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Original article

Elastoplastic problem for a plate with partially unknown boundary

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Abstract

In this paper there is considered the Elastoplastic problem for infinite plate, that is weakened by two identical square holes. The boundaries of the holes are partially unknown contours. The plate is in a stressed state, a region of plasticity contains only unknown parts of holes contours and does not spread inside of the plate. Applying the theory of functions of a complex variable and the conformal mapping theory the problem is reduced to a boundary value problem of the analytic function theory and the solution of this problem is obtained, the unknown parts of the holes contours are defined.

Keywords: Stressed state; Region of plasticity; Conformal mapping; A linear conjugation value problem

Let us consider a homogeneous isotropic infinite plate weakened by two identical square-shaped holes. Assume that two absolutely rigid square-shaped washers are inserted into the holes and the friction is ignored. Assume that the plate undergoes compression under the action of principal stresses $\sigma_x^\infty = A$, $\sigma_y^\infty = B$ acting at infinity. Since the friction is ignored and the washers are absolutely rigid, we have the conditions $\tau_{tn} = 0$, $u_n = 0$ on the hole contours. Under such conditions, the behavior of stresses near the vertices of the squares (holes) might be singular and, naturally, there exists a probability that the plate will develop cracks at these very points.

If we consider a plate of this kind but with cuts at the vertices of the squares along the smooth contours (see Fig. 1), the stress concentration will be a different one. It is obvious that the distribution of stresses along the hole contours depends on the cut configuration and dimension. Let us consider such a plate and denote by S the domain occupied by it in the complex plane $z = x + iy$. The smooth contours, along which the cuts are made, are unknown parts of the plate boundary and we denote them by l_1, while the remaining rectilinear part of the boundary consists of the known lines and we denote them by l_0. The entire boundary contour is denoted by l. It is assumed that in the XOY coordinate system in the complex plane $z = x + iy$, the line l_0 consists of the segments parallel to the OX- and OY-axes, while the domain S is symmetric with respect to the coordinate axes.

It is assumed that the principal stresses are the known values

$$\sigma_x^\infty = A, \quad \sigma_y^\infty = B, \quad \tau_{xy}^\infty = 0. \tag{1}$$

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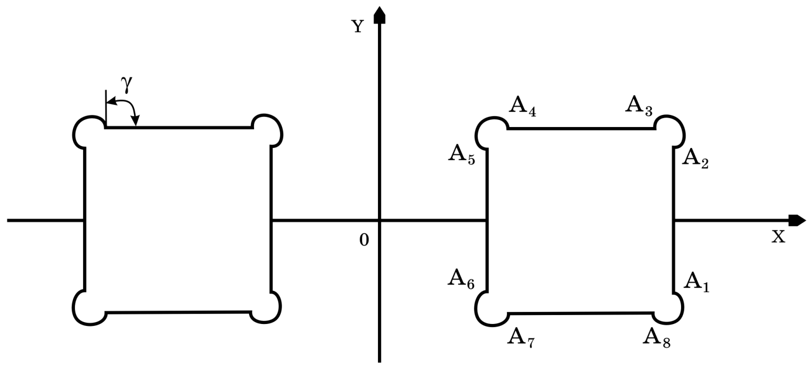

Fig. 1. The infinite plate weakened by two square-shaped holes.

The unknown part of the boundary is free from load, while the normal displacement on the rectilinear part is a constant value:

$$\sigma_n = 0, \quad t \in l_1, \tag{2}$$

$$u_n = const, \quad t \in l_0. \tag{3}$$

The friction is ignored throughout the boundary,

$$\tau_{tn} = 0, \quad t \in l. \tag{4}$$

Let us consider the following **problem**: Given conditions (1)–(4), define the shape of the sought line l_1, the part of the hole contour boundary of the considered plate and the stressed state of the plate with an additional assumption that the unknown part l_1 of the hole contour is in the plastic state, the plastic zone covering only the line l_1 and not spreading inwards the plate,

$$(\sigma_t - \sigma_n)^2 + 4\tau_{tn}^2 = 4b^2, \quad t \in l_1, \tag{5}$$

where σ_t is a tangential normal stress value.

After some elementary transformations on the basis of well-known Kolosov–Muskhelishvili formulae, we obtain the equalities

$$\sigma_n + i\tau_{tn} = \Phi(z) + \overline{\Phi(z)} - e^{-2i\alpha(t)}(z\overline{\Phi'(z)} + \overline{\Psi(z)}), \tag{6}$$

$$2\mu(u_t' - iu_n') = \varkappa\Phi(z) - \overline{\Phi(z)} + e^{-2i\alpha(t)}(z\overline{\Phi'(z)} + \overline{\Psi(z)}), \tag{7}$$

where $\varphi(z)$ and $\psi(z)$ or $\Phi(z)$ and $\Psi(z)$, ($\Phi(z) = \varphi'(z)$, $\Psi(z) = \psi'(z)$) are analytic functions in the domain S occupied by the body, $\alpha(t)$ is the angle between the OX-axis and the outward normal to the contour L at a point t.

By the plastic state equality (5), using conditions (2)–(4) we obtain the equality

$$4\,\mathrm{Re}\,\Phi(t) = \sigma_t + \sigma_n = 2b, \quad t \in l_1.$$

Using conditions (3), (4), from formulas (6), (7) we have the boundary condition

$$\mathrm{Im}\,\Phi(z) = 0, \quad t \in l_0.$$

In this case the analytic function $\Phi(z)$ have the form

$$\Phi(z) = \Gamma + \Phi_0(z), \tag{8}$$

where $\Phi_0(z)$ is a holomorphic function in the domain S that vanishes at the point at infinity,

$$4\,\mathrm{Re}\,\Gamma = \sigma_x^\infty + \sigma_y^\infty = A + B. \tag{9}$$

By equalities (8) and (9) we may conclude that the function $\Phi(z)$ is bounded at the point at infinity (the rotation angle at the point at infinity can be ignored since it does not influence the stressed state). Thus for the function $\Phi(z)$, which is holomorphic in the domain S and bounded at the point at infinity we obtain the following conditions

$$\operatorname{Re} \Phi(t) = p, \quad t \in l_1, \tag{10}$$

$$\operatorname{Im} \Phi(t) = 0, \quad t \in l_0, \tag{11}$$

$$|\Phi(t)| < C|z - A_k|^{-\varepsilon} \quad (k = 1, 2, \ldots, 8), \quad 0 \le \varepsilon < 1. \tag{12}$$

By the symmetry of the plate, for the vectors of stresses acting at the symmetric points z and $-z$ we have the equality

$$\vec{F}_n(z) = -\vec{F}_n(-z).$$

By virtue of this equality and taking into account the fact that the normals at the symmetric points can be regarded as lying in the opposite directions, we may conclude that the expressions $\sigma_n + i\tau_{tn}$ and $u_n + iu_t$ take equal values (at the symmetric points). Thus, using equalities (6), (7) we obtain the following equality for the function $\Phi(z)$,

$$\Phi(z) = \Phi(-z). \tag{13}$$

By the symmetry of the problem, the normal displacement and the tangential stress on the OY-axis are equal to zero and therefore it suffices to consider a part of the domain S, $\operatorname{Re} z > 0$. This part is denoted by D.

We denote by D_1 the external part of the unit circle of the plane ζ with center at the origin and cut along the real axis from the point $\zeta = m$ $(m > 1)$ to infinity.

Suppose the function $z = -i\sqrt{\omega(\zeta)}$ conformally maps the domain D onto the domain D_1, where $\omega(\zeta)$ is the analytic function in a domain $|\zeta| > 1$, equal to zero at the point $\zeta = m$ and having, for large $|\zeta|$, the form

$$\omega(\zeta) = R \cdot \zeta + O(\zeta^{-1}), \quad R > 0. \tag{14}$$

Also, assume that the points A_k (angular points) are mapped into the points a_k, $k = 1, 2, \ldots, 8$.

Denote the images of the contours l_0' and l_1' by L_0 and L_1, respectively (l_0' and l_1' denote respectively those parts of the contours l_0 and l_1 which lie in the domain D). By virtue of equality (13), the values of $\Phi_0(\zeta) = \Phi(-i\sqrt{\omega(\zeta)})$ on the cut of the domain D_1 from above and from below are equal to each other and thus the function $\Phi_0(\zeta)$ is analytic outside the unit circle in a domain $|\zeta| > 1$ and, by virtue of equalities (10) and (11), satisfies the conditions

$$\operatorname{Re} \Phi_0(\sigma) = p, \quad \sigma \in L_1, \tag{15}$$

$$\operatorname{Im} \Phi_0(\sigma) = 0, \quad \sigma \in L_0. \tag{16}$$

Define the function $\Phi_1(\zeta)$ by the rule

$$\Phi_1(\zeta) = \Phi_0(\zeta) - p.$$

Then the boundary conditions (15), (16) can be written as

$$\operatorname{Re} \Phi_1(\sigma) = 0, \quad \sigma \in L_1, \tag{17}$$

$$\operatorname{Im} \Phi_1(\sigma) = 0, \quad \sigma \in L_0. \tag{18}$$

Define the function $\Phi_2(\zeta)$ as follows

$$\Phi_2(\zeta) = \begin{cases} \Phi_1(\zeta), & |\zeta| > 1, \\ \overline{\Phi_1\left(\dfrac{1}{\bar{\zeta}}\right)}, & |\zeta| < 1. \end{cases} \tag{19}$$

From equality (18) it follows that for $\Phi_2(\zeta)$ the line L_0 is not a jump line, and by equality (17) the boundary condition on the line L_1 takes the form

$$\Phi_2^+(\sigma) + \Phi_2^-(\sigma) = 0, \quad \sigma \in L_1, \tag{20}$$

where L_1 is the union of separately lying arcs of the unit circle $|\zeta| = 1$.

At the ends of the line l, in the neighborhood of the points A_k the function $\omega(\zeta)$ can be represented as follows [1]

$$\omega(\zeta) - A_k = (\zeta - a_k)^{\alpha_k} \cdot \left\{c_0 + c_1(\zeta - a_k) + \cdots\right\} = (\zeta - a_k)^{\alpha_k} \cdot \omega^*(\zeta),$$

where $\omega^*(\zeta)$ is a nonzero function in the neighborhood of the points a_k. $\gamma = \alpha_k \pi$. The angle γ (see Fig. 1) is not larger than $\frac{\pi}{2}$ and therefore $0 < \alpha_k \leq \frac{1}{2}$. Thus, taking into account (12) we obtain

$$|\Phi_1(\zeta)| < const|\zeta - a_k|^{-\beta_k}, \quad \text{where } \; 0 \leq \beta_k < \frac{1}{2}.$$

So, we look for solutions of the boundary value problem (20) which are unbounded of order less than $\frac{1}{2}$ near the points a_k. In the class of such functions, problem (20) has only the zero solution $\Phi_2(\zeta) = 0$ and, finally, for the function $\Phi(z)$ we obtain

$$\Phi(z) = p. \tag{21}$$

Thus it remains to define the line l_1 and the function $\Psi(z)$.

By virtue of Eqs. (6), (7) and using conditions (2)–(4), the boundary conditions take the form

$$e^{2i\alpha(t)}\,\Psi(t) = b, \quad t \in l_1', \tag{22}$$

$$\operatorname{Im} e^{2i\alpha(t)}\,\Psi(t) = 0, \quad t \in l_0', \tag{23}$$

The angular points of the contour l are denoted by A_k $(k = 1, 2, \ldots, 8)$ as shown in the figure. $\alpha(t)$ is a piecewise-constant function on the contour l_0': $\alpha(t) = \alpha_k$ when $t \in A_k A_{k+1}$ ($k = 2n - 1$ or k is odd).

Together with equalities (22), (23), consider the equation of the contour l_0'

$$t - A_k = -i\rho \cdot e^{i\alpha}, \qquad \rho = |t - A_k|.$$

Hence we obtain

$$\operatorname{Re}(te^{-i\alpha(t)}) = \operatorname{Re}(A(t) \cdot e^{-i\alpha(t)}), \tag{24}$$

where $A(t) = A_k$ when $t \in A_k A_{k+1}$, $k = 1, 3, 5, 7$.

Taking into account (21) the function $\Psi(z)$ at the point at infinity can be written in the form $\Psi^{\infty}(z) = \frac{B-A}{2}$ and the condition

$$\frac{B - A}{2} < k$$

must be fulfilled since $\tau_{\max\infty} = \frac{\sigma_y - \sigma_x}{2} = \frac{B-A}{2}$; otherwise the entire plate will be in the plastic state.

During the conformal mapping of the domain D onto the domain D_1 by the functions $z = -i\sqrt{\omega(\zeta)}$, Eqs. (22)–(24) take the following form

$$e^{2i\alpha_0(\sigma)}\,\Psi_0(\sigma) = b, \quad \sigma \in L_1, \tag{25}$$

$$\operatorname{Im} e^{2i\alpha_0(\sigma)}\,\Psi_0(\sigma) = 0, \quad \sigma \in L_0, \tag{26}$$

$$\operatorname{Re}\Big(e^{-i\alpha_0(\sigma)}\big(-i\sqrt{\omega(\sigma)}\big)\Big) = \operatorname{Re}\big(A_0(\sigma) \cdot e^{-i\alpha_0(\sigma)}\big), \quad \sigma \in L_0, \tag{27}$$

where

$$\Psi_0(\sigma) = \Psi\big(-i\sqrt{\omega(\sigma)}\big), \qquad \alpha_0(\sigma) = \alpha\big(-i\sqrt{\omega(\sigma)}\big),$$

$\alpha_0(\sigma)$ is the known piecewise-constant function on the contour L_0 and the unknown function on L_1 since the contour itself is unknown,

$$A_0(\sigma) = A_k, \quad \sigma \in a_k a_{k+1}, \quad k = 1, 3, 5, 7.$$

To express $e^{2i\alpha_0(\sigma)}$ we have the equality

$$e^{2i\alpha_0(\sigma)} = -\frac{\sigma^2\omega'(\sigma)}{\sqrt{\omega(\sigma)}} \cdot \frac{\overline{\sqrt{\omega(\sigma)}}}{\overline{\omega'(\sigma)}}, \quad |\sigma| = 1. \tag{28}$$

Due to the cyclic symmetry of the plate, for the analytic function $\Psi(z)$ we have the equality

$$\Psi(ze^{i\beta}) = e^{-2i\beta}\,\Psi(z),$$

which in our case, in view of the fact that the angle of cyclic symmetry is equal to π $(\beta = \pi)$, can be written as

$$\Psi(-z) = \Psi(z), \quad z \in S. \tag{29}$$

When in the complex plane ζ approaches from above and from below to some point σ lying on the cut, the boundary values of the function $\Psi_0(\zeta)$ are $\Psi_0^-(\sigma)$ and $\Psi_0^+(\sigma)$, which in their turn represent $\Psi(t)$ and $\Psi(-t)$. By virtue of equality (29) we can conclude that the boundary values of $\Psi_0(\zeta)$ from above and from below on the cut of the plane ζ are equal to each other.

So, $\Psi_0(\zeta)$ is an analytic function in the external domain of the circle $|\zeta| = 1$.

If we use relation (28) in equality (25), then after differentiating equality (27) with respect to the variable ζ we obtain the boundary conditions

$$\frac{-\sigma^2 i\omega'(\sigma)}{2\sqrt{\omega(\sigma)}} \cdot \Psi_0(\sigma) = b \cdot \frac{\overline{i\omega'(\sigma)}}{\overline{2\sqrt{\omega(\sigma)}}}, \quad \sigma \in L_1, \tag{30}$$

$$\operatorname{Im}\Big(\sigma \cdot \Big(\frac{-i\omega'(\sigma)}{2\sqrt{\omega(\sigma)}}\Big) \cdot e^{-i\alpha_0(\sigma)}\Big) = 0, \quad \sigma \in L_0, \tag{31}$$

$$\operatorname{Im}\big(e^{2i\alpha_0(\sigma)}\,\Psi_0(\sigma)\big) = 0, \quad \sigma \in L_0. \tag{32}$$

Equality (30) can be written in the form

$$\frac{-\sigma^2 i\omega'(\sigma)}{2} \cdot \sqrt{\frac{\sigma - m}{\omega(\sigma)}} \cdot \Psi_0(\sigma) \cdot \overline{\sqrt{\sigma - m}} = \frac{b\overline{i\omega'(\sigma)}}{2} \cdot \overline{\sqrt{\frac{\sigma - m}{\omega(\sigma)}}} \cdot \sqrt{\sigma - m}. \tag{33}$$

Consider the function defined by the rule

$$F(\zeta) = \begin{cases} \dfrac{-\zeta^2 i\omega'(\zeta)}{2} \cdot \sqrt{\dfrac{\zeta - m}{\omega(\zeta)}} \cdot \Psi_0(\zeta) \cdot \overline{\sqrt{\dfrac{1}{\overline{\zeta}} - m}}, & |\zeta| > 1, \\ \dfrac{b\overline{i\omega'(\frac{1}{\overline{\zeta}})}}{2} \cdot \overline{\sqrt{\dfrac{\frac{1}{\overline{\zeta}} - m}{\omega(\frac{1}{\overline{\zeta}})}}} \cdot \sqrt{\zeta - m} & |\zeta| < 1. \end{cases} \tag{34}$$

Here $\zeta = m$ is a unique point in the external domain of a unit circle $|\zeta| > 1$, where the analytic function $\omega(\zeta)$ has a first order zero and therefore $\sqrt{\frac{\zeta - m}{\omega(\zeta)}}$ will be an analytic function in this domain. The function $F(\zeta)$ defined by equality (34) will be analytic inside and outside the unit circle $|\zeta| = 1$ and, by virtue of Eq. (33), will satisfy, on the part of the circle $|\zeta| = 1$, the boundary condition

$$F^+(\sigma) = F^-(\sigma), \quad \sigma \in L_1. \tag{35}$$

If we take into consideration equalities (31), (32) and (34), then for the analytic function $F(\zeta)$ in the domain cut along the line L_0 we obtain the boundary conditions

$$\operatorname{Im} \frac{F^\pm(\sigma)}{\sigma} e^{i\alpha} = 0, \quad \sigma \in L_0. \tag{36}$$

In the considered case, the expression $e^{-2i\alpha}$ on the contour L_0 gets the values equal to 1 or -1. Thus, if we multiply second of equalities (36) by $e^{-2i\alpha}$, then for the analytic function $F(\zeta)$ in the complex plane ζ cut along the line L_0 we obtain the boundary conditions

$$\operatorname{Im} \frac{F^\pm(\sigma)}{\sigma} e^{\pm i\alpha} = 0, \quad \sigma \in L_0. \tag{37}$$

The obtained equalities can be rewritten as follows

$$\frac{F^\pm(\sigma)}{\sigma} \cdot e^{\pm i\alpha} = \sigma \cdot \overline{F^\pm(\sigma)} \cdot e^{\mp i\alpha}, \quad \sigma \in L_0. \tag{38}$$

On the contour $|\zeta| = 1$, the positive direction is chosen so that when moving along this direction the domain $|\zeta| < 1$ remains on the left side.

We consider the function $F_*(\zeta)$ defined by

$$F_*(\zeta) = \overline{F(\frac{1}{\bar{\zeta}})} = \begin{cases} \overline{\dfrac{i\omega'(\frac{1}{\bar{\zeta}})}{2\zeta^2}} \cdot \overline{\sqrt{\dfrac{\frac{1}{\bar{\zeta}} - m}{\omega(\frac{1}{\bar{\zeta}})}}} \cdot \overline{\Psi_0\Big(\dfrac{1}{\bar{\zeta}}\Big)} \cdot \sqrt{\zeta - m}, & |\zeta| < 1, \\ \dfrac{-bi\omega'(\zeta)}{2} \cdot \sqrt{\dfrac{\zeta - m}{\omega(\zeta)}} \cdot \overline{\sqrt{\dfrac{1}{\bar{\zeta}} - m}}, & |\zeta| > 1, \end{cases} \tag{39}$$

and also consider the functions $W(\zeta)$ and $W_*(\zeta)$ defined by the equalities

$$W(\zeta) = \frac{1}{\zeta} F(\zeta), \tag{40}$$

$$W_*(\zeta) = \overline{W\Big(\frac{1}{\bar{\zeta}}\Big)}. \tag{41}$$

Further we introduce the function $\Omega(\zeta)$

$$\Omega(\zeta) = W(\zeta) + W_*(\zeta). \tag{42}$$

Boundary values of the function $\Omega(\zeta)$ are written in the form

$$\Omega^+(\sigma) = W^+(\sigma) + W_*^+(\sigma) = \frac{1}{\sigma} F^+(\sigma) + \sigma \overline{F^-(\sigma)}, \tag{43}$$

$$\Omega^-(\sigma) = W^-(\sigma) + W_*^-(\sigma) = \frac{1}{\sigma} F^-(\sigma) + \sigma \overline{F^+(\sigma)}. \tag{44}$$

Using equalities (35), (43) and (44) we have

$$\Omega^+(\sigma) = \Omega^-(\sigma), \quad \sigma \in L_1. \tag{45}$$

For the boundary values of the function $\Omega(\zeta)$ on the internal and the external side of the contour L_0, by virtue of condition (38) and equalities (43), (44) we obtain the equality

$$\Omega^+(\sigma) = e^{-2i\alpha} \Omega^-(\sigma), \quad \sigma \in L_0. \tag{46}$$

Let us introduce the function $T(\zeta)$ defined by the equality

$$T(\zeta) = W(\zeta) - W_*(\zeta). \tag{47}$$

Boundary values of the function $T(\zeta)$ are written in the form

$$T^+(\sigma) = W^+(\sigma) - W_*^+(\sigma) = \frac{1}{\sigma} F^+(\sigma) - \sigma \overline{F^-(\sigma)}, \tag{48}$$

$$T^-(\sigma) = W^-(\sigma) - W_*^-(\sigma) = \frac{1}{\sigma} F^-(\sigma) - \sigma \overline{F^+(\sigma)}. \tag{49}$$

In view of equalities (35), (48) and (49), for the boundary values of the function $T(\zeta)$ we have

$$T^+(\sigma) = T^-(\sigma), \quad \sigma \in L_1. \tag{50}$$

For the boundary values of the function $T(\zeta)$ on the internal and the external side of the contour L_0, by virtue of condition (38), and equalities (48), (49) we obtain the equality

$$T^+(\sigma) = -e^{-2i\alpha} \cdot T^-(\sigma), \quad \sigma \in L_0. \tag{51}$$

The expression $e^{-2i\alpha}$ on the contour L_0 gets the values

$$e^{-2i\alpha} = \begin{cases} 1, & \text{if } \sigma \in a_1a_2 \cup a_5a_6, \\ -1, & \text{if } \sigma \in a_3a_4 \cup a_7a_8. \end{cases} \tag{52}$$

With (52) taken into account, the boundary equalities (46) and (51) for $\Omega(\zeta)$ and $T(\zeta)$ can be rewritten respectively as follows

$$\begin{cases} \Omega^+(\sigma) = -\Omega^-(\sigma), & \sigma \in a_3a_4 \cup a_7a_8, \\ \Omega^+(\sigma) = \Omega^-(\sigma), & \sigma \in a_1a_2 \cup a_5a_6, \end{cases} \tag{53}$$

$$\begin{cases} T^+(\sigma) = -T^-(\sigma), & \sigma \in a_1a_2 \cup a_5a_6, \\ T^+(\sigma) = T^-(\sigma), & \sigma \in a_3a_4 \cup a_7a_8. \end{cases} \tag{54}$$

Equalities (53), (54) imply that for the function $\Omega(\zeta)$ the part of the contour L_0 $(a_1a_2 \cup a_5a_6)$ and the curve L_1 is not a jump line. For the function $T(\zeta)$ the part of the contour L_0 $(a_3a_4 \cup a_7a_8)$ and the curve L_1 is not the jump line.

The problem is thus reduced to a problem of finding analytic functions $\Omega(\zeta)$ and $T(\zeta)$ in the complex plane ζ cut along a part of the contour L_0 (the plane is cut along the lines $a_3a_4 \cup a_7a_8$ for the function $\Omega(\zeta)$, and along the lines $a_1a_2 \cup a_5a_6$ for the function $T(\zeta)$) with the conditions

$$\Omega^+(\sigma) = -\Omega^-(\sigma), \quad \sigma \in a_3a_4 \cup a_7a_8, \tag{55}$$

$$T^+(\sigma) = -T^-(\sigma), \quad \sigma \in a_1a_2 \cup a_5a_6. \tag{56}$$

By virtue of equalities (34), (39)–(42) and (47) we may conclude that the sought functions $\Omega(\zeta)$ and $T(\zeta)$ must satisfy the following additional conditions

$$\Omega(\zeta) = \overline{\Omega\Big(\frac{1}{\bar{\zeta}}\Big)}, \tag{57}$$

$$T(\zeta) = -\overline{T\Big(\frac{1}{\bar{\zeta}}\Big)}. \tag{58}$$

Problems (55), (56) are the particular cases of a linear conjugation problem, where the boundary consists of separately lying smooth contours. In particular the coefficient of the problem is $G(\sigma) = -1$.

We will seek unbounded solutions of order less than one near the nonsingular points a_k or, which is the same, solutions of the class h_0 [2].

A general solution of problem (55) has the form

$$\Omega(\zeta) = \chi_1(\zeta) \cdot P_1(\zeta), \tag{59}$$

where $P_1(\zeta)$ is a polynomial, the function $\chi_1(\zeta)$ is a canonical solution of the same problem that in the general case has the form

$$\chi(\zeta) = e^{\gamma(\zeta)} \prod_{k=1}^{n} (\zeta - a_k)^{\lambda_k}. \tag{60}$$

In our case, this formula can be written in the form

$$\chi_1(\zeta) = e^{\gamma(\zeta)}(\zeta - a_3)^{\lambda_3} \cdot (\zeta - a_4)^{\lambda_4} \cdot (\zeta - a_7)^{\lambda_7} \cdot (\zeta - a_8)^{\lambda_8},$$

$$\gamma(\zeta) = \frac{1}{2\pi i} \int_{a_3a_4} \frac{\pi i d\sigma}{\sigma - \zeta} + \frac{1}{2\pi i} \int_{a_7a_8} \frac{\pi i d\sigma}{\sigma - \zeta} = \frac{1}{2} \ln \frac{\zeta - a_4}{\zeta - a_3} + \frac{1}{2} \ln \frac{\zeta - a_8}{\zeta - a_7},$$

$$e^{\gamma(\zeta)} = \Big(\frac{\zeta - a_4}{\zeta - a_3}\Big)^{\frac{1}{2}} \cdot \Big(\frac{\zeta - a_8}{\zeta - a_7}\Big)^{\frac{1}{2}}.$$

Here under the expressions $(\frac{\zeta - a_4}{\zeta - a_3})^{\frac{1}{2}}$ and $(\frac{\zeta - a_8}{\zeta - a_7})^{\frac{1}{2}}$ we mean the holomorphic branches in the plane cut along the arcs a_3a_4 and a_7a_8 which at the point at infinity are equal to one.

$$\lambda_3 = \lambda_7 = 0, \quad \lambda_4 = \lambda_8 = -1.$$

For the index of the problem we obtain the equality

$$\varkappa_1 = -(\lambda_4 + \lambda_8) = 2. \tag{61}$$

For a canonical solution of the class h_0 we eventually obtain the expression

$$\chi_1(\zeta) = \frac{C_1^*}{\sqrt{R_1(\zeta)}}, \tag{62}$$

where C_1^* is constant different from zero,

$$R_1(\zeta) = (\zeta - a_3) \cdot (\zeta - a_4) \cdot (\zeta - a_7) \cdot (\zeta - a_8). \tag{63}$$

Under $\frac{1}{\sqrt{R_1(\zeta)}}$ we mean the holomorphic branch in the plane cut along the arcs a_3a_4 and a_7a_8, the expansion of which into decreasing powers ζ near the point at infinity has the form

$$\frac{1}{\sqrt{R_1(\zeta)}} = \zeta^{-2} + B_1'\zeta^{-3} + B_2'\zeta^{-4} + \cdots . \tag{64}$$

From equalities (34), (39), (40) and (42) we see that the function $\Omega(\zeta)$ at the points $\zeta = 0$ and $\zeta = \infty$ has a first order pole. Since the order of the canonical function $\chi_1(\zeta)$ is equal to $-\varkappa_1$ at the point at infinity, applying the above argumentation and equality (61), for the function $\Omega(\zeta)$ we obtain

$$\Omega(\zeta) = \chi_1(\zeta) \cdot \left(\frac{c_0'}{\zeta} + c_1' + c_2'\zeta + c_3'\zeta^2 + c_4'\zeta^3\right). \tag{65}$$

In view of equality (57) we may conclude that the constants $c_0', c_1', c_2', c_3', c_4'$ satisfy the conditions

$$c_0' = \overline{c_4'}, \quad c_1' = \overline{c_3'}, \quad c_2' = \overline{c_2'}. \tag{66}$$

By an analogous reasoning for problem (56) we obtain

$$\chi_2(\zeta) = \frac{C_2^*}{\sqrt{R_2(\zeta)}}, \tag{67}$$

where C_2^* is a constant different from zero,

$$R_2(\zeta) = (\zeta - a_1)(\zeta - a_2)(\zeta - a_5)(\zeta - a_6). \tag{68}$$

In this case, too, under $\frac{1}{\sqrt{R_2(\zeta)}}$ we mean that holomorphic branch on the plane cut along the arcs a_1a_2 and a_5a_6, the expansion of which near the point at infinity has the form

$$\frac{1}{\sqrt{R_2(\zeta)}} = \zeta^{-2} + B_1''\zeta^{-3} + B_2''\zeta^{-4} + \cdots , \tag{69}$$

$$\varkappa_2 = 2. \tag{70}$$

For the sought function $T(\zeta)$ we finally obtain

$$T(\zeta) = \chi_2(\zeta) \cdot \left(\frac{c_0''}{\zeta} + c_1'' + c_2''\zeta + c_3''\zeta^2 + c_4''\zeta^3\right), \tag{71}$$

where the constants $c_0'', c_1'', \ldots, c_4''$ satisfy the conditions

$$\begin{aligned} c_0'' &= \overline{c_4''}, \\ c_1'' &= \overline{c_3''}, \\ c_2'' &= \overline{c_2''}. \end{aligned} \tag{72}$$

The constants $c_0', c_1', c_2', c_3', c_4'$ and $c_0'', c_1'', c_2'', c_3'', c_4''$ in expressions (65) and (71) for the functions $\Omega(\zeta)$ and $T(\zeta)$ can be found if we use the known lengths of the linear parts of the plate boundary and fix some angular point.

After that, knowing the functions $\Omega(\zeta)$ and $T(\zeta)$, by virtue of equalities (34), (40), (42) and (47), we define the function $F(\zeta)$. Knowing the function $F(\zeta)$ and using equalities (34) and (39) we find the functions $f'(\zeta)$ and $\Psi_0(\zeta)$ $(z = f(\zeta) = -i\sqrt{\omega(\zeta)})$.

So, we have defined $\Psi_0(\zeta)$ and at the same time the function $\Psi(z)$, too, which together with the function $\Phi(z)$ describes the stressed state of the plate.

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Original article

Abstract formulations of some theorems on nonmeasurable sets

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Abstract

Here we give abstract formulations of some generalized versions of the classical Vitali theorem on Lebesgue nonmeasurable sets which are due to Kharazishvili and Solecki.

Keywords: Transformation group; Invariant (quasi-invariant) measure; Invariant set; k-additive measurable structure; k^+-saturation; *Ulam*(k, k^+)-matrix; k-independent (strictly k-independent) class; Almost invariant set; Small system

1. Introduction

Let (X, G) be a space equipped with a transformation group G. We say that G acts freely on X if $\{x \in X : gx = x\} = \emptyset$ for all $g \in G \setminus \{e\}$ where 'e' is the identity element of G (in fact, $e : X \to X$ is the identity transformation on X). For any $g \in G$ and $E \subseteq X$, we write gE for the set $\{gx : x \in E\}$ and call a nonempty family (or, class) $\mathcal{A}$ of subsets of X as G-invariant [1] if $gE \in \mathcal{A}$ for every $g \in G$ and $E \in \mathcal{A}$. If $\mathcal{A}$ is a σ-algebra, then a measure μ on $\mathcal{A}$ is called G-invariant [1] if $\mathcal{A}$ is a G-invariant class and $\mu(gE) = \mu(E)$ for every $g \in G$ and $E \in \mathcal{A}$. It is called G-quasiinvariant [1] if $\mathcal{A}$ and the σ-ideal generated by μ-null sets are both G-invariant classes. Obviously, every G-invariant measure is also G-quasiinvariant but not conversely. From a measure theoretic viewpoint, the concept of "G acting freely" can be suitably extended by saying that "G acts freely with respect to μ" (or, in short, μ-freely) on X if $\mu^*\{x \in X : gx = x\} = 0$ for every $g \in G \setminus \{e\}$ where μ^* is the outer measure induced by μ.

Given a subgroup H of G and an element x of X, a set of the form $Hx = \{hx : h \in H\}$ is called a H-orbit of x in X; and as x runs over X, the collection of all such H-orbits gives rise to a partition of X into mutually disjoint nonempty sets. A subset E of X is called invariant with respect to H (or, in short H-invariant) [1] if $g(E) = E$ for every $g \in H$. It may be easily checked that E is H-invariant if there exists a set $F \subseteq X$ such that $E = \bigcup_{x \in F} Hx$. A subset E of X is called a partial selector for H (or, in short, a partial H-selector) if $E \cap Hx$ consists of at most one point for each

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$x \in X$. If $E \cap Hx$ consists of exactly one point for each $x \in X$, then E is called a complete H-selector (or, simply, a H-selector) in X. A partial (resp. complete) H-selector is a subset Y of X which can be similarly defined by taking restriction of H-orbits on Y. Every partial selector in X is in fact a complete selector with respect to some subcollection (or, subfamily) of H-orbits and by axiom of choice, every partial H-selector can be extended to a complete H-selector.

Any H-selector in X can be taken as a generalized Definition of a $Vitali$ set in X corresponding to the subgroup H. In particular, the classical $Vitali$ set is a $\mathbb{Q}$-selector corresponding to the subgroup $\mathbb{Q}$ of rationals in $\mathbb{R}$.

Below we state some generalizations of the classical *Vitali* Theorem in spaces with transformation groups. The first two results (Theorems 1.1 and 1.2) are by $Kharazishvili$ [2,3] which deal with quasiinvariant measures and the third one deals with invariant measures and is due to $Solecki$ [4].

Theorem 1.1. *Let (X, G) be a space with transformation group G and let μ a σ-finite, G-quasiinvariant measure on X. Suppose G contains an uncountable subgroup Γ acting μ-freely on X. Then every μ-measurable set of positive measure contains a subset which is nonmeasurable with respect to μ.*

Theorem 1.2. *Let (X, G) be a space with transformation group G and let μ be a σ-finite, G-quasiinvariant measure on X. Suppose G contains an uncountable subgroup Γ acting μ-freely on X, and H be an arbitrary countable subgroup of Γ. Then there exists a subfamily of $\{Hx : x \in X\}$ such that its union is a μ-nonmeasurable subset of X. Consequently, all the H-selectors with respect to this subfamily are μ-nonmeasurable subsets of X.*

Theorem 1.3. *Let (X, G) be space with a transformation group and μ be a σ-finite, G-invariant measure on X. Suppose G is uncountable and acts μ-freely on X. Then every μ-measurable set E of positive measure in X contains a subset which is nonmeasurable with respect to every invariant extension of μ.*

Detailed proofs of Theorems 1.1 and 1.2 are based on $Ulam$'s transfinite matrix (or, $Ulam$ (ω, ω_1)-matrix) [5] but the same does not apply in the case of Theorem 1.3 where the proof is entirely independent of it. It may be noted here that $Ulam$'s transfinite matrix was developed by $Ulam$ for investigating various problems relating to the existence of nonmeasurable sets and sets not having the $Baire$ property (for details, see [6,7]).

In this article, we give abstract formulations of the three Theorems stated above. They are called abstract because they are free from any use of measure functions. Instead, we use a new type of structure which is introduced in the next section.

2. Preliminaries and results

Throughout the paper, we identify every infinite cardinal with the least ordinal representing it, and, every ordinal with the set of all ordinals preceding it. We write card(A) and card$(\mathcal{A})$ to denote the cardinals of any set A or any class $\mathcal{A}$ of sets and as is usually done else where, express the first infinite and first uncountable cardinals by the symbols ω_0 and ω_1 respectively. For any cardinal, we use symbols such as ξ, ϱ, η, k, etc. and write k^+ for the successor of k. In the entire discourse, we work within the framework of ZFC.

Definition 2.1. Let (X, G) be a space with a transformation group G and k be any arbitrary infinite cardinal such that card$(X) \geq k^+$. A pair $(\mathcal{S}, \mathcal{I})$ consisting of two classes $\mathcal{S}$ and $\mathcal{I}$ of subsets of X will be called a k-additive measurable structure on (X, G) if

(i) $\mathcal{S}$ is an algebra and $\mathcal{I}$ $(\subseteq \mathcal{S})$ a proper ideal in X.

(ii) Both $\mathcal{S}$ and $\mathcal{I}$ are k-additive in the sense that they are closed with respect to union of at most k number of sets.

and

(iii) $\mathcal{S}$ and $\mathcal{I}$ are G-invariant classes.

Henceforth, a k-additive algebra (resp. ideal) on (X, G) will mean that it is a k-additive algebra (resp. ideal) on X and also G-invariant. In particular, if G consists only of the identity transformation on X, then $(\mathcal{S}, \mathcal{I})$ is called a k-additive measurable structure on X.

Definition 2.2. A measurable structure $(\mathcal{S}, \mathcal{I})$ on (X, G) will be called k^+-saturated if the cardinality of any arbitrary collection of mutually disjoint sets from $\mathcal{S} \setminus \mathcal{I}$ is at most k.

The notion of a ω_0-additive measurable structure on a nonempty basic set E was defined by $Kharazishvili$ [8]. This was referred to as a measurable structure consisting a pair $(\mathcal{S}, \mathcal{I})$ where $\mathcal{S}$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ a proper σ-ideal of sets in E. If E is a group and $\mathcal{S}$, $\mathcal{I}$ are G-invariant classes, then $(\mathcal{S}, \mathcal{I})$ according to $Kharazishvili$ is a G-invariant measurable structure on E. Using this notion of measurable structures, $Kharazishvili$ proved several interesting results in commutative (and more generally) in solvable groups [8]. In [9], he used similar type of structures to generalize two classical results of $Sierpi\acute{n}ski$.

It may be noted that the notion of a k-additive, k^+-saturated measurable structure $(\mathcal{S}, \mathcal{I})$ on (X, G) lies somewhere in between a k-additive measurable structure on (X, G) satisfying countable chain condition (or, Suslin condition) and a ω_0-additive measurable structure on (X, G) which is k^+-saturated for it is weaker than the former whereas stronger than the later. We say that G acts $\mathcal{I}$-freely on X if the set $\{x \in X : gx = x\} \in \mathcal{I}$ for every $g \in G \setminus \{\mathrm{e}\}$. In fact, this notion is an extension of "G acts μ-freely on X" already stated in the introduction. If G acts freely on X, then it acts $\mathcal{I}$-freely on X for every ideal $\mathcal{I}$ on X.

The following Theorem is an abstract formulation of Theorems 1.1 and 1.2.

Theorem 2.3. *Let k be any arbitrary infinite cardinal and $(\mathcal{S}, \mathcal{I})$ be a k-additive measurable structure on (X, G) where card $(X) \geq k^+$. Also let card $(G) = k^+$, G acts $\mathcal{I}$-freely on X and $(\mathcal{S}, \mathcal{I})$ be k^+-saturated. Then every set $E \in \mathcal{S} \setminus \mathcal{I}$ contains a subset F which is $(\mathcal{S}, \mathcal{I})$-nonmeasurable. In particular, if E is G-invariant, then for every subgroup H of G having card $H = k$, there exists a subfamily of H-orbits in X all selectors with respect to which are $(\mathcal{S}, \mathcal{I})$-nonmeasurable.*

A proof of Theorem 2.3 can be established based on a similar line of argument as given for Theorem 8, Ch 4 [2] and Theorem 1, Theorem 2 [3] except that we need to replace ω_0-additivity, ω_1-saturation by k-additivity, k^+-saturation and $Ulam$'s (ω_0, ω_1)-matrix by a generalized form of the same as defined below.

Definition 2.4 *([6])*. Let E be an infinite set with card$(E) = k^+$. A double family $(E_{\xi,\zeta})_{\xi<k,\zeta<k^+}$ of subsets of E is called an Ulam (k, k^+)-matrix over E if the following two conditions are satisfied:
(i) card $(E \setminus \cup\{E_{\xi,\zeta} : \xi < k\}) \leq k$ for every $\zeta < k^+$
(ii) $E_{\xi,\zeta} \cap E_{\xi,\zeta'} = \emptyset$ for all $\xi < k$ and any two distinct ordinals $\zeta < k^+$ and $\zeta' < k^+$.

Theorem 2.3 can be further advanced using combinatorial approach. Combinatorial set theory plays a distinctive role in the construction of a maximal (in the sense of cardinality) family of independent sets in an infinite basic set and this was first observed by $Tarski$ [10]. Here based on the use of some combinatorial methods, we will show that under certain restrictions in any G-invariant set $E \in \mathcal{S} \setminus \mathcal{I}$, there exists a maximal k-independent family of $(\mathcal{S}, \mathcal{I})$-nonmeasurable sets. Apart from the use of generalized $Ulam$'s matrix, the proof also depends on the following set of Definitions and results.

Definition 2.5. A family $\{A_i : i \in \mathrm{I}\}$ of subsets of X is called k-independent (resp. strictly k-independent) if for each set $\mathrm{J} \subseteq \mathrm{I}$ having card$(\mathrm{J}) < k$ (resp. card$(\mathrm{J}) \leq k$) and every function $f : \mathrm{J} \to \{0, 1\}$, we have $\cap\{A_j^{f(j)} : j \in \mathrm{J}\} \neq \emptyset$ where $A_j^{f(j)} = A_j$ if $f(j) = 0$ and $A_j^{f(j)} = X \setminus A_j$ if $f(j) = 1$.

The definition of an independent or ω_0-independent (in the set theoretic sense) family is already given in [11]. The above definition is framed on this pattern. For another introduction to k-independent (resp. strictly k-independent) family see [12].

The existence of an ω_0-independent family of subsets of an infinite set, with maximal cardinality was solved by Tarski [10]. He showed that such a family exists is of cardinality $2^{card(E)}$. The result has many interesting applications. One such is its use in proving that the cardinality of all ultrafilters defined on an arbitrary infinite set E is $2^{2^{card(E)}}$. However, if the cardinality of the set E is that of the continuum, then the existence of a strictly independent family of subsets of E having cardinality 2^c can be proved where c is the cardinality of the continuum. The result has an application in the construction of a nonseparable invariant extension of the Lebesgue measure space [13].

Proposition 2.6 *([12])*. *Assume that the generalized continuum hypothesis holds. Then for any two infinite cardinals λ, k where $\lambda < k$ we have $k^\lambda = k$ provided λ is not cofinal with k.*

Proposition 2.7 *([12])*. *Let E be an infinite set satisfying the condition $card(E^k) = card(E)$, where k is an infinite cardinal. Then there exists a maximal strictly k-independent family $\{A_i : i \in \mathrm{I}\}$ of subsets of E such that card* $(\mathrm{I}) = 2^{cardE}$.

Definition 2.5 can be generalized using Definition 2.1, we say that

Definition 2.8. A family $\{A_i : i \in \mathrm{I}\}$ of subsets of X is k-independent (resp. strictly k-independent) with respect to any k-additive measurable structure $(\mathcal{S}, \mathcal{I})$ on (X, G) if for each set $\mathrm{J} \subseteq \mathrm{I}$ having card(J) $< k$ (resp. card(J) $\leq k$) and every function $f : \mathrm{J} \to \{0, 1\}$, $B \subseteq X \setminus \cap\{A_j^{f(j)} : j \in \mathrm{J}\}$ and $B \in \mathcal{S}$ implies that $B \in \mathcal{I}$, where $A_j^{f(j)}$ $(j \in \mathrm{J})$ has the same meaning as before.

Note that in the above Definition, condition (iii) of Definition 2.1 plays no role. So it may be conceived also with respect to any k-additive measurable structure $(\mathcal{S}, \mathcal{I})$ on X. The notion of an independent (resp. strictly independent) family with respect to a measure is already given in [14]. So the above definition is just an extension of this concept given in terms of k-additive measurable structures.

Definition 2.9 *([1])*. In a space (X, G) with transformation group G, a set $E \subseteq X$ is called almost G-invariant with respect to an ideal $\mathcal{I}$ if $gE \,\Delta E \in \mathcal{I}$ for every $g \in G$.

If the ideal is k-additive, then it can be easily checked that the class of all sets in X which are almost G-invariant with respect to $\mathcal{I}$ constitutes a k-additive algebra in X.

Definition 2.10. A set $E \subseteq X$ is called $(\mathcal{S}, \mathcal{I})$-thick if $B \subseteq X \setminus E$ and $B \in \mathcal{S}$ implies $B \in \mathcal{I}$.

Viewed in the above perspective, a family $\{A_i : i \in \mathrm{I}\}$ can be called k-independent (resp. strictly k-independent) with respect to $(\mathcal{S}, \mathcal{I})$ on X if for each set $\mathrm{J} \subseteq \mathrm{I}$ having card(J) $< k$ (resp. card(J) $\leq k$) and each function $f : \mathrm{J} \to \{0, 1\}$, the set $\cap\{A_j^{f(j)} : j \in \mathrm{J}\}$ is $(\mathcal{S}, \mathcal{I})$-thick in X.

Proposition 2.11. *Assume that the pair $(\mathcal{S}, \mathcal{I})$ is a k-additive measurable structure on a space (X, G) equipped with a transformation group G. Also let $(\mathcal{S}, \mathcal{I})$ be k^+-saturated and $E \subseteq X$ be almost G-invariant with respect to $\mathcal{I}$. Then $E \in \mathcal{S}$ implies either $E \in \mathcal{I}$ or $X \setminus E \in \mathcal{I}$. If $E \notin \mathcal{S}$, then both E and $X \setminus E$ are $(\mathcal{S}, \mathcal{I})$-thick in X.*

Proof. Let $E \in \mathcal{S}$. If $E \in \mathcal{I}$, then there is nothing to prove. Suppose $E \notin \mathcal{I}$. Then $X \setminus E \in \mathcal{I}$, for otherwise it is possible to generate by transfinite recursion a k-sequence $\{g_\alpha : \alpha < k\}$ in G such that $X \setminus \bigcup_{0 \leq \alpha < k} g_\alpha E \in \mathcal{I}$. But this contradicts the hypothesis.

Now let $E \notin \mathcal{S}$. Then $E \notin \mathcal{I}$ and also $X \setminus E \notin \mathcal{I}$. If E is not $(\mathcal{S}, \mathcal{I})$-thick, then there should exist $B \in \mathcal{S} \setminus \mathcal{I}$ such that $B \subseteq X \setminus E$. By a similar reasoning as given above, there exists a k-sequence $\{h_\alpha : \alpha < k\}$ in G such that $X \setminus \bigcup_{0 \leq \alpha < k} h_\alpha B \in \mathcal{I}$. But then from k-additivity of $\mathcal{I}$, there exists some $\alpha_0 < k$ such that $E \cap h_{\alpha_0} B \notin \mathcal{I}$. But this again contradicts the hypothesis.

Finally, we arrive at

Theorem 2.12. *Let k be any arbitrary infinite cardinal and $(\mathcal{S}, \mathcal{I})$ be k-additive measurable structure on (X, G) where card $(X) \geq k^+$. Also let card $(G) = k^+$, G acts freely on X and $(\mathcal{S}, \mathcal{I})$ be k^+-saturated. Then under the assumption of generalized continuum hypothesis, for every G-invariant set $E \in \mathcal{S} \setminus \mathcal{I}$ which contains at least one G-selector $L \in \mathcal{S}$, there exists a family $\{A_i : i \in \mathrm{I}\}$ of $(\mathcal{S}, \mathcal{I})$-nonmeasurable subsets of E which is strictly k-independent (and hence k-independent) with respect to $(\mathcal{S}, \mathcal{I})$ on E and having cardinality 2^{k^+}.*

Proof. We write G in the form $G = \bigcup_{\varrho < k^+} G_\varrho$ where $\{G_\varrho : \varrho < k^+\}$ is an increasing family of subgroups of G satisfying (i) $G_\varrho \neq \bigcup_{\eta < \varrho} G_\eta$ and (ii) card $G_\varrho \leq k$ for every $\varrho < k^+$ (for the above representation, see [11], Exercise 19, Ch 3).

Since G acts freely on X, the above increasing family yields a disjoint covering $\{\Omega_\gamma : \gamma < k^+\}$ of E where $\Omega_\gamma = (G_\gamma \setminus \bigcup_{\eta < \gamma} G_\eta)L$. Moreover, as $L \in \mathcal{S}$, G acts freely on X and $(\mathcal{S}, \mathcal{I})$ is k^+-saturated, so $gL \in \mathcal{I}$ for every $g \in G$.

Now we consider the *Ulam* (k, k^+)-matrix $(\Pi_{\xi,\varrho})_{\xi<k,\varrho<k^+}$ over k^+ and set $E_{\xi,\varrho} = \bigcup_{\gamma\in\Pi_{\xi,\varrho}} \Omega_\gamma$. Then there exists ξ_0 and a subset Ξ of k^+ having card $\Xi = k^+$ such that the sets $E_{\xi_0,\varrho} \notin \mathcal{I}$ for $\varrho \in \Xi$ and are mutually disjoint. This is so because $\mathcal{I}$ is k-additive and $E \notin \mathcal{I}$. Moreover, each $E_{\xi_0,\varrho}$ for $\varrho \in \Xi$ is almost G-invariant with respect to $\mathcal{I}$ which follows from the constructions of the sets Ω_γ.

Now note that k is not cofinal with k^+. This is so because k is not cofinal with 2^k and $2^k = k^+$ under the assumption of generalized continuum hypothesis. Hence according to Propositions 2.6 and 2.7 it follows that there exists a strictly k-independent family $\{\Xi_i : i \in \mathrm{I}\}$ of subsets of Ξ such that card(I) $= 2^{k^+}$. This means that for every set $\mathrm{J} \subseteq \mathrm{I}$ having card(J) $\leq k$ and every function $f : \mathrm{J} \to \{0, 1\}$, $\bigcap_{j\in \mathrm{J}} \Xi_j^{f(j)} \neq \emptyset$. Consequently, $\bigcap_{j\in \mathrm{J}} A_j^{f(j)} \neq \emptyset$ where $A_i = \bigcup_{\varrho\in\Xi_i} E_{\xi_0,\varrho}$ for $i \in \mathrm{I}$ making $\{A_i : i \in \mathrm{I}\}$ a strictly k-independent family of sets in E. Moreover, this family is strictly k-independent (and hence k-independent) with respect to $(\mathcal{S}, \mathcal{I})$ on E consisting only of $(\mathcal{S}, \mathcal{I})$-nonmeasurable sets since each $E_{\xi_0,\varrho}$ is $(\mathcal{S}, \mathcal{I})$-thick in E.

Hence the theorem.

In all the previous derivations, *Ulam* (k, k^+)-matrix (or, generalized *Ulam* matrix) played a decisive role. But such matrices cannot be applied in proving our next Theorem (which is an abstract formulation of Theorem 1.3). Instead, we pursue a different line of development, where we assume in advance the existence of a system of small sets (or, a small system) satisfying a definite set of axioms. The approach is a modified version of the one originally introduced by *Riećan* and *Neubrunn* [15] (see also [16–19]) in giving abstract formulations of some well-known classical results on measure and integration.

Let $\mathcal{S}$ be a k-additive algebra on (X, G) and

Definition 2.13. $\{\mathcal{N}_\alpha\}_{0<\alpha<k}$ be a k-sequence members which are classes of subsets of X satisfying the following set of conditions:

(i) $\emptyset \in \mathcal{N}_\alpha, \mathcal{N}_\alpha \subseteq \mathcal{S}$ and $\mathcal{S} \cap \mathcal{N}'_\alpha \neq \emptyset$ for $0 < \alpha < k$ where $\mathcal{N}'_\alpha = \{E \subseteq X : E \notin \mathcal{N}_\alpha\}$.

(ii) For every $\alpha, \beta < k$, there exists $\gamma > \alpha, \beta$ such that $\mathcal{N}_\gamma \subseteq \mathcal{N}_\alpha$ and $\mathcal{N}_\gamma \subseteq \mathcal{N}_\beta$. In other words, with respect to the inclusion relation (among classes of sets), the system $\{\mathcal{N}_\alpha\}_{0<\alpha<k}$ is directed.

(iii) For any $\alpha < k$, there exists $\alpha^* > \alpha$ such that for any one-to-one correspondence $\beta \to \mathcal{N}_\beta$ with $\beta > \alpha^*$, $\bigcup_\beta E_\beta \in \mathcal{N}_\alpha$ whenever $E_\beta \in \mathcal{N}_\beta$.

(iv) Each $\mathcal{N}_\alpha$ is a G-invariant class.

(v) If $E \in \mathcal{N}_\alpha$ and $F \subseteq E$, then $F \in \mathcal{N}_\alpha$. Thus every $\mathcal{N}_\alpha$ is a hereditary class. Moreover, if $\{E_\xi : \xi < k\}$ is a nested family of sets in $\mathcal{S}$ such that $\bigcap_\xi \mathrm{E}_\xi \in \mathcal{N}_\alpha$, then $E_\xi \in \mathcal{N}_\alpha$ for some $\xi < k$.

