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Original article

Harmonic analysis and integral transforms associated with a class of a system of partial differential operators

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Abstract

In this work, we consider a generalized system of partial differential operators, we define the related Fourier transform and establish some harmonic analysis results. We also investigate a wide class of integral transforms of Riemann–Liouville type. In particular we give a good estimate of these integrals kernels, inversion formula and a Plancherel theorem for the dual.

Keywords: Fourier transform; Convolution product; Integral operators; Mehler representation; Dual operator; Inversion formula

1. Introduction

The operator R_α defined by

$$R_\alpha(f)(r,x) = \begin{cases} \dfrac{2\alpha}{\Pi}\displaystyle\int_0^1\int_{-\Pi/2}^{\Pi/2} f(r\cos\theta, x + r\sin\theta)\cos^{2\alpha}\theta(1-s^2)^{\alpha-1}d\theta\, ds & \text{for } \alpha > 0 \\ \dfrac{1}{\Pi}\displaystyle\int_{-\Pi/2}^{\Pi/2} f(r\cos\theta, x + r\sin\theta)d\theta & \text{for } \alpha = 0, \end{cases}$$

and its dual ${}^tR_\alpha$ are of interest in several applications for example in image processing of the so-called aperture radar (SAR), data... [1], or in the linearized inverse scattering problems in acoustics [2,3]. These operators have been

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extensively studied in [4–14]. They arise in connection with the system

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}, \\ \Delta_2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}, \end{cases} \tag{1.1}$$

of partial differential operators [15,6,16–19].

The main aim of this article is to define and study a wide class of integral transforms which generalize the operators R_α and ${}^tR_\alpha$; $\alpha \geq 0$.

More precisely, we consider the singular partial differential operators Δ_1 and $\Delta_{2,A}$ such that

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}, \\ \Delta_{2,A} = \dfrac{\partial^2}{\partial r^2} + \dfrac{A'(r)}{A(r)}\dfrac{\partial}{\partial r} + \rho^2 - \dfrac{\partial^2}{\partial x^2}, \end{cases} \tag{1.2}$$

where ρ is non negative real number and A is a non negative function satisfying some properties.

First, we define a generalized Fourier transform $\mathcal{F}_A$ and generalized shift operator $\mathcal{T}_{(r,x)}$; $\Big((r,x) \in [0,\infty[\times\mathbb{R}\Big)$ related with $\Delta_{2,A}$. We give some harmonic analysis results associated with $\mathcal{F}_A$ and $\mathcal{T}_{(r,x)}$. Second we establish an integral representation of the eigenfunction of the operator $\Delta_{2,A}$. This result and by using the same techniques as Fitouhi [20,21], we define and study a wide class of integral transforms R_A and ${}^tR_A$ related with $\Delta_{2,A}$. More precisely we establish for these operators the same results given by Helgason [9], Ludwig [12] and Solmon [14] for the classical Radon transform on $\mathbb{R}^2$. Also, we define and characterize some spaces of functions on which R_A and ${}^tR_A$ are isomorphism.

The paper is arranged as follows. In Section 2, we recall some basic properties and results about the singular second order differential operator $\pounds_A = \frac{\partial^2}{\partial r^2} + \frac{A'(r)}{A(r)}\frac{\partial}{\partial r} + \rho^2$. In Section 3, we define a generalized Fourier transform $\mathcal{F}_A$ associated with the system (1.2) and we establish some harmonic analyses (inversion Formula, Paley–Wiener theorem and Plancherel theorem for $\mathcal{F}_A$). Also, we define and study a generalized shift operator $\mathcal{T}_{(r,x)}$; $(r,x) \in [0,\infty[\times\mathbb{R}$ and a generalized convolution product associated with $\mathcal{T}_{(r,x)}$. Section 4 deals with the integral representation of the eigenfunction related with $\Delta_{2,A}$ and the operator R_A and its dual ${}^tR_A$. In Section 5 we give an inversion formula for R_A, ${}^tR_A$ and Plancherel theorem for ${}^tR_A$.

2. Preliminaries of Chébli–Trimèche hypergroups

In this section we briefly recall some results of harmonic analysis related with the following second order singular differential operator on the half line:

$$\pounds = \pounds_A = \frac{\partial^2}{\partial r^2} + \frac{A'(r)}{A(r)}\frac{\partial}{\partial r} + \rho^2, \tag{2.3}$$

where A is continuous on $[0,\infty[$, twice continuously differentiable on $]0,\infty[$ and satisfies the conditions:

(1) $A(0) = 0$ and $A(x) > 0$ for $x > 0$.

(2) A is increasing and unbounded.

(3) $\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x)$ on a neighborhood of 0, where $\alpha > \frac{-1}{2}$ and B is an odd C^∞-function on $\mathbb{R}$.

(4) $\frac{A'(x)}{A(x)}$ is a decreasing C^∞-function on $]0,\infty[$ and $\lim_{+\infty} \frac{A'(x)}{A(x)} = 2\rho \geq 0$.

(5) There exists a constant $\delta > 0$, satisfying

$$\begin{cases} \dfrac{A'(r)}{A(r)} = 2\rho + F(r)\exp(-\delta r), & \text{for } \rho > 0. \\ \dfrac{A'(r)}{A(r)} = \dfrac{2\alpha+1}{r} + F(r)\exp(-\delta r), & \text{for } \rho = 0, \end{cases}$$

where F is an infinite function on $]0, \infty[$, bounded together with all their derivatives on all intervals $[r_0, \infty[$, $r_0 > 0$, [22–24,15,20,21,25,26,10,11,27–30].

From the properties of the function A, we deduce the following results:

Lemma 2.1 *([19])*.

(i) *For $\rho > 0$, we have*

$$A(x) \sim \exp(2\rho x), (x \longrightarrow +\infty).$$

(ii) *For $\rho = 0$, we have*

$$A(x) \sim x^{2\alpha+1}, (x \longrightarrow +\infty).$$

From [23,24,15,20,21,28,29,31,32] we have:

for $\lambda \in \mathbb{C}$ the equation

$$\pounds_A u = -\lambda^2 u \tag{2.4}$$

has a unique solution on $[0, \infty[$ satisfying $u(0) = 1$ and $u'(o) = 0$, which can be extended on $\mathbb{R}$ in an even C^∞ function denoted by φ_λ.

In the case of the Bessel operator that is $(A(x) = x^{2\alpha+1})$, this solution is $j_\alpha(\lambda x)$, where

$$\varphi_\lambda(x) = j_\alpha(\lambda x) = \begin{cases} \dfrac{2^\alpha \Gamma(\alpha+1)}{(\lambda x)^\alpha} J_\alpha(\lambda x) & \text{if } \lambda x \neq 0 \\ 1 & \text{if } \lambda x = 0. \end{cases} \tag{2.5}$$

In the case of the Jacobi operator that is $A(x) = (2\sinh(x))^{2\alpha+1}(2\cosh(x))^{2\beta+1}$, $\alpha \geq \beta > \frac{-1}{2}$;

$$\forall x \geq 0, \lambda \in \mathbb{C};\ \varphi_\lambda(x) = \varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha+1, 1 - \sinh^2 x\right),$$

where $\rho = \alpha + \beta + 1$ and ${}_2F_1$ is the Gauss hypergeometric function.

Eq. (2.4) also has two linear independent solutions ϕ_λ and we have

$$\varphi_\lambda(x) = c(\lambda)\phi_\lambda(x) + c(-\lambda)\phi_{-\lambda}(x),$$

where $c(.)$ is the Harish-Chandra type function [23,24,15,28,29].

Properties of the eigenfunction φ_λ, [23,24,20,28,29].

We have

- For

$$\rho = 0, \forall r \geq 0, \varphi_0(r) = 1. \tag{2.6}$$

- For $\rho > 0$, there exists $k > 0$ such that

$$\forall r > 0, \exp(-\rho r) \leq \varphi_0(r) \leq k(1+r)\exp(-\rho r). \tag{2.7}$$

- For $\lambda \in \mathbb{R}$,

$$|\varphi_\lambda(r)| \leq \varphi_0(r); \forall r \geq 0. \tag{2.8}$$

- For all $\lambda \in \mathbb{C}$ such that $|\Im m(\lambda)| \leq \rho$ and $r \geq 0$ we have

$$|\varphi_\lambda(r)| \leq 1. \tag{2.9}$$

• For all $\lambda \in \{\lambda \in \mathbb{C}; |\Im m(\lambda)| \leq \rho\}$

$$\sup_{r\geq 0} |\varphi_\lambda(r)| = 1. \tag{2.10}$$

• $\forall r > 0, \forall \lambda \in \mathbb{C}$,

$$\varphi_\lambda(r) = \int_0^r K(r,u)\cos\lambda u du, \tag{2.11}$$

where $K(r,.)$ is an even positive C^∞-function on $]-r,r[$ with support in $[-r,r]$.

• For all $s, r \in [0,\infty[$, and $\lambda \in \mathbb{C}$,

$$\varphi_\lambda(r)\varphi_\lambda(s) = \int_0^{+\infty} w(r,s,t)\varphi_\lambda(t)A(t)dt, \tag{2.12}$$

where $w(r,s,.)$ is a positive function with support in the interval $[|r-s|, r+s]$ and satisfying the following properties:

(i)

$$\int_0^\infty w(r,s,t)A(t)dt = 1, \tag{2.13}$$

(ii)

$$\begin{cases} \omega(r,s,t) = \omega(s,r,t), & \forall t > 0. \\ \omega(r,s,t) = \omega(r,t,s), & \forall t > 0. \end{cases} \tag{2.14}$$

Proposition 2.1 *([20,21]). There exist constants c_1, c_2 and c_3 such that, for every $x \geq 0$ and $\lambda \in \mathbb{C}$, we have*

$$|\varphi_\lambda(x)| \leq \frac{c_1}{\sqrt{B(x)}}\exp(|\Im m\lambda|x).$$
$$|\varphi'_\lambda(x)| \leq c_2|\lambda^2+\rho^2|\frac{x}{\sqrt{B(x)}}\exp(|\Im m\lambda|x|).$$
$$|\varphi''_\lambda(x)| \leq c_3|\lambda^2+\rho^2|\left(1+x\frac{A'(x)}{A(x)}\right)\frac{\exp(|\Im m\lambda|x)}{\sqrt{B(x)}}.$$

Proposition 2.2 *([20,21]). There exist analytic functions A_k such that*

$$\sqrt{A(x)}\varphi_\lambda(x) = \sum_{k=0}^m \sqrt{x}A_k(x)\frac{J_{\alpha+k}(\lambda x)}{\lambda^{\alpha+k}} + R_{m,\lambda}(x),$$

where

$$|R_{m,\lambda}(x)| \leq \frac{c_1}{\lambda^{\alpha+m+3/2}}\left(\int_0^x |A'_{m+1}(t)|dt\right)\exp\left(\frac{c_2}{\lambda}\int_0^x |Q(t)|dt\right),$$

with c_1 and c_2 two constants and

$$Q(t) = (2\alpha+1)\frac{B'(t)}{2tB(t)} + \frac{1}{2}\left(\frac{B'(t)}{B(t)}\right)' + \frac{1}{4}\left(\frac{B'(t)}{B(t)}\right)^2 - \rho^2.$$

Remark 2.1. From the above result, we can write

$$\varphi_\lambda(x) = \sum_{k=0}^m a_k(x)_{J\alpha+k}(\lambda x) + O\left(\frac{1}{\lambda^{\alpha+m+3/2}}\right) \tag{2.15}$$

where

$$a_k(x) = \frac{x^k A_k(x)}{2^{\alpha+k}\Gamma(\alpha+k+1)\sqrt{B(x)}}, \tag{2.16}$$

are continuous functions.

Properties of the Harish-Chandra function, [28].

(i) For $\lambda \in \mathbb{R}$; $c(-\lambda) = \overline{c(\lambda)}$.

(ii) The function $\lambda \mapsto |c(\lambda)|^{-2}$ is continuous on $[0, \infty[$ and there exist k_1, k_2, k_3 such that

- $\rho \geq 0, \alpha > \frac{-1}{2}, \forall \lambda \in \mathbb{C}; |\lambda| > k$
  $k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1}$,
- $\rho > 0, \alpha > \frac{-1}{2}, \forall \lambda \in \mathbb{C}; |\lambda| \leq k$
  $k_1|\lambda|^2 \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^2$,
- $\rho = 0, \alpha > 0, \forall \lambda \in \mathbb{C}; |\lambda| \leq k$
  $k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1}$.

(iii) $\lambda \mapsto c(\lambda)$ is different from zero on $\{\lambda \in \mathbb{C}^*/\Im m\lambda \leq 0\}$.

(iv) We suppose also, that the function $\lambda \longmapsto c(\lambda)$ is C^∞ on $]0, \infty[$ and for all $n \in \mathbb{N}$, $(\frac{d}{d\lambda})^n|c(\lambda)|^{-2}$ is different from zero on $]0, \infty[$ and there exist $p_n \in \mathbb{N}$ and $k_n > 0$ such that

- $\forall \lambda \geq 1; (\frac{d}{d\lambda})^n|c(\lambda)|^{-2} \leq k_n|\lambda|^{p_n}$,
- $(\frac{d}{d\lambda})^n|c(\lambda)|^{-2} \hookrightarrow a_n\lambda^{q_n}; a_n \in \mathbb{R}, q_n \in \mathbb{Z}$.

3. Harmonic analysis associated with a generalized system of partial differential operators

In this section we investigate harmonic analysis associated with the system (1.2), we define and study a generalized Fourier transform and convolution product linked with $\Delta_{2,A}$.

Proposition 3.3. *For all $(\mu, \lambda) \in \mathbb{C}^2$, the system*

$$\begin{cases} \Delta_1 u(r, x) = -i\lambda u(r, x), \\ \Delta_{2,A} u(r, x) = -\mu^2 u(r, x), \\ u(0, 0) = 1; \ \dfrac{\partial u}{\partial r}(0, x) = 0, \quad \forall x \in \mathbb{R} \end{cases} \tag{3.17}$$

has a unique solution given by

$$\psi_{(\mu,\lambda)}(r, x) = \varphi_{\sqrt{\mu^2+\lambda^2}}(r)\exp(-i\lambda x),$$

where φ_ν is the eigenfunction of the operator $\pounds_A$ such that $\varphi_\nu(0) = 1$ and $\varphi'_\nu(0) = 0$.

Proof. Let $\psi_{(\mu,\lambda)}$ be the solution of the system (3.17) and let us put

$$v_{(\mu,\lambda)}(r, x) = \psi_{(\mu,\lambda)}(r, x)\exp(i\lambda x).$$

Then

$$\frac{\partial v_{(\mu,\lambda)}(r, x)}{\partial x} = 0,$$

therefore $v_{(\mu,\lambda)}(r, x) = w_{(\mu,\lambda)}(r)$, with

$$\begin{cases} \pounds_A w(r) = -(\mu^2 + \lambda^2)w(r), \\ w(0) = 1, w'(0) = 0. \end{cases}$$

By Eq. (2.4), we have

$$w_{(\mu,\lambda)}(r) = \varphi_{\sqrt{\mu^2+\lambda^2}}(r)$$

which finishes the proof. □

Remark 3.2. Using the fact that for all $(r,x) \in \mathbb{R}^2$, $|\varphi_\nu(r)| \leq 1$, we deduce that for all $(\mu,\lambda) \in \Gamma$, we have

$$|\psi_{(\mu,\lambda)(r,x)}| \leq 1,$$

where Γ is the set given by

$$\Gamma = \mathbb{R}^2 \cup \{(i\mu,\lambda); (\mu,\lambda) \in \mathbb{R}^2; |\mu| \leq |\lambda|\}. \tag{3.18}$$

Notation

In the sequel of the paper we denote • $d\nu(r,x) = A(r)drdx$.

• $X = [0,\infty[\times\mathbb{R}$.

• $L^p_\nu(X)$: The space of measurable functions f on X satisfying

$$\|f\|_{p,\nu} = \left(\int_X |f(r,x)|^p d\nu(r,x)\right)^{\frac{1}{p}}, \text{ if } 1 \leq p < \infty$$

and

$$\|f\|_\infty = esssup_{(r,x)\in X}|f(r,x)|, \text{ if } p = \infty.$$

• $dm(\mu,\lambda) = ds(\mu)d\lambda$, where $ds(\mu) = \frac{d\mu}{|c(\mu)|^2}$.

• $d\gamma(\mu,\lambda)$ the measure defined on Γ, by

$$\int_\Gamma f(\mu,\lambda)d\gamma(\mu,\lambda) = \int_X f(\mu,\lambda)\mu\theta(\mu,\lambda)d\mu d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} f(i\mu,\lambda)\mu\theta(i\mu,\lambda)d\mu d\lambda,$$

where θ is the function defined on Γ, by

$$\theta(\mu,\lambda) = \frac{1}{\sqrt{\mu^2+|\lambda|^2}|c(\sqrt{\mu^2+|\lambda|^2})|^2}. \tag{3.19}$$

• L^p_m the space of measurable functions f on X satisfying

$$\|f\|_{p,m} = \left(\int_X |f(r,x)|^p dm(\mu,\lambda)\right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty$$

and

$$\|f\|_{\infty,m} = \|f\|_\infty = esssup_{(r,x)\in X}|f(r,x)| < \infty,\ p = \infty.$$

• $L_{p,\gamma}$ the space of measurable functions f on Γ satisfying

$$\|f\|_{p,\gamma} = \left(\int_\Gamma |f(\mu,\lambda)|^p d\gamma(\mu,\lambda)\right)^{\frac{1}{p}}, 1 \leq p < \infty$$

and

$$\|f\|_{\infty,\gamma} = \|f\|_\infty = esssup_{(\mu,\lambda)\in\Gamma}|f(\mu,\lambda)| < \infty,\ p = \infty.$$

• $\boldsymbol{\xi}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ even with respect to the first variable.

• $\mathcal{D}_*(\mathbb{R}^2)$ the space of C^∞ functions on $\mathbb{R}^2$, with compact support and even with respect to the first variable.

• $\mathbb{H}_*(\mathbb{C}^2)$ the space of entire functions $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ even with respect to the first variable rapidly decreasing of exponential type, that is there exists a positive constant M such that for all $k \in \mathbb{N}$,

$$\sup_{(\mu,\lambda)\in\mathbb{C}^2} (1+|\mu|^2+|\lambda|^2)^k|f(\mu,\lambda)|\exp(-\Im m(\mu,\lambda)M) < \infty,$$

where

$$\Im m(\mu,\lambda) = |\Im m\mu| + |\Im m\lambda|.$$

• $\mathbb{H}_{*,0}(\mathbb{C}^2)$ the subspace of $\mathbb{H}_*(\mathbb{C}^2)$ consisting of functions $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ such that for all $k \in \mathbb{N}$

$$\sup_{(\mu,\lambda)\in\mathbb{R}^2, |\mu|\leq|\lambda|} (1-\mu^2+2|\lambda|^2)^k |f(i\mu,\lambda)| < \infty.$$

• $\mathcal{H}_*(\mathbb{C}^2)$ the space of entire functions $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ even with respect to the first variable slowly increasing of exponential type, that is there exists a positive constant M and an integer k such that

$$\sup_{(\mu,\lambda)\in\mathbb{C}^2} (1+|\mu|^2+|\lambda|^2)^{-k} |f(\mu,\lambda)| \exp(-\Im m(\mu,\lambda)M) < \infty.$$

• $\mathcal{H}_{*,0}(\mathbb{C}^2)$ the subspace of $\mathcal{H}_*(\mathbb{C}^2)$ consisting of functions $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ such that there exists $k \in \mathbb{N}$

$$\sup_{(\mu,\lambda)\in\mathbb{C}^2, |\mu|\leq|\lambda|} (1-\mu^2+2|\lambda|^2)^{-k} |f(i\mu,\lambda)| < \infty.$$

• $\mathcal{S}_*(\mathbb{R}^2)$, the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives even with respect to the first variable;

•

$$\mathcal{S}^2_{*,\rho}(\mathbb{R}^2) = \begin{cases} \varphi_0(r)\mathcal{S}_*(\mathbb{R}^2), & \text{if } \rho > 0 \\ \mathcal{S}_*(\mathbb{R}^2), & \text{if } \rho = 0. \end{cases}$$

• $\mathcal{S}_*(\Gamma)$, the space of functions $g : \Gamma \longrightarrow \mathbb{R}$, even with respect to the first variable, infinitely differentiable and rapidly decreasing together with all derivatives.

Each of these spaces is equipped with usual topology.

Definition 3.1. The generalized Fourier transform associated with the system (1.2) is defined on L^1_ν by

$$\mathcal{F}(f)(\mu,\lambda) = \int_X f(r,x)\psi_{(\mu,\lambda)}(r,x)d\nu(r,x);\ (\mu,\lambda) \in \Gamma.$$

Proposition 3.4. *For every $f \in L^1_\nu$, we have*

$$\mathcal{F}(f) = (B \circ \tilde{\mathcal{F}})(f), \tag{3.20}$$

where B is the mapping defined on L^1_ν by

$$Bf(\mu,\lambda) = f(\sqrt{\mu^2+\lambda^2},\lambda),\ (\mu,\lambda) \in \Gamma \tag{3.21}$$

and

$$\tilde{\mathcal{F}}(f)(\mu,\lambda) = \int_X f(r,x)\varphi_\mu(r)\exp(-i\lambda x)d\nu(r,x),\ (\mu,\lambda) \in \mathbb{R}^2. \tag{3.22}$$

Using adequate change of variable, we deduce the following results.

Proposition 3.5. **(a)** *$f \in L^1_m$ if and only if $Bf \in L^1_\gamma$ and we have*

$$\|Bf\|_{1,\gamma} = \|f\|_{1,m}. \tag{3.23}$$

(b)

$$\int_\Gamma Bf(\mu,\lambda)d\gamma(\mu,\lambda) = \int_X f(\mu,\lambda)dm(\mu,\lambda). \tag{3.24}$$

Proposition 3.6. (a) *B is a linear continuous operator from $\mathcal{S}_*(\mathbb{R}^2)$ into $\mathcal{S}_*(\mathbb{R}^2)$.*
(b) *B is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ into $\mathcal{S}_*(\Gamma)$.*
(c) *B is an isomorphism from L^2_m into L^2_γ.*
(d) *B is an isomorphism from $\mathbb{H}_*(\mathbb{C}^2)$ respectively $\mathcal{H}_*(\mathbb{C}^2)$ into $\mathbb{H}_{*,0}(\mathbb{C}^2)$ respectively $\mathcal{H}_{*,0}(\mathbb{C}^2)$.*

From Proposition 3.5(a) and properties of the generalized Fourier transform associated with $\pounds_A$ [23,24,15,28], we deduce the following theorems.

Theorem 3.1 (*Inversion formula for $\mathcal{F}$*)**.** *Let $f \in L^1_\nu$ such that $\mathcal{F}(f) \in L^1_\gamma$, then for all almost every $(r, x) \in X$ we have*

$$f(r, x) = \int_\Gamma \mathcal{F}(f)(\mu, \lambda)\overline{\psi_{(\mu,\lambda)(r,x)}}d\gamma(\mu, \lambda).$$

Theorem 3.2. (1) *Plancherel formula for $\mathcal{F}$: for $f \in \mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$ we have*

$$\|\mathcal{F}(f)\|_{2,\gamma} = \|f\|_{2,\nu}.$$

(2) *Parseval formula for $\mathcal{F}$: for $f_1, f_2 \in \mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$ we have*

$$\int_X f_1(r, x)\overline{f_2(r, x)}A(r)drdx = \int_\Gamma \mathcal{F}(f_1)(\mu, \lambda)\overline{\mathcal{F}(f_2)(\mu, \lambda)}d\gamma(\mu, \lambda).$$

Using Proposition 3.6 and the density of $\mathcal{S}^2_*(\mathbb{R}^2)$ (resp $\mathcal{S}_*(\Gamma)$) in L^2_A (resp L^2_γ), we have

Theorem 3.3 (*of Plancherel*)**.** *The Fourier transform associated with the system* (1.2) *can be extended to an isometric isomorphism from L^2_ν onto L^2_ν.*

Theorem 3.4. (1) *(Paley–Wiener theorem): The generalized Fourier $\mathcal{F}$ is a topological isomorphism from $\mathcal{D}_*(\mathbb{R}^2)$ onto $\mathbb{H}_{*,0}(\mathbb{C}^2)$.*
(2) *(Schwartz theorem): The Fourier $\mathcal{F}$ associated with the system* (1.2) *is an isomorphism from $\mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$ onto $\mathcal{S}_*(\Gamma)$.*

Now we shall define the generalized shift operator and the convolution product associated with (1.2).

From relation (2.12), we obtain:
for all $(\mu, \lambda) \in \mathbb{C}^2$, $(r, x) \in X$,

$$\psi_{(\mu,\lambda)}(r, x) \cdot \psi_{(\mu,\lambda)}(s, y) = \int_0^\infty w(r, s, t)\psi_{(\mu,\lambda)}(t, x + y)A(t)dt. \tag{3.25}$$

So we have the following definition.

Definition 3.2. The generalized shift operator associated with the system (1.2) is defined on L^1_ν by,
$\forall(r, x), (s, y) \in X$,

$$\mathcal{T}_{(r,x)}f(s, y) = \begin{cases} \int_0^\infty w(r, s, t)f(t, x + y)A(t)dt, & \text{for } r > 0 \\ f(s, y), & \text{for } r = 0. \end{cases}$$

Definition 3.3. The convolution product associated with the system (1.2) of f, g in L^1_ν is defined by the following

$$\forall(r, x) \in X,\ f * g(r, x) = \int_X \mathcal{T}_{(r,-x)}\check{f}(s, y)g(s, y)d\nu(s, y), \tag{3.26}$$

where, $\check{f}(r, x) = f(r, -x)$.

Properties:

(1) $\forall(\mu,\lambda)\in\mathbf{C}^2$, $(r,x)\in X$, we have

$$\psi_{\mu,\lambda}(r,x)\psi_{\mu,\lambda}(s,y)=\mathcal{T}_{(r,x)}\psi_{\mu,\lambda}(s,y); \tag{3.27}$$

(2) if $f\in L^p_\nu$, $1\le p\le\infty$, then for all $(r,x)\in X$, $\mathcal{T}_{(r,x)}f$ belongs to L^p_ν, and we have

$$\|\mathcal{T}_{(r,x)}f\|_{p,\nu}\le\|f\|_{p,\nu}; \tag{3.28}$$

(3) Let $f\in L^p_\nu$, $p\in[1,\infty]$ and $g\in L^q_\nu$, $q\in[1,\infty]$; then $f*g$ belongs to L^r_ν, $r\in[1,\infty]$, such that $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$, and we have,

$$\|f*g\|_{r,\nu}\le\|f\|_{p,\nu}\|g\|_{q,\nu}; \tag{3.29}$$

(4) for all $f\in L^1_\nu$ and $(r,x)\in X$, we have $(\mu,\lambda)\in\Gamma$,

$$\mathcal{F}(\mathcal{T}_{(r,-x)}(f))(\mu,\lambda)=\psi_{(\mu,\lambda)}(r,x)\mathcal{F}(f)(\mu,\lambda); \tag{3.30}$$

(5) for $f,g\in L^1_\nu$

$$\mathcal{F}(f*g)(\mu,\lambda)=\mathcal{F}(f)(\mu,\lambda)\cdot\mathcal{F}(g)(\mu,\lambda). \tag{3.31}$$

4. Riemann–Liouville type transform and its dual transform associated with the system of partial differential operators (1.2)

In this section, using the following Mehler integral representation, (we refer to [33,32])

$$j_\alpha(s)=\frac{\Gamma(\alpha+1)}{\sqrt{\Pi}\,\Gamma(\alpha+1/2)}\int_{-1}^{1}(1-t^2)^{\alpha-1/2}\exp(-ist)dt \tag{4.32}$$

of the modified Bessel function j_α and by the same techniques as Fitouhi [20,21], we define a generalized Riemann–Liouville type transform R_A and its dual ${}^tR_A$ and we give some properties of these operators. In particular we give a nice estimates of there kernels which we will use in the coming paper to study these operators on weighted Lebesgue spaces L^p, $1<p<\infty$.

First, we can see that relation (2.15) allows us to get

$$\psi_{(\mu,\lambda)}(r,x)=\psi_{1,(\mu,\lambda)}(r,x)+\theta_{(\mu,\lambda)}(r,x),$$

where

$$\psi_{1,(\mu,\lambda)}(r,x)=\left(\sum_{k=0}^{m}a_k(r)j_{\alpha+k}(r\sqrt{\mu^2+\lambda^2})\right)\exp(-i\lambda x)$$

and

$$\theta_{(\mu,\lambda)}(r,x)=R_{m,\sqrt{\mu^2+\lambda^2}}(r)A^{-1/2}(r)\exp(-i\lambda x)$$

with $(a_k)_{0\le k\le m}$, respectively $R_{m,\lambda}$ are defined by relation (2.16), respectively Proposition 2.2.

Proposition 4.7. *The function $\psi_{1,(\mu,\lambda)}$ has the following Mehler integral representation*

$$\psi_{1,(\mu,\lambda)}(r,x)=\begin{cases}\displaystyle\int_{-1}^{1}\int_{-1}^{1}k_m(r,s,t)\cos(\mu rs\sqrt{1-t^2})\exp(-i\lambda(x+rt))dsdt, & \text{for } \alpha>0\\[2ex] \displaystyle\frac{a_0(r)}{\Pi}\int_{-1}^{1}\cos(r\mu\sqrt{1-t^2})\exp(-i\lambda t)\frac{dt}{\sqrt{1-t^2}} & \\ \displaystyle\quad+\int_{-1}^{1}\int_{-1}^{1}k_m^*(r,s,t)\cos(\mu rs\sqrt{1-t^2})\exp(-i\lambda(x+rt))dsdt, & \text{for } \alpha=0,\end{cases}$$

where

$$k_m(r,s,t) = (1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1}\sum_{k=0}^{m}\frac{a_k(r)(\alpha+k)}{\Pi}(1-t^2)^k(1-s^2)^k$$

and

$$k_m^*(r,s,t) = \sum_{k=1}^{m}\frac{ka_k(r)}{\Pi}(1-t^2)^{k-1/2}(1-s^2)^{k-1}.$$

Proof. From the following expansion of the function j_α

$$j_\alpha(r) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(r)}{r^\alpha} = \Gamma(\alpha+1)\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\Gamma(\alpha+i+1)}\left(\frac{r}{2}\right)^{2i},$$

we have

$$j_{\alpha+k}(r\sqrt{\mu^2+\lambda^2}) = \Gamma(\alpha+k+1)\sum_{p=0}^{\infty}\frac{(-1)^p}{p!\Gamma(\alpha+k+p+1)}\left(\frac{r\mu}{2}\right)^{2p} j_{\alpha+k+p}(r\lambda).$$

So relation (4.32), allows us

$$j_{\alpha+k}(r\sqrt{\mu^2+\lambda^2}) = \frac{\Gamma(\alpha+k+1)}{\sqrt{\Pi}\Gamma(\alpha+k+1/2)}\int_{-1}^{1} j_{\alpha+k-1/2}(r\mu\sqrt{1-t^2})(1-t^2)^{\alpha+k-1/2}\exp(-i\lambda rt)dt.$$

Then

• For $\alpha = 0$,

$$\psi_{1,(\mu,\lambda)} = f_1(r,x) + f_2(r,x),$$

where

$$f_1(r,x) = a_0(r)j_0(r\sqrt{\mu^2+\lambda^2})\exp(-i\lambda x)$$

and

$$f_2(r,x) = \sum_{k=1}^{m}a_k(r)j_k(r\sqrt{\mu^2+\lambda^2})\exp(-i\lambda x).$$

Thus

$$f_1(r,x) = \frac{a_0(r)}{\Pi}\int_{-1}^{1} j_{-1/2}(r\mu\sqrt{1-t^2})\exp(-i\lambda(rt+x))\frac{dt}{\sqrt{1-t^2}}.$$

But; $j_{-1/2}(x) = \cos x$, then

$$f_1(r,x) = \frac{a_0(r)}{\Pi}\int_{-1}^{1}\cos(r\mu\sqrt{1-t^2})\exp(-i\lambda(rt+x))\frac{dt}{\sqrt{1-t^2}}.$$

On the other hand

$$\begin{aligned} f_2(r,x) &= \sum_{k=1}^{m}a_k(r)j_k(r\sqrt{\mu^2+\lambda^2})\exp(-i\lambda x) \\ &= \sum_{k=1}^{m}\frac{a_k(r)\Gamma(k+1)}{\Gamma(k+\frac{1}{2})}\int_{-1}^{1} j_{k-1/2}(r\mu\sqrt{1-t^2})\exp(-i\lambda(rt+x))(1-t^2)^{k-1/2}dt. \end{aligned}$$

So, using again relation (4.32), we get the result for $\alpha = 0$.

• For $\alpha > 0$, we obtain the result by the same way as $\alpha = 0$. This completes the proof. □

Proposition 4.8. *The function defined by*

$$\psi_{2,(\mu,\lambda)}(r,x) = \psi_{(\mu,\lambda)}(r,x) - \psi_{1,(\mu,\lambda)}(r,x)$$

has the following Mehler integral representation

$$\psi_{2,(\mu,\lambda)}(r,x) = \int\int_{B_r^+} S_m(r,s,t)\cos(s\mu)\exp(-i\lambda(t+x))dsdt,$$

where

- $S_m(r,.,.)$ *is continuous function on* $\mathbb{R}^2$ *with support in* $B_r = \{(u,v)\in\mathbb{R}^2, u^2+v^2\leq r^2\}$ *even with each variables.*
- $B_r^+ = \{(u,v); u>0, u^2+v^2<r^2\}$.

Proof. The result is obtained by Proposition 2.2, Remark 2.1 and the classical Paley–Wiener theorem [34]. □

Corollary 4.1. *The function* $\psi_{(\mu,\lambda)}$ *has the following integral representation*

$$\psi_{(\mu,\lambda)}(r,x) = \begin{cases} \int_{-1}^{1}\int_{-1}^{1} k_m(r,s,t)\cos(\mu rs\sqrt{1-t^2})\exp(-i\lambda(x+rt))dsdt \\ \quad + \int\int_{B_r^+} S_m(r,s,t)\cos(\mu s)\exp(-i\lambda(x+t))dsdt, & \text{for } \alpha>0. \\ \frac{a_0(r)}{\Pi}\int_{-1}^{1}\cos(\mu r\sqrt{1-t^2})\exp(-i\lambda(x+rt))\frac{dt}{\sqrt{1-t^2}} \\ \quad + \int_{-1}^{1}\int_{-1}^{1} k_m(r,s,t)\cos(\mu rs\sqrt{1-t^2})\exp(-i\lambda(x+rt))dsdt \\ \quad + \int\int_{B_r^+} S_m(r,s,t)\cos(\mu s)\exp(-i\lambda(x+t))dsdt, & \text{for } \alpha=0, \end{cases}$$

where

$$k_m(r,s,t) = (1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1}\sum_{k=0}^{m}\frac{a_k(r)(\alpha+k)}{\Pi}(1-t^2)^k(1-s^2)^k$$

and

$$k_m^*(r,s,t) = \sum_{k=1}^{m}\frac{ka_k(r)}{\Pi}(1-t^2)^{k-1/2}(1-s^2)^{k-1},$$

and $S_m(r,.,.)$ *is the function defined by Proposition* 4.8.

Now, we give some estimates of the kernel S_m.

Theorem 4.5. *For* $\alpha\geq 0$ *and* $a>0$ *there exists a positive constant* $C_{\alpha,a}$ *such that* $s^2+(t-x)^2\leq r^2$; $(r,x)\in[0,a]\times[0,a]$

$$|h_m((r,x),(s,t))| \leq C_{\alpha,a} r^{1/2}A(r)^{-1/2}$$

where

$$h_m((r,x),(s,t)) = S_m((r,s),t-x).$$

Proof. From the fact that the functions

$$(s,t)\mapsto r^{\alpha-1/2}A(r)^{1/2}h_m((r,x),(s,t))$$

and

$$(\mu,\lambda)\mapsto r^{\alpha-1/2}A(r)^{1/2}\psi_{2,(\mu,\lambda)}(r,x)$$

are integrable, then by inversion formula for the classical Fourier transform, we have

$$r^{\alpha-1/2}A(r)^{1/2}h_m((r,x),(s,t)) = \frac{2}{\Pi^2}\int_0^{+\infty}\int_{\mathbb{R}} r^{\alpha-1/2}A(r)^{1/2}\psi_{2,(\mu,\lambda)}(r,x)\cos(\mu s)\exp(i\lambda t)d\mu d\lambda.$$

So,

$$|r^{\alpha-1/2}A(r)^{1/2}h_m((r,x),(s,t))| \le I_1(r,x) + I_2(r,x),$$

where

$$I_1(r,x) = \frac{2}{\Pi^2}\int\int_{\{\mu^2+\lambda^2\le 1\}} r^{\alpha-1/2}A(r)^{1/2}|\psi_{2,(\mu,\lambda)}(r,x)|d\mu d\lambda$$

and

$$I_2(r,x) = \frac{2}{\Pi^2}\int\int_{\{\mu^2+\lambda^2> 1\}} r^{\alpha-1/2}A(r)^{1/2}|\psi_{2,(\mu,\lambda)}(r,x)|d\mu d\lambda.$$

• Estimation of $I_1(r,x)$.

$$I_1(r,x) = \frac{2}{\Pi^2}r^{2\alpha-1}\int\int_{\{\mu^2+\lambda^2<1\}} f_{(\mu,\lambda)}(r,x)d\mu d\lambda,$$

where

$$f_{(\mu,\lambda)}(r,x) = r^{-\alpha+1/2}A(r)^{1/2}\psi_{2,(\mu,\lambda)}(r,x).$$

$f_{(\mu,\lambda)}(0,x) = 0$, then by Taylor formula, we deduce that there exists a positive constant $C_1(\alpha,a)$ such that

$$|f_{(\mu,\lambda)}(r,x)| \le C_1(\alpha,a)r.$$

Therefore, there exists a positive constant $C_2(\alpha,a)$ satisfying

$$|I_1(r,x)| \le C_2(\alpha,a)r^{2\alpha}.$$

• Estimation of $I_2(r,x)$.

Proposition 2.2 allows us that

$$I_2(r,x) \le \frac{2c_1}{\Pi^2}r^{\alpha-1/2}A(r)^{1/2}A(r)^{-1/2}\int\int_{\{\mu^2+\lambda^2>1\}} \frac{\chi(r)}{(\mu^2+\lambda^2)^{(2m+2\alpha+3)/4}}\exp\left(c_2\frac{\tilde{\chi}(r)}{\sqrt{\mu^2+\lambda^2}}\right),$$

where

$$\chi(r) = \int_0^r A'_{m+1}(t)dt,\ \tilde{\chi}(r) = \int_0^r Q(t)dt$$

with

$$Q(t) = (2\alpha+1)\frac{B'(t)}{2tB(t)} + \frac{1}{2}\left(\frac{B'(t)}{B(t)}\right)' + \frac{1}{4}\left(\frac{B'(t)}{B(t)}\right)^2 - \rho^2.$$

Thus, from the fact that $|\chi(r)| \le r\sup_{t\in[0,a]}|A'_m(t)|$ and by using change of variable, we deduce that there exists a positive constant $C_3(\alpha,a)$

$$I_2(r,x) \le C_3(\alpha,a)r^{\alpha+1/2}.$$

Hence, there exists a positive constant $C(\alpha,a)$ such that

$$|h_m((r,x),(s,t))| \le C(\alpha,a)(r^{1/2+\alpha}+r)A(r)^{-1/2},$$

which completes the proof. □

Definition 4.4. The generalized Riemann–Liouville type transform associated with $\triangle_1$ and $\triangle_{2,A}$ is the mapping defined on $C_*(\mathbb{R}^2)$ (the space of continuous function on $\mathbb{R}^2$ even with respect to the first variable) by the following. For all $(r, x) \in [0, \infty[\times\mathbb{R}$

$$R_A(f)(r,x) = \begin{cases} \displaystyle\int_{-1}^{1}\int_{-1}^{1} k_m(r,s,t) f(rs\sqrt{1-t^2}, x+rt)dsdt \\ \quad + \displaystyle\int\int_{B_r^+} S_m(r,s,t) f(s, t+x)dsdt, & \text{for } \alpha > 0 \\ \displaystyle\frac{a_0(r)}{\Pi}\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\frac{dt}{\sqrt{1-t^2}} \\ \quad + \displaystyle\int_{-1}^{1}\int_{-1}^{1} k_m^*(r,s,t) f(rs\sqrt{1-t^2}, x+rt)dsdt \\ \quad + \displaystyle\int\int_{B_r^+} S_m(r,s,t) f(s,t+x)dsdt, & \text{for } \alpha = 0. \end{cases}$$

Remark 4.3. (1) If you make adequate change of variables we obtain

$$R_A(f)(r,x) = \begin{cases} \displaystyle\int\int_{B_r^+(0,x)} N_m((r,x)), u, v) f(u,v)dudv, & \text{for } \alpha > 0 \\ \displaystyle\frac{a_0(r)}{\Pi}\int_{-\Pi/2}^{\Pi/2} f(r\cos\theta, x + r\sin\theta)d\theta \\ \quad + \displaystyle\int\int_{B_r^+(0,x)} N_m^*((r,x), u, v)dudv, & \text{for } \alpha = 0, \end{cases}$$

where

$$N_m((r,x),u,v) = 1_{B_r^+(0,x)}\Bigg(2r^{-2\alpha}(r^2-u^2-(v-x)^2)^{\alpha-1} \cdot \sum_{k=0}^{m}\frac{r^{-2k}(\alpha+k)}{\Pi}a_k(r)(r^2-u^2-(v-x)^2)^k + S_m(r,u,v-x)\Bigg) \tag{4.33}$$

and

$$N_m^*((r,x),u,v) = 2\sum_{k=1}^{m}\frac{kr^{-2k}}{\Pi}a_k(r)(r^2-u^2-(v-x)^2)^k + S_m(r,u,v-x). \tag{4.34}$$

(2) $\psi_{(\mu,\lambda)}(r,x) = R_A\big(\cos(\mu.)\exp(-i\lambda.)\big)(r,x)$.
(3) If $A(x) = x^{2\alpha+1}$; $R_A = R_\alpha$, with R_α is the integral operator defined in the introduction.

By using Fubini's theorem we have

Proposition 4.9. *For $f \in \mathcal{C}^*(\mathbb{R}^2) f$ bounded and $g \in \mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$, we have*

$$\int_X R_A(f)(r,x)g(r,x)d\nu(r,x) = \int_X f(r,x)\,{}^tR_A g(r,x)drdx,$$

where ${}^tR_A$ is an integral operator defined on $\mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$, called the dual operator of R_A.

Now, we give the connection between the generalized Fourier transform $\mathcal{F}$ and the dual operator ${}^tR_A$ associated with the system (1.2).

Proposition 4.10. *For every $f \in \mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$, we have*

$$\mathcal{F}(f) = \Lambda \circ {}^tR_A(f), \tag{4.35}$$

where Λ is a constant multiple of the classical Fourier transform on $\mathbb{R}^2$ defined by

$$\Lambda(f)(\mu,\lambda)=\int_X f(r,x)\cos(\mu r)\exp(-i\lambda x)drdx.$$

Proof. The result is obtained by using Definition 3.1 Remark 4.3(2) and Proposition 4.9. □

In the following, we will give analogous results to Weyl transform in Schwartz spaces, see [28].

Proposition 4.11. (a) *The operator ${}^tR_A$ associated with the partial differential operators* (1.2) *is linear and continuous from $\mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$ onto $\mathcal{S}_*(\mathbb{R}^2)$.*
(b) *${}^tR_A$ is not injective when applied to $\mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$.*
(c) ${}^tR_A(\mathcal{S}^2_{*,\rho}(\mathbb{R}^2))=\mathcal{S}_*(\mathbb{R}^2)$.

Proof. To prove (b) it suffices to consider a function $g\in\mathcal{S}_*(\mathbb{R}^2)$ such that $supp(g)=\{(r,x)\in\mathbb{R}^2;|r|<|x|\}$ and $g\neq 0$. Since the Fourier transform $\widetilde{\mathcal{F}}$ is an isomorphism from $\mathcal{S}^2_*(\mathbb{R}^2)$ into $\mathcal{S}_*(\mathbb{R}^2)$, then there exists $f\in\mathcal{S}_*(\mathbb{R}^2)$; $f\neq 0$ such that $\widetilde{\mathcal{F}}(f)=g$. This result leads us to get $\mathcal{F}(f)=0$, this means that there exists $f\neq 0$ such that $\Lambda\circ{}^tR_A(f)=0$. Thus the result is deduced from the fact that the Fourier Λ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ into itself.

The assertion (c) is obtained by using the same techniques as [4, pp. 218]. □

Theorem 4.6. *The operator ${}^tR_A$ satisfies the following properties:*
(1) *${}^tR_A$ maps injectively $\mathcal{D}_*(\mathbb{R}^2)$ into itself.*
(2) ${}^tR_A(\mathcal{D}_*(\mathbb{R}^2))\neq\mathcal{D}_*(\mathbb{R}^2)$.

In the following we determine subspaces of $\mathcal{S}_*(\mathbb{R}^2)$ and $\mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$ on which the Riemann–Liouville type operator R_A and its dual ${}^tR_A$ are bijective.

We denote by

- $\mathcal{N}$ the subspace of $\mathcal{S}_*(\mathbb{R}^2)$, consisting of functions f satisfying

$$\forall k\in\mathbb{N},\ and\ x\in\mathbb{R},\left(\frac{\partial f}{\partial r^2}\right)^k(0,x)=0.$$

- $\mathcal{S}_{*,0}(\mathbb{R}^2)$, the subspace of $\mathcal{S}_*(\mathbb{R}^2)$, consisting of functions f such that,

$$\forall k\in\mathbb{N},\ and\ x\in\mathbb{R},\int_0^{+\infty}f(r,x)r^{2k}dr=0.$$

- $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$, the subspace of $\mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$, consisting of functions f such that $supp\tilde{\mathcal{F}}(f)\subset\{(\mu,\lambda)\in\mathbb{R}^2/|\mu|>|\lambda|\}$.

It is easy to see that the space $\mathcal{N}$ can be written as

$$\left\{f\in\mathcal{S}_*(\mathbb{R}^2),\forall k\in\mathbb{N},\forall x\in\mathbb{R},\left(\frac{\partial f}{\partial r}\right)^{2k}(0,x)=0\right\}.$$

Lemma 4.2. *The classical Fourier transform Λ is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto $\mathcal{N}$.*

Theorem 4.7. *The Fourier transform $\mathcal{F}$ associated with the partial differential operator* (1.2) *is an isomorphism from $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$ onto $\mathcal{N}$.*

To prove this Theorem we need the following lemma.

Lemma 4.3. *For $f\in\mathcal{N}$, the function g defined by*

$$g(r,x)=\begin{cases}f(\sqrt{r^2-x^2},x) & if\ |r|>|x|\\ 0 & otherwise,\end{cases}$$

belongs to $\mathcal{S}_*(\mathbb{R}^2)$.

Proof. Let $f \in \mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$. So, $supp\tilde{\mathcal{F}}(f) \subset \{(\mu,\lambda) \in \mathbb{R}^2/|\mu| > |\lambda|\}$. Then by relation (3.20), we get

$$\left(\frac{\partial}{\partial \mu^2}\right)^k \mathcal{F}(f)(0,\lambda) = B\left(\left(\frac{\partial}{\partial \mu^2}\right)^k \tilde{\mathcal{F}}(f)\right)(0,\lambda) = \left(\frac{\partial}{\partial \mu^2}\right)^k \tilde{\mathcal{F}}(f)(|\lambda|,\lambda) = 0. \tag{4.36}$$

This means that for all $f \in \mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$, $\mathcal{F}(f)$ belongs to $\mathcal{N}$.

On the other hand since $\mathcal{F}$ is injective then $\mathcal{F}$ maps injectively $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$ onto $\mathcal{N}$. To achieve the proof it suffices to show that $\mathcal{F}$ is surjectively, from $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$ onto $\mathcal{N}$. Let $h \in \mathcal{N}$ and we consider g be a function defined by

$$g(r,x) = \begin{cases} h(\sqrt{r^2 - x^2}, x) & \text{if } |r| > |x| \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4.3 we have the function g belongs to $\mathcal{S}_{*,\rho}(\mathbb{R}^2)$. Therefore, since $\tilde{\mathcal{F}}$ is an isomorphism from $\mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$ onto $\mathcal{S}_*(\mathbb{R}^2)$, then there exists $f \in \mathcal{S}^2_*(\mathbb{R}^2)$ such that

$$\tilde{\mathcal{F}}(f) = g. \tag{4.37}$$

$supp\tilde{\mathcal{F}}(f) \subset \{(\mu,\lambda) \in \mathbb{R}^2/|\mu| > |\lambda|\}$. This result implies that $f \in \mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$. Furthermore, from identities (3.20) and (4.37) we get, $\mathcal{F}(f) = B \circ \tilde{\mathcal{F}}(f) = B(g) = h$. Thus $\mathcal{F}$ is surjectively operator from $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$ onto $\mathcal{N}$. □

By Lemma 4.2 and Theorem 4.7 we deduce the following result.

Corollary 4.2. *The dual transform* ${}^tR_A$ *is an isomorphism from* $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$ *onto* $\mathcal{S}_{*,0}(\mathbb{R}^2)$.

5. Inversion formula for R_A, ${}^tR_A$ and Plancherel theorem for ${}^tR_A$

In this section, we will give the inversion formula for R_A and its dual by using the following integral transform $\mathcal{K}_1$ and $\mathcal{K}_2$ defined by

$$\mathcal{K}_1(f)(r,x) = \Lambda^{-1}(h \cdot \Lambda(f))(r,x), \tag{5.38}$$

$$\mathcal{K}_2(f)(r,x) = \mathcal{F}^{-1}(h \cdot \mathcal{F}(g))(r,x), \tag{5.39}$$

where

$$h(\mu,\lambda) = C_\alpha |\mu| \theta(\mu,\lambda),$$

with

$$C_\alpha = \frac{\Pi}{2^{\alpha+1}[\Gamma(\alpha+2)]^2}, \alpha \geq 0.$$

We will give inversion formula for R_A and ${}^tR_A$.

