ISSN 2346-8092

ა. რაზმაძის მათემატიკის ინსტიტუტის შრომები

ივანე ჯავახიშვილის სახელობის თბილისის
სახელმწიფო უნივერსიტეტი

ტომი 171, N 3, 2017

Transactions of A. Razmadze Mathematical Institute is a continuation of Travaux de L' Institut Mathematique de Tbilisi, Vol. 1–15 (1937–1947), Trudy Tbilisskogo Matematicheskogo Instituta, Vol. 16–99 (1948–1989), Proceedings of A. Razmadze Mathematical Institute, Vol. 100–169 (1990–2015).

Transactions of A. Razmadze Mathematical Institute
Volume 171, Issue 3, December 2017

Contents

Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 171 (2017) 253–256
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original Article

Conservation of time scale for one-dimensional pulsating flow

G. Aptsiauri

Georgian Aviation University, 16, K. Tsamebuli str., Tbilisi, 0144, Georgia

Received 15 June 2017; accepted 5 July 2017
Available online 27 July 2017

Abstract

In the work by analysis of one-dimensional unsteady flows, based on the fundamental law of conservation with application of Fourier series is shown that in the presence of periodic, steady pulsations along the flow, the main frequency as well as all higher frequencies remain constant and only the amplitude of oscillations is changed that is in full agreement with the results of analysis of more complex three-dimensional flows. Thus, is confirmed the validity of the principle of conservation of frequencies or time scale along the flow. So, is obtained very interesting result for turbulence problem solution.

Keywords: Oscillation; Turbulence; Time scale; Conservation

1. Introduction

Integrating the Navier–Stokes differential equation, Osborne Reynolds admitted that:

$$\overline{\nabla F} = \nabla \left(\overline{F}\right). \tag{1.1}$$

In the work [1] was shown, that one of the main reasons of the Reynolds problem arising is this assumption.

If we have an arbitrary periodic function:

$$\overline{F} = \frac{1}{\tau_0} \int_0^{\tau_0} F\left(x, y, z, t\right) dt. \tag{1.2}$$

The following relations are valid:

$$\overline{\nabla F} = \nabla \left(\overline{F}\right) + \frac{\overline{F}}{\tau_0} \nabla \tau_0 = \nabla \left(\overline{F}\right) - \overline{F} A. \tag{1.3}$$

$$\overline{\nabla^2 F} = \nabla^2 \left(\overline{F}\right) - 2A \left[\nabla \left(\overline{F}\right)\right] - \overline{F}\left(\nabla A\right) + \overline{F} A^2, \tag{1.4}$$

where-τ_O is the duration of oscillation. $A = (1/f)\, grad\,(f) = grad\,(\ln f) = -grad\,(\ln \tau_0)$.

E-mail address: gegiaptsiauri@gmail.com.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.07.001

Integrating the differential equations of Navier–Stokes by taking in account (1.2)–(1.4), are obtained differential equations, that differ from the Reynolds equation. At the same time, the Reynolds equations express conservation laws for integral flows and they do not cause doubt. Consequently, the presence of two different systems makes it possible to obtain very important additional information on the turbulence problem. One of these results is the principle of conservation of frequencies (or time scales) along the flow.

In this paper, we prove what has been said on the example of a one-dimensional nonstationary periodic flow.

From acoustic theory it is well known that at propagation of acoustic waves, pressure fluctuation period and character at various locations of the perturbation region are qualitatively identical [2,3]. With increasing of distance from the source of vibration the amplitude of perturbations changes due to dissipation at perturbations spreading in a large space (in the case of spherical waves), however, the period of oscillation at this is not changing. Therefore, audio signals are not distorted, despite that they become weaker. In terms of acoustics theory, mathematically this would be easily explained, since perturbations that are propagating with the constant speed C should create the same pattern in different locations of space with shift in time *x/C* (see solutions of wave equations). Thus, we can say that for the case of acoustic disturbances, the preservation of oscillation frequency is observed. However, let us put the question of whether or not to preserve as constant the oscillation frequency along the flow, if we have arbitrary, strong periodic disturbances? The theory of acoustic waves in this issue does not help us, because, at significant perturbations, the wave propagation velocity is not constant due to its dependence on the changing of the environment state parameters.

However, as will be shown below, if in the one-dimensional flow are propagated periodic waves of arbitrary shape, the frequency of these oscillations in arbitrary cross section also will be the same. In other words, we show that conservation of frequency along certain lines is a property not only of acoustic disturbances, but also of arbitrary non-stationary periodic processes. Starting from simple examples, with the transition to a more general problem, we show that this property is a common feature of all periodic oscillatory processes. Therefore, this feature would be called as principle of conservation of frequencies (or time scales) along the vector of substance that is subject of periodic fluctuations.

2. Basic part

For obviousness, let us assume that in the straight channel receives periodic stream. If in the initial section of the channel we install the pressure sensor, it will register the oscillation process with the period of τ_0 (Fig. 1, line 1) or with the frequency $f = 1/\tau_0 = \omega/2\pi$. For these processes, there is a conventional, minimum angular velocity that will be determined from the equation $\omega = 2\pi f = 2\pi/\tau_0$.

The sensor located in a certain distance from the entrance section will also detect a certain periodic process with interval τ_x, and the perturbation amplitude will be relatively less (line 2). But third sensor that is located very far from the entrance, almost will not register vibrations due to dissipation and smoothing of the waves, the flow will gradually make stationary character (line 3).

We will show that, in arbitrary section of one-dimensional periodic flow, the oscillation period of the pulsating flow parameters must be the same not only for small but also for any perturbations ($\tau_0 = \tau_x = idem$). In other words, in any section, the period of the vibration and main angular velocity will be the same

$$\frac{\partial \omega}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial \tau_x}{\partial x} = 0 \tag{2.2}$$

to confirm the above mentioned, let us consider the instantaneous value of mass flow in an arbitrary cross-section of flow $G = \rho U F$. The instantaneous specific mass flow would be written as the sum of two functions, one of that depends on the coordinate x and the other is a periodic function, (and dependent on x and t)

$$g = \rho U = \eta(x) + \varphi(x, t), \tag{2.3}$$

thus, a periodic function is possible to express as a Fourier series [4,5]

$$\varphi(x, \tau) = \sum_{i=1,\infty} \left[a(x)_i \cos(i\omega t) + b(x)_i \sin(i\omega t) \right], \tag{2.4}$$

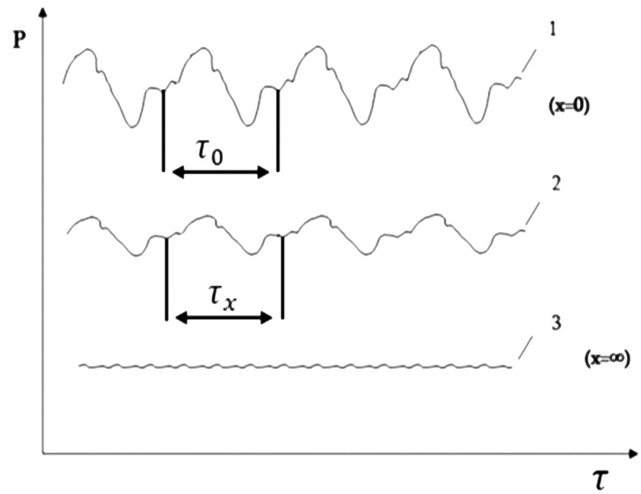

Fig. 1. The fluctuation of pressure in different cross-sections of one-dimensional unsteady flow.

wherein the amplitudes $a(x)_i$, $b(x)_i$ and main frequency ω as well are dependent on x. Knowing the instantaneous value of mass flow, the total amount of transferred per second substance is possible to determine from the expression

$$\begin{aligned}\overline{g} &= \frac{1}{\tau_x}\int_0^{\tau_x} g dt = \eta(x) + \frac{\omega}{2\pi}\int_0^{\tau_x}\varphi(x,t)\,dt \\ &= \eta(x) + \frac{1}{2\pi}\sum_{i=1,\infty}\left\{a(x)_i\int_0^{2\pi}\cos(i\omega t)\,d(\omega t) + b(x)_i\int_0^{2\pi}\sin(i\omega t)\,d(\omega t)\right\} = \eta(x).\end{aligned} \tag{2.5}$$

It is natural that, regardless of the nature of pulsation, in arbitrary section of the periodic flow will be passed the same amount of mass. Therefore we will have:

$$\frac{\partial\overline{g}}{\partial x} = \frac{\partial\eta}{\partial x} = 0. \tag{2.6}$$

Thus, the function η does not depend on the coordinate x and for the instantaneous mass flow we have

$$g = \rho U = C + \sum_{i=1,N}\left[a(x)_i\cos(i\omega t) + b(x)_i\sin(i\omega t)\right]. \tag{2.7}$$

And now, on the basis of the last expression, let us analyze the law of mass conservation. As it is known, for one-dimensional non-stationary flows this law is expressed grounded on the equation of continuity

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho U}{\partial x} = 0. \tag{2.8}$$

The integrating of this expression with respect to time, in an arbitrary section gives:

$$\int_0^{\tau_X}\frac{\partial\rho U}{\partial x}dt = 0, \tag{2.9}$$

if (2.7) will be differentiated along the x and substitute in (2.9) we obtain:

$$\begin{aligned}&\int_0^{\tau_x}\sum_{i=1,N}\left[\frac{\partial a(x)_i}{\partial x}\cos(i\omega t) + \frac{\partial b(x)_i}{\partial x}\sin(i\omega t)\right]dt \\ &\quad - \int_0^{\tau_x}\frac{\partial\omega}{\partial x}\sum_{i=1,N}\left[a(x)_i\,(it)\sin(i\omega t) - b(x)_i\,(it)\cos(i\omega t)\right]dt = 0,\end{aligned} \tag{2.10}$$

or, after introducing of integration sign within the summation sign:

$$\sum_{i=1,N}\left[\frac{\partial a(x)_i}{\partial x}\int_0^{\tau_x}\cos(i\omega t)\,dt+\frac{\partial b(x)_i}{\partial x}\int_0^{\tau_x}sin\,(i\omega t)\,dt\right]+$$
$$-\frac{1}{\omega^2}\frac{\partial\omega}{\partial x}\sum_{i=1,N}\left[a(x)_i\int_0^{2\pi}(i\omega t)\sin(i\omega t)\,d\,(\omega t)-b(x)_i\int_0^{2\pi}(i\omega t)\cos(i\omega t)\,d\,(\omega t)\right]=0. \quad (2.11)$$

It is easily seen that first two members of Eq. (2.11) are equal to zero, thus we have

$$\frac{\partial\omega}{\partial x}\sum_{i=1,N}\left[a(x)_i\int_0^{2\pi}(i\omega t)\sin(i\omega t)\,d\,(\omega t)-b(x)_i\int_0^{2\pi}(i\omega t)\cos(i\omega t)\,d\,(\omega t)\right]=0. \quad (2.12)$$

At the same time, from the theory of definite integrals follows:

$$\int_0^{2\pi}(i\omega t)\cos(i\omega t)\,d\,(\omega t)=0, \quad (2.13)$$
$$\int_0^{2\pi}(i\omega t)\sin(i\omega t)\,d\,(\omega t)=-2\pi. \quad (2.14)$$

Therefore, we will obtain

$$2\pi\frac{\partial\omega}{\partial x}\sum_{i=1,N}\left[a(x)_i\right]=0, \quad (2.15)$$

or

$$\frac{\partial\omega}{\partial x}=0. \quad (2.16)$$

Thus, we have shown that in any periodic mass flux the main frequency remains constant along the flow.

In the work [1] we have shown that the vibration frequency gradients are directed perpendicular to the vector of flow, that indicates the direction of propagation of physical substance. Obtained by us for one-dimensional flow result entirely coincides within the framework of this principle. I.e. we have shown that this result is valid for any flow. At this the stated judgments are valid not only for the mass flow, but also for energy flux and flows of various substances.

In the future we will show that — in the continuum where are processes with periodic disturbances, the minimum, main frequency and all higher oscillation frequencies of substances flows (mass, concentration, energy, etc.) remain constant in the direction of dissemination of substance (or along the streamlets of given substances).

3. Conclusion

Based on Eqs. (1.2)–(1.4) and above mentioned analysis of one-dimensional pulsating flow, is shown that, proposed by Professor Aptsiauri principle of conservation of frequencies (or the time scales) has convincing theoretical basis. At the same time, the system of obtained additional theoretical equations creates the possibility of solving the turbulence problem.

References

[1] A. Aptsiauri, G. Aptsiauri, Principle of frequency gradients and energy conservation laws for the integrated flux as a basis for solving the problem of turbulence, Int. Sci. J. IFToMM Probl. Mech. Tbilisi 2 (63) (2016) 50–55.
[2] R.W.B. Stephens, A.E. Bate, Acoustics and Vibrational Physics, second ed., Edward Arnold, London, 1966.
[3] О.В. Руденко, С.И. Солуян, Теоретические основы нелинейной акустики, in: Theoretical Foundations of Nonlinear Acoustics, Наука, Москва, 1975.
[4] А.Дж. Мак-Коннел, Ввещение в тензорный анализ с приложениями к геометрии, механике и физике, Introduction to Tensor Analysis with Applications to Geometry, Mechanics, and Physics, Физматлит, Москва, 1963.
[5] В.В. Жук, Г.и. Натансон, Тригонометрические рящы Фурье и элементы теории аппроксимации, Fourier Trigonometric Series and Elements of the Theory of Approximation, изд-во Ленингр ун-та, Денинград, 1983.

Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 171 (2017) 257–263
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

2-adic cofiltration of $SO_3(\mathbb{Q})$

Tengiz Bokelavadze[a,*], Raffaello Caserta[b]

[a] *Faculty of Exact and Natural Sciences, 59 Tamar Mefe Str., 4600 Kutaisi, GA, United States*
[b] *Dipartimento di Matematica e Informatica, Via Archirafi 34, I-90123 Palermo, Italy*

Received 1 February 2017; received in revised form 2 June 2017; accepted 7 June 2017
Available online 21 July 2017

Abstract

We prove that the group $SO_3(\mathbb{Q})$ of rational rotations is the inverse limit of a family of finite solvable groups of order $2^{3k-2} \cdot 3$, whose 2-Sylow subgroups have nilpotency class $2k - 3$, exponent 2^{k-1}, and Frattini subgroups coinciding with the commutator subgroups, and we give generators for these groups.

Keywords: Profinite groups; Hurwitz quaternions

This paper gives a presentation of the orthogonal group

$$SO_3(\mathbb{Q}) = \{X \in \mathbb{Q}^{3\times3} : XX^t = I_3\}$$

of rational isometries of the quadratic form $Q(x, y, z) = x^2 + y^2 + z^2$ as a profinite group.

Equivalently, we consider a set of infinitesimal generators for (a representative subgroup of) the inverse limit $SO_3(\mathbb{Q})$, and we notice that they are scalar multiples of infinitesimal generators for $SO_3(\mathbb{R})$ by a scalar factor of the form $\arcsin(2\theta)$, which has 2-adic norm greater than one. Thus, each term of the inverse limit is nilpotent, and the Baker–Campbell–Hausdorff formula has a finite number of terms, an effective tool for further computations.

1. $SO_3(\mathbb{Q})$

For the presentation of the elements of $SO_3(\mathbb{Q})$, it is natural to consider the $\mathbb{Q}$-algebra of Hurwitz quaternions

$$\mathbb{H} = \{z = m + n\mathbf{i} + p\mathbf{j} + q\mathbf{k} : \ m, n, p, q \in \mathbb{Q}, \ \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}; \mathbf{i}^2 = \mathbf{j}^2 = -1\}.$$

In fact, the subgroup N of Hurwitz quaternions of norm 1, by conjugation, on the 3-dimensional rational space of *vector* quaternions $z = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and this action gives back a representation of the group $SO_3(\mathbb{Q})$.

* Corresponding author.
E-mail addresses: bokel71@yahoo.com (T. Bokelavadze), raffaello.caserta@unipa.it (R. Caserta).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.06.001
2346-8092/

Representing the quaternion $z = m + n\mathbf{i} + p\mathbf{j} + q\mathbf{k}$ as a rational matrix of the shape

$$Z = \begin{pmatrix} m & -n & -p & -q \\ n & m & q & -p \\ p & -q & m & n \\ q & p & -n & m \end{pmatrix},$$

one sees that the norm is

$$\sqrt{\det(Z)} = \sqrt{m^2 + n^2 + p^2 + q^2}.$$

For the normalized quaternion w of norm 1, the elements $w\mathbf{i}w^{-1}$, $w\mathbf{j}w^{-1}$ and $w\mathbf{k}w^{-1}$ are respectively:

$$\frac{1}{m^2+n^2+p^2+q^2}\begin{pmatrix} 0 & q^2+p^2-n^2-m^2 & 2mq-2np & -2nq-2mp \\ -q^2-p^2+n^2+m^2 & 0 & 2nq+2mp & 2mq-2np \\ 2np-2mq & -2nq-2mp & 0 & -q^2-p^2+n^2+m^2 \\ 2nq+2mp & 2np-2mq & q^2+p^2-n^2-m^2 & 0 \end{pmatrix},$$

$$\frac{1}{m^2+n^2+p^2+q^2}\begin{pmatrix} 0 & -2mq-2np & q^2-p^2+n^2-m^2 & 2mn-2pq \\ 2mq+2np & 0 & 2pq-2mn & q^2-p^2+n^2-m^2 \\ -q^2+p^2-n^2+m^2 & 2mn-2pq & 0 & 2mq+2np \\ 2pq-2mn & -q^2+p^2-n^2+m^2 & -2mq-2np & 0 \end{pmatrix},$$

$$\frac{1}{m^2+n^2+p^2+q^2}\begin{pmatrix} 0 & 2mp-2nq & -2pq-2mn & -q^2+p^2+n^2-m^2 \\ 2nq-2mp & 0 & q^2-p^2-n^2+m^2 & -2pq-2mn \\ 2pq+2mn & -q^2+p^2+n^2-m^2 & 0 & 2nq-2mp \\ q^2-p^2-n^2+m^2 & 2pq+2mn & 2mp-2nq & 0 \end{pmatrix}.$$

If $m, n, p, q \in \mathbb{Z}$ and if we write the coordinates of the above conjugate elements of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in columns, then we obtain the following representation of $\mathrm{SO}_3(\mathbb{Q})$:

$$\frac{1}{m^2+n^2+p^2+q^2}\begin{pmatrix} m^2+n^2-p^2-q^2 & 2mq+2np & 2nq-2mp \\ 2np-2mq & m^2-n^2+p^2-q^2 & 2pq+2mn \\ 2nq+2mp & 2pq-2mn & m^2-n^2-p^2+q^2 \end{pmatrix}.$$

The above construction shows that, similarly to the classical case of Pythagorean triples, a primitive Pythagorean quadruple (a, b, c, d) is parametrized by (m, n, p, q), and, in fact, *all* 3×3 orthogonal matrices with rational coefficients are obtained in this manner (cf. [1]). The same result follows also from the Lebesgue identity (cf. [2])

$$(m^2+n^2+p^2+q^2)^2 = (m^2+n^2-p^2-q^2)^2 + (2mq+2np)^2 + (2nq-2mp)^2.$$

2. Cofiltration

For an odd prime p let $p = a^2+b^2+c^2+d^2$ and let $w = \frac{\sqrt{p}}{p}(a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d)$. The conjugation by w of a vector quaternion $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ produces the following rotation of $\mathbb{Q}^3$

$$\frac{1}{p}\begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 2ad+2bc & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2ab+2cd & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

This shows that, for any $p > 2$, there are elements in $\mathrm{SO}_3(\mathbb{Q})$ which cannot be reduced modulo p. For instance, let $p = 7$. In this case, $w = \frac{\sqrt{7}}{7}(2 + \mathbf{i}3 + \mathbf{j}6)$ gives in turn

$$\frac{1}{7}\begin{pmatrix} -3 & -2 & 6 \\ 6 & -3 & 2 \\ 2 & 6 & 3 \end{pmatrix} \in \mathrm{SO}_3(\mathbb{Q}).$$

On the contrary, for $p = 2$, we can see that, for each $k = 1, 2, \dots,$ it is possible to reduce modulo 2^k the arbitrary element

$$\frac{1}{m^2+n^2+p^2+q^2}\begin{pmatrix} m^2+n^2-p^2-q^2 & 2\,m\,q+2\,n\,p & 2\,n\,q-2\,m\,p \\ 2\,n\,p-2\,m\,q & m^2-n^2+p^2-q^2 & 2\,p\,q+2\,m\,n \\ 2\,n\,q+2\,m\,p & 2\,p\,q-2\,m\,n & m^2-n^2-p^2+q^2 \end{pmatrix}$$

of $\mathrm{SO}_3(\mathbb{Q})$, because the 2-adic valuation of each numerator is never smaller than the 2-adic valuation of the denominator $m^2 + n^2 + p^2 + q^2$, that is, the coefficients of each element in $\mathrm{SO}_3(\mathbb{Q})$ are always *rational numbers with odd denominators*.

In fact, it is well-known that the only prime where the Hurwitz quaternion ramify is $p = 2$, whereas for any odd prime p the algebra $\mathbb{Q}_p \otimes \mathbb{H}$ splits, becoming isomorphic to the $\mathbb{Q}_p$-algebra $\mathbb{Q}_p^{2\times 2}$ of 2×2 matrices (cf. e.g. [3]).

We notice that the reduction modulo 2^k induces a homomorphism

$$\mathrm{SO}_3(\mathbb{Q}) \longrightarrow \Xi_k \le \mathrm{SO}_3(\mathbb{Z}/2^k\mathbb{Z}),$$

which, for $k > 1$, is not surjective. In fact, putting for short

$$G_k = \mathrm{SO}_3(\mathbb{Z}/2^k\mathbb{Z}) = \{X \in (\mathbb{Z}/2^k\mathbb{Z})^{3\times 3} \;:\; XX^t = I\},$$

and denoting by P_k a 2-Sylow subgroup of G_k and by $\Phi(*)$ the Frattini subgroup, we find that

k	$\lvert G_k\rvert$	$\mathrm{ncl}(P_k)$	$\exp P_k$	$\Phi(P_k)$	P'_k	$Z(P_k)$	$P'_k \cap Z(P_k)$
2	$2^6 \cdot 3$	2	2^2	P'_k	$(\frac{\mathbb{Z}}{2\mathbb{Z}})^2$	$(\frac{\mathbb{Z}}{2\mathbb{Z}})^3$	P'_k
3	$2^9 \cdot 3$	3	2^2	P'_k	$(\frac{\mathbb{Z}}{2\mathbb{Z}})^4$	$(\frac{\mathbb{Z}}{2\mathbb{Z}})^3$	$(\frac{\mathbb{Z}}{2\mathbb{Z}})^2$

whereas, for $k \ge 4$ we always find $(\frac{\mathbb{Z}}{2\mathbb{Z}})^3 = Z(P_k) < P'_k$ and:

k	$\lvert G_k\rvert$	$\mathrm{ncl}(P_k)$	$\exp P_k$	$\Phi(P_k)$	$\mathrm{ncl}(P'_k)$	$\lvert P'_k\rvert$
4	$2^{12} \cdot 3$	5	2^3	P'_k	2	2^7
5	$2^{15} \cdot 3$	7	2^4	P'_k	3	2^{10}
6	$2^{18} \cdot 3$	9	2^5	P'_k	4	2^{13}

Hence we conjecture, for $k \ge 4$, $(\frac{\mathbb{Z}}{2\mathbb{Z}})^3 = Z(P_k) < P'_k$ and

$\lvert G_k\rvert$	$\mathrm{ncl}(P)$	$\exp P$	$\Phi(P)$	$\mathrm{ncl}(P')$	$\lvert P'\rvert$
$2^{3k} \cdot 3$	$2k - 3$	2^{k-1}	P'	$k - 2$	2^{3k-5}

For the image Ξ_k of $\mathrm{SO}_3(\mathbb{Q})$, denoting again by P_k a 2-Sylow subgroup of Ξ_k, we claim that

$\lvert\Xi_k\rvert$	$\mathrm{ncl}(P_k)$	$\exp P$	$\Phi(P_k)$	$\mathrm{ncl}(P'_k)$	$\lvert P'_k\rvert$
$2^{3k-2} \cdot 3$	$2k - 3$	2^{k-1}	P'	$k - 2$	2^{3k-6}

In fact, as a consequence of Hensel Lemma, the following elements are generators of Ξ_k:

$$A_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad B_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix};$$

$$B_y = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

and the following are generators of its 2-Sylow subgroup:

$$A_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad B_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix};$$

$$B_z = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad C = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix},$$

because one can directly check that they generate the first reduction modulo 2^4.

Remark. Since

$$\frac{1}{3} = \frac{1}{1+2} \equiv (1-2+4-8+\cdots) = \sum_{i=0}^{k-1} (-2)^i \pmod{2^k} \cdots,$$

the series $\sum_{k=0}^{\infty}(-2)^k$ converges to $\frac{1}{3}$ with respect to the 2-adic metric, and since manifestly $\sum_{k=0}^{\infty}(-2)^k \equiv 3 (\mathrm{mod}\ 8)$, the matrix

$$C = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \sum_{k=0}^{\infty}(-2)^k \cdot \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix} \in \mathrm{SO}_3(\mathbb{Q}),$$

corresponding to a rotation having the axis $(1, 0, 1)$, and the angle θ such that $\cos(\theta) = -\frac{1}{3}$ and $\sin(\theta) = \frac{2\sqrt{2}}{3}$, is mapped onto

$$3 \cdot \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -6 & 6 \\ 6 & -3 & -6 \\ 6 & 6 & 3 \end{pmatrix} \in \mathrm{SO}_3(\mathbb{Z}/8\mathbb{Z}),$$

which does not belong to the subgroup generated by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ and $\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

3. Logarithms in $\mathrm{SO}_3(\mathbb{Q})$

It is well-known that, in the case of $\mathrm{SO}_3(\mathbb{R})$, the Baker–Campbell–Hausdorff formula is weakened by the fact that it is an infinite series, rather than a polynomial (cf. [4]). This is also the case for $\mathrm{SO}_3(\mathbb{Q})$. In this last section, we consider the logarithmic image of the nilpotent group Ξ_k, in order to get the Baker–Campbell–Hausdorff formula as a polynomial. In passing, we also notice that this representation makes it possible to represent the elements in $\mathrm{SO}_3(\mathbb{R})$ as Witt vectors, an effective tool for an arithmetic analogue of ordinary differential equations, where the rôle of the derivative is played by the Fermat quotient operator $\partial(x) = \frac{x-x^2}{2}$ (cf. [5,6]).

Notice that, since the logarithm is defined on elements of the shape $I + 2X$, the element $\log(A)$ does not converge in our group. Consider therefore the subgroup H generated by B_x, B_z, C, having index 4 in the 2-Sylow subgroup P_k

of the projection of $SO_3(\mathbb{Q})$, that is,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}; \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Since $H \ni \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$, which has no logarithm, we have to split again

$$H = \langle B_x^2, B_z, C \rangle \rtimes \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It turns out that H and $\langle B_x^2, B_z, C \rangle$, modulo 2^k, have nilpotency class $k-1$.

Since

$$B_z = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{3+4i}{5} & 0 & 0 \\ 0 & \frac{3-4i}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we get

$$\begin{aligned} \mathfrak{b}_z = \log(B_z) &= \log \left\{ \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{3+4i}{5} & 0 & 0 \\ 0 & \frac{3-4i}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 2 \end{pmatrix} \log \begin{pmatrix} \frac{3+4i}{5} & 0 & 0 \\ 0 & \frac{3-4i}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -i\vartheta & 0 & 0 \\ 0 & i\vartheta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\vartheta & 0 \\ \vartheta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $\vartheta = \arcsin(4/5)$, and

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2n!!} \cdot \frac{x^{2n+1}}{2n+1} = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \cdots$$

(here the double factorial $k!!$ denotes the product of the integer from 1 to k having the same parity of k). Since, for $x \in 2\mathbb{Z}_2$,

$$\arcsin(x) \equiv x \pmod 2 \equiv x + \frac{1}{6} x^3 \pmod{2^2} \equiv x + \frac{1}{6} x^3 + \frac{3}{40} x^5 + \frac{15}{336} x^7 \pmod{2^3} \equiv \ldots,$$

we get for $\vartheta = \arcsin(4/5) \in \mathbb{Z}/2^k\mathbb{Z}$ the following values

mod	2	2^2	2^3	2^4	2^5	2^6
ϑ	0	0	4	4	20	20

The same computations yield

$$2\mathfrak{b}_x = \log(B_x^2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\vartheta \\ 0 & 2\vartheta & 0 \end{pmatrix}.$$

Finally, since

$$C = \frac{1}{4}\begin{pmatrix} 1 & 1 & 2 \\ -\sqrt{2}\,i & \sqrt{2}\,i & 0 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \dfrac{4\,i-\sqrt{2}}{3\sqrt{2}} & 0 & 0 \\ 0 & -\dfrac{4\,i+\sqrt{2}}{3\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2}\,i & -1 \\ 1 & -\sqrt{2}\,i & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

then

$$\begin{aligned} \mathfrak{c} = \log(C) &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 2 \\ -\sqrt{2}\,i & \sqrt{2}\,i & 0 \\ -1 & -1 & 2 \end{pmatrix} \log \begin{pmatrix} \dfrac{4\,i-\sqrt{2}}{3\sqrt{2}} & 0 & 0 \\ 0 & -\dfrac{4\,i+\sqrt{2}}{3\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2}\,i & -1 \\ 1 & -\sqrt{2}\,i & -1 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 2 \\ -\sqrt{2}\,i & \sqrt{2}\,i & 0 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -i\theta & 0 & 0 \\ 0 & i\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2}\,i & -1 \\ 1 & -\sqrt{2}\,i & -1 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \dfrac{\theta}{\sqrt{2}} & 0 \\ -\dfrac{\theta}{\sqrt{2}} & 0 & \dfrac{\theta}{\sqrt{2}} \\ 0 & -\dfrac{\theta}{\sqrt{2}} & 0 \end{pmatrix}, \end{aligned}$$

where for $\frac{\theta}{\sqrt{2}} = \frac{\arcsin(2\sqrt{2}/3)}{\sqrt{2}} \in \mathbb{Z}/2^k\mathbb{Z}$ we get the following values

mod	2	2^2	2^3	2^4	2^5	2^6
$\frac{\theta}{\sqrt{2}}$	0	2	6	14	14	46

For instance, for $k = 6$ we find

$$2\mathfrak{b}_x = \log \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}^2 \equiv 24 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \pmod{64},$$

$$\mathfrak{b}_z = \log \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv 44 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \pmod{64},$$

$$\mathfrak{c} = \log \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \equiv 18 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \pmod{64}.$$

It is worthwhile to remark that, even if $\mathfrak{c}$ is not zero modulo 4, the exponential series $\exp(\mathfrak{c})$ converges, because $(\mathfrak{c}/2)^5 \equiv 0$ modulo 4.

Acknowledgments

This research was supported by a Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme (PIRSES-GA-2012-317721). We thank Giovanni Falcone for many fruitful discussions on the matter of this paper.

References

[1] J. Cremona, Letters to the editor, Amer. Math. Monthly 94 (8) (1987) 757–758.
[2] T. Nagell, Introduction to Number Theory, John Wiley & Sons, Inc., New York, Almqvist & Wiksell, Stockholm, 1951.
[3] M.-F. Vignéras, Arithmétique Des Algèbres De Quaternions. (French) [Arithmetic of Quaternion Algebras], in: Lecture Notes in Mathematics, 800, Springer, Berlin, 1980.
[4] B.C. Hall, Lie Groups, Lie Algebras, and Representations. An Elementary Introduction, second ed., in: Graduate Texts in Mathematics, vol. 222, Springer, Cham, 2015.
[5] A. Di Bartolo, G. Falcone, Witt vectors and Fermat quotients, J. Number Theory 128 (5) (2008) 1376–1387.
[6] A. Buium, C.C. Ralph, S.R. Simanca, Arithmetic differential operators on $\mathbb{Z}_p$, J. Number Theory 131 (1) (2011) 96–105.

Further reading

[1] S.L. Altman, Rotations, Quaternions, and Double Groups, Dover Publications, 1986.

Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute 171 (2017) 264–292

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

Method of fundamental solutions for mixed and crack type problems in the classical theory of elasticity

Tengiz Buchukuri[a], Otar Chkadua[a,b], David Natroshvili[c,d,*]

[a] *A.Razmadze Mathematical Institute of I.Javakhishvili Tbilisi State University, 6 Tamarashvili st., Tbilisi 0177, Georgia*
[b] *Sokhumi State University, 9 A.Politkovskaia st., Tbilisi 0186, Georgia*
[c] *Georgian Technical University, 77 M.Kostava st., Tbilisi 0175, Georgia*
[d] *I.Javakhishvili Tbilisi State University, I.Vekua Institute of Applied Mathematics, 2 University st., Tbilisi 0186, Georgia*

Received 3 February 2017; received in revised form 20 April 2017; accepted 24 April 2017
Available online 20 May 2017

Abstract

We analyse some new aspects concerning application of the fundamental solution method to the basic three-dimensional boundary value problems, mixed transmission problems, and also interior and interfacial crack type problems for steady state oscillation equations of the elasticity theory. First we present existence and uniqueness theorems of weak solutions and derive the corresponding norm estimates in appropriate function spaces. Afterwards, by means of the columns of Kupradze's fundamental solution matrix special systems of vector functions are constructed explicitly. The linear independence and completeness of these systems are proved in appropriate Sobolev–Slobodetskii and Besov function spaces. It is shown that the problem of construction of approximate solutions to the basic and mixed boundary value problems and to the interior and interfacial crack problems can be reduced to the problems of approximation of the given boundary vector functions by elements of the linear spans of the corresponding complete systems constructed by the fundamental solution vectors. By this approach the approximate solutions of the boundary value and transmission problems are represented in the form of linear combinations of the columns of the fundamental solution matrix with appropriately chosen poles distributed outside the domain under consideration. The unknown coefficients of the linear combinations are defined by the approximation conditions of the corresponding boundary and transmission data.

Keywords: Method of fundamental solutions; Theory of elasticity; Elastic vibrations; Mixed boundary value problem; Mixed transmission problem; Crack problem; Approximate solutions

1. Introduction

The Method of Fundamental Solutions (MFS) for partial differential equations was first proposed by V. Kupradze in the 1960s (see the pioneering works in this direction by V. Kupradze and M. Alexidze, [1,2], [KuAl]). The main

* Corresponding author at: Georgian Technical University, 77 M.Kostava st., Tbilisi 0175, Georgia.
E-mail addresses: t_buchukuri@yahoo.com (T. Buchukuri), chkadua@rmi.ge (O. Chkadua), natrosh@hotmail.com (D. Natroshvili).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.04.004

idea of the MFS is to distribute the singularity poles $\{y^{(k)}\}_{k=1}^{\infty}$ of the fundamental solution $\Gamma(x-y)$ of a differential operator outside the domain under consideration, construct the set of functions $\{\Gamma(x-y^{(k)})\}_{k=1}^{\infty}$, prove its density properties in appropriate function spaces, and then approximate the sought-for solution by a linear combination of the fundamental solutions, $\sum_{k=1}^{N} C_k\, \Gamma(x-y^{(k)})$ with unknown coefficients C_k, which are to be determined by satisfying the corresponding boundary conditions.

Starting from the 1970s, the MFS gradually became a useful technique and is used to solve a large variety boundary value problems (BVP) arising in the mathematical models of physics, engineering, and biomedicine (see [1–18], and the references therein). However, it should be mentioned that until now it has not been worked out how to apply the MFS to crack type problems in solid mechanics, since the different approaches related to MFS described in the scientific literature are not applicable to crack type problems. To work out this problematic topic and to extend the MFS to crack type boundary-value problems are among the main goals of the present investigation. We will reformulate crack type problems in the form of mixed type transmission problems introducing an artificial interface boundary containing the crack faces and then substantiate mathematically the MFS on the basis of the results obtained for mixed transmission problems.

For the basic and mixed exterior boundary value problems, as well as for the crack and mixed transmission problems of steady state elastic oscillations, here we develop the approach which is applicable for all values of the oscillation frequency parameter.

We have to mention here that the main shortage of the MFS is its poor conditioning which should be alleviated, e.g., by preconditioning of the corresponding system matrix or by iterative refinement or by some other artificial approaches available for special particular cases (see, e.g. [19]).

However, the MFS features remarkable and unusual ease of implementation due to the following reasons (see, e.g. [3,17,19]): *"Uniform character of the trial functions, complete absence of singular integral evaluations, it does not require an elaborate discretization of the boundary, simplicity of finding values of approximate solution at inner points of the domain of interest, the derivatives of the MFS approximation can also be evaluated directly, extreme abundance of the set of trial functions that results in a high adaptivity of the method, MFS can be applied even in the case of domains with irregular boundaries (e.g., for domains with Lipschitz boundaries)"*. More detailed overview of the results related to the fundamental solution method can be found in [17] and the references therein.

In this paper we prove linear independence and density property of the appropriately chosen systems of vector functions constructed by the corresponding fundamental solutions (Kupradze's matrix of fundamental solutions). These systems are associated with particular type of problems and actually they reduce the solving procedure of boundary value problems to the approximation problems of the boundary data in the appropriate non-orthogonal complete systems of vector functions.

The paper is organized as follows. In Section 2, we introduce the notions of regular, semi-regular and weak solutions and formulate classical and weak settings of boundary value and transmission problems for steady state oscillation equations of the elasticity theory. We formulate also the corresponding uniqueness theorems for the problems under consideration in the class of vector functions satisfying the Sommerfeld–Kupradze radiation conditions at infinity. In Section 3, existence and uniqueness theorems are proved for weak solutions and the corresponding estimates are obtained in appropriate function spaces. Section 4 is devoted to the fundamental solution method for basic and mixed boundary value problems, as well as for the basic and mixed transmission problems containing crack type problems as special particular cases. Special systems of vector functions are constructed explicitly by means of the columns of Kupradze's fundamental solution matrix and their linear independence and completeness are proved in appropriate Sobolev–Slobodetskii and Besov function spaces. The problem of construction of approximate solutions to the boundary value and transmission problems are reduced to the approximation problems of the given boundary vector functions by linear combinations of the elements of the corresponding nonorthogonal, linearly independent, complete vector systems. In Appendix A, we collect some auxiliary material needed in the main text of the paper concerning properties of layer potentials and the corresponding boundary operators. In Appendix B, we present alternative integral representations of radiating solutions in unbounded regions. Finally, in Appendix C, we recall some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for proving existence theorems for mixed boundary, boundary-transmission, and crack type problems by the potential methods.

The approach developed in this paper can be successfully applied to boundary value problems of mathematical physics for homogeneous and piecewise homogeneous bounded and unbounded composite media containing interior or interfacial cuts where the screen or crack type conditions are prescribed. In particular, it can be applied to the interior and exterior problems of statics of the theory of elasticity, as well as to the interior BVP of steady state oscillations for bounded domains. As it is well-known, the interior problems of steady state oscillations have discrete (countable) sets of resonant frequencies for arbitrary bounded domain and the corresponding nonhomogeneous BVPs are not solvable for arbitrary data (see e.g., [12, Ch. 7], [20]). However, the approach described in the paper can be applied also to the interior problems if the oscillation parameter does not belong to the set of resonant frequencies, i.e., if the corresponding homogeneous boundary value and transmission problems of steady state oscillations possess only the trivial solutions.

2. Basic equations and operators, statement of problems, and uniqueness theorems

The basic equation of elastic vibrations in the case of isotropic solids reads as [12]

$$A(\partial,\omega)u(x) \equiv \mu\, \Delta u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x) + \varrho\,\omega^2\,u(x) = 0, \tag{2.1}$$

where $\Delta = \partial_1^2+\partial_2^2+\partial_3^2$ is the Laplace operator, $\partial = (\partial_1,\partial_2,\partial_3)$, $\partial_k := \partial/\partial x_k$, ϱ is the constant density of the homogeneous elastic solid under consideration, $\omega \in \mathbb{R}$ is the oscillation frequency parameter, $u = (u_1,u_2,u_3)^\top$ is the displacement vector (the amplitude), and $A(\partial,\omega)$ is the matrix differential operator,

$$A(\partial,\omega) = \big[\mu\,\delta_{kj}\,\Delta + (\lambda+\mu)\,\partial_k\partial_j + \varrho\,\omega^2\,\delta_{kj}\big]_{3\times 3},$$

δ_{kj} is the Kronecker symbol, λ and μ are the Lamé constants satisfying the inequalities $\mu > 0, \quad 2\lambda+3\mu > 0$. When $\omega = 0$, Eq. (2.1) coincides with the Lamé equilibrium equations of statics and generates the operator $A(\partial) := A(\partial,0)$. The principal homogeneous symbol matrix $\mathcal{A}(\xi) := [\mu\delta_{kj}|\xi|^2 + (\lambda+\mu)\xi_k\xi_j]_{3\times3}$ of the operators $-A(\partial,\omega)$ and $-A(\partial)$ is positive definite, $\mathcal{A}(\xi)\eta\cdot\eta \geq \delta_0\,|\xi|^2\,|\eta|^2, \quad \forall\xi\in\mathbb{R}^3, \quad \forall\eta\in\mathbb{C}^3$, where δ_0 is a positive constant, $a\cdot b$ denotes the scalar product of complex valued vectors $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$: $a\cdot b = \sum_{k=1}^3 a_k\,\overline{b_k}$; $\mathbb{R}^3$ and $\mathbb{C}^3$ stand for the set of real and complex 3-tuples respectively.

Let Ω^+ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with a boundary $S = \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+\cup S$, and $\Omega^- = \mathbb{R}^3\setminus\overline{\Omega^+}$. Throughout the paper, for simplicity, we assume that S is an infinitely smooth surface if not otherwise stated.

By $C^k(\overline{\Omega^\pm})$ we denote the subspace of functions from $C^k(\Omega^\pm)$ whose derivatives up to the order k are continuously extendable to S from $\Omega^\pm$.

The symbols $\{\,\cdot\,\}_S^+$ and $\{\,\cdot\,\}_S^-$ denote one-sided limits (traces) on S from Ω^+ and Ω^-, respectively. We often drop the subscript S if it does not lead to misunderstanding.

By L_p, $L_{p,loc}$, $L_{p,comp}$, W_p^r, $W_{p,loc}^r$, $W_{p,comp}^r$, H_p^s, and $B_{p,q}^s$ (with $r\geq 0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\leq q\leq\infty$) we denote the well-known Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [21,22]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r\geq 0$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. In our analysis we essentially employ also the following function spaces:

$$\widetilde{H}_p^s(M) := \big\{f:\ f\in H_p^s(M_0),\ \mathrm{supp} f\subset\overline{M}\big\},$$
$$\widetilde{B}_{p,q}^s(M) := \big\{f:\ f\in B_{p,q}^s(M_0),\ \mathrm{supp} f\subset\overline{M}\big\},$$
$$H_p^s(M) := \big\{r_{_M} f:\ f\in H_p^s(M_0)\big\},$$
$$B_{p,q}^s(M) := \big\{r_{_M} f:\ f\in B_{p,q}^s(M_0)\big\},$$

where M_0 is a closed manifold without boundary and M is an open proper submanifold of M_0 with nonempty smooth boundary $\partial M \neq \varnothing$; $r_{_M}$ is the restriction operator onto M.

Remark 2.1. Let a function f be defined on an open proper submanifold M of a closed manifold M_0 without boundary. Let $f\in B_{p,q}^s(M)$ and $\widetilde{f}$ be an extension of f by zero to $M_0\setminus M$. If the extension preserves the space, i.e., if $\widetilde{f}\in\widetilde{B}_{p,q}^s(M)$, then we write $f\in\widetilde{B}_{p,q}^s(M)$ instead of $f\in r_{_M}\widetilde{B}_{p,q}^s(M)$ when it does not lead to misunderstanding.

Now let us introduce some definitions (cf. [23]).

Definition 2.2. We say that w is a regular function in $\Omega^\pm$ if $w\in C^1(\overline{\Omega^\pm})\cap C^2(\Omega^\pm)$.

Definition 2.3. Let us consider the following smooth dissection of the boundary surface $S = \overline{S}_D \cup \overline{S}_N$, $S_D \cap S_N = \varnothing$, $\ell = \overline{S}_D \cap \overline{S}_N \in C^\infty$, and let $\widetilde{\Omega}^{\pm}_{\ell} := \overline{\Omega^{\pm}} \setminus \ell$.

We say that w is a semi-regular function in $\widetilde{\Omega}^{\pm}_{\ell}$ and write $w \in \mathbf{C}(\widetilde{\Omega}^{\pm}_{\ell};\delta)$ if

(i) w is continuous in $\overline{\Omega^{\pm}}$;

(ii) the first order derivatives of w are continuous in $\widetilde{\Omega}^{\pm}_{\ell}$ and there is a constant $\delta \in [0,1)$, such that at the collision curve ℓ the following estimates hold

$$|\partial_k w(x)| \leqslant C\,[\mathrm{dist}(x,\ell)]^{-\delta}, \quad x \in \widetilde{\Omega}^{\pm}_{\ell}, \quad C = const, \quad k = 1,2,3,$$

where $\mathrm{dist}(x,\ell)$ is the distance from the reference point x to the collision curve ℓ;

(iii) the second order derivatives of w are continuous in $\Omega^{\pm}$ and integrable over Ω^{+} and over any subdomain of Ω^{-} of finite diameter.

Evidently, $\mathbf{C}(\widetilde{\Omega}^{\pm}_{\ell};\delta) \subset \big[C(\overline{\Omega^{\pm}}) \cap C^1(\widetilde{\Omega}^{\pm}_{\ell}) \cap C^2(\Omega^{\pm})\big]$.

Definition 2.4. Let an elastic solid occupying the domain $\Omega^{\pm}$ contain an interior crack. We identify the crack surface as a two-dimensional, two-sided smooth manifold $\Sigma \subset \Omega^{\pm}$ with the crack edge $\ell_c := \partial\Sigma$. We assume that Σ is a proper submanifold of a closed surface S_0 surrounding a bounded domain $\overline{\Omega}_0$ which is a proper subdomain of $\Omega^{\pm}$. We choose the direction of the unit normal vector to the fictitious surface S_0 such that it is outward with respect to the domain Ω_0. This agreement defines uniquely the direction of the normal vector to the crack surface Σ. The symbols $\{\,\cdot\,\}^{+}_{\Sigma}$ and $\{\,\cdot\,\}^{-}_{\Sigma}$ denote the one-sided limits on Σ from Ω_0 and $\Omega^{\pm} \setminus \overline{\Omega}_0$, respectively.

Further, let $\Omega^{\pm}_{\Sigma} := \Omega^{\pm} \setminus \overline{\Sigma}$ and $\widetilde{\Omega}^{\pm}_{\Sigma} := \overline{\Omega^{\pm}} \setminus \overline{\Sigma}$ with $\overline{\Sigma} = \Sigma \cup \ell_c$.

We say that w is a semi-regular function in $\widetilde{\Omega}^{\pm}_{\Sigma}$ and write $w \in \mathbf{C}(\widetilde{\Omega}^{\pm}_{\Sigma};\delta)$ if

(i) w is continuous in $\widetilde{\Omega}^{\pm}_{\Sigma}$ and one-sided continuously extendable to $\overline{\Sigma}$ from Ω_0 and from $\Omega^{+} \setminus \overline{\Omega}_0$, i.e., w is continuous in the regions $\widetilde{\Omega}^{\pm}_{\Sigma}$, $\overline{\Omega^{\pm}} \setminus \Omega_0$, and $\overline{\Omega}_0$;

(ii) the first order derivatives of w are continuous in $\widetilde{\Omega}^{\pm}_{\Sigma}$ and one-sided continuously extendable to Σ from Ω_0 and from $\Omega^{\pm} \setminus \overline{\Omega}_0$, and there is a constant $\delta \in [0,1)$, such that at the crack edge $\ell_c = \partial\Sigma$ the following estimates hold

$$|\partial_k w(x)| \leqslant C\,[\mathrm{dist}(x,\ell_c)]^{-\delta}, \quad x \in \widetilde{\Omega}^{\pm}_{\Sigma}, \quad C = const, \quad k = 1,2,3;$$

(iii) the second order derivatives of w are continuous in $\Omega^{\pm}_{\Sigma}$ and integrable over Ω^{+}_{Σ} and over any subdomain of Ω^{-}_{Σ} of finite diameter.

Evidently, formally we can write $\mathbf{C}(\widetilde{\Omega}^{\pm}_{\Sigma};\delta) \subset \big[C(\overline{\Omega}_0) \cap C(\overline{\Omega^{\pm}} \setminus \Omega_0) \cap C^1(\widetilde{\Omega}^{\pm}_{\Sigma}) \cap C^2(\Omega^{\pm}_{\Sigma})\big]$, which is to be understood in the following sense: if $w \in \mathbf{C}(\widetilde{\Omega}^{\pm}_{\Sigma};\delta)$, then $r_{\overline{\Omega}_0}\, w \in C(\overline{\Omega}_0)$, $r_{\overline{\Omega^{\pm}} \setminus \Omega_0}\, w \in C(\overline{\Omega^{\pm}} \setminus \Omega_0)$, $w \in C^1(\widetilde{\Omega}^{\pm}_{\Sigma})$, $w \in C^2(\Omega^{\pm}_{\Sigma})$.

Definition 2.5. We say that a vector $u = (u_1,u_2,u_3)^{\top}$ in the exterior domain Ω^{-} satisfies the Sommerfeld–Kupradze type radiation conditions at infinity if u is representable in Ω^{-} as a sum of two metaharmonic vectors, the so called longitudinal $u^{(1)} \equiv u^{(p)}$ and transverse parts $u^{(2)} \equiv u^{(s)}$ (see, e.g., [12]),

$$u = u^{(1)} + u^{(2)} \text{ with } \Delta\, u^{(m)} + k_m^2\, u^{(m)} = 0, \quad m = 1,2,$$
$$k_1 \equiv k_p = \omega\sqrt{\frac{\varrho}{\lambda + 2\mu}}\,, \quad k_2 \equiv k_s = \omega\sqrt{\frac{\varrho}{\mu}}\,,$$

satisfying for sufficiently large $r = |x|$ the radiating conditions

$$\frac{\partial u^{(m)}(x)}{\partial r} - i\,k_m\,u^{(m)}(x) = o\,(r^{-1})\,, \quad m = 1,2.$$

Denote the Sommerfeld–Kupradze class of radiating vector functions by $Z(\Omega^{-})$.

Assume that the domains $\overline{\Omega^{\pm}}$ are occupied by an isotropic homogeneous elastic material.

Denote by $e_{kl} = e_{kl}(u)$ and $\sigma_{kl} = \sigma_{kl}(u)$ the strain and stress tensors respectively associated with the displacement vector u. Then the components of the stress vector $T(\partial,n)u$ acting upon a surface element with normal vector n read as [12]

$$\{T(\partial,n)u\}_k = \sigma_{kl}\,n_l, \;\; \sigma_{kl} = [\lambda\,\delta_{kl}\,\mathrm{div}\,u + 2\,\mu\,e_{kl}], \;\; e_{kl} = 2^{-1}\,(\partial_k u_l + \partial_l u_k).$$

Here $T(\partial, n)$ is the boundary stress operator,

$$T(\partial, n) := [T_{kl}(\partial, n)]_{3\times 3}, \quad T_{kl}(\partial, n) = \lambda\, n_k\, \partial_{x_l} + \mu\, n_l\, \partial_{x_k} + \mu\, \delta_{kl}\, \partial_n, \tag{2.2}$$

where $\partial_n = \partial/\partial n$ stands for the normal derivative.

Now we formulate the basic exterior BVPs of steady state elastic oscillations.

The Dirichlet problem (D)$_\omega^-$: Find a regular complex-valued solution vector $u \in [C^1(\overline{\Omega^-})]^3 \cap [C^2(\Omega^-)]^3 \cap Z(\Omega^-)$ to the steady state oscillation equation (2.1) in Ω^- satisfying the Dirichlet type boundary condition

$$\{u(x)\}^- = f(x), \quad x \in S, \tag{2.3}$$

where $f \in [C^1(S)]^3$ is a given smooth vector function on S.

The Neumann problem (N)$_\omega^-$: Find a regular complex-valued solution vector $u \in [C^1(\overline{\Omega^-})]^3 \cap [C^2(\Omega^-)]^3 \cap Z(\Omega^-)$ to the steady state oscillation equation (2.1) in Ω^- satisfying the Neumann type boundary condition

$$\{T(\partial, n)u(x)\}^- = F(x), \quad x \in S, \tag{2.4}$$

where $F \in [C(S)]^3$ is a given vector function on S.

Mixed type problem (M)$_\omega^-$: Find a semi-regular complex-valued solution vector $u \in [\mathbf{C}(\widetilde{\Omega}_\ell^-; \delta)]^3 \cap Z(\Omega^-)$ to the steady state oscillation equation (2.1) in Ω^- satisfying the mixed type boundary conditions:

$$\{u(x)\}^- = f^*(x), \quad x \in S_D, \tag{2.5}$$

$$\big\{T(\partial, n)u(x)\big\}^- = F^*(x), \quad x \in S_N, \tag{2.6}$$

where $f^* \in [C^1(S_D)]^3$ and $F^* \in [C(S_N)]^3$ are given vector functions.

Crack type problem (C)$_\omega^-$: Find a semi-regular complex-valued solution vector $u \in [\mathbf{C}(\widetilde{\Omega}_\Sigma^-; \delta)]^3 \cap Z(\Omega_\Sigma^-)$ to the steady state oscillation equation (2.1) in Ω_Σ^- satisfying either the Dirichlet or Neumann type boundary condition on S and the following crack type conditions on Σ:

$$\big\{T(\partial, n)u(x)\big\}^+ = F^{(+)}(x), \quad x \in \Sigma, \tag{2.7}$$

$$\big\{T(\partial, n)u(x)\big\}^- = F^{(-)}(x), \quad x \in \Sigma, \tag{2.8}$$

where $F^{(\pm)} \in [C(\Sigma)]^3$ are given vector functions.

Note that, if the mixed type boundary conditions are prescribed on the boundary surface S, then in addition we have to require that a solution is semi-regular in a neighbourhood of the collision curve ℓ.

Basic crack type problem (BC)$_\omega^-$: Find a semi-regular complex-valued solution vector $u \in [\mathbf{C}(\mathbb{R}^3_\Sigma; \delta)]^3 \cap Z(\mathbb{R}^3_\Sigma)$ to the steady state oscillation equation (2.1) in $\mathbb{R}^3_\Sigma := \mathbb{R}^3 \setminus \overline{\Sigma}$ satisfying the crack type conditions on Σ:

$$\big\{T(\partial, n)u(x)\big\}^+ = F^{(+)}(x), \quad x \in \Sigma, \tag{2.9}$$

$$\big\{T(\partial, n)u(x)\big\}^- = F^{(-)}(x), \quad x \in \Sigma, \tag{2.10}$$

where $F^{(\pm)} \in [C(\Sigma)]^3$ are given vector functions.

Now let us assume that the domains $\Omega^{(1)} = \Omega^+$ and $\Omega^{(2)} = \Omega^-$ are occupied by isotropic elastic materials with Lamé constants $\lambda^{(\kappa)}$, $\mu^{(\kappa)}$, and the density $\varrho^{(\kappa)}$, $\kappa = 1, 2$. In this case S is the interface of the composite elastic solid where various type transmission conditions are to be prescribed.

Basic transmission problem (BT)$_\omega$: Find regular complex-valued solution vectors $u^{(1)} \in [C^1(\overline{\Omega^+})]^3 \cap [C^2(\Omega^+)]^3$ and $u^{(2)} \in [C^1(\overline{\Omega^-})]^3 \cap [C^2(\Omega^-)]^3 \cap Z(\Omega^-)$ to the steady state oscillation equations

$$\begin{aligned} A^{(\kappa)}(\partial, \omega)u^{(\kappa)}(x) &\equiv \mu^{(\kappa)}\, \Delta u(x) + (\lambda^{(\kappa)} + \mu^{(\kappa)})\,\mathrm{grad}\,\mathrm{div}\, u(x) + \varrho^{(\kappa)}\, \omega^2\, u^{(\kappa)}(x) \\ &= 0, \quad x \in \Omega^{(\kappa)}, \quad \kappa = 1, 2, \end{aligned} \tag{2.11}$$

satisfying the rigid transmission conditions

$$\{u^{(1)}(x)\}^+ - \{u^{(2)}(x)\}^- = f(x), \quad x \in S, \tag{2.12}$$

$$\{T^{(1)}(\partial, n)u^{(1)}(x)\}^+ - \{T^{(2)}(\partial, n)u^{(2)}(x)\}^- = F(x), \quad x \in S, \tag{2.13}$$

where $f \in [C^1(S)]^3$ and $F \in [C(S)]^3$ are given vector functions on S and

$$T^{(\kappa)}(\partial, n) := [T^{(\kappa)}_{kl}(\partial, n)]_{3\times 3}, \;\; T^{(\kappa)}_{kl}(\partial, n) = \lambda^{(\kappa)} n_k \partial_{x_l} + \mu^{(\kappa)} n_l\, \partial_{x_k} + \mu^{(\kappa)} \delta_{kl} \partial_n. \tag{2.14}$$

If the interface S contains a crack along a subsurface $S_C \subset S$, then we have the following dissection $S = \overline{S_C} \cup \overline{S_T}$, where $S_T = S \setminus \overline{S_C}$ is the rigid transmission part of the interface, and $S_C \cap S_T = \varnothing$.

Basic mixed transmission problem (MT)$_\omega$: Find semi-regular complex-valued solution vectors $u^{(1)} \in [\mathbf{C}(\widetilde{\Omega}^+_\ell;\delta)]^3$ and $u^{(2)} \in [\mathbf{C}(\widetilde{\Omega}^-_\ell;\delta)]^3 \cap Z(\Omega^-)$, $\ell = \overline{S_C} \cap \overline{S_T}$, to the steady state oscillation equations (2.11) satisfying the rigid transmission conditions on S_T,

$$\{u^{(1)}(x)\}^+ - \{u^{(2)}(x)\}^- = f^{(T)}(x), \quad x \in S_T, \tag{2.15}$$

$$\{T^{(1)}(\partial,n)u^{(1)}(x)\}^+ - \{T^{(2)}(\partial,n)u^{(2)}(x)\}^- = F^{(T)}(x), \quad x \in S_T, \tag{2.16}$$

and the crack conditions on S_C,

$$\{T^{(1)}(\partial,n)u^{(1)}(x)\}^+ = F_C^{(+)}(x), \quad x \in S_C, \tag{2.17}$$

$$\{T^{(2)}(\partial,n)u^{(2)}(x)\}^- = F_C^{(-)}(x), \quad x \in S_C, \tag{2.18}$$

where $f^{(T)} \in [C^1(S_T)]^3$, $F^{(T)} \in [C(S_T)]^3$, and $F_C^{(\pm)} \in [C(S_C)]^3$ are given vector functions.

Weak setting of the problems. In the case of *weak formulation* of the above boundary value and boundary-transmission problems we look for *weak solution vectors* in the spaces $[W^1_p(\Omega^+)]^3$ and $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, $1 < p < +\infty$, respectively. In this case the differential equations (2.1) and (2.11) are understood in the distributional sense, the Dirichlet type conditions (2.3), (2.5), (2.12), and (2.15) are understood in the usual trace sense, while the Neumann type conditions (2.4), (2.6)–(2.10), (2.13), (2.16)–(2.18) are understood in the generalized functional trace sense, defined with the help of Green's identity (cf. [12,24]):

$$\langle \{Tu\}^\pm,\ \{\overline{v}\}^\pm \rangle_S = \pm \int_{\Omega^\pm} [\,E(u,\overline{v}) - \varrho\,\omega^2\, u\cdot v\,]dx, \tag{2.19}$$

where $u \in [W^1_p(\Omega^+)]^3$, $v \in [W^1_{p'}(\Omega^+)]^3$, or $u \in [W^1_{p,loc}(\Omega^-)]^3$, $v \in [W^1_{p',comp}(\Omega^-)]^3$ with $1/p + 1/p' = 1$, $1 < p < +\infty$, the over-bar denotes complex conjugation, the symbol $\langle\cdot,\ \cdot\rangle_S$ denotes bilinear duality brackets between the mutually adjoint spaces $[B^{-\frac{1}{p}}_{p,p}(S)]^3$ and $[B^{\frac{1}{p}}_{p',p'}(S)]^3$,

$$E(u,v) = \frac{3\,\lambda + 2\,\mu}{3}\operatorname{div} u \operatorname{div} v + \frac{\mu}{2}\sum_{k\neq l}(\partial_l u_k + \partial_k u_l)(\partial_l v_k + \partial_k v_l) + \frac{\mu}{3}\sum_{k,l=1}^{3}(\partial_k u_k - \partial_l u_l)(\partial_k v_k - \partial_l v_l)\,.$$

Note that by relations (2.19) the generalized traces $\{Tu\}^\pm \in [B^{-\frac{1}{p}}_{p,p}(S)]^3$ are well defined for weak solutions of the homogeneous differential equation of steady state oscillations, $u \in [W^1_p(\Omega^+)]^3$ and $u \in [W^1_{p,loc}(\Omega^-)]^3$.

In the case of weak setting, the boundary data belong to the natural Besov spaces:

$$f \in [B^{1-\frac{1}{p}}_{p,p}(S)]^3,\ \ f^* \in [B^{1-\frac{1}{p}}_{p,p}(S_D)]^3,\ \ f^{(T)} \in [B^{1-\frac{1}{p}}_{p,p}(S_T)]^3,\ \ F \in [B^{-\frac{1}{p}}_{p,p}(S)]^3,$$
$$F^* \in [B^{-\frac{1}{p}}_{p,p}(S_N)]^3,\ \ F^{(\pm)} \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3,\ \ F^{(+)} - F^{(-)} \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^3,$$
$$F^{(T)} \in [B^{-\frac{1}{p}}_{p,p}(S_T)]^3,\ \ F_C^{(\pm)} \in [B^{-\frac{1}{p}}_{p,p}(S_C)]^3.$$

With the help of the Rellich–Vekua lemma the following uniqueness theorem can be proved (for details see [12,20,25–28].

Theorem 2.6. *Let the manifolds $S = \partial\Omega^\pm$, S_D, S_N, S_T, S_C, and Σ be Lipschitz. Then the BVPs $(D)^-_\omega$, $(N)^-_\omega$, $(M)^-_\omega$, $(C)^-_\omega$, $(BT)_\omega$, and $(MT)_\omega$ possess at most one weak solution for $p = 2$ and for all values of the frequency parameter ω.*

3. Existence theorems

Here we employ the notation introduced in Appendices A–C and formulate basic existence results for weak solutions and prove representability of solutions by the layer potentials.

We apply a special representation of solutions by the layer potentials and reduce the above formulated BVPs of elastic oscillations to the corresponding uniquely solvable integral (pseudodifferential) equations for arbitrary value of the oscillation parameter ω. Similar approach for the Helmholtz equation has been developed in the Refs. [29–31].

Throughout the paper, $B(R)$ denotes the ball centred at the origin and radius R such that $\Omega^+ \subset B(R)$ and $\varkappa$ is a complex number

$$\varkappa = \varkappa_1 + i\,\varkappa_2 \in \mathbb{C}, \quad \varkappa_1, \varkappa_2 \in \mathbb{R}, \quad \varkappa_2 \neq 0.$$

Theorems 2.6, A.1, B.1 and B.2 directly lead to the following existence results for the exterior Dirichlet and Neumann type problems.

Theorem 3.1. *The Dirichlet problem $(D)^-_\omega$ with arbitrary boundary vector function $f \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ is uniquely solvable in the space $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, $p > 1$, and the solution is representable as a linear combination of the double and single layer potentials*

$$u(x) = W(g)(x) + \varkappa\, V(g)(x), \quad x \in \Omega^-,$$

where the density vector function $g \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ is defined by the uniquely solvable elliptic pseudodifferential equation

$$\mathcal{N}\, g \equiv [\,-2^{-1} I_3 + \widetilde{\mathcal{K}} + \varkappa\, \mathcal{H}\,]\, g = f \ \text{ on } \ S.$$

Moreover, the following estimate holds

$$\|u\|_{[W^1_p(\Omega^- \cap B(R))]^3} \leqslant C_D(R)\, \|f\|_{[B^{1-\frac{1}{p}}_{p,p}(S)]^3} \tag{3.20}$$

where $C_D(R)$ is a constant independent of f.

Theorem 3.2. *The Neumann problem $(N)^-_\omega$ with arbitrary boundary vector function $F \in \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^3$ is uniquely solvable in the space $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, $p > 1$, and the solution is representable as a linear combination of the double and single layer potentials*

$$u(x) = W(g)(x) + \varkappa\, V(g)(x), \quad x \in \Omega^-,$$

where the density vector $g \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ is defined by the uniquely solvable elliptic pseudodifferential equation

$$\mathcal{M}\, g \equiv \big[\mathcal{L} + \varkappa\,(2^{-1} I_3 + \mathcal{K})\big]\, g = F \ \text{ on } \ S.$$

Moreover, the following estimate holds

$$\|u\|_{[W^1_p(\Omega^- \cap B(R))]^3} \leqslant C_N(R)\, \|F\|_{[B^{-\frac{1}{p}}_{p,p}(S)]^3}$$

where $C_N(R)$ is a constant independent of F.

For the mixed problem we have the following assertion.

Theorem 3.3. *Let $4/3 < p < 4$. The mixed problem $(M)^-_\omega$ with arbitrary boundary data*

$$f^* \in \big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3, \qquad F^* \in \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3,$$

is uniquely solvable in the space $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, and the solution is representable as a linear combination of the double and single layer potentials

$$u(x) = W\big(\mathcal{N}^{-1}(f_e + \widetilde{g})\big)(x) + \varkappa\, V\big(\mathcal{N}^{-1}(f_e + \widetilde{g})\big)(x), \quad x \in \Omega^-, \tag{3.21}$$

where $f_e \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ is some fixed extension of the vector function f^* from S_D onto the whole of S, while $\widetilde{g} \in \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3$ is defined by the uniquely solvable elliptic pseudodifferential equation

$$r_{S_N} \mathcal{M}\mathcal{N}^{-1} g = F_0 \ \text{ on } \ S_N,$$

where

$$F_0 = F^* - r_{S_N} \mathcal{M}\mathcal{N}^{-1} f_e \in \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3, \quad \|f_e\|_{[B^{1-\frac{1}{p}}_{p,p}(S)]^3} \leqslant 2\|f^*\|_{[B^{1-\frac{1}{p}}_{p,p}(S_D)]^3}.$$

Moreover, the following estimate holds

$$\|u\|_{[W^1_p(\Omega^-\cap B(R))]^3} \leqslant C_M(R)\Big[\|f^*\|_{[B^{1-\frac{1}{p}}_{p,p}(S_D)]^3} + \|F^*\|_{[B^{-\frac{1}{p}}_{p,p}(S_N)]^3}\Big]$$

where $C_M(R)$ is a constant independent of f^* and F^*.

Proof. Invertibility of the operator

$$r_{S_N}\mathcal{M}\mathcal{N}^{-1} \,:\, [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^3 \to [B^{-\frac{1}{p}}_{p,p}(S_N)]^3 \tag{3.22}$$

for $4/3 < p < 4$ follows from Theorems B.2 and C.2, since the principal homogeneous symbol matrix of the operator $-\mathcal{M}\mathcal{N}^{-1}$,

$$\mathfrak{S}\big(-\mathcal{M}\mathcal{N}^{-1}; x, \xi\big) = -\mathbb{L}(x,\xi)\,[\widetilde{\mathbb{K}}_-(x,\xi)]^{-1}, \quad x \in S, \quad \xi = (\xi_1, \xi_2) \neq 0,$$

is positive definite due to (A.19) in Remark A.4 and the null-space of the operator (3.22) is trivial.