We further add that

Definition 2.14. A k-additive algebra $\mathcal{S}$ on (X, G) is ergodic (with respect to $\{\mathcal{N}_\alpha\}_{0<\alpha<k}$) if given $E \in \mathcal{S} \cap \mathcal{N}'_\alpha$ and $F \in \mathcal{S} \cap \mathcal{N}'_\beta$, there exist $g \in G$ and $\gamma > \alpha, \beta$ such that $gE \cap F \in \mathcal{S} \cap \mathcal{N}'_\gamma$.

Theorem 2.15. *Let k be an infinite regular cardinal and $\mathcal{S}$ be a k-additive algebra on (X, G) such that card $(G) = k^+ \leq card(X)$, G acts freely on X and $\{\mathcal{N}_\alpha\}_{0<\alpha<k}$ be as defined above. Moreover,*

(1) Let $\mathcal{S}$ be ergodic with respect to $\{\mathcal{N}_\alpha\}_{0<\alpha<k}$

(2) $X \notin \bigcap_{0<\alpha<k} \mathcal{N}_\alpha$ and $X = \bigcup_{0<\alpha<k} Y_\alpha$ where $Y_\alpha \in \mathcal{S}$ such that for no k-additive algebra $\mathfrak{T}$ on (X, G) containing $\mathcal{S}$, there can exist $\alpha_0 < k$ and a collection $\{E_\beta : \beta \in \mathcal{D}\}$ having card$(\mathcal{D}) = k$ of mutually disjoint sets $E_\beta \in \mathfrak{T} \cap \mathcal{N}'_{\alpha_0}$ which are all contained in some set in the given collection $\{Y_\alpha : \alpha < k\}$. Then every set E which belongs to $\mathcal{S}$ but not in $\bigcap_{0<\alpha<k} \mathcal{N}_\alpha$ contains a subset F that does not belong to any k-additive algebra on (X, G) which contains $\mathcal{S}$.

Proof. We set $\mathcal{N}_\infty = \bigcap_{0<\alpha<k} \mathcal{N}_\alpha$. From the conditions (**i**), (**iii**), (**iv**) and first part of (**v**), it is easy to check that $\mathcal{N}_\infty$ is a k-additive ideal on (X, G) so that the pair $(\mathcal{S}, \mathcal{N}_\infty)$ becomes a k-additive measurable structure on (X, G). Since $X \notin \mathcal{N}_\infty$, so without loss of generality, we may assume that $Y_\alpha \notin \mathcal{N}_\infty$ for $0 < \alpha < k$.

Now since $E \in \mathcal{S}$ and not in $\mathcal{N}_\infty$, we fix $\alpha_0 < k$ so that $E \in \mathcal{N}'_{\alpha_0}$. The justification for this follows from the Definition of $\mathcal{N}_\infty$. It is possible, by virtue of ergodicity to generate an injective mapping $\lambda : k \to k$ with the property that for each $\alpha < k$, there exists $g \in G$ such that $g^{-1}(Y_\alpha) \cap E \in \mathcal{N}'_{\lambda(\alpha)}$. Also by condition (**ii**) (defining $\{\mathcal{N}_\alpha\}_{0<\alpha<k}$), the family $\{\mathcal{N}_{\lambda(\alpha)} : \alpha < k\}$ can be chosen as a nested one with $\lambda(\alpha) > \alpha_0^*$.

We set $\Gamma_\alpha = \{g \in G : g^{-1}(Y_\alpha) \cap E \in \mathcal{N}'{}_{\lambda(\alpha)}\}$ and claim that there is some $\alpha_1 < k$ such that card $\Gamma_{\alpha_1} = k^+$. For otherwise, card $(\bigcup \Gamma_\alpha : \alpha < k) \leq k$ and so for any $g \in G \setminus \bigcup_{\alpha<k} \Gamma_\alpha$, $g^{-1}(Y_\alpha) \cap E \in \mathcal{N}_{\lambda(\alpha)}$ which consequently leads to the conclusion that $E = E \cap X = E \cap g^{-1}(\bigcup_{\alpha<k} \mathrm{Y}_\alpha) = \bigcup_{\alpha<k} g^{-1}(Y_\alpha) \cap E \in \mathcal{N}_{\alpha_0}$. But this contradicts the choice of the set E.

From Γ_{α_1} we choose a set $\{g_\alpha : \alpha < k\}$ of cardinality k. Then by first part of condition (**v**), $\bigcup_{\beta>\alpha} g_\beta^{-1}(Y_{\alpha_1}) \cap E \in \mathcal{N}'_{\lambda(\alpha_1)}$ and by an application of the second part of the same condition, the set $E_0 = \bigcap_{\alpha<k} \bigcup_{\beta>\alpha} g_\beta^{-1}(Y_{\alpha_1}) \cap E \in \mathcal{S} \cap \mathcal{N}'_{\lambda(\alpha_1)}$. We set $W_\alpha = \bigcup_{\beta>\alpha} g_\beta^{-1}(Y_{\alpha_1}) \cap E$ so that $E_0 = \bigcap_{\alpha<k} W_\alpha$.

Let H be a subgroup generated by $\{g_\alpha : \alpha < k\}$. Then card $H = k$. From the family of all H-orbits we extract out the subfamily members of which have nonempty intersection with E_0, and choose one selector (or, partial selector) V_0 corresponding to this subfamily such that $V_0 \subseteq E_0$. Let V be an H-selector in X which extends V_0 and we write $F = E \cap V$.

We claim that F cannot belong to any k-additive algebra on (X, G) which contains $\mathcal{S}$. If possible, let F belong to one such k-additive algebra $\mathfrak{T}$. Then $V_0 = F \cap E_0 \in \mathfrak{T}$ and therefore $E_0 \subseteq H(V_0)$. Let $V_\alpha = V_0 \cap W_\alpha$. Now as the action of G on X is free, so the collection $\{g_\alpha(V_\alpha) : \alpha < k\}$ consists of mutually disjoint sets. We claim that for every $\xi < k$, there exists $\alpha < k$ such that $V_\beta \in \mathcal{N}_\xi$ for $\beta > \alpha$. For otherwise, there would exist $\xi_0 < k$ and a cofinal set $\mathcal{D}$ of k such that $V_\alpha \in \mathcal{N}'_{\xi_0}$ for every $\alpha \in \mathcal{D}$ and $\{g_\alpha(V_\alpha) : \alpha \in \mathcal{D}\}$ is a family of mutually disjoint subsets of Y_{α_1}. As k is regular, this contradicts the hypothesis. Hence $V_0 = \bigcap_{\alpha<k} \bigcup_{\beta>\alpha} V_\beta \in \mathcal{N}_\infty$ and therefore $E_0 \in \mathfrak{T} \cap \mathcal{N}_\infty$. But this again contradicts our earlier derivation that $E_0 \in \mathcal{S} \cap \mathcal{N}'{}_{\lambda(\alpha_1)} \subseteq \mathfrak{T} \cap \mathcal{N}'{}_{\lambda(\alpha_1)}$.

This proves the theorem.

Remarks. The above result is an abstract generalization (without using measure) of Solecki's theorem. This may be easily observed if we choose $k = \omega_0$ and $(\mathbb{R}, \mathbb{R})$ as our space with transformation group $(\mathbb{R}, +)$; $\mathcal{S} = dom(\lambda)$ where λ is Lebesgue measure on $\mathbb{R}$, $\mathfrak{T} = dom(\mu)$ where μ is any translation invariant extension of λ and for any $n < \omega$ define $\mathcal{N}_n = \{E \in dom(\lambda) : \lambda(E) < \frac{1}{n}\}$.

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Original article

On Robinson's Energy Delay Theorem

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Abstract

An elementary proof of Robinson's Energy Delay Theorem on minimum-phase functions is provided. The situation in which the energy conservation property holds for an infinite number of lags is fully described.

Keywords: Minimum-phase functions; Hardy spaces

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be its boundary. The set of all analytic in $\mathbb{D}$ functions is denoted by $\mathcal{A}(\mathbb{D})$. The Hardy space $H^2 = H^2(\mathbb{D})$ consists of all the functions $f \in \mathcal{A}(\mathbb{D})$ the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

of which satisfy the condition

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

In engineering, these functions are known as *z-transforms* (resp. *transfer functions*) of discrete-time *causal signals* (resp. *filter impulse responses*) with a finite energy. It is well known that the boundary values of $f \in H^2$ exist a.e.,

$$f_+(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta}) \quad \text{for a.a. } \theta \in [0, 2\pi), \tag{1}$$

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and $f_+ \in L^2(\mathbb{T})$, the Lebesgue space of square integrable functions on $\mathbb{T}$. Furthermore, $f_+ \in L^2_+(\mathbb{T}) := \{f \in L^2(\mathbb{T}) : c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta = 0 \text{ for } n < 0\}$. Actually, there is a one-to-one correspondence between H^2 and $L^2_+(\mathbb{T})$, and therefore we may naturally identify these two classes.

For any function $f \in H^2$, the inequality

$$|f(0)| \leq \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f_+(e^{i\theta})|\,d\theta\right) \tag{2}$$

holds (see, e.g., [1, Th. 17.17]). The extreme functions for which (2) turns into an equality are called *outer*. In engineering they are also known as *minimum-phase*, or *optimal*, functions. According to the original definition of outer functions by Beurling [2], they admit the representation

$$f(z) = c \cdot \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|f_+(e^{i\theta})|\,d\theta\right), \tag{3}$$

where c is a unimodular constant. This representation easily implies that the equality holds in (2) for outer functions and it can be proved that the converse is also true. In particular, boundary values of the modulus of an outer function uniquely determine the function itself up to a constant multiple with absolute value 1.

The following property of minimum-phase functions, first observed by Robinson [3], plays an important role in several signal processing applications.

Theorem 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be H^2-functions satisfying*

$$|f_+(e^{i\theta})| = |g_+(e^{i\theta})| \text{ for a.e. } \theta. \tag{4}$$

If f is of minimum-phase, then for each N,

$$\sum_{n=0}^{N} |a_n|^2 \geq \sum_{n=0}^{N} |b_n|^2. \tag{5}$$

Robinson gave a physical interpretation to inequality (5) "that among all filters with the same gain, the outer filter makes the energy built-up as large as possible, and it does so for every positive time" [4] and found geological applications of minimum-phase waveforms. Consequently, the term *minimum-delay* [5, p. 211] functions is being used to describe optimal functions, and Theorem 1 is known as the Energy Delay Theorem within the geological community [6, p. 52].

Theorem 1 was further extended to the matrix polynomial case and used in MIMO communications in [7]. In [8], the theorem is formulated and proved for general operator valued functions in abstract Hilbert spaces.

In this paper, we provide a very short and simple proof of Theorem 1 based on classical facts from the theory of Hardy spaces. This is done in Section 3, while the modification of this proof fitting the matrix case is discussed in Section 4. In final Section 5, we treat the situation in which (5) turns into an equality for infinitely many values of N. The preliminary Section 2 contains some notation and known results, included for convenience of reference.

2. Notation

Let $L^p = L^p(\mathbb{T})$, $0 < p \leq \infty$, be the Lebesgue space of p-integrable complex functions f with the norm $\|f\|_{L^p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p\,d\theta\right)^{\frac{1}{p}}$ for $p \geq 1$ (with the standard modification for $p = \infty$), and let $H^p = H^p(\mathbb{D})$, $0 < p \leq \infty$, be the Hardy space

$$\left\{ f \in \mathcal{A}(\mathbb{D}) : \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta < \infty \right\}$$

with the norm $\|f\|_{H^p} = \sup_{r<1} \|f(re^{i\cdot})\|_{L^p}$ for $p \geq 1$ (H^∞ is the space of bounded analytic functions with the supremum norm). It is well known that boundary value function f_+ (see (1)) exists for every $f \in H^p$, $p > 0$, and belongs to L^p. Furthermore,

$$\|f\|_{H^p} = \|f_+\|_{L^p} \tag{6}$$

for every $p \geq 1$, and it follows from the standard Fourier series theory that

$$\left\|\sum_{n=0}^{\infty} a_n z^n\right\|_{H^2} = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2}. \tag{7}$$

Condition

$$\int_0^{2\pi} \log|f_+(e^{i\theta})|\,d\theta > -\infty \tag{8}$$

holds for every $f \in H^p$, and the function f is called outer if the representation (3) is valid. We have the equality (the optimality condition) in (2) if and only if f is outer (see [1, Th. 17.17]). One can check, using the Hölder inequality, that if f and g are outer functions from H^p and H^q, respectively, then the product fg is the outer function from $H^{pq/(p+q)}$.

A function $u \in \mathcal{A}(\mathbb{D})$ is called *inner* if $u \in H^\infty$ and

$$|u_+(e^{i\theta})| = 1 \quad \text{for a.a. } \theta \in [0, 2\pi). \tag{9}$$

If in addition $u(z) \neq 0$ for $z \in \mathbb{D}$, then it is called a *singular inner* function. Every $h \in H^p$ can be factorized as

$$h(z) = B(z)\mathcal{I}(z)f(z), \tag{10}$$

where $B(z) = z^m \prod_{n=1} \frac{|\omega_n|}{\omega_n} \frac{\omega_n - z}{1-\bar{z}_n z}$ is a Blaschke product, $\mathcal{I}$ is a singular inner function and f is an outer function from H^p. (Observe that $|h_+| = |f_+|$ a.e.) In these terms, a function is outer if and only if the inner factor in factorization (10) is constant, i.e., without loss of generality, $B \equiv \mathcal{I} \equiv 1$.

These definitions and factorization (10) are classical in mathematical theory of Hardy spaces. However, engineers frequently discard the middle term in the factorization (10): a singular inner factor, having the form

$$\mathcal{I}(z) = \exp\left(-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu_s(\theta)\right),$$

where μ_s is a singular measure on $[0, 2\pi)$, is trivial in case of rational f and thus not encountered in practice. So, they sometimes define a minimum-phase function $f \in H^2(\mathbb{D})$ by the condition $1/f \in \mathcal{A}(\mathbb{D})$ (i.e. $f(z) \neq 0$ for $z \in \mathbb{D}$). This definition can be used for rational functions, however, not for arbitrary analytic functions. As an example of a singular inner function $\mathcal{I}$ shows, the inequality in (2) might be strict in this case ($|\mathcal{I}(0)| < 1$, while $\int_0^{2\pi} \log|\mathcal{I}_+(e^{i\theta})|\,d\theta = 0$). So, the equality may not hold in (2) even if $f^{-1} \in \mathcal{A}(\mathbb{D})$, as it was incorrectly claimed in [9, p. 574].

We will make use of the following standard result from the theory of Hardy spaces (see [10, p. 109]).

Smirnov's Generalized Theorem: if $f = g/h$, where $g \in H^p$, $p > 0$, h is an outer function from H^q, $q > 0$, and $f_+ \in L^r$, $r > 0$, then $f \in H^r$.

For a positive integer N, let P_N be the projection operator on H^2 defined by

$$P_N\colon \sum_{n=0}^{\infty} a_n z^n \longmapsto \sum_{n=0}^{N} a_n z^n.$$

For $h(z) = \sum_{n=0}^{\infty} \gamma_n z^n \in \mathcal{A}(\mathbb{D})$, let $\operatorname{supp}(\hat{h}) = \{n \in \mathbb{N}_0 : \gamma_n \neq 0\}$.

Now we turn to matrices and matrix functions. For a given set X of scalars or scalar valued functions, let $X_{m\times n}$ stand for the set of $m \times n$ matrices with the entries from X. The elements of $L^p_{d\times d}$ (resp. $H^p_{d\times d}$) are assumed to be matrix functions with domain $\mathbb{T}$ (resp. $\mathbb{D}$) and range $\mathbb{C}_{d\times d}$, and of course $F_+ \in L^p_{d\times d}$ for $F \in H^p_{d\times d}$.

For $M \in \mathbb{C}_{d\times d}$, we consider the Frobenius norm of M:

$$\|M\|_2 = \left(\sum_{i=1}^{d}\sum_{j=1}^{d} |m_{ij}|^2\right)^{1/2} = \left(\operatorname{Tr}(MM^*)\right)^{1/2},$$

where $M^* = \overline{M}^T$, and for $F \in H^p_{d\times d}$, we define

$$\|F\|_{H^2_{d\times d}} = \left(\sum_{i=1}^{d}\sum_{j=1}^{d}|f_{ij}|^2_{H^2}\right)^{1/2}.$$

Similarly, we define $\|F_+\|_{L^2_{d\times d}}$ for $F_+ \in L^2_{d\times d}$. By virtue of (6), we have

$$\|F\|_{H^2_{d\times d}} = \|F_+\|_{L^2_{d\times d}} \tag{11}$$

and, as in (7),

$$\left\|\sum_{n=0}^{\infty} A_n z^n\right\|_{H^2_{d\times d}} = \left(\sum_{n=0}^{\infty}\|A_n\|_2^2\right)^{1/2} \tag{12}$$

for any sequence of matrix coefficients $A_0, A_1, \dots$ from $\mathbb{C}_{d\times d}$.

A matrix function $F \in H^2_{d\times d}$ is called *outer* if $\det F$ is an outer function from $H^{2/d}$. This definition is equivalent to number of other definitions of outer matrix functions (see, e.g., [11]). On the other hand, a matrix function $U \in \mathcal{A}(\mathbb{D})_{d\times d}$ is called *inner* if $U \in H^\infty_{d\times d}$ and U_+ is unitary a.e.:

$$U_+(e^{i\theta})U_+^*(e^{i\theta}) = I_d \quad \text{for a.a. } \theta \in [0, 2\pi). \tag{13}$$

3. Proof of Theorem 1

According to (7), the statement of Theorem 1 is equivalent to

$$\|P_N(f)\|_{H^2} \geq \|P_N(g)\|_{H^2}, \quad N \in \mathbb{N}_0. \tag{14}$$

For any bounded analytic function $u \in H^\infty$, we have

$$P_N(uf) = P_N\big(u \cdot P_N(f)\big) \tag{15}$$

since $P_N\big(u \cdot P_N(f)\big) = P_N\big(u(f - (f - P_N(f)))\big) = P_N(uf) - P_N\big(u(f - P_N(f))\big) = P_N(uf)$. Here we utilized the fact that the kernel of P_N is the set of functions in H^2 having zero as its root of multiplicity at least N, and thus invariant under multiplication by u.

Since (4) holds, by virtue of Beurling factorization (10), there exists an inner function u such that $g = uf$. Therefore, taking into account (6), (9), and (15), we get

$$\|P_N(f)\|_{H^2} = \|uP_N(f)\|_{H^2} \geq \|P_N\big(uP_N(f)\big)\|_{H^2} = \|P_N(uf)\|_{H^2} = \|P_N(g)\|_{H^2}. \tag{16}$$

Thus (14) follows, and Theorem 1 is proved.

4. The matrix case

In this section we prove the following matrix version of Theorem 1.

Theorem 2. *Let $F(z) = \sum_{n=0}^{\infty} A_n z^n$, $A_n \in \mathbb{C}_{d\times d}$, and $G(z) = \sum_{n=0}^{\infty} B_n z^n$, $B_n \in \mathbb{C}_{d\times d}$, be matrix functions from $H^2_{d\times d}$ satisfying*

$$F_+(e^{i\theta})\big(F_+(e^{i\theta})\big)^* = G_+(e^{i\theta})\big(G_+(e^{i\theta})\big)^* \quad \textit{for a.a. } \theta \in [0, 2\pi). \tag{17}$$

If F is optimal, then for each $N \in \mathbb{N}_0$,

$$\sum_{n=0}^{N}\|A_n\|_2^2 \geq \sum_{n=0}^{N}\|B_n\|_2^2. \tag{18}$$

Proof. Let $\mathbb{P}_N$ be the projection operator on $H^2_{d\times d}$ defined by

$$\mathbb{P}_N : \sum_{n=0}^{\infty} A_n z^n \longmapsto \sum_{n=0}^{N} A_n z^n.$$

By virtue of (12), we have to prove that

$$\|\mathbb{P}_N(F)\|_{H^2_{d\times d}} \geq \|\mathbb{P}_N(G)\|_{H^2_{d\times d}}. \tag{19}$$

Let

$$U(z) = F^{-1}(z)G(z). \tag{20}$$

It follows from (17) that (13) holds. Therefore, $U_+ \in L^\infty_{d\times d}$. Since, in addition, $F^{-1}(z) = \frac{1}{\det F(z)} Cof\big(F(z)\big)$, where $\det F(z)$ is an outer function, by the generalized Smirnov's theorem (see Section 2), we have $U \in H^\infty_{d\times d}$. Consequently, (20) is an inner matrix function.

Exactly in the same manner as (15) was proved, we can show that

$$\mathbb{P}_N(FU) = \mathbb{P}_N\big(\mathbb{P}_N(F)U\big). \tag{21}$$

Since unitary transformations preserve standard Euclidian norm on the space $\mathbb{C}^d$, it follows from (13) that, for any $V \in \mathbb{C}^{1\times d}$,

$$\|V\|_2 = \|V \cdot U_+(e^{i\theta})\|_2 \quad \text{for a.a. } \theta \in [0, 2\pi). \tag{22}$$

Therefore, by virtue of (11) and (22),

$$\|X\|_{H^2_{d\times d}} = \|X_+\|_{L^2_{d\times d}} = \|X_+U_+\|_{L^2_{d\times d}} = \|XU\|_{H^2_{d\times d}} \tag{23}$$

for any $X \in H^2_{d\times d}$. It follows now from (23), (21), and (20) that

$$\begin{aligned} \|\mathbb{P}_N(F)\|_{H^2_{d\times d}} &= \|\mathbb{P}_N(F)\cdot U\|_{H^2_{d\times d}} \geq \|\mathbb{P}_N\big(\mathbb{P}_N(F)\cdot U\big)\|_{H^2_{d\times d}} \\ &= \|\mathbb{P}_N(FU)\|_{H^2_{d\times d}} = \|\mathbb{P}_N(G)\|_{H^2_{d\times d}}. \end{aligned}$$

Thus (19) is true, and Theorem 2 is proved. □

5. An energy conservation property

As was mentioned in the Introduction, in the setting of Theorem 1 it can happen that the equality is attained in (5) for some values of N even when g is not a constant multiple of f. The next proposition describes exactly when it is possible. Though not very explicit, it will become instrumental when characterizing the case of (5) turning into an equality for infinitely many values of N.

Proposition 1. *Let $f, g \in H^2$ satisfy (4), with f being an outer function. Then*

$$\sum_{n=0}^{N} |a_n|^2 = \sum_{n=0}^{N} |b_n|^2 \tag{24}$$

holds for some $N \in \mathbb{N}$ if and only if

$$g = uf, \tag{25}$$

where u is a finite Blaschke product,

$$u(z) = cz^{m_0} \prod_{j=1}^{m_1} \frac{z-\alpha_j}{1-\overline{\alpha_j}z}, \quad |c| = 1,\ m_0, m_1 \in \mathbb{N}_0,\ 0 < |\alpha_j| < 1 \text{ for } j = 1, 2, \ldots, m_1, \tag{26}$$

the polynomial $P_N(f)$ has the degree

$$\deg(P_N(f)) \leq N - m_0 \tag{27}$$

and vanishes at $w_j := 1/\overline{\alpha_j},\ j = 1, 2, \ldots, m_1$:

$$P_N(f)(w_j) = 0, \quad j = 1, 2, \ldots, m_1. \tag{28}$$

Proof. It follows from (4) that (25) holds for some inner function u.

The chain of relations in (16) reveals that the equality

$$\|P_N(f)\|_{H_2} = \|P_N(g)\|_{H_2} \tag{29}$$

holds if and only if

$$\|uP_N(f)\|_{H_2} = \|P_N\big(uP_N(f)\big)\|_{H_2}.$$

Therefore (24), which is equivalent to (29), holds if and only if

$$uP_N(f) \text{ is a polynomial with } \deg\big(uP_N(f)\big) \le N. \tag{30}$$

Under the conditions (26), (27), and (28) the relation (30) holds since

$$\prod_{j=1}^{m_1} \frac{z-\alpha_j}{1-\overline{\alpha_j}z} P_N(f) \text{ is a polynomial of the same degree as } P_N(f) \tag{31}$$

and

$$\deg\big(uP_N(f)\big) = m_0 + \deg\big(P_N(f)\big). \tag{32}$$

Thus *sufficiency* is proved.

If now (30) holds, then $u = uP_N(f)/P_N(f)$ is a rational function and, being inner, it has to be of the form (26).

Furthermore, the polynomial $P_N(f)$ should be divisible by $\prod_{j=1}^{m_1}(1-\overline{\alpha_j}z)$. Therefore (28) holds and (31) follows. This implies that (32) holds and (27) follows by virtue of (30), thus proving the *necessity*. □

Note that conditions (27), (28) imply the inequality $N \ge m_0 + m_1 =: m$. In particular, $N = 0$ only if $m_0 = m_1 = 0$, that is, g is a scalar multiple of f. This is of course in agreement with the extremal property of outer functions, and guarantees (in a trivial way) that (24) holds for all $N \in \mathbb{N}$, and thus infinitely many times. The next theorem describes all the cases in which the latter phenomenon occurs.

Theorem 3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *be functions from* H^2 *satisfying* (4)*, with* f *being outer. The set* $\mathcal{N}$ *of those positive integers* N *for which* (24) *holds is infinite if and only if* (25)*,* (26) *hold and*

$$f = qh, \tag{33}$$

where

$$q(z) = \prod_{j=1}^{m_1}(z - w_j) \text{ with } w_j = 1/\overline{\alpha_j},\ j = 1, 2, \ldots, m_1, \tag{34}$$

and h is an outer "lacunary" analytic function with infinitely many gaps in its Fourier spectrum $\mathrm{supp}(\hat{h})$ *of length at least* $m = m_0 + m_1$*. Moreover,* $N \in \mathcal{N}$ *if and only if*

$$N - m + 1, \ldots, N \notin \mathrm{supp}(\hat{h}). \tag{35}$$

Proof. *Sufficiency.* Let g be defined by (25) and (26), and let (33) hold for the polynomial (34) of degree m_1 and an outer analytic function h satisfying (35) for some N. Then we have

$$P_N(f) = P_N(qh) = P_N\big(qP_N(h)\big) = q\sum_{n=0}^{N-m} \gamma_n z^n$$

due to (33), (15), and (35). Therefore,

$$\deg\big(P_N(f)\big) \le m_1 + N - m = N - m_0.$$

Hence $N \in \mathcal{N}$ by virtue of Proposition 1.

Necessity. By Proposition 1, g is given by (25), where the inner multiple (26), is such that (27), (28) hold for all $N \in \mathcal{N}$.

Labeling elements of $\mathcal{N}$ as an increasing sequence N_k, we thus have

$$P_{N_k}(f) = qh_k, \tag{36}$$

where polynomials h_k satisfy

$$\deg(h_k) \le N_k - m. \tag{37}$$

The function q is the same for all k as it is uniquely determined by (26).

Since $P_{N_k}(f) \to f$ in H_2 as $k \to \infty$, we have $qh_k \to f$. Therefore $(h_k)_+$ converges to f_+/q_+ in $L_2(\mathbb{T})$ (since $1/q_+$ is bounded on $\mathbb{T}$), and consequently h_k is convergent in H_2. Let h be the limit. Letting $k \to \infty$ in (36), we arrive at (33). Since f is outer, the function h is such as well.

Let now $N = N_k$ be an arbitrary element of $\mathcal{N}$. Because of (33) and (36), we have

$$f - P_N(f) = q(h - h_k).$$

Since $f - P_N(f)$ is divisible by z^{N+1} and 0 is not the root of q, we have $h - h_k = z^{N+1}\tilde{h}_k$ for some analytic function $\tilde{h}_k \in H^2$. Therefore $h = h_k + z^{N+1}\tilde{h}_k$ with $\deg(h_k) \le N - m$ (see (37)) and this implies that the coefficients with indices from $\{N - m + 1, N - m + 2, \dots, N\}$ are omitted in the power expansion of h. Thus (35) holds and the theorem is proved. $\square$

Corollary 1. *Let $\{N_1, N_2, \dots\} \subset \mathbb{N}$ be any infinite set. Then there exist functions $f, g \in H_2$ where f is an outer function such that*

$$\sum_{n=0}^{N} |a_n|^2 = \sum_{n=0}^{N} |b_n|^2 \tag{38}$$

if and only if $N \in \{N_1, N_2, \dots\}$.

Proof. Let $q(z) = z - w$ with $|w| > 1$, and let $h(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ be an outer function from H_2 such that $\gamma_n = 0$ if and only if $n \in \{N_1, N_2, \dots\}$ (the outerness of h can be achieved, for example, by making sure that $|\gamma_0| > \sum_{n=1}^{\infty} |\gamma_n|$). Define $f = qh$ and $g(z) = (1 - \overline{w}z)h(z)$. Then it follows from the proof of the theorem that (38) holds if and only if $N \in \{N_1, N_2, \dots\}$. $\square$

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Original article

Sharp weighted bounds for the Hilbert transform of odd and even functions

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Abstract

Our aim is to establish sharp weighted bounds for the Hilbert transform of odd and even functions in terms of the mixed type characteristics of weights. These bounds involve A_p and A_∞ type characteristics. As a consequence, we obtain weighted bounds in terms of so-called Andersen–Muckenhoupt type characteristics.

Keywords: Hilbert transform; Sharp weighted bound; One-weight inequality

1. Introduction

In this paper, we investigate sharp weighted bounds, involving A_p and A_∞ characteristics of weights, for the Hilbert transform of odd and even functions. Following general results we derive these sharp weighted A_p bounds in terms of so-called Andersen-Muckenhoupt characteristics. Let X and Y be two Banach spaces. Given a bounded operator $T : X \to Y$, we denote the operator norm by $\|T\|_{\mathcal{B}(X,Y)}$ which is defined in the standard way i.e. $\|T\|_{\mathcal{B}(X,Y)} = \sup_{\|f\|_X \le 1} \|Tf\|_Y$. If $X = Y$ we use the symbol $\|T\|_{\mathcal{B}(X)}$.

A non-negative locally integrable function (i.e. a weight function) w defined on $\mathbb{R}^n$ is said to satisfy the $A_p(\mathbb{R}^n)$ condition ($w \in A_p(\mathbb{R}^n)$) for $1 < p < \infty$ if

$$\|w\|_{A_p(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$ and supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. We call $\|w\|_{A_p(\mathbb{R}^n)}$ the A_p characteristic of w.

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In 1972, B. Muckenhoupt [1] showed that if $w \in A_p(\mathbb{R}^n)$, where $1 < p < \infty$, then the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy$$

is bounded in $L^p_w(\mathbb{R}^n)$. S. Buckley [2] investigated the sharp A_p bound for the operator M. In particular, he established the inequality

$$\|M\|_{L^p_w(\mathbb{R}^n)} \le C \|w\|^{\frac{1}{p-1}}_{A_p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{1.1}$$

Moreover, he showed that the exponent $\frac{1}{p-1}$ is best possible in the sense that we cannot replace $\|w\|^{\frac{1}{p-1}}_{A_p}$ by $\psi(\|w\|_{A_p})$ for any positive non-decreasing function ψ growing slowly than $x^{\frac{1}{p-1}}$. From here it follows that for any $\lambda > 0$,

$$\sup_{w \in A_p} \frac{\|M\|_{L^p_w}}{\|w\|^{\frac{1}{p-1}-\lambda}_{A_p}} = \infty.$$

Let H be the Hilbert transform given by

$$(Hf)(x) = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt, \quad x \in \mathbb{R}.$$

In 1973 R. Hunt, B. Muckenhoupt and R. L. Wheeden [3] solved the one-weight problem for the Hilbert transform in terms of Muckenhoupt condition. In particular, they established the inequality

$$\|Hf\|_{L^p_w(\mathbb{R})} \le c_p \|w\|^{\beta}_{A_p(\mathbb{R})} \|f\|_{L^p_w(\mathbb{R})} \tag{1.2}$$

for some positive constant β and some constant c_p depending on p. S. Petermichl showed that the value of the exponent $\beta = \max\{1, p'/p\}$ in (1.2) is sharp. In particular, the following statement holds (see [4] for $p = 2$, [5] for $p \neq 2$):

Theorem A. *Let $1 < p < \infty$ and let w be a weight function on $\mathbb{R}$. Then there is a positive constant c_p depending only on p such that*

$$\|H\|_{\mathcal{B}(L^p_w)} \le c_p \|w\|^{\beta}_{A_p(\mathbb{R})}, \tag{1.3}$$

where $\beta = \max\left\{1, \frac{p'}{p}\right\}$. Moreover, the exponent in (1.3) *is sharp.*

We say that $w \in A_\infty(\mathbb{R}^n)$ if $w \in A_p(\mathbb{R})$ for some $p > 1$. In what follows we will use the symbol $\|\rho\|_{A_\infty}$ for the A_∞ characteristic of a weight function ρ:

$$\|\rho\|_{A_\infty} = \sup_I \frac{1}{\rho(I)} \int_I M(\rho \chi_I)(x) dx.$$

This characteristic appeared first in the papers by Fiji [6] and Wilson [7,8] and is lower than that the one introduced by Hruščev [9]:

$$[\rho]_{A_\infty} = \sup_I \left(\frac{1}{|I|} \int_I \rho(x) dx \right) \exp\left(\frac{1}{|I|} \int_I \log \rho^{-1}(x) dx \right).$$

In 2012, Hytönen, Perez and Rela [10] improved Buckley's result and obtained a sharp weighted bound involving A_∞ constant:

$$\|M\|_{\mathcal{B}(L^p_w)} \le c_n \left(\frac{1}{p-1} \|w\|_{A_p} \|\sigma\|_{A_\infty} \right)^{1/p}, \quad 1 < p < \infty, \quad \sigma = w^{1-p'}.$$

Later, in [11], it was proved that the sharp weighted bound involving the A_∞ characteristic for the Calderón–Zygmund operator provides an improved estimate than the one obtained by Hytönen in his celebrated paper [12] about the A_2 conjecture. We recall the result of [10] for the Hilbert transform H in the following theorem.

Theorem B. *Let H be the Hilbert transform and let $p \in (1,\infty)$. Then if $w \in A_p(\mathbb{R}_+)$, we have*

$$\|H\|_{\mathcal{B}(L^p_w)} \le \begin{cases} \|w\|_{A_p}^{2/p}\|\sigma\|_{A_\infty}^{2/p-1}, & \text{if } p \in (1,2],\\ \|w\|_{A_p}^{2/p}\|w\|_{A_\infty}^{1-2/p}, & \text{if } p \in [2,\infty), \end{cases} \tag{1.4}$$

where $\sigma := w^{1-p'}$.

It is known (see [11]) that

$$c_n\|\rho\|_{A_\infty} \le [\rho]_{A_\infty} \le \|\rho\|_{A_p}. \tag{1.5}$$

It can be checked that

$$[\sigma]_{A_\infty}^{p-1} \le \|\sigma\|_{A'_p}^{p-1} = \|w\|_{A_p}.$$

In the sequel we will use the following relation between weights $w : \mathbb{R} \to \mathbb{R}_+$ and $W : \mathbb{R}_+ \to \mathbb{R}_+$ (resp. between $\sigma : \mathbb{R} \to \mathbb{R}_+$ and $\Sigma : \mathbb{R}_+ \to \mathbb{R}_+$)

$$w(x) := \frac{W(\sqrt{|x|})}{2\sqrt{|x|}} \quad \Big(\text{resp. } \sigma(x) := \frac{\Sigma(\sqrt{|x|})}{2\sqrt{|x|}}\Big),$$

where $x \ne 0$.

Finally we mention that weighted sharp estimates for one-sided operators on the real line in terms of one-sided Muckenhoupt characteristics were established in [13] (see also [14] for related topics regarding multiple integral operators).

The relation $A \approx B$ means that there are positive constants c_1 and c_2 (in general these constants will depend only on the space exponents r or p) such that $c_1 B \le A \le c_2 B$.

For a weight function ρ and a measurable set $E \subset \mathbb{R}$, we denote

$$\rho(E) := \int_E \rho(x)dx.$$

Constants will be denoted by c or C (the same notation will be used even if they can differ from line to line).

2. Preliminaries

Let $f : \mathbb{R} \to \mathbb{R}_+$ be odd. Then it is easy to check that Hf is even and given by $(Hf)(x) = (H_0 f)(x)$ for $x > 0$, where

$$(H_0 f)(x) = \frac{2}{\pi}\int_0^\infty \frac{tf(t)}{t^2 - x^2}dt, \quad x > 0.$$

If f is even, then Hf is odd and is given by $(Hf)(x) = (H_e f)(x)$ for $x > 0$, where

$$(H_e f)(x) = \frac{2}{\pi}\int_0^\infty \frac{xf(t)}{t^2 - x^2}dt.$$

Our aim is to investigate the sharp weighted bound of the type (1.4) for operators H_0 and H_e, and to derive sharp estimates of the type:

$$\|H_0 f\|_{L^p_W(\mathbb{R}_+)} \le c_p\|W\|^{\beta}_{A^0_p(\mathbb{R}_+)}\|f\|_{L^p_W(\mathbb{R}_+)}, \tag{2.1}$$

$$\|H_e f\|_{L^p_W(\mathbb{R}_+)} \le c_p\|W\|^{\gamma}_{A^e_p(\mathbb{R}_+)}\|f\|_{L^p_W(\mathbb{R}_+)} \tag{2.2}$$

where $1 < p < \infty$ and

$$\|W\|_{A^0_p(\mathbb{R}_+)} := \sup_{[a,b]\subset(0,\infty)} \left(\frac{1}{b^2-a^2}\int_a^b W(x)dx\right)\left(\frac{1}{b^2-a^2}\int_a^b x^{p'}W^{1-p'}(x)dx\right)^{p-1}$$

$$\|W\|_{A^e_p(\mathbb{R}_+)} := \sup_{[a,b]\subset(0,\infty)} \left(\frac{1}{b^2-a^2}\int_a^b x^p W(x)dx\right)\left(\frac{1}{b^2-a^2}\int_a^b W^{1-p'}(x)dx\right)^{p-1}.$$

K. Andersen [15] showed that if $1 < p < \infty$, then

(i) H_0 is bounded in $L^p_W(\mathbb{R}_+)$ if and only if $\|W\|_{A^0_p(\mathbb{R}_+)} < \infty$;

(ii) H_e is bounded in $L^p_W(\mathbb{R}_+)$ if and only if $\|W\|_{A^e_p(\mathbb{R}_+)} < \infty$.

The following lemma was proved in [15] but we give the proof because of the exponents of characteristics of weights.

Lemma 2.1. *Let $1 < r < \infty$ and let w be a non-negative measurable function on $(0, \infty)$. Then*

$$\|W\|_{A^0_r(\mathbb{R}_+)} \approx \|w\|_{A_r(\mathbb{R})}$$

with constants depending only on r.

Proof. First we show that

$$\|w\|_{A_r(\mathbb{R})} \le c_r \|W\|_{A^0_r(\mathbb{R}_+)}.$$

Let $[a, b] \subset (0, \infty)$. Then

$$\begin{aligned}\left(\int_a^b w(x)dx\right)\left(\int_a^b w^{1-r'}(x)dx\right)^{r-1} &= \left(\int_a^b W(\sqrt{x})\frac{dx}{2\sqrt{x}}\right)\left(\int_a^b W^{1-r'}(\sqrt{x})\frac{dx}{(2\sqrt{x})^{1-r'}}\right)^{r-1}\\ &= 2^r\left(\int_{\sqrt{a}}^{\sqrt{b}} W(x)dx\right)\left(\int_{\sqrt{a}}^{\sqrt{b}} x^{r'}W^{1-r'}(x)dx\right)^{r-1}.\end{aligned}$$

If $\|W\|_{A^0_r(\mathbb{R}_+)} < \infty$, then the latter expression is bounded by

$$2^r\|W\|_{A^0_r(\mathbb{R}_+)}((\sqrt{b})^2 - (\sqrt{a})^2)^r = 2^r\|W\|_{A^0_r(\mathbb{R}_+)}(b-a)^r.$$

This follows from the definition of $\|W\|_{A^0_r(\mathbb{R}_+)}$.

Suppose now that $[a, b] \subset (-\infty, 0)$. Arguing as before, we see that

$$\begin{aligned}\left(\int_a^b w(x)dx\right)\left(\int_a^b w^{1-r'}(x)dx\right)^{r-1} &= 2^r\left(\int_{\sqrt{-b}}^{\sqrt{-a}} W(x)dx\right)\left(\int_{\sqrt{-b}}^{\sqrt{-a}} x^{r'}W^{1-r'}(x)dx\right)^{r-1}\\ &\le 2^r\|W\|_{A^0_r(\mathbb{R}_+)}(b-a)^r.\end{aligned}$$

Now let $a < 0 < b$. Suppose that $c > 0$ is a number such that $[a, b] \subset [-c, c]$, and $[a, b]$ and $[-c, c]$ have at least one common endpoint. Then by using the above arguments we see that

$$\begin{aligned}\left(\int_a^b w(x)dx\right)\left(\int_a^b w^{1-r'}(x)dx\right)^{r-1} &\le 2^r\left(\int_0^c w(x)dx\right)\left(\int_0^c w^{1-r'}(x)dx\right)^{r-1}\\ &\le c_r\|W\|_{A^0_r(\mathbb{R}_+)}(b-a)^r\end{aligned}$$

where c_r is a positive constant depending only on r. Finally,

$$\|w\|_{A_r(\mathbb{R})} \le c_r\|W\|_{A^0_r(\mathbb{R}_+)}.$$

Inequality $\|W\|_{A^0_r(\mathbb{R}_+)} \le c_r\|w\|_{A_r(\mathbb{R})}$ follows from the arguments similar to those used above. □

Now we introduce Wilson type A_∞ characteristic for weights defined on $\mathbb{R}_+$. The classes A_∞^0 and A_∞^e are defined as follows:

$$A_\infty^0 = \cup_{p>1} A_p^0; \quad A_\infty^e = \cup_{p>1} A_p^e.$$

Let $\|W\|_{A_\infty^0}$ be the A_∞^0 characteristic of a W on $\mathbb{R}_+$ defined as follows:

$$\|W\|_{A_\infty^0} = \sup_{(a,b)\subset\mathbb{R}_+} \frac{1}{W([a,b])} \int_a^b x\big(\overline{M}(W\chi_{(a,b)})\big)(x)dx,$$

where

$$\overline{M} f(x) = \sup_{(c,d)\ni x} \frac{1}{d^2-c^2} \int_c^d W(t)dt. \tag{2.3}$$

Here the supremum is taken over all interval $(c,d) \subset \mathbb{R}_+$ containing x.

The next statement will be useful to prove the main Theorem.

Lemma 2.2. *Let w be a weight on $\mathbb{R}$. Then the following relation holds:*

$$\|w\|_{A_\infty(\mathbb{R})} \approx \|W\|_{A_\infty^0(\mathbb{R}_+)} \tag{2.4}$$

with constants independent of w.

Proof. At first suppose that $I := (a,b) \subset \mathbb{R}_+$. Then it is easy to see that

$$\frac{1}{w(I)} \int_I M(w\chi_I)(x)dx \approx \frac{1}{W([\sqrt{a},\sqrt{b}])} \int_{\sqrt{a}}^{\sqrt{b}} x\overline{M}\big(W\chi_{[\sqrt{a},\sqrt{b}]}\big)(x)dx, \tag{2.5}$$

with constants independent of I and w, where $\overline{M}$ is defined by formula (2.3).

Next, we use the following observation: let $x \in (a,b)$,

$$M\big(w\chi_{(a,b)}\big)(x) \approx \overline{M}\big(W\chi_{(\sqrt{|a|},\sqrt{|b|})}\big)(\sqrt{x})$$

which can be obtained from the relation between w and W. In a similar manner, if $I := (a,b) \subset \mathbb{R}_-$, we have

$$\frac{1}{w(I)} \int_a^b M(w\chi_I)(x)dx \approx \frac{1}{W([\sqrt{-a},\sqrt{-b}])} \int_{\sqrt{-b}}^{\sqrt{-a}} x\overline{M}\big(W\chi_{(\sqrt{a},\sqrt{b})}\big)(x)dx. \tag{2.6}$$

Let now $0 \in I$. Then we represent $I = (a,0] \cup (0,b)$ to get

$$\begin{aligned}\frac{1}{w(I)} \int_I M(w\chi_I)(x)dx \le{}& \frac{1}{w(I)} \int_{(a,0)} M(w\chi_{(a,0)})(x)dx \\ &+ \frac{1}{w(I)} \int_{(a,0)} M(w\chi_{(0,b)})(x)dx + \frac{1}{w(I)} \int_{(0,b)} M(w\chi_{(a,0)})(x)dx \\ &+ \frac{1}{w(I)} \int_{(0,b)} M(w\chi_{(0,b)})(x)dx := S_1 + S_2 + S_3 + S_4.\end{aligned}$$

We have to estimate S_2 and S_3. Estimates for S_1 and S_4 can be derived in a similar manner by using the estimates

$$\frac{1}{w(I)} \int_a^0 M(w\chi_{[a,0]})(x)dx \le \frac{1}{w([a,0])} \int_a^0 M(w\chi_{[a,0]})(x)dx$$

and

$$\frac{1}{w(I)} \int_0^b M(w\chi_{[0,b]})(x)dx \le \frac{1}{w([0,b])} \int_0^b M(w\chi_{[0,b]})(x)dx.$$

Simple observations lead us to the estimates:

$$S_i \le C \frac{1}{W([0,\sqrt{A}])} \int_0^{\sqrt{A}} x\overline{M}\big(W\chi_{[0,\sqrt{A}]}\big)(x)dx \le C\|W\|_{A_\infty^0}, \quad i = 2, 3,$$

where $A := \max\{|a|, |b|\}$. Finally we have that

$$\|w\|_{A_\infty(\mathbb{R})} \le C\|W\|_{A_\infty^0}$$

with a constant C independent of w. The reverse estimate can be obtained in a similar manner. □

The next lemma is a consequence of (1.5), Lemmas 2.2 and 2.1.

Lemma 2.3. *Let* $1 < p < \infty$. *Then*

$$\|W\|_{A_\infty^0} \le C\|W\|_{A_p^0}.$$

In the sequel we assume that $\sigma = w^{1-p'}$. Taking into account the definition of Σ,we have that

$$\Sigma(u) = W^{1-p'}(u)(2u)^{p'}. \tag{2.7}$$

Theorem 2.1. *Let* $1 < p < \infty$. *Then* (i)

$$\|H_0\|_{\mathcal{B}(L_W^p)} \le \begin{cases} \|W\|_{A_p^0}^{2/p}(\|\Sigma\|_{A_\infty^0})^{2/p-1}, & \text{if } p \in (1,2], \\ \|W\|_{A_p^0}^{2/p}(\|W\|_{A_\infty^0})^{1-2/p}, & \text{if } p \in [2,\infty), \end{cases} \tag{2.8}$$

(ii)

$$\|H_e\|_{\mathcal{B}(L_W^p)} \le \begin{cases} \|W\|_{A_p^e}^{2/p}(\|W^{1-p'}\|_{A_\infty^0})^{1-2/p'}, & \text{if } p \in (1,2], \\ \|W\|_{A_p^e}^{2/p}(\|W_p\|_{A_\infty^0})^{2/p'-1}, & \text{if } p \in [2,\infty), \end{cases} \tag{2.9}$$

where W and Σ are related by (2.7) *and* $W_p(x) = W(x)(2x)^p$.

Proof. Let us prove (i). The proof for (ii) is a consequence of the dual arguments and will be discussed afterwards.

Let us denote $g(x) := f(\sqrt{x})$, $x > 0$, $g(x) = 0$ otherwise. Suppose that w and W are related as in Lemma 2.1, we have

$$\int_{-\infty}^{+\infty} |g(x)|^p w(x)dx = \int_0^\infty |f(\sqrt{x})|^p w(x)dx = \int_0^\infty |f(\sqrt{x})|^p \frac{W(\sqrt{x})}{2\sqrt{x}}dx = \int_0^\infty |f(u)|^p W(u)du.$$

Furthermore, for $x > 0$,

$$(Hg)(x) = \frac{1}{\pi}\int_0^\infty \frac{f(\sqrt{t})}{t-x}dt = \frac{1}{\pi}\int_0^\infty \frac{2tf(t)}{t^2-x}dt = (H_0f)(\sqrt{x}).$$

By definition, we have

$$\begin{aligned} \|H_0f\|_{L_W^p(\mathbb{R}_+)}^p &= \int_0^\infty |(H_0f)(x)|^p W(x)dx = \int_0^\infty |(H_0f)(\sqrt{u})|^p W(\sqrt{u})\frac{du}{2\sqrt{u}} \\ &= \int_0^\infty |(H_0f)(\sqrt{u})|^p w(u)du \\ &= \int_0^\infty |(Hg)(u)|^p w(u)du \le \|Hg\|_{L_w^p(\mathbb{R})}^p. \end{aligned}$$

Let $1 < p \le 2$. Then by Theorem B and Lemmas 2.1 and 2.2 we have that

$$\|H\|_{\mathcal{B}(L_w^p(\mathbb{R}))} \le \|w\|_{A_p(\mathbb{R})}^{2/p}\|\sigma\|_{A_\infty(\mathbb{R})}^{2/p-1} \approx \|W\|_{A_p^0(\mathbb{R}_+)}^{2/p}\|\Sigma\|_{A_\infty^0(\mathbb{R}_+)}^{2/p-1}$$

where $\sigma = w^{1-p'}$, $\sigma(x) = \Sigma(\sqrt{|x|})/(2\sqrt{|x|})$. Observe that W and Σ are related also by (2.7). The case $p \geq 2$ follows analogously. Thus we have (2.8).

To prove (2.9) we use the duality arguments. First observe that the Riesz identity for the classical Hilbert transform H and the appropriate substitution of the variable yields that

$$\int_{\mathbb{R}_+} (H_0 f)(x) g(x) dx = -\int_{\mathbb{R}_+} (H_e g)(x) f(x) dx.$$

Hence, it follows that the adjoint of H_o is H_e with the equation

$$\|H_e\|_{\mathcal{B}(L^p_w(\mathbb{R}_+))} = \|H_o\|_{\mathcal{B}(L^{p'}_\sigma(\mathbb{R}_+))}.$$

By applying case (i) and Lemmas 2.1 and 2.2 we have the desired result also for (ii). □

The next statement gives sharp weighted bound in terms of A_p characteristics.

Theorem 2.2. *Let $1 < p < \infty$ and let W be a weight function on $\mathbb{R}_+$. Then the following estimates hold*

(a)

$$\|H_0\|_{L^p_W(\mathbb{R}_+)} \leq c_p \|W\|^\beta_{A^0_p(\mathbb{R})}; \tag{2.10}$$

(b)

$$\|H_e\|_{L^p_W(\mathbb{R}_+)} \leq C_p \|W\|^\beta_{A^e_p(\mathbb{R}_+)} \tag{2.11}$$

with some positive constants c_p and C_p, respectively, depending only on p, where $\beta = \max\{1, \frac{p'}{p}\}$. Moreover the exponent β in (2.10) and (2.11) is best possible.

Proof. We prove (a). The estimate (b) follows from the duality arguments. Let $1 < p \leq 2$. To show the validity of (a) we use (2.8), Lemma 2.1 and relations

$$\|\Sigma\|_{A^0_\infty(\mathbb{R}_+)} \approx \|\sigma\|_{A_\infty(\mathbb{R})} \leq \|\sigma\|_{A_{p'}(\mathbb{R})} = \|w\|^{p'-1}_{A_p(\mathbb{R})} \approx \|W\|^{p'-1}_{A^0_p(\mathbb{R}_+)}.$$

The case $p > 2$ follows from the estimates:

$$\|W\|_{A^0_\infty(\mathbb{R}_+)} \approx \|w\|_{A_\infty(\mathbb{R})} \leq \|w\|_{A_p(\mathbb{R})} \approx \|W\|_{A^0_p(\mathbb{R})}.$$

Sharpness: First we will show the sharpness for $p = 2$. Let

$$g(x) = x^{\varepsilon-1}\chi_{(0,1)}, \quad w(x) = |x|^{1-\varepsilon}.$$

Then (see [4]) the following estimate holds:

$$\|g\|_{L^2(\mathbb{R})} \approx \frac{1}{\varepsilon}; \quad \|w\|_{A_2(\mathbb{R})} \approx \frac{1}{\varepsilon}; \quad \|Hg\|_{L^2_w(\mathbb{R})} \geq 4\varepsilon^{-3}.$$

Let now

$$f(x) = x^{2(\varepsilon-1)}\chi_{(0,1)}, \quad W(x) = |x|^{3-\varepsilon}.$$

Hence by using the same changing of variable we find that

$$\|f\|^2_{L^2_W(\mathbb{R})} \approx \frac{1}{\varepsilon}; \quad \|H_0 f\|^2_{L^2_W(\mathbb{R}_+)} \geq \varepsilon^{-3}.$$

Consequently, if the exponent $1-\varepsilon$ is the best possible for the A^0_2 characteristic in the one-weight inequality for some $\lambda > 0$, we have

$$4\varepsilon^{-3} \leq \|H_0 f\|_{L^2_W(\mathbb{R}_+)} \leq C\|W\|^{1-\lambda}_{A^0_2} \|f\|_{L^2_W(\mathbb{R})} \leq C\|W\|^{1-\varepsilon}_{A^0_2} \leq C\varepsilon^{\lambda-3}.$$

Let $1 < p < 2$. Suppose that $0 < \epsilon < 1$ and that $w(x) = |x|^{(1-\epsilon)(p-1)}$. Then it is easy to check that (see also [4])

$$\|w\|^{1/(p-1)}_{A_p} \approx \frac{1}{\epsilon}.$$

Observe also, that for the function defined by

$$f(x) = x^{\epsilon-1}\chi_{(0,1)}, \tag{2.12}$$

the relation $\|f\|_{L^p_w} \approx \frac{1}{\epsilon^{\frac{1}{p}}}$ holds. Let

$$g(x) = x^{2(\varepsilon-1)}, \quad W(x) = |x|^{2(1-\varepsilon)(p-1)}.$$

Then the following estimates can be checked easily by using the appropriate change of variables:

$$\begin{aligned}\|H_0 g\|_{L^p_w(\mathbb{R}_+)} &= 2^{-1/p}\|Hf\|_{L^p_w(\mathbb{R})} \geq 2^{-1/p}\frac{1}{\epsilon}\|f\|_{L^p_w(\mathbb{R})} \\ &\approx \|w\|_{A_p}^{p'/p}\|f\|_{L^p_w(\mathbb{R})} \approx \|W\|_{A^0_p}^{p'/p}\|g\|_{L^p_W(\mathbb{R}_+)}\end{aligned}$$

are fulfilled. Thus we have sharpness in (2.10) for $1 < p < 2$.