Theorem 5.8. *Let* θ *be the function defined by relation* (3.19) *and* $l \in \mathbb{R}$. *Then the mappings*

(1)

$$f \longrightarrow s(\sqrt{\mu^2 + \lambda^2}) f,$$

(2)

$$f \longrightarrow \theta(\mu,\lambda) f,$$

(3)

$$f \longrightarrow |\mu|^l f$$

are an isomorphism from $\mathcal{N}$ *onto itself.*

Proof. Let $f \in \mathcal{N}$, $\partial_x^\beta = (\frac{\partial}{\partial x})^\beta$.
By Leibniz formula, we have

$$\left(\frac{\partial}{\partial r}\right)^{m_1} \partial_x^\beta [s(\sqrt{r^2+x^2}) f](r,x) = \sum_{k=0}^{m_1} \sum_{\gamma \le \beta} C_m^k C_\beta^\gamma P_k(r) P_\gamma(x) \left(\frac{\partial s}{\partial \lambda^2}\right)^{\gamma+k} (\sqrt{r^2+x^2}) \times \left(\frac{\partial}{\partial r}\right)^{m_1-k} \partial_x^{\beta-\gamma} f(r,x), \tag{5.40}$$

where

$$C_{m_1}^k = \frac{m_1!}{k!(m_1-k)!}, C_\beta^\gamma = \frac{\beta!}{\gamma!(\beta-\gamma)!},$$
$$\left(\frac{\partial}{\partial r}\right)^{m_1-k} f(r,x) = r^2 \int_0^1 (1-t) \left(\frac{\partial}{\partial r}\right)^{m_1-k+2} f(rt,x)dt = r^2 \int_1^\infty (1-t) \left(\frac{\partial}{\partial r}\right)^{m_1-k+2} f(rt,x)dt.$$

It follows

$$\left(\frac{\partial}{\partial r}\right)^{m_1-k} D_x^{\beta-\gamma} f(r,x) = r^2 \int_0^1 (1-t) \left(\frac{\partial}{\partial r}\right)^{m_1-k+2} \partial_x^\beta f(rt,x)dt = -r^2 \int_1^\infty (1-t) \left(\frac{\partial}{\partial r}\right)^{m_1-k+2} \partial_x^{\beta-\gamma} f(rt,x)dt.$$

Thus by properties of the Harish-Chandra function we deduce that there exists $m_{\gamma,k} \in \mathbb{R}$ such that

$$\left|\left(\frac{\partial s}{\partial \lambda^2}\right)^{\gamma+k} (\sqrt{r^2+x^2}) \left(\frac{\partial}{\partial r}\right)^{m_1-k} \partial_x^{\beta-\gamma} f(r,x)\right| \le r^2 (r^2+x^2)^{m_{\gamma,k}} \int_1^\infty (1-t) \left(\frac{\partial}{\partial r}\right)^{m_1-k+2} \partial_x^{\beta-\gamma} f(rt,x)dt. \tag{5.41}$$

Then relations (5.40) and (5.41) allow us to get that the function $s(\sqrt{r^2+x^2})f$ belongs to $\mathcal{N}$ and that the mapping $f \longrightarrow s(\sqrt{r^2+x^2})f$ is continuous from $\mathcal{N}$ into itself. The inverse mapping is given by

$$f \longrightarrow |c(\sqrt{r^2+x^2})|^2 f.$$

The assertions (2) and (3) are obtained by the same way as (1). □

Theorem 5.9. (1) *The operator* $\mathcal{K}_1$ *defined by relation* (5.38) *is an isomorphism from* $\mathcal{S}_{*,0}(\mathbb{R}^2)$ *onto itself.* (2) *The operator* $\mathcal{K}_2$ *defined by relation* (5.39) *is an isomorphism from* $\mathcal{S}_{*,\rho}^{2,0}(\mathbb{R}^2)$ *onto itself.*

Proof. The assertion (1) is obtained by Lemma 4.2 and Theorem 5.8. The assertion (2) is obtained by using Theorems 5.8 and 4.7. □

Theorem 5.10. (1) *For all* $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$ *and* $g \in \mathcal{S}_{*,\rho}^{2,0}(\mathbb{R}^2)$ *we have the following inversion formula for* R_A,

$$g = R_A \mathcal{K}_1 {}^t R_A(g), \; f = \mathcal{K}_1 {}^t R_A \; R_A(f)$$

(2) *For all* $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$ *and* $g \in \mathcal{S}_{*,\rho}^{2,0}(\mathbb{R}^2)$ *we have the following inversion formula for* ${}^t R_A$,

$$f = {}^t R_A \; \mathcal{K}_2 \; R_A(f), \; g = \mathcal{K}_2 \; R_A \; {}^t R_A(g).$$

Proof. Let $g \in \mathcal{S}_{*,\rho}^{2,0}(\mathbb{R}^2)$. By Theorem 4.7, we have $\mathcal{F}(g)$ belongs to $\mathcal{N}$, which implies that, $\mathcal{F}(g)$ belongs to $L^1(d\gamma(\mu,\lambda))$. Therefore, from the inversion formula for $\mathcal{F}$, we have

$$g(r,x) = \int_\Gamma \mathcal{F}(g)(\mu,\lambda) \overline{\psi_{(\mu,\lambda)}(r,x)} d\gamma(\mu,\lambda) = I_1(r,x) + I_2(r,x), \tag{5.42}$$

where

$$I_1(r,x) = \int_X \mathcal{F}(g)(\mu,\lambda)\overline{\psi_{(\mu,\lambda)}(r,x)}\mu\theta(\mu,\lambda)d\mu d\lambda$$

and

$$\begin{aligned} I_2(r,x) &= \int_{\mathbf{R}}\int_0^{|\lambda|} \mathcal{F}(g)(i\mu,\lambda)\overline{\psi_{(i\mu,\lambda)}(r,x)}\mu\theta(i\mu,\lambda)d\mu d\lambda \\ &= \int_{\mathbf{R}}\int_0^{|\lambda|} \tilde{\mathcal{F}}(g)(\sqrt{|\lambda^2-\mu^2|},\lambda)\overline{\psi_{(i\mu,\lambda)}(r,x)}\mu\theta(i\mu,\lambda)d\mu d\lambda. \end{aligned}$$

But $supp\tilde{\mathcal{F}}(g) \subset \{(\mu,\lambda) \in \mathbb{R}^2/|\mu| > |\lambda|\}$, so, we have

$$\forall (r,x) \in X,\ I_2(r,x) = 0. \tag{5.43}$$

We treat next $I_1(r,x)$. From Remark 4.3(2) and relation (4.35) we get

$$I_1(r,x) = \int_X \Lambda \circ^t R_A(g)(\mu,\lambda) R_A(\cos(\mu.)\exp(i\lambda.))(r,x)\mu\theta(\mu,\lambda)d\mu d\lambda,$$

where θ is the function defined by relation (3.19). Therefore, Fubini's theorem enables us to get

$$I_1(r,x) = R_A\left(\int_X \frac{\mu\cos(\mu.)\exp(i\lambda.)\Lambda\circ^t R_A(g)d\mu d\lambda}{\sqrt{\mu^2+\lambda^2}|c(\sqrt{\mu^2+\lambda^2})|^2}\right)(r,x).$$

Consequently

$$I_1(r,x) = R_A\left(\Lambda^{-1}\left(C_\alpha\frac{\mu}{\sqrt{\mu^2+\lambda^2}|c(\sqrt{\mu^2+\lambda^2})|^2}\Lambda\circ^t \mathcal{R}_{A,n}(g)\right)\right)(r,x).$$

This identity shows that,

$$I_1(r,x) = R_A \circ \mathcal{K}_1\, {}^t R_A(g)(r,x). \tag{5.44}$$

Thus, by relations (5.42)–(5.44), we deduce that for all $g \in \mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$, we have

$$g = R_A\mathcal{K}_1{}^t R_A(g). \tag{5.45}$$

We note that the relation (5.45), allows us to get that R_A is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$ and $\mathcal{K}_1\ {}^t R_A$ is its inverse. In particular, for all $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$, we have

$$f = \mathcal{K}_1\ {}^t R_A R_A(f). \tag{5.46}$$

Now, we shall prove the second part of the theorem.

Let $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$ and $g = R_A(f)$.

From the above note we have $g \in \mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$. Therefore, relation (5.46) implies that

$$R_A^{-1}(g) = \mathcal{K}_1\ {}^t R_A(g).$$

Thus, from the expression of the operator $\mathcal{K}_1$ we obtain

$$R_A^{-1}(g)(r,x) = \Lambda^{-1}\left(C_\alpha\frac{|\mu|}{\sqrt{\mu^2+\lambda^2}|c(\sqrt{\mu^2+\lambda^2})|^2}\Lambda\circ\ {}^t R_A(g)\right)(r,x).$$

This result leads to that

$${}^t R_A^{-1}\circ R_A^{-1}(g)(r,x) = {}^t R_A^{-1}\circ\Lambda^{-1}\left(C_\alpha\frac{|\mu|}{\sqrt{\mu^2+\lambda^2}|c(\sqrt{\mu^2+\lambda^2})|^2}\Lambda\circ\ {}^t R_A(g)\right)(r,x).$$

The result follows from relation (4.35) and Theorem 5.9. □

In the following we shall state a Plancherel theorem type for the usual transform ${}^tR_A$.

Proposition 5.12. *The operator $\mathcal{K}_3$ defined by*

$$\mathcal{K}_3(f)(r,x) = \Lambda^{-1}\left(C_\alpha|\mu|^{\frac{1}{2}}(\mu^2+\lambda^2)^{-\frac{1}{4}}\left[s\left(\sqrt{\mu^2+\lambda^2}\right)\right]^{\frac{1}{2}}\Lambda(f)\right)(r,x)$$

is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto itself, where $s(\lambda) = |c(\lambda)|^{-2}$.

Proof. The result is obtained by the same way as Theorem 5.9. □

Theorem 5.11. (1) *(Plancherel Formula) For $f \in \mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$, we have*

$$\int_X |f(r,x)|^2 d\nu(r,x) = \int_X |\mathcal{K}_3({}^t\mathcal{R}_A)(f)(r,x)|^2 drdx.$$

(2) *(Plancherel theorem) The operator $\mathcal{K}_3 \circ {}^tR_A$ can be extended to an isometric isomorphism from $L^2_{A,0}$ onto $L^2(X, drdx)$, where $L^2_{A,0}$ is the subspace of L^2_A consisting of functions f such that*

$$supp\widetilde{\mathcal{F}}(f) \subset \{(\mu,\lambda) \in \mathbb{R}^2/|\mu| \geq |\lambda|\}.$$

Proof. Let $f \in \mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$. By Plancherel Formula for $\mathcal{F}$ we have

$$\int_X |f(r,x)|^2 d\nu(r,x) = \int_\Gamma |\mathcal{F}(f)(\mu,\lambda)|^2 d\gamma(\mu,\lambda) = I_1 + I_2, \tag{5.47}$$

where

$$I_1 = \int_X |\mathcal{F}(f)(\mu,\lambda)|^2 \frac{|\mu|}{\sqrt{\mu^2+|\lambda|^2}} \frac{1}{|c(\sqrt{\mu^2+\lambda^2})|^2} d\mu d\lambda$$

and

$$I_2 = \int_{\mathbb{R}} \int_0^{|\lambda|} |\mathcal{F}(f)(i\mu,\lambda)|^2 \frac{|\mu|}{\sqrt{\lambda^2-\mu^2}} \frac{1}{|c(\sqrt{\lambda^2-\mu^2})|^2} d\mu d\lambda.$$

First by using the fact that $f \in \mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$, and relations (3.20), (3.21) we have,

$$I_2 = \int_{\mathbb{R}} \int_0^{|\lambda|} \left|\widetilde{\mathcal{F}}(f)\left(\sqrt{\lambda^2-\mu^2},\lambda\right)\right|^2 \frac{|\mu|}{\sqrt{\lambda^2-\mu^2}} \frac{1}{|c(\sqrt{\lambda^2-\mu^2})|^2} d\mu d\lambda = 0. \tag{5.48}$$

Second, (4.35) yields

$$I_1 = \int_X \left|\frac{|\mu|^{\frac{1}{2}}\Lambda \circ^t R_A(f)(\mu,\lambda)}{(\mu^2+\lambda^2)^{\frac{1}{4}}|c(\sqrt{\mu^2+\lambda^2})}\right|^2 d\mu d\lambda.$$

Therefore identities (5.47), (5.48) and Proposition 5.12 enable us to obtain

$$\int_X |f(r,x)|^2 d\nu(r,x) = \int_X |\Lambda \circ \mathcal{K}_3({}^tR_A)(f)|^2(\mu,\lambda) d\mu d\lambda.$$

Therefore, the first part of theorem is obtained by using Plancherel formula for the classical Fourier transform on $\mathbb{R}^2$, defined by Theorem 3.2. The second part of theorem follows from Plancherel theorem, Proposition 4.11 and the density of $\mathcal{S}_{*,0}(\mathbb{R}^2)$ respectively $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$ in $L^2(X, drdx)$ respectively $L^2_{A,0}$. □

In the following we shall prove that the Riemann–Liouville transform R_A and its dual operator are a permutation and transmutation integral operators.

Lemma 5.4. *For every* $f \in \xi_*(\mathbb{R}^2)$ *and* $g \in \mathcal{D}_*(\mathbb{R}^2)$*, we have*

$$\int_X \Delta_A(f)(r,x)g(r,x)d\nu(r,x) = \int_X f(r,x)\Delta_A(g)(r,x)d\nu(r,x).$$

Theorem 5.12. (1) *The integral transform* R_A *is a transmutation operator of* $\frac{\partial^2}{\partial r^2}, \partial_x$*, into* Δ_A, ∂_x *from* $\mathcal{S}_{*,0}(\mathbb{R}^2)$ *onto* $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$.

(2) *The dual transform* ${}^tR_A$ *is a transmutation operator of* Δ_A, ∂_x*, into* $\frac{\partial^2}{\partial r^2}, \partial_x$ *from* $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$ *onto* $\mathcal{S}_{*,0}(\mathbb{R}^2)$.

Proof. First we shall prove that:

$$ {}^tR_A(\Delta_A f) = \frac{\partial^2}{\partial r^2}({}^tR_A f), {}^tR_A(\partial_x f) = \partial_x({}^tR_A f), \tag{5.49}$$

for every $f \in \mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$

and

$$\Delta_A(R_A f) = R_A\left(\frac{\partial^2}{\partial r^2} f\right), R_A(\partial_x f) = \partial_x(R_A f), \tag{5.50}$$

for every $f \in \xi_*(\mathbb{R}^2)$,

where

$$\partial_x = \frac{\partial}{\partial x},$$

and

$$\Delta_A = \frac{\partial^2}{\partial r^2} + \frac{A'(r)}{A(r)}\frac{\partial}{\partial r} + \rho^2 - \partial_x^2.$$

It is well known that for all $\partial_x, \frac{\partial^2}{\partial r^2}$ are continuous mappings from $\mathcal{S}_*(\mathbb{R}^2)$ into itself. Also, we can see that Δ_A, ∂_x, is a linear mapping from $\mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$ into itself, and that the transform ${}^tR_A$ is a linear continuous mapping from $\mathcal{S}^2_{*,\rho}(\mathbb{R}^2)$ into $\mathcal{S}_*(\mathbb{R}^2)$. Thus, by applying the usual Fourier transform Λ, we have

$$\Lambda\big({}^tR_A(\Delta_A f)\big)(\mu,\lambda) = -\mu^2\mathcal{F}(f)(\mu,\lambda) = \Lambda\left(\frac{\partial^2}{\partial r^2}({}^tR_A f)\right)(\mu,\lambda),$$

and

$$\Lambda\big({}^tR_A(\partial_x f)\big)(\mu,\lambda) = -i\lambda\mathcal{F}(f)(\mu,\lambda) = \Lambda\big(\partial_x({}^tR_A f)\big)(\mu,\lambda),$$

where Λ is the usual Fourier transform on $\mathbb{R}^2$. Consequently, (5.49) follows from the fact that Λ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ into itself. The result (5.50) is obtained by using (5.49), Lemma 5.4 and Proposition 4.9.

Finally, using (5.49), Corollary 4.2 and the fact that R_A is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto $\mathcal{S}^{2,0}_{*,\rho}(\mathbb{R}^2)$ we deduce the result. □

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Original article

On a result of Bruckner relating to directional linear categorical density in Euclidean plane

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Abstract

Bruckner proved that with exception of a set of first category, all other points of any second category set having Baire property in the Euclidean plane are points of directional linear categorical density of the set in almost all directions in the sense of category. In this article, we investigate this result of Bruckner in relation to sets not necessarily having Baire property and with respect to a more general definition of directional linear categorical density frammed after the pattern originally introduced by Wilczyński for linear categorical density.

Keywords: Baire property; Directional linear categorical density; c-thick; c-contained; c-disjoint; c-saturated; Continuum hypothesis

1. Introduction and results

Throughout this paper, we use some standard notations such as

$\mathbb{R}^n (n \geq 1)$ for the Euclidean n-space $(n > 1)$ and $\mathbb{R}$ for the Real line.

$A \setminus B$ for the difference, $A \Delta B$ for the symmetric difference of sets A and B and χ_A for the characteristic function of A

$S(x; r)$ for the open sphere in $\mathbb{R}^n$ with centre at x and radius $r > 0$.

ω_1 for the first uncountable ordinal, and

E^y for the y-section of any set $E \subseteq X \times Y$, where X, Y are any two sets and $X \times Y$ represents their Cartesian product.

We also utilize the standard definition of Baire property of a set in any topological space X as introduced in [3].

Apart from these, we further define E to be c-thick in F, if $B \subseteq F \setminus E (E \subseteq F)$ and B having Baire-property implies that B is a set of first category; c-contained in F, c-disjoint from F if $B \subseteq E \setminus F$, $B \subseteq E \cap F$ and B having Baire property implies that B is a set of first category.

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For 'Ec-contained in F' and 'Ec-disjoint from F' we use symbols $E \subseteq_c F$ and $E \cap_c F = \emptyset$ respectively.

Bruckner and Rosenfeld [1] proved that given any measurable set M of positive Lebesgue measure in $\mathbb{R}^2$, almost all points of M are its points of directional linear metric density in almost all directions. More explicitly speaking, there is a small (in the sense of measure) exceptional set such that each point (x, y) in M and not belonging to this set is a one-dimensional metric density point in all but a small (in the sense of measure) set of directions. Later Bruckner [2] showed that the above mentioned result has an analogue for Baire category. He proved that if B is a second category set having Baire property in $\mathbb{R}^2$, then there is a small exceptional set (small in the sense of category) such that each point (x, y) in B and not in this set is a directional linear categorical density point in all but a small (in the sense of category) set of directions.

In defining 'directional linear categorical density', Bruckner made use of the notion of linear categorical density as introduced in [2]. But Wilczyński later pointed out that this classical definition of linear categorical density cannot be regarded as an exact analogue of the definition of linear metric density. In [4], using the idea Riesz convergence theorem (which describes convergence in measure without using measure), he introduced the notion of $\mathfrak{I}$-density point of any linear set $A \in \mathcal{S}$ where $\mathcal{S}$ is the σ-algebra of the sets having Baire property $\mathfrak{I}$ is the σ-ideal of first category sets in $\mathbb{R}$. In defining point of directional linear categorical density of a set in $\mathbb{R}^n$, we utilize this approach of Wilczyński with the only exception that here we do not assume the sets to possess the property of Baire.

Let $\Lambda^{(n)}$ denote the set of all directions in $\mathbb{R}^n$, i.e. $\Lambda^{(n)} = \{y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \|y\| = 1\}$ (where $\|.\|$ denotes the usual norm in $\mathbb{R}^n$). Let $\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, $\mathbb{R}^n_- = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n < 0\}$, $\Lambda^{(n)}_+ = \{y = (y_1, y_2, \ldots, y_n) \in \Lambda^{(n)} : y_n > 0\}$ i.e. $\Lambda^{(n)}_- = \{y = (y_1, y_2, \ldots, y_n) \in \Lambda^{(n)} : y_n < 0\}$ and write $L^{(y)}_0$ to represent the straight line in $\mathbb{R}^n$ passing through the origin such that the half line (or, the half ray) $L^{(y)}_0 \cap \mathbb{R}^n_+$ has direction $y \in \Lambda^{(n)}_+$. Now upon setting $L^{(y)}_{01} = L^{(y)}_0 \cap S(0; 1)$, we define

Definition 1.1. A point $x \in \mathbb{R}^n$ as a point of directional linear categorical density of a set $A \subseteq \mathbb{R}^n$ in the direction $y \in \Lambda^{(n)}_+$ if for every increasing sequence $\{t_j\}_j (t_j \in \mathbb{R}, t_j > 0)$ tending to infinity, there exists a subsequence $\{t_{j_k}\}_k$ such that $\chi_{t_{j_k}(A-x)\cap L^{(y)}_{01}} \longrightarrow \chi_{L^{(y)}_{01}}$ a.e. (category) which means that the set $\{z \in L^{(y)}_{01} : \chi_{t_{j_k}(A-x)\cap L^{(y)}_{01}}(z) \not\to \chi_{L^{(y)}_{01}}(z)\}$ is a set of first category in $L^{(y)}_{01}$. Equivalently, $L^{(y)}_{01} \setminus \liminf_{j\to\infty} t_j(A - x) \cap L^{(y)}_{01}$ is a set of first category in $L^{(y)}_{01}$.

Based on the above definition, we write

$A^*_c = \{x \in \mathbb{R}^n : x$ is a point of directional linear categorical density of A in almost all directions$\}$. Here almost all is meant in the sense of category or in the topological sense. In other words,

$A^*_c = \{x \in \mathbb{R}^n :$ there exists a set $\Lambda_x \subseteq \Lambda^{(n)}_+$ which is residual in $\Lambda^{(n)}_+$ such that x is a point of directional linear categorical density of A in the direction y for every $y \in \Lambda_x\}$.

Here we call a set $E \subseteq F$ residual in F if $F \setminus E$ is a set of first category. The following definition expresses A^*_c is a little more generalized form.

Definition 1.2. $\tilde{A}^*_c = \{x \in \mathbb{R}^n :$ there exists a set $\Lambda_x \subseteq \Lambda^{(n)}_+$ which is c-thick in $\Lambda^{(n)}_+$ such that x is a point of directional linear categorical density of A in the direction y for every $y \in \Lambda_x\}$.

However, based on the above definition of A^*_c, we now state and prove a variant formulation of Bruckner's theorem applicable for all sets in $\mathbb{R}^n$.

Theorem 1.3. *Let $A \subseteq \mathbb{R}^n$. Then for every subset B of A having the Baire property, $B \subseteq_c A^*_c$ and for every subset B of $\mathbb{R}^n \setminus A$ having Baire property, $B \cap_c A^*_c = \emptyset$. Thus, every subset of A having Baire property is c-contained in A^*_c, and every subset of $\mathbb{R}^n$ disjoint from A is c-disjoint from A^*_c.*

Lemma 1.4. *Let B be any second category set having Baire property in $\mathbb{R}^n$. Then $B \setminus B^*_c$ is a set of first category.*

Proof. According to the hypothesis, we may write $B = G \Delta P$ where $G(\neq\emptyset)$ is open and P is a set of first category. Now to prove the lemma, it is sufficient to establish that $G \setminus P \subseteq B^*_c$. Let $x \in G \setminus P$ and $\{t_j\}_j$ $(t_j \in \mathbb{R}, t_j > 0)$ be an arbitrary increasing sequence tending to infinity. Now consider the mappings

$\tau_1 : (0, \infty) \times \Lambda_+^{(n)} \longrightarrow \mathbb{R}_+^n$ defined by

$$x_1 = ry_1$$
$$x_2 = ry_2$$
$$\vdots$$
$$x_n = r_+\sqrt{1 - y_1^2 - y_2^2 - \cdots - y_{n-1}^2}$$

and $\tau_2 : (0, \infty) \times \Lambda_-^{(n)} \longrightarrow \mathbb{R}_-^n$ defined by

$$x_1 = ry_1$$
$$x_2 = ry_2$$
$$\vdots$$
$$x_n = r_-\sqrt{1 - y_1^2 - y_2^2 - \cdots - y_{n-1}^2}.$$

Since both of these mappings are homeomorphisms and $(P - x) \cap \mathbb{R}_+^n \cap S(0; 1)$, $(P - x) \cap \mathbb{R}_-^n \cap S(0; 1)$ are sets of first categories in $\mathbb{R}_+^n$, $\mathbb{R}_-^n$, so are the sets $\tau_1^{-1}((P - x) \cap \mathbb{R}_+^n \cap S(0; 1))$, $\tau_2^{-1}((P - x) \cap \mathbb{R}_-^n \cap S(0; 1))$ sets of first categories in the product topological spaces $(0, \infty) \times \Lambda_+^{(n)}$ and $(0, \infty) \times \Lambda_-^{(n)}$. Similarly, $\tau_1^{-1}((G - x) \cap \mathbb{R}_+^n \cap S(0; 1))$, $\tau_2^{-1}((G - x) \cap \mathbb{R}_-^n \cap S(0; 1))$ are also open sets in the product spaces $(0, \infty) \times \Lambda_+^{(n)}$, $(0, \infty) \times \Lambda_-^{(n)}$ respectively. So by Kuratowski–Ulam theorem, there exist sets $\Lambda_+ \subseteq \Lambda_+^{(n)}$, $\Lambda_- \subseteq \Lambda_-^{(n)}$ such that Λ_+, Λ_- are residual in $\Lambda_+^{(n)}$, $\Lambda_-^{(n)}$ and for every $y \in \Lambda_+$ (*resp*, $y \in \Lambda_-$), $(\tau_1^{-1}\{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k} t_j(P - x) \cap S(0; 1) \cap \mathbb{R}_+^n\})^y$ (*resp*, $(\tau_2^{-1}\{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k} t_j(P - x) \cap S(0; 1) \cap \mathbb{R}_-^n\})^y$) are first category sets in $(0, \infty)$.

But $\tau_1[(\tau_1^{-1}\{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k} t_j(P - x) \cap S(0; 1) \cap \mathbb{R}_+^n\})^y] = \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} t_j(P - x) \cap \mathbb{R}_+^n \cap L_{01}^{(y)}$ for $y \in \Lambda_+$
and $\tau_2[(\tau_2^{-1}\{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k} t_j(P - x) \cap S(0; 1) \cap \mathbb{R}_-^n\})^y] = \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} t_j(P - x) \cap \mathbb{R}_-^n \cap L_{01}^{(-y)}$ for $y \in \Lambda_-$.

Again, because for some m, $S(0; \frac{1}{t_j}) \subseteq G - x$ for all $j \geq m$,

so $(\tau_1^{-1}\{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k}\{S(0; 1) \cap \mathbb{R}_+^n \setminus t_j(G - x) \cap S(0; 1) \cap \mathbb{R}_+^n\}\})^y = \emptyset$ for every $y \in \Lambda_+^{(n)}$
and $(\tau_2^{-1}\{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k}\{S(0; 1) \cap \mathbb{R}_-^n \setminus t_j(G - x) \cap S(0; 1) \cap \mathbb{R}_-^n\}\})^y = \emptyset$ for every $y \in \Lambda_-^{(n)}$.

Therefore, $L_{01}^{(y)} \cap \mathbb{R}_+^n \setminus \bigcup_{k=1}^{\infty} \bigcap_{j \geq k} t_j(G - x) \cap \mathbb{R}_+^n \cap L_{01}^{(y)} = \tau_1[(\tau_1^{-1}\{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k}\{S(0; 1) \cap \mathbb{R}_+^n \setminus t_j(G - x) \cap S(0; 1) \cap \mathbb{R}_+^n\}\})^y] = \emptyset$ for $y \in \Lambda_+^{(n)}$
and $L_{01}^{(-y)} \cap \mathbb{R}_-^n \setminus \bigcup_{k=1}^{\infty} \bigcap_{j \geq k} t_j(G - x) \cap \mathbb{R}_-^n \cap L_{01}^{(-y)} = \tau_2[(\tau_2^{-1}\{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k}\{S(0; 1) \cap \mathbb{R}_-^n \setminus t_j(G - x) \cap S(0; 1) \cap \mathbb{R}_-^n\}\})^y] = \emptyset$ for $y \in \Lambda_-^{(n)}$.

Now upon setting $\Lambda_x = \{y \in \Lambda_+ : -y \in \Lambda_-\}$, we derive that
$L_{01}^{(y)} \cap \mathbb{R}_+^n \setminus \liminf_{j \to \infty} t_j(B - x) \cap L_{01}^{(y)} \cap \mathbb{R}_+^n \subseteq \bigcap_{k=1}^{\infty} \bigcup_{j \geq k}\{L_{01}^{(y)} \cap \mathbb{R}_+^n \setminus t_j(G - x) \cap \mathbb{R}_+^n \cap L_{01}^{(y)}\} \bigcup \{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k} t_j(P - x) \cap \mathbb{R}_+^n \cap L_{01}^{(y)}\}$ is a set of first category in $L_{01}^{(y)} \cap \mathbb{R}_+^n$ and likewise
$L_{01}^{(y)} \cap \mathbb{R}_-^n \setminus \liminf_{j \to \infty} t_j(B - x) \cap L_{01}^{(y)} \cap \mathbb{R}_-^n \subseteq \bigcap_{k=1}^{\infty} \bigcup_{j \geq k}\{L_{01}^{(y)} \cap \mathbb{R}_-^n \setminus t_j(G - x) \cap \mathbb{R}_-^n \cap L_{01}^{(y)}\} \bigcup \{\bigcap_{k=1}^{\infty} \bigcup_{j \geq k} t_j(P - x) \cap \mathbb{R}_-^n \cap L_{01}^{(y)}\}$ is a set of first category in $L_{01}^{(y)} \cap \mathbb{R}_-^n$ for every $y \in \Lambda_x$ where Λ_x is residual in $\Lambda_+^{(n)}$.

Finally, for every $y \in \Lambda_x$, $L_{01}^{(y)} \setminus \liminf_{j \to \infty} t_j(B - x) \cap L_{01}^{(y)}$ as a set of first category in $L_{01}^{(y)}$. Hence $G \setminus P \subseteq B_c^*$.
This proves the lemma.

Proof of the Theorem 1.3. Let $B \subseteq A$ and B possesses the Baire property. If B is a set of first category, there is nothing to prove. Otherwise if B is of second category, then by preceding lemma $B \setminus B_c^*$ is a set of first category. Again as $B_c^* \subseteq A_c^*$, therefore $B \subseteq_c A_c^*$.

Again, let $B \cap A = \emptyset$ and B possesses the property of Baire. If B is of first category, there is nothing to prove. Otherwise, let B be of second category, and suppose that there is a set C of second category having Baire property such that $C \subseteq B \cap A_c^*$. But $C \subseteq_c (\mathbb{R}^n \setminus A)_c^*$ by the preceding lemma. So there is a point $x \in A_c^* \cap (\mathbb{R}^n \setminus A)_c^*$

which means that there exists a set Λ_x residual in $\Lambda_+^{(n)}$ such that both the sets $L_{01}^{(y)} \setminus \liminf_{j\to\infty} t_j(A-x) \cap L_{01}^{(y)}$ and $L_{01}^{(y)} \setminus \liminf_{j\to\infty} t_j((\mathbb{R}^n \setminus A)-x) \cap L_{01}^{(y)}$ are sets of first category for every $y \in \Lambda_x$. But this is a contradiction.

Hence the theorem.

As a direct consequence of Theorem 1.3, we obtain the following corollary.

Corollary 1.5. *Let $A \subseteq \mathbb{R}^n$. Then no second category subset of $A\Delta A_c^*$ having Baire property can exist which can be expressed as $B = B_1 \cup B_2$, where $B_1 \subseteq A \setminus A_c^*$, $B_2 \subseteq A_c^* \setminus A$ and both have the property of Baire.*

Definition 1.6. A partition $\{A, B\}$ of $\mathbb{R}^n$ by disjoint nonempty sets is called c-admissible if both A and B are c-thick in $\mathbb{R}^n$. A set which together with its complement in $\mathbb{R}^n$ forms a c-admissible partition of $\mathbb{R}^n$ is called c-saturated.

Theorem 1.3 (and therefore of Corollary 1.5) is trivially valid for sets that are c-saturated. But unlike that of Bruckner's theorem (which states that $A\Delta A_c^*$ is a set of first category whenever A possesses the property of Baire), Theorem 1.3 has little to say regarding the nature of $A\Delta A_c^*$, for in fact for sets lacking the property of Baire, the nature of $A\Delta A_c^*$ can be quite bizarre. Below we give two examples showing that there exist sets E in $\mathbb{R}^2$ for which E_c^* (resp. $\tilde{E}_c^*$) can be c-thick and $E\Delta E_c^*$ (resp. $E\Delta\tilde{E}_c^*$) may be c-saturated as well.

Example 1.7. In [3] (Theorem 15.5, Ch15), Oxtoby showed that there exists a set in $\mathbb{R}^2$ which meets every second category G_δ set and no three points of which are collinear. Denoting the complement of this set by E, it is easy to verify that $E_c^* = \mathbb{R}^2$ and $E\Delta E_c^* = \mathbb{R}^2 \setminus E$. Thus E_c^* is a trivially c-thick and $E\Delta\tilde{E}_c^*$ is c-saturated in $\mathbb{R}^2$.

Example 1.8. Assuming continuum hypothesis, let us arrange the class of second category G_δ sets in $\mathbb{R}^2$ in the form of a well ordering $\{F_\alpha : \alpha < \omega_1\}$. Without any loss of generality, we may assume that each of the families $\{F_\alpha : \alpha \text{ is even}\}$ and $\{F_\alpha : \alpha \text{ is odd}\}$ consists of all sets belonging to the entire collection $\{F_\alpha : \alpha < \omega_1\}$. Let $\{\Omega_\alpha : \alpha < \omega_1\}$ be a well ordering of all second category G_δ sets in $[0, \pi)$. We now choose a point $p_0 \in F_0$, directions $\theta_0^{(0)} \in \Omega_0$, $\theta_1^{(0)} \in \Omega_1 \setminus \{\theta_0^{(0)}\}$ and straight lines $L_{p_0}^{\theta_0^{(0)}}$, $L_{p_0}^{\theta_1^{(0)}}$ through p_0 in the directions $\theta_0^{(0)}, \theta_1^{(0)}$ such that the sets $F_0 \cap L_{p_0}^{\theta_0^{(0)}}$, $F_0 \cap L_{p_0}^{\theta_1^{(0)}}$ are of second category and a point $t^{(0)}_{\theta_0^{(0)},p_0} (\neq p_0) \in F_0 \cap L_{p_0}^{\theta_0^{(0)}}$. Next we choose a point $p_1 \in F_1 \setminus L_{p_0}^{\theta_0^{(0)}} \cup L_{p_0}^{\theta_1^{(0)}}$, a direction $\theta_1^{(1)} \in \Omega_1 \setminus \{\theta_0^{(0)}, \theta_1^{(0)}\}$, a straight line $L_{p_1}^{\theta_1^{(1)}}$ through p_1 in the direction $\theta_1^{(1)}$ which does not pass through $p_0, t^{(0)}_{\theta_0^{(0)},p_0}$ and for which $F_1 \cap L_{p_1}^{\theta_1^{(1)}}$ is a set of second category, and also choose points $t^{(1)}_{\theta_0^{(0)},p_0} (\neq p_0) \in F_0 \cap L_{p_0}^{\theta_0^{(0)}}$, $t^{(1)}_{\theta_1^{(0)},p_0} (\neq p_0) \in F_0 \cap L_{p_0}^{\theta_1^{(0)}}$ and $t^{(1)}_{\theta_1^{(1)},p_1} (\neq p_1) \in F_1 \cap L_{p_1}^{\theta_1^{(1)}}$ such that $t^{(1)}_{\theta_0^{(0)},p_0} \notin L_{p_0}^{\theta_0^{(0)}} \cap L_{p_1}^{\theta_1^{(1)}}$, $t^{(1)}_{\theta_1^{(0)},p_0} \notin L_{p_0}^{\theta_1^{(0)}} \cap L_{p_1}^{\theta_1^{(1)}}$ and $t^{(1)}_{\theta_1^{(1)},p_1} \notin L_{p_0}^{\theta_0^{(0)}} \cap L_{p_1}^{\theta_1^{(1)}}, L_{p_0}^{\theta_1^{(0)}} \cap L_{p_1}^{\theta_1^{(1)}}$.

Thus each of the points lies on exactly one straight line and all the above choices are possible because of Theorem 1.3 and by virtue of some elementary properties of countable sets.

Now suppose that for any ordinal $\alpha < \omega_1$, we have already selected points $p_\beta \in F_\beta$, lines $L_{p_\beta}^{\theta_\gamma^{(\beta)}}$ through p_β in the direction $\theta_\gamma^{(\beta)}$ for $\beta < \alpha$ and $\beta \le \gamma < \alpha$ such that the sets $F_\beta \cap L_{p_\beta}^{\theta_\gamma^{(\beta)}}$ are of second category and also points $t^{(\delta)}_{\theta_\gamma^{(\beta)},p_\beta} \in F_\beta \cap L_{p_\beta}^{\theta_\gamma^{(\beta)}}$ for $\gamma \le \delta < \alpha$ such that the points $p_\beta, t^{(\delta)}_{\theta_\gamma^{(\beta)},p_\beta}$ lie on exactly one straight line in the entire collection $\{L_{p_\beta}^{\theta_\gamma^{(\beta)}} : 0 \le \beta < \alpha, \beta \le \gamma < \alpha\}$. Since we assume continuum hypothesis, so on account of similar reasonings as referred to in the previous paragraph, we may now select directions $\theta_\alpha^{(\beta)} \in \Omega_\alpha \setminus \bigcup_{0\le\xi<\beta} \bigcup_{\xi\le\gamma\le\alpha} \{\theta_\gamma^{(\xi)}\}$, straight lines $L_{p_\beta}^{\theta_\alpha^{(\beta)}}$ through p_β in the direction $\theta_\alpha^{(\beta)}$ such that the set $F_\beta \cap L_{p_\beta}^{\theta_\alpha^{(\beta)}}$ is of second category and $L_{p_\beta}^{\theta_\alpha^{(\beta)}}$ does not pass through any point $p_\xi, t^{(\delta)}_{\theta^{(\xi)}_\gamma, p_\xi}$ for $0 \leq \xi < \alpha$, $\xi \leq \gamma < \alpha$ and $\gamma \leq \delta < \alpha$ where $\xi \neq \beta$. Further, we select a point $p_\alpha \in F_\alpha \setminus \bigcup_{0\leq\beta<\alpha} \bigcup_{\beta\leq\gamma\leq\alpha} L^{\theta^{(\beta)}_\gamma}_{p_\beta}$, a direction $\theta^{(\alpha)}_\alpha \in \Omega_\alpha \setminus \bigcup_{0\leq\beta<\alpha} \bigcup_{\beta\leq\gamma\leq\alpha} \theta^{(\beta)}_\gamma$, a straight line $L^{\theta^{(\alpha)}_\alpha}_{p_\alpha}$ through p_α in the direction $\theta^{(\alpha)}_\alpha$ such that the set $F_\alpha \cap L^{\theta^{(\alpha)}_\alpha}_{p_\alpha}$ is of second category and points $t^{(\alpha)}_{\theta^{(\beta)}_\gamma, p_\beta} \in F_\beta \cap L^{\theta^{(\beta)}_\gamma}_{p_\beta}$ for $0 \leq \beta \leq \gamma \leq \alpha$ such that each point lies on exactly one straight line in the entire collection.

Now as any straight line L^θ_p through some point p in the direction θ can be rotated clockwise with p as fixed so that it becomes parallel to the x-axis and then properly translated so that it coincides with the x-axis with p as the origin, each such straight line can be identified with the real line and every point on it with the corresponding real number. This facilitates choosing points $t^{(\alpha)}_{\theta^{(\beta)}_\gamma, p_\beta}$ on the line $L^{\theta^{(\beta)}_\gamma}_{p_\beta}$ according to the following rule:

$t^{(\alpha)}_{\theta^{(\beta)}_\gamma, p_\beta} \in L^{\theta^{(\beta)}_\gamma}_{p_\beta} \setminus \{r^{-1}_\eta r_\sigma t^{(\delta)}_{\theta^{(\beta)}_\gamma, p_\beta} : \gamma \leq \eta, \sigma \leq \alpha\}$ where $\mathbb{R} = \{r_\alpha : 0 \leq \alpha < \omega_1\}$ can be considered as a well ordering of the set of all real numbers. We now set $E = \{\mathbb{R}^2 \setminus \{p_\beta, t^{(\alpha)}_{\theta^{(\beta)}_\gamma, p_\beta} : 0 \leq \alpha < \omega_1, \beta \leq \gamma < \omega_1, \gamma \leq \alpha < \omega_1\}\} \cup \{p_\beta : \beta \text{ is even}\}$. Then E is c-saturated and hence without the property of Baire. Moreover, it is easy to see that $\tilde{E}^*_c$ is c-thick and $E \triangle \tilde{E}^*_c$ is c-saturated.

Remark 1.9. In proving the categorical directional density theorem, Bruckner [2] used Kuratowski–Ulam theorem (a category analogue of Luzin's theorem) in the product space $B \times [0, \pi)$, where B is a set with Baire property in $\mathbb{R}^2$. But this technique of Bruckner is not applicable in the present situation, because in framming the definition of directional linear categorical density point in $\mathbb{R}^n$, we have used the approach of Wilczyńsky instead of using the classical definition as given in [2]. This is the reason why we use here the Kuratowski–Ulam theorem separately in the product spaces $(0, \infty) \times \Lambda^{(n)}_+$, $(0, \infty) \times \Lambda^{(n)}_-$ and also the homeomorphisms τ_1 and τ_2. However, we are not sure whether our process could be replicated in the measure theoretic case.

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Original article

A companion of Ostrowski type inequalities for mappings of bounded variation and some applications

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Abstract

In this paper, we establish a companion of Ostrowski type inequalities for mappings of bounded variation and the quadrature formula is also provided.

Keywords: Bounded variation; Ostrowski type inequalities; Riemann–Stieltjes integral

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \tag{1.1}$$

for all $x \in [a, b]$ [1]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality*.

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Definition 1. Let $P : a = x_0 < x_1 < \cdots < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$. Then $f(x)$ is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions. Let f be of bounded variation on $[a, b]$, and $\sum(P)$ denotes the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a, b]$. The number

$$\bigvee_a^b (f) := \sup\left\{\sum(P) : P \in P([a, b])\right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denote the family of partitions of $[a, b]$.

In [2], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then*

$$\left|\int_a^b f(t)dt - (b-a) f(x)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{1.2}$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

Dragomir gave the following trapezoid inequality in [3]:

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then we have the inequality*

$$\left|\frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(t)dt\right| \le \frac{1}{2}(b-a) \bigvee_a^b (f). \tag{1.3}$$

The constant $\frac{1}{2}$ is the best possible.

We introduce the notation $I_n : a = x_0 < x_1 < \cdots < x_n = b$ for a division of the interval $[a, b]$ with $h_i := x_{i+1} - x_i$ and $\upsilon(h) = \max\{h_i : i = 0, 1, \ldots, n-1\}$. Then we have

$$\int_a^b f(t)dt = A_T(f, I_n) + R_T(f, I_n) \tag{1.4}$$

where

$$A_T(f, I_n) := \sum_{i=0}^{n} \frac{f(x_i) + f(x_{i+1})}{2} h_i \tag{1.5}$$

and the remainder term satisfies

$$|R_T(f, I_n)| \le \frac{1}{2}\upsilon(h) \bigvee_a^b (f). \tag{1.6}$$

In [4], Dragomir proved the following companion Ostrowski type inequalities related functions of bounded variation:

Theorem 3. *Assume that the function $f : [a,b] \to R$ is of bounded variation on $[a,b]$. Then we have the inequalities:*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^{b}(f)\right] \\
&\leq \begin{cases}
\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\bigvee_a^b(f), \\
\left[2\left(\frac{x-a}{b-a}\right)^{\alpha}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\
\times\left[\left[\bigvee_a^x(f)\right]^{\beta}+\left[\bigvee_x^{a+b-x}(f)\right]^{\beta}+\left[\bigvee_{a+b-x}^{b}(f)\right]^{\beta}\right]^{\frac{1}{\beta}}, \quad \text{if } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1, \\
\left[\frac{x-a+\frac{b-a}{2}}{b-a}\right]\max\left\{\bigvee_a^x(f), \bigvee_x^{a+b-x}(f), \bigvee_{a+b-x}^{b}(f)\right\}
\end{cases}
\end{aligned}
\tag{1.7}
$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ where $\bigvee_c^d(f)$ denotes the total variation of f on $[c,d]$. The constant $\frac{1}{4}$ is the best possible in the first branch of second inequality in (1.7).

For recent results concerning the above Ostrowski's inequality and other related results see [1–26].

In this work, we obtain a new companion of Ostrowski type integral inequalities for functions of bounded variation. Then we give some applications for our results.

2. Main results

Now, we give a new companion of Ostrowski type integral inequalities for functions of bounded variation:

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then, we have the inequality*

$$
\begin{aligned}
&\left|\frac{b-a}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b f(t)dt\right| \\
&\leq \max\left\{\left|x-\frac{3a+b}{4}\right|, \left(\frac{a+b}{2}-x\right), \frac{x-a}{2}\right\}\bigvee_a^b(f)
\end{aligned}
\tag{2.1}
$$

where $x \in \left[a, \frac{a+b}{2}\right]$ and $\bigvee_c^d(f)$ denotes the total variation of f on $[c,d]$.

Proof. Consider the kernel $P(x,t)$ defined by Qayyum et al. in [7]

$$
P(x,t) = \begin{cases}
t-a, & t \in \left[x, \frac{a+x}{2}\right] \\
t-\frac{3a+b}{4}, & t \in \left(\frac{a+x}{2}, x\right] \\
t-\frac{a+b}{2}, & t \in (x, a+b-x] \\
t-\frac{a+3b}{4}, & t \in \left(a+b-x, \frac{a+2b-x}{2}\right] \\
t-b, & t \in \left[\frac{a+2b-x}{2}, b\right].
\end{cases}
$$

Integrating by parts, we get

$$\int_a^b P(x,t)df(t) = \frac{b-a}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \int_a^b f(t)dt. \qquad (2.2)$$

It is well known that if $g, f : [a,b] \to \mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exists and

$$\left|\int_a^b g(t)df(t)\right| \le \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (f). \qquad (2.3)$$

On the other hand, by using (2.3), we get

$$\begin{aligned}
&\left|\int_a^b P(x,t)df(t)\right| \\
&\le \left|\int_a^{\frac{a+x}{2}} (t-a)\,df(t)\right| + \left|\int_{\frac{a+x}{2}}^x \left(t - \frac{3a+b}{4}\right) df(t)\right| + \left|\int_x^{a+b-x} \left(t - \frac{a+b}{2}\right) df(t)\right| \\
&\quad + \left|\int_{a+b-x}^{\frac{a+b-x}{2}} \left(t - \frac{a+3b}{4}\right) df(t)\right| + \left|\int_{\frac{a+b-x}{2}}^b (t-b)\,df(t)\right| \\
&\le \sup_{t\in\left[a,\frac{a+x}{2}\right]} |t-a| \bigvee_a^{\frac{a+x}{2}}(f) + \sup_{t\in\left[\frac{a+x}{2},x\right]} \left|t - \frac{3a+b}{4}\right| \bigvee_{\frac{a+x}{2}}^x (f) + \sup_{t\in[x,a+b-x]} \left|t - \frac{a+b}{2}\right| \bigvee_x^{a+b-x} (f) \\
&\quad + \sup_{t\in\left[a+b-x,\frac{a+2b-x}{2}\right]} \left|t - \frac{a+3b}{4}\right| \bigvee_{a+b-x}^{\frac{a+2b-x}{2}} (f) + \sup_{t\in\left[\frac{a+2b-x}{2},b\right]} |t-b| \bigvee_{\frac{a+2b-x}{2}}^b (f) \\
&= \frac{x-a}{2} \bigvee_a^{\frac{a+x}{2}}(f) + \max\left\{\left|x - \frac{3a+b}{4}\right|, \frac{1}{2}\left(\frac{a+b}{2} - x\right)\right\} \bigvee_{\frac{a+x}{2}}^x (f) + \left(\frac{a+b}{2} - x\right) \bigvee_x^{a+b-x} (f) \\
&\quad + \max\left\{\left|x - \frac{3a+b}{4}\right|, \frac{1}{2}\left(\frac{a+b}{2} - x\right)\right\} \bigvee_{a+b-x}^{\frac{a+2b-x}{2}} (f) + \frac{x-a}{2} \bigvee_{\frac{a+2b-x}{2}}^b (f) \\
&\le \max\left\{\left|x - \frac{3a+b}{4}\right|, \left(\frac{a+b}{2} - x\right), \frac{x-a}{2}\right\} \bigvee_a^b (f).
\end{aligned}$$

This completes the proof. □

Remark 1. If we choose $x = a$ in Theorem 4, the inequality (2.1) reduces the inequality (1.3).

Corollary 1. *Under the assumption of Theorem* 4 *with* $x = \frac{a+b}{2}$, *then we have the following inequality*

$$\left|\frac{b-a}{4}\left[2f\left(\frac{a+b}{2}\right) + f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \int_a^b f(t)dt\right| \le \frac{1}{4}(b-a) \bigvee_a^b (f). \qquad (2.4)$$

The constant $\frac{1}{4}$ *is the best possible.*

Proof. For proof of the sharpness of the constant, assume that (2.4) holds with a constant $A > 0$, that is,

$$\left|\frac{b-a}{4}\left[2f\left(\frac{a+b}{2}\right) + f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \int_a^b f(t)dt\right| \le A(b-a) \bigvee_a^b (f). \qquad (2.5)$$

If we choose $f : [a, b] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1, & \text{if } x \in \left\{ \frac{a+b}{2}, \frac{3a+b}{4}, \frac{a+3b}{4} \right\} \\ 0, & \text{if } x \in [a, b] \,/ \left\{ \frac{a+b}{2}, \frac{3a+b}{4}, \frac{a+3b}{4} \right\} \end{cases}$$

then f is of bounded variation on $[a, b]$, and

$$2f\left(\frac{a+b}{2}\right) + f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) = 4, \ \int_a^b f(t)dt = 0, \text{ and } \bigvee_a^b (f) = 4,$$

giving in (2.5), $1 \le 4A$, thus $A \ge \frac{1}{4}$. □

Corollary 2. *Under the assumption of Theorem 4 with $x = \frac{3a+b}{4}$, then we get the inequality*

$$\left| \frac{b-a}{4} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] - \int_a^b f(t)dt \right| \le \frac{1}{8}(b-a) \bigvee_a^b (f). \tag{2.6}$$

The constant $\frac{1}{8}$ is the best possible.

Proof. For proof of the sharpness of the constant, assume that (3.4) holds with a constant $B > 0$, that is,

$$\left| \frac{b-a}{4} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] - \int_a^b f(t)dt \right| \le B(b-a) \bigvee_a^b (f). \tag{2.7}$$

If we choose $f : [a, b] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1, & \text{if } x \in \left\{ \frac{3a+b}{4}, \frac{a+3b}{4}, \frac{7a+b}{8}, \frac{a+7b}{8} \right\} \\ 0, & \text{if } x \in [a, b] \,/ \left\{ \frac{3a+b}{4}, \frac{a+3b}{4}, \frac{7a+b}{8}, \frac{a+7b}{8} \right\} \end{cases}$$

then f is of bounded variation on $[a, b]$, and

$$f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right) = 4,$$

$$\int_a^b f(t)dt = 0, \text{ and } \bigvee_a^b (f) = 8,$$

giving in (2.7), $1 \le 8B$, thus $B \ge \frac{1}{8}$. □

Corollary 3. *Let f be defined as in Theorem 4, and, additionally, if $f(x) = f(a+b-x)$, then we have*

$$\left| \frac{b-a}{4} \left[2f(x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \right] - \int_a^b f(t)dt \right| \le \max\left\{ \left| x - \frac{3a+b}{4} \right|, \left(\frac{a+b}{2} - x \right), \frac{x-a}{2} \right\} \bigvee_a^b (f). \tag{2.8}$$

Corollary 4. *If we choose $x = a$ in Corollary 3, then we have the inequality*

$$\left|\frac{3f(a)+f(b)}{4}(b-a)-\int_a^b f(t)dt\right| \leq \frac{1}{2}(b-a)\bigvee_a^b(f).$$

The constant $\frac{1}{2}$ is the best possible.

The sharpness of the constant can be proved similarly Corollaries 1 and 2, so it is omitted.

Corollary 5. *Under the assumption of Theorem 4, suppose that $f \in C^1[a,b]$. Then we have*

$$\left|\frac{b-a}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b f(t)dt\right|$$
$$\leq \max\left\{\left|x-\frac{3a+b}{4}\right|, \left(\frac{a+b}{2}-x\right), \frac{x-a}{2}\right\}\left\|f'\right\|_1$$

for all $x \in \left[a, \frac{a+b}{2}\right]$. Here as subsequently $\|.\|_1$ is the L_1-norm

$$\left\|f'\right\|_1 := \int_a^b f'(t)dt.$$

Corollary 6. *Under the assumption of Theorem 4, let $f : [a,b] \to \mathbb{R}$ be a Lipschitzian with the constant $L > 0$. Then*

$$\left|\frac{b-a}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b f(t)dt\right|$$
$$\leq \max\left\{\left|x-\frac{3a+b}{4}\right|, \left(\frac{a+b}{2}-x\right), \frac{x-a}{2}\right\}(b-a)L$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

3. Application to quadrature formula

We now introduce the intermediate points $\xi_i \in \left[x_i, \frac{x_i+x_{i+1}}{2}\right]$ $(i = 0, 1, \ldots, n-1)$ in the division $I_n : a = x_0 < x_1 < \cdots < x_n = b$. Let $h_i := x_{i+1} - x_i$ and $v(h) = \max\{h_i : i = 0, 1, \ldots, n-1\}$ and define the sum

$$A(f, I_n, \xi) := \frac{1}{4}\sum_{i=0}^{n} h_i\left[f(\xi_i)+f(x_i+x_{i+1}-\xi_i)+f\left(\frac{x_i+\xi_i}{2}\right)+f\left(\frac{x_i+2x_{i+1}-\xi_i}{2}\right)\right]. \tag{3.1}$$

Then the following theorem holds:

Theorem 5. *Let f be as Theorem 4. Then*

$$\int_a^b f(t)dt = A(f, I_n, \xi) + R(f, I_n, \xi) \tag{3.2}$$

where $A(f, I_n, \xi)$ is defined as above and the remainder term $R(f, I_n, \xi)$ satisfies

$$|R(f, I_n, \xi)| \leq \max_{i\in\{0,1,\ldots,n-1\}}\left[\max\left\{\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|, \left(\frac{x_i+x_{i+1}}{2}-\xi_i\right), \frac{\xi_i-x_i}{2}\right\}\right]\bigvee_a^b(f). \tag{3.3}$$

Proof. Applying Theorem 4 to the interval $\left[x_i, x_{i+1}\right]$ $(i = 0, 1, \ldots, n-1)$, we have

$$\left|\frac{h_i}{4}\left[f(\xi_i)+f(x_i+x_{i+1}-\xi_i)+f\left(\frac{x_i+\xi_i}{2}\right)+f\left(\frac{x_i+2x_{i+1}-\xi_i}{2}\right)\right]-\int_{x_i}^{x_{i+1}} f(t)dt\right|$$
$$\leq \max\left\{\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|, \left(\frac{x_i+x_{i+1}}{2}-\xi_i\right), \frac{\xi_i-x_i}{2}\right\}\bigvee_{x_i}^{x_{i+1}}(f) \tag{3.4}$$

for all $i \in \{0, 1, \ldots, n-1\}$. Summing the inequality (3.4) over i from 0 to $n-1$ and using the generalized triangle inequality, we have

$$
\begin{aligned}
|R(f, I_n, \xi)| &\leq \sum_{i=0}^{n} \max\left\{\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|, \left(\frac{x_i + x_{i+1}}{2} - \xi_i\right), \frac{\xi_i - x_i}{2}\right\} \bigvee_{x_i}^{x_{i+1}}(f) \\
&\leq \max_{i\in\{0,1,\ldots,n-1\}}\left[\max\left\{\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|, \left(\frac{x_i + x_{i+1}}{2} - \xi_i\right), \frac{\xi_i - x_i}{2}\right\}\right] \sum_{i=0}^{n} \bigvee_{x_i}^{x_{i+1}}(f) \\
&= \max_{i\in\{0,1,\ldots,n-1\}}\left[\max\left\{\left|\xi_i - \frac{3x_i + x_{i+1}}{4}\right|, \left(\frac{x_i + x_{i+1}}{2} - \xi_i\right), \frac{\xi_i - x_i}{2}\right\}\right] \bigvee_{a}^{b}(f)
\end{aligned}
$$

which completes the proof. □

Remark 2. If we choose $\xi_i = x_i$ in Theorem 5, we get (1.4) with (1.5) and (1.6).