Existence and uniqueness of a solution to the mixed problem and estimate (3.22) follow then from (3.21) and Theorems 2.6, B.1 and A.1. □

For the weak solution of the basic crack type problem the following existence result holds.

Theorem 3.4. *Let $4/3 < p < 4$ and $F^{(\pm)} \in \big[B^{-\frac{1}{p}}_{p,p}(\Sigma)\big]^3$ with $F^{(+)} - F^{(-)} \in \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)\big]^3$. Then the basic crack type problem $(BC)_\omega$ is uniquely solvable in the space $[W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^3 \cap Z(\mathbb{R}^3_\Sigma)$ and the solution is representable as a linear combination of the single and double layer potentials*

$$u(x) = W\big(g\big)(x) - V\big(F^{(+)} - F^{(-)}\big)(x), \quad x \in \mathbb{R}^3_\Sigma, \tag{3.23}$$

where $g \in \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)\big]^3$ is defined by the uniquely solvable elliptic pseudodifferential equation

$$r_{S_N}\mathcal{L}\,g = F_0 \;\; on \;\; \Sigma, \tag{3.24}$$

where

$$F_0 = \frac{1}{2}\big(F^{(+)} + F^{(-)}\big) + r_\Sigma \mathcal{K}\,\big(F^{(+)} - F^{(-)}\big) \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3.$$

Moreover, the following estimate holds

$$\|u\|_{[W^1_p(\mathbb{R}^3_\Sigma\cap B(R))]^3} \leqslant C_M(R)\big[\|F^{(+)} + F^{(-)}\|_{[B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3} + \|F^{(+)} - F^{(-)}\|_{[B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3}\big] \tag{3.25}$$

where $C_M(R)$ is a constant independent of $F^{(\pm)}$.

Proof. Let us rewrite the boundary conditions (2.9)–(2.10) of the crack problem $(\mathrm{BC})_\omega$ in the following equivalent form

$$\big\{T(\partial, n)u\big\}^+ - \big\{T(\partial, n)u\big\}^- = F^{(+)} - F^{(-)} \;\; \text{on} \;\; \Sigma, \tag{3.26}$$

$$\big\{T(\partial, n)u\big\}^+ + \big\{T(\partial, n)u\big\}^- = F^{(+)} + F^{(-)} \;\; \text{on} \;\; \Sigma. \tag{3.27}$$

The vector function (3.23) satisfies condition (3.26) automatically, while condition (3.27) leads to Eq. (3.24). Existence and uniqueness of a solution to the basic crack type problem and estimate (3.25) follow then from (3.23) and Theorems 2.6, A.1, A.3, B.2 and C.2. Indeed, the principal homogeneous symbol matrix $\mathbb{L}(x,\xi) := \mathfrak{S}(\mathcal{L}; x, \xi)$ of the operator $\mathcal{L}$ is positive definite (see Remark A.4) and the null-space of the operator

$$r_{S_N}\mathcal{L} \,:\, [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^3 \to [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3 \tag{3.28}$$

is trivial implying the invertibility of the operator (3.28). Thus Eq. (3.24) is uniquely solvable and the estimate (3.25) holds. □

In the case of transmission problems, we use the same notation for potentials and the corresponding integral operators as above but equipped with superscript $^{(\kappa)}$ which indicates that the layer potentials $V^{(\kappa)}$, $W^{(\kappa)}$ and the corresponding integral operators $\mathcal{H}^{(\kappa)}, \mathcal{K}^{(\kappa)}, \widetilde{\mathcal{K}}^{(\kappa)}, \mathcal{L}^{(\kappa)}, \mathcal{N}^{(\kappa)}$, and $\mathcal{M}^{(\kappa)}$ are constructed with the help of the fundamental solution $\Gamma^{(\kappa)}(x-y,\omega)$ associated with the operator $A^{(\kappa)}(\partial,\omega)$ and the stress operator is defined by (2.14).

Theorem 3.5. *The basic transmission problem* $(BT)_\omega$ *with arbitrary boundary vector functions* $f \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ *and* $F \in \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^3$ *is uniquely solvable in the class of vector functions* $[W^1_p(\Omega^+)]^3 \times \big([W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)\big)$, $p>1$, *and the solution pair* $\big(u^{(1)},u^{(2)}\big)$ *is representable by the layer potentials:*

$$u^{(1)}(x)=V^{(1)}(h)(x),\quad x\in\Omega^+=\Omega^{(1)}, \tag{3.29}$$
$$u^{(2)}(x)=W^{(2)}(g)(x)+\varkappa\,V^{(2)}(g)(x),\quad x\in\Omega^-=\Omega^{(2)}, \tag{3.30}$$

where the density vectors $h\in\big[B^{-\frac{1}{p}}_{p,p}(S)\big]^3$ *and* $g\in\big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ *are defined by the uniquely solvable elliptic system of pseudodifferential equations*

$$\mathcal{H}^{(1)}\,h-\mathcal{N}^{(2)}\,g=f\ \text{ on }\ S, \tag{3.31}$$
$$[\,-2^{-1}\,I_3+\mathcal{K}^{(1)}\,]\,h-\mathcal{M}^{(2)}\,g=F\ \text{ on }\ S. \tag{3.32}$$

Moreover, the following estimates hold

$$\|u^{(1)}\|_{[W^1_p(\Omega^+)]^3}\leqslant C^{(1)}_{BT}\Big(\|f\|_{[B^{1-\frac{1}{p}}_{p,p}(S)]^3}+\|F\|_{[B^{-\frac{1}{p}}_{p,p}(S)]^3}\Big), \tag{3.33}$$
$$\|u^{(2)}\|_{[W^1_p(\Omega^-\cap B(R))]^3}\leqslant C^{(2)}_{BT}(R)\Big(\|f\|_{[B^{1-\frac{1}{p}}_{p,p}(S)]^3}+\|F\|_{[B^{-\frac{1}{p}}_{p,p}(S)]^3}\Big), \tag{3.34}$$

where $C^{(1)}_{BT}$ *and* $C^{(2)}_{BT}(R)$ *are constants independent of* f *and* F.

Proof. The representations (3.29)–(3.30) lead to the system of pseudodifferential equations (3.31)–(3.32). Due to the invertibility property of the operators $\mathcal{N}^{(2)}$ and $\mathcal{M}^{(2)}$ (see Appendix B, Theorem B.1), we derive

$$g=\big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)}h-\big(\mathcal{N}^{(2)}\big)^{-1}f, \tag{3.35}$$
$$\mathcal{T}\,h=\big(\mathcal{N}^{(2)}\big)^{-1}f-\big(\mathcal{M}^{(2)}\big)^{-1}F, \tag{3.36}$$

where $\mathcal{T}=[\mathcal{T}_{kj}]_{3\times3}$ is the pseudodifferential operator of order -1 defined by the relation

$$\mathcal{T}:=\big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)}-\big(\mathcal{M}^{(2)}\big)^{-1}\big(-2^{-1}\,I_3+\mathcal{K}^{(1)}\big).$$

Rewrite system (3.35)–(3.36) in matrix form

$$\mathcal{Q}\,\Phi=\Psi, \tag{3.37}$$

where $\Phi=(g,h)^\top$, $\Psi:=\Big(-\big(\mathcal{N}^{(2)}\big)^{-1}f,\,\big(\mathcal{N}^{(2)}\big)^{-1}f-\big(\mathcal{M}^{(2)}\big)^{-1}F\Big)^\top$, and

$$\mathcal{Q}:=\begin{bmatrix} I_3 & -\big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)}\\ [0]_{3\times3} & \mathcal{T}\end{bmatrix}_{6\times6}.$$

Note that $\Psi=0$ if and only if $f=F=0$. Therefore the homogeneous equation (3.37) corresponds to the homogeneous basic transmission problem $(\mathrm{BT})_\omega$.

The principal homogeneous symbol matrix of the operator $\mathcal{T}$ reads as follows (see Appendix A, Remark A.4)

$$\mathfrak{S}\big(\mathcal{T};x,\xi\big)=\big(\widetilde{\mathbb{K}}^{(2)}_-\big)^{-1}\,\mathbb{H}^{(1)}-\big(\mathbb{L}^{(2)}\big)^{-1}\,\mathbb{K}^{(1)}_-=\big(\mathbb{L}^{(2)}\big)^{-1}\big[\mathbb{K}^{(1)}_-\,\big(\mathbb{H}^{(1)}\big)^{-1}-\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}^{(2)}_-\big)^{-1}\big]\big(-\mathbb{H}^{(1)}\big), \tag{3.38}$$

where the matrices $\mathbb{K}^{(1)}_-$, $\widetilde{\mathbb{K}}^{(2)}_-$, $\mathbb{H}^{(1)}$, and $\mathbb{L}^{(2)}$ are defined in (A.15). From the results stated in Remark A.4 it follows that the matrices $\big(\mathbb{L}^{(2)}\big)^{-1}$, $\mathbb{K}^{(1)}_-\,\big(\mathbb{H}^{(1)}\big)^{-1}-\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}^{(2)}_-\big)^{-1}$, and $\big(-\mathbb{H}^{(1)}\big)$ are positive definite nonsingular matrices for all $x\in S$ and $\xi\in\mathbb{R}^2\setminus\{0\}$. Consequently $\mathfrak{S}\big(\mathcal{T};x,\xi\big)$ is an elliptic symbol. Moreover, $\mathcal{T}$ is a composition of three

Fredholm operators with zero index and due to Atkinson's theorems (see, e.g., [32, Ch. 1, Theorem 3.3]) the index of the Fredholm operator

$$\mathcal{T} : \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3 \to \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$$

equals to zero. Therefore the operator

$$\mathcal{Q} : \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3 \times \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3 \to \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3 \times \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3 \tag{3.39}$$

is Fredholm with zero index as well.

From the uniqueness theorem for the basic transmission problem $(\mathrm{BT})_\omega$ for $p = 2$ and the general theory of pseudodifferential equations on smooth manifolds without boundary it follows that the null space of the operator (3.39) is trivial for $1 < p < \infty$. Thus the operator (3.39) is invertible and the system (3.31) is uniquely solvable implying the uniqueness and existence of a solution to the problem $(\mathrm{BT})_\omega$ for $1 < p < \infty$.

The estimates (3.33)–(3.34) follow then from Theorem A.1. □

Remark 3.6. From the arguments in the proof of Theorem 3.5 and relations (3.35) and (3.36) it follows that

$$h = \mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} f - \mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1} F, \tag{3.40}$$

$$g = \big(\mathcal{N}^{(2)}\big)^{-1}\big[\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} - I\big]f - \big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1} F. \tag{3.41}$$

Now we analyse the basic mixed transmission problem $(\mathrm{MT})_\omega$. To this end, rewrite the mixed transmission conditions (2.15)–(2.18) in the formulation of problem $(\mathrm{MT})_\omega$ in the following equivalent form

$$\{u^{(1)}\}^+ - \{u^{(2)}\}^- = f^{(T)} \quad \text{on} \quad S_T, \tag{3.42}$$

$$\{T^{(1)}(\partial, n)u^{(1)}\}^+ - \{T^{(2)}(\partial, n)u^{(2)}\}^- = F_0 \quad \text{on} \quad S, \tag{3.43}$$

$$\{T^{(1)}(\partial, n)u^{(1)}\}^+ + \{T^{(2)}(\partial, n)u^{(2)}\}^- = F_C^{(+)} + F_C^{(-)} \quad \text{on} \quad S_C, \tag{3.44}$$

where

$$F_0 = \begin{cases} F^{(T)} & \text{on} \quad S_T\,, \\ F_C^{(+)} - F_C^{(-)} & \text{on} \quad S_C\,, \end{cases} \tag{3.45}$$

and we assume that the following necessary compatibility condition is fulfilled

$$F_0 \in \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3. \tag{3.46}$$

Denote by f^* some fixed extension of the vector function $f^{(T)}$ from S_T onto the whole of S preserving the space, $f^* \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$. Evidently, an arbitrary extension has then the form $f = f^* + \widetilde{g}$, where $\widetilde{g} \in \big[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)\big]^3$.

Motivated by the existence result for the basic transmission problem described in Theorem 3.5, let us look for a solution to the basic mixed transmission problem (3.42)–(3.44) again in the form (3.29)–(3.30),

$$u^{(1)}(x) = V^{(1)}(h)(x), \quad x \in \Omega^+ = \Omega^{(1)}, \tag{3.47}$$

$$u^{(2)}(x) = W^{(2)}(g)(x) + \varkappa\, V^{(2)}(g)(x), \quad x \in \Omega^- = \Omega^{(2)}, \tag{3.48}$$

where (see (3.40)–(3.41))

$$h = \mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1}(f^* + \widetilde{g}) - \mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1} F_0, \tag{3.49}$$

$$g = \big(\mathcal{N}^{(2)}\big)^{-1}\big[\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} - I\big](f^* + \widetilde{g}) - \big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1} F_0, \tag{3.50}$$

F_0 is defined in (3.45), f^* is the above introduced fixed extension, and $\widetilde{g}$ is an unknown vector function.

Due to Theorem 3.5, we find that

$$\{u^{(1)}\}^+ - \{u^{(2)}\}^- = f^* + \widetilde{g} \quad \text{on} \quad S,$$

$$\{T^{(1)}(\partial, n)u^{(1)}\}^+ - \{T^{(2)}(\partial, n)u^{(2)}\}^- = F_0 \quad \text{on} \quad S,$$

implying that the transmission conditions (3.42)–(3.43) are satisfied. The remaining condition (3.44) leads to the following pseudodifferential equation for the unknown vector function $\widetilde{g}$ on S_C,

$$\big(-2^{-1}I_3 + \mathcal{K}^{(1)}\big)h + \mathcal{M}^{(2)}g = F_C^{(+)} + F_C^{(-)} \quad \text{on} \quad S_C,$$

which can be rewritten as

$$r_{S_C}\,\mathcal{P}\,\widetilde{g} = \Psi \quad \text{on} \quad S_C, \tag{3.51}$$

where

$$\mathcal{P} := \big(-2^{-1}I_3 + \mathcal{K}^{(1)}\big)\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} + \mathcal{M}^{(2)}\big(\mathcal{N}^{(2)}\big)^{-1}\big[\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} - I_3\big], \tag{3.52}$$

$$\begin{aligned}\Psi := F_C^{(+)} + F_C^{(-)} &- r_{S_C}\big\{\big(-2^{-1}I_3 + \mathcal{K}^{(1)}\big)\mathcal{T}^{-1}\big[\big(\mathcal{N}^{(2)}\big)^{-1} f^* - \big(\mathcal{M}^{(2)}\big)^{-1} F_0\big]\big\}\\ &- r_{S_C}\big\{\mathcal{M}^{(2)}\big(\mathcal{N}^{(2)}\big)^{-1}\big([\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} - I_3]f^* - \mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1}F_0\big)\big\}.\end{aligned} \tag{3.53}$$

Due to mapping properties of the operators involved in (3.52) and (3.53) we have (see Appendices A and B)

$$\Psi \in [B_{p,p}^{-\frac{1}{p}}(S_C)]^3 \tag{3.54}$$

and the operator

$$r_{S_C}\mathcal{P} \;:\; [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^3 \longrightarrow [B_{p,p}^{-\frac{1}{p}}(S_C)]^3 \tag{3.55}$$

is continuous.

In view of the relations derived in Appendix A, Remark A.4, and the equality (3.38), for the principal homogeneous symbol matrix $\mathfrak{S}\big(\mathcal{P};x,\xi\big)$ of the operator $\mathcal{P}$ we have:

$$\begin{aligned}\mathfrak{S}\big(\mathcal{P}\big) &= \mathbb{K}_-^{(1)}\big(-\mathbb{H}^{(1)}\big)^{-1}\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]^{-1}\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\\ &\quad + \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big\{-\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]^{-1}\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1} - I_3\big\}\\ &= -\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} + \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]^{-1}\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\\ &= \big\{\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} + \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]^{-1} + I_3\big\}\big[-\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big].\end{aligned}$$

Note that, due to the relations presented in Remark A.4, the matrices $\mathbf{A} := \mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1}$ and $\mathbf{B} := -\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}$ are positive definite and consequently they are self-adjoint. The symbol $\mathfrak{S}\big(\mathcal{P}\big)$ can be rewritten as

$$\begin{aligned}\mathfrak{S}\big(\mathcal{P}\big) &= \big\{\big(\mathbf{A}-\mathbf{B}\big)\big(\mathbf{A}+\mathbf{B}\big)^{-1} + I_3\big\}\,\mathbf{B} = \big\{\big(\mathbf{A}-\mathbf{B}\big)\big(\mathbf{A}+\mathbf{B}\big)^{-1} + \big(\mathbf{A}+\mathbf{B}\big)\big(\mathbf{A}+\mathbf{B}\big)^{-1}\big\}\,\mathbf{B}\\ &= 2\,\mathbf{A}\,\big(\mathbf{A}+\mathbf{B}\big)^{-1}\mathbf{B} = 2\,\big(\mathbf{A}^{-1}+\mathbf{B}^{-1}\big)^{-1},\end{aligned}$$

implying that the symbol $\mathfrak{S}\big(\mathcal{P}\big)$ is positive definite. Therefore by Remark C.1 and Theorem C.2, operator (3.55) is invertible if (see (C.1) with $s = 1-\frac{1}{p}$, $\nu = 1$, and $\delta_j = 0$, $j = 1,2,3$)

$$\frac{3}{4} < p < 4.$$

The above results lead to the following existence theorem.

Theorem 3.7. *Let* $\frac{3}{4} < p < 4$,

$$f^{(T)} \in [B_{p,p}^{1-\frac{1}{p}}(S_T)]^3, \quad F^{(T)} \in [B_{p,p}^{-\frac{1}{p}}(S_T)]^3, \quad F_C^{(\pm)} \in [B_{p,p}^{-\frac{1}{p}}(S_C)]^3\,,$$

and let the vector function F_0 *defined in* (3.45) *satisfy the inclusion* (3.46).

Then the basic mixed transmission problem $(MT)_\omega$ *is uniquely solvable in the class of vector functions* $[W_p^1(\Omega^+)]^3 \times \big([W_{p,loc}^1(\Omega^-)]^3 \cap Z(\Omega^-)\big)$, *and the solution pair* $\big(u^{(1)},u^{(2)}\big)$ *is representable by the layer potentials* (3.47)–(3.48) *with densities given by* (3.49)–(3.50), *where the unknown vector function* $\widetilde{g}$ *is defined by the uniquely solvable pseudodifferential equation* (3.51).

Moreover, the following estimates hold

$$\|u^{(1)}\|_{[W_p^1(\Omega^+)]^3} \leqslant C_{MT}^{(1)}\left(\|f^{(T)}\|_{[B_{p,p}^{1-\frac{1}{p}}(S)]^3} + \|F_0\|_{[B_{p,p}^{-\frac{1}{p}}(S)]^3} + \|F_C^{(+)} + F_C^{(-)}\|_{[B_{p,p}^{-\frac{1}{p}}(S_C)]^3}\right),$$

$$\|u^{(2)}\|_{[W_p^1(\Omega^-\cap B(R))]^3} \leqslant C_{MT}^{(2)}(R)\left(\|f^{(T)}\|_{[B_{p,p}^{1-\frac{1}{p}}(S)]^3} + \|F_0\|_{[B_{p,p}^{-\frac{1}{p}}(S)]^3} + \|F_C^{(+)} + F_C^{(-)}\|_{[B_{p,p}^{-\frac{1}{p}}(S_C)]^3}\right),$$

where $C_{MT}^{(1)}$ *and* $C_{MT}^{(2)}(R)$ *are constants independent of* $f^{(T)}$, F_0, *and* $F_C^{(+)} + F_C^{(-)}$.

Remark 3.8. The above formulated existence theorems with $p = 2$ remain valid also for Lipschitz domains, i.e., when the surfaces S, S_D, S_N, S_T, S_C, Σ, and their boundaries belong to Lipschitz continuous classes.

Remark 3.9. Applying the same arguments as in [23] for mixed and crack type problems, it can be shown that for sufficiently smooth data weak solutions to the Problems $(\mathrm{D})^-_\omega$, $(\mathrm{N})^-_\omega$, $(\mathrm{BT})^-_\omega$ actually are regular vector functions (see also [12]), while the weak solutions to the Problems $(\mathrm{M})^-_\omega$, $(\mathrm{BC})^-_\omega$, $(\mathrm{C})^-_\omega$, and $(\mathrm{MT})^-_\omega$ actually are semi-regular vector functions in the corresponding domains (cf. [25,27,33–36]). Therefore all the boundary, transmission, and crack type conditions can be understood in the classical pointwise sense.

Remark 3.10. Note that the crack problem can be considered as a particular case of the mixed transmission problem. Indeed, if we assume that in the formulation of the problem $(\mathrm{MT})_\omega$ both domains Ω^+ and Ω^- are occupied by the same type materials, i.e., all material constants in both domains are the same, and on S_T the homogeneous transmission conditions are prescribed (i.e. $f^{(T)} = 0$ and $F^{(T)} = 0$ on S_T in (2.15) and (2.16)), then the corresponding differential operators are the same in both domains and the transmission part S_T of the interface S becomes a formal interface since the continuity of the displacement and stress vectors across the surface S_T implies that in fact the differential equation is satisfied also at the points of the surface S_T and the corresponding solution actually is an analytic function in $\mathbb{R}^3 \setminus \overline{S_C}$. Evidently we arrive at the basic crack problem with $\Sigma = S_C$.

4. Method of fundamental solutions

Here we develop the Fundamental Solution Method for the above formulated boundary value and transmission problems for the elastic oscillation system for arbitrary values of the frequency parameter ω.

4.1. Auxiliary lemmata

Let Ω_0^+ be an arbitrary simply connected subdomain of Ω^+ such that $\overline{\Omega_0^+} \subset \Omega^+$ and denote $S_0^+ = \partial\Omega_0^+$.

Further, let Ω_0^- be an arbitrary simply connected bounded subdomain of Ω^- such that $\overline{\Omega_0^-} \subset \Omega^-$ and denote $S_0^- = \partial\Omega_0^-$.

We assume that S_0^+ and S_0^- are simply connected surfaces.

Let $\{z^{(k)}\}_{k=1}^\infty$ be an everywhere dense countable set of points in Ω_0^+ and $\{y^{(k)}\}_{k=1}^\infty$ be an everywhere dense countable set of points in Ω_0^-.

Denote by $\Gamma^{(j)}(x,\omega)$ the jth column of Kupradze's fundamental matrix $\Gamma(x,\omega)$ (see (A.1) in Appendix A).

Consider the systems of functions which can be employed for constructing approximate solutions to the Dirichlet problem,

$$\Phi_D^{(-)} := \left\{ \varphi^{(l)}(x) \right\}_{l=1}^\infty, \quad x \in \overline{\Omega^-}, \qquad \Phi_D^{(+)} := \left\{ \psi^{(l)}(x) \right\}_{l=1}^\infty, \quad x \in \overline{\Omega^+},$$

where

$$\varphi^{(l)}(x) := \begin{cases} \Gamma^{(1)}(x - z^{(k)}, \omega) & \text{for} \quad l = 3(k-1)+1, \\ \Gamma^{(2)}(x - z^{(k)}, \omega) & \text{for} \quad l = 3(k-1)+2, \quad k = 1,2,3,\dots, \quad z^{(k)} \in \Omega_0^+, \\ \Gamma^{(3)}(x - z^{(k)}, \omega) & \text{for} \quad l = 3k, \end{cases} \tag{4.56}$$

$$\psi^{(l)}(x) := \begin{cases} \Gamma^{(1)}(x - y^{(k)}, \omega) & \text{for} \quad l = 3(k-1)+1, \\ \Gamma^{(2)}(x - y^{(k)}, \omega) & \text{for} \quad l = 3(k-1)+2, \quad k = 1,2,3,\dots, \quad y^{(k)} \in \Omega_0^-. \\ \Gamma^{(3)}(x - y^{(k)}, \omega) & \text{for} \quad l = 3k, \end{cases} \tag{4.57}$$

Note that due to definition (4.56) to each point $z^{(k)}$ there corresponds the triplet of vector functions

$$\varphi^{(3(k-1)+1)}, \quad \varphi^{(3(k-1)+2)}, \quad \varphi^{(3k)}, \quad k = 1,2,\dots. \tag{4.58}$$

Similarly, in view of (4.57), to each point $y^{(k)}$ there corresponds the triplet of vector functions

$$\psi^{(3(k-1)+1)}, \quad \psi^{(3(k-1)+2)}, \quad \psi^{(3k)}, \quad k = 1,2,\dots. \tag{4.59}$$

Evidently, $\varphi^{(l)}$ are radiating, complex valued analytic vector functions in $\mathbb{R}^3 \setminus \overline{\Omega_0^+}$, while $\psi^{(l)}$ are radiating, complex valued analytic vector functions in $\mathbb{R}^3 \setminus \overline{\Omega_0^-}$. Moreover, $\varphi^{(l)}$ and $\psi^{(l)}$ solve the homogeneous equation (2.1) in the corresponding domains.

Now we prove several lemmas which play a crucial role in our further analysis.

Lemma 4.1. *The system $\Phi_D^{(-)}$ is linearly independent on S.*

Proof. We have to prove that any finite subsystem of $\Phi_D^{(-)}$ is linearly independent on S. Let m be an arbitrary natural number and for some complex valued constants C_l the following equality holds

$$u^{(m)}(x) = \sum_{l=1}^{m} C_l\, \varphi^{(l)}(x) = 0, \quad x \in S, \quad m \in \mathbb{N}. \tag{4.60}$$

Denote by $z^{(k)}$, $k = 1, 2, \dots, m_0$, the points involved in the expression (4.60). Without loss of generality we can assume that for each $z^{(k)}$, $k = 1, 2, \dots, m_0$, the expression (4.60) contains all three vector functions associated with the point $z^{(k)}$ (see (4.58)). If necessary, we can add the corresponding terms with zero coefficients. Therefore, in what follows we assume that m is multiple of 3, $m = 3m_0$.

Evidently, $u^{(m)}$ is a radiating analytic vector function in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$ which solves the homogeneous differential equation

$$A(\partial, \omega)u^{(m)}(x) = 0, \quad x \in \mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}. \tag{4.61}$$

Then in view of (4.60) and (4.61), we see that $u^{(m)}$ solves the homogeneous exterior Dirichlet problem $(\mathrm{D})_\omega^-$ and due to the existence and uniqueness Theorem 3.1 we conclude that $u^{(m)} = 0$ in Ω^-. By the analyticity then we get

$$u^{(m)}(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}. \tag{4.62}$$

Let $B(z^{(j)}, \varepsilon)$ be a ball centred at the point $z^{(j)}$ and radius ε such that $z^{(k)} \notin B(z^{(j)}, \varepsilon)$ for $k \leqslant m_0$ and $k \neq j$. Denote $\Sigma(z^{(j)}, \varepsilon) = \partial B(z^{(j)}, \varepsilon)$, $j = 1, 2, \dots, m_0$.

On the one hand, in view of (4.62) we have

$$\int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))u^{(m)}(x)\, dS = 0, \quad j = 1, \dots, m_0. \tag{4.63}$$

On the other hand, there holds the equality (see [23, Appendix D], [12, Ch.5])

$$\lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma(x - z^{(j)})\, dS = I_3,$$

where $\Gamma(x - z^{(j)})$ is Kelvin's matrix and I_3 is the 3×3 unit matrix. Therefore, in view of (A.2) in Appendix A, for $q = 1, 2, 3$, and $j = 1, 2, \dots, m_0$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(q)}(x - z^{(j)}, \omega)\, dS \\
&\quad = \lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(q)}(x - z^{(j)})\, dS = (\delta_{1q}, \delta_{2q}, \delta_{3q})^\top.
\end{aligned} \tag{4.64}$$

Keeping in mind (4.64) and passing to the limit in (4.63) as $\varepsilon \to 0$, we find

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))u^{(m)}(x)\, dS &= C_{3(j-1)+1} \lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(1)}(x - z^{(j)})\, dS \\
&\quad + C_{3(j-1)+2} \lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(2)}(x - z^{(j)})\, dS \\
&\quad + C_{3(j-1)+3} \lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(3)}(x - z^{(j)})\, dS \\
&= (C_{3(j-1)+1}, C_{3(j-1)+2}, C_{3(j-1)+3})^\top = 0,
\end{aligned}$$

for $j = 1, 2, \dots, m_0$, which implies that $C_l = 0$ for $l = 1, 2, \dots, 3m_0$. This completes the proof. $\square$

Lemma 4.2. *The system* $\Phi_D^{(-)}$ *is complete in* $\big[B_{p,\,p}^{1-\frac{1}{p}}(S)\big]^3$ *for* $p \in (1, +\infty)$.

Proof. We have to show that the linear span of the system $\Phi_D^{(-)}$ is dense in $\big[B_{p,\,p}^{1-\frac{1}{p}}(S)\big]^3$. To this end we will apply the following fact from the functional analysis which is a direct consequence of the Hahn–Banach theorem (see, e.g., [37, Ch. 1, Section 5]). Let **B** be a Banach space and $\mathbf{B}^*$ be its adjoint space. A subset $\mathbf{X} \subset \mathbf{B}$ is dense in **B** if and only if the relation

$$\langle f,\, x\rangle = 0 \ \text{ for all } \ x \in \mathbf{X}$$

with $f \in \mathbf{B}^*$ implies that f is the zero functional.

Thus to prove the density of the linear span of the system $\Phi_D^{(-)}$ in $[B_{p,\,p}^{1-\frac{1}{p}}(S)]^3$ it suffices to show that if a vector function χ belongs to the adjoint space, $\chi \in [B_{p',\,p'}^{-1+\frac{1}{p}}(S)]^3$ with $1/p + 1/p' = 1$, and

$$\langle \chi,\, \varphi^{(l)}\rangle_S = 0, \quad l = 1, 2, \ldots, \tag{4.65}$$

then $\chi = 0$. As above, here the symbol $\langle\cdot,\,\cdot\rangle_S$ denotes duality brackets between the mutually adjoint spaces $[B_{p,p}^{1-\frac{1}{p}}(S)]^3$ and $[B_{p',p'}^{-1+\frac{1}{p}}(S)]^3$.

Condition (4.65) can be rewritten as

$$\big\langle \chi,\, \Gamma^{(j)}(\cdot - z^{(k)}, \omega)\big\rangle_S = 0, \quad j = 1, 2, 3, \quad k = 1, 2, \ldots.$$

Due to the density of the set $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, we get

$$\big\langle \chi,\, \Gamma^{(j)}(\cdot - z, \omega)\big\rangle_S = 0, \quad z \in \Omega_0^+, \quad j = 1, 2, 3.$$

This implies that

$$V(\chi)(z) = \big\langle \chi,\, \Gamma(\cdot - z, \omega)\big\rangle_S = 0, \quad z \in \Omega_0^+, \tag{4.66}$$

where $V(\chi)$ is a single layer potential with the integration surface S and with the density χ (see (A.3)). Since the single layer potential is analytic in $\Omega^{\pm}$ we conclude from (4.66)

$$V(\chi)(z) = 0, \quad z \in \Omega^+. \tag{4.67}$$

Moreover, by Theorem A.1 we have

$$V(\chi) \in \big[\,W_{p'}^1(\Omega^+)\,\big]^3, \quad V(\chi) \in \big[W_{p',\,loc}^1(\Omega^-)\,\big]^3 \cap Z(\Omega^-), \quad p' > 1.$$

Further, by Theorem A.2, formula (A.11), and relation (4.67) it follows that the single layer potential $V(\chi)$ solves the homogeneous exterior Dirichlet problem in Ω^- and in accordance with Theorem 3.1 vanishes in Ω^-. Therefore due to Theorem A.2

$$\big\{T(\partial, n)V(\chi)\big\}^- - \big\{T(\partial, n)V(\chi)\big\}^+ = \chi = 0 \quad \text{on} \quad S,$$

which completes the proof. □

Now, let us introduce the following systems of functions on S which can be employed for constructing approximate solutions to the Neumann problem,

$$\Phi_N^{(-)} := \big\{\,T(\partial, n(x))\varphi^{(l)}(x)\,\big\}_{l=1}^{\infty}, \quad \Phi_N^{(+)} := \big\{\,T(\partial, n(x))\psi^{(l)}(x)\,\big\}_{l=1}^{\infty}, \quad x \in S,$$

where $T(\partial, n)$ is the boundary stress operator (2.2), $\varphi^{(l)}$ and $\psi^{(l)}$ are defined in (4.56) and (4.57) respectively.

Lemma 4.3. *The system* $\Phi_N^{(-)}$ *is linearly independent on* S.

Proof. We have to prove that any finite subsequence of the system $\Phi_N^{(-)}$ is linearly independent. Let $m \in \mathbb{N}$ be an arbitrary natural number and

$$\sum_{l=1}^{m} C_l\, T(\partial, n(x))\varphi^{(l)}(x) = 0, \quad x \in S, \tag{4.68}$$

where C_l are complex valued constants.

As in the proof of Lemma 4.1, we denote by $z^{(k)}$, $k = 1, 2, \dots, m_0$, the points involved in the expression (4.68) and without loss of generality we assume that for each $z^{(k)}$, $k = 1, 2, \dots, m_0$, the expression (4.68) contains all three vector functions associated with the point $z^{(k)}$ (see (4.58)) implying that $m = 3m_0$.

Now, let us set

$$u^{(m)}(x) := \sum_{l=1}^{m} C_l\, \varphi^{(l)}(x), \quad x \notin \overline{\Omega_0^+}. \tag{4.69}$$

Evidently, $u^{(m)}$ is a radiating analytic vector function in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$ which solves the homogeneous differential equation

$$A(\partial, \omega)u^{(m)}(x) = 0, \quad x \in \mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}. \tag{4.70}$$

In view of (4.69), (4.68), and (4.70) it then follows that $u^{(m)}$ solves the homogeneous exterior Neumann problem $(\mathrm{N})_\omega^-$ and due to uniqueness Theorem 2.6 we conclude that $u^{(m)} = 0$ in Ω^- and consequently, by the analyticity property, $u^{(m)} = 0$ in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$. By the same arguments as in the proof of Lemma 4.1 we show that all the constants C_l, $l = 1, 2, \dots, m$, equal to zero, which completes the proof. $\square$

Lemma 4.4. *The system $\Phi_N^{(-)}$ is complete in $\big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3$ for $p \in (1, +\infty)$.*

Proof. As in Lemma 4.2, to prove the density of the linear span of the system $\Phi_N^{(-)}$ in $[B_{p,p}^{-\frac{1}{p}}(S)]^3$ it suffices to show that if a vector function χ belongs to the adjoint space, $\chi \in [B_{p',p'}^{\frac{1}{p}}(S)]^3$ with $1/p + 1/p' = 1$, and

$$\langle \chi\,,\, T(\partial, n)\varphi^{(l)} \rangle_S = 0, \quad l = 1, 2, \dots, \tag{4.71}$$

then $\chi = 0$. Here the symbol $\langle \cdot, \cdot \rangle_S$ again denotes duality brackets between the mutually adjoint spaces $[B_{p,p}^{-\frac{1}{p}}(S)]^3$ and $[B_{p',p'}^{\frac{1}{p}}(S)]^3$.

Condition (4.71) can be rewritten as

$$\langle \chi\,,\, T(\partial, n)\Gamma^{(j)}(\cdot - z^{(k)}, \omega) \rangle_S = 0, \quad j = 1, 2, 3, \quad k = 1, 2, \dots$$

which is equivalent to the relation

$$W(\chi)(z^{(k)}) = 0, \quad k = 1, 2, \dots, \tag{4.72}$$

where $W(\chi)$ is the double layer potential with the integration surface S and with the density χ (see (A.4)). Due to the density property of the set $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, from (4.72) we deduce

$$W(\chi)(z) = 0, \quad z \in \Omega_0^+. \tag{4.73}$$

By analyticity property of the double layer potential in domains $\Omega^\pm$ we conclude

$$W(\chi)(z) = 0, \quad z \in \Omega^+.$$

Note that by Theorem A.1 we have

$$W(\chi) \in \big[\, W_{p'}^1(\Omega^+) \,\big]^3, \quad W(\chi) \in \big[W_{p',loc}^1(\Omega^-)\big]^3 \cap Z(\Omega^-), \quad p' > 1.$$

Further, by Theorem A.2, formula (A.12), and relation (4.73) it follows that the double layer potential $W(\chi)$ solves the homogeneous exterior Neumann problem in Ω^- and in accordance with Theorem 3.2 vanishes in Ω^-. Therefore due to Theorem A.2

$$\big\{W(\chi)\big\}^+ - \big\{W(\chi)\big\}^- = \chi = 0 \quad \text{on} \quad S,$$

which completes the proof. $\square$

Further, let us introduce the system which can be employed for constructing approximate solutions to the mixed Dirichlet–Neumann problem,

$$\Phi_M^{(-)} := \big\{\, \nu^{(l)}(x) \,\big\}_{l=1}^{\infty}, \quad x \in S, \tag{4.74}$$

where

$$\nu^{(l)}(x) := \begin{cases} \varphi^{(l)}(x) & \text{for} \quad x \in S_D\,, \\ T(\partial, n(x))\varphi^{(l)}(x) & \text{for} \quad x \in S_N\,, \end{cases} \tag{4.75}$$

where $\varphi^{(l)}$ is given in (4.56), S_D and S_N are the Dirichlet and Neumann parts in the mixed boundary value problem $(\mathrm{M})^-_\omega$.

It is evident that the vector $\nu^{(l)}$ can be considered as the following pair of restrictions

$$\widetilde{\nu}^{(l)} = \big(r_{S_D}\,\nu^{(l)}, r_{S_N}\,\nu^{(l)}\big) \equiv \big(r_{S_D}\,\varphi^{(l)}, r_{S_N}\,T(\partial, n(x))\varphi^{(l)}\big)\,.$$

Similarly, the system $\Phi^{(-)}_M$ defined in (4.74) can be identified with the system

$$\widetilde{\Phi}^{(-)}_M := \{\widetilde{\nu}^{(l)}\}_{l=1}^{\infty}.$$

Lemma 4.5. *The system $\Phi^{(-)}_M$ is linearly independent on S.*

Proof. Let $m \in \mathbb{N}$ be a natural number and

$$\sum_{l=1}^{m} C_l\,\nu^{(l)}(x) = 0, \quad x \in S, \tag{4.76}$$

where C_l are complex valued constants.

As in the proof of Lemma 4.1, we denote by $z^{(k)}$, $k = 1, 2, \dots, m_0$, the points involved in the expression (4.76) and without loss of generality we assume again that for each $z^{(k)}$, $k = 1, 2, \dots, m_0$, the expression (4.76) contains all three vector functions associated with the point $z^{(k)}$ (see (4.58)) implying that m is multiple of 3, $m = 3m_0$.

Now, let us construct the vector

$$u^{(m)}(x) := \sum_{l=1}^{m} C_l\,\varphi^{(l)}(x), \quad x \notin \overline{\Omega^+_0}. \tag{4.77}$$

Evidently, $u^{(m)}$ is a radiating analytic vector function in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$ which solves the homogeneous differential equation (4.70) in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$. In view of (4.77), (4.76), and (4.75) it then follows that $u^{(m)}$ solves the exterior homogeneous mixed problem $(\mathrm{M})^-_\omega$ and due to the existence and uniqueness Theorem 3.3 we conclude that $u^{(m)} = 0$ in Ω^- and consequently, by the analyticity property, $u^{(m)} = 0$ in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$. By the same arguments as in the proof of Lemma 4.1 we show that all the constants C_l, $l = 1, 2, \dots, m$, equal to zero, which completes the proof. $\square$

Lemma 4.6. *The system $\widetilde{\Phi}^{(-)}_M$ is complete in $\big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3$ for $4/3 < p < 4$.*

Proof. As in Lemma 4.2, to prove the density of the linear span of the system $\widetilde{\Phi}^{(-)}_M$ in $\big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3$ it suffices to show that if a pair of vector functions $\widetilde{\chi} = (\chi_D, \chi_N)$ belongs to the adjoint space, $\widetilde{\chi} = (\chi_D, \chi_N) \in \big[\widetilde{B}^{-1+\frac{1}{p}}_{p',p'}(S_D)\big]^3 \times \big[\widetilde{B}^{\frac{1}{p}}_{p',p'}(S_N)\big]^3$ with $1/p + 1/p' = 1$, and

$$\begin{aligned} \langle \widetilde{\chi}\,,\, \widetilde{\nu}^{(l)} \rangle_S &:= \langle \chi_D\,,\, \nu^{(l)} \rangle_{S_D} + \langle \chi_N\,,\, \nu^{(l)} \rangle_{S_N} \\ &= \langle \chi_D\,,\, \varphi^{(l)} \rangle_{S_D} + \langle T(\partial, n(x))\varphi^{(l)}\,,\, \chi_N \rangle_{S_N} = 0, \quad l = 1, 2, \dots, \end{aligned} \tag{4.78}$$

then $\widetilde{\chi} = 0$.

Here the symbols $\langle \cdot, \cdot \rangle_{S_D}$ and $\langle \cdot, \cdot \rangle_{S_N}$ denote duality brackets between the mutually adjoint pairs of Besov spaces $[\widetilde{B}^{-1+\frac{1}{p}}_{p,p}(S_D)]^3$ and $[B^{1-\frac{1}{p}}_{p',p'}(S_D)]^3$, and $[B^{-\frac{1}{p}}_{p,p}(S_N)]^3$ and $[\widetilde{B}^{\frac{1}{p}}_{p',p'}(S_N)]^3$, respectively.

Condition (4.78) can be rewritten as

$$\langle \chi_D\,,\, \Gamma^{(j)}(\cdot - z^{(k)}) \rangle_{S_D} + \langle T(\partial, n)\Gamma^{(j)}(\cdot - z^{(k)})\,,\, \chi_N \rangle_{S_N} = 0, \quad j = 1, 2, 3, \quad k = 1, 2, \dots$$

which is equivalent to the relation

$$V(\chi_D)(z^{(k)}) + W(\chi_N)(z^{(k)}) = 0, \quad k = 1, 2, \dots, \tag{4.79}$$

where $V(\chi_D)$ and $W(\chi_N)$ are the single and double layer potentials on S with the densities

$$\chi_D \in \big[\widetilde{B}^{-1+\frac{1}{p}}_{p',\,p'}(S_D)\big]^3, \qquad \chi_N \in \big[\widetilde{B}^{\frac{1}{p}}_{p',\,p'}(S_N)\big]^3. \tag{4.80}$$

Due to the density property of the set $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, from (4.79) we get

$$U(z) := V(\chi_D)(z) + W(\chi_N)(z) = 0, \quad z \in \Omega_0^+.$$

By analyticity property of the layer potentials in domains $\Omega^{\pm}$ we conclude

$$U(z) = V(\chi_D)(z) + W(\chi_N)(z) = 0, \quad z \in \Omega^+. \tag{4.81}$$

Note that if $4/3 < p < 4$, then $4/3 < p' < 4$, and by Theorem A.1 we have

$$U \in \big[\,W^1_{p'}(\Omega^+)\big]^3, \quad U \in \big[W^1_{p',\,loc}(\Omega^-)\big]^3 \cap Z(\Omega^-), \;\; 4/3 < p' < 4. \tag{4.82}$$

Further, by Theorem A.2 and relations (4.81) and (4.80) we find that

$$\big\{U\big\}^+ - \big\{U\big\}^- = \chi_N = 0 \text{ on } S_D, \quad \big\{T(\partial,n)U\big\}^+ - \big\{T(\partial,n)U\big\}^- = -\chi_D = 0 \text{ on } S_N.$$

Whence it follows that U belongs to the class (4.82) and solves the homogeneous exterior mixed problem $(\mathrm{M})_\omega^-$. In accordance with the existence and uniqueness Theorem 3.3 U vanishes in Ω^- implying $\chi_D = \chi_N = 0$ on S. □

Next, we introduce the system of vector functions which can be employed for constructing approximate solutions to the basic transmission problem $(\mathrm{BT})_\omega$.

By $\varphi^{(\kappa,l)}(x)$ and $\psi^{(\kappa,l)}(x)$ we denote the vector functions defined by formulas (4.56) and (4.57) respectively constructed by the columns $\Gamma^{(j,\kappa)}$, $\kappa = 1, 2$, $j = 1, 2, 3$, of the fundamental matrix $\Gamma^{(\kappa)}$ associated with the operator $A^{(\kappa)}(\partial, \omega)$. Here $\kappa = 1$ corresponds to the bounded domain $\Omega^+ = \Omega^{(1)}$, while $\kappa = 2$ corresponds to the exterior unbounded domain $\Omega^- = \Omega^{(2)}$. The set of points $\{z^{(k)}\}_{k=1}^{\infty} \subset \overline{\Omega_0^+} \subset \Omega^{(1)}$ and $\{y^{(k)}\}_{k=1}^{\infty} \subset \overline{\Omega_0^-} \subset \Omega^{(2)}$ are the same as above.

Let

$$\Phi_{BT} := \big\{\, \Psi^{(l,1)}(x)\,,\, \Phi^{(l,2)}(x)\big\}_{l=1}^{\infty}, \quad x \in S,$$

where $\Psi^{(l,1)}(x)$ and $\Phi^{(l,2)}(x)$ are six vectors defined on S by the relations

$$\Psi^{(l,1)}(x) = \big(\psi^{(l,1)}(x),\; T^{(1)}(\partial, n(x))\psi^{(l,1)}(x)\big)^{\top}, \quad x \in S, \tag{4.83}$$

$$\Phi^{(l,2)}(x) = \big(-\varphi^{(l,2)}(x),\; -T^{(2)}(\partial, n(x))\varphi^{(l,2)}(x)\big)^{\top}, \quad x \in S. \tag{4.84}$$

Lemma 4.7. *The system Φ_{BT} is linearly independent on S.*

Proof. Let $m_1, m_2 \in \mathbb{N}$ be arbitrary natural numbers and

$$\sum_{l=1}^{m_1} C_{l,1}\, \Psi^{(l,1)}(x) + \sum_{l=1}^{m_2} C_{l,2}\, \Phi^{(l,2)}(x) = 0, \quad x \in S, \tag{4.85}$$

where $C_{l,\kappa}$, $\kappa = 1, 2$, are complex valued constants.

Denote by $z^{(k)}$, $k = 1, 2, \ldots, p_2$, and $y^{(k)}$, $k = 1, 2, \ldots, p_1$, the points involved in the expression (4.85) and without loss of generality assume that for each $z^{(k)}$, $k = 1, 2, \ldots, p_2$, and for each $y^{(k)}$, $k = 1, 2, \ldots, p_1$, the expression (4.85) contains all three vector functions corresponding to the points $z^{(k)}$ and $y^{(k)}$ (see (4.58) and (4.59)) implying that m_2 and m_1 are multiples of 3, $m_2 = 3p_2$ and $m_1 = 3p_1$.

Now, let us construct the vectors

$$u^{(m_1,1)}(x) := \sum_{l=1}^{m_1} C_{l,1}\, \psi^{(l,1)}(x), \quad x \notin \overline{\Omega_0^-}, \tag{4.86}$$

$$u^{(m_2,2)}(x) := \sum_{l=1}^{m_2} C_{l,2}\, \varphi^{(l,2)}(x), \quad x \notin \overline{\Omega_0^+}. \tag{4.87}$$

Evidently, $u^{(m_1,1)}$ and $u^{(m_2,2)}$ are radiating analytic vector function in $\mathbb{R}^3\setminus\{y^{(k)}\}_{k=1}^{p_1}$ and $\mathbb{R}^3\setminus\{z^{(k)}\}_{k=1}^{p_2}$ respectively and solve the homogeneous differential equations

$$A^{(1)}(\partial,\omega)u^{(m_1,1)}(x)=0,\quad x\notin\overline{\Omega_0^-},\tag{4.88}$$

$$A^{(2)}(\partial,\omega)u^{(m_2,2)}(x)=0,\quad x\notin\overline{\Omega_0^+}.\tag{4.89}$$

In view of relations (4.83)–(4.89) it then follows that the pair $\big(u^{(m_1,1)},u^{(m_2,2)}\big)$ solves the homogeneous basic transmission problem $(\mathrm{BT})_\omega$ and due to the existence and uniqueness Theorem 3.5 we conclude that $u^{(m_1,1)}=0$ in Ω^+ and $u^{(m_2,2)}=0$ in Ω^-. Consequently, by the analyticity property, $u^{(m_1,1)}=0$ in $\mathbb{R}^3\setminus\{y^{(k)}\}_{k=1}^{p_1}$ and $u^{(m_2,2)}=0$ in $\mathbb{R}^3\setminus\{z^{(k)}\}_{k=1}^{p_2}$. Now, by the same arguments as in the proof of Lemma 4.1 we derive that $C_{l,1}=0$, $l=1,2,\dots,m_1$, and $C_{l,2}=0$, $l=1,2,\dots,m_2$, which completes the proof. □

Lemma 4.8. *The system Φ_{BT} is complete in $\big[B_{p,p}^{1-\frac1p}(S)\big]^3\times\big[B_{p,p}^{-\frac1p}(S)\big]^3$ for $p>1$.*

Proof. To prove the density property of the linear span of the system Φ_{BT} in $\big[B_{p,p}^{1-\frac1p}(S)\big]^3\times\big[B_{p,p}^{-\frac1p}(S)\big]^3$ it suffices to show that if a pair of vector functions $\chi=(g,h)$ belongs to the adjoint space, $\chi=(g,h)\in\big[B_{p',p'}^{-1+\frac1p}(S)\big]^3\times\big[B_{p',p'}^{\frac1p}(S)\big]^3$ with $1/p+1/p'=1$, and

$$\langle\chi\,,\,\Psi^{(l,1)}\rangle_S:=\langle g\,,\,\psi^{(l,1)}\rangle_S+\langle h\,,\,T^{(1)}(\partial,n)\psi^{(l,1)}\rangle_S=0,\quad l=1,2,\dots,\tag{4.90}$$

$$\langle\chi\,,\,\Phi^{(l,2)}\rangle_S:=\langle g\,,\,\varphi^{(l,2)}\rangle_S+\langle h\,,\,T^{(2)}(\partial,n)\varphi^{(l,2)}\rangle_S=0,\quad l=1,2,\dots,\tag{4.91}$$

then $\chi=(g,h)=0$.

Condition (4.90) and (4.91) can be rewritten as

$$\langle g\,,\,\Gamma^{(j,1)}(\cdot-y^{(k)})\rangle_S+\langle T^{(1)}(\partial,n)\Gamma^{(j,1)}(\cdot-y^{(k)})\,,\,h\rangle_S=0,\quad j=1,2,3,\quad k=1,2,\dots$$

$$\langle g\,,\,\Gamma^{(j,2)}(\cdot-z^{(k)})\rangle_S+\langle T^{(2)}(\partial,n)\Gamma^{(j,2)}(\cdot-z^{(k)})\,,\,h\rangle_S=0,\quad j=1,2,3,\quad k=1,2,\dots$$

which is equivalent to the relation

$$V^{(1)}(g)(y^{(k)})+W^{(1)}(h)(y^{(k)})=0,\quad k=1,2,\dots,\quad y^{(k)}\in\Omega_0^-,\tag{4.92}$$

$$V^{(2)}(g)(z^{(k)})+W^{(2)}(h)(z^{(k)})=0,\quad k=1,2,\dots,\quad z^{(k)}\in\Omega_0^+,\tag{4.93}$$

where $V^{(\kappa)}(g)$ and $W^{(\kappa)}(h)$ are the single and double layer potentials with the integration surface S constructed by the fundamental matrix $\Gamma^{(\kappa)}$ with the densities

$$g\in\big[B_{p',p'}^{-1+\frac1p}(S)\big]^3,\qquad h\in\big[B_{p',p'}^{\frac1p}(S)\big]^3,\quad p'>1.$$

Due to the density property of the sets $\{y^{(k)}\}_{k=1}^{\infty}$ in Ω_0^- and $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, from (4.92) and (4.93) we get

$$V^{(1)}(g)(z)+W^{(1)}(h)(z)=0,\quad z\in\Omega_0^-,\tag{4.94}$$

$$V^{(2)}(g)(z)+W^{(2)}(h)(z)=0,\quad z\in\Omega_0^+.\tag{4.95}$$

Due to analyticity of the layer potentials in domains $\Omega^\pm$ we conclude

$$U^{(1)}(z):=V^{(1)}(g)(z)+W^{(1)}(h)(z)=0,\quad z\in\Omega^-,\tag{4.96}$$

$$U^{(2)}(z):=-V^{(2)}(g)(z)-W^{(2)}(h)(z)=0,\quad z\in\Omega^+.\tag{4.97}$$

Note that by Theorem A.1 we have

$$U^{(1)}\in\big[W^1_{p'}(\Omega^+)\big]^3,\quad U^{(2)}\in\big[W^1_{p',loc}(\Omega^-)\big]^3\cap Z(\Omega^-),\quad p'>1.$$

Further, by Theorem A.2 and with the help of relations (4.94)–(4.97) we find that

$$\big\{U^{(1)}\big\}^+-\big\{U^{(2)}\big\}^-=0\text{ on }S,$$

$$\big\{T^{(1)}(\partial,n)U^{(1)}\big\}^+-\big\{T^{(2)}(\partial,n)U^{(2)}\big\}^-=0\text{ on }S.$$

Whence it follows that ($U^{(1)}$ and $U^{(2)}$) belong to the appropriate classes of vector functions and solve the homogeneous basic transmission problem $(\mathrm{BT})^-_\omega$. In accordance with the existence and uniqueness Theorem 3.5 then $U^{(1)}$ vanishes in Ω^+ and $U^{(2)}$ vanishes in Ω^- which along with (4.96) and (4.97) imply $g = h = 0$ on S. □

Finally, we introduce the system of vector functions which can be employed for constructing approximate solutions to the mixed transmission problem $(\mathrm{MT})_\omega$ which as a particular case covers the crack type problem.

Let us define the three vectors

$$\Lambda^{(l,1)}(x) := \begin{cases} \psi^{(l,1)}(x) & \text{for} \quad x \in S_T, \\ T^{(1)}(\partial, n(x))\psi^{(l,1)}(x) & \text{for} \quad x \in S_C, \end{cases} \qquad l = 1, 2, \dots \tag{4.98}$$

$$\Lambda^{(l,2)}(x) := \begin{cases} -\varphi^{(l,2)}(x) & \text{for} \quad x \in S_T, \\ T^{(2)}(\partial, n(x))\varphi^{(l,2)}(x) & \text{for} \quad x \in S_C, \end{cases} \qquad l = 1, 2, \dots \tag{4.99}$$

where $\psi^{(l,1)}$ and $\varphi^{(l,2)}$ are the same as in the above introduced system Φ_{BT} for the basic transmission problem.

Further, we define the six vectors

$$\Theta^{(l,1)}(x) = \big(\Lambda^{(l,1)}(x),\ T^{(1)}(\partial, n(x))\psi^{(l,1)}(x)\big)^\top, \quad x \in S, \tag{4.100}$$

$$\Theta^{(l,2)}(x) = \big(\Lambda^{(l,2)}(x),\ -T^{(2)}(\partial, n(x))\varphi^{(l,2)}(x)\big)^\top, \quad x \in S, \tag{4.101}$$

and set

$$\Phi_{MT} := \big\{\ \Theta^{(l,1)}(x)\ ,\ \Theta^{(l,2)}(x)\big\}_{l=1}^{\infty}, \quad x \in S. \tag{4.102}$$

Lemma 4.9. *The system Φ_{MT} is linearly independent on S.*

Proof. Let $m_1, m_2 \in \mathbb{N}$ be arbitrary natural numbers and let

$$\sum_{l=1}^{m_1} C_{l,1}\ \Theta^{(l,1)}(x) + \sum_{l=1}^{m_2} C_{l,2}\ \Theta^{(l,2)}(x) = 0, \quad x \in S, \tag{4.103}$$

with some complex constants $C_{l,\kappa}$, $\kappa = 1, 2,$.

Denote by $z^{(k)}$, $k = 1, 2, \dots, p_2$, and $y^{(k)}$, $k = 1, 2, \dots, p_1$, the points involved in the expression (4.103) and without loss of generality assume again that for each $z^{(k)}$, $k = 1, 2, \dots, p_2$, and for each $y^{(k)}$, $k = 1, 2, \dots, p_1$, the expression (4.85) contains all three vector functions corresponding to the points $z^{(k)}$ and $y^{(k)}$ implying that m_2 and m_1 are multiples of 3, $m_2 = 3p_2$ and $m_1 = 3p_1$.

Now, let us construct the vectors

$$u^{(m_1,1)}(x) := \sum_{l=1}^{m_1} C_{l,1}\ \psi^{(l,1)}(x), \quad x \notin \overline{\Omega_0^-}, \tag{4.104}$$

$$u^{(m_2,2)}(x) := \sum_{l=1}^{m_2} C_{l,2}\, \varphi^{(l,2)}(x), \quad x \notin \overline{\Omega_0^+}. \tag{4.105}$$

Evidently, $u^{(m_1,1)}$ and $u^{(m_2,2)}$ are radiating analytic vector function in $\mathbb{R}^3 \setminus \{y^{(k)}\}_{k=1}^{p_1}$ and $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{p_2}$ respectively and solve the homogeneous differential equations

$$A^{(1)}(\partial, \omega)u^{(m_1,1)}(x) = 0, \quad x \notin \overline{\Omega_0^-}, \tag{4.106}$$

$$A^{(2)}(\partial, \omega)u^{(m_2,2)}(x) = 0, \quad x \notin \overline{\Omega_0^+}. \tag{4.107}$$

In view of relations (4.98)–(4.107) it then follows that the pair $\big(u^{(m_1,1)}, u^{(m_2,2)}\big)$ solves the homogeneous mixed transmission problem $(\mathrm{MT})_\omega$ with equivalently transformed conditions (3.42)–(3.44) and due to the existence and uniqueness Theorem 3.7 we conclude that $u^{(m_1,1)} = 0$ in Ω^+ and $u^{(m_2,2)} = 0$ in Ω^-. Consequently, by the analyticity property, $u^{(m_1,1)} = 0$ in $\mathbb{R}^3 \setminus \{y^{(k)}\}_{k=1}^{p_1}$ and $u^{(m_2,2)} = 0$ in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{p_2}$. Now, by the same arguments as in the proof of Lemma 4.1 we derive that $C_{l,1} = 0, l = 1, 2, \dots, m_1$, and $C_{l,2} = 0, l = 1, 2, \dots, m_2$, which completes the proof. □

As in the case of the system $\Phi_M^{(-)}$, here we can identify the system Φ_{MT} with the system $\widetilde{\Phi}_{MT}$ defined as

$$\widetilde{\Phi}_{MT} := \{\widetilde{\Theta}^{(l,1)}, \widetilde{\Theta}^{(l,2)}\}_{l=1}^{\infty},$$

where

$$\widetilde{\Theta}^{(l,1)}(x) = \left(r_{S_T}\,\psi^{(l,1)},\ r_{S_C}\,T^{(1)}(\partial, n(x))\psi^{(l,1)},\ r_S T^{(1)}(\partial, n(x))\psi^{(l,1)}\right)^{\top},$$

$$\widetilde{\Theta}^{(l,2)}(x) = \left(r_{S_T}\,\varphi^{(l,2)},\ r_{S_C}\,T^{(2)}(\partial, n(x))\varphi^{(l,2)},\ r_S T^{(2)}(\partial, n(x))\varphi^{(l,2)}\right)^{\top}.$$

Lemma 4.10. *The system $\widetilde{\Phi}_{MT}$ is complete in the space*

$$\left[B_{p,p}^{1-\frac{1}{p}}(S_T)\right]^3 \times \left[B_{p,p}^{-\frac{1}{p}}(S_C)\right]^3 \times \left[B_{p,p}^{-\frac{1}{p}}(S)\right]^3 \tag{4.108}$$

for $4/3 < p < 4$.

Proof. To prove the density property of the linear span of the system $\widetilde{\Phi}_{MT}$ in the space (4.108) it suffices to show that if a pair of vector functions $\widetilde{\chi} = (f, g, h)$ belongs to the adjoint space,

$$\widetilde{\chi} = (f, g, h) \in \left[\left[\widetilde{B}_{p',p'}^{-1+\frac{1}{p}}(S_T)\right]^3 \times \left[\widetilde{B}_{p',p'}^{\frac{1}{p}}(S_C)\right]^3\right] \times \left[B_{p',p'}^{\frac{1}{p}}(S)\right]^3$$

with $1/p + 1/p' = 1$, and for all $l = 1, 2, \dots,$

$$\langle \widetilde{\chi}\,,\ \widetilde{\Theta}^{(l,1)}\rangle_S := \langle f\,,\ \psi^{(l,1)}\rangle_{S_T} + \langle g\,,\ T^{(1)}(\partial, n)\psi^{(l,1)}\rangle_{S_C} + \langle h\,,\ T^{(1)}(\partial, n)\psi^{(l,1)}\rangle_S = 0, \tag{4.109}$$

$$\langle \widetilde{\chi}\,,\ \widetilde{\Theta}^{(l,2)}\rangle_S := -\langle f\,,\ \varphi^{(l,2)}\rangle_{S_T} + \langle g\,,\ T^{(2)}(\partial, n)\varphi^{(l,2)}\rangle_{S_C} - \langle h\,,\ T^{(2)}(\partial, n)\varphi^{(l,2)}\rangle_S = 0, \tag{4.110}$$

then $\widetilde{\chi} = (f, g, h) = 0$.

Condition (4.109) and (4.110) can be rewritten as

$$\begin{aligned}
&\langle f\,,\ \Gamma^{(j,1)}(\cdot\,-y^{(k)})\rangle_{S_T} + \langle g\,,\ T^{(1)}(\partial, n)\Gamma^{(j,1)}(\cdot\,-y^{(k)})\rangle_{S_C} + \langle T^{(1)}(\partial, n)\Gamma^{(j,1)}(\cdot\,-y^{(k)})\,,\ h\rangle_S = 0,\\
&-\langle f\,,\ \Gamma^{(j,2)}(\cdot\,-z^{(k)})\rangle_{S_T} + \langle g\,,\ T^{(2)}(\partial, n)\Gamma^{(j,2)}(\cdot\,-z^{(k)})\rangle_{S_C} - \langle T^{(2)}(\partial, n)\Gamma^{(j,2)}(\cdot\,-z^{(k)})\,,\ h\rangle_S = 0,\\
&\quad j = 1, 2, 3, \quad k = 1, 2, \dots
\end{aligned}$$

which is equivalent to the relations

$$\begin{aligned}
&V^{(1)}(f)(y^{(k)}) + W^{(1)}(g)(y^{(k)}) + W^{(1)}(h)(y^{(k)}) = 0, \qquad k = 1, 2, \dots,\quad y^{(k)} \in \Omega_0^-,\\
&-V^{(2)}(f)(z^{(k)}) + W^{(2)}(g)(z^{(k)}) - W^{(2)}(h)(z^{(k)}) = 0, \quad k = 1, 2, \dots,\quad z^{(k)} \in \Omega_0^+,
\end{aligned}$$

where $V^{(\kappa)}(f)$, $V^{(\kappa)}(g)$, and $W^{(\kappa)}(h)$ are the single and double layer potentials constructed by the fundamental matrix $\Gamma^{(\kappa)}$ with the integration surface S and the densities

$$f \in \left[\widetilde{B}_{p',p'}^{-1+\frac{1}{p}}(S_T)\right]^3, \quad g \in \left[\widetilde{B}_{p',p'}^{\frac{1}{p}}(S_C)\right]^3, \quad h \in \left[B_{p',p'}^{\frac{1}{p}}(S)\right]^3, \quad 4/3 < p' < 4. \tag{4.111}$$

Due to the density property of the sets $\{y^{(k)}\}_{k=1}^{\infty}$ in Ω_0^- and $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, from (4.92) and (4.93) we get

$$\begin{aligned}
&V^{(1)}(f)(z) + V^{(1)}(g)(z) + W^{(1)}(h)(z) = 0, \quad z \in \Omega_0^-,\\
&-V^{(2)}(f)(z) + W^{(2)}(g)(z) - W^{(2)}(h)(z) = 0, \quad z \in \Omega_0^+.
\end{aligned}$$

Due to analyticity of the layer potentials in domains $\Omega^{\pm}$ we conclude

$$U^{(1)}(z) := V^{(1)}(f)(z) + W^{(1)}(g)(z) + W^{(1)}(h)(z) = 0, \quad z \in \Omega^-, \tag{4.112}$$

$$U^{(2)}(z) := -V^{(2)}(f)(z) + W^{(2)}(g)(z) - W^{(2)}(h)(z) = 0, \quad z \in \Omega^+. \tag{4.113}$$

Note that by Theorem A.1 we have

$$U^{(1)} \in \left[W_{p'}^1(\Omega^+)\right]^3, \quad U^{(2)} \in \left[W_{p',loc}^1(\Omega^-)\right]^3 \cap Z(\Omega^-), \quad 4/3 < p' < 4. \tag{4.114}$$

Further, by Theorem A.2 and relations (4.112) and (4.113) we find that

$$\left\{U^{(1)}\right\}^+ = g + h \text{ on } S, \tag{4.115}$$

$$\left\{T^{(1)}(\partial, n)U^{(1)}\right\}^+ = -f \text{ on } S, \tag{4.116}$$

$$\left\{U^{(2)}\right\}^- = -g + h \text{ on } S, \tag{4.117}$$

$$\left\{T^{(2)}(\partial, n)U^{(2)}\right\}^- = -f \text{ on } S. \tag{4.118}$$

Whence in view of (4.111) we find

$$\{U^{(1)}\}^+ - \{U^{(2)}\}^- = 0 \text{ on } S_T, \tag{4.119}$$

$$\{T^{(1)}(\partial, n)U^{(1)}\}^+ + \{T^{(2)}(\partial, n)U^{(2)}\}^- = 0 \text{ on } S_C, \tag{4.120}$$

$$\{T^{(1)}(\partial, n)U^{(1)}\}^+ - \{T^{(2)}(\partial, n)U^{(2)}\}^- = 0 \text{ on } S. \tag{4.121}$$

From (4.119)–(4.121) and (4.112)–(4.114) it follows that the pair $(U^{(1)}, U^{(2)})$ belongs to the appropriate class of functions and solves the homogeneous mixed transmission problem $(\mathrm{MT})^-_\omega$. In accordance with the existence and uniqueness Theorem 3.7 then $U^{(1)}$ vanishes in Ω^+ and $U^{(2)}$ vanishes in Ω^- which along with (4.115)–(4.118) imply $f = g = h = 0$ on S. □

4.2. *Construction of approximate solutions*

In this subsection we describe how to construct approximate solutions of the above considered boundary value problems. In what follows $\varphi^{(l)}$, $\psi^{(l)}$, $\varphi^{(l,\kappa)}$, $\psi^{(l,\kappa)}$, $\kappa = 1, 2$, are the vector functions introduced in the previous subsection.

4.2.1. *The Dirichlet problem*

Let us look for an approximate solution of the exterior Dirichlet problem $(\mathrm{D})^-_\omega$, (see (2.1), (2.3)) in the form

$$u^{(m)}(x) = \sum_{l=1}^{m} a_l\, \varphi^{(l)}(x), \quad x \in \Omega^-, \quad m \in \mathbb{N}, \tag{4.122}$$

where a_l are sought-for complex valued constants. These constants are to be chosen in such a way that the norm $\|u - u^{(m)}\|_{[W^1_p(\Omega^- \cap B(R))]^3}$ of the difference of the exact solution u and the approximate solution $u^{(m)}$ should be small.

Note that for all m the vector function $u^{(m)}$ solves the homogeneous differential equation (2.1) and is analytic and radiating in $\mathbb{R}^3 \setminus \overline{\Omega^+_0}$.