It remains to consider the case when $p > 2$. In the same manner as above, we can argue for the operator H_e and obtain the sharpness in (2.11) for $1 < p < \infty$. The duality arguments now imply the sharpness in (2.10) for $2 < p < \infty$. □

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Original article

Duality of fully measurable grand Lebesgue space

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Abstract

In this paper, we prove a Hölder's type inequality for fully measurable grand Lebesgue spaces, which involves the notion of fully measurable small Lebesgue spaces. It is proved that these spaces are non-reflexive rearrangement invariant Banach function spaces. Moreover, under certain continuity assumptions, along with several properties of fully measurable small Lebesgue spaces, we establish Levi's theorem for monotone convergence and that grand and small spaces are associated to each other.

Keywords: Banach function norm; Grand Lebesgue space; Associate space and Levi's theorem

1. Introduction

Let $I = (0, 1)$ and $1 < p < \infty$. The grand Lebesgue space $L^{p)}$ consists of measurable functions f defined on I for which

$$\|f\|_{L^{p)}} := \sup_{0<\epsilon<p-1} \left(\varepsilon \int_I |f(x)|^{p-\varepsilon} dx \right)^{1/(p-\varepsilon)} < \infty.$$

This space was originated in [1], and since then it has attained enormous attention. The people have studied this space for its basic properties like duality and convergence, for which one may refer to [2,3] and [4]. Further, the weighted version of this space was introduced in [5], and thereafter the boundedness of several integral operators has been studied on these spaces. One may refer to [6–8] and the references therein.

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In [9], Capone, Formica and Giova generalized the space $L^{p)}$, the new space being denoted by $L^{p),\delta}$, which consists of measurable functions f defined on I, for which

$$\|f\|_{L^{p),\delta}} := \operatorname*{ess\,sup}_{0<\epsilon<p-1} \left(\delta(\varepsilon)\int_I |f(x)|^{p-\varepsilon}dx\right)^{1/(p-\varepsilon)} < \infty, \tag{1.1}$$

where $0 \not\equiv \delta \in L^\infty(0, p-1)$.

In a very recent paper [10], Anatriello and Fiorenza have made a further generalization, replacing $p-\varepsilon$ in (1.1) by a general measurable function and called it as fully measurable grand Lebesgue space, denoted by $L^{p[\cdot],\delta(\cdot)}$ defined as follows:

Let $p(\cdot)$ be a measurable extended real valued function defined on I such that $p(\cdot) \geq 1$ almost everywhere (a.e.), $\delta \in L^\infty$, $\delta > 0$ a.e. and $0 < \|\delta\|_{L^\infty} \leq 1$. The space $L^{p[\cdot],\delta(\cdot)}$ consists of measurable functions f defined on I for which $\|f\|_{L^{p[\cdot],\delta(\cdot)}} := \rho_{p[\cdot],\delta(\cdot)}(|f|) < \infty$, where

$$\rho_{p[\cdot],\delta(\cdot)}(|f|) = \operatorname*{ess\,sup}_{x\in I} \rho_{p(x)}(\delta(x)|f(\cdot)|)$$

and

$$\rho_{p(x)}(\delta(x)|f(\cdot)|) = \begin{cases} \left(\displaystyle\int_I (\delta(x)|f(t)|)^{p(x)}dt\right)^{\frac{1}{p(x)}} & \text{if } 1 \leq p(x) < \infty; \\ \operatorname*{ess\,sup}_{t\in I}(\delta(x)|f(t)|) & \text{if } p(x) = \infty. \end{cases}$$

In [10], some properties of the space $L^{p[\cdot],\delta(\cdot)}$ have been established and moreover, Hardy inequality has been obtained in the framework of these spaces. The authors in [10] clearly pointed it out that the space $L^{p[\cdot],\delta(\cdot)}$ is different than the variable exponent Lebesgue space $L^{p(x)}$ which has been studied extensively during the recent past. A systematic treatment of the space $L^{p(x)}$ along with updated references can be found in [11].

The aim of the present paper is to investigate the duality for the fully measurable grand Lebesgue space $L^{p[\cdot],\delta(\cdot)}$. For the grand Lebesgue space $L^{p)}$, the duality was studied by Fiorenza [2]. In fact, he introduced the so called small Lebesgue space, denoted by $L^{p)'}$ and proved that this space is the associate space of $L^{p)}$, where $\frac{1}{p}+\frac{1}{p'}=1$. In order to define the space $L^{p)'}$, Fiorenza formulated an auxiliary space $L^{(p'}$ and then, its norm was used to define a norm on the space $L^{p)'}$. Moreover, in an other paper [12], it was shown that the norms defined on the spaces $L^{(p'}$ and $L^{p)'}$ are equivalent. In our case, under continuity assumptions for δ and p, we define fully measurable small Lebesgue space $L^{(p[\cdot]',\delta(\cdot)}$ as associate space of the space $L^{p[\cdot],\delta(\cdot)}$. Here, the novelty is that we do not go via intermediary auxiliary space.

The paper is organized as follows: In order not to disturb the flow of the paper, we collect certain prerequisites in Section 2 in the form of notations, conventions, known definitions and results. In Section 3, we define fully measurable small Lebesgue space and prove that it is a Banach space, and possesses lattice property. The fact that fully measurable small Lebesgue space is a Banach function space has been proved in Section 4, where we also prove Levi's theorem for monotone convergence and a Hölder type inequality for such spaces. Finally, in Section 5, we discuss the fully measurable small Lebesgue space as associate space of fully measurable grand Lebesgue space.

2. Prerequisites

Throughout the paper, we shall be using the following notations/conventions/considerations:

- $\mathbb{N}$:= set of natural numbers.
- $\mathcal{M}$:= set of extended real valued measurable functions defined on I.
- $\mathcal{M}^+$:= subset of $\mathcal{M}$, consisting of non-negative functions.
- $\mathcal{M}_0$:= set of finite a.e. measurable functions defined on I.
- $\mathcal{M}_0^+$:= subset of $\mathcal{M}_0$, consisting of non-negative functions.
- $p_+ := \operatorname{ess\,sup}_{x\in I} p(x)$.
- $|E|$:= Lebesgue measure of E, $E \subseteq I$.
- χ_E := the characteristic function on E, $E \subseteq I$.

- For a fixed $x \in I$, $\rho_{p(x)}(|f|)$ denotes the L^p-norm of f on I, i.e.,

$$\rho_{p(x)}(|f|) = \begin{cases} \left(\int_I |f(t)|^{p(x)} dt\right)^{\frac{1}{p(x)}} & \text{if } 1 \le p(x) < \infty; \\ \operatorname*{ess\,sup}_{t \in I} |f(t)| & \text{if } p(x) = \infty. \end{cases}$$

- For a fixed $x \in I$, $L^{p(x)}$ denotes the usual L^p- space with exponent $p = p(x)$.
- For a fixed $x \in I$, $p(x)'$ is the conjugate of $p(x)$,i.e., $\frac{1}{p(x)} + \frac{1}{p(x)'} = 1$.
- $\underline{\delta}(E) := \operatorname{ess\,inf}_{x \in E} \frac{1}{\delta(x)}$, where $E \subseteq I$, $|E| > 0$. In particular, $0 < \underline{\delta}(I) < \infty$.
- $f_n \uparrow f$ means that $\{f_n\}$ is nondecreasing sequence converging to f.
- C denotes a positive constant which may be different at different places.
- The relation $A \approx B$ means there exist positive constants c_1 and c_2, such that $c_1 A \le B \le c_2 A$.
- Unless specified otherwise, our discussion will be on the set $I = (0, 1)$ and all the functions will be extended real valued measurable, defined on I.

Below we collect certain definitions and results which can easily be found in the literature, e.g., one may refer to [13] and [14].

A mapping $\rho : \mathcal{M}_0^+ \to [0, \infty]$ is called a Banach function norm if for all $f, g, f_n \in \mathcal{M}_0^+$, $n \in \mathbb{N}$, for all constants $\lambda \ge 0$, and for all measurable subsets $E \subset I$, the following properties hold:

- $\rho(f) = 0$ if and only if $f = 0$ a.e. on I
- $\rho(\lambda f) = \lambda \rho(f)$
- $\rho(f + g) \le \rho(f) + \rho(g)$
- If $0 \le g \le f$ a.e. in I, then $\rho(g) \le \rho(f)$ (lattice property)
- If $0 \le f_n \uparrow f$ a.e. in I, then $\rho(f_n) \uparrow \rho(f)$ (Fatou property)
- $\rho(\chi_E) < \infty$
- $\int_E f(t)dt \le C_E\, \rho(f)$, for some constant $C_E < \infty$, depending upon E and ρ, but independent of f.

Note. In the above definition, one can take any measurable set $\Omega \subset \mathbb{R}$ in place of I.

If ρ is a Banach function norm, then the Banach space

$$X = X(\rho) := \{f \in \mathcal{M}_0 : \rho(|f|) < \infty\}$$

is called a Banach function space (BFS) with the norm $\|f\|_X := \rho(|f|)$.

A function f in a BFS X is said to have an absolutely continuous norm in X if $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ satisfying $E_n \to \emptyset$ a.e. The set of all those functions in X having absolutely continuous norm is denoted by X_a. If $X = X_a$,then the space X is said to have absolutely continuous norm.

Let X be a BFS, then the closure in X of the set of bounded functions is denoted by X_b.

Theorem A. *Let X be a BFS, then $X_a \subseteq X_b \subseteq X$.*

If ρ is a Banach function norm, then its associate norm ρ' is defined on $\mathcal{M}_0^+$ by

$$\rho'(g) := \sup_{f \in \mathcal{M}^+,\ \rho(f) \le 1} \int_I f(t)g(t)dt, \quad g \in \mathcal{M}_0^+.$$

Let ρ be a Banach function norm and $X = X(\rho)$ a BFS determined by ρ. Let ρ' be the associate norm of ρ. Then the BFS $X' = X'(\rho')$ determined by ρ' is called the associate space of X.

Theorem B. *Every BFS X, coincides with its second associate space X''.*

Theorem C. *The Banach space dual X^* of a BFS X, is isometrically isomorphic to the associate space X' if and only if X has absolutely continuous norm.*

Theorem D. *A BFS X is reflexive if and only if both X and its associate space X' have absolutely continuous norm.*

Theorem E. *Let X be a rearrangement invariant BFS and X' be its associate space, then X' is rearrangement invariant.*

3. Fully measurable small Lebesgue space

In this section, we shall define "fully measurable small Lebesgue space" which later, under continuity assumptions for δ and p, has been proved to be the associate space of $L^{p[\cdot],\delta(\cdot)}$.

Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) \geq 1$ a.e., $\delta \in L^\infty$ and $\delta > 0$ a.e. For $g \in \mathcal{M}_0^+,\ E \subseteq I$ and $|E| > 0$, define

$$\rho'_{p[\cdot]',\delta(\cdot),E}(g) := \inf_{\substack{g=\sum g_k \\ g_k \in \mathcal{M}_0}} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in E} \rho_{p(x)'}\left(\frac{1}{\delta(x)}|g_k(\cdot)|\right). \tag{3.1}$$

In particular, when $E = I$, we write $\rho'_{p[\cdot]',\delta(\cdot),E}$ as $\rho'_{p[\cdot]',\delta(\cdot)}$.

The following lemma was proved in [2]:

Lemma F. *If $f, g \in \mathcal{M}_0^+$ and $g \leq f = \sum_{k=1}^{\infty} f_k,\ \ f_k \geq 0,\ \ k \in \mathbb{N}$, then $g = \sum_{k=1}^{\infty}(f_k - h_k)$, where*

$$h_k = \left(f_k - \max\left\{g - \sum_{j=1}^{k-1} f_j,\ 0\right\}\right)\chi_{E_k},$$

$$E_k = \left\{x \in I : \sum_{j=1}^{k} f_j(x) > g(x)\right\} \quad \text{and} \quad 0 \leq h_k \leq f_k,\ \ \text{for all } k \in \mathbb{N}.$$

In the expression (3.1), g is composed of $g_k \in \mathcal{M}_0$. However, in view of Lemma F, following the steps as in Corollary 2.2 of [2], it can be proven that it is sufficient to have g_k's in $\mathcal{M}_0^+$. Precisely, we have the following:

Proposition 3.1. *For $g \in \mathcal{M}_0^+$, we have*

$$\rho'_{p[\cdot]',\delta(\cdot)}(g) = \inf_{\substack{g=\sum g_k \\ g_k \in \mathcal{M}_0^+}} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)}g_k(\cdot)\right).$$

Now onwards, the definition of $\rho'_{p[\cdot]',\delta(\cdot)}(\cdot)$ will be taken as that in Proposition 3.1.

Proposition 3.2. *If $p(x) = p_+$ for $x \in E \subseteq I$ and $|E| > 0$, then for $g \in \mathcal{M}_0^+$*

$$\rho'_{p[\cdot]',\delta(\cdot)}(g) \approx \rho_{(p_+)'}(g).$$

Proof. Let $g \in \mathcal{M}_0^+$, then

$$\begin{aligned}
\rho'_{p[\cdot]',\delta(\cdot)}(g) &\leq \operatorname*{ess\,inf}_{x\in E} \rho_{p(x)'}\left(\frac{1}{\delta(x)}g(\cdot)\right) \\
&= \operatorname*{ess\,inf}_{x\in E} \rho_{(p_+)'}\left(\frac{1}{\delta(x)}g(\cdot)\right) \\
&= \rho_{(p_+)'}(g) \operatorname*{ess\,inf}_{x\in E} \frac{1}{\delta(x)} = \underline{\delta}(E)\rho_{(p_+)'}(g).
\end{aligned} \tag{3.2}$$

For the reverse estimate, let $\sigma > 0$. Then there exists a decomposition $\{g_k\},\ g_k \in \mathcal{M}_0^+$ of g such that $g = \sum_{k=1}^{\infty} g_k$ and

$$\sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)}g_k(\cdot)\right) < \rho'_{p[\cdot]',\delta(\cdot)}(g) + \frac{\sigma}{2}. \tag{3.3}$$

Now, note that for each $k \in \mathbb{N},\ \frac{\sigma}{2^k} > 0$ and there exists $A_k^\sigma \subseteq I$ such that $|A_k^\sigma| > 0$, where

$$A_k^\sigma = \left\{x \in I : \rho_{p(x)'}\left(\frac{1}{\delta(x)}g_k(\cdot)\right) < \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)}g_k(\cdot)\right) + \frac{\sigma}{2^k}\right\}.$$

Therefore, for $x_k^\sigma \in A_k^\sigma$ with $0 < \delta(x_k^\sigma) < \infty$, we have

$$\sum_{k=1}^{\infty} \rho_{p(x_k^\sigma)'}\left(\frac{1}{\delta(x_k^\sigma)} g_k(\cdot)\right) < \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right) + \frac{\sigma}{2},$$

which on using (3.3) gives

$$\sum_{k=1}^{\infty} \rho_{p(x_k^\sigma)'}\left(\frac{1}{\delta(x_k^\sigma)} g_k(\cdot)\right) < \rho'_{p[\cdot]',\delta(\cdot)}(g) + \sigma,$$

i.e.,

$$\sum_{k=1}^{\infty} \frac{1}{\delta(x_k^\sigma)} \rho_{p(x_k^\sigma)'}(g_k) < \rho'_{p[\cdot]',\delta(\cdot)}(g) + \sigma.$$

Therefore,

$$\sum_{k=1}^{\infty} \rho_{p(x_k^\sigma)'}(g_k) < \frac{1}{\underline{\delta}(I)}\left[\rho'_{p[\cdot]',\delta(\cdot)}(g) + \sigma\right]. \tag{3.4}$$

Case I. If $p_+ = 1$, then $p(x) = 1$ a.e. on I. By (3.4), we obtain

$$\begin{aligned} \rho_{(p_+)'}(g) &= \rho_{(p_+)'}\left(\sum_{k=1}^{\infty} g_k\right) \\ &\le \sum_{k=1}^{\infty} \rho_{(p_+)'}(g_k) \\ &< \frac{1}{\underline{\delta}(I)}\left[\rho'_{p[\cdot]',\delta(\cdot)}(g) + \sigma\right] \end{aligned} \tag{3.5}$$

for all $\sigma > 0$. The assertion follows by (3.2) and (3.5).

Case II. If $p_+ > 1$, then $p(x) \le p_+$ a.e. on I, so that $p(x)' \ge (p_+)'$ a.e. on I. Therefore

$$\rho_{p(x)'}(g_k) \ge \rho_{(p_+)'}(g_k)$$

a.e. on I, for all $k \in \mathbb{N}$. Consequently,

$$\sum_{k=1}^{\infty} \rho_{p(x)'}(g_k) \ge \sum_{k=1}^{\infty} \rho_{(p_+)'}(g_k)$$

a.e. on I. In particular

$$\sum_{k=1}^{\infty} \rho_{p(x_k^\sigma)'}(g_k) \ge \sum_{k=1}^{\infty} \rho_{(p_+)'}(g_k). \tag{3.6}$$

Using (3.4) in (3.6), we get

$$\sum_{k=1}^{\infty} \rho_{(p_+)'}(g_k) < \frac{1}{\underline{\delta}(I)}\left[\rho'_{p[\cdot]',\delta(\cdot)}(g) + \sigma\right]$$

which on taking $\sigma \to 0$ gives

$$\rho_{(p_+)'}(g) \le \sum_{k=1}^{\infty} \rho_{(p_+)'}(g_k) < \frac{1}{\underline{\delta}(I)} \rho'_{p[\cdot]',\delta(\cdot)}(g). \tag{3.7}$$

The assertion now, follows from (3.2) and (3.7). $\square$

Remark 3.3. If $p(x) = \infty$ on a set of positive measures, then $p_+ = \infty$. Therefore, by Proposition 3.2, $\rho'_{p[\cdot]',\delta(\cdot)}(g) \approx \rho_1(g)$. Hence without loss of generality, we may assume that $p(x) < \infty$ a.e. on I.

Definition 3.4. For $p(\cdot) \in \mathcal{M},\ p(\cdot) \geq 1$ a.e., $\delta \in L^\infty$ and $\delta > 0$ a.e. on I, we define the "fully measurable small Lebesgue space" by

$$L^{(p[\cdot]',\delta(\cdot)} := \left\{ g \in \mathcal{M}_0 : \|g\|_{L^{(p[\cdot]',\delta(\cdot)}} = \rho'_{p[\cdot]',\delta(\cdot)}(|g|) < \infty \right\}.$$

We prove the following:

Theorem 3.5. *$L^{(p[\cdot]',\delta(\cdot)}$ is a Banach space.*

Proof. Without any loss of generality, we may assume that the members of $L^{(p[\cdot]',\delta(\cdot)}$ belong to $\mathcal{M}_0^+$.

It is obvious that $\rho'_{p[\cdot]',\delta(\cdot)}(g) \geq 0$ for all $g \in \mathcal{M}_0^+$ and that if $g = 0$, then $\rho'_{p[\cdot]',\delta(\cdot)}(g) = 0$. Assume that $\rho'_{p[\cdot]',\delta(\cdot)}(g) = 0$. We prove that $g = 0$.

Let $\sigma > 0$ be given. Then there exists a decomposition $\{g_k\},\ g_k \in \mathcal{M}_0^+$ of g such that $g = \sum_{k=1}^\infty g_k$ and $\sum_{k=1}^\infty \operatorname{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right) < \sigma$, i.e.,

$$\operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right) < \sigma, \quad k = 1, 2, \ldots$$

so that there exists $A_k^\sigma \subseteq I$ such that $|A_k^\sigma| > 0$, where

$$A_k^\sigma = \left\{ x \in I : \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right) < \sigma \right\}.$$

Therefore, for $x_k^\sigma \in A_k^\sigma$ such that $0 < \delta(x_k^\sigma) < \infty$, for each $k = 1, 2, \ldots$ we have

$$\rho_{p(x_k^\sigma)'}\left(\frac{1}{\delta(x_k^\sigma)} g_k(\cdot)\right) < \sigma, \quad \text{i.e.,} \ 0 \leq \frac{1}{\delta(x_k^\sigma)} \rho_{p(x_k^\sigma)'}(g_k) < \sigma.$$

Since $\sigma > 0$ is arbitrary, we have

$$\frac{1}{\delta(x_k^\sigma)} \rho_{p(x_k^\sigma)'}(g_k) = 0, \quad \text{i.e.,} \ \rho_{p(x_k^\sigma)'}(g_k) = 0$$

which gives that for all $k,\ g_k = 0$ a.e. Consequently,

$$g = \sum_{k=1}^\infty g_k = 0 \ \text{ a.e. on } I.$$

Next, let $\lambda > 0$ and $\{g_k\},\ g_k \in \mathcal{M}_0^+$ be a decomposition of g, so that $\{\lambda g_k\}$ is a decomposition of $\{\lambda g\}$. We have

$$\begin{aligned} \rho'_{p[\cdot]',\delta(\cdot)}(\lambda g) &\leq \sum_{k=1}^\infty \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} (\lambda g_k)(\cdot)\right) \\ &\leq \lambda \inf_{\substack{g=\sum g_k \\ g_k \in \mathcal{M}_0^+}} \sum_{k=1}^\infty \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} (g_k)(\cdot)\right) \\ &= \lambda\, \rho'_{p[\cdot]',\delta(\cdot)}(g). \end{aligned} \tag{3.8}$$

Again, let $\{h_k\},\ h_k \in \mathcal{M}_0^+$ be any decomposition of λg. Then $g = \sum_{k=1}^\infty \frac{1}{\lambda} h_k$ so that, we have

$$\begin{aligned} \rho'_{p[\cdot]',\delta(\cdot)}(g) &\leq \sum_{k=1}^\infty \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} \frac{1}{\lambda} h_k(\cdot)\right) \\ &\leq \frac{1}{\lambda} \inf_{\substack{\lambda g=\sum h_k \\ h_k \in \mathcal{M}_0^+}} \sum_{k=1}^\infty \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} h_k(\cdot)\right) \\ &= \frac{1}{\lambda} \rho'_{p[\cdot]',\delta(\cdot)}(\lambda g), \end{aligned}$$

which along with (3.8) gives that

$$\rho'_{p[\cdot]',\delta(\cdot)}(\lambda g) = \lambda\, \rho'_{p[\cdot]',\delta(\cdot)}(g) \quad \text{for all } \lambda > 0.$$

Next we prove the triangle inequality.

Let $g_1, g_2 \in \mathcal{M}_0^+$ and $\sigma > 0$ be given. Then there exist decompositions $\{g_{1,k}\}$ and $\{g_{2,k}\}$ of g_1 and g_2 respectively, such that

$$\sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_{1,k}(\cdot)\right) < \rho'_{p[\cdot]',\delta(\cdot)}(g_1) + \frac{\sigma}{2}$$

and

$$\sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_{2,k}(\cdot)\right) < \rho'_{p[\cdot]',\delta(\cdot)}(g_2) + \frac{\sigma}{2}.$$

Clearly, $g_1 + g_2$ exists a.e. and

$$g_1 + g_2 = \sum_{i=1}^{2}\sum_{k=1}^{\infty} g_{i,k} = \sum_{i,k}^{2,\infty} g_{i,k}.$$

Thus

$$\begin{aligned}
\rho'_{p[\cdot]',\delta(\cdot)}(g_1 + g_2) &\le \sum_{i,k}^{2,\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_{i,k}(\cdot)\right) \\
&= \sum_{i=1}^{2}\sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_{i,k}(\cdot)\right) \\
&< \sum_{i=1}^{2} \rho'_{p[\cdot]',\delta(\cdot)}(g_i) + \sigma.
\end{aligned}$$

Since the last inequality holds for all $\sigma > 0$, the triangle inequality follows.

Finally, in order to prove that $L^{(p[\cdot]',\delta(\cdot)}$ is a Banach space, in view of the well known Riesz–Fischer property, it suffices to prove that for any sequence $\{g_n\} \in L^{(p[\cdot]',\delta(\cdot)}$,

$$\rho'_{p[\cdot]',\delta(\cdot)}\left(\sum_{n=1}^{\infty} g_n\right) \le \sum_{n=1}^{\infty} \rho'_{p[\cdot]',\delta(\cdot)}(g_n),$$

which in fact, can easily be obtained on following the steps of triangle inequality being applied for $\sum_{n=1}^{\infty} g_n$. □

Proposition 3.6. *Lattice property holds in* $L^{(p[\cdot]',\delta(\cdot)}$.

Proof. Let $f, g \in \mathcal{M}_0^+$ such that $g \le f$ a.e. Let $f = \sum_{k=1}^{\infty} f_k$ for $f_k \in \mathcal{M}_0^+$. Then by Lemma F, $g = \sum_{k=1}^{\infty}(f_k - h_k)$, where $0 \le h_k \le f_k$ for all k. Therefore

$$\begin{aligned}
\rho'_{p[\cdot]',\delta(\cdot)}(f) &= \inf_{\substack{f=\sum f_k \\ f_k\in\mathcal{M}_0^+}} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} f_k(\cdot)\right) \\
&\ge \inf_{\substack{f=\sum f_k \\ f_k\in\mathcal{M}_0^+}} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} (f_k - h_k)(\cdot)\right) \ge \rho'_{p[\cdot]',\delta(\cdot)}(g). \quad \square
\end{aligned}$$

Theorem 3.7. *For* $\varepsilon > 0$, *the following continuous embeddings hold:*

$$L^{(p_+)'+\varepsilon} \subseteq L^{(p[\cdot]',\delta(\cdot)} \subseteq L^{(p_+)'} \quad a.e.\ on\ I.$$

Proof. The second embedding holds in view of (3.5) and (3.7). For the first one, let $\varepsilon > 0$ and $g \in L^{(p_+)'+\varepsilon}$. Note that

$$\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} \leq \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)}|g(\cdot)|\right). \tag{3.9}$$

Now, if $p_+ = 1$, then $p(x) \equiv 1$ a.e. on I, and (3.9) gives

$$\begin{aligned}\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} &\leq \operatorname*{ess\,inf}_{x\in I} \rho_\infty\left(\frac{1}{\delta(x)}|g(\cdot)|\right)\\ &= \rho_\infty(|g|)\operatorname*{ess\,inf}_{x\in I}\frac{1}{\delta(x)} = \underline{\delta}(I)\rho_\infty(|g|)\end{aligned}$$

which means that the desired embedding holds in this case.

On the other hand, let $p_+ \neq 1$. Observe that $p'_+ = \operatorname{ess\,inf}_{x\in I} p(x)'$. Let $\varepsilon > 0$ be given, then there exists $A_\varepsilon \subset I$ such that $|A_\varepsilon| > 0$, where

$$A_\varepsilon = \{x \in I : p(x)' < p'_+ + \varepsilon\}.$$

Since $|A_\varepsilon| > 0$, we may choose $x_\varepsilon \in A_\varepsilon$ such that

$$(p_+)' \leq p(x_\varepsilon)' < (p_+)' + \varepsilon$$

and $\delta(x_\varepsilon) > 0$. We get

$$\begin{aligned}\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} &\leq \rho_{p(x_\varepsilon)'}\left(\frac{1}{\delta(x_\varepsilon)}|g(\cdot)|\right)\\ &= \frac{1}{\delta(x_\varepsilon)}\rho_{p(x_\varepsilon)'}(|g|)\\ &\leq \frac{1}{\delta(x_\varepsilon)}\rho_{(p_+)'+\varepsilon}(|g|) < \infty.\end{aligned}$$

Thus, for $\varepsilon > 0$ a.e. on I, we have

$$\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} \leq \frac{1}{\delta(x_\varepsilon)}\rho_{(p_+)'+\varepsilon}(|g|)$$

and we are done. □

Remark 3.8. Note that, in particular $L^\infty \subseteq L^{(p[\cdot]',\delta(\cdot)}$.

4. Further properties of the space $L^{(p[\cdot]',\delta(\cdot)}$

In this section, we shall prove the Levi's theorem of monotone convergence for the fully measurable small Lebesgue space $L^{(p[\cdot]',\delta(\cdot)}$. In Section 3, it was proved that $L^{(p[\cdot]',\delta(\cdot)}$ is a Banach space. Here, we shall prove that the space $L^{(p[\cdot]',\delta(\cdot)}$ is, in fact, a BFS. We first prove the following:

Lemma 4.1. *Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) \geq 1$ a.e., $\delta \in L^\infty$ and $\delta > 0$ a.e. Then for $g \in \mathcal{M}_0^+$, the following holds for all $\tau \in [1, p_+)$.*

$$\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} \approx \rho'_{p[\cdot]',\delta(\cdot),p^{-1}[\tau,p_+]}(g). \tag{4.1}$$

Proof. If $p(\cdot) \equiv 1$, then $p_+ = 1$ and therefore equivalence in (4.1) makes sense only for $\tau = 1$. Since $p^{-1}(\{1\}) = I$, the equality holds in (4.1).

Let $p(\cdot) \not\equiv 1$. If $\tau = 1$, or if, $\tau \in (1, p_+)$ is such that $|p^{-1}[1,\tau)| = 0$, then again the equality holds in (4.1).

Thus we consider the case when $\tau \neq 1$ and $|p^{-1}[1,\tau)| > 0$. Set $X_\tau = p^{-1}([\tau, p_+])$ and $Y_\tau = p^{-1}([1,\tau))$. Let $\{g_k\}$, $g_k \in \mathcal{M}_0^+$ be a decomposition of g. For $x \in X_\tau$, we have $\rho_{p(x)'}(g_k) \leq \rho_{\tau'}(g_k)$ and for $x \in Y_\tau$, we have

$\rho_{\tau'}(g_k) \le \rho_{p(x)'}(g_k)$. For $y \in X_\tau$, we have

$$\begin{aligned}\operatorname*{ess\,inf}_{x\in Y_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right) &\ge \operatorname*{ess\,inf}_{x\in Y_\tau} \frac{1}{\delta(x)} \rho_{\tau'}(g_k)\\ &\ge \rho_{p(y)'}(g_k) \operatorname*{ess\,inf}_{x\in Y_\tau} \frac{1}{\delta(x)}. \end{aligned} \tag{4.2}$$

Now, for $\frac{\|\delta\|_{L^\infty(X_\tau)}}{2} > 0$, there exists $X_\tau^\delta \subseteq X_\tau$ such that $|X_\tau^\delta| > 0$ and $\delta(y) > \frac{\|\delta\|_{L^\infty(X_\tau)}}{2} > 0$ for all $y \in X_\tau^\delta$ a.e. Consequently, for $y \in X_\tau^\delta$, (4.2) gives

$$\begin{aligned}\operatorname*{ess\,inf}_{x\in Y_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right) &\ge \operatorname*{ess\,inf}_{x\in Y_\tau} \frac{1}{\delta(x)} \operatorname*{ess\,inf}_{y\in X_\tau^\delta}\left(\frac{1}{\delta(y)}\rho_{p(y)'}(g_k)\right)\cdot\delta(y)\\ &\ge \operatorname*{ess\,inf}_{x\in Y_\tau} \frac{1}{\delta(x)} \operatorname*{ess\,inf}_{y\in X_\tau^\delta}\left(\frac{1}{\delta(y)}\rho_{p(y)'}(g_k)\right)\frac{\|\delta\|_{L^\infty(X_\tau)}}{2}\\ &\ge C \operatorname*{ess\,inf}_{y\in X_\tau}\left(\rho_{p(y)'}\left(\frac{1}{\delta(y)} g_k(\cdot)\right)\right)\end{aligned} \tag{4.3}$$

where $C = \underline{\delta}(Y_\tau)\frac{\|\delta\|_{L^\infty(X_\tau)}}{2}$, which is independent of k and g, but depends on τ. Also, on using (4.3), we have

$$\begin{aligned}\operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right) &= \min\left\{\operatorname*{ess\,inf}_{x\in X_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right),\ \operatorname*{ess\,inf}_{x\in Y_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right)\right\}\\ &\ge \min\{1,\ C\}\operatorname*{ess\,inf}_{x\in X_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right).\end{aligned}$$

Therefore,

$$\sum_{k=1}^\infty \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right) \ge \min\{1,\ C\}\sum_{k=1}^\infty \operatorname*{ess\,inf}_{x\in X_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)} g_k(\cdot)\right)$$

for all decompositions $\{g_k\}$ of g, which implies that

$$\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} \ge \min\{1,\ C\}\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(g) \tag{4.4}$$

for all $\tau \in (1, p_+)$, where $C = \underline{\delta}(Y_\tau)\frac{\|\delta\|_{L^\infty(X_\tau)}}{2}$.

The reverse estimate holds trivially as $X_\tau \subseteq I$. □

We shall be using the following lemma (see [4]).

Lemma G. (i) *If* $a \ge b \ge 0,\ p \ge 1$, *then* $(a-b)^p \le a^p - b^p$.

(ii) *If* $0 \le b < a,\ r > 0,\ a \le (1+r)b,\ 0 < \alpha_0 \le \alpha < 1$, *then there exists a constant* $c = c(r, \alpha_0)$ *such that* $(a-b)^\alpha \le c(a^\alpha - b^\alpha)$ *with* $c = \frac{r^{\alpha_0}}{(1+r)^{\alpha_0}-1}$.

Now, we are ready to prove Levi's theorem of monotone convergence for fully measurable small Lebesgue space.

Theorem 4.2. *Let* $p(\cdot) \in \mathcal{M}$ *be such that* $p(\cdot) > 1$ *a.e.,* $\delta \in L^\infty$, $\delta > 0$ *a.e., and let* $\{f_m\}$, $f_m \in \mathcal{M}_0^+$ *be a nondecreasing sequence such that* $M = \sup_m \|f_m\|_{L^{(p[\cdot]',\delta(\cdot)}} < \infty$. *Then, the function* $f = \sup_m f_m$ *is such that*

(i) $f \in L^{(p[\cdot]',\delta(\cdot)}$;

(ii) $f_m \to f$ *in* $L^{(p[\cdot]',\delta(\cdot)}$ *and*

(iii) $f_m \uparrow f$ *a.e. on* I.

Proof. Choose $1 < \tau < p_+$ and set $X_\tau = p^{-1}([\tau, p_+])$. In view of Lemma 4.1, it is sufficient to prove the theorem with $\|\cdot\|_{L^{(p[\cdot]',\delta(\cdot)}}$ being replaced by $\rho'_{p[\cdot]',\delta(\cdot),X_\tau}$. Further, without loss of generality, we may assume that the sequence $\{\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_m)\}$ is convergent, since otherwise, there exists a convergent subsequence of it. Then, first the theorem can be proved for this subsequence and then by using the lattice property of $\|\cdot\|_{L^{(p[\cdot]',\delta(\cdot)}}$, we would get the assertion in general.

Let $\sigma > 0$ be given. Then there exists a decomposition $\{f_{r,k}\}$, $f_{r,k} \in \mathcal{M}_0^+$ of f_r, i.e., $f_r = \sum_{k=1}^{\infty} f_{r,k}$, so that

$$\sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x \in X_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)} f_{r,k}(\cdot)\right) < \rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r) + \frac{\sigma}{2}. \tag{4.5}$$

Also, for each $k = 1, 2, \ldots,$ there exists $A_{r,k} \subseteq X_\tau$, such that $|A_{r,k}| > 0$, and

$$\rho_{p(x)'}\left(\frac{1}{\delta(x)} f_{r,k}(\cdot)\right) < \operatorname*{ess\,inf}_{x \in X_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)} f_{r,k}(\cdot)\right) + \frac{\sigma}{2^k}$$

for all $x \in A_{r,k}$ a.e. In particular, we may choose $x_{r,k} \in A_{r,k} \subseteq X_\tau$ such that $1 < p(x_{r,k}) < \infty$ and $\frac{1}{\delta(x_{r,k})} \neq 0$ and finite. Therefore, we have

$$\rho_{p(x_{r,k})'}\left(\frac{1}{\delta(x_{r,k})} f_{r,k}(\cdot)\right) < \operatorname*{ess\,inf}_{x \in X_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)} f_{r,k}(\cdot)\right) + \frac{\sigma}{2^k}. \tag{4.6}$$

Using (4.5) and (4.6), we get

$$\sum_{k \in \mathbb{N}} \rho_{p(x_{r,k})'}\left(\frac{1}{\delta(x_{r,k})} f_{r,k}(\cdot)\right) < \rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r) + \sigma. \tag{4.7}$$

Since $s < r \Rightarrow f_s < f_r$, therefore, by Lemma F, there exists a decomposition $\{f_{s,k}\}$ of f_s such that $f_s = \sum_{k=1}^{\infty} f_{s,k}$ and $0 \le f_{s,k} \le f_{r,k}$ for all $k = 1, 2, \ldots$. Therefore, $f_r - f_s = \sum_{k=1}^{\infty}(f_{r,k} - f_{s,k})$. Now, as $1 < p(x_{r,k}) < \infty$, we have by using Lemma G(i)

$$\begin{aligned}
\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r - f_s) &\le \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x \in X_\tau} \rho_{p(x)'}\left(\frac{1}{\delta(x)}(f_{r,k} - f_{s,k})(\cdot)\right) \\
&\le \sum_{k=1}^{\infty} \rho_{p(x_{r,k})'}\left(\frac{1}{\delta(x_{r,k})}(f_{r,k} - f_{s,k})(\cdot)\right) \\
&= \sum_{k=1}^{\infty} \left(\int_I \left(\frac{1}{\delta(x_{r,k})}(f_{r,k} - f_{s,k})(t)\right)^{p(x_{r,k})'} dt\right)^{\frac{1}{p(x_{r,k})'}} \\
&\le \sum_{k=1}^{\infty} \Bigg[\int_I \left(\frac{1}{\delta(x_{r,k})} f_{r,k}(t)\right)^{p(x_{r,k})'} dt \\
&\qquad - \int_I \left(\frac{1}{\delta(x_{r,k})} f_{s,k}(t)\right)^{p(x_{r,k})'} dt\Bigg]^{\frac{1}{p(x_{r,k})'}}.
\end{aligned} \tag{4.8}$$

Now, for $0 < \gamma < 1$, consider the decomposition $\mathbb{N} = P_\gamma \cup Q_\gamma$, where

$$P_\gamma = \left\{k \in \mathbb{N} : \int_I \left(f_{r,k}(t)\right)^{p(x_{r,k})'} dt < (1+\gamma) \int_I \left(f_{s,k}(t)\right)^{p(x_{r,k})'} dt\right\};$$

and $Q_\gamma = \mathbb{N} \setminus P_\gamma$. Since $\|f_r\|_{L^{(p[\cdot]',\delta(\cdot)}} \le M$, we have by using (4.7) and (4.4)

$$\begin{aligned}
&\sum_{k \in P_\gamma} \left[\int_I \left(\frac{1}{\delta(x_{r,k})} f_{r,k}(t)\right)^{p(x_{r,k})'} dt - \int_I \left(\frac{1}{\delta(x_{r,k})} f_{s,k}(t)\right)^{p(x_{r,k})'} dt\right]^{\frac{1}{p(x_{r,k})'}} \\
&< \sum_{k \in P_\gamma} \frac{1}{\delta(x_{r,k})} \left(\gamma \int_I \left(f_{s,k}(t)\right)^{p(x_{r,k})'} dt\right)^{\frac{1}{p(x_{r,k})'}} \\
&\le \gamma^{\left(\frac{1}{\tau'}\right)} \sum_{k \in P_\gamma} \frac{1}{\delta(x_{r,k})} \left(\int_I \left(f_{s,k}(t)\right)^{p(x_{r,k})'} dt\right)^{\frac{1}{p(x_{r,k})'}}
\end{aligned}$$

$$
\begin{aligned}
&\le \gamma^{\left(\frac{1}{\tau'}\right)} \sum_{k\in\mathbb{N}} \rho_{p(x_{r,k})'}\left(\frac{1}{\delta(x_{r,k})} f_{r,k}(\cdot)\right)\\
&< \gamma^{\left(\frac{1}{\tau'}\right)} \left(\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r)+\sigma\right)\\
&\le \gamma^{\left(\frac{1}{\tau'}\right)} \frac{M}{C_\tau},
\end{aligned} \tag{4.9}
$$

where $C_\tau = \min\left\{1,\ \underline{\delta}(Y_\tau)^{\frac{\|\delta\|_{L^\infty(X_\tau)}}{2}}\right\}$, since $\sigma > 0$ is arbitrary.

On the other hand, for $k \in Q_\gamma$, by Lemma G(ii), there exists $C(\gamma, \frac{1}{\tau'})$ such that

$$
\begin{aligned}
&\sum_{k\in Q_\gamma}\left[\int_I \left(\frac{1}{\delta(x_{r,k})} f_{r,k}(t)\right)^{p(x_{r,k})'} dt - \int_I \left(\frac{1}{\delta(x_{r,k})} f_{s,k}(t)\right)^{p(x_{r,k})'} dt\right]^{\frac{1}{p(x_{r,k})'}}\\
&\quad\le C\left(\gamma,\ \frac{1}{\tau'}\right)\sum_{k\in Q_\gamma}\frac{1}{\delta(x_{r,k})}\left[\left(\int_I \left(f_{r,k}(t)\right)^{p(x_{r,k})'} dt\right)^{\frac{1}{p(x_{r,k})'}} - \left(\int_I \left(f_{s,k}(t)\right)^{p(x_{r,k})'} dt\right)^{\frac{1}{p(x_{r,k})'}}\right]\\
&\quad= C\left(\gamma,\ \frac{1}{\tau'}\right)\sum_{k\in Q_\gamma}\left[\rho_{p(x_{r,k})'}\left(\frac{1}{\delta(x_{r,k})} f_{r,k}(\cdot)\right) - \rho_{p(x_{r,k})'}\left(\frac{1}{\delta(x_{r,k})} f_{s,k}(\cdot)\right)\right]\\
&\quad\le C\left(\gamma,\ \frac{1}{\tau'}\right)\left[\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r)+\sigma-\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_s)\right]\\
&\quad= C\left(\gamma,\ \frac{1}{\tau'}\right)\left[\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r)-\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_s)\right]
\end{aligned} \tag{4.10}
$$

on using (4.7), and the fact that $\sigma > 0$ is arbitrary. By using (4.9) and (4.10) in (4.8), we get

$$
\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r - f_s) \le \gamma^{\frac{1}{\tau'}}\frac{M}{C_\tau} + C\left(\gamma,\ \frac{1}{\tau'}\right)\left[\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r)-\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_s)\right]. \tag{4.11}
$$

Let $\varepsilon > 0$ be given. Since $\lim_{\gamma\to 0}(\gamma^{\frac{1}{\tau'}}) \to 0$, there exists $\eta_\varepsilon > 0$ such that

$$
\gamma^{\frac{1}{\tau'}}\frac{M}{C_\tau} < \frac{\varepsilon}{2} \tag{4.12}
$$

whenever $0 < \gamma < \eta_\varepsilon$. Since $\{\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_m)\}$ is convergent, for $\frac{\varepsilon}{2} > 0$ there exists a positive integer N_ε such that

$$
C\left(\gamma,\ \frac{1}{\tau'}\right)\left[\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r)-\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_s)\right] < \frac{\varepsilon}{2} \tag{4.13}
$$

for all $r > s \ge N_\varepsilon$. Using (4.12) and (4.13) in (4.11), we get

$$
\rho'_{p[\cdot]',\delta(\cdot),X_\tau}(f_r - f_s) \le \varepsilon,
$$

for $r > s,\ r, s \in \mathbb{N}$, and for all $1 < \tau < p_+$, which means that the sequence $\{f_m\}$ is Cauchy in $L^{(p[\cdot]',\delta(\cdot)}$ and hence convergent, say, to $f \in L^{(p[\cdot]',\delta(\cdot)}$. Hence (i) and (ii) are done.

Further, since $L^1 \supseteq L^{(p[\cdot]',\delta(\cdot)}$ and $f_m \uparrow f$ a.e., it follows that the limit f coincides a.e. with $\sup_m f_m$, which is also the a.e. limit of $\{f_m\}$. □

Theorem 4.3 (*Fatou Property*)**.** *If* $0 \le g_n \uparrow g$ *a.e. on* I*, then* $\|g_n\|_{L^{(p[\cdot]',\delta(\cdot)}} \uparrow \|g\|_{L^{(p[\cdot]',\delta(\cdot)}}$.

Proof. By lattice property of $\|\cdot\|_{L^{(p[\cdot]',\delta(\cdot)}}$, the sequence $\{\|g_n\|_{L^{(p[\cdot]',\delta(\cdot)}}\}$ is nondecreasing and

$$
\lim_{n\to\infty}\|g_n\|_{L^{(p[\cdot]',\delta(\cdot)}} \le \|g\|_{L^{(p[\cdot]',\delta(\cdot)}}. \tag{4.14}
$$

Now if $g \in L^{(p[\cdot]',\delta(\cdot)}$, then $\sup_n \|g_n\|_{L^{(p[\cdot]',\delta(\cdot)}} < \infty$, and the assertion follows from Theorem 4.2.

On the other hand, let $g \notin L^{(p[\cdot]',\delta(\cdot)}$. Then $\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} = \infty$. On the contrary, if

$$\lim_{n\to\infty} \|g_n\|_{L^{(p[\cdot]',\delta(\cdot)}} \neq \|g\|_{L^{(p[\cdot]',\delta(\cdot)}},$$

then it follows that

$$\lim_{n\to\infty} \|g_n\|_{L^{(p[\cdot]',\delta(\cdot)}} < \infty$$

which, by Theorem 4.2 gives that $g \in L^{(p[\cdot]',\delta(\cdot)}$, a contradiction. Hence the assertion follows in this case too. □

Theorem 4.4 (*Hölder's Type Inequality*)**.** *For $f \in L^{p[\cdot],\delta(\cdot)}$ and $g \in L^{(p[\cdot]',\delta(\cdot)}$, the following holds:*

$$\int_I f(t)g(t)dt \leq \|f\|_{L^{p[\cdot],\delta(\cdot)}} \|g\|_{L^{(p[\cdot]',\delta(\cdot)}}.$$

Proof. The result is trivially true if $f = 0$ a.e. So assume that $f \neq 0$. Let $|g| = \sum_{k=1}^{\infty} g_k,\ g_k \in \mathcal{M}_0^+$ be a decomposition of $|g|$. Then for each $k \in \mathbb{N}$ and for each fixed $x \in I$, by applying L^p-Hölder's inequality on the index $p(x)$, we have

$$\int_I f(t)g_k(t)dt \leq \int_I |f(t)g_k(t)|dt \leq \|f\|_{L^{p(x)}} \|g\|_{L^{p(x)'}} \quad \text{a.e. on } I.$$

Since $\delta(x) > 0$ a.e. on I, for $x \in I$ such that $\delta(x) \neq 0$, for each $k \in \mathbb{N}$, we have

$$\begin{aligned}
\|f\|_{L^{p(x)}} \|g_k\|_{L^{p(x)'}} &\leq \left(\frac{1}{\delta(x)} \|g_k\|_{L^{p(x)'}}\right) \operatorname*{ess\,sup}_{x\in I} \left(\delta(x)\|f\|_{L^{p(x)}}\right) \\
&= \rho_{p[\cdot],\delta(\cdot)}(|f|) \left(\frac{1}{\delta(x)} \|g_k\|_{L^{p(x)'}}\right) \quad \text{a.e. on } I \\
&\leq \rho_{p[\cdot],\delta(\cdot)}(|f|) \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'} \left(\frac{1}{\delta(x)} g_k(\cdot)\right).
\end{aligned}$$

Thus, using the above estimates, we have

$$\begin{aligned}
\int_I f(t)g(t)dt &\leq \int_I |f(t)||g(t)|dt \\
&= \int_I |f(t)| \left(\sum_{k=1}^{\infty} g_k(t)\right) dt \\
&= \sum_{k=1}^{\infty} \int_I |f(t)|g_k(t)dt \\
&\leq \sum_{k=1}^{\infty} \rho_{p[\cdot],\delta(\cdot)}(|f|) \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'} \left(\frac{1}{\delta(x)} g_k(.)\right) \\
&= \rho_{p[\cdot],\delta(\cdot)}(|f|) \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'} \left(\frac{1}{\delta(x)} g_k(.)\right)
\end{aligned}$$

which holds for all decompositions $\{g_k\}$ of $|g|$. Taking the infimum over all such decompositions, the assertion follows. □

Theorem 4.5. *For $p(\cdot) \in \mathcal{M},\ p(\cdot) > 1$ a.e., $\delta \in L^\infty, \delta > 0$ a.e., the fully measurable small Lebesgue space $L^{(p[\cdot]',\delta(\cdot)}$ is a BFS.*

Proof. For any $E \subseteq I$, we have

$$\begin{aligned}
\|\chi_E\|_{L^{(p[\cdot]',\delta(\cdot)}} &\leq \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'} \left(\frac{1}{\delta(x)} \chi_E(.)\right) \\
&= \operatorname*{ess\,inf}_{x\in I} \frac{1}{\delta(x)} \|\chi_E\|_{L^{p(x)'}} \leq \underline{\delta}(I) < \infty,
\end{aligned}$$

and also by Theorem 4.4

$$\begin{aligned}\int_E f(t)dt &= \int_I f(t)\chi_E(t)dt \\ &\leq \|\chi_E\|_{L^{p[\cdot],\delta(\cdot)}}\|f\|_{L^{(p[\cdot]',\delta(\cdot)}} \\ &\leq \operatorname*{ess\,sup}_{x\in I}\delta(x)\|f\|_{L^{(p[\cdot]',\delta(\cdot)}} = C(\delta)\|f\|_{L^{(p[\cdot]',\delta(\cdot)}},\end{aligned}$$

where the constant $C(\delta) = \text{ess sup}_{x\in I}\delta(x)$ is independent of f. Now, in view of Theorems 4.3 and 3.5 and Proposition 3.6, it follows that $L^{(p[\cdot]',\delta(\cdot)}$ is a Banach function space (BFS). □

5. Associate space of $L^{p[\cdot],\delta(\cdot)}$

We begin with the following:

Theorem 5.1. *$L^{(p[\cdot]',\delta(\cdot)} = L_a^{(p[\cdot]',\delta(\cdot)}$, i.e., the BFS $L^{(p[\cdot]',\delta(\cdot)}$ has an absolutely continuous norm.*

Proof. Let $E_n \subseteq I$, $n \in \mathbb{N}$ be such that $\chi_{E_n} \downarrow 0$ a.e. on I and $g \in L^{(p[\cdot]',\delta(\cdot)}$,which without any loss of generality can be assumed to be non-negative. Define

$$g_n = g - g\chi_{E_n} = \begin{cases} 0, & x \in E_n \\ g(x), & x \notin E_n. \end{cases}$$

Since $\chi_{E_n} \downarrow 0$, we find that $\{g_n\}$ is a nondecreasing sequence such that $g_n \leq g$ for all n so that

$$\|g_n\|_{L^{(p[\cdot]',\delta(\cdot)}} \leq \|g\|_{L^{(p[\cdot]',\delta(\cdot)}} < \infty.$$

Therefore by Theorem 4.2, $g_n \to g$ in $L^{(p[\cdot]',\delta(\cdot)}$, which gives

$$\|g\chi_{E_n}\|_{L^{(p[\cdot]',\delta(\cdot)}} = \|g - g_n\|_{L^{(p[\cdot]',\delta(\cdot)}} \to 0. \quad \square$$

Theorem 5.2. *$L^{(p[\cdot]',\delta(\cdot)} = L_b^{(p[\cdot]',\delta(\cdot)}$, i.e., the set of bounded functions is dense in fully measurable small Lebesgue space.*

Proof. It can be obtained in view of Theorems A and 5.1. □

Lemma 5.3. *Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) > 1$ a.e. and continuous. Let $\delta \in L^\infty$, continuous and $\delta > 0$ a.e. with $\lim_{x\to 0^+}\delta(x) = 0$. Let $0 \neq f \in L^\infty$, then there exists $g \in L^\infty$ such that the following holds*

$$\int_I f(t)g(t)dt = \|f\|_{L^{p[\cdot],\delta(\cdot)}}\|g\|_{L^{(p[\cdot]',\delta(\cdot)}}.$$

Proof. Since $0 \neq f \in L^\infty$, we have $\|f\|_{L^\infty} \neq 0$, so that

$$\lim_{x\to 0^+}\rho_{p(x)}\left(\delta(x)|f(\cdot)|\right) = \lim_{x\to 0^+}\delta(x)\|f\|_{L^{p(x)}} = 0.$$

Therefore,

$$\|f\|_{L^{p[\cdot],\delta(\cdot)}} = \operatorname*{ess\,sup}_{x\in I}\rho_{p(x)}\left(\delta(x)|f(\cdot)|\right) = \rho_{p(x_0)}\left(\delta(x_0)|f(\cdot)|\right) \tag{5.1}$$

for some $x_0 \in I$. For index $p(x_0)$, define $g = |f|^{p(x_0)'-1}\cdot sgnf$ on I, where $sgnf(t) := 1, 0, -1$ accordingly as $f(t) > 0, = 0, < 0$ respectively. Now, for indices $p(x_0)$ and $p(x_0)'$, we obtain that

$$\int_I f(t)g(t)dt = \|f\|_{L^{p(x_0)}}\|g\|_{L^{p(x_0)'}}. \tag{5.2}$$

By Theorem 4.4, Eqs. (5.1) and (5.2) we have

$$\begin{aligned}\int_I f(t)g(t)dt &\leq \|f\|_{L^{p[\cdot],\delta(\cdot)}} \|g\|_{L^{(p[\cdot]',\delta(\cdot)}} \\ &\leq \|f\|_{L^{p[\cdot],\delta(\cdot)}} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)'}\left(\frac{1}{\delta(x)}|g(\cdot)|\right) \\ &\leq \|f\|_{L^{p[\cdot],\delta(\cdot)}} \rho_{p(x_0)'}\left(\frac{1}{\delta(x_0)}|g(\cdot)|\right) \\ &= \rho_{p(x_0)}(\delta(x_0)|f(\cdot)|)\, \rho_{p(x_0)'}\left(\frac{1}{\delta(x_0)}|g(\cdot)|\right) \\ &= \|f\|_{L^{p(x_0)}} \|g\|_{L^{p(x_0)'}} = \int_I f(t)g(t)dt\end{aligned}$$

and we are done. □

Theorem 5.4. *Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) > 1$ a.e. and continuous. Let $\delta \in L^\infty$, continuous and $\delta > 0$ a.e. with $\lim_{x\to 0^+} \delta(x) = 0$. Let $g \in L^{(p[\cdot]',\delta(\cdot)}$. Then*

$$\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} = \sup_{0\neq f\in L^{p[\cdot],\delta(\cdot)}} \frac{\int_I fg}{\|f\|_{L^{p[\cdot],\delta(\cdot)}}}.$$

Proof. By Theorem 4.4, we have

$$\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} \geq \sup_{0\neq f\in L^{p[\cdot],\delta(\cdot)}} \frac{\int_I fg}{\|f\|_{L^{p[\cdot],\delta(\cdot)}}}. \tag{5.3}$$

It is sufficient to prove the result for $g \in L^\infty$, since the assertion would then follow from Theorem 5.2. So, let $g \in L^\infty$. Then by Lemma 5.3, there exists $f \in L^\infty$ such that

$$\int_I fg = \|f\|_{L^{p[\cdot],\delta(\cdot)}} \|g\|_{L^{(p[\cdot]',\delta(\cdot)}}. \tag{5.4}$$

Therefore, for $f \in L^\infty \subseteq L^{p[\cdot],\delta(\cdot)}$, we have by (5.4) and (5.3)

$$\begin{aligned}\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} &= \frac{\int_I fg}{\|f\|_{L^{p[\cdot],\delta(\cdot)}}} \\ &\leq \sup_{0\neq f\in L^{p[\cdot],\delta(\cdot)}} \frac{\int_I fg}{\|f\|_{L^{p[\cdot],\delta(\cdot)}}} \leq \|g\|_{L^{(p[\cdot]',\delta(\cdot)}},\end{aligned}$$

i.e.,

$$\|g\|_{L^{(p[\cdot]',\delta(\cdot)}} = \sup_{0\neq f\in L^{p[\cdot],\delta(\cdot)}} \frac{\int_I fg}{\|f\|_{L^{p[\cdot],\delta(\cdot)}}}$$

and we are done. □

In view of Theorem 5.4, we immediately have the following:

Theorem 5.5. *Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) > 1$ a.e. and continuous. Let $\delta \in L^\infty$, continuous and $\delta > 0$ a.e. with $\lim_{x\to 0^+} \delta(x) = 0$. Then the associate space of $L^{p[\cdot],\delta(\cdot)}$ is $L^{(p[\cdot]',\delta(\cdot)}$.*

Remark 5.6. In view of Theorem B, it follows that under the continuity of δ and p, $L^{p[\cdot],\delta(\cdot)}$ is associate space of $L^{(p[\cdot]',\delta(\cdot)}$.