Corollary 7. *If we choose* $\xi_i = \frac{x_i + x_{i+1}}{2}$ *in Theorem* 5*, then we have*

$$\int_a^b f(t)dt = A(f, I_n) + R(f, I_n)$$

where

$$A(f, I_n) := \frac{1}{4}\sum_{i=0}^{n} h_i \left[2f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\frac{3x_i + x_{i+1}}{2}\right) + f\left(\frac{x_i + 3x_{i+1}}{2}\right)\right]$$

and the remainder term $R(f, I_n)$ *satisfies*

$$|R(f, I_n)| \leq \frac{1}{4}\upsilon(h) \bigvee_a^b(f).$$

Corollary 8. *If we choose* $\xi_i = \frac{3x_i + x_{i+1}}{4}$ *in Theorem* 5*, then we have*

$$\int_a^b f(t)dt = A(f, I_n) + R(f, I_n)$$

where

$$A(f, I_n) := \frac{1}{4}\sum_{i=0}^{n} h_i \left[f\left(\frac{3x_i + x_{i+1}}{2}\right) + f\left(\frac{x_i + 3x_{i+1}}{2}\right) + f\left(\frac{7x_i + x_{i+1}}{8}\right) + f\left(\frac{x_i + 7x_{i+1}}{8}\right)\right]$$

and the remainder term $R(f, I_n)$ *satisfies*

$$|R(f, I_n)| \leq \frac{1}{8}\upsilon(h) \bigvee_a^b(f).$$

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Original Article

Fractional integrals and solution of fractional kinetic equations involving k-Mittag-Leffler function

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Abstract

In this paper, our main objective is to establish certain new fractional integral by applying the Saigo hypergeometric fractional integral operators and by employing some integral transforms on the resulting formulas, we presented their image formulas involving the product of the generalized k-Mittag-Leffler function. Furthermore, We develop a new and further generalized form of the fractional kinetic equation involving the product of the generalized k-Mittag-Leffler function. The manifold generality of the generalized k-Mittag-Leffler function is discussed in terms of the solution of the fractional kinetic equation and their graphical interpretation is interpreted in the present paper. The results obtained here are quite general in nature and capable of yielding a very large number of known and (presumably) new results.

Keywords: k-Pochhammer symbol; k-gamma function; Generalized k-Mittag-Leffler function; Laplace transform; Fractional kinetic equations; MATLAB

1. Introduction and preliminaries

In 2006, Diaz and Pariguan [1] introduced the k-Pochhammer symbol and k-gamma function defined as follows:

$$(\gamma)_{n,k} := \begin{cases} \dfrac{\Gamma_k(\gamma + nk)}{\Gamma_k(\gamma)} & (k \in \mathbb{R}; \gamma \in \mathbb{C} \setminus \{0\}) \\ \gamma(\gamma + k)...(\gamma + (n-1)k) & (n \in \mathbb{N}; \gamma \in \mathbb{C}), \end{cases} \tag{1.1}$$

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and the relation with the classical Euler's gamma function as:

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right) \tag{1.2}$$

where $\gamma \in \mathbb{C}, k \in \mathbb{R}$ and $n \in \mathbb{N}$.

When $k = 1$, (1.1) reduces to the classical Pochhammer symbol and Euler's gamma function respectively.

Also let $\gamma \in \mathbb{C}, k, s \in \mathbb{R}$, then the following identity holds

$$\Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1}\Gamma_k\left(\frac{k\gamma}{s}\right), \tag{1.3}$$

in particular,

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right), \tag{1.4}$$

Further, let $\gamma \in \mathbb{C}, k, s \in \mathbb{R}$ and $\gamma \in \mathbb{C}$, then the following identity holds

$$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq}\left(\frac{k\gamma}{s}\right)_{nq}, \tag{1.5}$$

in particular,

$$(\gamma)_{nq,k} = (k)^{nq}\left(\frac{\gamma}{k}\right)_{nq}, \tag{1.6}$$

For more details of k-Pochhammer symbol, k-special function and fractional Fourier transform one can refer to the papers by Romero et al. [2,3].

Let $k \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ and $q \in \mathbb{R}^+$, then the generalized k-Mittag-Leffler function, denoted by $E^{\gamma,q}_{k,\alpha,\beta}(z)$, is defined as

$$E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{nq,k}z^n}{\Gamma_k(n\alpha+\beta)n!} \tag{1.7}$$

where $(\gamma)_{nq,k}$ denotes the k-Pochhammer symbol given by Eq. (1.6) and $\Gamma_k(\gamma)$ is the k-gamma function given by Eq. (1.4) (also see [4]).

Particular cases of $E^{\gamma,q}_{k,\alpha,\beta}(z)$

(i) For $q = 1$, Eq. (1.7) yields k-Mittag-Leffler function (Dorrego and Cerutti [5]), defined as:

$$E^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}z^n}{\Gamma_k(n\alpha+\beta)n!} = E^{\gamma}_{k,\alpha,\beta}(z) \tag{1.8}$$

(ii) For $k = 1$, Eq. (1.7) yields Mittag-Leffler function, defined as (Shukla and Prajapati [6])

$$E^{\gamma,q}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{nq}z^n}{\Gamma(n\alpha+\beta)n!} = E^{\gamma,q}_{\alpha,\beta}(z), \tag{1.9}$$

(iii) For $q = 1$ and $k = 1$, Eq. (1.7) gives Mittag-Leffler function, defined as (Dorrego and Cerutti [5])

$$E^{\gamma,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(n\alpha+\beta)n!} = E^{\gamma}_{\alpha,\beta}(z) \tag{1.10}$$

(iv) For $q = 1, k = 1$ and $\gamma = 1$, Eq. (1.7) gives Mittag-Leffler function (Wiman [?]), defined as

$$E^{1,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}(z) \tag{1.11}$$

(v) For $q = 1, k = 1, \gamma = 1$ and $\beta = 1$, Eq. (1.7) gives Mittag-Leffler function (Mittag-Leffler [7]), defined as

$$E^{1,1}_{1,\alpha,1}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)} = E_{\alpha}(z). \tag{1.12}$$

The results given by Kiryakova [8], Miller and Ross [9], Srivastava et al., [10] can be referred for some basic results on fractional calculus. The Fox–Wright function ${}_p\Psi_q$ is defined as (see, for details, Srivastava and Karlsson 1985, [11])

$$\begin{aligned} {}_p\Psi_q[z] &= {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\ (b_1,\beta_1),\ldots,(b_q,\beta_q);\end{matrix} z\right] \\ &= {}_p\Psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p};\\ (b_j,\beta_j)_{1,q};\end{matrix} z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}, \end{aligned} \tag{1.13}$$

where the coefficients $\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q\in\mathbb{R}^+$ such that

$$1+\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i\geq 0. \tag{1.14}$$

2. Fractional integration

In this section, we will establish some fractional integral formulas for the generalized k-Mittag-Leffler function. To do this, we need to recall the following pair of Saigo hypergeometric fractional integral operators.

For $x>0$, $\lambda,\sigma,\vartheta\in\mathbb{C}$ and $\Re(\lambda)>0$, we have

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta}f(t)\right)(x)=\frac{x^{-\lambda-\sigma}}{\Gamma(\lambda)}\int_0^x(x-t)^{\lambda-1}{}_2F_1\left(\lambda+\sigma,-\vartheta;\lambda;1-\frac{t}{x}\right)f(t)\,dt \tag{2.1}$$

and

$$\left(J_{x,\infty}^{\lambda,\sigma,\vartheta}f(t)\right)(x)=\frac{1}{\Gamma(\lambda)}\int_x^{\infty}(t-x)^{\lambda-1}t^{-\lambda-\sigma}{}_2F_1\left(\lambda+\sigma,-\vartheta;\lambda;1-\frac{x}{t}\right)f(t)\,dt \tag{2.2}$$

where the ${}_2F_1(.)$, a special case of the generalized hypergeometric function, is the Gauss hypergeometric function.

The operator $I_{0,x}^{\lambda,\sigma,\vartheta}(.)$ contains the Riemann–Liouville $R_{0,x}^{\lambda}(.)$ fractional integral operators by means of the following relationships:

$$\left(R_{0,x}^{\lambda}f(t)\right)(x)=\left(I_{0,x}^{\lambda,-\lambda,\vartheta}f(t)\right)(x)=\frac{1}{\Gamma(\lambda)}\int_0^x(x-t)^{\lambda-1}f(t)\,dt \tag{2.3}$$

$$\left(W_{x,\infty}^{\lambda}f(t)\right)(x)=\left(J_{x,\infty}^{\lambda,-\lambda,\vartheta}f(t)\right)(x)=\frac{1}{\Gamma(\lambda)}\int_x^{\infty}(t-x)^{\lambda-1}f(t)\,dt. \tag{2.4}$$

It is noted that the operator (2.2) unifies the Erdêlyi–Kober fractional integral operators as follows:

$$\left(E_{0,x}^{\lambda,\vartheta}f(t)\right)(x)=\left(I_{0,x}^{\lambda,0,\vartheta}f(t)\right)(x)=\frac{x^{-\lambda-\vartheta}}{\Gamma(\lambda)}\int_0^x(x-t)^{\lambda-1}t^{\eta}f(t)\,dt \tag{2.5}$$

$$\left(K_{x,\infty}^{\lambda,\vartheta}f(t)\right)(x)=\left(J_{x,\infty}^{\lambda,0,\vartheta}f(t)\right)(x)=\frac{x^{\vartheta}}{\Gamma(\lambda)}\int_x^{\infty}(t-x)^{\lambda-1}t^{-\lambda-\vartheta}f(t)\,dt. \tag{2.6}$$

The following lemmas proved in Kilbas and Sebastin (2008) [12] are useful to prove our main results.

Lemma 1 (*Kilbas and Sebastian 2008*)**.** *Let $\lambda,\sigma,\vartheta\in\mathbb{C}$ be such that $\Re(\lambda)>0$, $\Re(\rho)>max[0,\Re(\sigma-\vartheta)]$, then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)}x^{\rho-\sigma-1}. \tag{2.7}$$

Lemma 2 (*Kilbas and Sebastian 2008*)**.** *Let $\lambda,\sigma,\vartheta\in\mathbb{C}$ be such that $\Re(\lambda)>0$, $\Re(\rho)<1+min[\Re(\sigma),\Re(\vartheta)]$, then*

$$\left(J_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}\right)(x)=\frac{\Gamma(\sigma-\rho+1)\Gamma(\vartheta-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma+\vartheta-\rho+1)}x^{\rho-\sigma-1}. \tag{2.8}$$

The main results are given in the following theorem.

Theorem 1. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *and* $q_i \in \mathbb{R}^+$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$, *then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(t)\right)(x) = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}\, {}_{r+2}\Psi_{r+2}\left[\begin{matrix} (\gamma_1/k_1, q_1), \ldots, (\gamma_r/k_r, q_r), \\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), \end{matrix} \begin{matrix} (\rho, r), (\rho+\vartheta-\sigma, r) \\ (\rho-\sigma, r), (\rho+\lambda+\vartheta, r) \end{matrix} \middle| k_1^{(q_1-\alpha_1/k_1)} \ldots k_r^{(q_r-\alpha_r/k_r)} x^r \right]. \tag{2.9}$$

Proof. For convenience, we denote the left-hand side of the result (2.9) by $\mathscr{I}$. Using (1.7), and then changing the order of integration and summation, which is valid under the conditions of Theorem 1, then

$$\mathscr{I} = \prod_{i=1}^{r} \left\{ \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{1}{n!} \right\} \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{nr+\rho-1} \right)(x), \tag{2.10}$$

applying the result (2.7), Eq. (2.10) reduces to

$$\mathscr{I} = \prod_{i=1}^{r} \left\{ \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{1}{n!} \right\} \times \frac{\Gamma(\rho+nr)\Gamma(\rho+\vartheta-\sigma+nr)}{\Gamma(\rho-\sigma+nr)\Gamma(\rho+\lambda+\vartheta+nr)} x^{\rho+nr-\sigma-1}, \tag{2.11}$$

after simplification, Eq. (2.11) reduces to

$$\mathscr{I} = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma(\gamma_i/k_i + q_i n)}{\Gamma(\beta_i/k_i + \alpha_i n/k_i)} \times \frac{\Gamma(\rho+nr)\Gamma(\rho+\vartheta-\sigma+nr)}{\Gamma(\rho-\sigma+nr)\Gamma(\rho+\lambda+\vartheta+nr)} \frac{1}{n!} x^{nr} k_i^{(q_i-\alpha_i/k_i)} \right\}, \tag{2.12}$$

interpreting the above equation with the help of (1.13), we have the required result. □

Theorem 2. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *and* $q_i \in \mathbb{R}^+$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$, *Then*

$$\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(1/t)\right)(x) = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}\, {}_{r+2}\Psi_{r+2}\left[\begin{matrix} (\gamma_1/k_1, q_1), \ldots, (\gamma_r/k_r, q_r), \\ (\beta_1/k_1, \alpha_1/k_1,) \ldots, (\beta_r/k_r, \alpha_r/k_r), \end{matrix} \begin{matrix} (\sigma-\rho+1, r), (\vartheta-\rho+1, r) \\ (1-\rho, r), (\lambda+\sigma+\vartheta-\rho+1, r) \end{matrix} \middle| \frac{k_1^{(q_1-\alpha_1/k_1)} \ldots k_r^{(q_r-\alpha_r/k_r)}}{x^r} \right]. \tag{2.13}$$

Proof. Proof is parallel to Theorem 1. □

2.1. Special cases

k-Mittag-Leffler function is the generalized form of the Mittag-Leffler function. By assigning the suitable values to the parameters, we have the following particular cases.

Setting $\sigma = 0$ in Theorems 1 and 2 and employing the relations (2.9) and (2.13) yield certain interesting results asserted by the following corollaries.

Corollary 1. *Let $\lambda, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$ (where $i = 1, \ldots, r$), such that $\Re(\rho) > \Re(\vartheta)$, then*

$$\left(E_{0,x}^{\lambda,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(t)\right)(x) = x^{\rho-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\gamma_1/k_1, q_1), \ldots, (\gamma_r/k_r, q_r), (\rho, r), (\rho+\vartheta, r)\\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), (\rho, r), (\rho+\lambda+\vartheta, r)\end{matrix}\,\middle|\, k_1^{(q_1-\alpha_1/k_1)} \ldots k_r^{(q_r-\alpha_r/k_r)} x^r\right]. \tag{2.14}$$

Corollary 2. *Let $\lambda, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$ (where $i = 1, \ldots, r$), such that $\Re(\rho) < 1 + \Re(\vartheta)$, then*

$$\left(K_{x,\infty}^{\lambda,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(1/t)\right)(x) = x^{\rho-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\gamma_1/k_1, q_1), \ldots, (\gamma_r/k_r, q_r), (1-\rho, r), (\vartheta-\rho+1, r)\\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), (1-\rho, r), (\lambda+\vartheta-\rho+1, r)\end{matrix}\,\middle|\, \frac{k_1^{(q_1-\alpha_1/k_1)} \ldots k_r^{(q_r-\alpha_r/k_r)}}{x^r}\right]. \tag{2.15}$$

Further, if we replace σ with $-\lambda$ in Theorems 1 and 2 reduced to the following form

Corollary 3. *Let $\lambda, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$ (where $i = 1, \ldots, r$), such that $\min\{\Re(\lambda), \Re(\rho)\} > 0$, then*

$$\left(R_{0,x}^{\lambda} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(t)\right)(x) = x^{\rho-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\gamma_1/k_1, q_1), \ldots, (\gamma_r/k_r, q_r), (\rho, r), (\rho, r)\\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), (\rho, r), (\rho+\lambda, r)\end{matrix}\,\middle|\, k_1^{(q_1-\alpha_1/k_1)} \ldots k_r^{(q_r-\alpha_r/k_r)} x^r\right]. \tag{2.16}$$

Corollary 4. *Let $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$ (where $i = 1, \ldots, r$), such that $\min\{\Re(\lambda), \Re(\rho)\} > 0$, Then*

$$\left(W_{x,\infty}^{\lambda} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(1/t)\right)(x) = x^{\rho-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\gamma_1/k_1, q_1), \ldots, (\gamma_r/k_r, q_r), (1-\rho, r), (1-\rho, r)\\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), (1-\rho, r), (\lambda-\rho+1, r)\end{matrix}\,\middle|\, \frac{k_1^{(q_1-\alpha_1/k_1)} \ldots k_r^{(q_r-\alpha_r/k_r)}}{x^r}\right]. \tag{2.17}$$

When $q_i = 1$ (where $i = 1, \ldots, r$), the k-Mittag-Leffler function reduced to $E_{k_i,\alpha_i,\beta_i}^{\gamma_i}(.)$ (see Eq. (1.8)) then the results in (2.9) and (2.13) reduced to the following form:

Corollary 5. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$, *then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i}(t)\right)(x) = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\gamma_1/k_1, 1), \ldots, (\gamma_r/k_r, 1), (\rho, r), (\rho+\vartheta-\sigma, r)\\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), (\rho-\sigma, r), (\rho+\lambda+\vartheta, r)\end{matrix}\middle| k_1^{(1-\alpha_1/k_1)} \ldots k_r^{(1-\alpha_r/k_r)} x^r\right]. \tag{2.18}$$

Corollary 6. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$, *then*

$$\left(I_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i}(1/t)\right)(x) = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\gamma_1/k_1, 1), \ldots, (\gamma_r/k_r, 1), (\sigma-\rho+1, r), (\vartheta-\rho+1, r)\\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), (1-\rho, r), (\lambda+\sigma+\vartheta-\rho+1, r)\end{matrix}\middle| \frac{k_1^{(1-\alpha_1/k_1)} \ldots k_r^{(1-\alpha_r/k_r)}}{x^r}\right]. \tag{2.19}$$

When $k_i = 1$ (where $i = 1, \ldots, r$), the k-Mittag-Leffler function reduced to $E_{\alpha_i,\beta_i}^{\gamma_i,q_i}(.)$ (see Eq. (1.9)) then the results in Eqs. (2.9) and (2.13) reduced to the following form:

Corollary 7. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *and* $q_i \in \mathbb{R}^+$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$, *then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i,q_i}(t)\right)(x) = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{1}{\Gamma(\gamma_i)} \times {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho, r), (\rho+\vartheta-\sigma, r)\\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\rho-\sigma, r), (\rho+\lambda+\vartheta, r)\end{matrix}\middle| x^r\right]. \tag{2.20}$$

Corollary 8. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *and* $q_i \in \mathbb{R}^+$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$, *then*

$$\left(I_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i,q_i}(1/t)\right)(x) = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{1}{\Gamma(\gamma_i)} \times {}_{r+2}\Psi_{r+2}\left[\begin{matrix}(\gamma_1, q_1), \ldots (\gamma_r, q_r), (\sigma-\rho+1, r), (\vartheta-\rho+1, r)\\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (1-\rho, r), (\lambda+\sigma+\vartheta-\rho+1, r)\end{matrix}\middle| \frac{1}{x^r}\right]. \tag{2.21}$$

When $k_i = q_i = 1$ (where $i = 1, \ldots, r$), the k-Mittag-Leffler function reduced to $E_{\alpha_i,\beta_i}^{\gamma_i}(.)$ (see Eq. (1.10)) then the results in Eqs. (2.9) and (2.13) reduced to the following form:

Corollary 9. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$, *then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i}(t)\right)(x) = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{1}{\Gamma(\gamma_i)} \times {}_{r+2}\Psi_{r+2}\left[\begin{matrix} (\gamma_1, 1), \ldots, (\gamma_r, 1), (\rho, r), (\rho+\vartheta-\sigma, r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\rho-\sigma, r), (\rho+\lambda+\vartheta, r) \end{matrix} \middle| x^r \right]. \tag{2.22}$$

Corollary 10. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$*, then*

$$\left(I_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i}(1/t)\right)(x) = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{1}{\Gamma(\gamma_i)} \times {}_{r+2}\Psi_{r+2}\left[\begin{matrix} (\gamma_1, 1), \ldots, (\gamma_r, 1), (\sigma-\rho+1, r), (\vartheta-\rho+1, r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (1-\rho, r), (\lambda+\sigma+\vartheta-\rho+1, r) \end{matrix} \middle| \frac{1}{x^r} \right]. \tag{2.23}$$

When $k_i = q_i = \gamma_i = 1$ (where $i = 1, \ldots, r$), the k-Mittag-Leffler function reduced to $E_{\alpha_i,\beta_i}(.)$ (see Eq. (1.11)) then the results in Eqs. (2.9) and (2.13) reduced to the following form:

Corollary 11. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i \in \mathbb{C}$, $\min\{\Re(\alpha_i), \Re(\beta_i)\} > 0$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) > \max[0, \Re(\sigma-\vartheta)]$*, Then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{\alpha_i,\beta_i}(t)\right)(x) = x^{\rho-\sigma-1} \times {}_3\Psi_{r+2}\left[\begin{matrix} (1, 1), (\rho, r), (\rho+\vartheta-\sigma, r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\rho-\sigma, r), (\rho+\lambda+\vartheta, r) \end{matrix} \middle| x^r \right]. \tag{2.24}$$

Corollary 12. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i \in \mathbb{C}$, $\min\{\Re(\alpha_i), \Re(\beta_i)\} > 0$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$*, then*

$$\left(I_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{\alpha_i,\beta_i}(1/t)\right)(x) = x^{\rho-\sigma-1} \times {}_3\Psi_{r+2}\left[\begin{matrix} (1, 1), (\sigma-\rho+1, r), (\vartheta-\rho+1, r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (1-\rho, r), (\lambda+\sigma+\vartheta-\rho+1, r) \end{matrix} \middle| \frac{1}{x^r} \right]. \tag{2.25}$$

When $k_i = q_i = \gamma_i = \beta_i = 1$ (where $i = 1, \ldots, r$), the k-Mittag-Leffler function reduced to $E_{\alpha_i,\beta_i}(.)$ (see Eq. (1.12)) then the results in Eqs. (2.9) and (2.13) reduced to the following form:

Corollary 13. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i \in \mathbb{C}$ *and* $\Re(\alpha_i) > 0$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) > \max[0, \Re(\sigma-\vartheta)]$*, then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{\alpha_i}(t)\right)(x) = x^{\rho-\sigma-1} \times {}_3\Psi_{r+2}\left[\begin{matrix} (1, 1), (\rho, r), (\rho+\vartheta-\sigma, r) \\ (1, \alpha_1), \ldots, (1, \alpha_r), (\rho-\sigma, r), (\rho+\lambda+\vartheta, r) \end{matrix} \middle| x^r \right]. \tag{2.26}$$

Corollary 14. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i \in \mathbb{C}$ *and* $\Re(\alpha_i) > 0$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$*, then*

$$\left(I_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{\alpha_i}(1/t)\right)(x) = x^{\rho-\sigma-1} \times {}_3\Psi_{r+2}\left[\begin{matrix} (1, 1), (\sigma-\rho+1, r), (\vartheta-\rho+1, r) \\ (1, \alpha_1), \ldots, (1, \alpha_r), (1-\rho, r), (\lambda+\sigma+\vartheta-\rho+1, r) \end{matrix} \middle| \frac{1}{x^r} \right]. \tag{2.27}$$

If $r = 1$ and $k_i = k, \alpha_i = \alpha, \beta_i = \beta, \gamma_i = \gamma, q_i = q$, then the results in Eqs. (2.9) and (2.13) reduced to the following form:

Corollary 15. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha, \beta, \gamma \in \mathbb{C}, k \in \mathbb{R}$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ *and* $q \in \mathbb{R}^+$, *such that* $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$, *then*

$$\begin{aligned}\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} E_{k,\alpha,\beta}^{\gamma,q}(t)\right)(x) &= x^{\rho-\sigma-1} \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \\ &\times {}_3\Psi_3\left[\begin{matrix}(\gamma/k, q), (\rho, 1), (\rho+\vartheta-\sigma, 1) \\ (\beta/k, \alpha/k), (\rho-\sigma, 1), (\rho+\lambda+\vartheta, 1)\end{matrix}\middle| k^{(q-\alpha/k)} x\right].\end{aligned} \tag{2.28}$$

Corollary 16. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha, \beta, \gamma \in \mathbb{C}, k \in \mathbb{R}$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ *and* $q \in \mathbb{R}^+$, *such that* $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$, *then*

$$\begin{aligned}\left(I_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} E_{k,\alpha,\beta}^{\gamma,q}(1/t)\right)(x) &= x^{\rho-\sigma-1} \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \\ &\times {}_3\Psi_3\left[\begin{matrix}(\gamma/k, q), (\sigma-\rho+1, 1), (\vartheta-\rho+1, 1) \\ (\beta/k, \alpha/k), (1-\rho, 1), (\lambda+\sigma+\vartheta-\rho+1, 1)\end{matrix}\middle| \frac{k^{(q-\alpha/k)}}{x}\right].\end{aligned} \tag{2.29}$$

Remark 1. If we assign the values to parameters involving in the k-Mittag-Leffler function, then all the results in Eqs. (2.28) and (2.29) reduced to the particular cases given in Eqs. (1.8)–(1.12).

3. Image formulas associated with integral transform

In this section, we establish certain theorems involving the results obtained in previous section associated with the integral transforms like, Beta transform, Laplace transform and Whittaker transform.

3.1. Beta transform

The Beta transform of $f(z)$ is defined as [13]:

$$B\{f(z) : a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z) dz. \tag{3.1}$$

Theorem 3. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(l), \Re(m), \Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *and* $q_i \in \mathbb{R}^+$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$, *then*

$$\begin{aligned}B&\left\{\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(t)\right)(x) : l, m\right\} = \Gamma(m) x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} \\ &\times {}_{r+3}\Psi_{r+3}\left[\begin{matrix}(\gamma_1/k_1, q_1), \ldots, (\gamma_r/k_r, q_r), (\rho, r), \\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), (\rho-\sigma, r),\end{matrix}\right. \\ &\qquad \left.\begin{matrix}(\rho+\vartheta-\sigma, r), (l, r) \\ (\rho+\lambda+\vartheta, r), (l+m, r)\end{matrix}\middle| k_1^{(q_1-\alpha_1/k_1)} \ldots k_r^{(q_r-\alpha_r/k_r)} t^r\right].\end{aligned} \tag{3.2}$$

Proof. For convenience, we denote the left-hand side of the result (3.2) by $\mathscr{B}$. Using the definition of beta transform, the LHS of (3.2) becomes:

$$\mathscr{B} = \int_0^1 z^{l-1}(1-z)^{m-1} \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(tz)\right)(x) dz, \tag{3.3}$$

further using (1.7) and then changing the order of integration and summation, which is valid under the conditions of Theorem 1, then

$$\mathscr{B} = \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{1}{n!} \left(I_{0+}^{\lambda,\sigma,\vartheta} t^{nr+\rho-1} \right)(x) \int_0^1 z^{l+nr-1}(1-z)^{m-1} dz \tag{3.4}$$

applying the result (2.7), after simplification Eq. (3.4) reduced to

$$\begin{aligned} \mathscr{B} = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma_i/k_i + q_i n)}{\Gamma(\beta_i/k_i + \alpha_i n/k_i)} \frac{\Gamma(\rho + nr)}{\Gamma(\rho - \sigma + nr)} \\ \times \frac{\Gamma(\rho + \vartheta - \sigma + nr)}{\Gamma(\rho + \lambda + \vartheta + nr)} \frac{x^{nr} k_i^{n(q_i - \alpha_i/k_i)}}{n!} \int_0^1 z^{l+nr-1}(1-z)^{m-1} dz, \end{aligned} \tag{3.5}$$

applying the definition of beta transform, Eq. (3.5) reduced to

$$\begin{aligned} \mathscr{B} = x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma_i/k_i + q_i n)}{\Gamma(\beta_i/k_i + \alpha_i n/k_i)} \frac{\Gamma(\rho + nr)}{\Gamma(\rho - \sigma + nr)} \\ \times \frac{\Gamma(\rho + \vartheta - \sigma + nr)}{\Gamma(\rho + \lambda + \vartheta + nr)} \frac{x^{nr} k_i^{n(q_i - \alpha_i/k_i)}}{n!} \frac{\Gamma(l + nr)\Gamma(m)}{\Gamma(l + m + nr)} \end{aligned} \tag{3.6}$$

interpreting the above equation with the help of (1.13), we have the required result. □

Theorem 4. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(l), \Re(m), \Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ *and* $q_i \in \mathbb{R}^+$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$*, then*

$$\begin{aligned} B\left\{ \left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(z/t) \right)(x) : l, m \right\} = \Gamma(m) x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} \\ \times {}_{3+r}\Psi_{3+r} \left[\begin{matrix} (\gamma_1/k_1, q_1), \ldots, (\gamma_r/k_r, q_r), (\sigma - \rho + 1, r), \\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), (1 - \rho, r), \end{matrix} \right. \\ \left. \begin{matrix} (\vartheta - \rho + 1, r), (l, r) \\ (\lambda + \sigma + \vartheta - \rho + 1, r), (l + m, r) \end{matrix} \middle| \frac{k_1^{(q_1 - \alpha_1/k_1)} \ldots k_r^{(q_r - \alpha_r/k_r)}}{t^r} \right]. \end{aligned} \tag{3.7}$$

Proof. The proof of this theorem is the same as that of Theorem 3. □

3.2. Laplace transform

The Laplace transform of $f(z)$ is defined as [13]:

$$L\{f(z)\} = \int_0^{\infty} e^{-sz} f(z) dz. \tag{3.8}$$

Theorem 5. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\Re(\alpha_i) > 0, \Re(\beta_i) > 0, \Re(\gamma_i) > 0$ *and* $q_i \in \mathbb{R}^+$ *(where* $i = 1, \ldots, r$*), such that* $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$*, then*

$$\begin{aligned} L\left\{ z^{l-1} \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(tz) \right)(x) \right\} = \frac{x^{\rho-\sigma-1}}{s^l} \prod_{i=1}^{r} \frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)} \\ \times {}_{r+3}\Psi_{r+2} \left[\begin{matrix} (\gamma_1/k_1, q_1), \ldots, (\gamma_r/k_r, q_r), (\rho, r), \\ (\beta_1/k_1, \alpha_1/k_1), \ldots, (\beta_r/k_r, \alpha_r/k_r), \end{matrix} \right. \\ \left. \begin{matrix} (\rho + \vartheta - \sigma, r), (l, r) \\ (\rho - \sigma, r), (\rho + \lambda + \vartheta, r) \end{matrix} \middle| k_1^{(q_1 - \alpha_1)/k_1} \ldots k_r^{(q_r - \alpha_r)/k_r} \left(\frac{t}{s} \right)^r \right]. \end{aligned} \tag{3.9}$$

Proof. For convenience, we denote the left-hand side of the result (3.9) by $\mathscr{L}$. Then applying the Laplace, we have:

$$\mathscr{L} = \int_0^{\infty} e^{-sz} z^{l-1} \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(tz) \right)(x) dz \tag{3.10}$$

further using (1.7) and then changing the order of integration and summation, which is valid under the conditions of Theorem 1, then

$$\begin{aligned}\mathscr{L} &= x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma_i/k_i+q_i n)}{\Gamma(\beta_i/k_i+\alpha_i n/k_i)}\frac{\Gamma(\rho+nr)}{\Gamma(\rho-\sigma+nr)}\\ &\quad\times\frac{\Gamma(\rho+\vartheta-\sigma+nr)}{\Gamma(\rho+\lambda+\vartheta+nr)}\frac{x^{nr}k_i^{n(q_i-\alpha_i/k_i)}}{n!}\int_0^{\infty}e^{-sz}z^{nr+l-1}dz\\ &= x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma_i/k_i+q_i n)}{\Gamma(\beta_i/k_i+\alpha_i n/k_i)}\frac{\Gamma(\rho+nr)}{\Gamma(\rho-\sigma+nr)}\\ &\quad\times\frac{\Gamma(\rho+\vartheta-\sigma+nr)}{\Gamma(\rho+\lambda+\vartheta+nr)}\frac{\Gamma(nr+l)}{s^{nr+l}}\frac{x^{nr}k_i^{n(q_i-\alpha_i/k_i)}}{n!},\end{aligned} \tag{3.11}$$

interpreting the above equation with the help of (1.13), we have the required result. □

Theorem 6. *Let $\lambda,\sigma,\vartheta,\rho,\alpha_i,\beta_i,\gamma_i\in\mathbb{C}, k_i\in\mathbb{R}$, $\Re(\alpha_i)>0, \Re(\beta_i)>0, \Re(\gamma_i)>0$ and $q_i\in\mathbb{R}^+$ (where $i=1,\dots,r$), such that $\Re(\rho)<1+\min[\Re(\sigma),\Re(\vartheta)]$, then*

$$\begin{aligned}&L\left\{z^{l-1}\left(J_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}\prod_{i=1}^{r}E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(z/t)\right)(x)\right\}=\frac{x^{\rho-\sigma-1}}{s^l}\prod_{i=1}^{r}\frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}\\ &\quad\times{}_{r+3}\Psi_{r+2}\left[\begin{matrix}(\gamma_1/k_1,q_1),\dots,(\gamma_r/k_r,q_r),(\sigma-\rho+1,r),\\ (\beta_1/k_1,\alpha_1/k_1),\dots,(\beta_r/k_r,\alpha_r/k_r),\end{matrix}\right.\\ &\quad\left.\begin{matrix}(\vartheta-\rho+1,r),(l,r)\\ (1-\rho,r),(\lambda+\sigma+\vartheta-\rho+1,r)\end{matrix}\;\middle|\;\frac{k_1^{(q_1-\alpha_1/k_1)}\dots k_r^{(q_r-\alpha_r/k_r)}}{(st)^r}\right].\end{aligned} \tag{3.12}$$

Proof. The proof of this theorem would run parallel as that of Theorem 5. □

3.3. Whittaker transform

Theorem 7. *Let $\lambda,\sigma,\vartheta,\rho,\alpha_i,\beta_i,\gamma_i\in\mathbb{C}, k_i\in\mathbb{R}$, $\min\{\Re(\alpha_i),\Re(\beta_i),\Re(\gamma_i)\}>0$; $\Re(\xi\pm\omega)>\frac{-1}{2}$ and $q_i\in\mathbb{R}^+$ (where $i=1,\dots,r$), such that $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, then*

$$\begin{aligned}&\int_0^{\infty}z^{\xi-1}e^{-\delta z/2}W_{\tau,\omega}(\eta z)\left\{\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}\prod_{i=1}^{r}E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(tz)\right)(x)\right\}dz\\ &\quad=\frac{x^{\rho-\sigma-1}}{\eta^{\xi-1}}\prod_{i=1}^{r}\frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}{}_{r+4}\Psi_{r+3}\left[\begin{matrix}(\gamma_1/k_1,q_1),\dots,\\ (\beta_1/k_1,\alpha_1/k_1),\dots,\end{matrix}\right.\\ &\quad\begin{matrix}(\gamma_r/k_r,q_r),(\rho,r),(\rho+\vartheta-\sigma,r),(1/2+\omega+\xi,r),\\ (\beta_r/k_r,\alpha_r/k_r),(\rho-\sigma,r),(\rho+\lambda+\vartheta,r),\end{matrix}\\ &\quad\left.\begin{matrix}(1/2-\omega+\xi,r)\\ (1/2-\tau+\xi,r)\end{matrix}\;\middle|\;k_1^{(q_1-\alpha_1/k_1)}\dots k_r^{(q_r-\alpha_r/k_r)}\left(\frac{x}{\eta}\right)^r\right].\end{aligned} \tag{3.13}$$

Proof. For convenience, we denote the left-hand side of the result (3.13) by $\mathscr{W}$. Then using the result from (2.12), after changing the order of integration and summation, we get:

$$\begin{aligned}\mathscr{W} &= x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma_i/k_i+q_i n)}{\Gamma(\beta_i/k_i+\alpha_i n/k_i)}\\ &\quad\times\frac{\Gamma(\rho+nr)\Gamma(\rho+\vartheta-\sigma+nr)}{\Gamma(\rho-\sigma+nr)\Gamma(\rho+\lambda+\vartheta+nr)}\\ &\quad\times\frac{x^{nr}k_i^{n(q_i-\alpha_i/k_i)}}{n!}\int_0^{\infty}z^{nr+\xi-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)dz,\end{aligned} \tag{3.14}$$

by substituting $\eta z = \varsigma$, (3.14) becomes:

$$\begin{aligned}\mathscr{W} &= x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma_i/k_i+q_in)}{\Gamma(\beta_i/k_i+\alpha_in/k_i)}\\ &\times\frac{\Gamma(\rho+nr)\Gamma(\rho+\vartheta-\sigma+nr)}{\Gamma(\rho-\sigma+nr)\Gamma(\rho+\lambda+\vartheta+nr)}\\ &\times\frac{x^{nr}k_i^{n(q_i-\alpha_i/k_i)}}{n!}\frac{1}{\eta^{nr+\xi-1}}\int_0^{\infty}\varsigma^{nr+\xi-1}e^{-\varsigma/2}W_{\tau,\omega}(\varsigma)d\varsigma.\end{aligned}\tag{3.15}$$

Now we use the following integral formula involving Whittaker function

$$\int_0^{\infty}t^{\nu-1}e^{-t/2}W_{\tau,\omega}(t)dt = \frac{\Gamma(1/2+\omega+\nu)\,\Gamma(1/2-\omega+\nu)}{\Gamma(1/2-\tau+\nu)},\quad\left(\Re(\nu\pm\omega)>\frac{-1}{2}\right).\tag{3.16}$$

Then we have

$$\begin{aligned}\mathscr{W} &= \frac{x^{\rho-\sigma-1}}{\eta^{\xi-1}}\prod_{i=1}^{r}\frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma_i/k_i+q_in)}{\Gamma(\beta_i/k_i+\alpha_in/k_i)}\\ &\times\frac{\Gamma(\rho+nr)\Gamma(\rho+\vartheta-\sigma+nr)\Gamma(1/2+\omega+\xi+nr)}{\Gamma(\rho-\sigma+nr)\Gamma(\rho+\lambda+\vartheta+nr)}\\ &\times\frac{\Gamma(1/2-\omega+\xi+nr)}{\Gamma(1/2-\tau+\xi+nr)}\frac{k_i^{(q_i-\alpha_i/k_i)}}{n!}\left(\frac{x}{\eta}\right)^{nr},\end{aligned}\tag{3.17}$$

interpreting the above equation with the help of (1.13), we have the required result. □

Theorem 8. *Let $\lambda, \sigma, \vartheta, \rho, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}, k_i \in \mathbb{R}$, $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$; $\Re(\xi \pm \omega) > \frac{-1}{2}$ and $q_i \in \mathbb{R}^+$ (where $i = 1, \ldots, r$), such that $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$, Then*

$$\begin{aligned}&\int_0^{\infty}z^{\xi-1}e^{-\delta z/2}W_{\tau,\omega}(\eta z)\left\{\left(J_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}\prod_{i=1}^{r}E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(z/t)\right)(x)\right\}dz\\ &= \frac{x^{\rho-\sigma-1}}{\eta^{\xi-1}}\prod_{i=1}^{r}\frac{k_i^{1-\beta_i/k_i}}{\Gamma(\gamma_i/k_i)}{}_{r+4}\Psi_{r+3}\left[\begin{matrix}(\gamma_1/k_1,q_1),\ldots,\\(\beta_1/k_1,\alpha_1/k_1),\ldots,\end{matrix}\right.\\ &\quad\begin{matrix}(\gamma_r/k_r,q_r),(\sigma-\rho+1,r),(\vartheta-\rho+1,r),(1/2+\omega+\xi,r),\\(\beta_r/k_r,\alpha_r/k_r),(1-\rho,r),(\lambda+\sigma+\vartheta-\rho+1,r),\end{matrix}\\ &\quad\left.\begin{matrix}(1/2-\omega+\xi,r)\\(1/2-\tau+\xi,r)\end{matrix}\right|\left.\frac{k_1^{(q_1-\alpha_1/k_1)}\ldots k_r^{(q_r-\alpha_r/k_r)}}{(x\eta)^r}\right].\end{aligned}\tag{3.18}$$

Proof. The proof of this theorem would run parallel as those of Theorem 7. □

4. Fractional kinetic equations

The importance of fractional differential equations in the field of applied science has gained more attention not only in mathematics but also in physics, dynamical systems, control systems and engineering, to create the mathematical model of many physical phenomena. Especially, the kinetic equations describe the continuity of motion of substance. The extension and generalization of fractional kinetic equations involving many fractional operators were found in [14–27].

In view of the effectiveness and a great importance of the kinetic equation in certain astrophysical problems the authors develop a further generalized form of the fractional kinetic equation involving generalized k-Mittag-Leffler function.

The fractional differential equation between rate of change of the reaction, the destruction rate and the production rate was established by Haubold and Mathai [20] given as follows:

$$\frac{dN}{dt} = -d(N_t) + p(N_t),\tag{4.1}$$

where $N = N(t)$ the rate of reaction, $d = d(N)$ the rate of destruction, $p = p(N)$ the rate of production and N_t denotes the function defined by $N_t(t^*) = N(t - t^*)$, $t^* > 0$.

The special case of (4.1) for spatial fluctuations and inhomogeneities in $N(t)$ the quantities are neglected, that is the equation

$$\frac{dN}{dt} = -c_i N_i(t), \tag{4.2}$$

with the initial condition that $N_i(t = 0) = N_0$ is the number density of the species i at time $t = 0$ and $c_i > 0$. If we remove the index i and integrate the standard kinetic equation(4.2), we have

$$N(t) - N_0 = -c_0 D_t^{-1} N(t) \tag{4.3}$$

where ${}_0D_t^{-1}$ is the special case of the Riemann–Liouville integral operator ${}_0D_t^{-\nu}$ defined as

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s)ds, \quad (t > 0,\ R(\nu) > 0). \tag{4.4}$$

The fractional generalization of the standard kinetic equation (4.3) is given by Haubold and Mathai [20] as follows:

$$N(t) - N_0 = -c^\nu {}_0D_t^{-1} N(t) \tag{4.5}$$

and obtained the solution of (4.5) as follows:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (ct)^{\nu k}. \tag{4.6}$$

Further, Saxena and Kalla [25] considered the following fractional kinetic equation:

$$N(t) - N_0 f(t) = -c^\nu {}_0D_t^{-\nu} N(t), \quad (\Re(v) > 0), \tag{4.7}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$, c is a constant and $f \in \mathcal{L}(0, \infty)$.

By applying the Laplace transform to (4.7) (see [21]),

$$L\{N(t); p\} = N_0 \frac{F(p)}{1 + c^\nu p^{-\nu}} = N_0 \left(\sum_{n=0}^{\infty} (-c^\nu)^n p^{-\nu n} \right) F(p), \quad \left(n \in N_0, \left| \frac{c}{p} \right| < 1 \right) \tag{4.8}$$

where the Laplace transform [28] is given by

$$F(p) = L\{N(t); p\} = \int_0^\infty e^{-pt} f(t)dt, \quad (\mathcal{R}(p) > 0). \tag{4.9}$$

5. Solution of generalized fractional kinetic equations

In this section, we investigated the solutions of the generalized fractional kinetic equations by considering generalized k-Mittag-Leffler function.

Remark 2. The solutions of the fractional kinetic equations in this section are obtained in terms of the generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$ (Mittag-Leffler [7]), which is defined as:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \Re(\alpha) > 0, \quad \Re(\beta) > 0. \tag{5.1}$$

Theorem 9. *If $a > 0, d > 0, \nu > 0$, $k_i \in \mathbb{R}, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{5.2}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i} \Gamma(\nu n + 1)}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{5.3}$$

Proof. Laplace transform of Riemann–Liouville fractional integral operator is given by (Erdelyi et al. [29], Srivastava and Saxena [30]):

$$L\left\{{}_0D_t^{-\nu} f(t);\, p\right\} = p^{-\nu} F(p) \tag{5.4}$$

where $F(p)$ is defined in (4.9). Now, applying Laplace transform on (5.2) gives,

$$L\{N(t);\, p\} = N_0 L\left\{\prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(d^\nu t^\nu);\, p\right\} - a^\nu L\left\{{}_0D_t^{-\nu} N(t);\, p\right\} \tag{5.5}$$

$$i.e. \quad N(p) = N_0 \left(\int_0^\infty e^{-pt} \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu t^\nu)^n}{n!} dt\right) - a^\nu p^{-\nu} N(p) \tag{5.6}$$

interchanging the order of integration and summation in (5.6), we have

$$N(p) + a^\nu p^{-\nu} N(p) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu)^n}{n!} \int_0^\infty e^{-pt} t^{\nu n} dt \tag{5.7}$$

$$= N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu)^n}{n!} \frac{\Gamma(\nu n + 1)}{p^{\nu n+1}} \tag{5.8}$$

this leads to

$$\begin{aligned} N(p) &= N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu)^n}{n!} \\ &\quad \times \Gamma(\nu n + 1) \left\{p^{-(\nu n+1)} \textstyle\sum_{l=0}^{\infty} \left[-\left(\frac{p}{a}\right)^{-\nu}\right]^l\right\}. \end{aligned} \tag{5.9}$$

Taking Laplace inverse of (5.9), and by using

$$L^{-1}\left\{p^{-\nu};\, t\right\} = \frac{t^{\nu-1}}{\Gamma(\nu)}, \, (R(\nu) > 0) \tag{5.10}$$

we have,

$$\begin{aligned} L^{-1}\{N(p)\} &= N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu)^n}{n!} \\ &\quad \times \Gamma(\nu n + 1) L^{-1}\left\{\sum_{l=0}^{\infty} (-1)^l a^{\nu l} p^{-[\nu(n+l)+1]}\right\} \end{aligned} \tag{5.11}$$

$$\begin{aligned} i.e. \quad N(t) &= N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu)^n}{n!} \Gamma(\nu n + 1) \\ &\quad \times \left\{\sum_{l=0}^{\infty} (-1)^l a^{\nu l} \frac{t^{\nu(n+l)}}{\Gamma(\nu(n+l)+1)}\right\} \end{aligned} \tag{5.12}$$

$$\begin{aligned} &= N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i}}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu t^\nu)^n}{n!} \Gamma(\nu n + 1) \\ &\quad \times \left\{\sum_{l=0}^{\infty} (-1)^l \frac{(a^\nu t^\nu)^l}{\Gamma(\nu(n+l)+1)}\right\}. \end{aligned} \tag{5.13}$$

Eq. (5.13) can be written as

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i} \Gamma(\nu n + 1)}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-a^\nu t^\nu). \qquad \square \tag{5.14}$$

Theorem 10. *If $d > 0, \nu > 0, k_i \in \mathbb{R}, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.15}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i} \Gamma(\nu n + 1)}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-d^\nu t^\nu). \tag{5.16}$$

Theorem 11. *If $d > 0, \nu > 0, k_i \in \mathbb{R}, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i,q_i}(t) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.17}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i,k_i} \Gamma(n + 1)}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(t)^n}{n!} E_{\nu,n+1}(-d^\nu t^\nu). \tag{5.18}$$

5.1. Special cases

k-Mittag-Leffler function is the generalized form of the Mittag-Leffler function. By assigning the suitable values to the parameters, we have the following particular cases.

When $q_i = 1$ (where $i = 1, \ldots, r$), the k-Mittag-Leffler function reduced to $E_{k_i,\alpha_i,\beta_i}^{\gamma_i}(.)$ (see Eq. (1.8)) then the results in (5.2), (5.15) and (5.17) and their solutions reduced to the following form:

Corollary 17. *If $a > 0, d > 0, \nu > 0, k_i \in \mathbb{R}, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{5.19}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{n,k_i} \Gamma(\nu n + 1)}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{5.20}$$

Corollary 18. *If $d > 0, \nu > 0, k_i \in \mathbb{R}, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.21}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{n,k_i} \Gamma(\nu n + 1)}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-d^\nu t^\nu). \tag{5.22}$$

Corollary 19. *If $d > 0, \nu > 0, k_i \in \mathbb{R}, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{k_i,\alpha_i,\beta_i}^{\gamma_i}(t) = -d^{\nu}{}_0D_t^{-\nu} N(t) \tag{5.23}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{n,k_i}\, \Gamma(n+1)}{\Gamma_{k_i}(n\alpha_i + \beta_i)} \frac{(t)^n}{n!} E_{\nu,n+1}(-d^{\nu} t^{\nu}). \tag{5.24}$$

When $k_i = 1$ (where $i = 1, \dots, r$), the k-Mittag-Leffler function reduced to $E_{\alpha_i,\beta_i}^{\gamma_i,q_i}(.)$ (see Eq. (1.9)) then the results in Eqs. (5.2), (5.15) and (5.17) and their solutions reduced to the following form:

Corollary 20. *If $a > 0, d > 0, \nu > 0, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i,q_i}(d^{\nu} t^{\nu}) = -a^{\nu}{}_0D_t^{-\nu} N(t) \tag{5.25}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i}\, \Gamma(\nu n+1)}{\Gamma(n\alpha_i + \beta_i)} \frac{(d^{\nu} t^{\nu})^n}{n!} E_{\nu,\nu n+1}(-a^{\nu} t^{\nu}). \tag{5.26}$$

Corollary 21. *If $d > 0, \nu > 0, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i,q_i}(d^{\nu} t^{\nu}) = -d^{\nu}{}_0D_t^{-\nu} N(t) \tag{5.27}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i}\, \Gamma(\nu n+1)}{\Gamma(n\alpha_i + \beta_i)} \frac{(d^{\nu} t^{\nu})^n}{n!} E_{\nu,\nu n+1}(-d^{\nu} t^{\nu}). \tag{5.28}$$

Corollary 22. *If $d > 0, \nu > 0, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$ and $q_i \in \mathbb{R}^+$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i,q_i}(t) = -d^{\nu}{}_0D_t^{-\nu} N(t) \tag{5.29}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_{nq_i}\, \Gamma(n+1)}{\Gamma(n\alpha_i + \beta_i)} \frac{(t)^n}{n!} E_{\nu,n+1}(-d^{\nu} t^{\nu}). \tag{5.30}$$

When $k_i = q_i = 1$ (where $i = 1, \dots, r$), the k-Mittag-Leffler function reduced to $E_{\alpha_i,\beta_i}^{\gamma_i}(.)$ (see Eq. (1.10)) then the results in Eqs. (5.2), (5.15) and (5.17) and their solutions reduced to the following form:

Corollary 23. *If $a > 0, d > 0, \nu > 0, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i}(d^{\nu} t^{\nu}) = -a^{\nu}{}_0D_t^{-\nu} N(t) \tag{5.31}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_n \Gamma(\nu n+1)}{\Gamma(n\alpha_i+\beta_i)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{5.32}$$

Corollary 24. *If* $d > 0, \nu > 0, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}; \min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$, *then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.33}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_n \Gamma(\nu n+1)}{\Gamma(n\alpha_i+\beta_i)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-d^\nu t^\nu). \tag{5.34}$$

Corollary 25. *If* $d > 0, \nu > 0, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}; \min\{\Re(\alpha_i), \Re(\beta_i), \Re(\gamma_i)\} > 0$, *then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i}(t) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.35}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(\gamma_i)_n \Gamma(n+1)}{\Gamma(n\alpha_i+\beta_i)} \frac{(t)^n}{n!} E_{\nu,n+1}(-d^\nu t^\nu). \tag{5.36}$$

When $k_i = q_i = \gamma_i = 1$ (where $i = 1, \dots, r$), the k-Mittag-Leffler function reduced to $E_{\alpha_i,\beta_i}(.)$ (see Eq. (1.11)) then the results in Eqs. (5.2), (5.15) and (5.17) and their solutions reduced to the following form:

Corollary 26. *If* $a > 0, d > 0, \nu > 0, \alpha_i, \beta_i \in \mathbb{C}; \min\{\Re(\alpha_i), \Re(\beta_i)\} > 0$, *then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i,\beta_i}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{5.37}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{\Gamma(\nu n+1)}{\Gamma(n\alpha_i+\beta_i)} (d^\nu t^\nu)^n E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{5.38}$$

Corollary 27. *If* $d > 0, \nu > 0, \alpha_i, \beta_i \in \mathbb{C}; \min\{\Re(\alpha_i), \Re(\beta_i)\} > 0$, *then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i,\beta_i}^{\gamma_i}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.39}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{\Gamma(\nu n+1)}{\Gamma(n\alpha_i+\beta_i)} (d^\nu t^\nu)^n E_{\nu,\nu n+1}(-d^\nu t^\nu). \tag{5.40}$$

Corollary 28. *If* $d > 0, \nu > 0, \alpha_i, \beta_i \in \mathbb{C}; \min\{\Re(\alpha_i), \Re(\beta_i)\} > 0$, *then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i,\beta_i}(t) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.41}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(t)^n}{\Gamma(n\alpha_i+\beta_i)} E_{\nu,n+1}(-d^\nu t^\nu). \tag{5.42}$$

When $k_i = q_i = \gamma_i = \beta_i = 1$ (where $i = 1, \ldots, r$), the k-Mittag-Leffler function reduced to $E_{\alpha_i,\beta_i}(.)$ (see Eq. (1.12)) then the results in Eqs. (5.2), (5.15) and (5.17) and their solutions reduced to the following form:

Corollary 29. *If $a > 0, d > 0, \nu > 0, \alpha_i \in \mathbb{C}; \Re(\alpha_i) > 0$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{5.43}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{\Gamma(\nu n + 1)}{\Gamma(n\alpha_i + 1)} (d^\nu t^\nu)^n E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{5.44}$$

Corollary 30. *If $d > 0, \nu > 0, \alpha_i \in \mathbb{C}; \Re(\alpha_i) > 0$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i}^{\gamma_i}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.45}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{\Gamma(\nu n + 1)}{\Gamma(n\alpha_i + 1)} (d^\nu t^\nu)^n E_{\nu,\nu n+1}(-d^\nu t^\nu). \tag{5.46}$$

Corollary 31. *If $d > 0, \nu > 0, \alpha_i \in \mathbb{C}; \Re(\alpha_i) > 0$, then the solution of the equation*

$$N(t) - N_0 \prod_{i=1}^{r} E_{\alpha_i}(t) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.47}$$

is given by the following formula

$$N(t) = N_0 \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(t)^n}{\Gamma(n\alpha_i + 1)} E_{\nu,n+1}(-d^\nu t^\nu). \tag{5.48}$$

If $r = 1$ and $k_i = k, \alpha_i = \alpha, \beta_i = \beta, \gamma_i = \gamma, q_i = q$, (5.2), (5.15) and (5.17) and their solutions reduced to the following form:

Corollary 32. *If $a > 0, d > 0, \nu > 0, k \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ and $q \in \mathbb{R}^+$, then the solution of the equation*

$$N(t) - N_0 E_{k,\alpha,\beta}^{\gamma,q}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{5.49}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} \Gamma(\nu n + 1)}{\Gamma_k(n\alpha + \beta)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{5.50}$$

Corollary 33. *If $d > 0, \nu > 0, k \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ and $q \in \mathbb{R}^+$, then the solution of the equation*

$$N(t) - N_0 E_{k,\alpha,\beta}^{\gamma,q}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.51}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} \Gamma(\nu n + 1)}{\Gamma_k(n\alpha + \beta)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\nu n+1}(-d^\nu t^\nu). \tag{5.52}$$

Table 1
Numerical solutions of KE containing 100 terms of KMLF.

t	$\upsilon = 2.0$	$\upsilon = 2.2$	$\upsilon = 2.4$	$\upsilon = 2.6$	$\upsilon = 2.8$
0	6.516846	6.516846	6.516846	6.516846	6.516846
0.2	6.461213	6.484693	6.498794	6.50705	6.511758
0.4	6.29689	6.370156	6.421997	6.457636	6.481496
0.6	6.031509	6.163597	6.268155	6.348104	6.407487
0.8	5.677478	5.864749	6.028106	6.164575	6.274797
1	5.251521	5.480041	5.699526	5.898869	6.072628
1.2	4.77407	5.021969	5.286818	5.549145	5.795762
1.4	4.268521	4.508722	4.801438	5.121049	5.446635
1.6	3.760396	3.963786	4.262314	4.629152	5.03807
1.8	3.276435	3.415445	3.696246	4.098657	4.59685
2	2.843676	2.896159	3.138273	3.567422	4.168487
2.2	2.488539	2.44185	2.631986	3.088403	3.823734
2.4	2.235976	2.091115	2.229861	2.732748	3.667753
2.6	2.108714	1.884414	1.993663	2.593842	3.853343
2.8	2.126626	1.863299	1.995042	2.792817	4.600487
3	2.306273	2.069749	2.316491	3.486243	6.225768

Corollary 34. *If $d > 0$, $\nu > 0$, $k \in \mathbb{R}$, $\alpha, \beta, \gamma_i \in \mathbb{C}$; $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ and $q \in \mathbb{R}^+$, then the solution of the equation*

$$N(t) - N_0 E^{\gamma,q}_{k,\alpha,\beta}(t) = -d^{\nu}\,{}_0D_t^{-\nu} N(t) \tag{5.53}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}\,\Gamma(n+1)}{\Gamma_k(n\alpha+\beta)} \frac{(t)^n}{n!} E_{\nu,n+1}(-d^{\nu}t^{\nu}). \tag{5.54}$$

6. Numerical solutions of fractional kinetic equations

In this section, we establish database for numerical solutions of the kinetic equation (5.2) by employing Eq. (5.3) for particular values of the parameters, which are given in Tables 1–3; their graphs are plotted in Figs. 1–3 and Mesh-plot is also established in Figs. 4 and 5. For this purpose, we denote the solution of Eq. (5.2) for $r = 2$ (i.e. kinetic equation involving the product of two k-Mittag-Leffler functions) as

$$N(t) = N(N_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \gamma_2, q_1, q_2, d, a, \nu, k_1, k_2, t)$$

and then we develop the program in MATLAB. Employing the program, we establish database, graphs and mesh-plot.

