously, and for each $(x_k)_{k\in N} \in (0,1)^\infty \setminus E_f$ the sentences 1–3 are not true or false simultaneously.

Note also that both sets D_f and E_f have full ℓ_1^∞ measure.

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Original article

Some problems of oscillation and stability of prestressed shells of rotation close to cylindrical ones, with an elastic filler and under the action of temperature

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Abstract

The present paper investigates natural oscillations and stability of shells of revolution which are close by their form to cylindrical ones, with elastic filler and under the action of meridional forces, external pressure and temperature. The shell is assumed to be thin and elastic. A filler is simulated by an elastic base. The shells of positive and negative Gaussian curvature are considered. Formulas for finding the least frequencies and a form of wave formation are written out. The questions dealing with the higher frequencies and stability of shells of revolution are studied, and formulas for critical loadings are also written out.

Keywords: Oscillation; Stability; Shells; Critical load; Filler; Temperature; Lowest; Meridional forces; Frequency

We investigate natural oscillations and stability of closed shells of revolution which are close by their form to cylindrical ones, with an elastic filler and under the action of meridional forces distributed uniformly over the end-walls, external pressure and temperature. We consider a light filler for which tangential stresses on the contact area and inertia forces may be neglected. The shell is assumed to be thin and elastic. Temperature is uniformly distributed in the shell body. The elastic filler is simulated by the Winkler's base, its extension upon heating is neglected. We investigate the shells of middle length; the form of their middle surface generatrix is described by the parabolic function. We consider the shells of positive and negative Gaussian curvatures. The boundary conditions on the end-walls correspond to à free support admitting radial displacements in the initial state. The obtained formulas and universal curves show that the least frequency and a form of wave formation depend on meridional loadings, external pressure, temperature, elastic filler rigidity and on the deviation amplitude of the shell from the cylinder. It is shown that in the presence of prestresses and of an elastic filler, temperature affects the lowest frequencies and a form of wave formation differently, depending on the sign of Gaussian curvature of the shell. We consider the problem on the highest frequencies and stability of the shell of revolution and derive formulas for finding critical loadings.

We consider the shell whose middle surface is formed by the rotation of square parabola around the z-axis of the rectangular system of coordinates x, y, z with the origin in the midsegment of the axis of revolution. It is assumed

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that the radius R of the midsurface cross-section is defined by the equality

$$R = r + \delta_0\big[1 - \xi^2(r/\ell)^2\big],$$

where r is the radius of the end-wall cross-section, δ_0 is a maximal deviation from the cylindrical form (for $\delta_0 > 0$ the shell is convex and for $\delta_0 < 0$ it is concave), $L = 2\ell$ is the shell length, $\xi = z/r$.

We consider the shells of middle length [1] and assume that

$$\big(\delta_0/r\big)^2, \quad \big(\delta_0/\ell\big)^2 \ll 1. \tag{1}$$

For the basic equations of oscillations we adopt equations of the theory of shallow shells [2]. For the shells of middle length, the forms of oscillations corresponding to the lowest frequencies vary weakly in the longitudinal direction compared with circumferential one. Therefore the relation

$$\frac{\partial^2 f}{\partial \xi^2} \ll \frac{\partial^2 f}{\partial \varphi^2} \quad (f = w, \psi) \tag{2}$$

is valid, where w and ψ are, respectively, the function of radial displacement and the stress function. As a result, the system of equations for the shells under consideration is reduced to the following resolving equation (owing to the adopted assumption, temperature terms are equal to zero [3]):

$$\begin{gathered}
\varepsilon\frac{\partial^8 w}{\partial \varphi^8} + \frac{\partial^4 w}{\partial \xi^4} + 4\,\delta\frac{\partial^4 w}{\partial \xi^2\,\partial \varphi^2} + 4\,\delta^2\frac{\partial^4 w}{\partial \varphi^4} - t_1^0\frac{\partial^6 w}{\partial \xi^2\,\partial \varphi^4} - t_2^0\frac{\partial^6 w}{\partial \varphi^6} \\
- 2s^0\frac{\partial^6 w}{\partial \xi\,\partial \varphi^5} + \gamma\frac{\partial^4 w}{\partial \varphi^4} + \frac{\rho r^2}{E}\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial \varphi^4}\Big) = 0, \\
\varepsilon = h^2/12(1-\nu^2)r^2, \quad \delta = \delta_0 r/\ell^2, \quad t_i^0 = T_i^0/Eh \quad (i = 1, 2), \quad s^0 = S^0/Eh, \quad \gamma = \beta r^2/Eh,
\end{gathered} \tag{3}$$

where T_1^0 and T_2^0 are, respectively, meridional and circumferential normal forces of the initial state; S^0 is shearing force of the initial state; E and ν are, respectively, the modulus of elasticity and the Poisson coefficient; h is the shell thickness, ρ is density of the shell material; β is the "bed" coefficient of the elastic filler (characterizing elastic rigidity); t is time.

The initial state is assumed to be momentless. On the basis of the corresponding solution, with regard for the filler reaction and due to inequality (1), we obtain the following approximate expressions:

$$\begin{aligned}
T_1^0 &= P_1\Big[1 + \frac{\delta_0}{r}\big(\xi^2(r/\ell)^2 - 1\big)\Big] + q\delta_0\big(\xi^2(r/\ell)^2 - 1\big], \\
T_2^0 &= -2\,P_1\delta_0 r/\ell^2 - qr + \beta_0 r w_0, \quad S^0 = 0,
\end{aligned} \tag{4}$$

where w_0, β_0 are, respectively, deflection and the "bed" coefficient of the filler in the initial state. Taking into account that

$$\big|\xi^2(r/\ell)^2 - 1\big|\,\frac{\partial^2 w}{\partial \xi^2} \ll 2(r/\ell)^2\frac{\partial^2 w}{\partial \varphi^2}, \quad \frac{\delta_0}{r}\,\big|\xi^2(r/\ell)^2 - 1\big|\,\frac{\partial^2 w}{\partial \xi^2} \ll \frac{\partial^2 w}{\partial \varphi^2},$$

the expressions (4) after substitution into (3) can be simplified, and they take the form

$$\frac{T_1^0}{Eh} = \frac{P_1}{Eh}, \quad \frac{T_2^0}{Eh} = -2\frac{P_1}{Eh}\,\delta - \frac{qr}{Eh} + w_0\frac{\beta_0 r}{Eh}, \quad T_i^0 = \sigma_i^0 h \quad (i = 1, 2). \tag{4$'$}$$

Taking into account the fact that in the initial state the shell deformation in the circumferential direction ε_φ^0 is defined by the equalities

$$\varepsilon_\varphi^0 = \frac{\sigma_2^0 - \nu\sigma_1^0}{E} + \alpha T, \quad \varepsilon_\varphi^0 = -\frac{w_0}{r},$$

where α is the linear extension coefficient, and T is temperature, we have

$$w_0 = \big(-\sigma_2^0 + \nu\sigma_1^0\big)\frac{r}{E} - \alpha T r. \tag{5}$$

Substituting (5) into (4′), we get

$$\frac{T_2^0}{Eh} = \frac{\sigma_2^0}{E} = -\frac{qr}{Eh} - 2\frac{P_1}{Eh}\,\delta + \frac{\beta_0 r^2}{Eh}\left(-\sigma_2^0 + \nu\sigma_1^0\right)\frac{1}{E} - \frac{\alpha T\beta_0 r^2}{Eh}$$

whence

$$\frac{\sigma_2^0}{E}\left(1 + \frac{\beta_0 r^2}{Eh}\right) = -\frac{qr}{Eh} - 2\frac{P_1}{Eh}\,\delta + \nu\frac{\sigma_1^0}{E}\,\frac{\beta_0 r^2}{Eh} - \alpha T\,\frac{\beta_0 r^2}{Eh}.$$

Introducing the notation

$$\frac{qr}{Eh} = \overline{q}, \quad \frac{P_1}{Eh} = -p, \quad \frac{\beta_0 r^2}{Eh} = \gamma_0, \quad 1 + \gamma_0 = g$$

the expressions (4′) take the form

$$-\frac{\sigma_1^0}{E} = p, \quad -\frac{\sigma_2^0}{E} = \left(\overline{q} - 2\,p\delta + \nu p\gamma_0 + \alpha T\gamma_0\right)g^{-1}. \tag{5′}$$

It should be noted that since R is close to r, therefore in the expressions (5′) for stresses we adopt $R \approx r$.