Due to Theorem 3.1 and estimate (3.20), if the trace of $u^{(m)}$ on the boundary S approximates the boundary function f with a sufficiently good accuracy in the space $[B^{1-1/p}_{p,\,p}(S)]^3$, then the norm $\|u - u^{(m)}\|_{[W^1_p(\Omega^- \cap B(R))]^3}$ for fixed R will also be sufficiently small and $u^{(m)}$ can be considered as a good approximation of the exact solution u in the region $\Omega^- \cap B(R)$).

Lemmas 4.1 and 4.2 show that a good approximation on S of a boundary vector function $f \in [B^{1-1/p}_{p,\,p}(S)]^3$ is possible within an arbitrary accuracy by the linear combinations of type (4.122):

$$\sum_{l=1}^{m} a_l\, \varphi^{(l)}(x) \approx f(x), \quad x \in S. \tag{4.123}$$

Thus, construction of an approximate solution of the Dirichlet BVP is reduced to the approximation problem for the boundary vector function into the linearly independent complete system of vector functions $\Phi^{(-)}_D$ explicitly constructed by the columns of the fundamental solution matrix.

This approximation can be practically carried out by choosing finite sets of functions from the system $\Phi^{(-)}_D$ appropriately and then applying some well-known methods, e.g., Galerkin, collocation, least square, adaptive cross approximation etc. However, this is a very serious problem which needs a special investigation from the point of view of numerical analysis (cf. [16–19,38,39]).

Similar approach with word for word arguments can be applied to all BVP considered in Section 3. Therefore, below we will write down schematically only the expressions of approximate solutions in the corresponding domains and formulate the desired boundary approximation problems (counterparts of (4.122) and (4.123)).

4.2.2. *The Neumann problem*

Approximate solution of the exterior Neumann problem (see (2.1), (2.4))

$$u^{(m)}(x) = \sum_{l=1}^{m} a_l\, \varphi^{(l)}(x), \quad x \in \Omega^-, \quad m \in \mathbb{N},$$

where a_l are sought-for complex valued constants.

Desired boundary approximation of the vector function $F \in [B_{p,p}^{-1/p}(S)]^3$ in the system $\Phi_N^{(-)}$ (see Theorem 3.2 and Lemmas 4.3 and 4.4):

$$\sum_{l=1}^{m} a_l\, T(\partial, n(x))\varphi^{(l)} \approx F \quad \text{on} \quad S.$$

4.2.3. The mixed problem

Approximate solution of the exterior mixed problem (see (2.1), (2.5), (2.6))

$$u^{(m)}(x) = \sum_{l=1}^{m} a_l\, \varphi^{(l)}(x), \quad x \in \Omega^-, \quad m \in \mathbb{N},$$

where a_l are sought-for complex valued constants.

Desired boundary approximation on S_D and S_N of the vector functions $f^* \in [B_{p,p}^{1-1/p}(S_D)]^3$ and $F^* \in [B_{p,p}^{-1/p}(S_N)]^3$ in the system $\widetilde{\Phi}_M^{(-)}$ (see Theorem 3.3 and Lemmas 4.5 and 4.6):

$$\sum_{l=1}^{m} a_l\, \varphi^{(l)} \approx f^* \quad \text{on} \quad S_D,$$
$$\sum_{l=1}^{m} a_l\, T(\partial, n)\varphi^{(l)} \approx F^* \quad \text{on} \quad S_N.$$

4.2.4. The basic transmission problem

Approximate solution of the basic transmission problem (see (2.1), (2.12)–(2.13))

$$u^{(m_1,1)}(x) := \sum_{l=1}^{m_1} a_{l,1}\, \psi^{(l,1)}(x), \quad x \in \Omega^+, \quad m_1 \in \mathbb{N},$$
$$u^{(m_2,2)}(x) := \sum_{l=1}^{m_2} a_{l,2}\, \varphi^{(l,2)}(x), \quad x \in \Omega^-, \quad m_2 \in \mathbb{N},$$

where $a_{l,1}$ and $a_{l,2}$ are sought-for complex valued constants.

Desired boundary approximation on S_D and S_N of the vector functions $f \in [B_{p,p}^{1-1/p}(S)]^3$ and $F \in [B_{p,p}^{-1/p}(S)]^3$ in the system Φ_{BT} (see Theorem 3.5 and Lemmas 4.7 and 4.8):

$$\sum_{l=1}^{m_1} a_{l,1}\, \psi^{(l,1)} - \sum_{l=1}^{m_2} a_{l,2}\, \varphi^{(l,2)} \approx f \quad \text{on} \quad S,$$
$$\sum_{l=1}^{m_1} a_{l,1}\, T^{(1)}(\partial, n)\psi^{(l,1)} - \sum_{l=1}^{m_2} a_{l,2}\, T^{(2)}(\partial, n)\varphi^{(l,2)} \approx F \quad \text{on} \quad S.$$

4.2.5. The mixed transmission problem

Approximate solution of the mixed transmission problem (see (2.1), (2.15)–(2.18), and (3.42)–(3.45))

$$u^{(m_1,1)}(x) := \sum_{l=1}^{m_1} a_{l,1}\, \psi^{(l,1)}(x), \quad x \in \Omega^+, \quad m_1 \in \mathbb{N},$$
$$u^{(m_2,2)}(x) := \sum_{l=1}^{m_2} a_{l,2}\, \varphi^{(l,2)}(x), \quad x \in \Omega^-, \quad m_2 \in \mathbb{N},$$

where $a_{l,1}$ and $a_{l,2}$ are sought-for complex valued constants.

Desired boundary approximation on S_T, S_C, and S of the vector functions

$$f^{(T)} \in [B_{p,p}^{1-1/p}(S_T)]^3, \quad F^{(+)} + F^{(-)} \in [B_{p,p}^{-1/p}(S_C)]^3, \quad F_0 \in [B_{p,p}^{-1/p}(S)]^3$$

in the system $\widetilde{\Phi}_{MT}$ (see Theorem 3.7 and Lemmas 4.9 and 4.10):

$$\sum_{l=1}^{m_1} a_{l,1}\,\psi^{(l,1)} - \sum_{l=1}^{m_2} a_{l,2}\,\varphi^{(l,2)} \approx f^{(T)} \ \text{ on } \ S_T,$$
$$\sum_{l=1}^{m_1} a_{l,1}\,T^{(1)}(\partial,n)\psi^{(l,1)} + \sum_{l=1}^{m_2} a_{l,2}\,T^{(2)}(\partial,n)\varphi^{(l,2)} \approx F^{(+)} + F^{(-)} \ \text{ on } \ S_C,$$
$$\sum_{l=1}^{m_1} a_{l,1}\,T^{(1)}(\partial,n)\psi^{(l,1)} - \sum_{l=1}^{m_2} a_{l,2}\,T^{(2)}(\partial,n)\varphi^{(l,2)} \approx F_0 \ \text{ on } \ S.$$

Due to Remark 3.10 the crack problem is a particular case of a special mixed transmission problem and its approximate solution can be constructed in accordance with the approach described in the present subsection.

Acknowledgements

This research was supported by the Shota Rustaveli National Science Foundation (SRNSF) Grant No. FR/286/5-101/13.

Appendix A. Layer potentials and their properties

Here we collect some auxiliary material needed in the main text of the paper concerning properties of layer potentials and the corresponding boundary operators.

Denote by $\Gamma(x,\omega)$ and $\Gamma(x)$ respectively Kupradze's and Kelvin's matrices of fundamental solutions of the differential operator of elastic oscillations $A(\partial,\omega)$ and its principal homogeneous part $A(\partial)$ (Lamé's operator)

$$A(\partial,\omega)\Gamma(x,\omega) = I_3\,\delta(x), \qquad A(\partial)\Gamma(x) = I_3\,\delta(x),$$

where $\delta(x)$ is the Dirac delta function. These matrices read as (see [12, Ch. 2])

$$\Gamma(x,\omega) = [\Gamma_{kj}(x,\omega)]_{3\times3}, \qquad \Gamma_{kj}(x,\omega) = \sum_{l=1}^{2} (\delta_{kj}\alpha_l + \beta_l\,\partial_k\,\partial_j)\,\frac{e^{ik_l\,|x|}}{|x|}\,, \tag{A.1}$$
$$\alpha_l = -\frac{\delta_{2l}}{4\pi\,\mu}\,, \quad \beta_l = \frac{(-1)^{l+1}}{4\pi\,\varrho\,\omega^2}\,, \quad k_1 \equiv k_p = \omega\sqrt{\frac{\varrho}{\lambda+2\,\mu}}\,, \quad k_2 \equiv k_s = \omega\sqrt{\frac{\varrho}{\mu}}\,,$$
$$\Gamma(x) = [\Gamma_{kj}(x)]_{3\times3}, \qquad \Gamma_{kj}(x) = \frac{\delta_{kj}\,\lambda'}{|x|} + \frac{\mu'\,x_k\,x_j}{|x|^3}\,,$$
$$\lambda' = -\frac{\lambda+3\,\mu}{8\,\pi\,\mu\,(\lambda+2\,\mu)}\,, \quad \mu' = -\frac{\lambda+\mu}{8\,\pi\,\mu\,(\lambda+2\,\mu)}\,.$$

The following relations hold true

$$\begin{array}{ll} \Gamma(x,\omega) = \Gamma(-x,\omega) = [\Gamma(x,\omega)]^\top, & \Gamma(x) = \Gamma(-x) = [\Gamma(x)]^\top, \\ |\Gamma_{pq}(x,\omega)| \le c(\lambda,\mu)\,|x|^{-1}, & |\Gamma({}_{pq}x,\omega) - \Gamma_{pq}(x)| \le |\omega|\,c(\lambda,\mu), \\ |\partial_j\Gamma_{pq}(x,\omega) - \partial_j\Gamma_{pq}(x)| \le |\omega|^2 c(\lambda,\mu), & |\partial_j\partial_l\Gamma_{pq}(x,\omega) - \partial_j\partial_l\Gamma_{pq}(x)| \le c(\lambda,\mu,\omega)|x|^{-1}, \end{array} \tag{A.2}$$

where $c(\lambda,\mu)$ and $c(\lambda,\mu,\omega)$ are positive constants. These relations show that the Kelvin matrix of statics $\Gamma(x)$ is the principal singular homogeneous part of Kupradze's matrix $\Gamma(x,\omega)$. It is evident that the entries of $\Gamma(x,\omega)$ and $\Gamma(x)$ are analytic functions of the real variable $x \in \mathbb{R}^3 \setminus \{0\}$ and, moreover, the columns of $\Gamma(x,\omega)$ satisfy the Sommerfeld–Kupradze radiation conditions at infinity.

Introduce the single and double layer potentials of elastic oscillations

$$V(g)(x) := \int_S \Gamma(x-y,\omega)\,g(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S\,, \tag{A.3}$$

$$W(h)(x) := \int_S [T(\partial_y,n(y))\Gamma(x-y,\omega)]^\top\,h(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{A.4}$$

where $g = (g_1,g_2,g_3)^\top$ and $h = (h_1,h_2,h_3)^\top$ are densities of the potentials.

By standard arguments and with the help of Green's second formula one can derive the following integral representation formula for a regular solution u to the homogeneous equation $A(\partial,\omega)u = 0$ in the domain Ω^+,

$$W(\{u\}^+)(x) - V(\{Tu\}^+)(x) = \begin{cases} u(x) & \text{in} \quad \Omega^+, \\ 0 & \text{in} \quad \Omega^-. \end{cases} \tag{A.5}$$

Similarly, for a radiating regular solution of the homogeneous equation $A(\partial,\omega)u = 0$ in the domain Ω^- we have an analogous representation formula (see [12,27])

$$-W(\{u\}^-)(x) + V(\{Tu\}^-)(x) = \begin{cases} 0 & \text{in} \quad \Omega^+, \\ u(x) & \text{in} \quad \Omega^-. \end{cases} \tag{A.6}$$

These representation formulae can be extended to the classes $[W^1_p(\Omega^+)]^3$ and $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, and to Lipschitz domains. From these formulae it is evident that any solution to the homogeneous equation is actually an analytic vector function of the real variable $x \in \Omega^\pm$. Further, if u solves the homogeneous equation $A(\partial,\omega)u = 0$ in Ω^+ and Ω^-, and $r_{\Omega^+} u \in [W^1_p(\Omega^+)]^3$, $r_{\Omega^-} u \in [W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$ then by adding formulae (A.5) and (A.6) we get

$$\begin{aligned} &u(x) = W\big([u]_S\big)(x) - V\big([Tu]_S\big)(x) \text{ in } \Omega^+ \cup \Omega^- \\ &\quad \text{with} \quad [u]_S := \{u\}^+_S - \{u\}^-_S, \qquad [Tu]_S := \{Tu\}^+_S - \{Tu\}^-_S, \end{aligned}$$

which shows that if on some open part $S_1 \subset S$ of the common boundary S of the adjacent domains Ω^+ and Ω^- the jumps of the Cauchy data equal to zero, i.e., $r_{S_1}[\{u\}^+ - \{u\}^-] = 0$ and $r_{S_1}[\{Tu\}^+ - \{Tu\}^-] = 0$, then the vector-function $\widetilde{u}$ defined by the equality

$$\widetilde{u} := \begin{cases} u(x) & \text{for} \quad x \in \Omega^+, \\ u(x) & \text{for} \quad x \in \Omega^-, \\ \{u(x)\}^+ & \text{for} \quad x \in S_1, \end{cases}$$

is an analytic vector function in the connected domain $\mathbb{R}^3 \setminus \overline{S_2}$ with $S_2 = S \setminus \overline{S_1}$.

Further we introduce the boundary operators generated by the single and double layer potentials,

$$(\mathcal{H}\, g)(x) := \int_S \Gamma(x-y,\omega)\, g(y)\, dS_y\,, \qquad x \in S, \tag{A.7}$$

$$(\mathcal{K}\, g)(x) := \int_S \big[\, T(\partial_x, n(x))\Gamma(x-y,\omega)\,\big]\, g(y)\, dS_y, \qquad x \in S, \tag{A.8}$$

$$(\widetilde{\mathcal{K}}\, h)(x) := \int_S \big[\, T(\partial_y, n(y))\, \Gamma(x-y,\omega)\,\big]^\top\, h(y)\, dS_y, \qquad x \in S, \tag{A.9}$$

$$(\mathcal{L}h)(x) := \big\{\, T(\partial_x, n(x)) W(h)(x)\,\big\}^\pm, \qquad x \in S. \tag{A.10}$$

The boundary operators $\mathcal{H}$ and $\mathcal{L}$ are pseudodifferential operators of order -1 and 1, respectively, while the operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are mutually adjoint singular integral operators–pseudodifferential operators of order 0 (for details see [12,40–45]).

We will employ the same notation equipped with subscript "0" for the elastostatic potentials constructed by the Kelvin matrix $\Gamma(x-y)$ and the corresponding boundary operators.

Now we describe the basic mapping and jump properties of the above introduced layer potentials. They can be found in [24–26,35,40,41,43–55].

Theorem A.1. *Let S be C^∞-smooth and $1 < p < \infty$, $1 \le t \le \infty$, and $s \in \mathbb{R}$. The operators*

$$\begin{aligned}
V \ &: \ \big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H_p^{s+1+\frac{1}{p}}(\Omega^+)\big]^3 \qquad \Big[\big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big], \\
&: \ \big[B^s_{p,t}(S)\big]^3 \longrightarrow \big[B_{p,t}^{s+1+\frac{1}{p}}(\Omega^+)\big]^3 \qquad \Big[\big[B^s_{p,t}(S)\big]^3 \longrightarrow \big[B_{p,t,loc}^{s+1+\frac{1}{p}}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big], \\
W \ &: \ \big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H_p^{s+\frac{1}{p}}(\Omega^+)\big]^3 \qquad \Big[\big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big], \\
&: \ \big[B^s_{p,t}(S)\big]^3 \longrightarrow \big[B_{p,t}^{s+\frac{1}{p}}(\Omega^+)\big]^3 \qquad \Big[\big[B^s_{p,t}(S)\big]^3 \longrightarrow \big[B_{p,t,loc}^{s+\frac{1}{p}}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big],
\end{aligned}$$

are continuous.

If S is Lipschitz, then the operators

$$\begin{aligned} V &: \left[H_2^{-\frac12}(S)\right]^3 \longrightarrow \left[H_2^1(\Omega^+)\right]^3 \qquad \left[\left[H_2^{-\frac12}(S)\right]^3 \longrightarrow \left[H^1_{2,\,loc}(\Omega^-)\right]^3 \cap Z(\Omega^-)\right],\\ W &: \left[H_2^{\frac12}(S)\right]^3 \longrightarrow \left[H_2^1(\Omega^+)\right]^3 \qquad \left[\left[H_2^{\frac12}(S)\right]^3 \longrightarrow \left[H^1_{2,\,loc}(\Omega^-)\right]^3 \cap Z(\Omega^-)\right], \end{aligned}$$

are continuous.

Theorem A.2. *Let S be C^∞-smooth and $1 < p < \infty$, $1 \le t \le \infty$, and $g \in \left[B_{p,t}^{-\frac1p}(S)\right]^3$, $h \in \left[B_{p,t}^{1-\frac1p}(S)\right]^3$. Then*

$$\{V(g)\}^+ = \{V(g)\}^- = \mathcal{H}\, g \quad on \quad S, \tag{A.11}$$

$$\{T(\partial,n)V(g)\}^\pm = \left[\mp 2^{-1} I_3 + \mathcal{K}\right] g \quad on \quad S,$$

$$\{W(h)\}^\pm = \left[\pm 2^{-1} I_3 + \widetilde{\mathcal{K}}\right] h \quad on \quad S,$$

$$\{T(\partial,n)W(h)\}^+ = \{T(\partial,n)W(h)\}^- = \mathcal{L}\, h \quad on \quad S. \tag{A.12}$$

The same relations hold for a Lipschitz boundary S and $p = t = 2$.

Theorem A.3. (i) *Let S be C^∞-smooth and $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbb{R}$. The operators*

$$\begin{aligned} \mathcal{H} &: \left[H_p^s(S)\right]^3 \longrightarrow \left[H_p^{s+1}(S)\right]^3 \qquad \left[\left[B_{p,t}^s(S)\right]^3 \longrightarrow \left[B_{p,t}^{s+1}(S)\right]^3\right],\\ \pm 2^{-1} I_3 + \mathcal{K},\ \pm 2^{-1} I_3 + \widetilde{\mathcal{K}} &: \left[H_p^s(S)\right]^3 \longrightarrow \left[H_p^s(S)\right]^3 \qquad \left[\left[B_{p,t}^s(S)\right]^3 \longrightarrow \left[B_{p,t}^s(S)\right]^3\right],\\ \mathcal{L} &: \left[H_p^{s+1}(S)\right]^3 \longrightarrow \left[H_p^s(S)\right]^3 \qquad \left[\left[B_{p,t}^{s+1}(S)\right]^3 \longrightarrow \left[B_{p,t}^s(S)\right]^3\right], \end{aligned}$$

are continuous Fredholm operators with zero index. The principal homogeneous symbol matrices of these operators are non-degenerate. Moreover, the principal homogeneous symbol matrices of the operators $-\mathcal{H}$ and $\mathcal{L}$ are positive definite.

(ii) *If S is Lipschitz, then the operators*

$$\begin{aligned} \mathcal{H} &: \left[H_2^{-\frac12}(S)\right]^3 \longrightarrow \left[H_2^{\frac12}(S)\right]^3,\\ \pm 2^{-1} I_3 + \mathcal{K} &: \left[H_2^{-\frac12}(S)\right]^3 \longrightarrow \left[H_2^{-\frac12}(S)\right]^3,\\ \pm 2^{-1} I_3 + \widetilde{\mathcal{K}} &: \left[H_2^{\frac12}(S)\right]^3 \longrightarrow \left[H_2^{\frac12}(S)\right]^3,\\ \mathcal{L} &: \left[H_2^{\frac12}(S)\right]^3 \longrightarrow \left[H_2^{-\frac12}(S)\right]^3, \end{aligned}$$

are continuous Fredholm operators with zero index, and moreover, there exist positive constants C_k, $k = 1, 2, 3, 4$, such that

$$\begin{aligned} &\langle h\,,\, -\mathcal{H}h\rangle_S \ge C_1\, \|h\|^2_{[H_2^{-\frac12}(S)]^3} - C_2\, \|\mathcal{T}h\|^2_{[H_2^{-\frac12}(S)]^3} \quad for\ all\ \ h \in [H_2^{-\frac12}(S)]^3,\\ &\langle \mathcal{L}g\,,\, g\rangle_S \ge C_3\, \|g\|^2_{[H_2^{\frac12}(S)]^3} - C_4\, \|g\|^2_{[H_2^0(S)]^3} \quad for\ all\ \ g \in [H_2^{\frac12}(S)]^3, \end{aligned} \tag{A.13}$$

where the symbol $\langle\cdot,\cdot\rangle_S$ denotes the duality brackets between the mutually adjoint spaces $[H_2^{-\frac12}(S)]^3$ and $[H_2^{\frac12}(S)]^3$, and $\mathcal{T} : [H_2^{-\frac12}(S)]^3 \longrightarrow [H_2^{-\frac12}(S)]^3$ is a compact operator.

(iii) *The following operator equalities hold in appropriate function spaces:*

$$\widetilde{\mathcal{K}}\,\mathcal{H} = \mathcal{H}\,\mathcal{K}, \quad \mathcal{L}\,\widetilde{\mathcal{K}} = \mathcal{K}\,\mathcal{L}, \quad \mathcal{L}\,\mathcal{H} = -4^{-1} I_3 + \mathcal{K}^2, \quad \mathcal{H}\,\mathcal{L} = -4^{-1} I_3 + \widetilde{\mathcal{K}}^2. \tag{A.14}$$

Remark A.4. In the static case, i.e., for the operators constructed by the Kelvin fundamental matrix $\Gamma(x-y)$, the operators $\mathcal{H}_0$, $2^{-1} I_3 + \widetilde{\mathcal{K}}_0$ and $2^{-1} I_3 + \mathcal{K}_0$ in items (i) and (ii) of Theorem A.3 are invertible. Moreover, $\widetilde{\mathcal{K}}_0$ and $\mathcal{K}_0$ are mutually adjoint singular integral operators and the inequality (A.13) holds with $C_2 = 0$ [36].

In view of the relations (A.2), it is evident that the operators of elastostatics $\mathcal{H}_0$, $\pm 2^{-1} I_3 + \mathcal{K}_0$, $\pm 2^{-1} I_3 + \widetilde{\mathcal{K}}_0$, $\mathcal{L}_0$, and elasto-oscillations $\mathcal{H}$, $\pm 2^{-1} I_3 + \mathcal{K}$, $\pm 2^{-1} I_3 + \widetilde{\mathcal{K}}$, $\mathcal{L}$ have the same principal homogeneous symbol matrices respectively:

$$\begin{aligned}
&\mathbb{H}(x,\xi) := \mathfrak{S}(\mathcal{H}_0; x,\xi) = \mathfrak{S}(\mathcal{H}; x,\xi),\\
&\mathbb{K}_\pm(x,\xi) := \mathfrak{S}(\pm 2^{-1} I_3 + \mathcal{K}_0)(x,\xi) = \mathfrak{S}(\pm 2^{-1} I_3 + \mathcal{K})(x,\xi),\\
&\mathbb{L}(x,\xi) := \mathfrak{S}(\mathcal{L}_0; x,\xi) = \mathfrak{S}(\mathcal{L}; x,\xi),\\
&\widetilde{\mathbb{K}}_\pm(x,\xi) := \mathfrak{S}(\pm 2^{-1} I_3 + \widetilde{\mathcal{K}}_0)(x,\xi) = \mathfrak{S}(\pm 2^{-1} I_3 + \widetilde{\mathcal{K}})(x,\xi),\\
&x \in S, \quad \xi \in \mathbb{R}^2 \setminus \{0\}.
\end{aligned} \tag{A.15}$$

The matrices $-\mathbb{H}(x,\xi)$ and $\mathbb{L}(x,\xi)$ are positive definite matrices with entries being real valued even functions in ξ, while the matrices $\mathbb{K}_\pm(x,\xi)$ and $\widetilde{\mathbb{K}}_\pm(x,\xi)$ are non-degenerate and mutually adjoint, i.e., $\widetilde{\mathbb{K}}_\pm(x,\xi) = [\,\overline{\mathbb{K}_\pm(x,\xi)}\,]^\top$. The following matrices related to the so called Steklov–Poincaré operators of statics $[-2^{-1} I_3 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}$ and $-[2^{-1} I_3 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}$ corresponding to the interior and exterior domains, respectively,

$$\mathfrak{S}\big([-2^{-1} I_3 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}; x,\xi\big) = \mathbb{K}_-(x,\xi)\,[\mathbb{H}(x,\xi)]^{-1}, \tag{A.16}$$

$$-\,\mathfrak{S}\big([2^{-1} I_3 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}; x,\xi\big) = -\,\mathbb{K}_+(x,\xi)\,[\mathbb{H}(x,\xi)]^{-1}, \tag{A.17}$$

are positive definite as well. Moreover, from (A.14) it follows that

$$\begin{aligned}
&\widetilde{\mathbb{K}}_\pm(x,\xi)\,\mathbb{H}(x,\xi) = \mathbb{H}(x,\xi)\,\mathbb{K}_\pm(x,\xi), \quad \mathbb{L}(x,\xi)\,\widetilde{\mathbb{K}}_\pm(x,\xi) = \mathbb{K}_\pm(x,\xi)\,\mathbb{L}(x,\xi),\\
&\mathbb{L}(x,\xi)\,\mathbb{H}(x,\xi) = \mathbb{K}_+(x,\xi)\,\mathbb{K}_-(x,\xi), \quad \mathbb{H}(x,\xi)\mathbb{L}(x,\xi) = \widetilde{\mathbb{K}}_+(x,\xi)\,\widetilde{\mathbb{K}}_-(x,\xi).
\end{aligned} \tag{A.18}$$

Note that the matrices $\widetilde{\mathbb{K}}_-(x,\xi)$ and $\widetilde{\mathbb{K}}_+(x,\xi)$, as well as the matrices $\mathbb{K}_-(x,\xi)$ and $\mathbb{K}_+(x,\xi)$ commute each other. Therefore from (A.18) we derive

$$\begin{aligned}
\pm\,\mathbb{L}(x,\xi)\,[\widetilde{\mathbb{K}}_\pm(x,\xi)]^{-1} &= \pm\,[\mathbb{H}(x,\xi)]^{-1}\,\widetilde{\mathbb{K}}_+(x,\xi)\,\widetilde{\mathbb{K}}_-(x,\xi)\,[\widetilde{\mathbb{K}}_\pm(x,\xi)]^{-1}\\
&= \pm\,[\mathbb{H}(x,\xi)]^{-1}\,\widetilde{\mathbb{K}}_\mp(x,\xi) = \pm\,\mathbb{K}_\mp(x,\xi)\,[\mathbb{H}(x,\xi)]^{-1},
\end{aligned} \tag{A.19}$$

implying that the matrices $\pm\,\mathbb{L}(x,\xi)\,[\widetilde{\mathbb{K}}_\pm(x,\xi)]^{-1}$ are positive definite in view of the positive definiteness of the matrices (A.16) and (A.17).

Moreover, it can be shown that the entries of the matrices $\mathbb{H}(x,\xi)$ and $\mathbb{L}(x,\xi)$ are real valued functions, while $\mathbb{K}_\pm(x,\xi) = \pm 2^{-1} I_3 + i\,\mathbf{K}(x,\xi)$ and $\widetilde{\mathbb{K}}_\pm(x,\xi) = \pm 2^{-1} I_3 + i\,\widetilde{\mathbf{K}}(x,\xi)$, where the entries of the matrices $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ are real valued odd functions in ξ (see Appendix C in [23]).

Appendix B. Alternative representations of radiating solutions

Let Ω^+, Ω^-, and S be the same as in Appendix A, and consider the following linear combination of the single and double layer potentials

$$u(x) = W(g)(x) + \varkappa\,V(g)(x), \quad x \in \Omega^-,$$

where $\varkappa = \varkappa_1 + i\,\varkappa_2 \in \mathbb{C}$ with $\varkappa_1, \varkappa_2 \in \mathbb{R}$ and $\varkappa_2 \neq 0$, and $g \in (g_1, g_2, g_3)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^3$ is a density vector.

Evidently, $u \in [W^1_{p,\,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$ in view of Theorem A.1, while by Theorem A.2 we have

$$\{u\}^- = [\,-2^{-1}\,I_3 + \widetilde{\mathcal{K}} + \varkappa\,\mathcal{H}\,]\,g \equiv \mathcal{N}\,g, \qquad \{Tu\}^- = \mathcal{L}\,g + \varkappa\,[\,2^{-1}\,I_3 + \mathcal{K}\,]\,g \equiv \mathcal{M}\,g,$$

where $\mathcal{H}$, $\mathcal{K}$, $\widetilde{\mathcal{K}}$ and $\mathcal{L}$ are given by equalities (A.7)–(A.10) respectively.

In Refs. [25] and [27] the following assertions are proved.

Theorem B.1. *Let $S \in C^\infty$, $s \in \mathbb{R}$, $1 < p < +\infty$, and $1 \le t \le +\infty$. Then the operators*

$$\mathcal{N} : [H_p^{s+1}(S)]^3 \longrightarrow [H_p^{s+1}(S)]^3 \quad \Big[\,[B_{p,t}^{s+1}(S)]^3 \longrightarrow [B_{p,t}^{s+1}(S)]^3\,\Big], \tag{B.1}$$

$$\mathcal{M} : [H_p^{s+1}(S)]^3 \longrightarrow [H_p^{s}(S)]^3 \qquad \Big[\,[B_{p,t}^{s+1}(S)]^3 \longrightarrow [B_{p,t}^{s}(S)]^3\,\Big], \tag{B.2}$$

are invertible.

For a Lipschitz manifold S these operators are also invertible for $p = t = 2$ and $s = -1/2$ and, moreover, there are positive constants C_1 and C_2 such that the following inequality holds true

$$Re\,\big\langle -\mathcal{M}[\mathcal{N}]^{-1} g,\ \overline{g}\big\rangle_S \geqslant C_1\,\|g\|^2_{[H_2^{\frac{1}{2}}(S)]^3} - C_2\,\|g\|^2_{[H_2^0(S)]^3} \ \textit{for all } g \in [H_2^{\frac{1}{2}}(S)]^3.$$

Theorem B.2. *If $u \in [W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, $1 < p < +\infty$, solves the homogeneous equation $A(\partial,\omega)u(x) = 0$ in Ω^-, then u can be represented uniquely in the following two equivalent to each other forms*

$$u(x) = W(\mathcal{N}^{-1}g)(x) + \varkappa\, V(\mathcal{N}^{-1}g)(x), \quad x \in \Omega^-,$$
$$u(x) = W(\mathcal{M}^{-1}h)(x) + \varkappa\, V(\mathcal{M}^{-1}h)(x), \quad x \in \Omega^-,$$

where $\mathcal{N}^{-1}$ and $\mathcal{M}^{-1}$ are the operators inverse to $\mathcal{N}$ and $\mathcal{M}$ respectively defined in (B.1) and (B.2)*, while the densities g and h, are related to the vector u by the equalities*

$$g = \{u\}^-_S \in [B^{1-\frac{1}{p}}_{p,p}(S)]^3\,, \qquad h = \{Tu\}^-_S \in [B^{-\frac{1}{p}}_{p,p}(S)]^3\,.$$

In the case of a Lipschitz surface S, the same assertion holds true with $p = 2$.

Appendix C. Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary

Here we recall some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for proving existence theorems for mixed boundary, boundary-transmission, and crack problems by the potential methods. They can be found in [56–58].

Let $\overline{M} \in C^\infty$ be a compact, n-dimensional, nonintersecting manifold with boundary $\partial M \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{M}$. Denote by $\mathfrak{S}(\mathcal{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{M}$, $\xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \dots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\mathfrak{S}(\mathcal{A}; x, 0, \dots, 0, +1)\,]^{-1}[\,\mathfrak{S}(\mathcal{A}; x, 0, \dots, 0, -1)\,], \quad x \in \partial\overline{M},$$

and introduce the notation $\delta_j(x) = \operatorname{Re}\big[(2\pi\, i)^{-1} \ln \lambda_j(x)\big]$, $j = 1, \dots, N$, where $\ln \zeta$ denotes the branch of the logarithm function analytic in the complex plane cut along $(-\infty, 0]$. Due to the strong ellipticity of $\mathcal{A}$ we have the strong inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{M}$, $j = 1, 2, \dots, N$. The numbers $\delta_j(x)$ do not depend on a particular choice of the local coordinate system at a fixed pint $x \in \partial M$.

Remark C.1. Note that if $\mathfrak{S}(\mathcal{A}; x, \xi)$ is a positive definite matrix for every $x \in \overline{M}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\delta_j(x) = 0$ for $j = 1, \dots, N$, since all the eigenvalues $\lambda_j(x)$ ($j = 1, \dots, N$) are positive numbers for any $x \in \overline{M}$. The same holds if $\mathfrak{S}(\mathcal{A}; x, \xi)$ is representable in the form

$$\mathfrak{S}(\mathcal{A}; x, \xi) = Q^{(1)}(x, \xi)\, Q(x, \xi)\, Q^{(2)}(x, \xi),$$

where $Q(x, \xi) = \|Q_{kj}(x, \xi)\|_{N\times N}$ and $Q^{(m)}(x, \xi) = \|Q^{(m)}_{kj}(x, \xi)\|_{N\times N}$, $m = 1, 2$, are positive definite matrices and, in addition, the entries $Q^{(m)}_{kj}(x, \xi)$ are even functions in ξ.

The Fredholm properties of strongly elliptic pseudo-differential operators on manifolds with boundary are characterized by the following theorem.

Theorem C.2. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant C_0 such that $\operatorname{Re} \mathfrak{S}(\mathcal{A}; x, \xi)\, \eta \cdot \eta \ge C_0\, |\eta|^2$ for $x \in \overline{M}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.*

Then the operators

$$\mathcal{A} : \big[\widetilde{H}^s_p(M)\big]^N \longrightarrow \big[H^{s-\nu}_p(M)\big]^N \quad \Big[\big[\widetilde{B}^s_{p,t}(M)\big]^N \longrightarrow \big[B^{s-\nu}_{p,t}(M)\big]^N\Big], \tag{C.1}$$

are Fredholm with zero index if

$$\frac{1}{p} - 1 + \sup_{x\in\partial M,\, 1\le j\le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial M,\, 1\le j\le N} \delta_j(x). \tag{C.2}$$

Moreover, the null-spaces and indices of the operators (C.1) *are the same (for all values of the parameter $t \in [1, +\infty]$) provided p and s satisfy inequality* (C.2).

References

[1] V.D. Kupradze, On a method of solving approximately the limiting problems of mathematical physics, Z. Vychisl. Mat. i Mat. Fiz. 4 (1964) 1118–1121.
[2] V.D. Kupradze, Approximate solution of problems of mathematical physics, Uspkhi Mat. Nauk 22 (2) (1967) 59–67.
[3] A. Bogomolny, Fundamental solutions method for elliptic boundary value problems, SIAM J. Numer. Anal. 22 (4) (1985) 644–669.
[4] F. Browder, Approximation by solutions of partial differential equations, Amer. J. Math. 84 (1962) 134–160.
[5] W. Chen, F.Z. Wang, A method of fundamental solutions without fictitious boundary, Eng. Bound. Elem. 34 (2010) 530–532.
[6] W. Chen, J. Lin, F. Wang, Regularized meshless method for nonhomogeneous problems, Eng. Anal Bound. Elem. 35 (2011) 253–257.
[7] C.M. Fan, C.S. Chen, J. Monroe, The method of fundamental solutions for solving convection- diffusion equations with variable coefficients, Adv. Appl. Math. Mech. 1 (2009) 215–230.
[8] A. Karageorghis, D. Lesnic, Application of the MFS to inverse obstacle scattering problems, Eng. Anal. Bound. Elem. 35 (2011) 631–638.
[9] A. Karageorghis, D. Lesnic, L. Marin, The MFS for the detection of inner boundaries in linear elasticity, in: Boundary Elements and other Mesh Reduction Methods, vol. 33, 2011, pp. 229–239.
[10] T. Kitagava, On the numerical stability of the method of fundamental solution applied to the Dirichlet problem, Japan J. Appl. Math. 5 (1) (1988) 123–133.
[11] V.D. Kupradze, M.A. Aleksidze, The method of functional equations for the approximate solution of certain boundary value problems, USSR Comp. Meth. Math. Phys. 4 (4) (1964) 82–126.
[12] V. Kupradze, T. Gegelia, M. Basheleishvili, T. Burchuladze, Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, North-Holland, Amsterdam, 1979.
[13] L. Marin, D. Lesnic, The method of fundamental solutions for nonlinear functionally graded materials, Int. J. Solids Struct. 44 (2007) 6878–6890.
[14] R. Mathon, R.L. Johnston, The approximate solution of elliptic boundary value problems by fundamental solutions, SIAM J. Numer. Anal. 14 (4) (1977) 638–650.
[15] D. Natroshvili, Method of generalized series in anisotropic elasticity, Proc. I. Vekua Inst. Appl. Math. Tbilisi State Univ. 39 (1990) 110–132.
[16] Y.-S. Smyrlis, The method of fundamental solutions: a weighted least-squares approach, BIT 46 (1) (2006) 163–194.
[17] Y.-S. Smyrlis, Applicability and applications of the method of fundamental solutions, Math. Comp. 78 (267) (2009) 1399–1434.
[18] T. Ushijima, F. Chiba, Error estimates for a fundamental solution method applied to reduce wave problems in a domain exterior to a disc, J. Comput. Appl. Math. 159 (1) (2003) 137–148.
[19] M. Fleck, R. Grzhibovskis, S. Rjasanow, A new fundamental solution method based on the adaptive cross approximation, Mem. Differential Equations Math. Phys. 65 (2015) 93–111.
[20] I.N. Vekua, On metaharmonic functions, Proc. Tbilisi Mathem. Inst. of Acad. Sci. Georgian SSR 12 (1943) 105–174.
[21] J.-L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol. I (Translated from the French), in: Die Grundlehren der mathematischen Wissenschaften, Band, vol. 181, Springer-Verlag, New York–Heidelberg, 1972.
[22] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North Holland, Amsterdam, 1978.
[23] T. Buchukuri, O. Chkadua, D. Natroshvili, Mathematical problems of generalized thermo-electro-magneto-elasticity theory, Mem. Diff. Equations Math. Phys. 68 (2016) 1–166.
[24] T. Buchukuri, O. Chkadua, R. Duduchava, D. Natroshvili, Interface crack problems for metallic-piezoelectric composite structures, Mem. Differential Equations Math. Phys. 55 (2012) 1–150.
[25] C.E. Athanasiadis, D. Natroshvili, V. Sevroglou, I.G. Stratis, A boundary integral equations approach for mixed impedance problems in elasticity, J. Integral Equ. Appl. 23 (2011) 183–222.
[26] C.E. Athanasiadis, D. Natroshvili, V. Sevroglou, I.G. Stratis, Mixed impedance transmission problems for vibrating layered elastic bodies, Math. Methods Appl. Sci. 38 (2015) 3264–3294.
[27] D. Natroshvili, Boundary integral equation method in the steady state oscillation problems for anisotropic bodies, Math. Methods Appl. Sci. 20 (2) (1997) 95–119.
[28] F. Rellich, Über das asymptotische Verhalten der Lösungen von $(\Delta + k^2)u = 0$ in anendlichen Gebieten, Jber. Deutsch. Math. Verein 53 (1943) 57–65.
[29] H. Brakhage, P. Werner, Uber das Dirichletsche Auβenraumproblem für die Helmholtzsche Schwingungsgleichung, Arch. Math. 16 (1965) 325–329.
[30] R. Leis, Zur Dirichletschen Randwertaufgabe des Aussenraums der Schwingungsgleichung. Mat. Z. 90 (1965) 205–211.
[31] O.I. Panich, On solvability of external boundary value problems for wave equation and Maxwell system, Usp. Mat. Nauk 20 (1965) 221–226.
[32] S. Mikhlin, S. Prössdorf, Singular Integral Operators, Springer-Verlag, Berlin, 1986.
[33] O. Chkadua, R. Duduchava, Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotic, Math. Nachr. 222 (2001) 79–139.
[34] O. Chkadua, R. Duduchava, Asymptotic of functions represented by potentials, Russ. J. Math. Phys. 7 (1) (2000) 15–47.
[35] R. Duduchava, D. Natroshvili, E. Shargorodsky, Boundary value problems of the mathematical theory of cracks, Proc. I. Vekua Inst. Appl. Math. Tbilisi State University 39 (1990) 68–84.
[36] D. Natroshvili, O. Chkadua, E. Shargorodsky, Mixed boundary value problems of the anisotropic elasticity, Proc. I. Vekua Inst. Appl. Math. Tbilisi State Univ. 39 (1990) 133–181.
[37] R.M. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New York–London, 1980.
[38] A. Karageorghis, Efficient MFS algorithms for inhomogeneous polyharmonic problems, J. Sci. Comput. 46 (2011) 519–541.
[39] A. Karageorghis, D. Lesnic, L. Marin, A survey of applications of the MFS to inverse problems, Inverse Probl. Sci. Eng. 19 (3) (2011) 309–336.

[40] M.S. Agranovich, Spectral properties of potential type operators for a class of strongly elliptic systems on smooth and Lipschitz surfaces, Trans. Moscow Math. Soc. 62 (2001) 1–47.
[41] M.S. Agranovich, Spectral problems for second-order strongly elliptic systems in smooth and non-smooth domains, Russian Math. Surveys 57 (2002) 847–920.
[42] M. Costabel, W.L. Wendland, Strong ellipticity of boundary integral operators, J. Reine Angew. Math. 372 (1986) 34–63.
[43] G.C. Hsiao, W.L. Wendland, Boundary Integral Equations, Applied Mathematical Sciences, Springer-Verlag, Berlin-Heidelberg, 2008.
[44] L. Jentsch, D. Natroshvili, W.L. Wendland, General transmission problems in the theory of elastic oscillations of anisotropic bodies (Basic interface problems), J. Math. Anal. Appl. 220 (1998) 397–433.
[45] L. Jentsch, D. Natroshvili, W.L. Wendland, General transmission problems in the theory of elastic oscillations of anisotropic bodies (Mixed interface problems), J. Math. Anal. Appl. 235 (1999) 418–434.
[46] M. Costabel, Boundary integral operators on Lipschitz domains: elementary results, SIAM J. Math. Anal. 19 (1988) 613–626.
[47] R. Duduchava, The Green formula and layer potentials, Integral Equations Operator Theory 41 (2) (2001) 127–178.
[48] R. Duduchava, D. Natroshvili, E. Shargorodsky, Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts, I, Georgian Math. J. 2 (2) (1995) 123–140.
[49] R. Duduchava, D. Natroshvili, E. Shargorodsky, Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts, II, Georgian Math. J. 2 (3) (1995) 259–276.
[50] W. Gao, Layer potentials and boundary value problems for elliptic systems in Lipschitz domains, J. Funct. Anal. 95 (1991) 377–399.
[51] L. Jentsch, D. Natroshvili, Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies, Part I, Mem. Differential Equations Math. Phys. 17 (1999) 7–126.
[52] L. Jentsch, D. Natroshvili, Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies, Part II, Mem. Differential Equations Math. Phys. 18 (1999) 1–50.
[53] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations, Cambridge University Press, Cambridge, 2000.
[54] D. Mitrea, M. Mitrea, J. Pipher, Vector potential theory on non-smooth domains in R^3 and applications to electromagnetic scattering, J. Fourier Anal. Appl. 3 (1997) 1419–1448.
[55] R.T. Seeley, Singular integrals and boundary value problems, Amer. J. Math. 88 (1966) 781–809.
[56] G. Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations, in: Transl. of Mathem. Monographs, 52, Amer. Math. Soc., Providence, Rhode Island, 1981.
[57] G. Grubb, Pseudodifferential boundary problems in L_p spaces, Comm. Partial Differential Equations 15 (1990) 289–340.
[58] E. Shargorodsky, An analogue of the Vishik-Eskin theory, Mem. Differential Equations Math. Phys. 2 (1994) 41–146.

Further Reading

[1] L. Boutet de Monvel, Boundary problems for pseudo-differential operators, Acta Math. 126 (1971) 11–51.
[2] D. Natroshvili, Mathematical Problems of Thermo-Electro-Magneto-Elasticity, in: Lecture Notes of TICMI, vol. 12, Tbilisi University Press, Tbilisi, 2011.
[3] D. Natroshvili, T. Buchukuri, O. Chkadua, Mathematical modelling and analysis of interaction problems for piezoelectric composites, in: Rendiconti Academia Nazionale delle Scienze detta dei XL, Memorie di Matematica e Applicazioni 124, Vol. XXX, in: Fasc., vol. 1, 2006, pp. 159–190.
[4] D. Natroshvili, I. Stratis, S. Zazashvili, Interface crack problems for metallic-piezoelectric composite structures, Math. Methods Appl. Sci. 33 (4) (2010) 539–562.
[5] D. Natroshvili, Z. Tediashvili, Crack problems for composite structures, Oper. Theory Adv. Appl. 193 (2009) 227–243.

Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 171 (2017) 293–303

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

New generalizations of Popoviciu type inequalities via new green functions and Fink's identity

Saad Ihsan Butt[a,*], Nasir Mehmood[a], Josip Pečarić[b]

[a] *Department of Mathematics, COMSATS, Institute of Information Technology, Lahore, Pakistan*
[b] *Faculty of Textile Technology, University of Zagreb, 10000 Zagreb, Croatia*

Received 12 January 2017; received in revised form 28 March 2017; accepted 25 April 2017
Available online 17 May 2017

Abstract

We formulate new identities involving new Green functions. Inequality of Popoviciu, which was improved by Vasić and Stanković (1976), is generalized by using newly introduced Green functions. We utilize Fink's identity along with new Green's function to generalize the known Popoviciu's inequality from convex functions to higher order convex functions. Then we construct linear functionals from the generalized identities and formulate the monotonicity of these functionals utilizing the recent theory of inequalities for n-convex functions at a point. New upper bounds of Grüss and Ostrowski type are computed.

Keywords: Popoviciu inequality; Fink's identity; Abel–Gontscharoff interpolating polynomial; New Green functions; Grüss upper bounds; Ostrowski type bounds

1. Introduction and preliminary results

Systematic study of convex functions started over the period 1905–1906 by thought provoking ideas and fascinating work of Jensen. However there also exists some literature about convex functions even before Jensen because one may find the existence of the roots of their definition in the work of O. Hölder (1889) and J. Hadamard (1893). The study of convex functions is used as a major tool to solve optimization problems in analysis. However the impact of inequalities involving convex functions is magical as it solves many problems in different branches of mathematics with considerable high rate. That is why the study of such inequalities has been given great importance in literature.

Higher order convexity was introduced by Popoviciu, who defined it under the context of divided differences of a function (see Ch.1, [1]). Inequalities of higher order convex functions are very important and many physicists used it while dealing in higher dimensions. It is interesting to note that results for convex functions may not be true for convex

* Corresponding author.
E-mail address: saadihsanbutt@gmail.com (S.I. Butt).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.04.003
2346-8092/

functions of higher order. There are remarkable changes in the results, which forces to think about the existence of such results. S. I. Butt and J. Pečarić pay tribute to Professor T. Popoviciu in their book [2] in 2015 at the 50th years to Popoviciu's inequality. They generalize Popoviciu's inequality for higher order convex functions and give its applications. Also in 2015 [3], a new class of n-convex functions at a point is introduced by J. Pečarić, M. Praljak and A. Witkowski. They developed a remarkable theory to investigate linear operator inequalities with the help of the functions, which are n-convex at a point. This theory leads to many interesting and fascinating results with lot of applications in operator theory and statistics.

A characterization of convex function established by T. Popoviciu [4] is studied by many people (see [1,5] and references within). In recent years the inequality of Popoviciu is studied in [6–9]. The following form of Popoviciu's inequality is by Vasić and Stanković in [5] (see page 173 [1]):

Theorem 1.1. *Let $[\delta_1, \delta_2]$ be interval in $\mathbb{R}$, for integers $s \geq 3$, $2 \leq m \leq s-1$, consider the tuples $\mathbf{z} = (z_1, \ldots, z_s) \in [\delta_1, \delta_2]^s$, $\mathbf{q} = (q_1, \ldots, q_s)$ be a positive s-tuple along with the condition that $\sum_{i=1}^{s} q_i = 1$. Then for $\psi : [\delta_1, \delta_2] \to \mathbb{R}$ being convex function*

$$\psi_{m,s}(\mathbf{z}, \mathbf{q}) \leq \frac{s-m}{s-1}\psi_{1,s}(\mathbf{z}, \mathbf{q}) + \frac{m-1}{s-1}\psi_{s,s}(\mathbf{z}, \mathbf{q}) \tag{1}$$

holds, where

$$\psi_{m,s}(\mathbf{z}, \mathbf{q}) := \frac{1}{C_{m-1}^{s-1}} \sum_{1 \leq i_1 < \cdots < i_m \leq s} \left(\sum_{j=1}^{m} q_{i_j} \right) \psi \left(\frac{\sum_{j=1}^{m} q_{i_j} z_{i_j}}{\sum_{j=1}^{m} q_{i_j}} \right)$$

is the linear functional with respect to ψ.

By inequality (1), we write

$$\mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi] := \frac{s-m}{s-1}\psi_{1,s}(\mathbf{z}, \mathbf{q}) + \frac{m-1}{s-1}\psi_{s,s}(\mathbf{z}, \mathbf{q}) - \psi_{m,s}(\mathbf{z}, \mathbf{q}). \tag{2}$$

Remark 1.2. Under the assumptions of Theorem 1.1, $\mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi] \geq 0$ for ψ being convex function and zero for constant and linear function.

In the current paper, we need the following results of our interest. For $\psi^{(n-1)}$ to be absolutely continuous on $[\delta_1, \delta_2] \subset \mathbb{R}$, A. M. Fink in [10] proved the following famous identity:

$$\begin{aligned}\psi(z) = {} & \frac{n}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \psi(\xi)\, d\xi + \sum_{\zeta=1}^{n-1} \left(\frac{n-\zeta}{\zeta!} \right) \left(\frac{\psi^{(\zeta-1)}(\delta_2)(z-\delta_2)^{\zeta} - \psi^{(\zeta-1)}(\delta_1)(z-\delta_1)^{\zeta}}{\delta_2 - \delta_1} \right) \\ & + \frac{1}{(n-1)!\,(\delta_2 - \delta_1)} \int_{\delta_1}^{\delta_2} (z-\xi)^{n-1} F_{\delta_1}^{\delta_2}(\xi, z)\, \psi^{(n)}(\xi)\, d\xi,\end{aligned} \tag{3}$$

where

$$F_{\delta_1}^{\delta_2}(\xi, z) = \begin{cases} \xi - \delta_1, & \xi \leq z \leq \delta_2, \\ \xi - \delta_2, & z < \xi \leq \delta_2. \end{cases} \tag{4}$$

The complete reference about Abel–Gontscharoff polynomial and theorem for 'two-point right focal' problem is given in [11]. As a special choice the Abel–Gontscharoff polynomial for 'two-point right focal' interpolating polynomial for $n = 2$ can be given as:

$$\psi(z) = \psi(\delta_1) + (z - \delta_1)\psi'(\delta_2) + \int_{\delta_1}^{\delta_2} G_{\Lambda,2}(z, w)\psi''(w)dw, \tag{5}$$

where $G_{\Lambda,2}(z, w)$ is the Green function for 'two-point right focal problem' given as

$$G_1(z, w) = G_{\Lambda,2}(z, w) = \begin{cases} \delta_1 - w, & \delta_1 \leq w \leq z, \\ \delta_1 - z, & z \leq w \leq \delta_2. \end{cases} \tag{6}$$

In the next section, we will present our main results by introducing some new types of Green functions.

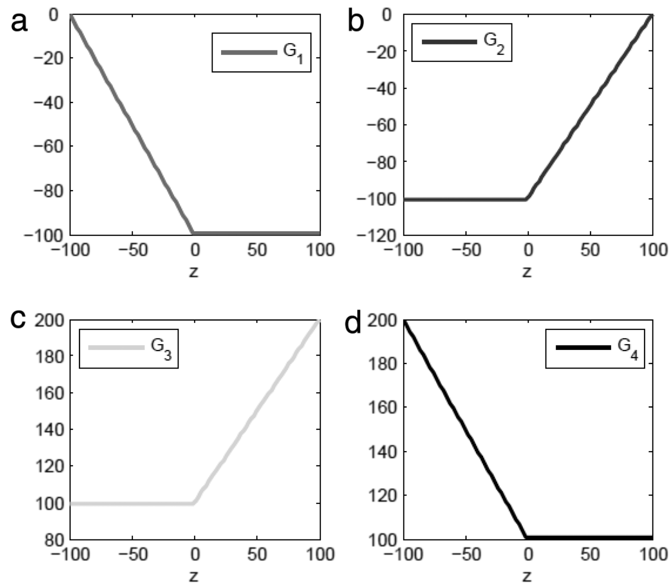

Fig. 1. Graph of Green functions for fix w.

2. New generalizations of Popoviciu's inequality

We start this section by our nice observation to Abel–Gontscharoff identity (5) and the related Green's function for 'two-point right focal problem'. Therefore keeping in view Abel–Gontscharoff Green's function for 'two-point right focal problem' we would like to introduce, some new types of Green functions $G_k : [\delta_1, \delta_2] \times [\delta_1, \delta_2] \to \mathbb{R}$, $(k = 2, 3, 4,)$ defined as:

$$G_2(z, w) = \begin{cases} z - \delta_2, & \delta_1 \le w \le z, \\ w - \delta_2, & z \le w \le \delta_2. \end{cases} \tag{7}$$

$$G_3(z, w) = \begin{cases} z - \delta_1, & \delta_1 \le w \le z, \\ w - \delta_1, & z \le w \le \delta_2. \end{cases} \tag{8}$$

$$G_4(z, w) = \begin{cases} \delta_2 - w, & \delta_1 \le w \le z, \\ \delta_2 - z, & z \le w \le \delta_2. \end{cases} \tag{9}$$

The graphical representations of $G_k, \ \ k = 1, 2, 3, 4$, are depicted in Fig. 1 which shows that all four Green functions are continuous and symmetric. Moreover, all functions are convex with respect to the both variables z and w. These new Green functions enable us to introduce some new identities, stated in the form of following lemma:

Lemma 2.1. *Let $\psi : [\delta_1, \delta_2] \to \mathbb{R}$ be a twice differentiable function and $G_k, \ \ (k = 1, 2, 3, 4)$ be the new Green functions defined above then along with* (5) *the following identities hold:*

$$\psi(z) = \psi(\delta_2) + (\delta_2 - z)\psi'(\delta_1) + \int_{\delta_1}^{\delta_2} G_2(z, w)\psi''(w)dw, \tag{10}$$

$$\psi(z) = \psi(\delta_2) - (\delta_2 - \delta_1)\psi'(\delta_2) + (z - \delta_1)\psi'(\delta_1) + \int_{\delta_1}^{\delta_2} G_3(z, w)\psi''(w)dw, \tag{11}$$

$$\psi(z) = \psi(\delta_1) + (\delta_2 - \delta_1)\psi'(\delta_1) - (\delta_2 - z)\psi'(\delta_2) + \int_{\delta_1}^{\delta_2} G_4(z, w)\psi''(w)dw. \tag{12}$$

Proof. We can give the proofs of above identities by following same integrating scheme, therefore we would like to give the proof of (12) only:

As

$$\begin{aligned}
\int_{\delta_1}^{\delta_2} G_4(z,w)\psi''(w)dw &= \int_{\delta_1}^{z} G_4(z,w)\psi''(w)dw + \int_{z}^{\delta_2} G_4(z,w)\psi''(w)dw \\
&= \int_{\delta_1}^{z} (\delta_2 - w)\psi''(w)dw + \int_{z}^{\delta_2} (\delta_2 - z)\psi''(w)dw \\
&= (\delta_2 - w)\psi'(w)|_{\delta_1}^{z} - \int_{\delta_1}^{z} -1.\psi'(w)dw + (\delta_2 - z)[\psi'(\delta_2) - \psi'(z)] \\
&= (\delta_2 - z)\psi'(z) - (\delta_2 - \delta_1)\psi'(\delta_1) + \psi(z) - \psi(\delta_1) + (\delta_2 - z)\psi'(\delta_2) - (\delta_2 - z)\psi'(z) \\
&= (\delta_2 - z)\psi'(\delta_2) - (\delta_2 - \delta_1)\psi'(\delta_1) - \psi(\delta_1) + \psi(z).
\end{aligned}$$

Now by simplifying terms, we will get our identity (12). □

Remark 2.2. Lemma 2.1 gives another proof of special case of Abel–Gontscharoff identity (5). G_3 and G_4 are new Green functions but results are not so simple as in other two cases.

Next we formulate generalized identities with the help of identities defined in Lemma 2.1 and Fink's identity:

Theorem 2.3. *Let $\psi : [\delta_1, \delta_2] \to \mathbb{R}$ be such that for $n \geq 3$, $\psi \in C^n[\delta_1, \delta_2]$ such that $\psi^{(n-1)}$ is absolutely continuous and let $s, m \in \mathbb{N}$, $s \geq 3$, $2 \leq m \leq s-1$, $\mathbf{z} \in [\delta_1, \delta_2]^m$, $\mathbf{q}$ be a real s-tuple such that $\sum_{j=1}^{m} q_{i_j} \neq 0$ for any $1 \leq i_1 < \cdots < i_m \leq s$ and $\sum_{i=1}^{s} q_i = 1$. Also let $\frac{\sum_{j=1}^{m} q_{i_j} z_{i_j}}{\sum_{j=1}^{m} q_{i_j}} \in [\delta_1, \delta_2]$ for any $1 \leq i_1 < \cdots < i_m \leq s$ with $F_{\delta_1}^{\delta_2}(\xi, \cdot)$ and $G_k(\cdot, w)$, $(k = 1, 2, 3, 4)$ be the same as defined in (4) and Lemma 2.1 respectively. Then we have the following new identities for $k = 1, 2, 3, 4$*

$$\begin{aligned}
\mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi(z)] &= (n-2)\left(\frac{\psi^{(1)}(\delta_2) - \psi^{(1)}(\delta_1)}{\delta_2 - \delta_1}\right)\int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z,w)]dw \\
&+ \frac{1}{(\delta_2 - \delta_1)} \int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z,w)] \\
&\times \left(\sum_{\zeta=1}^{n-3} \left(\frac{n-2-\zeta}{\zeta!}\right)\left(\psi^{(\zeta+1)}(\delta_2)(w-\delta_2)^{\zeta} - \psi^{(\zeta+1)}(\delta_1)(w-\delta_1)^{\zeta}\right)\right)dw \\
&+ \frac{1}{(n-3)!(\delta_2 - \delta_1)} \int_{\delta_1}^{\delta_2} \psi^{(n)}(\xi)\left(\int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z,w)](w-\xi)^{n-3} F_{\delta_1}^{\delta_2}(\xi, w)\,dw\right)d\xi. \qquad (13)
\end{aligned}$$

Proof. Fix $k = 1, 2, 3, 4$. Applying Popoviciu's functional (2) to identities (5), (10), (11), (12) along with their respective new Green functions and following properties of $\mathbb{POP}[\mathbf{z}, \mathbf{q}; \cdot]$, we get

$$\mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi] = \int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z,w)]\psi''(w)dw. \qquad (14)$$

Differentiating (3), twice with respect to variable w, we get

$$\begin{aligned}
\psi''(w) &= \sum_{\zeta=0}^{n-3} \left(\frac{n-2-\zeta}{\zeta!}\right)\left(\frac{\phi^{(\zeta+1)}(\delta_2)(w-\delta_2)^{\zeta} - \phi^{(\zeta+1)}(\delta_1)(w-\delta_1)^{\zeta}}{\delta_2 - \delta_1}\right) \\
&+ \frac{1}{(n-3)!(\delta_2 - \delta_1)} \int_{\delta_1}^{\delta_2} (w-\xi)^{n-3} F_{\delta_1}^{\delta_2}(\xi, w)\,\phi^{(n)}(\xi)\,d\xi \\
&= \sum_{\zeta=1}^{n-2} \left(\frac{n-1-\zeta}{(\zeta-1)!}\right)\left(\frac{\phi^{(\zeta)}(\delta_2)(w-\delta_2)^{\zeta-1} - \phi^{(\zeta)}(\delta_1)(w-\delta_1)^{\zeta-1}}{\delta_2 - \delta_1}\right) \\
&+ \frac{1}{(n-3)!(\delta_2 - \delta_1)} \int_{\delta_1}^{\delta_2} (w-\xi)^{n-3} F_{\delta_1}^{\delta_2}(\xi, w)\,\phi^{(n)}(\xi)\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&= (n-2)\left(\frac{\psi^{(1)}(\delta_2)-\psi^{(1)}(\delta_1)}{\delta_2-\delta_1}\right) \\
&\quad + \sum_{\zeta=2}^{n-2}\left(\frac{n-1-\zeta}{(\zeta-1)!}\right)\left(\frac{\phi^{(\zeta)}(\delta_2)(w-\delta_2)^{\zeta-1}-\phi^{(\zeta)}(\delta_1)(w-\delta_1)^{\zeta-1}}{\delta_2-\delta_1}\right) \\
&\quad + \frac{1}{(n-3)!(\delta_2-\delta_1)}\int_{\delta_1}^{\delta_2}(w-\xi)^{n-3}F_{\delta_1}^{\delta_2}(\xi,w)\,\phi^{(n)}(\xi)\,d\xi. \qquad (15)
\end{aligned}
$$

Substituting (15) in (14) and executing Fubini's Theorem in the obtained terms we get (13) respectively for $k = 1, 2, 3, 4$.

On the other hand, we rewrite (3) considering function ψ'' and replacing n by $n-2$ ($n \geq 3$), to get

$$
\begin{aligned}
\psi''(w) &= (n-2)\left(\frac{\psi^{(1)}(\delta_2)-\psi^{(1)}(\delta_1)}{\delta_2-\delta_1}\right) \\
&\quad + \sum_{\zeta=1}^{n-3}\left(\frac{n-1-\zeta}{\zeta!}\right)\left(\frac{\phi^{(\zeta+1)}(\delta_2)(w-\delta_2)^{\zeta}-\phi^{(\zeta+1)}(\delta_1)(w-\delta_1)^{\zeta}}{\delta_2-\delta_1}\right) \\
&\quad + \frac{1}{(n-3)!(\delta_2-\delta_1)}\int_{\delta_1}^{\delta_2}(w-\xi)^{n-3}F_{\delta_1}^{\delta_2}(\xi,w)\,\phi^{(n)}(\xi)\,d\xi \\
&= (n-2)\left(\frac{\psi^{(1)}(\delta_2)-\psi^{(1)}(\delta_1)}{\delta_2-\delta_1}\right) \\
&\quad + \sum_{\zeta=2}^{n-2}\left(\frac{n-1-\zeta}{(\zeta-1)!}\right)\left(\frac{\phi^{(\zeta)}(\delta_2)(w-\delta_2)^{\zeta-1}-\phi^{(\zeta)}(\delta_1)(w-\delta_1)^{\zeta-1}}{\delta_2-\delta_1}\right) \\
&\quad + \frac{1}{(n-3)!(\delta_2-\delta_1)}\int_{\delta_1}^{\delta_2}(w-\xi)^{n-3}F_{\delta_1}^{\delta_2}(\xi,w)\,\phi^{(n)}(\xi)\,d\xi. \qquad (16)
\end{aligned}
$$

Now employing Fubini's Theorem in the last term obtained by putting (16) in (14), we get (13) respectively for $k = 1, 2, 3, 4$. □

The next theorem gives artistic generalization of Popoviciu's type inequalities for n-convex functions involving new Green functions.

Theorem 2.4. *Assuming the conditions of Theorem* 2.3 *be true, let for* $n \geq 3$

$$
\int_{\delta_1}^{\delta_2}\mathbb{POP}[\mathbf{z},\mathbf{q};G_k(z,w)](w-\xi)^{n-3}F_{\delta_1}^{\delta_2}(\xi,w)\,dw \geq 0, \quad \xi \in [\delta_1,\delta_2], \qquad (17)
$$

for all $k = 1, 2, 3, 4$. *If* ψ *is* n*-convex function such that* $\psi^{(n-1)}$ *is absolutely continuous, then for* $k = 1, 2, 3, 4$,

$$
\begin{aligned}
\mathbb{POP}[\mathbf{z},\mathbf{q};\psi(z)] &\geq (n-2)\left(\frac{\psi^{(1)}(\delta_2)-\psi^{(1)}(\delta_1)}{\delta_2-\delta_1}\right)\int_{\delta_1}^{\delta_2}\mathbb{POP}[\mathbf{z},\mathbf{q};G_k(z,w)]dw \\
&\quad + \frac{1}{(\delta_2-\delta_1)}\int_{\delta_1}^{\delta_2}\mathbb{POP}[\mathbf{z},\mathbf{q};G_k(z,w)] \\
&\quad \times\left(\sum_{\zeta=1}^{n-3}\left(\frac{n-2-\zeta}{\zeta!}\right)\left(\psi^{(\zeta+1)}(\delta_2)(w-\delta_2)^{\zeta}-\psi^{(\zeta+1)}(\delta_1)(w-\delta_1)^{\zeta}\right)\right)dw. \qquad (18)
\end{aligned}
$$

Proof. Fix $k = 1, 2, 3, 4$. Since $\psi^{(n-1)}$ is absolutely continuous on $[\delta_1, \delta_2]$, $\psi^{(n)}$ exists almost everywhere. As ψ is n-convex, so $\psi^{(n)}(z) \geq 0$ for almost everywhere on $[\delta_1, \delta_2]$ (see [1], p. 16). Hence we can apply Theorem 2.3 to obtain (18). □

Now we state the final result of this section in the form of following theorem:

Theorem 2.5. *Let in addition to the assumptions of Theorem* 2.3*,* $\mathbf{q} = (q_1, \ldots, q_s)$ *be a positive* s*-tuple such that* $\sum_{i=1}^{s} q_i = 1$*, and* $\psi : [\delta_1, \delta_2] \to \mathbb{R}$ *be an* n*-convex function.*

(i) *For fix* $k = 1, 2, 3, 4$*, inequality* (18) *holds provided that* n *is even and greater than 3.*

(ii) *Let* (18) *be satisfied for all fix* $k = 1, 2, 3, 4$ *and*

$$\sum_{\zeta=0}^{n-3} \left(\frac{n-2-\zeta}{\zeta!} \right) \left(\psi^{(\zeta+1)} (\delta_2) (w - \delta_2)^{\zeta} - \psi^{(\zeta+1)} (\delta_1) (w - \delta_1)^{\zeta} \right) \geq 0. \tag{19}$$

Then we have

$$\mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi(z)] \geq 0. \tag{20}$$

Proof.

(i) It is clear from Fig. 1 that Green's function $G_k(z, w)$ are convex for all $k = 1, 2, 3, 4$ and the weights are assumed to be positive. Therefore applying Theorem 1.1 and taking into account Remark 1.2, we can obtain $\mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)] \geq 0$ for all $k = 1, 2, 3, 4$. Moreover, the function σ

$$\sigma(w) := (w - \xi)^{n-3} F_{\delta_1}^{\delta_2} (\xi, w) = \begin{cases} (w - \xi)^{n-3} (\xi - \delta_1), & \delta_1 \leq \xi \leq w \leq \delta_2, \\ (w - \xi)^{n-3} (\xi - \delta_2), & \delta_1 \leq w < \xi \leq \delta_2, \end{cases}$$

is positive for even $n > 3$, as a result (17) is established. Now employing Theorem 2.4, (18) is established.