In our investigation, particular values to the parameters involving in the solution of the fractional kinetic equation are selected as $N_0 = 2$; $\alpha_1 = \beta_1 = 1$; $\alpha_2 = \beta_2 = 2$, $\gamma_1 = \gamma_2 = q_1 = q_2 = 0.1$; $d = a = 1$; $\nu = 2.0 : 0.2 : 2.8$; $k_1 = k_2 = 2$ for $0 \leq t \leq 3$. Solutions of Kinetic equations are involving with the generalized Mittag-Leffler function, which contain infinite number of terms, further solution of the fractional kinetic equation also contains the summation of infinite terms with r times product, which makes the complexity for numerical solutions. For critical analysis of the numerical solutions, we investigate by taking different range of terms occurring in the solution of fractional kinetic equation in three stages as follows.

- At the first stage, we choose first 100 terms of k-Mittag-Leffler function and 50 terms of the summation of Eq. (5.3), the Data-Base and graphs are established in Table 1 and Fig. 1 respectively. We found that $N(t) \geq 0$ for all different values of the parameters for $t \geq 0$ and $N(t) \to \infty$ as $t \to \infty$.
- At the first and the third stage, we choose first 500 and 1000 terms of k-Mittag-Leffler function and 50 terms of the summation of Eq. (5.3), the Data-Base and graphs are established in Tables 2, 3 and Figs. 2, 3 respectively, which can be easily observed from these tables and graphs that solution remains same for any interval of t convergent values exist as we increase the number of terms in Mittag-Leffler function. Also the convergent values decrease as we increase the number of terms, which can be easily observed from Tables 2, 3 and Figs. 2, 3.

Table 2
Numerical solutions of KE containing 500 terms of KMLF.

t	$v = 2.0$	$v = 2.2$	$v = 2.4$	$v = 2.6$	$v = 2.8$
0	6.516846	6.516846	6.516846	6.516846	6.516846
0.2	6.461213	6.484693	6.498794	6.50705	6.511758
0.4	6.29689	6.370156	6.421997	6.457636	6.481496
0.6	6.031509	6.163597	6.268155	6.348104	6.407487
0.8	5.677478	5.864749	6.028106	6.164575	6.274797
1	5.251521	5.480041	5.699526	5.898869	6.072628
1.2	4.77407	5.021969	5.286818	5.549145	5.795762
1.4	4.268521	4.508722	4.801438	5.121049	5.446635
1.6	3.760396	3.963786	4.262314	4.629152	5.03807
1.8	3.276435	3.415445	3.696246	NaN	NaN
2	2.843676	NaN	NaN	NaN	NaN
2.2	NaN	NaN	NaN	NaN	NaN
2.4	NaN	NaN	NaN	NaN	NaN
2.6	NaN	NaN	NaN	NaN	NaN
2.8	NaN	NaN	NaN	NaN	NaN
3	NaN	NaN	NaN	NaN	NaN

Table 3
Numerical solutions of KE containing 1000 terms of KMLF.

t	$v = 2.0$	$v = 2.2$	$v = 2.4$	$v = 2.6$	$v = 2.8$
0	6.516846	6.516846	6.516846	6.516846	6.516846
0.2	6.461213	6.484693	6.498794	6.50705	6.511758
0.4	6.29689	6.370156	6.421997	6.457636	6.481496
0.6	6.031509	6.163597	6.268155	6.348104	6.407487
0.8	5.677478	5.864749	6.028106	6.164575	6.274797
1	5.251521	5.480041	5.699526	5.898869	6.072628
1.2	4.77407	5.021969	5.286818	5.549145	5.795762
1.4	4.268521	NaN	NaN	NaN	NaN
1.6	NaN	NaN	NaN	NaN	NaN
1.8	NaN	NaN	NaN	NaN	NaN
2	NaN	NaN	NaN	NaN	NaN
2.2	NaN	NaN	NaN	NaN	NaN
2.4	NaN	NaN	NaN	NaN	NaN
2.6	NaN	NaN	NaN	NaN	NaN
2.8	NaN	NaN	NaN	NaN	NaN
3	NaN	NaN	NaN	NaN	NaN

In the above discussion, we find that $N(t) \geq 0$ for all different values of the parameters for $t \geq 0$ for different number of terms occurring in the solutions of fractional kinetic equation. Mesh-Plot for first 100 and 500 terms of k-Mittag-Leffler function and 50 terms of the summation of Eq. (5.3) is also established in Figs. 4 and 5, from which we can easily interpret the behavior of the solution of fractional kinetic equation.

7. Concluding remarks

We can also present a large number of special cases of our main fractional integral formulas, images formulas and solutions of the generalized fractional kinetic equations.

If we setting $r = 1; \alpha_1 = \alpha, \beta_1 = \beta, \gamma_1 = \gamma, k_1 = k, q_1 = q$ in Theorems 1 and 2. Here, we illustrate the following formulas.

Corollary 35. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha, \beta, \gamma, b, c \in \mathbb{C}, k \in \mathbb{R}$, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ *and* $q \in \mathbb{R}^+$, *such that* $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$, *then*

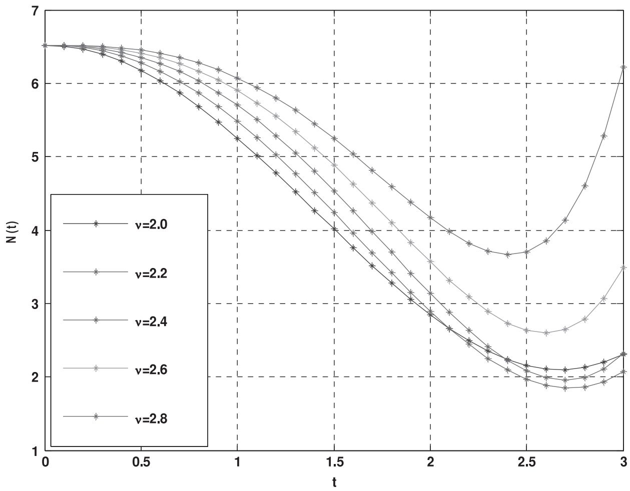

Fig. 1. Graphical solutions of KE containing 100 terms of KMLF.

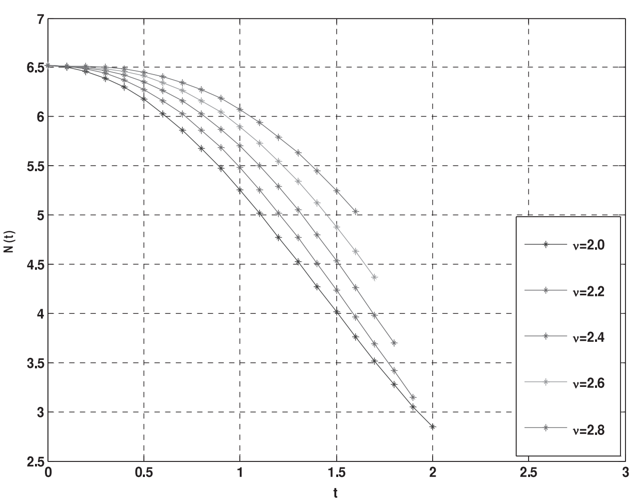

Fig. 2. Graphical solutions of KE containing 500 terms of KMLF.

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} E_{k,\alpha,\beta}^{\gamma,q}(t)\right)(x) = x^{\rho-\sigma-1} \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \times {}_3\Psi_3\left[\begin{matrix} (\gamma/k, q/k), (\rho, 1), (\rho+\vartheta-\sigma, 1) \\ (\beta/k, \alpha/k), (\rho-\sigma, 1), (\rho+\lambda+\vartheta, 1) \end{matrix} \middle| k^{(q-\alpha)/k} x \right]. \tag{7.1}$$

Corollary 36. *Let* $\lambda, \sigma, \vartheta, \rho, \alpha, \beta, \gamma, b, c \in \mathbb{C}, k \in \mathbb{R}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$ *and* $q \in \mathbb{R}^+$*, such that* $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$*, then*

$$\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} E_{k,\alpha,\beta}^{\gamma,q}(1/t)\right)(x) = x^{\rho-\sigma-1} \frac{k^{1-\beta/k}}{\Gamma(\gamma/k)} \times {}_3\Psi_3\left[\begin{matrix} (\gamma/k, q/k), (\sigma-\rho+1, 1), (\vartheta-\rho+1, 1) \\ (\beta/k, \alpha/k), (1-\rho, 1), (\lambda+\sigma+\vartheta-\rho+1, 1) \end{matrix} \middle| \frac{k^{(q-\alpha)/k}}{x} \right]. \tag{7.2}$$

The above two results in Eqs. (7.1) and (7.2) are involving pair of Saigo hypergeometric fractional integral operators, using the relations given in Eqs. (2.3), (2.4), (2.5) and (2.6), these formulas in Eqs. (7.1) and (7.2) reduced

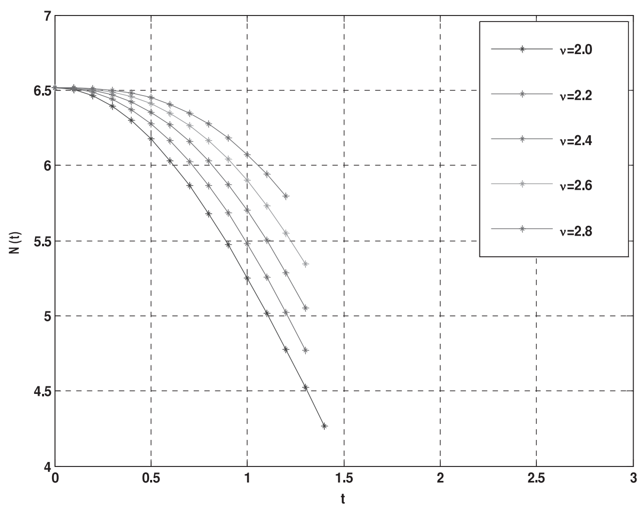

Fig. 3. Graphical solutions of KE containing 1000 terms of KMLF.

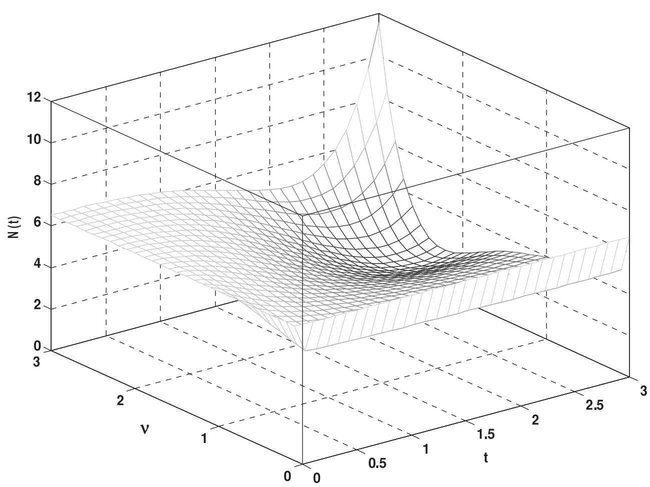

Fig. 4. Mesh-plot of KE containing 100 terms of KMLF.

to the type of Riemann–Liouville and Erdêlyi-Kober fractional integrals involving k-Mittag-Leffler function. Further by employing the particular cases to the k-Mittag-Leffler function we obtain more special cases of all the fractional integrals in Section 2 and their images formulas in Section 3.

We may also emphasize that results derived in this paper are of general character and can specialize to give further interesting and potentially useful formulas involving integral transform and fractional calculus. Also we give a new fractional generalization of the standard kinetic equation and derived solution for the same. From the close relationship of the generalized k-Mittag-Leffler function with many special functions, we can easily construct various known and new fractional kinetic equations. Also from the numerical solutions established in Tables 1–3 and their graphical interpretation in Figs. 1–5 for product of two k-Mittag-Leffler functions, we came to the conclusion that the solutions of the fractional Kinetic equations are always positive ($N(t) \geq 0$ for all values of the parameters). In our investigation, we choose $r = 2$. The reader can choose any value of r for further more analysis of the solutions of fractional kinetic equations.

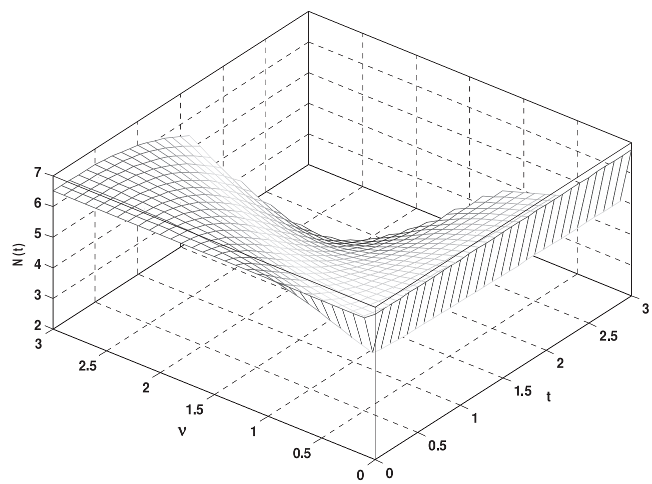

Fig. 5. Mesh-plot of KE containing 500 terms of KMLF.

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Original article

One-dimensional Fourier series of a function of many variables☆

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Abstract

It is well known that to each summable in the n-dimensional cube $[-\pi, \pi]^n$ function f of variables $x_1, \dots, x_n$ there corresponds one n-multiple trigonometric Fourier series $S[f]$ with constant coefficients.

In the present paper, with the function f we associate n one-dimensional Fourier series $S[f]_1, \dots, S[f]_n$, with respect to variables $x_1, \dots, x_n$, respectively, with nonconstant coefficients and announce the preliminary results. In particular, if a continuous function f is differentiable at some point $x = (x_1, \dots, x_n)$, then all one-dimensional Fourier series $S[f]_1, \dots, S[f]_n$ converge at x to the value $f(x)$.

For illustration we consider the well known example of Ch. Fefferman's function $F(x, y)$ whose double trigonometric Fourier series $S[F]$ diverges everywhere in the sense of Prinsheim. Namely, we establish the simultaneous convergence of the one-dimensional Fourier series $S[F]_1$ and $S[F]_2$ at almost all points $(x, y) \in [-\pi, \pi]^2$ to the values $F(x, y)$.

Keywords: One-dimensional Fourier series; Nonconstant Fourier coefficients

1. Notions of one-dimensional fourier series of a function of many variables

Let some function f of variables $x_1, \dots, x_n$ be defined and summable in the n-dimensional cube $[-\pi, \pi]^n$ and, in addition, be 2π-periodic with respect to each variable.

By Fubini's theorem we know that f is summable on $[-\pi, \pi]$ as a function of one variable x_1 for almost all $(x_2, x_3, \dots, x_n) \in [-\pi, \pi]^{n-1}$. We denote by E^1 the set of such $(x_2, x_3, \dots, x_n)$ and by X^1 the point $(x_2, x_3, \dots, x_n)$, i.e. $X^1 = (x_2, x_3, \dots, x_n)$, $X^1 \in E^1$.

Thus we have the function $f(x_1, X^1)$ which is summable with respect to the variable x_1 on $[-\pi, \pi]$ for each $X^1 \in E^1$.

☆ The results of this paper were announced in the author's report on one-dimensional Fourier Series of Several Variable Functions, Book of Abstracts, VIIth International Joint Conference of the Georgian Mathematical Union and Georgian Mechanical Union Dedicated to the 125th Birthday Anniversary of Academician N. Muskhelishvii, September 5–9, 2016, Batumi, Georgia, p. 118.
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Let us consider a Fourier series corresponds to the function $f(x_1, X^1)$ with respect to the variable x_1 on $[-\pi, \pi]$ and we denote it by $S[f]_1$, i.e.

$$S[f]_1 = \frac{1}{2}a_0(X^1) + \sum_{k=1}^{\infty} a_k(X^1)\cos kx_1 + b_k(X^1)\sin kx_1,$$

where the coefficients $a_0(X^1)$, $a_k(X^1)$ and $b_k(X^1)$ are defined by the Fourier formulas

$$\begin{aligned} a_0(X^1) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t, X^1)dt, \quad a_k(X^1) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t, X^1)\cos ktdt, \\ b_k(X^1) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t, X^1)\sin ktdt, \quad k = 1, 2, \ldots. \end{aligned} \tag{1}$$

In these relations, anyone of the variables $x_2, x_3, \ldots, x_n$ may play the role of x_1.

Therefore to each summable function f in the n-dimensional cube $[-\pi, \pi]^n$ there correspond one-dimensional Fourier series $S[f]_1, \ldots, S[f]_n$ with nonconstant coefficients.

In what follows we will discuss only the series $S[f]_1$.

2. Necessary and sufficient condition for the convergence of a one-dimensional fourier series of a function of many variables

Let us consider the partial sum of the one-dimensional Fourier series $S[f]_1$

$$S_m(f; (x_1, X^1)) = \frac{1}{2}a_0(X^1) + \sum_{k=1}^{m} a_k(X^1)\cos kx_1 + b_k(X^1)\sin kx_1,$$

which, after substituting in it the coefficients (1), takes the form

$$S_m(f; (x_1, X^1)) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t, X^1)D_m(t - x_1)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_1 + y_1, X^1)D_m(y_1)dy_1,$$

where D_m is the Dirichlet kernel, i.e.

$$D_m(t) = \frac{\sin(m + \frac{1}{2})t}{2\sin\frac{t}{2}} \quad \text{for} \quad t \neq 2k\pi$$

and

$$D_m(2k\pi) = m + \frac{1}{2} \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots.$$

Since the function f is summable with respect to the variable x_1 on $[-\pi, \pi]$ for any $X^1 \in E^1$, the well known necessary and sufficient condition for the Fourier series $S[\varphi]$ of a function $\varphi \in L[-\pi, \pi]$ to be convergent at some point $t \in [-\pi, \pi]$ to the value $\varphi(t)$ (see [1], Ch. I, §37, equality (37.5); [2], p.55)

$$\lim_{m\to\infty}\int_0^{\delta}[\varphi(t + u) + \varphi(t - u) - 2\varphi(t)]\frac{\sin mu}{u}du = 0 \tag{2}$$

takes in our case the form

$$\lim_{m\to\infty}\int_0^{\delta}[f(x_1 + y_1, X^1) + f(x_1 - y_1, X^1) - 2f(x_1, X^1)]\frac{\sin my_1}{y_1}dy_1 = 0, \quad X^1 \in E^1.$$

Hence we can formulate

Proposition 2.1. *For a one-dimensional Fourier series $S[f]_1$ to converge at a point (x_1, X^1) to the value $f(x_1, X^1)$ for some $x_1 \in [-\pi, \pi]$ and $X^1 \in E^1$ it is necessary and sufficient that the equality*

$$\lim_{m\to\infty}\int_0^{\delta}\frac{f(x_1 + y_1, X^1) + f(x_1 - y_1, X^1) - 2f(x_1, X^1)}{y_1}\sin my_1dy_1 = 0 \tag{3}$$

be fulfilled.

3. Sufficient conditions for the convergence of a one-dimensional Fourier series of a function of many variables

As far back as 1853 B. Riemann considered the problem of representation of functions by trigonometric series. In connection with this problem Riemann introduced into consideration a function, say, φ with the property ([3], p. 245; [1], Ch. I, §66)

$$\lim_{h\to 0} \frac{\varphi(x_0+h)+\varphi(x_0-h)-2\varphi(x_0)}{h} = 0 \tag{4}$$

at a point x_0.

Later, A. Zygmund called the function φ having the property (4) a smooth function at the point x_0 ([4]; [2], p. 43).

It is obvious that a smooth function φ at a point x_0 has the property $\varphi(x_0+h)+\varphi(x_0-h)-2\varphi(x_0) \to 0$ as $h \to 0$ which is called the symmetry of the function φ at x_0.

It is the well-established fact that almost all points of symmetry of any function is the point of its continuity ([5], p. 266) and the converse statement is obvious.

Therefore almost all points of smoothness of any function is the point of its continuity. In addition, a smooth function at separate points may be discontinuous, for example, a discontinuous odd function.

It should be said that if the function φ has the finite derivative $\varphi'(x_0)$ at some point x_0, then φ is smooth at x_0 ([3], p. 43; [1], Ch.I, §66), but the converse statement is not true ([2], p. 48).

Note that if a 2π-periodic and summable function on $[-\pi, \pi]$ is smooth at some point x_0, in particular if φ has the finite derivative $\varphi'(x_0)$, then the Fourier series $S[\varphi]$ of the function φ converges at the point x_0 to the value $\varphi(x_0)$ (see the equality (2)).

Following Riemann, we introduce the following notion of smoothness of a function of many variables (the case $n = 2$ is considered in [6]).

Definition 3.1. A function f of n variables $x_1, \ldots, x_n$ is called smooth at a point $x = (x_1, \ldots, x_n)$ if the equality

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{|h|} = 0 \tag{5}$$

is fulfilled, where $h = (h_1, \ldots, h_n)$ and $|h| = |h_1| + \cdots + |h_n|$.

Proposition 3.2. *If a function f is differentiable at some point x, then f is smooth at x.*

Indeed, that this is so follows from the equality

$$\begin{aligned}
&\frac{f(x_1+h_1,\ldots,x_n+h_n)+f(x_1-h_1,\ldots,x_n-h_n)-2f(x_1,\ldots,x_n)}{|h_1|+\cdots+|h_n|}\\
&\quad= \frac{f(x_1+h_1,\ldots,x_n+h_n)-f(x_1,\ldots,x_n)-A_1(h_1)-\cdots-A_n(h_n)}{|h_1|+\cdots+|h_n|}\\
&\qquad+ \frac{f(x_1-h_1,\ldots,x_n-h_n)-f(x_1,\ldots,x_n)-A_1(-h_1)-\cdots-A_n(-h_n)}{|-h_1|+\cdots+|-h_n|}.
\end{aligned}$$

The converse to Proposition 3.2 is not true (for the case $n = 2$ see [6]).

Proposition 3.3. *If a function f is smooth at a point x, then it is smooth at x with respect to each variable x_j, $1 \le j \le n$.*

To verify that this is so it suffices to put (5) $h_i = 0$ for all $i \neq j$.

Proposition 3.4. *If a function f has at a point x the finite partial derivative $\frac{\partial f}{\partial x_j}$ with respect to the variable x_j, then f is smooth at x with respect to the same variable x_j.*

That this is so follows from the corresponding statement for functions of one variable.

Proposition 3.5. *If a function f is smooth with respect to the variable x_1 at the point (x_1, X^1) for some $x_1 \in [-\pi, \pi]$ and $X^1 \in E^1$, then the Fourier series $S[f]_1$ converges at (x_1, X^1) to the value $f(x_1, X^1)$.*

This assertion follows from the equality (3).

Propositions 3.3 and 3.5 give rise to

Theorem 3.6. *If a continuous on* $[-\pi, \pi]^n$ *function* f *is smooth at a point* x*, in particular if* f *is differentiable at* x*, then all one-dimensional Fourier series* $S[f]_1, \dots, S[f]_n$ *converge at the point* x *to one and the same value* $f(x)$*.*

Indeed, the function f as a function of the variable x_j is summable on $[-\pi, \pi]$ for any point $X^j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ from $[-\pi, \pi]^{n-1}$. By virtue of Propositions 3.3 and 3.5, the one-dimensional Fourier series $S[f]_j$ converges at the point $(x_j, X^j) = (x_1, \dots, x_n)$ to the value $f(x_j, X^j) = f(x_1, \dots, x_n)$.

4. Almost everywhere convergence of one-dimensional Fourier series $S[F]_1$ and $S[F]_2$ for Ch. Fefferman's function F

It is well known that there exists an everywhere continuous function $F(x, y)$ of two variables and a 2π-periodic with respect to x and y double trigonometric Fourier series $S[F]$ which diverges everywhere in the Prinsheim sense [7].

The function $F(x, y)$ as function of the variable $x_1 \in [-\pi, \pi]$ belongs to the class $L^2[-\pi, \pi]$ for each $y \in [-\pi, \pi]$. Therefore by L. Carleson's theorem [8] we have

Proposition 4.1. *A one-dimensional Fourier series* $S[F]_1$ *converges to values* $F(x, y)$ *for almost all* $x \in [-\pi, \pi]$ *and all* $y \in [-\pi, \pi]$*.*

Analogously, the following assertion is true.

Proposition 4.2. *The one-dimensional Fourier series* $S[F]_2$ *converges to the values* $F(x, y)$ *for all* $x \in [-\pi, \pi]$ *and almost all* $y \in [-\pi, \pi]$*.*

Propositions 4.1 and 4.2 give rise to

Theorem 4.3. *The one-dimensional Fourier series* $S[F]_1$ *and* $S[F]_2$ *simultaneously converges to the values* $F(x, y)$ *for almost all* $(x, y) \in [-\pi, \pi]^2$*.*

Finally, Propositions 4.1, 4.2 and Theorem 4.3 can be made stronger as follows.

Theorem 4.4. *For any function* $f \in L^2[-\pi, \pi]^2$ *there exist measurable sets* E_1*,* E_2 *and* E_3 *from the square* $[-\pi, \pi]^2$ *with the properties* $|E_1| = |E_2| = |E_3| = 4\pi^2$*, at whose points the following equalities are fulfilled:*

$S[f]_1(x, y) = f(x, y)$ *for* $(x, y) \in E_1$*,*

$S[f]_2(x, y) = f(x, y)$ *for* $(x, y) \in E_2$*,*

$S[f]_1(x, y) = f(x, y) = S[f]_2(x, y)$ *for* $(x, y) \in E_3$*.*

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Original article

Homogeneous functions: New characterization and applications

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Abstract

Positive homogeneous functions on $\mathbb{R}$ of a negative degree are characterized by a new counterpart of the Euler's homogeneous function theorem using quantum calculus and replacing the classical derivative operator by Jackson derivative. As application we start by characterizing the harmonic functions associated to Jackson derivative. Then, the solution of the Cauchy problem associated to the analogue of the Euler operator is given. Using this solution we study the associated ν-potential. Its Markovianity property is treated.

Keywords: Homogeneous functions; Euler's theorem; Quantum calculus; Cauchy problem; Markovian semigroups

1. Introduction and preliminaries

The notion of a homogeneous function arises in connection with the spherical harmonic functions. The solid harmonic also can be defined as homogeneous functions that obey Laplace's equation. The Euler theorem is used in proving that the Hamiltonian is equal to the total energy. In thermodynamics, extensive thermodynamic functions are homogeneous functions. In this context, Euler's theorem is applied in thermodynamics by taking Gibbs free energy. Also, Euler's theorem is of value in analytical mechanics and has been widely implemented as a theoretical basis for the reversal of wide magnetic and gravity data sets in terms of single sources, see [1–3]. In mathematics, a homogeneous function is a function f with multiplicative scaling behavior, i.e, if the argument is multiplied by a factor α, then the result is multiplied by some power λ of this factor. Positive homogeneous functions are characterized by Euler's homogeneous function theorem which consists of: f is positive homogeneous of degree $\lambda \in \mathbb{R}$ if and only

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if

$$\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} f(x) = \lambda f(x).$$

The operator $\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$ is called the Euler operator (see [4]). In microeconomics, they use homogeneous production functions, including the function of Cobb–Douglas, developed in 1928, the degree of such homogeneous functions can be negative which was interpreted as decreasing returns to scale. The concept of homogeneity in methods for enforcement finds modeling physical phenomena and, in particular, for directly solving inverse problems for potential fields, see [3], where the gravity field of a mass point has the potential V. The tests of homogeneity for this potential can be implemented using equation of Euler's theorem to study the homogeneity for the gravity potential V with a negative degree of homogeneity ($\lambda = -1$), see [3]. Also, Gel'fand and Shilov [4] studied the homogeneous distributions of negative integer degree λ on $\mathbb{R}$.

At the last quarter of the XX century, q-calculus appeared as a connection between mathematics and physics (see for more details [5–9]). It has a lot of applications in different basic hyper-geometric functions and other sciences as quantum theory, mechanics and theory of relativity. We shall briefly recall some of the concepts, notations and known results on q-calculus as given in [5–11]. Let $q \in (0, 1)$. A q-number $[n]_q$ is defined by

$$[n]_q := 1 + q + + \cdots + q^n, \; n \in \mathbb{N}.$$

Generally a q-complex number is given by $[a]_q$ is $[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C}$. The factorial of a number $[n]_q$ is defined by

$$[0]_q! = 1, \; [n]_q! = [n]_q [n-1]_q ... [1]_q, \; n \in \mathbb{N}.$$

The q-derivative also referred to as Jackson derivative [5] is defined as follows

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & q \in (0,1), \; x \neq 0 \\ f'(0), & x = 0, \end{cases} \tag{1}$$

such that, $\lim_{q \longrightarrow 1} D_q f(x) = \frac{df(x)}{dx}$, if f is differentiable at x. This derivative (2.1) verifies the following q-derivation property

$$D_q(f.g)(x) = g(qx) D_q f(x) + f(x) D_q g(x) = f(qx) D_q g(x) + g(x) D_q f(x). \tag{2}$$

And the high q-derivatives are $D_q^0 f := f, \; D_q^n f := D_q(D_q^{n-1} f), \; n \in \mathbb{N}$. Notice, that a continuous function on an interval, which does not include 0 is continuous q-differentiable. The q-derivative operator D_q and the operator X defined by $Xf(x) = xf(x)$ give a bounded representation of

$$aa^* - qa^*a = 1$$

on $\mathcal{H}^2(B_q, \mu_q)$ which is the completion of the analytic functions on $B_q = \{z \in \mathbb{C} : \|z\|^2 < \frac{1}{1-q}\}$ with respect to the inner product defined by a measure μ_q on the complex plane that replaces the Lebesgue measure on the unit circle, for more details see [12]. As q tends to 1, μ_q tends to the Gauss measure on the complex plane. This representation generalizes the Bargmann representation of analytic functions on the complex plane. The operator $D_q + X$, viewed as a non-commutative (or quantum) random variable, has a q-Gaussian distribution in the vacuum state. The operator XD_q will be called the q-Euler operator. This operator has a q-Poisson distribution in the vacuum state. It is obvious that XD_q is the q-deformation of the operator $X\frac{d}{dx}$ verifying: as q tends to 1, XD_q tends to $X\frac{d}{dx}$. Now, if we replace $X\frac{d}{dx}$ by XD_q, what are the q-analogues of the Euler's theorem?

In this paper we give a response to this above question as follows: positive homogeneous functions f on $\mathbb{R}$ of a negative degree λ are characterized by a new counterpart of the Euler's homogeneous function theorem using the q-Euler operator, i.e, f is homogeneous of degree λ if and only if

$$\Delta_{E,q} f(x) = [\lambda]_q f(x),$$

for $q \in (0, 1)$ and $x \in \mathbb{R}_+$. As application we start by characterizing the q-harmonic functions. Then, the solution of the Cauchy problem associated to the q-Euler operator is given. Using this solution we study the associated ν-potential. Its Markovianity property is treated.

2. The q-analogues of the Euler theorem

Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$. We say that f is homogeneous of degree $\lambda \in \mathbb{R}$, if for all $x \in \mathbb{R}$ and for all $\alpha > 0$

$$f(\alpha x) = \alpha^{\lambda} f(x).$$

Definition 2.1. For $q \in (0, 1)$, we define the q-Euler operator by

$$\Delta_{E,q} = X.D_q.$$

Proposition 2.1. *Let $f, g : \mathbb{R}_+ \longrightarrow \mathbb{R}$, then for $q \in (0, 1)$, we have*

$$\Delta_{E,q}(f.g)(x) = g(x)\Delta_{E,q} f(x) + f(qx)\Delta_{E,q} g(x). \tag{3}$$

Proof. By definition, we have, for $x \neq 0$

$$\begin{aligned}\Delta_{E,q}(f.g)(x) &= xD_q(f.g)(x)\\ &= x\frac{f(x)g(x) - f(qx)g(qx)}{(1-q)x}\\ &= x\frac{f(x) - f(qx)}{(1-q)x}g(x) + xf(qx)\frac{g(x) - g(qx)}{(1-q)x}\\ &= x\Big(D_q f\Big)(x)g(x) + xf(qx)\Big(D_q g\Big)(x)\\ &= g(x)\Delta_{E,q} f(x) + f(qx)\Delta_{E,q} g(x).\end{aligned}$$

Now, for $x = 0$, we know that

$$D_q(f.g)(x)_{|_{x=0}} = (f.g)'(0) = f'(0)g(0) + g'(0)f(0).$$

Then, applying the multiplication operator x, we obtain

$$\Delta_{E,q}(f.g)(0) = g(0)\Big(\Delta_{E,q} f\Big)(0) + f(q.0)\Big(\Delta_{E,q} g\Big)(0),$$

which completes the proof. □

Theorem 2.2. *If f is homogeneous of degree $\lambda \in \mathbb{R}$, then we have*

$$\Delta_{E,q} f(x) = [\lambda]_q f(x), \quad q \in (0, 1).$$

Proof. Let f be a homogeneous function of degree $\lambda \in \mathbb{R}$. Then for $x \neq 0$, we have

$$\begin{aligned}D_q(f)(x) &= \frac{f(x) - f(qx)}{(1-q)x}\\ &= \frac{f(x) - q^{\lambda} f(x)}{(1-q)x}\\ &= \frac{(1-q^{\lambda})}{(1-q)}\frac{f(x)}{x}\\ &= [\lambda]_q \frac{f(x)}{x}.\end{aligned}$$

Therefore, we get

$$\Delta_{E,q} f(x) = [\lambda]_q f(x).$$

Now, for $x = 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$, we have

$$(X.D_q f)(0) = 0 \text{ and } f'(0) = 0.$$

But, we know that f is homogeneous of degree λ then $f(\alpha x) = \alpha^\lambda f(x)$. In particular for $x = 0$, $f(0) = \alpha^\lambda f(0)$ for $\alpha > 0$, which implies that $f(0) = 0$ for $\lambda \in \mathbb{R}^*$. Then, we get

$$(X.D_q f)(0) = 0 = [\lambda]_q f(0).$$

Now, for $\lambda = 0$ and $x = 0$

$$(X.D_q f)(0) = 0 = [0]_q f(0).$$

Hence, for $x \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$, we obtain

$$\Delta_{E,q} f(x) = [\lambda]_q f(x),$$

which gives the desired statement, which completes the proof. □

Theorem 2.3. *Let $q \in (0, 1)$, $x \in \mathbb{R}$ and $\lambda < 0$. If we have*

$$\Delta_{E,q} f(x) = [\lambda]_q f(x).$$

Then, f is homogeneous of degree λ.

Proof. Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $\Delta_{E,q} f(x) = [\lambda]_q f(x)$. For all $x \in \mathbb{R}_+$, define $g : (0, \infty) \longrightarrow \mathbb{R}$ by

$$g(\alpha) = f(\alpha x) - \alpha^\lambda f(x).$$

Then, for $\alpha > 0$, we get

$$\begin{aligned} D_q(g)(\alpha) &= \frac{f(\alpha x) - f(q\alpha x)}{(1-q)\alpha} - [\lambda]_q \alpha^{\lambda-1} f(x) \\ &= \frac{x}{\alpha} \frac{f(\alpha x) - f(q\alpha x)}{(1-q)x} - [\lambda]_q \alpha^{\lambda-1} f(x) \\ &= \frac{1}{\alpha} \Delta_{E,q} f(\alpha x) - [\lambda]_q \alpha^{\lambda-1} f(x). \end{aligned}$$

Then, we can obtain

$$\begin{aligned} \alpha D_q g(\alpha) &= \Delta_{E,q} f(\alpha x) - [\lambda]_q \alpha^\lambda f(x) \\ &= [\lambda]_q \left(f(\alpha x) - \alpha^\lambda f(x)\right) \\ &= [\lambda]_q g(\alpha). \end{aligned}$$

Since α is arbitrary, g satisfies the following q-differential equation

$$D_q g(\alpha) - \frac{[\lambda]_q}{\alpha} g(\alpha) = 0. \tag{4}$$

Eq. (4) is equivalent to

$$\frac{g(\alpha) - g(q\alpha)}{(1-q)\alpha} = \frac{[\lambda]_q}{\alpha} g(\alpha).$$

Then, we get

$$g(\alpha) = q^{-\lambda} g(q\alpha).$$

By a simple iteration, we obtain

$$g(\alpha) = q^{-\lambda n} g(q^n \alpha).$$

Then, for $n \longrightarrow \infty$, we get $g(\alpha) = 0$, which is equivalent to

$$f(\alpha x) = \alpha^{\lambda} f(x).$$

This completes the proof. □

Combining Theorem 2.2 with Theorem 2.3 above, we obtain the following theorem, which will be called q-analogues of Euler's Theorem:

Theorem 2.4. *Let $\lambda < 0$. Then, f is homogeneous of degree λ if and only if*

$$\Delta_{E,q} f(x) = [\lambda]_q f(x),$$

for $q \in (0, 1)$ and $x \in \mathbb{R}_+$.

Remark 1. It is obvious from Theorem 2.4 that, as q tends to 1, we refind the classical Euler's theorem for $\lambda < 0$.

3. Applications of the q-Euler operator

3.1. q-harmonic function

This subsection deals with the study of a link taken homogeneous function and a new notion of q-harmonic functions.

Definition 3.1. f is called q-harmonic function if $D_q^2(f) = 0$.

Theorem 3.1. *Let f be λ-homogeneous. Then f is q-harmonic if and only if $\lambda = 0$.*

Proof. "⇒" Let f be λ-homogeneous and q-harmonic. Then from Theorem 2.2, we have

$$D_q^2(f)(x) = 0$$

and from Theorem 2.2

$$X.D_q(f)(x) = [\lambda]_q f(x). \tag{5}$$

Then we get

$$(X.D_q)^2(f)(x) = [\lambda]_q^2 f(x).$$

But we know that,

$$D_q X - qXD_q = 1.$$

Then,

$$D_q X = 1 + qXD_q$$

from which we get

$$X(1 + q\ X\ D_q)^2\ D_q(f)(x) = [\lambda]_q^2 f(x)$$
$$X.D_q(f)(x) + q\ X^2\ D_q^2 f(x) = [\lambda]_q^2 f(x).$$

Thus

$$[\lambda]_q f(x) = [\lambda]_q^2 f(x).$$

This gives

$$\lambda = 0.$$

"$\Leftarrow$" Let f be 0-homogeneous. Then for $\lambda = 0$, we have $f(\alpha x) = f(x)$. Therefore, we get

$$\begin{aligned} D_q f(x) &= \frac{f(x) - f(q\,x)}{(1-q)x} \\ &= \frac{f(x) - f(x)}{(1-q)x} \\ &= 0. \end{aligned}$$

Then f is q-harmonic, which completes the proof. □

3.2. Cauchy problem associated to the q-Euler operator

Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be homogeneous function of degree λ where $0 < \lambda \leq 1$. Consider, the following Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} U(t,x) = \Delta_{E,q} U(t,x), \ x \in \mathbb{R}_+ \\ U(0,x) = f(x). \end{cases} \tag{6}$$

Theorem 3.2. *The Cauchy problem* (6) *admits a unique solution given by*

$$U(t,x) = f(x) + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(-1)^k t^n [f(q^k x) - f(q^{k+1}x)]}{n(n-1-k)!k!(1-q)^n}. \tag{7}$$

Proof. We start by verifying that

$$U(t,x) := f(x) + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(-1)^k t^n [f(q^k x) - f(q^{k+1}x)]}{n(n-1-k)!k!(1-q)^n}$$

is a solution of the system (6). On the one hand, we have

$$\frac{\partial U(t,x)}{\partial t} = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(-1)^k t^{n-1} [f(q^k x) - f(q^{k+1}x)]}{(n-1-k)!k!(1-q)^n}.$$

On the other hand, we have

$$\begin{aligned} \Delta_{E,q} U(t,x) &= x D_q U(t,x) \\ &= \frac{U(t,x) - U(t,qx)}{1-q} \\ &= \frac{f(x) - f(qx)}{1-q} \\ &\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(-1)^k t^n [f(q^k x) - f(q^{k+1}x)]}{n(n-1-k)!k!(1-q)^{n+1}} \\ &\quad - \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(-1)^k t^n [f(q^{k+1} x) - f(q^{k+2}x)]}{n(n-1-k)!k!(1-q)^{n+1}}. \end{aligned}$$

By indices change in the right sums of the above equation, we obtain

$$\begin{aligned} \Delta_{E,q} U(t,x) &= \frac{f(x) - f(qx)}{1-q} \\ &\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(-1)^k t^n [f(q^k x) - f(q^{k+1}x)]}{n(n-1-k)!k!(1-q)^{n+1}} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^k t^n[f(q^k x)-f(q^{k+1}x)]}{n(n-k)!(k-1)!(1-q)^{n+1}}\\
&= xD_q f(x)+\sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k t^n[f(q^k x)-f(q^{k+1}x)]}{(n-k)!k!(1-q)^{n+1}}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k t^n[f(q^k x)-f(q^{k+1}x)]}{(n-k)!k!(1-q)^{n+1}}\\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{(-1)^k t^{n-1}[f(q^k x)-f(q^{k+1}x)]}{(n-1-k)!k!(1-q)^{n}}\\
&= \frac{\partial U(t,x)}{\partial t},
\end{aligned}
$$

which shows that $U(t,x)$ is a solution of (6). Let us show by recursion on $n \in \mathbb{N}^\star$ that

$$
\Delta^n_{E,q} f(x) = \frac{1}{(1-q)^n}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\Big(f(q^k x)-f(q^{k+1}x)\Big). \tag{8}
$$

We have for, $n = 1$

$$
\Delta_{E,q} f(x) := xD_q f(x) = \frac{1}{1-q}\Big(f(x)-f(qx)\Big), \quad x\in\mathbb{R}_+.
$$

Now, suppose that Eq. (7) is verified, then we get

$$
\begin{aligned}
\Delta^{n+1}_{E,q} f(x) &= \Delta_{E,q}(\Delta^n_{E,q} f(x))\\
&= xD_q(\Delta^n_{E,q} f(x))\\
&= \frac{1}{1-q}\Big[\Delta^n_{E,q} f(x) - \Delta^n_{E,q} f(qx)\Big]\\
&= \frac{1}{(1-q)^n}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k q^k xD_q f(q^k x)\\
&\quad - \frac{1}{(1-q)^n}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k q^{k+1} xD_q f(q^{k+1} x).
\end{aligned}
$$

By indices change in the right sums of the above equation, we obtain

$$
\begin{aligned}
\Delta^{n+1}_{E,q} f(x) &= \frac{1}{(1-q)^n}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k q^k xD_q f(q^k x)\\
&\quad + \frac{1}{(1-q)^n}\sum_{k=1}^{n}\binom{n-1}{k-1}(-1)^k q^k xD_q f(q^k x)\\
&= \frac{1}{(1-q)^n}\Big(xD_q f(x) + (-1)^n q^n xD_q f(q^n x)\Big)\\
&\quad + \frac{1}{(1-q)^n}\sum_{k=1}^{n-1}\left\{\binom{n-1}{k}+\binom{n-1}{k-1}\right\}(-1)^k q^k xD_q f(q^k x)\\
&= \frac{1}{(1-q)^n}\Big(xD_q f(x) + (-1)^n q^n xD_q f(q^n x)\Big)\\
&\quad + \frac{1}{(1-q)^n}\sum_{k=1}^{n-1}\binom{n}{k}(-1)^k q^k xD_q f(q^k x)
\end{aligned}
$$

$$= \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^k x D_q f(q^k x)$$

$$= \frac{1}{(1-q)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Big(f(q^k x) - f(q^{k+1} x) \Big). \tag{9}$$

This shows Eq. (6) for all $n \in \mathbb{N}$. Then, using Eq. (6) we get

$$\begin{aligned} Q_t f(x) &:= \sum_{n=0}^{\infty} \frac{t^k}{n!} \Delta_{E,q}^n f(x) \\ &= f(x) + \sum_{n=1}^{\infty} \frac{t^k}{n!} \Delta_{E,q}^n f(x) \\ &= f(x) + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^k t^n \Big(f(q^k x) - f(q^{k+1} x) \Big)}{n(n-1-k)!k!(1-q)^n} \\ &= U(t,x). \end{aligned} \tag{10}$$

Finally, we show the uniqueness of the above solution. Let $V(t,x)$ be another solution of Eq. (6), we set $W(t,x) = Q_{-t} V(t,x)$. Then

$$\begin{aligned} \frac{\partial W(t,x)}{\partial t} &= -\Delta_{E,q} W(t,x) + Q_{-t}(\Delta_{E,q} V(t,x)) \\ &= -\Delta_{E,q} W(t,x) + \Delta_{E,q} Q_{-t} V(t,x) \\ &= 0 \end{aligned}$$

from which, we deduce that

$$W(t,x) = W(0,x) = V(0,x) = f(x).$$

This implies that

$$V(t,x) = Q_t f(x) = U(t,x),$$

which completes the proof. □

3.3. ν–q-potential

Using the semigroup $\{Q_t\}_t$ we come to the following.

Definition 3.2. For $\nu > 0$, we define the ν–q-potential by:

$$H_{\nu,q}\ f(x) = \int_0^{\infty} e^{-\nu t} \Big(Q_t(f)(x) - f(x) \Big) dt.$$

Theorem 3.3. *The ν–q-potential is the unique solution of the following Poisson equation:*

$$(\nu I - \Delta_{E,q})\ F = \frac{1}{\nu} D_q.$$

Proof. By Definition 3.2 and Eq. (10) we have

$$\begin{aligned} H_{\nu,q}\ f(x) &= \int_0^{\infty} e^{-\nu t} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^k t^n \Big(f(q^k x) - f(q^{k+1} x) \Big)}{n\,(n-1-k)!k!(1-q)^n} dt \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^k \Big(f(q^k x) - f(q^{k+1} x) \Big)}{n\,(n-1-k)!k!(1-q)^n} \int_0^{\infty} e^{-\nu t} t^n dt. \end{aligned}$$

One can show easily that

$$\int_0^\infty e^{-\nu t} t^n dt = \frac{n!}{\nu^{n+1}}.$$

Then,

$$H_{\nu,q}\ f(x) = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(-1)^k t^n \Big(f(q^k x) - f(q^{k+1}x)\Big)}{(n-k)!k!(1-q)^n}\Big(\frac{(n-1)!}{\nu^{n+1}}\Big). \tag{11}$$

On the other hand we have

$$\begin{aligned} X.D_q Q_t f(x) &= \Delta_{E,q}\ e^{t\Delta_{E,q}}\ f(x). \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\ \Delta_{E,q}^{n+1}\ f(x). \\ &= \Delta_{E,q}\ f(x) + \sum_{n=1}^{\infty} \frac{t^n}{n!}\ \Delta_{E,q}^{n+1}\ f(x). \end{aligned}$$

Using Eq. (9), we get

$$\Delta_{E,q}\ Q_t\ f(x) = \Delta_{E,q}\ f(x) + \sum_{n=1}^{\infty}\frac{t^n}{n!}\frac{1}{(1-q)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\Big(f(q^k x) - f(q^{k+1}x)\Big)$$

from which we obtain

$$\Delta_{E,q}(Q_t f - f)(x) = \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{t^n(-1)^k}{k!(n-k)!(1-q)^{n+1}}\Big(f(q^k x) - f(q^{k+1}x)\Big).$$

Then, from Definition 3.2, we get

$$\Delta_{E,q}\ H_{\nu,q}\ f(x) = \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{n!(-1)^k}{\nu^{n+1}k!(n-k)!(1-q)^{n+1}}\Big(f(q^k x) - f(q^{k+1}x)\Big). \tag{12}$$

By the change indices $(n-1=j)$ in the right sums of Eq. (11), we have

$$H_{\nu,q}\ f(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{j!(-1)^k}{\nu^{n+2}k!(j-k)!(1-q)^{j+1}}\Big(f(q^k x) - f(q^{k+1}x)\Big).$$

Using Eq. (12) we obtain

$$\begin{aligned} H_{\nu,q}\ f(x) &= \frac{1}{\nu^2(1-q)}\Big(f(x) - f(qx)\Big) \\ &\quad + \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{j!(-1)^k}{\nu^{n+2}k!(j-k)!(1-q)^{j+1}}\Big(f(q^k x) - f(q^{k+1}x)\Big) \\ &= \frac{1}{\nu^2(1-q)}\Big(f(x) - f(qx) + \frac{1}{\nu}\ \Delta_{E,q}\ H_{\nu,q}\ f(x)\Big), \end{aligned}$$

which is equivalent to

$$\nu\ H_{\nu,q}\ f(x) - \Delta_{E,q}\ H_{\nu,q}\ f(x) = \frac{1}{\nu}\ D_q\ f(x).$$

This implies that

$$(\nu I - \Delta_{E,q})H_{\nu,q} = \frac{1}{\nu}D_q,$$

which completes the proof. □

3.4. Markovianity property

Recall that from [13] $\{P_t\}_{t\geq 0}$ is called a Markov semigroup if it satisfies

(**a**) $P_0 = Id$
(**b**) $P_{t+s} = P_t\, P_s$ for $s, t \geq 0$
(**c**) Strong continuity : $P_t f \to f$ as $t \to 0$ for all f.
(**d**) $P_t\, f \geq 0$ whenever $f \geq 0$
(**e**) $P_t 1 = 1$ for all $t \geq 0$.

Note that conditions (d) and (e) imply Contraction Property: $\|P_t f\| \leq \|f\|$ for all f and t.

Theorem 3.4. *The family* $\{Q_t\}_{t\geq 0}$ *is Markov semigroup.*

Proof. (**a**) It is obvious that $Q_0 = Id$.
(**b**) Let $s, t \geq 0$, then

$$\begin{aligned} Q_{t+s} &= e^{(s+t)\Delta_{E,q}} \\ &= e^{s\Delta_{E,q}} e^{t\Delta_{E,q}} \\ &= Q_t Q_s. \end{aligned}$$

(**c**) Let $t \geq 0$, then

$$\begin{aligned} \|Q_t f - f\| &\leq \sum_{n=1}^{\infty} t^n \frac{\|\Delta_{E,q}\|^n \|f\|}{n!} \\ &= \left(e^{t\|\Delta_{E,q}\|} - 1\right)\|f\| \to 0 \text{ as } t \to 0. \end{aligned}$$

(**d**) Using Theorem 2.2, we get

$$\Delta_{E,q} f(x) = [\lambda]_q f(x).$$

Similarly using Theorem 2.2, we obtain

$$(\Delta_{E,q})^n f(x) = ([\lambda]_q)^n f(x).$$

Then,

$$Q_t f(x) = \sum_{n=o}^{\infty} \frac{t^n}{n!} [\lambda]_q^n f(x) = e^{t[\lambda]_q} f(x).$$

Hence, when $f \geq 0$, we obtain

$$Q_t f \geq 0.$$

(**e**) Using Eq. (10), we get

$$Q_t 1 = 1, \text{ for all } t \geq 0.$$

This completes the proof. □

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Original article

Fixed point results for generalized (ψ, ϕ)-weak contractions with an application to system of non-linear integral equations

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Abstract

The aim of this paper is to prove fixed point results under (ψ, ϕ)-weak contractive condition for continuous weak compatible mappings in ordered b-metric spaces. The results proved herein generalize, modify and unify some recent results of the existing literature. An application demonstrating the usability of our established results is also discussed besides furnishing an illustrative example.