As a result, Eq. (3) has the form

$$\varepsilon\frac{\partial^8 w}{\partial\xi^8} + \frac{\partial^4 w}{\partial\xi^4} + 4\,\delta\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2} + 4\left(\delta^2 + \gamma/4\right)\frac{\partial^4 w}{\partial\varphi^4} + \left(\overline{q} - 2\,p\delta + \nu p\gamma_0 + \alpha T\gamma_0\right)g^{-1}\frac{\partial^6 w}{\partial\varphi^6}$$
$$+ p\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4} + \frac{\rho r^2}{E}\,\frac{\partial^2}{\partial t^2}\left(\frac{\partial^4 w}{\partial\varphi^4}\right) = 0. \tag{6}$$

We consider harmonic oscillations. For the given boundary conditions of free support and for Eq. (6), the solution

$$w = A_{mn}\cos\lambda_m\xi\,\sin n\varphi\cos\omega t, \quad \lambda_m = m\pi r/2\ell \ \ (m = 2i+1,\ i = 0, 1, 2, \ldots) \tag{7}$$

is satisfied.

Substituting the expression (7) into Eq. (6), for the natural frequencies we obtain the following equality:

$$\omega^2 = \frac{E}{\rho r^2}\left[\varepsilon n^4 + \lambda_m^4 n^{-4} + 4\,\delta\lambda_m^2 n^{-2} + 4(\delta^2 + \gamma/4) - p(\lambda_m^2 - 2\,\widetilde{\delta}n^2) - (\overline{q} + \gamma_0\alpha T)g^{-1}n^2\right]. \tag{8}$$

Introduce the notation

$$\overline{\delta}^{\,2} = \delta^2 + \gamma/4, \quad \widetilde{\delta} = \left(\delta - \frac{1}{2}\,\nu\gamma_0\right)g^{-1}, \quad \widetilde{q} = (\overline{q} + \gamma_0\alpha T)g^{-1},$$

then

$$\omega^2 = \frac{E}{\rho r^2}\left[\varepsilon n^4 + \lambda_m^4 n^{-4} + 4\,\delta\lambda_m^2 n^2 + 4\,\overline{\delta}^{\,2} - p(\lambda_m^2 - 2\,\widetilde{\delta}n^2) - \widetilde{q}\,n^2\right]. \tag{8′}$$

It is evident that for $p = 0$, $\delta > 0$, to the least frequency there corresponds the value $m = 1$. It can also be shown that this condition holds for $\delta < 0$ if we take into account inequality (1) and the fact that $\omega^2 > 0$. Therefore, we will first consider the forms of oscillations under which along the shell length there arises only one half-wave ($m = 1$), while in the circumferential direction we have n waves. For the contraction $p > 0$, and for the tension $p < 0$; q is the normal pressure which is assumed to be positive if it is external one.

We represent the expression (8′) for $m = 1$ in a dimensionless form. To this end, we introduce dimensionless values

$$N = n^2/n_0^2, \quad P = p/p_{0*}, \quad \widetilde{Q} = \widetilde{q}/\widetilde{q}_{0*}, \quad p_{0*} = \frac{(1-\nu^2)^{-1/2}}{\sqrt{3}}\,\frac{h}{r}, \quad \overline{q}_{0*} = 0,855(1-\nu^2)^{-3/4}\left(\frac{h}{r}\right)^{3/2}\frac{r}{L},$$
$$\delta_* = \delta\varepsilon_*^{-1/2}. \ \ \varepsilon_* = (1-\nu^2)^{-1/2}\left(\frac{r}{L}\right)^2\frac{h}{r}, \quad n_0^2 = \lambda_1\varepsilon^{-1/4}, \quad \lambda_1 = \pi r/L, \quad \overline{\delta}_*^{\,2} = \delta_*^2 + \gamma_*/4, \quad \gamma_* = \gamma\varepsilon_*^{-1}, \tag{9}$$
$$\widetilde{\delta}_* = (\delta - 0,5\,\nu\gamma_0)\varepsilon_*^{-1/2}g^{-1}, \quad \omega_8^2 = 2\lambda_1^2\varepsilon^{1/2}\frac{E}{\rho r^2}, \quad \frac{\widetilde{q}}{\overline{q}_{0*}} = \left(\frac{\overline{q}}{\overline{q}_{0*}} + \frac{\gamma_0\alpha T}{\overline{q}_{0*}}\right)g^{-1},$$

where ω_*, p_{0*} and $\overline{q}_{0*}$ are, respectively, the least frequency, critical contraction loading and critical pressure for the cylindrical midlength shell [1,4]. As a result, the equality (8′) can be written in the dimensionless form as follows:

$$\omega^2(N)/\omega_*^2 = 0,5\big[N^2 + N^{-2} + 2,37\,\delta_* N^{-1} + 1,404\,\overline{\delta}_*^{\,2} - 2P(1 - 1,185\,\widetilde{\delta}_* N) - 1,755\,\widetilde{Q}N\big]. \tag{10}$$

The least frequency (for $\omega^2(N) > 0$) is derived from the condition $[\omega^2(N)]' = 0$. Thus we obtain either

$$0,8775\,\widetilde{Q} - 1,185\,\widetilde{\delta}_* P = N - 1,185\,\delta_* N^{-2} - N^{-3}$$

or

$$N^4 - (0,8775\,\widetilde{Q} - 1,185\,\widetilde{\delta}_* P)N^3 - 1,185\,\delta_* N - 1 = 0. \tag{11}$$

This implies that for $P = \widetilde{Q} = 0$, we have the known equation

$$N^4 - 1,185\,\delta_* N - 1 = 0,$$

whose roots have been obtained explicitly in [5]. Moreover, from (11), for $\delta_* = 0$, $\widetilde{Q} = 0$ ($\delta = \gamma_0 = 0$, $q = 0$) we obtain the equation $N^4 - 1 = 0$ whose positive root $N = 1$. Consequently, for the cylindrical shell of middle length, the least frequency is realized for $N = 1$, independently of P, what fully agrees with [6].

For $\omega = 0$, equality (10) yields

$$1,755\,\overline{Q} = N + N^{-3} + 2,37\,\delta_* N^{-2} + 1,404\,\overline{\delta}_*^{\,2} N^{-1} - 2P(N^{-1} - 1,185\,\widetilde{\delta}_*). \tag{12}$$

The least $\widetilde{Q} > 0$, depending on N, is realized for $\widetilde{Q}'_N = 0$. This implies

$$N^4 + cN^2 + dN + e = 0, \quad c = 2P - 1,404\,\overline{\delta}_*^{\,2}, \quad d = -4,74\,\delta_*, \quad e = -3. \tag{13}$$

The roots of Eq. (13) coincide with those of the following two quadratic equations:

$$N^2 + \frac{A_{1,2}}{2}N + \left(y_1 - \frac{d}{A_{1,2}}\right) = 0, \quad A_{1,2} = \pm\sqrt{8\alpha},$$

$$N_{1,2} = -\sqrt{\frac{\alpha}{2}} \pm \sqrt{\frac{d}{\sqrt{8\alpha}} - \frac{\alpha_1}{2}}, \quad N_{3,4} = \sqrt{\frac{\alpha}{2}} \pm \sqrt{\frac{-d}{\sqrt{8\alpha}} - \frac{\alpha_1}{2}}, \tag{14}$$

$$\alpha = y_1 - \frac{c}{2}, \quad \alpha_1 = y_1 + \frac{c}{2}, \tag{15}$$

where y_1 is any real root of the cubic equation

$$y^3 - \frac{c}{2}y^2 - ey + \left(\frac{ce}{2} - \frac{d^2}{8}\right) = 0 \tag{16}$$

or

$$z^3 + 3pz + 2q = 0 \quad (z = y - c/6), \tag{17}$$

$$p = 1 - \big(2P - 1,404\,\overline{\delta}_*^{\,2}\big)^2/36, \quad q = -\frac{1}{2}\big(2P + 1,404\,\overline{\delta}_*^{\,2}\big)\left[1 - \frac{(2P - 1,404\,\overline{\delta}_*^{\,2})^3}{108(2P + 1,404\,\overline{\delta}_*^{\,2})}\right]. \tag{18}$$