In [10], Anatriello and Fiorenza mentioned that the space $L^{p[\cdot],\delta(\cdot)}$ is rearrangement invariant. Consequently, by Theorem E, we have the following:

Theorem 5.7. *Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) > 1$ a.e. and continuous. Let $\delta \in L^\infty$, continuous and $\delta > 0$ a.e. with $\lim_{x\to 0^+} \delta(x) = 0$. Then the space $L^{(p[\cdot]',\delta(\cdot)}$ is a rearrangement invariant BFS.*

Theorem 5.8. *Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) > 1$ a.e. and continuous. Let $\delta \in L^\infty$, continuous and $\delta > 0$ a.e. with $\lim_{x\to 0^+} \delta(x) = 0$. Then the Banach space dual of the BFS $L^{(p[\cdot]',\delta(\cdot)}$ is canonically isometrically isomorphic to its associate space $L^{p[\cdot],\delta(\cdot)}$, i.e.,*

$$\left(L^{(p[\cdot]',\delta(\cdot)}\right)^* \cong \left(L^{(p[\cdot]',\delta(\cdot)}\right)' \cong L^{p[\cdot],\delta(\cdot)}.$$

Proof. It follows from Theorems C, 5.1 and Remark 5.6. □

Towards the end of the paper, we show that fully measurable grand Lebesgue space and its associate space are not reflexive. For this purpose, the following theorem is required.

Theorem 5.9. *Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) > 1$ a.e. and continuous. Let $\delta \in L^\infty$, continuous and $\delta > 0$ a.e. with $\lim_{x\to 0^+} \delta(x) = 0$. If $f \in L_b^{p[\cdot],\delta(\cdot)}$, then $\lim_{x\to 0^+} \rho_{p(x)}(\delta(x) f(\cdot)) = 0$.*

Proof. If $f \in L_b^{p[\cdot],\delta(\cdot)}$, then there exists a sequence $\{f_n\}$ of bounded functions such that $f_n \to f$ in $L^{p[\cdot],\delta(\cdot)}$. Let $\varepsilon > 0$ be given. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\|f_{n_0} - f\|_{L^{p[\cdot],\delta(\cdot)}} < \frac{\varepsilon}{2}. \tag{5.5}$$

By using the monotonicity of $\|\cdot\|_{L^{p(x)}}$ with respect to the exponent $p(x)$, we have

$$\|f_{n_0}\|_{L^{p(x)}} \le \|f_{n_0}\|_{L^{p+}}$$

a.e. on I. For $x \in I$ such that $\delta(x) \neq 0$, multiplying the above inequality by $\delta(x)$ and letting $x \to 0^+$, we get

$$\lim_{x\to 0^+} \delta(x)\|f_{n_0}\|_{L^{p(x)}} = 0.$$

Therefore, for $\varepsilon > 0$, there exists $\eta_0 > 0$ such that

$$\delta(x)\|f_{n_0}\|_{L^{p(x)}} < \frac{\varepsilon}{2} \tag{5.6}$$

whenever $0 < x < \eta_0$. Thus for $0 < x < \eta_0$, we have by using (5.6) and (5.5)

$$\begin{aligned}
\rho_{p(x)}(\delta(x) f(\cdot)) &= \delta(x)\|f\|_{L^{p(x)}} \\
&\le \delta(x)\|f - f_{n_0}\|_{L^{p(x)}} + \delta(x)\|f_{n_0}\|_{L^{p(x)}} \\
&< \delta(x)\|f - f_{n_0}\|_{L^{p(x)}} + \frac{\varepsilon}{2} \\
&\le \operatorname*{ess\,sup}_{x\in I} \delta(x)\|f - f_{n_0}\|_{L^{p(x)}} + \frac{\varepsilon}{2} \\
&= \|f_{n_0} - f\|_{L^{p[\cdot],\delta(\cdot)}} + \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

and the assertion follows. □

Remark 5.10. (i) The set of bounded functions is not dense in $L^{p[\cdot],\delta(\cdot)}$, *i.e.*, $L^{p[\cdot],\delta(\cdot)} \neq L_b^{p[\cdot],\delta(\cdot)}$. For example, consider $p(x) = 2 - x,\ x \in I,\ \delta(x) = x^{\frac{1}{2-x}}, x \in I,\ f(t) = t^{-1/2},\ t \in I$, then

$$\begin{aligned}
\|f\|_{L^{p[\cdot],\delta(\cdot)}} &= \operatorname*{ess\,sup}_{x\in I} \left(x^{\frac{1}{2-x}} \|f\|_{L^{p(x)}}\right) \\
&= \operatorname*{ess\,sup}_{x\in I} x^{\frac{1}{2-x}} \left(\int_I t^{\frac{-1}{2}(2-x)} dt\right)^{\frac{1}{2-x}} \\
&= \operatorname*{ess\,sup}_{x\in I} x^{\frac{1}{2-x}} \left(\frac{2}{x}\right)^{\frac{1}{2-x}} = \sqrt{2} < \infty,
\end{aligned}$$

i.e., $f \in L^{p[\cdot],\delta(\cdot)}$. But

$$\begin{aligned}\lim_{x\to 0^+} \rho_{p(x)}(\delta(x)f(\cdot)) &= \lim_{x\to 0^+} x^{\frac{1}{2-x}} \left(\int_I t^{\frac{-1}{2}(2-x)} dt\right)^{\frac{1}{2-x}} \\ &= \lim_{x\to 0^+} 2^{\frac{1}{2-x}} \nrightarrow 0,\end{aligned}$$

so that by Theorem 5.9, $f \notin L_b^{p[\cdot],\delta(\cdot)}$.

(ii) In view of Theorem A and the remark above, $L^{p[\cdot],\delta(\cdot)}$ does not have absolutely continuous norm.

In light of Remark 5.10(ii) and Theorem D, we have the following:

Theorem 5.11. *The spaces $L^{p[\cdot],\delta(\cdot)}$ and $L^{(p[\cdot]',\delta(\cdot)}$ are not reflexive.*

Remark 5.12. The associate space of $L^{p[\cdot],\delta(\cdot)}$ is not isometrically isomorphic to its dual space. According to Theorem 5.8, the dual of fully measurable small Lebesgue space $L^{(p[\cdot]',\delta(\cdot)}$ coincides with its associate space which is $L^{p[\cdot],\delta(\cdot)}$. However, since $L^{p[\cdot],\delta(\cdot)}$ does not have absolutely continuous norm, its dual and associate spaces are not the same, i.e.,

$$\left(L^{p[\cdot],\delta(\cdot)}\right)' \ncong \left(L^{p[\cdot],\delta(\cdot)}\right)^*.$$

Note. Recently the authors learnt that the same definition of fully measurable small Lebesgue spaces has been considered, independently, also by Anatriello, Formica and Giova [15]. During the revision of the present paper, the authors take this opportunity to acknowledge their work.

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Original article

Estimation of multianisotropic kernels and their application to the embedding theorems

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Abstract

In the current paper we consider an integral representation of functions and embedding theorems of multianisotropic Sobolev spaces in the three-dimensional case when the completely regular polyhedron has an arbitrary number of anisotropic vertices. This work generalizes results obtained in Karapetyan (in press) and Karapetyan (2016). Particularly, in Karapetyan (in press) the two-dimensional case was fully solved and in Karapetyan (2016) analogous results were obtained for the case of one anisotropic vertex. The problem takes root from various works of Sobolev, particularly, Sobolev (1938) and Sobolev (0000) [4,5]. Related results were obtained by many authors and can be found in Besov et al. (1967), Reshetnyak (1971), Smith (1961), Nikolsky (0000) and Il'in (1967) [6–10]. The monograph (Besov, 1978) contains an overview of the problem. The results obtained in this paper are based on a generalization of regularization by a quasi-homogeneous polynomial (see Uspenskii (1972) and Karapetyan (1990) [11,12]).

Keywords: Integral representation; Embedding theorem; Multianisotropic spaces; Completely regular polyhedron

1. Estimation of integrals containing the multianisotropic polynomial

Let $\mathbb{R}^3$-be the three-dimensional Euclidean space, $\mathbb{Z}_+^3$ be the set of multi-indices. For $\xi, \eta \in \mathbb{R}^3$, $\alpha \in \mathbb{Z}_+^3$, $t > 0$ denote by $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$, $t^\eta = (t^{\eta_1}, t^{\eta_2}, t^{\eta_3})$. Let $D_k = \frac{1}{i}\frac{\partial}{\partial x_k}$, $(k = 1, 2, 3)$, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$ denote the weak derivative. A polyhedron $\mathfrak{N}$ is said to be completely regular if it has a vertex at the origin and further vertices on each of the coordinate axes; the components of the outer-normals of all two-dimensional non-coordinate faces are positive. Let $\alpha^1, \alpha^2, \ldots, \alpha^n \in \mathbb{Z}_+^3$ be the vertices of a completely regular polyhedron $\mathfrak{N}$ (excluding the origin), where $\alpha^1 = (l_1, 0, 0)$, $\alpha^2 = (0, l_2, 0)$, $\alpha^3 = (0, 0, l_3)$ lie on the coordinate axes, while the others are in the positive octant. We call points of the latter type anisotropic. For a completely regular polyhedron $\mathfrak{N}$ denote by $\mathfrak{N}_i^2$ $(i = 1, \ldots, M)$ the two-dimensional non-coordinate faces with corresponding outer normal μ^i, so that the

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equation of that face is given by $\left(\alpha, \mu^i\right) = 1$. Suppose that $\mathfrak{N}_i^2$ for $i = 1, 2, 3$ contains the vertices $\{\alpha^1, \alpha^2, \alpha^3\} \setminus \{\alpha^i\}$. Let γ be the point of intersection of the planes passing through $\mathfrak{N}_1^2$, $\mathfrak{N}_2^2$ and $\mathfrak{N}_3^2$ correspondingly. Since $\mathfrak{N}$ is completely regular, $\left(\gamma, \mu^i\right) \geq 1$.

For $\nu > 0$ and positive integer k the multianisotropic polynomial $P(\nu, \xi)$ is defined as

$$P_{\mathfrak{N}}(\nu, \xi) = \sum_{i=1}^{n} \left(\nu \xi^{\alpha^i}\right)^{2k}, \tag{1}$$

where $\xi \in \mathbb{R}^3$ and $\xi^\alpha = (\xi_1^{\alpha_1}, \xi_2^{\alpha_2}, \xi_3^{\alpha^3})$. Let $m = (m_1, m_2, m_3) \in \mathbb{Z}_+^3$. Consider the following integral

$$I(\nu) = \int_{\mathbb{R}^3} \xi^m e^{-P_{\mathfrak{N}}(\nu,\xi)} d\xi. \tag{2}$$

We are interested in its behaviour for $0 < \nu < 1$. Let Ω be a domain in $\mathbb{R}^3$. Consider the integral

$$I_\Omega(\nu) = \int_\Omega \xi^m e^{-P_{\mathfrak{N}}(\nu,\xi)} d\xi. \tag{3}$$

Definition 1. We call the substitution $\xi = \nu^{-\mu^i}\eta = (\nu^{-\mu_1^i}\eta_1, \nu^{-\mu_2^i}\eta_2, \nu^{-\mu_3^i}\eta_3)$ through the vertices $\beta^1, \beta^2, \beta^3$ lying on the non-coordinate face $\mathfrak{N}_i^2$ feasible for the multi-index $m = (m_1, m_2, m_3)$ if there exists $p = (p_1, p_2, p_3)$, such that $p_k \geq -1$ and the relation

$$\prod_{k=1}^{3} \left(\eta^{\beta^k}\right)^{p_k/\beta_k^k} = \eta^m \prod_{k=1}^{3} \prod_{\substack{j=1 \\ j \neq k}}^{3} \eta_j^{-\beta_j^k/\beta_k^k} \tag{4}$$

holds. Equivalently, we can state the condition of feasibility in terms of existence of a non-negative solution to the system of linear equations

$$A_{\beta^1,\beta^2,\beta^3} \cdot p' = \begin{pmatrix} m_1 + 1 \\ m_2 + 1 \\ m_3 + 1 \end{pmatrix}, \tag{5}$$

where $p'_k = (1 + p_k)/\beta_k^k$ and $A_{\beta^1,\beta^2,\beta^3}$ is defined as

$$A_{\beta^1,\beta^2,\beta^3} = \begin{pmatrix} \beta_1^1 & \beta_1^2 & \beta_1^3 \\ \beta_2^1 & \beta_2^2 & \beta_2^3 \\ \beta_3^1 & \beta_3^2 & \beta_3^3 \end{pmatrix}.$$

If m and $\beta_1, \beta_2, \beta_3$ are clear from the context, we refer to the substitution as μ^i-transformation.

Note that if there exists a feasible μ^i-transformation of (3) for the given m, then by applying the μ^i-transformation and afterwards the change of variables

$$\tau_k = \prod_{j=1}^{n} \eta_j^{\frac{\beta_j^{k,i}}{\beta_k^{k,i}}} \quad (k = 1, \ldots, n) \tag{6}$$

we can make an estimate of (3)

$$I_\Omega(\nu) \leq C\nu^{-\left(|\mu^i| + (m,\mu^i)\right)} \int_{\Omega^*} \tau^p e^{-Q(\tau)} d\tau,$$

where $\Omega*$ is the image of Ω under the transformations, C is independent of ν and $Q(\tau)$ is

$$Q(\tau) = \tau_1^{2k\beta_1^1} + \tau_2^{2k\beta_2^2} + \tau_3^{2k\beta_3^3}.$$

Note that the solution is non-negative if and only if the point $(m_1 + 1, m_2 + 1, m_3 + 1)$ lies in the conic hull generated by the points β^1, β^2 and β^3, which we denote by $\mathrm{Cone}(\{\beta^1, \beta^2, \beta^3\})$.

Lemma 1. *Let $A = \{\alpha^1, \alpha^2, \dots, \alpha^k\} \subset \mathbb{R}^3$, $\alpha^i \neq 0$. Suppose all of them lie on a plane p. Let $\beta \in \mathrm{Cone}(A)\backslash\{0\}$. Denote $A_i = \{\alpha^i, \alpha^{i+1}\beta\}$ where $\alpha^{k+1} = \alpha^1$. Then*

$$\mathrm{Cone}(A) = \bigcup_{i=1}^{k} \mathrm{Cone}(A_i).$$

Proof. Since $\beta \in \mathrm{Cone}(A)$, it is apparent that $\bigcup_{i=1}^{k} \mathrm{Cone}(A_i) \subseteq \mathrm{Cone}(A)$, so we need to show the inverse inclusion. First, we show that it is sufficient to consider the case $\beta \in \mathrm{Conv}(A)$. Since $\beta \in \mathrm{Cone}(A)$, there are $b_i \in \mathbb{R}_+$ such that

$$\beta = \sum_{i=1}^{k} b_i \alpha^i.$$

Let $s = \sum_{i=1}^{k} b_i$. Since $\beta \neq 0$ and $b_i \geq 0$, then $s > 0$ (otherwise $\beta = 0$). Then $\frac{1}{s}\beta \in \mathrm{Conv}(A)$. Finally, note that $A_i = \mathrm{Cone}(\alpha^i, \alpha^{i+1}, \beta) = \mathrm{Cone}(\alpha^i, \alpha^{i+1}, \frac{1}{s}\beta)$. So considering the case when $\beta \in \mathrm{Conv}(A)$ is sufficient.

Now let $\beta \in \mathrm{Conv}(A)$, then β also lies on the plane p. Thus, we have $\mathrm{Conv}(A) = \bigcup_{i=1}^{k} \mathrm{Conv}(A_i)$, because β is inside $\mathrm{Conv}(A)$. Let $a \in \mathrm{Cone}(A)$, then there is a constant $t > 0$, such that $ta \in \mathrm{Conv}(A)$. It follows that ta lies in one of the $\mathrm{Conv}(A_i)$, so $a \in \mathrm{Cone}(A_i)$.

Lemma 2. *Let $\mathfrak{N}$ be a completely regular polyhedron with at least one anisotropic vertex. Then any such vertex lies inside the conic hull generated by its neighbouring vertices.*

Proof. Let β be an anisotropic vertex of $\mathfrak{N}$ and $A = \{\alpha^1, \alpha^2, \dots, \alpha^k\}$ be the set of its neighbours where α^i-s are ordered in such a way that there is a face of $\mathfrak{N}$ passing through the points β, α^i and α^{i+1} for $i = 1, \dots, k$ (here α^{k+1} is equal to α^1). We need to show that $\beta \in \mathrm{Cone}(A)$. Let $A_i = \{\alpha^i, \alpha^{i+1}, \beta\}$. Without loss of generality, suppose that the neighbours of β lie on a plane p (otherwise we can multiply each α^i by some small enough positive number so that they do lie on one plane). Let μ^0 be the outer-normal of the plane p passing through the points of A and let μ^i be the outer-normal of the face passing through the points of the set A_i. As $\mathfrak{N}$ is completely regular, p separates β and the origin, so $t = \left(\beta, \mu^0\right) > 1$. Also $\left(\alpha^i, \mu^j\right) \leq 1$ for $i, j = 1, \dots, k$. It means that $\mathrm{Conv}(A \cup \{\beta\})$ can be represented as an intersection of half-spaces

$$\mathrm{Conv}(A \cup \{\beta\}) = \left(\bigcap_{i=1}^{k} \left\{x \mid \left(\mu^i, x\right) \leq 1\right\}\right) \cap \left\{x \mid \left(\mu^0, x\right) \geq 1\right\}.$$

If we show that $\beta' = \frac{1}{t}\beta \in \mathrm{Conv}(A)$ then $\beta \in \mathrm{Cone}(A)$, since $t > 1$. Due to the choice of t, we have $\left(\beta', \mu^0\right) = 1$. Note that $\left(\beta, \mu^i\right) = 1$ for $i = 1, \dots, k$, because β lies on each of the faces corresponding to these outer-normals. Consequently, $\left(\beta', \mu^i\right) = \frac{1}{t}\left(\beta, \mu^i\right) = \frac{1}{t} < 1$. Thus, β' lies in each of the half-spaces $\{x \mid \left(\mu^i, x\right) \leq 1\}$ and in the half-space $\{x \mid \left(\mu^0, x\right) \geq 1\}$. As β' also lies on the plane p we have

$$\begin{aligned}\beta' &\in \left(\bigcap_{i-1}^{k} \left\{x \mid \left(\mu^i, x\right) \leq 1\right\}\right) \cap \left\{x \mid \left(\mu^0, x\right) \geq 1\right\} \cap p \\ &= \mathrm{Conv}(A \cup \{\beta\}) \cap p = \mathrm{Conv}(A).\end{aligned}$$

Corollary 1. *For a given $m \in \mathbb{Z}_+^3$ and any completely regular polyhedron $\mathfrak{N}$ there is at least one feasible μ^i-transformation of the integral* (3).

Proof. The proof is by induction on the number of anisotropic points of $\mathfrak{N}$ (denoted by n).

Base case: $n = 0$. When there are no anisotropic points the only non-coordinate face contains the points $\alpha^1 = (l_1, 0, 0)$, $\alpha^2 = (0, l_2, 0)$ and $\alpha^3 = (0, 0, l_3)$. The solution to the system (5) over the points α^1, α^2 and α^3 for any $m \in \mathbb{Z}_+^3$ is $(\frac{m_1+1}{l_1}, \frac{m_2+1}{l_2}, \frac{m_3+1}{l_3})$, which is positive. Hence, the transformation over the outer-normal of that face is feasible.

Inductive step: Suppose that the claim holds for a given $m \in \mathbb{Z}_+^3$ and any completely regular polyhedron with n anisotropic points, such that those vertices are also vertices of $\mathfrak{N}$. Let $\mathfrak{N}$ be any completely regular polyhedron with $n+1$ anisotropic vertices. It is possible to cut $\mathfrak{N}$ in such a way, that the resulting polyhedron $\mathfrak{N}'$ is still completely regular and has exactly n anisotropic points. Call the left out anisotropic point β. By the inductive step the claim holds for $\mathfrak{N}'$, so there exists a face of $\mathfrak{N}'$, such that the transformation over its outer-normal is feasible. If it is also a face of $\mathfrak{N}$, then we are finished. Otherwise, the points that lie on that face are neighbours of β. By applying Lemma 2 and then Lemma 1 we get that m lies in the conic hull generated by vertices of a face passing through β and two of its neighbours, so the solution to the system (5) over those points is non-negative.

Lemma 3. *For any $m \in \mathbb{Z}_+^3$ there are constants c_0, c_1, c_2 which are independent of m and $\mathfrak{N}$, such that for any $\nu \in (0,1)$*

$$|I(\nu)| \le (c_2(\ln\nu)^2 + c_1|\ln\nu| + c_0)\nu^{-\max\limits_{i=1,\dots,M}\left(|\mu^i|+(m,\mu^i)\right)}. \tag{7}$$

Proof. By Corollary 1 there exists a feasible μ^i-transformation. Consider the p-vector of the feasible transformation.

Case 1. All $p_k > -1$. Let $\Omega = \mathbb{R}_+^3$ then by applying Corollary 1 we get

$$I_{\mathbb{R}_+^3}(\nu) \le C\nu^{-\left(|\mu^i|+(m,\mu^i)\right)}\int_{\mathbb{R}_+^3}\tau^p e^{-Q(\tau)}d\tau \le C\nu^{-\left(|\mu^i|+(m,\mu^i)\right)},$$

since the integral converges due to $p_i > -1$.

Case 2. Some of $p_k = -1$. As we have noted before, $p \ne (-1,-1,-1)$, so either one or two of p's coordinates equal -1. Let $\mu^0 = (\mu_1^0, \mu_2^0, \mu_3^0)$ be such that $\mu_j^0 = \min_{i=1,\dots,M}\mu_j^i$. Consider $I_{\mathbb{R}_+^3}$. We can represent it as a sum of integrals

$$\begin{aligned} I_{\mathbb{R}_+^3} &= I_1 + I_2 + \dots + I_8 = \int_0^{\nu^{-\mu_1^0}}\int_0^{\nu^{-\mu_2^0}}\int_0^{\nu^{-\mu_3^0}} \xi^m e^{-P_{\mathfrak{N}}(\nu,\xi)}d\xi \\ &+ \int_{\nu^{-\mu_1^0}}^{\infty}\int_0^{\nu^{-\mu_2^0}}\int_0^{\nu^{-\mu_3^0}} \xi^m e^{-P_{\mathfrak{N}}(\nu,\xi)}d\xi + I_3 + I_4 \\ &+ \dots + \int_{\nu^{-\mu_1^0}}^{\infty}\int_{\nu^{-\mu_2^0}}^{\infty}\int_0^{\nu^{-\mu_3^0}} \xi^m e^{-P_{\mathfrak{N}}(\nu,\xi)}d\xi + \int_{\nu^{-\mu_1^0}}^{\infty}\int_{\nu^{-\mu_2^0}}^{\infty}\int_{\nu^{-\mu_3^0}}^{\infty} \xi^m e^{-P_{\mathfrak{N}}(\nu,\xi)}d\xi. \end{aligned}$$

Let us estimate each summand separately. If we make the substitution $\xi = \nu^{-\mu^i}\eta$ in I_1 for some $i = 1,\dots,M$, then we get

$$I_1 \le C\nu^{-\left(|\mu^i|+(m,\mu^i)\right)},$$

since $\mu_j^i - mu_j^0 \ge 0$.

To estimate I_2, we apply the substitution $\xi = \nu^{-\mu^2}\eta$ and get

$$I_2 \le C\nu^{-\left(|\mu^i|+(m,\mu^i)\right)}\int_0^{\infty}\eta_1^{m_1}e^{-\eta_1^{2kl_1}}d\eta_1\int_0^1\eta_2^{m_2}d\eta_2\int_0^1\eta_3^{m_3}d\eta_3.$$

I_3 and I_4 can be estimated analogously.

Let $\mathfrak{M} = \mathfrak{N}\cap\{z=0\}$. Then $\mathfrak{M}$ is a completely regular polyhedron in $\mathbb{R}^2$. Referring to [1] (see Lemma 1.1 in particular), we deduce that there is a one-dimensional face of $\mathfrak{M}$ passing through some points α^j, α^{j+1}, such that the transformation over its outer-normal is feasible for the integral

$$\int_{\mathbb{R}_+^2}\xi_1^{m_1}\xi_2^{m_2}e^{-P_{\mathfrak{M}}(\nu,\xi_1,\xi_2)}d\xi_1 d\xi_2.$$

Now consider a face of $\mathfrak{N}$ passing through the points α^j, α^{j+1} and let μ^i be the outer normal of that face. By applying $\xi = \nu^{-\mu^i}\eta$ to I_5 and taking Lemma 1.1 of [1] into account, we get

$$I_5 \le C\nu^{-(|\mu^i|+(m,\mu^i))} \int_{\nu^{\mu_1^i-\mu_1^0}}^{\infty} \int_{\nu^{\mu_2^i-\mu_2^0}}^{\infty} \eta_1^{m_1}\eta_2^{m_2} e^{-\eta_1^{2k\alpha_1^j}\eta_2^{2k\alpha_2^j}-\eta_1^{2k\alpha_1^{j+1}}\eta_2^{2k\alpha_2^{j+1}}} d\eta_1 d\eta_2$$
$$\le (c_1|\ln\nu| + c_0)\,\nu^{-(|\mu^i|+(m,\mu^i))}.$$

I_6 and I_7 can be handled in a similar fashion.

Now consider I_8. By Corollary 1 there is a feasible μ^i-transformation of I_8. By applying it to I_8 we get

$$I_8 \le C\nu^{-(|\mu^i|+(m,\mu^i))} \int_{\nu^{\mu_1^i-\mu_1^0}}^{\infty} \tau_1^{p_1} e^{-\tau_1^{2k\beta_1^1}} d\eta_1 \int_{\nu^{\mu_2^i-\mu_2^0}}^{\infty} \tau_2^{p_2} e^{-\tau_1^{2k\beta_2^2}} d\eta_2$$
$$\cdot \int_{\nu^{\mu_3^i-\mu_3^0}}^{\infty} \tau_3^{p_3} e^{-\tau_3^{2k\beta_3^3}} d\eta_3 \le \nu^{-(|\mu^i|+(m,\mu^i))}\left(c_2|\ln\nu|^2 + c_1|\ln\nu| + c_0\right),$$

because if $p_j > -1$, then the integral is convergent, if $p_j = -1$ then

$$\int_{\nu^{\mu_j^i-\mu_j^0}}^{\infty} \frac{e^{-\tau_j^{2k\beta_j^j}}}{\eta_j} d\eta_j \le (c_1|\ln\nu| + c_0).$$

Combining the estimates for each summand, the claim follows.

2. Multianisotropic kernels and the integral representation by them

Denote by $G_0(\xi,\nu)$ and $G_{1,j}$ (see [1] and [2]) the multianisotropic kernels

$$G_0(\xi,\nu) = e^{-P_{\mathfrak{N}}(\nu,\xi)}, \tag{8}$$
$$G_{1,j}(\xi,\nu) = 2k(\nu\xi^{\alpha^j})^{2k-1} e^{-P_{\mathfrak{N}}(\nu,\xi)} \qquad j = 1,\dots,n. \tag{9}$$

Let $\hat{G}_0(\xi,\nu)$ and $\hat{G_{1,j}}(\xi,\nu)$ be the respective Fourier transforms of $G_0(\xi,\nu)$ and $G_{1,j}(\xi,\nu)$. It is apparent, that these functions belong to the Schwartz space $S(\mathbb{R}^3)$ of rapidly decreasing functions.

Let $\gamma = (\gamma_1,\gamma_2,\gamma_3)$ be defined as previously. Suppose that $\gamma_1 < \gamma_2 < \gamma_3$. Then let $\sigma = (\sigma_1,\sigma_2,0)$ be the point of intersection of the x–y plane and the planes passing through $\mathfrak{N}_1^2$ and $\mathfrak{N}_2^2$. An easy calculation shows that $\sigma_1 = \frac{\gamma_1 l_3}{l_3-\gamma_3}$ and $\sigma_2 = \frac{\gamma_2 l_3}{l_3-\gamma_3}$. Since $\gamma_1 < \gamma_2$, $\sigma_1 < \sigma_2$. Let $\delta = (\delta_1,0,0)$ be the point of intersection of the x-axis and the plane passing through $\mathfrak{N}_1^2$. If a positive integer N is such that $N\gamma$, $N\sigma$, and $N\delta \in 2\mathbb{Z}_+^3$, we will call such N straightening.

Lemma 4. *Let $\gamma_1 < \gamma_2 < \gamma_3$ and $\nu \in (0,1)$. Then for any $m = (m_1,m_2,m_3) \in \mathbb{Z}_+^3$ and a straightening N there are constants $c_i\,(i = 0,1,2)$, such that*

$$|D^m \hat{G_{1,j}}(t,\nu)| \le \nu^{-\max_{i=1,\dots,M}(|\mu^i|+(m,\mu^i))} \frac{c_2(\ln\nu)^2 + c_1|\ln\nu| + c_0}{1+\nu^{-N}\left(t^{N\gamma}+t^{N\sigma}+t^{N\delta}\right)}. \tag{10}$$

It is an analogue of Lemma 1.1 of [1] and has a similar proof. Furthermore, analogues of Lemma 1.2–1.6 of [1] are true as well. Let us formulate them.

Lemma 5. *Let $\gamma_1 < \gamma_2 < \gamma_3$, then there is a constant $C > 0$, such that for any $\nu \in (0,1)$*

$$\int_{\mathbb{R}^3} \frac{dt}{1+\nu^{-N}\left(t^{N\gamma}+t^{N\sigma}+t^{N\delta}\right)} \le C\nu^{|\mu^1|}. \tag{11}$$

Lemma 6. *Let $\gamma_1 < \gamma_2 = \gamma_3$ and $\nu \in (0,1)$. Then for any multi-index $m = (m_1, m_2, m_3)$ and a straightening N there are constants $c_i (i = 0, 1, 2)$, such that*

$$|D^m \hat{G_{1,j}}(t,\nu)| \le \nu^{-\max\limits_{i=1,\dots,M}\left(|\mu^i| + (m,\mu^i)\right)} \left(c_2 (\ln \nu)^2 + c_1 |\ln \nu| + c_0\right) \cdot \frac{1}{1+\nu^{-N}\left(t^{N\gamma} + t^{N\sigma} + t^{N\delta}\right)} \cdot \frac{1}{1+\nu^{-N}\left(t^{N\gamma} + t^{Nr} + t^{N\delta}\right)}. \tag{12}$$

Here $\sigma = (\sigma_1, \sigma_2, 0)$ is the point of intersection of the line passing through the points α^3 and γ and the xOy plane. $r = (r_1, 0, r_2)$ is the point of intersection of line passing through the points α^2 and γ and the xOy plane. $\delta = (\delta_1, 0, 0)$?

Lemma 7. *Let $\gamma_1 < \gamma_2 = \gamma_3$. Then there is a constant $C > 0$, such that for any $\nu \in (0,1)$*

$$\int_{\mathbb{R}^3} \frac{dt}{\left(1+\nu^{-N}\left(t^{N\gamma} + t^{N\sigma} + t^{N\delta}\right)\right)\left(1+\nu^{-N}\left(t^{N\gamma} + t^{Nr} + t^{N\delta}\right)\right)} \le C\nu^{|\mu^1|}. \tag{13}$$

Lemma 8. *Let $\gamma_1 = \gamma_2 = \gamma_3$ and $\nu \in (0,1)$. Then for any multi-index $m = (m_1, m_2, m_3)$ and a positive straightening integer N there are constants $c_i (i = 0, 1, 2)$, such that*

$$\begin{aligned}|D^m \hat{G_{1,j}}(t,\nu)| \le{}& \nu^{-\max\limits_{i=1,\dots,M}\left(|\mu^i| + (m,\mu^i)\right)} \left(c_2 (\ln \nu)^2 + c_1 |\ln \nu| + c_0\right) \\ &\cdot \frac{1}{1+\nu^{-N}\left(t^{N\gamma} + t^{N\sigma} + t^{N\delta}\right)} \cdot \frac{1}{1+\nu^{-N}\left(t^{N\gamma} + t^{Nr} + t^{Nq}\right)} \\ &\cdot \frac{1}{1+\nu^{-N}\left(t^{N\gamma} + t^{Nk} + t^{Nm}\right)},\end{aligned} \tag{14}$$

where $\sigma = (\sigma_1, \sigma_2, 0)$ is the point of intersection of the xOy plane and the planes passing through the faces of $\mathfrak{N}_1^2$ and $\mathfrak{N}_2^2$, $r = (r_1, 0, r_2)$ is the point of intersection of the xOz plane and the planes passing through the faces $\mathfrak{N}_1^2$ and $\mathfrak{N}_3^2$. $k = (0, k_1, k_2)$ is the point of intersection of the yOz plane and planes passing through faces $\mathfrak{N}_2^2$ and $\mathfrak{N}_3^2$.

Lemma 9. *Let $\gamma_1 = \gamma_2 = \gamma_3$. Then there are constants $c_i (i = 0, 1, 2)$, such that*

$$\begin{aligned}\int_{\mathbb{R}^3} & \frac{1}{\left(1+\nu^{-N}\left(t^{N\gamma} + t^{N\sigma} + t^{N\delta}\right)\right)\left(1+\nu^{-N}\left(t^{N\gamma} + t^{Nr} + t^{Nq}\right)\right)} \\ &\cdot \frac{dt}{1+\nu^{-N}\left(t^{N\gamma} + t^{Nk} + t^{Nm}\right)} \le \nu^{\min\limits_{i=1,2,3}|\mu^i|}\left(c_2 (\ln \nu)^2 + c_1 |\ln \nu| + c_0\right).\end{aligned} \tag{15}$$

Lemma 10. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$. Then there is a constant c, such that for $\nu \in (0,1)$*

$$|\hat{G}_0(t,\nu)| \le c\nu^{-|\lambda| - \max\limits_{i=1,\dots,n}\left((\lambda,\alpha^i)-1\right)} \frac{1}{1+\nu^{-N}\left(t_1^{Nl_1} + t_2^{Nl_2} + t_3^{Nl_3}\right)}. \tag{16}$$

As in [1], for any function U consider a regularization with the kernel $\hat{G}_0(t,\nu)$

$$U_\nu(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} U(t) \hat{G}_0(t - x, \nu) dt. \tag{17}$$

Such a regularization has some useful properties.

Lemma 11. *If $f \in L_p(\mathbb{R}^3)$, then $f_\nu \in L_p(\mathbb{R}^3)$, and* $\lim_{\nu \to 0} \|f_\nu - f\|_{L_p(\mathbb{R}^3)} = 0$.

For proof we refer to Lemma 2.2 of [1].

Using (17) we can get an integral representation of functions by the multianisotropic kernels $G_{1,j}$.

Theorem 1. *Let $1 \leq p < \infty$. Let f be such that $D^{\alpha^i} f \in L_p(\mathbb{R}^3)$, where α^i are the vertices of a completely regular polyhedron $\mathfrak{N}$. Let $h > 0$ be fixed. Then almost everywhere*

$$f(x) = f_h(x) + \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^{3/2}} \sum_{i=1}^{n} \int_{\varepsilon}^{h} d\nu \int_{\mathbb{R}^3} D^{\alpha^i} f(t) \hat{G_{1,i}}(t - x, \nu) dt. \tag{18}$$

Proof. By the Fundamental Theorem of Calculus and the integral representation (17)

$$\begin{aligned} f_h(x) - f_\varepsilon(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\varepsilon}^{h} \frac{\partial}{\partial \nu} d\nu \int_{\mathbb{R}^3} f(x + t) \hat{G}_0(t, \nu) dt \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\varepsilon}^{h} d\nu \int_{\mathbb{R}^3} f(x + t) \frac{\partial}{\partial \nu} \hat{G}_0(t, \nu) dt \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\varepsilon}^{h} d\nu \int_{\mathbb{R}^3} f(x + t) \sum_{i=1}^{n} D_t^{\alpha^i} G_{1,i}(t, \nu) dt \\ &= \frac{1}{(2\pi)^{3/2}} \sum_{i=1}^{n} \int_{\varepsilon}^{h} d\nu \int_{\mathbb{R}^3} D^{\alpha^i} f(x + t) \hat{G_{1,i}}(t, \nu) dt. \end{aligned} \tag{19}$$

The claim follows from the properties of L_p convergence.

3. Embedding theorems for multianisotropic spaces

Let $\mathfrak{N}$ be a completely regular polyhedron with vertices $\alpha^1, \alpha^2, \ldots, \alpha^n$. The space of functions $W_p^{\mathfrak{N}}(\mathbb{R}^3)$ where

$$W_p^{\mathfrak{N}}\left(\mathbb{R}^3\right) = \left\{ f : f \in L_p\left(\mathbb{R}^3\right); D^{\alpha^i} f \in L_p\left(\mathbb{R}^3\right) 1 \leq i \leq n \right\}$$

is called the multianisotropic Sobolev space. It is a generalization of the isotropic and anisotropic Sobolev spaces.

Theorem 2. *Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and suppose $\gamma_1 \leq \gamma_2 \leq \gamma_3$. Denote by l the number of equal components in the vector γ minus one ($0 \leq l \leq 2$). Let p and q be such that $1 \leq p \leq q < \infty$ or $1 \leq p < \infty$ and $q = \infty$. Let $m = (m_1, m_2, m_3)$ be a multi-index. Denote by $\varkappa$*

$$\varkappa = \max_{i=1,\ldots,M} \left(|\mu^i| + \left(m, \mu^i \right) \right) - \min_{i=1,\ldots,l+1} |\mu^i| \left(1 - \frac{1}{p} + \frac{1}{q} \right).$$

If $\varkappa < 1$ then $D^m W_p^{\mathfrak{N}}(\mathbb{R}^3) \hookrightarrow L_q(\mathbb{R}^3)$, and the following inequality holds

$$\begin{aligned} \|D^m f\|_{L_q(\mathbb{R}^3)} &\leq h^{1-\varkappa} \left(a_{l+2} |\ln h|^{l+2} + \cdots + a_0 \right) \sum_{i=1}^{n} \|D^{\alpha^i} f\|_{L_p(\mathbb{R}^3)} \\ &\quad + h^{-\varkappa} \left(b_{l+2} |\ln h|^{l+2} + \cdots + b_0 \right) \|f\|_{L_p(\mathbb{R}^3)}. \end{aligned} \tag{20}$$

Proof. By (19) we have

$$D^m f_h(x) - D^m f_\varepsilon(x) = \frac{1}{(2\pi)^{3/2}} \sum_{i=1}^{n} \int_{\varepsilon}^{h} d\nu \int_{\mathbb{R}^3} D^{\alpha^i} f(t) D^m \hat{G_{1,i}}(t - x, \nu) dt. \tag{21}$$

By applying Young's inequality we get

$$\begin{aligned} &\|D^m f_h - D^m f_\varepsilon\|_{L_q(\mathbb{R}^3)} \leq C \sum_{i=1}^{n} \int_{\varepsilon}^{h} d\nu \\ &\|D^{\alpha^i} f\|_{L_p(\mathbb{R}^3)} \|D^m \hat{G_{1,i}}(\cdot, \nu)\|_{L_r(\mathbb{R}^3)}, \end{aligned} \tag{22}$$

where $1-\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$. We can estimate $\|D^m \hat{G_{1,i}}(\cdot,\nu)\|_{L_r(\mathbb{R}^3)}$ by applying either one of Lemmas 4–9 depending on how components of γ relate to each other. We consider only the case $\gamma_1=\gamma_2=\gamma_3$, since the other cases can be handled analogously.

$$\|\hat{G_{1,i}}(\cdot,\nu)\|_{L_r(\mathbb{R}^3)} \le \nu^{-\max\limits_{i=1,\dots,M}\left(|\mu^i|+(m,\mu^i)\right)}(c_2(\ln\nu)^2+c_1|\ln\nu|+c_0)$$
$$\cdot\int_{\mathbb{R}^3}\frac{1}{1+\nu^{-N}\left(t^{N\gamma}+t^{N\sigma}+t^{N\delta}\right)}\cdot\frac{1}{1+\nu^{-N}\left(t^{N\gamma}+t^{Nr}+t^{Nq}\right)}$$
$$\cdot\frac{1}{1+\nu^{-N}\left(t^{N\gamma}+t^{Nk}+t^{Nm}\right)}dt. \tag{23}$$

Now we can apply Lemma 9 to the right-hand side of (23)

$$\|\hat{G_{1,i}}(\cdot,\nu)\|_{L_r(\mathbb{R}^3)} \le \nu^{-\varkappa}(c_{l+2}|\ln\nu|^{l+2}+\cdots c_0).$$

We can use the above estimate in (22) to get

$$\|D^m f_h - D^m f_\varepsilon\|_{L_q(\mathbb{R}^3)} \le h^{1-\varkappa}(c_{l+2}|\ln h|^{l+2}+\cdots c_0)\sum_{i=1}^{n}\|D^{\alpha^i} f\|_{L_p(\mathbb{R}^3)}. \tag{24}$$

The right-hand side tends to 0 when $h\to 0$, so $D^m f_h$ is a Cauchy sequence in $L_q(\mathbb{R}^3)$. By the properties of Sobolev weak derivative (see Lemma 6.2 of [3]) and by Lemma 11 it follows that the Sobolev weak derivative $D^m f$ exists, $D^m f\in L_q(\mathbb{R}^3)$ and $\|D^m f - D^m f_\varepsilon\|_{L_q(\mathbb{R}^3)}\to 0$ when $\varepsilon\to 0$. Consequently, we get

$$\begin{aligned}\|D^m f\|_{L_q(\mathbb{R}^3)} &\le \|D^m f_h\|_{L_q(\mathbb{R}^3)} + \|D^m f - D^m f_h\|_{L_q(\mathbb{R}^3)}\\ &\le \|D^m f_h\|_{L_q(\mathbb{R}^3)} + h^{1-\varkappa}(a_{l+2}|\ln h|^{l+2}+\cdots c_0)\sum_{i=1}^{n}\|D^{\alpha^i} f\|_{L_p(\mathbb{R}^3)}.\end{aligned}$$

Now let us estimate $\|D^m f_h\|_{L_q(\mathbb{R}^3)}$. By the integral representation and Young's inequality we get

$$\|D^m f_h\|_{L_q(\mathbb{R}^3)} \le C\|f\|_{L_p(\mathbb{R}^3)}\|D^m\hat{G}_0(\cdot,h)\|_{L_r(\mathbb{R}^3)}.$$

By Lemma 4 for $\hat{G}_0(t,\nu)$ we get

$$\|D^m\hat{G}_0(\cdot,h)\|_{L_r(\mathbb{R}^3)} \le \nu^{-\max\limits_{i=1,\dots,M}\left(|\mu^i|+(m,\mu^i)\right)}\left(c_2(\ln h)^2+c_1|\ln h|+c_0\right)$$
$$\cdot\int_{\mathbb{R}^3}\frac{1}{1+\nu^{-N}\left(t^{N\gamma}+t^{N\sigma}+t^{N\delta}\right)}\cdot\frac{1}{1+\nu^{-N}\left(t^{N\gamma}+t^{Nr}+t^{Nq}\right)}$$
$$\cdot\frac{1}{1+\nu^{-N}\left(t^{N\gamma}+t^{Nk}+t^{Nm}\right)}dt.$$

Again, by Lemma 9 we get

$$\|D^m f_h\|_{L_q(\mathbb{R}^3)} \le h^{-\varkappa}(b_{l+2}|\ln h|^{l+2}+\cdots b_0).$$

Remark 1. If $q=\infty$ then as a consequence of Theorem 2 we obtain the embedding $D^m W_p^{\mathfrak{N}}(\mathbb{R}^3)\hookrightarrow C(\mathbb{R}^3)$.

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Original article

Recursive estimation procedures for one-dimensional parameter of statistical models associated with semimartingales

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Abstract

The recursive estimation problem of a one-dimensional parameter for statistical models associated with semimartingales is considered. The asymptotic properties of recursive estimators are derived, based on the results on the asymptotic behavior of a Robbins–Monro type SDE. Various special cases are considered.

Keywords: Stochastic approximation; Robbins–Monro type SDE; Semimartingale statistical models; Recursive estimation; Asymptotic properties

0. Introduction

Beginning from the paper [1] of A. Albert and L. Gardner a link between Robbins–Monro (RM) stochastic approximation algorithm (introduced in [2]) and recursive parameter estimation procedures was intensively exploited. Later on recursive parameter estimation procedures for various special models (e.g., i.i.d. models, non i.i.d. models in discrete time, etc.) have been studied by a number of authors using methods of stochastic approximation (see, e.g., [3–12]). It would be mentioned the fundamental book [13] by M.B. Nevelson and R.Z. Khas'minski (1972) between them.

In 1987 by N. Lazrieva and T. Toronjadze a heuristic algorithm of a construction of the recursive parameter estimation procedures for statistical models associated with semimartingales (including both discrete and continuous time semimartingale statistical models) was proposed [14]. These procedures could not be covered by the generalized stochastic approximation algorithm with martingale noises (see, e.g., [15]), while in discrete time case the classical RM algorithm contains recursive estimation procedures.

To recover the link between the stochastic approximation and recursive parameter estimation in [16–18] by Lazrieva, Sharia and Toronjadze the semimartingale stochastic differential equation was introduced, which naturally

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includes both generalized RM stochastic approximation algorithms with martingale noises and recursive parameter estimation procedures for semimartingale statistical models.

In the present work we are concerning with the construction of recursive estimation procedures for semimartingale statistical models asymptotically equivalent to the MLE and M-estimators, embedding these procedures in the Robbins–Monro type equation. For this reason in Section 1 we shortly describe the Robbins–Monro type SDE and give necessary objects to state results concerning the asymptotic behavior of recursive estimator procedures.

In Section 2 we give a heuristic algorithm of constructing recursive estimation procedures for one-dimensional parameter of semimartingale statistical models. These procedures provide estimators asymptotically equivalent to MLE. To study the asymptotic behavior of these procedures we rewrite them in the form of the Robbins–Monro type SDE. Besides, we give a detailed description of all objects presented in this SDE, allowing us separately study special cases (e.g. discrete time case, diffusion processes, point processes, etc.).

In Section 4 we formulate main results concerning the asymptotic behavior of recursive procedures, asymptotically equivalent to the MLE.

In Section 5, we develop recursive procedures, asymptotically equivalent to M-estimators.

Finally, in Section 6, we give various examples demonstrating the usefulness of our approach.

1. The Robbins–Monro type SDE

Let on the stochastic basis $(\Omega, \mathcal{F}, F = (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions the following objects be given:

(a) the random field $H = \{H_t(u), t \geq 0, u \in R^1\} = \{H_t(\omega, u), t \geq 0, \omega \in \Omega, u \in R^1\}$ such that for each $u \in R^1$ the process $H(u) = (H_t(u))_{t\geq 0} \in \mathcal{P}$ (i.e. is predictable);

(b) the random field $M = \{M(t, u), t \geq 0, u \in R^1\} = \{M(\omega, t, u), \omega \in \Omega, t \geq 0, u \in R^1\}$ such that for each $u \in R^1$ the process $M(u) = (M(t, u))_{t\geq 0} \in \mathcal{M}^2_{\mathrm{loc}}(P)$;

(c) the predictable increasing process $K = (K_t)_{t\geq 0}$ (i.e. $K \in \mathcal{V}^+ \cap \mathcal{P}$).

In the sequel we restrict ourselves to the consideration of the following particular case: for each $u \in R^1 M(u) = \varphi(u) \cdot m + W(u) * (\mu - \nu)$, where $m \in \mathcal{M}^c_{\mathrm{loc}}(P)$, μ is an integer-valued random measure on $(R \times E, \mathcal{B}(R_+) \times \mathcal{E})$, ν is its P-compensator, $(E, \mathcal{E})$ is the Blackwell space, $W(u) = (W(t, x, u), t \geq 0, x \in E) \in \mathcal{P} \otimes \mathcal{E}$. Here we also mean that all stochastic integrals are well-defined.[1]

Later on by the symbol $\int_0^t M(ds, u_s)$, where $u = (u_t)_{t\geq 0}$ is some predictable process, we denote the following stochastic line integrals:

$$\int_0^t \varphi(s, u_s)\, dm_s + \int_0^t \int_E W(s, x, u_s)(\mu - \nu)(ds, dx)$$

provided the latters are well-defined.

Consider the following semimartingale stochastic differential equation

$$z_t = z_0 + \int_0^t H_s(z_{s-})\, dK_s + \int_0^t M(ds, z_{s-}), \quad z_0 \in \mathcal{F}_0. \tag{1.1}$$

We call SDE (1.1) the Robbins–Monro (RM) type SDE if the drift coefficient $H_t(u)$, $t \geq 0$, $u \in R^1$ satisfies the following conditions: for all $t \in [0, \infty)$ P-a.s.

$$\text{(A)} \quad \begin{array}{l} H_t(0) = 0, \\ H_t(u)u < 0 \quad \text{for all } u \neq 0. \end{array}$$

The question of strong solvability of SDE (1.1) is well-investigated (see, e.g., [20]).

We assume that there exists a unique strong solution $z = (z_t)_{t\geq 0}$ of Eq. (1.1) on the whole time interval $[0, \infty)$ and such that $\widetilde{M} \in \mathcal{M}^2_{\mathrm{loc}}(P)$, where

$$\widetilde{M}_t = \int_0^t M(ds, z_{s-}).$$

Sufficient conditions for the latter can be found in [20].