(ii) Putting (19) in (18), we get (20) for all $k = 1, 2, 3, 4$. □

3. Related inequalities for $(n + 1)$-convex functions at a point

In the present section we will give related results for the class of $(n + 1)$-convex functions at a point introduced in [3].

Definition 1. Let $I \subseteq \mathbb{R}$ be an interval, $c \in I^o$ and $n \in \mathbb{N}$. A function $\psi : I \to \mathbb{R}$ is said to be $(n+1)$-convex at point c if there exists a constant Z_c such that the function

$$\Psi(z) = \psi(z) - \frac{Z_c}{n!} z^n \tag{21}$$

is n-concave on $I \cap (-\infty, c]$ and n-convex on $I \cap [c, \infty)$. A function ψ is said to be $(n + 1)$-concave at point c if the function $-\psi$ is $(n + 1)$-convex at point c.

A function is $(n + 1)$-convex on an interval if and only if it is $(n + 1)$-convex at every point of the interval (see [3]). Pečarić, Praljak and Witkowski in [3] study necessary and sufficient conditions on two linear functionals $\Omega : C([\delta_1, c]) \to \mathbb{R}$ and $\Lambda : C([c, \delta_2]) \to \mathbb{R}$ so that the inequality $\Omega(\psi) \leq \Lambda(\psi)$ holds for every function ψ that is $(n + 1)$-convex at point c. In the present section we will give inequalities of such type for the particular linear functionals obtained from the inequalities in the previous section. Let σ_i denote the monomials $\sigma_i(z) = z^i$, $i \in \mathbb{N}$. For the rest of this section, $\Omega_k(\psi)$ and $\Lambda_k(\psi)$ where $k = 1, 2, 3, 4$, will denote the linear functionals obtained as the difference of the L. H. S. and R. H. S. of inequality (18), applied to the intervals $[\delta_1, c]$ and $[c, \delta_2]$ respectively, i.e., for $\mathbf{z} \in [\delta_1, c]^s$, $\mathbf{q} \in \mathbb{R}^s$, $\mathbf{y} \in [c, \delta_2]^{\bar{s}}$ and $\bar{\mathbf{q}} \in \mathbb{R}^{\bar{s}}$ let

$$\begin{aligned} \Omega_k(\psi) := \mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi(z)] &- (n-2) \left(\frac{\psi^{(1)}(c) - \psi^{(1)}(\delta_1)}{c - \delta_1} \right) \int_{\delta_1}^{c} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)] dw \\ &- \frac{1}{(c - \delta_1)} \int_{\delta_1}^{c} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)] \\ &\times \left(\sum_{\zeta=1}^{n-3} \left(\frac{n-2-\zeta}{\zeta!} \right) \left(\psi^{(\zeta+1)}(c) (w - c)^{\zeta} - \psi^{(\zeta+1)}(\delta_1) (w - \delta_1)^{\zeta} \right) \right) dw, \end{aligned} \tag{22}$$

$$\Lambda_k(\psi) := \mathbb{POP}[\mathbf{y}, \bar{\mathbf{q}}; \psi(y)] - (n-2)\left(\frac{\psi^{(1)}(\delta_2) - \psi^{(1)}(c)}{\delta_2 - c}\right)\int_c^{\delta_2} \mathbb{POP}[\mathbf{y}, \bar{\mathbf{q}}; G_k(y, w)]dw$$
$$- \frac{1}{(\delta_2 - c)}\int_c^{\delta_2} \mathbb{POP}[\mathbf{y}, \bar{\mathbf{q}}; G_k(y, w)]$$
$$\times \left(\sum_{\zeta=1}^{n-3}\left(\frac{n-2-\zeta}{\zeta!}\right)\left(\psi^{(\zeta+1)}(\delta_2)(w-\delta_2)^{\zeta} - \psi^{(\zeta+1)}(c)(w-c)^{\zeta}\right)\right)dw. \tag{23}$$

It is important to notify that by introducing new linear functionals for $k = 1, 2, 3, 4$, $\Omega_k(\psi)$ and $\Lambda_k(\psi)$, identity (13) applied to the respective intervals $[\delta_1, c]$ and $[c, \delta_2]$ takes the shape:

$$\Omega_k(\psi) = \frac{1}{(n-3)!\,(c-\delta_1)}\int_{\delta_1}^{c} \psi^{(n)}(\xi)\left(\int_{\delta_1}^{c} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)](w-\xi)^{n-3}F_{\delta_1}^{c}(\xi, w)\,dw\right)d\xi, \tag{24}$$

$$\Lambda_k(\psi) = \frac{1}{(n-3)!\,(\delta_2 - c)}\int_{c}^{\delta_2} \psi^{(n)}(\xi)\left(\int_{c}^{\delta_2} \mathbb{POP}[\mathbf{y}, \bar{\mathbf{q}}; G_k(y, w)](w-\xi)^{n-3}F_{c}^{\delta_2}(\xi, w)\,dw\right)d\xi. \tag{25}$$

Now we are ready to state the following theorem for inequalities involving $(n+1)$-convex function at a point:

Theorem 3.1. *Let $\mathbf{z} \in [\delta_1, c]^s$, $\mathbf{q} \in \mathbb{R}^s$, $\mathbf{y} \in [c, \delta_2]^{\bar{s}}$ and $\bar{\mathbf{q}} \in \mathbb{R}^{\bar{s}}$ in such a way that for $k = 1, 2, 3, 4$*

$$\int_{\delta_1}^{c} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)](w-\xi)^{n-3}F_{\delta_1}^{c}(\xi, w)\,dw \geq 0, \quad \xi \in [\delta_1, c], \tag{26}$$

$$\int_{c}^{\delta_2} \mathbb{POP}[\mathbf{y}, \bar{\mathbf{q}}; G_k(y, w)](w-\xi)^{n-3}F_{c}^{\delta_2}(\xi, w)\,dw \geq 0, \quad \xi \in [c, \delta_2], \tag{27}$$

$$\int_{\delta_1}^{c}\left(\int_{\delta_1}^{c} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)](w-\xi)^{n-3}F_{\delta_1}^{c}(\xi, w)\,dw\right)d\xi = \left(\frac{c-\delta_1}{\delta_2 - c}\right)$$
$$\times \int_{c}^{\delta_2}\left(\int_{c}^{\delta_2} \mathbb{POP}[\mathbf{y}, \bar{\mathbf{q}}; G_k(y, w)](w-\xi)^{n-3}F_{c}^{\delta_2}(\xi, w)\,dw\right)d\xi, \tag{28}$$

where $F_{\delta_1}^{\delta_2}(\xi, \cdot)$, $G_k(\cdot, w)$, $(k = 1, 2, 3, 4)$ be the same as defined in (4) *and Lemma* 2.1 *respectively, and let $\Omega_k(\psi)$, $\Lambda_k(\psi)$ be the linear functionals given by* (22) and (23)*. If $\psi : [\delta_1, \delta_2] \to \mathbb{R}$ is $(n+1)$-convex at point c, then we get the monotonicity*

$$\Omega_k(\psi) \leq \Lambda_k(\psi), \quad \textit{for} \;\; k = 1, 2, 3, 4. \tag{29}$$

If the inequalities in (26) and (27) *are reversed, then* (29) *holds with the reversed sign of inequality.*

Proof. Using Definition 1, construct function $\Psi(z) = \psi(z) - \frac{Z_c}{n!}\sigma_n$ is such a way that the function Ψ is n-concave on $[\delta_1, c]$ and n-convex on $[c, \delta_2]$. Fix $k = 1, 2, 3, 4$, and apply Theorem 2.4 to Ψ on the interval $[\delta_1, c]$, we have

$$0 \geq \Omega_k(\Psi) = \Omega_k(\psi) - \frac{Z_c}{n!}\Omega_k(\sigma_n). \tag{30}$$

Analogously applying Theorem 2.4 to Ψ on the interval $[c, \delta_2]$, we get

$$0 \leq \Lambda_k(\Psi) = \Lambda_k(\psi) - \frac{Z_c}{n!}\Lambda_k(\sigma^n). \tag{31}$$

Moreover, identities (26) and (27) applied to the function σ^n gives

$$\Omega_k(\sigma^n) = \frac{n^3 - 3n^2 + 2n}{(c-\delta_1)}\int_{\delta_1}^{c}\left(\int_{\delta_1}^{c} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)](w-\xi)^{n-3}F_{\delta_1}^{c}(\xi, w)\,dw\right)d\xi, \tag{32}$$

$$\Lambda_k(\sigma^n) = \frac{n^3 - 3n^2 + 2n}{(\delta_2 - c)}\int_{c}^{\delta_2}\left(\int_{c}^{\delta_2} \mathbb{POP}[\mathbf{y}, \bar{\mathbf{q}}; G_k(y, w)](w-\xi)^{n-3}F_{c}^{\delta_2}(\xi, w)\,dw\right)d\xi. \tag{33}$$

Therefore assumption (28) is equivalent to

$$\Omega_k(\sigma^n) = \Lambda_k(\sigma^n).$$

So form (30) and (31), we obtained the desired result. $\square$

Remark 3.2. In the proof of Theorem 3.1, we have shown that for $k = 1, 2, 3, 4$

$$\Omega_k(\psi) \leq \frac{Z_c}{n!}\Omega_k(\sigma^n) = \frac{Z_c}{n!}\Lambda_k(\sigma^n) \leq \Lambda_k(\psi). \tag{34}$$

More importantly, inequality (29) still holds if we replace assumption (28) with the weaker assumption that is $Z_c\big(\Lambda_k(\sigma^n) - \Omega_k(\sigma^n)\big) \geq 0$.

4. New upper bounds of Grüss and Ostrowski type for generalized identities

In the present section we use Čebyšev functional defined for Lebesgue integrable functions $\mathbb{F}_1, \mathbb{F}_2 : [\delta_1, \delta_2] \to \mathbb{R}$ as

$$\mathbb{C}(\mathbb{F}_1, \mathbb{F}_2) = \frac{1}{\delta_2 - \delta_1}\int_{\delta_1}^{\delta_2} \mathbb{F}_1(\xi)\mathbb{F}_2(\xi)d\xi - \frac{1}{\delta_2 - \delta_1}\int_{\delta_1}^{\delta_2} \mathbb{F}_1(\xi)d\xi.\frac{1}{\delta_2 - \delta_1}\int_{\delta_1}^{\delta_2} \mathbb{F}_2(\xi)d\xi,$$

to construct some new upper bounds.

The following inequalities of Grüss type were given in [12].

Theorem 4.1. *Let $\mathbb{F}_1 \in L[\delta_1, \delta_2]$ and $\mathbb{F}_2 : [\delta_1, \delta_2] \to \mathbb{R}$ be an absolutely-continuous function along with $(. - \delta_1)(\delta_2 - .)[\mathbb{F}_2']^2 \in L[\delta_1, \delta_2]$. Then the inequality*

$$|\mathbb{C}(\mathbb{F}_1, \mathbb{F}_2)| \leq \frac{1}{\sqrt{2}}\left[\frac{\mathbb{C}(\mathbb{F}_1, \mathbb{F}_1)}{(\delta_2 - \delta_1)}\right]^{\frac{1}{2}}\left(\int_{\delta_1}^{\delta_2} (z - \delta_1)(\delta_2 - z)[\mathbb{F}_2'(z)]^2 dz\right)^{\frac{1}{2}}, \tag{35}$$

holds with $\frac{1}{\sqrt{2}}$ be the best possible constant.

Theorem 4.2. *Let $\mathbb{F}_1 : [\delta_1, \delta_2] \to \mathbb{R}$ be an absolutely continuous with $\mathbb{F}_1' \in L_\infty[\delta_1, \delta_2]$ and $\mathbb{F}_2 : [\delta_1, \delta_2] \to \mathbb{R}$ is monotonic nondecreasing function. Then the inequality*

$$|\mathbb{C}(\mathbb{F}_1, \mathbb{F}_2)| \leq \frac{\|\mathbb{F}_1'\|_\infty}{2(\delta_2 - \delta_1)}\int_{\delta_1}^{\delta_2} (z - \delta_1)(\delta_2 - z)d\mathbb{F}_2(z), \tag{36}$$

holds with best possible constant $\frac{1}{2}$.

In the sequel, we consider above theorems to construct new estimations of generalized identities proved earlier. In what follows we let for $k = 1, 2, 3, 4$,

$$\mathfrak{O}_k(\xi) = \int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)](w - \xi)^{n-3} F_{\delta_1}^{\delta_2}(\xi, w)\, dw, \quad \xi \in [\delta_1, \delta_2]. \tag{37}$$

First we express some Ostrowski type inequalities affiliated with our generalized Popoviciu's inequality.

Theorem 4.3. *Consider the suppositions of Theorem 2.3 be satisfied. Let $|\psi^{(n)}|^r : [\delta_1, \delta_2] \to \mathbb{R}$ be a R-integrable function with $r, r' \in [1, \infty]$ such that $\frac{1}{r} + \frac{1}{r'} = 1$. Then for $k = 1, 2, 3, 4$, we have*

$$\begin{aligned} &\Bigg|\mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi(z)] - \frac{1}{(\delta_2 - \delta_1)}\int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)] \\ &\quad \times \left(\sum_{\zeta=0}^{n-3}\left(\frac{n-2-\zeta}{\zeta!}\right)\left(\psi^{(\zeta+1)}(\delta_2)(w - \delta_2)^\zeta - \psi^{(\zeta+1)}(\delta_1)(w - \delta_1)^\zeta\right)\right)dw\Bigg| \\ &\leq \frac{1}{(n-3)!(\delta_2 - \delta_1)}\|\psi^{(n)}\|_r\left(\int_{\delta_1}^{\delta_2}\left|\int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)](w - \xi)^{n-3} F_{\delta_1}^{\delta_2}(\xi, w)\, dw\right|^{r'} d\xi\right)^{1/r'}. \end{aligned} \tag{38}$$

The constant on the R.H.S. of (38) is sharp for $1 < r \leq \infty$ and the best possible for $r = 1$.

Proof. Fix $k = 1, 2, 3, 4$. Rearranging identity (13) in such a way that

$$\Bigg|\mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi(z)] - \frac{1}{(\delta_2 - \delta_1)} \int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)] \times \left(\sum_{\zeta=1}^{n-3} \left(\frac{n-2-\zeta}{\zeta!}\right) \left(\psi^{(\zeta+1)}(\delta_2)(w-\delta_2)^{\zeta} - \psi^{(\zeta+1)}(\delta_1)(w-\delta_1)^{\zeta}\right)\right) dw\Bigg| = \left|\int_{\delta_1}^{\delta_2} \tilde{\mathfrak{O}}_k(\xi) \psi^{(n)}(\xi)\, d\xi\right|, \tag{39}$$

where $\tilde{\mathfrak{O}_k}(\xi) = \frac{\mathfrak{O}_k(\xi)}{(\delta_2-\delta_1)(n-3)!}$. Employing the classical Holder's inequality to R. H. S. of (39) yields (38). The proof for sharpness is similar to the Theorem 3.5 in [13] (see also [14]). □

Next we give some upper bounds of Grüss type.

Theorem 4.4. *Consider the suppositions of Theorem 2.3 be fulfilled. Also let $\psi^{(n)}$ is absolutely continuous with $(. - \delta_1)(\delta_2 - .)[\psi^{(n+1)}]^2 \in L[\delta_1, \delta_2]$ such that $\mathfrak{O}_k$ $(k = 1, 2, 3, 4)$ defined in (37). Then the remainder $Rem(\delta_1, \delta_2, \mathfrak{O}_k, \psi^{(n)})$ given in the following identity*

$$\mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi(z)] - \frac{1}{(\delta_2 - \delta_1)} \int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)] \times \left(\sum_{\zeta=0}^{n-3} \left(\frac{n-2-\zeta}{\zeta!}\right) \left(\psi^{(\zeta+1)}(\delta_2)(w-\delta_2)^{\zeta} - \psi^{(\zeta+1)}(\delta_1)(w-\delta_1)^{\zeta}\right)\right) dw - \frac{\psi^{(n-1)}(\delta_2) - \psi^{(n-1)}(\delta_1)}{(\delta_2 - \delta_1)^2 (n-3)!} \int_{\delta_1}^{\delta_2} \mathfrak{O}_k(\xi) d\xi = Rem(\delta_1, \delta_2, \mathfrak{O}_k, \psi^{(n)}), \tag{40}$$

satisfies the bound

$$|Rem(\delta_1, \delta_2, \mathfrak{O}_k, \psi^{(n)})| \le \frac{1}{\sqrt{2}(n-3)!} \left[\frac{\mathbb{C}(\mathfrak{O}_k, \mathfrak{O}_k)}{(\delta_2 - \delta_1)}\right]^{\frac{1}{2}} \left|\int_{\delta_1}^{\delta_2} (\xi - \delta_1)(\delta_2 - \xi)[\psi^{(n+1)}(\xi)]^2 d\xi\right|^{\frac{1}{2}}.$$

Proof. Fix $k = 1, 2, 3, 4$. Using Čebyšev functional for $\mathbb{F}_1 = \mathfrak{O}_k, \mathbb{F}_2 = \psi^{(n)}$ and by comparing (40) with (13), we have

$$Rem(\delta_1, \delta_2, \mathfrak{O}_k, \psi^{(n)}) = \frac{1}{(n-3)!} \mathbb{C}(\mathfrak{O}_k, \psi^{(n)}).$$

Now applying Theorem 4.1 for the corresponding functions, we get the required bound. □

Theorem 4.5. *Let $\psi : [\delta_1, \delta_2] \to \mathbb{R}$ be such that for $n \ge 3$, $\psi^{(n)}$ is absolutely continuous and let $\psi^{(n+1)} \ge 0$ on $[\delta_1, \delta_2]$ with $\mathfrak{O}_k$ $(k = 1, 3, 4)$ defined in (37). Then in the representation (40) the remainder $Rem(\delta_1, \delta_2, \mathfrak{O}_k, \psi^{(n)})$ satisfies the estimate*

$$|Rem(\delta_1, \delta_2, \mathfrak{O}_k, \psi^{(n)})| \le \frac{\|\mathfrak{O}_k'\|_\infty}{(n-3)!} \left[\frac{\psi^{(n-1)}(\delta_2) + \psi^{(n-1)}(\delta_1)}{2} - \frac{\psi^{(n-2)}(\delta_2) - \psi^{(n-2)}(\delta_1)}{\delta_2 - \delta_1}\right]. \tag{41}$$

Proof. Fix $k = 1, 2, 3, 4$. Since, we have established

$$Rem(\delta_1, \delta_2, \mathfrak{O}_k, \psi^{(n)}) = \frac{1}{(n-3)!} \mathbb{C}(\mathfrak{O}_k, \psi^{(n)}).$$

Now applying Theorem 4.2 for $\mathbb{F}_1 \to \mathfrak{O}_k$ and $\mathbb{F}_2 \to \psi^{(n)}$, we have

$$|Rem(\delta_1, \delta_2, \mathfrak{O}_k, \psi^{(n)})| = \frac{1}{(n-3)!} |\mathbb{C}(\mathfrak{O}_k, \psi^{(n)})| \le \frac{\|\mathfrak{O}_k'\|_\infty}{2(\delta_2 - \delta_1)(n-3)!} \int_{\delta_1}^{\delta_2} (\xi - \delta_1)(\delta_2 - \xi) \psi^{(n+1)}(\xi). \tag{42}$$

Simplifying the integral on R. H. S. of (42), we get the estimation in (41). □

5. Mean value theorems and *n*-exponential convexity

In the present section, we construct a positive linear functionals and then give mean value theorems of Lagrange and Cauchy type.

Remark 5.1. In virtue of Theorem 2.4, we can define the positive linear functionals from (18) ($k = 1, 2, 3, 4$), with respect to n-convex function ψ as follows

$$\Omega_k(\psi) := \mathbb{POP}[\mathbf{z}, \mathbf{q}; \psi(z)] - \frac{1}{(\delta_2 - \delta_1)} \int_{\delta_1}^{\delta_2} \mathbb{POP}[\mathbf{z}, \mathbf{q}; G_k(z, w)] \times \left(\sum_{\zeta=0}^{n-3} \left(\frac{n-2-\zeta}{\zeta!} \right) \left(\psi^{(\zeta+1)}(\delta_2)(w-\delta_2)^{\zeta} - \psi^{(\zeta+1)}(\delta_1)(w-\delta_1)^{\zeta} \right) \right) dw \geq 0. \tag{43}$$

Lagrange and Cauchy type mean value theorems related to above functionals are given in the following theorems.

Theorem 5.2. *Let $\psi : [\delta_1, \delta_2] \to \mathbb{R}$ be such that $\psi \in C^n[\delta_1, \delta_2]$. If the inequality in* (17) ($k = 1, 2, 3, 4$) *holds, then there exist $\xi_k \in [\delta_1, \delta_2]$ such that*

$$\Omega_k(\psi) = \psi^{(n)}(\xi_k)\Omega_k\left(\frac{z^n}{n!}\right), \quad k = 1, 2, 3, 4,$$

where $\Omega_k(\cdot)$ is defined by (43).

Proof. Similar to the proof of Theorem 4.1 in [15] (see also [16]). □

Theorem 5.3. *Let $\psi, \mu : [\delta_1, \delta_2] \to \mathbb{R}$ be such that $\psi, \mu \in C^n[\delta_1, \delta_2]$. If the inequality in* (17) ($k = 1, 2, 3, 4$)*, holds, then there exist $\xi_k \in [\delta_1, \delta_2]$ such that*

$$\frac{\Omega_k(\psi)}{\Omega_k(\mu)} = \frac{\psi^{(n)}(\xi_k)}{\mu^{(n)}(\xi_k)}, \quad k = 1, 2, 3, 4,$$

provided that the denominators are non-zero, where $\Omega_k(\cdot)$ is defined by (43).

Proof. Similar to the proof of Corollary 4.2 in [15] (see also [16]). □

Theorem 5.3 enables us to define Cauchy means for ($k = 1, 2, 3, 4$), in fact

$$\xi_k = \left(\frac{\psi^{(n)}}{\mu^{(n)}}\right)^{-1} \left(\frac{\Omega_k(\psi)}{\Omega_k(\mu)}\right),$$

means that ξ is the mean of δ_1, δ_2 for given functions ψ and μ.

We conclude our paper with the following remark.

Remark 5.4. One can construct the non trivial examples of n-exponentially and exponentially convex functions from positive linear functionals $\Omega_k(\cdot)$ ($k = 1, 2, 3, 4$), by following the n-exponentially method introduced by Pečarić et al. in [17] and [18] (see also [13,19] and [14]). As an application it enables us to construct large families of functions which are exponentially convex. Moreover by considering the class of 2-convex functions we can get the log-convexity of these functionals and new Cauchy means, which are monotonic in nature.

Acknowledgments

The research of first and second authors has been fully supported by H. E. C. Pakistan under NRPU project. The research of third author has been fully supported by Croatian Science Foundation under the project 5435.

References

[1] J. Pečarić, F. Proschan, Y.L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, New York, 1992.
[2] S.I. Butt, J. Pečarić, Popoviciu' S Inequality for N-Convex Functions, Lap Lambert Academic Publishing, ISBN: 978-3-659-81905-6, 2016.

[3] J. Pečarić, M. Praljak, A. Witkowski, Linear operator inequality for n-convex functions at a point, Math. Inequal. Appl. 18 (2015) 1201–1217.
[4] T. Popoviciu, Sur certaines inegalites qui caracterisent les fonctions convexes (Romanian), An. ŞTiinŢ. Univ. Al. I. Cuza, IaŞI. Mat. 11B (1965) 155–164.
[5] P.M. Vasić, L.j.R. Stanković, Some inequalities for convex functions, Math. Balkanica 6 (44) (1976) 281–288.
[6] M. Bencze, C.P. Niculescu, F. Popovici, Popoviciu's inequality for functions of several variables, J. Math. Anal. Appl. 365 (1) (2010) 399–409.
[7] M.V. Mihai, F.C. Mitroi-Symeonidis, New extensions of Popovicius inequality, Mediterr. J. Math. 13 (2016) 3121. http://dx.doi.org/10.1007/s00009-015-0675-3.
[8] C.P. Niculescu, The Integral Version of Popoviciu's Inequality, J. Math. Inequal. 3 (3) (2009) 323–328.
[9] C.P. Niculescu, F. Popovici, A Refinement of Popoviciu's Inequality, Bull. Soc. Sci. Math. Roum. 49 (97) (2006) 285–290.
[10] A.M. Fink, Bounds of the deviation of a function from its avereges, Czechoslovak Math. J. 42 (117) (1992) 289–310.
[11] R.P. Agarwal, P.J.Y. Wong, Error Inequalities in Polynomial Interpolation and Their Applications, Kluwer Academic Publishers, Dordrecht, 1993.
[12] P. Cerone, S.S. Dragomir, Some new Ostrowski-type bounds for the Čebyšev functional and applications, J. Math. Inequal. 8 (1) (2014) 159–170.
[13] S.I. Butt, K.A. Khan, J. Pečarić, Further generalization of Popoviciu inequality for higher order convex functions via Taylor polynomial, Turkish J. Math. 40 (2016) 333–349.
[14] S.I. Butt, K.A. Khan, J. Pečarić, Popoviciu type inequalities via Green function and generalized montgomery identity, Math. Inequal. Appl. 18 (4) (2015) 1519–1538.
[15] J. Jakšetić, J. Pečarić, A. Perušić, Steffensen inequality, higher order convexity and exponential convexity, Rend. Circ. Mat. Palermo 63 (1) (2014) 109–127.
[16] S.I. Butt, J. Pečarić, Generalized Hermite-Hadamard's inequality, Proc. a. Razmadze Math. Inst. 163 (2013) 9–27.
[17] J. Jakšetic, J. Pečarić, Exponential convexity method, J. Convex Anal. 20 (1) (2013) 181–197.
[18] J. Pečarić, J. Perić, Improvement of the Giaccardi and the Petrović inequality and related Stolarsky type means, An. Univ. Craiova Ser. Mat. Inform. 39 (1) (2012) 65–75.
[19] S.I. Butt, J. Pečarić, A. Vukelić, Generalization of Popoviciu type inequalities Via Fink's identity, Mediterr. J. Math. 13 (4) (2016) 1495–1511.

Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 171 (2017) 304–315
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

Impulsive integro-differential equations with nonlocal conditions in Banach spaces

Mouhamadou Alpha Diallo[a], Khalil Ezzinbi[b], Abdoulaye Séne[c,*]

[a] *Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, UCAD Dakar, Sénégal*
[b] *Université Cadi Ayyad, Faculté des Sciences Semlalia Département de Mathématiques, B.P. 2390 Marrakech, Morocco*
[c] *Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, UMI Ummisco 209 UCAD Dakar, Sénégal*

Received 31 May 2016; received in revised form 6 December 2016; accepted 11 December 2016
Available online 16 January 2017

Abstract

In this work, we give sufficient conditions for the existence of a mild solution for some impulsive integro-differential equations in Banach spaces. We study the existence without assuming the Lipschitz condition on the nonlinear term f. The compactness on the C_0-*semigroup* $(T(t))_{t\geq 0}$ in a Banach space is not needed. We use Hausdorff's measure of noncompactness, resolvent operators and Darbo's fixed point Theorem to obtain the main result of this work.

Keywords: Impulsive integro-differential equations; Mild solutions; Sadovskii's fixed point Theorem; Resolvent operators; Noncompactness measures

1. Introduction

In this work, we investigate the existence of mild solutions for the following impulsive integro-differential equations with nonlocal conditions

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t C(t-s)u(s)ds + f(t, u(t)) \text{ for } t \in J = [0, b] \text{ and } t \neq t_i \\ \Delta u(t_i) = I_i(u(t_i)) \text{ for } i = 1, \dots, p \text{ and } 0 < t_1 < t_2 < \cdots < t_p < b \\ u(0) = g(u), \end{cases} \tag{1}$$

where A and $C(t)$ are closed linear operators defined on a Banach space X with fixed domain which is denoted by $D(A)$ while f and g are functions that will be given later and $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitutes the impulsive

* Corresponding author.
E-mail address: abdoulaye.sene@ucad.edu.sn (A. Séne).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2016.12.002
2346-8092/

condition. The impulsive differential equations describe the evolution systems whose state changes rapidly in some times, which cannot be modeled by traditional initial value problems. In mathematical modeling in those processes, the change of state takes place momentarily and is represented by jumps (discontinuities) in the state of the modeled systems. Processes of such characters are observed in several areas of applied sciences (mechanics, population dynamics, biology, etc.). An impulsive differential equation consists of a continuous part and a discrete part by representing the impulses.

In the case where $C \equiv 0$, Eq. (1) is widely studied by several authors. Liu [1] discussed the classic initial problem when f is Lipschitz continuous with respect to second variable and the impulsive functions I_i are Lipschitz continuous. Cardinali and Rubbioni [2] studied the multivalued impulsive semilinear differential equations by means of the Hausdorff's measure of noncompactness. Liang et al. [3] investigated the nonlocal impulsive problems for nonlinear differential equations in Banach spaces first by assuming that f and g are Lipschitz, second by assuming that g is compact and f is not compact and not Lipschitz. L. Zhu, Q. Dong and G. Li [4] investigated the impulsive differential equations with nonlocal condition without the compactness condition and without the condition of Lipschitz continuous on f but by using Hausdorff's measure of noncompactness and Darbo's fixed point Theorem.

The aim of this work is to use Hausdorff's measure of noncompactness, Darbo's fixed point Theorem and the resolvent operators to prove the existence of at least one mild solution of Eq. (1).

The organization of this work is as follows, in Section 2, we give some preliminary results on the noncompactness measures, the impulsive differential equations and on the resolvent operators, in Section 3, we show the existence of mild solution of Eq. (1). Finally, in Section 4, we study an example to illustrate the abstract result of this work.

2. Hausdorff's measure of noncompactness and integro-differential equations in Banach spaces

2.1. Noncompactness measures

Compactness plays an essential role in the proof of Schauder's fixed point Theorem and other fixed point Theorems.

Definition 2.1 (*[5]*). Let X be a Banach space and $\mathcal{B}$ be a family of bounded subsets of X. For every $\Omega \in \mathcal{B}$, we define Hausdorff's measure of noncompactness α in the following way:

$$\alpha(\Omega) = \inf\{\varepsilon > 0 : \Omega \subset \cup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \varepsilon \text{ for } i = 1, \ldots, n\}.$$

The Hausdorff's measure of noncompactness satisfies the following properties.

Theorem 2.2 (*[5]*).

(i) *Regularity :* $\phi(\Omega) = 0 \Longleftrightarrow \overline{\Omega}$ *is compact,*
(ii) *Monotonicity : if* $\Omega_1 \subset \Omega_2$*, then* $\phi(\Omega_1) \le \phi(\Omega_2)$ *for all* Ω_1 *and* Ω_2 *in* $\mathcal{B}$,
(iii) *semi-homogeneity :* $\phi(\lambda\Omega) = |\lambda|\phi(\Omega)$ *for all* λ *in* $\mathbb{K}$ *and* Ω *in* $\mathcal{B}$,
(iv) *semi-additivity :* $\phi(\Omega_1 \cup \Omega_2) = \max\{\phi(\Omega_1), \phi(\Omega_2)\}$ *for all* Ω_1 *and* Ω_2 *in* $\mathcal{B}$.

Theorem 2.3 (*[5]*). *Let* $B(0, 1)$ *be the unit ball in a Banach space* X *of infinite dimension. Then* $\alpha(B(0, 1)) = 1$.

Theorem 2.4 (*[5]*). *Let* A *be a bounded subset of a Banach space* X *of infinite dimension, then for all* $r > 0$

$$\alpha(B(A, r)) = \alpha(A) + r,$$

where $B(A, r) = \cup_{x\in A} B(x, r) = A + B(0, r) = A + rB(0, 1)$.

Finally we prove a generalized Arzela–Ascoli's Theorem by using the measure of noncompactness of Kuratowski.

Theorem 2.5 (*[5]*). *Let* X *be a Banach space,* $K \subset \mathbb{R}^n$ *be compact. Denote by* $\mathcal{C}(K, X)$ *the Banach space of all continuous functions from* K *to* X *and let* $B \subset \mathcal{C}(K, X)$ *be bounded and equicontinuous. Then*

$$\alpha(B) = \sup_{t\in K} \alpha(\{x(t) : x \in B\}).$$

Theorem 2.6 (*[4] (Sadovskii's Fixed Point Theorem)*). *Let X be a Banach space, F be a nonempty, bounded, closed and convex subset of X. Suppose that $T : F \to F$ is a continuous map such that*

$$\alpha(T(B)) < \alpha(B) \text{ for all } B \subset F.$$

Then T has at least one fixed point on F.

Throughout this work, $(X, \|\cdot\|)$ is a Banach space. Denote by $\mathcal{C}(J, X)$ the Banach space of all continuous functions from J into X with the norm $|u| = \sup\{\|u(t)\|, t \in J\}$, by $L^1(J, X)$ the Banach space of all X-valued integrable functions defined on J with the norm $|u|_1 = \int_0^T \|u(t)\|dt$, by $PC(J, X)$ the space of all continuous functions at $t \in J, t \neq t_i$ and left continuous at $t = t_i$ and the right limit $u(t^+)$ exists in X for $i = 1, \dots, p$ and

$$PC^1(J, X) = \left\{u : J \to X \text{ differentiable at } t \neq t_i, \frac{du}{dt} \in PC(J, X)\right\}.$$

$PC(J, X)$ is a Banach space with the norm $\|u\|_{pc} = \sup\{\|u(t)\|, t \in J\}$ and

$$\mathcal{C}(J, X) \subset PC(J, X) \subset L^1(J, X).$$

2.2. Resolvent operators

We recall some basic results about the resolvent operators for the following integro-differential equation

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t C(t-s)u(s)ds \text{ for } t \geq 0, \\ u(0) = u_0, \end{cases} \tag{2}$$

where A and $C(\cdot)$ are closed linear operators on X.

Denote by $(Y, |\cdot|)$ the Banach space $D(A)$ equipped with the graph norm defined by $|y| = \|Ay\| + \|y\|$.

Definition 2.7 (*[6]*). A resolvent operator for the linear Equation (2) is a family of bounded linear operators $(R(t))_{t\geq 0}$ on X such that:

(i) $R(0) = Id_X$ and there are $M > 0$ and $w \in \mathbb{R}$ such that $\|R(t)\| \leq Me^{wt}$ for all $t \geq 0$
(ii) for all $x \in X$, the map $t \to R(t)x$ is strongly continuous from $\mathbb{R}^+$ to X
(iii) for all $t \in \mathbb{R}^+$, $R(t)Y \subset Y$
(iv) for all $y \in Y$, $R(\cdot)y \in \mathcal{C}(\mathbb{R}^+, Y) \cap \mathcal{C}^1(\mathbb{R}^+, X)$ and satisfies

$$R'(t)y = AR(t)y + \int_0^t C(t-s)R(s)yds \tag{3}$$

$$= R(t)Ay + \int_0^t R(t-s)C(s)yds. \tag{4}$$

In the next, we give some conditions on A and $C(\cdot)$ which ensure the existence of the resolvent operators for the linear equation (2).

($\mathbf{H}_1$): A generates a C_0-$semigroup$ $(T(t))_{t\geq 0}$ on X.
($\mathbf{H}_2$): For each $t \in [0, b]$, $C(t)$ is a bounded closed linear operator from Y to X. Moreover for all $y \in Y$, the application $t \mapsto B(t)y$ is in $W^{1,1}(J, X)$ and there exists $\xi \in \mathcal{L}^1(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\|B'(t)y\| \leq \xi(t)|y| \text{ for all } y \in Y \text{ and } t \in J.$$

Theorem 2.8 (*[7]*). *Let the assumptions* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *be satisfied. Then there is a unique resolvent operator for the linear Equation* (2)*.*

Lemma 2.9 (*[7]*). *Let the assumptions* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *be satisfied. Then for all* $a > 0$*, there exists* $C = C(a)$ *such that*

$$\|R(t+h) - R(t)R(h)\| \leq Ch \text{ for all } 0 < h \leq t \leq a.$$

Definition 2.10 (*[4]*). A C_0-*semigroup* $(T(t))_{t\geq 0}$ is said to be uniformly continuous for $t > 0$ if

$$\lim_{h\to 0} \|T(t+h) - T(t)\| = 0 \text{ for all } t > 0.$$

Theorem 2.11. *Let* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *be satisfied. If the* C_0-*semigroup* $(T(t))_{t\geq 0}$ *is uniformly continuous for* $t > 0$*, then the resolvent operator* $(R(t))_{t\geq 0}$ *is uniformly continuous for* $t > 0$.

For the proof of Theorem 2.11, we need to use the following Theorem.

Theorem 2.12 (*[8]*). *Let* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *be satisfied, then*

$$R(t)x = T(t)x + \int_0^t T(t-s)Q(s)x ds,$$

where

$$Q(s)x = \int_0^s C'(s-u)\int_0^u R(v)x dv du + C(0)\int_0^s R(u)x du.$$

Moreover, the operators Q are uniformly bounded on bounded intervals, and for each $x \in X$, $Q(\cdot)x \in \mathcal{C}(J, X)$.

Proof of Theorem 2.11. Let $t_0 > 0$ and $h > 0$. Then for $\|x\| \leq 1$

$$\begin{aligned} R(t_0+h)x - R(t_0)x &= T(t_0+h)x + \int_0^{t_0+h} T(t_0+h-s)Q(s)x ds - T(t_0)x - \int_0^{t_0} T(t_0-s)Q(s)x ds \\ &= \Big[T(t_0+h)x - T(t_0)x\Big] + \int_{t_0}^{t_0+h} T(t_0+h-s)Q(s)x ds \\ &\quad + \int_0^{t_0}\Big[T(t_0+h-s) - T(t_0-s)\Big]Q(s)x ds. \end{aligned}$$

Thus

$$\begin{aligned} \|R(t_0+h)x - R(t_0)x\| &\leq \|T(t_0+h) - T(t_0)\|\|x\| + \int_{t_0}^{t_0+h} \|T(t_0+h-s)Q(s)x\| ds \\ &\quad + \int_0^{t_0} \|T(t_0+h-s) - T(t_0-s)\|\|Q(s)x\| ds. \end{aligned}$$

By Theorem 2.12 $\|Q(s)x\|$ are uniformly bounded, then there exists $c > 0$ such that $\|Q(s)x\| \leq c$ for s bounded and $\|x\| \leq 1$. Thus

$$\|R(t_0+h) - R(t_0)\| \leq \|T(t_0+h) - T(t_0)\| + cMh + c\int_0^{t_0} \|T(t_0+h-s) - T(t_0-s)\| ds,$$

where $M = \sup\{\|T(t)\|, t \in J\}$. Since $(T(t))_{t\geq 0}$ is uniformly continuous for $t > 0$, then $\|T(t_0+h) - T(t_0)\| \to 0$ as $h \to 0$ and by the dominated convergence Theorem, we deduce that

$$\int_0^{t_0} \|T(t_0+h-s) - T(t_0-s)\| ds \to 0 \text{ as } h \to 0 \text{ for } t_0 \neq s.$$

Then

$$\|R(t_0+h) - R(t_0)\| \to 0 \text{ as } h \to 0.$$

We get the same estimate when $t_0 > 0, h < 0$ such that $t_0 + h > 0$, which allows us to conclude that $(R(t))_{t\geq 0}$ is uniformly continuous for $t > 0$.

Theorem 2.13. *Let* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *be satisfied. If the resolvent operator* $(R(t))_{t\geq 0}$ *is uniformly continuous for* $t > 0$, *let* $w \in L^1(J, \mathbb{R}^+)$. *Then the following set*

$$H = \left\{ \int_0^{\bullet} R(\bullet - s)u(s)ds : u \in W_w \right\}$$

is equicontinuous J, *where* $W_w = \{u : \|u(s)\| \leq w(s) \text{ for a.e.} s \in J\}$.

Proof. Let us pose $Hu(t) = \int_0^t R(t-s)u(s)ds$ for $t \in J$ and $u \in W_w$.

Let $t_0 \in J$ and $h > 0$ such that $t_0 + h \in J$.

$$\|Hu(t_0+h) - Hu(t_0)\| \leq \int_0^{t_0} \|R(t_0+h-s) - R(t_0-s)\| w(s)ds + \int_{t_0}^{t_0+h} \|R(t_0+h-s)\| w(s)ds.$$

Since $(R(t))_{t\geq 0}$ is uniformly continuous for $t > 0$, then by the dominated convergence Theorem, we deduce that

$$\int_0^{t_0} \|R(t_0+h-s) - R(t_0-s)\| w(s)ds \to 0 \text{ as } h \to 0 \text{ for } t_0 \neq s.$$

The same proof works for $h < 0$. Then

$$\lim_{h\to 0} \sup_{u\in W_w} \|Hu(t_0+h) - Hu(t_0)\| = 0.$$

Lemma 2.14 *([4]). If* $W \subset \mathcal{C}(J, X)$ *is bounded, then for all* $t \in J$

$$\alpha(W(t)) \leq \alpha(W),$$

where $W(t) = \{u(t) : u \in W\} \subset X$. Furthermore, if W is equicontinuous on J, then

$$\alpha(W) = \alpha(W(J)) = \sup\{\alpha(W(t)) : t \in J\},$$

where $W(J) = \{u(t) : u \in W, t \in J\}$. We consider now the problem of exchanging noncompactness measure and integral. Let $S : L^1([0,b]; X) \to \mathcal{C}([0,b]; X)$ be an abstract operator satisfying the following conditions.

$(\mathbf{S}_1)$: There exists $D > 0$ such that

$$\|Sf(t) - Sg(t)\| \leq D \int_0^t \|f(s) - g(s)\| ds$$

for every $f, g \in L^1([0,b]; X)$ and $t \in [0,b]$.

$(\mathbf{S}_2)$: Let $K \subset X$ be compact and sequence $(g_n)_{n\geq 1} \subset L^1([0,b]; X)$ such that $\{g_n(t); n \geq 1\} \subset K$ for a.e. $t \in [0,b]$. Then the weak convergence $g_n \rightharpoonup g_0$ implies the strong convergence $Sg_n \to Sg_0$.

Theorem 2.15 *([9]). If* S *satisfies* $(\mathbf{S}_1)$ *and* $(\mathbf{S}_2)$. *Let* $(f_n)_{n\geq 1} \subset L^1([0,b]; X)$ *and there exists* $v \in L^1([0,b]; \mathbb{R}^+)$ *such that* $\|f_n(t)\| \leq v(t)$ *for a.e.* $t \in [0,b]$, *and for all* $n \geq 1$. *Then*

$$\alpha(\{S_{f_n}(t) : n \geq 1\}) \leq 2D \int_0^t \alpha(\{f_n(s) : n \geq 1\})ds \text{ for } t \in [0,b].$$

Let $(R(t))_{t\geq 0}$ be the resolvent operator of Eq. (2). We define the operator

$$G : L^1(J; X) \to C(J; X)$$

by

$$Gf(t) = \int_0^t R(t-s)f(s)ds \text{ for } t \in J.$$

Theorem 2.16. *Let* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *be satisfied. Let* $(f_n)_{n\geq 1} \subset L^1([0,b];X)$ *and there exists* $v \in L^1([0,b];\mathbb{R}^+)$ *such that* $\|f_n(t)\| \leq v(t)$ *for a.e.* $t \in [0,b]$, *and for all* $n \geq 1$. *Then*

$$\alpha(\{G_{f_n}(t) : n \geq 1\}) \leq 2R_b \int_0^t \alpha(\{f_n(s) : n \geq 1\})ds \text{ for } t \in [0,b],$$

where $R_b = \sup_{t\in[0,b]} \|R(t)\|$.

Proof. It suffices to show that the operator G satisfies $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$. Since $\|R(t)\| \leq R_b$ for all $t \in [0,b]$, then condition $(\mathbf{S}_1)$ is automatically satisfied.

We claim that $(\mathbf{S}_2)$ condition is satisfied. Since G is a bounded linear operator, then $f_n \rightharpoonup f_0$ implies $Gf_n \rightharpoonup Gf_0$. By using Ascoli–Arzela's Theorem, show that $\{Gf_n, n \geq 1\}$ is relatively compact in $\mathcal{C}(J,X)$. It suffices to show that for each $t \in J$ fixed, the set $Y(t) = \{Gf_n(t), n \geq 1\}$ is relatively compact in X and $\{Gf_n, n \geq 1\}$ is equicontinuous on J.

Let the compact $K \subset X$, $t \in J$ fixed and the set $Q_t \subset X$ be defined by

$$Q_t = \cup_{s\in[0,t]} R(t-s)K.$$

The set Q_t is relatively compact. In fact, let us define functions $g_x : [0,t] \to X$ defined by $g_x(s) = R(t-s)x$ for all $x \in K$ and $s \in [0,t]$, then

$$Q_t = \{g_x([0,t]), x \in K\}.$$

Note that $\{g_x(\cdot), x \in K\}$ is equicontinuous on $[0,t]$. In fact, let $s_0 \in [0,t]$, $h > 0$ such that $s_0 + h \in [0,t]$

$$\begin{aligned}\|g_x(s_0+h) - g_x(s_0)\| &= \|R(t-s_0-h)x - R(t-s_0)x\| \\ &\leq \|R(t-s_0-h) - R(t-s_0)\|\|x\|.\end{aligned}$$

Since $(R(t))_{t\geq 0}$ is uniformly continuous for $t > 0$, then

$$\|R(t-s_0-h) - R(t-s_0)\| \to 0 \text{ as } h \to 0.$$

Thus $\|g_x(s_0+h)-g_x(s_0)\| \to 0$ as $h \to 0$ uniformly for $x \in K$. The same proof works for $h < 0$. Then $\{g_x(s), x \in K\}$ is equicontinuous on $[0,t]$. By using Lemma 2.14, since $R(t-s)K$ is compact, then

$$\begin{aligned}\alpha(Q_t) &= \alpha(\{g_x[0,t], x \in K\}) = \sup_{s\in[0,t]} \alpha\big(\{g_x(s), x \in K\}\big) = \sup_{s\in[0,t]} \alpha(R(t-s)K) \\ &\leq \sup_{s\in[0,t]} \|R(t-s)\|\alpha(K) = 0.\end{aligned}$$

Then Q_t is relatively compact.

Lemma 2.17 (*[10] Lemma 1.3*)**.** *If* $f \in L^1(J,X)$, *then*

$$\int_a^b f(t)dt \in (b-a)\overline{co}(\{f(t) : t \in [a,b]\})$$

for all $a, b \in J$ *with* $a < b$.

For all $n_0 \geq 1$, $f_{n_0}(s) \in K$ for all $s \in J$. Then for $s \in [0,t]$,

$$R(t-s)f_{n_0}(s) \in R(t-s)K \subset \cup_{\tau\in[0,t]} R(t-\tau)K = Q_t.$$

Then by Lemma 2.17

$$Gf_{n_0}(t) = \int_0^t R(t-s)f_{n_0}(s)ds \in t\overline{co}\Big(\{R(t-s)f_{n_0}(s) : s \in [0,t]\}\Big) \subset t\overline{co}(Q_t).$$

Since $f_n(t) \in K$ for all $n \geq 1$ and t fixed, then $\{Gf_n(t), n \geq 1\} \subset t\overline{co}(Q_t)$. We have

$$\alpha(\{Gf_n(t), n \geq 1\}) \leq \alpha(t\overline{co}(Q_t)) = t\alpha(\overline{co}(Q_t)) = t\alpha(co(Q_t)) = t\alpha(Q_t) = 0.$$

Thus $\{Gf_n(t), n \geq 1\}$ is relatively compact in X.

Let $t_0 \in J$ and $h > 0$ such that $t_0 + h \in J$.

$$\|Gf_n(t_0+h) - Gf_n(t_0)\| \le \int_0^{t_0} \|R(t_0+h-s)f_n(s) - R(t_0-s)f_n(s)\|ds + \int_{t_0}^{t_0+h} \|R(t_0+h-s)\|\|f_n(s)\|ds.$$

Since for all $t > 0$, $R(t)$ is a bounded linear operator and K compact. Then by using Lemma 2.9 and the fact that $R(t)K$ is compact, we deduce that

$$\lim_{h\to 0}\sup_n \|R(t_0+h-s)f_n(s) - R(t_0-s)f_n(s)\| = 0.$$

Then by the dominated convergence Theorem, we deduce that

$$\int_0^{t_0} \|R(t_0+h-s)f_n(s) - R(t_0-s)f_n(s)\|ds \to 0 \text{ as } h \to 0 \text{ for } t_0 \neq s.$$

The same proof works for $h < 0$. Then

$$\lim_{h\to 0} \|Gf_n(t_0+h) - Gf_n(t_0)\| = 0 \text{ uniformly for } n \ge 1.$$

Then $\{Gf_n, n \ge 0\}$ is equicontinuous. By applying now Ascoli–Arzela's Theorem, we get the relative compactness of $\{Gf_n, n \ge 1\}$. We have $Gf_n \rightharpoonup Gf_0$, the relative compactness of $\{Gf_n, n \ge 1\}$ provides that the last convergence is in the norm of the space $\mathcal{C}(J, X)$.

Let S_f be the unique mild solution of the following integro-differential equation

$$\begin{cases} u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t) \text{ for } t \in J, \\ u(0) = u_0. \end{cases} \tag{5}$$

Now we give the following result about α-estimation of the mild solutions.

Corollary 2.18. *Let* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *be satisfied. Let* $(f_k)_{k\ge 1}$ *be a sequence of functions. Assume that there exists a function* $\varphi \in L^1(J, \mathbb{R}^+)$ *such that*

$$\|f_k(t)\| \le \varphi(t) \textit{ for a.e. } t \in J \textit{ and } k \ge 1.$$

Then for all $t \in J$

$$\alpha(\{S_{f_k}(t) : k \ge 1\}) \le 2R_b \int_0^t \alpha(\{f_k(s) : k \ge 1\})ds.$$

3. Main results

Definition 3.1. A mild solution of Eq. (1) is a function $u \in PC(J, X)$ such that

$$u(t) = R(t)g(u) + \int_0^t R(t-s)f(s, u(s))ds + \sum_{0<t_i<t} R(t-t_i)I_i(u(t_i)) \text{ for } t \in J.$$

Let $r > 0$ and $W_r = \{u \in PC(J, X) : \|u\|_{pc} \le r\}$.

In order to prove the existence of mild solution, we assume the following assumptions.

$(\mathbf{H}_3)$: The C_0-$semigroup$ $(T(t))_{t\ge 0}$ generated by A is uniformly continuous for $t > 0$.

$(\mathbf{H}_4)$: $f : J \times X \to X$ satisfies the following conditions:

(i) $f(\cdot, x) : J \to X$ is strongly measurable for $x \in X$

(ii) $f(t, \cdot) : X \to X$ is continuous for $t \in J$

(iii) There exists $l \in L^1(J, \mathbb{R}^+)$ such that

$$\alpha(f(t, B)) \leq l(t)\alpha(B)$$

for all $t \in J$ and every bounded subset $B \subset X$.

($\mathbf{H}_5$): $I_i : X \to X$ is Lipschitz continuous with Lipschitz constants k_i, $i = 1, \ldots, p$.

($\mathbf{H}_6$): There exists a constant $k \in (0, m)$ such that:

$$\|g(u) - g(v)\| \leq k\|u - v\|_{pc} \text{ for all } u, v \in PC(J, X),$$

where $m = \frac{1}{R_b} - \sum_{i=1}^{p} k_i$.

($\mathbf{H}_7$): There exists $r > 0$ such that

$$R_b\Big(\|g(0)\| + \sum_{i=1}^{p} \|I_i(0)\| + b \cdot \sup_{t\in J, u\in W_r} \|f(t, u)\|\Big) \leq \Big(1 - R_b\Big(k + \sum_{i=1}^{p} k_i\Big)\Big)r.$$

Let $l_1 = \int_0^b l(s)ds$.

Theorem 3.2. *Assume that the conditions* ($\mathbf{H}_1$)–($\mathbf{H}_7$) *are satisfied. Then the Eq.* (1) *has at least one mild solution on J provided that*

$$R_b\left(4l_1 + k + \sum_{i=1}^{p} k_i\right) < 1.$$

Proof. We consider the operator H defined on $PC(J, X)$ by

$$(Hu)(t) = \int_0^t R(t - s)f(s, u(s))ds \text{ for } t \in J.$$

Let $B_r = \{x \in X : \|x\| \leq r\}$ and $W_r = \{u \in PC(J, X) : \|u\|_{pc} \leq r\}$.

We claim that H is continuous on $PC(J, X)$. Let $(u_n)_{n\in\mathbb{N}} \subset PC(J, X)$ be such that $u_n \to u$ in $PC(J, X)$, then there exists an integer r such that $\|u_n\| \leq r$ for all $n \in \mathbb{N}$, then $u_n \in W_r$ and $u \in W_r$. Since $f(t, .)$ is continuous on X, it follows that

$$\lim_{n\to+\infty} \|f(s, u_n(s)) - f(s, u(s))\| = 0.$$

By using ($\mathbf{H}_7$) hypothesis and the dominated convergence Theorem, we deduce that

$$\|Hu_n - Hu\| \leq R_b \int_0^b \|f(s, u_n(s)) - f(s, u(s))\|ds \to 0 \text{ when } n \to +\infty.$$

Thus H is continuous on $PC(J, X)$. Furthermore, by the assumption ($\mathbf{H}_7$) and Theorem 2.13, we know that $H(W_r)$ is bounded, equicontinuous on J. The following Lemma is needed.

Lemma 3.3 (*[4]*)**.** *If the condition* ($\mathbf{H}_7$) *holds, then for an arbitrary bounded set* $W \subset W_r$

$$\alpha(HW(t)) \leq 4R_b \int_0^t \alpha f(s, W(s))ds \ \ for\ t \in J.$$

Let $Q : PC(J, X) \to PC(J, X)$ be defined by

$$(Qu)(t) = u(t) - R(t)g(u) - \sum_{0<t_i<t} R(t - t_i)I_i(u(t_i)) \text{ for } t \in J.$$

The fixed point of $Q^{-1}H$ is the mild solution of Eq. (1). We prove that $Q^{-1}H$ has a fixed point. Let $u_1, u_2 \in PC(J, X)$, then

$$\begin{aligned}\|(Qu_1)(t) - (Qu_2)(t) &\leq \|u_1(t) - u_2(t)\| + \|R(t)g(u_1) - R(t)g(u_2)\| \\ &\quad + \sum_{i=1}^{p} \|R(t - t_i)I_i(u_1(t_i)) - R(t - t_i)I_i(u_2(t_i))\|\end{aligned}$$

$$\leq \|u_1 - u_2\|_{PC} + R_b k \|u_1 - u_2\|_{PC} + R_b \sum_{i=1}^{p} k_i \|u_1 - u_2\|_{PC}$$
$$\leq \Big(1 + R_b\Big(k + \sum_{i=1}^{p} k_i\Big)\Big)\|u_1 - u_2\|_{PC}.$$

We claim that Q is bijective. For this purpose, for any fixed $v \in PC(J, X)$, we prove there exists a unique $u \in PC(J, X)$ such that

$$(Qu)(t) = v(t) \text{ for } t \in J.$$

We define the operator $L : PC(J, X) \to PC(J, X)$ by

$$(Lu)(t) = R(t)g(u) + \sum_{0<t_i<t} R(t - t_i)I_i(u(t_i)) + v(t) \text{ for } t \in J.$$

The existence and uniqueness of a fixed point of L for any $v \in PC(J, X)$ implies that Q is bijective. Let $u_1, u_2 \in PC(J, X)$, then

$$\begin{aligned}\|(Lu_1)(t) - (Lu_2)(t)\| &\leq \|R(t)g(u_1) - R(t)g(u_2)\| \\ &\quad + \sum_{i=1}^{p} \|R(t - t_i)I_i(u_1(t_i)) - R(t - t_i)I_i(u_2(t_i))\| \\ &\leq R_b k\|u_1 - u_2\|_{PC} + R_b \sum_{i=1}^{p} k_i \|u_1 - u_2\|_{PC} \\ &\leq R_b\Big(k + \sum_{i=1}^{p} k_i\Big)\|u_1 - u_2\|_{PC}.\end{aligned}$$

From the condition ($\mathbf{H}_6$), we find that $R_b\Big(k + \sum_{i=1}^{p} ki\Big) < 1$, that is L is a contraction operator on $PC(J, X)$. By Banach's fixed point Theorem, we deduce that L has a unique fixed point. Thus, Q is bijective.

We claim that Q^{-1} is Lipschitz continuous. Let $v_1, v_2 \in PC(J, X)$, then

$$\begin{aligned}\|(Q^{-1}v_1)(t) - (Q^{-1}v_2)(t)\| &\leq \|v_1 - v_2\| + \|R(t)g(Q^{-1}v_1) - R(t)g(Q^{-1}v_2)\| \\ &\quad + \sum_{i=1}^{p} \|R(t - t_i)I_i(Q^{-1}v_1)(t_i) - R(t - t_i)I_i(Q^{-1}v_2)(t_i)\| \\ &\leq \|v_1 - v_2\|_{PC} + R_b k\|Q^{-1}v_1 - Q^{-1}v_2\|_{PC} \\ &\quad + R_b \sum_{i=1}^{p} k_i \|Q^{-1}v_1 - Q^{-1}v_2\|_{PC} \\ &\leq \|v_1 - v_2\|_{PC} + R_b\left(k + \sum_{i=1}^{p} k_i\right)\|Q^{-1}v_1 - Q^{-1}v_2\|_{PC} \\ &\leq \frac{1}{1 - R_b\left(k + \sum_{i=1}^{p} k_i\right)}\|v_1 - v_2\|_{PC}.\end{aligned}$$

We claim that $(Q^{-1}H)(W_r) \subset W_r$. In fact, for any $u \in W_r \subset PC(J, X)$, let $w = (Q^{-1}H)(u)$, from the hypotheses ($\mathbf{H}_5$)–($\mathbf{H}_7$), we have

$$\begin{aligned}\|w(t)\| &\leq \|R(t)g(u)\| + \sum_{i=1}^{p} \|R(t - t_i)I_i(u)(t_i)\| + \int_0^t \|R(t - s)\| \sup_{s\in J, u\in W_r} \|f(s, u(s))\|ds \\ &\leq R_b\Big(\Big(k + \sum_{i=1}^{p} k_i\Big)\|u\|_{PC} + \|g(0)\| + \sum_{i=1}^{p} \|I_i(0)\| + b \cdot \sup_{s\in J, u\in W_r} \|f(s, u(s))\|\Big) \\ &\leq r,\end{aligned}$$

we infer that $\|w\|_{PC} \leq r$. Thus, $(Q^{-1}H)(W_r) \subset W_r$.

Finally, we prove that $Q^{-1}H$ is a α-contraction. As Q^{-1} is Lipschitz continuous and H is continuous on $PC(J, X)$, we have $Q^{-1}H$ is continuous on $PC(J, X)$. Actually, since $H(W_r)$ is bounded and equicontinuous on J, we can even deduce that $(Q^{-1}H)(W_r) \subset PC(J, X)$ is equicontinuous on each J_i, $i = 0, 1, \dots, p$, where $J_0 = (0, t_1]$; $J_i = (t_i, t_{i+1}]$, $i = 1, \dots, p$.

As Q^{-1} is Lipschitz with constant $\frac{1}{1-R_b(k+\sum_{i=1}^{p} k_i)}$ for $W \subset W_r$, we obtain that

$$\alpha(Q^{-1}HW) < \frac{1}{1 - R_b\Big(k + \sum\limits_{i=1}^{p} k_i\Big)} \alpha(HW).$$

On the other hand, from Lemma 3.3 for $t \in J$, we know that

$$\alpha(HW(t)) \leq 4R_b \int_0^t \alpha f(s, W(s))ds \leq 4R_b \int_0^t l(s)\alpha(W(s))ds.$$

From Lemma 2.14 and Lemma 3.3, we deduce that

$$\alpha(HW) \leq 4l_1 R_b \alpha(W).$$

Consequently,

$$\alpha(Q^{-1}HW) \leq \frac{4l_1 R_b}{1 - R_b\Big(k + \sum\limits_{i=1}^{p} k_i\Big)} \alpha(W).$$

Since $R_b(4l_1 + k + \sum_{i=1}^{p} k_i) < 1$, the mapping $Q^{-1}H$ is a α-contraction in W_r, by Theorem 2.6 the operator $Q^{-1}H$ has a fixed point in W_r which is just the mild solution of Eq. (1). This completes the proof.

4. Application

Consider the following partial impulsive integro-differential equation

$$\begin{cases} \dfrac{\partial}{\partial t} z(t, x) = \dfrac{\partial^2}{\partial x^2} z(t, x) + \displaystyle\int_0^t \gamma(t-s) \dfrac{\partial^2}{\partial x^2} z(s, x) ds \\ \qquad\qquad + \lambda(t)\phi(z(t, x)) \text{ for } t \in [0, b] \text{ and } x \in [0, \pi], \\ z(t, 0) = z(t, \pi) = 0, \\ z(0, x) = \displaystyle\sum_{i=1}^{p} c_i z(t_i, x), 0 < t_1 < \cdots < t_p < b, x \in [0, \pi], \\ \Delta z(t_i, x) = k_i z(t_i, x), i = 1, \dots, p; k_i \in \mathbb{R} \text{ for any } i \end{cases} \tag{6}$$

where $\gamma \in \mathcal{C}^1([0, b], \mathbb{R})$, $\lambda : \mathbb{R}^+ \to \mathbb{R}$ is continuous, $\phi : \mathbb{R} \to \mathbb{R}$ is such that there exists $a_1 > 0$ such that $|\phi(x) - \phi(y)| \leq a_1|x - y|$ for all $x, y \in \mathbb{R}$ and $c_i, k_i \in \mathbb{R}$ for $i = 1, \dots, p$. To rewrite Eq. (6) in the abstract form, we introduce the space $X = L^2(0, \pi)$. Let $A : D(A) \to X$ defined by

$$\begin{cases} D(A) = H_0^1(0, \pi) \cap H^2(0, \pi) \\ Ay = y''. \end{cases} \tag{7}$$

Let $C(t) : D(A) \to X$ defined by $C(t)y = \gamma(t)Ay$.

Let $f : [0, b] \times X \to X$ defined by

$$f(t, v)(x) = \lambda(t)\phi(v(x)) \; for \; t \in [0, b] \; and \; x \in [0, \pi].$$

Let $g : PC([0, b], X) \to X$ defined by

$$g(u)(x) = \sum_{i=1}^{p} c_i u(t_i)(x) \; for \; 0 < t_1 < \cdots < t_p < b, \; and \; x \in [0, \pi].$$

Let $I_i : X \to X$ defined by

$$I_i(v)(x) = k_i v(x) \ for\ 0 < t_1 < \cdots < t_p < b,\ and\ x \in [0, \pi],$$

where $u(t)(x) = z(t, x)$. Let us suppose $u(t) = z(t, x)$, Eq. (6) takes the following abstract form

$$\begin{cases} u'(t) &= Au(t) + \int_0^t C(t-s)u(s)ds + f(t, u(t)) \text{ for } t \in J = [0, b] \text{ and } t \neq t_i \\ \Delta u(t_i) &= I_i(u(t_i)) \text{ for } i = 1, \ldots, p \text{ and } 0 < t_1 < t_2 < \cdots < t_p < b \\ u(0) &= g(u). \end{cases} \tag{8}$$

It is well known that A generates a C_0-$semigroup$ that is uniformly continuous for $t > 0$, which implies that $(\mathbf{H}_1)$ and $(\mathbf{H}_3)$ are satisfied. Moreover $(\mathbf{H}_2)$ is true, it follows that the linear Equation (2) has a resolvent operator $(R(t))_{t \geq 0}$. Claim that the application $f : [0, b] \times E \to E$ defined by $f(t, v)(x) = \lambda(t)\phi(v(x))$ for $t \in [0, b]$ and $x \in [0, \pi]$ is continuous. Let $(v_n)_{n \geq 0} \subset L^2(0, \pi)$ such that $v_n \to v$ in $L^2(0, \pi)$,

$$\begin{aligned} \|f(t, v_n) - f(t, v)\|_{L^2(0,\pi)} &= \Big(\int_0^\pi |f(t, v_n)(x) - f(t, v)(x)|^2 dx\Big)^{\frac{1}{2}} \\ &= \Big(\int_0^\pi |\lambda(t)\phi(v_n(x)) - \lambda(t)\phi(v(x))|^2 dx\Big)^{\frac{1}{2}} \\ &= |\lambda(t)|\Big(\int_0^\pi |\phi(v_n(x)) - \phi(v(x))|^2 dx\Big)^{\frac{1}{2}}. \end{aligned}$$

Lemma 4.1 (*[11]*)**.** *Let $(f_n) \subset L^2(0, \pi)$ and $f \in L^2(0, \pi)$ such that $f_n \to f$ in $L^2(0, \pi)$. Then there exist a subsequence (f_{n_k}) of (f_n) and a function $\rho \in L^2(0, \pi)$ such that*

(i) *$f_{n_k}(x) \to f(x)$ as $k \to +\infty$ a.e. $x \in [0, \pi]$*
(ii) *$|f_{n_k}(x)| \leq \rho(x)$ for all k and for a.e. $x \in [0, \pi]$.*

By using the above Lemma 4.1. Then there exists a subsequence v_{n_k} of v_n such that $v_{n_k}(z) \to v(z)$ for a.e. $z \in [0, \pi]$ and $\rho \in L^2((0, \pi), \mathbb{R}^+)$ such that $|v_{n_k}(z)| \leq \rho(z)$ for a.e. $z \in [0, \pi]$.

By the continuity of ϕ, $|\phi(v_{n_k}(x)) - \phi(v(x))| \to 0$ for a.e. $x \in [0, \pi]$ and $|\phi(v_{n_k}(x))| \leq a_1|v_{n_k}(x)| + |\phi(0)| \leq a_1\rho(x) + |\phi(0)| \in L^2(0, \pi)$. By the dominated convergence theorem $\phi(v_{n_k}(\cdot)) \to \phi(v(\cdot))$ in $L^2(0, \pi)$ for any subsequence $v_{n_k}(\cdot)$ of $v_n(\cdot)$. Then $\phi(v_n(\cdot)) \to \phi(v(\cdot))$ in $L^2(0, \pi)$ thus $\|f(t, v_n) - f(t, v)\|_{L^2(0,\pi)} \to 0$ as $n \to \infty$ then $(\mathbf{H}_4)$ is satisfied. The conditions $(\mathbf{H}_5)$, $(\mathbf{H}_6)$ and $(\mathbf{H}_7)$ are satisfied with $k = \sum_{i=1}^p |c_i|$ and addition, if the inequality

$$R_b\left(4l_1 + k + \sum_{i=1}^p k_i\right) < 1$$

held, where $l_1 = \sup_{x \in [0,\pi]} \int_0^b |\rho(t)\phi(z(t, x))|dt$. Then due to Theorem 3.2 Eq. (6) has at least one mild solution u.

Acknowledgment

The authors thank the reviewer for these valuable remarks.

References

[1] J. Liu, Nonlinear impulsive evolution equations, Dyn. Contin. Discrete Impuls. Syst. 6 (1) (1999) 77–85.
[2] T. Cardinali, P. Rubbioni, Impulsive semilinear differential inclusions: topological structure of the solution set and solutions on noncompact domains, Nonlinear Anal. TMA 69 (1) (2008) 73–84.
[3] J. Liang, J. Liu, T. Xiao, Nonlocal impulsive problems for nonlinear differential equations in Banach spaces, Math. Comput. Modeling 49 (3) (2009) 798–804.
[4] Q. Don, G. Li, L. Zhu, Impulsive differential equations with nonlocal condition in general Banach spaces, Adv. Difference Equ. 2012 (1) (2012) 1–11.