If we adopt that

$$\big(2P - 1,404\,\overline{\delta}_*^{\,2}\big)^2/36 \ll 1$$

then the expressions (18) take the form $p = 1$, $q = -\frac{1}{2}(2P + 1,404\,\overline{\delta}_*^{\,2})$. Since the discriminant of Eq. (8) $D = q^2 + p^3 > 0$, we have one real root

$$z = \big(-q + \sqrt{q^2 + p^3}\big)^{1/3} + \big(-q - \sqrt{q^2 + p^3}\big)^{1/3}. \tag{19}$$

If we adopt that

$$\left(2\,P + 1,404\,\overline{\delta}_*^{\,2}\right)^2/36 \ll 1 \tag{20}$$

and expand into series the expressions appearing in (19), neglecting here the values of second order smallness, we get $z = [(2\,P + 1,404\,(\delta_*^2 - \gamma_*/4))]/3$. Then, by virtue of (13), (15) and (17), we obtain

$$\alpha = z - c/3 = 2 \cdot 1,404\,\overline{\delta}_*^{\,2}/3, \quad \alpha_1 = z + \frac{2}{3}c = 2\,P + 1,404\Big(\delta_*^2 + \frac{3}{4}\gamma_*\Big)\Big/3. \tag{21}$$

Taking into account that y_1 is the root of Eq. (16), we have

$$\frac{d^2}{8(y_1 - c/2)} = y_1^2 - e,$$

whence it follows that

$$\frac{|d|}{\sqrt{8\,\alpha}} = \sqrt{y_1^2 - e} > y_1 = \frac{y_1}{2} + \frac{y_1}{2} + \frac{c}{4} - \frac{c}{4} = \frac{1}{2}\Big(y_1 - \frac{c}{2}\Big) + \frac{1}{2}\Big(y_1 + \frac{c}{2}\Big).$$

Consequently,

$$\frac{|d|}{\sqrt{8\,\alpha}} - \frac{\alpha_1}{2} > \frac{\alpha}{2}. \tag{22}$$

Since $N^2 = n^2/n_0^2$, of our interest are only the positive roots of Eq. (13). Bearing in mind inequality (22), we can see that positive for $\delta_* < 0$ ($d > 0$) is only the root N_1, and for $\delta_* > 0$ ($d < 0$), only the root N_3. Substituting the values d, α, α_1, according to equalities (13) and (21), into the expressions (14), we obtain

$$\begin{aligned} N_* &= \sqrt{\sqrt{3} + 0,234\Big(\delta_*^2 + \frac{3}{4}\gamma_*\Big) - P} - 0,684\,|\delta_*| \quad (\delta_* < 0), \\ N_* &= \sqrt{\sqrt{3} + 0,234\Big(\delta_*^2 + \frac{3}{4}\gamma_*\Big) - P} + 0,684\,\delta_* \quad (\delta_* > 0). \end{aligned} \tag{23}$$

Thus we obtain

$$n_{1,2}^2 = \left(\sqrt{\sqrt{3} + 0,2703\,\varepsilon^{-1/2}\Big[\Big(\frac{\delta_0}{\ell}\Big)^2 + \frac{\gamma}{4}\Big(\frac{\ell}{r}\Big)^2\Big] - P} \pm 0,735\,\varepsilon^{-1/4}\frac{|\delta_0|}{\ell}\right)\lambda_1\varepsilon^{1/4}, \tag{24}$$

where the indices (1) and (2) correspond to $\delta_0 > 0$ and $\delta_0 < 0$, respectively. In particular, for $\delta_0 = \gamma_0 = p = 0$, we obtain the known formula for a critical number of waves of the cylindrical midlength shell $n_*^2 = \sqrt[4]{3}\,\lambda_1\varepsilon^{-1/4}$ [1].

From formula (24), it is not difficult to notice that under the action of contractive forces, a critical number of waves in the circumferential direction decreases, whereas under tensile forces it increases.

As is mentioned above, formula (23) holds when condition (20) is fulfilled. If, however, this condition is not fulfilled, we have to proceed from the full expressions (18).

Defining in such a way the values of N_* (for fixed δ_*, γ_*, P) and substituting them into (12), we obtain the corresponding critical value of $\widetilde{Q}_*$. In an expanded form, formula (12) for critical pressure has the form

$$\overline{q}_{kp} = 0,57\big[N_* + N_*^{-3} + 2,37\,\delta_* N_*^{-2} + 1,404(\delta_*^2 + \gamma_*/4)N_*^{-1} - 2P(N_*^{-1} - 1,185\,\widetilde{\delta}_*)\big]g\overline{q}_{0*} - \gamma_0\alpha T.$$

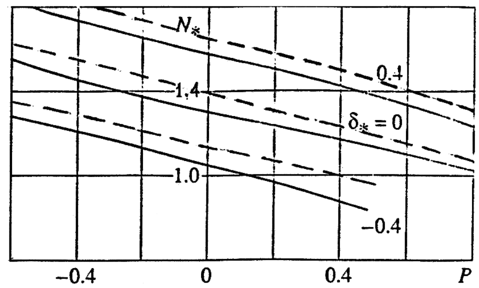

Fig. 1.

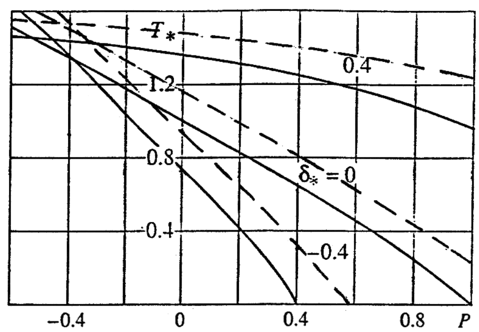

Fig. 2.

In Fig. 1, in dimensionless form are given critical values of N_* depending on P for $\delta_* = -0,4;\ 0;\ 0,4$ and for $\gamma_* = 0;\ 1,272$. The corresponding charts for $\gamma_* = 0$ are represented by firm curves and for $\gamma_* = 1,272$ by dotted curves. The values of $\widetilde{Q}_*(\delta_*, \gamma_*, P)$ denoted, respectively, by firm and dotted curves are given in Fig. 2. It should be noted that the curve $\widetilde{Q}_*(\delta_* = 0, \gamma_* = 0)$ for $P > 0$ given in Fig. 2 coincides practically with the corresponding curve in [7].

Let us consider now Eq. (11) and write it in the form

$$N^4 + bN^3 + dN + e = 0, \quad b = 1,185\,\widetilde{\delta}_* P - 0,8775\,\widetilde{Q}, \quad d = -1,185\,\delta_*, \quad e = -1. \tag{25}$$

The roots of that equation coincide the roots of the two equations

$$N^2 + (b + B_{1,2})\frac{N}{2} + \left(y_1 + \frac{by_1 - d}{B_{1,2}}\right) = 0 \text{ and } B_{1,2} = \pm\sqrt{8(y_1 + b^2/8)}.$$

Introduce the notation

$$\gamma_1 = y_1 + b^2/8, \quad \gamma_2 = y_1 - b^2/8.$$

Then the roots of these equations take the form

$$N_{1,2} = -\frac{\sqrt{8\,\gamma_1} + b}{4} \pm \sqrt{-\frac{by_1 - d}{\sqrt{8\,\gamma_1}} - \frac{b\sqrt{8\,y_1 - 4\,\gamma_2}}{8}}, \tag{26}$$

$$N_{3,4} = \frac{\sqrt{8\,\gamma_1} - b}{4} \pm \sqrt{\frac{by_1 - d}{\sqrt{8\,\gamma_1}} - \frac{b\sqrt{8\,y_1 + 4\,\gamma_2}}{8}}, \tag{27}$$

where y_1 is any real root of the cubic equation $y^3 + 3py + 2q = 0$,

$$3p = 1 - \frac{1,185^2\,\widetilde{\delta}_*^{\,2} PM}{4}, \quad 2q = -\frac{1,185^2\,\widetilde{\delta}_*^{\,2}(1 - P^2M^2)}{8}, \quad M = 1 - 0,7405\,\widetilde{Q}/\widetilde{\delta}_*\,P$$

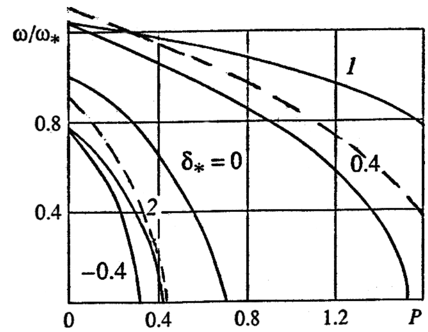

Fig. 3.

for

$$\frac{1,185^2\,\widetilde{\delta}_*^2|PM|}{4} \ll 1 \;\; (\widetilde{\delta}_* \le 0,5,\;\; PM \le 0,5), \quad p = \frac{1}{3}. \;\; q = -1,185^2\,\widetilde{\delta}_*^2(1 - P^2M^2)/16. \tag{28}$$

Since the discriminant of that equation $D > 0$, we have one real root

$$y_1 = (-q + \sqrt{D})^{1/3} + (-q - \sqrt{D})^{1/3}, \quad \sqrt{D} = \sqrt{1 + 0,208\,\widetilde{\delta}_*^4(1 - P^2M^2)^2}/3^{3/2}.$$

If we take

$$0,208\,\widetilde{\delta}_*^4(1 - P^2M^2) \ll 1 \tag{29}$$

then in analogy with the above-said, we obtain $y_1 = 0,1755\,\widetilde{\delta}_*^2(1 - P^2M^2)$. Under the restrictions (28), inequality (29) is all the more fulfilled.