[1] See [19] for basic concepts and notations.

The unique solution $z = (z_t)_{t\geq 0}$ of RM type SDE (1.1) can be viewed as a semimartingale stochastic approximation procedure.

In [16,17], the asymptotic properties of the process $z = (z_t)_{t\geq 0}$ as $t \to \infty$ are investigated, namely, convergence ($z_t \to 0$ as $t \to \infty$ P-a.s.), rate of convergence (that means that for all $\delta < \frac{1}{2}$, $\gamma_t^\delta z_t \to 0$ as $t \to \infty$ P-a.s., with the specially chosen normalizing sequence $(\gamma_t)_{t\geq 0}$) and asymptotic expansion

$$\chi_t^2 z_t^2 = \frac{L_t}{\langle L\rangle_t^{1/2}} + R_t$$

with the specially chosen normalizing sequence χ_t^2 and martingale $L = (L_t)_{t\geq 0}$, where $R_t \to 0$ as $t \to \infty$ (see [16,17] for definition of objects χ_t^2, L_t and R_t).

2. Basic model and regularity

Our object of consideration is a parametric filtered statistical model

$$\mathcal{E} = (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \ \{P_\theta; \theta \in R\})$$

associated with one-dimensional $\mathbb{F}$-adapted RCLL process $X = (X_t)_{t\geq 0}$ in the following way: for each $\theta \in R^1$ P_θ is assumed to be the unique measure on $(\Omega, \mathcal{F})$ such that under this measure X is a semimartingale with predictable characteristics $(B(\theta), C(\theta), \nu_\theta)$ (w.r.t. standard truncation function $h(x) = xI_{\{|x|\leq 1\}}$). For simplicity assume that all P_θ coincide on $\mathcal{F}_0$.

Suppose that for each pair (θ, θ') $P_\theta \overset{\text{loc}}{\sim} P_{\theta'}$. Fix some $\theta_0 \in R$ and denote $P = P_{\theta_0}$, $B = B(\theta_0)$, $C = C(\theta_0)$, $\nu = \nu_{\theta_0}$.

Let $\rho(\theta) = (\rho_t(\theta))_{t\geq 0}$ be a local density process (likelihood ratio process)

$$\rho_t(\theta) = \frac{dP_{\theta,t}}{dP_t},$$

where for each θ $P_{\theta,t} := P_\theta|\mathcal{F}_t$, $P_t := P|\mathcal{F}_t$ are restrictions of measures P_θ and P on $\mathcal{F}_t$, respectively.

As it is well-known (see, e.g., [21, Ch. III, §3d, Th. 3.24]) for each θ there exists a $\widetilde{\mathcal{P}}$-measurable positive function

$$Y(\theta) = \{Y(\omega, t, x; \theta), \ (\omega, t, x) \in \Omega \times R_+ \times R\},$$

and a predicable process $\beta(\theta) = (\beta_t(\theta))_{t\geq 0}$ with

$$|h(Y(\theta) - 1)| * \nu \in \mathcal{A}^+_{\text{loc}}(P), \qquad \beta^2(\theta) \circ C \in \mathcal{A}^+_{\text{loc}}(P),$$

and such that

$$\begin{aligned} &(1)\ B(\theta) = B + \beta(\theta) \circ C + h(Y(\theta) - 1) * \nu, \\ &(2)\ C(\theta) = C, \qquad (3)\ \nu_\theta = Y(\theta) \cdot \nu. \end{aligned} \tag{2.1}$$

In addition, the function $Y(\theta)$ can be chosen in such a way that

$$a_t := \nu(\{t\}, R) = 1 \Longleftrightarrow a_t(\theta) := \nu_\theta(\{t\}, R) = \int Y(t, x; \theta)\nu(\{t\})dx = \widehat{Y}_t(\theta) = 1.$$

We give a definition of the regularity of the model based on the following representation of the density process as exponential martingale:

$$\rho(\theta) = \mathcal{E}(M(\theta)),$$

where

$$M(\theta) = \beta(\theta) \cdot X^c + \left(Y(\theta) - 1 + \frac{\widehat{Y}(\theta) - a}{1 - a}\, I_{\{0<a<1\}}\right) * (\mu - \nu) \in \mathcal{M}_{\text{loc}}(P), \tag{2.2}$$

$\mathcal{E}_t(M)$ is the Dolean exponential of the martingale M (see, e.g., [19]). Here X^c is a continuous martingale part of X under measure P.

We say that the model is regular if for almost all (ω, t, x) the functions $\beta : \theta \to \beta_t(\omega; \theta)$ and $Y : \theta \to Y(\omega, t, x; \theta)$ are differentiable (notation $\dot\beta(\theta) := \frac{\partial}{\partial\theta}\beta(\theta)$, $\dot Y(\theta) := \frac{\partial}{\partial\theta}Y(\theta)$) and differentiability under integral sign is possible. Then

$$\frac{\partial}{\partial\theta} \ln \rho(\theta) = L(\dot M(\theta), M(\theta)) := L(\theta) \in \mathcal{M}_{\mathrm{loc}}(P_\theta),$$

where $L(m, M)$ is the Girsanov transformation defined as follows: if $m, M \in \mathcal{M}_{\mathrm{loc}}(P)$ and $Q \ll P$ with $\frac{dQ}{dP} = \mathcal{E}(M)$, then

$$L(m, M) := m - (1 + \Delta M)^{-1} \circ [m, M] \in \mathcal{M}_{\mathrm{loc}}(Q).$$

It is not hard to verify that

$$L(\theta) = \dot\beta(\theta) \cdot (X^c - \beta(\theta) \circ C) + \Phi(\theta) * (\mu - \nu_\theta), \tag{2.3}$$

where

$$\Phi(\theta) = \frac{\dot Y(\theta)}{Y(\theta)} + \frac{\dot a(\theta)}{1 - a(\theta)}$$

with $I_{\{a(\theta)=1\}}\dot a(\theta) = 0$, and $0/0 = 0$ (recall that $\frac{\partial}{\partial\theta}\widehat{Y}(\theta) = \dot a(\theta)$).

Indeed, due to the regularity of the model, we have

$$\dot M(\theta) = \dot\beta(\theta) \cdot X^c + \left(\dot Y(\theta) - \frac{\dot a(\theta)}{1 - a} I_{(0<a<1)}\right) * (\mu - \nu)$$

and (2.3) simply follows from (1.16)–(1.18) of [22, Part I] with

$$g(\theta) = Y(\theta) - 1 + \frac{a(\theta) - a}{1 - a} I_{(0<a<1)},$$

$$\psi(\theta) = \dot Y(\theta) - \frac{\dot a(\theta)}{1 - a} I_{(0<a<1)}.$$

The empirical Fisher information process is $\widehat{I}_t(\theta) = [L(\theta), L(\theta)]_t$ and if we assume that for each $\theta \in R^1 L(\theta) \in \mathcal{M}^2_{\mathrm{loc}}(P_\theta)$, then the Fisher information process is

$$I_t(\theta) = \langle L(\theta), L(\theta)\rangle_t.$$

3. Recursive estimation procedure for MLE

In [14], a heuristic algorithm was proposed for the construction of recursive estimators of unknown parameter θ asymptotically equivalent to the maximum likelihood estimator (MLE).

This algorithm was derived using the following reasons:

Consider the MLE $\widehat\theta = (\widehat\theta_t)_{t\ge 0}$, where $\widehat\theta_t$ is a solution of estimational equation

$$L_t(\theta) = 0.$$

The question of solvability of this equation is considered in [22, Part II].

Assume that

(1) for each $\theta \in R^1$, $I_t(\theta) \to \infty$ as $t \to \infty$, P_θ-a.s., the process $(\widehat{I}_t(\theta))^{1/2}(\widehat\theta_t - \theta)$ is P_θ-stochastically bounded and, in addition, the process $(\widehat\theta_t)_{t\ge0}$ is a P_θ-semimartingale;
(2) for each pair (θ', θ) the process $L(\theta') \in \mathcal{M}^2_{\mathrm{loc}}(P_{\theta'})$ and is a P_θ-special semimartingale;
(3) the family $(L(\theta), \theta \in R^1)$ is such that the Itô–Ventzel formula is applicable to the process $(L(t, \widehat\theta_t))_{t\ge0}$ w.r.t. P_θ for each $\theta \in R^1$;
(4) for each $\theta \in R^1$ there exists a positive increasing predictable process $(\gamma_t(\theta))_{t\ge0}$, $\gamma_0 > 0$, asymptotically equivalent to $\widehat{I}_t^{-1}(\theta)$, i.e.

$$\gamma_t(\theta)\widehat{I}_t(\theta) \xrightarrow{P_\theta} 1 \quad \text{as } t \to \infty.$$

Under these assumptions using the Ito–Ventzel formula for the process $(L(t,\widehat{\theta}_t))_{t\geq 0}$ we get an "implicit" stochastic equation for $\widehat{\theta} = (\widehat{\theta}_t)_{t\geq 0}$. Analyzing the orders of infinitesimality of terms of this equation and rejecting the high order terms we get the following SDE (recursive procedure)

$$d\theta_t = \gamma_t(\theta_{t-})L(dt,\theta_{t-}), \tag{3.1}$$

where $L(dt,u_t)$ is a stochastic line integral w.r.t. the family $\{L(t,u), u\in R^1, t\in R_+\}$ of P_θ-special semimartingales along the predictable curve $u=(u_t)_{t\geq 0}$.

Note that in many cases under consideration one can choose $\gamma_t(\theta) = (I_t^{-1}(\theta)+1)^{-1}$, or in ergodic situations such as i.i.d. case, ergodic diffusion one can replace $I_t(\theta)$ by another process equivalent to them (see examples).

To give an explicit form to the SDE (3.1) for the statistical model associated with the semimartingale X assume for a moment that for each (u,θ) (including the case $u=\theta$)

$$|\Phi(u)| * \mu \in \mathcal{A}^+_{\mathrm{loc}}(P_\theta). \tag{3.2}$$

Then for each pair (u,θ) we have

$$\Phi(u) * (\mu - \nu_u) = \Phi(u) * (\mu - \nu_\theta) + \Phi(u)\left(1 - \frac{Y(u)}{Y(\theta)}\right) * \nu_\theta.$$

Based on this equality one can obtain the canonical decomposition of P_θ-special semimartingale $L(u)$ (w.r.t. measure P_θ):

$$L(u) = \dot{\beta}(u)\circ(X^c - \beta(\theta)\circ C) + \Phi(u)*(\mu-\nu_\theta) + \dot{\beta}(u)(\beta(\theta)-\beta(u))\circ C + \Phi(u)\left(1-\frac{Y(u)}{Y(\theta)}\right)*\nu_\theta. \tag{3.3}$$

Now, using (3.3) the meaning of $L(dt,u_t)$ is

$$\begin{aligned}\int_0^t L(ds,u_{s-}) &= \int_0^t \dot{\beta}_s(u_{s-})d(X^c - \beta(\theta)\circ C)_s + \int_0^t\int \Phi(s,x,u_{s-})(\mu-\nu_\theta)(ds,dx)\\ &\quad + \int_0^t \dot{\beta}_s(u_s)(\beta_s(\theta)-\beta_s(u_s))dC_s + \int_0^t\int \Phi(s,x,u_{s-})\left(1-\frac{Y(s,x,u_{s-})}{Y(s,x,\theta)}\right)\nu_\theta(ds,dx).\end{aligned}$$

Finally, the recursive SDE (3.1) takes the form

$$\begin{aligned}\theta_t &= \theta_0 + \int_0^t \gamma_s(\theta_{s-})\dot{\beta}_s(\theta_{s-})d(X^c-\beta(\theta)\circ C)_s + \int_0^t\int \gamma_s(\theta_{s-})\Phi(s,x,\theta_{s-})(\mu-\nu_\theta)(ds,dx)\\ &\quad + \int_0^t \gamma_s(\theta)\dot{\beta}_s(\theta_s)(\beta_s(\theta)-\beta_s(\theta_s))dC_s\\ &\quad + \int_0^t\int \gamma_s(\theta_{s-})\Phi(s,x,\theta_{s-})\left(1-\frac{Y(s,x,\theta_{s-})}{Y(s,x,\theta)}\right)\nu_\theta(ds,dx).\end{aligned} \tag{3.4}$$

Remark 3.1. One can give more accurate than (3.2) sufficient conditions (see, e.g., [21,19]) to ensure the validity of decomposition (3.3).

Assume that there exists a unique strong solution $(\theta_t)_{t\geq 0}$ of the SDE (3.4).

Fix arbitrary $\theta\in R^1$. To investigate the asymptotic properties, under measure P_θ, of recursive estimators $(\theta_t)_{t\geq 0}$ as $t\to\infty$, namely, a strong consistency, rate of convergence and asymptotic expansion we reduce the SDE (3.4) to the Robbins–Monro type SDE.

For this aim denote $z_t = \theta_t - \theta$. Then (3.4) can be rewritten as

$$\begin{aligned}z_t &= z_0 + \int_0^t \gamma_s(\theta+z_{s-})\dot{\beta}(\theta+z_{s-})(\beta_s(\theta)-\beta_s(\theta+z_{s-}))dC_s\\ &\quad + \int_0^t\int \gamma_s(\theta+z_{s-})\Phi(s,x,\theta+z_{s-})\left(1-\frac{Y(s,x,\theta+z_{s-})}{Y(s,x,\theta)}\right)\nu_\theta(ds,dx)\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \gamma_s(\theta+z_s)\dot\beta_s(\theta+z_s)d(X^c-\beta(\theta)\circ C)_s \\
&+\int_0^t\int \gamma_s(\theta+z_{s-})\Phi(s,x,\theta+z_{s-})(\mu-\nu_\theta)(ds,dx). \qquad (3.5)
\end{aligned}
$$

For the definition of the objects K^θ, $\{H^\theta(u),\ u\in R^1\}$ and $\{M^\theta(u),\ u\in R^1\}$ we consider such a version of characteristics (C,ν_θ) that

$$
\begin{aligned}
C_t &= c^\theta\circ A^\theta_t,\\
\nu_\theta(\omega,dt,dx) &= dA^\theta_t B^\theta_{\omega,t}(dx),
\end{aligned}
$$

where $A^\theta=(A^\theta_t)_{t\ge0}\in\mathcal{A}^+_{\rm loc}(P_\theta)$, $c^\theta=(c^\theta_t)_{t\ge0}$ is a nonnegative predictable process, and $B^\theta_{\omega,t}(dx)$ is a transition kernel from $(\Omega\times R_+,\mathcal{P})$ in $(R,\mathcal{B}(R))$ with $B^\theta_{\omega,t}(\{0\})=0$ and

$$\Delta A^\theta_t B^\theta_{\omega,t}(R)\le 1$$

(see [21, Ch. 2, §2, Prop. 2.9]).

Put $K^\theta_t=A^\theta_t$,

$$H^\theta_t(u)=\gamma_t(\theta+u)\left\{\dot\beta_t(\theta+u)(\beta_t(\theta)-\beta_t(\theta+u))c^\theta_t+\int\Phi(t,x,\theta+u)\left(1-\frac{Y(t,x,\theta+u)}{Y(t,x,\theta)}\right)B^\theta_{\omega,t}(dx)\right\}, \qquad (3.6)$$

$$M^\theta(t,u)=\int_0^t\gamma_s(\theta+u)\dot\beta_s(\theta+u)d(X^c-\beta(\theta)\circ C)_s+\int_0^t\int\gamma_s(\theta+u)\Phi(s,x,\theta+u)(\mu-\nu_\theta)(ds,dx). \qquad (3.7)$$

Assume that for each u, $u\in R$, $M^\theta(u)=(M^\theta(t,u))_{t\ge0}\in\mathcal{M}^2_{\rm loc}(P_\theta)$. Then

$$
\begin{aligned}
\langle M^\theta(u)\rangle_t &= \int_0^t(\gamma_s(\theta+u)\dot\beta_s(\theta+u))^2c^\theta_s dA^\theta_s+\int_0^t\gamma^2_s(\theta+u)\left(\int\Phi^2(s,x,\theta+u)B^\theta_{\omega,s}(dx)\right)dA^{\theta,c}_s\\
&+\int_0^t\gamma^2_s(\theta+u)B^\theta_{\omega,t}(R)\left\{\int\Phi^2(s,x,\theta+u)q^\theta_{\omega,s}(dx)\right.\\
&\left.-\,a_s(\theta)\left(\int\Phi(s,x,\theta+u)q^\theta_{\omega,s}(dx)\right)^2\right\}dA^{\theta,d}_s,
\end{aligned}
$$

where $a_s(\theta)=\Delta A^\theta_s B^\theta_{\omega,s}(R)$, $q^\theta_{\omega,s}(dx)I_{\{a_s(\theta)>0\}}=\frac{B^\theta_{\omega,s}(dx)}{B^\theta_{\omega,s}(R)}I_{\{a_s(\theta)>0\}}$.

Now we give a more detailed description of $\Phi(\theta)$, $I(\theta)$, $H^\theta(u)$ and $\langle M^\theta(u)\rangle$. This allows us to study the special cases separately (see Remark 3.2 below). Denote

$$\frac{d\nu^c_\theta}{d\nu^c}:=F(\theta),\qquad \frac{q^\theta_{\omega,t}(dx)}{q_{\omega,t}(dx)}:=f_{\omega,t}(x,\theta)\quad(:=f_t(\theta)).$$

Then

$$Y(\theta)=F(\theta)I_{\{a=0\}}+\frac{a(\theta)}{a}f(\theta)I_{\{a>0\}}$$

and

$$\dot Y(\theta)=\dot F(\theta)I_{\{a=0\}}+\left(\frac{\dot a(\theta)}{a}f(\theta)+\frac{a(\theta)}{a}\dot f(\theta)\right)I_{\{a>0\}}.$$

Therefore

$$\Phi(\theta)=\frac{\dot F(\theta)}{F(\theta)}I_{\{a=0\}}+\left\{\frac{\dot f(\theta)}{f(\theta)}+\frac{\dot a(\theta)}{a(\theta)(1-a(\theta))}\right\}I_{\{a>0\}} \qquad (3.8)$$

with $I_{\{a(\theta)>0\}}\int\frac{\dot f(\theta)}{f(\theta)}q^\theta(dx)=0$.

Remark 3.2. Denote $\dot{\beta}(\theta) = \ell^c(\theta)$, $\frac{\dot{F}(\theta)}{F(\theta)} := \ell^\pi(\theta)$, $\frac{\dot{f}(\theta)}{f(\theta)} := \ell^\delta(\theta)$, $\frac{\dot{a}(\theta)}{a(\theta)(1-a(\theta))} := \ell^b(\theta)$.

Indices $i = c, \pi, \delta, b$ carry the following loads: "c" corresponds to the continuous part, "π" to the Poisson type part, "δ" to the predictable moments of jumps (including a main special case—the discrete time case), "b" to the binomial type part of the likelihood score $\ell(\theta) = (\ell^c(\theta), \ell^\pi(\theta), \ell^\delta(\theta), \ell^b(\theta))$.

In these notations we have for the Fisher information process:

$$
\begin{aligned}
I_t(\theta) = {} & \int_0^t (\ell_s^c(\theta))^2 dC_s + \int_0^t \int (\ell_s^\pi(x;\theta))^2 B_{\omega,s}^\theta(dx) dA_s^{\theta,c} \\
& + \int_0^t B_{\omega,s}^\theta(R)\left[\int (\ell_s^\delta(x;\theta))^2 q_{\omega,s}^\theta(dx)\right] dA_s^{\theta,d} + \int_0^t (\ell_s^b(\theta))^2 (1-a_s(\theta)) dA_s^{\theta,d}.
\end{aligned} \tag{3.9}
$$

For the random field $H^\theta(u)$ we have

$$
\begin{aligned}
H_t^\theta(u) = {} & \gamma_t(\theta+u)\Bigg\{\ell_t^c(\theta+u)(\beta_t(\theta)-\beta_t(\theta+u))c_t^\theta \\
& + \int \ell_t^\pi(x;\theta+u)\left(1-\frac{F_t(x;\theta+u)}{F_t(x;\theta)}\right)\Bigg\} B_{\omega,t}^\theta(dx) I_{\{\Delta A_t^\theta=0\}} \\
& + \left\{\int \ell_t^\delta(x;\theta+u) q_{\omega,t}^\theta(dx) \ell_t^b(\theta+u)\,\frac{a_t(\theta)-a_t(\theta+u)}{a_t(\theta)}\right\} B_{\omega,t}^\theta(R) I_{\{\Delta A_t^\theta>0\}}.
\end{aligned} \tag{3.10}
$$

Finally, we have for $\langle M^\theta(u)\rangle$:

$$
\begin{aligned}
\langle M^\theta(u)\rangle_t = {} & \big(\gamma(\theta+u)\ell^c(\theta+u)\big)^2 c^\theta \circ A_t^\theta + \int_0^t \gamma_s^2(\theta+u)\int (\ell_s^\pi(x;\theta+u))^2 B_{\omega,s}^\theta(dx) dA_s^{\theta,c} \\
& + \int_0^t \gamma_s^2(\theta+u) B_{\omega,s}^\theta(R)\Bigg\{\int (\ell_s^\delta(x;\theta+u)+\ell_s^b(\theta+u))^2 q_{\omega,s}^\theta(dx) \\
& - a_s(\theta)\left(\int (\ell_s^\delta(x;\theta+u)+\ell_s^b(\theta+u)) q_{\omega,s}^\theta(dx)\right)^2\Bigg\} dA_s^{\theta,d}.
\end{aligned} \tag{3.11}
$$

Thus, we reduced SDE (3.5) to the Robbins–Monro type SDE with $K_t^\theta = A_t^\theta$, and $H^\theta(u)$ and $M^\theta(u)$ defined by (3.6) and (3.7), respectively.

As it follows from (3.6), (3.10)

$$H_t^\theta(0) = 0 \quad \text{for all } t \geq 0,\ P_\theta\text{-a.s.}$$

As for condition (A) to be satisfied it is enough to require that for all $t \geq 0$, $u \neq 0$ P_θ-a.s.

$$
\begin{aligned}
& \dot{\beta}_t(\theta+u)(\beta_t(\theta)-\beta_t(\theta+u)) < 0, \\
& \left(\int \frac{\dot{F}(t,x,\theta+u)}{F(t,x,\theta+u)}\left(1-\frac{F(t,x;\theta+u)}{F(t,x;\theta)}\right) B_{\omega,t}^\theta(dx)\right) I_{\{\Delta A_t^\theta=0\}} u < 0, \\
& \left(\int \frac{\dot{f}(t,x;\theta+u)}{f(t,x;\theta+u)}\, q_t^\theta(dx)\right) I_{\{\Delta A_t^\theta>0\}} u < 0, \\
& \dot{a}_t(\theta+u)(a_t(\theta)-a_t(\theta+u))u < 0,
\end{aligned}
$$

and the simplest sufficient conditions for the latter ones are the strong monotonicity (P-a.s.) of functions $\beta(\theta)$, $F(\theta)$, $f(\theta)$ and $a(\theta)$ w.r.t. θ.

4. Main results

We are ready to formulate main results about asymptotic properties of recursive estimators $\{\theta_t,\ t \geq 0\}$ as $t \to \infty$, (P_θ-a.s.), which is the same of solution z_t, $t \geq 0$, of Eq. (3.5).

For simplicity we restrict ourselves by the case when semimartingale $X = (X_t)_{t\geq 0}$ is left quasi-continuous, so $\nu(\omega; \{t\}, R) = 0$ for all $t \geq 0$, P-a.s., and $A^\theta = (A_t^\theta)_{t\geq 0}$ is a continuous process. In this case

$$H_t^\theta(u) = \gamma_t(\theta+u)\left\{\dot\beta_t(\theta+u)(\beta_t(\theta)-\beta_t(\theta+u))c_t^\theta + \int \frac{\dot F_t(x;\theta+u)}{F_t(x;\theta+u)}\left(1-\frac{\dot F_t(x;\theta+u)}{F_t(x;\theta)}\right)B_{\omega,t}^\theta(dx)\right\}, \tag{4.1}$$

$$\langle M^\theta(u)\rangle_t = \int_0^t (\gamma_s(\theta+u)\dot\beta_s(\theta+u))^2 dA_s^\theta + \int_0^t \gamma_s^2(\theta+u)\left(\int\left(\frac{\dot F_s(x;\theta+u)}{F_s(x;\theta+u)}\right)^2 B_{\omega,s}^\theta(dx)\right)dA_s^\theta, \tag{4.2}$$

$$I_t(\theta) = \int_0^t (\dot\beta_s(\theta))^2 c_s^\theta dA_s^\theta + \int_0^t\int\left(\frac{\dot F_s(x;\theta)}{F_s(x;\theta)}\right)^2 B_{\omega,s}(dx)dA_s^\theta. \tag{4.3}$$

Theorem 4.1 (*Strong Consistency*)**.** *Let for all $t \geq 0$, P_θ-a.s. the following conditions be satisfied:*

(A) $H_t^\theta(0) = 0$, $H_t^\theta(u)u < 0$, $u \neq 0$,

(B) $h_t^\theta(u) \leq B_t^\theta(1+u^2)$, *where $B^\theta = (B_t^\theta)_{t\geq 0}$ is a predictable process, $B_t^\theta \geq 0$, $B^\theta \circ A_\infty^\theta < \infty$,*

$$h_t^\theta(u) = \frac{d\langle M^\theta(u)\rangle_t}{dA_t^\theta}, \tag{4.4}$$

(C) *for each ε, $\varepsilon > 0$,*

$$\inf_{\varepsilon\leq|u|\leq\frac{1}{\varepsilon}} |H^\theta(u)u| \circ A_\infty^\theta = \infty.$$

Then for each $\theta \in R^1$

$$\widehat{\theta}_t \to 0 \quad (or \quad z_t \to 0), \quad as\ t\to\infty,\ P_\theta\text{-a.s.}$$

Proof. Immediately follows from conditions of Theorem 3.1 of [16] applied to prespecified by (4.1)–(4.3) objects. □

In the sequel we assume that for each $\theta \in R^1$

$$P_\theta\left(\lim_{t\to\infty}\frac{\widehat{I}_t(\theta)}{I_t(\theta)} = 1\right) = 1,$$

from which it follows that $\gamma_t(\theta) = I_t^{-1}(\theta)$. Denote

$$g_t^\theta = \frac{dI_t(\theta)}{dA_t^\theta} = (\dot\beta_t(\theta))^2 c_t^\theta + \int\left(\frac{\dot F_t(x;\theta)}{F_t(x;\theta)}\right)^2 B_{\omega,t}(dx). \tag{4.5}$$

We assume also that $z_t \to 0$ as $t\to\infty$, P_θ-a.s.

Theorem 4.2 (*Rate of Convergence*)**.** *Suppose that for each δ, $0<\delta<1$, the following conditions are satisfied:*

$$\text{(i)}\ \int_0^\infty \left[\delta\,\frac{g_t^\theta}{I_t^\theta} - 2\beta_t^\theta(z_t)\right]^+ dA_t^\theta < \infty, \quad P_\theta\text{-a.s., where } \beta_t^\theta(u) = \begin{cases} -\dfrac{H_t^\theta(u)}{u}, & u\neq 0,\\ -\lim\limits_{u\to 0}\dfrac{H_t^\theta(u)}{u}, & u = 0,\end{cases} \tag{4.6}$$

$$\text{(ii)}\ \int_0^\infty (I_t(\theta))^\delta h_t^\theta(z_t)dA_t^\theta < \infty, \quad P_\theta\text{-a.s.}$$

Then for each $\theta \in R^1$, δ, $0<\delta<1$,

$$I_t^\delta(\theta)z_t^2 \to 0 \quad as\ t\to\infty,\ P_\theta\text{-a.s.}$$

Proof. It is enough to note that conditions (2.3) and (2.4) of Theorem 2.1 from [17] are satisfied with $I_t(\theta)$ instead of γ_t, $\delta g_t^\theta / I_t(\theta)$ instead of r_t^δ and $\beta_t^\theta(u)$ instead of $\beta_t(u)$. $\square$

In the sequel we assume that for all δ, $0 < \delta < \frac{1}{2}$,

$$I_t^\delta(\theta) z_t \to 0 \quad \text{as } t \to \infty, \; P_\theta\text{-a.s.}$$

It is not hard to verify that the following expansion holds true

$$I_t^{1/2}(\theta) z_t = \frac{L_t^\theta}{\langle L^\theta \rangle_t^{1/2}} + R_t^\theta, \tag{4.7}$$

where L_t^θ, R_t^θ will be specified below.

Indeed, according to "Preliminary and Notation" section of [17]

$$\overline{\beta}_t^\theta = -\lim_{u \to 0} \frac{H_t^\theta(u)}{u} = -I_t^{-1}(\theta) g_t^\theta.$$

Further,

$$-\overline{\beta}^\theta \circ A_t^\theta = \int_0^t I_s^{-1}(\theta) \frac{dI_s(\theta)}{dA_s(\theta)} dA_s^\theta = \ln I_t(\theta).$$

Therefore

$$\Gamma_t^\theta = \varepsilon_t^{-1}(-\overline{\beta}^\theta \circ A_t^\theta) = I_t(\theta) \tag{4.8}$$

and

$$L_t^\theta = \int_0^t \Gamma_s^\theta dM^\theta(s, 0)$$

with

$$\langle L^\theta \rangle_t = \int_0^t (\Gamma_s^\theta)^2 d\langle M^\theta(0) \rangle_s = \int_0^t I_s^2(\theta) I_s^{-2}(\theta) dI_s(\theta) = I_t(\theta). \tag{4.9}$$

Finally, we obtain

$$\chi_t^\theta = \Gamma_t^\theta \langle L^\theta \rangle_t^{-1/2} = I_t^{1/2}(\theta). \tag{4.10}$$

As for R_t^θ, one can use the definition of R_t from the same section by replacing of objects by the corresponding objects with upperscripts "θ", e.g. $\overline{\beta}_t$ by $\overline{\beta}_t^\theta$, L_t by L_t^θ, etc.

Theorem 4.3 (*Asymptotic Expansion*)**.** *Let the following conditions be satisfied:*

(i) $\langle L^\theta \rangle_t$ *is a deterministic process,* $\langle L^\theta \rangle_\infty = \infty$,

(ii) *there exists* ε, $0 < \varepsilon < \frac{1}{2}$, *such that*

$$\frac{1}{\langle L^\theta \rangle_t} \int_0^t |\beta_s^\theta - \beta_s^\theta(z_s)| I_s^{-\varepsilon}(\theta) \langle L^\theta \rangle_s dA_s^\theta \to 0 \quad \text{as } t \to \infty, \; P_\theta\text{-a.s.},$$

(iii)

$$\frac{1}{\langle L^\theta \rangle_t} \int_0^t I_t^2(\theta)(h_s^\theta(z_s, z_s) - 2h_s^\theta(z_s, 0) + h_s(0, 0)) dA_s^\theta \xrightarrow{P_\theta} 0 \quad \text{as } t \to \infty,$$

where

$$h_t^\theta(u, v) = \frac{d\langle M^\theta(u), M^\theta(v) \rangle}{dA_t^\theta}. \tag{4.11}$$

Then in Eq. (4.7) *for each* $\theta \in R$

$$R_t^\theta \xrightarrow{P_\theta} 0 \quad \text{as } t \to \infty.$$

Proof. It is not hard to verify that all conditions of Theorem 3.1 from [17] are satisfied with $\langle L^\theta \rangle_t$ instead of $\langle L \rangle_t$, $\beta_s^\theta(u)$ instead of $\beta_s(u)$, $I_\theta^{-1}(\theta)$ instead of γ_t, A_t^θ instead of χ_t, Γ_s^θ instead Γ_s, and $I_t^{1/2}(\theta)$ instead of χ_t, $h_t^\theta(u, v)$ instead of $h_t(u, v)$, and, finally, P^θ instead of P. $\square$

Remark. It follows from Eq. (4.7) and Theorem 4.3 that, using the Central Limit Theorem for martingales

$$I_t^{1/2}(\theta)(\theta_t - \theta) \xrightarrow{d} N(0, 1).$$

5. Recursive procedure for M-estimators

As stated in previous section the maximum likelihood equation has the form

$$L_t(\theta) = L_t(\dot{M}_\theta, M_\theta) = 0.$$

This equation is the special member of the following family of estimational equations

$$L_t(m_\theta, M_\theta) = 0 \tag{5.1}$$

with certain P-martingales m_θ, $\theta \in R_1$. These equations are of the following sense: their solutions are viewed as estimators of unknown parameter θ, so-called M-estimators. To preserve the classical terminology we shall say that the martingale m_θ defines the M-estimator, and P_θ-martingale $L(m_\theta, M_\theta)$ is the influence martingale.

As it is well known M-estimators play the important role in robust statistics, besides they are sources to obtain asymptotically normal estimators.

Since for each $\theta \in R_1 P_\theta$ is a unique measure such that under this measure $X = (X_t)_{t \geq 0}$ is a semimartingale with characteristics $(B(\theta), c(\theta), \nu_\theta)$ all P_θ-martingales admit an integral representation property w.r.t. continuous martingale part and martingale measure $(\mu - \nu_\theta)$ of X. In particular, the P-martingale M_θ has the form (see Eq. (2.2))

$$M_\theta = \beta(\theta) \circ X^s + \psi * (\mu - \nu), \tag{5.2}$$

where

$$\psi(s, x, \theta) = Y(t, x, \theta) - 1 + \frac{\widehat{Y}(t, \theta) - a}{1 - a} I_{(0<a<1)}$$

and $m_\theta \in \mathcal{M}_{\text{loc}}(P)$ can be represented as

$$m(\theta) = g(\theta) \circ X^c + G(\theta) * (\mu - \nu) \tag{5.3}$$

with certain functions $g(\theta)$ and $G(\theta)$.

It can be easily shown that P_θ-martingale $L(m_\theta, M_\theta)$ can be represented as

$$L(m_\theta, M_\theta) = \varphi_m(\theta) \cdot (X^c - \beta(\theta) \circ C) + \Phi_m(\theta) * (\mu - \nu_\theta), \tag{5.4}$$

where the functions φ_m and Φ_m are expressed in terms of functions $\beta(\theta)$, $\psi(\theta)$, $g(\theta)$ and $G(\theta)$.

On the other hand, it can be easily shown that each P_θ-martingale $\widetilde{M}_\theta$ can be expressed as $L(\widetilde{m}_\theta, M_\theta)$ with P-martingale $\widetilde{m}_\theta$ defined as

$$\widetilde{m}_\theta = L(\widetilde{M}_\theta, L(-M_\theta, M_\theta)) \in \mathcal{M}_{\text{loc}}(P)$$

(since $\frac{dP}{dP_\theta} = \mathcal{E}(L(-M_\theta, M_\theta))$, according to the generalized Girsanov theorem $L(\widetilde{M}_\theta, L(-M_\theta, M_\theta)) \in \mathcal{M}_{\text{loc}}(P)$).

Therefore without loss of generality one can consider the M-estimator associated with the parametric family $(\widetilde{M}_\theta,\ \theta \in R)$ of P_θ-martingale as the solution of the estimational equation

$$\widetilde{M}_t(\theta) = 0. \tag{5.5}$$

In the sequel we assume that for each $\theta \in R_1$, $\widetilde{M}_\theta \in \mathcal{M}^2_{\mathrm{loc}}(P_\theta)$. Assume also that there exists a positive decreasing predictable process $\widetilde{\gamma}_t(\theta)$ with $\widetilde{\gamma}_0(\theta) = 1$ such that $\widetilde{\gamma}_t(\theta)\langle \widetilde{M}_\theta\rangle_t \xrightarrow{P_\theta} 1$ as $t \to \infty$.

Now using the same arguments as in Section 3 we introduce the following recursive procedure for constructing estimator $(\widetilde{\theta}_t,\ t \geq 0)$ asymptotically equivalent to the M-estimator defined by relation (5.5) as the solution of the following SDE

$$d\widetilde{\theta}_t = \widetilde{\gamma}_t(\theta)\widetilde{M}(dt, \widetilde{\theta}_{t-}). \tag{5.6}$$

To obtain the explicit form of the last SDE, recall that $\widetilde{M}_\theta$ has an integral representation property

$$\widetilde{M}_t(\theta) = \widetilde{\varphi}(\theta) \circ (X^c - \beta(\theta) \circ \langle X^c\rangle) + \widetilde{\Phi}(\theta) * (\mu - \nu_\theta).$$

We can obtain the canonical decomposition of P_θ-semimartingale $\widetilde{M}_t(u)$, $u \in R^1$ (w.r.t. measure P_θ)

$$\widetilde{M}(u) = \widetilde{\varphi}(u) \circ (X^c - \beta(\theta) \circ C) + \widetilde{\Phi}(u) * (\mu - \nu_\theta)$$
$$+ [\widetilde{\varphi}(u)(\beta(\theta) - \beta(u))] \circ C + \widetilde{\Phi}(u)\left(1 - \frac{y(u)}{y(\theta)}\right) * (\mu - \nu_\theta).$$

Based on the last expression we can derive the explicit form of SDE (5.5)

$$\begin{aligned}\theta_t = \theta_0 &+ \int_0^t \widetilde{\gamma}_s(\widetilde{\theta}_{s-})\widetilde{\varphi}(s, \theta_{s-})d(X^c - \beta(\theta) \circ C) + \int_0^t \int \widetilde{\gamma}_s(\theta_{s-})\widetilde{\Phi}(s, x, \widetilde{\theta}_{s-})(\mu - \nu_\theta)(ds, dx)\\ &+ \int_0^t \widetilde{\gamma}_s(\theta_{s-})\widetilde{\varphi}(s, \widetilde{\theta}_{s-})(\beta_s(\theta) - \beta_s(\theta_{s-}))dC_s\\ &+ \int_0^t \int \gamma_s(\theta_{s-})\widetilde{\Phi}(s, x, \widetilde{\theta}_{s-})\left(1 - \frac{Y(s, x, \widetilde{\theta}_{s-})}{Y(s, x, \theta)}\right)\nu_\theta(ds, dx).\end{aligned} \tag{5.7}$$

To study the asymptotic properties of the solution of this equation $(\widetilde{\theta}_t,\ t \geq 0)$ (e.g. consistency, rate of convergence, asymptotic normality) is more convenient to rewrite this equation as $(z_t = \widetilde{\theta}_t - \theta)$

$$\begin{aligned}z_t = z_0 &+ \int_0^t \widetilde{\gamma}_s(\theta + z_{s-})\widetilde{\varphi}(s, \theta + z_{s-})d(X^c - \beta(\theta) \circ C)\\ &+ \int_0^t \int \widetilde{\gamma}_s(\theta + z_{s-})\widetilde{\Phi}(s, x, \theta + z_{s-})(\mu - \nu_\theta)(ds, dx)\\ &+ \int_0^t \widetilde{\gamma}_s(\theta + z_{s-})\widetilde{\varphi}(s, \theta + z_{s-})(\beta_s(\theta) - \beta_s(\theta_s + z_{s-}))dC_s\\ &+ \int_0^t \int \widetilde{\gamma}_s(\theta + z_{s-})\widetilde{\Phi}(s, x, \theta + z_{s-})\left(1 - \frac{Y(s, x, \theta + z_{s-})}{Y(s, x, \theta)}\right)\nu_\theta(ds, dx).\end{aligned} \tag{5.8}$$

6. Examples

To make the things more clear let us begin with the simplest case of i.i.d. observations.

Example 1. Let $\{p_\theta,\ \theta \in R_1\}$ be the family of probability measures defined on some measurable space $(X, \mathcal{B})$ such that for each pair θ, θ', $p_\theta \sim p_{\theta'}$.

Put $\Omega = X^\infty$, $\mathcal{F}_n = \mathcal{B}(X^n)$, $\mathcal{F} = \mathcal{B}(X^\infty)$, $P_\theta = p_\theta \times p_\theta \times \cdots$. Then for θ, θ', $P_\theta \overset{\mathrm{loc}}{\sim} P_{\theta'}$. Fix some $\theta_0 \in R_1$ and denote $p = p_{\theta_0}$. Let $dp_\theta/dp = f(x, \theta)$. Then the local density process

$$\rho_n(\theta) = \frac{dP_{n,\theta}}{dP_n} = \prod_{i=1}^n f(X_i, \theta) = \mathcal{E}_n(M_\theta), \tag{6.1}$$

where

$$M(\theta) = \sum_{i=1}^{n} (f(X_i, \theta) - 1)$$

is a P-martingale. Here $(X_n)_{n\geq 1}$ is a coordinate process, $X_n(\omega) = x_n$.

Assume that for all x, $f(x, \theta)$ is continuous differentiable in θ and denote $\frac{\partial}{\partial\theta} f(X, \theta) = \dot{f}(X, \theta)$. Assume also that $\frac{\partial}{\partial\theta} \int f(x, \theta) p(dx) = \int \dot{f}(x, \theta) p(dx)$. Then $\dot{M}_n(\theta) = \sum_{i=1}^{n} \dot{f}(X_i, \theta)$ is a P-martingale.

In these notation the MLE takes the form

$$L_n(\dot{M}(\theta), M_\theta) = \sum_{i=1}^{n} \frac{\dot{f}(X_i, \theta)}{f(X_i, \theta)} = 0.$$

The Fischer information process

$$I_n(\theta) = \langle L(\dot{M}_\theta, M_\theta) \rangle = nI(\theta), \tag{6.2}$$

where $I(\theta) = E_\theta \big(\frac{\dot{f}(\cdot,\theta)}{f(\cdot,\theta)}\big)^2$, assuming that the last integral is finite.

The recursive estimation procedure to obtain the estimator θ_n, asymptotically equivalent to MLE is well known:

$$\theta_n = \theta_{n-1} + \frac{1}{nI(\theta_{n-1})} \frac{\dot{f}(X_n, \theta_{n-1})}{f(X_n, \theta_{n-1})}. \tag{6.3}$$

Let us derive this equation from the general recursive SDE.

For this aim consider the process $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. This process is a semimartingale with the jump measure

$$\mu(\omega, [0, n] \times B) = \sum_{i\leq n} I_{\{X_i \in B\}}$$

and its P_θ-compensator is

$$\nu_\theta(\omega, [0, n] \times B) = \sum_{i\leq n} P_\theta(X_i \in B) = n \int_B f(x, \theta) p(dx).$$

Note that $a_n(\theta) = \nu(\omega, \{n\}; X) = 1$ for all $n \geq 1$ and $\theta \in R_1$.

It is obvious that $\nu_\theta = Y \cdot \nu$, where $Y_\theta(\omega, n, x) \equiv f(x, \theta)$. Besides,

$$\Phi(\theta) = \frac{\dot{Y}(\theta)}{Y(\theta)} + \frac{\dot{a}(\theta)}{1 - a(\theta)} = \frac{\dot{f}(\cdot, \theta)}{f(\cdot, \theta)}.$$

At the same time the general recursive SDE for this special case can be written as

$$\theta_n = \theta_{n-1} + \frac{1}{nI(\theta_{n-1})} \frac{\dot{f}(x_n, \theta_{n-1})}{f(x_n, \theta_{n-1})} - \frac{1}{nI(\theta_{n-1})} \int \frac{\dot{f}(x, u)}{f(x, u)} \frac{f(x, u)}{f(x, \theta)} f(x, \theta)\, d\mu|_{u=\theta_{n-1}}.$$

But $\int \dot{f}(x, u)\, d\mu = 0$ and thus the last term equals zero and we come to Eq. (6.3).

In terms of $z_n = \theta_n - \theta$ Eq. (6.3) takes the form

$$z_n = z_{n-1} + \frac{1}{nI(\theta + z_{n-1})} b(\theta, z_{n-1}) + \frac{1}{nI(\theta + z_{n-1})} \Delta m_n,$$

where

$$b(\theta, u) = \int \frac{\dot{f}(x, u)}{f(x, u)} f(x, \theta)\, d\mu, \qquad \Delta m_n = \Delta m_n(u), \qquad \Delta m_n = \frac{\dot{f}(x, u)}{f(x, u)} - b(\theta, u).$$

Concerning M-estimators recall that by the definition the estimational equation is

$$L_n(m(\theta), M(\theta)) = 0, \tag{6.4}$$

where $m(\theta)$ is some P-martingale, $m_n(\theta) = \sum_{i\leq n} g(X_i, \theta)$ with $\int g(x, \theta)\, dp = 0$.

Eq. (6.4) can be written as

$$\sum_{i \le n} \frac{g(X_i, \theta)}{f(X_i, \theta)} = 0.$$

Thus, without loss of generality, we can define M-estimator as the solution of the equation

$$\widetilde{M}_n(\theta) = \sum_{i \le n} \psi(X_i, \theta) = 0, \tag{6.5}$$

where

$$\int \psi(x_i, \theta) f(x_i, \theta)\, \mu(dx) = 0, \qquad \langle \widetilde{M}(\theta) \rangle_n = n \int \psi^2(x, \theta) f(x, \theta)\, \mu(dx) = n I_\psi(\theta).$$

Now using the same arguments as in the case of MLE we obtain the following recursive procedure for constructing the estimator asymptotically equivalent to the M-estimator defined by (6.5)

$$\theta_n = \theta_{n-1} + \frac{1}{n I_\psi(\theta_{n-1})}\, \psi(X_n, \theta_{n-1}).$$

Example 2. Discrete time case.

Let $X_0, X_1, \dots, X_n, \dots$ be observations taking values in some measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ such that the regular conditional densities of distributions (w.r.t. some measure p) $f_i(x_i, \theta | x_{i-1}, \dots, x_0)$, $i \le n$, $n \ge 1$ exist, $f_0(x_0, \theta) \equiv f_0(x_0)$, $\theta \in R^1$ is the parameter to be estimated. Denote P_θ corresponding distribution on $(\Omega, \mathcal{F}) := (\mathcal{X}^\infty, \mathcal{B}(\mathcal{X}^\infty))$. Identify the process $X = (X_i)_{i \ge 0}$ with coordinate process and denote $\mathcal{F}_0 = \sigma(X_0)$, $\mathcal{F}_n = \sigma\ (X_i,\ i \le n)$. If $\psi = \psi(X_i, X_{i-1}, \dots, X_0)$ is a r.v., then under $E_\theta(\psi | \mathcal{F}_{i-1})$ we mean the following version of conditional expectation

$$E_\theta(\psi \mid \mathcal{F}_{i-1}) := \int \psi(z, X_{i-1}, \dots, X_0) f_i(z, \theta \mid X_{i-1}, \dots, X_0) \mu(dz),$$

if the last integral exists.

Assume that the usual regularity conditions are satisfied and denote

$$\frac{\partial}{\partial \theta} f_i(x_i, \theta \mid x_{i-1}, \dots, x_0) := \dot{f}_i(x_i, \theta \mid x_{i-1}, \dots, x_0),$$

the maximum likelihood scores

$$l_i(\theta) := \frac{\dot{f}_i}{f_i} (X_i, \theta \mid X_{i-1}, \dots, X_0)$$

and the empirical Fisher information

$$I_n(\theta) := \sum_{i=1}^n E_\theta(l_i^2(\theta) \mid \mathcal{F}_{i-1}).$$

Denote also

$$b_n(\theta, u) := E_\theta(l_n(\theta + u) \mid \mathcal{F}_{n-1})$$

and indicate that for each $\theta \in R^1$, $n \ge 1$

$$b_n(\theta, 0) = 0 \quad (P_\theta\text{-a.s.}). \tag{6.6}$$

Using the same arguments as in the case of i.i.d. observations we come to the following recursive procedure

$$\theta_n = \theta_{n-1} + I_n^{-1}(\theta_{n-1}) l_n(\theta_{n-1}), \quad \theta_0 \in \mathcal{F}_0.$$

Fix θ, denote $z_n = \theta_n - \theta$ and rewrite the last equation in the form

$$\begin{aligned} z_n &= z_{n-1} + I_n^{-1}(\theta + z_{n-1}) b_n(\theta, z_{n-1}) + I_n^{-1}(\theta + z_{n-1}) \Delta m_n, \\ z_0 &= \theta - \theta, \end{aligned} \tag{6.7}$$

where $\Delta m_n = \Delta m(n, z_{n-1})$ with $\Delta m(n, u) = l_n(\theta + u) - E_\theta(l_n(\theta + u) | \mathcal{F}_{n-1})$.

Note that the algorithm (6.7) is embedded in SDE (1.1) with

$$H_n(u) = I_n^{-1}(\theta + u) b_n(\theta, u) \in \mathcal{F}_{n-1}, \quad \Delta K_n = 1,$$
$$\Delta M(n, u) = I_n^{-1}(\theta + u) \Delta m(n, u).$$

This example clearly shows the necessity of consideration of random fields $H_n(u)$ and $M(n, u)$.

The discrete time case was considered by T. Sharia in [10,11].

Example 3. Recursive parameter estimation in the trend coefficient of a diffusion process.

Here we consider the problem of recursive estimation of the one-dimensional parameter in the trend coefficient of a diffusion process $\xi = \{\xi_t,\ t \geq 0\}$ with

$$d\xi_t = a(\xi_t, \theta)\, dt + \sigma(\xi_t)\, dw_t, \quad \xi_0, \tag{6.8}$$

where $w = \{w_t,\ t \geq 0\}$ is a standard Wiener process, $a(\cdot, \theta)$ is the known function, $\theta \in \Theta \subseteq R$ is a parameter to be estimated, Θ is some open subset of R, $\sigma^2(\cdot)$ is the known diffusion coefficient.

We assume that there exists a unique weak solution of Eq. (6.8).

For each $\theta \in \Theta$ denote by P^θ the distribution of the process ξ on $(C_{[0,\infty)}, \mathcal{B})$.

Let $X = \{X_t,\ t \geq 0\}$ be the coordinate process, that is, for each $x = \{x_t, t \geq 0\} \in C_{[0,\infty)}$, $X_t(x) = x_t, t \geq 0$.

Fix some $\theta \in \Theta$ and assume that for each $\theta' \in \Theta$, $P^\theta \overset{(loc)}{\sim} P^{\theta'}$. Then the density process $\rho_t(X, \theta)$ can be written as

$$\rho_t(X, \theta) := \frac{dP_t^\theta}{dP_t^{\theta'}}(X) = \exp\left\{ \int_0^t \frac{a(X_s, \theta) - a(X_s, \theta')}{\sigma(X_s)}\, \frac{(dX_s - a(X_s, \theta')ds)}{\sigma(X_s)} \right\}$$
$$- \frac{1}{2} \int_0^t \left(\frac{a(X_s, \theta) - a(X_s, \theta')}{\sigma(X_s)} \right)^2 ds.$$

Recall that if for all $t \geq 0$ P^θ-a.s.

$$\int_0^1 \sigma^2(X_s)\, ds < \infty, \tag{6.9}$$

then the process $\left\{ X_t - \int_0^t a(X_s, \theta)\, ds,\ t \geq 0 \right\} \in M_{\mathrm{loc}}^2(P^\theta)$ with the square characteristic $\int_0^t \sigma^2(X_s)\, ds$.

Under suitable regularity conditions if we assume that for all $t \geq 0$ P^θ-a.s.

$$\int_0^t \left(\frac{\dot{a}(X_s, \theta)}{\sigma(X_s)} \right)^2 ds < \infty, \tag{6.10}$$

we will have

$$\left\{ \frac{\partial}{\partial \theta} \ln \rho_t(X, \theta) = \int_0^t \left(\frac{\dot{a}(X_s, \theta)}{\sigma(X_s)} \right) d(X_s - a(X_s, \theta)ds),\ t \geq 0 \right\} \in M_{\mathrm{loc}}^2(P^\theta),$$

where $\dot{a}(\cdot, \theta)$ denotes the derivative of $a(\cdot, \theta)$ w.r.t. θ.

Below we assume that conditions (6.9) and (6.10) are satisfied.

Introduce the Fisher information process

$$I_t(\theta) = \int_0^t \left(\frac{\dot{a}(X_s, \theta)}{\sigma(X_s)} \right)^2 ds.$$

Then, according to Eq. (3.4), the SDE for constructing the recursive estimator $(\theta_t, t \geq 0)$ has the form

$$d\theta_t = I_t(\theta_t) \left[\frac{\dot{a}(X_t, \theta_t)}{\sigma^2(X_s)}\, dX_t^c + \frac{\dot{a}(X_t, \theta_t)}{\sigma^2(X_t)}\ (a(X_t, \theta) - a(X_t, \theta_t))\, dt \right]. \tag{6.11}$$

Fix some $\theta \in \Theta$. To study the asymptotic properties of the recursive estimator $\{\theta_t, t \geq 0\}$ as $t \to \infty$ under measure P^θ let us denote $z_t = \theta_t - \theta$ and rewrite (6.11) in the following form:

$$dz_t = I_t(\theta + z_t)\left[\frac{\dot{a}(X_t, \theta + z_t)}{\sigma^2(X_s)}\, dX_t^c + \frac{\dot{a}(X_t, \theta + z_t)}{\sigma^2(X_t)}\, (a(X_t, \theta) - a(X_t, \theta + z_t))\, dt\right]. \tag{6.12}$$

In the sequel we assume that there exists a unique strong solution of Eq. (6.12) such that

$$\left\{\int_0^t I_s(\theta + z_s)\,\frac{\dot{a}(X_s, \theta + z_s)}{\sigma^2(X_s)}\, dX_s^c,\ t \geq 0\right\} \in M_{\text{loc}}^2(P_\theta),$$

that is, for each $t \geq 0$ P^θ-a.s.

$$\int_0^t I_s^2(\theta + z_s)\left(\frac{\dot{a}(X_s, \theta + z_s)}{\sigma(X_s)}\right)^2 ds < \infty.$$

To study the asymptotic properties of the process $z = \{z_t,\ t \geq 0\}$ as $t \to \infty$ (under the measure P^θ) one can use the results of Theorems 4.1–4.3 concerning the asymptotic behavior of solutions of the Robbins–Monro type SDE

$$z_t = z_0 + \int_0^t H_s(z_{s-})\, dK_s + \int_0^t M(ds, z_{s-}). \tag{6.13}$$

Note that Eq. (6.13) covers Eq. (6.12) with $K_t = t$,

$$H_t(u) := H_t^\theta(u) = I_t(\theta + u)\,\frac{\dot{a}(X_t, \theta + u)}{\sigma^2(X_t)}\,(a(X_t, \theta) - a(X_t, \theta + u))\,, \quad H_t^\theta(0) = 0, \tag{6.14}$$

$$M(u) := M^\theta(u) = \left\{M^\theta(t, u) = \int_0^t I_s(\theta + u)\,\frac{\dot{a}(X_t, \theta + u)}{\sigma^2(X_t)}\, dX_s^c,\ t \geq 0\right\}. \tag{6.15}$$

Let for each $u \in R$ the process $M^\theta(u) \in M_{\text{loc}}^2(P^\theta)$. Then

$$\langle M^\theta(u), M^\theta(v)\rangle_t = \int_0^t h_s(u, v)\, ds,$$

where

$$h_t(u, v) = h_t^\theta(u, v) = I_t(\theta + u) I_t(\theta + v)\,\frac{\dot{a}(X_t, \theta + u)\dot{a}(X_t, \theta + v)}{\sigma^2(X_t)}. \tag{6.16}$$

This problem is fully studied by Lazrieva and Toronjadze in [14].