Keywords: Coincidence point; (ψ, ϕ)-weak contraction; Weak compatible mapping and ordered b-metric spaces

1. Introduction and preliminaries

Alber and Guerre-Delabriere [1] established a novel fixed point result for weak contraction in Hilbert spaces. Rhoudes [2] extended this result to metric spaces and also showed the generality of such results besides deducing Banach contraction principle. In [3], Zhang and Song replaced the idea of ϕ-weak contraction with generalized ϕ-weak contraction and obtained their fixed point results in complete metric spaces. Dutta and Choudhury [4] proved some fixed point results in complete metric spaces under (ψ, ϕ)-weak contractive condition whereas Doric [5] extended some fixed point results of [4,3] to generalized (ψ, ϕ)-weak contraction. Abbas and Doric [6] proved similar results on fixed point in complete metric spaces involving four mappings while Murthy et al. [7] obtained fixed point results in complete metric spaces under (ψ, ϕ)-generalized weak contractive condition.

The origin of existence results on fixed points in partially ordered metric spaces is often traced back to Ran and Reurring [8]. Using generalized weak contraction, Radenovic and Kadelburg [9] established certain fixed point results in partially ordered metric spaces. Radenovic et al. [10] and Salimi et al. [11] proved fixed point results besides

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discussing possible applications under cyclic contraction and cyclic α–ψ–ϕ-contraction respectively. In [12] Aghajani and Arab also discussed coupled coincidence point results under generalized (ψ, ϕ, θ)-almost contractive condition in ordered b-metric spaces. Furthermore, Roshan et al. [13] proved results on coincidence point for almost generalized and generalized (ψ, ϕ)-weak contractions in partially ordered b-metric spaces. Aghajani et al. [14] utilized generalized weak contraction to prove their results on fixed points involving four mappings in partially ordered b-metric spaces. Huang et al. [15] established coincidence point results in partially ordered b-metric spaces without using a special lemma as employed in [13].

The aim of this paper is to prove common fixed point results for pair of weak compatible mappings satisfying generalized (ψ, ϕ)-weak contractive condition in partially ordered b-metric spaces.

Throughout this paper, $\mathbb{R}^+$ stands for the set of non-negative real numbers.

Definition 1.1 (*[16]*). Let $P, Q : Y \to Y$ be a pair of mappings on the partial order set Y. The pair (P, Q) is called
(a) weakly increasing if $Pu \preceq Q(Pu)$ and $Qu \preceq P(Qu)$, $\forall\, u \in Y$,
(b) partially weakly increasing if, $\forall\, u \in Y$ $Pu \preceq Q(Pu)$.

Definition 1.2 (*[17]*). Let $P, Q, S : Y \to Y$ be three mappings on the partial order set $(Y, \preceq)$ such that $P(Y) \subseteq S(Y)$ and $Q(Y) \subseteq S(Y)$. The pair (P, Q) is called

(i) weakly increasing with respect to $S \Leftrightarrow \forall\, u \in Y$, $Pu \preceq Qw, \forall\, w \in S^{-1}(Pu)$ and $Qu \preceq Pw$ for all $w \in S^{-1}(Qu)$,
(ii) partially weakly increasing with respect to $S \Leftrightarrow Pu \preceq Qw, \forall\, w \in S^{-1}(Pu)$.

Definition 1.3 (*[18]*). Let $P, Q : Y \to Y$ be a pair of mappings on a metric space (Y, d). The pair (P, Q) is said to be compatible if and only if

$$\lim_{m\to\infty} d(PQu_m, QPu_m) = 0,$$

whenever $\{u_m\}$ is a sequence such that,

$$\lim_{m\to\infty} Pu_m = \lim_{m\to\infty} Qu_m = r \text{ with } r \in Y.$$

Definition 1.4 (*[19]*). Let $P, Q : Y \to Y$ be a pair of mappings on metric space (Y, d). The pair (P, Q) is weakly compatible when the pair (P, Q) commutes on the set of coincidence points (*i.e.*, $PQu = QPu$ when $Pu = Qu$).

Definition 1.5 (*[20]*). Let $d_1 : Y \times Y \to \mathbb{R}^+$ be a mapping, where Y is non-empty set. Then d_1 is called a b-metric if and only if ($\forall u$, w and $v \in Y$ and $s \geq 1$) the following conditions are fulfilled:

(b_1) $d_1(w, u) = 0$ if and only if $w = u$;
(b_2) $d_1(w, u) = d_1(u, w)$;
(b_3) $d_1(w, v) \leq s\big(d_1(w, u) + d_1(u, v)\big)$.

The pair (Y, d_1) is called a b-metric space, where d_1 is termed as b-metric defined on a partial order set $(Y, \preceq)$. Such a b-metric space is called a partially ordered b-metric space.

Definition 1.6 (*[13]*). A sequence $\{u_m\}$ is called b-Cauchy in (Y, d_1) if and only if

$$\lim_{m,n\to\infty} d_1(u_m, u_n) = 0.$$

Definition 1.7 (*[13]*). A sequence $\{w_m\}$ is called b-convergent in a b-metric space (Y, d_1) if and only if there is $w \in Y$ such that $\lim_{m\to\infty} d(w_m, w) = 0$ (*i.e.*, $\lim_{m\to\infty} w_m = w$).

Lemma 1.8 (*[13]*). *Suppose that the sequences $\{u_m\}$ and $\{v_m\}$ are b-convergent to u_1 and v_1 respectively in b-metric space (Y, d_1) with $s \geq 1$. Then,*

$$\frac{1}{s^2} d_1(u, v) \leq \lim_{m\to\infty} \inf d_1(u_m, v_m) \leq \lim_{m\to\infty} \sup d_1(u_m, v_m) \leq s^2 d_1(u, v).$$

In particular, $\lim_{m\to\infty} d_1(u_m, v_m) = 0$*, for* $u = v$*. Also,* $\forall\, w \in Y$*,*

$$\frac{1}{s} d_1(u, w) \le \lim_{m\to\infty} \inf d_1(u_m, w) \le \lim_{m\to\infty} \sup d_1(u_m, w) \le s d_1(u, w).$$

2. Main results

In this section, employing the idea of generalized (ψ, ϕ)-weak contraction, some fixed point results for continuous self-mappings defined on partially ordered b-metric spaces are established. The existing literature contains several fixed point results in the setting of b-metric spaces, where in Lemma 1.8 is used in their proofs. In our present paper, we do not employ Lemma 1.8.

Throughout this paper, we write $(\forall\ u, w\ \in Y)$;

$$M(u, w) := \max\left\{ d_1(Su, Tw), \frac{d_1(Su, Pu) + d_1(Tw, Qw)}{2}, \frac{d_1(Su, Qw) + (Tw, Pu)}{2s} \right\},$$

and

$$N(u, w) := \min\left\{ d_1(Su, Tw), \frac{d_1(Su, Pu) + d_1(Tw, Qw)}{2}, \frac{d_1(Su, Qw) + (Tw, Pu)}{2s} \right\}.$$

Now, we are equipped to prove our main result:

Theorem 2.1. *Let* $P, Q, S, T : Y \to Y$ *be continuous self-mappings on a partially ordered complete b-metric space* $(Y, \preceq, d_1)$ *such that* $P(Y) \subset T(Y)$ *and* $Q(Y) \subset S(Y)$*. Suppose that the pairs* (P, S) *and* (Q, T) *are compatible while the pairs* (P, Q) *and* (Q, P) *are partially weakly increasing with respect to* T *and* S *respectively. Assume for altering distance function* $\psi : [0, \infty) \to [0, \infty)$ *and lower semi continuous* $\phi : [0, \infty) \mapsto [0, \infty)$ *which is discontinuous at* $v = 0$ *and satisfies* $\phi(v) > 0, \forall\, v > 0$*, and* $\phi(0) = 0$*, the following condition holds:*

$$\psi(sd_1(Pu, Qw)) \le \psi(M(u, w)) - \phi(N(u, w)),\ \forall\ u, w\ \in Y. \tag{2.1}$$

Then there exists a unique common fixed point of P, Q, T *and* S*.*

Proof. Take $u_0 \in\ Y$. Since $P(Y) \subset T(Y)$ and $Q(Y) \subset S(Y)$, there exists a point $u_1 \in Y$ such that $Pu_0 = Tu_1$, where $u_2 \in\ Y$ such that $Qu_1 = Su_2$. In this way inductively, we are in the receipt of a sequence such that

$$w_{2m+1} = Pu_{2m} = Tu_{2m+1},$$
$$w_{2m+2} = Qu_{2m+1} = Su_{2m+2},\ \ for\ \ m = 0, 1, 2, \ldots.$$

As $u_1 \in T^{-1}(Pu_0)$ and $u_2 \in S^{-1}(Qu_1)$ and the pairs (P, Q) and (Q, P) are partially weakly increasing with respect to T and S respectively, we have $Tu_1 = Pu_0 \preceq Qu_1 = Su_2 \preceq Pu_2 = Tu_3$. Repeating this process inductively, one gets;

$$w_{2m+1} \preceq w_{2m+2},\ \ \forall\ \ m\ \in \mathbb{N} \cup \{0\}.$$

Now, we wish to prove that $d_1(w_m, w_{m+1}) \longrightarrow 0$ as $m \longrightarrow \infty,\ \forall\, m \in \mathbb{N} \cup \{0\}$. Write $u = u_{2m}$ and $w = u_{2m+1}$. Taking this into account in (2.1), we get the inequity

$$\psi(sd_1(Pu_{2m}, Qu_{2m+1})) = \psi(sd_1(w_{2m+1}, w_{2m+2})) \le \psi(M(u_{2m}, u_{2m+1})) - \phi(N(u_{2m}, u_{2m+1})), \tag{2.2}$$

where

$$M(u_{2m}, u_{2m+1}) = \max\left\{ d_1(w_{2m}, w_{2m+1}), \frac{d_1(w_{2m}, w_{2m+1}) + d_1(w_{2m+2}, w_{2m+1})}{2}, \right.$$
$$\left. \frac{d_1(w_{2m}, w_{2m+2}) + d_1(w_{2m+1}, w_{2m+1})}{2s} \right\}$$

and

$$N(u_{2m}, u_{2m+1}) = \min\left\{d_1(w_{2m}, w_{2m+1}), \frac{d_1(w_{2m}, w_{2m+1}) + d_1(w_{2m+2}, w_{2m+1})}{2}, \frac{d_1(w_{2m}, w_{2m+2}) + d_1(w_{2m+1}, w_{2m+1})}{2s}\right\}.$$

Write, $d_k = d_1(w_k, w_{k+1})$. Suppose $d_{k_0} = 0$ for some k_0, then $w_{k_0} = w_{k_{0+1}}$. In case $k_0 = 2m$, we have $w_{2m} = w_{2m+1}$. Now, we have

$$\begin{aligned} M(u_{2m}, u_{2m+1}) &= \max\left\{0, \frac{d_1(w_{2m+1}, w_{2m+2}) + 0}{2}, \frac{0 + d_1(w_{2m}, w_{2m+2})}{2s}\right\} \\ &= \max\left\{0, \frac{d_1(w_{2m+2}, w_{2m+1})}{2}, \frac{d_1(w_{2m+2}, w_{2m})}{2s}\right\} = \frac{d_1\big(w_{2m+2}, w_{2m+1}\big)}{2}. \end{aligned}$$

Due to triangle inequality we see that

$$\begin{aligned} d_1(w_{2m+2}, w_{2m}) &\leq s(d_1(w_{2m}, w_{2m+1}) + d_1(w_{2m+1}, w_{2m+2})) = sd_1(w_{2m+1}, w_{2m+2}). \\ &\Rightarrow \frac{d_1(w_{2m}, w_{2m+2})}{s} \leq d_1(w_{2m+1}, w_{2m+2}), \end{aligned}$$

which together with Eq. (2.2) gives

$$\psi\big(sd_1(w_{2m+1}, w_{2m+2})\big) \leq \psi\left(\frac{d_1(w_{2m+1}, w_{2m+2})}{2}\right).$$

Since ψ is non-decreasing, one can write

$$sd_1(w_{2m+1}, w_{2m+2}) \leq \frac{d_1(w_{2m+1}, w_{2m+2})}{2}$$

which is possible only when $d_1(w_{2m+1}, w_{2m+2}) = 0$. Hence $w_{2m+1} = w_{2m+2}$. Thus in all, $w_{2m} = w_{2m+1} = w_{2m+2}$.

Similarly, if $k_0 = 2m + 1$, then $w_{2m+1} = w_{2m+2}$, which gives $w_{2m+2} = w_{2m+3}$. Consequently, the sequence $\{w_k\}$ reduces to a constant sequence whenever $k \geq k_0$ so that

$$\lim_{m\to\infty} d_1(w_m, w_{m+1}) = 0.$$

In case, $w_{2m} \neq w_{2m+1}$, then $d_1(w_{2m}, w_{2m+1}) > 0$, for all $m \in \mathbb{N} \cup \{0\}$. Since

$$M(u_{2m}, u_{2m+1}) = \max\left\{d_1(w_{2m}, w_{2m+1}), \frac{d_1(w_{2m}, w_{2m+1}) + d_1(w_{2m+1}, w_{2m+2})}{2}, \frac{d_1(w_{2m}, w_{2m+2})}{2s}\right\},$$

using triangle inequality we find that

$$\begin{aligned} M(u_{2m}, u_{2m+1}) &\leq \max\left\{d_1(w_{2m}, w_{2m+1}), \frac{d_1(w_{2m}, w_{2m+1}) + d_1(w_{2m+1}, w_{2m+2})}{2}, \frac{s(d_1(w_{2m}, w_{2m+1}) + d_1(w_{2m+1}, w_{2m+2}))}{2s}\right\} \\ &= \max\left\{d_1(w_{2m}, w_{2m+1}), \frac{d_1(w_{2m}, w_{2m+1}) + d_1(w_{2m+1}, w_{2m+2})}{2}, \frac{(d_1(w_{2m}, w_{2m+1}) + d_1(w_{2m+1}, w_{2m+2}))}{2}\right\}. \end{aligned} \tag{2.3}$$

On the contrary, assume that $d_1(w_{2m}, w_{2m+1}) < d_1(w_{2m+1}, w_{2m+2})$. Then

$$M(u_{2m}, u_{2m+1}) \leq d_1(w_{2m+1}, w_{2m+2}) \leq sd_1(w_{2m+1}, w_{2m+2}), \tag{2.4}$$

$$N(u_{2m}, u_{2m+1}) = \min\left\{d_1(w_{2m}, w_{2m+1}), \frac{d_1(w_{2m}, w_{2m+1}) + d_1(w_{2m+1}, w_{2m+2})}{2}, \frac{d_1(w_{2m}, w_{2m+2}) + d_1(w_{2m+1}, w_{2m+1})}{2s}\right\}.$$

Since $w_{2m} \neq w_{2m+1}$, so that $N(u_{2m}, u_{2m+1}) > 0$. Therefore, $\phi\big(N(u_{2m}, u_{2m+1})\big) > 0$. By using Eq. (2.2), we have

$$\psi\big(sd_1(w_{2m+1}, w_{2m+2})\big) \leq \psi\big(M(w_{2m}, w_{2m+1})\big).$$

As ψ is non-decreasing, we see that

$$sd_1(w_{2m+1}, w_{2m+2}) \leq M(u_{2m}, u_{2m+1}). \tag{2.5}$$

From (2.4) and (2.5) we find that

$$M(u_{2m}, u_{2m+1}) = sd_1(w_{2m+1}, w_{2m+2}). \tag{2.6}$$

Using $\phi(N(u_{2m}, u_{2m+1})) > 0$ and taking Eqs. (2.6) and (2.2) into account, we can write that

$$\begin{aligned}\psi\big(sd_1(w_{2m+1}, w_{2m+2})\big) &\leq \psi\big(M(u_{2m}, u_{2m+1})\big) - \phi\big(N(u_{2m}, u_{2m+1})\big),\\ &< \psi\big(sd_1(w_{2m+1}, w_{2m+2})\big)\end{aligned}$$

which is a contradiction. Hence

$$d_1(w_{2m+1}, w_{2m+2}) \leq d_1(w_{2m}, w_{2m+1}). \tag{2.7}$$

It is clear from (2.7) that sequence of non-negative real numbers $\{d_1(w_{2m}, w_{2m+1})\}$, is monotonically decreasing, therefore there exists a number $r \geq 0$, such that;

$$\lim_{m\to\infty} d_1(w_{2m}, w_{2m+1}) = r.$$

We assert that $r = 0$. Let us assume the contrary that $r > 0$, then

$$\begin{aligned}\lim_{m\to\infty} \psi\big(sd_1(w_{2m+1}, w_{2m+2})\big) &\leq \lim_{m\to\infty} \psi\big(d_1(w_{2m}, w_{2m+1})\big) - \lim_{m\to\infty} \phi\big(N(u_{2m}, u_{2m+1})\big).\\ &\Rightarrow \psi(sr) \leq \psi(r) - \lim_{m\to\infty} \phi\big(N(u_{2m}, u_{2m+1})\big).\end{aligned}$$

In view of the lower semi continuity of ϕ, the second term on the right hand side of the preceding inequality is non zero, therefore $\psi(sr) < \psi(r)$ implies that $sr < r$, which is a contradiction to our supposition. Hence $r = 0$.

Thus,

$$\lim_{m\to\infty} d_1(w_{2m}, w_{2m+1}) = 0.$$

Similarly, on putting $u = u_{2m+1}$ and $w = w_{2m+2}$ in (2.1), one gets;

$$\lim_{m\to\infty} d_1(w_{2m+1}, w_{2m+2}) = 0.$$

Therefore in all, we conclude that;

$$\lim_{m\to\infty} d_1(w_m, w_{m+1}) = 0,\ \forall\ m \in \mathbb{N} \cup \{0\}. \tag{2.8}$$

Now, we show that $\{w_m\}$ is a b-Cauchy sequence. To accomplish this, it is sufficient to show that $\{w_{2m}\}$, is a b-Cauchy. Let on contrary that $\{w_{2m}\}$be not a b-Cauchy sequence. Suppose that for sequences $\{2m(t)\}$ and $\{2n(t)\}$ with $2m(t) > 2n(t) > 2t, \forall\, t \in \mathbb{N}$, there exists $\epsilon > 0$ such that,

$$d_1(w_{2n(t)}, w_{2m(t)}) \geq \epsilon. \tag{2.9}$$

Moreover, suppose that corresponding to integer $2n(t)$, $2m(t)$ is the smallest integer such that condition (2.9) is satisfied. Then, we have

$$d_1(w_{2n(t)}, w_{2m(t)-1}) < \epsilon. \tag{2.10}$$

By taking $u = u_{2n(t)-1}$ and $w = w_{2m(t)-1}$ in (2.1), we can have

$$\psi\big(sd_1(w_{2n(t)}, w_{2m(t)})\big) \leq \psi\big(M(u_{2n(t)-1}, u_{2m(t)-1})\big) - \phi\big(N(u_{2n(t)-1}, u_{2m(t)-1})\big), \tag{2.11}$$

where,

$$M(u_{2n(t)-1}, u_{2m(t)-1}) = \max\left\{d_1(w_{2n(t)-1}, w_{2m(t)-1}), \frac{d_1(w_{2n(t)-1}, w_{2n(t)}) + d_1(w_{2m(t)-1}, w_{2m(t)})}{2}, \frac{d_1(w_{2n(t)-1}, w_{2m(t)}) + d_1(w_{2m(t)-1}, w_{2n(t)})}{2s}\right\},$$

and

$$N(u_{2n(t)-1}, u_{2m(t)-1}) = \min\left\{d_1(w_{2n(t)-1}, w_{2m(t)-1}), \frac{d_1(w_{2n(t)-1}, w_{2n(t)}) + d_1(w_{2m(t)-1}, w_{2m(t)})}{2}, \frac{d_1(w_{2n(t)-1}, w_{2m(t)}) + d_1(w_{2m(t)-1}, w_{2n(t)})}{2s}\right\}.$$

Using triangle inequality, one can write

$$d(w_{2n(t)}, w_{2m(t)}) \leq s(d_1(w_{2n(t)}, w_{2m(t)-1}) + d_1(w_{2m(t)-1}, w_{2m(t)})),$$

which by using (2.8), (2.10) and letting $t \to \infty$ in the preceding inequality we find that

$$d_1(w_{2n(t)}, w_{2m(t)}) \leq s\epsilon. \tag{2.12}$$

Similarly, using triangle inequality we have that

$$d_1(w_{2n(t)-1}, w_{2m(t)-1}) \leq s(d_1(w_{2n(t)-1}, w_{2n(t)}) + d_1(w_{2n(t)}, w_{2m(t)-1})).$$

Now using (2.8), (2.10) and taking $t \to \infty$, the preceding inequality yields:

$$d_1(w_{2n(t)-1}, w_{2m(t)-1}) \leq s\epsilon. \tag{2.13}$$

Applying once more triangle inequality, we have

$$d_1(w_{2n(t)-1}, w_{2m(t)}) \leq s(d_1(w_{2n(t)-1}, w_{2m(t)-1}) + d_1(w_{2m(t)-1}, w_{2m(t)})).$$

Applying (2.8), (2.13) and letting $t \to \infty$, the latter inequality yields

$$d_1(w_{2n(t)-1}, w_{2m(t)}) \leq s(s\epsilon) = s^2\epsilon. \tag{2.14}$$

Since,

$$M(u_{2n(t)-1}, u_{2m(t)-1}) = \max\left\{d_1(w_{2n(t)-1}, w_{2m(t)-1}), \frac{d_1(w_{2n(t)-1}, w_{2n(t)}) + d_1(w_{2m(t)-1}, w_{2m(t)})}{2}, \frac{d_1(w_{2n(t)-1}, w_{2m(t)}) + d_1(w_{2m(t)-1}, w_{2m(t)})}{2s}\right\},$$

which together with (2.8), (2.13) and (2.14) gives:

$$\lim_{t\to\infty} M(u_{2n(t)-1}, u_{2m(t)-1}) \leq s\epsilon. \tag{2.15}$$

Similarly, one can also show that $\lim_{t\to\infty} N(u_{2n(t)-1}, u_{2m(t)-1}) = 0$.

Taking, $t \to \infty$ and using (2.9) and (2.15) in (2.11), we can have

$$\psi(s\epsilon) \leq \psi(s\epsilon) - \lim_{t\to\infty} \phi(N(u_{2n(t)-1}, u_{2m(t)-1})),$$

which leads to contradiction due to lower semi continuity of ϕ and the fact that $\phi(v) = 0$ at $v = 0$. Hence $\{w_m\}$ is a b-Cauchy sequence.

Since (Y, d_1) is complete b-metric space, there exists $a \in Y$ such that the b-Cauchy sequence $\{w_n\}$ is b-convergent to a. Consequently, the subsequences $Pu_{2m} \to a$, $Qu_{2m+1} \to a$, $Tu_{2m+1} \to a$ and $Su_{2m} \to a$, where $a \in Y$.

Next, we show that a is a coincidence point of P and S. Since

$$\lim_{m\to\infty} Pu_{2m} = \lim_{m\to\infty} w_{2m+1} = a,$$

and

$$\lim_{m\to\infty} Su_{2m+2} = \lim_{m\to\infty} w_{2m+2} = a,$$

so that $\lim_{m\to\infty} d_1(Pu_{2m}, a) = 0$ and $\lim_{m\to\infty} d_1(Su_{2m}, a) = 0$.

Since P and S are continuous mappings, we have

$$\lim_{m\to\infty} d_1(SPu_{2m}, Sa) = \lim_{m\to\infty} d_1(PSu_{2m}, Pa) = 0. \tag{2.16}$$

As the pair (P, S) is compatible, one can write

$$\lim_{m\to\infty} d_1(SPu_{2m}, PSu_{2m}) = 0. \tag{2.17}$$

Using triangle inequality, we have

$$d_1(Sa, Pa) \le s(d_1(Sa, SPu_{2m}) + d_1(SPw_{2m}, Pa)).$$

Again, using triangle inequality on the second term of the right hand side, we have

$$d_1(Sa, Pa) \le sd_1(Sa, SPu_{2m}) + s^2(d_1(SPu_{2m}, PSu_{2m}) + d_1(PSu_{2m}, Pa)).$$

By using (2.17), (2.16) and taking limit $m \to \infty$ in above inequality, we have

$$\lim_{m\to\infty} d_1(Sa, Pa) \le 0.$$

Hence, $d_1(Sa, Pa) = 0$ which implies that $Pa = Sa$. Thus, a is a coincidence point of P and S. Similarly we can show that $Qa = Ta$.

Further, we show that $Pa = Sa = a$. For this purpose, we set $u = a$ and $w = u_{2m+1}$ in (2.1) so that

$$\psi(sd_1(Pa, Qu_{2m+1})) \le \psi(M(a, u_{2m+1})) - \phi(N(a, u_{2m+1})), \tag{2.18}$$

where,

$$\begin{aligned} M(a, u_{2m+1}) = \max\Big\{ & d_1(Sa, Tu_{2m+1}), \frac{d_1(Sa, Pa) + d_1(Tu_{2m+1}, Qu_{2m+1})}{2}, \\ & \frac{d_1(Sa, Qu_{2m+1}) + (Tu_{2m+1}, Pa)}{2s} \Big\}, \end{aligned}$$

and

$$\begin{aligned} N(a, u_{2m+1}) = \min\Big\{ & d_1(Sa, Tu_{2m+1}), \frac{d_1(Sa, Pa) + d_1(Tu_{2m+1}, Qu_{2m+1})}{2}, \\ & \frac{d_1(Sa, Qu_{2m+1}) + (Tu_{2m+1}, Pa)}{2s} \Big\}, \end{aligned}$$

together with

$$\lim_{m\to\infty} M(a, u_{2m+1}) = M(a, a) = d_1(Sa, a). \tag{2.19}$$

By taking the limit as $m \to \infty$on both the sides of (2.18) and using Eqs. (2.19), we get

$$\psi(sd_1(Pa, a)) \le \psi(M(a, a)) - \lim_{m\to\infty} \phi(N(a, u_{2m+1})).$$
$$\psi(sd_1(Sa, a)) \le \psi(d_1(Sa, a)) - \lim_{m\to\infty} \phi(N(a, u_{2m+1})).$$

Since, ϕ is lower semi continuous, then one can write

$$\psi(sd_1(Sa, a)) < \psi(d_1(Sa, a)),$$

which is contradiction. Hence, $d_1(Sa, a) = 0 \Rightarrow Sa = a \Rightarrow Sa = Pa = a$.

Similarly, we can show that $Ta = Qa = a$. Hence, $Sa = Pa = Ta = Qa = a$.

Finally, we prove that a is a unique common fixed point of P, Q, S and T. Suppose z is another common fixed point of P, Q, S and T. On setting $u = a$ and $w = z$ in the definitions of M and N, we have

$$M(u, w) = d(a, z),$$

and

$$N(u, w) = \frac{d_1(a, z)}{2}.$$

Hence

$$\psi\big(sd_1(a, z)\big) \leq \psi\big(d_1(a, z)\big) - \phi\left(\frac{d_1(a, z)}{2}\right).$$

It is possible only when $d(a, z) = 0 \Rightarrow a = z$. Hence a is unique common fixed point of P, Q, S and T. This concludes the proof of this theorem. □

By setting, $\psi(r) = r$, in Theorem 2.1 one can get the following corollary:

Corollary 2.2. *Let $P, Q, S, T : Y \to Y$ be continuous self-mappings on partially ordered complete b-metric space $(Y, \preceq, d_1)$ such that $P(Y) \subset T(Y)$ and $Q(Y) \subset S(Y)$. Suppose that the pairs (P, S) and (Q, T) are compatible while the pairs (P, Q) and (Q, P) are partially weakly increasing with respect to T and S respectively. Assume for lower semi continuous $\phi : [0, \infty) \mapsto [0, \infty)$ which is discontinuous at $v = 0$ and satisfies $\phi(v) > 0$, $\forall\, v > 0$, and $\phi(0) = 0$, the following condition holds:*

$$sd_1(Pu, Qw) \leq M(u, w) - \phi(N(u, w)),\ \forall\ u, w\ \in Y. \tag{2.20}$$

Then there exists a unique common fixed point of P, Q, T and S.

By setting $S = T$ in Theorem 2.1, we have that

$$M(u, w) := \max\left\{d_1(Su, Tw), \frac{d_1(Su, Pu) + d_1(Tw, Qw)}{2}, \frac{d_1(Su, Qw) + (Tw, Pu)}{2s}\right\},$$

and

$$N(u, w) := \min\left\{d_1(Su, Tw), \frac{d_1(Su, Pu) + d_1(Tw, Qw)}{2}, \frac{d_1(Su, Qw) + (Tw, Pu)}{2s}\right\}.$$

Thus, we can have the following theorem involving three mappings.

Theorem 2.3. *Let P, Q and $S : Y \to Y$ be continuous self-mappings on partially ordered complete b-metric space $(Y, \preceq, d_1)$ such that $P(Y) \subset S(Y)$ and $Q(Y) \subset S(Y)$. Suppose that the pairs (P, S) and (Q, S) are compatible while the pair (P, Q) is partially weakly increasing and (Q, P) is partially weakly increasing with respect to S. Assume for altering distance function $\psi : [0, \infty) \to [0, \infty)$ and lower semi continuous $\phi : [0, \infty) \mapsto [0, \infty)$, which is discontinuous at $v = 0$ and satisfies $\phi(v) > 0$, $\forall\, v > 0$, and $\phi(0) = 0$, the following condition holds:*

$$\psi(sd_1(Pu, Qw)) \leq \psi(M(u, w)) - \phi(N(u, w)),\ \forall\ u, w\ \in Y. \tag{2.21}$$

Then there exists a unique common fixed point of P, Q and S.

By setting $\psi(r) = r$ in Theorem 2.3 one can prove easily the following corollary.

Corollary 2.4. *Let P, Q and $S : Y \to Y$ be continuous self-mappings on partially ordered complete b-metric space $(Y, \preceq, d_1)$ such that $P(Y) \subset S(Y)$ and $Q(Y) \subset S(Y)$. Suppose that the pairs (P, S) and (Q, S) are compatible while the pair (P, Q) is partially weakly increasing and (Q, P) is partially weakly increasing with respect to S. Assume for lower semi continuous $\phi : [0, \infty) \mapsto [0, \infty)$, which is discontinuous at $v = 0$ and satisfies $\phi(v) > 0$, $\forall\, v > 0$, and $\phi(0) = 0$, the following condition holds:*

$$sd_1(Pu, Qw) \leq M(u, w) - \phi(N(u, w)),\ \forall\ u, w\ \in Y. \tag{2.22}$$

Then there exists a unique common fixed point of P, Q and S.

By setting, $S = T = I$ identity mapping in Theorem 2.1 we find that

$$M(u, w) := \max\left\{d_1(u, w), \frac{d_1(u, Pu) + d_1(w, Qw)}{2}, \frac{d_1(u, Qw) + (w, Pu)}{2s}\right\},$$

and

$$N(u, w) := \min\left\{d_1(u, w), \frac{d_1(u, Pu) + d_1(w, Qw)}{2}, \frac{d_1(u, Qw) + (w, Pu)}{2s}\right\},$$

the following theorem for two mappings can be easily obtained.

Theorem 2.5. *Let P and $Q : Y \to Y$ be continuous self-mappings on partially ordered complete b-metric space $(Y, \preceq, d_1)$. Suppose that the pairs (P, Q) and (Q, P) are partially weakly increasing. Assume for altering distance function $\psi : [0, \infty) \to [0, \infty)$ and lower semi continuous $\phi : [0, \infty) \mapsto [0, \infty)$, which is discontinuous at $v = 0$ and satisfies $\phi(v) > 0$, $\forall\, v > 0$, and $\phi(0) = 0$, the following condition holds:*

$$\psi(sd_1(Pu, Qw)) \le \psi(M(u, w)) - \phi(N(u, w)), \quad \forall\ u, w \in Y. \tag{2.23}$$

Then there exists a unique common fixed point of P and Q.

By setting, $\psi(r) = r$ in Theorem 2.5 one can get the following corollary.

Corollary 2.6. *Let P and $Q : Y \to Y$ be continuous self-mappings on partially ordered complete b-metric space $(Y, \preceq, d_1)$. Suppose that the pairs (P, Q) and (Q, P) are partially weakly increasing. Assume for lower semi continuous $\phi : [0, \infty) \mapsto [0, \infty)$, which is discontinuous at $v = 0$ and satisfies $\phi(v) > 0$, $\forall\, v > 0$, and $\phi(0) = 0$, the following condition holds:*

$$sd_1(Pu, Qw) \le M(u, w) - \phi(N(u, w)), \ \forall\ u, w \in Y. \tag{2.24}$$

Then there exists a unique common fixed point of P and Q.

By setting, $P = Q$ in Theorem 2.5 then,

$$M(u, w) := \max\left\{d_1(u, w), \frac{d_1(u, Pu) + d_1(w, Pw)}{2}, \frac{d_1(u, Pw) + (w, Pu)}{2s}\right\},$$

and

$$N(u, w) := \min\left\{d_1(u, w), \frac{d_1(u, Pu) + d_1(w, Pw)}{2}, \frac{d_1(u, Pw) + (w, Pu)}{2s}\right\}.$$

One can obtain the following unique fixed point theorem.

Theorem 2.7. *Let $P : Y \to Y$ be continuous self mapping on partially ordered complete b-metric space $(Y, \preceq, d_1)$ with $P(u) \preceq P(P(u))$. Assume for altering distance function $\psi : [0, \infty) \to [0, \infty)$ and lower semi continuous $\phi : [0, \infty) \mapsto [0, \infty)$, which is discontinuous at $v = 0$ and satisfies $\phi(v) > 0$, $\forall\, v > 0$, and $\phi(0) = 0$, the following condition holds:*

$$\psi(sd_1(Pu, Pw)) \le \psi(M(u, w)) - \phi(N(u, w)), \ \forall\ u, w \in Y. \tag{2.25}$$

Then, P has a unique common fixed point in Y.

By putting, $\psi(r) = r$, Theorem 2.7 yields the following corollary:

Corollary 2.8. *Let $P : Y \to Y$ be continuous self mapping on partially ordered complete b-metric space $(Y, \preceq, d_1)$ with $P(u) \preceq P(P(u))$. Assume for lower semi continuous $\phi : [0, \infty) \mapsto [0, \infty)$, which is discontinuous at $v = 0$ and satisfies $\phi(v) > 0$, $\forall\, v > 0$ and $\phi(0) = 0$, the following condition holds:*

$$sd_1(Pu, Pw) \le M(u, w) - \phi(N(u, w)), \ \forall\ u, w \in Y. \tag{2.26}$$

Then P has a unique common fixed point in Y.

Example 2.9. Let us define partial ordering $\preceq$ on $Y = [1, 2]$ as follows;

$$u \preceq w \Leftrightarrow w \leq u, \forall\, u, w \in Y.$$

Let $d_1(u, w) = |u - w|^2$, $\forall\ u, w\ \in Y$, then clearly d_1 is a partially order b-metric on Y with partial ordering $\preceq$. Suppose P, Q, T and S are continuous mappings on $Y = [1, 2]$ defined by:

$$P(u) = \frac{u+4}{5},\quad Q(u) = \frac{2u+3}{5},$$
$$T(u) = \frac{4u}{5} + \frac{1}{5},\quad S(u) = \frac{3u}{5} + \frac{2}{5}.$$

Evidently,

$$P(Y) = \left[1, \frac{6}{5}\right] \text{ and } S(Y) = \left[1, \frac{8}{5}\right],\quad P(Y) \subset S(Y)$$
$$Q(Y) = \left[1, \frac{7}{5}\right] \text{ and } T(Y) = \left[1, \frac{9}{5}\right],\quad Q(Y) \subset T(Y).$$

Define, $\psi(v) = v$ and $\phi(v) = \begin{cases} \frac{v}{5}, & \text{when } v \neq 0; \\ 0, & \text{when } v = 0. \end{cases}$ The compatibility of the pairs (P, S) and (Q, T) is straightforward. Indeed, with a sequence $\{u_m\}$ in Y such that for some $v \in Y$,

$$\lim_{m\to\infty} d_1(v, Pu_m) = \lim_{m\to\infty} d_1(v, Su_m) = 0,$$

we have

$$\lim_{m\to\infty} \left|\frac{u_m}{5} + \frac{4}{5} - v\right|^2 = \lim_{m\to\infty} \left|\frac{3u_m}{5} + \frac{2}{5} - v\right|^2 = 0.$$

Since P and S are continuous, then one can write

$$\lim_{m\to\infty} |u_m - (5v - 4)|^2 = \lim_{v\to\infty} \left|u_m - \frac{5v-2}{3}\right|^2 = 0.$$

But the limit is unique. Therefore, $5v - 4 = \frac{5v-2}{3} \Longleftrightarrow v = 1$. From continuity of P and S we have

$$\lim_{m\to\infty} d_1(PSu_m, SPu_m) = \lim_{m\to\infty} |PSu_m - SPu_m|^2 = 0$$

which shows the compatibility of the pair (P, S). In the same way the compatibility of the pair (Q, T) can also be shown.

Next, we show that pair (P, Q) is partially weakly increasing with respect to T. Let $w \in T^{-1}(Pu)$, for $u, w \in Y$. Then

$$P(u) = T(w) \Rightarrow P(u) = \frac{4w}{5} + \frac{1}{5} \geq \frac{2w}{5} + \frac{3}{5} = Q(w),$$

$P(u) \geq Q(w)$ so that $P(u) \preceq Q(w)$.

Also, the pair (Q, P) is partially weakly increasing with respect to S. Consequently,

$$Q(u) = S(w) \Rightarrow Q(u) = \frac{3w+2}{5} \geq \frac{w+4}{5} = P(w).$$

Thus, $Q(u) \preceq P(w)$.

Now, we have to show that,

$$\psi(sd_1(Pu, Qw)) \leq \psi(M(u, w)) - \phi(N(u, w)),\ \ \forall\ u, w\ \in Y,$$

where $M(u, w)$ and $N(u, w)$ are described earlier. Herein;

$$d_1(Pu, Qw) = \left|\frac{2w}{5} - \frac{u}{5} - \frac{1}{5}\right|^2,\ d_1(Su, Pu) = \left|-\frac{2u}{5} + \frac{2}{5}\right|^2\ d_1(Tw, Qw) = \left|-\frac{2w}{5} + \frac{2}{5}\right|^2,$$
$$d_1(Su, Qw) = \left|\frac{2w}{5} - \frac{3u}{5} + \frac{1}{5}\right|^2,\ d_1(Tw, Pu) = \left|\frac{u}{5} - \frac{4w}{5} + \frac{3}{5}\right|^2,\text{ and } d_1(Su, Tw) = \left|\frac{4w}{5} - \frac{3u}{5} - \frac{1}{5}\right|^2.$$

Case (1): When $u = w$, then

$$M(u, w) = \frac{\left|-\frac{2u}{5}+\frac{2}{5}\right|^2 + \left|-\frac{2w}{5}+\frac{2}{5}\right|^2}{2},$$

and

$$N(u, w) = d_1(Su, Tw) = \left|\frac{4w}{5} - \frac{3u}{5} - \frac{1}{5}\right|^2.$$

Clearly it is satisfied the following contraction: condition,

$$\psi\left(s\left|\frac{2w}{5} - \frac{u}{5} - \frac{1}{5}\right|^2\right) \le \psi\left(\frac{|-\frac{2u}{5}+\frac{2}{5}|^2 + |-\frac{2w}{5}+\frac{2}{5}|^2}{2}\right) - \phi\left(\left|\frac{4w}{5} - \frac{3u}{5} - \frac{1}{5}\right|^2\right), \quad \text{for all } u = w \in Y.$$

Case (2): For $u \neq w$, there arise two cases.

If $\left|\frac{4w}{5} - \frac{3u}{5} - \frac{1}{5}\right|^2$ is maximum, then $\frac{\left|-\frac{2u}{5}+\frac{2}{5}\right|^2 + \left|-\frac{2w}{5}+\frac{2}{5}\right|^2}{2}$ is minimum. Therefore,

$$M(u, w) = \left|\frac{4w}{5} - \frac{3u}{5} - \frac{1}{5}\right|^2,$$

and

$$N(u, w) = \frac{\left|-\frac{2u}{5}+\frac{2}{5}\right|^2 + \left|-\frac{2w}{5}+\frac{2}{5}\right|^2}{2}.$$

Evidently, the following contraction is satisfied

$$\psi\left(s\left|\frac{2w}{5} - \frac{u}{5} - \frac{1}{5}\right|^2\right) \le \psi\left(\left|\frac{4w}{5} - \frac{3u}{5} - \frac{1}{5}\right|^2\right) - \phi\left(\frac{|-\frac{2u}{5}+\frac{2}{5}|^2 + |-\frac{2w}{5}+\frac{2}{5}|^2}{2}\right).$$

If $\frac{\left|-\frac{2u}{5}+\frac{2}{5}\right|^2 + \left|-\frac{2w}{5}+\frac{2}{5}\right|^2}{2}$ is maximum, then $\left|\frac{4w}{5} - \frac{3u}{5} - \frac{1}{5}\right|^2$, is minimum. Therefore,

$$M(u, w) = \frac{\left|-\frac{2u}{5}+\frac{2}{5}\right|^2 + \left|-\frac{2w}{5}+\frac{2}{5}\right|^2}{2},$$

and

$$N(u, w) = \left|\frac{4w}{5} - \frac{3u}{5} - \frac{1}{5}\right|^2.$$

Obviously one can write,

$$\psi\left(s\left|\frac{2w}{5} - \frac{u}{5} - \frac{1}{5}\right|^2\right) \le \psi\left(\frac{|-\frac{2u}{5}+\frac{2}{5}|^2 + |-\frac{2w}{5}+\frac{2}{5}|^2}{2}\right) - \phi\left(\left|\frac{4w}{5} - \frac{3u}{5} - \frac{1}{5}\right|^2\right).$$

Hence, all conditions of Theorem 2.1 are satisfied. Thus, 1 is the unique common fixed point of P, Q, S and T.

3. Application to the system of non-linear integral equations

Let us take the system of integral equations given below:

$$\begin{Bmatrix} u(a) = F(a) + \int_t^r K_1\big(a, v, u(v)\big)dv, \\ u(a) = G(a) + \int_t^r K_2\big(a, v, u(v)\big)dv, \end{Bmatrix} \tag{3.1}$$

where K_1 *and* $K_2 : [t, r] \times [t, r] \times \mathbb{R} \to \mathbb{R}$.

Let $P, Q : Y \to Y$, and $F, G : [t, r] \to \mathbb{R}$ be continuous mappings. By redefining the above system of integral equation one has the following system,

$$\begin{cases} P(u(a)) = F(a) + \int_t^r K_1\Big(a, v, u(v)\Big)dv. \\ Q(u(a)) = G(a) + \int_t^r K_2\Big(a, v, u(v)\Big)dv. \end{cases} \tag{3.2}$$

For all $u \in Y$ and $a \in [t, r]$.

Evidently, the existence of a solution of (3.1) that belongs to $Y = C[t, r]$ is equivalent to the existence of a common fixed point of P and Q.

Define partial ordering on Y by $u \preceq w \Leftrightarrow u(a) \le w(a)$. Also define a b-metric as (for all $u, w \in Y$)

$$d_1(u(a), w(a)) = \max_{a \in [t,\, r]} \big|u(a) - w(a)\big|^p.$$

Theorem 3.1. *Suppose the conditions given below are satisfied:*

(i) *K_1 and $K_2 : [t, r] \times [t, r] \times \mathbb{R} \to \mathbb{R}$ are continuous;*

(ii) *For all $a, v \in [t, r]$ and $u \in Y$ we have*

$$F(a) + K_1\Big(a, v, u(v)\Big) \le G(F(a)) + K_2\Big(a, v, \int_t^r K_1(v, a, u(a))da\Big)$$

and

$$G(a) + K_2\Big(a, v, u(v)\Big) \le F(G(a)) + K_1\Big(a, v, \int_t^r K_2(v, a, u(a))da\Big).$$

(iii) *For all $a, v \in [t, r]$ and $u, w \in Y$ with $u \preceq w$ we have*

$$\big|K_1(a, v, u(v)) - K_2(a, v, w(v))\big|^p \le R(a, v)Ln(1 + \big|u(v) - w(v)\big|^p),$$

and R is continuous function satisfying the condition,

$$\sup_{a \in [t,\, r]} \int_t^r R\big(a, v\big)dv < \frac{1}{(3)^p\big(r - t\big)^{p-1}};$$

(iv) $\sup_{a \in [t,\, r]} |F(a) - G(a)|^p \le \frac{\left|u(v) - w(v)\right|^p - 1}{(3)^p}$, *for all $a, v \in [t, r]$.*

Then system (3.1) *of non linear integral equation has a unique solution.*

Proof. Clearly from condition (ii), the pairs (P, Q) and (Q, P) are partially weakly increasing. Let $1 \le p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ from condition (iii) and (iv) for all $a \in [t, r]$ we have

$$\begin{aligned} 3^p\big|P(u(a)) - Q(w(a))\big|^p &\le 3^p\left[\left|F(a) + \int_t^r K_1(a, v, u(v))dv - G(a) - \int_t^r K_2\big(a, v, w(v)\big)dv\right|^p\right] \\ &\le 3^p\left[\Big|F(a) - G(a)\Big|^p + \left(\int_t^r \Big|(1)(K_1(a, v, u(v)) - K_2(a, v, w(v)))\Big|dv\right)^p\right] \\ &\le 3^p\left[\Big|F(a) - G(a)\Big|^p + \left(\left(\int_t^r 1^q ds\right)^{\frac{1}{q}}\right)^p\left(\left(\int_t^r \big|K_1(a, v, u(v)) - K_2(a, v, w(v))\big|^p dv\right)^{\frac{1}{p}}\right)^p\right] \\ &\le 3^p\Big|F(a) - G(a)\Big|^p + 3^p(r - t)^{\frac{p}{q}}\int_t^r R(a, v)Ln\Big(1 + |u(v) - w(v)|^p dv\Big) \\ &\le \Big|u(v) - w(v)\Big|^p - 1 + 3^p(r - t)^{p-1}\sup_{a \in [t,\, r]}\Big(Ln(1 + \big|u(v) - w(v)\big|^p)\Big)\int_t^r R(a, v)dv \end{aligned}$$

$$\begin{aligned}
&\le d_1\big(u(v), w(v)\big) - 1 + Ln\Big(1 + d_1\big(u(v), w(v)\big)\Big)\\
&\le M(u, w) - 1 + Ln\Big(1 + M(u, w)\Big)\\
&\le M(u, w) - 1 + 1 + M(u, w)\\
&\le 2M(u, w) + M(u, w) - N(u, w)\\
&= 3M(u, w) - N(u, w).
\end{aligned}$$

Hence,

$$3sd_1(P(u), Q(w)) \le 3M(u, w) - N(u, w).$$

Define $\psi(z) = 3z$ and $\phi(z) = z$, where $s = 3^{P-1}$. Then

$$\psi\Big(sd_1(P(u), Q(w))\Big) \le \psi\Big(M(u, w)\Big) - \phi\Big(N(u, w)\Big).$$

Thus, by Theorem 2.5, system (3.2) has a unique solution. Consequently (3.1) has a unique solution in Y. □

Competing Interest

The authors declare that they have no competing interests.

Authors contributions

All authors contribute equally to the writing of this manuscript. All authors read and approved the final version.

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Original article

MHD flow and heat transfer characteristics of Williamson nanofluid over a stretching sheet with variable thickness and variable thermal conductivity

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Abstract

The magneto hydrodynamic boundary layer flow with heat and mass transfer of Williamson nanofluid over a stretching sheet with variable thickness and variable thermal conductivity under the radiation effect is examined. It is assumed that the sheet is non-flat. The governing partial differential equations are reduced to nonlinear coupled ordinary differential equations by applying the suitable similarity transformations. These nonlinear coupled ordinary differential equations, subject to the appropriate boundary conditions, are then solved by using spectral quasi-linearisation method (SQLM). The effects of the physical parameters on the flow, heat transfer and nanoparticle concentration characteristics of the problem are presented through graphs and are discussed in detailed. Numerical values of skin friction co-efficient and Nusselt number with different parameters were computed and analysed.

Keywords: MHD; SQLM; Variable thermal conductivity; Variable thickness; Williamson fluid

1. Introduction

Nanofluid is a liquid filled with nanometre-sized particles with diameter less than 100 nm called nanoparticles. These particles are made up of metals such as (Al, Cu), oxides (Al2O3), carbides (SiC), nitrides (AlN, SiN) or nonmetals (Graphite, carbon nanotubes). Choi [1] experimentally verified that addition of small amount of these particles in the base fluid results in the appreciable increase in the effective thermal conductivity of the base fluid.

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Nomenclature

A, b	Constants
B	Magnetic field (T)
C	Nanoparticle volume fraction (kg m^{-3})
C_f	Skin friction coefficient
C_p	Specific heat at constant temperature(J kg^{-1} K)
D_B	Brownian diffusion (m^2 s^{-1})
D_T	Thermophoretic diffusion coefficient (m^2 s^{-1})
f	Dimensionless stream function
I	Identity tensor
k	Thermal conductivity
κ^*	Mean absorption coefficient (m^1)
M	Magnetic parameter
Nbt	Ratio of diffusivity
Nc	Ratio of heat capacities
Pr	Prandtl number
q_r	Radiative heat flux (W m^{-2})
R	Radiation Parameter
Sc	Schmidt number
S	Cauchy stress tensor (N m^{-2})
Sh	Local Sherwood number
U_w	Velocity of the stretching sheet
u, v	Velocity components in x and y directions (m s^{-1})
x, y	Cartesian coordinates
α	Wall thickness parameter
η	Similarity independent variable
λ_1	Williamson parameter
ε	Thermal conductivity parameter
ν	Kinematic viscosity (m^2 s^{-1})
ϕ	Dimensionless nanoparticle volume fraction
ψ	Stream function
σ	Electrical conductivity (Sm^{-1})
ρ	Density (kg m^{-3})
$(\rho c)_f$	Heat capacities of nanofluid
$(\rho c)_p$	Effective heat capacity of the nanoparticle

Subscripts

∞	Ambient condition
w	Conditions at the wall

Recently, researchers have used this concept of nanofluid as a route to enhance the performance of heat transfer rate in liquids. Non-homogeneous equilibrium model proposed by Buongiorno [2] reveals that this abnormal increase in the thermal conductivity occurs due to the presence of two main effects namely the Brownian motion and thermophoretic diffusion of nanoparticles. Excellent reviews on the flows of nanofluids have been conducted by Daungthongsuk and Wongwises [3], Wang and Mujumdar [4,5] and Kakac and Pramuanjaroenkij [6]. Boundary layer flow of nanofluid over a flat plate has been analysed by Kuznetsov and Nield [7]. In another paper, Nield and Kuznetsov [8] addressed the Cheng–Mincowcz problem for flow of nanofluid through a porous medium. Flow of nanofluid over a moving flat plate with uniform free stream has been investigated by Bachok et al. [9]. Rashidi et al. [10] investigate magnetic field effect on mixed convection heat transfer of nanofluid in a channel with sinusoidal walls. Recently, various attempts dealing with the boundary layer flow of nanofluid over stationary or moving surfaces have been made (see Khan and Pop [11], Rana and Bhargava [12], Makinde and Aziz [13] and Mustafa et al. [14] etc.). Ramesh et al. [15] studied MHD Stagnation Point Flow of Nanofluid Towards a Stretching Surface with the effects of variable thickness and thermal radiation. Heat transfer of a steady, incompressible water based nanofluid flow over a stretching sheet in the presence of magnetic field with thermal radiation and buoyancy effects are investigated by Rashidi et al. [16].

Numerous applications of boundary layer flow and heat transfer over a stretching sheet have been found in engineering processes such as in the extraction of polymer sheets, wire drawing, paper production, and glass-fibre production and thereby are considered significant. During the manufacturing process, a stretching sheet interacts with the ambient fluid both thermally and mechanically. The study of boundary layer flow caused by a stretching surface was initiated by Crane [17] who gave an exact similarity solution in closed form. Mahapatra and Gupta [18] reconsidered the steady stagnation point flow towards a stretching sheet taking different stretching and stagnation point velocities and observed two different kinds of boundary layer structure near the sheet. Mukhopadhyay [19] has studied the effects of Casson fluid flow and heat transfer over a nonlinearly stretching surface. Mahmud et al. [20] investigate the transient MHD laminar free convection flow of nanofluid past a vertical stretching surface. MHD Boundary-Layer Viscoelastic Fluid Flow over Continuously Moving Stretching Surface by considering PST and PHF case are studied by Rashidi et al. [21].

Williamson fluid is characteristic of a non-Newtonian fluid model with shear thinning property. This model was proposed by Williamson [22] and later on used by several authors (Dapra and Scarpi [23]; Vasudev et al. [24]; Nadeem and Akbar [25]; Nadeem and Hussain [26,27]) to investigate fluid flow by using Homotopy Analysis Method (OHAM) to solve the governing system of equation for Williamson nanofluid flow. Very recently Gorla and Gireesha [28] studied stagnation-point flow and heat transfer of a Williamson nanofluid on a linear stretching/shrinking sheet with convective boundary condition. Most of the above mentioned studies investigated the boundary layer flow and heat transfer analysis restricted for only flat stretching sheet. Study of flow and heat transfer of viscous fluids over stretching sheet with a variable thickness (non-flatness) can be more relevant to the situation in practical applications. For the first time Fang et al. [29] obtained an elegant analytical and numerical solution to the two-dimensional boundary layer flow due to a non-flatness stretching sheet. Further this problem was extended by Subhashini et al. [30] by including the energy equation and found that thermal boundary layer thicknesses for the first solution were thinner than those of the second solution. Numerical solution for the flow of a Newtonian fluid over a stretching sheet with a power law surface velocity, slip velocity and variable thickness was studied by Khader et al. [31]. Khader and Meghad [32] studied numerical solution for the flow of a Newtonian fluid over an impermeable stretching sheet embedded in a porous medium with the power law surface velocity and variable thickness in the presence of thermal radiation. Some recent studies on above model can be found in [33–36].