Substituting the values $y_1,\ b,\ d,\ \gamma_1,\ \gamma_2$ into (26) and (27) and taking into account inequality (28), we find that for $d > 0$ ($\delta_* < 0$), positive is only the root N_1, whereas for $d < 0$ ($\delta_* > 0$), only the root N_3. As a result, we have

$$N_1 = \left[1 + 0,1755\,\widetilde{\delta}_*^2 PM_1(1 - P^2M_1^2) - 0,0877\,\widetilde{\delta}_*^2(1 + 2\,PM_1 - 2\,P^2M_1^2)\right]^{1/2} + 0,2962\,\widetilde{\delta}_*(1 - PM_1) \qquad (\delta_* > 0), \tag{30}$$

$$N_3 = \left[1 + 0,1755\,\widetilde{\delta}_*^2 PM_2(1 - P^2M_2^2) - 0,0877\,\widetilde{\delta}_*^2(1 + 2\,PM_2 - 2\,P^2M_2^2)\right]^{1/2} - 0,2962\,|\widetilde{\delta}_*|(1 - PM_2) \qquad (\delta_* < 0), \tag{31}$$

$$M_1 = 1 - 0,7405\,\widetilde{Q}/\widetilde{\delta}_* P, \quad M_2 = 1 + 0,7405\,\widetilde{Q}/|\widetilde{\delta}_*|P.$$

For $\widetilde{\delta}_* > 0$, $P/\widetilde{Q} > 0$, the value $M_1 = 0$, if $\widetilde{\delta}_* = 0,7405\,P/\widetilde{Q}$; for $\widetilde{\delta}_* < 0$, $P/\widetilde{Q} < 0$, the value $M_2 = 0$, if $|\widetilde{\delta}_*| = -0,7405\,P/\widetilde{Q}$. In addition, formulas (30) and (31) take the form

$$N = \sqrt{1 - 0,0877\,\widetilde{\delta}_*^2} + 0,2962\,\widetilde{\delta}_* \quad (\delta_* > 0),$$

$$N = \sqrt{1 - 0,0877\,\widetilde{\delta}_*^2} - 0,2962\,|\widetilde{\delta}_*| \quad (\delta_* < 0).$$

Note that this case with definite values $\widetilde{\delta}_*$ corresponds to the cases where the normal circumferential stresses under the action of meridional loading, external pressure and temperature neutralize each other.

For $\gamma_0 = 0$, we have $\widetilde{\delta}_* = \delta_*$, and hence for N we obtain formulas given in [7]. In the sequel, we adopt that $\gamma = \gamma_0$.

Substituting the values of N, owing to formulas (30) and (31) for the fixed values $(\widetilde{\delta}_*,\ P,\ \widetilde{Q},\ \gamma)$ into (10), we obtain the least value for dimensionless frequency ω/ω_*. In Fig. 3 are given the values ω/ω_* depending on P for the relation $\widetilde{Q} = 0,54\,P$ (for $\delta_* = 0,4;\ 0;\ -0,4$) and ($\gamma_* = 0;\ 1,272$). The corresponding dependencies for $\gamma_* = 0$ are given by firm curves and for $\gamma_* = 1,272$ by dotted ones. Moreover, for comparison, the curves of dependence of the least frequency on P, when $\widetilde{Q} = 0$, $\gamma = 0$ for $\delta_* = 0,4;\ -0,4$ are denoted, respectively, by 1 and 2.

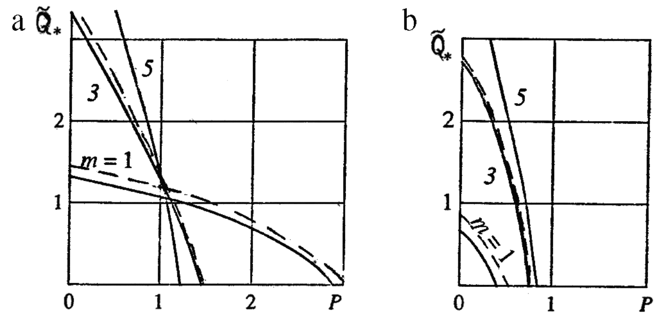

Fig. 4.

Next, consider the values $m > 1$. Using notation (9), we can represent formula (8) in the form

$$\omega^2/\omega_*^2 = 0,5\,m^2\big[\theta^2 + \theta^{-2} + 2,37\,\delta_*\theta^{-1}m^{-1} + 1,404\,\widetilde{\delta}_*^{\,2}m^{-2} - 2P(1 - 1,185\,\widetilde{\delta}_*\theta m^{-1}) - 1,755\,\widetilde{Q}\theta m^{-1}\big], \tag{32}$$

$$\theta = N/m. \tag{33}$$

For $\delta_* = 0$, formula (32) takes the form

$$\omega^2/\omega_*^2 = 0,5\,m^2\big[\theta^2 + \theta^{-2} + 1,404\,\gamma_*/4m^2 - 2P(1 + 1,185\,\nu\gamma_* g^{-1}\theta/m) - 1,755\,\widetilde{Q}\theta m^{-1}\big]. \tag{34}$$

For $\omega = 0$, $\gamma_* = 0$ we obtain

$$1,755\,\widetilde{Q} = m(\theta + \theta^{-3} - 2P\theta^{-1}).$$

Thus we can see that for the cylindrical shell the least value $\widetilde{Q} > 0$ depending on m will be for $m = 1$ (when $P < 1$), whereas depending on θ, it will be for

$$\theta^2 = \sqrt{3 + P^2} - P$$

which completely coincides with the result in [7].

Let us define how ω^2 varies as m varies for $\delta_* = 0$. Towards this end, we represent (34) in the form

$$\omega^2/\omega_*^2 = 0,5\,m^2\Big[(\theta^2 + \theta^{-2}) - 2P\Big(1 + 0,5925\,\nu\gamma_* g^{-1}\frac{\theta}{m}\Big) - 1,755\,\widetilde{Q}\frac{\theta}{m}\Big] + 1,404\,\gamma_*/4. \tag{34$'$}$$

When m in square brackets increases, the last two terms in (34′) decrease ($P > 0$, $\widetilde{Q} > 0$). Consequently, the expression in the square brackets increases, and moreover, the factor m^2 increases. Therefore the least value ω is realized when $m = 1$. The least value of ω from θ (for fixed m) will be defined below as a particular case of a more general case, when $\delta_* \neq 0$.

Consider now the expression allowing us to define $\widetilde{Q} > 0$ for $\delta_* \neq 0$. The right-hand side in the relation (32) vanishes for

$$1,755\,\widetilde{Q} = m(\theta + \theta^{-3} - 2\,P\theta^{-1}) + 2,37\,\delta_*\theta^{-2} + 1,404\,\overline{\delta}_*^{\,2}\theta^{-1}m^{-1} + 2,37\,\widetilde{\delta}_* P.$$

In the sequel, taking into account inequality (1), we restrict ourselves to the consideration of $|\delta_*| \leq 1$.