Example 4. Let $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, P, P_\theta, \theta \in R_1)$ be filtered probability space and $M = (M_t)_{t \geq 0}$ be a P-martingale with the deterministic characteristic $\langle M\rangle_t$, $\langle M\rangle_\infty = \infty$. Let for each $\theta \in R_1$ P_θ be unique measure on $(\Omega, \mathcal{F})$ such that the process $X(t)$ follows the equation

$$X_t = X_0 + a(\theta)\langle M\rangle_t + M_t,$$

where $a(\theta)$ is known function depending on the unknown parameter θ. Then for each pair (θ, θ'), $P_\theta \overset{\text{loc}}{\sim} P_{\theta'}$. Fix some $\theta_0 \in R_1$. Then the local density process

$$\rho_t(\theta) = \frac{dP_{\theta,t}}{dP_{\theta_0,t}} = \mathcal{E}_t(M(\theta)),$$

where

$$M_t(\theta) = (a(\theta) - a(\theta_0))(X_t - a(\theta_0)\langle M\rangle_t). \tag{6.17}$$

Assume that $a(\theta)$ is strongly monotone function continuously differentiable in θ. Then

$$L_t(\theta) = \frac{\partial}{\partial\theta}\ln \rho_t(\theta) = L_t(\dot{M}(\theta), M(\theta)) = \dot{a}(\theta)(X_t - a(\theta)\langle M\rangle_t)$$

and the Fischer information process is

$$I_t(\theta) = \langle L(\theta), L(\theta)\rangle_t = [\dot{a}(\theta)]^2 \langle M\rangle_t.$$

Put $\gamma_t(\theta) = [\dot{a}(\theta)]^{-2} \frac{1}{\langle M\rangle_t + 1} = [\dot{a}(\theta)]^{-2}\gamma_t^{-1}$ (with the obvious notation $\gamma_t = \langle M\rangle_t + 1$). Therefore the recursive estimation procedure to obtain estimator asymptotically equivalent to the MLE θ_t is

$$\theta_t = \theta_0 + \int_0^t \frac{1}{\langle M\rangle_s + 1} \frac{a(\theta) - a(\theta_s)}{\dot{a}(\theta_s)} d\langle M\rangle_s + \int_0^t \frac{1}{1 + \langle M\rangle_s} \frac{1}{\dot{a}(\theta_s)} d(X_s - a(\theta)\langle M\rangle_s). \tag{6.18}$$

Denote $z_t = \theta_t - \theta$ and rewrite the last equation

$$dz_t = \frac{1}{\langle M\rangle_t + 1} \frac{a(\theta) - a(\theta + z_t)}{\dot{a}(\theta + z_t)} d\langle M\rangle_t + \frac{1}{\langle M\rangle_t + 1} \frac{1}{\dot{a}(\theta + z_t)} d(X_t - a(\theta)\langle M\rangle_t). \tag{6.19}$$

Further, denote

$$H_t(\theta, u) = \frac{1}{\langle M\rangle_t + 1} \frac{a(\theta) - a(\theta + z_t)}{\dot{a}(\theta + z_t)},$$

$$M_t(\theta, u) = \int_0^t \frac{1}{\langle M\rangle_s + 1} \frac{1}{\dot{a}(\theta + u)} d(X_s - a(\theta)\langle M\rangle_t).$$

In these notation Eq. (6.19) is the Robbins–Monro type equation

$$dz_t = H_t(\theta, z_t) d\langle M\rangle_t + dM_t(\theta, z_t). \tag{6.20}$$

Indeed, condition (A) of Theorem 4.1 is satisfied since

$$H_t(\theta, 0) = 0 \quad \text{and} \quad H_t(\theta, u)u < 0 \quad \text{for all } u \neq 0.$$

We study the asymptotic behavior of z_t as $t \to \infty$ under measure P_θ.

(1) Convergence: $z_t \to 0$ as $t \to \infty$ P_θ-a.s. or $\theta_t \to \theta$ as $t \to \infty$ P_θ-a.s. (strong consistency).

Proposition 6.1. *Let the following condition be satisfied*

$$[\dot{a}(\theta + u)]^2 (1 + u^2) \geq c, \tag{6.21}$$

where c is some constant depending on θ. Then

$$z_t \to 0 \quad \text{as } t \to \infty \ P_\theta\text{-a.s.}$$

Proof. Let us check conditions (A), (B), (C) of Theorem 4.1. (A) is evident. Concerning condition (B) note that

$$\langle M(\theta, u)\rangle_t = \frac{1}{(\dot{a}(\theta + u))^2} \int_0^t \frac{1}{(\langle M\rangle_s + 1)^2} d\langle M\rangle_s$$

and

$$h_t(\theta, u) = \frac{1}{(\dot{a}(\theta + u))^2} \frac{1}{(\langle M\rangle_t + 1)^2}.$$

Then if we denote $B_t = \frac{1}{(\langle M\rangle_t + 1)^2}$, taking into account Eq. (6.21) we simply obtain

$$h_t(\theta, u) \leq B_t(1 + u^2) \quad \text{with } B \circ \langle M\rangle_\infty < \infty.$$

As for condition (C), we have to verify that for each $\varepsilon > 0$

$$\inf_{\varepsilon \leq u \leq \frac{1}{\varepsilon}} \left| \frac{a(\theta) - a(\theta + u)}{\dot{a}(\theta + u)} \right| \int_0^\infty \frac{d\langle M\rangle_t}{\langle M\rangle_t + 1} = \infty.$$

The last condition is satisfied if for each $\varepsilon > 0$

$$\inf_{\varepsilon \leq |u| \leq \frac{1}{\varepsilon}} \left| \frac{a(\theta) - a(\theta + u)}{\dot{a}(\theta + u)} \right| > 0,$$

which holds since $\dot{a}(\theta)$ is continuous. $\square$

(2) Rate of convergence. Here we assume that $z_t \to 0$ as $t \to \infty$ P_θ-a.s.

Proposition 6.2. *For all δ, $0 < \delta < \frac{1}{2}$, we have*

$$\gamma_t^\delta z_t = (\langle M\rangle_t + 1)^\delta z_t \to 0 \quad \text{as } t \to \infty, \ P_\theta\text{-a.s.}$$

Proof. We have to check conditions (i) and (ii) of Theorem 4.2.

Condition (ii) is satisfied. Indeed, for all $0 < \delta < 1$

$$\int_0^\infty (\langle M\rangle_t + 1)^\delta [\dot a(\theta+u)]^{-2} \frac{1}{(\langle M\rangle_t + 1)^2}\, d\langle M\rangle_t < \infty.$$

As for condition (i), it is enough to verify that for all δ, $0 < \delta < \frac{1}{2}$,

$$\int_0^\infty \frac{1}{\langle M\rangle_t + 1}\left[\delta - I_{(z_t=0)} - \frac{a(\theta) - a(\theta+z_t)}{z_t\, \dot a(\theta+z_t)}\right]^+ d\langle M\rangle_t < \infty.$$

But $\left[\delta - I_{(z_t=0)} - \frac{a(\theta)-a(\theta+z_t)}{z_t\,\dot a(\theta+z_t)} I_{\{z_t\neq 0\}}\right]^+ = 0$ eventually since $z_t \to 0$. □

(3) Asymptotic expansion. Here we assume that for all δ, $0 < \delta < \frac{1}{2}$, $\gamma_t^\delta z_t \to 0$ as $t \to \infty$ P_θ-a.s.

Proposition 6.3. *Let there exist some $\varepsilon > 0$, $\gamma > 0$ and $c(\theta)$ such that*

$$|\dot a(\theta+u) - \dot a(\theta+v)| \le c|u-v|^\gamma \tag{6.22}$$

for all $(u, v) \in O_\varepsilon(0)$, then all conditions of Theorem 4.3 are satisfied and the following asymptotic expansion holds true

$$(1 + \langle M\rangle_t)^{1/2} \dot a(\theta) z_t = \frac{L_t}{\langle L\rangle_t^{1/2}} + R_t,$$

where $R_t \to 0$ as $t \to \infty$ P-a.s., $L_t = [\dot a(\theta)]^{-1}(X_t - a(\theta)\langle M\rangle_t)$.

Example 5 (*Point Process with Continuous Compensator*)**.** Let Ω be a space of piecewise constant functions $x = (x_t)_{t\ge 0}$ such that $x_0 = 0$, $x_t = x_{t-} + (0 \text{ or } 1)$, $\mathcal{F} = \sigma\{x : x_s, s \ge 0\}$ and $\mathcal{F}_t = \sigma\{x : x_s, 0 < s \le t\}$. Let for $x \in \Omega$

$$\tau_n(x) = \inf\{s : s > 0,\ x_s = n\}$$

setting $\tau_n(\infty) = \infty$ if $\lim_{t\to\infty} x_t < n$. Let $\tau_\infty(x) = \lim_{n\to\infty} \tau_n(x)$.

Note that $x = (x_t)_{t\ge 0}$ can be written as

$$x_t = \sum_{n\ge 1} I_{\{\tau_n(x)\le t\}},$$

and so $(x_t)_{t\ge 0}$ and the family of σ-algebras $(\mathcal{F}_t)_{t\ge 0}$ are right-continuous.

Let for each $\theta \in R_1$ P_θ be a probability measure on $(\Omega, \mathcal{F})$ such that under this measure the coordinate process $X_t(\omega) = x_t$ if $\omega = (x_t)_{t\ge 0}$ is a point process with compensator $A_t(\theta) = A(\theta)A(t)$, where $A(t) = A(t, \omega)$ is an increasing process with continuous trajectories (P_θ-a.s.), $A(0) = 0$, $P_\theta\{A_\infty = \infty\} = 1$, and for each $t > 0$ $P_{\dot\theta}(A_t < \infty) = 1$, $A(\theta)$ is a strongly monotone deterministic function, $A(\theta) > 0$, and $A(\theta)$ is continuously differentiable (denote $\dot A(\theta) = \frac{d}{d\theta} A(\theta)$).

Assume that for each pair (θ, θ'), $P_\theta \overset{loc}{\sim} P_{\theta'}$. Fix as usual some $\theta_0 \in R_1$. Then the local density process $\rho_t(\theta) = \frac{dP_{\theta,t}}{dP_{\theta_0,t}}$ can be represented as

$$\rho_t(\theta) = \mathcal{E}_t(M(\theta)),$$

where

$$M_t(\theta) = \left(\frac{A(\theta)}{A(\theta_0)} - 1\right)(X_t - A(\theta_0)A_t).$$

Therefore $L_t(\theta) = \frac{\partial}{\partial\theta} \ln \rho_t(\theta)$ has the form

$$L_t(\theta) = L_t(\dot{M}(\theta), M(\theta)) = \frac{\dot{A}(\theta)}{A(\theta)} (X_t - A(\theta)A(t)).$$

The Fisher information process is

$$I_t(\theta) = \langle L(\dot{M}(\theta), M(\theta))\rangle_t = \left[\frac{\dot{A}(\theta)}{A(\theta)}\right]^2 A(\theta)A(t).$$

Put $\gamma_t(\theta) = \frac{A(\theta)}{[\dot{A}(\theta)]^2} \frac{1}{A(t)+1}$. It is evident that

$$\lim_{t\to\infty} \gamma_t(\theta)I_t(\theta) = 1.$$

Note that the process $(X_t)_{t\geq 0}$ is a P_θ-semimartingale with the triplet of characteristics $(A(\theta)A(t), 0, A(\theta)A(t))$. Therefore, according to Section 3,

$$F(\theta) = F(\omega, t, x, \theta) = \frac{A(\theta)}{A(\theta_0)}, \quad \Phi(\theta) = \frac{\dot{A}(\theta)}{A(\theta)},$$
$$\ell^c(\theta) = \ell^\delta(\theta) = \ell^b(\theta) = 0, \quad \ell^\pi(\theta) = \frac{\dot{A}(\theta)}{A(\theta)}.$$

Thus from (3.10) we obtain

$$H_t^\theta(u) = \frac{1}{A(t)+1} \frac{A(\theta) - A(\theta+u)}{\dot{A}(\theta+u)},$$
$$M^\theta(t, u) = \frac{1}{\dot{A}(\theta+u)} \int_0^t \frac{1}{A(s)+1} d(X_s - A(\theta)A(s)),$$

and the equation for $z_t = \theta_t - \theta$ is

$$dz_t = \frac{1}{A(t)+1} \frac{A(\theta) - A(\theta+z_t)}{\dot{A}(\theta+z_t)} dA(t) + \frac{1}{A(t)+1} \frac{1}{\dot{A}(\theta+z_t)} d(X_t - A(\theta)A(t)), \tag{6.23}$$

where $(\theta_t)_{t\geq 0}$ is recursive estimation satisfying the equation

$$d\theta_t = \frac{1}{A(t)+1} \frac{A(\theta) - A(\theta_t)}{\dot{A}(\theta_t)} dA(t) + \frac{1}{A(t)+1} \frac{1}{\dot{A}(\theta_t)} d(X_t - A(\theta)A(t)).$$

As one can see Eq. (6.23) is quite similar to (6.19) with $A(\theta)$ instead of $a(\theta)$ and $A(t)$ instead of $\langle M\rangle_t$.

Now if conditions (6.21) and (6.22) with $A(\theta)$ instead of $a(\theta)$ and $A(t)$ instead of $\langle M\rangle_t$ are satisfied, then the asymptotic expansion holds true

$$(A(t)+1)^{1/2}\dot{A}(\theta)z_t = \frac{L_t}{\langle L\rangle_t^{1/2}} + R_t,$$

where $R_t \to 0$ as $t \to \infty$ P_θ-a.s., $L_t = [\dot{A}(\theta)]^{-1}(X_t - A(\theta)A(t))$.

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Original article

Stochastic differential equations in a Banach space driven by the cylindrical Wiener process

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Abstract

Generalized stochastic integral from predictable operator-valued random process with respect to a cylindrical Wiener process in an arbitrary Banach space is defined. The question of existence of the stochastic integral in a Banach space is reduced to the problem of decomposability of the generalized random element. The sufficient condition of existence of the stochastic integral in terms of p-absolutely summing operators is given. The stochastic differential equation for generalized random processes is considered and existence and uniqueness of the solution is developed. As a consequence, the corresponding results of the stochastic differential equations in an arbitrary Banach space are given.

Keywords: Ito stochastic integrals and stochastic differential equations; Wiener processes; Covariance operators in Banach spaces

1. Introduction

First results on the infinite dimensional stochastic differential equations started to appear in the mid 1960s. The traditional finite dimensional methods gave desired results for Hilbert space case (see [1,2]), but they turned out deadlock in the general Banach space case. Then, researchers began to develop the problem in such Banach spaces, the geometry of which is close to the geometry of Hilbert space (see for example [3,4]). Important results are received in the case, when the Banach space has UMD property (see [5–7]). But the class of UMD Banach spaces is very narrow—they are reflexive Banach spaces. Stochastic analysis in UMD spaces intensively developed after the end of the eighties of the lust century, but the class of Banach spaces, where the traditional methods give desired results, has not yet extended. Numerous works are dedicated to this problem (see [8–10,6]). Therefore, it is greatly interesting to develop the stochastic differential equations in an arbitrary Banach space.

The first step to investigate this direction is to construct the Ito stochastic integral in an arbitrary separable Banach space. Stochastic integral for Banach space valued non random function by one dimensional Wiener process (the Wiener integral) is constructed in [11]. Stochastic integral from operator-valued non-random process by the Banach

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space valued Wiener process is considered in [12]. In [13] is constructed the stochastic integral from operator-valued (from Hilbert space to Banach space) non random function by the cylindrical Wiener process. There are also considered the traditional conditions of the existence of the stochastic integral with relation to the geometry of Banach space. The Ito stochastic integral in 2-uniformly smooth Banach spaces is considered in [3,14–16]. In [17] is shown, that the property of definition of 2-uniformly smooth Banach space is equivalent to the martingale type 2 property. Stochastic integral in UMD Banach spaces is constructed in [18,19,7]. In [20] is considered linear stochastic evolution equations on some special Banach spaces. We define the generalized stochastic integral in an arbitrary Separable Banach space for a wide class of non-anticipating operator-valued random processes by the cylindrical Wiener process, which is a generalized random element (a random linear function or a cylindrical random element), and if there exists the corresponding random element, that is, if this generalized random element is decomposable by the Banach space valued random element, then we say that this random element is the stochastic integral. Thus, the problem of existence of the stochastic integral in an arbitrary separable Banach space is reduced to the well known problem of decomposability of the generalized random element. We give the sufficient condition of existence of the stochastic integral using the L. Schwartz's and S. Kwapien's result in terms of p-absolutely summing operator (see [21,22]).

The second main problem to develop the stochastic differential equations in a Banach space is to estimate the stochastic integral, which is necessary for the iteration procedure to prove the existence and uniqueness of the solution. Such estimation is yet impossible in an arbitrary Banach space case. We consider the Banach space of generalized random elements and introduce there the stochastic differential equation for the generalized random process. For this situation, it is possible to use traditional methods to develop the problem of existence and uniqueness of the solution as a generalized random process. Afterward, from the main stochastic differential equation in an arbitrary Banach space we produce the equation for a generalized random process. As we have proved the existence and uniqueness of the solution of this equation, we receive the generalized random process as a solution of the produced stochastic differential equation. If this generalized random process is decomposable, then the corresponding Banach space valued random process will be the solution of the main stochastic differential equation in a Banach space. Therefore, we have also reduced the problem of existence of the solution of the stochastic differential equation in an arbitrary Banach space to the problem of decomposability of the generalized random element.

The investigation of the stochastic differential equations in a Banach space takes place in three directions. They can be described by means of the corresponding stochastic integrals in the equation. In the first (relatively) direction, the integrand non-anticipating process takes its values in a Banach space and the stochastic integral is taken by the scalar Wiener process. We considered this case in the paper [23]. In the second direction the integrand non-anticipating process is operator-valued (from Banach space to Banach space) and the stochastic integral is taken by the Wiener process in a Banach space. This case we investigated in the papers [24–26]. In the third direction the integrand is an operator-valued non-anticipating random function from Hilbert space to Banach space while the stochastic integral is taken by the cylindrical Wiener process in a Hilbert space. This article is devoted to this direction.

Now we give some definitions and preliminary results to realize our approach.

Let X be a real separable Banach space. X^*—its conjugate, $\mathcal{B}(X)$—the Borel σ-algebra of X, $(\Omega, \mathcal{B}, P)$—a probability space. The continuous linear operator $L : X^* \to L_2(\Omega, \mathcal{B}, P)$ is called a generalized random element (GRE). (Sometimes the terms: linear random function or cylindrical random element are used). We consider such GRE, which maps X^* to a fixed closed separable subspace $G \subset L_2(\Omega, \mathcal{B}, P)$. Denote $M_1 := L(X^*, G)$—the Banach space of GRE with the norm $\|L\| = \sup_{\|x^*\|\leq 1} \|Lx^*\|_{L_2}$. A random element (measurable map) $\xi : \Omega \to X$ is said to have a weak second order, if, for all $x^* \in X^*$, $E\langle \xi, x^*\rangle^2 < \infty$. ξ we can realize as an element of $M_1 : L_\xi x^* = \langle \xi, x^*\rangle$. But in infinite dimensional spaces not every GRE may be represented by the Banach space valued random element. The problem of finding the conditions under which the GRE is represented by the Banach space valued random element is well known, otherwise also called the problem of decomposability of the GRE. This is the reason why we allot the superiority to the term GRE; GRE is a generalization of the random element in the infinite dimensional spaces. In the finite dimensional spaces every GRE is decomposable, thus, it is a random element. This term was early used by many authors (see for example [27,28,2,22] p. 140). Likewise, the problem of decomposability of the GRE is equal to the problem of extension of the finite additive (cylindrical) measure to the σ-additive measure. This is a reason why the term "cylindrical random element" appears.

Denote by M_2 the linear space of all random elements of the weak second order with the norm $\|\xi\| = \|L_\xi\|$. Therefore, we can assume $M_2 \subseteq M_1$.

Let $L \in M_1$. Consider the map $m_L : X^* \to R^1$, $m_L x^* = ELx^*$. m_L is linear and bounded, therefore $m_L \in X^{**}$, which is called the mean of the GRE L. When $L \in M_2$, that is, if there exists $\xi : \Omega \to X$ such that $Lx^* = \langle \xi, x^* \rangle$, then $m \in X$ (see [22] Th.2.3.1), and it is the Pettis integral of ξ. Further we consider the GRE with the mean 0.

The covariance operator of $L \in M_1$ is a symmetric and positive operator $R_L : X^* \to X^{**}$, $\langle R_L x^*, y^* \rangle = ELx^*Ly^*$ for all x^* and y^* from X^*. $R_L = L^*L$. It is known that if $L = L_\xi \in M_2$, then R_L maps X^* to X (see [22, Th.3.2.1]), and if R is a positive and symmetric linear operator from X^* to X, then there exist $(x_k^*)_{k\in N} \subset X^*$ and $(x_k)_{k\in N} \subset X$ such that $\langle Rx_k^*, x_j^* \rangle = \delta_{kj}$, $Rx_k^* = x_k$, and for $x^* \in X^*$, $Rx^* = \sum_{k=1}^{\infty} \langle x_k, x^* \rangle x_k$ (see [22, Lemma 3.1.1]). In general, for a positive and symmetric linear operator $R_L : X^* \to X^{**}$ (as G is a separable subspace of $L_2(\Omega, \mathcal{B}, P)$), there exist $(x_k^*)_{k\in N} \subset X^*$ and $(x_k^{**})_{k\in N} \subset X^{**}$ such that $\langle Rx_k^*, x_j^* \rangle = \delta_{kj}$, $Rx_k^* = x_k^{**}$, and for $x^* \in X^*$, $Rx^* = \sum_{k=1}^{\infty} \langle x_k^{**}, x^* \rangle x_k^{**}$.

Proposition 1. *Let T be a GRE. There exist $(x_k^*)_{k\in N} \subset X^*$ and $(x_k^{**})_{k\in N} \subset X^{**}$ such that for all $x^* \in X^*$, $Tx^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**} \rangle Tx_k^*$, $ETx_k^*Tx_j^* = \langle R_T x_k^*, x_j^* \rangle = \delta_{kj}$, $R_T x_k^* = x_k^{**}$, $R_T x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**} \rangle x_k^{**}$. Therefore, if T is a Gaussian, then Tx_k^*, $k = 1, 2, \ldots$ are independent, standard Gaussian random variables.*

Proof. Consider the covariance operator of the GRE T, $R_T : X^* \to X^{**}$, $R_T = T^*T$. Let $(x_k^*)_{k\in N} \subset X^*$ and $(x_k^{**})_{k\in N} \subset X^{**}$ be such that $\langle R_T x_k^*, x_j^* \rangle = \delta_{kj}$, $R_T x_k^* = x_k^{**}$, $R_T x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**} \rangle x_k^{**}$, for all $x^* \in X^*$. If we take up $T_n x^* = \sum_{k=1}^{n} \langle x^*, x_k^{**} \rangle Tx_k^*$, then $E(Tx^* - T_n x^*)^2 = E(Tx^*)^2 - 2ETx^*T_n x^* + E(T_n x^*)^2 = \sum_{k=1}^{\infty} \langle x_k^{**}, x^* \rangle^2 - 2\sum_{k=1}^{n} \langle x_k^{**}, x^* \rangle^2 + \sum_{k=1}^{n} \langle x_k^{**}, x^* \rangle^2 = \sum_{k=n+1}^{\infty} \langle x_k^{**}, x^* \rangle^2 \to 0$.

Therefore $Tx^* = \sum_{k=1}^{\infty} \langle x^*, x_k \rangle Tx_k^*$.

If T is a Gaussian GRE, then Tx_k and Tx_m are independent for all $k \neq m$ as $ETx_k^*Tx_m^* = \langle R_T x_k^*, x_m^* \rangle = \delta_{k,m} = 0$. □

A family of GRE $(L_t)_{t\in[0,1]}$ is called a generalized random process (GRP). A weak second order Banach space valued random process $(\xi_t)_{t\in[0,1]}$ can be represented as a GRP: $L_{\xi_t} x^* = \langle \xi_t, x^* \rangle$. The GRP is called Gaussian, if for all $t_1, t_2, \ldots, t_n$ and $x_1^*, x_2^* \ldots, x_n$, the n-dimensional vector $(L_{t_1} x_1^*, L_{t_2} x_2^*, \ldots, L_{t_n} x_n^*)$ is a Gaussian vector in R^n.

Definition 1. The Gaussian generalized random process $(W_H(t))_{t\in[0,1]}$ in a separable Hilbert space H is called a cylindrical Wiener process, if for all h and g from H, and t, s, from $[0, 1]$, $EW_H(t)hW_H(s)g = \min(t, s)\langle h, g \rangle$.

Proposition 2. *Let $(W_H(t))_{t\in[0,1]}$ be a cylindrical Wiener process in H. For any orthonormal basis $(e_k)_{k\in N}$ in H, there exists the sequence of independent, standard, real valued Wiener processes $w_k(t)$ such that $W_H(t)h = \sum_{k=1}^{\infty} \langle e_k, h \rangle w_k(t)$.*

Proof. For any orthonormal basis $(e_k)_{k\in N}$ the random processes $W_H(t)e_k$, $k = 1, 2, \ldots$ are standard, one dimensional, independent Wiener processes in H. Therefore, $W_H(t)h = W_H(t) \sum_{k=1}^{\infty} \langle h, e_k \rangle e_k = \sum_{k=1}^{\infty} \langle h, e_k \rangle W_H(t)e_k = \sum_{k=1}^{\infty} \langle e_k, h \rangle w_k(t)$, where $w_k(t) \equiv W_H(t)e_k$, $k = 1, 2, \ldots$. □

Definition 2. The Gaussian GRP $(T_t)_{t\in[0,1]}$ is called a generalized Wiener process in a Banach space X, if, for all $x^* \in X^*$, $T_t x^*$ is one dimensional Wiener process and for all t, s from $[0, 1]$ and $y^* \in X^*$, $ET_t x^* T_s y^* = \min(t, s)\langle Rx^* y^* \rangle$, where $R : X^* \to X^{**}$ is the covariance operator of the GRE T_1.

Let R be the covariance operator of the GRE T_1, $R : X^* \to X^{**}$, by the factorization lemma (see [22, Lemma 3.1.1]) we have $R = A^*A$, where $A : H \to X^{**}$, H is a real separable Hilbert space.

Proposition 3. *Let $(T_t)_{t\in[0,1]}$ be a generalized Wiener process and R be the covariance operator of T_1, $R = AA^*$. $A : H \to X^{**}$. There exists the cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$, in H such that $T_t = AW_H(t) = \sum_{k=1}^{\infty} Ae_k w_k(t)$, where $(e_k)_{k\in N}$ is an orthonormal basis in H and $w_k(t)$, $k = 1, 2, \ldots$ is a sequence of one dimensional independent Wiener processes. Therefore every generalized Wiener process in X is the "image" of the cylindrical Wiener process in a separable Hilbert space H.*

Proof. Let $R = AA^*$ be the covariance operator of the GRE T_1. We have $(x_k^*)_{k\in N} \subset X^*$ and $(x_k^{**})_{k\in N} \subset X^{**}$ such that, $\langle Rx_k^*, x_j^* \rangle = \delta_{kj}$, $Rx_k^* = x_k^{**}$ and for $x^* \in X^*$, $Rx^* = \sum_{k=1}^{\infty} \langle x_k^{**}, x^* \rangle x_k^{**}$. By the definition of the generalized Wiener process, $T_t x_k^*$, $k = 1, 2, \ldots$ are one dimensional Wiener processes, and for all t, s from $[0, 1]$ and x_j^*, $ET_t x_k^* T_s x_j^* = \min(t, s)\langle Rx_k^* x_j^* \rangle = \delta_{k,j}$. Therefore $T_t x_k^* := w_k(t)$, $k = 1, \ldots$ is a sequence of one dimensional independent Wiener processes.

Denote $T_n(t) = \sum_{k=1}^n Ae_k w_k(t) = \sum_{k=1}^n x_k^{**} T_t x_k^*$. Then, for any $x^* \in X^*$,

$$\begin{aligned} E(T_t x^* - T_n(t)x^*)^2 &= E(T_t x^*)^2 - 2ET_t x^* T_n(t)x^* + E(T_n(t)x^*)^2 \\ &= t\langle Rx^*, x^*\rangle - 2t\sum_{k=1}^n \langle Rx_k^*, x^*\rangle + t\sum_{k=1}^n \langle Rx_k^*, x_k^*\rangle \\ &= t\sum_{k=1}^\infty \langle x_k^{**}, x^*\rangle^2 - 2\sum_{k=1}^n \langle x_k^{**}, x^*\rangle^2 + \sum_{k=1}^n \langle x_k^{**}, x^*\rangle^2 = \sum_{k=n+1}^\infty \langle x_k^{**}, x^*\rangle^2 \to 0. \end{aligned}$$

That is, $T_t x^* = \lim T_n(t)x^* = \sum_{k=1}^\infty \langle Ae_k, x^*\rangle w_k(t) = \lim\langle A(\sum_{k=1}^n e_k w_k(t)), x^*\rangle = \langle AW_H(t), x^*\rangle$. □

Remark 1. In [29] we have analyzed the definition of the Wiener processes in a Banach space, where we have used the term "canonical generalized Wiener Process" instead of the term "cylindrical Wiener process". The term "cylindrical random element" appeared in relation to the cylindrical measures in vector spaces, as cylindrical random element (generalized random element) induces the finitely additive measure in a Banach space, which is naturally defined in the cylindrical algebra. We mentioned above the reason why we use the term GRE. In our opinion this term better responds to the purpose of the definition than the term "cylindrical random element". As the term "cylindrical Wiener process" is widely applied in literature, we also use this term here and intend to continue discussions on the terminology.

Remark 2. If $H = R^n$ and $(W_H(t))_{t\in[0,1]}$ is n-dimensional standard Wiener process $W_H(t) = (W_H(t)e_1, \ldots, W_H(t)e_n) = (w_1(t), w_2(t), \ldots, w_n(t))$, then, for all linear operators $A : R^n \to R^n$, $(AW_H(t))_{t\in[0,1]}$ is a Wiener process in R^n with covariance operator $R = AA^*$. For infinite dimensional H and bounded linear operator $A : H \to H$, $(AW_H(t))_{t\in[0,1]}$, $AW_H(t) = \sum_{k=1}^\infty Ae_k w_k(t)$ is a Hilbert space valued Wiener process with the covariance operator $R = AA^*$, if, and only if, A is a Hilbert–Schmidt operator. The generalized Wiener process in X, $(W_t)_{t\in[0,1]} \equiv (AW_H(t))_{t\in[0,1]}$, $A : H \to X$, is X-valued Wiener process, if, and only if, $R = AA^*$ is a Gaussian covariance. The sum $W_t = \sum_{k=1}^\infty Ae_k w_k(t)$ converges a.s. uniformly for t in X (see [30,31,25]).

Remark 3. Wiener process in a Banach space was first considered by L. Gross [32]. He introduced for it a special term—the measurable pseudonorm. The definition of the Wiener process introduced by L. Gross is unnatural in comparison with the definition of the finite dimensional Wiener process. The definition of the covariance operator of the Banach space valued random elements (see [33,22]) allows to consider Wiener process in a Banach space analogous to the finite dimensional case.

2. Stochastic integrals

2.1. Stochastic integral of the Hilbert space valued random function by the cylindrical Wiener process

Let $(W_H(t))_{t\in[0,1]}$ be a cylindrical Wiener process in H, $(F_t)_{t\in[0,1]}$—be the increasing family of σ-algebras such that (a) for all $h \in H$, $W_H(t)h$ is F_t-measurable for all $t \in [0, 1]$; (b) $W_H(s)h - W_H(t)h$ is independent to the σ-algebra F_t for all $s > t$. F_t contains all P-null sets from $\mathcal{B}$. We say that $(W_H(t))_{t\in[0,1]}$ is adapted to the family $(F)_{t\in[0,1]}$. Consider the non-anticipating function $\varphi : [0, 1] \times \Omega \to H$, that is, φ is $B([0, 1]) \times \mathcal{B}(\Omega)$-measurable and $\varphi(t)$ is F_t-measurable for all $t \in [0, 1]$.

We define the stochastic integral for a non-anticipating function $\varphi : [0, 1] \times \Omega \to H$, $\int_0^1 \int_\Omega \|\varphi\|^2 dt dP < \infty$ by the cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$.

If $\varphi(t, \omega)$ is a step function, $\varphi(t, \omega) = \sum_{k=0}^{n-1} \varphi(t_k)\chi_{[t_k,t_{k+1})}$, $0 = t_0 < t_1 < \cdots < t_n = 1$, $\varphi_{t_k} : \Omega \to H$, $k = 0, 1, \ldots, (n-1)$, then the stochastic integral of φ by the $(W_H(t))_{t\in[0,1]}$ is defined by the equality $\int_0^1 \varphi(t)dW_H(t) = \sum_{k=0}^{n-1} \langle W_H(t_{k+1}) - W_H(t_k), \varphi(t_k)\rangle$.

Let $(h_i)_{i\in N}$ be any orthonormal basis in H, then $W_H(t)h_i \equiv w_i(t)$ are independent F_t-adapted standard real valued Wiener processes and $\int_0^1 \varphi(t)dW_H(t) = \sum_{k=0}^{n-1}\sum_{i=1}^\infty \langle h_i, \varphi(t_k)\rangle(w_i(t_{k+1}) - w_i(t_k))$.

We have $E(\int_0^1 \varphi(t)dW_H(t))^2 = \sum_{k=0}^{n-1}(t_{k+1} - t_k)\sum_{i=1}^\infty E\langle\varphi_{t_k}, h_i\rangle^2 = \sum_{k=0}^{n-1} E\|\varphi(t_k)\|^2(t_{k+1} - t_k) = \int_0^1 \int_\Omega \|\varphi(t, \omega)\|^2 dt dP$.

The following lemma will be used to define the stochastic integral of non-anticipating function from $L_2([0,1] \times \Omega, H)$.

Lemma 1. *For any non-anticipating function $\varphi(t,\omega) \in L_2([0,1] \times \Omega, H)$ there exists a sequence of non-anticipating step functions $\varphi_n(t,\omega) \in L_2([0,1] \times \Omega, H)$ such that $\varphi_n \to \varphi$ in $L_2([0,1] \times \Omega, H)$.*

Proof. Define $\phi_n(t,\omega) = \sum_{k=1}^{n} \langle \varphi(t,\omega), h_k \rangle h_k$. We have

$$\int_0^1 E\|\phi_n - \varphi\|^2 dt = \int_0^1 E\left\|\sum_{k=1}^{n} \langle \varphi(t,\omega), h_k\rangle h_k - \sum_{k=1}^{\infty} \langle \varphi(t,\omega), h_k\rangle h_k\right\|^2 dt$$

$$= \int_0^1 E\left\|\sum_{k=n+1}^{\infty} \langle \varphi(t,\omega), h_k\rangle h_k\right\|^2 dt = \int_0^1 E\sum_{k=n+1}^{\infty} \langle \varphi(t,\omega), h_k\rangle^2 dt \to 0.$$

For a fixed $k \in N$, let $(\varphi_{km})_{m\in N}$ be a sequence of real valued non-anticipating step functions such that $\varphi_{km} \to \langle \varphi, h_k\rangle$ in $L_2([0,1]\times\Omega)$, when $m \to \infty$. Let $\phi_{nm} = \sum_{k=1}^{n} \varphi_{km} h_k$. Then $\|\phi_{nm} - \phi_n\|_{L_2}^2 = \sum_{k=1}^{n} \int_0^1 \int_\Omega (\varphi_{km} - \langle \varphi, h_k\rangle)^2 dt dP \to 0$. Therefore we can choose a subsequence $(\varphi_n)_{n\in N}$ of $((\phi)_{nm})_{n,m\in N}$ converging to φ in $L_2([0,1] \times \Omega, H)$. Lemma 1 is proved. □

Let $\varphi(t,\omega) \in L_2([0,1] \times \Omega, H)$ be a non-anticipating function. By Lemma 1, there exists the sequence of step functions $(\varphi_n)_{n\in N}$ converging to φ in $L_2([0,1] \times \Omega, H)$. Then as $E(\int_0^1 \varphi_n(t) dW_H(t) - \varphi_m(t) dW_H(t))^2 = \int_0^1 E\|\varphi_n - \varphi_m\|^2 dt \to 0, n, m \to \infty$, we can define the stochastic integral for an arbitrary non-anticipating function $\varphi(t,\omega) \in L_2([0,1] \times \Omega, H)$.

Definition 3. Let $\varphi(t,\omega) \in L_2([0,1] \times \Omega, H)$ be a non-anticipating function. The limit of the sequence of the random variables $\int_0^1 \varphi_n(t) dW_H(t)$ in $L_2(\Omega)$ is called the stochastic integral of φ by the cylindrical Wiener process in H, and is denoted by $\int_0^1 \varphi(t) dW_H(t)$.

We can naturally define the stochastic integral $\int_0^t \varphi(s) dW_H(s)$ for all $t \in [0,1]$. It is easy to see that $\int_0^t \varphi(s) dW_H(s) = \sum_{k=1}^{\infty} \int_0^t \langle \varphi(s), e_k\rangle dw_k(s)$, where $(e_k)_{k\in N}$ is an arbitrary orthonormal basis in H and $(w_k(t) = W_H(t)e_k)_{t\in[0,1]}, k = 1, 2\ldots$ are independent one-dimensional standard Wiener processes.

2.2. Stochastic integral of operator valued random process by the cylindrical Wiener process

Let $(F)_{t\in[0,1]}$ be a filtration, $(\Omega, \mathcal{B}, P)$, $(W_H(t))_{t\in[0,1]}$ be the cylindrical Wiener process in H adapted to $(F_t)_{t\in[0,1]}$, X be a real separable Banach space. Consider the Banach space of linear bounded operators $L(H, X)(L(X^*, H))$ from H to X (from X^* to H).

Definition 4. A function $\varphi(t,\omega) : [0,1] \times \Omega \to L(H, X)$ is called non-anticipating with respect to $(F)_{t\in[0,1]}$, if

1. For all $h \in H$ the function $(t \times \omega) \to \varphi(t,\omega)h$ is measurable;
2. For all $h \in H$, $t \in [0,1]$ the function $\omega \to \varphi(t,\omega)h$ is F_t-measurable.

Definition 5. We say that a non-anticipating function $\varphi(t,\omega) : [0,1] \times \Omega \to L(H, X)$ belongs to the class $G(L(H, X))$ if

$$\sup_{\|x^*\|\le 1} \int_0^1 \int_\Omega \|\varphi^*(t,\omega)x^*\|^2 dt dP < \infty,$$

where $\varphi^*(t,\omega)$ is the conjugate of the operator $\varphi(t,\omega)$. We can define the norm in the linear space $G(L(H, X))$: $\|\varphi\|_G^2 \equiv \sup_{\|x^*\|\le 1} \int_0^1 \int_\Omega \|\varphi^*(t,\omega)x^*\|^2 dt dP$.

Let $\varphi \in G(L(H, X))$ and take any $x^* \in X^*$. $\varphi^* x^*$ maps $[0,1] \times \Omega$ into H, $\int_0^1 \int_\Omega \|\varphi^* x^*\|^2 dt dP < \infty$ and it is non-anticipating. Therefore, we can define the stochastic integral $\int_0^1 \varphi^*(t,\omega)x^* dW_H(t)$ which is a real random variable with variance $\int_0^1 \int_\Omega \|\varphi^*(t,\omega)x^*\|^2 dt dP$. Consider the map $T_\varphi : X^* \to L_2(\Omega, \mathcal{B}, P)$, $T_\varphi x^* = \int_0^1 \varphi^*(t,\omega)x^* dW_H(t)$. T_φ is a GRE.

Definition 6. Let $\varphi \in G(L(H, X))$. The generalized random element $T_\varphi : X^* \to L_2(\Omega, \mathcal{B}, P)$, $T_\varphi x^* = \int_0^1 \varphi^*(t, \omega) x^* dW_H(t)$ is called the generalized stochastic integral of the operator-valued random function φ with respect to the cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$.

Accordingly, we define the generalized stochastic integral $T_\varphi(t)x^* = \int_0^t \varphi^*(s, \omega)x^* dW_H(s)$, for all $t \in [0, 1]$.

We have $\int_0^t \varphi^*(s, \omega)x^* dW_H(s) = \sum_{k=1}^{\infty} \int_o^t \langle \varphi^*(s, \omega)x^*, e_k\rangle dw_k(t)$, where $w_k(t) = \langle W_H(t), e_k\rangle$, $k = 1, 2, \ldots$ are one dimensional independent standard Wiener processes.

For any $\varphi \in G(L(H, X))$ the generalized stochastic integral as a GRE exists.

Let $\varphi \in G(L(H, X))$, $T_\varphi : X^* \to L_2(\Omega, \mathcal{B}, P)$ be a generalized stochastic integral of φ. Denote by $L_\varphi : X^* \to X^{**}$ the covariance operator of the GRE T_φ. It is easy to see that $L_\varphi = T_\varphi^* T_\varphi$.

Theorem 1. *The covariance operator of the generalized stochastic integral of an operator-valued random function $\varphi \in G(L(H, X))$ with respect to the cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$ has the form $L_\varphi x^* = \int_0^1 \int_\Omega \varphi\varphi^* x^* dt dP$ and maps X^* to X (the double integral is meant in the sense of Pettis).*

Proof. Let us find the value of the operator L_φ on $x^* \in X^*$. For any $x_1^* \in X^*$, we have

$$\begin{aligned}\langle L_\varphi x^*, x_1^*\rangle &= ET_\varphi x^* T_\varphi x_1^* = E\int_0^1 \varphi(t, \omega)^* x^* dW_H(t) \int_0^1 \varphi(t, \omega)^* x_1^* dW_H(t) \\ &= \int_0^1 \int_\Omega \langle \varphi^*(t, \omega)x^*, \varphi^*(t, \omega)x_1^*\rangle_H dt dP = \int_0^1 \int_\Omega \langle \varphi\varphi^* x^*, x_1^*\rangle dt dP.\end{aligned}$$

Therefore the Pettis integral $\int_0^1 \int_\Omega \varphi\varphi^* x^* dt dP$ as an element of X^{**} exists for all $x^* \in X^*$.

Let $(h_k)_{k\in N}$ be an orthonormal basis in H. Then

$$\begin{aligned}L_\varphi x^* &= \int_0^1 \int_\Omega \varphi(t, \omega)\varphi^*(t, \omega)x^* dt dP = \int_0^1 \int_\Omega \varphi(t, \omega) \left(\sum_{k=1}^{\infty} \langle \varphi^*(t, \omega)x^*, h_k\rangle h_k\right) dt dP \\ &= \int_0^1 \int_\Omega \sum_{k=1}^{\infty} \langle \varphi(t, \omega)h_k, x^*\rangle \varphi(t, \omega)h_k dt dP.\end{aligned}$$

Denote $L_\varphi^{(n)} = \int_0^1 \int_\Omega \sum_{k=1}^{n} \langle \varphi(t, \omega)h_k, x^*\rangle \varphi(t, \omega)h_k dt dP$. Consider the random element $\varphi h_k : [0, 1] \times \Omega \to X$, $k = 1, 2, \ldots$. As φh_k is a random element of the weak second order, its covariance operator maps X^* to X and equals $L_k x^* = \int_0^1 \int_\Omega \langle \varphi(t, \omega)h_k, x^*\rangle \varphi(t, \omega)h_k dt dP$.

Therefore, for all n and x^*, $L_\varphi^{(n)} x^*$ belongs to X. As X is a closed subspace of X^{**}, it is enough to prove the convergence of the sequence $L_\varphi^{(n)} x^*$, $n = 1, 2, \ldots$, to the $L_\varphi x^*$ in X^{**} for all $x^* \in X^*$. We have

$$\begin{aligned}\|L_\varphi x^* - L_\varphi^{(n)} x^*\| &= \sup_{\|x_1^*\|\le 1} \int_0^1 \int_\Omega \sum_{k=n+1}^{\infty} \langle \varphi(t, \omega)h_k, x^*\rangle \langle \varphi(t, \omega)h_k, x_1^*\rangle dt dP \\ &\le \sup_{\|x_1^*\|\le 1} \left(\int_0^1 \int_\Omega \sum_{k=n+1}^{\infty} \langle \varphi(t, \omega)h_k, x_1^*\rangle^2 dt dP\right)^{1/2} \\ &\quad \times \left(\int_0^1 \int_\Omega \sum_{k=n+1}^{\infty} \langle \varphi(t, \omega)h_k, x^*\rangle^2 dt dP\right)^{1/2} \to 0.\end{aligned}$$

As we have

$$\left(\int_0^1 \int_\Omega \sum_{k=n+1}^{\infty} \langle \varphi(t, \omega)h_k, x^*\rangle^2 dt dP\right)^{1/2} \to 0 \quad \text{and}$$

$$\sup_{\|x_1^*\|\le 1} \left(\int_0^1 \int_\Omega \sum_{k=n+1}^{\infty} \langle \varphi(t, \omega)h_k, x_1^*\rangle^2 dt dP\right)^{1/2} \le \|L_\varphi x^*\| < \infty.$$

Therefore $L_\varphi^{(n)} x^* \to L_\varphi x^*$, $n \to \infty$. That is $L_\varphi x^* \in X$. Theorem 1 is proved. □

We defined the generalized stochastic integral for a wide class of non-anticipating operator-valued random functions $G(L(H, X))$. The generalized stochastic integral from $\varphi \in G(L(H, X))$ is GRE. This GRE T_φ is not always decomposable. That is, there does not always exist a random element $\xi : \Omega \to X$ such that $T_\varphi x^* = \langle \xi, x^* \rangle$, $x^* \in X^*$.

Definition 7. Let $\varphi \in G(L(H, X))$ be an operator-valued non-anticipating random function. We say that a random element $\xi : \Omega \to X$ (if such element exists) is the stochastic integral of φ with respect to a cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$ if for all $x^* \in X^* T_\varphi x^* = \langle \xi, x^* \rangle$ a.s. and write $\xi = \int_0^1 \varphi(t, \omega) dW_H(t)$.

Thus, the question of the existence of the stochastic integral is reduced to the problem of decomposability of the GRE. This problem is equivalent to the problem of extension of the weak second order cylindrical measure to the countable-additive measure. Therefore, to study the problem of the existence of the stochastic integral we can use the results in the mentioned fields.

Now we give a sufficient condition of existence of the stochastic integral from the operator-valued non-anticipating random process by the cylindrical Wiener process using the term of p-absolutely summing operators. In case of the Banach spaces the role of the Hilbert–Schmidt operator plays the p-absolutely summing operator.

Definition 8. A linear operator $A : H \to X$ is called p-absolutely summing, if there exist a constant $c > 0$ such that for all $n \in N$ and $h_1, h_2, \dots, h_n$ from H

$$\left(\sum_{i=1}^n \|Ah_i\|^p\right)^{1/p} \le c \sup_{\|h\|\le 1}\left(\sum_{i=1}^n \langle h_i, h\rangle^p\right)^{1/p}.$$

If X is a Hilbert space, then for any $p \ge 1$ the class of the p-absolutely summing operators from H to H coincides with the class of the Hilbert–Schmidt operators (see [34, Corr. 3.16 and Th.4.10]].

By the factorization lemma, the covariance operator L_φ factorized through separable Hilbert space $L_\varphi = AA^*$, $A : H \to X$, if $(e_k)_{k\in N}$ is the orthonormal basis in H, then there exists $(x_k)_{k\in N}$ and $(x_k^*)_{k\in N}$ such that $Ae_k = x_k$, $\langle x_k, x_j^* \rangle = \delta_{k,j}$ and $L_\varphi = \sum_{k=1}^\infty \langle x_k, x^* \rangle x_k$.

Theorem 2. *Let $\varphi \in G(L(H, X))$ be an operator-valued non-anticipating random process, $L_\varphi x^* = \int_0^1 \int_\Omega \varphi\varphi^* x^* dt dP$ be the covariance operator of the generalized stochastic integral of φ with respect to the cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$. If $L_\varphi = AA^*$ be such, that $A : H \to X$ is the p-absolutely summing operator for any $p \ge 2$, there exists the closed subspace $S \subset L_2(\Omega, B, P)$ such that for all $x^* \in X^* T_\varphi x^* \in S$ and $S \subset L_p(\Omega, B, P) \subset L_2(\Omega, B, P)$, then the stochastic integral $\xi = \int_0^1 \varphi(t, \omega) dW_H(t)$ exists, $E\|\xi\|^p < \infty$, $\xi = \sum_{k=1}^\infty x_k \int_0^1 \varphi^*(t, \omega) x_k^* dW_H(t)$ and the convergence is in $L_p(\Omega, X)$, where $Ae_k = x_k$, $\langle x_k, x_j^* \rangle = \delta_{k,j}$ and $L_\varphi = \sum_{k=1}^\infty \langle x_k, x^* \rangle x_k$.*

Proof. By Proposition 1, for any $x^* \in X^*$, we have $T_\varphi x^* = \int_0^1 \varphi^*(t, \omega) x^* dW_H(t) = \sum_{k=1}^\infty \langle x_k, x^* \rangle \int_0^1 \varphi^*(t, \omega) x_k^* dW_H(t)$. Since $T_\varphi x^* \in S$ and $S \subset L_p(\Omega, B, P)$, we can consider the identical map $I : S \to L_p(\Omega, B, P)$. By the closed graph theorem, I is a bounded operator, therefore, there exists $c > 0$, such that $(E(T_\varphi x^*)^p)^{\frac{1}{p}} \le c(E(T_\varphi x^*)^2)^{\frac{1}{2}}$. In a Hilbert space H consider the sum $\eta_n \equiv \sum_{k=1}^n e_k \int_0^1 \varphi^*(t, \omega) x_k^* dW_H(t)$. For all $h \in H$, $\langle \eta_n, h \rangle$ converges in $L_p(\Omega, B, P)$ as

$$\begin{aligned}(E(\langle \eta_n, h\rangle - \langle \eta_m, h\rangle)^p)^{\frac{1}{p}} &= \left(E\left(\sum_{k=n}^m \langle e_k, h\rangle \int_0^1 \varphi^*(t, \omega) x_k^* dW_H(t)\right)^p\right)^{\frac{1}{p}} \\ &= \left(E\left(\int_0^1 \varphi^*(t, \omega)\left(\sum_{k=n}^m \langle e_k, h\rangle x_k^*\right)\right)^p dt\right)^{\frac{1}{p}} dt \\ &\le c\left(E\left(\int_0^1 \varphi^*(t, \omega)\left(\sum_{k=n}^m \langle e_k, h\rangle x_k^*\right)\right)^2 dt\right)^{\frac{1}{2}} = \left(\sum_{k=n}^m \langle h, e_k\rangle^2\right)^{\frac{1}{2}} \to 0.\end{aligned}$$

Here we used the following equalities: $\langle Lx_i^*, x_j^*\rangle = \delta_{i,j} = ET_\varphi x_i^* T_\varphi x_j^* = \int_0^1 \int_\Omega \langle \varphi^*(t,\omega)x_i^* \varphi^*(t,\omega)x_j^*\rangle dt dP$. That is, $(\eta_n)_{n\in N}$ is a sequence of the weak pth order random elements in H such that, for all $h \in H$, the sequence $\langle \eta_n, h\rangle$ converges in $L_p(\Omega, B, P)$. As A is a p-absolutely summing operator, by the lemma 6.5.2 of [22], the sequence $\sum_{k=1}^n Ae_k \int_0^1 \varphi^*(t,\omega)x_k^* dW_H(t)$ converges in $L_p(\Omega, X)$. Therefore, the stochastic integral $\xi = \int_0^1 \varphi(t,\omega) dW_H(t)$ exists, $\xi = \sum_{k=1}^\infty x_k \int_0^1 \varphi^*(t,\omega)x_k^* dW_H(t)$ and $E\|\xi\|^p < \infty$. □

Remark 4. Stochastic integral of operator-valued non-anticipating random process by the Wiener process in an arbitrary Banach spaces we considered in [24] (see also [25]), where we gave the sufficient condition of existence of the stochastic integral using p-absolutely summing operators.