To the best of authors knowledge not much attention has been paid to investigate the flow and heat transfer characteristic of Williamson nanofluid over a variable thickness stretching sheet. Hence the problem studied here is an extension of the work done by Khader [31] wherein we have considered Williamson nanofluid over a stretching sheet with variable thickness under the influence of magnetic field, thermal radiation with variable thermal conductivity.

2. Mathematical formulation

Consider a MHD two-dimensional steady laminar flow of Williamson nanofluid over a stretching sheet. The origin is located at a slit through which the sheet is drawn through the fluid medium. The sheet is stretching with velocity $U_w = U_0(x+b)^m$. The x-axis is along the stretching surface in the direction of the sheet motion and the y-axis is perpendicular to it. Assume that the sheet is not flat, and its thickness varies as $y = A(x+b)^{\frac{1m}{2}}$, where A is a very small constant so that the sheet is sufficiently thin and m is the velocity power index, for $m = 1$ the problem reduces to flat sheet. Magnetic field B is applied along the transverse direction of flow. The fluid is assumed to be slightly conducting, so that the magnetic Reynolds number is much less than unity and hence the induced magnetic field is negligible in comparison to the applied magnetic. The coordinate system and flow regime are illustrated in Fig. 1. For Williamson fluid model Cauchy stress tensor $\mathbf{S}$ is defined in [23] as

$$\mathbf{S} = -p\mathbf{I} + \tau$$
$$\tau = \left(\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 - \Gamma\dot{\gamma}}\right) A_1$$

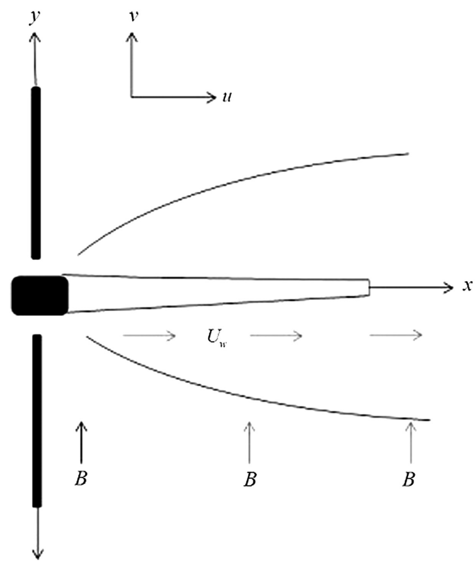


Fig. 1. Flow configuration and co-ordinate system.

where **S** is extra stress tensor, μ_0 is limiting viscosity at zero shear rate and μ_∞ is limiting viscosity at infinite shear rate, $\Gamma > 0$ is a time constant, A_1 is the first Rivlin–Erickson tensor and $\dot{\gamma}$ is defined as follows:

$$\dot{\gamma} = \sqrt{\frac{1}{2}\pi}$$

$$\pi = trace(A_1^2).$$

Here we considered the case for which $\mu_\infty = 0$ and $\Gamma\dot{\gamma} < 1$. Thus τ can be written as

$$\tau = \left(\frac{\mu_0}{1 - \Gamma\dot{\gamma}}\right) A_1.$$

By using binomial expansion we get

$$\tau = \mu_0 \left(1 - \Gamma\dot{\gamma}\right) A_1.$$

The two dimensional boundary layer equations governing the flow are given by [26]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \sqrt{2}\nu\Gamma\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho}u \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} + \frac{(\rho c)_p}{(\rho c)_f}\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} \tag{4}$$

with the boundary conditions [26]

$$u = u_w(x) = U_0(x+b)^m, \quad v = 0, \quad T = T_w, \quad C = C_w \quad at \quad y = (x+b)^{\frac{1-m}{2}}, \tag{5a}$$

$$u = 0, \quad T = T_\infty, \quad C = C_\infty \quad at \quad y \to \infty \tag{5b}$$

where u and v are the velocity components in the x and y directions, respectively. Further, ρ is the density, g is the force of gravity, μ is the viscosity, ν is the kinematic viscosity, C_p is the specific heat at constant pressure, B is the magnetic field. T and C are fluid temperature and nanoparticle fraction, respectively. T_w and T_∞ are the temperature of the fluid at the wall and ambient temperature when $y \to \infty$. D_B and D_T are respectively the Brownian diffusion coefficient and thermophoretic diffusion coefficient. $\tau = \frac{(\rho c)_p}{(\rho c)_f}$ is the ratio between the effective heat capacity of the nanoparticles material and heat capacity of the fluid. κ is the temperature dependent thermal conductivity. We consider the temperature dependent thermal conductivity in the following form Chaim [37]

$$\kappa = \kappa_\infty \left[1 + \varepsilon \frac{T - T_\infty}{T_w - T_\infty}\right]. \tag{6}$$

The Rosseland approximation for radiation is

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}, \tag{7}$$

where σ^* and k^* are the Stefan–Boltzmann constant and the mean absorption coefficient, respectively. It is assumed that the temperature differences within the flow, such as the term T^4, may be expressed as a linear function of temperature.We get the Taylor series expansion for T^4 at a free stream temperature T_∞ after neglecting higher-order terms as

$$T^4 = 4T_\infty^4 T - 3T_\infty^4. \tag{8}$$

Using (7) and (8), we obtain

$$\frac{\partial q_r}{\partial y} = -\frac{16\sigma^* T_\infty^3}{3k^*}\frac{\partial^2 T}{\partial y^2}. \tag{9}$$

Introducing the following similarity transformations

$$\begin{gathered} \eta = \sqrt{\frac{U_0(m+1)}{2\nu}}\left(y(x+b)^{\frac{m-1}{2}} - A\right),\ \psi = \sqrt{\frac{2\nu U_0}{m+1}}(x+b)^{\frac{m+1}{2}} f, \\ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty},\ \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}. \end{gathered} \tag{10}$$

Using the above similarity transformations, the governing equations Eqs. (1)–(4) are reduced to

$$f''' + \lambda_1 f'' f''' + f f'' - \frac{2m}{m+1} f'^2 - M f' = 0 \tag{11}$$

$$\left(1 + \frac{4R}{3}\right)\left((1+\varepsilon\theta)\,\theta'' + \varepsilon\theta'^2\right) + Prf\theta' + \frac{Nc}{Le}\theta'\phi' + \frac{Nc}{LeNbt}\theta'^2 = 0 \tag{12}$$

$$\phi'' + LePrf\phi' + \frac{1}{Nbt}\theta'' = 0. \tag{13}$$

Using Eq. (9), the boundary conditions become,

$$f' = 1,\ \ f = \alpha\frac{1-m}{m+1},\ \ \theta = 1,\ \ \phi = 1 \quad \text{at} \quad \eta = 0 \tag{14a}$$

$$f' \to 0,\ \ \theta \to 0,\ \ \phi \to 0 \quad \text{as} \quad \eta \to \infty \tag{14b}$$

where $\alpha = A\sqrt{\frac{U_0(m+1)}{2\nu}}$ is the wall thickness parameter, $M = \frac{2\sigma B_0^2}{U_0\rho(m+1)}$ is the magnetic field parameter, $Pr = \frac{\nu}{\alpha}$ is Prandtl number, ϵ is the thermal conductivity parameter, $R = \frac{4\sigma^* T_\infty^3}{k_\infty k^*}$ is the radiation parameter, $Nc = \frac{\rho_p c_p}{\rho c}(C_w - C_\infty)$(nanoparticles heat capacity/nanofluid heat capacity), $Nbt = \frac{D_B T_\infty (C_w - C_\infty)}{D_T (T_w - T_\infty)}$ (Brownian diffusivity/thermophoretic diffusivity), $Le = \frac{\alpha}{D_B}$ the Lewis number and $\lambda_1 = \Gamma\sqrt{\frac{U_0^3 (x+b)^{3*m-1}(m+1)}{\nu}}$ is Williamson fluid parameter. Expressions for the local skin friction co-efficient C_{f_x} and local Nusselt number $Nu_x Sh_x$ are defined as,

$$C_{f_x} = \frac{\tau_w}{\rho U_w^2},\ \ Nu_x = \frac{Xq_w}{k_\infty(T_w - T_\infty)} \quad \text{and} \quad Sh_x = \frac{Xq_m}{k(T_w - T_\infty)} \tag{15}$$

where k is the thermal conductivity of the nanofluid, and q_w and q_m are the heat flux and mass flux, respectively given by

$$q_w = -k_\infty \left(\frac{\partial T}{\partial y}\right)_{y=0}, q_m = -D_B \left(\frac{\partial C}{\partial y}\right)_{y=0}. \tag{16}$$

Applying similarity transformations (6) for skin friction coefficient, Nusselt number and Sherwood numbers are converted to

$$\begin{aligned}
&Re_x^{1/2} C_{f_x} = \sqrt{\frac{(m+1)}{2}} \left(f''(0) + \frac{\lambda}{2} f''(0)^2\right),\\
&Re_x^{-1/2} Nu_x = -\sqrt{\frac{(m+1)}{2}} \left(1 + \frac{4R}{3}\right) \theta'(0),\\
&Re_x^{-1/2} Sh_x = -\sqrt{\frac{(m+1)}{2}} \phi'(0)
\end{aligned} \tag{17}$$

where $Re_x = U_w(x) X/\upsilon$ is local Reynolds number and $X = x + b$.

3. Method of solution

The nonlinear coupled ordinary differential equations (11)–(13) subject to the boundary conditions (14) have been solved numerically using the spectral quasilinearization method (SQLM). This quasilinearization method (QLM) is a generalisation of the Newton–Raphson method and was first proposed by Bellman and Kalaba [38] for solving nonlinear boundary value problems. The quasilinearisation method is employed to linearise the equations before they are solved iteratively using the Chebyshev spectral collocation method. Applying the quasilinearisation procedure on Eqs. (11)–(14), the resultant equations are

$$\alpha_{1,r} f'''_{r+1} + \alpha_{2,r} f''_{r+1} + \alpha_{3,1} f'_{r+1} + \alpha_{4,r} f_{r+1} = R_1, \tag{18}$$

$$\beta_{1,r} f_{r+1} + \beta_{2,r} \theta''_{r+1} + \beta_{3,r} \theta'_{r+1} + \beta_{4,r} \theta_{r+1} + \beta_{5,r} \phi_{r+1} = R_2, \tag{19}$$

$$\gamma_{1,r} f_{r+1} + \gamma_{2,r} \theta''_{r+1} + \gamma_{3,r} \phi''_{r+1} + \gamma_{4,r} \phi'_{r+1} = R_3, \tag{20}$$

and the boundary conditions are

$$\begin{aligned}
&f_{r+1} = \alpha \frac{1-m}{1+m}, \quad f'_{r+1} = 1, \quad \theta_{r+1} = 1, \quad \phi_{r+1} = 0, \text{ at } \eta = 0,\\
&f'_{r+1} = 0, \quad \theta_{r+1} = 0, \quad \phi_{r+1} = 0, \text{ at } \eta \to \infty
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&\alpha_{1,r} = \left(1 + \lambda_1 f''_r\right), \quad \alpha_{2,r} = f_r + \lambda_1 f'''_r, \quad \alpha_{3,r} = -M - 4\frac{m}{m+1} f'_r, \quad \alpha_{4,r} = f''_r,\\
&\beta_{1,r} = Pr\theta'_r, \quad \beta_{2,r} = \left(1 + \frac{4R}{3}\right)(1 + \varepsilon\theta_r),\\
&\beta_{3,r} = \left(1 + \frac{4R}{3}\right) 2\varepsilon\theta'_r + \frac{Nc}{Le}\phi'_r + 2\frac{Nc}{LeNbt}\theta'_r + Prf_r, \quad \beta_{4,r} = \left(1 + \frac{4R}{3}\right)\varepsilon\theta''_r,\\
&\beta_{5,r} = \frac{Nc}{Le}\theta'_r,\\
&\gamma_{1,r} = PrLe\phi'_r, \quad \gamma_{2,r} = \frac{1}{Nbt}, \quad \gamma_{3,r} = 1, \quad \gamma_{4,r} = LePrf_r\\
&R_1 = \lambda_1 f''_r f'''_r - \frac{2m}{m+1}(f'_r)^2 + f_r f''_r,\\
&R_2 = \left(1 + \frac{4R}{3}\right)\varepsilon\theta_r\theta''_r + \left(1 + \frac{4R}{3}\right)\varepsilon(\theta'_r)^2 + Prf_r\theta'_r + \frac{Nc}{LeNbt}(\theta'_r)^2 + \frac{Nc}{Le}\theta'_r\phi'_r,\\
&R_3 = LePrf_r\phi'_r.
\end{aligned}$$

The above system (19)–(21) constitute a linear system of coupled differential equations with variable coefficients and can be solved iteratively using any numerical method for $r = 1, 2, 3, \ldots$. In this work, as we discussed below, the Chebyshev spectral collocation method was used to solve the QLM scheme (19)–(21) (for more details, refer to the works of Motsa et al. [39]). Before applying the spectral method, it is convenient to transform the domain in the η direction is approximated to $[0, L]$ where L is the edge of the boundary limit (large enough), use the transformation of algebraic mapping $\eta = \frac{(\tau+1)L}{2}$ to map the physical domain in to the computational domain $[-1, 1]$. This basic idea of this method is approximating the unknown functions by the Chebyshev interpolating polynomials in such a way that they are collocated at the Gauss–Lobatto points defined as

$$\tau_i = \cos\left(\frac{\pi i}{N}\right), \qquad -1 \le \tau \ge 1, \qquad i = 0, 1, 2, \ldots, N \tag{22}$$

where N is the number of collocation points. The derivative of f_{r+1} at the collocation points is represented as

$$\frac{\partial^p f_{r+1}}{\partial \eta^p} = \left(\frac{2}{L}\right)^p \sum_{k=0}^{N} D^p_{N,k} f_{r+1}(\tau_k) = \mathbf{D}^p \mathbf{F} \tag{23}$$

where $\mathbf{D} = \frac{2}{L} D$ and D is the Chebyshev spectral differentiation matrix (for details [40]), $\mathbf{F} = [f(\tau_0), f(\tau_1), \ldots f(\tau_N)]$. Similarly the derivatives of θ, and ϕ are given by $\theta^p = \mathbf{D}^p \Theta$ and $\phi^p = \mathbf{D}^p \Phi$ where p is the order of derivative, and $\mathbf{D}$ is the matrix of order $(N + 1) \times (N + 1)$. Substituting (23)–(24) in Eqs. (19)–(21) we obtain

$$\left[\boldsymbol{\alpha}_{1,r}\mathbf{D}^3 + \boldsymbol{\alpha}_{2,r}\mathbf{D}^2 + \boldsymbol{\alpha}_{3,r}\mathbf{D} + \boldsymbol{\alpha}_{4,r}\right]\mathbf{F}_{r+1} = \mathbf{R}_1, \tag{24}$$

$$\left[\boldsymbol{\beta}_{1,r}\mathbf{D}\right]\mathbf{F}_{r+1} + \left[\boldsymbol{\beta}_{2,r}\mathbf{D}^2 + \boldsymbol{\beta}_{3,r}\mathbf{D} + \boldsymbol{\beta}_{4,r}\right]\boldsymbol{\Theta}_{r+1} + \left[\boldsymbol{\beta}_{5,r}\mathbf{D}\right]\boldsymbol{\Phi}_{r+1} = \mathbf{R}_2, \tag{25}$$

$$\left[\boldsymbol{\gamma}_{1,r}\right]\mathbf{F}_{r+1} + \left[\boldsymbol{\gamma}_{2,r}\mathbf{D}^2\right]\boldsymbol{\Theta}_{r+1} + \left[\boldsymbol{\gamma}_{3,r}\mathbf{D}^2 + \boldsymbol{\gamma}_{4,r}\mathbf{D}\right]\boldsymbol{\Phi}_{r+1} = \mathbf{R}_3. \tag{26}$$

Applying spectral method on the boundary conditions gives

$$f_{r+1}(\tau_N) = \alpha\frac{1-m}{1+m}, \quad \sum_{k=0}^{N} D_{N,k} f_{r+1}(\tau_k) = 1, \quad \theta_{r+1}(\tau_N) = 1, \quad \phi_{r+1}(\tau_N) = 1, \tag{27}$$

$$\sum_{k=0}^{N} D_{0,k} f_{r+1}(\tau_k) = 0, \quad \theta_{r+1}(\tau_0) = 0, \quad \phi_{r+1}(\tau_0) = 0. \tag{28}$$

The above system of equations can be written in the matrix form as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{r+1} \\ \boldsymbol{\Theta}_{r+1} \\ \boldsymbol{\Phi}_{r+1} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix} \tag{29}$$

where

$$A_{11} = diag[\alpha_{1,r}]\mathbf{D}^3 + diag[\boldsymbol{\alpha}_{2,r}]\mathbf{D}^2 + diag[\boldsymbol{\alpha}_{3,r}]\mathbf{D} + diag[\boldsymbol{\alpha}_{4,r}]\boldsymbol{I},$$
$$A_{12} = diag[\boldsymbol{\alpha}_{5,r}]\mathbf{D} + diag[\boldsymbol{\alpha}_{6,r}]\boldsymbol{I},$$
$$A_{13} = diag[\boldsymbol{\alpha}_{7,r}]\boldsymbol{I},$$
$$A_{21} = diag[\boldsymbol{\beta}_{1,r}]\mathbf{D} + diag[\boldsymbol{\beta}_{2,r}]\boldsymbol{I},$$
$$A_{22} = diag[\boldsymbol{\beta}_{3,r}]\mathbf{D}^2 + diag[\boldsymbol{\beta}_{4,r}]\mathbf{D} + diag[\boldsymbol{\beta}_{5,r}]\boldsymbol{I},$$
$$A_{23} = diag[\boldsymbol{\beta}_{6,r}]\mathbf{D},$$
$$A_{31} = diag[\boldsymbol{\gamma}_{1,r}]\boldsymbol{I},$$
$$A_{32} = diag[\boldsymbol{\gamma}_{2,r}]\mathbf{D}^2,$$
$$A_{33} = diag[\boldsymbol{\gamma}_{3,r}]\mathbf{D}^2 + diag[\boldsymbol{\gamma}_{4,r}]\mathbf{D},$$

Table 1
Computational time to compute $f''(0)$ and $-\theta'(0)$ when $L = 20$.

Iterations	N	$f''(0)$	$-\theta'(0)$	Time
10	60	−1.675061856704815	−0.391980686936591	0.063
50	60	−1.675061856721641	−0.391980686934517	0.205
500	60	−1.675061856703678	−0.391980686936421	1.566
100	5	−0.407168548592312	−0.430424821728138	0.069
100	10	−1.129453810591957	−0.428313095185532	0.086
100	15	−1.528592346289352	−0.396286877252487	0.102
100	20	−1.622932208457137	−0.392879499783479	0.118
100	30	−1.667391603629412	−0.392040943052731	0.148
100	60	−1.675061856697084	−0.391980686936577	0.357
100	100	−1.675081390323612	−0.391980647682374	0.767
100	200	−1.675081384571968	−0.391980647558285	3.018

Table 2
Comparison results for skinfriction $-f''(0)$ for m values when $\alpha_1 = 0.5$, $M = 0$ and $\lambda_1 = 0$.

m	Fang [29]	Khader[31]	Present results
10	1.0603	1.0603	1.0603432
9	1.0589	1.0588	1.0589342
7	1.0550	1.0551	1.0550628
5	1.0486	1.0486	1.0486285
3	1.0359	1.0358	1.0358835
2	1.0234	1.0234	1.0234206
1	1.0000	1.0000	1.0000084
0.5	0.9799	0.9798	0.9799497
0	0.9576	0.9577	0.9576443

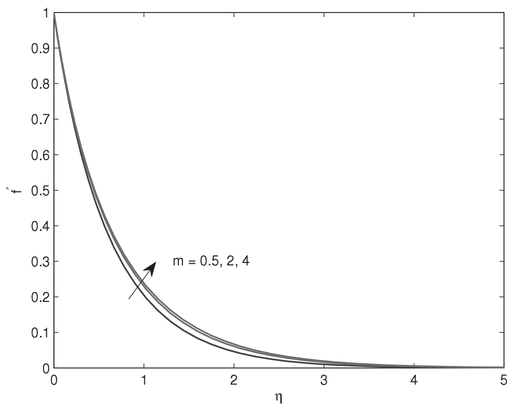
Table 3
Comparison results for $-\theta'(0)$ when $m = 0.5$, $\alpha = 0.2$, $Pr = 1$, $\varepsilon = 0.1$ and $R = 0.375$.

m	α	Pr	ε	R	Khader [31]	Present results
0.5	0.2	1	0.1	0.375	0.441 845 1	0.441838369

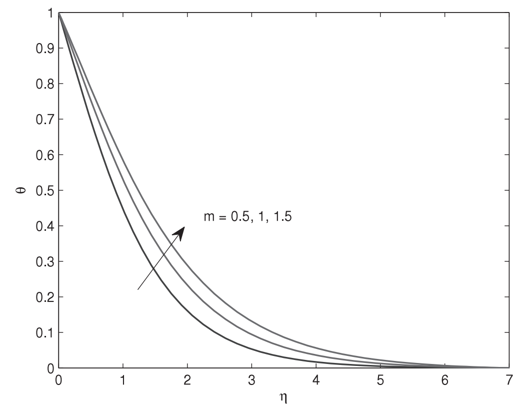
where α, β and γ are $(N + 1) \times (N + 1)$ diagonal matrices, $\mathbf{I}$ is a $(N + 1) \times (N + 1)$ unit matrix. The approximate solutions for $\mathbf{F}$, $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$ are obtained by solving the matrix system (30). In this spectral method, a finite computational domain of extent $L = 20$ was chosen in the η-direction. Through numerical computation, this value was found to give accurate results for all the selected governing physical parameters used in the generation of results. Increasing the value of η did not change the results to a significant extent. The number of collocation points used in the spectral method discretisation was $N = 70$ in all cases. We remark that the SQLM algorithm is based on the computation of the value of some quantity, say F_{r+1}^{n+1}, at each time step. This is achieved by iterating using the quasilinearization method using a known value at the previous time step, n as initial approximation. The iteration calculations are carried out till the desired tolerance level 10^{-6} is attained, and the computational time is also given in the Table 1.

4. Results and discussion

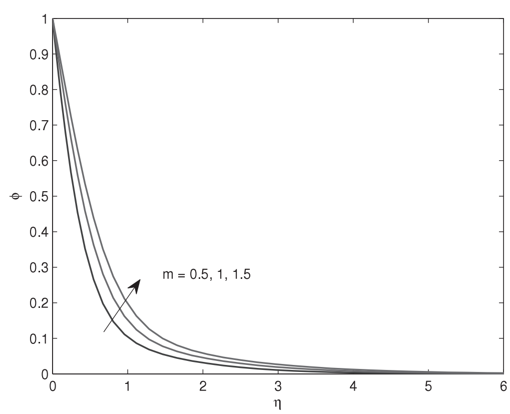
Numerical solution for the effects of physical parameters on magneto hydrodynamic boundary layer flow of Williamson nanofluid over a stretching sheet with variable thickness and variable thermal conductivity are investigated. The computational results obtained by the present method (spectral quasi-linearisation method) are compared with the available results of Fang [29] and Khader [31] for some limiting conditions. The present results is found to be in good agreement as shown in Tables 2 and 3. The effects of various physical parameters such as velocity



(a) Velocity.



(b) Temperature.



(c) Nanoparticle volumefraction.

Fig. 2. Effect of m on velocity, temperature and nanoparticle volumefraction profiles.

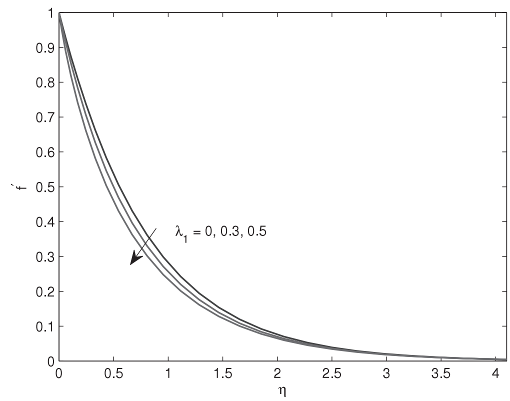
power index parameter m, Williamson parameter λ_1, magnetic field parameter M, wall thickness parameter α, thermal conductivity parameter ε, radiation parameter R, Prandtl number Pr, heat capacity parameter Nc, diffusion parameter Nbt and Lewis number Le on velocity (f'), temperature (θ) and nanoparticle volume fraction (ϕ) profiles are shown in Figs. 2–15.

Fig. 2 illustrates the effects of velocity power index parameter m on velocity, temperature and nanoparticle volume fraction profiles respectively. It is noticed that increase in velocity power index parameter m increases the velocity, temperature and nanoparticle volume fraction profiles. This implies that momentum boundary layer thickness and thermal boundary layer thickness become smaller as m increases.

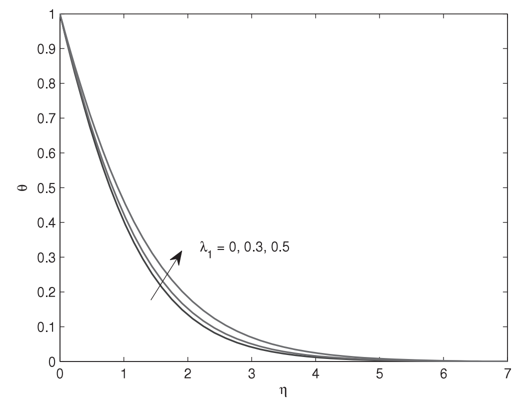
Figs. 3(a)–3(c) show the effects of Williamson parameter λ_1 on velocity, temperature and nanoparticle volume fraction profiles respectively. The effect of Williamson parameter λ_1 reduces the velocity profiles while, increases the temperature and nanoparticle volume fraction profiles.

From Fig. 4 it is observed that the influence of magnetic field parameter M is to reduce the velocity profiles as well as thermal boundary layer thickness. It is also noticed that the temperature and nanoparticle volume fraction profiles increase with the increase of M. This is due to when the value of M increases it exits fluid particles motion which will diffuses quickly into the neighbouring fluid layers.

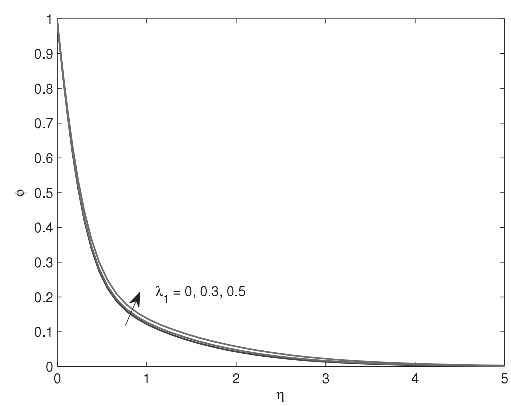
Fig. 5 describes the effect of wall thickness parameter α on velocity, temperature and nanoparticle volume fraction profiles for the cases $m > 1$ and $m < 1$. It is noticed that the velocity near the plate decreases as the thickness parameter α increases for $m < 1$ and reverse is true for $m > 1$. It is also noted that the increase in wall thickness parameter α causes a reduction in thermal boundary layer near the plate for $m < 1$ while, the reverse phenomena is noticed for $m > 1$. It is obvious from Fig. 5(c) that the nanoparticle volume fraction profiles reduce with increase in



(a) Velocity.



(b) Temperature.



(c) Nanoparticle volumefraction.

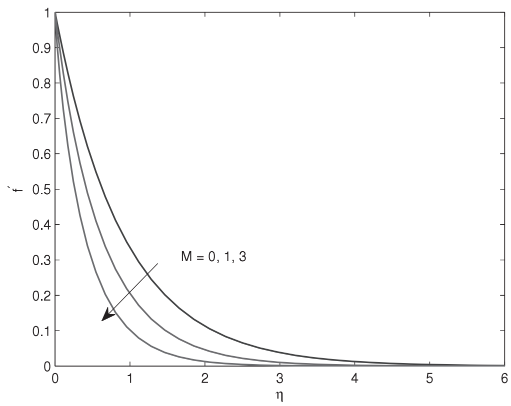
Fig. 3. Effect of λ_1 on velocity, temperature and nanoparticle volumefraction profiles.

wall thickness parameter α for $m < 1$, and increase for $m > 1$. It is obvious that the boundary layer becomes thinner for higher values of α for $m < 1$ while thicker for $m > 1$. This is due to induced mass transfer.

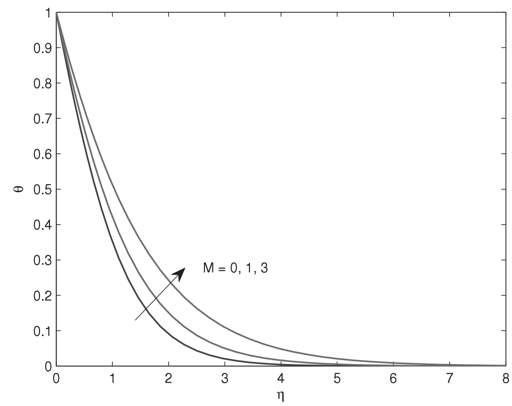
The effect of variable thermal conductivity parameter ε on temperature and nanoparticle volume fraction profiles are shown in Figs. 6(a) and 6(b) respectively. It is noticed that effect of variable thermal conductivity parameter ε is to enhance the temperature profiles significantly, while the reverse phenomena is observed for nanoparticle volume fraction profiles. Therefore, the assumption of temperature dependent thermal conductivity suggests a reduction in the magnitude of the transverse velocity by a quantity $\frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right)$ in Eq. (3). The rate of cooling is much faster for the coolant material having small thermal conductivity parameter.

Fig. 7 is drawn to examine the effects of radiation parameter R on temperature and nanoparticle volume fraction profiles. It is noted that the temperature distribution enhance significantly with the increase of R because an increase in the radiation parameter provides more heat to fluid that causes an enhancement in the temperature and thermal boundary layer thickness. As the radiation parameter R increases the nanoparticle volume fraction profiles decrease as seen in Fig. 7(b). In Figs. 8(a)–8(b) temperature and nanoparticle volume fraction profiles are evaluated at different values of Prandtl number Pr. It is observed that as the Prandtl number increases both temperature and nanoparticle volume fraction decrease. Since by definition of Pr, thus by increasing Prandtl number Pr thermal conductivity of fluid decreases which decreases temperature profile. Additionally, an increase in Prandtl number Pr reduces thermal boundary layer thickness. So heat rapidly transfers which causes a drop in fluid temperature.

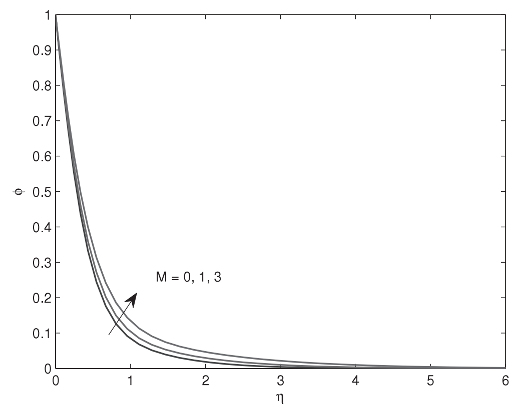
The effect of Lewis number Le and diffusion parameter Nbt on temperature and nanoparticle volume fraction profiles are plotted in Figs. 9 and 10. It is found that with increase in Le and Nbt decreases the temperature profiles. It is also noted that the thermal boundary layer thickness decreases with increase in Le and Nbt. Nanoparticle volume



(a) Velocity.



(b) Temperature.



(c) Nanoparticle volumefraction.

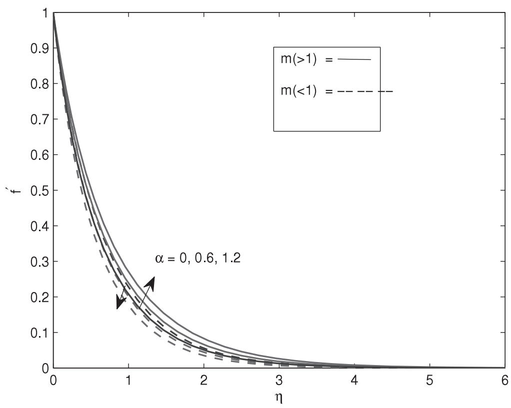
Fig. 4. Effect of Magnetic parameter M on velocity, temperature and nanoparticle volumefraction profiles.

fraction profiles decreases with the increase in Le and Nbt significantly. Since Nbt is the ratio of Brownian diffusivity to thermophoretic diffusivities, increase in Nbt means greater activity of nanofluid particles. Physically Le cannotbe equal to zero since it is the ratio of thermal diffusivity to Brownian diffusion. It is observed that temperature profile and thermal boundary layer decrease with increase in Nbt. When Brownian diffusivity is very large as compared to thermophoretic diffusivity, temperature profiles show only very small variation.

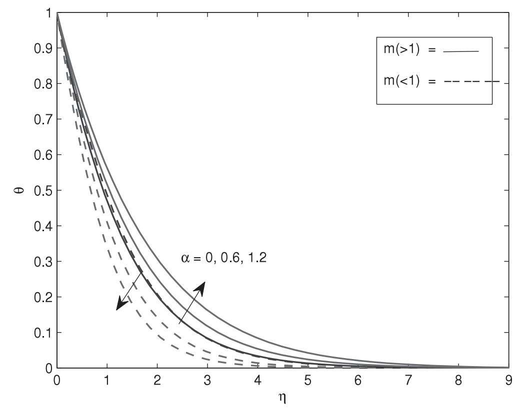
Fig. 11 depicts the effects of Nc on temperature profiles. It is found that an increase in Nc enhances temperature profiles and hence thicker the boundary layer thickness. If we look at the definition of Nc, it is the ratio of heat capacity of nanoparticles and nanofluid. Usually the specific heat c_p of nanoparticles is less than that of liquids. So addition of solid particles will decrease the specific heat of base fluid, hence temperature profile decreases.

Table 4 shows that effects of skin friction co-efficient for different values of wall thickness parameter α, M, λ_1 and m. Increasing the wall thickness parameter leads to an increase in the local skin-friction coefficient α while decreases with Williamson parameter λ_1. Skin friction co-efficient increases with increased values of magnetic field parameter M and velocity power index parameter m. Fig. 12 shows that local Nusselt number decreases by uplifting the magnetic parameter M, velocity power index m, but $-\theta'(0)$ increases with increase in wall thickness parameter. Fig. 13 explores the variation of local Nusselt number for various values of υ and R, local nusselt number decreases with increase in thermal conductivity parameter υ and radiation parameter R.

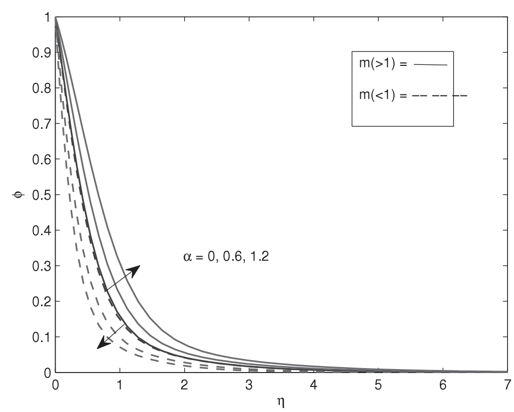
Impacts of Nc, Nbt and Le on local Nusselt number and local Sherwood number are shown in Figs. 14 and 15 respectively. Fig. 14 exhibits that as the values of Nc increase the local Nusselt number decreases. However it enhances as the values of Nbt and Le. It is observed from Fig. 15 that heat capacity parameter Nc boosts the growth of local Sherwood number, also same trend is observed with Nbt and Le.



(a) Velocity.

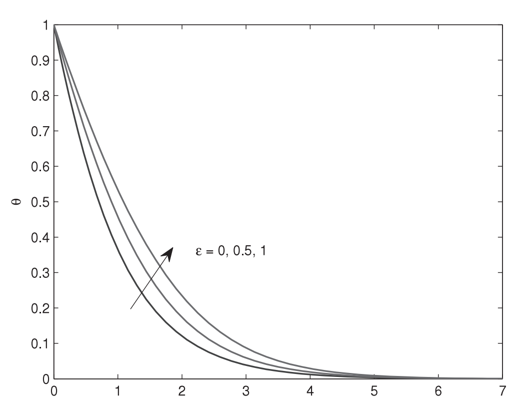


(b) Temperature.

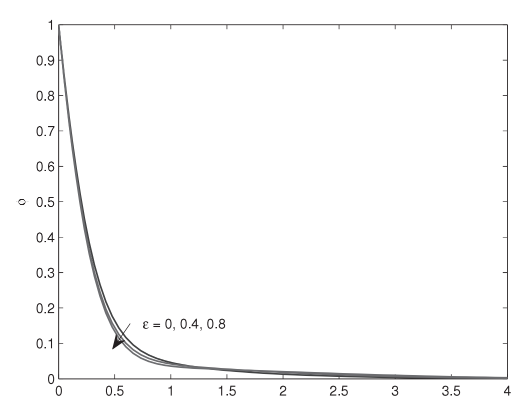


(c) Nanoparticle volume fraction.

Fig. 5. Effect of α on velocity, temperature and nanoparticle volumefraction profiles.



(a) Temperature.

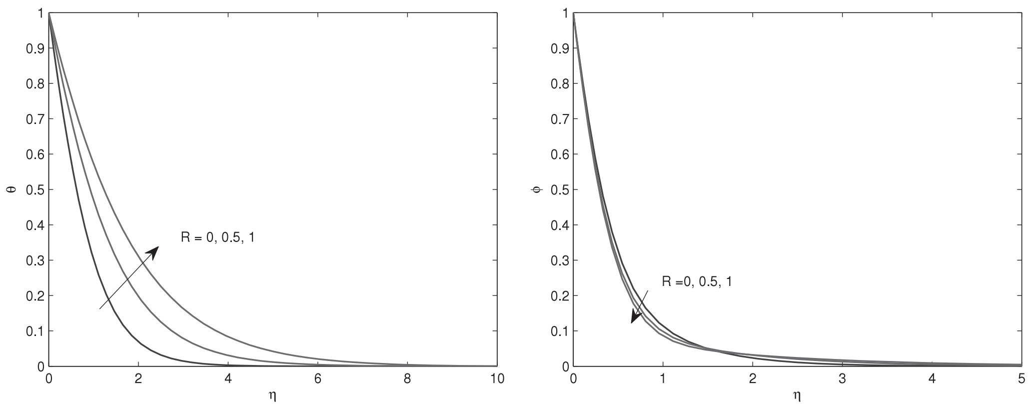


(b) Nanoparticle volumefraction.

Fig. 6. Effect of ϵ on temperature and nanoparticle volumefraction profiles.

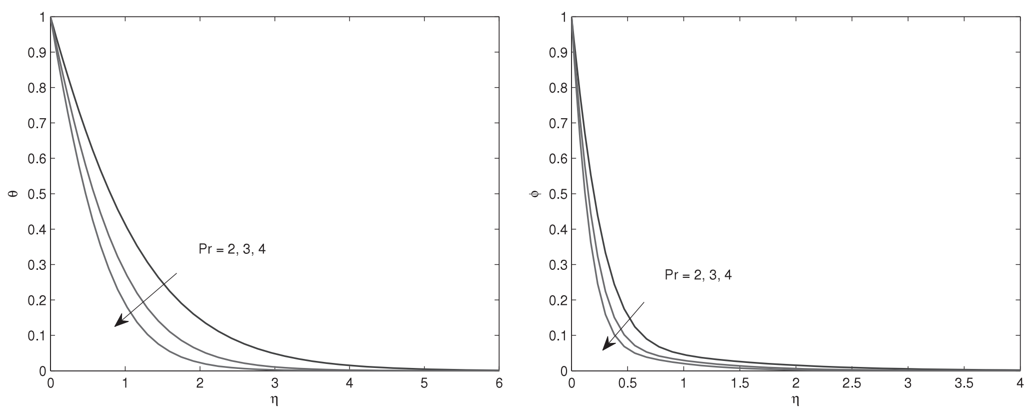
5. Conclusions

In the present study we investigated the influence of thermal radiation on MHD boundary layer flow of Williamson nanofluid over a stretching sheet with variable thickness and variable thermal conductivity. The governing equations



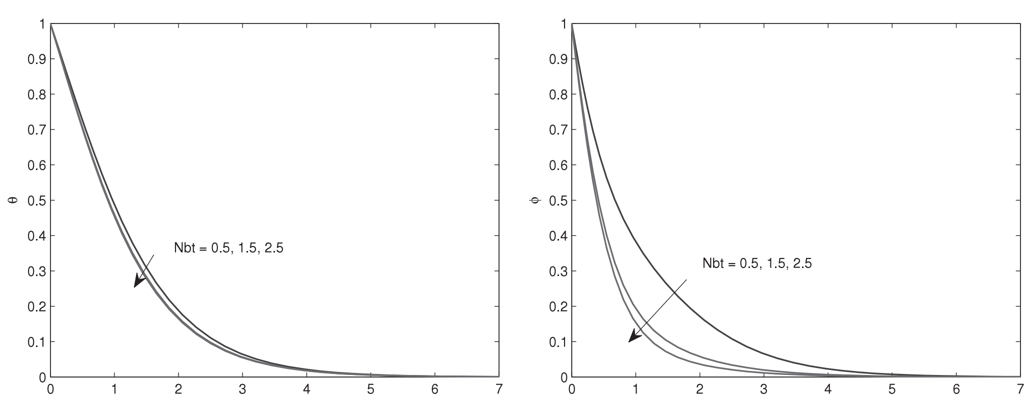
(a) Temperature. (b) Nanoparticle volumefraction.

Fig. 7. Effect of R on temperature and nanoparticle volumefraction profiles.



(a) Temperature. (b) Nanoparticle volume fraction.

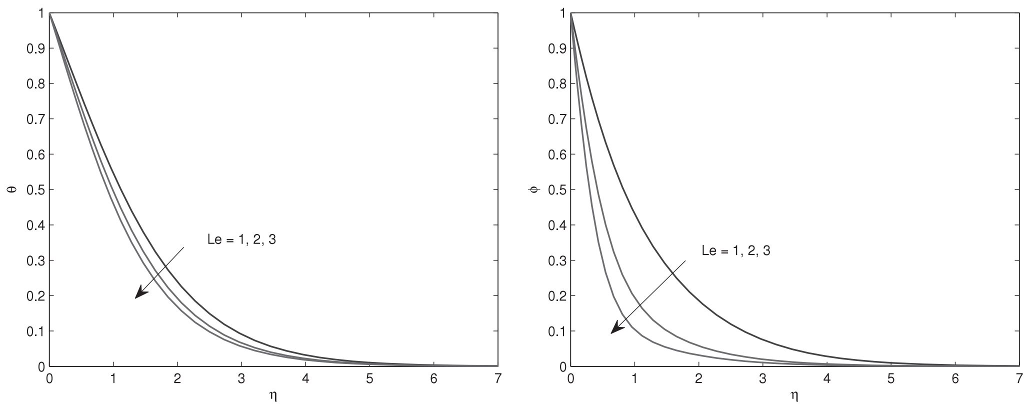
Fig. 8. Effect of Pr on temperature and nanoparticle volumefraction profiles.



(a) Temperature. (b) Nanoparticle volume fraction.

Fig. 9. Effect of Nbt on temperature and nanoparticle volumefraction profiles.

were transformed to the corresponding ordinary differential equations by using appropriate similarity transformations. These ordinary differential equations were further solved numerically by spectral quasilinearization method. From the numerical results obtained, some of the interesting conclusions are as follows:



(a) Temperature. (b) Nanoparticle volume fraction.

Fig. 10. Effect of Le on temperature and nanoparticle volumefraction profiles.

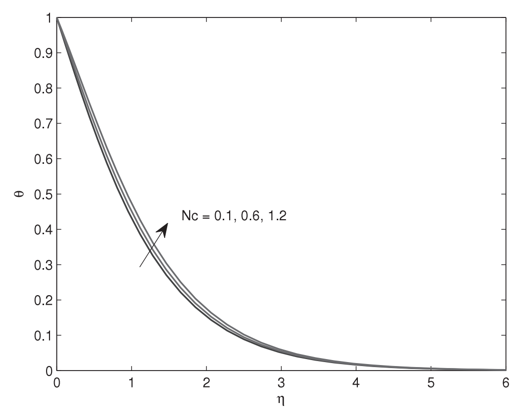


Fig. 11. Temperature profiles for various values of Nc.

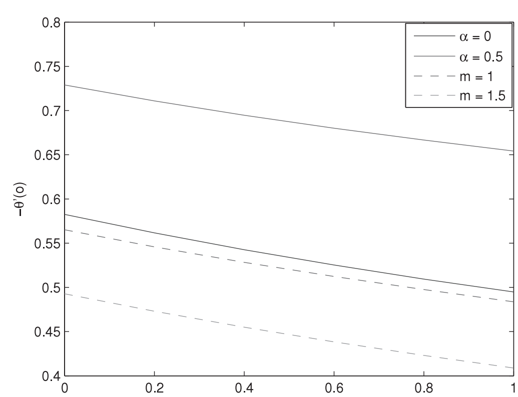


Fig. 12. Heat transfer co-efficient for various values of Lewis number M, m and α.

- Velocity profile decreases for increase in wall thickness parameter when $m < 1$, reverse trend can be seen for $m > 1$.
- Increasing magnetic field parameter M decreases the velocity profiles whereas increase the temperature and nanoparticle volume fraction profiles.

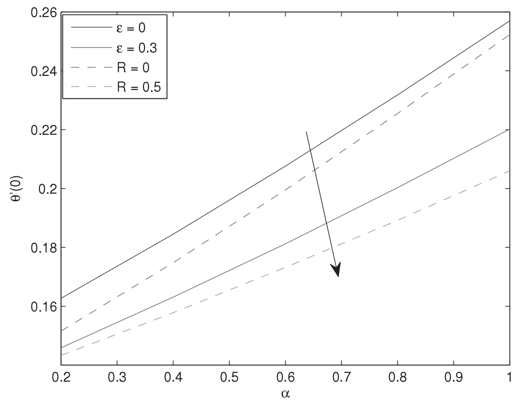


Fig. 13. Heat transfer co-efficient for various values of R, α and ϵ.

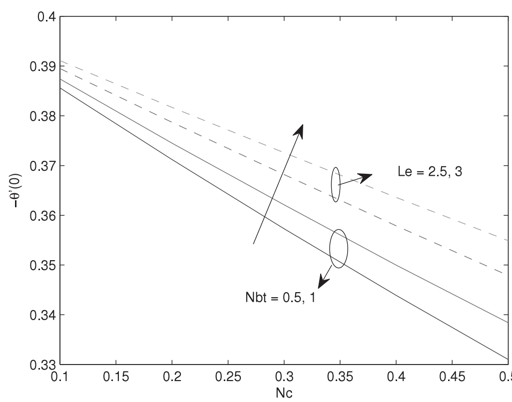


Fig. 14. Heat transfer co-efficient for various values of Lewis number Le, Nbt and Nc.

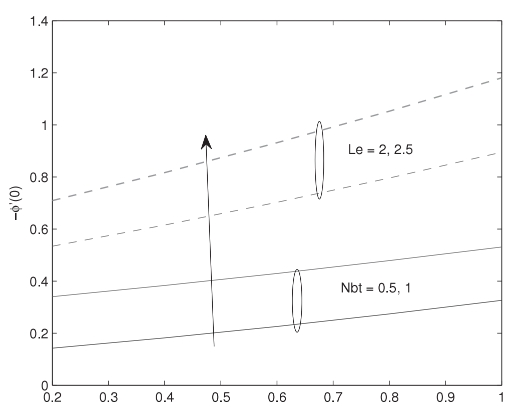


Fig. 15. Mass transfer co-efficient for various values of Lewis number Le, Nbt and Nc.

- Increase in Williamson parameter λ_1 decreases the velocity where as temperature and nanoparticle volume fraction profiles increase.
- With the increase of radiation parameter R the nanoparticle volume fraction decreases in the boundary region and the opposite effect is seen far away from the boundary sheet.
- The effect of variable thermal conductivity is to decrease the heat transfer co-efficient $-\theta'(0)$,
- With the effect of Nbt, Nc and Le the co-efficient of sherwood number $-\phi'(0)$ is increase.

Table 4
Skinfriction co-efficient for different values of M, m, α and λ_1.

M	m	λ_1	α	$-\left(f''(0)+\frac{\lambda_1}{2}f''^2\right)$
0	0.4	0.3	0.3	0.8760
0.5				1.0946
1				1.2661
1	0.2			1.2542
	0.4			1.2661
	0.6			1.2751
	0.4	0		1.3796
		0.3		1.3466
		0.6		1.3095
		0.3	0	1.2043
			0.2	1.2452
			0.4	1.2873

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Original article

On non-absolute type spaces and their Köthe-Toeplitz duals

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Abstract

In this paper, we introduce and study some non-absolute type spaces $l_\infty(u, \lambda, \Delta_v^m)$, $c_0(u, \lambda, \Delta_v^m)$ and $c(u, \lambda, \Delta_v^m)$, which are BK-spaces. Moreover, we prove that these spaces are linearly isomorphic to the spaces l_∞, c_0 and c. We also make an effort to establish some inclusion relations between these spaces. Furthermore, we find the Schauder basis for these spaces and also determine the α-, β- and γ-duals of these spaces.

Keywords: Sequence spaces; Difference sequence spaces; BK-spaces; α-, β- and γ- duals; Schauder basis

1. Introduction and preliminaries

Let w be the vector space of all real or complex sequences. Any vector subspace of w is called a sequence space. We shall write c, c_0 and l_∞ for the sequence spaces of all convergent, null and bounded sequences. Moreover, we write bs, cs, l_1 and l_p for the spaces of all bounded, convergent, absolutely and p-convergent series, respectively. Let X be a sequence space. If X is a Banach Space and

$$\tau_k : X \to C, \qquad \tau_k(x) = x_k \ \ (k = 1, 2 \dots)$$

is a continuous for all k, X is called a BK-space.

The sequence spaces c, c_0 and l_∞ are BK-spaces with the norm given by

$$\|x\|_\infty = \sup_k |x_k| \text{ for all } k \in \mathbb{N}.$$

Also, we use the conventions that $e = (1, 1, 1, \dots)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$.

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Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix transformation from X into Y and we denote it by writing $A : X \to Y$ if for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$ and the A-transform of x is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \quad (n \in \mathbb{N}). \tag{1}$$

By (X, Y) we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right-hand side of (1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax \in Y$ for all $x \in X$ (see, [1]). The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}. \tag{2}$$

The notion of difference sequence spaces was introduced by Kızmaz [2], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [3] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Let m, n be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1}x_k - \Delta_n^{m-1}x_{k+1})$ and $\Delta_n^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+nv}.$$

Taking $n = 1$, we get the spaces which were studied by Et and Çolak [3]. Taking $m = n = 1$, we get the spaces which were introduced and studied by Kızmaz [2]. For more details about sequence spaces see, [4–7], etc.

We shall denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors. They introduced the sequence spaces $(l_\infty)_{N_q}$ and c_{N_q} (see, [8]), $(l_p)_{C_1} = X_p$ and $(l_\infty)_{C_1} = X_\infty$ (see, [9]), $(l_\infty)_{R^t} = r_\infty^t$, $(c)_{R^t} = r_c^t$ and $(c_0)_{R^t} = r_0^t$ (see, [10]), $(l_p)_{R^t} = r_p^t$ (see, [11]), $(c_0)_{E^r} = e_0^r$ and $(c)_{E^r} = e_c^r$ (see, [12]), $(l_p)_{E^r} = e_p^r$ and $(l_\infty)_{E^r} = e_\infty^r$ (see, [13,14]), $(c_0)_{A^r} = a_0^r$ and $(c)_{A^r} = a_c^r$ (see, [15]), $[c_0(u, p)]_{A^r} = a_0^r(u, p)$ and $[c(u, p)]_{A^r} = a_c^r(u, p)$ (see, [16]), $(l_p)_{A^r} = a_p^r$ and $(l_\infty)_{A^r} = a_\infty^r$ (see, [17]), $(c_0)_{C_1} = \hat{c_0}$, $(c_{C_1}) = \hat{c}$ (see, [18]), $\mu_G = Z(u, v, \mu)$ (see [19]), where N_q, C_1 R^t and E^r denotes the Nörland, Cesàro, Riesz and Euler means, respectively, A^r and C are respectively defined in [19,20], $\mu = \{c_0, c, l_p\}$ and $1 \le p < \infty$. Also $c_0(u, p)$ and $c(u, p)$ denote the sequence spaces generated from the Maddox's spaces $c_0(p)$ and $c(p)$ by Basarir (see, [20]).

A sequence space X with a linear topology is called a K-space if each map $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space X is called an FK-space provided X is complete linear metric space. An FK-space whose topology is normable is called a BK-space. In [21] Gaine and Sheikh introduced the sequence spaces $c_0(\Delta_u^\lambda)$ and $c(\Delta_u^\lambda)$ and derive some inclusion relations. Furthermore, they determine the α-, β- and γ-duals of these spaces. In the last they have characterized some matrix classes concerning these spaces.

Mursaleen and Noman [22] introduced the sequence spaces l_∞^λ, c^λ and c_0^λ as a set of λ-bounded, λ-convergent and λ-null sequences, respectively, that is

$$l_\infty^\lambda = \{x \in w : \sup_{n\to\infty} |\Lambda_n(x)| < \infty\}$$

$$c^\lambda = \{x \in w : \lim_{n\to\infty} \Lambda_n(x) \text{ exists}\}$$

and

$$c_0^\lambda = \{x \in w : \lim_{n\to\infty} \Lambda_n(x) = 0\}$$

where $\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})x_k, \quad k \in \mathbb{N}$.

Mursaleen and Noman [23] introduced the sequence spaces $c^{\lambda}(\Delta)$ and $c_0^{\lambda}(\Delta)$, respectively, that is

$$c^{\lambda}(\Delta) = \{x \in w : \lim_{n\to\infty} \tilde{\Lambda}_n(x) \text{ exists}\}$$

and

$$c_0^{\lambda}(\Delta) = \{x \in w : \lim_{n\to\infty} \tilde{\Lambda}_n(x) = 0\}$$

where $\tilde{\Lambda}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(x_k - x_{k-1}), \quad k \in \mathbb{N}$.