The value of θ realizing the least value of $\widetilde{Q}$ (for fixed m) can be defined by means of the positive root of the equation

$$\theta^4 + (2\,P - 1,404\,\overline{\delta}_\nu^{\,2})\theta^2 - 4,74\,\delta_\nu\theta - 3 = 0, \quad \delta_\nu = \delta_*/m, \quad \overline{\delta}_\nu^{\,2} = (\delta_*^2 + \gamma/4)m^2.$$

Analogously to the above-said, taking into account inequality (20) (with δ_* replaced by δ_ν), we obtain

$$\theta = \sqrt{\sqrt{3} + 0,234\Big(\delta_*^2 + \frac{3}{4}\gamma_*\Big)\Big/m^2 - P} + 0,684\,\delta_*/m \quad (\delta_* > 0),$$

$$\theta = \sqrt{\sqrt{3} + 0,234\Big(\delta_*^2 + \frac{3}{4}\gamma_*\Big)\Big/m^2 - P} - 0,684\,|\delta_*|/m \quad (\delta_* < 0).$$

In the case if the values δ_*, γ_*, P do not satisfy these inequalities, we have to proceed from the complete expressions for the roots θ_1 and θ_3. They are of the same form as N_1 and N_3 defined by equalities (14), where δ_* is replaced by δ_ν, and $\overline{\delta}_*^{\,2}$ by $\overline{\delta}_\nu^{\,2}$.

In Fig. 4, we can see critical values of $\widetilde{Q}(m, P)$ when $m = 1, 3, 5$ for ($\gamma_* = 0$; $1, 272$), when $\delta_* = 0, 4$ (Fig. 4a) and $\delta_* = -0, 4$ (Fig. 4b). The corresponding dependencies for $\gamma_* = 0$ are presented by firm curves, and for $\gamma_* = 1, 272$ by dotted ones. It is easily seen that for $\delta_* > 0$ and $P < 1$, the least value of $\widetilde{Q}_*$ is realized, independently of γ_*, for $m = 1$; whereas for P, approaching from the above to unity, the critical value of $\widetilde{Q}_*$ is realized for large m. For $\delta_* < 0$, the least value of $\widetilde{Q}_*$ is realized for $m = 1$, when $0 \le P \le P_*$ (P_* is the critical value of P for $\widetilde{Q} = 0$).

Consider now the expression (32). The least value ω^2 with respect to θ (for the fixed m) is defined by the condition

$$(\omega^2)'_\theta = 0, 5\, m^2\big(2\,\theta - 2\,\theta^{-1} - 2, 37\,\delta_\nu\theta^{-2} + 2, 37\,P\delta_\nu - 1, 755\,\widetilde{Q}_\nu\big) = 0, \quad \widetilde{Q}_\nu = \widetilde{Q}/m, \quad \delta_\nu = \delta_*/m$$

from which it follows that

$$\theta^4 + b\theta^3 + d\theta + e = 0, \quad b = 1, 185\,\widetilde{\delta}_\nu P - 0, 8775\,\widetilde{Q}_\nu, \quad d = -1, 185\,\delta_\nu, \quad e = -1.$$

This equation is of the same form as (25), where δ_* is replaced by δ_ν, $\widetilde{\delta}_*$ by $\widetilde{\delta}_\nu$ and $\widetilde{Q}$ by $\widetilde{Q}_\nu$. Therefore, similarly to the above-said, under the restrictions (34), we obtain

$$\theta = \sqrt{1 + 1, 755\,\widetilde{\delta}_\nu^{\,2} P M_1(1 - P^2 M_1^2) - 0, 08775\,\widetilde{\delta}_\nu^{\,2}(1 + 2\,P M_1 - 2\,P^2 M_1^2)} + 0, 2962\,\widetilde{\delta}_*(1 - P M_1), \quad (\delta_* > 0), \tag{35}$$

$$\theta = \sqrt{1 + 1, 755\,\widetilde{\delta}_\nu^{\,2} P M_2(1 - P^2 M_2^2) - 0, 08775\,\widetilde{\delta}_\nu^{\,2}(1 + 2\,P M_2 - 2\,P^2 M_2^2)} - 0, 2962\,|\widetilde{\delta}_*|(1 - P M_2) \quad (\delta_* < 0), \tag{36}$$

where

$$M_{1,2} = 1 \mp \big(0, 7405\,\widetilde{Q}/|\widetilde{\delta}_*|P\big) \quad \widetilde{\delta}_\nu = \widetilde{\delta}_*/m. \tag{37}$$

The indices (1) and (2) correspond to $\delta_* > 0$ and $\delta_* < 0$, respectively. In the case if the values δ_ν, P, $\widetilde{Q}$ do not satisfy inequality (29) (where δ_* has to be replaced by δ_ν), then we have to proceed from the full expressions for the roots θ_1 and θ_3. They have the same form as N_1 and N_3 which are defined by equalities (26) and (27), where δ_* has to be replaced by δ_ν, and $M_{1,2}$ have the form (37), since $\widetilde{Q}_\nu/|\widetilde{\delta}_\nu| = Q/|\delta_*|$.

For $\delta_* = 0$, $P = \widetilde{Q} = \gamma = 0$ these formulas yield $\theta = N = 1$ ($m = 1$). For $\delta_* \neq 0$, we get formulas derived in [5].

For $\delta_* \neq 0$, $P \neq 0$, $\widetilde{Q} = \gamma = 0$ we obtain $M_{1,2} = 1$, and formulas (35) and (36) will have the form

$$\theta = \sqrt{1 + 1, 755\,\delta_\nu^2 P(1 - P^2) - 0, 08775\,\delta_\nu^2(1 + 2\,P - 2\,P^2)} + 0, 2962\,\delta_\nu(1 - P) \quad (\delta_* > 0),$$

$$\theta = \sqrt{1 + 1, 755\,\delta_\nu^2 P(1 - P^2) - 0, 08775\,\delta_\nu^2(1 + 2\,P - 2\,P^2)} - 0, 2962\,|\delta_\nu|(1 - P) \quad (\delta_* < 0).$$

For $\delta_* \neq 0$, $Q \neq 0$, $P = \gamma = 0$, formulas (35) and (36) take the form

$$\theta = \sqrt{1 - 0, 1295\,\delta_\nu\widetilde{Q} - 0, 08775\,(\delta_\nu^2 - 1, 48\,\delta_\nu\widetilde{Q} - 1, 097\,\widetilde{Q}^{\,2})} + 0, 2962\,(\delta_\nu + 0, 7405\,\widetilde{Q}) \quad (\delta_* > 0),$$

$$\theta = \sqrt{1 - 0, 1295\,\delta_\nu\widetilde{Q} - 0, 08775\,(\delta_\nu^2 + 1, 48\,|\delta_\nu|\widetilde{Q} - 1, 097\,\widetilde{Q}^{\,2})} - 0, 2962\,(|\delta_\nu| - 0, 7405\,\widetilde{Q}) \quad (\delta_* < 0).$$

In Fig. 5 we can see the least values of frequencies $\omega(m, P, \widetilde{Q})$, when $\widetilde{Q} = 0, 54\,P$ for $m = 1, 3, 5$; $\gamma_* = 0; 1, 272$ for $\delta_* = 0, 4$ (Fig. 5a) and $\delta_* = -0, 4$ (Fig. 5b).

The corresponding dependencies for $\gamma_* = 0$ are presented by firm curves and for $\gamma_* = 1, 272$ by dotted ones. It is not difficult to see that for $\delta_* > 0$, when P varies in the interval $0 \le P < 1$, the least is the frequency for $m = 1$, whereas for P, approaching from the above to unity, the least frequency is realized for large values m. For $\delta < 0$, when P varies in the interval $0 \le P \le P_*$ (P_* is the critical value of P for $\widetilde{Q} = 0$), the least frequency is realized for $m = 1$.