Denote by $M_1^H := L(X^*, L_2(\Omega, B, P, H))$ the Banach space of generalized random elements with the norm $\|T\|^2 = \sup_{\|x^*\|\le 1} \int_\Omega \|Tx^*\|^2 dP < \infty$. If $\varphi : \Omega \to L(H, X)$ is such that for all $x^* \in X^*$, $\int_\Omega \|\varphi^* x^*\|^2 dP < \infty$, then, by the closed graph theorem, $T_\varphi : X^* \to L_2(\Omega, B, P, H)$, $T_\varphi x^* = \varphi^* x^*$ belongs to the space M_1^H. Denote by M_2^H the subspace of M_1^H of such GRE T_φ, that $\varphi : \Omega \to L(H, X)$ and $\int_\Omega \|\varphi^* x^*\|^2 dP < \infty$, for all $x^* \in X^*$. Consider now the family of linear bounded operators $(T_t)_{t\in[0,1]}$, $T_t : X^* \to L_2(\Omega, B, P, H)$ such that for all $x^* \in X^*$ the random process $T_t x^*$ is non-anticipating and $\sup_{\|x^*\|\le 1} \int_0^1 \int_\Omega \|T_t x^*\|^2 dt dP < \infty$. Denote by TM_1^H the Banach space of such family of operators $(T_t)_{t\in[0,1]}$. We can define the generalized stochastic integral from $(T_t)_{t\in[0,1]} \in TM_1^H$.

Definition 9. Consider the GRP $(T_t)_{t\in[0,1]} \in TM_1^H$. The stochastic integral from $(T_t)_{t\in[0,1]}$ by the cylindrical Wiener process in H is the GRE defined by $I_T x^* = \int_0^1 T_t x^* dW_H(t)$, for all $x^* \in X^*$.

It is easy to see, that

$$I_T x^* = \int_0^1 T_t x^* dW_H(t) = \sum_{k=1}^\infty \int_0^1 \langle T_t x^*(\omega), e_k\rangle dw_k(t).$$

We have

$$\|I_T\|^2 = \sup_{\|x^*\|\le 1} E\left(\int_0^1 T_t x^* dW_H(t)\right)^2 = \sup_{\|x^*\|\le 1} \int_0^1 \int_\Omega \|T_t x^*\|^2 dt dP.$$

Accordingly, we have the isometrical operator

$$I : TM_1^H \to M_1,\ I((T_t)_{t\in[0,1]}) = \sum_{k=1}^\infty \int_0^1 \langle T_t x^*(\omega), e_k\rangle dw_k(t).$$

3. Stochastic differential equations

3.1. Stochastic differential equation for generalized random process driven by the cylindrical Wiener process

Consider now the Banach space of GRE M_1 and the stochastic differential equation for generalized random process in it:

$$dT_t = a(t, T_t)dt + B(t, T_t)dW_H(t), \tag{1}$$

with F_0-measurable initial condition $T_0 = L$, where $a : [0,1] \times M_1 \to M_1$ and $B : [0,1] \times M_1 \to M_1^H$.

Definition 10. A GRP $(T_t)_{t\in[0,1]}$ is called the strong generalized solution of Eq. (1) with the F_0-measurable initial condition $T_0 = L$, if the following assertions are true:

for all $x^* \in X^*$, $a(t, T_t)x^*$ and $B(t, T_t)x^*$ are $B[0,1] \times F_t$ measurable;

$E\int_0^1 (a(t, T_t)x^*)^2 dt + E\int_0^1 \|B(t, T_t)x^*\|^2 dt < \infty$; $T_t x^*$ is continuous, F_t-adapted random process and for all $t \in [0,1]$ and $x^* \in X^*$,

$T_t x^* = T_0 x^* + \int_0^t a(s, T_s)x^* ds + \int_0^1 B(s, T_s)x^* dW_H(s)$ a.s.

Definition 11. We say that the stochastic differential equation (1) has a unique strong generalized solution, if $(T_t)_{t\in[0,1]}$ and $(\overline{T_t})_{t\in[0,1]}$ are two solutions, then for each $x^* \in X^*$,

$$P(\omega : T_t(\omega)x^* = \overline{T_t}(\omega)x^* \quad \text{for all } t \in [0,1]) = 1.$$

The following theorem gives the sufficient conditions of existence and uniqueness of a strong generalized solution to a stochastic differential equation for GRP.

Theorem 3. *Suppose that the coefficients of the stochastic differential equation* (1) *satisfy the following conditions:*
1. $\|a(t,T)\|^2_{M_1} + \|B(t,T)\|^2_{M_1^H} \le K^2(1 + \|T\|^2_{M_1})$,
2. $\|a(t,T) - a(t,S)\|^2_{M_1} + \|B(t,T) - B(t,S)\|^2_{M_1^H} \le K^2\|T - S\|^2_{M_1}$.

Then there exists a unique strong generalized solution $(T_t)_{t\in[0,1]}$ to (1) with initial condition $T_0 = L$, $L \in M_1$ and for all $x^* \in X^*$, Lx^* is F_0-measurable. The GRP $T : [0,1] \to M_1$ is continuous.

Proof. To prove this Theorem we use the one dimensional technique which works here successfully. For all $t \in [0,1]$, let $T_t^{(0)} = L$ and

$$\begin{aligned}
T_t^{(n)}x^* &= T_t^{(0)}x^* + \int_0^t a(s, T_s^{(n-1)})x^*ds + \int_0^t B^*(s, T_s^{(n-1)})x^*dW_H(s),\\
\|T_t^{(n+1)} - T_t^{(n)}\|^2_{M_1} &\le 2\sup_{\|x^*\|\le 1} E\left(\int_0^t (a(s, T_s^{(n)}) - a(s, T_s^{(n-1)}))x^*\,ds\right)^2\\
&\quad + 2\sup_{\|x^*\|\le 1}\left(E\left(\int_0^t (B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*dW_H(s)\right)\right)^2\\
&\le 2\int_0^t \|a(s, T_s^{(n)}) - a(s, T_s^{(n-1)})\|^2_{M_1}ds + 2\int_0^t \|B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)})\|^2_{M_1^H}ds\\
&\le 2K^2\int_0^t \|T_s^{(n)} - T_s^{(n-1)}\|^2_{M_1}ds.
\end{aligned}$$

Then we have

$$\|T_t^{(n+1)} - T_t^{(n)}\|^2_{M_1} \le (2K^2)^{(n-1)}\int_0^t \frac{(t-s)^{(n-1)}}{(n-1)!}\|T_s^{(1)} - T_s^{(0)}\|^2_{M_1}ds,$$

$$\|T_s^{(1)} - T_s^{(0)}\|^2_{M_1} \le 2\left\|\int_0^t a(s, T_s^{(0)})ds\right\|^2_{M_1} + 2\left\|\int_0^t (B^*s, T_s^{(0)})dW_H(s)\right\|^2_{M_1^H} \le 2K^2(1 + \|T_0\|^2_{M_1}).$$

Consequently, $\|T_t^{(n+1)} - T_t^{(n)}\|^2_{M_1} \le pC^n/n!$ for any positive p and C.
It is easy to see, that $\int_0^t (B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)})x^*)dW_H(s)$ is a martingale for all fixed $x^* \in X^*$ and therefore,

$$\begin{aligned}
&E\sup_{0\le t\le 1}\left|\int_0^t (B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*\,dW_H(s)\right|^2 \le 4\int_0^1 E\|(B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*\|^2ds\\
&\le 4\sup_{\|x^*\|\le 1}\int_0^1 E\|(B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*\|^2ds\\
&\le 4\int_0^1 E\|(B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))\|^2_{M_1^H}ds.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&E\sup_{0\le t\le 1}|(T_t^{(n+1)} - T_t^{(n)})x^*|^2 \le 2E\sup_{0\le t\le 1}\int_0^t ((a(s, T_s^{(n)}) - a(s, T_s^{(n-1)}))x^*)^2ds\\
&\quad + 2E\sup_{0\le t\le 1}\left|\int_0^t (B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*\,dW_H(s)\right|^2
\end{aligned}$$

$$\leq 2\int_0^1 \|a(s,T_s^{(n)})-a(s,T_s^{(n-1)})\|_{M_1}^2 ds + 8\int_0^1 \|B^*(s,T_s^{(n)})-B^*(s,T_s^{(n-1)})\|_{M_1}^2 ds$$
$$\leq \frac{10pC^{n-1}}{(n-1)!}.$$

Then we have

$$\sum_{n=1}^{\infty} P\Big(\sup_{0\leq t\leq 1}|(T_t^{(n+1)}-T_t^{(n)})x^*|\geq \frac{1}{n^2}\Big)\leq \sum_{n=1}^{\infty} n^4 E\left(\sup_{0\leq t\leq 1}|(T_t^{(n+1)}-T_t^{(n)})x^*|^2\leq 10p\sum_{n=1}^{\infty}\frac{n^4C^{n-1}}{(n-1)!}\right).$$

By the Borel–Cantelli lemma, the series $T_t^{(0)}x^*(\omega)+\sum_{n=1}^{\infty}(T_t^{(n)}(\omega)-T_t^{(n-1)}(\omega))x^*$ converges uniformly for t (P-a.s.) to the continuous random process which we denote by T_tx^*, $x^*\in X^*$. Therefore, we get GRP $T_t : X^*\to L_2(\Omega, B, P)$. From Eq. (2) we obtain

$$T_tx^* = T^{(0)}x^* + \int_0^t a(s,T_s)x^*ds + \int_0^t B^*(s,T_s)x^*dW_H(s) \quad \text{a.s.}$$

Therefore, the GRP (T_t), $t\in[0,1]$, constructed above, is a strong generalized solution of Eq. (1).

The uniqueness of the solution we can prove similarly to the finite dimensional case. □

3.2. Stochastic differential equation in an arbitrary Banach space driven by the cylindrical Wiener process

Let us now consider the stochastic differential equation in an arbitrary Banach space

$$d\xi_t = a(t,\xi_t)dt + B(t,\xi_t)dW_H(t), \tag{2}$$

where $a:[0,1]\times X\to X$ and $B:[0,1]\times X\to L(H,X)$ are such functions that $a(t,\xi)\in M_2$ and $B^*(t,\xi)\in M_2^H$ for all $t\in[0,1]$ and for all weak second order random elements ξ; and the following inequalities hold at that:

1′. $\|a(t,\xi)\|_{M_1}^2+\|B^*(t,\xi)\|_{M_1^H}^2\leq K^2(1+\|\xi\|_{M_1}^2)$,

2′. $\|a(t,\xi)-a(t,\eta)\|_{M_1}^2+\|B^*(t,\xi)-B^*(t,\eta)\|_{M_1^H}^2\leq K^2\|\xi-\eta\|_{M_1}^2$, where ξ and η are weak second order X-valued random elements.

We can extend the coefficients a and B on $\overline{M_2}\subset M_1$ correspondingly: Let $T\in\overline{M_2}$, there exists $(\xi_n)_{n\in N}\subset M_2$ such that $\|\xi_n-T\|_{M_1}\to 0$. Then $\|a(t,\xi_n)-a(t,\xi_m)\|_{M_1}^2\leq K^2\|\xi_n-\xi_m\|_{M_1}^2\to 0$ and $\|B(t,\xi_n)h-B(t,\xi_m)h\|_{M_1}^2\leq K^2\|h\|^2\|\xi_n-\xi_m\|_{M_1}^2\to 0$. $\|B^*(t,\xi_n)-B^*(t,\xi_m)\|_{M_1^H}^2\leq K^2\|\xi_n-\xi_m\|_{M_1}^2\to 0$. Denote $a(t,T):=\lim_{n\to\infty}a(t,\xi_n)$, $B(t,T)h:=\lim_{n\to\infty}B(t,\xi_n)h$ and $B^*(t,T):=\lim B^*(t,\xi_n)$. $a(t,T)\in\overline{M_2}$, $B(t,T)h\in\overline{M_2}$ and $B^*(t,T)\in\overline{M_2^H}\subset M_1^H$. Therefore, we receive from Eq. (2) the corresponding stochastic differential equation for GRP:

$$dT_t = a(t,T_t)dt + B^*(t,T_t)dW_H(t), \tag{3}$$

with initial condition $T_0x^*=\langle\xi_0,x^*\rangle$. It is easy to see that the coefficients of this equation satisfy the conditions 1 and 2 of Theorem 2.

Remember that we have the condition $B^*(t,\xi)\in M_1^H$, that is $\sup_{\|x^*\|\leq 1}E\|B^*x^*\|^2=\sup_{\|x^*\|\leq 1}\sum_{k=1}^{\infty}E\langle B^*(t,\xi)x^*,e_k\rangle^2<\infty$. Further we need the following assertion:

$$\sup_{\|x^*\|\leq 1}\sum_{k=n}^{\infty}E\langle B^*(t,\xi)x^*,e_k\rangle^2\to 0. \tag{4}$$

It is easy to see, that if $B^*(t,\xi)$ satisfies the condition (4) for all $\xi\in M_2$, then this condition is true for all $T\in\overline{M_2}$.

Then we have the following theorem:

Theorem 4. *If the coefficients of Eq.* (2) *satisfy the conditions* 1′, 2′, (4) *and for all* $\xi\in M_2$, $a(.,\xi)$ *from* $[0,1]$ *to* $\overline{M_2}$ *and* $B^*(.,\xi)$ *from* $[0,1]$ *to* $\overline{M_2^H}$ *are continuous, then the corresponding stochastic differential equation* (3) *possesses a unique strong generalized solution with initial condition* $T_0x^*=\langle\xi_0,x^*\rangle$. *The solution* $(T_t)_{t\in[0,1]}$ *is such that* $T_t\in\overline{M_2}$ *for all* $t\in[0,1]$.

Proof. To use Theorem 2, it is enough to prove that in the iteration formula

$$T_t^{(n)}x^* = T_t^{(0)}x^* + \int_0^t a(s, T_s^{(n-1)})x^*ds + \int_0^t B^*(s, T_s^{(n-1)})x^*dW_H(s), \tag{5}$$

the members $\int_0^t a(s, T_s^{(n-1)})x^*ds$ and $\int_0^t B^*(s, T_s^{(n-1)})x^*dW_H(s)$ of the formula (5) belong to the space $\overline{M_2}$, where $T^{(0)}x^* = \langle \xi_0, x^*\rangle$. As we showed above, $a(t, T)$ and $B(t, T)h$ belong to $\overline{M_2}$ for all $h \in H$.

In [23] we defined the generalized stochastic integral from the non-anticipating weak second order Banach space valued random processes (from the non-anticipating GRP) by one dimensional standard Wiener process. If $\varphi(t, \omega) \in G(L(H, X))$ is a non-anticipating function, $(W_H(t))_{t\in[0,1]}$, $W_H(t) = \sum_{k=1}^{\infty} e_k w_k(t)$ is the cylindrical Wiener process for any $(e_k)_{k\in N}$ orthonormal basis of H, then $\varphi(t, \omega)e_k$ is X-valued non-anticipating random process for all $k \in N$. The generalized stochastic integral $\int_0^t \varphi(t, \omega)e_k dw_k(t)$ exists. This stochastic integral belongs to $\overline{M_2}$; moreover, if $L : [0, 1] \to \overline{M_2}$, is continuous, $\int_0^1 \|L(t)\|_{M_1}^2 < \infty$, then $\int_0^1 L(t)dw_t \in \overline{M_2}$ (see [23], Theorem 2). We will use this result to prove that $I(t) : X^* \to L_2(\Omega, B, P)$, $I(t)x^* = \int_0^t B^*(s, T_s^{(n-1)})x^*dW_H(s)$ belongs to $\overline{M_2}$ for all $n \in N$: $x^* \to \int_0^t \langle B^*(s, T_s^0)x^*, e_k\rangle dw_k(s)$ belongs to $\overline{M_2}$. If $T_s^{(n-1)}$ belongs to $\overline{M_2}$, then $x^* \to \int_0^t \langle B^*(s, T_s^{(n-1)})x^*, e_k\rangle dw_k(s)$ belongs to $\overline{M_2}$. Therefore, $I_m(t) : X^* \to L_2(\Omega, B, P)$, $I_m(t)x^* := \sum_{k=1}^m \int_0^t \langle B^*(s, T_s^{(n-1)})x^*, e_k\rangle dw_k(s)$ belongs to $\overline{M_2}$; $\|I(t)-I_m(t)\|_{M_1}^2 = \| \sum_{k=m+1}^{\infty} \int_0^t B(s, T_s^{(n-1)})e_k dw_k(s)\|_{M_1}^2 = \sup_{\|x^*\|\le 1} \sum_{k=m+1}^{\infty} \int_0^t E\langle B^*(t, T^{(n-1)})x^*, e_k\rangle^2 \to 0$ by the condition (4) and Lebesgue Theorem, as $\sup_{\|x^*\|\le 1} \sum_{k=m+1}^{\infty} E\langle B^*(t, T^{(n-1)})x^*, e_k\rangle^2 \le \sup_{\|x^*\|\le 1} \sum_{k=1}^{\infty} E\langle B^*(t, T^{(n-1)})x^*, e_k\rangle^2 = \sup_{\|x^*\|\le 1} E\|B^*x^*\|^2 = \|B^*(t, T^{(n-1)})x^*\|_{M_1^H}^2 \le K^2(1 + \|T^{(n-1)}\|^2) < \infty$, we have $I(t) \in \overline{M_2}$. □

Consequently, we receive the GRP $(T_t)_{t\in[0,1]} \in \overline{M_2}$,

$$T_t x^* = \langle \xi_0, x^*\rangle + \int_0^t \langle a(s, T_s), x^*\rangle ds + \int_0^t B^*(s, T_s)x^*dW_H(s) \tag{6}$$

as a generalized solution of the stochastic differential equation (3) corresponding to the stochastic differential equation (2) in an arbitrary separable Banach space.

Consider now the members of the equality (6): denote $T_t'x^* = \int_0^t \langle a(s, T_s), x^*\rangle ds + \int_0^t B^*(s, T_s)x^*dW_H(s)$. Let L_1' be the covariance operator of the GRE T_1'. By Theorem 1, the operator L_1' maps X^* to X. Let $L_1' = A'A'^*$ be the factorization of the covariance operator L_1', $A' : H \to X$. From Theorems 2 and 4 we receive the following:

Corollary 1. *If the GRE T_1' satisfies the conditions of Theorem 2, in particular, if the operator $A' : H \to X$ is 2-absolutely summing, then there exists the X-valued random process $(\xi_t)_{t\in[0,1]}$ such that $E\|\xi_t\|^2 < \infty$ and $\xi_t = \xi_0 + \int_0^t a(s, \xi_s)ds + \int_0^t B(s, \xi_s)dW_H(s)$, that is, $(\xi_t)_{t\in[0,1]}$ is the solution of the stochastic differential equation (2) in an arbitrary separable Banach space.*

Consider now a linear stochastic differential equation in a separable Banach space.

$$d\xi_t = A(t)\xi_t dt + B(t)\xi_t dW_H(t), \tag{7}$$

where $A : [0, 1] \to L(X, X)$ and $B : [0, 1] \to L(X, L(H, X))$ are continuous and $B(t, x)$ is such, that there exists $(e_k)_{k\in N}$ the orthonormal basis in H with the property $\sup_{t\in[0,1]} \sup_{\|x^*\|\le 1} \sum_{k=n}^{\infty} \|B(t)^*\delta_{e_k,x^*}\|^2 \to 0$, where δ_{e_k,x^*} is an element of $L(H, X)^*$, $\langle C, \delta_{e_k,x^*}\rangle = \langle Ce_k, x^*\rangle$, for all $C \in L(H, X)$. Denote $D \equiv \sup_{t\in[0,1]} \sup_{\|x^*\|\le 1} \sum_{k=1}^{\infty} \|B(t)^*\delta_{e_k,x^*}\|^2$. Then $\max_{t\in[0,1]}(\|A(t)\|, D) \equiv M < \infty$. For all weak second order random elements ξ, we have

$$\begin{aligned}\|A(t)\xi\|_{M_1}^2 &= \sup_{\|x^*\|\le 1} E\langle A(t)\xi, x^*\rangle^2 = \sup_{\|x^*\|\le 1} E\langle \xi, A^*(t)x^*\rangle^2 \\ &= \|A^*(t)\|^2 \sup_{\|x^*\|\le 1} E\left\langle \xi, \frac{A^*(t)x^*}{\|A^*(t)\|}\right\rangle^2 \le \|A^*(t)\|^2 \sup_{\|x^*\|\le 1} E\langle \xi, x^*\rangle^2 \le M^2(1 + \|\xi\|_{M_1}^2),\end{aligned}$$

and for all weak second order random elements ξ and η we have also

$$\begin{aligned}\|A(t)\xi - A(t)\eta\|^2_{M_1} &= \sup_{\|x^*\|\le 1} E\langle A(t)(\xi-\eta), x^*\rangle^2 \\ &= \sup_{\|x^*\|\le 1} E\langle(\xi-\eta), A^*(t)x^*\rangle^2 = \|A^*(t)\|^2 \sup_{\|x^*\|\le 1} E\left\langle(\xi-\eta), \frac{A^*(t)x^*}{\|A^*(t)\|}\right\rangle^2 \\ &\le M^2 \sup_{\|x^*\|\le 1} E\langle(\xi-\eta), x^*\rangle^2 = M^2\|\xi-\eta\|^2_{M_1}.\end{aligned}$$

Further, for all weak second order random elements ξ

$$\begin{aligned}\|B(t)\xi\|^2_{M_1^H} &= \sup_{\|x^*\|\le 1} E\|(B(t)\xi)^*x^*\|^2 \\ &= \sup_{\|x^*\|\le 1} E\sum_{k=1}^{\infty}\langle(B(t)\xi)^*x^*, e_k\rangle^2 = \sup_{\|x^*\|\le 1} E\sum_{k=1}^{\infty}\langle\xi, B(t)^*\delta_{x^*,e_k}\rangle^2 \\ &\le \sup_{\|x^*\|\le 1}\sum_{k=1}^{\infty}\|B(t)^*\delta_{e_k,x^*}\|^2 E\left\langle\xi, \frac{B(t)^*\delta_{x^*,e_k}}{\|B(t)^*\delta_{e_k,x^*}\|}\right\rangle^2 \\ &\le \sup_{\|x^*\|\le 1}\sum_{k=1}^{\infty}\|B(t)^*\delta_{e_k,x^*}\|^2 \sup_{\|x^*\|\le 1} E\langle\xi, x^*\rangle^2 \le M^2\|\xi\|^2_{M_1}.\end{aligned}$$

Analogously, we can receive the inequality $\|B(t)\xi - B(t)\eta\|^2_{M_1^H} \le M^2\|\xi-\eta\|^2_{M_1}$.

Further,

$$\begin{aligned}\sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty} E\langle B^*(t,\xi)x^*, e_k\rangle^2 &= \sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty} E\langle(B(t)\xi)^*x^*, e_k\rangle^2 \\ &= \sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty}\|B(t)^*\delta_{x^*e_k}\|^2 E\left\langle\xi, \frac{B(t)^*\delta_{x^*,e_k}}{\|B(t)^*\delta_{x^*e_k}\|}\right\rangle^2 \\ &\le \|\xi\|^2_{M_1}\cdot\sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty}\|B(t)^*\delta_{x^*e_k}\|^2 \to 0.\end{aligned}$$

Therefore, if there exists $(e_k)_{k\in N}$ the orthonormal basis in H, such that $\sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty}\|B(t)^*\delta_{e_k,x^*}\|^2 \to 0$, then for the linear stochastic differential equation (7) the conditions $1'$ and $2'$ and (4) are satisfied. Thus, by Theorem 4 we have the following:

Theorem 5. *For the linear stochastic differential equation* (7)*, if there exists* $(e_k)_{k\in N}$ *the orthonormal basis in* H *with the property* $\sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty}\|B(t)^*\delta_{e_k,x^*}\|^2 \to 0$*, then there exists the unique generalized solution of this equation* $(T_t)_{t\in[0,1]}$*,* $T_t \in \overline{M_2}$ *for all* $t \in [0,1]$ *with the initial condition* $T_0x^* = \langle\xi_0, x^*\rangle$*, where* $\xi_0 \in M_2$ *is* F_0*-measurable.*

In [26] we considered the stochastic differential equation driven by the Wiener process in a Banach space. If $R = UU^*$ is a Gaussian covariance in a Banach space, then $W_t \equiv UW_H(t) = \sum_{k=1}^{\infty} Ue_k w_k(t)$, $t \in [0,1]$ is a Wiener process in a Banach space for all orthonormal bases in H and convergence we have in $C([0,1], X)$. If $A : [0,1] \to L(X,X)$ and $B : [0,1] \to L(X, L(X,X))$ are continuous, then by Theorem 2 of [26], we have the following.

Corollary 2. *For the linear stochastic differential equation* $d\xi_t = A(t)\xi_t dt + (B(t)\xi_t)UdW_H(t)$*, where* $A : [0,1] \to L(X,X)$ *and* $B : [0,1] \to L(X, L(X,X))$ *are continuous and* $R = UU^*$ *is a Gaussian covariance, there exists the unique generalized solution* $(T_t)_{t\in[0,1]}$*,* $T_t \in \overline{M_2}$ *for all* $t \in [0,1]$ *with the initial condition* $T_0x^* = \langle\xi_0, x^*\rangle$*, where* $\xi \in M_2$ *is* F_0*-measurable.*

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Original article

Sobolev regularity of the Bergman projection on certain pseudoconvex domains

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Abstract

In this paper we study the Sobolev regularity of the Bergman projection B and the $\overline{\partial}$-Neumann operator N on a certain pseudoconvex domain. We show that if Ω is a domain with Lipschitz boundary, which is relatively compact in an n-dimensional compact Kähler manifold and satisfies some "log δ-pseudoconvexity" condition, the operators B, N and $\overline{\partial}^* N$ are regular in the Sobolev spaces $W^k_{r,s}(\Omega, E)$ for forms with values in a holomorphic vector bundle E and for any $k < \eta/2$, $0 < \eta < 1$, $0 \le r \le n$, $0 \le s \le n-1$.

Keywords: $\overline{\partial}$-Neumann operator; Bergman projection; Kähler manifold; Pseudoconvex domain

1. Introduction

Let X be an n-dimensional Kähler manifold and Ω be a relatively compact domain in X. Let δ be the boundary distance function of Ω with respect to the Kähler form ω associated to the Kähler metric σ on X, then Ω is $\log \delta$-pseudoconvex if $\partial\overline{\partial}(-\log \delta + h) \ge c\,\omega$ for some $c > 0$ and some bounded function h on Ω.

For example, if X is a Stein manifold, then any relatively compact domain Ω in X, which is locally Stein, satisfies the $\log \delta$-pseudoconvexity condition (see [12]). The same is true if X has positive holomorphic bisectional curvature, that is $T^{1,0}X$ is positive in the sense of Griffiths (see [22,12,23]).

In this paper, we consider a $\log \delta$-pseudoconvex domain Ω with Lipschitz boundary in a compact Kähler manifold X of complex dimension n. We show that, for any $\eta \in (0, 1)$, the Bergman projection B, the $\overline{\partial}$-Neumann operator N and the canonical solution operator $\overline{\partial}^* N$ are regular in the Sobolev spaces $W^k_{r,s}(\Omega, E)$, $k < \eta/2$, $0 \le r \le n$, $0 \le s \le n-1$, for forms with values in a holomorphic vector bundle E. This result generalizes the well known results of Boas–Straube [4], Berndtsson–Charpentier [2], Cao–Shaw–Wang [6], Harrington [15] and Saber [20] in the case of $\log \delta$-pseudoconvex domain in a compact Kähler manifold for forms with values in a holomorphic vector bundle E.

Indeed, when Ω is smooth pseudoconvex domain in $\mathbb{C}^n$ admitting a defining function that is plurisubharmonic on the boundary $b\Omega$ of Ω, Boas–Straube [4] proved that B maps $W^k(\Omega)$ to itself for any $k > 0$. For a pseudoconvex

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domain Ω with C^2 boundary in $\mathbb{C}^n$, Berndtsson–Charpentier [2] (see also Kohn [18]) obtained the Sobolev regularity for B. If Ω is a locally Stein in the complex projective space, Cao–Shaw–Wang [6] obtained the Sobolev regularity of the operators N, $\overline{\partial}N$, $\overline{\partial}^*N$ and B. Harrington [15] proved this result on a bounded pseudoconvex domain with Lipschitz boundary in $\mathbb{C}^n$. In [20], Saber proved the Sobolev regularity of the operators N, $\overline{\partial}^*N$ and B on a weakly q-convex domain Ω with smooth boundary in $\mathbb{C}^n$.

2. Notations and preliminaries

Let X be an n-dimensional Kähler manifold with Kähler metric σ and $\pi : E \longrightarrow X$ be a holomorphic vector bundle, of rank p, over X. Let TX be the tangent bundle of X and ω be the Kähler form associated to the Kähler metric σ. Let $\{U_j\}$, $j \in J$, be an open covering of X such that $E|_{U_j}$ is trivial, namely $\pi^{-1}(U_j) = U_j \times \mathbb{C}^p$, and $(z_j^1, z_j^2, \dots, z_j^n)$ be local coordinates on U_j. Let (ρ_j) be a partition of unity subordinate to U_j. A Hermitian metric $h = \{h_j\}$ along the fibers of E is defined by specifying on each U_j a positive definite Hermitian matrix h_j whose entries we require to be differentiable functions and on $U_j \cap U_k$ we have $h_k = {}^t\overline{f}_{jk} h_j f_{jk}$, where $\{f_{jk}\}$ is the system of transition functions of E and ${}^t f_{jk}$ is the transpose of f_{jk}. For an orthonormal basis $e_1, e_2, \dots, e_p$ on the fiber $E_z = \pi^{-1}(z)$, over z, we express h_j as $h_j = (h_{ja\overline{b}})$; $h_{ja\overline{b}} = h_j(e_a, e_b)$. Let $(h_j^{a\overline{b}})$ be the inverse matrix of $(h_{ja\overline{b}})$. Thus every E-valued differential (r, s)-form u on X can be written locally, on U_j, as $u(z) = \sum_{a=1}^p u^a(z)\, e_a(z)$, where u^a are the components of the restriction of u on U_j. Let $C^\infty_{r,s}(X, E)$ be the complex vector space of E-valued differential forms of class C^∞ and of type (r, s) on X. Let $\# : C^\infty_{r,s}(X, E) \longrightarrow C^\infty_{s,r}(X, E^\star)$ be the operator defined locally by $(\#u)_j = \overline{h_j\, u_j}$. For $u, f \in C^\infty_{r,s}(X, E)$, we define a local inner product (u, f) with respect to σ and h by

$$(u, f)\, dV = \sum_{a=1}^{p} u^a \wedge \star\, \overline{(h\, f)^a} = {}^t u \wedge \star\, \#\, f,$$

where dV is the volume element with respect to σ, $\star : C^\infty_{r,s}(X, E) \longrightarrow C^\infty_{n-s,n-r}(X, E)$ is the Hodge star operator defined by σ. Let Ω be a relatively compact domain in X and

$$C^\infty_{r,s}(\overline{\Omega}, E) = \left\{u\,|_{\overline{\Omega}};\, u \in C^\infty_{r,s}(X, E)\right\}$$

be the subspace of $C^\infty_{r,s}(\Omega, E)$ whose elements can be extended smoothly up to the boundary $b\Omega$ of Ω. Let $\mathcal{D}_{r,s}(\Omega, E)$ be the subspace of $C^\infty_{r,s}(\overline{\Omega}, E)$ whose elements have compact support disjoint from $b\Omega$. For $u, f \in C^\infty_{r,s}(\overline{\Omega}, E)$, the associated global inner product $\langle u, f\rangle_\phi$ and the L^2-norm $\|u\|_\Omega$, with respect to σ, h and the weight function ϕ, are defined by

$$\langle u, f\rangle_\phi = \int_\Omega (u, f)\, e^{-\phi} dV,$$

$$\|u\|_\phi^2 = \langle u, u\rangle_\phi = \int_\Omega e^{-\phi}\, |u|^2\, dV,$$

where $|u|^2 = (u, u)$. We shall consider the weighted L^2-spaces

$$L^2_{r,s}(\Omega, e^{-\phi}, E) = \{f : \|f\|_\phi < \infty\}$$

of E-valued differential forms of various degrees. Let $\overline{\partial} : L^2_{r,s}(\Omega, e^{-\phi}, E) \longrightarrow L^2_{r,s+1}(\Omega, e^{-\phi}, E)$ be the maximal closed extension of the original $\overline{\partial}$ and $\overline{\partial}^*_\phi : L^2_{r,s}(\Omega, e^{-\phi}, E) \longrightarrow L^2_{r,s-1}(\Omega, e^{-\phi}, E)$ be its Hilbert space adjoint. Let $\Box_\phi = \overline{\partial}\,\overline{\partial}^*_\phi + \overline{\partial}^*_\phi\overline{\partial}$ be the associated complex Laplace operator. Let N_ϕ be the $\overline{\partial}$-Neumann operator on (r, s)-forms (cf. [13]), solving

$$N_\phi \Box_\phi f = f$$

for any (r, s)-form f in $L^2_{r,s}(\Omega, e^{-\phi}, E)$. We denote by $B_{s,\phi}$ the Bergman operator, mapping a (r, s)-form in $L^2_{r,s}(\Omega, e^{-\phi}, E)$ to its orthogonal projection in the closed subspace of $\overline{\partial}$-closed forms. In particular, for $s = 0$, $B_{0,\phi}$ maps a section to a holomorphic section. By a classical result, if f is $\overline{\partial}$-closed, then

$$u = \overline{\partial}^*_\phi N_\phi f$$

is the solution to $\overline{\partial} u = f$ of minimal norm in $L^2_{r,s}(\Omega, e^{-\phi}, E)$. If $\phi = 0$ we shall omit subscripts and write simply $\overline{\partial}^*_\phi = \overline{\partial}^*$, $\Box_\phi = \Box$ etc.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multiindices, that is, $\alpha_1, \dots, \alpha_n$ are nonnegative integers. For $x \in \mathbb{R}^n$, we define $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. Let $\mathcal{D}^\beta$ be the operator defined by

$$\mathcal{D}^\beta = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\beta_1} \dots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\beta_n}.$$

Denote by $\mathfrak{T}$ the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$, that is, $\mathfrak{T}$ consists of all smooth functions f on $\mathbb{R}^n$ with $\sup_{x\in\mathbb{R}^n} |x^\alpha \mathcal{D}^\beta f(x)| < \infty$ for all multiindices α, β. We define the Fourier transform $\hat{f}$ of a function $f \in \mathfrak{T}$ by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\, e^{-i\, x.\xi}\, dx,$$

where $x.\xi = \sum_{j=1}^n x_j\, \xi_j$ and $dx = dx_1 \wedge \dots \wedge dx_n$ with $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$. If $f \in \mathfrak{T}$, then $\hat{f} \in \mathfrak{T}$ (cf. [21], Chapter 14, Theorem 1.1). The Sobolev space $W^k(\mathbb{R}^n)$, $k \in \mathbb{R}$, is the completion of $\mathfrak{T}$ under the Sobolev norm

$$\|f\|^2_{W^k(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2\, d\xi.$$

Suppose that X is a compact complex manifold of complex dimension n. Choose finite covering $\{U_j\}$, $j \in J$ by domains of the charts $\eta_j : U_j \longrightarrow V_j \subset \mathbb{R}^n$ and let $\phi_i : E|_{U_j} \longrightarrow V_j \times \mathbb{C}^p$ be a collection of trivializations. Let ϕ_i^* be an induced map $\phi_j^* \xi = \phi_j \circ \xi \circ \eta_j^{-1}$ acting from $C^\infty(U_j, E|U_j)$ to $C^\infty(V_j, \mathbb{C}^p)$ which can be identified with $C^\infty(V_i)^p$. Let $(\rho_j)_j$ be a smooth partition of unity subordinate to $(U_j)_j$ and put

$$\|f\|_{W^k(X,E)} = \sum_j \|\phi_j^* \rho_j f\|_{W^k(\mathbb{R}^n)}, \tag{2.1}$$

where on the right hand side we have usual Sobolev k-norm defined as in the Euclidean case. Then, the Sobolev k-space, $W^k(X, E)$, is defined as the completion of the space of all $f \in C^\infty(X, E)$ such that (2.1) is finite. We denote by $W^k(\Omega, E)$, $k \geq 0$, the space of the restriction of all sections $u \in W^k(X, E)$ to Ω. Denote by

$$\|f\|_{W^k(\Omega,E)} = \inf\left\{\|u\|_{W^k(X,E)}, u \in W^k(X, E), u|_\Omega = f\right\}$$

the $W^k(\Omega, E)$-norm. Let $W_0^k(\Omega, E)$ be the completion of $\mathcal{D}(\Omega, E)$ under the $W^k(\Omega, E)$-norm. If Ω is a Lipschitz domain, then $C^\infty(\overline{\Omega}, E)$ is dense in $W^k(\Omega, E)$ with respect to the $W^k(\Omega, E)$-norm. If $0 \leq k \leq 1/2$, we also have that $\mathcal{D}(\Omega, E)$ is dense in $W^k(\Omega, E)$ (cf. [14]; Theorem 1.4.2.4). For $k > 0$, we define $W^{-k}(\Omega, E)$ to be the dual of $W_0^k(\Omega, E)$. For $k > 0$, we define $W^{-k}(\Omega, E)$ to be the dual of $W_0^k(\Omega, E)$ and the norm of $W^{-k}(\Omega, E)$ is defined by

$$\|u\|_{W^{-k}(\Omega,E)} = \sup \frac{|\langle u, f\rangle_\Omega|}{\|f\|_{W^k(\Omega,E)}},$$

where the supremum is taken over all nonzero sections $f \in \mathcal{D}(\Omega, E)$. We denote by $W^k_{r,s}(\Omega, E)$ the Hilbert spaces of (r, s)-forms with $W^k(\Omega, E)$-coefficients and their norms are denoted by $\|\,.\,\|_{W^k_{r,s}(\Omega,E)}$. It is verified that, if $\mathfrak{T}^\star$ is the adjoint map of $\mathfrak{T}$ with respect to the L^2-norm, then

$$\begin{aligned}\|\mathfrak{T} f\|_{W^{k/2}_{r,s}(\Omega,E)} &= \sup_{g\in L^2} \frac{\langle \mathfrak{T} f, g\rangle_\Omega}{\|g\|_{W^{k/2}_{r,s}(\Omega,E)}} = \sup_{g\in\Omega} \frac{\langle f, \mathfrak{T}^\star g\rangle_{L^2}}{\|g\|_{W^{-k/2}_{r,s}(\Omega,E)}} \\ &\leq \|\mathfrak{T}^\star\|_{W^{-k/2}_{r,s}(\Omega,E)} \|g\|_{W^{k/2}_{r,s}(\Omega,E)}.\end{aligned} \tag{2.2}$$

Let V be a vector space of finite dimension. We call $\wedge^\alpha V$ the α-th exterior product of V. Elements of $\wedge^\alpha V$ are written in the form $u_1 \wedge \dots \wedge u_\alpha$, where $u_1, \dots, u_\alpha \in V$. Let $\mathrm{Hom}_\mathbb{R}(TX, \mathbb{C})$ be the complex vector space of complex-valued real-linear mappings of TX to $\mathbb{C}$. We denote by

$$\wedge \operatorname{Hom}_{\mathbb{R}}(TX, \mathbb{C}) = \sum_{t=0}^{2n} \sum_{r+s=t} \wedge^{r,s} T^*X,$$

the $\mathbb{C}$-linear exterior algebra of $\operatorname{Hom}_{\mathbb{R}}(TX, \mathbb{C})$. A linear mapping $L : \wedge\operatorname{Hom}_{\mathbb{R}}(TX, \mathbb{C}) \longrightarrow \wedge\operatorname{Hom}_{\mathbb{R}}(TX, \mathbb{C})$ is defined by $L\phi = e(\omega)\phi = \omega \wedge \phi$, for $\phi \in \wedge^{r,s} T^*X$, i.e., $L : \wedge^{r,s} T^*X \longrightarrow \wedge^{r+1,s+1} T^*X$. The formal adjoint operator $\Lambda : \wedge^{r,s} T^*X \longrightarrow \wedge^{r-1,s-1} T^*X$ of the operator L is defined locally by:

$$\Lambda\phi = (-1)^{r+s} \star L \star \phi.$$

Let $\theta = \{\theta_j\}$; $\theta_j = (\theta^c_{ja})$, $\theta^c_{ja} = \sum_{\alpha=1}^n \sum_{b=1}^p h_j^{c\bar{b}} \frac{\partial h_{ja\bar{b}}}{\partial z_j^\alpha} dz_j^\alpha = \sum_{\alpha=1}^n \mu^c_{ja\alpha} dz_j^\alpha$, be the $(1,0)$-form of the connection associated to h. Put $\Theta^c_{ja\alpha\overline{\beta}} = -\frac{\partial \mu^c_{ja\alpha}}{\partial \bar{z}_j^\beta}$. Since the curvature form, associated to h, is defined by $\Theta = \{\Theta_j\}$; $\Theta_j = i\,\overline{\partial}\theta_j = i\,\partial\overline{\partial}\log h_j$. Then $\Theta_j = \{\Theta^c_{ja}\}$; $\Theta^c_{ja} = i\sum_{\alpha,\beta=1}^n \Theta^c_{ja\alpha\overline{\beta}} dz_j^\alpha \wedge d\bar{z}_j^\beta$; $0 \le a \le p$ and $0 \le c \le p$. Let

$$\Pi = \left(\Pi_{j\overline{b}\overline{\beta},c\alpha}\right) = \left(\sum_{a=1}^p h_{ja\bar{b}}\, \Theta^a_{jc\alpha\overline{\beta}}\right) \tag{2.3}$$

be the associated curvature matrix. For $0 \le r \le n$, we define

$$m_r(\Omega; E) = \sup\{m \in \mathbb{R} |\ \Theta(\wedge^{n-r} T\Omega \otimes E) \ge m\,\omega \otimes \mathrm{Id}_{\wedge^{n-r} T\Omega\otimes E}\}, \tag{2.4}$$

where $\Theta(\wedge^{n-r} T\Omega \otimes E)$ and $\mathrm{Id}_{\wedge^{n-r} T\Omega\otimes E}$ are the curvature form and the identity homomorphism of the holomorphic vector bundle $\wedge^{n-r} T\Omega \otimes E$, respectively.

Definition 1. Let T and E be complex vector spaces of dimensions n, p respectively, and let Θ be a Hermitian form on $T \otimes E$.

(a) A tensor $\xi \in TX \otimes E$ is said to be of rank m if m is the smallest positive integer such that we can write $\xi(z) = \sum_{j=1}^m t_j \otimes e_j$, $t_j \in T_zX$, $e_j \in E_z$.

(b) Π is said to be m-semi-positive ($\Pi \ge_m 0$), m an integer ≥ 1, if

$$\Pi(\xi, \xi) = \sum \Pi_{\overline{b}\,\overline{\beta},c\,\alpha} \xi^c_b\, \xi^\alpha_\beta \ge 0,$$

for any $\xi \in T_zX \otimes E_z$ and of rank $\le m$.

(c) Π is said to be m-positive ($\Pi >_m 0$) if $\Pi(\xi, \xi) > 0$ for any tensor $\xi \in T_zX \otimes E_z$; $\xi \ne 0$, and of rank $\le m$.

Let ϕ be a real $(1,1)$-form with values in the vector bundle $\mathrm{Herm}(E; E) = E^* \otimes E$ satisfies $\phi \ge_{n-s+1} 0$. For $\phi \in \wedge^{n,s} T^*X \otimes E$, we put

$$|f|^2_\phi = \sup_{\substack{u\in \wedge^{n,s} T^*X\otimes E,\\ u\ne 0}} \frac{|(f,u)|^2}{(\phi \wedge \Lambda u, u)}.$$

Definition 2. Let X be an n-dimensional Kähler manifold and $\Omega \Subset X$ be an open set. Let $\delta(z)$ be the distance from $z \in \Omega$ to the boundary $b\Omega$ of Ω with respect to the metric σ. We say that Ω is $\log\delta$-pseudoconvex, if there exists a smooth bounded function h on Ω such that

$$i\partial\overline{\partial}(-\log\delta + h) \ge c\,\omega \ \text{ in } \ \Omega, \tag{2.5}$$

for some $c > 0$, where ω is the Kähler form associated to the Kähler metric σ.

In particular, every $\log\delta$-pseudoconvex domain admits a strictly plurisubharmonic exhaustion function, therefore is a Stein manifold.

Example 2.1. Let X be a Stein manifold and let $\Omega \Subset X$ be a domain which is locally Stein, i.e. for every $x \in b\Omega$, there exists a neighborhood U_x of x in X such that U_x is Stein. It was shown in [12] that there exists a Kähler metric σ on X such that is $\log \delta$-pseudoconvex.

In particular, every bounded weakly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$ is $\log \delta$-pseudoconvex.

Example 2.2. Let (X, σ) be a Kähler manifold with positive holomorphic bisectional curvature, that is $T^{1,0}X$ is positive in the sense of Griffiths. Then every domain $\Omega \Subset X$, which is locally Stein, is $\log \delta$-pseudoconvex (see [23] for the case $X = \mathbb{P}^n$, [12,22]).

Example 2.3. Let X be a complex manifold such that there exists a continuous strongly plurisubharmonic function on X and $\Omega \Subset X$ a locally Stein domain. It was shown in [12] that there exists a Kähler metric on X such that Ω is $\log \delta$-pseudoconvex.

In particular, every locally Stein domain in a Stein manifold is $\log \delta$-pseudoconvex.

Definition 3. (a) A Riemannian manifold (X, σ) is said to be complete if (X, σ) is complete as a metric space.

(b) A continuous function $\psi : X \longrightarrow \mathbb{R}$ is said to be exhaustive if for every $c \in \mathbb{R}$ the sublevel set $X_c = \{x \in X;\, \psi(x) < c\}$ is relatively compact in X.

(c) A sequence $(K_\nu)_{\nu\in\mathbb{N}}$ of compact subsets of X is said to be exhaustive if $X = \bigcup K_\nu$ and if K_ν is contained in the interior of $K_{\nu+1}$ for all ν (so that every compact subset of X is contained in some K_ν).

Lemma 2.1 (*cf. [5]*)**.** *Let (X, σ) be a Kähler manifold and E be a holomorphic vector bundle, of rank p $(p \geq 1)$, over X. Let $h = \{h_j\}$ be a Hermitian metric along the fibers of E and Θ be the associated curvature form. Then, for $f \in C^\infty_{r,s}(X, E)$, at any point, we have*

$$\left(\left(\Box_{r,s} - \star^{-1}\Box_{n-s,n-r}\star\right) f, f\right) = (A^{r,s}_{E,\sigma} f, f), \tag{2.6}$$

where $A^{r,s}_{E,\sigma} = [i\,\Theta(E), \Lambda]$ acting on $\wedge^{r,s} T^*X \otimes E$ and $\Box_{r,s} = \overline{\partial}\,\vartheta + \vartheta\,\overline{\partial}$.

Lemma 2.2 (*cf. [9]*)**.** *Let σ_1, σ_2 be two Hermitian metrics on X such that $\sigma_2 \geq \sigma_1$. For every $u \in \wedge^{n,s} T^*X \otimes E$, $s \geq 1$, we have*

$$|u|^2_{\sigma_2}\, dV_{\sigma_2} \leq |u|^2\, dV,$$
$$((A^{n,s}_{E,\sigma_2})^{-1}u, u)_{\sigma_2}\, dV_{\sigma_2} \leq ((A^{n,s}_{E,\sigma_1})^{-1}u, u)\, dV,$$

where an index σ_2 means that the corresponding term is computed in terms of σ_2 instead of σ_1.

Lemma 2.3 (*cf. [9]*)**.** *The (n, n)-form $|f|^2_\phi\, dv$ is a decreasing function of ω. Also, for any real number $c \geq 0$ such that $\Pi \geq_{n-s+1} c\,\omega \otimes Id_E$, and for each $f \in \wedge^{n,s} T^*X \otimes E$, we have*

$$|f|^2_\phi \leq \frac{1}{s\,c}|f|^2.$$

Finally, let η be a $(0, 1)$-form on X, then we get

$$|\,\eta \wedge f|_\phi \leq |\eta|\; |f|_\phi.$$

Lemma 2.4 (*cf. [10]*)**.** *The following properties are equivalent:*

(i) *(X, σ) is complete;*

(ii) *there exists an exhaustive function $\phi \in C^\infty(X, \mathbb{R})$ such that $|d\phi|_\sigma \leq 1$;*

(iii) *there exists an exhaustive sequence $(K_\nu)_{\nu\in\mathbb{N}}$ of compact subsets of X and functions $\phi_\nu \in C^\infty(X, \mathbb{R})$ such that $\phi_\nu = 1$ in a neighborhood of K_ν, supp $\phi_\nu \subset K^\circ_{\nu+1}$, $0 \leq \phi_\nu \leq 1$ and $|d\phi_\nu|_\sigma \leq 2^{-\nu}$.*

Lemma 2.5 (*[10]; Theorem 5.2*)**.** *Every weakly pseudoconvex Kähler manifold (X, σ) carries a complete Kähler metric $\tilde{\sigma}$.*

3. L^2 estimates for solutions of $\overline{\partial}$-equations

Our goal here is to prove a central L^2 existence theorem, which is essentially due to Hörmander [16], and Andreotti–Vesentini [1]. We will only outline the main ideas, referring e.g. to Demailly [9] for a more detailed exposition of the technical situation considered here.

Theorem 3.1 (*cf. [10]*)**.** *Let (X, σ) be complete Kähler manifold of complex dimension n. Let E be a holomorphic vector bundle over X. Suppose $A^{r,s}_{E,\sigma}$ is a positive Hermitian operator on $\wedge^{r,s} T^*X \otimes E$, and let $f \in L^2_{r,s}(X, E)$ with $s \geq 1$ satisfying $\overline{\partial} f = 0$ and*

$$\int_X ((A^{r,s}_{E,\sigma})^{-1} f, f)\, dV_\sigma < +\infty,$$

there exists $u \in L^2_{r,s-1}(X, E)$ such that $\overline{\partial} u = f$ and

$$\int_X |u|^2 dV_\sigma \leq \int_X ((A^{r,s}_{E,\sigma})^{-1} f, f)\, dV_\sigma.$$

Proof. Consider the Hilbert space orthogonal decomposition

$$L^2_{r,s}(X, E) = Ker\,\overline{\partial} \oplus (Ker\,\overline{\partial})^\perp,$$

observing that $Ker\,\overline{\partial}$ is weakly (hence strongly) closed. Let $v = v_1 + v_2$ be the decomposition of a smooth form $v \in \mathcal{D}_{r,s}(X, E)$ with compact support according to this decomposition (v_1, v_2 do not have compact support in general). Since $(Ker\,\overline{\partial})^\perp = \overline{Im\,\overline{\partial}^*} \subset Ker\,\overline{\partial}^*$ and $f, v_1 \in Ker\,\overline{\partial}$ by hypothesis, we get $\overline{\partial}^* v_2 = 0$ and by the Cauchy–Schwarz inequality, we have

$$|\langle f, v\rangle|^2 = |\langle f, v_1\rangle|^2 \leq \int_X ((A^{r,s}_{E,\sigma})^{-1} f, f)\, dV_\sigma \int_X (A^{r,s}_{E,\sigma} v_1, v_1)\, dV_\sigma.$$

By using a priori inequality

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 \geq (A^{r,s}_{E,\sigma} u, u)$$

for every $u \in Dom\overline{\partial} \cap Dom\overline{\partial}^*$ of bidegree (r, s) if $A^{r,s}_{E,\sigma}$ acting on $\wedge^{r,s} T^*X \otimes E$ is semi-positive. Applying (2.6) to $u = v_1$ yields

$$\int_X (A^{r,s}_{E,\sigma} v_1, v_1)\, dV_\sigma \leq \|\overline{\partial} v_1\|^2 + \|\overline{\partial}^* v_1\|^2 = \|\overline{\partial}^* v_1\|^2 = \|\overline{\partial}^* v\|^2.$$

Combining both inequalities, we find

$$|\langle f, v\rangle|^2 \leq \left(\int_X ((A^{r,s}_{E,\sigma})^{-1} f, f)\, dV_\sigma\right) \|\overline{\partial}^* v\|^2$$

for every smooth (r, s)-form v with compact support. This shows that we have a well defined linear form

$$w = \overline{\partial}^* v \longmapsto (v, f), \quad L^2_{r,s-1}(X, E) \supset \overline{\partial}^*(\mathcal{D}_{r,s}(X, E)) \longmapsto \mathbb{C}$$

on the range of $\overline{\partial}^*$. This linear form is continuous in L^2 norm and has norm $\leq C$ with

$$C = \left(\int_X ((A^{r,s}_{E,\sigma})^{-1} f, f)\, dV_\sigma\right) \|\overline{\partial}^* v\|^2.$$

By the Hahn–Banach theorem, there is an element $u \in L^2_{r,s-1}(X, E)$ with $\|u\| \leq C$, such that $\langle v, f\rangle = \langle \overline{\partial}^* v, u\rangle$ for every v, hence $\overline{\partial} u = f$ in the sense of distributions. The inequality $\|u\| \leq C$ is equivalent to the last estimate in the theorem. □

If we apply the main L^2 existence theorem (Theorem 3.1) to a sequence σ_ε of complete Kähler metrics, we see, by passing to the limit, that the theorem even applies to non necessarily complete metrics if our manifold is pseudoconvex.