The main purpose of this paper is to study some non-absolute type difference sequence spaces $l_\infty(u, \lambda, \Delta_v^m)$, $c_0(u, \lambda, \Delta_v^m)$ and $c(u, \lambda, \Delta_v^m)$. We have proved that these spaces are BK-spaces and linearly isomorphic to the spaces l_∞, c_0 and c respectively. Furthermore some inclusion relations between these spaces are established and finally we have determined the α-, β- and γ-duals of these spaces.

2. Non-absolute type sequence spaces

In the present section we introduce and study the sequence spaces $l_\infty(u, \lambda, \Delta_v^m)$, $c_0(u, \lambda, \Delta_v^m)$ and $c(u, \lambda, \Delta_v^m)$ of non-absolute type as follows:

$$l_\infty(u, \lambda, \Delta_v^m) = \{x \in w : \sup_{n\to\infty} |\hat{\Lambda}_n(x)| < \infty\},$$
$$c^{\lambda}(u, \lambda, \Delta_v^m) = \{x \in w : \lim_{n\to\infty} \hat{\Lambda}_n(x) \text{ exists}\}$$

and

$$c_0^{\lambda}(u, \lambda, \Delta_v^m) = \{x \in w : \lim_{n\to\infty} \hat{\Lambda}_n(x) = 0\},$$

where

$$\hat{\Lambda}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^m x_k, \quad k, m \in \mathbb{N},$$
$$= \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k(\Delta_v^{m-1}x_k - \Delta_v^{m-1}x_{k-1}), \quad k, m \in \mathbb{N}.$$

Let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to infinity, that is,

$$0 < \lambda_0 < \lambda_1 < \cdots\cdots$$

and $\lambda_k \to \infty$ as $k \to \infty$. Here in sequel, we use the convection that any term with a negative subscript is equal to naught, e.g. $\lambda_{-1} = 0$ and $x_{-1} = 0$. We define the matrix $\hat{\Lambda} = (\hat{\lambda}_{nk})$ for all $n, k \in \mathbb{N}$ by

$$\hat{\lambda}_{nk} = \begin{cases} \sum_{i=k}^{n}\binom{m}{i-k}(-1)^{i-k}\frac{\lambda_i - \lambda_{i-1}}{\lambda_n}u_k, & k \le n; \\ 0, & k > n, \end{cases}$$

$\hat{\Lambda} = (\hat{\lambda}_{nk})$ equality can be easily seen from

$$\hat{\Lambda}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^m x_k \tag{3}$$

for all $m, v \in \mathbb{N}$ and every $x = (x_k) \in w$. Then it leads together with (1) to the fact that

$$l_\infty(u, \lambda, \Delta_v^m) = (l_\infty)_{\hat{\Lambda}}, \quad c_0(u, \lambda, \Delta_v^m) = (c_0)_{\hat{\Lambda}} \quad \text{and} \quad c(u, \lambda, \Delta_v^m) = (c)_{\hat{\Lambda}}. \tag{4}$$

The matrix $\hat{\Lambda} = (\hat{\lambda}_{nk})$ is a triangle, that is $\hat{\lambda}_{nn} \neq 0$ and $\hat{\lambda}_{nk} = 0 (k > n)$ for all $n, k \in \mathbb{N}$. Further for any sequence $x = (x_k)$ we define the sequence $y(\lambda) = \{y_k(\lambda)\}$ as the $\hat{\Lambda}$-transform of x, i.e. $y(\lambda) = \hat{\Lambda}(x)$ so we have that

$$y(\lambda) = \sum_{j=0}^{k} \sum_{i=j}^{k} (-1)^{i-j} \binom{m}{i-j} \frac{\lambda_i - \lambda_{i-1}}{\lambda_k} u_j x_j \tag{5}$$

for all $k \in \mathbb{N}$. Then summation running from 0 to $k-1$ is equal to zero when $k = 0$.

Theorem 1. *The spaces $l_\infty(u, \lambda, \Delta_v^m)$, $c_0(u, \lambda, \Delta_v^m)$ and $c(u, \lambda, \Delta_v^m)$ are BK-spaces with the norm*

$$\|x\|_{l_\infty(u,\lambda,\Delta_v^m)} = \|x\|_{c_0(u,\lambda,\Delta_v^m)} = \|x\|_{c(u,\lambda,\Delta_v^m)} = \sup_n |\hat{\Lambda}_n(x)|.$$

Proof. The proof is a routine verification, so is left as an easy exercise to readers (see, [24,25]). □

Remark 2. The absolute property does not hold on $l_\infty(u, \lambda, \Delta_v^m)$, $c_0(u, \lambda, \Delta_v^m)$ and $c(u, \lambda, \Delta_v^m)$ spaces. For instance, if we take $|x| = (|x_k|)$ then we have $\|x\|_{l_\infty(u,\lambda,\Delta_v^m)} \neq \||x|\|_{l_\infty(u,\lambda,\Delta_v^m)}$, similarly other hold. Thus, the spaces $l_\infty(u, \lambda, \Delta_v^m)$, $c_0(u, \lambda, \Delta_v^m)$ and $c(u, \lambda, \Delta_v^m)$ are BK-spaces of non-absolute type.

Theorem 3. *The spaces $l_\infty(u, \lambda, \Delta_v^m)$, $c_0(u, \lambda, \Delta_v^m)$ and $c(u, \lambda, \Delta_v^m)$ of non-absolute type are linearly isomorphic to the spaces l_∞, c_0 and c, respectively, that is $l_\infty(u, \lambda, \Delta_v^m) \cong l_\infty$, $c_0(u, \lambda, \Delta_v^m) \cong c_0$ and $c(u, \lambda, \Delta_v^m) \cong c$.*

Proof. We only consider the case $c_0(u, \lambda, \Delta_v^m) \cong c_0$ and others will follow similarly. To prove the theorem, we must show the existence of linear bijection between the spaces $c_0(u, \lambda, \Delta_v^m)$ and c_0. For this we consider the transformation T defined, with the notation (5), from $c_0(u, \lambda, \Delta_v^m)$ to c_0 by $x \to y(\lambda) = Tx$. Then $Tx = y(\lambda) = \hat{\Lambda}(x) \in c_0$ for every $x \in c_0(u, \lambda, \Delta_v^m)$. The linearity of T is obvious. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective. Next, let $y = (y_k) \in c_0$ and define the sequence $x = \{x_k(\lambda)\}$ by

$$x_k(\lambda) = \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{u_j(\lambda_j - \lambda_{j-1})} y_i \tag{6}$$

so we have

$$\Delta_v^m x_k = \sum_{i=k-1}^{k} (-1)^{k-i} \frac{\lambda_i}{u_k(\lambda_k - \lambda_{k-1})} y_i.$$

Thus, for every $k \in \mathbb{N}$, we have by (5) that

$$\begin{aligned} \hat{\Lambda}_n(x) &= \frac{1}{\lambda_n} \sum_{k=0}^{n} \sum_{i=k-1}^{k} (-1)^{k-i} \lambda_i y_i \\ &= \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k y_k - \lambda_{k-1} y_{k-1}) = y_n. \end{aligned}$$

This shows that $\hat{\Lambda}(x) = y$ and since $y \in c_0$, we obtain that $\hat{\Lambda}(x) \in c_0$. Thus, we deduce that $x \in c_0(u, \lambda, \Delta_v^m)$ and $Tx = y$. Hence, T is surjective. Further, we have for every $x \in c_0(u, \lambda, \Delta_v^m)$ that

$$\|Tx\|_{c_0} = \|Tx\|_{l_\infty} = \|y(\lambda)\|_{l_\infty} = \|\hat{\Lambda}(x)\|_{l_\infty} = \|x\|_{c_0(u,\lambda,\Delta_v^m)}$$

which means that $c_0(u, \lambda, \Delta_v^m)$ and c_0 are linearly isomorphic. □

Theorem 4. *Suppose $\lambda = (\lambda_n)$ is a strictly increasing sequence of positive real numbers tends to infinity and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the inclusion $c_0(u, \lambda, \Delta_v^m) \subset c(u, \lambda, \Delta_v^m)$ strictly holds.*

Proof. It is clear that the inclusion $c_0(u,\lambda,\Delta_v^m) \subset c(u,\lambda,\Delta_v^m)$ holds. Further to show strict, consider the sequence $x = (x_k)$ defined by $x_k = \frac{k^m}{u_k}$, for all $k \in \mathbb{N}$. Then we obtain that

$$\hat{\Lambda}(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^m x_k = m!$$

for $n \in \mathbb{N}$ and clearly $\hat{\Lambda}(x) \in c\backslash c_0$. Thus the sequence x is in $c(u,\lambda,\Delta_v^m)$ but not in $c_0(u,\lambda,\Delta_v^m)$. Hence the inclusion $c_0(u,\lambda,\Delta_v^m) \subset c(u,\lambda,\Delta_v^m)$ is strict and this completes the proof. □

Theorem 5. *Suppose $\lambda = (\lambda_n)$ is a strictly increasing sequence of positive real numbers tends to infinity and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the inclusion $c \subset c_0(u,\lambda,\Delta_v^m)$ strictly holds.*

Proof. Let $x \in c$. Then $\hat{\Lambda}(x) \in c_0$. This shows that $x \in c_0(u,\lambda,\Delta_v^m)$. Hence the inclusion holds. Now consider the sequence $y = (y_k)$ defined by $y_k = \sqrt{k+1}$ for $k \in \mathbb{N}$. It is trivial that $y \notin c$. On the other hand it can easily seen that $\hat{\Lambda}(y) \in c_0$ and $y \in c_0(u,\lambda,\Delta_v^m)$. Consequently, the sequence y is in $c_0(u,\lambda,\Delta_v^m)$ but not in c. We therefore deduce that the inclusion $c \subset c_0(u,\lambda,\Delta_v^m)$ is strict. This completes the proof. □

Theorem 6. *Let $\lambda = (\lambda_n)$ be a strictly increasing sequence of positive real numbers tends to infinity and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the inclusion $c_0(u,\lambda,\Delta_v^m) \subset c_0(u,\lambda,\Delta_v^{m+1})$ strictly holds.*

Proof. Let $x \in c_0(u,\lambda,\Delta_v^m)$. Then we have

$$\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^{m+1}x_k\right| \le \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^{m}x_k\right| + \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^{m}x_{k-1}\right|$$

for $k \to \infty$ from the inequality above we conclude that $x \in c_0(u,\lambda,\Delta_v^{m+1})$. To show strictness consider the sequence $x = (x_k)$ defined by $x_k = k^m$. Then it can be easily seen that $x \in c_0(u,\lambda,\Delta_v^{m+1})$ and $x \notin c_0(u,\lambda,\Delta_v^m)$. □

Theorem 7. *If $\lambda = (\lambda_n)$ is a strictly increasing sequence of positive real numbers tends to infinity and $u = (u_k)$ be a sequence of strictly positive real numbers, then the inclusion $c(u,\lambda,\Delta_v^{m-1}) \subset c(u,\lambda,\Delta_v^m)$ strictly holds.*

Proof. Let $x \in c(u,\lambda,\Delta_v^{m-1})$. Then we have

$$\hat{\Lambda}(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^{m-1}x_k \to l \ \ (k \to \infty).$$

Furthermore, we obtain the inequality that $x \in c(u,\lambda,\Delta_v^m)$. Hence the inclusion $c(u,\lambda,\Delta_v^{m-1}) \subset c(u,\lambda,\Delta_v^m)$ holds as

$$\begin{aligned}\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^{m}x_k\right| &\le \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^{m-1}x_k - l\right| \\ &\quad + \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^{m-1}x_{k-1} - l\right| \\ &\to 0. \quad \square\end{aligned}$$

Theorem 8. *Suppose $\lambda = (\lambda_n)$ is a strictly increasing sequence of positive real numbers tends to infinity and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the inclusion $l_\infty(u,\lambda,\Delta_v^{m-1}) \subset l_\infty(u,\lambda,\Delta_v^m)$ strictly holds.*

Proof. Let $x \in l_\infty(u,\lambda,\Delta_v^{m-1})$. Then we have

$$\hat{\Lambda}(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k\Delta_v^{m-1}x_k \le K,$$

for $K > 0$. We obtain the following inequality that $x \in l_\infty(u, \lambda, \Delta_v^m)$. Hence the inclusion $l_\infty(u, \lambda, \Delta_v^{m-1}) \subset l_\infty(u, \lambda, \Delta_v^m)$ holds as

$$\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k \Delta_v^m x_k\right| \le \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k \Delta_v^{m-1} x_k\right| + \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k \Delta_v^{m-1} x_{k-1}\right|.$$

To show strict, we consider $x = (x_k)$ defined by $x = (k^m)$, then we obtain $x \in l_\infty(u, \lambda, \Delta_v^m)$ but $x \notin l_\infty(u, \lambda, \Delta_v^{m-1})$. $\square$

3. Basis and α-, β-, γ-duals of the spaces $c(u, \lambda, \Delta_v^m)$ and $c_0(u, \lambda, \Delta_v^m)$

If the normed space X contains a sequence (b_n) with the property that for every $x \in X$, there is a unique sequence of scalars (α_n) such that

$$\lim_n \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots\cdots + \alpha_n b_n)\| = 0$$

then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and is written as $x = \sum_k \alpha_k b_k$.

Theorem 9. *Define the sequence $b^{(k)}(u, \lambda, \Delta_v^m) = \{b_n^{(k)}(u, \lambda, \Delta_v^m)\}_{k=0}^{\infty}$ for every fixed $k, m \in \mathbb{N}$ and by*

$$b^{(k)}(u, \lambda, \Delta_v^m) = \begin{cases} \binom{m+n-k-1}{n-k}\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)u_k - \binom{m+n-k-2}{n-k-1}\left(\frac{\lambda_k}{\lambda_{k+1} - \lambda_k}\right)u_k, & n > k; \\ \left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)u_k, & n = k; \\ 0, & n < k. \end{cases}$$

Then the sequence $\{b_n^{(k)}(u, \lambda, \Delta_v^m)\}$ is a basis for the space $c_0(u, \lambda, \Delta_v^m)$ and every $x \in c_0(u, \lambda, \Delta_v^m)$ has a unique representation of the form

$$x = \sum_k \alpha_k(\lambda) b^{(k)}(u, \lambda, \Delta_v^m),$$

where $\alpha_k(\lambda) = \hat{\Lambda}(x)$, for all $k \in \mathbb{N}$.

Theorem 10. *The sequence $\{b, b^0(u, \lambda, \Delta_v^m), b^{(1)}(u, \lambda, \Delta_v^m), \ldots\ldots.\}$ is a basis for the space $c(u, \lambda, \Delta_v^m)$ and every $x \in c(u, \lambda, \Delta_v^m)$ has a unique representation of the form*

$$x = la + \sum_k [\alpha_k(\lambda) - l] b^{(k)}(u, \lambda, \Delta_v^m);$$

where $\alpha_k(\lambda) = \hat{\Lambda}(x)$, for all $k \in \mathbb{N}$. The sequence $b = (b_k)$ is defined by

$$b = (b_k) = \sum_{j=0}^{k} \binom{m+k-j-1}{k-j}.$$

Corollary 11. *The difference sequence spaces $c(u, \lambda, \Delta_v^m)$ and $c_0(u, \lambda, \Delta_v^m)$ are separable.*

To determine α-, β- and γ-duals of non-absolute type spaces $c(u, \lambda, \Delta_v^m)$ and $c_0(u, \lambda, \Delta_v^m)$, we shall use the following result:

For the sequence spaces X and Y, the set

$$M(X : Y) = \{a = (a_k) \in w : ax = (a_k x_k) \in Y \text{ for all } x = (x_k) \in w\} \tag{7}$$

is known as multiplier space of X and Y. With the notion of (7), α-, β- and γ-duals of the space X respectively denoted by X^{α}, X^{β} and X^{γ} and defined by

$$X^{\alpha} = M(X, l_1),\ \ X^{\beta} = M(X, cs) \text{ and } X^{\gamma} = M(X, bs).$$

We now state following lemmas which we shall use to prove further theorems.

Lemma 1. *$A \in (c_0 : l_1) = (c : l_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \Big| \sum_{k\in K} a_{nk} \Big| < \infty.$$

Lemma 2. *$A \in (c_0 : c)$ if and only if*

$$\lim_n a_{nk} \ \text{ exists for each } \ k \in \mathbb{N} \text{ and} \tag{8}$$

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{9}$$

Lemma 3. *$A \in (c : c)$ if and only if (8) and (9) hold, and*

$$\lim_n \sum_k a_{nk} \ \text{ exists.} \tag{10}$$

Lemma 4. *$A \in (c_0 : l_\infty) = (c : l_\infty)$ if and only if (9) holds.*

Lemma 5. *$A \in (l_\infty : c)$ if and only if and (8) holds and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k |\alpha_k|.$$

Theorem 12. *The α-dual of the spaces $c(u, \lambda, \Delta_v^m)$ and $c_0(u, \lambda, \Delta_v^m)$ is the set*

$$b_1^{\lambda} = \left\{ a = (a_k) \in w : \sup_{K\in\mathcal{F}} \left| \sum_{k\in K} b_{nk}(u, \lambda, \Delta_v^m) \right| < \infty \right\};$$

where the matrix $B^{\lambda} = (b_{nk}^{\lambda})$ is defined via the sequence $a = (a_k)$ by

$$b^{(k)}(u, \lambda, \Delta_v^m) = \begin{cases} \left(\binom{m+n-k-1}{n-k} \left(\dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}} \right) - \binom{m+n-k-2}{n-k-1} \left(\dfrac{\lambda_k}{\lambda_{k+1} - \lambda_k} \right) \right) \dfrac{a_n}{u_k}, & n > k; \\ \left(\dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}} \right) \dfrac{a_n}{u_k}, & n = k; \\ 0, & n < k. \end{cases}$$

Proof. We prove the theorem for the space $c_0(u, \lambda, \Delta_v^m)$. If $a = (a_k) \in w$, then we have the equality

$$a_k x_k = \sum_{k=0}^{n} \binom{m+n-k-1}{n-k} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j}{u_k(\lambda_j - \lambda_{j-1})} y_j = B_n^{\lambda}(y); \quad n \in \mathbb{N}. \tag{11}$$

Thus, we observe by (11) that $ax = (a_k x_k) \in l_1$, whenever $x = (x_k) \in c_0(u, \lambda, \Delta_v^m)$ or $c(u, \lambda, \Delta_v^m)$ if and only if $B^{\lambda}(y) \in l_1$ whenever $y = (y_k) \in c_0$ or c. This means that the sequence $a = (a_k)$ is in the α-duals of the spaces $c(u, \lambda, \Delta_v^m)$ or $c_0(u, \lambda, \Delta_v^m)$ if and only if $B^{\lambda} \in (c_0 : l_1) = (c : l_1)$. We therefore obtain from Lemma 1 with B^{λ} instead of A that $a \in \{c_0(u, \lambda, \Delta_v^m)\}^{\alpha} = \{c(u, \lambda, \Delta_v^m)\}^{\beta}$ if and only if

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} b_{nk}(u, \lambda, \Delta_v^m) \right| < \infty$$

which leads us to the consequences that $\{c_0(u, \lambda, \Delta_v^m)\}^{\alpha} = \{c(u, \lambda, \Delta_v^m)\}^{\beta} = b_1^{\lambda}$. This completes the proof. □

Theorem 13. *The β-dual of the spaces $c_0(u,\lambda,\Delta_v^m)$ and $c(u,\lambda,\Delta_v^m)$ that is $\{c_0(u,\lambda,\Delta_v^m)\}^\beta = b_2^\lambda \cap b_3^\lambda \cap b_4^\lambda$ and $\{c(u,\lambda,\Delta_v^m)\}^\beta = b_2^\lambda \cap b_3^\lambda \cap b_4^\lambda \cap b_5^\lambda$, where*

$$b_2^\lambda = \left\{a=(a_k)\in w : \sum_{j=k}^{\infty}\binom{m+n-j-1}{n-j}a_j \text{ exists for each } k\in\mathbb{N}\right\},$$

$$b_3^\lambda = \left\{a=(a_k)\in w : \sup_{n\in\mathbb{N}}\sum_{k=0}^{n-1}|a_k(n)|<\infty\right\},$$

$$b_4^\lambda = \left\{a=(a_k)\in w : \sup_k\left|\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}u_k^{-1}a_k\right|<\infty\right\},$$

$$b_5^\lambda = \left\{a=(a_k)\in w : \lim_{n\to\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{m+k-j-1}{k-j}a_k \text{ exists}\right\}$$

and $a_k(n)$ is defined as

$$a_k(n) = \lambda_k u_k^{-1}\left[\frac{1}{\lambda_k-\lambda_{k-1}}\sum_{k=0}^{n}\binom{m+j-k-1}{j-k}a_j - \frac{1}{\lambda_{k+1}-\lambda_k}\sum_{k=0}^{n}\binom{m+j-k-2}{j-k-1}a_j\right]y_k$$

for $k<n$.

Proof. We have from (6)

$$\begin{aligned}\sum_{k=0}^{n}a_kx_k &= \sum_{j=0}^{k}\left[\sum_{j=0}^{k}\binom{m+k-j-1}{k-j}\sum_{i=j-1}^{j}(-1)^{j-i}\frac{\lambda_i}{u_j(\lambda_j-\lambda_{j-1})}y_i\right]a_k\\ &= \sum_{k=0}^{n-1}\lambda_k\left[\frac{\sum_{j=k}^{n}\binom{m+j-k-1}{j-k}a_j}{\lambda_k-\lambda_{k-1}} - \frac{\sum_{j=k+1}^{n}\binom{m+j-k-2}{j-k-1}a_j}{\lambda_{k+1}-\lambda_k}\right]y_k + \frac{a_n\lambda_n}{\lambda_n-\lambda_{n-1}}y_n\\ &= \sum_{k=0}^{n-1}a_k(n)y_k + \frac{a_n\lambda_n}{\lambda_n-\lambda_{n-1}}y_n = (D_n^\lambda)(y); \quad (n\in\mathbb{N})\end{aligned}$$

where $(D_n^\lambda)=(d_{nk}^\lambda)$ is defined by

$$(d_{nk}^\lambda) = \begin{cases} a_k(n), & n>k;\\ \left(\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\right)\frac{a_n}{u_n}, & n=k;\\ 0, & n<k.\end{cases}$$

Thus, we derive that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in c_0(u,\lambda,\Delta_v^m)$ if and only if $D^\lambda y\in c$ whenever $y=(y_k)\in c_0$. This means that $a=(a_k)\in\{c_0(u,\lambda,\Delta_v^m)\}^\beta$ if and only if $D^\lambda y\in(c_0,c)$. Therefore by using Lemma 2, we obtain

$$\sum_{j=k}^{\infty}\binom{m+k-j-1}{k-j}a_j \text{ exists for each } k\in\mathbb{N},$$

$$\sup_{n\in\mathcal{N}}\sum_{k=0}^{n-1}|a_k(n)|<\infty$$

$$\sup_{n\in\mathcal{N}}\sum_{k=0}^{n-1}\left|\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_k\right|<\infty.$$

Hence, we conclude that $\{c_0(u, \lambda, \Delta_v^m)\}^\beta = b_2^\lambda \cap b_3^\lambda \cap b_4^\lambda$. Finally we ended this section with the following theorem which determines the γ-duals of sequence spaces $c(u, \lambda, \Delta_v^m)$, $c_0(u, \lambda, \Delta_v^m)$ and $l_\infty(u, \lambda, \Delta_v^m)$. □

Theorem 14. $\{c_0(u, \lambda, \Delta_v^m)\}^\gamma = \{c(u, \lambda, \Delta_v^m)\}^\gamma = \{l_\infty(u, \lambda, \Delta_v^m)\}^\gamma = b_3^\lambda \cap b_4^\lambda$.

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Original article

Unique fixed point results on closed ball for dislocated quasi G-metric spaces

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Abstract

The aim of this paper is to introduce the new concept of ordered complete dislocated quasi G-metric space. The notion of dominated mappings is applied to approximate the unique solution of non linear functional equations. In this paper, we find the fixed point results for mappings satisfying the locally contractive conditions on a closed ball in an ordered complete dislocated quasi G-metric space. Our results improve several well known classical results.

Keywords: Fixed point; Banach mapping; Kannan mapping; Closed ball; Dominated mapping; Ordered complete dislocated quasi metric spaces

1. Introduction and preliminaries

Let $T : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of T if $x = Tx$. Let x_0 be an arbitrary chosen point in X. Define a sequence $\{x_n\}$ in X by a simple iterative method given by $x_{n+1} = Tx_n$, where $n \in \{0, 1, 2, 3, \ldots\}$. Such a sequence is called a picard iterative sequence and its convergence plays a very important role in proving existence of a fixed point of a mapping T. A self mapping T on a metric space X is said to be a Banach contraction mapping if,

$$d(Tx, Ty) \leq kd(x, y)$$

holds for all $x, y \in X$ where $0 \leq k < 1$. Recently, many results appeared related to fixed point theorem in complete metric spaces endowed with a partial ordering in literature. Ran and Reurings [1] proved an analogue of Banach's fixed

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point theorem in metric space endowed with partial order and gave applications to matrix equations. Recently, Arshad et al. [2] proved a result concerning the existence of fixed points of a mapping satisfying a contractive conditions on closed ball in a complete dislocated metric space. For further results on closed ball we refer the reader to [3–7] and references therein. Subsequently, Nieto et al. [8] extended the results of [1] for non decreasing mappings and applied this results to obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions. On the other hand in 2005, Mustafa and Sims in [9] introduce the notion of a generalized metric space as generalization of the usual metric space. Mustafa and others studied fixed point theorems for mappings satisfying different contractive conditions for further useful results can be seen in [10–15]. Recently, Agarwal and Karapinar introduced some coupled fixed point theorems in G metric space [16]. Azam and Nayyar proved fixed point theorems for multivalued mappings in G-cone metric space see [17]. Further latest fixed point results on G metric space can be seen in [18–20]. The dominated mapping [21] which satisfies the condition $fx \preceq x$ occurs very naturally in several practical problems. For example x denotes the total quantity of food produced over a certain period of time and $f(x)$ gives the quantity of food consumed over the same period in a certain town, then we must have $fx \preceq x$.

In this paper we have obtained fixed point theorems for a contractive dominated self-mapping in an ordered complete dislocated quasi G-metric space on a closed ball to generalize, extend and improve some classical fixed point results. We have used weaker contractive condition and weaker restrictions to obtain unique fixed point.

Definition 1. Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(i) If $G(x, y, z) = G(y, z, x) = G(z, x, y) = 0$, then $x = y = z$,

(ii) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the pair (X, G) is called the dislocated quasi G-metric space. It is clear that if

$G(x, y, z) = G(y, z, x) = G(z, x, y) = 0$ then from (i) $x = y = z$. But if $x = y = z$ then $G(x, y, z)$ may not be 0. It is observed that if $G(x, y, z) = G(y, z, x) = G(z, x, y)$ for all $x, y, z \in X$, then (X, G) becomes a dislocated G-metric space.

Example 2. If $X = R^+ \cup \{0\}$ then $G(x, y, z) = x + \max\{x, y, z\}$ defines a dislocated quasi metric on X.

Definition 3. Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x. Thus, if $x_n \to x$ in a dislocated quasi G-metric space (X, G), then for any $\in > 0$, there exists $n, m \in N$ such that $G(x, x_n, x_m) < \in$, for all $n, m \geq N$.

Definition 4. Let (X, G) be a dislocated quasi G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if, for each $\in > 0$ there exists a positive integer $n^\star \in N$ such that $G(x_n, x_{m,} x_l) < \in$ for all $n, l, m \geq n^\star$; i.e. if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 5. A dislocated quasi G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 6. *Let (X, G) be a dislocated quasi G-metric space, then the following are equivalent:*

(1) $\{x_n\}$ *is G convergent to x.*
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$.
(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$.
(4) $G(x_n, x_m, x) \to 0$ *as* $m\ n \to \infty$.

Definition 7. Let (X, G) be a G-metric space then for $x_0 \in X$, $r > 0$, the G-ball with centre x_0 and radius r is,

$$\overline{B(x_0, r)} = \{y \in X : G(x_0, y, y) \leq r\}.$$

Definition 8 *([21])*. Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 9 *([21])*. Let $(X, \preceq)$ be a partially ordered set. A self mapping f on X is called dominated if $fx \preceq x$ for each x in X.

Example 10 *([21])*. Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in N$. Since $fx = x^n \leq x$ for all $x \in X$, therefore f is a dominated map.

2. Main results

Theorem 11. *Let $(X, \preceq, G)$ be an ordered complete dislocated quasi G-metric space, $S : X \to X$ be a dominated mapping and x_0 be any arbitrary point in X. Suppose there exists $k \in [0, 1)$ with,*

$$G(Sx, Sy, Sz) \leq kG(x, y, z), \text{ for all } x, \ y \text{ and } z \in Y = \overline{B(x_0, r)}, \tag{2.1}$$

and

$$G(x_0, Sx_0, Sx_0) \leq (1 - k)r. \tag{2.2}$$

If for a nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point $x^\star$ in $\overline{B(x_0, r)}$ such that $x^\star = Sx^\star$ and $G(x^\star, x^\star, x^\star) = 0$. Moreover if for any three points x, y and z in $\overline{B(x_0, r)}$ such that there exists a point $v \in \overline{B(x_0, r)}$ such that $v \preceq x$, $v \preceq y$ and $v \preceq z$, that is, every three of elements in $\overline{B(x_0, r)}$ has a lower bound, then the point $x^\star$ is unique.

Proof. Consider a picard sequence $x_{n+1} = Sx_n$ with initial guess x_0. As $x_{n+1} = Sx_n \preceq x_n$ for all $n \in \{0\} \cup N$. Now by inequality (2.2) we have

$$G(x_0, x_1, x_1) \leq r,$$

which implies that $x_1 \in \overline{B(x_0, r)}$. By rectangular inequality

$$G(x_0, x_2, x_2) \leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2)$$

then we get,

$$\begin{aligned} G(x_0, x_2, x_2) &\leq G(x_0, Sx_0, Sx_0) + G(Sx_0, Sx_1, Sx_1) \\ &\leq (1 - k)r + k(1 - k)r \\ &\leq (1 - k^2)r \leq r \\ &\leq r. \end{aligned}$$

Thus, $x_2 \in \overline{B(x_0, r)}$. We suppose that $x_3, \ldots, x_j \in \overline{B(x_0, r)}$, for some $j \in N$. Now using (2.1) we get,

$$\begin{aligned} G(x_j, x_{j+1}, x_{j+1}) &= G(Sx_{j-1}, Sx_j, Sx_j) \leq k[G(x_{j-1}, x_j, x_j)] \\ &\leq k^2[G(x_{j-2}, x_{j-1}, x_{j-1})] \\ &\ \ \vdots \\ &\leq k^j[G(x_0, x_1, x_1)]. \end{aligned} \tag{2.3}$$

By using inequalities (2.1) and (2.3) we have,

$$\begin{aligned} G(x_0, x_{j+1}, x_{j+1}) &\leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2) + \cdots + G(x_j, x_{j+1}, x_{j+1}) \\ &\leq (1 - k)r + rk(1 - k) + \cdots + rk^j(1 - k) \\ &= r(1 - k)[1 + k + k^2 + \cdots + k^j] \\ &\leq r(1 - k)\frac{(1 - k^{j+1})}{(1 - k)} \leq r. \end{aligned}$$

Thus, $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. Now inequality (2.3) can be written as,

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1). \tag{2.4}$$

Using inequality (2.4) we get,

$$\begin{aligned} G(x_n, x_{n+i}, x_{n+i}) &\le G(x_n, x_{n+1}, x_{n+1}) + \cdots + G(x_{n+i-1}, x_{n+i}, x_{n+i}) \\ &\le k^n \frac{(1-k^i)}{(1-k)} G(x_0, x_1, x_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This proves that the sequence $\{x_n\}$ is a G-Cauchy sequence in $(\overline{B(x_0, r)}, G)$. Therefore there exists a point $x^\star \in \overline{B(x_0, r)}$ with,

$$\lim_{n\to\infty} G(x_n, x^\star, x^\star) = 0.$$

Similarly, it can be proved that

$$\lim_{n\to\infty} G(x^\star, x^\star, x_n) = 0.$$

Therefore

$$\lim_{n\to\infty} G(x_n, x^\star, x^\star) = \lim_{n\to\infty} G(x^\star, x^\star, x_n) = 0. \tag{2.5}$$

Now,

$$G(x^\star, Sx^\star, Sx^\star) \le G(x^\star, x_n, x_n) + G(x_n, Sx^\star, Sx^\star).$$

By assumption $x^\star \preceq x_n \preceq x_{n-1,}$ therefore,

$$\begin{aligned} G(x^\star, Sx^\star, Sx^\star) &\le G(x^\star, x_n, x_n) + G(Sx_{n-1}, Sx^\star, Sx^\star) \\ &\le G(x^\star, x_n, x_n) + kG(x_{n-1,} x^\star, x^\star) \\ &\le \lim_{n\to\infty} [G(x^\star, x_n, x_n) + kG(x_{n-1}, x^\star, x^\star)] \\ &\le 0 \\ &\Rightarrow G(x^\star, Sx^\star, Sx^\star) = 0. \end{aligned}$$

Therefore, $x^\star = Sx^\star$. Similarly, $G(Sx^\star, Sx^\star, x^\star) \le 0$, and hence $x^\star = Sx^\star$. Now,

$$G(x^\star, x^\star, x^\star) = G(Sx^\star, Sx^\star, Sx^\star) \le kG(x^\star, x^\star, x^\star).$$

Since, $k \in [0, 1)$, then $G(x^\star, x^\star, x^\star) = 0$.

Uniqueness: Let $y^\star$ be another point in $\overline{B(x_0, r)}$ such that $y^\star = Sy^\star$, if $x^\star$ and $y^\star$ are comparable then,

$$G(x^\star, x^\star, y^\star) = G(Sx^\star, Sx^\star, Sy^\star) \le kG(x^\star, x^\star, y^\star).$$

Therefore,

$$G(y^\star, x^\star, x^\star) \le 0.$$

This shows that $x^\star = y^\star$. Now if $x^\star$ and $y^\star$ are not comparable then there exists a point $v \in \overline{B(x_{0,} r)}$ which is the lower bound of both $x^\star$ and $y^\star$ that is $v \preceq x^\star$ and $v \preceq y^\star$. Moreover by assumption $x^\star \preceq x_n$ as $x_n \to x^\star$. Therefore $v \preceq x^\star \preceq x_n \preceq \cdots \preceq x_0$.

$$\begin{aligned} G(x_0, Sv, Sv) &\le G(x_0, x_1, x_1) + G(x_1, Sv, Sv) \\ &\le G(x_0, Sx_0, Sx_0) + G(Sx_0, Sv, Sv) \\ &\le (1-k)r + kG(x_0, v, v) \\ &\le (1-k)r + kr \quad \text{(by (2.1) and (2.2))}. \end{aligned}$$

But, x_0 and $v \in \overline{B(x_0, r)}$), then $G(x_0, Sv, Sv) \le r - rk + rk \le r \Rightarrow G(x_0, Sv, Sv) \le r$.

It follows that $Sv \in \overline{B(x_0, r)}$. Now we will prove that $S^n v \in \overline{B(x_0, r)}$, by using mathematical induction. Let $S^2 v, S^3 v, \ldots, S^j v \in \overline{B(x_0, r)}$ for some $j \in N$. As $S^j v \preceq S^{j-1} v \preceq \cdots \preceq v \preceq x^\star \preceq x_n \cdots \preceq x_0$, then,

$$G(x_{j+1}, S^{j+1} v, S^{j+1} v) = G(Sx_j, S(S^j v), S(S^j v)).$$

Thus by (2.1),

$$G(x_{j+1}, S^{j+1}v, S^{j+1}v) \leq kG(x_j, S^j v, S^j v) \leq \cdots \leq k^{j+1}G(x_0, v, v). \tag{2.6}$$

Now,

$$\begin{aligned} G(x_0, S^{j+1}v, S^{j+1}v) &\leq G(x_0, x_1, x_1) + \cdots + G(x_{j,}x_{j+1}, x_{j+1}) + G(x_{j+1}, S^{j+1}v, S^{j+1}v) \\ &\leq G(x_0, x_1, x_1) + \cdots + k^j G(x_0, x_1, x_1) + k^{j+1}G(x_0, v, v) \\ &\leq G(x_0, x_1, x_1)[1 + k + k^2 + \cdots + k^j] + rk^{j+1} \quad \text{by (2.6)} \\ &\leq (1-k)r\frac{(1-k^{j+1})}{(1-k)} + rk^{j+1} = r \end{aligned}$$

$\Rightarrow G(x_0, S^{j+1}v, S^{j+1}v) \leq r$. It follows that $S^{j+1}v \in \overline{B(x_0, r)}$ and hence $S^n v \in \overline{B(x_o, r)}$ for all n. Now

$$G(x^\star, y^\star, y^\star) \leq G(S^n x^\star, S^{n-1}v, S^{n-1}v) + G(S^{n-1}v, S^n y^\star, S^n y^\star).$$

As $S^{n-1}v \preceq S^{n-2}v \preceq \cdots \preceq v \preceq x^\star$ and $S^{n-1}v \preceq y^\star$ for all $n \in N$ as $S^n x^\star = x^\star$ and $S^n y^\star = y^\star$ for all $n \in N$. Then by (2.1)

$$\begin{aligned} G(x^\star, y^\star, y^\star) &\leq kG(S^{n-1}x^\star, S^{n-2}v, S^{n-2}v) + kG(S^{n-2}v, S^{n-1}y^\star, S^{n-1}y^\star) \\ &\vdots \\ G(x^\star, y^\star, y^\star) &\leq k^n G(x^\star, Sv, Sv) + k^n G(Sv, y^\star, y^\star) \to 0 \text{ as } n \to \infty \\ G(x^\star, y^\star, y^\star) &\leq 0, \text{ hence } x^\star = y^\star. \end{aligned}$$

Similarly,

$$G(y^\star, x^\star, x^\star) \leq 0, \text{ hence } y^\star = x^\star.$$

This proves the uniqueness of the fixed point. □

Theorem 12. *Let $(X, \preceq, G)$ be an ordered complete dislocated quasi G-metric space $S : X \to R$ be a mapping and x_0 be an arbitrary point in X. Suppose there exists $k \in \left[0, \frac{1}{2}\right)$ with*

$$G(Sx, Sy, Sz) \leq k(G(x, Sx, Sx) + G(y, Sy, Sy) + G(z, Sz, Sz)) \tag{2.7}$$

for all comparable elements $x, y, z \in \overline{B(x_{0,}r)}$ and

$$G(x_0, Sx_0, Sx_0) \leq (1-\theta)r, \tag{2.8}$$

where $\theta = \frac{k}{1-2k}$. If for nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point $x^\star$ in $\overline{B(x_0, r)}$ such that $x^\star = Sx^\star$ and $G(x^\star, x^\star, x^\star) = 0$. Moreover, if for any three points $x, y, z \in \overline{B(x_o, r)}$, there exists a point v in $\overline{B(x_0, r)}$ such that $v \preceq x$ and $v \preceq y$, $v \preceq z$, where

$$G(x_0, Sx_0, Sx_0) + G(v, Sv, Sv) + G(v, Sv, Sv) \leq G(x_0, v, v) + G(Sx_0, Sv, Sv) + G(Sx_0, Sv, Sv) \tag{2.9}$$

then the point $x^\star$ is unique.

Proof. Consider a picard sequence $x_{n+1} = Sx_n$ with initial guess x_0. Then $x_{n+1} = Sx_n \preceq x_n$ for all $n \in \{0\} \cup N$ and by using inequality (2.8), we have,

$$G(x_0, Sx_0, Sx_0) \leq (1-\theta)r \leq r.$$

Therefore, $x_1 \in \overline{B(x_0, r)}$. Let $x_1, \ldots, x_j \in \overline{B(x_0, r)}$ for some $j \in N$. Thus by using inequality (2.7) we have,

$$\begin{aligned} G(x_{j,}x_{j+1}, x_{j+1}) &= G(Sx_{j-1}, Sx_j, Sx_j) \\ &\leq k[G(x_{j-1,}Sx_{j-1}, Sx_{j-1}) + G(x_j, Sx_j, Sx_j) + G(x_j, Sx_j, Sx_j)], \end{aligned}$$

which implies that,

$$\begin{aligned} G(x_j, x_{j+1}, x_{j+1}) &\le \theta G(x_{j-1}, x_j, x_j) \le \theta^2 G(x_{j-2}, x_{j-1}, x_{j-1}) \\ &\ \vdots \\ &\le \theta^j G(x_0, x_1, x_1) \end{aligned}$$

then,

$$G(x_j, x_{j+1}, x_{j+1}) \le \theta^j G(x_0, x_1, x_1). \tag{2.10}$$

Now by using the inequality (2.8) and (2.10) we have,

$$\begin{aligned} G(x_0, x_{j+1}, x_{j+1}) &\le G(x_0, x_1, x_1) + G(x_1, x_2, x_2) + \cdots + G(x_j, x_{j+1}, x_{j+1}) \\ &\le (1-\theta)r[1+\theta+\theta^2+\cdots+\theta^j] \\ &\le (1-\theta)r\frac{(1-\theta^{j+1})}{(1-\theta)} \le r, \end{aligned}$$

which gives, $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. It implies that inequality (2.10) can be written as,

$$G(x_n, x_{n+1}, x_{n+1}) \le \theta^n G(x_0, x_1, x_1). \tag{2.11}$$

Now by using inequality (2.11) we have,

$$\begin{aligned} G(x_n, x_{n+i}, x_{n+i}) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{n+i-1}, x_{n+i}, x_{n+i}) \\ G(x_n, x_{n+i}, x_{n+i}) &\le \theta^n \frac{(1-\theta^i)}{(1-\theta)} G(x_0, x_1, x_1) \to 0 \ \text{ as } m, n \to \infty. \end{aligned}$$

Notice that the sequence $\{x_n\}$ is G-Cauchy sequence in $(\overline{B(x_0, r)}, G)$. Therefore there exists a point $x^\star \in \overline{B(x_0, r)}$ with $\lim_{n\to\infty} x_n = x^\star$. Also

$$\lim_{n\to\infty} G(x_n, x^\star, x^\star) = \lim_{n\to\infty} G(x^\star, x^\star, x_n) = 0. \tag{2.12}$$

Now,

$$G(x^\star, Sx^\star, Sx^\star) \le G(x^\star, x_n, x_n) + G(x_n, Sx^\star, Sx^\star).$$

By assumption $x^\star \preceq x_n \preceq x_{n-1,}$ therefore,

$$\begin{aligned} G(x^\star, Sx^\star, Sx^\star) \le \lim_{n\to\infty} [&G(x^\star, x_n, x_n) + k\{G(x_{n-1}, Sx_{n-1,} Sx_{n-1}) \\ &+ G(x^\star, Sx^\star, Sx^\star) + G(x^\star, Sx^\star, Sx^\star)\}]. \end{aligned}$$

Thus, $(1-2k)G(x^\star, Sx^\star, Sx^\star) \le 0 \Rightarrow G(x^\star, Sx^\star, Sx^\star) = 0$. Similarly,

$$G(Sx^\star, Sx^\star, x^\star) \le 0,$$

and hence $x^\star = Sx^\star$. Now

$$\begin{aligned} G(x^\star, x^\star, x^\star) &= G(Sx^\star, Sx^\star, Sx^\star) \\ &\le k[G(x^\star, Sx^\star, Sx^\star) + G(x^\star, Sx^\star, Sx^\star) + G(x^\star, Sx^\star, Sx^\star)] \end{aligned}$$

which implies that,

$$(1-3k)G(x^\star, x^\star, x^\star) \le 0.$$

This implies that,

$$G(x^\star, x^\star, x^\star) = 0. \tag{2.13}$$

Uniqueness: Now we show that $x^\star$ is unique. Let $y^\star$ be another point in $\overline{B(x_0, r)}$ such that $y^\star = Sy^\star$. By following similar arguments as in inequality (2.12) we obtain,

$$G(y^\star, y^\star, y^\star) = 0. \tag{2.14}$$

Now if $x^\star \preceq y^*$, then,

$$\begin{aligned} G(x^\star, y^\star, y^\star) &= G(Sx^\star, Sy^\star, Sy^\star) \\ &\leq k[G(x^\star, Sx^\star, Sx^\star) + G(y^\star, Sy^\star, Sy^\star) + G(y^\star, Sy^\star, Sy^\star)] \end{aligned}$$

then, $G(x^\star, y^\star, y^\star) = 0$ by using (2.13) and (2.14). Similarly,

$$G(y^\star, y^\star, x^\star) = 0.$$

Hence, we have $x^\star = y^\star$. Now if $x^\star$ and $y^\star$ are not comparable then there exists a point $v \in \overline{B(x_0, r)}$ which is a lower bound of both $x^\star$ and $y^\star$. Now we will prove that $S^n v \in \overline{B(x_0, r)}$. Moreover by assumptions $v \preceq x^\star \preceq x_n \cdots \preceq x_0$. Now by using inequality (2.7), we have,

$$\begin{aligned} G(Sx_0, Sv, Sv) &\leq k[G(x_0, Sx_0, Sx_0) + G(v, Sv, Sv) + G(v, Sv, Sv)] \\ &\leq k[G(x_0, x_1, x_1) + G(v, Sv, Sv) + G(v, Sv, Sv)] \\ &\leq k[G(x_0, v, v) + G(Sx_0, Sv, Sv) + G(Sx_0, Sv, Sv)] \text{ by using (2.9).} \end{aligned}$$

Hence,

$$G(Sx_0, Sv, Sv) \leq k[G(x_0, v, v) + G(x_1, Sv, Sv) + G(x_1, Sv, Sv)].$$

Thus,

$$G(x_1, Sv, Sv) \leq \theta G(x_0, v, v). \tag{2.15}$$

Now,

$$\begin{aligned} G(x_0, Sv, Sv) &\leq G(x_0, x_1, x_1) + G(x_1, Sv, Sv) \\ &\leq G(x_0, x_1, x_1) + \theta G(x_0, v, v), \text{ by using (2.15)} \\ &\leq (1-\theta)r + \theta r (\text{since } G(x_0, v, v) \leq r). \end{aligned}$$

Thus, $G(x_0, Sv, Sv) \leq r$, then it follows that $Sv \in \overline{B(x_0.r)}$. Now we will prove that $S^n v \in \overline{B(x_0, r)}$. By using the mathematical induction to apply inequality (2.7). Let $S^2 v, \ldots, S^j v \in \overline{B(x_0, r)}$ for some $j \in N$. As

$$S^j v \preceq S^{j-1} v \preceq \cdots \preceq v \preceq x^\star \preceq x_n \preceq \cdots \preceq x_0,$$

then,

$$\begin{aligned} G(S^j v, S^{j+1} v, S^{j+1} v) &= G(S(S^{j-1} v), S(S^j v), S(S^j v)) \\ &\leq k[G(S^{j-1} v, S^j v, S^j v) + G(S^j v, S^{j+1} v, S^{j+1} v) + G(S^j v, S^{j+1} v, S^{j+1} v)] \end{aligned}$$

which implies that,

$$\begin{aligned} G(S^j v, S^{j+1} v, S^{j+1} v) &\leq \theta G(S^{j-1} v, S^j v, S^j v) \\ &\leq \theta^2 G(S^{j-2} v, S^{j-1} v, S^{j-1} v) \\ &\quad \vdots \\ &\leq \theta^j G(v, Sv, Sv). \end{aligned} \tag{2.16}$$

Now,

$$\begin{aligned} G(x_{j+1}, S^j v, S^{j+1} v) &= G(Sx_j, S(S^j v), S(S^j v)) \\ &\leq k\{G(x_j, Sx_j, Sx_j) + G(S^j v, S^{j+1} v, S^{j+1} v) + G(S^j v, S^{j+1} v, S^{j+1} v)\}. \end{aligned}$$

By (2.10) and (2.16), we get

$$\begin{aligned} G(x_{j+1,} S^{j+1} v, S^{j+1} v) &\leq k[\theta^j G(x_0, x_1, x_1) + \theta^j G(v, Sv, Sv) + \theta^j G(v, Sv, Sv)] \\ &\leq k\theta^j [G(x_0, x_1, x_1) + G(v, Sv, Sv) + G(v, Sv, Sv)] \\ &\leq k\theta^j [G(x_0, v, v) + G(x_1, Sv, Sv) + G(x_1, Sv, Sv)] \end{aligned}$$

$$G(x_{j+1,}S^{j+1}v, S^{j+1}v) \leq k\theta^j[G(x_0, v, v) + \theta G(x_0, v, v) + \theta G(x_0, v, v)]$$
$$G(x_{j+1}, S^{j+1}v, S^{j+1}v) = \theta^{j+1}G(x_0, v, v). \tag{2.17}$$

Now,

$$\begin{aligned} G(x_0, S^{j+1}v, S^{j+1}v) &\leq G(x_0, x_1, x_1) + \cdots + G(x_j, x_{j+1}, x_{j+1}) + G(x_{j+1}, S^{j+1}v, S^{j+1}v) \\ &\leq G(x_0, x_1, x_1) + \theta G(x_0, x_1, x_1) + \cdots + \theta^{j+1}G(x_0, v, v) \\ G(x_0, S^{j+1}v, S^{j+1}v) &\leq G(x_0, x_1, x_1)[1 + \theta + \theta^2 + \cdots + \theta^j] + \theta^{j+1}r \\ G(x_0, S^{j+1}v, S^{j+1}v) &\leq (1-\theta)r\frac{(1-\theta^{j+1})}{(1-\theta)} + \theta^{j+1}r = r. \end{aligned}$$

It follows that $S^{j+1}v \in \overline{B(x_0, r)}$ and hence $S^n v \in \overline{B(x_0, r)}$. Now inequality (2.16) can be written as,

$$G(S^n v, S^{n+1}v, S^{n+1}v) \leq \theta^n G(v, Sv, Sv) \to 0 \text{ as } n \to \infty. \tag{2.18}$$

Now,

$$\begin{aligned} G(x^\star, y^\star, y^\star) &= G(Sx^\star, Sy^\star, Sy^\star) \leq G(Sx^\star, S^{n+1}v, S^{n+1}v) + G(S^{n+1}v, Sy^\star, Sy^\star) \\ G(x^\star, y^\star, y^\star) &\leq k[G(x^\star, Sx^\star, Sx^\star) + G(S^n v, S^{n+1}v, S^{n+1}v) \\ &\quad + G(S^n v, S^{n+1}v, S^{n+1}v) + k\{G(S^n v, S^{n+1}v, S^{n+1}v) + 2G(y^\star, Sy^\star, Sy^\star) \\ G(x^\star, y^\star, y^\star) &\leq kG(x^\star, x^\star, x^\star) + 3kG(S^n v, S^{n+1}v, S^{n+1}v) + 2kG(y^\star, y^\star, y^\star) \\ G(x^\star, y^\star, y^\star) &\leq 0 \quad \text{(by (2.13), (2.14) and (2.18))}. \end{aligned}$$

Similarly,

$$G(y^\star, x^\star, x^\star) = 0.$$

Thus, $x^\star = y^\star$. □

The following example exhibits the superiority of our Theorem 12. The mapping is contractive on the closed ball instead on the whole space.

Example 13. Let $X = R^+ \cup \{0\}$ be endowed with usual order and $G : X \times X \times X \to X$ be an ordered complete dislocated quasi G-metric space defined by,

$$G(x, y, z) = \frac{x}{2} + y + z.$$

Let $S : X \to X$ be defined by,

$$Sx = \left\{ \begin{array}{c} \frac{x}{8} \text{ if } x \in \left[0, \frac{1}{2}\right] \\ x - \frac{1}{2} \text{ if } x \in [1, \infty) \end{array} \right\}.$$

Clearly, S is a dominated mappings. Then for $x_0 = \frac{1}{2}$, $r = \frac{3}{2}$, $\theta = \frac{3}{8}$, $\overline{B(x_0, r)} = [0, \frac{1}{2}]$ and for $k = \frac{3}{10}$

$$(1-\theta)r = \left(1 - \frac{3}{8}\right)\frac{3}{2} = \frac{15}{16}$$

and,

$$\begin{aligned} G(x_0, Sx_0, Sx_0) &= G\left(\frac{1}{2}, S\frac{1}{2}, S\frac{1}{2}\right) \\ &= \frac{1}{4} + \frac{1}{16} + \frac{1}{16} = \frac{3}{8} \\ &\Rightarrow G(x_0, Sx_0, Sx_0) \leq (1-\theta)r \\ \frac{3}{8} &\leq \frac{15}{16} \Rightarrow 48 \leq 120. \end{aligned}$$

Also if x, y and $z \in (1, \infty)$. We assume that $x < y$ and $y < z$, then

$$
\begin{aligned}
5x + 10y + 10z &\geq \frac{15}{2}x + \frac{15}{2}y + \frac{15}{2}z + \frac{7}{2} \\
5x + 10y + 10z - 5 - 5 - \frac{5}{2} &\geq \frac{15}{2}x + \frac{15}{2}y + \frac{15}{2}z - 9 \\
10\left[\left(\frac{x}{2} - \frac{1}{4}\right) + \left(y - \frac{1}{2}\right) + \left(z - \frac{1}{2}\right)\right] &\geq 3\left[\left(\frac{x}{2} + x + x - 1\right) + \left(\frac{y}{2} + y + y - 1\right)\right. \\
&\quad \left. + \left(\frac{z}{2} + z + z - 1\right)\right] \\
G(Sx, Sy, Sz) &\geq k\left[\left(\frac{x}{2} + x - \frac{1}{2} + x - \frac{1}{2}\right) + \left(\frac{y}{2} + y - \frac{1}{2} + y - \frac{1}{2}\right)\right. \\
&\quad \left. + \left(\frac{z}{2} + z - \frac{1}{2} + z - \frac{1}{2}\right)\right] \\
G(Sx, Sy, SZ) &\geq k[G(x, Sx, Sx) + G(y, Sy, Sy) + G(z, Sz, Sz)].
\end{aligned}
$$

So the contractive condition does not hold in X. Now if x, y and $z \in \overline{B(x_0, r)}$ then

$$
\begin{aligned}
G(Sx, Sy, Sz) &= \frac{x}{16} + \frac{y}{8} + \frac{z}{8} = \frac{1}{8}\left\{\frac{x}{2} + y + z\right\} \\
&\leq \frac{3}{10}\left\{\frac{x}{2} + \frac{y}{2} + \frac{z}{2}\right\} \leq \frac{3}{10}\left\{\left(\frac{x}{2} + \frac{x}{8} + \frac{x}{8}\right) + \left(\frac{y}{2} + \frac{y}{8} + \frac{y}{8}\right) + \left(\frac{z}{2} + \frac{z}{8} + \frac{z}{8}\right)\right\} \\
G(Sx, Sy, Sz) &\leq \frac{3}{10}\{G(x, Sx, Sx) + G(y, Sy, Sy) + G(z, Sz, Sz)\} \\
G(Sx, Sy, Sz) &\leq k\{G(x, Sx, Sx) + G(y, Sy, Sy) + G(z, Sz, Sz)\}.
\end{aligned}
$$

Hence it satisfies all the requirements of Theorem 12.