It follows from the above formulas (35), (36) and (33) that for $m > 1$ ($0 \le P < 1$) the values θ are close to unity, i.e., when $n^2 \approx \lambda_m \varepsilon^{-1/4}$. Therefore the obtained result is valid only for sufficiently thin shells, when $\varepsilon^{-1/4} \gg \lambda_m^2$;

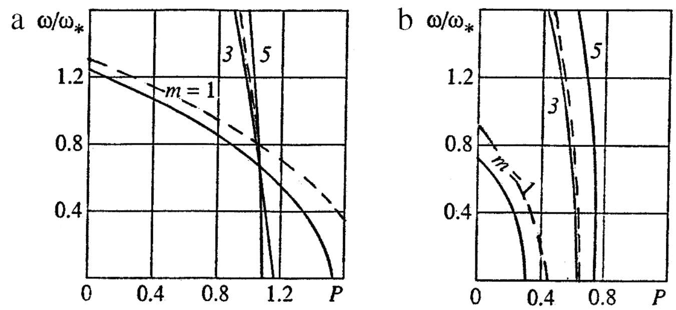

Fig. 5.

then the relation $n^2 \gg \lambda_m^2$ holds, and hence the given theory is valid. Moreover, on the basis of formula (32) we find that for comparatively large m, when $\theta \approx 1$, $\omega^2/\omega_*^2 \approx 0$, $5\,m^2(\theta^2+\theta^{-2}-2P) \approx m^2(1-P)$, i.e., the influence of δ_* and $\widetilde{Q}$ may practically be neglected.

Thus we have shown that if stresses arise in the considered shells, from the action of external pressure, temperature and filler constraint, change essentially the lowest frequencies, then the influence of these factors on comparatively higher frequencies is practically inessential. At the same time, the influence of meridional loading is essential both for the lowest and for the highest frequencies.

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Original article

An approximate solution of one class of singular integro-differential equations

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Abstract

The problem of definition of mechanical field in a homogeneous plate supported by finite inhomogeneous inclusion is considered. The contact between the plate and inclusion is realized by a thin glue layer. The problem is reduced to the boundary value problem for singular integro-differential equations. Asymptotic analysis is carried out. Using the method of orthogonal polynomials, the problem is reduced to the solution of an infinite system of linear algebraic equations. The obtained system is investigated for regularity.

Keywords: Elastic inclusion; Integro-differential equation; Orthogonal polynomials

1. Statement of the Problem and its Reduction to a Singular Integro-Differential Equation (SIDE)

Let an elastic plane with the modulus of elasticity E_2 and the Poisson coefficient ν_2 on a finite interval $[-1, 1]$ of the ox-axis be reinforced by an inclusion in the form of a cover plate of small thickness $h_1(x)$, with the modulus of elasticity $E_1(x)$ and the Poisson coefficient ν_1, loaded by tangential force of intensity $\tau_0(x)$, and the plate at infinity towards to the ox and oy-axes be subjected to uniformly stretching forces of intensities p and q, respectively.

Under the conditions of plane deformation we are required to determine contact stresses acting in the interval of the inclusion and plate joint. An inclusion will be assumed to be a thin plate free from bending rigidity, and the contact between the plate and inclusion is realized by a thin glue layer with thickness h_0 and modulus of shear G_0.

Equation of equilibrium of differential element of inclusion has the form [1]

$$\frac{d}{dx}\Big(E(x)\frac{du_1(x)}{dx}\Big) = \tau_-(x) - \tau_+(x) - \tau_0(x), \quad |x| < 1, \tag{1}$$

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where $\tau_{\pm}(x)$ are unknown tangential contact stresses at the upper and lower contours of the inclusion, $u_1(x)$ is horizontal displacement of inclusion points towards the ox-axis, $E(x) = \frac{E_1(x)h_1(x)}{1-\nu_1^2}$. Introducing the notation $\tau(x) := \tau_-(x) - \tau_+(x)$ and based on Eq. (1), deformation of points of inclusion can be expressed as

$$\varepsilon_x^{(1)} := \frac{du_1(x)}{dx} = \frac{1}{E(x)} \int_{-1}^{x} [\tau(t) - \tau_0(t)]dt, \quad |x| < 1. \tag{2}$$

The condition of equilibrium of the inclusion has the form

$$\int_{-1}^{1} [\tau(t) - \tau_0(t)]dt = 0. \tag{3}$$

Assuming that every element of the glue layer is under the conditions of pure shear, the contact condition has the form [2]

$$u_1(x) - u_2(x, 0) = k_0\tau(x), \quad |x| \le 1, \tag{4}$$

where $u_2(x, y)$ are displacement of the plate points along the ox-axis, $k_0 := h_0/G_0$.

On the basis of the well-known results (see, e.g., [3]), the deformation $\varepsilon_x^{(2)} := \frac{du_2(x,0)}{dx}$ of the plane point along the ox-axis caused by the force factors $\tau(x)$, p and q is represented in the form

$$\varepsilon_x^{(2)} = \frac{\aleph}{2\pi\mu_2(1+\aleph)} \int_{-1}^{1} \frac{\tau(t)dt}{t-x} + \frac{\aleph+1}{8\mu_2} p + \frac{\aleph-3}{8\mu_2} q, \tag{5}$$

where $\aleph = 3 - 4\nu_2$, while λ_2 and μ_2 are the Lamé parameters.

Taking into account (2) and (5), from the contact conditions (4), we get

$$\frac{1}{E(x)} \int_{-1}^{x} [\tau(t)dt - \tau_0(t)]dt - \frac{\aleph}{2\pi\mu_2(1+\aleph)} \int_{-1}^{1} \frac{\tau(t)dt}{t-x} - \frac{\aleph+1}{8\mu_2} p - \frac{\aleph-3}{8\mu_2} q = k_0\tau'(x), \quad |x| < 1. \tag{6}$$

In the notations

$$\varphi(x) = \int_{-1}^{x} [\tau(t) - \tau_0(t)]dt, \quad \lambda = \frac{\aleph}{2\mu_2(1+\aleph)},$$

$$g(x) = \frac{\lambda}{\pi} \int_{-1}^{1} \frac{\tau_0(t)dt}{t-x} + k_0\tau_0'(x) + \frac{\aleph+1}{8\mu_2} p + \frac{\aleph-3}{8\mu_2} q,$$

we rewrite Eq. (6) in the form

$$\frac{\varphi(x)}{E(x)} - \frac{\lambda}{\pi} \int_{-1}^{1} \frac{\varphi'(t)dt}{t-x} - k_0\varphi''(x) = g(x), \quad |x| < 1. \tag{7}$$

Thus the equilibrium condition (3) takes the form

$$\varphi(1) = 0. \tag{8}$$

Thus the above posed boundary contact problem is reduced to the solution of SIDE (7) with the condition (8). From the symmetry of the problem, we assume, that function $E(x)$ is even and external load $\tau_0(x)$ is uneven, the solution of Eq. (7) under the condition (8) can be sought in the class of even functions. Moreover, we assume that the function is continuous and has a continuous first order derivative on the interval $[-1, 1]$.

2. Asymptotic investigation

Under the assumption that

$$E(x) = (1-x^2)^{\gamma} b_0(x), \quad \gamma \ge 0, \quad b_0(x) = b_0(-x), \quad b_0 \in C([-1, 1]), \tag{9}$$
$$b_0(x) \ge c_0 = \text{const} > 0$$

a solution of problem (7), (8) will be sought in the class of even functions whose derivatives are representable in the form

$$\varphi'(x) = (1-x^2)^{\alpha} g_0(x), \quad \alpha > -1, \tag{10}$$

where $g_0(x) = -g_0(-x)$, $g_0 \in C'([-1,1])$, $g_0(x) \neq 0$, $x \in [-1,1]$.

Taking into account the following asymptotic formulas [4], for $-1 < \alpha < 0$, we have

$$\int_{-1}^{1} \frac{(1-t^2)^{\alpha} g_0(t)dt}{t-x} = \mp\pi \operatorname{ctg} \pi\alpha\, g_0(\mp 1) 2^{\alpha} (1 \pm x)^{\alpha} + \Phi_{\mp}(x), \quad x \to \mp 1,$$

where $\Phi_{\mp}(x) = \Phi^*_{\mp}(x)(1 \pm x)^{\alpha_{\mp}}$, $\Phi^*_{\mp}$ belongs to the class H in the neighbourhoods of the points $x = \mp 1$, $\alpha_{\mp} = \text{const} > \alpha$;

If $\alpha = 0$, we have

$$\int_{-1}^{1} \frac{g_0(t)dt}{t-x} = \mp g_0(\mp 1) \ln(1 \pm x) + \widetilde{\Phi}_{\mp}(x), \quad x \to \mp 1,$$

where $\widetilde{\Phi}_{\pm}(x)$ satisfies the H condition in the neighbourhoods of the points $x = \mp 1$, respectively.