[5] G. Acedo, T. Benavides, J. Toledano, Measure of NonCompactness in Metric Fixed Point Theory, Springer Science and Business Media, 1997.
[6] K. Ezzinbi, H. Toure, I. Zabsorne, Existence and regularity of solutions for some partial functional integro-differential equations in Banach spaces, Nonlinear Anal. 70 (7) (2009) 2761–2771.
[7] J. Liang, J. Liu, T. Xiao, Nonlocal problems for integro-differential equations, Dyn. Contin. Discrete Impuls. Syst. Ser. A 15 (6) (2008) 815–824.
[8] W. Desch, R. Grimmer, W. Scharppacher, Some considerations for linear integro-differential equation, J. Math. Anal. Appl. 104 (1) (1984) 219–234.
[9] M. Kamenskii, V. Obukhovskii, P. Zecca, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, Walter de Gruyter, 2001.
[10] R. Martin, Nonlinear Operators and Differential Equations in Banach Spaces, John Wiley and Sons, 1976.
[11] H. Brezis, Opérateurs Maximaux Monotones et Semi-groupes de Contraction dans les Espaces de Hilbert, vol. 5, Elsevier, 1973.

Further reading

[1] D. Bahatta, L. Debnath, Integral Transforms and Their Applications, CRC Press, 2007.
[2] K. Balachandran, R.V. Kumar, Existence of solutions of integro-differential evolution equations with time varying delays, Appl. Math. E-Notes (7) (2007) 1–8.
[3] H. Bouzahir, X. Fu, Controllability of neutral functional differential equation with infinite delay, Acta Math. Sci. 31 (1) (2011) 73–80.
[4] H.S. Ding, J. Lian, T. Xiao, Pseudo almost periodic solutions to integro-differential equations of heat conduction in materials with memory, Nonlinear Anal. RWA 13 (2012) 2659–2670.
[5] L. Górniewicz, S.K. Ntouyas, D. O'regan, Controllability of semilinear differential equation and inclusions via semigroup theory in Banach spaces, Rep. Math. Phys. 56 (3) (2005) 437–470.
[6] R. Grimmer, Resolvent opérateurs for integral equation in a Banach space, Trans. Amer. Math. Soc. 273 (1) (1982) 333–349.
[7] R. Grimmer, A.J. Pritchard, Analytic resolvent operator for integral equation in Banach space, J. Differential Equations 50 (2) (1983) 234–259.

Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute 171 (2017) 316–327

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

Some generalized Ostrowski type inequalities for functions whose second derivatives absolute values are convex and applications

Yusuf Erdem[a], Hüseyin Budak[b,*], Hasan Öğünmez[a]

[a] *Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyonkarahisar, Turkey*
[b] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Turkey*

Received 27 December 2016; received in revised form 6 May 2017; accepted 12 June 2017
Available online 24 June 2017

Abstract

We first establish some Ostrowski type inequalities for mappings whose second derivatives absolute values are convex. Then we give some special cases of these inequalities which provide extensions of those given in earlier works. Finally, some applications of these inequalities for special means are also provided.

Keywords: Ostrowski inequality; Convex functions; Special means

1. Introduction

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of scientists, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of A. M. Ostrowski [1] is the following classical integral inequality associated with the differentiable mappings:

Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality (1.1) has applications in numerical analysis, probability and optimization theory, stochastic, statistics, information and integral operator theory, see for example [2–20].

* Corresponding author.
E-mail addresses: mateyus26@gmail.com (Y. Erdem), hsyn.budak@gmail.com (H. Budak), hogunmez@aku.edu.tr (H. Öğünmez).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.06.002

The remainder of this work is organized as follows: In this section, we present definition of convex function and give an important identity which will be used to establish our main results. In Section 2, some new Ostrowski type integral inequalities are proved for function whose second derivatives absolute values are convex. These inequalities are provided for special means in Section 3. At the end some conclusions of research are discussed in Section 4.

Definition 1. The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Dragomir and Pearce proved the following identity and using this identity they gave important trapezoid inequalities.

Lemma 1 ([5]). *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a, b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t) f''(ta+(1-t)b)dt. \tag{1.2}$$

Sarikaya et al. gave the following identity for twice differentiable mapping:

Lemma 2 ([17]). *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a, b]$, then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)^2}{4}\int_0^1 m(t)\left[f''(ta+(1-t)b) + f''(tb+(1-t)a)\right]dt, \end{aligned} \tag{1.3}$$

where

$$m(t) = \begin{cases} t^2, & t \in \left[0, \frac{1}{2}\right] \\ (1-t)^2, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Theorem 1 ([17]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$be twice differentiable function on I with $f'' \in L[a, b]$. If $\left|f''\right|$ is convex on $[a, b]$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{24}\left[\frac{\left|f''(a)\right| + \left|f''(b)\right|}{2}\right]. \tag{1.4}$$

In [10], Erden et al. gave the following important equality for twice differentiable function:

Lemma 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L[a, b]$, the interior of the interval I, where $a, b \in I^\circ$with $a < b$. Then the following identity holds:*

$$\begin{aligned} &\frac{1}{2(b-a)}\int_a^b P_h(x,t)\, f''(t)\,dt \\ &= \frac{h-2}{2}\left(x - \frac{a+b}{2}\right) f'(x) + f(x) - \frac{f(b)-f(a)}{2(b-a)} m_h(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &=: S_{x,h}(f) \end{aligned} \tag{1.5}$$

for

$$P_h(x,t) := \begin{cases} (a-t)(t-a-m_h(x)), & a \le t < x \\ (b-t)(t-b-m_h(x)), & x \le t \le b \end{cases}$$

where $m_h(x) = \left(x - \frac{a+b}{2}\right) h$, $h \in [0,2]$ *and* $x \in [a,b]$.

Using the convexity of $|f''|$ and identity (1.5), we establish some generalized Ostrowski type inequalities.

2. Main results

Now, we establish our main theorems and also give some results related to these theorems.

Theorem 2. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be twice differentiable function on* I°, *the interior of the interval* I, *where* $a, b \in I^\circ$ *with* $a < b$. *If* $|f''|$ *is a convex mapping on* $[a,b]$, *then the following inequalities hold:*

$$\begin{aligned} &\left|S_{x,h}(f)\right| \\ &\le \frac{1}{2(b-a)^2}\Bigg\{\left|f''(a)\right|\Bigg[\frac{(b-x)^4-(x-a)^4}{4} + m_h(x)\frac{(x-a)^3+(b-x)^3}{3} \\ &+ (b-a)\frac{(x-a)^3}{3} - (b-a)\,m_h(x)\frac{(x-a)^2}{2} + \frac{[m_h(x)]^4}{6}\Bigg] \\ &+ \left|f''(b)\right|\Bigg[\frac{(x-a)^4-(b-x)^4}{4} - m_h(x)\frac{(x-a)^3+(b-x)^3}{3} \\ &+ (b-a)\frac{(b-x)^3}{3} + (b-a)\,m_h(x)\frac{(b-x)^2}{2} - \frac{[m_h(x)]^4}{6} - (b-a)\frac{[m_h(x)]^3}{3}\Bigg]\Bigg\} \end{aligned} \tag{2.1}$$

for all $a \le x \le \frac{a+b}{2}$ *with* $h \in [0,2]$ *and*

$$\begin{aligned} &\left|S_{x,h}(f)\right| \\ &\le \frac{1}{2(b-a)^2}\Bigg\{\left|f''(a)\right|\Bigg[\frac{(b-x)^4-(x-a)^4}{4} + m_h(x)\frac{(x-a)^3+(b-x)^3}{3} \\ &+ (b-a)\frac{(x-a)^3}{3} - (b-a)\,m_h(x)\frac{(x-a)^2}{2} - \frac{[m_h(x)]^4}{6} + (b-a)\frac{[m_h(x)]^3}{3}\Bigg] \\ &+ \left|f''(b)\right|\Bigg[\frac{(x-a)^4-(b-x)^4}{4} - m_h(x)\frac{(x-a)^3+(b-x)^3}{3} \\ &+ (b-a)\frac{(b-x)^3}{3} + (b-a)\,m_h(x)\frac{(b-x)^2}{2} + \frac{[m_h(x)]^4}{6}\Bigg]\Bigg\} \end{aligned} \tag{2.2}$$

for all $\frac{a+b}{2} \le x \le b$ *with* $h \in [0,2]$, *where* $m_h(x) = h\left(x - \frac{a+b}{2}\right)$.

Proof. Taking modulus of equality given in (1.5) and using the triangle inequality for integrals, we find that

$$\begin{aligned} \left|S_{x,h}(f)\right| &= \frac{1}{2(b-a)}\left|\int_a^b P_h(x,t)\, f''(t)\, dt\right| \\ &\le \frac{1}{2(b-a)}\int_a^b \left|P_h(x,t)\right| \left|f''(t)\right| dt \end{aligned}$$

$$= \frac{1}{2(b-a)} \left[\int_a^x |a-t| \, |t-a-m_h(x)| \left| f''(t) \right| dt + \int_x^b |b-t| \, |t-b-m_h(x)| \left| f''(t) \right| dt \right].$$

Since $\left|f''\right|$ is a convex mapping on $[a, b]$,we have

$$\left|f''(t)\right| = \left| f''\left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right| \leq \frac{b-t}{b-a} \left|f''(a)\right| + \frac{t-a}{b-a} \left|f''(b)\right|. \tag{2.3}$$

Using (2.3), we get

$$\begin{aligned}
&\left|S_{x,h}(f)\right| \\
&\leq \frac{1}{2(b-a)^2} \left[\int_a^x |a-t| \, |t-a-m_h(x)| \left[(b-t) \left|f''(a)\right| + (t-a) \left|f''(b)\right| \right] dt \right. \\
&\quad \left. + \int_x^b |b-t| \, |t-b-m_h(x)| \left[(b-t) \left|f''(a)\right| + (t-a) \left|f''(b)\right| \right] dt \right] \\
&= \frac{1}{2(b-a)^2} \left\{ \left|f''(a)\right| \left[\int_a^x |a-t| \, |t-a-m_h(x)| \, (b-t) \, dt \right. \right. \\
&\quad \left. + \int_x^b |b-t| \, |t-b-m_h(x)| \, (b-t) \, dt \right] \\
&\quad + \left|f''(b)\right| \left[\int_a^x |a-t| \, |t-a-m_h(x)| \, (t-a) \, dt \right. \\
&\quad \left. \left. + \int_x^b |b-t| \, |t-b-m_h(x)| \, (t-a) \, dt \right] \right\} \\
&= \frac{1}{2(b-a)^2} \left[\left|f''(a)\right| (I_1 + I_2) + \left|f''(b)\right| (I_3 + I_4) \right].
\end{aligned} \tag{2.4}$$

We calculate integrals I_i, $i = 1, \ldots, 4$, for the cases $a \leq x \leq \frac{a+b}{2}$ and $\frac{a+b}{2} \leq x \leq b$;
In case of $a \leq x \leq \frac{a+b}{2}$, using the fact that $m_h(x) \leq 0$, we get

$$\begin{aligned}
I_1 &= \int_a^x (t-a)(t-a-m_h(x))(b-t) \, dt \\
&= (b-a) \int_a^x (t-a)(t-a-m_h(x)) \, dt - \int_a^x (t-a)^2 (t-a-m_h(x)) \, dt \\
&= (b-a) \frac{(x-a)^3}{3} - (b-a) \, m_h(x) \frac{(x-a)^2}{2} - \frac{(x-a)^4}{4} + m_h(x) \frac{(x-a)^3}{3},
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
I_2 &= \int_x^b (b-t)^2 \, |t-b-m_h(x)| \, dt \\
&= \int_x^{b+m_h(x)} (b-t)^2 (m_h(x)+b-t) \, dt + \int_{b+m_h(x)}^b (b-t)^2 (t-b-m_h(x)) \, dt \\
&= \frac{[m_h(x)]^4}{6} + m_h(x) \frac{(b-x)^3}{3} + \frac{(b-x)^4}{4},
\end{aligned} \tag{2.6}$$

$$I_3 = \int_a^x (t-a)^2 (t-a-m_h(x)) \, dt = \frac{(x-a)^4}{4} - m_h(x) \frac{(x-a)^3}{3} \tag{2.7}$$

and

$$I_4 = \int_x^b (b-t) \, |t-b-m_h(x)| \, (t-a) \, dt$$

$$
\begin{aligned}
&= \int_{x}^{b+m_h(x)} (b-t)\,(m_h(x)+b-t)\,(t-a)\,dt \\
&\quad + \int_{x}^{b} (b-t)\,(t-b-m_h(x))\,(t-a)\,dt \\
&= -\frac{[m_h(x)]^4}{6} - (b-a)\,\frac{[m_h(x)]^3}{3} - \frac{(b-x)^4}{4} - m_h(x)\frac{(b-x)^3}{3} \\
&\quad + (b-a)\,m_h(x)\frac{(b-x)^2}{2} + (b-a)\,\frac{(b-x)^3}{3}. \qquad (2.8)
\end{aligned}
$$

If we substitute the equalities (2.5)–(2.6) in (2.4), then we obtain the required inequality (2.1).

In case of $\frac{a+b}{2} \le x \le b$, using the fact that $m_h(x) \ge 0$, we get

$$
\begin{aligned}
I_1 &= \int_{a}^{x} (t-a)\,|t-a-m_h(x)|\,(b-t)\,dt \\
&= -\frac{[m_h(x)]^4}{6} + (b-a)\,\frac{[m_h(x)]^3}{3} - \frac{(x-a)^4}{4} + m_h(x)\frac{(x-a)^3}{3} \\
&\quad - (b-a)\,m_h(x)\frac{(x-a)^2}{2} + (b-a)\,\frac{(x-a)^3}{3} \qquad (2.9)
\end{aligned}
$$

$$
I_2 = \int_{x}^{b} (b-t)^2\,(m_h(x)+b-t)\,dt = m_h(x)\frac{(b-x)^3}{3} + \frac{(b-x)^4}{4}, \qquad (2.10)
$$

$$
I_3 = \int_{a}^{x} (t-a)^2\,|t-a-m_h(x)|\,dt = \frac{[m_h(x)]^4}{6} + \frac{(x-a)^4}{4} - m_h(x)\frac{(x-a)^3}{3} \qquad (2.11)
$$

and

$$
\begin{aligned}
I_4 &= \int_{x}^{b} (b-t)\,(m_h(x)+b-t)\,(t-a)\,dt \\
&= -m_h(x)\frac{(b-x)^3}{3} - \frac{(b-x)^4}{4} + (b-a)\,m_h(x)\frac{(b-x)^2}{2} + (b-a)\,\frac{(b-x)^3}{3}. \qquad (2.12)
\end{aligned}
$$

If we substitute the equalities (2.9)–(2.12) in (2.4), then we obtain the desired inequality (2.2). The proof is thus completed. □

Remark 1. If we choose $x = \frac{a+b}{2}$ in Theorem 2, then the inequalities (2.1) and (2.2) reduce to the inequality (1.4).

Remark 2. If we choose $h = 0$ in the inequalities (2.1) and (2.2), then we have the following inequality

$$
\begin{aligned}
&\left| f(x) - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\
&\le \frac{b-a}{2}\left\{ \left|f''(a)\right| \left[\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{a+b}{2} - x\right) + \frac{(x-a)^3}{3(b-a)^2}\right] \right. \\
&\quad \left. + \left|f''(b)\right| \left[\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left(x - \frac{a+b}{2}\right) + \frac{(b-x)^3}{3(b-a)^2}\right] \right\}
\end{aligned}
$$

for $x \in [a, b]$.

Corollary 1. *Let us substitute $x = a$ and $x = b$ in Theorem 2. Subsequently, if we add the obtained result and use the triangle inequality for the modulus, we get the inequality*

$$\left|\frac{h-2}{2}\frac{b-a}{4}\left(f'(b)-f'(a)\right)+\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{8}\left[\frac{2}{3}-\frac{h}{2}+\frac{h^3}{12}\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Remark 3. If we take $h = 0$ in Corollary 1, then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{b-a}{4}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{6}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right]. \tag{2.13}$$

Particularly, if $|f(x)| < M$, $x \in [a, b]$, then the inequality reduces the inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{b-a}{4}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{M(b-a)^2}{6}$$

which was given by Liu in [12].

Remark 4. If we take $h = 2$ in Corollary 1, then we have the trapezoid inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right] \tag{2.14}$$

which was given by Kiris and Sarikaya in [11].

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$. If $\left|f''\right|^q$, $q > 1$, is a convex mapping on $[a, b]$, then the following inequalities hold:*

$$\begin{aligned}
&\left|S_{x,h}(f)\right| \\
&\leq \frac{1}{2(b-a)^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg\{\left((x-a-m_h(x))^{p+1}+(-1)^p[m_h(x)]^{p+1}\right)^{\frac{1}{p}} \\
&\quad\times\left[\left(\frac{(b-a)(x-a)^{q+1}}{q+1}-\frac{(x-a)^{q+2}}{q+2}\right)\left|f''(a)\right|^q+\frac{(x-a)^{q+2}}{q+2}\left|f''(b)\right|^q\right]^{\frac{1}{q}} \\
&\quad+\left((m_h(x)+b-x)^{p+1}+(-1)^{p+1}[m_h(x)]^{p+1}\right)^{\frac{1}{p}} \\
&\quad\times\left[\frac{(b-x)^{q+2}}{q+2}\left|f''(a)\right|^q+\left(\frac{(b-a)(b-x)^{q+1}}{q+1}-\frac{(b-x)^{q+2}}{q+2}\right)\left|f''(b)\right|^q\right]^{\frac{1}{q}}\Bigg\}
\end{aligned} \tag{2.15}$$

for $a \leq x \leq \frac{a+b}{2}$, and

$$\begin{aligned}
&\left|S_{x,h}(f)\right| \\
&\leq \frac{1}{2(b-a)^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg\{\left([m_h(x)]^{p+1}+(x-a-m_h(x))^{p+1}\right)^{\frac{1}{p}} \\
&\quad\times\left[\left(\frac{(b-a)(x-a)^{q+1}}{q+1}-\frac{(x-a)^{q+2}}{q+2}\right)\left|f''(a)\right|^q+\frac{(x-a)^{q+2}}{q+2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}
\end{aligned}$$

$$+\left((m_h(x)+b-x)^{p+1}-[m_h(x)]^{p+1}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{(b-x)^{q+2}}{q+2}\left|f''(a)\right|^q+\left(\frac{(b-a)(b-x)^{q+1}}{q+1}-\frac{(b-x)^{q+2}}{q+2}\right)\left|f''(b)\right|^q\right]^{\frac{1}{q}}\Bigg\} \tag{2.16}$$

for $\frac{a+b}{2}\le x\le b$ *with* $h\in[0,2]$, *where* $m_h(x)=h\left(x-\frac{a+b}{2}\right)$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking the modulus of equality given in Lemma 3 and then using the well-known Hölder's inequality, we have

$$\begin{aligned}
&\left|S_{x,h}(f)\right| \\
&\le\frac{1}{2(b-a)}\int_a^b\left|P_h(x,t)\right|\left|f''(t)\right|dt \\
&=\frac{1}{2(b-a)}\left[\int_a^x|a-t|\,|t-a-m_h(x)|\left|f''(t)\right|dt\right. \\
&\quad\left.+\int_x^b|b-t|\,|t-b-m_h(x)|\left|f''(t)\right|dt\right] \\
&\le\frac{1}{2(b-a)}\left[\left(\int_a^x|t-a-m_h(x)|^p dt\right)^{\frac{1}{p}}\left(\int_a^x(t-a)^q\left|f''(t)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_x^b|t-b-m_h(x)|^p dt\right)^{\frac{1}{p}}\left(\int_x^b(b-t)^q\left|f''(t)\right|^q dt\right)^{\frac{1}{p}}\right].
\end{aligned} \tag{2.17}$$

Since $\left|f''\right|^q$ is a convex mapping on $[a,b]$,we get

$$\left|f''(t)\right|^q=\left|f''\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\le\frac{b-t}{b-a}\left|f''(a)\right|^q+\frac{t-a}{b-a}\left|f''(b)\right|^q. \tag{2.18}$$

Using (2.18), we have

$$\begin{aligned}
&\int_a^x(t-a)^q\left|f''(t)\right|^q dt \\
&\le\frac{1}{b-a}\int_a^x(t-a)^q\left[(b-t)\left|f''(a)\right|^q+(t-a)\left|f''(b)\right|^q\right] \\
&=\frac{1}{b-a}\left\{\left[\frac{(b-a)(x-a)^{q+1}}{q+1}-\frac{(x-a)^{q+2}}{q+2}\right]\left|f''(a)\right|^q+\frac{(x-a)^{q+2}}{q+2}\left|f''(b)\right|^q\right\}
\end{aligned} \tag{2.19}$$

and similarly,

$$\begin{aligned}
&\int_x^b(b-t)^q\left|f''(t)\right|^q dt \\
&\le\frac{1}{b-a}\int_x^b(b-t)^q\left[(b-t)\left|f''(a)\right|^q+(t-a)\left|f''(b)\right|^q\right] \\
&=\frac{1}{b-a}\left\{\frac{(b-x)^{q+2}}{q+2}\left|f''(a)\right|^q+\left[\frac{(b-a)(b-x)^{q+1}}{q+1}-\frac{(b-x)^{q+2}}{q+2}\right]\left|f''(b)\right|^q\right\}.
\end{aligned} \tag{2.20}$$

Moreover, we obtain

$$\int_a^x|t-a-m_h(x)|^p dt=\frac{(x-a-m_h(x))^{p+1}+(-1)^p[m_h(x)]^{p+1}}{p+1} \tag{2.21}$$

for $a \le x \le \frac{a+b}{2}$, and

$$\int_a^x |t-a-m_h(x)|^p dt = \frac{[m_h(x)]^{p+1} + (x-a-m_h(x))^{p+1}}{p+1} \tag{2.22}$$

for $\frac{a+b}{2} \le x \le b$.

Using the similar way we also have,

$$\int_x^b |t-b-m_h(x)|^p dt = \frac{(m_h(x)+b-x)^{p+1} + (-1)^{p+1}[m_h(x)]^{p+1}}{p+1} \tag{2.23}$$

for $a \le x \le \frac{a+b}{2}$, and

$$\int_x^b |t-b-m_h(x)|^p dt = \frac{(m_h(x)+b-x)^{p+1} - [m_h(x)]^{p+1}}{p+1} \tag{2.24}$$

for $\frac{a+b}{2} \le x \le b$.

Using the identities (2.19)–(2.21) and (2.23) for the case $a \le x \le \frac{a+b}{2}$ and using the identities (2.19), (2.20), (2.22) and (2.24) for the case $\frac{a+b}{2} \le x \le b$, we obtain required results (2.15) and (2.16). □

Corollary 2. *If we choose $x = \frac{a+b}{2}$ in Theorem 3, then we have the inequality*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\le \frac{(b-a)^2}{2^{4+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
&\quad\times \left\{ \left[\frac{(q+3)\left|f''(a)\right|^q + (q+1)\left|f''(b)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}} + \left[\frac{(q+1)\left|f''(a)\right|^q + (q+3)\left|f''(b)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Corollary 3. *If we choose $h = 0$ in Theorem 3 , then we have the following inequality for $a \le x \le b$*

$$\begin{aligned}
&\left| f(x) - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\
&\le \frac{1}{2(b-a)^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
&\quad\times \left\{ (x-a)^3 \left[\left(\frac{b-a}{q+1} - \frac{x-a}{q+2}\right)\left|f'(a)\right|^q + \frac{x-a}{q+2}\left|f'(b)\right|^q\right]^{\frac{1}{q}} \right. \\
&\quad\left. + (b-x)^3 \left[\frac{b-x}{q+2}\left|f'(a)\right|^q + \left(\frac{b-a}{q+1} - \frac{b-x}{q+2}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \right\}
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 4. *Let us $x = a$ and $x = b$ in Theorem 3. Subsequently, if we add the obtained result and use the triangle inequality for the modulus, we get the inequality for $h \in [0, 2]$*

$$\begin{aligned}
&\left| \frac{h-2}{2}\frac{b-a}{4}\left(f'(b) - f'(a)\right) + \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\le \frac{(b-a)^2}{2^{3+\frac{1}{p}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left((2-h)^{p+1} + h^{p+1}\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left\{\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}+\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 5. If we take $h=0$ in Corollary 4 , then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{(b-a)}{4}\left[f'(a)+f'(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{(b-a)^2}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}+\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 6. If we take $h=2$ in Corollary 4, then we have following inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq\frac{(b-a)^2}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}+\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

3. Applications to some special means

Let us recall the following means:

(a) The *Arithmetic mean:*

$$A=A(a,b):=\frac{a+b}{2},\qquad a,b\geq 0$$

(b) The *Geometric mean*:

$$G=G(a,b):=\sqrt{ab},\qquad a,b\geq 0$$

(c) The *Harmonic mean*:

$$H=H(a,b):=\frac{2}{\frac{1}{a}+\frac{1}{b}},\qquad a,b>0$$

(d) The *Logarithmic mean*:

$$L=L(a,b):=\begin{cases} a & \text{if } a=b \\ \dfrac{b-a}{\ln b-\ln a} & \text{if } a\neq b\end{cases},\qquad a,b>0$$

(e) The *Identric mean*:

$$I=L(a,b):=\begin{cases} a & \text{if } a=b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b\end{cases},\qquad a,b>0$$

(f) The *p-logarithmic mean*:

$$L_p=L_p(a,b):=\begin{cases} a & \text{if } a=b \\ \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\neq b\end{cases},\qquad a,b>0$$

where $p\in\mathbb{R}\setminus\{-1,0\}$.

The following simple relationships are known in literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 = I$ and $L_{-1} = L$.

Proposition 1. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $n \in Z$ and $|n(n-1)| \geq 3$. Then, we have*

$$\begin{aligned}
&\left| \frac{n\,(h-2)}{2}\,(x-A)\,x^{n-1} + x^n - \frac{n.h}{2} L_{n-1}^{n-1}\,(x-A) - L_n^n \right| \\
&\leq \frac{1}{2(b-a)^2} \left\{ |n(n-1)|\,a^{n-2} \left[\frac{(b-x)^4-(x-a)^4}{4} + h(x-A)\frac{(x-a)^3+(b-x)^3}{3} \right.\right.\\
&\quad \left. + (b-a)\,\frac{(x-a)^3}{3} - (b-a)\,h(x-A)\frac{(x-a)^2}{2} + \frac{[h(x-A)]^4}{6} \right] \\
&\quad + |n(n-1)|\,b^{n-2} \left[\frac{(x-a)^4-(b-x)^4}{4} - h(x-A)\frac{(x-a)^3+(b-x)^3}{3} \right. \\
&\quad \left.\left. + (b-a)\,\frac{(b-x)^3}{3} + (b-a)\,h(x-A)\frac{(b-x)^2}{2} - \frac{[h(x-A)]^4}{6} - (b-a)\,\frac{[h(x-A)]^3}{3} \right] \right\}
\end{aligned}$$

for all $a \leq x \leq A$ with $h \in [0, 2]$ and

$$\begin{aligned}
&\left| \frac{n\,(h-2)}{2}\,(x-A)\,x^{n-1} + x^n - \frac{n.h}{2} L_{n-1}^{n-1}\,(x-A) - L_n^n \right| \\
&\leq \frac{1}{2(b-a)^2} \left\{ |n(n-1)|\,a^{n-2} \left[\frac{(b-x)^4-(x-a)^4}{4} + h(x-A)\frac{(x-a)^3+(b-x)^3}{3} \right.\right.\\
&\quad \left. + (b-a)\,\frac{(x-a)^3}{3} - (b-a)\,h(x-A)\frac{(x-a)^2}{2} - \frac{[h(x-A)]^4}{6} + (b-a)\,\frac{[h(x-A)]^3}{3} \right] \\
&\quad + |n(n-1)|\,b^{n-2} \left[\frac{(x-a)^4-(b-x)^4}{4} - h(x-A)\frac{(x-a)^3+(b-x)^3}{3} \right. \\
&\quad \left.\left. \times\ (b-a)\,\frac{(b-x)^3}{3} + (b-a)\,h(x-A)\frac{(b-x)^2}{2} + \frac{[h(x-A)]^4}{6} \right] \right\}
\end{aligned}$$

for all $A \leq x \leq b$ with $h \in [0, 2]$.

Proof. The proof is immediate from Theorem 2 applied for $f(x) = x^n$, $x \in \mathbb{R}, n \in Z, |n(n-1)| \geq 3$. □

Remark 7. If we choose $h = 0$ in Proposition 1, then we have the inequality

$$\begin{aligned}
&\left| x^n - n\,(x-A)\,x^{n-1} - L_n^n \right| \\
&\leq \frac{1}{2(b-a)^2} \left\{ |n(n-1)|\,a^{n-2} \left[\frac{(b-x)^4-(x-a)^4}{4} + + (b-a)\,\frac{(x-a)^3}{3} \right] \right. \\
&\quad \left. + |n(n-1)|\,b^{n-2} \left[\frac{(x-a)^4-(b-x)^4}{4} + (b-a)\,\frac{(b-x)^3}{3} \right] \right\}
\end{aligned}$$

for $x \in [a, b]$.

Proposition 2. *Let $a, b \in (0, \infty)$ and $a < b$. Then, we have*

$$\begin{aligned}
&\left|\ln I + \frac{h\,(x-A)}{2L} - \frac{(h-2)\,(x-A)}{2x} - \ln x\right| \\
&\le \frac{1}{2(b-a)^2}\left\{\frac{1}{a^2}\left[\frac{(b-x)^4-(x-a)^4}{4} + h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right.\right. \\
&\quad \left. + (b-a)\,\frac{(x-a)^3}{3} - (b-a)\,h(x-A)\frac{(x-a)^2}{2} + \frac{[h(x-A)]^4}{6}\right] \\
&\quad + \frac{1}{b^2}\left[\frac{(x-a)^4-(b-x)^4}{4} - h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right. \\
&\quad \left.\left. + (b-a)\,\frac{(b-x)^3}{3} + (b-a)\,h(x-A)\frac{(b-x)^2}{2} - \frac{[h(x-A)]^4}{6} - (b-a)\,\frac{[h(x-A)]^3}{3}\right]\right\}
\end{aligned}$$

for all $a \le x \le A$ with $h \in [0, 2]$ and

$$\begin{aligned}
&\left|\ln I + \frac{h\,(x-A)}{2L} - \frac{(h-2)\,(x-A)}{2x} - \ln x\right| \\
&\le \frac{1}{2(b-a)^2}\left\{\frac{1}{a^2}\left[\frac{(b-x)^4-(x-a)^4}{4} + h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right.\right. \\
&\quad \left. + (b-a)\,\frac{(x-a)^3}{3} - (b-a)\,h(x-A)\frac{(x-a)^2}{2} - \frac{[h(x-A)]^4}{6} + (b-a)\,\frac{[h(x-A)]^3}{3}\right] \\
&\quad + \frac{1}{b^2}\left[\frac{(x-a)^4-(b-x)^4}{4} - h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right. \\
&\quad \left.\left. (b-a)\,\frac{(b-x)^3}{3} + (b-a)\,h(x-A)\frac{(b-x)^2}{2} + \frac{[h(x-A)]^4}{6}\right]\right\}
\end{aligned}$$

for all $A \le x \le b$ with $h \in [0, 2]$.

Proof. The assertion follows from Theorem 2 applied to the mapping $f : (0, \infty) \to (-\infty, 0)$, $f(x) = -lnx$ and the details are omitted. □

Remark 8. If we choose $h = 0$ in Proposition 2, then we have the inequality,

$$\begin{aligned}
&\left|\ln I + \frac{(x-A)}{x} - \ln x\right| \\
&\le \frac{1}{2(b-a)^2}\left\{\frac{1}{a^2}\left[\frac{(b-x)^4-(x-a)^4}{4} + (b-a)\,\frac{(x-a)^3}{3}\right]\right. \\
&\quad \left. + \frac{1}{b^2}\left[\frac{(x-a)^4-(b-x)^4}{4} + (b-a)\,\frac{(b-x)^3}{3}\right]\right\}
\end{aligned}$$

for $x \in [a, b]$.

4. Concluding Remarks

In this study, first of all, using practical identity for twice differentiable functions proved by Erden et al., we present some new upper bounds for generalized Ostrowski type inequalities by taking advantage of mappings whose second derivatives absolute values are convex. Moreover, we provide these inequalities for special means.

References

[1] A.M. Ostrowski, Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, Comment. Math. Helv. 10 (1938) 226–227.

[2] N.S. Barnett, S.S. Dragomir, An ostrowski type inequality for double integrals and applications for cubature formulae, Soochow J. Math. 27 (1) (2001) 109–114.

[3] Huseyin Budak, M. Zeki Sarikaya, On generalized ostrowski type inequalities for functions whose first derivatives absolute value are convex, Turkish J. Math. 40 (2016) 1193–1210.

[4] P. Cerone, S.S. Dragomir, J. Roumeliotis, An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications, in: RGMIA Research Report Collection, vol. 1, No. 1, 1998 Art. 4.

[5] S.S. Dragomir, C.E.M. Pearce, Selected topics on hermite–hadamard inequalities and applications, in: RGMIA Monographs, Victoria University, 2000. Online: http://www.sta.vu.edu.au/RGMIA/monographs/hermite_hadamard.html.

[6] S.S. Dragomir, R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11 (5) (1998) 91–95.

[7] S.S. Dragomir, P. Cerone, J. Roumeliotis, A new generalization of Ostrowski's integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means, in: RGMIA Research Report Collection, vol. 2, No. 1, 1999.

[8] S.S. Dragomir, A. Sofo, An integral inequality for twice differentiable mappings and applications, Tamk. J. Math. 31 (4) (2000).

[9] S.S. Dragomir, P. Cerone, A. Sofo, Some remarks on the midpoint rule in numerical integration, in: RGMIA Research Report Collection, vol. 1, No. 2, 1998 Art. 4.

[10] S. Erden, H. Budak, M.Z. Sarikaya, An Ostrowski type inequality for twice differentiable mappings and applications, Math. Model. Anal. 21 (4) (2016) 522–532.

[11] M.E. Kiris, M.Z. Sarikaya, On Ostrowski type inequalities and Čebysev type inequalities with applications, Filomat 29 (8) (2015) 1695–1713.

[12] Z. Liu, Some Ostrowski type inequalities, Math. Comput. Modelling 48 (2008) 949–960.

[13] B.G. Pachpatte, Analytic Inequalities: Recent Advances, Atlantis Pres, Paris, France, 2012.

[14] A. Qayyum, A generalized inequality of Ostrowski type for twice differentiable bounded mappings and applications, Appl. Math. Sci. 8 (38) (2014) 1889–1901.

[15] M.Z. Sarikaya, On the Ostrowski type integral inequality, Acta Math. Univ. Comenian. LXXIX (1) (2010) 129–134.

[16] M.Z. Sarikaya, On the Ostrowski type integral inequality for double integrals, Demonstratio Math. XLV (3) (2012) 533–540.

[17] M.Z. Sarikaya, A. Saglam, H. Yildirim, New inequalities of hermite-hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, Int. J. Open Probl. Comput. Sci. Math. 5 (3) (2012) 1–14.

[18] M.Z. Sarikaya, H. Yildirim, Some new integral inequalities for twice differentiable convex mappings, Nonlinear Ana. Forum 17 (2012) 1–14.

[19] F. Zafar, F.A. Mir, A generalized integral inequality for twice differentiable mappings, Kragujevac J. Math. 32 (2009) 81–96.

[20] S.S. Dragomir, N.S. Barnett, An Ostrowski type inequality for mappings whose second derivatives are bounded and applications, in: RGMIA Research Report Collection, V.U.T., vol. 1, 1999, pp. 67–76.

Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 171 (2017) 328–331
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

On Riesz summability factors of Fourier series

Şebnem Yildiz

Department of Mathematics, Ahi Evran University, Kırşehir, Turkey

Received 22 February 2017; received in revised form 8 June 2017; accepted 15 June 2017
Available online 4 July 2017

Abstract

In this paper, a main theorem dealing with $|\bar{N}, p_n|_k$ summability method has been generalized for $\varphi - |\bar{N}, p_n; \delta|_k$ summability by using different and general summability factors of Fourier series.

Keywords: Summability factors; Absolute matrix summability; Fourier series; Infinite series; Hölder inequality; Minkowski inequality

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots. \tag{1.1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1.2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (P_n \neq 0), \tag{1.3}$$

E-mail address: sebnemyildiz@ahievran.edu.tr.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.06.003

defines the sequence (σ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [1]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\Delta\sigma_{n-1}|^k < \infty. \tag{1.4}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $\left|\bar{N}, p_n\right|_k$ summability is the same as $|C, 1|_k$ (resp. $\left|\bar{N}, p_n\right|$) summability.

The $\varphi - \left|\bar{N}, p_n; \delta\right|_k$ summability method is defined by Seyhan (see [3]). The series $\sum a_n$ is said to be summable $\varphi - \left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1$ and $\delta \geq 0$, if

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|\sigma_n - \sigma_{n-1}|^k < \infty. \tag{1.5}$$

If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - \left|\bar{N}, p_n; \delta\right|_k$ summability is the same as $\left|\bar{N}, p_n\right|_k$ summability.

Let f be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$.

Without loss of generality we may assume that the constant term in the Fourier series of f is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0, \tag{1.6}$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} C_n(t). \tag{1.7}$$

2. Known result

Many papers dealing with $\left|\bar{N}, p_n\right|_k$ summability factors and $\varphi - \left|\bar{N}, p_n; \delta\right|_k$ summability factors of Fourier series have been done (see [4–10]). Among them, Bor [5] has proved the following theorem.

Theorem A. *If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$, and $\sum_{v=1}^{n} P_v C_v(t) = O(P_n)$, then the series $\sum C_n(t)P_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.*

3. Main result

The aim of this paper is to prove a more general theorem which includes the above mentioned result as special cases. Now, we shall prove the following theorem.

Theorem B. *Let (p_n) and (λ_n) be sequences satisfying the conditions of Theorem* A *and let (φ_n) be a sequence of positive real numbers such that*

$$\varphi_n p_n = O(P_n), \tag{3.1}$$

$$\sum_{n=v+1}^{\infty}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}} = O\left(\varphi_v^{\delta k}\frac{1}{P_v}\right), \tag{3.2}$$

$$\sum_{n=1}^{m}\varphi_n^{\delta k} p_n\lambda_n = O(1) \quad as \quad m \to \infty, \tag{3.3}$$

$$\sum_{n=1}^{m}\varphi_n^{\delta k} P_n\Delta\lambda_n = O(1) \quad as \quad m \to \infty. \tag{3.4}$$

Then the series $\sum C_n(t)P_n\lambda_n$ is summable $\varphi - \left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1$ and $0 \leq \delta k < 1$.

We need the following lemma for the proof of Theorem B.

Lemma 1 (*[5]*). *If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$, then $P_n\lambda_n = O(1)$ as $n \to \infty$ and $\sum P_n\Delta\lambda_n < \infty$.*

Remark 1. It should be noted that if we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$ in this theorem, (3.4) is satisfied by Lemma 1. Condition (3.3) is satisfied by a hypothesis of Theorem A. Also in this case conditions (3.1) and (3.2) are obvious.

4. Proof of Theorem B

Let $I_n(t)$ be the sequence of $(\bar{N}, p_n)$ means of the series $\sum C_n(t)P_n\lambda_n$. Then, by definition, we have

$$I_n(t) = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} C_i(t)P_i\lambda_i = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})C_v(t)P_v\lambda_v.$$

Then, for $n \geq 1$, we have

$$I_n(t) - I_{n-1}(t) = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}C_v(t)P_v\lambda_v.$$

By Abel's transformation, we have

$$\begin{aligned}
I_n(t) - I_{n-1}(t) &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta(P_{v-1}\lambda_v)\sum_{r=1}^{v}P_rC_r(t) + \frac{p_n}{P_n}\lambda_n\sum_{v=1}^{n}P_vC_v(t)\\
&= O(1)\left\{\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}(P_v\lambda_v - p_v\lambda_v - P_v\lambda_{v+1})P_v\right\} + O(1)p_n\lambda_n\\
&= O(1)\left\{\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vp_v\lambda_v + p_n\lambda_n\right\}\\
&= O(1)\left\{I_{n,1}(t) + I_{n,2}(t) + I_{n,3}(t)\right\}.
\end{aligned}$$

To prove Theorem B, by Minkowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|I_{n,r}(t)|^k < \infty, \quad for \quad r = 1, 2, 3.$$

First, using the hypotheses of Theorem B, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,1}(t)|^k &= \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\right|^k\\
&\leq \sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\\
&= O(1)\sum_{v=1}^{m}P_vP_v\Delta\lambda_v\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}P_v\Delta\lambda_v = O(1) \quad as \quad m \to \infty.
\end{aligned}$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k' where $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,2}(t)|^k = \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vp_v\lambda_v\right|^k$$

$$\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v^k p_v \lambda_v^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$
$$= O(1) \sum_{v=1}^{m} P_v^k \lambda_v^k p_v \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} P_v^k \lambda_v^k p_v \varphi_v^{\delta k} \frac{1}{P_v}$$
$$= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} (P_v \lambda_v)^{k-1} p_v \lambda_v$$
$$= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} p_v \lambda_v = O(1) \quad as \quad m \to \infty,$$

by virtue of the hypotheses of Theorem B and Lemma 1. Finally, using the fact that $P_n\lambda_n = O(1)$, by Lemma 1, we obtain that

$$\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} |I_{n,3}(t)|^k = \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} |p_n \lambda_n|^k$$
$$\leq \sum_{n=1}^{m} \varphi_n^{\delta k} \varphi_n^{k-1} (p_n \lambda_n)^{k-1} (p_n \lambda_n)$$
$$= \sum_{n=1}^{m} \varphi_n^{\delta k} (\varphi_n p_n)^{k-1} \lambda_n^{k-1} (p_n \lambda_n)$$
$$= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k} (P_n \lambda_n)^{k-1} (p_n \lambda_n)$$
$$= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k} (p_n \lambda_n) = O(1) \quad as \quad m \to \infty,$$

by virtue of the hypotheses of Theorem B. This completes the proof of Theorem B.

Acknowledgment

The author would like to express her sincerest thanks to the referee for valuable suggestions for the improvement of this paper.

References

[1] G.H. Hardy, Divergent Series, Oxford Univ. Press, Oxford, 1949.
[2] H. Bor, On two summability methods, Math. Proc. Cambridge Philos. Soc. 97 (1985) 147–149.
[3] H. Seyhan, (Ph.D thesis), Erciyes University, Kayseri, 1995.
[4] H. Bor, Multipliers for $\left|\bar{N}, p_n\right|_k$ summability of Fourier series, Bull. Inst. Math. Acad. Sin. (N.S.) 17 (1989) 285–290.
[5] H. Bor, On the absolute summability factors of Fourier series, J. Comput. Anal. Appl. 8 (2006) 223–227.
[6] H. Bor, Some new results on infinite series and Fourier series, Positivity 19 (2015) 467–473.
[7] H. Bor, Some new results on absolute Riesz summability of infinite series and Fourier series, Positivity 20 (2016) 599–605.
[8] K.N. Mishra, Multipliers for $\left|\bar{N}, p_n\right|_k$ summability of Fourier series, Bull. Inst. Math. Acad. Sin. (N.S.) 14 (1986) 431–438.
[9] H.S. Özarslan, On the local property of $\phi - |\bar{N}, p_n; \delta|_k$ summability of factored Fourier series, Bull. Inst. Math. Acad. Sin. (N.S.) 25 (4) (1997) 311–316.
[10] H.S. Özarslan, A note on $\left|\bar{N}, p_n\right|_k$ summability factors, Int. J. Pure Appl. Math. 13 (2004) 485–490.

Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute 171 (2017) 332–349

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

Coupled systems of Caputo type fractional Δ-difference boundary value problems at resonance

Yousef Gholami*, Kazem Ghanbari

Department of Applied Mathematics, Sahand University of Technology, P. O. Box: 51335-1996, Tabriz, Iran

Received 6 July 2016; received in revised form 27 March 2017; accepted 29 March 2017
Available online 4 May 2017

Abstract

The main research line of this paper is concerned with the existence and uniqueness of solutions for a certain class of coupled systems of Caputo type fractional Δ-difference boundary value problems at resonance. To this aim, we use coincidence degree theory to obtain existence results and impose growth controlling conditions on nonlinearities, uniqueness results will be concluded. At the end by means of an illustrative example the obtained main results will be implemented.

Keywords: Fractional sums and differences; Coincidence degree theory; Existence and uniqueness; Resonance

1. Introduction

The main objective of this paper is devoted to study of the following coupled system of higher order Caputo type fractional Δ-difference boundary value problems at resonance

$$\begin{cases} \Delta_*^\alpha y(t) = f(t+N-\alpha-2, z, \Delta z, \Delta^2 z, \dots, \Delta^{N-1} z), \\ \Delta_*^\alpha z(t) = g(t+N-\alpha-2, y, \Delta y, \Delta^2 y, \dots, \Delta^{N-1} y), \end{cases} \quad t = a, a+1, \dots, b, \tag{1.1}$$

$$\begin{cases} \Delta^{N-i} y(a+N-\alpha-2) = 0,\ \Delta^{N-1} y(a+N-\alpha-2) = \Delta^{N-1} y(b+N-\alpha-1),\ i = 2, 3, \dots, N, \\ \Delta^{N-i} z(a+N-\alpha-2) = 0,\ \Delta^{N-1} z(a+N-\alpha-2) = \Delta^{N-1} z(b+N-\alpha-1),\ i = 2, 3, \dots, N, \end{cases} \tag{1.2}$$

where $N-1 < \alpha \le N$, $N \in \mathbb{N}_2$, and $a \in \mathbb{Z}_1$, $b \in \mathbb{Z}_2$ with $a < b$. Δ_*^α denotes the Caputo type fractional Δ-difference of order $\alpha > 0$. In this paper we will assume that $f, g : \mathbb{N}_{a+N-\alpha-2}^{b+N-\alpha-1} \times \mathbb{R}^N \to \mathbb{R}$ are continuous functions.

* Corresponding author.
E-mail addresses: y_gholami@sut.ac.ir (Y. Gholami), kghanbari@sut.ac.ir (K. Ghanbari).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.03.005
2346-8092/

With a sharp overview on the theory of fractional calculus, it is easy to check that discrete versions of the fractional order operators do not follow the standard structure of corresponding continuous ones. More precisely in general, all of the Riemann–Liouville based fractional operators have an impulse response function $h_\gamma(t) = \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)}$ as their kernels. In discrete versions of these operators, corresponding kernels take a different forms such that there is no the power term like one appeared in h_γ, any more. Instead, these new kernels are closely related with just the Euler's gamma function $\Gamma(.)$.

The concept memory in fractional order operators turn back to these impulse response functions. Namely, according to the varying of order γ in $h_\gamma(t)$, fractional operators keep or lose the memory. So fractional order operators possess full, null or one sided memory. For more details see [1–5].

On the other hand since less than a decade earlier by now, the theory of discrete fractional calculus is taking its standard shape. Unerring, keeping the memory can be considered as one of the extraordinary properties of the newly defined fractional operators (fractional Δ and ∇ difference operators). In discrete fractional operators, the impulse response functions are of the form of fractional falling or, rising functions $\underline{h}_\gamma(t) = \frac{(t-\tau-1)^{\underline{\gamma-1}}}{\Gamma(\gamma)}$ or, $\overline{h}_\gamma(t) = \frac{(t-\tau+1)^{\overline{\gamma-1}}}{\Gamma(\gamma)}$, respectively. For more details about discrete fractional calculus, see [6–11].

Besides this advantage, establishing solvability of discrete fractional order boundary value problems is one of the most popular research areas in discrete fractional calculus. As pioneering works, we suggest the collection of papers due to P.W. Eloe and F.M. Atici [7–9,12], works due to C. Goodrich [10,11,13,14] and Y. Gholami and K. Ghanbari [15–17]. The most applied technique in the mentioned references is fixed point theory (fixed point theorems such as Krasnoselśkii, Krasnoselśkii–Zabreiko, nonlinear alternative of Leray–Schauder and Banach). In this paper, we are going to apply a different technique to obtain existence and uniqueness of solutions for coupled resonant system (1.1) and (1.2), that is the coincidence degree theory due to Jean Mawhin. For an eager follower of the resonant problems, we suggest the references [10,11,13,14,18–29] and references cited therein.

W. Rui in [26], considered the following two-point Caputo fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f(t, x(t), x'(t), x''(t)), \quad t \in [0,1], \ 2 < \alpha \le 3, \\ x(0) = x'(0) = 0, \quad x''(0) = x''(1), \end{cases}$$

where $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous. The author used coincidence degree theory to obtain at least one solution for fractional boundary value problems. This paper together with [21] are the main motivation of this work.

The rest of this paper is organized as follows. In Section 2, we present necessary requirements of the discrete fractional calculus and a quick overview of the coincidence degree theory. In Section 3, first we apply coincidence degree theory for the existence at least one solution for coupled resonant system (1.1), (1.2) and then by means of nonlinearities growth restriction, an uniqueness criterion will be presented. In Section 4, implementing the theoretical obtained results, we present an illustrative example.

2. Preliminaries

We begin this section with basic definitions and lemmas of fractional Δ-difference calculus based on the references [6,8,10]. We then give an overview to the J. Mawhin's coincidence degree theory [25,30].

Definition 2.1. The fractional falling function is defined by

$$t^{\underline{\alpha}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \quad t \in \mathbb{R} \setminus \{..., \alpha-3, \alpha-2, \alpha-1\}, \ \alpha \in \mathbb{R}, \tag{2.1}$$

such that

(i) $t^{\underline{\alpha}} = 0$, provided that $\{t+1-\alpha\} \in \mathbb{Z}_- = \{..., -2, -1, 0\}$, $\alpha \in \mathbb{R}$,
(ii) $t^{\underline{0}} = 1$,
(iii) $\alpha^{\underline{\alpha}} = \Gamma(\alpha+1)$.

We will use the following notation.

$$\begin{aligned} &\mathbb{N}_a = \{a, a+1, a+2, \ldots\}, \ {}_b\mathbb{N} = \{..., b-2, b-1, b\}, \\ &\mathbb{N}_c^d = \{c, c+1, \ldots, d-1, d\}, \ a, b \in \mathbb{R}, \ c, d \in \mathbb{Z}. \end{aligned} \tag{2.2}$$

As can be seen, the fractional falling functions, make the main structures of kernels for discrete fractional order operators. So we can now define these operators as follows.

Definition 2.2 (*c.f. [10], Sec. 2.3, Def. 2.25, p. 101*)**.** The left sided fractional Δ-sum of order $\alpha > 0$ for $f : \mathbb{N}_a \to \mathbb{R}$ is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t - \sigma(s))^{\underline{\alpha-1}} f(s), \tag{2.3}$$

where $\alpha > 0,\ \sigma(s) = s + 1$.

Remark 2.3. The left sided fractional Δ-sum of order $\alpha > 0$, defined by (2.3) has the following property:

- $\Delta_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$.

Definition 2.4 (*c.f [6], Sec. 3, Def. 13, p. 1607*)**.** The left sided Caputo type fractional Δ-difference of order $\alpha > 0$ for $f : \mathbb{N}_a \to \mathbb{R}$ is given by

$$\Delta_*^{\alpha} f(t) = \Delta^{-(n-\alpha)} \Delta_t^n f(t), \quad t \in \mathbb{N}_{a+n-\alpha}, \tag{2.4}$$

such that $\alpha > 0,\ n - 1 < \alpha \le n,\ n \in \mathbb{N}$.

In the following lemma, we give the composition and power rules for Caputo type fractional Δ-difference operators, that will be needed to obtain the main results.

Lemma 2.5. *Assume that f is a real-valued function defined on $\mathbb{N}_a$ and $\alpha > 0,\ 0 \le n - 1 < \nu \le n$. Then*

(Q_1) $\Delta_a^{-\alpha} \Delta_*^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\underline{k}}}{k!} \Delta^k f(a)$.
(Q_2) $\Delta_*^{\alpha} \Delta_a^{-\alpha} f(t) = f(t)$.
(Q_3) $\Delta_a^{-\alpha} (t-a)^{\underline{\nu}} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} (t-a)^{\underline{\nu+\alpha}},\ \nu + \alpha + 1 \notin \mathbb{Z}_-$.
(Q_4) $\Delta^M \Delta_a^{-\alpha} f(t) = \Delta^{M-\alpha} f(t), \quad M \in \mathbb{N}_1$.

Next we discuss the coincidence degree theory, see [25] and chapters IV and V in [30].

Definition 2.6. Assume that $\mathfrak{B}$ and $\mathfrak{D}$ are real normed spaces. A linear mapping $L : domL \subset \mathfrak{B} \to \mathfrak{D}$ is called a Fredholm mapping provided that the following conditions hold:

(i) $\ker L$ has a finite dimension,
(ii) ImL is closed and has a finite codimension.

Let L be a Fredholm mapping. Then its $index$ is given by

$$IndL = dim\ kerL - codim\ ImL.$$

Assume that L is a Fredholm mapping with index zero and there exist continuous projectors $P : \mathfrak{B} \to \mathfrak{B}$ and $Q : \mathfrak{D} \to \mathfrak{D}$ such that

$$ImP = \ker L, \quad \ker Q = ImL, \quad \mathfrak{B} = \ker L \oplus \ker P, \quad \mathfrak{D} = ImL \oplus ImQ.$$

It follows that the mapping

$$L|_{domL \cap \ker P} : domL \cap \ker P \to ImL$$

is invertible. Let us denote the inverse by $K_P : ImL \to domL \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q} : Z \to domL \cap \ker P$ is defined by $K_{P,Q} = K_P(I - Q)$.

If L is a Fredholm mapping of index zero, then for every isomorphism $J : ImQ \to \ker L$, the mapping $JQ + K_{P,Q} : Z \to domL$ is an isomorphism and, for every $u \in domL$,

$$(JQ + K_{P,Q})^{-1} u = (L + J^{-1}P)u.$$

Definition 2.7. Let $L : domL \subset \mathfrak{B} \to \mathfrak{D}$ be a Fredholm mapping, E be a metric space, and $\mathcal{N} : E \to \mathfrak{D}$ be a mapping. $\mathcal{N}$ is to be called L-compact on E provided that, $Q\mathcal{N} : E \to \mathfrak{D}$ is continuous and $K_{P,Q} : E \to \mathfrak{B}$ is compact on E. In addition, we say that, $\mathcal{N}$ is L-completely continuous if it is L-compact on every bounded $E \subset \mathfrak{B}$.

Theorem 2.8. *Let $\Omega \subset \mathfrak{B}$ be open and bounded, L be a Fredholm mapping of index zero and $\mathcal{N}$ be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lu \neq \lambda \mathcal{N} u$ *for every* $(u, \lambda) \in ((domL \setminus \ker L) \cap \partial\Omega) \times (0, 1)$;
(ii) $\mathcal{N} u \notin ImL$ *for every* $u \in \ker L \cap \partial\Omega$;
(iii) $\deg(JQ\mathcal{N}|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$ *with* $Q : \mathfrak{D} \to \mathfrak{D}$ *a continuous projector such that* $\ker Q = ImL$ *and* $J : ImQ \to \ker L$ *is an isomorphism.*

Then the equation $Lu = \mathcal{N} u$ *has at least one solution in* $domL \cap \overline{\Omega}$.

Before beginning the main body of our work, we describe the resonant nature of the discrete fractional coupled system (1.1), (1.2). The operator $Ly = \Delta^{\alpha}_{a*} y$ in the homogeneous fractional Δ-difference boundary value problem

$$\begin{cases} \Delta^{\alpha}_{*} y(t) = 0, \quad t \in \mathbb{N}^{b}_{a},\ N-1 < \alpha \leq N,\ N \in \mathbb{N}_2, \\ \Delta^{N-i} y(a + N - \alpha - 2) = 0,\ i = 2, 3, \ldots, N, \end{cases} \tag{2.5}$$

is said to be resonant, provided that the fractional boundary value problem (2.5) has a nontrivial solution and L is said to be non-resonant otherwise. On the other hand by property (Q_1) in Lemma 2.5, it follows that

$$\Delta^{\alpha}_{*} y(t) = 0 \quad \text{iff} \quad y(t) = \sum_{k=0}^{N-1} c_k (t-a)^{\underline{k}}.$$

Thus

$$\Delta^{N-j} y(t) = \sum_{k=0}^{N-1} c_k \Delta^{N-j} (t-a)^{\underline{k}}, \qquad j = 2, 3, \ldots, N.$$

So, using the power rule (Q_4) in Lemma 2.5 and taking $t = a + N - \alpha - 2$ one has

$$\Delta^{N-j} y(a + N - \alpha - 2) = \sum_{k=0}^{N-1} c_k \frac{\Gamma(k+1)}{\Gamma(k+j-N+1)} \frac{\Gamma(N-\alpha-1)}{\Gamma(2N-k-j-1-\alpha)}, \qquad j = 2, 3, \ldots, N.$$

Equivalently, we have the following:

$$\begin{cases} j = 2 \Rightarrow C_{N-2} \dfrac{(N-2)!}{0!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-1)} + C_{N-1} \dfrac{(N-1)!}{1!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-2)} = 0, \\ j = 3 \Rightarrow C_{N-3} \dfrac{(N-3)!}{0!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-1)} + C_{N-2} \dfrac{(N-2)!}{1!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-2)} \\ \quad + C_{N-1} \dfrac{(N-1)!}{2!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-3)} = 0, \\ . \\ . \\ . \\ j = N-2 \Rightarrow C_2 \dfrac{2!}{0!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-1)} + C_3 \dfrac{3!}{1!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-2)} + \cdots + C_{N-1} \dfrac{(N-1)!}{(N-3)!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(2-\alpha)} = 0, \\ j = N-1 \Rightarrow C_1 \dfrac{1!}{0!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-1)} + C_2 \dfrac{2!}{1!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-2)} + \cdots + C_{N-1} \dfrac{(N-1)!}{(N-2)!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(1-\alpha)} = 0, \\ j = N \Rightarrow C_0 \dfrac{0!}{0!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-1)} + C_1 \dfrac{1!}{1!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(N-\alpha-2)} + \cdots + C_{N-1} \dfrac{(N-1)!}{(N-1)!} . \dfrac{\Gamma(N-\alpha-1)}{\Gamma(-\alpha)} = 0. \end{cases}$$

Therefore, the sequence of above equalities based on the first one ($j = 2$), ensure that the boundary value problem (2.5) has a nontrivial solution. We are concerned with the resonance case.

At the end of this section, we introduce the appropriate Banach spaces as follows. Our basic Banach space is

$$E = C\left(\mathbb{N}^{b+N-\alpha-1}_{a+N-\alpha-2}, \mathbb{R}\right), \qquad N-1 < \alpha \leq N,\ N \in \mathbb{N}_2, \tag{2.6}$$

equipped with the standard max-norm

$$\|\mathfrak{f}\|_E = \max|\mathfrak{f}(t)|, \qquad t \in \mathbb{N}^{b+N-\alpha-1}_{a+N-\alpha-2}.$$

Now we define the Banach space

$$X=\left\{u\,\middle|\, u\in E,\ \Delta^{N-i}u\in E,\ i=1,2,\ldots,N-1,\ N\in\mathbb{N}_2\right\}, \tag{2.7}$$

with corresponding norm

$$\|u\|_X=\max\left\{\|u\|_E,\ \left\|\Delta^{N-i}u\right\|_E;\ i=1,2,\ldots,N-1,\ N\in\mathbb{N}_2\right\}. \tag{2.8}$$

At last our desired Banach spaces are defined by

$$(\mathfrak{B},\|.\|_{\mathfrak{B}}),\quad \begin{cases}\mathfrak{B}=X\times X,\\ \|(u,v)\|_{\mathfrak{B}}=\max\{\|u\|_X,\|v\|_X\},\end{cases} \tag{2.9}$$

and

$$(\mathfrak{D},\|.\|_{\mathfrak{D}}),\quad \begin{cases}\mathfrak{D}=E\times E,\\ \|(u,v)\|_{\mathfrak{D}}=\max\{\|u\|_E,\|v\|_E\}.\end{cases} \tag{2.10}$$

3. Main results

We begin the main results with constructing preparatory tools for applying coincidence degree theory as follows. Let us take $L_1:\operatorname{dom}L_1\cap X\to E$ as

$$L_1y=\Delta_*^{\alpha}y, \tag{3.1}$$

where

$$\operatorname{dom}L_1=\left\{y\in X\,\middle|\,\Delta^{N-i}y(a+N-\alpha-2)=0,\right.$$
$$\left.\Delta^{N-1}y(a+N-\alpha-2)=\Delta^{N-1}y(b+N-\alpha-1),\ i=2,3,\ldots,N\right\}. \tag{3.2}$$

Similarly, we define $L_2:\operatorname{dom}L_2\cap X\to E$ as

$$L_2z=\Delta_*^{\alpha}z, \tag{3.3}$$

where

$$\operatorname{dom}L_2=\left\{z\in X\,\middle|\,\Delta^{N-i}z(a+N-\alpha-2)=0,\right.$$
$$\left.\Delta^{N-1}z(a+N-\alpha-2)=\Delta^{N-1}z(b+N-\alpha-1),\ i=2,3,\ldots,N\right\}. \tag{3.4}$$

Therefore we can define $L:\operatorname{dom}L\cap\mathfrak{B}\to\mathfrak{D}$ as

$$L(y,z)=(L_1y,L_2z), \tag{3.5}$$

where

$$\operatorname{dom}L=\{(y,z)\in\mathfrak{B}|\ y\in\operatorname{dom}L_1,\ z\in\operatorname{dom}L_2\}. \tag{3.6}$$

Also we define $\mathcal{N}:\mathfrak{B}\to\mathfrak{D}$ as below

$$\mathcal{N}(y,z)=(\mathcal{N}_1z,\mathcal{N}_2y), \tag{3.7}$$

where $\mathcal{N}_k:X\to E$ for $k=1,2$, are defined as

$$\begin{aligned}\mathcal{N}_1z&=f(t+N-\alpha-2,z,\Delta z,\Delta^2z,\ldots,\Delta^{N-1}z),\\ \mathcal{N}_2y&=g(t+N-\alpha-2,y,\Delta y,\Delta^2y,\ldots,\Delta^{N-1}y).\end{aligned} \tag{3.8}$$

Therefore considering (1.1), (1.2), (3.1)–(3.8) lead us to the $L(y,z)=\mathcal{N}(y,z)$.

To obtain claimed solvability results for the coupled system (1.1), (1.2), we shall prepare ourselves to apply the coincidence degree theory. So, first we prove that the mapping L defined by (3.5) is a Fredholm operator of index zero and then the mapping $\mathcal{N}$ defined by (3.7) is L-compact.

Lemma 3.1. *The mapping $L : dom\ L \cap \mathfrak{B} \to \mathfrak{D}$ defined by* (3.1)–(3.5) *is a Fredholm operator having index zero.*

Proof. Using (Q_1) in Lemma 2.5, it follows that $\ker L = \left(c_1(t-a)^{\underline{N-1}}, d_1(t-a)^{\underline{N-1}}\right)$. So $\ker L \cong \mathbb{R}^2$. Suppose that $(u, v) \in \operatorname{Im} L$. Thus there exists $(y, z) \in \operatorname{dom} L$ such that $L(y, z) = (u, v)$. Equivalently by means of property (Q_1) in Lemma 2.5, we deduce that

$$\begin{aligned} y(t) &= \Delta_a^{-\alpha} u(t) + c_1(t-a)^{\underline{N-1}} + c_2(t-a)^{\underline{N-2}} + \cdots + c_N, \\ z(t) &= \Delta_a^{-\alpha} v(t) + d_1(t-a)^{\underline{N-1}} + d_2(t-a)^{\underline{N-2}} + \cdots + d_N. \end{aligned}$$

The definition of the dom L in (3.2)–(3.6), implies that $c_i = d_i = 0,\ i = 2, 3, \dots, n$. Hence

$$\begin{aligned} y(t) &= \Delta_a^{-\alpha} u(t) + c_1(t-a)^{\underline{N-1}}, \\ z(t) &= \Delta_a^{-\alpha} v(t) + d_1(t-a)^{\underline{N-1}}. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \Delta^{N-1} y(t) &= \Delta^{N-1}\left(\Delta_a^{-\alpha} u\right)(t) + c_1 \Gamma(N) \\ \Delta^{N-1} z(t) &= \Delta^{N-1}\left(\Delta_a^{-\alpha} v\right)(t) + d_1 \Gamma(N). \end{aligned}$$

Equivalently, we have

$$\begin{aligned} \Delta^{N-1} y(t) &= \Delta_a^{N-\alpha-1} u(t) + c_1 \Gamma(N) \\ \Delta^{N-1} z(t) &= \Delta_a^{N-\alpha-1} v(t) + d_1 \Gamma(N). \end{aligned} \tag{3.9}$$

Let us take a look once again to the boundary conditions

$$\begin{aligned} \Delta^{N-1} y(a+N-\alpha-2) &= \Delta^{N-1} y(b+N-\alpha-1), \\ \Delta^{N-1} z(a+N-\alpha-2) &= \Delta^{N-1} z(b+N-\alpha-1). \end{aligned}$$

Imposing these boundary conditions on (3.9), we get the following

$$\sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}} u(s) = 0, \quad \sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}} v(s) = 0.$$

Assume given (u, v) satisfies the recent equalities. If we take $y(t) = \Delta_a^{-\alpha} u(t)$ and $z(t) = \Delta_a^{-\alpha} v(t)$, then immediately one may derive that $(y, z) \in \operatorname{dom} L$. Thus we have

$$\operatorname{Im} L = \left\{ (u, v) \middle| \sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}} u(s) = 0, \quad \sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}} v(s) = 0 \right\}. \tag{3.10}$$

We now define the operators $Q_k : E \to E,\ k = 1, 2$ as

$$Q_1 u(t) = \frac{\sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}} u(s)}{\sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}}}, \quad Q_2 v(t) = \frac{\sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}} v(s)}{\sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}}}. \tag{3.11}$$

Clearly $Q(u, v) = (Q_1 u, Q_2 v) \cong \mathbb{R}^2$. It is easy to check that for $u, v \in E$, the following properties hold:

$$Q_1^2 u(t) = Q_1 u(t), \quad Q_2^2 v(t) = Q_2 v(t).$$

Consequently we conclude that $Q^2(u, v) = Q(u, v)$. Since $(u, v) = (u, v) - Q(u, v) + Q(u, v)$, one can deduce that $\mathfrak{D} = \operatorname{Im} L + \operatorname{Im} Q$. In addition as a result of $\operatorname{Im} L \cap \operatorname{Im} Q = \{(0, 0)\}$, we find that $\mathfrak{D} = \operatorname{Im} L \oplus \operatorname{Im} Q$. Finally by means of Definition 2.6, one has

$$\operatorname{Ind} L = \dim \ker L - \operatorname{codim} \operatorname{Im} L = \dim \ker\ L - [\dim \mathfrak{D} - \dim \operatorname{Im} L] = 2 - [4 - 2] = 0.$$

Therefore the operator L defined above is a Fredholm operator of index zero. This completes the proof. □

In this position, we define the operators $P_k : X \to X,\ k = 1, 2$ by

$$P_1 u(t) = \frac{\Delta^{N-1} u(a)}{(N-1)!}(t-a)^{\underline{N-1}}, \quad P_2 v(t) = \frac{\Delta^{N-1} v(a)}{(N-1)!}(t-a)^{\underline{N-1}}. \tag{3.12}$$

In this case, using the property (Q$_4$) in Lemma 2.5, we conclude that $P_1^2 u = P_1 u$ and $P_2^2 v = P_2 v$. Now let us define $P : \mathfrak{B} \to \mathfrak{B}$ as $P(u, v) = (P_1 u, P_2 v)$. So we have

$$\ker P = \left\{ (u, v) \middle| \; \Delta^{N-1} u(a) = 0, \; \Delta^{N-1} v(a) = 0 \right\}.$$

Since $(u, v) = (u, v) - P(u, v) + P(u, v)$, it is easy to check that $\mathfrak{B} = \ker P + \ker L$, and because of $\ker P \cap \ker L = \{(0, 0)\}$, we deduce that $\mathfrak{B} = \ker P \oplus \ker L$.