Theorem 3.2 (*cf. [10]*)**.** *Let (X, σ) be a Kähler manifold (σ is not assumed to be complete). Assume that X is weakly pseudo-convex. Let E be a holomorphic vector bundle over X and assume that there exists a positive continuous function $\gamma : X \longrightarrow \mathbb{R}$ such that*

$$\Theta(E) \geq \gamma\, \omega \otimes Id_E.$$

Then, for $f \in L^2_{loc}(X, \wedge^{n,s} T^* X \otimes E)$, $s \geq 1$, satisfying $\overline{\partial} f = 0$ and $\int_X \gamma^{-1}|f|^2\, dV_\sigma < +\infty$, there exists a solution $u \in L^2(X, \wedge^{n,s-1} T^* X \otimes E)$ to the equation $\overline{\partial} u = f$ such that

$$\int_X |u|^2 dV_\sigma \leq \int_X \gamma^{-1}|f|^2\, dV_\sigma.$$

Proof. Indeed, under the assumption on E, we have

$$(A^{n,s}_{E,\sigma} f, f)_\sigma \geq \gamma |f|^2_\sigma,$$

hence $((A^{n,s}_{E,\sigma})^{-1} f, f)_\sigma \leq \gamma^{-1}|f|^2_\sigma$. The assumption that $f \in L^2_{\mathrm{loc}}(X, \wedge^{n,s} T^* X \otimes E)$ instead of $f \in L^2(X, \wedge^{n,s} T^* X \otimes E)$ is not a real problem, since we may restrict ourselves to $X_c = \{x \in X : \rho(x) < c\} \Subset X$, where ρ is a plurisubharmonic exhaustion function on X. Then X_c is itself weakly pseudoconvex (with plurisubharmonic exhaustion function $\rho_c = 1/(c - \rho)$), hence X_c can be equipped with a complete Kähler metric $\sigma_{c,\varepsilon} = \sigma + \varepsilon\, i\partial\overline{\partial}(\rho_c^2)$.

For each (c, ε), Theorem 3.1 yields a solution $u_{c,\varepsilon} \in L^2_{\sigma_{c,\varepsilon}}(X_c, \wedge^{n,s-1} T^* X \otimes E)$ to the equation $\overline{\partial} u_{c,\varepsilon} = f$ on X_c such that

$$\int_{X_c} |u_{c,\varepsilon}|^2_{\sigma_{c,\varepsilon}} dV_{\sigma_{c,\varepsilon}} \leq \int_{X_c} ((A^{n,s}_{E,\sigma_{c,\varepsilon}})^{-1} f, f)_{\sigma_{c,\varepsilon}}\, dV_{\sigma_{c,\varepsilon}}.$$

From Lemma 2.2, we obtain

$$\begin{aligned}\int_{X_c} ((A^{n,s}_{E,\sigma_{c,\varepsilon}})^{-1} f, f)_{\sigma_{c,\varepsilon}}\, dV_{\sigma_{c,\varepsilon}} &\leq \int_{X_c} ((A^{n,s}_{E,\sigma})^{-1} f, f)_\sigma\, dV_\sigma \\ &\leq \int_X \gamma^{-1}|f|^2_\sigma\, dV_\sigma < +\infty.\end{aligned}$$

Thus, the solutions $\psi_{c,\varepsilon}$ are uniformly bounded in L^2 norm on every compact subset of X. Since the closed unit ball of an Hilbert space is weakly compact, we can extract a subsequence

$$u_{c_m,\,\varepsilon_m} \longrightarrow u \in L^2_{\mathrm{loc}}$$

converging weakly in L^2 on any compact subset $K \subset X$, for some $c_m \longrightarrow +\infty$ and $\varepsilon_m \longrightarrow 0$. By the weak continuity of differentiations, we get again in the limit $\overline{\partial} u = f$. Also, for every compact set $K \subset X$, we get

$$\int_K |\psi|^2_\sigma dV_\sigma \leq \liminf_{m \longrightarrow \infty} \int_K |u_{c_m,\varepsilon_m}|^2_{\sigma_{c_m,\varepsilon_m}} dV_{\sigma_{c_m,\varepsilon_m}}$$

by weak L^2_{loc} convergence. Finally, we let K increase to X and conclude that the desired estimate holds on all of X. □

Theorem 3.3 (*cf. [9]*)**.** *Let X be an n-dimensional Kähler manifold. Assume that X is weakly pseudoconvex. Let E be a holomorphic vector bundle over X and $\phi \in L^1_{loc}$ be a weight function which is plurisubharmonic and of class C^2 in X. Suppose that the curvature form $\Theta(E)$ and ϕ satisfy the inequality*

$$\Theta(E) + i\partial\overline{\partial}\phi \otimes Id_E \geq \gamma\omega \otimes Id_E,$$

where γ is a positive continuous function on X. Then, for $f \in L^2_{n,s}(X, loc, E)$ with $s \geq 1$ satisfying $\overline{\partial} f = 0$ and $\int_X |f|^2_{i\partial\overline{\partial}\phi} e^{-\phi} dv < +\infty$, there exists $u \in L^2_{n,s-1}(X, loc, E)$ such that $\overline{\partial} u = f$ and

$$\int_X |u|^2 e^{-\phi} dv \leq \int_X |f|^2_{i\partial\overline{\partial}\phi} e^{-\phi} dv.$$

Proof. Apply the general estimates to the bundle E deduced from E by multiplication of the metric by $e^{-\phi}$; we have

$$i\,\Theta(E_\phi) = i\,\Theta(E) + i\partial\overline{\partial}\phi.$$

It is not necessary here to assume in addition that $u \in L^2_{n,s-1}(X, E_\phi)$. In fact, u is in L^2_{loc} and we can exhaust X by the relatively compact weakly pseudoconvex domains $\{X_c = x \in X; \psi(x) < c\}$, where $\psi \in C^\infty(X, \mathbb{R})$ is a plurisubharmonic exhaustion function (note that $-\log(c - \psi)$ is also such a function on X_c). We get therefore solutions f_c on X_c with uniform L^2 bounds; any weak limit f gives the desired solution. □

Remark 3.4. To obtain the same result of Theorem 3.3 for (r, s)-form as well, we just observe that we have a canonical duality pairing $\wedge^m T\Omega \otimes \wedge^m T^*\Omega \longrightarrow \mathbb{C}$, hence a (r, s)-form with values in E can be viewed as a section of

$$\wedge^{r,s} T^*\Omega \otimes E = \wedge^{0,s} T^*\Omega \otimes \wedge^r T^*\Omega \otimes E = \wedge^{n,s} T^*\Omega \otimes \tilde{E},$$

where $\tilde{E}$ is the holomorphic vector bundle

$$\tilde{E} = \wedge^n T\Omega \otimes \wedge^r T^*\Omega \otimes E = \wedge^{n-r} T\Omega \otimes E,$$

through the contraction pairing

$$\wedge^n T\Omega \otimes \wedge^r T^*\Omega \simeq \wedge^{n-r} T\Omega.$$

Thus $L^2_{r,s}(\Omega, E) = L^2_{n,s}(\Omega, \wedge^{n-r} T\Omega \otimes E)$.

4. Sobolev regularity of the Bergman projection

In this section we prove the main results of this paper.

Lemma 4.1. *Let $\Omega \Subset X$ be a* $\log\delta$*-pseudoconvex domain in an n-dimensional Kähler manifold X. Let $\psi_k = -k\log\delta$, where k is a positive constant. Then, there exists $\alpha \in (0, 1)$ small enough such that*

$$i\partial\psi_k \wedge \overline{\partial}\psi_k < \left(\frac{k}{\alpha}\right) i\partial\overline{\partial}\psi_k \text{ on } \Omega. \tag{4.1}$$

Proof. As in Ohsawa and Sibony [19] and Cao and Shaw [7], by using (2.5), there exists a constant $\alpha \in (0, 1)$ such that

$$i\partial\overline{\partial}(-\delta^\alpha) > 0 \quad \text{on } \Omega.$$

Since

$$i\partial\overline{\partial}(-\delta^\alpha) = \alpha\,\delta^\alpha\left((1-\alpha)\frac{i\partial\delta \wedge \overline{\partial}\delta}{\delta^2} + \frac{i\partial\overline{\partial}(-\delta)}{\delta}\right),$$

then

$$(1-\alpha)\frac{i\partial\delta \wedge \overline{\partial}\delta}{\delta^2} + \frac{i\partial\overline{\partial}(-\delta)}{\delta} > 0 \quad \text{on } \Omega. \tag{4.2}$$

But

$$i\partial\overline{\partial}(-\log\delta) = \frac{i\partial\delta \wedge \overline{\partial}\delta}{\delta^2} + \frac{i\partial\overline{\partial}(-\delta)}{\delta}.$$

It follows, from (4.2), that

$$i\partial\overline{\partial}(-\log\delta) > \alpha\frac{i\partial\delta \wedge \overline{\partial}\delta}{\delta^2}. \tag{4.3}$$

Since $\partial\psi_k = -k\frac{\partial\delta}{\delta}$ and $\overline{\partial}\psi_k = -k\frac{\overline{\partial}\delta}{\delta}$, then

$$i\partial\psi_k \wedge \overline{\partial}\psi_k = k^2\frac{i\partial\delta \wedge \overline{\partial}\delta}{\delta^2}.$$

Thus, from (4.3), we obtain

$$i\partial\overline{\partial}(-\log\delta) > \left(\frac{\alpha}{k^2}\right) i\partial\psi_k \wedge \overline{\partial}\psi_k.$$

Thus (4.1) follows. □

Theorem 4.2. *Let X be an n-dimensional Kähler manifold and E be a holomorphic vector bundle over X. Let $\Omega \Subset X$ be a $\log\delta$-pseudoconvex domain and $\phi_\beta = -\beta\log\delta$, where $\beta \geq 0$ and δ is the function defined in Definition 2. Let $m_r(\Omega; E)$ be defined as in (2.4) such that $m_r(\Omega; E) > 0$. Then, for $f \in L^2_{r,s}(\Omega, \delta^\beta, E)$, $1 \leq s \leq n$, with $\overline{\partial} f = 0$, there exists $u \in L^2_{r,s-1}(\Omega, \delta^\beta, E)$ such that $\overline{\partial}u = f$ and*

$$\int_\Omega |u|^2\delta^\beta dv \leq \int_\Omega |f|^2_{i\partial\overline{\partial}\phi_\beta}\delta^\beta dv. \tag{4.4}$$

Proof. Since $m_r(\Omega; E) > 0$ and by using (2.4) and (2.5), there exists a positive constant m such that

$$\Theta(\wedge^{n-r} T\Omega \otimes E) + \beta\, i\, \partial\overline{\partial}(-\log\delta) \otimes \mathrm{Id}_{\wedge^{n-r} T\Omega\otimes E} \geq \; [m + \beta\, C]\, \omega \otimes \mathrm{Id}_{\wedge^{n-r} T\Omega\otimes E}.$$

Thus, according to Remark 3.4, by using the solution to the $\overline{\partial}$-equation for (n, s)-forms of Theorem 3.3 with values in the holomorphic vector bundle $\wedge^{n-r} T\Omega \otimes E$ and with the weight function $\phi_\beta = -\beta\log\delta$, there exists $u \in L^2_{r,s-1}(\Omega, \delta^\beta, E)$ such that $\overline{\partial}u = f$ and

$$\int_\Omega |u|^2\, \delta^\beta\, dV \leq \int_\Omega |f|^2_{i\partial\overline{\partial}\phi_\beta}\delta^\beta\, dV.$$

Thus the proof follows. □

Remark 4.3. One can always select the solution u of Theorem 4.2 satisfying the additional property $u \in \left(\ker(\overline{\partial}, E)\right)^\perp$ (otherwise, just replace u by its orthogonal projection on $\left(\ker(\overline{\partial}, E)\right)^\perp$). The solution u satisfies the additional property $u \in L^2_{r,s-1}(\Omega, e^{-\phi_\beta}, E) \cap \left(\ker(\overline{\partial}, E)\right)^\perp$, i.e., satisfies the following

$$\int_\Omega e^{-\phi_\beta}\ {}^t u \wedge \star\overline{h\,\upsilon} = 0, \tag{4.5}$$

for any $\overline{\partial}$-closed form $\upsilon \in L^2_{r,s-1}(\Omega, e^{-\phi_\beta}, E)$. Hence the theorem implies that if u is any form which is orthogonal to $L^2_{r,s-1}(\Omega, e^{-\phi_\beta}, E) \cap \ker(\overline{\partial}, E)$, u satisfies

$$\int_\Omega |u|^2 e^{-\phi_\beta}\, dV \leq \int_\Omega |\overline{\partial}u|^2_{i\partial\overline{\partial}\phi_\beta} e^{-\phi_\beta}\, dV. \tag{4.6}$$

Theorem 4.4. *Let X, Ω and E be the same as in Theorem 4.2. Let ϕ and ψ be plurisubharmonic and of class C^2 in Ω, and assume $\psi_k \geq 0$ satisfies (4.1) with $r < 1$. Let $0 \leq r \leq n$ such that $m_r(\Omega; E) > 0$. Let $f \in L^2_{r,s}(\Omega, \delta^{\beta-k}, E)$, $1 \leq s \leq n$, with $\overline{\partial} f = 0$ and let $u = \overline{\partial}^*_\beta N^\beta f$ be the solution to the equation $\overline{\partial}u = f$ in $L^2_{r,s}(\Omega, \delta^\beta, E)$. Thus, there exists a positive constant C such that*

$$\int_\Omega |u|^2\delta^{\beta-k} dV \leq C \int_\Omega |f|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\, \delta^{\beta-k} dV. \tag{4.7}$$

Proof. Since $f \in L^2_{r,s}(\Omega, \delta^\beta, E)$, thus by Theorem 4.2 there is a solution $u \in L^2_{r,s-1}(\Omega, \delta^\beta, E) \cap (\ker(\overline{\partial}, E))^\perp$. Put $g = u\, e^{\psi_k} = u\, \delta^{-k}$, then

$$\int_\Omega |u|^2\delta^{\beta-k} dV = \int_\Omega |g|^2\delta^{\beta+k} dV. \tag{4.8}$$

Thus, from (4.5), we have

$$0=\int_{\Omega}e^{-\phi_\beta}\ {}^t u\wedge\star\#_E\upsilon=\int_{\Omega}e^{-(\psi_k+\phi_\beta)}\ {}^t g\wedge\star\#_E\upsilon$$
$$=\int_{\Omega}\delta^{\beta+k}\ {}^t g\wedge\star\#_E\upsilon.$$

Thus, u is orthogonal to all $\overline{\partial}$-closed forms of $L^2_{r,s-1}(\Omega,\delta^{\beta+k},E)$, so by (4.6) we have

$$\int_{\Omega}|u|^2\delta^{\beta+k}dV\leq\int_{\Omega}|\overline{\partial}u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta+k}dV.$$

Thus, from (4.8), we obtain

$$\int_{\Omega}|u|^2\delta^{\beta-k}dV\leq\int_{\Omega}|\overline{\partial}g|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta+k}dV. \tag{4.9}$$

Since, for any two real numbers a and b, and for every $\varepsilon>0$, we have

$$2|a|\,|b|\leq\varepsilon|a|^2+\frac{1}{\varepsilon}|b|^2,$$

and since $\overline{\partial}g=\delta^{-k}\,\overline{\partial}u+\delta^{-k}\,\overline{\partial}\psi_k\wedge u$. Thus, from (4.9), we obtain

$$\begin{aligned}\int_{\Omega}|u|^2\delta^{\beta-k}dV&\leq\int_{\Omega}|\,\overline{\partial}u+\overline{\partial}\psi_k\wedge u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV\\&\leq\int_{\Omega}|\overline{\partial}u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV+|\overline{\partial}\psi_k\wedge u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV\\&\quad+2|\overline{\partial}u|_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}|\overline{\partial}\psi_k\wedge u|_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV\\&\leq\left(1+\frac{1}{\varepsilon}\right)\int_{\Omega}|f|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV+(1+\varepsilon)\int_{\Omega}|\,\overline{\partial}\psi_k\wedge u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV.\end{aligned}$$

Since $i\partial\psi_k\wedge\overline{\partial}\psi_k<t\,i\,\partial\overline{\partial}\psi_k$ is valid for $0<t<1$. This means that the norm of the form $\overline{\partial}\psi_k$, measured in the metric with Kähler form $i\partial\overline{\partial}\psi_k$ is smaller than t at any point. Also, we can improve the estimate (4.4) by replacing $|f|_{i\partial\overline{\partial}\phi_\beta}e^{-\phi_\beta}$ by $|f|_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}e^{-\phi_\beta}$ without having to change the weight function from ϕ_β to $\psi_k+\phi_\beta$. Thus

$$|\,\overline{\partial}\psi_k\wedge u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\leq|\,\overline{\partial}\psi_k|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\,|u|^2\leq|\overline{\partial}\psi_k|^2_{i\partial\overline{\partial}\psi_k}\,|u|^2\leq t|u|^2. \tag{4.10}$$

By choosing ε so small such that $(1+\varepsilon)t<1$, we obtain

$$\int_{\Omega}|u|^2\delta^{\beta-k}dV\leq C\int_{\Omega}|f|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV,$$

with $C=\frac{\left(1+\frac{1}{\varepsilon}\right)}{[1-(1+\varepsilon)t]}$. □

We are now ready to prove the main theorem of this section.

Theorem 4.5. *Let $\Omega\Subset X$ be a $\log\delta$-pseudoconvex domain with Lipschitz boundary in an n-dimensional compact Kähler manifold and E be a holomorphic vector bundle over X. Let $0\leq r\leq n$ such that $m_r(\Omega;E)>0$. Then, for $\eta\in(0,1)$, the operators B, N and $\overline{\partial}^*N$ are exact regular in the Sobolev spaces $W^k_{r,s}(\Omega,E)$ for $0<k<\eta/2$, $0\leq s\leq n-1$. In other words, B, N and $\overline{\partial}^*N$ are continuous in $W^k_{r,s}(\Omega,E)$, $k<\eta/2$ and satisfies the following estimates:*

$$\|Bu\|^2_{W^{k/2}_{r,s}(\Omega,E)}\leq c_1\|u\|^2_{W^{k/2}_{r,s}(\Omega,E)}, \tag{4.11}$$

$$\|Nu\|_{W^{k/2}_{r,s}(\Omega,E)}\leq c_2\|u\|_{W^{k/2}_{r,s}(\Omega,E)}, \tag{4.12}$$

$$\|\overline{\partial}^*Nu\|_{W^{k/2}_{r,s}(\Omega,E)}\leq c_3\|u\|_{W^{k/2}_{r,s}(\Omega,E)}, \tag{4.13}$$

where c_1, c_2 and c_3 are positive constants depend only on k.

Proof. From the Kohn's formula, we have the following:

$$B^{\beta} = I - \overline{\partial}^{*}_{\beta} N^{\beta}_{r,s+1} \overline{\partial}. \tag{4.14}$$

For $u \in L^2_{r,s}(\Omega, \delta^{\beta-k}, E)$ and for $f \in L^2_{r,s}(\Omega, \delta^{\beta-k}, E) \cap \ker(\overline{\partial}, E)$, we have from (4.14) that

$$\begin{aligned}
\langle B^{\beta} u, f\rangle_{\beta,\,\Omega} &= \langle u - \overline{\partial}^{*}_{\beta} N^{\beta} \overline{\partial} u, f\rangle_{\beta,\,\Omega} \\
&= \langle u, f\rangle_{\beta,\Omega} - \langle \overline{\partial}^{*}_{\beta} N^{\beta} \overline{\partial} u, f\rangle_{\beta,\,\Omega} \\
&= \langle \delta^{-k} u, f\rangle_{\beta+k,\Omega} \\
&= \langle \delta^{-k} u, f\rangle_{\beta+k,\,\Omega} - \langle \overline{\partial}^{*}_{\beta+k} N^{\beta+k} \overline{\partial}(\delta^{-k} u), f\rangle_{\beta+k,\,\Omega} \\
&= \langle (I - \overline{\partial}^{*}_{\beta+k} N^{\beta+k} \overline{\partial})(\delta^{-k} u), f\rangle_{\beta+k,\,\Omega} \\
&= \langle B^{\beta+k}(\delta^{-k} u), f\rangle_{\beta+k,\,\Omega} \\
&= \langle \delta^{k} B^{\beta+k}(\delta^{-k} u), f\rangle_{\beta,\,\Omega}.
\end{aligned}$$

Thus we have $B^{\beta}(\delta^{k} B^{\beta+k}(\delta^{-k} u)) = B^{\beta} u$. Using (4.14), we get

$$\begin{aligned}
B^{\beta} u &= B^{\beta}(\delta^{k} B^{\beta+k}(\delta^{-k} u)) \\
&= (I - \overline{\partial}^{*}_{\beta} N^{\beta} \overline{\partial}) \delta^{k} B^{\beta+k}(\delta^{-k} u) \\
&= \delta^{k} B^{\beta+k}(\delta^{-k} u) - \overline{\partial}^{*}_{\beta} N^{\beta}(\overline{\partial} \delta^{k} \wedge B^{\beta+k}(\delta^{-k} u)) \\
&= \delta^{k} B^{\beta+k}(\delta^{-k} u) - k\, \overline{\partial}^{*}_{\beta} N^{\beta} \left(\frac{\overline{\partial} \delta}{\delta} \wedge \delta^{k} B^{\beta+k}(\delta^{-k} u) \right),
\end{aligned} \tag{4.15}$$

because $\overline{\partial} B^{\beta+k} = 0$. For simplicity, we write $\eta = \delta^{k}\, B^{\beta+k}(\delta^{-k}\, u)$. Then, for $u \in L^2_{r,s}(\Omega, \delta^{\beta-k}, E)$, we have

$$\begin{aligned}
\int_{\Omega} |\eta|^2\, \delta^{\beta-k}\, dV &= \int_{\Omega} \left| \delta^{k}\, B^{\beta+k}(\delta^{-k}\, u) \right|^2 \delta^{\beta-k}\, dV \\
&= \int_{\Omega} \left| B^{\beta+k}(\delta^{-k}\, u) \right|^2 \delta^{\beta+k}\, dV \\
&\leq \int_{\Omega} \left| \delta^{-k}\, u \right|^2 \delta^{\beta+k}\, dV \\
&= \int_{\Omega} |u|^2\, \delta^{\beta-k}\, dV.
\end{aligned} \tag{4.16}$$

Thus, from (4.7), we obtain

$$\int_{\Omega} \left| \overline{\partial}^{*}_{\beta} N^{\beta}(\overline{\partial}\psi_k \wedge \eta) \right|^2 \delta^{\beta-k}\, dV \leq c_1 \int_{\Omega} \left| \overline{\partial}\psi_k \wedge \eta \right|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)} \delta^{\beta-k}\, dV. \tag{4.17}$$

From (4.10), we obtain

$$\left| \overline{\partial}\psi_k \wedge \eta \right|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)} \leq \left| \overline{\partial}\psi_k \wedge \eta \right|^2_{i\partial\overline{\partial}\psi_k} \leq t\, |\eta|^2\, . \tag{4.18}$$

Substituting (4.16) and (4.18) into (4.17), we obtain

$$\int_{\Omega} \left| \overline{\partial}^{*}_{\beta} N^{\beta}(\overline{\partial}\psi_k \wedge \eta) \right|^2 \delta^{\beta-k}\, dV \leq c_1 t \int_{\Omega} |u|^2\, \delta^{\beta-k}\, dV. \tag{4.19}$$

Thus, by using (4.15), (4.18) and (4.19), we obtain

$$\left\| B^{\beta} u \right\|^2_{\beta-k,\,\Omega} \leq c_2\, \|u\|^2_{\beta-k,\,\Omega}\, . \tag{4.20}$$

Thus, the Bergman projection B^{β} maps $L^2_{r,s}(\Omega, \delta^{\beta-k}, E)$ boundedly to itself. Since $B^{\beta} u = (I - \overline{\partial}^{*}_{\beta} N^{\beta} \overline{\partial}) u$ and $\overline{\partial}^{*}_{\beta} N^{\beta} u = N^{\beta} \overline{\partial}^{*}_{\beta} u$, then $\overline{\partial}^{*}_{\beta} N^{\beta} u = \overline{\partial}^{*}_{\beta} N^{\beta} B^{\beta} u$ and we already know that B^{β} is bounded on $L^2_{r,s}(\Omega, \delta^{\beta-k}, E)$ we may as well assume from the start that $\overline{\partial} f = 0$. Then, by using (4.7) and (4.20), we obtain

$$\left\|\overline{\partial}^*_\beta N^\beta u\right\|^2_{\beta-k,\,\Omega} = \left\|\overline{\partial}^*_\beta N^\beta B^\beta u\right\|^2_{\beta-k,\,\Omega} \le c_1 \|B^\beta u\|^2_{\beta-k,\,\Omega} \le c_1 c_2\, \|u\|^2_{\beta-k,\,\Omega}\,. \tag{4.21}$$

Thus, the operator $\overline{\partial}^*_\beta N^\beta$ maps $L^2_{r,s}(\Omega, \delta^{\beta-k}, E)$ boundedly to itself. Thus by taking $\beta = 0$ and by using (4.20) and (4.21), we obtain

$$\begin{aligned} \|Bu\|^2_{-k} &\le c_3 \|u\|^2_{-k} \\ \|\overline{\partial}^* N u\|^2_{-k} &\le c_3 \|u\|^2_{-k}. \end{aligned} \tag{4.22}$$

By [14, Theorem 1.4.4.3], for $0 < k < \frac{1}{2}$, the space $W^{k/2}(\Omega, E)$ is continuously embedded in $L^2(\Omega, \delta^{-k}, E)$. Also since any harmonic section in $L^2(\Omega, \delta^{-k}, E)$ also lies in $W^{k/2}(\Omega, E)$ (see [17, Theorem 4.2], [11, Lemma 1] and also [8, Lemma 6.5.4 and Theorem C.4]). Then, from (4.22), we obtain

$$\|B\psi\|^2_{W^{k/2}(\Omega,E)} \le \|Bu\|^2_{-k} \le c_3 \|u\|^2_{-k} \le c_3 \|u\|^2_{W^{k/2}(\Omega,E)}. \tag{4.23}$$

It follows that the Bergman projection B is continuous in $W^k(\Omega, E)$, $0 < k < \eta/2$. Since $B = I - \overline{\partial}^* N \overline{\partial}$ and $\overline{\partial}^* N = N \overline{\partial}^*$, then $\overline{\partial}^* N u = \overline{\partial}^* N B u$ and

$$\|\overline{\partial}^* N u\|_k = \|\overline{\partial}^* N B u\|_k \le c_1 \|Bu\|_k \le c_1 c_3 \|u\|_k. \tag{4.24}$$

Using (4.24) and as in (4.23), we obtain that $\overline{\partial}^* N$ is bounded operator on $W^{k/2}_{r,s}(\Omega, E)$ for any $s \ge 1$ and satisfies

$$\|\overline{\partial}^* N u\|^2_{W^{k/2}(\Omega,E)} \le c_2 \|u\|^2_{W^{k/2}(\Omega,E)}.$$

Then $\overline{\partial}^* N$ is continuous in $W^k(\Omega, E)$, $0 < k < \eta/2$. Due to the result of Boas–Straube [3], the $\overline{\partial}$-Neumann operator N is regular if and only if the Bergman projection B is regular. Thus the exact regularity of N follows. □

Corollary 1. *Under the same assumption of Theorem* 4.5 *and for* $0 \le s \le n-1$*, the operators* N, $\overline{\partial}^* N$ *and* B *are exact regular in the Sobolev space* $W^{-k}_{r,s}(\Omega, E)$ *for* $0 < k < \eta/2$, $0 \le s \le n-1$ *and satisfy the following estimates:*

$$\begin{aligned} \|Bu\|^2_{W^{-k/2}_{r,s}(\Omega,E)} &\le c_4 \|u\|^2_{W^{-k/2}_{r,s}(\Omega,E)}, \\ \|Nu\|_{W^{-k/2}_{r,s}(\Omega,E)} &\le c_5 \|u\|_{W^{-k/2}_{r,s}(\Omega,E)}, \\ \|\overline{\partial}^* N u\|_{W^{-k/2}_{r,s}(\Omega,E)} &\le c_6 \|u\|_{W^{-k/2}_{r,s}(\Omega,E)}, \end{aligned}$$

where c_4, c_5 *and* c_6 *are positive constants depend only on* k.

Proof. By using (2.2), (4.11), (4.12) and (4.13) the result follows. □

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Original article

Investigation and numerical solution of some 3D internal Dirichlet generalized harmonic problems in finite domains

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Abstract

A Dirichlet generalized harmonic problem for finite right circular cylindrical domains is considered. The term "generalized" indicates that a boundary function has a finite number of first kind discontinuity curves. It is shown that if a finite domain is bounded by several surfaces and the curves are placed in arbitrary form, then the generalized problem has a unique solution depending continuously on the data. The problem is considered for the simple case when the curves of discontinuity are circles with centers situated on the axis of the cylinder. An algorithm of numerical solution by a probabilistic method is given, which in its turn is based on a computer simulation of the Wiener process. A numerical example is considered to illustrate the effectiveness and simplicity of the proposed method.

Keywords: Dirichlet generalized problem; Harmonic function; Cylindrical domain; Discontinuity curve; Probabilistic solution

1. Introduction

It is known (see e.g., [1–5]) that in practical stationary problems (for example, for the determination of the temperature of the thermal field or the potential of the electric field, and so on) there are cases when the corresponding boundary function has a finite number of first kind discontinuity points (in the case of 2D) or curves (in the case of 3D). Problems of such type are known as Dirichlet generalized problems [1], and their solutions represent generalized solutions, respectively. In general, it is known (see [3,6]) that methods used to obtain an approximate solution to ordinary boundary problems are less suitable (or not suitable at all) for solving boundary problems with singularities.

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In particular, the convergence is very slow in the neighborhood of boundary singularities and, consequently, the accuracy of the approximate solution of the generalized problem is very low.

The choice and construction of the computational schemes (algorithms) mainly depend on the problem class, its dimension, geometry and location of singularities on the boundary. e.g., plane Dirichlet generalized problems for harmonic functions with concrete location of discontinuity points in the cases of simply connected domains are considered in [3,7], and general cases for finite and infinite domains are studied in [8–12].

In the case of spatial (3D) harmonic generalized problems, due to their higher dimension, the difficulties become more significant. On the other hand, the study of such problems from the viewpoint of correctness and approximate solution is of certain interest, since, some processes occur whose investigation is reduced to solution of problems of the indicated type (see e.g., [3,4]). In the 3D case, there does not exist a standard scheme which can be applied to a wide class of domains. In the classical literature, simplified, or so called "solvable" generalized problems (problems whose "exact" solutions can be constructed by series, whose terms are represented by special functions) are considered, and for their solution the classical method of separation of variables is mainly applied and therefore the accuracy of the solution is rather low. In particular, in the mentioned problems, the boundary functions (conditions) are mainly constants, and in the general case, the analytic form of the "exact" solution is so difficult in the sense of numerical implementation, that it only has theoretical significance (see e.g., [5]).

As a consequence of the above, from our viewpoint, the construction of high accuracy and effectively realizable computational schemes for the approximate solution of 3D generalized harmonic problems (whose application is possible to a wide class of domains) has both theoretical and practical importance.

2. Statement of the problem and properties of its solution

Let D be a finite right circular cylindrical domain in the Euclidian space E_3, bounded by a surface S. Without loss of generality we assume that the coordinate axis ox_3 of the Cartesian coordinates $ox_1x_2x_3$ is the axis of the cylinder D. We consider the Dirichlet generalized problem for the Laplace equation.

Problem A. Function $g(y)$ is given on the boundary S of the domain D and is continuous everywhere, except a finite number of circles $l_1, l_2, \ldots, l_n$ which represent discontinuity curves of the first kind for the function $g(y)$. Besides, it is assumed that the centers of these circles are situated on the axis of the cylinder D. It is required to find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D} \setminus \bigcup_{k=1}^{n} l_k)$ satisfying the conditions:

$$\Delta u(x) = 0, \quad x \in D, \tag{2.1}$$

$$u(y) = g(y), \quad y \in S,\ y \overline{\in} l_k \ (k = \overline{1, n}), \tag{2.2}$$

$$|u(y)| < c, \quad y \in \overline{D}, \tag{2.3}$$

where $\varDelta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and c is a real constant.

For the sake of simplicity, in the following we assume that the circles $l_k (k = \overline{1, n})$ are situated on S preserving the order of succession in the direction of axis ox_3. It is evident that the surface S is divided into parts $S_k (k = \overline{1, n+1})$ by the circles l_k or $S = \bigcup_{k=1}^{n+1} S_k$. On the basis of the above, the boundary function $g(y)$ has the following form

$$g(y) = \begin{cases} g_1(y), & y \in S_1, \\ g_2(y), & y \in S_2, \\ \ldots\ldots\ldots\ldots\ldots \\ g_{n+1}(y), & y \in S_{n+1}, \end{cases} \tag{2.4}$$

where the functions $g_k(y) = g_k(y_1, y_2, y_3)$, $y \in S_k$ are continuous on the parts S_k of S, respectively.

Note that the additional requirement (2.3) of boundedness concerns actually only the neighborhoods of the discontinuity curves of the function $g(y)$ and it plays an important role in the extremum principle (see Theorem 1).

Remark 1. If inside the surface S there is a vacuum then we have the generalized problem with respect to a right circular cylindrical shell.

In order to study the properties of the solution of Problem ((2.1), (2.2), (2.3)), we will first prove the generalized extremum principle in a more general case. Let us consider a finite domain D in the space E_3 with surface S (D may be bounded by several surfaces).

Theorem 1. *If the function $u(x)$ is harmonic in D, bounded in $\overline{D}$ and takes a value $g(y)$ on the boundary S, which is continuous on S everywhere, except a finite number of curves $l_1, l_2, \ldots, l_n$ (with discontinuities of first kind), then*

$$\min_{x\in S} u(x) < \underset{x\in D}{u(x)} < \max_{x\in S} u(x), \tag{2.5}$$

where for $x \in S$ it is meant that $x \overline{\in} l_k$ $(k = \overline{1,n})$.

Proof. Let $M = \max u(x)$, $x \in S'$, $S' = S \setminus \bigcup_{k=1}^{n} l_k$ and consider function

$$v(x) = M + \varepsilon \sum_{k=1}^{n} \frac{1}{r_k}, \quad x \in D. \tag{2.6}$$

In (2.6): ε is an arbitrary positive number, r_k is the minimal distance from the considered point x to the kth curve of discontinuity l_k or $r_k = \min \rho(x; y^k)$, where y^k-is a point on the curve l_k. Evidently, the function $v(x)$ is harmonic and larger than M in D, continuous in $\overline{D}$ everywhere, except curves l_k and $\lim v(x) = \infty$ for $x \to l_k$. Assume that $C(y^k, \delta)$ are kernels with radius δ and with centers at points y^k of the curves $l_k (k = \overline{1,n})$. At passing by point y^k the line l_k by the kernel $C(y^k, \delta)$ we obtain certain domain T_k, respectively. It is evident that $T_k \to l_k$ when $\delta \to 0$.

Let us consider the closed domain $\overline{D_\delta} = \overline{D} \setminus \bigcup_{k=1}^{n} T_k$. The function $v(x) - u(x)$ is continuous in $\overline{D}_\delta$, harmonic in D_δ and $v(x) - u(x) > 0$ on the common part of the boundaries D and D_δ. For sufficiently small δ the above inequality is also valid on the surfaces of the domains T_k (since the function $u(x)$ is bounded in $\overline{D}$ and for $\delta \to 0$ the values $v(x)$ increase infinitely on the surfaces of domains T_k). Thus, from the usual extremum principle we have $u(x) < v(x)$, $x \in D_\delta$, and consequently

$$u(x) < v(x), \quad x \in D. \tag{2.7}$$

Indeed, any point x in the domain D belongs to some domain D_δ for arbitrarily small δ.

Since $u(x)$ does not depend on ε, from (2.7) we obtain $u(x) < M$, $x \in D$ or

$$u(x)_{x\in D} < \max_{x\in S'} u(x)$$

for any fixed point x in the domain D when $\varepsilon \to 0$.

Now, if in the role of function $v(x)$ we take

$$v(x) = m - \varepsilon \sum_{k=1}^{n} \frac{1}{r_k}, \quad x \in D,$$

where $m = \min u(x)$, $x \in S'$, then the inequality

$$u(x)_{x\in D} > \min_{x\in S'} u(x),$$

can be proved in a similar way.

Thus, for the solution of Problem A, the generalized extremum principle (2.5) is valid. □

It should be noted that the following results can be obtained from Theorem 1.

Corollary 1. *If the generalized functions (in the sense of Theorem 1) $u(x)$ and $v(x)$ are harmonic in D, continuous in $D' = \overline{D} \setminus \bigcup_{k=1}^{n} l_k$ and if $u(x) \le v(x)$ on S', then $u(x) \le v(x)$, $x \in D$.*

Indeed, the function $v(x) - u(x)$ is continuous on S' and harmonic in D and $v(x) - u(x) \ge 0$ on S'. Due to Theorem 1 $v(x) - u(x) \ge 0$, $x \in D$ or $u(x) \le v(x)$, $x \in D$.

Corollary 2. *If the functions $u(x)$ and $v(x)$ are harmonic in D and continuous in D', and if $|u(x)| \le v(x)$ on S', then $|u(x)| \le v(x)$, $x \in D$.*

From the conditions it follows that $-v(x) \le u(x) \le v(x)$, $x \in S'$.

Applying twice Corollary 1, we have $-v(x) \le u(x) \le v(x)$, $x \in D$ or $|u(x)| \le v(x)$, $x \in D$.

Corollary 3. *For the function $u(x)$ which is harmonic in D and continuous in D' the inequality $|u(x)| \leq \max|u||_{S'}$, $x \in D'$ is valid.*

In order to prove this we put $v = \max|u||_{S'}$ and use Corollary 2.

Now the theorem for the uniqueness of solution of boundary Problem A can be easily proved.

Theorem 2. *The generalized spatial inner Dirichlet problem for the Laplace equation cannot have two different solutions.*

Proof. Assume that there exist two different functions $u_1(x)$ and $u_2(x)$, satisfying the conditions of the problem. Their difference $u(x) = u_1(x) - u_2(x)$ is harmonic in the domain D, bounded in $\overline{D}$ and $u(x) = 0$, $x \in S'$, From Theorem 1 it follows, that $u(x) \equiv 0$, $x \in D$, i.e. $u_1(x) = u_2(x)$, $x \in D$. The theorem is thus proved. □

Theorem 3. *The solution of the generalized spatial inner Dirichlet problem for the Laplace equation depends continuously on the boundary data.*

Proof. It is known [2], that a problem is called physically definite (or stable), if a small change in the conditions, determining the problem solution (boundary conditions in the given case), causes a small change of the solution itself.

Let $u_1(x)$ and $u_2(x)$ be generalized solutions of the problem and which satisfy the condition

$$|u_1(x) - u_2(x)| \leq \varepsilon, \quad x \in S'. \tag{2.8}$$

Then the same inequality is true in D. Indeed, the functions $u(x) = u_1(x) - u_2(x)$ and $v(x) = \varepsilon$ are harmonic in D and continuous in D', therefore due to Corollary 2 of Theorem 1, inequality (2.8) is valid in D.

Thus the theorem is proved. □

3. A method of probabilistic solution

It is known [13] that a relation between the theory of probability and the Dirichlet problem for Laplace's equation was observed long before the general theory of Markov's processes arose (the works by G. Phillips and N. Wiener (1923), R. Courant, K. Fredrichs and Kh. Levi (1928)). This idea was further developed in the works of A.Ya. Khintchin (1933) and I.G. Petrovski (1934).

This idea obtained a completed form by E.B. Duenkin [13]. He obtained a formula which expresses the relation between a solution of a Dirichlet ordinary (or generalized) boundary problem for Laplace's equation and the Wiener (diffusion) process, when the problem dimension is $n \geq 2$.

In particular, E.B. Duenkin proved a general theorem which for $n = 3$ states:

Theorem 4. *If a finite domain $D \in E_3$ is bounded by a piecewise smooth surface S and $g(y)$ is a continuous (or discontinuous) bounded function on S, then the solution of the Dirichlet ordinary (or generalized) boundary problem for the Laplace equation at the fixed point $x \in D$ has the form*

$$u(x) = M_x g(x(t)). \tag{3.1}$$

In (3.1): $M_x g(x(t))$ is the mathematical expectation of the values of the boundary function $g(y)$ at the random intersection points of the Wiener process and the boundary S; t is the moment of first exit of the Wiener process $x(t) = (x_1(t), x_2(t), x_3(t))$ from the domain D. It is assumed that the starting point of the Wiener process is always $x(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0)) \in D$, where the value of the desired function is being determined. If the number N of the random intersection points $y^i = (y_1^i, y_2^i, y_3^i) \in S (i = \overline{1, N})$ is sufficiently large, then according to the law of large numbers, from (3.1) we have

$$u(x) \approx u_N(x) = \frac{1}{N} \sum_{i=1}^{N} g(y^i) \tag{3.2}$$

or $u(x) = \lim u_N(x)$ for $N \to \infty$, in the probabilistic sense. Thus, in the presence of the Wiener process the approximate value of the probabilistic solution to Problem A at a point $x \in D$ is calculated by formula (3.2).

Thus, on the basis of Theorem 4, the existence of solution of the Dirichlet generalized problem in the case of Laplace's equation for a sufficiently wide class of domains is shown. Besides, we have also an explicit formula giving such a solution.

Remark 2. If the finite domain D is bounded by several surfaces (or $S = \bigcup_{k=1}^{m} S^k$ and $S^k \cap S^j = \varnothing$ for $k \neq j$), then instead of formula (3.2) we have the following formula

$$u(x) \approx u_N(x) = \frac{1}{N} \sum_{k=1}^{m} \sum_{i=1}^{N_k} g^k(y^{k,i}). \tag{3.3}$$

In (3.3): $N = N_1 + N_2 + \cdots + N_m$; $g^k(y)$ is a boundary function on S^k; N_k is the number of the intersection points $y^{k,i}$ ($k = \overline{1,m}$; $i = \overline{1,N_k}$) of the Wiener process and the surface S^k. It is evident, that it is not necessary for discontinuity curves to be situated on all S^k.

Analogously to the considered cases (see [14–18]), on the basis of Theorem 4, the probabilistic solution of Problem A consists in the realization of the Wiener process using the three-dimensional generator, which gives three independent values $w_1(t), w_2(t), w_3(t)$. In our case the Wiener process is realized by computer simulation. In particular, for the computer simulation of the Wiener process we use the following recursion relations:

$$\begin{aligned} x_1(t_k) &= x_1(t_{k-1}) + w_1(t_k)/kv, \\ x_2(t_k) &= x_2(t_{k-1}) + w_2(t_k)/kv, \\ x_3(t_k) &= x_3(t_{k-1}) + w_3(t_k)/kv, \quad (k = 1, 2, \ldots), \end{aligned} \tag{3.4}$$

with the help of which the coordinates of the point $x(t_k) = (x_1(t_k),\ x_2(t_k), x_3(t_k))$ are being determined. In (3.4): $w_1(t_k), w_2(t_k), w_3(t_k)$ are three normally distributed independent random numbers for the kth step, with zero means and variances one; kv is a quantification number and when $kv \to \infty$, then the discrete Wiener process approaches the continuous Wiener process. In the implementation, the random process is simulated at each step of the walk and continues until it crosses the boundary.

It is known that there exist two principles for generating random numbers, physical and programmatic:

1. The physical principle of generation gives real random numbers but its realization is connected with computationally expensive, especially in the multidimensional case, and therefore its application is not practical.
2. In spite of a great number of methods the generating random numbers, they also have disadvantages which are contained in the generating principle itself. Firstly, they are pseudo-random, and not real random numbers. Besides, we can observe periodicity at generating such numbers. In particular, when solving the Dirichlet boundary problems for Laplace's equation it is possible to use pseudo-random numbers. In our computations generation of pseudo-random numbers is done in MATLAB.

4. Numerical example

We consider a numerical example from [3,4] where it is solved by the method of separation of variables. In particular, Problem A is considered for the finite right circular cylinder $D(0 \le r \le a, 0 \le x_3 \le h)$, in which $n = 2$ and l_1, l_2 are the circles of the bases of the cylinder. Besides, it is assumed that the boundary function $g(y)$ (potential) has the form

$$g(y) = \begin{cases} 0, & y \in S_1, \\ v = const, & y \in S_2, \\ 0, & y \in S_3, \end{cases} \tag{4.1}$$

where S_1, S_3 are the bases and S_2 is the lateral surface of the cylinder, respectively. In [3] it is noted that fields of these types occur in electron-optical apparatuses.

(a) In the conditions (4.1) the "exact" solution to Problem A obtained by G. Grinberg and W.R. Smythe has the following form (in cylindrical coordinates)

$$W(r, x_3) = \frac{4v}{\pi} \sum_{k=0}^{\infty} \frac{I_o\left[\frac{(2k+1)\pi r}{h}\right]}{I_o\left[\frac{(2k+1)\pi a}{h}\right]} \frac{\sin\frac{(2k+1)\pi x_3}{h}}{2k+1} \equiv \sum_{k=0}^{\infty} \omega_k(r, x_3), \tag{4.2}$$

Table 1

(r, x_3) $r = 0$	$w_m(r, x_3)$ $m = 10$	(r, x_3) $r = 0.5$	$w_m(r, x_3)$ $m = 1000$	$w_m(r, x_3)$ $m = 4000$
0.0001	0.000252825	0.0001	0.391704	1.13433
0.0005	0.00126412	0.0005	1.17898	0.94994
0.001	0.00252824	0.001	0.902825	0.974749
0.005	0.0126403	0.005	0.979786	0.994937
0.01	0.0252753	0.01	0.989891	0.997472
0.05	0.125528	0.05	0.998065	0.999516
0.1	0.245902	0.1	0.999167	0.999792
0.2	0.454543	0.2	0.999832	0.999958
0.3	0.604858	0.3	1.00012	1.00003
0.4	0.69272	0.4	1.00027	1.00007
0.5	0.721326	0.5	1.00032	1.00008

where $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$, h is a height of the cylinder, and a is a radius of the bases. In (4.2) $I_0(x)$ is Bessel's function of order zero. Namely,

$$\begin{aligned} I_0(x) &\equiv J_0(ix) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}, \quad \text{where } x \in R, \\ I_0(0) &= 1 \quad \text{and} \quad I_0(x) \to \frac{e^x}{\sqrt{2\pi x}} \quad \text{for } x \to \infty. \end{aligned} \tag{4.3}$$

It is evident, that for the solution $W(r, x)$ the boundary conditions are satisfied on the bases S_1 and S_3 or $W(r, 0) = W(r, h) = 0$, where $0 \le r \le a$.

From (4.3), it is easy to see that the series (4.2) converges rapidly for all points $(r, x_3) \in D$, when $0 \le r < a$, especially for $r = 0$. If $r = a$, then the rate of convergence becomes worse on S_2, especially in the neighborhood of curves l_1 and l_2 (i.e., when $(r, x_3) \in S_2$ and $x_3 \to 0$ or $x_3 \to h$). In particular, the convergence is very slow and consequently, the accuracy in the satisfaction of boundary condition on S_2 is very low. This is caused by the fact that, when $x_3 \to 0$ or $x_3 \to h$, all terms of the series (4.2) tend to zero.

Besides, it should be noted that the methods which are considered in [3,4], can be applied to solution of Problem A only when the discontinuity curves are the circles of the bases of the cylinder. In particular, if $n = 2$, then l_1, l_2 are the circles of bases of the cylinder, and if $n = 1$, then l_1 is one of these circles.

Since boundary condition (4.1) is independent of the angle of rotation with respect to ox_3 and symmetric with respect to the plane $x_3 = \frac{h}{2}$, the potential has the same properties. In the numerical experiments we took: $v = 1$, $h = 1$, $a = 0.5$.

In Table 1 the results of the calculations for the sum of the first $m + 1$ terms of the series (4.2) (which is denoted by $w_m(r, x_3)$) are given.

In Table 1, because of the above-mentioned, $w_m(r, x_3)$ is calculated at the points (r, x_3) ($r = 0$, 0.5 and $0 < x_3 \le 0.5$) which represent a certain interest. The numerical calculations have shown that practically $w_{10}(0, x_3) = w_m(0, x_3)$ when $m > 10$, therefore in Table 1 the results of calculations are given only for $m = 10$. For example, $\omega_{11}(0; 0.0001) \approx 0.1 * 10^{-17}$, and $\omega_{101}(0; 0.5) \approx -0.7 * 10^{-139}$ (see (4.2)).

It should be noted that in spite of the low accuracy of the solution $W(r, x_3)$ (on the basis of the extremum principle and condition (4.1)) $|u(x) - W(r, x_3)|$ is minimal on the axis, where $u(x)$ is the exact solution of Problem A.

(b) In order to determine the intersection points $y^i = (y_1^i, y_2^i, y_3^i)(i = \overline{1, N})$ of the Wiener process and of the surface S, we operate in the following way. During the implementation of the Wiener process, for each current point $x(t_k)$, defined from (3.4), its location with respect to S is checked. In particular: if $x(t_k) \in D$ then the Wiener process is continued by (3.4); if $x(t_k) \in S$ then $y^i = x(t_k)$, in this case, if $y^i \in l_1$ or $y^i \in l_2$ then we always assume that $y^i \in S_1$ or $y^i \in S_2$, respectively.

Let $x(t_{k-1}) \in D$ for the moment $t = t_{k-1}$ and $x(t_k) \overline{\in} \overline{D}$ for the moment $t = t_k$. In this case, for the approximate determination of the point y^i, an equation of a line l passing through the points $x(t_{k-1})$ and $x(t_k)$ is first obtained. For

Table 2

$u_N(0, 0, x_3)$				
$(0, 0, x_3)$	$kv = 200$ $N = 50\,000$	$kv = 200$ $N = 100\,000$	$kv = 200$ $N = 200\,000$	$kv = 400$ $N = 200\,000$
(0, 0, 0.0001)	0.0096	0.0090	0.0092	0.0041
(0, 0, 0.0005)	0.0097	0.0095	0.0098	0.0058
(0, 0, 0.001)	0.0110	0.0108	0.0106	0.0064
(0, 0, 0.005)	0.0200	0.0197	0.0197	0.0157
(0, 0, 0.01)	0.0325	0.0322	0.0324	0.0285
(0, 0, 0.05)	0.1313	0.1322	0.1312	0.1287
(0, 0, 0.1)	0.2511	0.2515	0.2514	0.2477
(0, 0, 0.2)	0.4586	0.4587	0.4576	0.4554
(0, 0, 0.3)	0.6078	0.6037	0.6063	0.6055
(0, 0, 0.4)	0.6914	0.6961	0.6940	0.6908
(0, 0, 0.5)	0.7222	0.7194	0.7197	0.7210

Table 3

$u_N(0, 0, x_3)$, $N = 200\,000$				
$(0, 0, x_3)$	$kv = 1000$	$kv = 2000$	$kv = 4000$	$kv = 8000$
(0, 0, 0.0001)	0.0019	0.0010	0.0005	0.00034
(0, 0, 0.0005)	0.0028	0.0018	0.0016	0.0015
(0, 0, 0.001)	0.0037	0.0033	0.0031	0.0027
(0, 0, 0.005)	0.0141	0.0133	0.0121	
(0, 0, 0.01)	0.0268	0.0254	0.0253	
(0, 0, 0.05)	0.1262			
(0, 0, 0.1)	0.2455			

the intersection point y^i we have three cases: (1) $y^i = l \cap S_1$; (2) $y^i = l \cap S_3$; (3) $y^i = l \cap S_2$. In this case, if we have two intersection points x^* and x^{**} of the line l and the surface S_2, then in the role of the point y^i we choose the one (from x^* and x^{**}) for which $|x(t_k) - x|$ is minimal.

The results of the probabilistic solution to Problem A for cylinder D with boundary function (4.1) (calculated by formula (3.2)) are given in Tables 2 and 3. The numerical solutions $u_N(0, 0, x_3)$ are found at the same points of the axis for various N and kv, where N is the number of the implementation of the Wiener process, and kv is the number of the quantification.

The analysis of the results of numerical experiments show the following (see Tables 2 and 3): if the point $x(t_o)$ (at which the approximate solution of Problem A must be determined) is situated at a small distance from surface S, then the current point $x(t_k)$ must be under the condition of a random walk in D until it crosses S. To get this, the number kv must be taken sufficiently large.

Although, we have solved Problem A for $n = 2$, its solution under condition (2.4) is not difficult. Indeed, after finding the intersection point y^i of the Wiener process and the surface S, it is easy to establish the part of S in which the point y^i is situated. Moreover, in general, we can solve Problem A for all such locations of discontinuity curves, which give the possibility to of establishing the part of surface S where the intersection point is located.

From Tables 1–3 and the above mentioned it is clear that the results obtained by the probabilistic method are reliable, and this method is effective for numerical solution of problems of type A. In particular, the algorithm is sufficiently simple for numerical implementation.

It should be noted that if we apply the method of parallel programming to probabilistic solution of Problem A, then we will avoid that difficulty which is noted in point 2 of Section 3. Consequently, significantly less time will be needed for numerical realization and besides the accuracy of the obtained results will improve.

5. Concluding remarks

1. The method is suitable for the approximate solution of both ordinary and generalized Dirichlet problems for a rather wide class of domains, in the case of Laplace's equation. The results obtained using this method are reliable and characterized by an accuracy which is sufficient for many problems (see [14–18]).
2. The method is very simple and does not require sophisticated numerical methods and programming. Accordingly, it satisfies modern requests to numerical methods and algorithms.

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