If $\alpha > 0$, the function $\Phi_0(x) := \int_{-1}^{1} \frac{(1-t^2)^{\alpha} g_0(t)dt}{t-x}$ belongs to the class H in the neighbourhoods of the points $x = \pm 1$. Moreover, we have [5]

$$\int_{-1}^{x} (1-t^2)^{\alpha} g_0(t)dt = \frac{2^{\alpha}(1 \pm x)^{\alpha+1}}{\alpha+1} g_0(\mp 1) F(\alpha+1, -\alpha, 2+\alpha, (1 \pm x)/2) + G_{\mp}(x), \quad x \to \mp 1,$$

where $F(a,b,c,x)$ is the Gaussian hypergeometric function, $\lim_{x\to\mp 1} G_{\mp}(x)(1 \pm x)^{\alpha+1} = 0$.

In the case of the condition $-1 < \alpha < 0$, Eq. (7) in the neighbourhoods of the points $x = -1$ takes the form

$$\lambda \operatorname{ctg} \pi\alpha\, g_0(-1) 2^{\alpha} (1+x)^{\alpha} - \frac{\lambda}{\pi} \Phi_{-}(x) + \frac{2^{\alpha}(1+x)^{\alpha+1} g_0(-1)}{2^{\gamma}(\alpha+1)(1+x)^{\gamma} b_0(-1)} + G_{-}(x)$$
$$- k_0 2^{\alpha} (1+x)^{\alpha-1} \tilde{g}_0(-1) = g(-1), \quad \tilde{g}_0(x) = (1-x^2) g_0'(x) - 2x g_0(x)$$

which in the neighbourhoods of the points $x = -1$ is not satisfied. In the condition $-1 < \alpha < 0$, Eq. (7) has no solutions. Note, that the negative value of the index α contradicts the physical meaning of condition (4).

Let $0 \le \alpha \le 1$, then we have

$$\frac{\lambda}{\pi} g_0(-1) \ln(1+x) - \frac{\lambda}{\pi} \widetilde{\Phi}_{-}(x) + \frac{(1+x) g_0(-1)}{2^{\gamma}(1+x)^{\gamma} b_0(-1)} + G_{-}(x)$$
$$- k_0 (1+x)^{-1} \tilde{g}_0(-1) = g(-1), \tag{11}$$

for $\alpha = 0$, and

$$-\frac{\lambda}{\pi} \Phi_0(x) + \frac{2^{\alpha}(1+x)^{\alpha+1} g_0(-1)}{2^{\gamma}(\alpha+1)(1+x)^{\gamma} b_0(-1)} + G_{-}(x) - k_0 2^{\alpha} (1+x)^{\alpha-1} \tilde{g}_0(-1) = g(-1) \tag{12}$$

for $0 < \alpha \le 1$.

Multiplying now both sides of relations (11) $(1+x)^{1+\varepsilon}$ and (12) by $(1+x)^{1+\varepsilon-\alpha}$ (ε is an arbitrarily small positive number), we obtain

$$\lambda g_0(-1)(1+x)^{1+\varepsilon} \ln(1+x) - \frac{\lambda}{\pi}(1+x)^{1+\varepsilon} \widetilde{\Phi}_{-}(x)$$
$$+ \frac{(1+x)^{2+\varepsilon} g_0(-1)}{2^{\gamma}(1+x)^{\gamma} b_0(-1)} + G_{-}(x)(1+x)^{1+\varepsilon} - k_0 2^{\alpha} (1+x)^{\varepsilon} \tilde{g}_0(-1)$$
$$= g(-1)(1+x)^{1+\varepsilon}$$

and

$$-\frac{\lambda}{\pi}(1+x)^{1+\varepsilon-\alpha}\Phi_0(x)+\frac{2^{\alpha}(1+x)^{2+\varepsilon}g_0(-1)}{2^{\gamma}(\alpha+1)(1+x)^{\gamma}b_0(-1)}+G_-(x)(1+x)^{1+\varepsilon-\alpha}$$
$$-k_0 2^{\alpha}(1+x)^{\varepsilon}\tilde{g}_0(-1)=g(-1)(1+x)^{1+\varepsilon-\alpha}.$$

When passing to the limit $x\to-1$, analysis of the obtained equalities shows that the inequality $2+\varepsilon>\gamma$, i.e. $\gamma\le 2$, needs to be fulfilled.

If $\alpha>1$, then from relation (12) it follows that $\alpha=\gamma-1$.

Analogous result is obtained in the neighbourhoods of the points $x=1$.

Thus we have proved the following statement: When fulfilling condition (9), if problem (7), (8) has a solution whose derivative is representable in the form (10), then we have: if $\gamma>2$, then $\alpha=\gamma-1$, $(\alpha>1)$; if $\gamma\le 2$, then $0\le\alpha\le 1$.

From the relation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{(1-t)^{\alpha}(1+t)^{\beta}P_m^{(\alpha,\beta)}(t)dt}{t-x}=\operatorname{ctg}\pi\alpha(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)$$
$$-\frac{2^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta+m+1)}{\pi\Gamma(\alpha+\beta+m+1)}F(m+1,\ -\alpha-\beta-m,\ 1-\alpha,\ (1-x)/2)$$

obtained by Tricomi [6] for orthogonal Jacobi polynomials $P_m^{(\alpha,\beta)}$ and from the well-known equality (see, e.g., [7])

$$m!P_m^{(\alpha,\beta)}(1-2x)=\frac{\Gamma(\alpha+m+1)}{\Gamma(1+\alpha)}F(\alpha+\beta+m+1,\ -m,\ 1+\alpha,\ x)$$

we get the following spectral relation for the Hilbert singular operator

$$\int_{-1}^{1}\frac{(1-t^2)^{n-1/2}P_m^{(n-1/2,n-1/2)}(t)dt}{t-x}=-2^{2n-1}\Gamma(n-1/2)\Gamma(3/2-n)P_{m+2n-1}^{(1/2-n,1/2-n)}(x), \tag{13}$$

where $\Gamma(z)$ is the known Gamma function.

If the inclusion rigidity varies by the law

$$E(x)=(1-x^2)^{n+\frac{1}{2}}b_0(x),$$

where $b_0(x)>0$ for $|x|\le 1$, $b_0(x)=b_0(-x)$, $n\ge 0$ is integer, then following from the above asymptotic analysis, we obtain $\alpha=n-\frac{1}{2}$ for $n=2,3,\ldots$ and $0<\alpha<1$ for $n=0$ or $n=1$ (the same result is obtained for $E(x)=b_0(x)>0$, or $E(x)=\text{const}$, $|x|\le 1$).

3. An approximate solution of SIDE (7)

On the basis of the above asymptotic analysis performed in the cases $n=0$, $n=1$, $E(x)=b_0(x)>0$, $E(x)=\text{const}$, $|x|\le 1$ a solution of Eq. (7) will be sought in the form

$$\varphi'(x)=\sqrt{1-x^2}\sum_{k=1}^{\infty}X_kP_k^{(1/2,1/2)}(x), \tag{14}$$

where the numbers X_k have to be defined, $k=1,2,\ldots$.

Using the relations arising from (13) and from the Rodrigue formula (see [8, p. 107]), for the orthogonal Jacobi polynomials, we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}P_k^{(1/2,1/2)}(t)dt}{t-x}=-2\pi P_{k+1}^{(-1/2,-1/2)}(x),$$

$$\varphi(x)=-(1-x^2)^{3/2}\sum_{k=1}^{\infty}\frac{X_k}{2k}P_{k-1}^{(3/2,3/2)}(x),\quad \varphi''(x)=-2(1-x^2)^{-1/2}\sum_{k=1}^{\infty}kX_kP_{k+1}^{(-1/2,-1/2)}(x). \tag{15}$$

Substituting relations (14), (15) into Eq. (7), we have

$$-\frac{(1-x^2)^{3/2}}{E(x)}\sum_{r=1}^{\infty}\frac{X_k}{2k}P_{k-1}^{(3/2,3/2)}(x)-2\lambda\sum_{k=1}^{\infty}X_kP_{k+1}^{(-1/2,-1/2)}(x)$$
$$+2k_0(1-x^2)^{-1/2}\sum_{k=1}^{\infty}kX_kP_{k+1}^{(-1/2,-1/2)}(x)=g(x),\quad |x|\le 1. \tag{16}$$