In the sequel we define the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ as follows:

$$K_P(u, v) = \left(\Delta_a^{-\alpha} u, \Delta_a^{-\alpha} v \right). \tag{3.13}$$

Thus for each $(u, v) \in ImL$, we have

$$LK_P(u, v) = L\left(\Delta_a^{-\alpha} u, \Delta_a^{-\alpha} v \right) = \left(\Delta_*^{\alpha} \Delta_a^{-\alpha} u, \Delta_*^{\alpha} \Delta_a^{-\alpha} v \right) = (u, v). \tag{3.14}$$

On the other hand, since for every $(u, v) \in \operatorname{dom} L \cap \ker P$, we have $\Delta^{N-1} u(a) = 0$ and $\Delta^{N-1} v(a) = 0$, hence in the identities

$$\begin{aligned} u(t) &= \Delta_a^{-\alpha} \Delta_*^{\alpha} u(t) + c_1 (t-a)^{\underline{N-1}} + c_2 (t-a)^{\underline{N-2}} + \cdots + c_N, \\ v(t) &= \Delta_a^{-\alpha} \Delta_*^{\alpha} v(t) + d_1 (t-a)^{\underline{N-1}} + d_2 (t-a)^{\underline{N-2}} + \cdots + d_N, \end{aligned}$$

all of the coefficients $c_i, d_i = 0$ for $i = 1, 2, \ldots, N$. This implies that

$$K_P L(u, v) = \left(\Delta_a^{-\alpha} \Delta_{a*}^{\alpha} u, \Delta_a^{-\alpha} \Delta_{a*}^{\alpha} v \right) = (u, v). \tag{3.15}$$

So, using (3.14) and (3.15), we conclude that $K_P = (L_{\operatorname{dom} L \cap \ker P})^{-1}$.

We are now ready to prove the second step, i.e. prove that $\mathcal{N}$ defined by (3.7), (3.8) is an L-compact operator.

Lemma 3.2. *Assume that Ω is an open and bounded subset of $\mathfrak{B}$ such that $domL \cap \overline{\Omega} \neq \varnothing$. Then the operator $\mathcal{N}$ defined by* (3.7)*,* (3.8) *is L-compact.*

Proof. Continuity of $f, g : \mathbb{N}_{a+N-\alpha-2}^{b+N-\alpha-1} \times \mathbb{R}^N \to \mathbb{R}$ ensures that $Q\mathcal{N}(\overline{\Omega})$ and $K_P(I - Q)\mathcal{N}(\overline{\Omega})$ are bounded. So, using the *Arzela–Ascoli* theorem, it is sufficient to prove that $K_P(I - Q)\mathcal{N}(\overline{\Omega}) \subset \mathfrak{B}$ is equicontinuous. The discrete nature of the fractional delta difference operators easily proves it. Hence, the proof is completed. $\square$

Lemma 3.3. *Assume that $N - 1 < \alpha \leq N, \; N \in \mathbb{N}_2$. Then*

$$\left\| (t-a)^{\underline{N-1}} \right\|_X = \max\left\{ \mathcal{O}_{N-1}^{N-1}, \mathcal{O}_{i-1}^{i-1}, \; i = 1, 2, \ldots, N-1 \right\},$$

where

$$\begin{aligned} \mathcal{O}_{N-1}^{N-1} &= \max (t-a)^{\underline{N-1}} = \max\left\{ ((N-1) - \alpha - 1)^{\underline{N-1}}, (\alpha + b - a + (N-1))^{\underline{N-1}} \right\}, \\ \mathcal{O}_{i-1}^{i-1} &= \max \frac{(N-1)!}{i!} \left\{ ((i-1) - \alpha - 1)^{\underline{i-1}}, (\alpha + b - a + (i-1))^{\underline{i-1}} \right\}, \; i = 1, 2, \ldots, N-1. \end{aligned}$$

Note that the $N-1$ as superscript in $\mathcal{O}_{N-1}^{N-1}$ depends on the $N-1$ in the falling exponent of the falling function $(\ldots)^{\underline{N-1}}$ and another one as subscript refers to the basis in the corresponding falling functions.

Proof. According to (2.8), we have

$$\left\| (t-a)^{\underline{N-1}} \right\|_X = \max\left\{ \left\| (t-a)^{\underline{N-1}} \right\|_E, \; \left\| \Delta^{N-i} (t-a)^{\underline{N-1}} \right\|_E ; \; i = 1, 2, 3, \ldots, N-1 \right\}.$$

A direct calculation indicates that

$$\Delta (t-a)^{\underline{N-1}} = \frac{\Gamma(t-a+1)}{\Gamma(t-a-N+3)} (N-1), \quad t \in \mathbb{N}_{a+N-\alpha-2}^{b+N-\alpha-1}.$$

So we have the following cases.

(i) If $t \geq a+N-3$, we have $\Delta_t (t-a)^{\underline{N-1}} \geq 0$. Therefore $\max(t-a)^{\underline{N-1}} = (\alpha+b-a+(N-1))^{\underline{N-1}}$.

(ii) If $t < a+N-3$, we have the following subcases:

$$\begin{cases} 1.\ t-a-N+3<0, & \Gamma(t-a-N+3)>0,\\ 2.\ t-a-N+3<0, & \Gamma(t-a-N+3)<0. \end{cases}$$

Thus

$$\begin{cases} 1.\ t-a-N+3 \in (m, m+1), & m=-2k,\ k\in\mathbb{N},\\ 2.\ t-a-N+3 \in (m, m+1), & m=-2k+1,\ k\in\mathbb{N}. \end{cases}$$

Hence, it follows that:

$$\begin{cases} 1.\ \Delta(t-a)^{\underline{N-1}} \geq 0,\\ 2.\ \Delta(t-a)^{\underline{N-1}} \leq 0. \end{cases}$$

Finally we deduce that

$$\mathcal{O}^{N-1}_{N-1} = \max(t-a)^{\underline{N-1}} = \max\left\{(N-\alpha-2)^{\underline{N-1}}, (\alpha+b-a+(N-1))^{\underline{N-1}}\right\}. \tag{3.16}$$

On the other hand, (Q_3) in Lemma 2.5, gives us the following

$$\Delta^{N-i}(t-a)^{\underline{N-1}} = \frac{(N-1)!}{i!}(t-a)^{\underline{i-1}}, \quad i=1,2,3,\dots,N-1.$$

So as calculated in (3.16), we have

$$\begin{aligned}\mathcal{O}^{i-1}_{i-1} &= \left\|\Delta^{N-i}(t-a)^{\underline{N-1}}\right\|_E\\ &= \max\frac{(N-1)!}{i!}\left\{((i-1)-\alpha-1)^{\underline{i-1}}, (\alpha+b-a+(i-1))^{\underline{i-1}}\right\},\ i=1,2,\dots,N-1.\end{aligned}$$

Therefore

$$\left\|(t-a)^{\underline{N-1}}\right\|_X = \max\left\{\mathcal{O}^{N-1}_{N-1}, \mathcal{O}^{i-1}_{i-1},\ i=1,2,\dots,N-1\right\}.$$

The proof is complete. □

Remark 3.4. By an analogous analysis as presented in Lemma 3.3, one has

$$\left\|(t-a-1)^{\underline{N-1}}\right\|_X = \max\left\{\mathcal{O}^{N-1}_{N-2}, \mathcal{O}^{i-1}_{i-2},\ i=1,2,\dots,N-1\right\}. \tag{3.17}$$

Remark 3.5. Given $(u,v)\in\mathfrak{B}$, by means of Lemma 3.3 we have

$$\begin{aligned}\|P(u,v)\|_{\mathfrak{B}} &= \|(P_1(u), P_1(v))\|_{\mathfrak{B}} = \max\left\{\|P_1(u)\|_X, \|P_2(v)\|_X\right\}\\ &= \max\left\{\frac{\left|\Delta^{N-1}u(a)\right|}{(N-1)!}\left\|(t-a)^{\underline{N-1}}\right\|_X, \frac{\left|\Delta^{N-1}v(a)\right|}{(N-1)!}\left\|(t-a)^{\underline{N-1}}\right\|_X\right\}\\ &\leq \Lambda_1\max\left\{\left|\Delta^{N-1}u(a)\right|, \left|\Delta^{N-1}v(a)\right|\right\},\end{aligned} \tag{3.18}$$

where

$$\Lambda_1 = \frac{\max\left\{\left|\mathcal{O}^{N-1}_{N-1}\right|, \left|\mathcal{O}^{i-1}_{i-1}\right|,\ i=1,2,\dots,N-1\right\}}{(N-1)!}. \tag{3.19}$$

Also, since $\Delta_s(t-s-1)^{\underline{\alpha-1}} \leq 0$ with $t-\alpha \geq s$ and $\Delta_t(t-s-1)^{\underline{\alpha-1}} \geq 0$, it follows that

$$\begin{aligned} &\max_{a\leq s\leq t-\alpha} (t-s-1)^{\underline{\alpha-1}} = (t-a-1)^{\underline{\alpha-1}}, \\ &\max_{t\in\mathbb{N}_{a+N-\alpha-2}^{b+N-\alpha-1}} (t-a-1)^{\underline{\alpha-1}} = (\alpha+b-a+(N-2))^{\underline{\alpha-1}}. \end{aligned} \tag{3.20}$$

Hence, for $(u,v)\in ImL$ we have

$$\begin{aligned} \|K_P(u,v)\|_{\mathfrak{B}} &= \left\| \left(\Delta_a^{-\alpha}u, \Delta_a^{-\alpha}v\right) \right\|_{\mathfrak{B}} = \max\left\{ \left\|\Delta_a^{-\alpha}u\right\|_X, \left\|\Delta_a^{-\alpha}v\right\|_X \right\} \\ &\leq \Lambda_2 \max\left\{ \|u\|_E, \|v\|_E \right\}, \end{aligned} \tag{3.21}$$

where

$$\Lambda_2 = \frac{\max\left\{ \left|\mathcal{O}_{N-2}^{N-1}\right|, \left|\mathcal{O}_{i-2}^{i-1}\right|,\ i=1,2,\ldots,N-1 \right\}}{\Gamma(\alpha)}. \tag{3.22}$$

The forthcoming hypotheses will enable us to obtain the main results.

(C_1) The continuous functions f, g satisfy in the following properties:

$$f: \mathbb{N}_{a+N-\alpha-2}^{b+N-\alpha-1} \times \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\},\ \ or \quad f: \mathbb{N}_{a+N-\alpha-2}^{b+N-\alpha-1} \times \mathbb{R}^n \to \mathbb{R}^- \cup \{0\}, \tag{3.23a}$$

and

$$g: \mathbb{N}_{a+N-\alpha-2}^{b+N-\alpha-1} \times \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\},\ \ or \quad g: \mathbb{N}_{a+N-\alpha-2}^{b+N-\alpha-1} \times \mathbb{R}^n \to \mathbb{R}^- \cup \{0\}. \tag{3.23b}$$

(C_2) There exist positive real constants b_k, c_k, d_1, d_2 for $k=1,2,\ldots,N$ and real constants $\theta_k, \lambda_k \in [0,1]$ with $k=1,2,\ldots,N$ such that for all $(x_1,x_2,\ldots,x_N)\in\mathbb{R}^N$,

$$|f(t+N-\alpha-2, x_1, x_2, \ldots, x_N)| \leq d_1 + \sum_{k=1}^{N} b_k |x_k|^{\theta_k}, \quad t\in\mathbb{N}_a^b, \tag{3.24a}$$

and

$$|g(t+N-\alpha-2, x_1, x_2, \ldots, x_N)| \leq d_2 + \sum_{k=1}^{N} c_k |x_k|^{\lambda_k}, \quad t\in\mathbb{N}_a^b. \tag{3.24b}$$

(C_3) There exists a positive real constant B such that for any $w_i, z_i\in\mathbb{R},\ i=1,2,\ldots,N$, if $\min\{|w_N|,|z_N|\} > B$, one has either

$$\begin{aligned} &z_N.f\,(t+N-\alpha-2, w_1, w_2, \ldots, w_N) > 0,\ \ or \\ &z_N.f\,(t+N-\alpha-2, w_1, w_2, \ldots, w_N) < 0, \quad t\in\mathbb{N}_a^b, \end{aligned} \tag{3.25a}$$

and

$$\begin{aligned} &w_N.g\,(t+N-\alpha-2, z_1, z_2, \ldots, z_N) > 0,\ \ or \\ &w_N.g\,(t+N-\alpha-2, z_1, z_2, \ldots, z_N) < 0, \quad t\in\mathbb{N}_a^b. \end{aligned} \tag{3.25b}$$

(C_4)

$$(\Lambda_1+\Lambda_2)\sum_{i=1}^{N} x_i < 1, \quad x=b,c, \tag{3.26a}$$

$$\Lambda_1\sum_{i=1}^{N} c_i + \Lambda_2\sum_{i=1}^{N} b_i < 1, \tag{3.26b}$$

$$\Lambda_1\sum_{i=1}^{N} b_i + \Lambda_2\sum_{i=1}^{N} c_i < 1. \tag{3.26c}$$

As a fundamental step to achieve the existence of at least one solution for the coupled resonant system (1.1), (1.2), we shall prove boundedness of the following sets:

$$\Omega_1 = \left\{(u,v) \in domL \setminus \ker L \,\middle|\, L(u,v) = \lambda \mathcal{N}(u,v),\ \lambda \in [0,1]\right\}, \tag{3.27a}$$

$$\Omega_2 = \left\{(u,v) \in \ker L \,\middle|\, \mathcal{N}(u,v) \in \operatorname{Im} L\right\}, \tag{3.27b}$$

$$\Omega_3 = \left\{(u,v) \in \ker L \,\middle|\, \lambda(u,v) + (1-\lambda) Q\mathcal{N}(u,v) = (0,0),\ \lambda \in [0,1]\right\}, \tag{3.27c}$$

$$\Omega_4 = \left\{(u,v) \in \ker L \,\middle|\, -\lambda(u,v) + (1-\lambda) Q\mathcal{N}(u,v) = (0,0),\ \lambda \in [0,1]\right\}. \tag{3.27d}$$

Lemma 3.6. *Ω_1 defined by* (3.27a) *is bounded.*

Proof. Taking a look at the Ω_1, we have that $\lambda \neq 0$. On the other hand $L(u,v) = \lambda \mathcal{N}(u,v) \in \operatorname{Im} L = \ker Q$, that is

$$\frac{\lambda \sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}} f\left(s+N-\alpha-2, v, \Delta v, \Delta^2 v, \ldots, \Delta^{N-1} v\right)}{\sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}}} = 0,$$
$$\frac{\lambda \sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}} g\left(s+N-\alpha-2, u, \Delta u, \Delta^2 u, \ldots, \Delta^{N-1} u\right)}{\sum_{s=a}^{b} (b+N-\alpha-1-s)^{\underline{\alpha-N}}} = 0.$$

Therefore by means of property (C_1), there exist $t_0, t_1 \in \mathbb{N}_{a+N-\alpha-2}^{b+N-\alpha-1}$ such that

$$f\left(t_1, v, \Delta v, \Delta^2 v, \ldots, \Delta^{N-1} v\right) = 0,$$
$$g\left(t_0, u, \Delta u, \Delta^2, \ldots, \Delta^{N-1} u\right) = 0.$$

Thus according to the property (C_3), we conclude that $|\Delta^{N-1} u(t_0)| \le B$ and $|\Delta^{N-1} v(t_1)| \le B$.

$(u,v) \in \Omega_1$, implies that $(u,v) \in \operatorname{dom} L \setminus \ker L$. Hence, since $P^2 = P$, we conclude that $(I-P)(u,v) \in \operatorname{dom} L \cap \ker P$ and $LP(u,v) = (0,0)$. So, by (3.21) it follows that

$$\begin{aligned}
\|(I-P)(u,v)\|_{\mathfrak{B}} &= \|K_P L (I-P)(u,v)\|_{\mathfrak{B}} = \|K_P (L_1 u, L_2 v)\|_{\mathfrak{B}} = \left\| \left(\Delta_a^{-\alpha} L_1 u, \Delta_a^{-\alpha} L_2 v\right) \right\|_{\mathfrak{B}} \\
&\le \lambda \Lambda_2 \max\left\{ \|\mathcal{N}_1 v\|_E, \|\mathcal{N}_2 u\|_E \right\} \\
&\le \Lambda_2 \max\left\{ \|\mathcal{N}_1 v\|_E, \|\mathcal{N}_2 u\|_E \right\}.
\end{aligned} \tag{3.28}$$

On the other hand we notice that

$$\begin{aligned}
& L(u,v) = \lambda \mathcal{N}(u,v), \quad (u,v) \in domL \\
\iff & \begin{cases} L_1 u = \lambda \mathcal{N}_1 v, \\ L_2 v = \lambda \mathcal{N}_2 u. \end{cases} \\
\iff & \begin{cases} u(t) = \lambda \Delta_a^{-\alpha} \mathcal{N}_1 v + \displaystyle\sum_{k=0}^{N-1} \frac{(t-a)^{\underline{k}}}{k!} \Delta^k u(a), \\[2ex] v(t) = \lambda \Delta_a^{-\alpha} \mathcal{N}_2 u + \displaystyle\sum_{k=0}^{n-1} \frac{(t-a)^{\underline{k}}}{k!} \Delta^k v(a). \end{cases} \\
\iff & \begin{cases} \Delta^{N-1} u(t) = \lambda \Delta_a^{N-\alpha-1} \mathcal{N}_1 v + \Delta^{N-1} u(a), \\[1ex] \Delta^{N-1} v(t) = \lambda \Delta_a^{N-\alpha-1} \mathcal{N}_2 u + \Delta^{N-1} v(a). \end{cases}
\end{aligned}$$

Now substituting $t = t_0$ in the first equality of the recent coupled equalities and $t = t_1$ in second one, since $|\Delta^{N-1}u(t_0)| \leq B$ and $|\Delta^{N-1}v(t_1)| \leq B$, one has

$$\begin{cases} |\Delta^{N-1}u(a)| \leq B + \lambda \dfrac{\sum_{s=a}^{t_0-(\alpha-N+1)}(t_0-s-1)^{\underline{\alpha-N}}\left|f(s+N-\alpha-2, v(s), \Delta v(s), \Delta^2 v(s), \dots, \Delta^{N-1}v(s))\right|}{\Gamma(\alpha-N)}, \\ |\Delta^{N-1}v(a)| \leq B + \lambda \dfrac{\sum_{s=a}^{t_1-(\alpha-N+1)}(t_1-s-1)^{\underline{\alpha-N}}\left|g(s+N-\alpha-2, u(s), \Delta u(s), \Delta^2 u(s), \dots, \Delta^{N-1}u(s))\right|}{\Gamma(\alpha-N)}. \end{cases}$$

Equivalently, (3.20) implies that

$$\begin{cases} |\Delta^{N-1}u(a)| \leq B + \Lambda_2\left|f(s+N-\alpha-2, v(s), \Delta v(s), \Delta^2 v(s), \dots, \Delta^{N-1}v(s))\right|, \\ |\Delta^{N-1}v(a)| \leq B + \Lambda_2\left|g(s+N-\alpha-2, u(s), \Delta u(s), \Delta^2 u(s), \dots, \Delta^{N-1}u(s))\right|. \end{cases}$$

Finally applying the hypothesis (C_2) represented by (3.24a) and (3.24b), we get the following

$$\begin{cases} |\Delta^{N-1}u(a)| \leq B + \Lambda_2\left(d_1 + b_1\|v\|_E^{\theta_1} + \sum_{i=2}^{N} b_i \left\|\Delta^{N-i+1}v\right\|_E^{\theta_i}\right), \\ |\Delta^{N-1}v(a)| \leq B + \Lambda_2\left(d_2 + c_1\|u\|_E^{\lambda_1} + \sum_{i=2}^{N} c_i \left\|\Delta^{N-i+1}u\right\|_E^{\lambda_i}\right). \end{cases} \tag{3.29}$$

Let us consider Remark 3.5. As a result of (3.28), we have

$$\begin{aligned} \|(u,v)\|_{\mathfrak{B}} &= \|P(u,v)+(I-P)(u,v)\|_{\mathfrak{B}} \leq \|P(u,v)\|_{\mathfrak{B}} + \|(I-P)(u,v)\|_{\mathfrak{B}} \\ &\leq \max\Big\{\Big\{\Lambda_1\left|\Delta^{N-1}u(a)\right| + \Lambda_2\|\mathcal{N}_1 v\|_E\Big\},\ \Big\{\Lambda_1\left|\Delta^{N-1}v(a)\right| + \Lambda_2\|\mathcal{N}_2 u\|_E\Big\}, \\ &\qquad \Big\{\Lambda_1\left|\Delta^{N-1}u(a)\right| + \Lambda_2\|\mathcal{N}_2 u\|_E\Big\},\ \Big\{\Lambda_1\left|\Delta^{N-1}v(a)\right| + \Lambda_2\|\mathcal{N}_1 v\|_E\Big\}\Big\}. \end{aligned} \tag{3.30}$$

In the sequel, we divide the remainder of proof into the four cases as following:

(i) Using (C_2) and (3.29), we conclude that

$$\begin{aligned} \|(u,v)\|_{\mathfrak{B}} &\leq \Lambda_1\left|\Delta^{N-1}u(a)\right| + \Lambda_2\|\mathcal{N}_1 v\|_E \\ &\leq \Lambda_1 B + (\Lambda_1+\Lambda_2)\left(d_1 + b_1\|v\|_E^{\theta_1} + \sum_{i=2}^{N} b_i\left\|\Delta^{N-i+1}v\right\|_E^{\theta_i}\right). \end{aligned}$$

(ii) Once again using (C_2) and (3.29), similarly we can derive

$$\begin{aligned} \|(u,v)\|_{\mathfrak{B}} &\leq \Lambda_1\left|\Delta^{N-1}v(a)\right| + \Lambda_2\|\mathcal{N}_2 u\|_E \\ &\leq \Lambda_1 B + (\Lambda_1+\Lambda_2)\left(d_2 + c_1\|u\|_E^{\lambda_1} + \sum_{i=2}^{N} c_i\left\|\Delta^{N-i+1}u\right\|_E^{\lambda_i}\right). \end{aligned}$$

(iii) In the third case, we have the following

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\le \Lambda_1\left|\Delta^{N-1}u(a)\right| + \Lambda_2\|\mathcal{N}_2 u\|_E\\ &\le \Lambda_1\left\{B + \left(d_1 + b_1\|v\|_E^{\theta_1} + \sum_{i=2}^{N} b_i\left\|\Delta^{N-i+1}v\right\|_E^{\theta_i}\right)\right\}\\ &\quad + \Lambda_2\left(d_2 + c_1\|u\|_E^{\lambda_1} + \sum_{i=2}^{N} c_i\left\|\Delta^{N-i+1}u\right\|_E^{\lambda_i}\right).\end{aligned}$$

(iv) In the last case, similar with case (iii) it follows that

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\le \Lambda_1\left|\Delta^{N-1}v(a)\right| + \Lambda_2\|\mathcal{N}_1 v\|_E\\ &\le \Lambda_1\left\{B + \left(d_2 + c_1\|u\|_E^{\lambda_1} + \sum_{i=2}^{N} c_i\left\|\Delta^{N-i+1}u\right\|_E^{\lambda_i}\right)\right\}\\ &\quad + \Lambda_2\left(d_1 + b_1\|v\|_E^{\theta_1} + \sum_{i=2}^{N} b_i\left\|\Delta^{N-i+1}v\right\|_E^{\theta_i}\right).\end{aligned}$$

Interlacing the above inequalities (3.26a)–(3.26c) in the hypothesis (C_4), gives us the following:

(i)

$$\|(u,v)\|_{\mathfrak{B}} \le \frac{\Lambda_1 B + d_1(\Lambda_1+\Lambda_2)}{1-(\Lambda_1+\Lambda_2)\sum_{i=1}^{N} b_i}.$$

(ii)

$$\|(u,v)\|_{\mathfrak{B}} \le \frac{\Lambda_1 B + d_2(\Lambda_1+\Lambda_2)}{1-(\Lambda_1+\Lambda_2)\sum_{i=1}^{N} c_i}.$$

(iii)

$$\|(u,v)\|_{\mathfrak{B}} \le \frac{\Lambda_1 B + (\Lambda_1 d_1+\Lambda_2 d_2)}{1-\left[\Lambda_1\sum_{i=1}^{N} b_i + \Lambda_2\sum_{i=1}^{N} c_i\right]}.$$

(iv)

$$\|(u,v)\|_{\mathfrak{B}} \le \frac{\Lambda_1 B + (\Lambda_1 d_2+\Lambda_2 d_1)}{1-\left[\Lambda_1\sum_{i=1}^{N} c_i + \Lambda_2\sum_{i=1}^{N} b_i\right]}.$$

Finally, these results in the above four cases guarantee the boundedness of Ω_1 defined by (3.27a), that is our desired result. □

Lemma 3.7. *Ω_2 defined by* (3.27b) *is bounded.*

Proof. Assume that $(u,v)\in\Omega_2$. Then $u=c_1(t-a)^{\underline{N-1}}$, $v=c_2(t-a)^{\underline{N-1}}$, $c_1,c_2\in\mathbb{R}$. On the other hand $\mathcal{N}(u,v)=(\mathcal{N}_1 v,\mathcal{N}_2 u)\in \operatorname{Im} L=\ker Q$, implies the following

$$\frac{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}} f\left(s+N-\alpha-2, c_2(s-a)^{\underline{N-1}}, c_2\Delta(s-a)^{\underline{N-1}},\dots,c_2\Delta^{N-1}(s-a)^{\underline{N-1}}\right)}{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}}}=0,$$

$$\frac{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}} f\left(s+N-\alpha-2, c_1(s-a)^{\underline{N-1}}, c_1\Delta(s-a)^{\underline{N-1}},\dots,c_1\Delta^{N-1}(s-a)^{\underline{N-1}}\right)}{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}}}=0.$$

So, using hypothesis (C_1), there exist constants $t_0, t_1 \in \mathbb{N}_a^b$ such that

$$f\left(t_1+N-\alpha-2, c_2(t_1-a)^{\underline{N-1}}, c_2\Delta(t_1-a)^{\underline{N-1}}, \ldots, c_2\Delta^{N-1}(t_1-a)^{\underline{N-1}}\right)=0,$$
$$f\left(t_0+N-\alpha-2, c_1(t_0-a)^{\underline{N-1}}, c_1\Delta(t_0-a)^{\underline{N-1}}, \ldots, c_1\Delta^{N-1}(t_0-a)^{\underline{N-1}}\right)=0.$$

Consequently, because of the condition (C_3), we have

$$|c_1|, |c_2| \le \frac{B}{(N-1)!}.$$

Recent inequalities ensure that Ω_2 is bounded. This completes the proof. □

Lemma 3.8. *Ω_3 defined by* (3.27c) *is bounded.*

Proof. Suppose that $(u, v) \in \Omega_3$. So $(u, v) = (c_1(t-a)^{\underline{N-1}}, c_2(t-a)^{\underline{N-1}})$, $c_1, c_2 \in \mathbb{R}$. Therefore $\lambda(u, v) + (1-\lambda)Q\mathcal{N}(u, v) = (0, 0)$ consequences the following hold

$$c_1\lambda(t-a)^{\underline{N-1}} + \frac{(1-\lambda)\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}} f\left(s+N-\alpha-2, c_2(s-a)^{\underline{N-1}}, c_2\Delta(s-a)^{\underline{N-1}}, \ldots, c_2\Delta^{N-1}(s-a)^{\underline{N-1}}\right)}{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}}} = 0,$$
$$c_2\lambda(t-a)^{\underline{N-1}} + \frac{(1-\lambda)\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}} f\left(s+N-\alpha-2, c_1(s-a)^{\underline{N-1}}, c_1\Delta(s-a)^{\underline{N-1}}, \ldots, c_1\Delta^{N-1}(s-a)^{\underline{N-1}}\right)}{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}}} = 0.$$

If $\lambda = 0$, then a similar argument as given in Lemma 3.7 yields the boundedness of Ω_3. Hence, let us consider $\lambda \in (0, 1]$. In this case the hypothesis (C_3) and more precisely the first parts of (3.25a) and (3.25b), enable us to achieve to the desired result. □

Applying the counter part of hypothesis (C_3) that applied in Lemma 3.8, one can deduce the following lemma.

Lemma 3.9. *Ω_4 defined by* (3.27d) *is bounded.*

Now we are ready to state and prove our main existence result.

Theorem 3.10. *Assume that the hypotheses* (C_1)–(C_4) *hold. Then the coupled resonant system* (1.1)*,* (1.2) *has at least one solution in* $\mathfrak{B}$.

Proof. Let $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i}\cup\{0\}$ $\left(or,\ \Omega \supset \cup_{i=1}^{2}\overline{\Omega_i} \cup \Omega_4 \cup \{0\}\right)$ be a bounded open subset f $\mathfrak{B}$. It follows from Lemma 3.2 that $\mathcal{N}$ is a L-compact operator on Ω. Also by means of Lemmas 3.6–3.9, it follows that:

(1) $L(u, v) = \lambda\mathcal{N}(u, v)$ for every $((u, v), \lambda) \in [\text{dom } L \setminus \ker\ L \cap \partial\Omega] \times (0, 1)$.
(2) $\mathcal{N}(u, v) \notin \text{Im } L$ for every $(u, v) \in \ker\ L \cap \partial\Omega$.
So we just need to prove:
(3) $\deg\left(JQ\mathcal{N}|_{\ker L},\ \Omega \cap \ker\ L,\ 0\right) \neq 0$.

Define

$$H((u, v), \lambda) = \pm\lambda \mathrm{Id}(u, v) + (1-\lambda)JQ\mathcal{N}(u, v).$$

By the degree property of invariance under a homotopy, if $u \in \ker L \cap \partial\Omega$, then

$$\begin{aligned}
&\deg\left(JQ\mathcal{N}|_{\ker L},\ \Omega \cap \ker\ L,\ 0\right)\\
&\quad = \deg\left(H(., 0),\ \Omega \cap \ker\ L,\ 0\right)\\
&\quad = \deg\left(H(., 1),\ \Omega \cap \ker\ L,\ 0\right)\\
&\quad = \deg\left(\pm \mathrm{Id},\ \Omega \cap \ker\ L,\ 0\right) \neq 0.
\end{aligned}$$

Hence, the assumption (iii) in Theorem 2.8 is fulfilled that completes the proof. □

So far, we have been studied only existence of solutions for the fractional Δ-difference coupled resonant system (1.1) and (1.2). So as we promised above, it is time to establish the uniqueness results.

Theorem 3.11. *Assume that the condition* (C_2) *is replaced with the following conditions:*

($C'_{2,1}$) *There exist positive constants* $(a_i, b_i) \in \mathbb{R}^2,\ i = 1, \ldots, N$, *such that for all* $\big((x_i)_1^N, (y_i)_1^N\big) \in \mathbb{R}^N \times \mathbb{R}^N$, *one has*

$$|f(t+N-\alpha-2, x_1, x_2, \ldots, x_N) - f(t+N-\alpha-2, y_1, y_2, \ldots, y_N)| \leq \sum_{i=1}^{N} a_i |x_i - y_i|, \quad t \in \mathbb{N}_a^b, \tag{3.31a}$$

$$|g(t+N-\alpha-2, x_1, x_2, \ldots, x_N) - f(t+N-\alpha-2, y_1, y_2, \ldots, y_N)| \leq \sum_{i=1}^{N} b_i |x_i - y_i|, \quad t \in \mathbb{N}_a^b. \tag{3.31b}$$

($C'_{2,2}$) *There exist positive constants* $(k_i, l_i) \in \mathbb{R}^2,\ i = 1, \ldots, N$, *such that for all* $\big((x_i)_1^N, (y_i)_1^N\big) \in \mathbb{R}^N \times \mathbb{R}^N$, *one has*

$$|f(t+N-\alpha-2, x_1, x_2, \ldots, x_N) - f(t+N-\alpha-2, y_1, y_2, \ldots, y_N)| \geq k_N |x_N - y_N| - \sum_{i=1}^{N-1} k_i |x_i - y_i|, \quad t \in \mathbb{N}_a^b, \tag{3.32a}$$

$$|g(t+N-\alpha-2, x_1, x_2, \ldots, x_N) - f(t+N-\alpha-2, y_1, y_2, \ldots, y_N)| \geq l_N |x_N - y_N| - \sum_{i=1}^{N-1} l_i |x_i - y_i|, \quad t \in \mathbb{N}_a^b. \tag{3.32b}$$

Then the coupled resonant system (1.1) and (1.2) *has exactly one solution in* $\mathfrak{B}$ *provided that*

$$\Lambda_1 \left[\sum_{i=1}^{N-2} \frac{l_{N-i}}{l_N} + \frac{l_1}{l_N} \right] + (\Lambda_1 + \Lambda_2) \sum_{i=1}^{N} |b_i| > 1, \tag{3.33a}$$

$$\Lambda_1 \left[\sum_{i=1}^{N-2} \frac{k_{N-i}}{k_N} + \frac{k_1}{k_N} \right] + (\Lambda_1 + \Lambda_2) \sum_{i=1}^{N} |a_i| > 1, \tag{3.33b}$$

$$\Lambda_1 \left[\sum_{i=1}^{N-2} \frac{l_{N-i}}{l_N} + \frac{l_1}{l_N} \right] + \left[\Lambda_1 \sum_{i=1}^{N} |b_i| + \Lambda_2 \sum_{i=1}^{N} |a_i| \right] > 1, \tag{3.33c}$$

$$\Lambda_1 \left[\sum_{i=1}^{N-2} \frac{k_{N-i}}{k_N} + \frac{k_1}{k_N} \right] + \left[\Lambda_1 \sum_{i=1}^{N} |a_i| + \Lambda_2 \sum_{i=1}^{N} |b_i| \right] > 1. \tag{3.33d}$$

Proof. Considering $y_i = 0,\ i = 1, 2, \ldots, n$ and defining

$$d_1 = \max f(t+N-\alpha-2, 0, 0, \ldots, 0), \quad d_2 = \max g(t+N-\alpha-2, 0, 0, \ldots, 0), \quad t \in \mathbb{N}_a^b,$$

we deduce that the condition (C_2) is satisfied. Thus by Theorem 3.10 the existence of at least one solution for the coupled resonant system (1.1) and (1.2) is immediate. The uniqueness of solution will be proved as follows.

Assume that $(u_i, v_i) \in \mathfrak{B}$ for $i = 1, 2$ are two solutions of fractional resonant system (1.1) and (1.2). So we have

$$\Delta_*^\alpha u_i(t) = f\left(t+N-\alpha-2, v_i, \Delta v_i, \Delta^2 v_i, \ldots, \Delta^{N-1} v_i\right),$$

$$\Delta_*^\alpha v_i(t) = g\left(t+N-\alpha-2, u_i, \Delta u_i, \Delta^2 u_i, \ldots, \Delta^{N-1} u_i\right).$$

Denoting $u = u_1 - u_2,\ v = v_1 - v_2$, it follows that

$$\begin{aligned} \Delta_*^\alpha u(t) &= f\left(t+N-\alpha-2, v_1, \Delta v_1, \ldots, \Delta^{N-1} v_1\right) - f\left(t+N-\alpha-2, v_2, \Delta v_2, \ldots, \Delta^{N-1} v_2\right), \\ \Delta_*^\alpha v(t) &= g\left(t+N-\alpha-2, u_1, \Delta u_1, \ldots, \Delta^{N-1} u_1\right) - g\left(t+N-\alpha-2, u_2, \Delta u_2, \ldots, \Delta^{N-1} u_2\right). \end{aligned} \tag{3.34}$$

Because of equality $\mathrm{Im}L = \ker Q$, we conclude that

$$\frac{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}}\left\{f\left(s+N-\alpha-2, v_1, \Delta v_1, \dots, \Delta^{N-1}v_1\right) - f\left(s+N-\alpha-2, v_2, \Delta v_2, \dots, \Delta^{N-1}v_2\right)\right\}}{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}}} = 0,$$

$$\frac{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}}\left\{g\left(s+N-\alpha-2, u_1, \Delta u_1, \dots, \Delta^{N-1}u_1\right) - g\left(s+N-\alpha-2, u_2, \Delta u_2, \dots, \Delta^{N-1}u_2\right)\right\}}{\sum_{s=a}^{b}(b+N-\alpha-1-s)^{\underline{\alpha-N}}} = 0.$$

Accordingly the hypothesis (C_1) implies that there exist $t_2, t_3 \in \mathbb{N}_a^b$ such that

$$f\left(t_3+N-\alpha-2, v_1, \Delta v_1, \dots, \Delta^{N-1}v_1\right) = f\left(t_3+N-\alpha-2, v_2, \Delta v_3, \dots, \Delta^{N-1}v_2\right),$$
$$g\left(t_2+N-\alpha-2, u_1, \Delta u_1, \dots, \Delta^{N-1}u_1\right) = g\left(t_2+N-\alpha-2, u_2, \Delta u_2, \dots, \Delta^{\alpha-1}u_2\right).$$

Now, by ($C'_{2,2}$) we have

$$\begin{aligned} 0 &= \left|f\left(t_3+N-\alpha-2, v_1, \Delta v_1, \dots, \Delta^{N-1}v_1\right) - f\left(t_3+N-\alpha-2, v_2, \Delta v_2, \dots, \Delta^{N-1}v_2\right)\right| \\ &\geq k_N\left|\Delta^{N-1}v(t_3)\right| - \sum_{i=1}^{N-2}k_{N-i}\left|\Delta^{N-i-1}v(t_3)\right| - k_1|v(t_3)|. \end{aligned}$$

Therefore

$$\left|\Delta^{N-1}v(t_3)\right| \leq \sum_{i=1}^{N-2}\frac{k_{N-i}}{k_N}\left|\Delta^{N-i-1}v(t_3)\right| + \frac{k_1}{k_N}|v(t_3)|.$$

So, it follows that

$$\left|\Delta^{N-1}v(t_3)\right| \leq \sum_{i=1}^{N-2}\frac{k_{N-i}}{k_N}\left\|\Delta^{N-i-1}v\right\|_{E_i} + \frac{k_1}{k_N}\|v\|_E \leq \left[\sum_{i=1}^{N-2}\frac{k_{N-i}}{k_N} + \frac{k_1}{k_N}\right]\|v\|_X. \tag{3.35}$$

Similarly one can derive

$$\left|\Delta^{N-1}u(t_2)\right| \leq \left[\sum_{i=1}^{N-2}\frac{l_{N-i}}{l_N} + \frac{l_1}{l_N}\right]\|u\|_X. \tag{3.36}$$

Considering (3.34), we obtain

$$\begin{aligned} \Delta^{N-1}u(t) &= \Delta_a^{N-\alpha-1}\Big\{f\left(t+N-\alpha-2, v_1, \Delta v_1, \dots, \Delta^{N-1}v_1\right) \\ &\qquad - f\left(t+N-\alpha-2, v_2, \Delta v_2, \dots, \Delta^{N-1}v_2\right)\Big\} + \Delta^{N-1}u(a), \\ \Delta^{N-1}v(t) &= \Delta_a^{N-\alpha-1}\Big\{g\left(t+N-\alpha-2, u_1, \Delta u_1, \dots, \Delta^{N-1}u_1\right) \\ &\qquad - g\left(t+N-\alpha-2, u_2, \Delta u_2, \dots, \Delta^{N-1}u_2\right)\Big\} + \Delta^{N-1}v(a). \end{aligned}$$

Substituting $t = t_2$ in the first equality and $t = t_3$ in second one and then applying the hypothesis ($C'_{2,1}$), we can derive the following

$$\left|\Delta N - 1u(a)\right| \leq \left|\Delta^{N-1}u(t_2)\right| + \Lambda_2\sum_{i=1}^{N}a_i\left\|\Delta^{N-i+1}v\right\|_E, \tag{3.37a}$$

$$\left|\Delta^{N-1}v(a)\right| \leq \left|\Delta^{N-1}v(t_3)\right| + \Lambda_2\sum_{i=1}^{N}b_i\left\|\Delta^{N-i+1}u\right\|_E. \tag{3.37b}$$

Hence, using (3.35) and (3.36), we have

$$\left|\Delta^{N-1}u(a)\right| \leq \left[\sum_{i=1}^{N-2}\frac{l_{N-i}}{l_N}+\frac{l_1}{l_N}\right]\|u\|_X+\Lambda_2\sum_{i=1}^{N}b_i\|v\|_X, \tag{3.38a}$$

$$\left|\Delta^{N-1}v(a)\right| \leq \left[\sum_{i=1}^{N-2}\frac{k_{N-i}}{k_N}+\frac{k_1}{k_N}\right]\|v\|_X+\Lambda_2\sum_{i=1}^{N}a_i\|u\|_X. \tag{3.38b}$$

Let us recall once again (3.30). So, we have

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &= \|P(u,v)+(I-P)(u,v)\|_{\mathfrak{B}} \leq \|P(u,v)\|_{\mathfrak{B}}+\|(I-P)(u,v)\|_{\mathfrak{B}}\\ &\leq \max\left\{\left\{\Lambda_1\left|\Delta^{N-1}u(a)\right|+\Lambda_2\|\mathcal{N}_1v\|_E\right\},\ \left\{\Lambda_1\left|\Delta^{N-1}v(a)\right|+\Lambda_2\|\mathcal{N}_2u\|_E\right\},\right.\\ &\left.\left\{\Lambda_1\left|\Delta^{N-1}u(a)\right|+\Lambda_2\|\mathcal{N}_2u\|_E\right\},\ \left\{\Lambda_1\left|\Delta^{N-1}v(a)\right|+\Lambda_2\|\mathcal{N}_1v\|_E\right\}\right\}.\end{aligned} \tag{3.39}$$

Using (3.39) with (3.38a) and (3.38b), one can prove the following inequalities:

(i)

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leq \Lambda_1\left|\Delta^{N-1}u(a)\right|+\Lambda_2\|\mathcal{N}_1v\|_E\\ &\leq \frac{\Lambda_2|d_1|}{1-\left\{\Lambda_1\left[\sum_{i=1}^{N-2}\frac{l_{N-i}}{l_N}+\frac{l_1}{l_N}\right]+(\Lambda_1+\Lambda_2)\sum_{i=1}^{N}|b_i|\right\}}.\end{aligned}$$

(ii)

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leq \Lambda_1\left|\Delta^{N-1}v(a)\right|+\Lambda_2\|\mathcal{N}_2u\|_E\\ &\leq \frac{\Lambda_2|d_2|}{1-\left\{\Lambda_1\left[\sum_{i=1}^{N-2}\frac{k_{N-i}}{k_N}+\frac{k_1}{k_N}\right]+(\Lambda_1+\Lambda_2)\sum_{i=1}|a_i|\right\}}.\end{aligned}$$

(iii)

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leq \Lambda_1\left|\Delta^{N-1}u(a)\right|+\Lambda_2\|\mathcal{N}_2u\|_E\\ &\leq \frac{\Lambda_2|d_2|}{1-\left\{\Lambda_1\left[\sum_{i=1}^{N-2}\frac{l_{N-i}}{l_N}+\frac{l_1}{l_N}\right]+\left[\Lambda_1\sum_{i=1}^{N}|b_i|+\Lambda_2\sum_{i=1}^{N}|a_i|\right]\right\}}.\end{aligned}$$

(iv)

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leq \Lambda_1\left|\Delta^{N-1}v(a)\right|+\Lambda_2\|\mathcal{N}_1v\|_E\\ &\leq \frac{\Lambda_2|d_1|}{1-\left\{\Lambda_1\left[\sum_{i=1}^{N-2}\frac{k_{N-i}}{k_N}+\frac{k_1}{k_N}\right]+\left[\Lambda_1\sum_{i=1}^{N}|a_i|+\Lambda_2\sum_{i=1}^{N}|b_i|\right]\right\}}.\end{aligned}$$

Implying the hypotheses (3.33a)–(3.33d) in the recent inequalities, we conclude that $u = v = 0$. Equivalently, we have $(u_1, v_1) = (u_2, v_2)$. Therefore, we have proved that the fractional Δ-difference resonant system (1.1) and (1.2) has exactly one solution. □

4. An application

Consider the fractional Δ-difference resonant system

$$\begin{cases} \Delta_*^{\frac{5}{2}} y(t) = f\left(t - \dfrac{3}{2}, z, \Delta z, \Delta^2 z\right), & \\ & t \in \mathbb{N}_1^5, \\ \Delta_*^{\frac{5}{2}} z(t) = g\left(t - \dfrac{3}{2}, y, \Delta y, \Delta^2 y\right), & \end{cases} \tag{4.1}$$

$$\begin{cases} y\left(\dfrac{1}{2}\right) = 0, \quad \Delta y\left(\dfrac{1}{2}\right) = 0, \quad \Delta^2 y\left(\dfrac{1}{2}\right) = \Delta^2 y\left(\dfrac{9}{2}\right), \\ z\left(\dfrac{1}{2}\right) = 0, \quad \Delta z\left(\dfrac{1}{2}\right) = 0, \quad \Delta^2 z\left(\dfrac{1}{2}\right) = \Delta^2 z\left(\dfrac{9}{2}\right). \end{cases} \tag{4.2}$$

Indeed, the aforementioned system is reduced by the primitive resonant system (1.1) under selection of $N = 3,\ \alpha = \frac{5}{2}$ and $a = 1,\ b = 5$. Also the functions $f, g : \mathbb{N}_{\frac{7}{2}}^{\frac{9}{2}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ in system (4.1) read as follows

$$f(x, u, v, w) = 1 - \sin\left(\frac{\pi}{2}\left[x - \frac{7}{2}\right]\right) + \frac{|u| + |v| + |w|}{1000}, \tag{4.3}$$

$$g(x, u, v, w) = 1 - \sin\left(\frac{\pi}{2}\left[x - \frac{7}{2}\right]\right) + \frac{|u| + |v| + |w - 2|}{1000}. \tag{4.4}$$

Choosing $d_1 = d_2 = 2,\ b_k = c_k = \frac{1}{1000},\ k = 1, 2, 3$ and $\theta_i = 1,\ $ for $i = 1, 2$ and $\lambda_1 = 1,\ \lambda_2 = \frac{1}{2}$, it is easy to check that the hypotheses (C_1) and (C_2) are satisfied. Also because of the nonnegative nature of f and g for given positive parameter $B > 2$, the hypothesis (C_3) is also satisfied. On the other hand, case $\alpha = \frac{5}{2}$ and $N = 3$, with a direct calculation it follows that the parameters Λ_1 and Λ_2 defined by (3.19) and (3.22), respectively, satisfy

$$\Lambda_1 =\approx 31.875, \quad \Lambda_2 \approx 266.48444. \tag{4.5}$$

Therefore (4.5)) ensures that the hypothesis (C_4) holds. So, based on Theorem 3.10, the coupled system (4.1) and (4.2) admits at least one solution in $\mathfrak{B}$.

For uniqueness, choosing $a_i = b_i = \frac{1}{1000},\ i = 1, 2, 3, k_3 = l_3 = \frac{1}{1000}$ and $k_i = l_i = 1,\ i = 1, 2$ we conclude that the hypotheses ($C'_{2,1}$), ($C'_{2,2}$) and (3.33a)–(3.33d) hold. So the coupled resonant system (4.1) and (4.2) has a unique solution in $\mathfrak{B}$.

Acknowledgments

The authors would like to express their sincere gratitude to both anonymous referees for invaluable comments on the first draft of the current paper.

References

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: Mathematics Studies, vol. 204, North-Holland, Elsevier Science, 2006.
[2] K.S. Miller, B. Ross, An Introduction to Fractional Calculus and Fractioal Differential Equation, John Wiley, New York, 1993.
[3] M. Moshrefi-Torbati, J.K. Hammond, Physical and geometrical interpretation of fractional operators, J. Franklin. Inst. 335B (6) (1998) 1077–1086.
[4] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[5] I. Podlubny, Fractional Differential Equations, in: Mathematics in Science and Applications, vol. 19, Academic Press, New York, 1999.
[6] Thabet Abdeljawad, On Riemann and Caputo fractional differences, Comput. Math. Appl. 62 (2011) 1602–1611.
[7] Ferhan M. Atici, Paul W. Eloe, A transform method in discrete fractional calculus, Int. J. Differ. Equ. 2 (2) (2007) 165–176.
[8] Ferhan M. Atici, Paul W. Eloe, Two-point boundary value problems for finite fractional difference equations, J. Difference Equ. Appl. 17 (4) (2011) 445–456.
[9] Ferhan M. Atici, Paul W. Eloe, Initial value problems in discrete fractional calculus, Proc. Amer. Math. Soc. 137 (3) (2009) 981–989.
[10] C. Goodrich, A.C. Peterson, Discrete Fractional Calculus, Springer International Publishing, 2015. http://dx.doi.org/10.1007/978-3-319-25562-0.

[11] Christopher S. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions, Comput. Math. Appl. 61 (2011) 191–202.
[12] Ferhan M. Atici, Paul W. Eloe, Discrete fractional calculus with the nabla operator, Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I (3) (2009) 1–12.
[13] Christopher S. Goodrich, Systems of discrete fractional boundary value problems with nonlinearities satisfying no growth conditions, J. Difference Equ. Appl. 21 (5) (2015) 437–453.
[14] Christopher S. Goodrich, On a discrete fractional three-point boundary value problem, J. Difference Equ. Appl. 18 (3) (2012) 397–415.
[15] Yousef Gholami, Kazem Ghanbari, Existence and uniqueness of positive solutions for coupled systems of fractional Δ-difference boundary value problems, J. Fract. Calculus Appl. 7 (2) (2016) 88–100.
[16] Yousef Gholami, Kazem Ghanbari, Coupled systems of fractional ∇-difference boundary value problems, J. Difference Equ. Appl. 8 (4) (2016) 459–470.
[17] Kazem Ghanbari, Yousef Gholami, New classes of Lyapunov type inequalities of fractional Δ-difference Sturm-Liouville problems with applications, Bull. Iranian Math. Soc. (2017) in press.
[18] Z. Bai, Solvability for a class of fractional m-point boundary value problem at resonance, Comput. Math. Appl. 62 (3) (2011) 1292–1302.
[19] Z. Hu, W. Liu, T. Chen, Existence of solutions for a coupled system of fractional differential equations at resonance, Bound. Value Probl. 2012 (98) (2012) 1–13.
[20] Z. Hu, W. Liu, T. Chen, Two-point boundary value problems for fractional differential equations at resonance, Bull. Malays. Math. Sci. Soc. 36 (3) (2013) 747–755.
[21] L. Hu, S. Zhang, Existence and uniqueness of solutions for a higher-order coupled fractional differential equations at resonance, Adv. Differential Equations 202 (2015) 1–14.
[22] Weihua Jiang, The existence of solutions to boundary value problems of fractional differential equations at resonance, Nonlinear Anal. 74 (2011) 1987–1994.
[23] Nickolai Kosmatov, Multi-point boundary value problems on time scales at resonance, J. Math. Anal. Appl. 323 (2006) 253–266.
[24] Nickolai Kosmatov, A singular non-local problem at resonance, J. Math. Anal. Appl. 394 (2012) 425–431.
[25] Nickolai Kosmatov, A boundary value problem of fractional order at resonance, Electron. J. Differential Equations 2010 (135) (2010) 1–10.
[26] Wenjuan Rui, Existence of solutions of nonlinear fractional differential equations at resonance, Electron. J. Qual. Theory Differ. Equ. 66 (2011) 1–12.
[27] N. Xu, W. Liu, L. Xiao, The existence of solutions for nonlinear fractional multipoint boundary value problems at resonance, Bound. Value Probl. 2012 (65) (2012) 1–14.
[28] Y. Zhang, Z. Bai, T. Feng, Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance, Comput. Math. Appl. 61 (2011) 1032–1047.
[29] Y. Zhang, Z. Bai, Existence of solutions for nonlinear fractional three-point boundary value problems at resonance, J. Appl. Math. Comput. 36 (1) (2011) 417–440.
[30] R.E. Gaines, J. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Springer, 1977.

Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute 171 (2017) 350–378

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

Mathematical problems of thermoelasticity of bodies with microstructure and microtemperatures

L. Giorgashvili, S. Zazashvili*

Georgian Technical University, Department of Mathematics, Kostava str. 77, Tbilisi 0175, Georgia

Received 6 March 2017; received in revised form 18 April 2017; accepted 20 April 2017
Available online 5 May 2017

Abstract

The paper deals with the linear theory of thermoelasticity for elastic isotropic microstretch materials with microtemperatures and microdilatations. For the differential equations of pseudo-oscillations the fundamental matrix is constructed explicitly in terms of elementary functions. With the help of the corresponding Green identities the general integral representation formula of solutions by means of generalized layer and Newtonian potentials are derived. The basic Dirichlet and Neumann type boundary value problems are formulated in appropriate function spaces and the uniqueness theorems are proved. The existence theorems for classical solutions are established by using the potential method.

Keywords: Elastic bodies with microstructure; Thermoelasticity with microtemperatures; Potential theory; Integral equations

1. Introduction

The main goal of our investigation is analysis of the basic boundary value problems for the pseudo-oscillation equations of the theory of thermoelasticity for isotropic materials with microstructure, whose microelements possess microtemperatures.

A theory of thermoelasticity with microtemperatures, in which the microelements can stretch and contract independently of their translations has been studied by Ieşan [1]. This is the simplest thermomechanical theory of elastic bodies that takes into account the microtemperatures and the inner structure of the materials. This model has been investigate by various authors (see e.g., [2–4]).

The mathematical model of a linear theory of thermodynamics for microstretch elastic solids with microtemperatures, using the results established by Grot [5] has been proposed by Ieşan [6]. This theory introduces three extra degrees of freedom over the theory presented in [1]. An interesting aspect in this theory is the coupling of

* Corresponding author.
E-mail addresses: lgiorgashvili@gmail.com (L. Giorgashvili), zaza-ude@hotmail.com (S. Zazashvili).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.04.002
2346-8092/

microrotation vector with the microtemperatures even for isotropic bodies. This effect is different from the classical theory of Cosserat thermoelasticity for isotropic bodies [7], where the microrotation vector is independent of the thermal field. In the model [6] a material particle is equipped with 11 degrees of freedom (3 displacement components, 3 microrotation components, 3 microtemperature components, 1 microdilatation and 1 temperature).

The system of differential equations of thermodynamics for isotropic elastic materials with microstructure, with respect to the displacement vector, microrotation vector, microtemperature vector, microdilation function, and temperature function, represents a coupled complex system of second order partial differential equations (see [6]).

If the mechanical and thermal characteristics are time harmonic dependent (i.e. they are represented as the product of the time dependent exponential function $exp(-i\sigma t)$ with a complex parameter $\sigma = \sigma_1 + i\sigma_2, \sigma_1 \in R, \sigma_2 > 0$ and a function of the spatial variable $x \in \mathbb{R}^3$), then we have the so called *pseudo-oscillation equations*. The corresponding simultaneous equations generate 11×11 strongly elliptic formally non-self-adjoint matrix differential operator with constant coefficients.

The present paper is devoted to investigation of the basic boundary value problems for the system of pseudo-oscillations. First, we collect the field equations, derive the corresponding Green's identities and formulate the basic boundary value problems. Further, we construct the matrix of fundamental solutions explicitly in terms of elementary functions for the differential operator of pseudo-oscillations and establish the asymptotic properties near the origin and at infinity. Applying the potential method and the theory of singular integral equations we investigate the basic boundary value problems of pseudo-oscillations (cf. [8–13] and the references therein).

2. Basic differential equations

The pseudo-oscillation equations of the thermoelasticity theory of microstretch materials with microtemperatures and microdilatations in the case of isotropic homogeneous bodies according to [6] have the form

$$(\mu + \varkappa)\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u + \rho\sigma^2 u + \varkappa \operatorname{rot} \omega + \mu_0 \operatorname{grad} v - \beta_0 \operatorname{grad}\theta = -\rho H(x), \tag{2.1}$$

$$\varkappa \operatorname{rot} u + \gamma \Delta\omega + (\alpha + \beta) \operatorname{grad} \operatorname{div} \omega + \delta\omega - \mu_1 \operatorname{rot} w = -\rho g(x), \tag{2.2}$$

$$\varkappa_6 \Delta w + (\varkappa_4 + \varkappa_5) \operatorname{grad} \operatorname{div} w + \varkappa_0 w - i\sigma\mu_1 \operatorname{rot} \omega + i\sigma\mu_2 \operatorname{grad} v - \varkappa_3 \operatorname{grad}\theta = \rho G(x), \tag{2.3}$$

$$-\mu_0 \operatorname{div} u - \mu_2 \operatorname{div} w + a_0 \Delta v + \eta_0 v + \beta_1\theta = -\rho l(x), \tag{2.4}$$

$$i\beta_0 T_0 \sigma \operatorname{div} u + \varkappa_1 \operatorname{div} w + i\beta_1 T_0 \sigma v + \varkappa_7 \Delta\theta + i\sigma c\theta = -\rho S^*(x), \tag{2.5}$$

where $\alpha, \beta, \gamma, \lambda, \mu, \varkappa, \eta, \beta_0, \beta_1, \mu_0, \mu_1, \mu_2, a, b, a_0, b_0, \mathcal{I}, \mathcal{I}_1, \varkappa_j, \quad j = 1, 2, 3, 4, 5, 6, 7$, are the real constants characterizing the mechanical and thermal properties of the body, ρ is the mass density, $\delta = \mathcal{I}_1\sigma^2 - 2\varkappa$, $\varkappa_0 = i\sigma b - \varkappa_2$, $\eta_0 = \mathcal{I}\sigma^2 - \eta$, σ is a frequency parameter, $\sigma = \sigma_1 + i\sigma_2$, $\sigma_2 > 0$, $\sigma_1 \in \mathbb{R}$, Δ is the Laplace operator, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, $\omega = (\omega_1, \omega_2, \omega_3)^\top$ is the microrotation vector, $w = (w_1, w_2, w_3)^\top$ is the microtemperature vector, v is the microdilatation function, θ is the temperature, measured from a fixed absolute temperature T_0 $(T_0 > 0)$, $c = aT_0$; $H = (H_1, H_2, H_3)^\top$, $g = (g_1, g_2, g_3)^\top$, and $G = (G_1, G_2, G_3)^\top$ are complex-valued vector functions, connected with the body force, the body couple density, and the first heat supply moment vector, respectively; l and S^* are complex-valued functions connected with the external microstretch body load and the heat supply per unit mass, respectively; the superscript $(\cdot)^\top$ denotes transposition operation.

Let us introduce the matrix differential operator of order 11×11 generated by the left hand side expressions in system (2.1)–(2.5)

$$L(\partial, \sigma) := \begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(6)}(\partial,\sigma) & L^{(11)}(\partial,\sigma) & L^{(16)}(\partial,\sigma) & L^{(21)}(\partial,\sigma) \\ L^{(2)}(\partial,\sigma) & L^{(7)}(\partial,\sigma) & L^{(12)}(\partial,\sigma) & L^{(17)}(\partial,\sigma) & L^{(22)}(\partial,\sigma) \\ L^{(3)}(\partial,\sigma) & L^{(8)}(\partial,\sigma) & L^{(13)}(\partial,\sigma) & L^{(18)}(\partial,\sigma) & L^{(23)}(\partial,\sigma) \\ L^{(4)}(\partial,\sigma) & L^{(9)}(\partial,\sigma) & L^{(14)}(\partial,\sigma) & L^{(19)}(\partial,\sigma) & L^{(24)}(\partial,\sigma) \\ L^{(5)}(\partial,\sigma) & L^{(10)}(\partial,\sigma) & L^{(15)}(\partial,\sigma) & L^{(20)}(\partial,\sigma) & L^{(25)}(\partial,\sigma) \end{bmatrix}_{11\times 11}, \tag{2.6}$$

where

$$
\begin{aligned}
&L^{(1)}(\partial,\sigma):=((\mu+\varkappa)\Delta+\rho\sigma^2)I_3+(\lambda+\mu)Q(\partial),\quad L^{(2)}(\partial,\sigma):=\varkappa R(\partial),\\
&L^{(3)}(\partial,\sigma):=[0]_{3\times 3},\quad L^{(4)}(\partial,\sigma):=-\mu_0\nabla,\quad L^{(5)}(\partial,\sigma):=i\beta_0\sigma T_0\nabla,\\
&L^{(6)}(\partial,\sigma):=\varkappa R(\partial),\quad L^{(7)}(\partial,\sigma):=(\gamma\Delta+\delta)I_3+(\alpha+\beta)Q(\partial),\\
&L^{(8)}(\partial,\sigma):=-i\sigma\mu_1R(\partial),\quad L^{(9)}(\partial,\sigma):=[0]_{1\times 3},\quad L^{(10)}(\partial,\sigma):=[0]_{1\times 3},\\
&L^{(11)}(\partial,\sigma):=[0]_{3\times 3},\quad L^{(12)}(\partial,\sigma):=-\mu_1R(\partial),\\
&L^{(13)}(\partial,\sigma):=(\varkappa_6\Delta+\varkappa_0)I_3+(\varkappa_4+\varkappa_5)Q(\partial),\quad L^{(14)}(\partial,\sigma):=-\mu_2\nabla,\\
&L^{(15)}(\partial,\sigma):=\varkappa_1\nabla,\quad L^{(16)}(\partial,\sigma):=\mu_0\nabla^\top,\quad L^{(17)}(\partial,\sigma):=[0]_{3\times 1},\\
&L^{(18)}(\partial,\sigma):=i\sigma\mu_2\nabla^\top,\quad L^{(19)}(\partial,\sigma):=a_0\Delta+\eta_0,\quad L^{(20)}(\partial,\sigma):=i\sigma\beta_1T_0,\\
&L^{(21)}(\partial,\sigma):=-\beta_0\nabla^\top,\quad L^{(22)}(\partial,\sigma):=[0]_{3\times 1},\quad L^{(23)}(\partial,\sigma):=-\varkappa_3\nabla^\top,\\
&L^{(24)}(\partial,\sigma):=\beta_1,\quad L^{(25)}(\partial,\sigma):=\varkappa_7\Delta+i\sigma c.
\end{aligned}
\tag{2.7}
$$

Here and in the sequel I_k stands for the $k\times k$ unit matrix and

$$
R(\partial):=\begin{bmatrix}0&-\partial_3&\partial_2\\ \partial_3&0&-\partial_1\\ -\partial_2&\partial_1&0\end{bmatrix}_{3\times 3},\quad Q(\partial):=[\partial_k\partial_j]_{3\times 3},\ \nabla:=[\partial_1,\partial_2,\partial_3],\ \partial_k=\partial/\partial_{x_k}.
\tag{2.8}
$$

It is easy to see that for $V=(V_1,V_2,V_3)^\top$

$$
\begin{aligned}
&R(\partial)V=\operatorname{rot}V,\qquad Q(\partial)V=\operatorname{grad}\operatorname{div}V,\\
&R(-\partial)=-R(\partial)=[R(\partial)]^\top,\qquad Q(\partial)R(\partial)=R(\partial)Q(\partial)=0,\\
&Q(\partial)=[Q(\partial)]^\top,\ \ [R(\partial)]^2=Q(\partial)-I_3\Delta,\ \ [Q(\partial)]^2=Q(\partial)\Delta.
\end{aligned}
\tag{2.9}
$$

Due to the above notation, system (2.1)–(2.5) can be rewritten in the matrix form as

$$
L(\partial,\sigma)U(x)=\Phi(x),\quad U=(u,\omega,w,v,\theta)^\top,
$$

where $\Phi(x)=(-\rho H(x),\,-\rho g(x),\,\rho G(x),\,-\rho l(x),\,-\rho S^*(x))$. Note that $L(\partial,\sigma)$ is not formally self-adjoint differential operator.

Further let us introduce the generalized thermo-stress operator [6],

$$
P(\partial,n):=\begin{bmatrix}
P^{(1)}(\partial,n)&P^{(2)}(\partial,n)&[0]_{3\times 3}&\mu_0n^\top&-\beta_0n^\top\\
[0]_{3\times 3}&P^{(3)}(\partial,n)&P^{(4)}(\partial,n)&-b_0S^\top(\partial,n)&[0]_{3\times 1}\\
[0]_{3\times 3}&[0]_{3\times 3}&P^{(5)}(\partial,n)&[0]_{3\times 1}&[0]_{3\times 1}\\
[0]_{1\times 3}&b_0S(\partial,n)&-\mu_2n&a_0\partial_n&0\\
[0]_{1\times 3}&[0]_{1\times 3}&\varkappa_1n&0&\varkappa_7\partial_n
\end{bmatrix}_{11\times 11},
\tag{2.10}
$$

where

$$
\begin{aligned}
&P^{(l)}(\partial,n)=\left[P^{(l)}_{kj}(\partial,n)\right]_{3\times 3},\ l=1,2,3,4,5,\\
&P^{(1)}_{kj}(\partial,n)=(\mu+\varkappa)\delta_{kj}\partial_n+\lambda n_k\partial_j+\mu n_j\partial_k,\quad P^{(2)}_{kj}(\partial,n)=\varkappa\sum_{p=1}^{3}\varepsilon_{pjk}n_p,\\
&P^{(3)}_{kj}(\partial,n)=\gamma\delta_{kj}\partial_n+\alpha n_k\partial_j+\beta n_j\partial_k,\quad P^{(4)}_{kj}(\partial,n)=\mu_1\sum_{p=1}^{3}\varepsilon_{kjp}n_p,\\
&P^{(5)}_{kj}(\partial,n)=\varkappa_6\delta_{kj}\partial_n+\varkappa_4n_k\partial_j+\varkappa_5n_j\partial_k,\quad S(\partial,n)=(\partial S_1,\partial S_2,\partial S_3),\\
&\qquad \partial S_1=n_2\partial_3-n_3\partial_2,\quad \partial S_2=n_3\partial_1-n_1\partial_3,\quad \partial S_3=n_1\partial_2-n_2\partial_1,
\end{aligned}
\tag{2.11}
$$

ε_{kjp} is the permutation (Levi-Civita) symbol, $\partial_n=\partial/\partial n$ is the normal derivative, $n=(n_1,n_2,n_3)$.