Remark 14. In the above example, $\max\{G(x, y, y), G(y, x, x)\} = \frac{5}{2}(x + y)$, is not a metric space. So our results cannot be obtained from metric fixed point results by adopting the technique given in [16,12,22].

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Original article

Approximation in mean on homogeneous compact spaces

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Abstract

Jackson's type theorem on approximation of square integrable functions is proved for functions defined on homogeneous spaces with a compact transitive transformation group actions. An example is proved which illustrates the theorem.

Keywords: Homogeneous space; Compact group; Quotient group; L^2 space; Fourier series

1. Definition and notations

Let $\mathfrak{W}$ be a homogeneous space, and G be a compact transitive transformation group of $\mathfrak{W}$ with respect to 1 at $\int_G dg = 1$. Let a be a fixed point from $\mathfrak{W}$, and consider its stationary subgroup $H = \{h \in G : \ ha = a\}$. There exists the following one-to-one correspondence φ between $\mathfrak{W}$ and the quotient space G/H: if $w \in \mathfrak{W}$ and $g \in G$ transforms a to $w \in \mathfrak{W}$, then the corresponding element $\varphi(w) \in G/H$ is the class gH; conversely, the corresponding to a class gH element in $\mathfrak{W}$ is $w = \varphi^{-1}(gH) = ga$. H is a closed subgroup of G and there exists a G-invariant Radon measure μ on G/H, that is, a Radon measure μ such that $\mu(xE) = \mu(E)$ for every $x \in G$, $E \subset G/H$. ([1], 2.49–2.53, 2.7). μ is unique up to a constant factor, and if this factor is suitably chosen, then

$$\int_G f(g)dg = \int_{G/H} \int_H f(g\xi)d\xi d\mu(gH) \tag{1}$$

for any function f integrable on G with respect to the Haar measure.

A representation T is of class 1 with respect to H, if its carrier space $\mathcal{L}$ contains non-zero vectors invariant with respect to all operators Th, $h \in H$ and all these operators are unitary [2, p. 103]. In what follows, A_H will stand for a set of indices for which $(T_l)_{l \in A_H}$ is the family of all pairwise nonequivalent irreducible representations of G which

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are of class 1 with respect to H. Below, we consider the case when the stationary group H is massive. This means that, for any representation of the group G of class 1 with respect to H, the subspace of vectors in $\mathcal{L}_l$, invariant for H, is one-dimensional ([2], p. 103). Let $\{e_i^l\}$ be an orthonormal basis of $\mathcal{L}_l$ such that $T_l(e_1^l) = e_1^l$.

A measure $d_{\mathfrak{W}}$ on $\mathfrak{W}$ can be introduced by a G-invariant Radon measure $d_{G/H}$ of the compact quotient space G/H as follows: if the set $\varphi(W) \subset G/H$ corresponds to a set $W \subset \mathfrak{W}$, then their measures are equal to each other. Every function f given on $\mathfrak{W}$ with a transformation group G can be regarded as a function on G which is constant on the left cosets with respect to the stationary subgroup H with respect to a point $a \in \mathfrak{W}$. Namely, if the class gH corresponds to a $w \in \mathfrak{W}$, then the function $f_a(g) = f_a(gH) = f(w)$ defined on G corresponds to a function $f(w)$. We say that a defined on $\mathfrak{W}$ function $f(w)$ belongs to the space $L^2(\mathfrak{W})$, if $f_a(\varphi(w)) \in L^2(G/H)$ or $f_a(g) \in L^2(G)$. Conversely, if a function $f_a \in L^2(G)$ is constant on the left cosets with respect to H, then its corresponding function $f(w)$ belongs to $L^2(\mathfrak{W})$. $L^2(\mathfrak{W})$ is a Hilbert space with respect to the usually norm. It is clear that an expansion of a function $f \in L^2(\mathfrak{W})$ on a homogeneous space $\mathfrak{W}$ can be obtained by means of the Fourier expansion of $f_a \in L^2(G)$, which is invariant under left shifts by elements of the corresponding to $a \in \mathfrak{W}$ subgroup H.

Let us denote by $L_H^2(G)$ the subspace in $L^2(G)$ which is invariant under left shifts by elements of the massive stationary subgroup H of a fixed $a \in \mathfrak{W}$. Any function f_a from $L_H^2(G)$ can be expanded into the Fourier series of the form ([2], p. 105)

$$f_a(g) = \sum_{l \in A_H} Y_l(f_a, g), \quad \text{where} \quad Y_l(f_a, g) = \sum_{m=1}^{d_l} c_m^l t_{m1}^l(g). \tag{2}$$

The integer d_l in (2) is the dimension of the carrier space $\mathcal{L}_l$ of the representation $T_l(g)$ and $t_{m1}^l(g)$, $1 \le m \le d_l$ are the matrix functions of the representation T_l. Together with $f_a(g)$, the matrix functions $t_{m1}^l(g)$ are also invariant under left shifts by elements of the subgroup H ([2], p. 104). The coefficients c_m^l, according to (1), are given by the equalities

$$\begin{aligned} c_m^l &= d_l \int_G f_a(g)\overline{t_{m1}^l(g)}dg = d_l \int_{G/H}\int_H f_a(g\xi)\overline{t_{m1}^l(g\xi)}d\xi d\mu(gH) \\ &= d_l \text{mes} H \int_{G/H} f_a(u)\overline{t_{m1}^l(u)}d_\mu(u) = d_l \text{mes} H \int_{\mathfrak{W}} f(w)\overline{t_{m1}^l(\varphi(w))}dw, \end{aligned} \tag{3}$$

where mes$H$ is Haar measure of subgroup $H \subset G$.

The index l in (2) becomes a countable number of values, for which $c_m^l \ne 0$. List them as $\{l_1, \dots, l_n, \dots\}$. Following [3], the symbol $l < n$ is interpreted as $l \in \{l_1, \dots, l_n\}$, and $l \ge n$ denotes that $l \in A_H \setminus \{l_1, \dots, l_n\}$.

Thus, if $f \in L^2(\mathfrak{W})$ and H is a massive stationary subgroup of an element $a \in \mathfrak{W}$ in G, then f can be expanded in the series of the form

$$f(w) = \sum_{l \in A_H} \sum_{m=1}^{d_l} c_m^l t_{m1}^l(\varphi(w)),$$

where the coefficients c_m^l are defined by (3).

If

$$S_n(f, w) = \sum_{l \in A_H, l \le n} \sum_{m=1}^{d_l} c_m^l t_{m1}^l(\varphi(w)), n \in \mathbb{N},$$

is the nth partial sum of this series, and $S_n(f_a, g) - n$th partial sum of the series (2), then

$$f(w) - S_n(f, w) = f_a(g) - \sum_{l \in A_H;\ l \le n} Y_l(f_a, g). \tag{4}$$

It follows from this that the sum $S_n(f, w)$ is the unique element of best approximation in $L^2(\mathfrak{W})$ by means of sums $\sum_{l\in A_H, l\le n}\sum_{m=1}^{d_l} a_m^l t_{m1}^l(\varphi(w))$, i.e.

$$
\begin{aligned}
E_n(f)_2 &:= \|f - S_n(f)\|_{L^2(\mathfrak{W})} \\
&= \inf \|f(w) - \sum_{l\in A_H, l\le n}\sum_{m=1}^{d_l} a_m^l t_{m1}^l(\varphi(w))\|_{L^2(\mathfrak{W})},
\end{aligned} \tag{5}
$$

where inf is taken with respect to complex numbers $a_m^l,\ 1 \le m \le d_l$.

Let $\{U_n\},\ n \in \mathbb{N}$ be a sequence of neighborhoods of unity $e \in G$, such that $\text{mes}U_n \to 0$ if $n \to \infty$. For $l \in A_H$, we denote by χ_l the character and by d_l the dimension of the representation T_l of the group G.

Definition. Let $f \in L^2(\mathfrak{W}),\ a \in \mathfrak{W}$ and χ_l be the character of the representation T_l which corresponds to the index l from expansion (2). Let $\{U_n\},\ n \in \mathbb{N}$, be a sequence of neighborhoods of unity $e \in G$, such that $\text{mes}U_n \to 0$ if $n \to \infty$, and k be a fixed natural number. We say that $\{U_n\}$ satisfy the condition (k, n_0, r, θ), if there exist some positive numbers $n_0,\ r$ and θ such that for any natural number $n \ge n_0$ the inequality

$$
(\text{mes}U_n)^{-1}\int_{U_n} |1 - \chi_l(g)/d_l|^{2k} dg \ge \theta^2, \tag{6}
$$

is true for all $l \ge rn$.

Remark. Using Bernoulli's inequality $(1 + x)^k \ge 1 + kx$ for $x \ge -1$, we obtain that

$$
\begin{aligned}
|1 - \chi_l(g)/d_l|^{2k} &\ge (1 - 2\text{Re}\ \chi_l(g)/d_l + |\chi_l(g)|^2/d_l^2)^k \\
&\ge (1 - 2k\text{Re}\chi_l(g))/d_l + k|\chi_l(g)|^2/d_l^2.
\end{aligned}
$$

Therefore, the condition (6) will be satisfied, if

$$
\{\text{mes}\ U_n\}^{-1}\int_{U_n}(1 - 2k\text{Re}\ \chi_l(g)/d_l + k|\chi_l(g)|^2/d_l^2)dg \ge \theta^2.
$$

For a function $f \in L^2(\mathfrak{W})$, a natural number $k \in \mathbb{N}$, and a neighborhood U of unity $e \in G$, we consider the following quantity

$$
\omega_k(f, U)_2 := \omega_k(f, U)_{L^2(\mathfrak{W})} := \left((\text{mes}U)^{-1}\int_U \|\Delta_u^k f\|^2_{L^2(\mathfrak{W})} du\right)^{1/2}, \tag{7}
$$

where

$$
\Delta_u f(w) := f(w) - \int_G f(tut^{-1}w)dt,\ \Delta_u^k f = \Delta_u(\Delta_u^{k-1} f)\ u \in G,\ k \in \mathbb{N}.
$$

Note, that for a function $f^* \in L^2(G)$ the quantity

$$
\Delta_u f^*(g) := f^*(g) - \int_G f^*(tut^{-1}g)dt
$$

was considered in [3]. We call the quantity (7) as the kth average modulus of smoothness of a function $f \in L^2(\mathfrak{W})$ (according to the neighborhood U). For a neighborhood $U_n,\ n \in \mathbb{N}$ of unity $e \in G$, we will use the notation $\omega_k(f, n^{-1})_2 := \omega_k(f, U_n)_2$.

Let $H \subset G$ be the stationary subgroup of a point $a \in \mathfrak{W}$ and let $f_a \in L^2(G)$ be the corresponding function to the $f \in L^2(\mathfrak{W})$. If we recall that f_a is constant on the left cosets with respect to H, then we obtain with the help of (1) that

$$
\omega_k(f, n^{-1})_{L^2(\mathfrak{W})} := \left((\text{mes}U_n)^{-1}\text{mes}H\int_{U_n} \|\Delta_u^k f_a\|^2_{L^2(G)} du\right)^{1/2}.
$$

Fix $k \in \mathbb{N}$; it is easy to verify the following properties of kth average modulus of smoothness:

(a) If for a sequence U_n, $n \in \mathbb{N}$ of neighborhoods of $e \in G$ we have $\text{mes}U_n \to 0$ as $n \to \infty$ then $\lim \omega_k(f, n^{-1})_2 = 0$ for each $f \in L^2(\mathfrak{W})$;

(b)
$$\omega_k(f_1 + f_2, n^{-1})_2 \le \omega_k(f_1, n^{-1})_2 + \omega_k(f_2, n^{-1})_2;$$

(c)
$$\omega_{k+l}(f, n^{-1})_2 \le 2^l \omega_k(f, n^{-1})_2.$$

We note that $\omega_k(f, n^{-1})_2 \le \omega'_k(f, n^{-1})_2 := \sup\{\|\Delta_u^k f\|_2,\ u \in U_n\}$, where ω'_k have the well-known properties of modulus of smoothness (the properties (a), (b), (c) and the property $\omega'_k(f, n^{-1})_2 \le \omega'_k(f, (n+1)^{-1})_2$ for $U_n \subset U_{n+1}$ [3]).

2. Results

Theorem 1. *Let $\mathfrak{W}$ be a homogeneous space, G be a compact transitive transformation group of $\mathfrak{W}$ with the normalized Haar measure dg and $f \in L^2(\mathfrak{W})$. Suppose that G contains a massive stationary subgroup of a fixed point $a \in \mathfrak{W}$. Let $E_n(f)_2$ (resp. $\omega_k(f, n)_2$) are defined by (5) (resp. (7)) and for a sequence of neighborhoods $\{U_n\}$ the condition (k, n_0, r, θ) of Definition is fulfilled. Then, the following inequality holds*

$$E_{rn}(f)_2 = \|f - S_{rn}(f)\|_{L^2(\mathfrak{W})} \le \theta^{-1} \omega_k(f, n^{-1})_{L^2(\mathfrak{W})},\ n \ge n_0.$$

Proof. Let H be the stationary subgroup of $a \in \mathfrak{W}$ and f_a be the corresponding to f function according to the above mentioned correspondence. Recall that f_a is constant on the left cosets with respect to H. Then, according to (1), (4), and Parseval's equality, we obtain from (4) that

$$(\text{mes}H)\|f - S_n(f)\|^2_{L^2(\mathfrak{W})} = \|f_a - S_n(f_a)\|^2_{L^2(G)} = \sum_{l \in A_H; l \ge n} d_l^{-1} \sum_{m=1}^{d_l} |c_m^l|^2. \tag{8}$$

It follows from the equality $\int_G t^l_{m1}(tut^{-1}g)dt = d_l^{-1}\chi_l(u)t^l_{m1}(g)$ ([3], Lemma 3.1) that

$$(Y_l^k(\Delta f_a))(g) = (1 - \chi_l(u)/d_l)^k (Y_l f_a)(g),\ l \in A_H. \tag{9}$$

Consequently,

$$(\Delta_u^k f_a)(g) = \sum_{l \in A_H} d_l^{-1} \sum_{m=1}^{d_l} (1 - \chi_l(u)/d_l)^k c_m^l t^l_{m1}(g).$$

Let neighborhoods U_n and an integer r are chosen according to Definition. By application of Parseval's equality, we obtain from (9)

$$\begin{aligned}\|\Delta_u^k f_a(g)\|^2_{L^2(G)} &= \sum_{l \in A_H} d_l^{-1} \sum_{m=1}^{d_l} |1 - \chi_l(u)/d_l|^{2k} |c_m^l|^2 \\ &\ge \sum_{l \in A_H, l \ge rn} d_l^{-1} \sum_{m=1}^{d_l} |1 - \chi_l(u)/d_l|^{2k} |c_m^l|^2.\end{aligned}$$

Integrating this inequality on the neighborhood U_n and applying the property (6) of Definition and (8), we get

$$\begin{aligned}&(\text{mes}U_n)^{-1} \int_{U_n} \|\Delta_u^k f_a(g)\|^2_{L^2(G)} du \\ &\ge \sum_{l \in A_H} (d_l \text{mes}U_n)^{-1}) \sum_{m=1}^{d_l} \int_{U_n} |1 - \chi_l(u)/d_l|^{2k} |du| c_m^l|^2 \\ &\ge \theta^2 \sum_{l \in A_H; l \ge rn} d_l^{-1} \sum_{m=1}^{d_l} |c_m^l|^2 = \theta^2 \|f_a - S_{rn}(f_a)\|^2_{L^2(G)}.\end{aligned} \tag{10}$$

Because of the functions f_a and t_m^l are constant on the left cosets with respect to H, we have that $\|\Delta_u^k f_a\|^2_{L^2(G)} = (\mathrm{mes} H)\|\Delta_u^k f\|^2_{L^2(\mathfrak{W})}$ and $\|f_a - S_{rn}(f_a)\|^2_{L^2(G)} = (\mathrm{mes} H)\|f - S_{rn}(f)\|^2_{L^2(\mathfrak{W})}$. Therefore, Theorem 1 follows from (10). □

For an illustration of Theorem 1, we consider the following example. Let $\mathfrak{W} = \mathfrak{S}^2$ be the unit sphere in the three dimensional space $\mathbb{R}^3$. $\mathfrak{S}^2$ is the homogeneous space. The compact transitive transformation group $G = SU(2)$ operates on it ([2], p. 269). This group consists of unimodular unitary matrices of the second order, i.e. of matrices

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$. $SU(2)$ operates on $\mathfrak{S}^2$ in the following way. A matrix $\left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right) \in SU(2)$ takes the point $(x, y, z) \in \mathfrak{S}^2$ to $(x', y', z') \in \mathfrak{S}^2$ according to the following equation ([4], p. 32)

$$\begin{pmatrix} z' & x' + iy' \\ x' - iy' & -z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} z & x + iy \\ x - iy & -z \end{pmatrix} \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix}.$$

The stationary subgroup, which corresponds to a fixed point $a \in \mathfrak{S}^2$, is the subgroup of rotations around the axes passing through this point. We will use the stationary subgroup H of the points $(0, 0, 1)$. This group consists of the diagonal matrices of the form

$$\begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix},$$

which corresponds to the rotations around the axes Oz by the angle t. Since the group $SU(2)$ is compact, there exists an invariant scalar product on the carrier space $\mathcal{L}_l$ of its finite dimensional representation T_l ([2], p. 278–279). If $\mathcal{L}_l$ is the space of the $2l$th order polynomials, then $T_l(h)x^{l-k} = e^{-kt}x^{l-k}$ for all $h \in H$. It follows from this that this subgroup H is massive. α and β may be represented by three real parameters, for example by φ, θ, and ψ, called Euler angles. These parameters are connected with $|\alpha|$, $\arg\alpha$, and $\arg\beta$ by $|\alpha| = \cos\theta/2$, $\mathrm{Arg}\alpha = (\varphi + \phi)/2$, and $\mathrm{Arg}\beta = (\varphi - \phi + \pi)/2$. If $0 \le \varphi < 2\pi$, $0 < \theta < \pi$, and $-2\pi \le \psi < 2\pi$, then the correspondence $(\alpha, \beta) \to (\varphi, \theta, \psi)$, where $\alpha\beta \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$, is one-to-one. If $\alpha\beta = 0$, then for the uniqueness of the correspondence, we assume that the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ corresponds to the triple $(0, 0, 0)$, the matrix $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$ to the triple $(0, \pi, 0)$ and so on ([4], p. 28). Thus, the parametrization φ, θ, ψ is determined almost everywhere on $SU(2)$. The character $\chi_l(g)$, $g \in G$, of the group $SU(2)$ in the carrier space $\mathfrak{L}_l$ is the sum $\sum_{m=-l}^{l} t^l_{mm}(g)$, or, in terms of the Eulers angles, the sum ([2], p. 358)

$$\chi_l(\varphi, \theta, \psi) = \sum_{m=-l}^{l} e^{-im(\varphi+\psi)} P^l_{mm}(\cos\theta).$$

The form of functions P^l_{mm} is given in [2] (p. 347). However, this formula is not convenient since the character there is represented as a function of three variables. Every class of conjugate elements is given by one parameter t, $-2\pi \le t \le 2\pi$, and t and $-t$ define the same class. Therefore, we can assume that the characters are the functions of parameter t, varying from 0 to 2π. Moreover, $\cos\frac{t}{2} = \cos\frac{\theta}{2}\cos\frac{\varphi+\psi}{2}$. It is proved in [2] (p. 359), that $\chi_l(g) = (\sin\frac{t}{2})^{-1}\sin(l+\frac{1}{2})t$. According to the described correspondence, the image of $[0, \pi/n]$ is some set $U_n \subset G$, moreover $e \in U_n$ and $U_n \to e$, if $n \to \infty$. It was proved in [2] (p. 362) that if the function $f(g)$ is constant on classes of conjugate elements, i.e. depends on t only: $f(g) := F(t)$, then $\int_G f(g)dg = \pi^{-1}\int_0^{2\pi} F(t)\sin^2\frac{t}{2}dt$. Thus, for the expression in (6), we obtain

$$\begin{aligned} \int_{U_n} \left|1 - \frac{\chi_l(g)}{d_l}\right|^2 dg &= \pi^{-1}\int_0^{\pi/n} \left|1 - \frac{\sin(l+1/2)t}{d_l \sin t/2}\right|^2 \sin^2 t/2dt \\ &= \mathrm{mes}U_n - \frac{2}{\pi d_l}\int_0^{\pi/n} \sin(l+1/2)t\sin t/2dt + \frac{1}{\pi d_l^2}\int_0^{\pi/n} \sin^2(l+1/2)dt \\ &= \mathrm{mes}U_n - J_1 + J_2. \end{aligned} \tag{11}$$

Now, we estimate J_1 and J_2.

$$\begin{aligned} J_1 &= \frac{2}{\pi d_l}\int_0^{\pi/n}\sin(l+1/2)t\sin t/2dt \\ &= \frac{1}{\pi d_l}\left(\left(\frac{1}{l}-\frac{1}{l+1}\right)\sin\frac{l\pi}{n}+\frac{1}{l+1}\left(\sin\frac{l\pi}{n}-\sin\frac{(l+1)\pi}{n}\right)\right) \\ &= \frac{1}{\pi d_l}\left(\frac{1}{l(l+1)}\sin\frac{l\pi}{n}-\frac{2}{l+1}\sin\frac{\pi}{2n}\cos\frac{(2l+1)\pi}{2n}\right) \\ &\le \frac{1}{\pi d_l}\left(\frac{1}{l(l+1)}+\frac{2}{l+1}\sin\frac{\pi}{2n}\right)\le\frac{1}{\pi d_l(l+1)}\left(\frac{1}{l}+\frac{\pi}{c}c\,()\,n\right). \end{aligned} \tag{12}$$

$$J_2=\frac{1}{2\pi d_l^2}\int_0^{\pi/n}(1-\cos(2l+1)t)dt\ge\frac{1}{2\pi d_l^2}\left(\frac{\pi}{n}-\frac{1}{2l+1}\right). \tag{13}$$

It follows from the Taylor well-known formula that

$$\frac{t^3}{6}\cos\frac{\pi}{n}\le t-\sin t\le\frac{t^3}{6},\ 0\le t\le\frac{\pi}{n},\ n\ge 2.$$

Therefore, we have for $\mathrm{mes}U_n=\frac{1}{\pi}\int_0^{\pi/n}\sin^2\frac{t}{2}dt$ that

$$\frac{12n^3}{\pi^2}\le(\mathrm{mes}U_n)^{-1}\le\frac{12n^3}{\pi\cos\frac{\pi}{n}},\ n\ge 3. \tag{14}$$

From (11)–(14), we obtain for $l\ge rn$

$$\begin{aligned}(\mathrm{mes}U_n)^{-1}\int_{U_n}\left|1-\frac{\chi_l(g)}{d_l}\right|^2dg &\ge 1+\frac{6n^3}{\pi^3d_l^2}\left(\frac{\pi}{n}-\frac{1}{2l+1}\right)-\frac{12n^3}{\pi^3d_l(l+1)}\left(\frac{1}{l}+\frac{\pi}{n}\right)\frac{1}{\cos\frac{\pi}{n}} \\ &> 1-\frac{3n^2}{\pi^2d_l(l+1)}\left(\frac{4}{\cos\frac{\pi}{n}}-1\right)-\frac{6n^3}{\pi^3d_l^3}-\frac{12n^3}{\pi^3d_l(l+1)l\cos\frac{\pi}{n}} \\ &\ge 1-\frac{3}{2\pi^2r^2}\left(\frac{4}{\cos\frac{\pi}{n}}-1\right)-\frac{3}{4\pi^3r^3}-\frac{6}{\pi^3r^3\cos\frac{\pi}{n}},\ n\ge 3,\ r\ge 1.\end{aligned}$$

Performing similar calculations, one can obtain in the considered example that it is possible to take the following constants in Theorem 1 for $n\ge 6$:

1. If $r=1$, then $\theta^{-1}=2.2226$;
2. If $r=1.5$, then $\theta^{-1}=1.2108$;
3. If $r=2$, then $\theta^{-1}=1.0966$;
4. If $r=3$, then $\theta^{-1}=1.0371$.

Due to above reasoning, we can formulate the following.

Theorem 2. *Let $\mathfrak{S}^2$ be the unit sphere in the three dimensional space $\mathbb{R}^3$, and f be a function, from $L^2(\mathfrak{S}^2)$. Then, for $n\ge 3,\ r\ge 1$, in* (6) *and in Theorem* 1*, we can take*

$$\theta=1-\frac{3}{2\pi^2r^2}\left(\frac{4}{\cos\frac{\pi}{n}}-1\right)-\frac{3}{4\pi^3r^3}-\frac{6}{\pi^3r^3\cos\frac{\pi}{n}},\ n\ge 3,\ r\ge 1.$$

Let $f(w)$ be a function from $L^2(\mathfrak{S}^2)$, and $f_a\in L^2(SU(2))$ be the corresponding function to f function, whose Fourier expansion is written in the form $f_a(g)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_m^lt_{m1}^l(g),\ g\in SU(2)$. Then, the expansion for $f(w),\ w\in\mathfrak{S}^2$, coincides with the Fourier–Laplace series ([2], p. 367) and has the form $f(w)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_m^le^{im\varphi}P_l^m(\cos(\theta))$, where $\varphi,\theta,\ 0\le\varphi\le 2\pi,\ 0<\theta<\pi$, are the spherical coordinates of w, $P_l^m(\cos(\theta))$ are the Legendre adjoint functions and

$$c_m^l=\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\int_0^{2\pi}\int_0^{\pi}f(\varphi,\theta,0)e^{im\varphi}P_l^m(\cos(\theta))\sin\theta d\theta d\varphi.$$

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Original article

Higher-order commutators of parametrized Littlewood–Paley operators on Herz spaces with variable exponent

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Abstract

Let $\Omega \in L^2(\mathrm{S}^{n-1})$ be a homogeneous function of degree zero and b be a BMO or Lipschitz function. In this paper, we obtain some boundedness of the parametrized Littlewood–Paley operators and their high-order commutators on Herz spaces with variable exponent.

Keywords: Herz space; Variable exponent; Commutator; Parametrized area integral; Parametrized Littlewood–Paley g_λ^* function

1. Introduction

The theory of function spaces with variable exponent has been extensively studied by researchers since the work of Kováčik and Rákosník [1] appearing in 1991. In [2–5] and [6], the authors proved the boundedness of some integral operators on variable L^p spaces.

Given an open set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : E \longrightarrow [1, \infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\lambda > 0$,

$$\int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg–Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

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These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

The space $L^{p(\cdot)}_{\text{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(E) := \{f : f \in L^{p(\cdot)}(F) \text{ for all compact subsets } F \subset E\}.$$

Define $\mathcal{P}(E)$ to be the set of $p(\cdot) : E \longrightarrow [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in E\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in E\} < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$.

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy,$$

where $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A, respectively.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies*

$$|p(x) - p(y)| \le \frac{C}{-\log(|x - y|)}, \quad |x - y| \le 1/2 \tag{1.1}$$

and

$$|p(x) - p(y)| \le \frac{C}{\log(|x| + e)}, \quad |y| \ge |x|, \tag{1.2}$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, that is the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 1.2 ([1])**.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is called the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.3 ([4])**.** *Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1} \text{ and } \frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Throughout this paper δ_1 and δ_2 are the same as in Lemma 1.3.

Lemma 1.4 ([4])**.** *Suppose $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C.$$

Next we recall the definition of the Herz-type spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.

Definition 1.1 (*[4]*). Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

The non-homogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha p} \|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n (n \ge 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in L^1(\mathbb{R}^n)$, be a homogeneous function of degree zero and

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.3}$$

where $x' = x/|x|$ for any $x \ne 0$. The parametrized Littlewood–Paley area integral $\mu^{\rho}_{\Omega,S}$ and g^*_{λ} function $\mu^{*,\rho}_{\Omega,\lambda}$ are defined by

$$\mu^{\rho}_{\Omega,S}(f)(x) = \left(\iint_{\Gamma(x)} \left|\frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$\mu^{*,\rho}_{\Omega,\lambda}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$, $\rho > 0$ and $\lambda > 1$.

For an integer $m \ge 1$, let b be a locally integrable function on $\mathbb{R}^n$, the commutators $[b^m, \mu^{\rho}_{\Omega,S}]$ and $[b^m, \mu^{*,\rho}_{\Omega,\lambda}]$ are defined by

$$[b^m, \mu^{\rho}_{\Omega,S}](f)(x) = \left(\iint_{\Gamma(x)} \left|\frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} [b(y)-b(z)]^m f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} [b(y)-b(z)]^m f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}.$$

In [7], the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the parametrized Littlewood–Paley operators and their commutators was given by Wang and Tao. Motivated by [8,9], we will study the boundedness for the parametrized Littlewood–Paley operators and their commutators on the Herz space with variable exponent, where $\Omega \in L^2(\mathrm{S}^{n-1})$.

2. Estimate for the parametrized Littlewood–Paley operator

In this section we will prove the boundedness of the parametrized Littlewood–Paley area integral $\mu^{\rho}_{\Omega,S}$ and g^*_{λ} function $\mu^{*,\rho}_{\Omega,\lambda}$ on Herz spaces with variable exponent.

Let $\Omega \in L^s(\mathrm{S}^{n-1})$ with $s \ge 1$ be homogeneous of degree zero on $\mathbb{R}^n$. The definition of the integral modulus $\omega_s(\delta)$ of continuity of order s of Ω is defined by

$$\omega_s(\delta) = \sup_{\|\rho\| \le \delta} \left(\int_{\mathrm{S}^{n-1}} \left|\Omega(\rho x') - \Omega(\rho x')\right|^s dx'\right)^{1/s}$$

and ρ is a rotation on S^{n-1} and $\|\rho\| = \sup_{x' \in \mathrm{S}^{n-1}} |\rho x' - x'|$.

Theorem 2.1. *Suppose that* $0 < p \le \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\rho > n/2$, $\lambda > 2$, $\Omega \in L^2(\mathrm{S}^{n-1})$ *satisfying* (1.3) *and the following condition*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^\sigma d\delta < \infty, \quad \text{for } \sigma > 2. \tag{2.1}$$

If $-n\delta_1 < \alpha < n\delta_2$*, then the parametrized Littlewood–Paley* g_λ^* *function* $\mu_{\Omega,\lambda}^{*,\rho}$ *is bounded on* $\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ *and* $K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$.

Proof. We only prove the homogeneous case. The non-homogeneous case can be proved in the same way. We suppose $0 < p < \infty$, since the proof of the case $p = \infty$ is easier. Let $f \in \dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$. Denote $f_j = f\chi_j$ for each $j \in \mathbb{Z}$, we decompose $f(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Then we have

$$\begin{aligned}
\|\mu_{\Omega,\lambda}^{*,\rho}(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|\mu_{\Omega,\lambda}^{*,\rho}(f)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\|\mu_{\Omega,\lambda}^{*,\rho}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{k+1}\|\mu_{\Omega,\lambda}^{*,\rho}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\|\mu_{\Omega,\lambda}^{*,\rho}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&=: CI_1 + CI_2 + CI_3.
\end{aligned} \tag{2.2}$$

We first estimate I_2, by the $(L^{q(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $\mu_{\Omega,\lambda}^{*,\rho}$ we have

$$I_2 \le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p} = C\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \tag{2.3}$$

Now we estimate I_1. By the Minkowski inequality we have

$$\begin{aligned}
|\mu_{\Omega,\lambda}^{*,\rho}(f_j)(x)| &= \left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
&= \left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
&\le \int_{\mathbb{R}^n}|f_j(z)|\left(\int_0^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz \\
&\le \int_{\mathbb{R}^n}|f_j(z)|\left(\int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz \\
&\quad + \int_{\mathbb{R}^n}|f_j(z)|\left(\int_{|x-z|}^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz.
\end{aligned} \tag{2.4}$$

Note that $z \in A_j$ and $|y-z| < t$, so we know that $|y-z| \sim |y|$. Then for $\Omega \in L^2(\mathrm{S}^{n-1})$, we have

$$\begin{aligned}
\int_{|y-z|<t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy &\le \int_{|y|<t}\frac{|\Omega(y)|^2}{|y|^{2n-2\rho}}dy \\
&\le \int_0^t r^{2\rho-n-1}dr\int_{\mathrm{S}^{n-1}}|\Omega(y')|^2 d\sigma(y') \\
&\le t^{2\rho-n}\|\Omega\|_{L^2(\mathrm{S}^{n-1})}^2.
\end{aligned} \tag{2.5}$$

For $\lambda > 2$, we take $0 < \theta < (\lambda - 2)n$. Since $|x - z| \le |x - y| + |y - z| \le |x - y| + t$, by (2.5) we have

$$\begin{aligned}
&\int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\\
&\le \int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n-2n-\theta}\frac{1}{|x-z|^{2n+\theta}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho-n-\theta+1}}\\
&\le \frac{1}{|x-z|^{2n+\theta}}\int_0^{|x-z|}\int_{|y-z|<t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho-n-\theta+1}}\\
&\le \frac{\|\Omega\|^2_{L^2(S^{n-1})}}{|x-z|^{2n+\theta}}\int_0^{|x-z|}t^{\theta-1}dt\\
&\le C|x-z|^{-2n}.
\end{aligned} \tag{2.6}$$

Similarly, noting that $|y - z| \sim |y|$, by (2.5) we have

$$\begin{aligned}
\int_{|x-z|}^{\infty}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}} &\le \int_{|x-z|}^{\infty}\int_{|y-z|<t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\\
&\le \|\Omega\|^2_{L^2(S^{n-1})}\int_{|x-z|}^{\infty}t^{-2n-1}dt\\
&\le C|x-z|^{-2n}.
\end{aligned} \tag{2.7}$$

Note that $x \in A_k,\ z \in A_j$ and $j \le k - 2$. By (2.5), (2.6) and the generalized Hölder inequality we have

$$\begin{aligned}
|\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)| &\le C\int_{\mathbb{R}^n}\frac{|f_j(z)|}{|x-z|^n}dz\\
&\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{2.8}$$

By Lemmas 1.3 and 1.4 we have

$$\begin{aligned}
\|\mu^{*,\rho}_{\Omega,\lambda}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\
&\le C2^{(j-k)n\delta_2}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
I_1 &\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_2}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&= C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\alpha)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.
\end{aligned}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned}
I_1 &\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\alpha)p/2}\right)\times\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\alpha)p'/2}\right)^{p/p'}\right\}^{1/p}\\
&\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p}\\
&= C\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p}
\end{aligned}$$

$$\begin{aligned}&\le C\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}\\&= C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{2.9}$$

If $0 < p \le 1$, then we have

$$\begin{aligned}I_1 &\le C\left\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{j\alpha p}2^{(j-k)(n\delta_2-\alpha)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}\\&= C\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\alpha)p}\right)\right\}^{1/p}\\&\le C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{2.10}$$

Let us now estimate I_3. Note that $x \in A_k,\ y \in A_j$ and $j \ge k+2$, so we have $|y-z| \sim |y|$. By (2.3)–(2.6) and the generalized Hölder inequality we have

$$\begin{aligned}|\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)| &\le C\int_{\mathbb{R}^n}\frac{|f_j(z)|}{|x-z|^n}dz\\&\le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{2.11}$$

By Lemmas 1.3 and 1.4 we have

$$\begin{aligned}\|\mu^{*,\rho}_{\Omega,\lambda}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\le C\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\&\le C2^{(k-j)n\delta_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}I_3 &\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\&= C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1+\alpha)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.\end{aligned}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_1 + \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned}I_3 &\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\alpha)p/2}\right)\times\left(\sum_{j=-\infty}^{k-2}2^{(k-j)(n\delta_1+\alpha)p'/2}\right)^{p/p'}\right\}^{1/p}\\&\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\alpha)p/2}\right)\right\}^{1/p}\\&= C\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\alpha)p/2}\right)\right\}^{1/p}\\&\le C\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}\\&= C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{2.12}$$

If $0 < p \le 1$, then we have

$$\begin{aligned} I_3 &\le C\left\{\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty} 2^{j\alpha p}2^{(k-j)(n\delta_1+\alpha)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\ &= C\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\alpha)p}\right)\right\}^{1/p} \\ &\le C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{2.13}$$

Therefore, by (2.2), (2.3), (2.9), (2.10), (2.12) and (2.13) we complete the proof of Theorem 2.1.

Since $\mu^{\rho}_{\Omega,S}(f)(x) \le C_\lambda \mu^{*,\rho}_{\Omega,\lambda}(f)(x)$, we easily obtain the following theorem.

Theorem 2.2. *Suppose that* $0 < p \le \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\rho > n/2$, $\Omega \in L^2(S^{n-1})$ *satisfying* (1.3) *and* (2.1). *If* $-n\delta_1 < \alpha < n\delta_2$, *then the parametrized Littlewood–Paley area integral* $\mu^{\rho}_{\Omega,S}$ *is bounded on* $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *and* $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

3. BMO estimate for the commutators of parametrized Littlewood–Paley operators

Let us first recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_* = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx < \infty,$$

where $f_Q = |Q|^{-1}\int_Q f(y)dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

Next, we will give the BMO estimate for the commutators $[b^m, \mu^{\rho}_{\Omega,S}]$ and $[b^m, \mu^{*,\rho}_{\Omega,\lambda}]$ on Herz spaces with variable exponent.

Theorem 3.1. *Suppose that* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $m \in \mathbb{Z}_+$, $0 < p \le \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\rho > n/2$, $\lambda > 2$, $\Omega \in L^2(S^{n-1})$ *satisfying* (1.3) *and* (2.1). *If* $-n\delta_1 < \alpha < n\delta_2$, *then* $[b^m, \mu^{*,\rho}_{\Omega,\lambda}]$ *is bounded on* $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *and* $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

In the proof of Theorem 3.1, we also need the following lemma.

Lemma 3.2 *([5])*. *Let* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, *m be a positive integer and B be a ball in* $\mathbb{R}^n$. *Then we have that for all* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and all* $j, i \in \mathbb{Z}$ *with* $j > i$,

$$\frac{1}{C}\|b\|^m_* \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)^m\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|^m_*,$$

$$\|(b-b_{B_i})^m\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(j-i)^m\|b\|^m_*\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $B_i = \{x \in \mathbb{R}^n : |x| \le 2^i\}$ *and* $B_j = \{x \in \mathbb{R}^n : |x| \le 2^j\}$.

Proof of Theorem 3.1. Similar to Theorem 2.1, we only prove the homogeneous case and still suppose $0 < p < \infty$. Let $f \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, and we write $f(x) = \sum_{j=-\infty}^{\infty} f\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Then we have

$$\begin{aligned} \|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\ &\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{k+1}\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \end{aligned}$$

$$+C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\|[b^m,\mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}$$
$$=: CJ_1+CJ_2+CJ_3. \tag{3.1}$$

Noting that $[b^m,\mu^{*,\rho}_{\Omega,\lambda}]$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, so we have

$$J_2\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|f_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}=C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{3.2}$$

Now we estimate J_1. By the Minkowski inequality we have

$$\begin{aligned}
&|[b^m,\mu^{*,\rho}_{\Omega,\lambda}](f_j)(x)|\\
&=\left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&=\left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&\le\int_{\mathbb{R}^n}|b(x)-b(z)|^m|f_j(z)|\left(\int_0^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz\\
&\le\int_{\mathbb{R}^n}|b(x)-b(z)|^m|f_j(z)|\left(\int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz\\
&\quad+\int_{\mathbb{R}^n}|b(x)-b(z)|^m|f_j(z)|\left(\int_{|x-z|}^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz.
\end{aligned} \tag{3.3}$$

Note that $x\in A_k,\ z\in A_j$ and $j\le k-2$. By (2.6), (2.7) and the generalized Hölder inequality we have

$$\begin{aligned}
|[b^m,\mu^{*,\rho}_{\Omega,\lambda}](f_j)(x)|&\le C\int_{\mathbb{R}^n}\frac{|f_j(z)|}{|x-z|^n}|b(x)-b(z)|^m dz\\
&\le C\left(|b(x)-b_{B_j}|^m\int_{A_j}\frac{|f_j(z)|}{|x-z|^n}dz+\int_{A_j}\frac{|f_j(z)|}{|x-z|^n}|b_{B_j}-b(z)|^m dz\right)\\
&\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(|b(x)-b_{B_j}|^m\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right).
\end{aligned} \tag{3.4}$$

By Lemmas 1.3, 1.4 and 3.2 we have

$$\begin{aligned}
&\|[b^m,\mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|(b(\cdot)-b_{B_j})^m\chi_k(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)}+\|b\|_*^m\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)\\
&\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left((k-j)^m\|b\|_*^m\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)}+\|b\|_*^m\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)\\
&\le C2^{-kn}(k-j)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C(k-j)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\
&\le C2^{(j-k)n\delta_2}(k-j)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned} J_1 &\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)n\delta_2}(k-j)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\ &= C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha}2^{(j-k)(n\delta_2-\alpha)}(k-j)^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}. \end{aligned}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} J_1 &\le C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\alpha)p/2}\right)\times\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2-\alpha)p'/2}(k-j)^{mp'}\right)^{p/p'}\right\}^{1/p} \\ &\le C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p} \\ &= C\|b\|_*^m\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p} \\ &\le C\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\ &= C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.5}$$

If $0 < p \le 1$, then we have

$$\begin{aligned} J_1 &\le C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2} 2^{j\alpha p}2^{(j-k)(n\delta_2-\alpha)p}(k-j)^{mp}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\ &= C\|b\|_*^m\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=j+2}^{\infty} 2^{(j-k)(n\delta_2-\alpha)p}(k-j)^{mp}\right)\right\}^{1/p} \\ &\le C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.6}$$

Let us now estimate J_3. Note that $x \in A_k,\ y \in A_j$ and $j \ge k+2$, so we have $|y-z| \sim |y|$. Similar to (3.4), we get

$$|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)(x)| \le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(|b(x)-b_{B_k}|^m\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \|(b_{B_k}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right). \tag{3.7}$$

By Lemmas 1.3, 1.4 and 3.2 we have

$$\begin{aligned} &\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\|b\|_*^m\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \|(b_{B_k}-b(\cdot))^m\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big) \\ &\quad\le C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\|b\|_*^m\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)} + (j-k)^m\|b\|_*^m\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big) \\ &\quad\le C2^{-jn}(j-k)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C(j-k)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \\ &\quad\le C2^{(k-j)n\delta_1}(j-k)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Thus we obtain

$$J_3 \le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_1}(j-k)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}$$
$$= C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty} 2^{j\alpha}2^{(k-j)(n\delta_1+\alpha)}(j-k)^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_1 + \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned} J_3 &\le C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty} 2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p 2^{(k-j)(n\delta_1+\alpha)p/2}\right)\times\left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_1+\alpha)p'/2}(j-k)^{mp'}\right)^{p/p'}\right\}^{1/p} \\ &\le C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty} 2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p 2^{(k-j)(n\delta_1+\alpha)p/2}\right)\right\}^{1/p} \\ &= C\|b\|_*^m\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\alpha)p/2}\right)\right\}^{1/p} \\ &\le C\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p} \\ &= C\|f\|_{\dot K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned} \tag{3.8}$$

If $0 < p \le 1$, then we have

$$\begin{aligned} J_3 &\le C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty} 2^{j\alpha p}2^{(k-j)(n\delta_1+\alpha)p}(j-k)^{mp}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p} \\ &= C\|b\|_*^m\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\alpha)p}(j-k)^{mp}\right)\right\}^{1/p} \\ &\le C\|f\|_{\dot K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned} \tag{3.9}$$

Therefore, by (3.1), (3.2), (3.5), (3.6), (3.8), (3.9) we complete the proof of Theorem 3.1.

Since $[b^m, \mu_{\Omega,S}^{\rho}](f)(x) \le C_\lambda[b^m, \mu_{\Omega,\lambda}^{*,\rho}](f)(x)$, we easily obtain the following theorem.

Theorem 3.2. *Suppose that* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $m \in \mathbb{Z}_+$, $0 < p \le \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\rho > n/2$, $\Omega \in L^2(\mathrm{S}^{n-1})$ *satisfying* (1.3) *and* (2.1). *If* $-n\delta_1 < \alpha < n\delta_2$, *then* $[b^m, \mu_{\Omega,S}]$ *is bounded on* $\dot K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ *and* $K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$.

4. Lipschitz estimate for the commutators of parametrized Littlewood–Paley operators

For $0 < \beta \le 1$, the Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is defined as

$$\mathrm{Lip}_\beta(\mathbb{R}^n) = \left\{f : \|f\|_{\mathrm{Lip}_\beta} = \sup_{x,y\in\mathbb{R}^n; x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty\right\}.$$

Next, we will give the Lipschitz estimate for the commutators $[b^m, \mu_{\Omega,S}^{\rho}]$ and $[b^m, \mu_{\Omega,\lambda}^{*,\rho}]$ on Herz spaces with variable exponent.

Theorem 4.1. *Let* $m \in \mathbb{Z}_+$, $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, $0 < p \le \infty$, $\rho > n/2$, $\lambda > 2$, $\Omega \in L^2(\mathrm{S}^{n-1})$ *satisfy* (1.3) *and* $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *be such that* $q_1^+ < \frac{n}{m\beta}$. *If* $0 < \beta < \min\{1, n/m\}$, $-n\delta_1 + m\beta < \alpha < n\delta_2$,

$q_2(\cdot)(n-m\beta)/n \in \mathcal{B}(\mathbb{R}^n)$ and

$$\frac{1}{q_1(x)} - \frac{1}{q_2(x)} = \frac{m\beta}{n},$$

then $[b^m, \mu^{*,\rho}_{\Omega,\lambda}]$ is bounded from $\dot{K}^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)$(or $K^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)$) to $\dot{K}^{\alpha,p}_{q_2(\cdot)}(\mathbb{R}^n)$(or $K^{\alpha,p}_{q_2(\cdot)}(\mathbb{R}^n)$).

In the proof of Theorem 4.1, we also need the following lemma.

Lemma 4.1 ([2]). *Let $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $q_1^+ < n/\nu$ and $1/q_1(x) - 1/q_2(x) = \nu/n$. If $q_2(\cdot)(n-\nu)/n \in \mathcal{B}(\mathbb{R}^n)$, then $\|I_\nu f\|_{q_2(\cdot)} \le C\|f\|_{q_1(\cdot)}$, where I_ν is the fractional integral operator with $0 < \nu < n$.*

Proof of Theorem 4.1. Similar to Theorem 2.1, we only prove the homogeneous case and still suppose $0 < p < \infty$. Let $f \in \dot{K}^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)$, and we write $f(x) = \sum_{j=-\infty}^{\infty} f\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Then we have

$$\begin{aligned}
\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f)\|_{\dot{K}^{\alpha,p}_{q_2(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f)\chi_k\|^p_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\
&\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{k+1}\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&=: CU_1 + CU_2 + CU_3.
\end{aligned} \tag{4.1}$$

In [7], the authors proved that $[b^m, \mu^{*,\rho}_{\Omega,\lambda}]$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_2(\cdot)}(\mathbb{R}^n)$. So we have

$$U_2 \le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_k\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} = C\|f\|_{\dot{K}^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)}. \tag{4.2}$$

Now we estimate U_1. By the Minkowski inequality we have

$$\begin{aligned}
&|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)(x)| \\
&= \left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
&= \left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f_j(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
&\le C\|b\|^m_{\mathrm{Lip}_\beta}\int_{\mathbb{R}^n}|x-z|^{m\beta}|f_j(z)|\left(\int_0^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz \\
&\le C\|b\|^m_{\mathrm{Lip}_\beta}\int_{\mathbb{R}^n}|x-z|^{m\beta}|f_j(z)|\left(\int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz \\
&\quad + C\|b\|^m_{\mathrm{Lip}_\beta}\int_{\mathbb{R}^n}|x-z|^{m\beta}|f_j(z)|\left(\int_{|x-z|}^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz.
\end{aligned} \tag{4.3}$$

Note that $x \in A_k,\ z \in A_j$ and $j \le k-2$. By (2.6), (2.7) and the generalized Hölder inequality we have

$$\begin{aligned}
|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)(x)| &\le C\|b\|^m_{\mathrm{Lip}_\beta}\int_{\mathbb{R}^n}\frac{|f_j(z)|}{|x-z|^{n-m\beta}}dz \\
&\le C\|b\|^m_{\mathrm{Lip}_\beta}2^{-k(n-m\beta)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{4.4}$$

Take $\nu = m\beta$, since

$$I_{m\beta}(\chi_{B_k})(x) \geq \int_{B_k} \frac{dy}{|x-y|^{n-m\beta}} \chi_{B_k}(x) \geq C2^{km\beta}\chi_{B_k}(x), \tag{4.5}$$

by Lemmas 1.3, 1.4 and 4.1 we have

$$\begin{aligned}
\|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq C2^{-k(n-m\beta)}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\leq C2^{-kn}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|I_{m\beta}(\chi_{B_k})\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\leq C2^{-kn}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \\
&\leq C2^{(j-k)n\delta_2}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
U_1 &\leq C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)n\delta_2}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&= C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha}2^{(j-k)(n\delta_2-\alpha)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.
\end{aligned}$$

If $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$\begin{aligned}
U_1 &\leq C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\alpha)p/2}\right)\times\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\alpha)p'/2}\right)^{p/p'}\right\}^{1/p} \\
&\leq C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p} \\
&= C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p} \\
&\leq C\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\
&= C\|f\|_{\dot{K}^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{4.6}$$

If $0 < p \leq 1$, then we have

$$\begin{aligned}
U_1 &\leq C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2} 2^{j\alpha p}2^{(j-k)(n\delta_2-\alpha)p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\
&= C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\alpha)p}\right)\right\}^{1/p} \\
&\leq C\|f\|_{\dot{K}^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{4.7}$$

Let us now estimate U_3. Note that $x \in A_k,\ y \in A_j$ and $j \geq k+2$, so we have $|y-z| \sim |y|$. Similar to (4.4), we get

$$|[b^m, \mu^{*,\rho}_{\Omega,\lambda}](f_j)(x)| \leq C\|b\|^m_{\mathrm{Lip}_\beta}2^{-j(n-m\beta)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}. \tag{4.8}$$

By (4.5), Lemmas 1.3, 1.4 and 4.1 we have

$$\begin{aligned}\|[b^m,\mu^{*,\rho}_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\le C2^{-j(n-m\beta)}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\ &\le C2^{-jn+(j-k)m\beta}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|I_{m\beta}(\chi_{B_k})\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\ &\le C2^{-jn+(j-k)m\beta}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\ &\le C2^{(j-k)m\beta}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}\\ &\le C2^{(k-j)(n\delta_1-m\beta)}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}U_3 &\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_1-m\beta)}\|b\|^m_{\mathrm{Lip}_\beta}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\ &= C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1-m\beta+\alpha)}(j-k)^m\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.\end{aligned}$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_1-m\beta+\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}U_3 &\le C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1-m\beta+\alpha)p/2}\right)\right.\\ &\qquad\left.\times\left(\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_1-m\beta+\alpha)p'/2}(j-k)^{mp'}\right)^{p/p'}\right\}^{1/p}\\ &\le C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1-m\beta+\alpha)p/2}\right)\right\}^{1/p}\\ &= C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1-m\beta+\alpha)p/2}\right)\right\}^{1/p}\\ &\le C\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}\\ &= C\|f\|_{\dot K^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{4.9}$$

If $0<p\le 1$, then we have

$$\begin{aligned}U_3 &\le C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty}2^{j\alpha p}2^{(k-j)(n\delta_1-m\beta+\alpha)p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}\\ &= C\|b\|^m_{\mathrm{Lip}_\beta}\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1-m\beta+\alpha)p}\right)\right\}^{1/p}\\ &\le C\|f\|_{\dot K^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{4.10}$$

Therefore, by (4.1), (4.2), (4.6), (4.7), (4.9), (4.10) we complete the proof of Theorem 4.1.

Since $[b^m,\mu^{\rho}_{\Omega,S}](f)(x)\le C_\lambda[b^m,\mu^{*,\rho}_{\Omega,\lambda}](f)(x)$, we easily obtain the following theorem.

Theorem 4.2. *Let* $m \in \mathbb{Z}_+$, $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, $0 < p \le \infty$, $\rho > n/2$, $\Omega \in L^2(\mathrm{S}^{n-1})$ *satisfy* (1.3) *and* $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *be such that* $q_1^+ < \frac{n}{m\beta}$. *If* $0 < \beta < \min\{1, n/m\}$, $-n\delta_1 + m\beta < \alpha < n\delta_2$, $q_2(\cdot)(n - m\beta)/n \in \mathcal{B}(\mathbb{R}^n)$ *and*

$$\frac{1}{q_1(x)} - \frac{1}{q_2(x)} = \frac{m\beta}{n},$$

then $[b^m, \mu^\rho_{\Omega,S}]$ *is bounded from* $\dot{K}^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)$*(or* $K^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)$*) to* $\dot{K}^{\alpha,p}_{q_2(\cdot)}(\mathbb{R}^n)$*(or* $K^{\alpha,p}_{q_2(\cdot)}(\mathbb{R}^n)$*).*

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