Multiplying both parts of equality (16) by $P_{m+1}^{(-1/2,-1/2)}(x)$ and integrating in the interval $(-1, 1)$, we obtain an infinite system of linear algebraic equations of the type

$$k_0m\left(\frac{\Gamma(m+3/2)}{\Gamma(m+2)}\right)^2X_m-\sum_{k=1}^{\infty}\left(R_{mk}^{(1)}+\frac{R_{mk}^{(2)}}{k}\right)X_k=g_m,\quad m=1,2,\ldots, \tag{17}$$

where

$$R_{mk}^{(1)}=-2\lambda\int_{-1}^{1}P_{k+1}^{(-1/2,-1/2)}(x)P_{m+1}^{(-1/2,-1/2)}(x)dx,$$
$$R_{mk}^{(2)}=\frac{1}{2}\int_{-1}^{1}\frac{(1-x^2)^{3/2}}{E(x)}P_{k-1}^{(3/2,3/2)}(x)P_{m+1}^{(-1/2,-1/2)}(x)dx,$$
$$g_m=\int_{-1}^{1}g(x)P_{m+1}^{(-1/2,-1/2)}(x)dx.$$

Investigating system (17) for regularity in the class of bounded sequences and using the known relations for the Chebyshev first order polynomials and for the function $\Gamma(z)$ (see [5, pp. 584, 83]),

$$P_m^{(-1/2,-1/2)}(x)=\frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)}T_m(x),\quad T_m(\cos\theta)=\cos m\theta$$
$$\lim_{m\to\infty}m^{b-a}\frac{\Gamma(m+a)}{\Gamma(m+b)}=1,$$

we obtain

$$\begin{aligned}R_{mk}^{(1)}&=-\frac{2\lambda\alpha(k)\beta(m)}{\pi\sqrt{(k+1)m+1}}\int_0^{\pi}\cos(k+1)\theta\cos(m+1)\theta\sin\theta d\theta\\
&=-\frac{2\lambda\alpha(k)\beta(m)}{\pi\sqrt{(k+1)(m+1)}}\\
&\quad\times\begin{cases}1-\dfrac{1}{(2m+3)(2m+1)}, & k=m,\\ -\dfrac{(-1)^{k+m}+1}{2}\left[\dfrac{1}{(k+m+3)(k+m+1)}+\dfrac{1}{(k-m+1)(k-m-1)}\right], & k\ne m,\end{cases}\\
&=\begin{cases}O(m^{-1}), & k=m,\quad m\to\infty,\\ O(m^{-5/2}),\quad O(k^{-5/2}), & k\ne m,\quad m\to\infty,\quad k\to\infty,\end{cases}\end{aligned}$$

$\alpha(k), \beta(m)\to 1$ for $k, m\to\infty$. Introducing the notation $\widetilde{X}_m=\omega_mX_m$, where $\omega_m=m\left(\frac{\Gamma(m+3/2)}{\Gamma(m+2)}\right)^2\to 1, m\to\infty$, system (17) will take the form

$$k_0\widetilde{X}_m-\sum_{k=1}^{\infty}\left(\frac{R_{mk}^{(1)}}{\omega_k}+\frac{R_{mk}^{(2)}}{\omega_kk}\right)\widetilde{X}_k=g_m,\quad m=1,2,\ldots. \tag{18}$$

By virtue of the Darboux asymptotic formula (see [8, p. 175]), we obtain analogous estimates likewise for $R_{mk}^{(2)}$, and the right-hand side g_m of Eq. (18) satisfies at least the estimate

$$g_m = O(m^{1/2}), \quad m \to \infty.$$

However, if $n = 2$, a solution of Eq. (7) will be sought in the form

$$\varphi'(x) = (1 - x^2)^{3/2} \sum_{k=1}^{\infty} Y_k P_k^{(3/2,3/2)}(x), \tag{19}$$

where the numbers Y_k are to be defined, $k = 1, 2, \dots$.

Using the relations arising from (13) and from the Rodrigue formula for the orthogonal Jacobi polynomials, we get

$$\frac{1}{\pi} \int_{-1}^{1} \frac{(1 - x^2)^{3/2} P_k^{(3/2,3/2)}(t)dt}{t - x} = -2\pi P_{k+1}^{(-3/2,-3/2)}(x),$$

$$\varphi(x) = -(1 - x^2)^{5/2} \sum_{k=1}^{\infty} \frac{Y_k}{2k} P_{k-1}^{(5/2,5/2)}(x), \quad \varphi''(x) = -2(1 - x^2)^{1/2} \sum_{k=1}^{\infty} k Y_k P_{k+1}^{(1/2,1/2)}(x). \tag{20}$$

Substituting relations (19), (20) into Eq. (7) we obtain

$$-\frac{1}{b_0(x)} \sum_{r=1}^{\infty} \frac{Y_k}{2k} P_{k-1}^{(5/2,5/2)}(x) - \frac{2\lambda \Gamma^2\big(1/2\big)}{\pi} \sum_{k=1}^{\infty} Y_k P_{k+1}^{(-3/2,-3/2)}(x)$$
$$+ 2k_0 (1 - x^2)^{1/2} \sum_{k=1}^{\infty} k Y_k P_{k+1}^{(1/2,1/2)}(x) = g(x), \quad |x| \le 1. \tag{21}$$

Reasoning analogous to that carried out for system (18), from (21) we obtain

$$4k_0 m \Big(\frac{\Gamma(m + 5/2)}{\Gamma(m + 3)}\Big)^2 Y_m - \sum_{k=1}^{\infty} \Big(R_{mk}^{(3)} + \frac{R_{mk}^{(4)}}{k}\Big) Y_k = \tilde{g}_m, \quad m = 1, 2, \dots, \tag{22}$$

where

$$R_{mk}^{(3)} = -2\lambda \int_{-1}^{1} P_{k+1}^{(-3/2,-3/2)}(x) P_{m+1}^{(1/2,1/2)}(x)dx,$$
$$R_{mk}^{(4)} = \frac{1}{2} \int_{-1}^{1} \frac{1}{b_0(x)} P_{k-1}^{(5/2,5/2)}(x)dx\, P_{m+1}^{(1/2,1/2)}(x)dx,$$
$$\tilde{g}_m = \int_{-1}^{1} g(x) P_{m+1}^{(1/2,1/2)}(x)dx.$$

Introducing the notation $\widetilde{Y}_m = \delta_m Y_m$, where $\delta_m = m\Big(\frac{\Gamma(m+5/2)}{\Gamma(m+3)}\Big)^2 \to 1$, $m \to \infty$, system (22) will take the form

$$4k_0 \widetilde{Y}_m - \sum_{k=1}^{\infty} \Big(\frac{R_{mk}^{(1)}}{\delta_k} + m \frac{R_{mk}^{(2)}}{\delta_k k}\Big) \widetilde{Y}_k = \widetilde{g}_m, \quad m = 1, 2, \dots. \tag{23}$$

Using again the Darboux formula, and the known relation for the Chebyshev second order polynomial (see [5, p. 584])

$$P_m^{(1/2,1/2)}(x) = \frac{\Gamma(m + 3/2)}{\sqrt{\pi}\,\Gamma(m + 2)} U_m(x), \quad U_m(\cos\theta) = \frac{\sin(n + 1)\theta}{\sin\theta},$$

for $R_{mk}^{(3)}$ and $R_{mk}^{(4)}$, we obtain the following estimates:

$$R_{mk}^{(3)} = \begin{cases} O(m^{-1}), & k = m, \quad m \to \infty, \\ O(m^{-5/2}), \quad O(k^{-5/2}), & k \neq m, \quad m \to \infty, \quad k \to \infty, \end{cases}$$

$$R_{mk}^{(4)} = \begin{cases} O(m^{-1}), & k = m, \quad m \to \infty, \\ O(m^{-1/2}), \quad O(k^{-1/2}), & k \neq m, \quad m \to \infty, \quad k \to \infty, \end{cases}$$

and for the right-hand side $\tilde{g}_m$ of Eq. (23) we have at least the estimate

$$\tilde{g}_m = O(m^{-1/2}), \quad m \to \infty.$$

Thus systems (18) and (23) are quasi-completely regular for any positive values of parameters k_0 and λ in the class of bounded sequences.

On the basis of the Hilbert alternatives [9,10], if the determinants of the corresponding finite systems of linear algebraic equations are other than zero, then systems (18) and (23) will have unique solutions in the class of bounded sequences. Therefore, by the equivalence of systems (18), (23) and SIDE (7) the latter has likewise a unique solution.

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