The generalized thermo-stress vector has the form

$$
P(\partial,n)U=(T^{(1)}(\partial,n)U,\ T^{(2)}(\partial,n)U,\ T^{(3)}(\partial,n)U,\ T^{(4)}(\partial,n)U,\ T^{(5)}(\partial,n)U)^\top,
$$

where

$$
\begin{aligned}
T^{(1)}(\partial,n)U &= P^{(1)}(\partial,n)u + P^{(2)}(\partial,n)\omega + (\mu_0 v - \beta_0\theta)n \\
&= (2\mu+\varkappa)\frac{\partial u}{\partial n} + \lambda n \operatorname{div} u + \mu[n\times \operatorname{rot}\, u] + \varkappa[n\times\omega] + (\mu_0 v - \beta_0\theta)n, \\
T^{(2)}(\partial,n)U &= P^{(3)}(\partial,n)\omega + P^{(4)}(\partial,n)w - b_0 S^\top(\partial,n)v = \\
&= (\beta+\gamma)\frac{\partial\omega}{\partial n} + \alpha\, n \operatorname{div}\omega + \beta[n\times \operatorname{rot}\,\omega] - \mu_1[n\times w] - b_0[n\times \operatorname{grad} v], \\
T^{(3)}(\partial,n)U &= P^{(5)}(\partial,n)w = (\varkappa_5+\varkappa_6)\frac{\partial w}{\partial n} + \varkappa_4\, n \operatorname{div} w + \varkappa_5[n\times \operatorname{rot}\, w], \\
T^{(4)}(\partial,n)U &= b_0 S^\top(\partial,n)\omega - \mu_2\, n\cdot w + a_0\frac{\partial v}{\partial n} = a_0\frac{\partial v}{\partial n} - \mu_2\, n\cdot w + b_0\, n\cdot \operatorname{rot}\,\omega, \\
T^{(5)}(\partial,n)U &= \varkappa_1\, n\cdot w + \varkappa_7\frac{\partial\theta}{\partial n}.
\end{aligned}
$$

We recall, that the central dot denotes the real scalar product $a\cdot b = \sum_{k=1}^{N} a_k b_k$ for $a, b \in \mathbb{C}^N$, and $[c\times d]$ denotes the vector product of two vectors $c, d \in \mathbb{C}^3$.

Further, let us introduce the associated boundary operator which occurs in Green's formulas and is related to the adjoint differential operator $L^*(\partial,\sigma) := L^\top(-\partial,\sigma)$,

$$
P^*(\partial,n) := \begin{bmatrix}
P^{(1)}(\partial,n) & P^{(2)}(\partial,n) & [0]_{3\times3} & \mu_0 n^\top & -i\beta_0 T_0\sigma\, n^\top \\
[0]_{3\times3} & P^{(3)}(\partial,n) & i\sigma P^{(4)}(\partial,n) & -b_0 S^\top(\partial,n) & [0]_{3\times1} \\
[0]_{3\times3} & [0]_{3\times3} & P^{(5)}(\partial,n) & [0]_{3\times1} & [0]_{3\times1} \\
[0]_{1\times3} & b_0 S(\partial,n) & -i\sigma\mu_2 n & a_0\partial_n & 0 \\
[0]_{1\times3} & [0]_{1\times3} & \varkappa_3 n & 0 & \varkappa_7\partial_n
\end{bmatrix}_{11\times11}, \tag{2.12}
$$

where $P^{(j)}(\partial,n)$, $j = 1,2,3,4,5$, are given by (2.11).

3. Green's formulae

Let Ω^+ be a finite three-dimensional region bounded by the Lyapunov surface $\partial\Omega^+$, $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

Definition 3.1. A vector function $U = (u,\omega,w,v,\theta)^\top$ is said to be regular in a domain $\Omega^+ \subset \mathbb{R}^3$ if $U \in C^2(\Omega^+)\cap C^1(\overline{\Omega^+})$.

For the regular vector functions $U = (u,\omega,w,v,\theta)^\top$ and $U' = (u',\omega',w',v',\theta')^\top$ in the domain Ω^+, we have the following Green formulae

$$
\int_{\Omega^+} U'\cdot L(\partial,\sigma)U\,dx = \int_{\partial\Omega^+}\{U'\}^+\cdot\{P(\partial,n)U\}^+ ds - \int_{\Omega^+} E(U',U)\,dx, \tag{3.13}
$$

where the operators $L(\partial,\sigma)$ and $P(\partial,n)$ are given by (2.6) and (2.10) respectively, n is the outward unit normal vector to $\partial\Omega^+$; the symbols $\{\cdot\}^\pm$ denote one-sided limiting values on $\partial\Omega^+$ from $\Omega^\pm$ respectively; $E(\cdot,\cdot)$ is the so called energy bilinear form

$$
\begin{aligned}
E(U',U) = {} & E^{(1)}(u',u) + E^{(2)}(\omega',\omega) + E^{(3)}(w',w) - \rho\sigma^2 u'\cdot u - \delta\,\omega'\cdot\omega - \eta_0\, v'v - i\sigma\, c\,\theta'\theta \\
& - \varkappa\,(\omega'\cdot \operatorname{rot} u + \omega\cdot \operatorname{rot} u') + \mu_1\,(w\cdot \operatorname{rot}\omega' + i\sigma\, w'\cdot \operatorname{rot}\omega) - \mu_2\,(w\cdot \operatorname{grad} v' + i\sigma\, w'\cdot \operatorname{grad} v) \\
& + \varkappa_3\, w'\cdot \operatorname{grad}\theta + \varkappa_1\, w\cdot \operatorname{grad}\theta' - \varkappa_0\, w'\cdot w + \mu_0\,(v \operatorname{div} u' + v' \operatorname{div} u) - \beta_0\,(\theta \operatorname{div} u' \\
& + i\sigma\, T_0\,\theta' \operatorname{div} u) - \beta_1\,(v'\theta + {+i\sigma}\, T_0\, v\theta') - b_0\,(\operatorname{rot}\omega'\cdot \operatorname{grad} v + \operatorname{rot}\omega\cdot \operatorname{grad} v') + a_0 \operatorname{grad} v' \\
& \cdot \operatorname{grad} v + \varkappa_7 \operatorname{grad}\theta'\cdot \operatorname{grad}\theta,
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
E^{(1)}(u',u) = {} & \frac{3\lambda+2\mu+\varkappa}{3}\operatorname{div} u' \operatorname{div} u + \frac{\varkappa}{2}\operatorname{rot} u' \cdot \operatorname{rot} u + \frac{\mu+\varkappa}{3}\sum_{k,j=1}^{3}\left(\frac{\partial u_k'}{\partial x_k}-\frac{\partial u_j'}{\partial x_j}\right)\left(\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}\right) \\
& + \frac{2\mu+\varkappa}{4}\sum_{k,j=1,\,k\neq j}^{3}\left(\frac{\partial u_k'}{\partial x_j}+\frac{\partial u_j'}{\partial x_k}\right)\left(\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\right), \\
E^{(2)}(\omega',\omega) = {} & \frac{3\alpha+\beta+\gamma}{3}\operatorname{div}\omega'\operatorname{div}\omega + \frac{\gamma-\beta}{2}\operatorname{rot}\omega'\cdot\operatorname{rot}\omega \\
& + \frac{\gamma+\beta}{4}\sum_{k,j=1,\,k\neq j}^{3}\left(\frac{\partial \omega_k'}{\partial x_j}+\frac{\partial \omega_j'}{\partial x_k}\right)\left(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right) \\
& + \frac{\gamma+\beta}{6}\sum_{k,j=1}^{3}\left(\frac{\partial \omega_k'}{\partial x_k}-\frac{\partial \omega_j'}{\partial x_j}\right)\left(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right), \\
E^{(3)}(w',w) = {} & \frac{3\varkappa_4+\varkappa_5+\varkappa_6}{3}\operatorname{div} w'\operatorname{div} w + \frac{\varkappa_6-\varkappa_5}{2}\operatorname{rot} w'\cdot\operatorname{rot} w \\
& + \frac{\varkappa_5+\varkappa_6}{4}\sum_{k,j=1,\,k\neq j}^{3}\left(\frac{\partial w_k'}{\partial x_j}+\frac{\partial w_j'}{\partial x_k}\right)\left(\frac{\partial w_k}{\partial x_j}+\frac{\partial w_j}{\partial x_k}\right) \\
& + \frac{\varkappa_5+\varkappa_6}{6}\sum_{k,j=1}^{3}\left(\frac{\partial w_k'}{\partial x_k}-\frac{\partial w_j'}{\partial x_j}\right)\left(\frac{\partial w_k}{\partial x_k}-\frac{\partial w_j}{\partial x_j}\right).
\end{aligned}
$$

We assume that the constitutive coefficients satisfy the following inequalities [6]

$$
\begin{gathered}
\rho>0,\ \ \mathcal{I}>0,\ \ \mathcal{I}_1>0,\ \ a_0>0,\ \ a>0,\ \ b>0,\ \ \mu>0,\ \ 3\lambda+2\mu>0, \\
\varkappa>0,\ \ (3\lambda+2\mu+\varkappa)\eta-3\mu_0^2\geq 0,\ \ \varkappa_6\pm\varkappa_5\geq 0,\ \ 3\varkappa_4+\varkappa_5+\varkappa_6\geq 0, \\
\varkappa_7>0,\ \ (\varkappa_1+\varkappa_3T_0)^2\leq 4T_0\varkappa_2\varkappa_7,\ \ \gamma+\beta\geq 0,\ \ 3\alpha+\beta+\gamma\geq 0, \\
a_0(\gamma-\beta)-2b_0^2\geq 0.
\end{gathered}
\tag{3.15}
$$

With the help of relations (3.13) and (3.14) we can show that the following second Green identity holds

$$
\begin{aligned}
& \int_{\Omega^+}\left[U'\cdot L(\partial,\sigma)U - U\cdot L^*(\partial,\sigma)U'\right]dx \\
& \quad = \int_{\partial\Omega^+}\left[\left\{U'\right\}^+\cdot\{P(\partial,n)U\}^+ - \{U\}^+\cdot\left\{P^*(\partial,n)U'\right\}^+\right]ds,
\end{aligned}
\tag{3.16}
$$

where the differential operator $L(\partial,\sigma)$ is given by (2.6), $L^*(\partial,\sigma) = L^\top(-\partial,\sigma)$ is the formally adjoint operator to $L(\partial,\sigma)$, the boundary operators $P(\partial,n)$ and $P^*(\partial,n)$ are defined by (2.10), (2.11) and (2.12) respectively.

The corresponding Green identities hold true in the case of an exterior unbounded domain Ω^- if regular vector functions U and U' satisfy decay conditions at infinity.

Let us remark that the differential operator

$$
L(\partial) := L(\partial,0) \tag{3.17}
$$

corresponds to the static equilibrium case, while the formally self-adjoint differential operator

$$
L_0(\partial) := \begin{bmatrix}
L_0^{(1)}(\partial) & [0]_{3\times 3} & [0]_{3\times 3} & [0]_{3\times 1} & [0]_{3\times 1} \\
[0]_{3\times 3} & L_0^{(7)}(\partial) & [0]_{3\times 3} & [0]_{3\times 1} & [0]_{3\times 1} \\
[0]_{3\times 3} & [0]_{3\times 3} & L_0^{(13)}(\partial) & [0]_{3\times 1} & [0]_{3\times 1} \\
[0]_{1\times 3} & [0]_{1\times 3} & [0]_{1\times 3} & a_0\Delta & 0 \\
[0]_{1\times 3} & [0]_{1\times 3} & [0]_{1\times 3} & 0 & \varkappa_7\Delta
\end{bmatrix}_{11\times 11}, \tag{3.18}
$$

with

$$
\begin{aligned}
& L_0^{(1)}(\partial) := (\mu+\varkappa)\Delta I_3 + (\lambda+\mu)Q(\partial), \\
& L_0^{(7)}(\partial) := \gamma\Delta I_3 + (\alpha+\beta)Q(\partial), \\
& L_0^{(13)}(\partial) := \varkappa_6\Delta I_3 + (\varkappa_4+\varkappa_5)Q(\partial),
\end{aligned}
\tag{3.19}
$$

represents the principal homogeneous part of operators (2.6) and (3.17).

Note that the differential operators $L_0(\partial)$ and $L(\partial,\sigma)$ are *strongly elliptic* and the following inequality *(the accretivity condition)* holds (cf., e.g., [14], Part I, §5)

$$C_2|\xi|^2|\eta|^2 \geq L_0(\xi)\eta\cdot\eta = \sum_{k,j=1}^{11} L_0(\xi)_{kj}\eta_j\overline{\eta_k} \geq C_1|\xi|^2|\eta|^2 \tag{3.20}$$

with some constants $C_k > 0$ $(k = 1, 2)$ for arbitrary $\xi \in \mathbb{R}^3$ and arbitrary complex vector $\eta \in \mathbb{C}^{11}$.

4. Representation of solutions by metaharmonic functions

Let us introduce the following differential operators

$$\begin{aligned}
\Lambda_1(\Delta) &= \frac{1}{d_1}\det\begin{bmatrix} \lambda_0\Delta+\rho\sigma^2 & 0 & \mu_0\Delta & -\beta_0\Delta \\ 0 & l_0\Delta+\varkappa_0 & i\sigma\mu_2\Delta & -\varkappa_3\Delta \\ -\mu_0 & -\mu_2 & a_0\Delta+\eta_0 & \beta_1 \\ i\sigma\beta_0T_0 & \varkappa_1 & i\sigma\beta_1T_0 & \varkappa_7\Delta+i\sigma c \end{bmatrix}_{4\times4} \\
&= (\Delta+k_1^2)(\Delta+k_2^2)(\Delta+k_3^2)(\Delta+k_4^2), \\
\Lambda_2(\operatorname{rot}) &= \frac{1}{p_0}\det\begin{bmatrix} \rho\sigma^2-(\mu+\varkappa)\operatorname{rot}\operatorname{rot} & \varkappa\operatorname{rot} & 0 \\ \varkappa\operatorname{rot} & \delta-\gamma\operatorname{rot}\operatorname{rot} & -\mu_1\operatorname{rot} \\ 0 & -i\sigma\mu_1\operatorname{rot} & \varkappa_0-\varkappa_6\operatorname{rot}\operatorname{rot} \end{bmatrix}_{3\times3} \\
&= -(\operatorname{rot}\operatorname{rot}-k_5^2)(\operatorname{rot}\operatorname{rot}-k_6^2)(\operatorname{rot}\operatorname{rot}-k_7^2),
\end{aligned}$$

where $d_1 = a_0\lambda_0 l_0\varkappa_7$, $\lambda_0 = \lambda+2\mu+\varkappa$, $l_0 = \varkappa_4+\varkappa_5+\varkappa_6$, $p_0 = \gamma(\mu+\varkappa)\varkappa_6$ and $-k_j^2$, $j = 1, 2, 3, 4$ are the roots of the equation

$$\Lambda_1(t) = 0, \tag{4.21}$$

while k_j^2, $j = 5, 6, 7$, are the roots of the following equation

$$\begin{aligned}
&\gamma(\mu+\varkappa)\varkappa_6\, t^3 - \big[(\mu+\varkappa)(\delta\varkappa_6+\gamma\varkappa_0+i\sigma\mu_1^2)+\varkappa_6(\rho\sigma^2\gamma+\varkappa^2)\big]t^2 \\
&\quad+\big[\rho\sigma^2(\delta\varkappa_6+\gamma\varkappa_0+i\sigma\mu_1^2)+\varkappa_0(\delta(\mu+\varkappa)+\varkappa^2)\big]t-\delta\varkappa_0\rho\sigma^2 = 0.
\end{aligned}\tag{4.22}$$

Let Ω be a bounded region in $\mathbb{R}^3$.

Theorem 4.1. *A vector $U = (u, \omega, w, v, \theta)^\top \in C^2(\Omega)$ is a solution of the homogeneous system $L(\partial,\sigma)U = 0$ in a domain $\Omega \subset \mathbb{R}^3$ if and only if U is representable in the form*

$$\begin{aligned}
u(x) &= \sum_{j=1}^{7} u^{(j)}(x), \quad \omega(x) = u^{(8)}(x) + \sum_{j=5}^{7}\gamma_j \operatorname{rot} u^{(j)}(x), \\
w(x) &= \sum_{j=1}^{7}\alpha_j u^{(j)}(x), \quad v(x) = \sum_{j=1}^{4}\gamma_j \operatorname{div} u^{(j)}(x), \quad \theta(x) = \sum_{j=1}^{4}\delta_j \operatorname{div} u^{(j)}(x),
\end{aligned}\tag{4.23}$$

where

$$\begin{aligned}
&(\Delta+k_j^2)u^{(j)}(x) = 0, \quad j = 1, 2, \ldots, 8, \quad \operatorname{rot} u^{(j)}(x) = 0, \quad j = 1, 2, 3, 4, 8, \\
&\qquad \operatorname{div} u^{(j)}(x) = 0, \quad j = 5, 6, 7, \quad k_8^2 = \delta/\alpha_0, \quad \alpha_0 = \alpha+\beta+\gamma,
\end{aligned}\tag{4.24}$$

$$\begin{aligned}
\alpha_j &= \frac{1}{a_1}\Big[(\lambda_0k_j^2-\rho\sigma^2)(a_0\varkappa_3k_j^2-\eta_0\varkappa_3-i\sigma\beta_1\mu_2)-\mu_0(\mu_0\varkappa_3-i\sigma\beta_0\mu_2)k_j^2\Big], \\
\gamma_j &= \frac{1}{a_1k_j^2}\Big[(\lambda_0k_j^2-\rho\sigma^2)\big[\beta_1(l_0k_j^2-\varkappa_0)-\mu_2\varkappa_3k_j^2\big]-\beta_0\mu_0(l_0k_j^2-\varkappa_0)k_j^2\Big], \\
\delta_j &= \frac{1}{a_1k_j^2}\Big[(\lambda_0k_j^2-\rho\sigma^2)\big[(l_0k_j^2-\varkappa_0)(a_0k_j^2-\eta_0)-i\sigma\mu_2^2k_j^2\big]-\mu_0^2(l_0k_j^2-\varkappa_0)k_j^2\Big], \qquad j = 1, 2, 3, 4,
\end{aligned}\tag{4.25}$$

$$\begin{aligned}&\alpha_j=\frac{i\sigma\mu_1((\mu+\varkappa)k_j^2-\rho\sigma^2)}{\varkappa(\varkappa_0-\varkappa_6k_j^2)},\quad \gamma_j=\frac{(\mu+\varkappa)k_j^2-\rho\sigma^2}{\varkappa k_j^2},\quad j=5,6,7,\\ &a_1=a_0\beta_0l_0k_j^4-\Big[\beta_0(a_0\varkappa_0+l_0\eta_0+i\sigma\mu_2^2)+\mu_0(l_0\beta_1-\mu_2\varkappa_3)\Big]k_j^2+\varkappa_0(\mu_0\beta_1+\beta_0\eta_0).\end{aligned}\tag{4.26}$$

Proof. Assume that a vector $U=(u,\omega,w,v,\theta)^\top\in C^2(\Omega)$ is a solution of the homogeneous system $L(\partial,\sigma)U=0$. Homogeneous equations (2.1)–(2.3) (with $H=0$, $g=0$, $G=0$) can be rewritten in the form

$$u(x)=u'(x)+u''(x),\quad \omega(x)=\omega'(x)+\omega''(x),\quad w(x)=w'(x)+w''(x)\tag{4.27}$$

where

$$u'(x)=\frac{1}{\rho\sigma^2}\operatorname{grad}(-\lambda_0\operatorname{div}u(x)-\mu_0v(x)+\beta_0\theta(x)),\tag{4.28}$$

$$u''(x)=\frac{1}{\rho\sigma^2}\operatorname{rot}((\mu+\varkappa)\operatorname{rot}u(x)-\varkappa\omega(x)),\tag{4.29}$$

$$\omega'(x)=-\frac{1}{k_8^2}\operatorname{grad}\operatorname{div}\omega(x),\tag{4.30}$$

$$\omega''(x)=\frac{1}{\delta}\operatorname{rot}(\gamma\operatorname{rot}\omega(x)-\varkappa u(x)+\mu_1w(x)),\tag{4.31}$$

$$w'(x)=\frac{1}{\varkappa_0}\operatorname{grad}(-l_0\operatorname{div}w(x)-i\sigma\mu_2v(x)+\varkappa_3\theta(x)),\tag{4.32}$$

$$w''(x)=\frac{1}{\varkappa_0}\operatorname{rot}(\varkappa_6\operatorname{rot}w(x)+i\sigma\mu_1\omega(x)).\tag{4.33}$$

Applying the operator div to both sides of the homogeneous equations (2.1)–(2.3), and take into account the homogeneous equations (2.4), (2.5) (with $l=0$, $S^*=0$), with respect to the vector $(\operatorname{div}u,\operatorname{div}w,v,\theta)^\top$ we find

$$\begin{aligned}&\Lambda_1(\Delta)(\operatorname{div}u,\operatorname{div}w,v,\theta)^\top=(\Delta+k_1^2)(\Delta+k_2^2)(\Delta+k_3^2)(\Delta+k_4^2)(\operatorname{div}u,\operatorname{div}w,v,\theta)^\top=0,\\ &(\Delta+k_8^2)\operatorname{div}\omega=0.\end{aligned}$$

In view of the equalities (4.27)–(4.33) we establish that vectors $(u',w',v,\theta)^\top$ and ω' are solutions of the following equations

$$\begin{aligned}&(\Delta+k_1^2)(\Delta+k_2^2)(\Delta+k_3^2)(\Delta+k_4^2)(u',w',v,\theta)^\top=0,\\ &(\Delta+k_8^2)\operatorname{div}\omega'=0,\quad \operatorname{rot}u'=0,\quad \operatorname{rot}\omega'=0,\quad \operatorname{rot}w'=0.\end{aligned}$$

Therefore the vectors u', ω', w' and the functions v,θ can be represented as follows:

$$\begin{aligned}&u'(x)=\sum_{j=1}^4u^{(j)}(x),\quad w'(x)=\sum_{j=1}^4w^{(j)}(x),\quad v(x)=\sum_{j=1}^4v^{(j)}(x),\\ &\theta(x)=\sum_{j=1}^4\theta^{(j)}(x),\quad \omega'(x)=u^{(8)}(x),\end{aligned}\tag{4.34}$$

where

$$(u^{(j)},w^{(j)},v^{(j)},\theta^{(j)})^\top=\prod_{q=1,\,q\neq j}^4\frac{\Delta+k_q^2}{k_q^2-k_j^2}(u',w',v,\theta)^\top,\quad j=1,2,3,4.\tag{4.35}$$

With the help of relations (4.34) and (4.35), we derive

$$\begin{aligned}&(\Delta+k_j^2)u^{(j)}(x)=0,\quad \operatorname{rot}u^{(j)}(x)=0,\\ &(\Delta+k_j^2)w^{(j)}(x)=0,\quad \operatorname{rot}w^{(j)}(x)=0,\\ &(\Delta+k_j^2)v^{(j)}(x)=0,\quad (\Delta+k_j^2)\theta^{(j)}(x)=0,\quad j=1,2,3,4,\\ &(\Delta+k_8^2)u^{(8)}(x)=0,\quad \operatorname{rot}u^{(8)}(x)=0.\end{aligned}\tag{4.36}$$

Since

$$\operatorname{div} u = \operatorname{div} u', \quad \operatorname{div} \omega = \operatorname{div} \omega', \quad \operatorname{div} w = \operatorname{div} w', \quad \operatorname{rot} u' = 0, \quad \operatorname{rot} \omega' = 0, \quad \operatorname{rot} w' = 0,$$

we find that

$$\begin{aligned} &\operatorname{grad}\operatorname{div} u' = \Delta u' + \operatorname{rot}\operatorname{rot} u' = \Delta u', \\ &\operatorname{grad}\operatorname{div} \omega' = \Delta \omega', \\ &\operatorname{grad}\operatorname{div} w' = \Delta w'. \end{aligned}$$

Therefore from (4.27) and the homogeneous equations (2.4) and (2.5), with the help of equalities (4.35) and (4.36), we get

$$\begin{aligned} &(\rho\sigma^2 - \lambda_0 k_j^2) u^{(j)} + \mu_0 \operatorname{grad} v^{(j)} - \beta_0 \operatorname{grad} \theta^{(j)} = 0, \\ &(\varkappa_0 - l_0 k_j^2) w^{(j)} + i\sigma\mu_2 \operatorname{grad} v^{(j)} - \varkappa_3 \operatorname{grad} \theta^{(j)} = 0, \\ &-\mu_0 \operatorname{div} u^{(j)} - \mu_2 \operatorname{div} w^{(j)} + (\eta_0 - a_0 k_j^2) v^{(j)} + \beta_1 \theta^{(j)} = 0, \\ &i\beta_0 T_0 \sigma \operatorname{div} u^{(j)} + \varkappa_1 \operatorname{div} w^{(j)} + i\beta_1 T_0 \sigma v^{(j)} + (i\sigma c - k_j^2 \varkappa_7)\theta^{(j)} = 0, \quad j = 1, 2, 3, 4. \end{aligned} \tag{4.37}$$

Using (4.21) we can show that the solution of the homogeneous system (4.37) has the form

$$\begin{aligned} &w^{(j)}(x) = \alpha_j u^{(j)}(x), \quad v^{(j)}(x) = \gamma_j \operatorname{div} u^{(j)}(x), \\ &\theta^{(j)}(x) = \delta_j \operatorname{div} u^{(j)}(x), \quad \operatorname{rot} u^{(j)}(x) = 0, \quad j = 1, 2, 3, 4, \end{aligned} \tag{4.38}$$

where constants α_j, γ_j, δ_j are given by (4.25), (4.26) and $u^{(j)}(x)$ is arbitrary vector, satisfying the equations in the first raw (4.36).

Keeping in mind equalities (4.38), from (4.34) we find

$$\begin{aligned} &u'(x) = \sum_{j=1}^{4} u^{(j)}(x), \quad \omega'(x) = u^{(8)}(x), \quad w'(x) = \sum_{j=1}^{4} \alpha_j u^{(j)}(x), \\ &v(x) = \sum_{j=1}^{4} \gamma_j \operatorname{div} u^{(j)}(x), \quad \theta(x) = \sum_{j=1}^{4} \delta_j \operatorname{div} u^{(j)}(x), \quad \operatorname{rot} u^{(j)}(x) = 0, \quad j = 1, 2, 3, 4, 8. \end{aligned} \tag{4.39}$$

Further, if we apply the operator rot to both sides of homogeneous equations (2.1)–(2.3), we arrive at the following equation with respect to the vector $(\operatorname{rot} u,\ \operatorname{rot}\omega,\ \operatorname{rot} w)^\top$,

$$\Lambda_2(\operatorname{rot})(\operatorname{rot} u,\ \operatorname{rot}\omega,\ \operatorname{rot} w)^\top = (\Delta + k_5^2)(\Delta + k_6^2)(\Delta + k_7^2)(\operatorname{rot} u,\ \operatorname{rot}\omega,\ \operatorname{rot} w)^\top = 0. \tag{4.40}$$

By using Eq. (4.40) we can show that the vectors u'', ω'' and w'', defined by relations (4.29), (4.31), and (4.33) satisfy the following equations

$$\begin{aligned} &(\Delta + k_5^2)(\Delta + k_6^2)(\Delta + k_7^2)(u'', \omega'', w'')^\top = 0, \\ &\operatorname{div} u'' = 0, \quad \operatorname{div} \omega'' = 0, \quad \operatorname{div} w'' = 0. \end{aligned} \tag{4.41}$$

The vectors u'', ω'', and w'', as solutions of Eq. (4.41), can be represented in the form of the following sums,

$$u''(x) = \sum_{j=5}^{7} u^{(j)}(x), \quad \omega''(x) = \sum_{j=5}^{7} \omega^{(j)}(x), \quad w''(x) = \sum_{j=5}^{7} w^{(j)}(x), \tag{4.42}$$

where

$$(u^{(j)}, \omega^{(j)}, w^{(j)})^\top = \prod_{q=5,\, q\neq j}^{7} \frac{\Delta + k_q^2}{k_q^2 - k_j^2} (u'', \omega'', w'')^\top, \quad j = 5, 6, 7.$$

Applying Eqs. (4.41), for the vectors $u^{(j)}$, $\omega^{(j)}$, $w^{(j)}$, $j = 5, 6, 7$, we derive

$$\begin{aligned} &(\Delta + k_j^2) u^{(j)}(x) = 0, \quad \operatorname{div} u^{(j)}(x) = 0, \\ &(\Delta + k_j^2) \omega^{(j)}(x) = 0, \quad \operatorname{div} \omega^{(j)}(x) = 0, \\ &(\Delta + k_j^2) w^{(j)}(x) = 0, \quad \operatorname{div} w^{(j)}(x) = 0, \quad j = 5, 6, 7. \end{aligned} \tag{4.43}$$

On the other hand, since

$$\operatorname{rot} u = \operatorname{rot} u'', \quad \operatorname{rot} \omega = \operatorname{rot} \omega'', \quad \operatorname{rot} w = \operatorname{rot} w'',$$
$$\operatorname{div} u'' = 0, \quad \operatorname{div} \omega'' = 0, \quad \operatorname{div} w'' = 0$$

and

$$\operatorname{rot}\operatorname{rot} u'' = -\Delta u'', \quad \operatorname{rot}\operatorname{rot} \omega'' = -\Delta \omega'', \quad \operatorname{rot}\operatorname{rot} w'' = -\Delta w'',$$

from (4.29), (4.31), and (4.33) we have

$$\begin{aligned}
&(\rho\sigma^2 - (\mu + \varkappa)k_j^2)u^{(j)}(x) + \varkappa \operatorname{rot} \omega^{(j)}(x) = 0,\\
&\varkappa \operatorname{rot} u^{(j)}(x) + (\delta - \gamma k_j^2)\omega^{(j)}(x) - \mu_1 \operatorname{rot} w^{(j)}(x) = 0,\\
&-i\sigma\mu_1 \operatorname{rot} \omega^{(j)}(x) + (\varkappa_0 - \varkappa_6 k_j^2)w^{(j)}(x) = 0, \quad j = 5, 6, 7.
\end{aligned} \tag{4.44}$$

Take into account (4.22) it is easy to verify that the vectors

$$(u^{(j)},\ \omega^{(j)},\ w^{(j)})^\top = (u^{(j)},\ \gamma_j \operatorname{rot} u^{(j)},\ \alpha_j\, u^{(j)})^\top, \quad j = 5, 6, 7, \tag{4.45}$$

where α_j, γ_j, $j = 5, 6, 7$, are given by (4.26), are solutions of the homogeneous system (4.44) for arbitrary $u^{(j)}$ satisfying Eqs. (4.43).

Substituting the expressions of $\omega^{(j)}$ and $w^{(j)}$ from (4.45) into (4.42), we get

$$\begin{gathered}
u''(x) = \sum_{j=5}^{7} u^{(j)}(x), \qquad \omega''(x) = \sum_{j=5}^{7} \gamma_j \operatorname{rot} u^{(j)}(x), \quad w''(x) = \sum_{j=5}^{7} \alpha_j\, u^{(j)}(x),\\
\operatorname{div} u^{(j)}(x) = 0, \quad j = 5, 6, 7.
\end{gathered} \tag{4.46}$$

Finally, formulas (4.27), (4.39), and (4.46) prove the first part of the theorem.

The sufficient part of the theorem we can prove by substituting the vector $U = (u, \omega, w, v, \theta)^\top$ represented by (4.23)–(4.26) into the homogeneous system $L(\partial, \sigma)U = 0$. $\square$

Throughout the paper we assume that

$$\operatorname{Im} k_j > 0, \quad j = 1, 2, \ldots, 8. \tag{4.47}$$

Definition 4.2. A vector $U = (u, \omega, w, v, \theta)^\top$ is said to be regular in Ω^- if it is representable in the form (4.23), and:
(i) $U \in C^1(\overline{\Omega^-}) \cap C^2(\Omega^-)$;
(ii) for $|x| \gg 1$,

$$\left(\frac{\partial}{\partial |x|} - i k_j\right) u_l^{(j)}(x) = e^{i k_j |x|} \mathcal{O}(|x|^{-2}), \quad l = 1, 2, 3, \quad j = 1, 2, \ldots, 8. \tag{4.48}$$

Remark 4.3. If vectors $u^{(j)}$, $j = 1, 2, \ldots, 8$, satisfy equations $(\Delta + k_j^2)u^{(j)} = 0$ and conditions (4.48) then for sufficiently large $|x|$ the following decay conditions hold true [21]

$$u_l^{(j)}(x) = e^{i k_j |x|} O(|x|^{-1}), \quad j = 1, 2, \ldots, 8. \tag{4.49}$$

Keeping in mind relations (4.47) and (4.49), for sufficiently large $|x|$ from (4.23) we have

$$U = (u, \omega, w, v, \theta)^\top = e^{-\varsigma |x|} O(|x|^{-1}),$$

where $\varsigma = \min\limits_{1 \leqslant j \leqslant 8} (\operatorname{Im} k_j) > 0$.

Therefore a regular vector U and its partial derivatives $\partial^l U$ decay exponentially as $|x| \to +\infty$ for arbitrary multi-index $l = (l_1, l_2, l_3)$.

By the standard limiting procedure it can be shown that for regular vectors Green's identities hold in an unbounded domain Ω^-. In particular, for regular vectors U and U' in the domain Ω^- we have the following Green formulae

$$\int_{\Omega^-} U' \cdot L(\partial, \sigma) U dx = -\int_{\partial\Omega^-} \{U'\}^- \cdot \{P(\partial, n)U\}^- ds - \int_{\Omega^-} E(U', U) dx. \tag{4.50}$$

5. Boundary value problems and uniqueness theorems

Let us formulate the basic interior and exterior boundary value problems for the domains Ω^+ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, $S = \partial\Omega^+$. We assume that $S \in C^{1,\gamma'}$, $0 < \gamma' \le 1$. In what follows, $n(z)$ stands for the outward unit normal vector at the point $z \in S$ with respect to the domain Ω^+.

Problem $(I^{(\sigma)})^{\pm}$ (Dirichlet problem). Find a regular solution vector $U = (u, \omega, w, v, \theta)^\top$ to the differential equation

$$L(\partial, \sigma)\, U(x) = \Phi^{\pm}(x), \quad x \in \Omega^{\pm}, \tag{5.51}$$

satisfying the boundary condition

$$\{U(z)\}^{\pm} = f(z), \quad z \in S. \tag{5.52}$$

Problem $(II^{(\sigma)})^{\pm}$ (Neumann problem). Find a regular solution vector $U = (u, \omega, w, v, \theta)^\top$ to the Eq. (5.51), satisfying the boundary condition

$$\{P(\partial, n)\, U(z)\}^{\pm} = F(z), \quad z \in S. \tag{5.53}$$

We assume that the data of the boundary value problems belong to the appropriate classes,

$$\Phi^{\pm} \in C^{0,\alpha'}(\overline{\Omega})^{\pm}, \qquad f \in C^{1,\alpha'}(S), \qquad F \in C^{0,\alpha'}(S), \quad 0 < \alpha' < \gamma' \le 1;$$

in addition, in the case of exterior problems we assume that the vector-function Φ^- is compactly supported in Ω^-.

Now we prove the following uniqueness theorem.

Theorem 5.1. *Let $\sigma = \sigma_1 + i\sigma_2$, with $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$. Then the homogeneous boundary value problems $(I^{(\sigma)})^{\pm}$ and $(II^{(\sigma)})^{\pm}$ have only the trivial solution in the class of regular vector-function.*

Proof. Let $U = (u, \omega, w, v, \theta)^\top$ be a regular solution of the homogeneous boundary value problem $(I^{(\sigma)})^{\pm}$ or $(II^{(\sigma)})^{\pm}$. Since $L(\partial, \sigma)U = 0$, we can apply Green's formula of type (3.13) or (4.50) for the vector function U and its complex conjugate $\overline{U}$. In particular, let us perform the following operations:
(i) multiply the homogeneous equations (2.1), (2.2), (2.4) by the vectors $i\,\overline{\sigma}\,\overline{u}$, $i\,\overline{\sigma}\,\overline{\omega}$, $i\,\overline{\sigma}\,\overline{v}$ respectively;
(ii) multiply the complex conjugate of homogeneous equations (2.3) and (2.5) by w and $\frac{1}{T_0}\theta$ respectively;
(iii) Sum the results and integrate over the domain Ω^+ or Ω^-.

We arrive at the following relation

$$\begin{aligned} \pm \int_{\partial\Omega^{\pm}} \Big[& i\,\overline{\sigma}\,\overline{u}(x) \cdot T^{(1)}(\partial, n)U(x) + i\,\overline{\sigma}\,\overline{\omega}(x) \cdot T^{(2)}(\partial, n)U(x) + w(x) \cdot T^{(3)}(\partial, n)\overline{U}(x) \\ & + i\,\overline{\sigma}\,\overline{v}(x) T^{(4)}(\partial, n)U(x) + \frac{1}{T_0}\theta(x) T^{(5)}(\partial, n)\overline{U}(x) \Big] ds - \int_{\Omega^{\pm}} E^*(\overline{U}, U) dx = 0, \end{aligned} \tag{5.54}$$

where

$$\begin{aligned} E^*(\overline{U}, U) = {} & i\,\overline{\sigma}\widetilde{E}^{(1)}(\overline{u}, u) + i\,\overline{\sigma}\frac{3\lambda + 2\mu + \varkappa}{3}|\operatorname{div} u|^2 + \frac{i\,\overline{\sigma}\,\varkappa}{2}|\operatorname{rot} u|^2 - i\,\sigma\rho|\sigma|^2|u|^2 + i\,\overline{\sigma}\mu_0(v \operatorname{div}\overline{u} \\ & + \overline{v}\operatorname{div} u) - i\,\overline{\sigma}\varkappa(\overline{\omega} \cdot \operatorname{rot} u + \omega \cdot \operatorname{rot}\overline{u}) + i\,\overline{\sigma}\widetilde{E}^{(2)}(\overline{\omega}, \omega) - i\,\overline{\sigma}\delta|\omega|^2 + i\,\overline{\sigma} b_0(\operatorname{rot}\overline{\omega} \cdot \operatorname{grad} v \\ & + \operatorname{rot}\omega \cdot \operatorname{grad}\overline{v}) + E^{(3)}(\overline{w}, w) - \overline{\varkappa}_0|w|^2 + \varkappa_3 w \cdot \operatorname{grad}\overline{\theta} + \frac{\varkappa_1}{T_0}\overline{w} \cdot \operatorname{grad}\theta + i\,\overline{\sigma} a_0|\operatorname{grad} v|^2 \\ & - i\overline{\sigma}\eta_0|v|^2 + \frac{\varkappa_7}{T_0}|\operatorname{grad}\theta|^2 + \frac{i\,\overline{\sigma} c}{T_0}|\theta|^2 + i\,\overline{\sigma}\frac{\gamma - \beta}{2}|\operatorname{rot}\omega|^2, \end{aligned}$$

$$\begin{aligned}
\widetilde{E}^{(1)}(\overline{u},u) &= \frac{\mu+\varkappa}{3}\sum_{k,j=1}^{3}\left|\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}\right|^2+\frac{2\mu+\varkappa}{4}\sum_{k,j=1,\,k\neq j}^{3}\left|\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\right|^2,\\
\widetilde{E}^{(2)}(\overline{\omega},\omega) &= \frac{3\alpha+\beta+\gamma}{3}|\operatorname{div}\omega|^2+\frac{\gamma+\beta}{4}\sum_{k,j=1,\,k\neq j}^{3}\left|\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right|^2+\frac{\gamma+\beta}{6}\sum_{k,j=1}^{3}\left|\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right|^2,\\
E^{(3)}(\overline{w},w) &= \frac{3\varkappa_4+\varkappa_5+\varkappa_6}{3}|\operatorname{div}w|^2+\frac{\varkappa_6-\varkappa_5}{2}|\operatorname{rot}w|^2\\
&\quad+\frac{\varkappa_5+\varkappa_6}{4}\sum_{k,j=1,\,k\neq j}^{3}\left|\frac{\partial w_k}{\partial x_j}+\frac{\partial w_j}{\partial x_k}\right|^2+\frac{\varkappa_5+\varkappa_6}{6}\sum_{k,j=1}^{3}\left|\frac{\partial w_k}{\partial x_k}-\frac{\partial w_j}{\partial x_j}\right|^2.
\end{aligned}$$

Since $U=(u,\omega,w,v,\theta)^\top$ solves the homogeneous boundary value problem $(I^{(\sigma)})^\pm$ or $(II^{(\sigma)})^\pm$, we see that in the left hand side of (5.54) the surface integral vanishes and we get

$$\int_{\Omega^\pm}E^*(\overline{U},U)dx=0.$$

The real part of this equation reads as

$$\begin{aligned}
&\int_{\Omega^\pm}\Big\{\sigma_2\widetilde{E}^{(1)}(\overline{u},u)+\sigma_2\rho|\sigma|^2|u|^2+\sigma_2\Big[\frac{1}{3}(3\lambda+2\mu+\varkappa)|\operatorname{div}u|^2+\mu_0(v\operatorname{div}\overline{u}+\overline{v}\operatorname{div}u)+\eta|v|^2\Big]\\
&\quad+\sigma_2\widetilde{E}^{(2)}(\overline{\omega},\omega)+\mathcal{I}_1\sigma_2|\sigma|^2|\omega|^2+\frac{1}{2}\sigma_2\varkappa\Big[|\operatorname{rot}u|^2-2(\omega\cdot\operatorname{rot}\overline{u}+\overline{\omega}\cdot\operatorname{rot}u)+4|\omega|^2\Big]\\
&\quad+\sigma_2\Big[\frac{1}{2}(\gamma-\beta)|\operatorname{rot}\omega|^2+b_0(\operatorname{rot}\overline{\omega}\cdot\operatorname{grad}v+\operatorname{rot}\omega\cdot\operatorname{grad}\overline{v})+a_0|\operatorname{grad}v|^2\Big]\\
&\quad+E^{(3)}(\overline{w},w)+\sigma_2 b|w|^2+\frac{1}{T_0}\Big[\varkappa_2T_0|w|^2+\frac{1}{2}(\varkappa_1+T_0\varkappa_3)(w\cdot\operatorname{grad}\overline{\theta}+\overline{w}\cdot\operatorname{grad}\theta)+\varkappa_7|\operatorname{grad}\theta|^2\Big]\\
&\quad+\mathcal{I}\sigma_2|\sigma|^2|v|^2+\sigma_2a|\theta|^2\Big\}dx=0.
\end{aligned}\tag{5.55}$$

Taking into account conditions (3.15) and the following inequalities

(1) $$\begin{aligned}&\frac{3\lambda+2\mu+\varkappa}{3}|\operatorname{div}u|^2+\mu_0(v\operatorname{div}\overline{u}+\overline{v}\operatorname{div}u)+\eta|v|^2\\
&\quad=\frac{\alpha_0'\eta-\mu_0^2}{\alpha_0'}|v|^2+\frac{1}{\alpha_0'}|\mu_0v+\alpha_0'\operatorname{div}u|^2\geq0,\quad \alpha_0'=\frac{3\lambda+2\mu+\varkappa}{3},\end{aligned}$$

(2) $$|\operatorname{rot}u|^2-2(\omega\cdot\operatorname{rot}\overline{u}+\overline{\omega}\cdot\operatorname{rot}u)+4|\omega|^2=|\operatorname{rot}u-2\omega|^2\geq0,$$

(3) $$\begin{aligned}&\frac{\gamma-\beta}{2}|\operatorname{rot}\omega|^2+b_0(\operatorname{rot}\overline{\omega}\cdot\operatorname{grad}v+\operatorname{rot}\omega\cdot\operatorname{grad}\overline{v})+a_0|\operatorname{grad}v|^2\\
&\quad=\frac{a_0(\gamma-\beta)-2b_0^2}{2a_0}|\operatorname{rot}\omega|^2+\frac{1}{a_0}|b_0\operatorname{rot}\omega+a_0\operatorname{grad}v|^2\geq0,\end{aligned}$$

(4) $$\begin{aligned}&\varkappa_2T_0|w|^2+\frac{1}{2}(\varkappa_1+T_0\varkappa_3)(w\cdot\operatorname{grad}\overline{\theta}+\overline{w}\cdot\operatorname{grad}\theta)+\varkappa_7|\operatorname{grad}\theta|^2\\
&\quad=\frac{4T_0\varkappa_2\varkappa_7-(\varkappa_1+T_0\varkappa_3)^2}{4\varkappa_7}|w|^2+\frac{1}{4\varkappa_7}\big|(\varkappa_1+T_0\varkappa_3)w+2\varkappa_7\operatorname{grad}\theta\big|^2\geq0,\end{aligned}$$

we derive

$$\operatorname{Re}E^*(\overline{U},U)\geq0,\quad x\in\Omega^\pm.$$

Therefore, from (5.55) it follows that $U=0$ for $x\in\Omega^\pm$. $\square$

6. Fundamental matrix of solutions

Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized Fourier transform in the space of tempered distributions (Schwartz space $\mathcal{S}'(\mathbb{R}^3)$), which for regular summable functions f and $\widehat{f}$ read as follows

$$\mathcal{F}_{x\to\xi}[f]=\int_{\mathbb{R}^3}f(x)\,e^{i\,x\cdot\xi}dx=\widehat{f}(\xi),\quad \mathcal{F}^{-1}_{\xi\to x}[\widehat{f}]=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\widehat{f}(\xi)\,e^{-i\,x\cdot\xi}d\xi=f(x),\tag{6.56}$$

where $x = (x_1, x_2, x_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$. Note that for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $f \in \mathcal{S}'(\mathbb{R}^3)$

$$\mathcal{F}[\partial^\alpha f] = (-i\,\xi)^\alpha \mathcal{F}[f], \quad \mathcal{F}^{-1}[\xi^\alpha \widehat{f}\,] = (i\,\partial)^\alpha \mathcal{F}^{-1}[\widehat{f}\,], \tag{6.57}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\xi^\alpha = \xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\,\xi_3^{\alpha_3}$.

Denote by $\varGamma(x, \sigma) = [\varGamma_{kj}(x, \sigma)]_{11\times 11}$ the matrix of fundamental solutions of the operator $L(\partial, \sigma)$ (see (2.6), (2.7))

$$L(\partial, \sigma)\,\varGamma(x, \sigma) = \delta(x)\,I_{11}; \tag{6.58}$$

here $\delta(\,\cdot\,)$ is Dirac's delta distribution. We assume that the frequency parameter σ is complex, in general,

$$\sigma = \sigma_1 + i\,\sigma_2, \quad \sigma_1, \sigma_2 \in \mathbb{R}.$$

We represent the matrix $\varGamma(x, \sigma)$ in the block wise form

$$\varGamma(x,\sigma) = \begin{bmatrix} \varGamma^{(1)}(x,\sigma) & \varGamma^{(2)}(x,\sigma) & \varGamma^{(3)}(x,\sigma) & \varGamma^{(4)}(x,\sigma) & \varGamma^{(5)}(x,\sigma) \\ \varGamma^{(6)}(x,\sigma) & \varGamma^{(7)}(x,\sigma) & \varGamma^{(8)}(x,\sigma) & \varGamma^{(9)}(x,\sigma) & \varGamma^{(10)}(x,\sigma) \\ \varGamma^{(11)}(x,\sigma) & \varGamma^{(12)}(x,\sigma) & \varGamma^{(13)}(x,\sigma) & \varGamma^{(14)}(x,\sigma) & \varGamma^{(15)}(x,\sigma) \\ \varGamma^{(16)}(x,\sigma) & \varGamma^{(17)}(x,\sigma) & \varGamma^{(18)}(x,\sigma) & \varGamma^{(19)}(x,\sigma) & \varGamma^{(20)}(x,\sigma) \\ \varGamma^{(21)}(x,\sigma) & \varGamma^{(22)}(x,\sigma) & \varGamma^{(23)}(x,\sigma) & \varGamma^{(24)}(x,\sigma) & \varGamma^{(25)}(x,\sigma) \end{bmatrix}_{11\times 11},$$

where

$$\begin{aligned} &\varGamma^{(j)}(x,\sigma) = \big[\varGamma^{(j)}_{pq}(x,\sigma)\big]_{3\times 3}, \quad j = 1, 2, 3, 6, 7, 8, 11, 12, 13, \\ &\varGamma^{(j)}(x,\sigma) = \big[\varGamma^{(j)}_{pq}(x,\sigma)\big]_{3\times 1}, \quad j = 4, 5, 9, 10, 14, 15, \\ &\varGamma^{(j)}(x,\sigma) = \big[\varGamma^{(j)}_{pq}(x,\sigma)\big]_{1\times 3}, \quad j = 16, 17, 18, 21, 22, 23, \end{aligned}$$

and $\varGamma^{(19)}(x, \sigma)$, $\varGamma^{(20)}(x, \sigma)$, $\varGamma^{(24)}(x, \sigma)$, and $\varGamma^{(25)}(x, \sigma)$ are scalar functions.

By $\widehat{\varGamma}(\xi, \sigma)$ and $\widehat{\varGamma}^{(k)}(\xi, \sigma)$ we denote the Fourier transforms of the matrices $\varGamma(x, \sigma)$ and $\varGamma^{(k)}(x, \sigma)$, $k = 1, 2, \ldots, 25$.

Applying the Fourier transform to Eq. (6.58) and taking into consideration (6.57) and the equality $\mathcal{F}[\delta(\cdot)] = 1$, we get

$$L(-i\,\xi, \sigma)\,\widehat{\varGamma}(\xi, \sigma) = I_{11}. \tag{6.59}$$

We have to determine $\widehat{\varGamma}(\xi, \sigma)$ from (6.59) and afterwards with the help of the inverse Fourier transform construct the fundamental matrix $\varGamma(x, \sigma)$ explicitly in terms of standard elementary functions. Evidently, first of all we have to represent the matrix $\widehat{\varGamma}(\xi, \sigma) = [L(-i\,\xi, \sigma)]^{-1}$ in such form which is convenient for calculation of the inverse Fourier transform.

To this end, we proceed as follows. We set $r := |\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$ and introduce the notation

$$\begin{aligned} &A(\xi) := L^{(1)}(-i\xi, \sigma) = [\rho\,\sigma^2 - (\mu + \varkappa)\,r^2]\,I_3 - (\lambda + \mu)\,Q(\xi), \\ &B(\xi) := L^{(7)}(-i\xi, \sigma) = (\delta - \gamma\,r^2)\,I_3 - (\alpha + \beta)Q(\xi), \\ &C(\xi) := L^{(13)}(-i\,\xi, \sigma) = (\varkappa_0 - \varkappa_6 r^2)\,I_3 - (\varkappa_4 + \varkappa_5)\,Q(\xi), \end{aligned} \tag{6.60}$$

where $Q(\cdot)$ is defined by (2.8). In view of (2.6)–(2.8) from (6.59) we easily derive

$$\begin{aligned} &A(\xi)\,\widehat{\varGamma}^{(j)}(\xi,\sigma) - i\varkappa R(\xi)\,\widehat{\varGamma}^{(j+5)}(\xi,\sigma) - i\,\mu_0\,\xi^\top\,\widehat{\varGamma}^{(j+15)}(\xi,\sigma) + i\,\beta_0\,\xi^\top\,\widehat{\varGamma}^{(j+20)}(\xi,\sigma) = \delta_{1j} I_3, \\ &-i\varkappa R(\xi)\,\widehat{\varGamma}^{(j)}(\xi,\sigma) + B(\xi)\,\widehat{\varGamma}^{(j+5)}(\xi,\sigma) + i\,\mu_1\,R(\xi)\,\widehat{\varGamma}^{(j+10)}(\xi,\sigma) = \delta_{2j} I_3, \\ &-\sigma\mu_1 R(\xi)\,\widehat{\varGamma}^{(j+5)}(\xi,\sigma) + C(\xi)\,\widehat{\varGamma}^{(j+10)}(\xi,\sigma) + \sigma\,\mu_2\,\xi^\top\,\widehat{\varGamma}^{(j+15)}(\xi,\sigma) + i\varkappa_3\xi^\top\,\widehat{\varGamma}^{(j+20)}(\xi,\sigma) = \delta_{3j} I_3, \\ &i\mu_0\xi\,\widehat{\varGamma}^{(j)}(\xi,\sigma) + i\mu_2\xi\,\widehat{\varGamma}^{(j+10)}(\xi,\sigma) + (\eta_0 - a_0\,r^2)\,\widehat{\varGamma}^{(j+15)}(\xi,\sigma) + \beta_1\,\widehat{\varGamma}^{(j+20)}(\xi,\sigma) = \delta_{4j}, \\ &\beta_0 T_0\sigma\,\widehat{\varGamma}^{(j)}(\xi,\sigma) - i\varkappa_1\xi\,\widehat{\varGamma}^{(j+10)}(\xi,\sigma) + i\sigma\beta_1 T_0\,\widehat{\varGamma}^{(j+15)}(\xi,\sigma) + (i\sigma c - \varkappa_7\,r^2)\,\widehat{\varGamma}^{(j+20)}(\xi,\sigma) = \delta_{5j}, \\ &\qquad\qquad j = 1, 2, \ldots, 5, \end{aligned} \tag{6.61}$$

where $R(\cdot)$ is defined by (2.8).

Applying the relations (2.8), (2.9) and (6.60) we can easily show that

$$A(\xi)=A(-\xi)=A^\top(\xi),\quad B(\xi)=B(-\xi)=B^\top(\xi),\quad C(\xi)=C(-\xi)=C^\top(\xi),$$
$$Q(\xi)=Q^\top(\xi),\quad R^\top(\xi)=-R(\xi)=R(-\xi),\quad Q(\xi)R(\xi)=R(\xi)Q(\xi)=[0]_{3\times 3},$$
$$[Q(\xi)]^2=r^2Q(\xi),\quad [R(\xi)]^2=Q(\xi)-r^2I_3,$$

and the matrices A, B and C commute to each other.

By direct calculations, we can show that the elements of the matrix $\widehat{\Gamma}(\xi,\sigma)$ from the system (6.61) have the form

$$\begin{aligned}
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}[a_j(\xi)I_3+b_j(\xi)Q(\xi)],\quad j=1,3,7,11,13,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}c_j(\xi)R(\xi),\quad j=2,6,8,12,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}c_j(\xi)\xi^\top,\quad j=4,5,14,15,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}c_j(\xi)\xi,\quad j=16,18,21,23,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}a_j(\xi),\quad j=19,20,24,25,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=[0]_{3\times 1}\quad j=9,10,\\
\widehat{\Gamma}^{(j)}(\xi,\sigma)&=[0]_{1\times 3}\quad j=17,22.
\end{aligned}$$

Here

$$\Lambda(\xi)=\det L(-i\xi,\sigma)=\varkappa_6\,(l_0r^2-\varkappa_0)a(\xi)\big(a(\xi)+b(\xi)r^2\big)\Delta'(\xi)=d_1d_2d_3\prod_{j=1}^{11}(r^2-\lambda_j^2);\tag{6.62}$$
$$d_1=\lambda_0l_0\varkappa_7a_0,\quad d_2=\alpha_0\lambda_0\gamma\,(\mu+\varkappa)\varkappa_6^2,\quad d_3=\varkappa_6l_0,$$
$$\lambda_1^2=\rho\sigma^2/\lambda_0,\quad \lambda_2^2=\varkappa_0/l_0,\quad \lambda_7^2=\varkappa_0/\varkappa_6,\quad \lambda_{11}^2=\delta/\alpha_0,$$

λ_8^2, λ_9^2, λ_{10}^2 are the roots of the equation $a(\xi)=0$, where

$$\begin{aligned}a(\xi)&=\varkappa_6(r^2-\lambda_7^2)\big[(\rho\sigma^2-(\mu+\varkappa)r^2)(\gamma r^2-\delta)+\varkappa^2r^2\big]+i\sigma\mu_1^2\big[(\mu+\varkappa)r^2-\rho\sigma^2\big]r^2\\
&=-\gamma(\mu+\varkappa)\varkappa_6(r^2-\lambda_8^2)(r^2-\lambda_9^2)(r^2-\lambda_{10}^2);\end{aligned}\tag{6.63}$$

$$\begin{aligned}b(\xi)=&-\varkappa_6(r^2-\lambda_7^2)\big[\alpha_0(\lambda+\mu)(r^2-\lambda_{11}^2)+(\alpha+\beta)((\mu+\varkappa)r^2-\rho\sigma^2)+\varkappa^2\big]\\
&-i\sigma\mu_1^2\big[(\mu+\varkappa)r^2-\rho\sigma^2\big],\end{aligned}\tag{6.64}$$

$$\Delta_2'(\xi)=a(\xi)+b(\xi)r^2=-\alpha_0\lambda_0\varkappa_6(r^2-\lambda_1^2)(r^2-\lambda_7^2)(r^2-\lambda_{11}^2),\tag{6.65}$$

λ_j^2, $j=3,4,5,6$ are roots of the equation $\Delta'(\xi)=0$,

$$\begin{aligned}\Delta'(\xi)=&\big[\mu_0^2\,\varkappa_1\,\varkappa_3-\beta_0^2\,\mu_2^2\,T_0\,\sigma^2-i\sigma\beta_0\,\mu_0\,\mu_2\,(\varkappa_1+T_0\varkappa_3)\big]r^4\\
&-\lambda_0\big[i\sigma\mu_2^2(\varkappa_7r^2-i\sigma c)+\varkappa_1\varkappa_3(a_0r^2-\eta_0)-i\sigma\beta_1\mu_2(\varkappa_1+T_0\varkappa_3)\big](r^2-\lambda_1^2)r^2\\
&-l_0\big[\mu_0^2(\varkappa_7r^2-i\sigma c)+i\beta_0^2T_0\sigma(a_0r^2-\eta_0)-2i\sigma\beta_0\beta_1T_0\mu_0\big](r^2-\lambda_2^2)r^2\\
&+\lambda_0l_0\big[(\varkappa_7r^2-i\sigma c)(a_0r^2-\eta_0)-i\sigma\beta_1^2T_0\big](r^2-\lambda_1^2)(r^2-\lambda_2^2)\\
=&\,d_1(r^2-\lambda_3^2)(r^2-\lambda_4^2)(r^2-\lambda_5^2)(r^2-\lambda_6^2),\end{aligned}\tag{6.66}$$

$$\Delta_2(\xi)=a(\xi)\big[a(\xi)+b(\xi)r^2\big]=d_2(r^2-\lambda_1^2)\prod_{j=7}^{11}(r^2-\lambda_j^2),\tag{6.67}$$

$$\Delta_3(\xi)=d_3(r^2-\lambda_2^2)(r^2-\lambda_7^2),\tag{6.68}$$

$$\Delta_2'(\xi)=a(\xi)+b(\xi)r^2=-\alpha_0\lambda_0\varkappa_6(r^2-\lambda_1^2)(r^2-\lambda_7^2)(r^2-\lambda_{11}^2),\tag{6.69}$$

$$\begin{aligned}
a_1(\xi) &= d_3\Delta'(\xi)\Delta_2'(\xi)(r^2-\lambda_2^2)\big[\varkappa_6(\gamma\, r^2-\delta)(r^2-\lambda_7^2)-i\sigma\mu_1^2r^2\big],\\
a_3(\xi) &= -\mu_1\varkappa\, d_3\Delta'(\xi)\Delta_2'(\xi)(r^2-\lambda_2^2)r^2,\\
a_7(\xi) &= \varkappa_6\, d_3\big[(\mu+\varkappa)r^2-\rho\sigma^2\big]\Delta'(\xi)\Delta_2'(\xi)(r^2-\lambda_2^2)(r^2-\lambda_7^2),\\
a_{11}(\xi) &= -i\sigma\mu_1\varkappa d_3r^2(r^2-\lambda_2^2)\Delta'(\xi)\Delta_2'(\xi),\\
a_{13}(\xi) &= \frac{l_0}{r^2-\lambda_7^2}(r^2-\lambda_2^2)\Delta'(\xi)\big[i\sigma\mu_1^2((\mu+\varkappa)r^2-\rho\sigma^2)r^2\Delta_2'(\xi)-\Delta_2(\xi)\big],
\end{aligned}\tag{6.70}$$

$$\begin{aligned}
a_{19}(\xi) &= d_3(r^2-\lambda_2^2)\Delta_2(\xi)\alpha_{22}(\xi),\\
a_{20}(\xi) &= -d_3(r^2-\lambda_2^2)\Delta_2(\xi)\alpha_{12}(\xi),\\
a_{24}(\xi) &= -d_3(r^2-\lambda_2^2)\Delta_2(\xi)\alpha_{21}(\xi),\\
a_{25}(\xi) &= d_3(r^2-\lambda_2^2)\Delta_2(\xi)\alpha_{11}(\xi),
\end{aligned}\tag{6.71}$$

$$\begin{aligned}
\alpha_{11}(\xi) &= \mu_0^2l_0(r^2-\lambda_2^2)r^2+i\sigma\mu_2^2\lambda_0(r^2-\lambda_1^2)r^2+\lambda_0l_0(\eta_0-a_0r^2)(r^2-\lambda_1^2)(r^2-\lambda_2^2),\\
\alpha_{12}(\xi) &= -\mu_0\beta_0l_0(r^2-\lambda_2^2)r^2-\mu_2\lambda_0\varkappa_3(r^2-\lambda_1^2)r^2+\beta_1\lambda_0l_0(r^2-\lambda_1^2)(r^2-\lambda_2^2),\\
\alpha_{21}(\xi) &= -i\mu_0\beta_0T_0l_0\sigma(r^2-\lambda_2^2)r^2-i\varkappa_1\mu_2\lambda_0\sigma(r^2-\lambda_1^2)r^2+i\sigma\beta_1T_0\lambda_0l_0(r^2-\lambda_1^2)(r^2-\lambda_2^2),\\
\alpha_{22}(\xi) &= i\sigma\beta_0^2T_0l_0(r^2-\lambda_2^2)r^2+\varkappa_1\varkappa_3\lambda_0(r^2-\lambda_1^2)r^2-\lambda_0l_0(\varkappa_7r^2-i\sigma c)(r^2-\lambda_1^2)(r^2-\lambda_2^2),
\end{aligned}\tag{6.72}$$

$$\begin{aligned}
b_1(\xi) &= \alpha_0\varkappa_6d_3(r^2-\lambda_2^2)(r^2-\lambda_7^2)(r^2-\lambda_{11}^2)\big\{\beta_{11}(\xi)a(\xi)\\
&\quad-\big[\lambda_0\varkappa_6(\alpha+\beta)(r^2-\lambda_1^2)(r^2-\lambda_7^2)+i\sigma\lambda_0\mu_1^2(r^2-\lambda_1^2)+b(\xi)\big]\Delta'(\xi)\big\},\\
b_3(\xi) &= \alpha_0\varkappa_6(r^2-\lambda_{11}^2)\Delta_3(\xi)\big[a(\xi)\beta_{13}(\xi)-\lambda_0\mu_1\varkappa(r^2-\lambda_1^2)\Delta'(\xi)\big],\\
b_7(\xi) &= -\varkappa_6\Delta'(\xi)\Delta_3(\xi)\big[\lambda_0b(\xi)(r^2-\lambda_1^2)-(\lambda+\mu)\Delta_2'(\xi)\big],\\
b_{11}(\xi) &= \varkappa_6\big[i\sigma\mu_1l_0\varkappa(r^2-\lambda_2^2)\Delta_2'(\xi)\Delta'(\xi)+(\sigma\mu_2\gamma_{41}(\xi)+i\varkappa_3\gamma_{51}(\xi))\Delta_2(\xi)\big],\\
b_{13}(\xi) &= i\sigma\mu_1^2\alpha_0l_0\lambda_0\varkappa_6(r^2-\lambda_1^2)(r^2-\lambda_2^2)(r^2-\lambda_{11}^2)((\mu+\varkappa)r^2-\rho\sigma^2)\Delta'(\xi)\\
&\quad+\varkappa_6(\sigma\mu_2\gamma_{43}(\xi)+i\varkappa_3\gamma_{53}(\xi))\Delta_2(\xi)+d_2(\varkappa_4+\varkappa_5)(r^2-\lambda_1^2)\Delta'(\xi)\prod_{j=8}^{11}(r^2-\lambda_j^2),
\end{aligned}\tag{6.73}$$

$$\begin{aligned}
c_2(\xi) &= -i\varkappa\varkappa_6\Delta'(\xi)\Delta_2'(\xi)\Delta_3(\xi),\\
c_4(\xi) &= \alpha_0\varkappa_6a(\xi)\beta_{14}(\xi)(r^2-\lambda_{11}^2)\Delta_3(\xi),\\
c_5(\xi) &= \alpha_0\varkappa_6a(\xi)\beta_{15}(\xi)(r^2-\lambda_{11}^2)\Delta_3(\xi),\\
c_6(\xi) &= -i\varkappa\varkappa_6\Delta'(\xi)\Delta_2'(\xi)\Delta_3(\xi),\\
c_8(\xi) &= i\mu_1d_3(r^2-\lambda_2^2)((\mu+\varkappa)r^2-\rho\sigma^2)\Delta'(\xi)\Delta_2'(\xi),\\
c_{12}(\xi) &= -\sigma\mu_1d_3((\mu+\varkappa)r^2-\rho\sigma^2)(r^2-\lambda_2^2)\Delta'(\xi)\Delta_2'(\xi),\\
c_{14}(\xi) &= \varkappa_6(\sigma\mu_2\alpha_{22}(\xi)-i\varkappa_3\alpha_{21}(\xi))\Delta_2(\xi),\\
c_{15}(\xi) &= -\varkappa_6(\sigma\mu_2\alpha_{12}(\xi)-i\varkappa_3\alpha_{11}(\xi))\Delta_2(\xi),\\
c_{16}(\xi) &= d_3(r^2-\lambda_2^2)\Delta_2(\xi)\gamma_{41}(\xi),\\
c_{18}(\xi) &= d_3(r^2-\lambda_2^2)\Delta_2(\xi)\gamma_{43}(\xi),\\
c_{21}(\xi) &= d_3(r^2-\lambda_2^2)\Delta_2(\xi)\gamma_{51}(\xi),\\
c_{23}(\xi) &= d_3(r^2-\lambda_2^2)\Delta_2(\xi)\gamma_{53}(\xi),
\end{aligned}\tag{6.74}$$

$$\begin{aligned}
\beta_{11}(\xi) &= i\mu_0\gamma_{41}(\xi)-i\beta_0\gamma_{51}(\xi),\qquad \beta_{13}(\xi)=i\mu_0\gamma_{43}(\xi)-i\beta_0\gamma_{53}(\xi),\\
\beta_{14}(\xi) &= i\mu_0\alpha_{22}(\xi)+i\beta_0\alpha_{21}(\xi),\quad \beta_{15}(\xi)=-i\mu_0\alpha_{12}(\xi)-i\beta_0\alpha_{11}(\xi),
\end{aligned}\tag{6.75}$$

$$\begin{aligned}
\gamma_{41}(\xi) &= (i\mu_0\varkappa_1\varkappa_3+\sigma\beta_0\mu_2\varkappa_3T_0)r^2-\big[i\mu_0l_0(\varkappa_7r^2-i\sigma c)+\beta_0\beta_1l_0\sigma T_0\big](r^2-\lambda_2^2),\\
\gamma_{43}(\xi) &= -\beta_0(\sigma\mu_2\beta_0T_0+i\mu_0\varkappa_1)r^2-i\lambda_0\big[\mu_2(\varkappa_7r^2-i\sigma c)-\varkappa_1\beta_1\big](r^2-\lambda_1^2),\\
\gamma_{51}(\xi) &= \sigma\mu_2(i\sigma\beta_0\mu_2T_0-\mu_0\varkappa_1)r^2-l_0T_0\sigma\big[\beta_0(a_0r^2-\eta_0)-\mu_0\beta_1\big](r^2-\lambda_2^2),\\
\gamma_{53}(\xi) &= -\mu_0(i\varkappa_1\mu_0+\sigma\mu_2\beta_0T_0)r^2+\lambda_0\big[i\varkappa_1(a_0r^2-\eta_0)+\sigma\mu_2\beta_1T_0\big](r^2-\lambda_1^2).
\end{aligned}\tag{6.76}$$

Therefore, we can represent the matrix $\widehat{\Gamma}(\xi,\sigma)$ in the form

$$\widehat{\Gamma}(\xi,\sigma)=[L(-i\,\xi,\sigma)]^{-1}=\frac{1}{\Lambda(\xi)}\,\mathcal{M}(\xi,\sigma), \tag{6.77}$$

where

$$\begin{aligned}\mathcal{M}(\xi,\sigma) \;:=\; & \begin{bmatrix} a_1(\xi)\,I_3 & [0]_{3\times3} & a_3(\xi)\,I_3 & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & a_7(\xi)\,I_3 & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\ a_{11}(\xi)\,I_3 & [0]_{3\times3} & a_{13}(\xi)\,I_3 & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & a_{19}(\xi) & a_{20}(\xi)\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & a_{24}(\xi) & a_{25}(\xi) \end{bmatrix}\\ &+\begin{bmatrix} b_1(\xi)\,Q(\xi) & [0]_{3\times3} & b_3(\xi)\,Q(\xi) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & b_7(\xi)\,Q(\xi) & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\ b_{11}(\xi)\,Q(\xi) & [0]_{3\times3} & b_{13}(\xi)\,Q(\xi) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0 \end{bmatrix}\\ &+\begin{bmatrix} [0]_{3\times3} & c_2(\xi)\,R(\xi) & [0]_{3\times3} & c_4(\xi)\,\xi^\top & c_5(\xi)\,\xi^\top\\ c_6(\xi)\,R(\xi) & [0]_{3\times3} & c_8(\xi)\,R(\xi) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & c_{12}(\xi)\,R(\xi) & [0]_{3\times3} & c_{14}(\xi)\,\xi^\top & c_{15}(\xi)\,\xi^\top\\ c_{16}(\xi)\,\xi & [0]_{1\times3} & c_{18}(\xi)\,\xi & 0 & 0\\ c_{21}(\xi)\,\xi & [0]_{1\times3} & c_{23}(\xi)\,\xi & 0 & 0 \end{bmatrix}. \end{aligned} \tag{6.78}$$

It is easy to see that the entries of the 11×11 matrix $\mathcal{M}(\xi,\sigma)$ are polynomials in ξ. Therefore to invert the Fourier transform and find an explicit form for the fundamental matrix $\Gamma(x,\sigma)$ we need the roots of the equation

$$\Lambda(r)=\det L(-i\,\xi,\sigma)=0 \quad \text{with} \quad r=|\xi|.$$

Due to the evenness of the function $\Lambda(r)$ with respect to r, it is clear that if $r=r_0$ is a root of the equation $\Lambda(r)=0$, then so is $r=-r_0$.

In view of (6.62) the roots of the equation $\Lambda(r)=0$ are $\pm\lambda_j$, $j=1,2,\dots,11$. For simplicity we assume that (see Appendix) $\lambda_j\neq\lambda_k$, for $j\neq k$, $\operatorname{Im}\lambda_j>0$, and if $\operatorname{Im}\lambda_j=0$, then $\lambda_j>0$.

Therefore in view of (6.77) we can represent the fundamental solution as

$$\Gamma(x,\sigma)=\mathcal{F}^{-1}_{\xi\to x}\Big[\widehat{\Gamma}(\xi,\sigma)\Big]=\frac{1}{d_1\,d_2\,d_3}\,\mathcal{F}^{-1}_{\xi\to x}\Big[\mathcal{M}(\xi,\sigma)\,\frac{1}{\Phi(r)}\Big]=\frac{1}{d_1\,d_2\,d_3}\,\mathcal{M}(i\,\partial,\sigma)\,\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{\Phi(r)}\Big], \tag{6.79}$$

where

$$\Phi(r)=\prod_{j=1}^{11}(r^2-\lambda_j^2).$$

Note that

$$\frac{1}{\Phi(r)}=\sum_{j=1}^{11}\frac{p_j}{r^2-\lambda_j^2},$$

where the parameters $p_1,p_2,\dots,p_{11}$ solve the system of linear algebraic equations

$$\begin{aligned}&\lambda_1^{2m}\,p_1+\lambda_2^{2m}\,p_2+\cdots+\lambda_{11}^{2m}\,p_{11}=0, \quad m=0,1,\dots,9,\\ &\lambda_1^{20}\,p_1+\lambda_2^{20}\,p_2+\cdots+\lambda_{11}^{20}\,p_{11}=1.\end{aligned}$$

They can be represented as follows

$$p_j=\Bigg[\prod_{l=1,\,l\neq j}^{11}(k_l^2-\lambda_j^2)\Bigg]^{-1}.$$

Note that, if $\operatorname{Im}\lambda_j \geq 0$, then

$$\mathcal{F}^{-1}_{\xi\to x}\left[\frac{1}{r^2-\lambda_j^2}\right] = \frac{e^{i\,\lambda_j|x|}}{4\pi\,|x|}.$$

Therefore

$$\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{\Phi(r)}\Big] = \frac{1}{4\pi}\sum_{j=1}^{11} p_j\,\frac{e^{i\,\lambda_j|x|}}{|x|}. \tag{6.80}$$

Now from (6.79) and (6.80) it follows that

$$\Gamma(x,\sigma) = \frac{1}{4\,\pi\,d_1\,d_2\,d_3}\,\mathcal{M}(i\,\partial,\sigma)\sum_{j=1}^{11} p_j\,\frac{e^{i\,\lambda_j|x|}}{|x|} \tag{6.81}$$

or

$$\Gamma(x,\sigma) = \frac{1}{4\,\pi\,d_1\,d_2\,d_3}\mathcal{M}(i\,\partial,\sigma)\,\Psi(x,\sigma),$$

where the differential operator $\mathcal{M}(i\,\partial,\sigma)$ is given by (6.78) with $i\,\partial$ for ξ and

$$\Psi(x,\sigma) = \sum_{j=1}^{11} p_j\,\frac{e^{i\,\lambda_j|x|}}{|x|}.$$

We can calculate the expression $\mathcal{M}(i\,\partial,\sigma)\,\Psi(x,\sigma)$ and rewrite the fundamental solution in a more explicit form. To this end let us note that

$$\Delta\,\frac{e^{i\,\lambda_j|x|}}{|x|} = -\lambda_j^2\,\frac{e^{i\,\lambda_j|x|}}{|x|},\quad |x|\neq 0,$$

and apply formulas (6.63)–(6.76), to obtain

$$a(i\,\partial)\,\Psi(x,\sigma) = \sum_{j=1}^{11} p_j\,a_j^*\,\frac{e^{i\,\lambda_j|x|}}{|x|},$$

$$b(i\,\partial)\,\Psi(x,\sigma) = \sum_{j=1}^{11} p_j\,b_j^*\,\frac{e^{i\,\lambda_j|x|}}{|x|},$$

$$a_l(i\,\partial)\,\Psi(x,\sigma) = \sum_{j=1}^{11} p_j a_{lj}^*\,\frac{e^{i\,\lambda_j|x|}}{|x|},\quad l = 1, 3, 7, 11, 13, 19, 20, 24, 25,$$

$$b_l(i\,\partial)\,\Psi(x,\sigma) = \sum_{j=1}^{11} p_j b_{lj}^*\,\frac{e^{i\,\lambda_j|x|}}{|x|},\quad l = 1, 3, 7, 11, 13,$$

$$c_l(i\,\partial)\,\Psi(x,\sigma) = \sum_{j=1}^{11} p_j c_{lj}^*\,\frac{e^{i\,\lambda_j|x|}}{|x|},\quad l = 2, 4, 5, 6, 8, 12, 14, 15, 16, 18, 21, 23,$$

where

$$a_j^* = \gamma(\mu+\varkappa)\varkappa_6\prod_{l=8}^{10}(\lambda_l^2-\lambda_j^2),$$

$$b_j^* = -\varkappa_6(\lambda_j^2-\lambda_7^2)\big[\alpha_0(\lambda+\mu)(\lambda_j^2-\lambda_{11}^2)+(\alpha+\beta)((\mu+\varkappa)\lambda_j^2-\rho\sigma^2)+\varkappa^2\big]-i\sigma\mu_1^2((\mu+\varkappa)\lambda_j^2-\rho\sigma^2),$$

$$a_{1j}^* = -\alpha_0\lambda_0\varkappa_6 d_1\,d_3\,(\lambda_j^2-\lambda_{11}^2)\big[\varkappa_6(\gamma\lambda_j^2-\delta)(\lambda_j^2-\lambda_7^2)-i\sigma\mu_1^2\lambda_j^2\big]\prod_{l=1}^{7}(\lambda_j^2-\lambda_l^2),$$

$$a_{3j}^* = -\mu_1\varkappa\alpha_0\lambda_0\varkappa_6 d_1\,d_3\,\lambda_j^2(\lambda_j^2-\lambda_{11}^2)\prod_{l=1}^{7}(\lambda_j^2-\lambda_l^2),$$

$$a^*_{7j} = -\alpha_0\lambda_0\varkappa_6^2 d_1 d_3((\mu+\varkappa)\lambda_j^2 - \rho\sigma^2)(\lambda_j^2 - \lambda_7^2)(\lambda_j^2 - \lambda_{11}^2)\prod_{l=1}^{7}(\lambda_j^2 - \lambda_l^2),$$

$$a^*_{11j} = i\sigma\mu_1\varkappa\alpha_0\lambda_0\varkappa_6 d_1\, d_3\,\lambda_j^2(\lambda_j^2 - \lambda_{11}^2)\prod_{l=1}^{7}(\lambda_j^2 - \lambda_l^2),$$

$$a^*_{13j} = -l_0 d_1\prod_{l=1}^{6}(\lambda_j^2 - \lambda_l^2)\Big[i\sigma\mu_1^2\alpha_0\lambda_0\varkappa_6(\lambda_j^2 - \lambda_{11}^2)((\mu+\varkappa)\lambda_j^2 - \rho\sigma^2)\lambda_j^2 + d_2\prod_{l=8}^{11}(\lambda_j^2 - \lambda_l^2)\Big],$$

$$a^*_{19j} = -d_2\, d_3\,(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2)\alpha_{22}^{(j)},$$

$$a^*_{20j} = -d_2\, d_3\,(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2)\alpha_{12}^{(j)},$$

$$a^*_{24j} = -d_2\, d_3\,(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2)\alpha_{21}^{(j)},$$

$$a^*_{25j} = d_2\, d_3\,(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2)\alpha_{11}^{(j)},$$

$$\alpha_{11}^{(j)} = l_0\mu_0^2\,(\lambda_j^2 - \lambda_2^2)\lambda_j^2 + i\sigma\mu_2^2\lambda_0(\lambda_j^2 - \lambda_1^2)\lambda_j^2 + l_0\lambda_0(\eta_0 - a_0\lambda_j^2)(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2),$$
$$\alpha_{12}^{(j)} = -\mu_0\beta_0 l_0\,(\lambda_j^2 - \lambda_2^2)\lambda_j^2 - \mu_2\lambda_0\varkappa_3(\lambda_j^2 - \lambda_1^2)\lambda_j^2 + \beta_1\lambda_0 l_0(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2),$$
$$\alpha_{21}^{(j)} = -i\mu_0\sigma l_0\beta_0 T_0\,(\lambda_j^2 - \lambda_2^2)\lambda_j^2 - i\varkappa_1\mu_2\sigma\lambda_0(\lambda_j^2 - \lambda_1^2)\lambda_j^2 + i\sigma\beta_1\lambda_0 l_0 T_0(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2),$$
$$\alpha_{22}^{(j)} = i\sigma l_0\beta_0^2 T_0\,(\lambda_j^2 - \lambda_2^2)\lambda_j^2 + \varkappa_1\varkappa_3\lambda_0(\lambda_j^2 - \lambda_1^2)\lambda_j^2 - \lambda_0 l_0(\varkappa_7\lambda_j^2 - i\sigma c)(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2),$$

$$b^*_{1j} = \alpha_0\varkappa_6 d_3(\lambda_j - \lambda_2^2)(\lambda_j^2 - \lambda_7^2)(\lambda_j^2 - \lambda_{11}^2)\Big\{a^*_j\beta_{11}^{(j)}$$
$$- d_1\Big[\lambda_0\varkappa_6(\alpha+\beta)(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_7^2) + i\sigma\lambda_0\mu_1^2(\lambda_j^2 - \lambda_1^2) + b^*_j\Big]\prod_{l=3}^{6}(\lambda_j^2 - \lambda_l^2)\Big\},$$

$$b^*_{3j} = -\alpha_0\varkappa_6 d_3(\lambda_j - \lambda_2^2)(\lambda_j^2 - \lambda_7^2)(\lambda_j^2 - \lambda_{11}^2)\Big[a^*_j\beta_{13}^{(j)} - \lambda_0\mu_1\varkappa d_1(\lambda_j^2 - \lambda_1^2)\prod_{l=3}^{6}(\lambda_j^2 - \lambda_l^2)\Big],$$

$$b^*_{7j} = -\varkappa_6 d_1 d_3\Big[\lambda_0 b^*_j + (\lambda+\mu)\alpha_0\lambda_0\varkappa_6(\lambda_j^2 - \lambda_7^2)(\lambda_j^2 - \lambda_{11}^2)\Big]\prod_{l=1}^{7}(\lambda_j^2 - \lambda_l^2),$$

$$b^*_{11j} = \varkappa_6\Big[-i\sigma\mu_1\varkappa\alpha_0\lambda_0 d_1 d_3(\lambda_j^2 - \lambda_{11}^2)\prod_{l=1}^{7}(\lambda_j^2 - \lambda_l^2) + d_2(\sigma\mu_2\gamma_{41}^{(j)} + i\varkappa_3\gamma_{51}^{(j)})(\lambda_j^2 - \lambda_1^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2)\Big],$$

$$b^*_{13j} = i\sigma\mu_1^2\alpha_0\lambda_0 d_1 d_3(\lambda_j^2 - \lambda_{11}^2)((\mu+\varkappa)\lambda_j^2 - \rho\sigma^2)\prod_{l=1}^{6}(\lambda_j^2 - \lambda_l^2)$$
$$+ d_1 d_2(\varkappa_4 + \varkappa_5)(\lambda_j^2 - \lambda_1^2)\prod_{l=3}^{6}(\lambda_j^2 - \lambda_l^2)\prod_{l=8}^{11}(\lambda_j^2 - \lambda_l^2) + \varkappa_6 d_2(\sigma\mu_2\gamma_{43}^{(j)} + i\varkappa_3\gamma_{53}^{(j)})\prod_{l=8}^{11}(\lambda_j^2 - \lambda_l^2),$$

$$\gamma_{41}^{(j)} = (i\mu_0\varkappa_1\varkappa_3 + \sigma\beta_0\mu_2\varkappa_3 T_0)\lambda_j^2 - \big[i\mu_0 l_0(\varkappa_7\lambda_j^2 - i\sigma c) + \beta_0\beta_1 l_0\sigma T_0\big](\lambda_j^2 - \lambda_2^2),$$
$$\gamma_{43}^{(j)} = -\beta_0(\sigma\mu_2\beta_0 T_0 + i\mu_0\varkappa_1)\lambda_j^2 - i\lambda_0\big[\mu_2(\varkappa_7\lambda_j^2 - i\sigma c) - \varkappa_1\beta_1\big](\lambda_j^2 - \lambda_1^2),$$
$$\gamma_{51}^{(j)} = \sigma\mu_2(i\sigma\beta_0\mu_2 T_0 - \mu_0\varkappa_1)\lambda_j^2 - l_0 T_0\sigma\big[\beta_0(a_0\lambda_j^2 - \eta_0) - \mu_0\beta_1\big](\lambda_j^2 - \lambda_2^2),$$
$$\gamma_{53}^{(j)} = -\mu_0(i\varkappa_1\mu_0 + \sigma\mu_2\beta_0 T_0)\lambda_j^2 + \lambda_0\big[i\varkappa_1(a_0\lambda_j^2 - \eta_0) + \sigma\mu_2\beta_1 T_0\big](\lambda_j^2 - \lambda_1^2),$$

$$c^*_{2j} = i\alpha_0\lambda_0\varkappa_6^2\varkappa d_1 d_3(\lambda_j^2 - \lambda_7^2)(\lambda_j^2 - \lambda_{11}^2)\prod_{l=1}^{7}(\lambda_j^2 - \lambda_l^2),$$
$$c^*_{4j} = \alpha_0\varkappa_6 d_3 a_j^*\beta_{14}^{(j)}(\lambda_j^2 - \lambda_2^2)(\lambda_j^2 - \lambda_7^2)(\lambda_j^2 - \lambda_{11}^2),$$
$$c^*_{5j} = \alpha_0\varkappa_6 d_3 a_j^*\beta_{15}^{(j)}(\lambda_j^2 - \lambda_2^2)(\lambda_j^2 - \lambda_7^2)(\lambda_j^2 - \lambda_{11}^2),$$
$$c^*_{6j} = i\alpha_0\lambda_0\varkappa\varkappa_6^2 d_1 d_3(\lambda_j^2 - \lambda_7^2)(\lambda_j^2 - \lambda_{11}^2)\prod_{l=1}^{7}(\lambda_j^2 - \lambda_l^2),$$
$$c^*_{8j} = -i\mu_1\alpha_0\lambda_0\varkappa_6 d_1 d_3(\lambda_j^2 - \lambda_{11}^2)((\mu+\varkappa)\lambda_j^2 - \rho\sigma^2)\prod_{l=1}^{7}(\lambda_j^2 - \lambda_l^2),$$
$$c^*_{12j} = \sigma\mu_1\alpha_0\lambda_0\varkappa_6 d_1 d_3(\lambda_j^2 - \lambda_{11}^2)((\mu+\varkappa)\lambda_j^2 - \rho\sigma^2)\prod_{l=1}^{7}(\lambda_j^2 - \lambda_l^2),$$
$$c^*_{14j} = \varkappa_6 d_2(\sigma\mu_2\alpha_{22}^{(j)} - i\varkappa_3\alpha_{21}^{(j)})(\lambda_j^2 - \lambda_1^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2),$$
$$c^*_{15j} = -\varkappa_6 d_2(\sigma\mu_2\alpha_{12}^{(j)} - i\varkappa_3\alpha_{11}^{(j)})(\lambda_j^2 - \lambda_1^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2),$$
$$c^*_{16j} = d_2 d_3\gamma_{41}^{(j)}(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2),$$
$$c^*_{18j} = d_2 d_3\gamma_{43}^{(j)}(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2),$$
$$c^*_{21j} = d_2 d_3\gamma_{51}^{(j)}(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2),$$
$$c^*_{23j} = d_2 d_3\gamma_{53}^{(j)}(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\prod_{l=7}^{11}(\lambda_j^2 - \lambda_l^2),$$
$$\beta_{11}^{(j)} = i\mu_0\gamma_{41}^{(j)} - i\beta_0\gamma_{51}^{(j)}, \qquad \beta_{13}^{(j)} = i\mu_0\gamma_{43}^{(j)} - i\beta_0\gamma_{53}^{(j)},$$
$$\beta_{14}^{(j)} = i\mu_0\alpha_{22}^{(j)} + i\beta_0\alpha_{21}^{(j)}, \qquad \beta_{15}^{(j)} = -i\mu_0\alpha_{12}^{(j)} - i\beta_0\alpha_{11}^{(j)}.$$

From (6.78) and (6.79) with the help of the above relations we get the following representation of the fundamental matrix

$$\Gamma(x,\sigma) = \frac{1}{4\pi\, d_1\, d_2\, d_3}\left\{\begin{bmatrix} \Psi_1(x,\sigma)\, I_3 & [0]_{3\times3} & \Psi_3(x,\sigma)\, I_3 & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{3\times3} & \Psi_7(x,\sigma)\, I_3 & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ \Psi_{11}(x,\sigma)\, I_3 & [0]_{3\times3} & \Psi_{13}(x,\sigma) I_3 & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & \Psi_{19}(x,\sigma) & \Psi_{20}(x,\sigma) \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & \Psi_{24}(x,\sigma) & \Psi_{25}(x,\sigma) \end{bmatrix}\right.$$
$$+\begin{bmatrix} Q(\partial)\widetilde{\Psi}_1(x,\sigma) & [0]_{3\times3} & Q(\partial)\widetilde{\Psi}_3(x,\sigma) & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{3\times3} & Q(\partial)\widetilde{\Psi}_7(x,\sigma) & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ Q(\partial)\widetilde{\Psi}_{11}(x,\sigma) & [0]_{3\times3} & Q(\partial)\widetilde{\Psi}_{13}(x,\sigma) & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0 \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0 \end{bmatrix}$$
$$\left.+\begin{bmatrix} [0]_{3\times3} & R(\partial)\Psi'_2(x,\sigma) & [0]_{3\times3} & \nabla^\top\Psi'_4(x,\sigma) & \nabla^\top\Psi'_5(x,\sigma) \\ R(\partial)\Psi'_6(x,\sigma) & [0]_{3\times3} & R(\partial)\Psi'_8(x,\sigma) & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{3\times3} & R(\partial)\Psi'_{12}(x,\sigma) & [0]_{3\times3} & \nabla^\top\Psi'_{14}(x,\sigma) & \nabla^\top\Psi'_{15}(x,\sigma) \\ \nabla\Psi'_{16}(x,\sigma) & [0]_{1\times3} & \nabla\Psi'_{18}(x,\sigma) & 0 & 0 \\ \nabla\Psi'_{21}(x,\sigma) & [0]_{1\times3} & \nabla\Psi'_{23}(x,\sigma) & 0 & 0 \end{bmatrix}\right\}, \quad (6.82)$$

where

$$\Psi_l(x,\sigma)=\sum_{j=1}^{11} p_j\, a^*_{lj}\,\frac{e^{i\,\lambda_j|x|}}{|x|},\qquad l=1,3,7,11,13,19,20,24,25,$$

$$\widetilde{\Psi}_l(x,\sigma)=-\sum_{j=1}^{11} p_j\, b^*_{lj}\,\frac{e^{i\,\lambda_j|x|}}{|x|},\qquad l=1,3,7,11,13,$$

$$\Psi'_l(x,\sigma)=i\sum_{j=1}^{11} p_j\, c^*_{lj}\,\frac{e^{i\,\lambda_j|x|}}{|x|},\quad l=2,4,5,6,8,12,14,15,16,18,21,23.$$

Remark 6.1. Note that (6.81) can be rewritten in the form

$$\Gamma(x,\sigma)=\sum_{j=1}^{11}\Phi^{(j)}(x,\sigma), \tag{6.83}$$

where

$$\Phi^{(j)}(x,\sigma)=-\frac{p_j}{4\pi\, d_1\, d_2\, d_3}\;\mathcal{M}(i\,\partial,\sigma)\,\frac{e^{i\lambda_j|x|}}{|x|}, \tag{6.84}$$

and $\mathcal{M}(i\,\partial,\sigma)$ is defined by (6.78). Since $\mathcal{M}(i\,\partial,\sigma)$ is a matrix differential operator with constant coefficients from the representation (6.84) it follows that the entries of the matrix $\Phi^{(j)}(x,\sigma)=\left[\Phi^{(j)}_{pq}(x,\sigma)\right]_{11\times 11}$ are metaharmonic functions corresponding to the wave number λ_j, i.e., solutions of the Helmholtz equation

$$(\Delta+\lambda_j^2)\,\Phi^{(j)}_{pq}(x,\sigma)=0,\quad |x|\neq 0,$$

and decay exponentially at infinity:

$$\frac{\partial}{\partial|x|}\,\Phi^{(j)}_{pq}(x,\sigma)-i\,\lambda_j\,\Phi^{(j)}_{pq}(x,\sigma)=\exp\{-\mathrm{Im}\,\lambda_j\,|x|\}\,O(|x|^{-2}),\quad p,q=\overline{1,11},$$

as $|x|\to+\infty$.

The entries of the matrix $\Phi^{(j)}(x,\sigma)$ and its derivatives satisfy also the following decay conditions at infinity [21]

$$\Phi^{(j)}_{pq}(x,\sigma)=\exp\{-\mathrm{Im}\,\lambda_j\,|x|\}\,O(|x|^{-1}),$$

$$\frac{\partial}{\partial x_l}\,\Phi^{(j)}_{pq}(x,\sigma)-i\,\lambda_j\,\frac{x_l}{|x|}\,\Phi^{(j)}_{pq}(x,\sigma)=\exp\{-\mathrm{Im}\,\lambda_j\,|x|\}\,O(|x|^{-2}),\quad l=1,2,3.$$

These asymptotic equalities can be differentiated any times with respect to the variable x.

In accordance with formulas (6.83), (6.84) and Corollary A.2 (see Appendix) we see that for $\mathrm{Im}\,\sigma=\sigma_2>0$ the entries of the matrix $\Gamma(x,\sigma)$ decay exponentially as $|x|\to\infty$ since $\mathrm{Im}\,\lambda_j>0$, $j=\overline{1,11}$.

Remark 6.2. Note that the matrix $\Gamma^*(x,\sigma):=[\Gamma(-x,\sigma)]^\top$ represents a fundamental solution to the formally adjoint differential operator $L^*(\partial,\sigma)\equiv\left[L(-\partial,\sigma)\right]^\top$,

$$L^*(\partial,\sigma)\left[\Gamma(-x,\sigma)\right]^\top=I_{11}\,\delta(x).$$

In the case of repeated roots the fundamental solution can be obtained from (6.81) by the standard limiting procedure.

7. Principal singular part of the Fundamental matrix

The principal singular part $\Gamma_0(x)$ of the fundamental matrix (6.82) represents a 11×11 fundamental matrix of the operator $L_0(\partial)$ defined by (3.18), (3.19) and solves the equation:

$$L_0(\partial)\Gamma_0(x)=\delta(x)I_{11}. \tag{7.85}$$

It is easy to show that

$$\Gamma_0(x) = \begin{bmatrix} \Gamma_0^{(1)}(x) & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{3\times3} & \Gamma_0^{(7)}(x) & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{3\times3} & [0]_{3\times3} & \Gamma_0^{(13)}(x) & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & \Gamma_0^{(19)}(x) & 0 \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & \Gamma_0^{(25)}(x) \end{bmatrix}_{11\times11}, \tag{7.86}$$

where

$$\begin{aligned} \Gamma_0^{(1)}(x) &= -\frac{1}{8\pi\,(\mu+\varkappa)}\Big\{\frac{2}{|x|}\,I_3 - \frac{\lambda+\mu}{\lambda_0}\,Q(\partial)|x|\Big\}, \\ \Gamma_0^{(7)}(x) &= -\frac{1}{8\pi\,\gamma}\Big\{\frac{2}{|x|}\,I_3 - \frac{\alpha+\beta}{\alpha_0}\,Q(\partial)|x|\Big\}, \\ \Gamma_0^{(13)}(x) &= -\frac{1}{8\pi\,\varkappa_6}\Big\{\frac{2}{|x|}\,I_3 - \frac{\varkappa_4+\varkappa_5}{l_0}\,Q(\partial)|x|\Big\}, \\ \Gamma_0^{(19)}(x) &= -\frac{1}{4\pi\,a_0\,|x|}, \\ \Gamma_0^{(25)}(x) &= -\frac{1}{4\pi\,\varkappa_7\,|x|}. \end{aligned} \tag{7.87}$$

Note that $\Gamma_0(x) = \Gamma_0^{\top}(x) = \Gamma_0(-x)$ and the entries of the matrix $\Gamma_0(x)$ are homogeneous functions of order -1. For an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and an arbitrary complex number σ it can easily be shown that in a neighborhood of the origin (i.e., for small $|x|$)

$$\partial^\alpha[\Gamma(x,\sigma) - \Gamma_0(x)] = \mathcal{O}(|x|^{-\alpha}), \quad |\alpha| = \alpha_1+\alpha_2+\alpha_3,$$

which shows that $\Gamma_0(x)$ is a principal singular part of the matrix $\Gamma(x,\sigma)$.

8. Integral representation formulae of solutions

Let us introduce the generalized single and double layer potentials, and the Newton type volume potential

$$V(\varphi)(x) = \int_S \Gamma(x-y,\sigma)\,\varphi(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{8.88}$$

$$W(\varphi)(x) = \int_S \big[P^*(\partial_y, n(y))\Gamma^\top(x-y,\sigma)\big]^\top \varphi(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{8.89}$$

$$N_{\Omega^\pm}(\psi)(x) = \int_{\Omega^\pm} \Gamma(x-y,\sigma)\,\psi(y)\,dy, \quad x \in \mathbb{R}^3, \tag{8.90}$$

where $\Gamma(\cdot,\sigma)$ is the fundamental matrix given by (6.81) or (6.82), $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{11})^\top$ is a density vector-function defined on S, while a density vector-function $\psi = (\psi_1, \ldots, \psi_{11})^\top$ is defined on $\Omega^\pm$ and we assume that in the case of Ω^- the support of the density vector-function ψ of the Newtonian potential (8.90) is a compact set, $P^*(\partial_y, n(y))$ is the boundary differential operator defined by (2.12).

Due to the equality

$$\begin{aligned} &\sum_{j=1}^{11} L_{kj}(\partial_x,\sigma)\Big(\big[P^*(\partial_y,n(y))\Gamma^\top(x-y,\sigma)\big]^\top\Big)_{jp} \\ &\quad = \sum_{j,q=1}^{11} L_{kj}(\partial_x,\sigma)P^*_{pq}(\partial_y,n(y))\Gamma_{jq}(x-y,\sigma) \\ &\quad = \sum_{j,q=1}^{11} P^*_{pq}(\partial_y,n(y))L_{kj}(\partial_x,\sigma)\Gamma_{jq}(x-y,\sigma) = 0, \quad x \neq y, \quad k,p = \overline{1,11}, \end{aligned}$$

it can easily be checked that the potentials defined by (8.88) and (8.89) are C^∞–smooth in $\mathbb{R}^3 \setminus S$ and solve the homogeneous equation $L(\partial,\sigma)U = 0$ in $\mathbb{R}^3 \setminus S$ for an arbitrary L_p-summable vector function φ. The volume potential solves the nonhomogeneous equation

$$L(\partial,\sigma)N_{\Omega^{\pm}}(\psi)=\psi \quad \text{in} \quad \Omega^{\pm} \quad \text{for} \quad \psi\in C^{0,\alpha}(\overline{\Omega^{\pm}}). \tag{8.91}$$

The relation (8.91) holds true for an arbitrary $\psi\in L_p(\Omega^{\pm})$ with $1<p<+\infty$.

Theorem 8.1. *Let $S=\partial\Omega^+$ be $C^{1,\gamma'}$smooth with $0<\gamma'\leq 1$, $\sigma=\sigma_1+i\,\sigma_2$ with $\sigma_2>0$, and let U be a regular vector function of the class $C^2(\overline{\Omega^+})$. Then there holds the integral representation formula*

$$W(\{U\}^+)(x)-V(\{PU\}^+)(x)+N_{\Omega^+}(L(\partial,\sigma)U)(x)=\begin{cases}U(x) & \text{for} \quad x\in\Omega^+,\\ 0 & \text{for} \quad x\in\Omega^-.\end{cases} \tag{8.92}$$

Proof. It follows from Green's formula (3.16) with the domain of integration $\Omega^+\setminus B(x,\varepsilon')$, where $x\in\Omega^+$ is treated as a fixed parameter, $B(x,\varepsilon')$ is a ball centered at the point x and radius $\varepsilon'>0$ and $\overline{B(x,\varepsilon')}\subset\Omega^+$. One needs to take the jth column of the fundamental matrix $\Gamma^*(y-x,\sigma)$ for U', calculate the surface integrals over the sphere $\Sigma(x,\varepsilon'):=\partial B(x,\varepsilon')$ and pass to the limit as $\varepsilon'\to 0$ (see [15], Appendix D). □

Similar representation formula holds in the exterior domain Ω^- if a vector U and its derivatives possess some asymptotic properties at infinity. In particular, the following assertion holds.

Theorem 8.2. *Let $S=\partial\Omega^-$ be $C^{1,\gamma'}$smooth with $0<\gamma'\leq 1$ and let U be a regular vector of the class $C^2(\overline{\Omega^-})$, such that for any multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ with $0\leq|\alpha|=\alpha_1+\alpha_2+\alpha_3\leq 2$, the function $\partial^\alpha U_j$ is polynomially bounded at infinity, i.e., for sufficiently large $|x|$*

$$|\partial^\alpha U_j(x)|\leq C_0\,|x|^m, \quad j=1,2,\dots,11, \tag{8.93}$$

with some constants m and $C_0>0$. Then there holds the integral representation formula

$$-W(\{U\}^-)(x)+V(\{PU\}^-)(x)+N_{\Omega^-}(L(\partial,\sigma)U)(x)=\begin{cases}0 & \text{for} \quad x\in\Omega^+,\\ U(x) & \text{for} \quad x\in\Omega^-,\end{cases}$$

where $\sigma=\sigma_1+i\,\sigma_2$ with $\sigma_2>0$.

Proof. The proof immediately follows from Theorem 8.1 and Remark 6.1. Indeed, one needs to write the integral representation formula (8.92) for bounded domain $\Omega^-\cap B(0,R)$, send then R to $+\infty$ and take into consideration that the surface integral over $\Sigma(0,R)$ tends to zero due to the conditions (8.93) and the exponential decay of the fundamental matrix at infinity.

Remark 8.3. Let $\sigma=\sigma_1+i\,\sigma_2$ with $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$, and U be a solution to the homogeneous equations $L(\partial,\sigma)U=0$ in $\Omega^{\pm}$ satisfying the condition (8.93) and $U\in C^{1,\gamma'}(\overline{\Omega^{\pm}})$ with some $0<\gamma'\leq 1$. Then the following representation formula holds

$$U(x)=W([U]_S)(x)-V([PU]_S)(x), \quad x\in\Omega^{\pm},$$

where $[U]_S=\{U\}^+-\{U\}^-$ and $[PU]_S=\{PU\}^+-\{PU\}^-$ on S. The proof immediately follows from Theorems 8.1 and 8.2.

9. Properties of layer potentials

Here we consider the mapping and regularity properties of the single and double layer potentials and the boundary pseudodifferential operators generated by them in the Hölder $C^{m,\gamma'}$ spaces. They can be established by standard methods (see [8,15–17]). We remark only that the layer potentials corresponding to the fundamental matrices with different values of the parameter σ have the same smoothness properties and possess the same jump relations. Therefore, using the word for word arguments given in [8,10,16,18,19,12,13,15,17] we can prove the following theorems concerning the above introduced layer potentials.

If not otherwise stated, for simplicity, we assume that

$$\begin{aligned}&S=\partial\Omega^{\pm}\in C^{m,\gamma'} \text{ with integer } m\geq 2 \quad \text{and} \quad 0<\gamma'\leq 1;\\ &\sigma=\sigma_1+i\,\sigma_2, \quad \sigma_1\in\mathbb{R}, \quad \operatorname{Im}\sigma=\sigma_2>0.\end{aligned} \tag{9.94}$$

Theorem 9.1. *Let S, m, and γ' be as in* (9.94), $0 < \delta' < \gamma'$, *and let $k \le m-1$ be integer. Then the operators*

$$V \ : \ C^{k,\delta'}(S) \to C^{k+1,\delta'}(\overline{\Omega^{\pm}}), \quad W \ : \ C^{k,\delta'}(S) \to C^{k,\delta'}(\overline{\Omega^{\pm}}) \tag{9.95}$$

are continuous.

For any $g \in C^{0,\delta'}(S)$, $h \in C^{1,\delta'}(S)$, and any $x \in S$

$$[V(g)(x)]^{\pm} = V(g)(x) = \mathcal{H}\, g(x), \tag{9.96}$$

$$\{P(\partial_x, n(x))\, V(g)(x)\}^{\pm} = \big[\mp 2^{-1} I_{11} + \mathcal{K}\big]\, g(x), \tag{9.97}$$

$$\{W(g)(x)\}^{\pm} = \big[\pm 2^{-1} I_{11} + \mathcal{N}\big]\, g(x), \tag{9.98}$$

$$\{P(\partial_x, n(x))\, W(h)(x)\}^{+} = \{P(\partial_x, n(x))\, W(h)(x)\}^{-} = \mathcal{L}\, h(x), \tag{9.99}$$

where

$$\mathcal{H}\, g(x) := \int_S \Gamma(x-y,\sigma)\, g(y)\, dS_y, \tag{9.100}$$

$$\mathcal{K}\, g(x) := \int_S \big[P(\partial_x, n(x))\, \Gamma(x-y,\sigma)\big]\, g(y)\, dS_y, \tag{9.101}$$

$$\mathcal{N}\, g(x) := \int_S \big[P^*(\partial_y, n(y))\, \Gamma^{\top}(x-y,\sigma)\big]^{\top}\, g(y)\, dS_y, \tag{9.102}$$

$$\mathcal{L}\, h(x) := \lim_{\Omega^{\pm} \ni z \to x \in S} P(\partial_z, n(x)) \int_S \big[P^*(\partial_y, n(y))\, \Gamma^{\top}(z-y,\sigma)\big]^{\top}\, h(y)\, dS_y. \tag{9.103}$$

Proof. The proof of the relations (9.95)–(9.98) can be performed by standard arguments (see, e.g., [8,10]). We demonstrate here only a simplified proof of the relation (9.99), the so called *Lyapunov–Tauber type theorem*. Let $h \in C^{1,\delta'}(S)$ and consider de double layer potential $U := W(h) \in C^{1,\delta'}(\overline{\Omega^{\pm}})$. Then by Remark 8.3 and the jump relations (9.98), we have

$$U(x) = W([U]_S)(x) - V([PU]_S)(x), \quad x \in \Omega^{\pm},$$

i.e.,

$$W(h)(x) = W(h)(x) - V([PW(h)]_S)(x), \quad x \in \Omega^{\pm},$$

since $[U]_S = \{W(h)\}^+ - \{W(h)\}^- = h$ on S due to (9.98). Therefore $V([PW(h)]_S) = 0$ in $\Omega^{\pm}$ and in view of (9.97) we conclude

$$\{PV([PW(h)]_S)\}^- - \{PV([PW(h)]_S)\}^+ = [PW(h)]_S = \{PW(h)\}^+ - \{PW(h)\}^- = 0$$

on S, which completes the proof. □

With the help of the explicit form of the fundamental matrix $\Gamma(x-y,\sigma)$ it can easily be shown that the operators $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators, $\mathcal{H}$ is a smoothing (weakly singular) integral operator, while $\mathcal{L}$ is a singular integro-differential operator.

Theorem 9.2. *Let S, m, γ', δ' and k be as in Theorem* 9.1. *Then the operators*

$$\mathcal{H} : C^{k,\delta'}(S) \to C^{k+1,\delta'}(S), \tag{9.104}$$

$$\mathcal{K} : C^{k,\delta'}(S) \to C^{k,\delta'}(S), \tag{9.105}$$

$$\mathcal{N} : C^{k,\delta'}(S) \to C^{k,\delta'}(S), \tag{9.106}$$

$$\mathcal{L} : C^{k,\delta'}(S) \to C^{k-1,\delta'}(S), \tag{9.107}$$

are continuous. Moreover,

(1) *the principal homogeneous symbol matrices of the operators $\pm 2^{-1} I_{11} + \mathcal{K}$ and $\pm 2^{-1} I_{11} + \mathcal{N}$ are non-degenerate, while the principal homogeneous symbol matrices of the operators $-\mathcal{H}$ and $\mathcal{L}$ are positive definite;*

(2) *the operators $\mathcal{H}$, $\pm 2^{-1} I_{11} + \mathcal{K}$, $\pm 2^{-1} I_{11} + \mathcal{N}$, and $\mathcal{L}$ are elliptic pseudodifferential operators (of order* -1, 0, 0, *and* 1, *respectively) with zero index;*

(3) *the following equalities hold in appropriate function spaces:*

$$\begin{aligned} &\mathcal{N}\mathcal{H} = \mathcal{H}\mathcal{K}, && \mathcal{L}\mathcal{N} = \mathcal{K}\mathcal{L}, \\ &\mathcal{H}\mathcal{L} = -4^{-1} I_{11} + \mathcal{N}^2, && \mathcal{L}\mathcal{H} = -4^{-1} I_{11} + \mathcal{K}^2. \end{aligned} \tag{9.108}$$

Proof. The mapping properties (9.104)–(9.107) are standard and can be proved as their counterparts in [8,12,13,19].

The item (3) follows from the jump relations for the layer potentials and the general integral representation formulas of solutions to the homogeneous equation $L(\partial, \sigma)U = 0$.

Proofs of items (1) and (2) are based on the positive definiteness of the potential energy functional and positive definiteness of the symbol matrix $L_0(\xi)$ for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \setminus \{0\}$, see (3.18), (3.19) (cf. [20,13,17,19], and [15]). □

10. Existence results for boundary value problems

Now we apply the potential method and prove existence theorems for the Dirichlet and Neumann type boundary value problems for pseudo-oscillation equations (see Section 5). We reduce the original boundary value problems to the equivalent integral equations on the boundary of the elastic body under consideration and investigate their Fredholm properties. In particular, we show that the corresponding integral operators are invertible. Without loss of generality we consider the boundary value problems for the homogeneous differential equation $L(\partial, \sigma)\, U = 0$, since a particular solution to the nonhomogeneous equation (5.51) can be written explicitly in the form of volume potential $N_{\Omega^\pm}(\Phi^\pm)$, see (8.91).

Moreover, throughout this section we assume that the conditions (9.94) are fulfilled if not otherwise stated.

10.1. Investigation of the interior and exterior Dirichlet problems

These problems are formulated in Section 5. We assume that $\Phi^{(\pm)} = 0$ and look for solutions in $\Omega^\pm$ in the form of double layer potential $U = W(h)$ (see (8.89)). Applying the jump relations for the double layer potential (see Theorem 9.1) and taking into consideration the boundary conditions (5.52), for the unknown density vector function $h = (h_1, h_2, \dots, h_{11})^\top$ we get the following boundary integral equations,

$$\left[2^{-1} I_{11} + \mathcal{N}\right] h = f \quad \text{on} \quad S, \tag{10.109}$$

in the case of Problem $(I^{(\sigma)})^+$, and

$$\left[-2^{-1} I_{11} + \mathcal{N}\right] h = f \quad \text{on} \quad S, \tag{10.110}$$

in the case of Problem $(I^{(\sigma)})^-$.

Here the operator $\mathcal{N}$ is given by (9.102). Due to Theorem 9.2, the operators $\pm 2^{-1} I_{11} + \mathcal{N}$ are singular integral operators of normal type with index zero. This leads to the following existence theorems.

Theorem 10.1. *Let $S \in C^{2,\nu}$ and $f \in C^{1,\tau}(S)$ with $0 < \tau < \nu \le 1$. Then the boundary value problem $(I^{(\sigma)})^+$ is uniquely solvable in the space $C^{1,\tau}(\overline{\Omega^+})$ and the solution is represented by the double layer potential $W(h)$ defined by* (8.89)*, where density $h \in C^{1,\tau}(S)$ is a unique solution of the integral equation* (10.109).

Proof. The uniqueness follows from Theorems 9.1 and 5.1. It remains to show that the singular integral operator

$$2^{-1} I_{11} + \mathcal{N} \;:\; C^{1,\tau}(S) \to C^{1,\tau}(S) \tag{10.111}$$

is invertible.

Due to Theorem 9.2, we conclude that (10.111) is a Fredholm operator with zero index. Further, we show that $\ker\left[2^{-1} I_{11} + \mathcal{N}\right]$ is trivial. Indeed, let h_0 solve the homogeneous equation

$$\left[2^{-1} I_{11} + \mathcal{N}\right] h_0 = 0 \quad \text{on} \quad S. \tag{10.112}$$

Construct the double layer potential $W(h_0)$. Since $h_0 \in C^{1,\tau}(S)$, we have $W(h_0) \in C^{1,\tau}(\overline{\Omega^{\pm}})$. In view of Eq. (10.112), we see, that then $\{W(h_0)(x)\}^+ = 0$ for $x \in S$ and by the uniqueness Theorem 5.1 we get $W(h_0)(x) = 0$ for $x \in \Omega^+$. Consequently, $\{P(\partial, n)\, W(h_0)(x)\}^+ = 0$ for $x \in S$. By the Lyapunov–Tauber theorem (see Theorem 9.1)

$$\{P(\partial, n)\, W(h_0)(x)\}^+ = \{P(\partial, n)\, W(h_0)(x)\}^- = 0, \quad x \in S,$$

i.e., $W(h_0)$ solves the homogeneous exterior Neumann type boundary value problem $(II^{(\sigma)})^-$ and decays at infinity exponentially. Therefore, $W(h_0)(x) = 0$ in Ω^- by Theorem 5.1. Since

$$\{W(h_0)(x)\}^+ - \{W(h_0)(x)\}^- = 2\, h_0(x), \qquad x \in S,$$

we conclude that $h_0 = 0$ on S, which shows that null space of the operator $2^{-1} I_{11} + \mathcal{N}$ is trivial. Therefore, (10.111) is invertible. $\square$

Quite similarly, by the word for word arguments and with the help of Theorem 5.1, we can show that the operator

$$-\,2^{-1} I_{11} + \mathcal{N} \;:\; C^{1,\tau}(S) \to C^{1,\tau}(S) \tag{10.113}$$

is invertible, which leads to the existence theorem for the Dirichlet type exterior boundary value problem.

Theorem 10.2. *Let $S \in C^{2,\nu}$ and $f \in C^{1,\nu}(S)$ with $0 < \tau < \nu \le 1$. Then the boundary value problem $(I^{(\sigma)})^-$ is uniquely solvable in the class of vector functions belonging to the space $C^{1,\tau}(\overline{\Omega^-})$ and decaying at infinity, and the solution is represented by the double layer potential $W(h)$ defined by* (8.89)*, where $h \in C^{1,\tau}(S)$ is a unique solution of the integral equation* (10.110).

10.2. Investigation of the interior and exterior Neumann problems

These problems are formulated in Section 5 as problems $(II^{(\sigma)})^+$ and $(II^{(\sigma)})^-$. As above, we assume that $\Phi^{(\pm)} = 0$ and look for solutions in $\Omega^{\pm}$ in the form of the single layer potential $U = V(g)$ (see 8.1) and taking into consideration the boundary conditions (5.53), for the unknown density vector function $g = (g_1, g_2, \dots, g_{11})^\top$ we get the boundary integral equations,

$$\big[-2^{-1} I_{11} + \mathcal{K}\big]\, g = F \;\text{ on }\; S, \tag{10.114}$$

in the case of Problem $(II^{(\sigma)})^+$, and

$$\big[2^{-1} I_{11} + \mathcal{K}\big]\, g = F \;\text{ on }\; S, \tag{10.115}$$

in the case of Problem $(II^{(\sigma)})^-$.

Here the operator $\mathcal{K}$ is given by (9.101). Due to Theorem 9.2, the operators $\pm 2^{-1} I_{11} + \mathcal{K}$ are singular integral operators of normal type with index zero. This yields the following existence theorems.

Theorem 10.3. *Let $S \in C^{1,\nu}$ and $F \in C^{0,\tau}(S)]$ with $0 < \tau < \nu \le 1$. Then the boundary value problem $(II^{(\sigma)})^+$ is uniquely solvable in the space $C^{1,\tau}(\overline{\Omega^+})$ and the solution is represented by the single layer potential $V(g)$ defined by* (8.88)*, where $g \in C^{0,\tau}(S)$ is a unique solution of the integral equation* (10.114).

Proof. The uniqueness is a consequence of Theorems 9.1 and 5.1. Now, we show that the operator

$$-\,2^{-1} I_{11} + \mathcal{K} \;:\; C^{0,\tau}(S) \to C^{0,\tau}(S) \tag{10.116}$$

is invertible.

Due to Theorem 9.2, the operator (10.116) is a Fredholm operator with zero index. Therefore, it remains to show that the null space of the operator $-2^{-1} I_{11} + \mathcal{K}$ is trivial. Let $g_0 \in C^{0,\tau}(S)$ solve the homogeneous equation

$$[-\,2^{-1} I_{11} + \mathcal{K}\,]\, g_0 = 0 \;\text{ on }\; S.$$

Construct the single layer potential $V(g_0)$. Evidently, $V(g_0) \in C^{1,\tau}(\overline{\Omega^+})$ due to Theorem 9.1. Moreover, $V(g_0)$ solves the homogeneous Problem $(II^{(\sigma)})^+$ and therefore it vanishes identically in Ω^+, due to Theorem 5.1. Further, by

Theorem 9.1 we have $\{V(g_0)(x)\}^+ = \{V(g_0)(x)\}^- = 0$ for $x \in S$, and since it exponentially decays at infinity, by the uniqueness theorem for the Dirichlet exterior boundary value problem, we conclude, that $V(g_0)(x) = 0$ for $x \in \Omega^-$. Finally, with the help of equality

$$\{P(\partial, n)\, V(g_0)(x)\}^- - \{P(\partial, n)\, V(g_0)(x)\}^+ = 2\, g_0\,(x), \quad x \in S,$$

we derive $g_0 = 0$ on S. Thus, the operator (10.116) is invertible. $\square$

By the word for word arguments we can prove that the operator

$$2^{-1} I_{11} + \mathcal{K} \;:\; C^{0,\tau}(S) \to C^{0,\tau}(S) \tag{10.117}$$

is invertible, which leads to the existence theorem for the Neumann type exterior boundary value problem.

Theorem 10.4. *Let $S \in C^{1,\nu}$ and $F \in C^{0,\tau}(S)$ with $0 < \tau < \nu \le 1$. Then the boundary value problem $(II^{(\sigma)})^-$ is uniquely solvable in the class of vector functions belonging to the space $C^{1,\tau}(\overline{\Omega^-})$ and decaying at infinity, and the solution is represented by the single layer potential $V(g)$ defined by* (8.88)*, where $g \in C^{0,\tau}(S)$ is a unique solution of the integral equation* (10.115)*.*

10.3. Investigation of the basic boundary value problems by the first kind integral equations

Here we apply an alternative approach and reduce the basic interior and exterior boundary value problem, considered in the previous subsections, to the first kind integral equations (cf. [13]). These results play a crucial role in the study of mixed boundary value problems.

9.3.1. Investigation of the Dirichlet problem with the help of the first kind integral equations. We look for a solution to the problems $(I^{(\sigma)})^+$ and $(I^{(\sigma)})^-$ (see (5.51)–(5.52) with $\Phi^{(\pm)} = 0$) in the form of the single layer potential $U = V(g)$ (see (8.88)). In both cases, for the interior and exterior boundary value problems, we arrive at the equation

$$\mathcal{H}\, g = f \quad \text{on} \quad S, \tag{10.118}$$

where $\mathcal{H}$ is defined by (9.100).

We have the following existence theorem.

Theorem 10.5. *Let $S \in C^{2,\nu}$ and $f \in C^{1,\tau}(S)$ with $0 < \tau < \nu \le 1$. Then the boundary value problems $(I^{(\sigma)})^\pm$ are uniquely solvable in the class of vector functions belonging to the space $C^{1,\tau}(\overline{\Omega^\pm})$ and decaying at infinity, and the solution is represented by the single layer potential $V(g)$ defined by* (8.88)*, where $g \in C^{0,\tau}(S)$ is a unique solution of the integral equation* (10.118)*.*

Proof. The uniqueness follows from Theorems 9.1 and 5.1. Evidently, it remains to show the invertibility of the operator

$$\mathcal{H} \;:\; C^{0,\tau}(S) \to C^{1,\tau}(S). \tag{10.119}$$

To this end, we apply the operator $\mathcal{L}$ (see (9.103)) to both sides of Eq. (10.118) and take into consideration the operator equalities (9.108),

$$\mathcal{L}\mathcal{H}g \equiv \left[-4^{-1} I_1 + \mathcal{K}^2\right] g = \mathcal{L} f \quad \text{on} \quad S. \tag{10.120}$$

Clearly, $\mathcal{L} f \;\in C^{0,\tau}(S)$ due to Theorem 9.2. Since the operators (10.116) and (10.117) are invertible, we conclude that the singular integral operator

$$\mathcal{L}\mathcal{H} = \left[-2^{-1} I_{11} + \mathcal{K}\right]\left[2^{-1} I_{11} + \mathcal{K}\right] \;:\; C^{0,\tau}(S) \to C^{0,\tau}(S)$$

is invertible as well. Therefore, from (10.120) we get the following representation of a solution of Eq. (10.118)

$$g = \left[-4^{-1} + \mathcal{K}^2\right]^{-1} \mathcal{L} f \in C^{0,\tau}(S).$$

With the help of the uniqueness Theorem 5.1, one can easily show that the operators

$$\mathcal{H} \;:\; C^{0,\tau}(S) \to C^{1,\tau}(S), \quad \mathcal{L} \;:\; C^{1,\tau}(S) \to C^{0,\tau}(S), \tag{10.121}$$

are injective. Therefore, Eqs. (10.118) and (10.120) are equivalent and the operator (10.119) is invertible, which completes the proof. □

Corollary 10.6. *A solution $U \in C^{1,\tau}(\overline{\Omega^{\pm}})$ of the boundary value problems $(I^{(\sigma)})^{\pm}$ with $\Phi^{(\pm)} = 0$ is uniquely representable in the form*

$$U(x) = V(\mathcal{H}^{-1} f)(x), \quad x \in \Omega^{\pm},$$

where $f = \{U\}^{\pm}$ on S and

$$\mathcal{H}^{-1} : C^{1,\tau}(S) \to C^{0,\tau}(S)$$

is the inverse to the operator (10.119).

This representation plays a crucial role in the investigation of mixed boundary value problems (cf. [13]).

9.3.2. Investigation of the Neumann problem with the help of the first kind integral equations. We look for a solution to the problems $(II^{(\sigma)})^{+}$ and $(II^{(\sigma)})^{-}$ (see (5.51), (5.53) with $\Phi^{\pm} = 0$) in the form of the double layer potential $U = W(h)$ (see (8.89)). In both cases, for the interior and exterior boundary value problems, we arrive at the equation

$$\mathcal{L} h = F \quad \text{on} \quad S, \tag{10.122}$$

where $\mathcal{L}$ is defined by (9.103).

We have the following existence theorem.

Theorem 10.7. *Let $S \in C^{2,\nu}$ and $F \in C^{0,\tau}(S)$ with $0 < \tau < \nu \le 1$. Then the boundary value problems $(II^{(\sigma)})^{\pm}$ are uniquely solvable in the class of vector functions belonging to the space $C^{1,\tau}(\overline{\Omega^{\pm}})$ and decaying at infinity, and the solution is represented by the double layer potential $W(h)$ defined by* (8.89)*, where $h \in C^{1,\tau}(S)$ is a unique solution of the integral equation* (10.122).

Proof. The uniqueness follows from Theorems 9.1 and 5.1. Evidently, it remains to show the invertibility of the operator

$$\mathcal{L} : C^{1,\tau}(S) \to C^{0,\tau}(S). \tag{10.123}$$

To this end, we apply the operator $\mathcal{H}$ (see (9.100)) to both sides of Eq. (10.122) and take into consideration the operator equalities (9.108),

$$\mathcal{H}\mathcal{L}h \equiv \left[-4^{-1} I_{11} + \mathcal{N}^2\right] h = \mathcal{H}F \quad \text{on} \quad S. \tag{10.124}$$

Clearly, $\mathcal{H}F \in C^{1,\tau}(S)$ due to Theorem 9.2. Since the operators (10.111) and (10.113) are invertible, we conclude that the singular integral operator

$$\mathcal{H}\mathcal{L} = \left[-2^{-1} I_{11} + \mathcal{N}\right]\left[2^{-1} I_{11} + \mathcal{N}\right] : C^{1,\tau}(S) \to C^{1,\tau}(S)$$

is invertible as well. Therefore, from (10.124) we get the following representation formula of a solution of Eq. (10.122)

$$h = \left[-4^{-1} I_{11} + \mathcal{N}^2\right]^{-1} \mathcal{H}F \in C^{1,\tau}(S).$$

Since the operators (10.121) are injective, we conclude that Eqs. (10.122) and (10.124) are equivalent and the operator (10.123) is invertible, which completes the proof. □

Corollary 10.8. *A solution $U \in C^{1,\tau}(\overline{\Omega^{\pm}})$ of the boundary value problems $(II^{(\sigma)})^{\pm}$ with $\Phi^{\pm} = 0$ is uniquely representable in the form*

$$U(x) = W(\mathcal{L}^{-1} F)(x), \quad x \in \Omega^{\pm},$$

where $F = \{P(\partial, n)U\}^{\pm}$ on S and

$$\mathcal{L}^{-1} : C^{0,\tau}(S) \to C^{1,\tau}(S)$$

is the inverse to the operator (10.123).

Appendix. Properties of the characteristic roots

Here we investigate the properties of roots of Eq. (6.62) with respect to r. In particular we prove the following assertion.

Lemma A.1. *Let us assume that $\sigma = \sigma_1 + i\,\sigma_2$ is a complex parameter where $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$. Then*

$$\det L(-i\,\xi, \sigma) \neq 0$$

for arbitrary $\xi \in \mathbb{R}^3$.

Proof. We prove the lemma by contradiction. Let $\det L(-i\,\xi, \sigma) = 0$, $\xi \in \mathbb{R}^3$. Then the system of linear equations $L(-i\,\xi, \sigma)\,X = 0$ has a nontrivial solution $X \in \mathbb{C}^{11} \setminus \{0\}$ which can be written as $X = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}, X^{(5)})^\top$, where $X^{(j)} = (X_1^{(j)}, X_2^{(j)}, X_3^{(j)})^\top \in \mathbb{C}^3$, $j = 1, 2, 3$ and $X^{(j)} \in \mathbb{C}$, $j = 4, 5$, are scalars. Taking into consideration (2.6), the system $L(-i\,\xi, \sigma)\,X = 0$ can be rewritten as follows:

$$\begin{aligned} &L^{(j)}(-i\,\xi, \sigma)\,X^{(1)} + L^{(j+5)}(-i\,\xi, \sigma)\,X^{(2)} + L^{(j+10)}(-i\,\xi, \sigma)\,X^{(3)} + L^{(j+15)}(-i\,\xi, \sigma)\,X^{(4)} \\ &\quad + L^{(j+20)}(-i\,\xi, \sigma)\,X^{(5)} = 0, \\ &j = 1, 2, 3, 4, 5, \end{aligned}$$

implying

$$\big[(-(\mu + \varkappa)\,|\xi|^2 + \varrho\,\sigma^2)\,I_3 - (\lambda + \mu)\,Q(\xi)\big]\,X^{(1)} - i\varkappa R(\xi) X^{(2)} - i\,\mu_0 \xi^\top X^{(4)} + i\beta_0 \xi^\top X^{(5)} = 0, \tag{A.1}$$

$$-\,i\varkappa R(\xi) X^{(1)} + \big[(-\gamma|\xi|^2 + \delta)\,I_3 - (\alpha + \beta)\,Q(\xi)\big] X^{(2)} + i\,\mu_1\,R(\xi)\,X^{(3)} = 0, \tag{A.2}$$

$$-\,\sigma\mu_1 R(\xi)\,X^{(2)} + \big[(-\varkappa_6|\xi|^2 + \varkappa_0)\,I_3 - (\varkappa_4 + \varkappa_5)\,Q(\xi)\big] X^{(3)} + \sigma\,\mu_2\,\xi^\top\,X^{(4)} + i\varkappa_3 \xi^\top X^{(5)} = 0, \tag{A.3}$$

$$i\,\mu_0 \xi \cdot X^{(1)} + i\,\mu_2 \xi \cdot X^{(3)} + (-a_0|\xi|^2 + \eta_0)\,X^{(4)} + \beta_1\,X^{(5)} = 0, \tag{A.4}$$

$$\beta_0 T_0 \sigma \xi \cdot X^{(1)} - i\varkappa_1 \xi \cdot X^{(3)} + i\beta_1 T_0 \sigma X^{(4)} + (-\varkappa_7|\xi|^2 + i\sigma c)\,X^{(5)} = 0. \tag{A.5}$$

Let us take the dot products of Eqs. (A.1) and (A.2) by the vectors $-i\overline{\sigma}\,\overline{X^{(1)}}$ and $-i\overline{\sigma}\,\overline{X^{(2)}}$ respectively, multiply equality (A.4) by the function $-i\overline{\sigma}\,\overline{X^{(4)}}$, then multiply complex conjugates of Eqs. (A.3) and (A.5) by the vector $-X^{(3)}$ and the function $-\frac{1}{T_0} X^{(5)}$ respectively and sum up the results to obtain

$$\begin{aligned} &i\overline{\sigma}\big[(\mu + \varkappa)\,|\xi|^2 - \rho\,\sigma^2\big]\,\big|X^{(1)}\big|^2 + i\,\overline{\sigma}\,(\lambda + \mu)\,\big|\xi \cdot X^{(1)}\big|^2 - \varkappa\,\overline{\sigma}\big(\big[\xi \times X^{(2)}\big] \cdot \overline{X^{(1)}} + \big[\xi \times X^{(1)}\big] \cdot \overline{X^{(2)}}\big) \\ &\quad + \mu_0 \overline{\sigma}\big[(\xi \cdot X^{(1)})\overline{X^{(4)}} - (\xi \cdot \overline{X^{(1)}}) X^{(4)}\big] + i\,\overline{\sigma}\big(\gamma\big|\xi\big|^2 - \delta\big)\big|X^{(2)}\big|^2 + i\,\overline{\sigma}(\alpha + \beta)\big|\xi \cdot X^{(2)}\big|^2 + \big(\varkappa_6\big|\xi\big|^2 \\ &\quad - \overline{\varkappa_0}\big)\,\big|X^{(3)}\big|^2 + (\varkappa_4 + \varkappa_5)\,\big|\,\xi \cdot X^{(3)}\,\big|^2 + i\,\varkappa_3(\,\xi \cdot X^{(3)})\overline{X^{(5)}} - \frac{i\,\varkappa_1}{T_0}(\xi \cdot \overline{X^{(3)}}) X^{(5)} \\ &\quad - i\,\overline{\sigma}\big(-a_0\,\big|\xi\big|^2 + \eta_0\big)\big|X^{(4)}\big|^2 + \frac{1}{T_0}\big(\varkappa_7\big|\xi\big|^2 + i\,\overline{\sigma}c\big)\big|X^{(5)}\big|^2 = 0. \end{aligned}$$

By separating the real part from this equation, we deduce

$$\begin{aligned} &\sigma_2\big[(\mu + \varkappa)\,\big|\xi\big|^2 + \varrho\,\big|\sigma\big|^2\big]\,\big|X^{(1)}\big|^2 + \sigma_2(\lambda + \mu)\,\big|\xi \cdot X^{(1)}\big|^2 \\ &\quad - 2\varkappa\sigma_2\,\mathrm{Im}\,\big(\big[\xi \times X^{(2)}\big] \cdot \overline{X^{(1)}}\big) + 2\mu_0\sigma_2\,\mathrm{Im}\,\big((\xi \cdot X^{(1)})\,\overline{X^{(4)}}\big) \\ &\quad + \sigma_2\big(\gamma\,\big|\xi\big|^2 + \mathcal{I}_1\big|\sigma\big|^2 + 2\varkappa\big)\big|X^{(2)}\big|^2 + \sigma_2(\alpha + \beta)\big|\xi \cdot X^{(2)}\big|^2 + \big(\varkappa_6\big|\xi\big|^2 \\ &\quad + \sigma_2 b + \varkappa_2\big)\,\big|X^{(3)}\big|^2 + (\varkappa_4 + \varkappa_5)\,\big|\,\xi \cdot X^{(3)}\,\big|^2 + \frac{\varkappa_1 + \varkappa_3 T_0}{T_0}\,\mathrm{Im}\,\big(\big(\xi \cdot \overline{X^{(3)}}\big) X^{(5)}\big) \\ &\quad + \sigma_2\big(a_0\,\big|\xi\big|^2 + \mathcal{I}\big|\sigma\big|^2 + \eta\big)\big|X^{(4)}\big|^2 + \frac{1}{T_0}\big(\varkappa_7\big|\xi\big|^2 + \sigma_2 c\big)\big|X^{(5)}\big|^2 = 0. \end{aligned} \tag{A.6}$$

With the help of inequalities (3.15) and also using the following relations

$$
\begin{aligned}
&|\xi|^2 |X^{(j)}|^2 - |\xi \cdot X^{(j)}|^2 = \big| [\xi \times X^{(j)}] \big|^2, \quad j = 1, 2, 3,\\
&\gamma |\xi|^2 |X^{(2)}|^2 + (\alpha + \beta) |\xi \cdot X^{(2)}|^2 = (\alpha + \beta + \gamma) |\xi \cdot X^{(2)}|^2 + \gamma \big| [\xi \times X^{(2)}] \big|^2 \geq 0,\\
&\varkappa_6 |\xi|^2 |X^{(3)}|^2 + (\varkappa_4 + \varkappa_5) |\xi \cdot X^{(3)}|^2 = (\varkappa_4 + \varkappa_5 + \varkappa_6) |\xi \cdot X^{(3)}|^2 + \varkappa_6 \big| [\xi \times X^{(3)}] \big|^2 \geq 0,\\
&\big| [\xi \times X^{(1)}] \big|^2 - 2 \operatorname{Im} \big([\xi \times X^{(2)}] \cdot \overline{X^{(1)}} \big) + |X^{(2)}|^2 = \big| [\xi \times \overline{X^{(1)}}] + i \overline{X^{(2)}} \big|^2 \geq 0,\\
&T_0 \varkappa_2 |X^{(3)}|^2 + (\varkappa_1 + T_0 \varkappa_3) \operatorname{Im} \big[(\xi \cdot \overline{X^{(3)}}) X^{(5)} \big] + \varkappa_7 |\xi|^2 |X^{(5)}|^2\\
&\quad = \frac{4 T_0 \varkappa_2 \varkappa_7 - (\varkappa_1 + T_0 \varkappa_3)^2}{4 \varkappa_7} |X^{(3)}|^2 + \frac{1}{4 \varkappa_7} \big| (\varkappa_1 + T_0 \varkappa_3) X^{(3)} - 2 i \varkappa_7 \xi X^{(5)} \big|^2 \geq 0,\\
&\lambda_0 |\xi \cdot X^{(1)}|^2 + 2 \mu_0 \operatorname{Im} \big[(\xi \cdot X^{(1)}) \overline{X^{(4)}} \big] + \eta |X^{(4)}|^2 = \frac{\lambda_0 \eta - \mu_0^2}{\lambda_0} |X^{(4)}|^2 + \frac{1}{\lambda_0} \big| \mu_0 X^{(4)} - i \lambda_0 \xi \cdot X^{(1)} \big|^2 \geq 0,\\
&\qquad\qquad \lambda_0 = \lambda + 2\mu + \varkappa > 0, \qquad \lambda_0 \eta - \mu_0^2 > 0,
\end{aligned}
$$

from (A.6) we conclude

$$X^{(j)} = 0, \qquad j = 1, 2, 3, 4, 5.$$

Thus, the system $L(-i\,\xi, \sigma)\,X = 0$ possesses only the trivial solution for arbitrary $\xi \in \mathbb{R}^3$. This contradiction proves the lemma. □

Corollary A.2. *Let $\sigma = \sigma_1 + i\,\sigma_2$ be a complex parameter with $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$. Then the equation with respect to $r = |\xi|$*

$$\Lambda(r) = \det L(-i\,\xi, \sigma) = 0$$

possesses complex roots $\pm\lambda_j$, $j = \overline{1, 11}$ with $\operatorname{Im} \lambda_j > 0$, $j = \overline{1, 11}$.

References

[1] D. Ieşan, On a theory of micromorfic elastic solids with microtemperatures, J. Therm. Stresses 24 (2001) 737–752.

[2] P. Casas, R. Quintanilla, Exponential stability in thermoelasticity with microtemperatures, Internat. J. Engrg. Sci. 43 (2005) 33–47.

[3] A. Scalia, M. Svanadze, On the representation of solutions of the theory of thermoelasticity with microtemperatures, J. Therm. Stresses 29 (2006) 849-863.

[4] M. Svanadze, Fundamental solutions of the equations of the theory of thermoelasticity with microstructures, J. Therm. Stresses 27 (2004) 151–170.

[5] R. Grot, Thermodynamics of a continuum mechanics, Internat. J. Engrg. Sci. 3 (1969) 801–814.

[6] D. Ieşan, Thermoelasticity of bodies with microstructure and microtemperatures, Int. J. Solids Struct. 44 (2007) 8648–8662.

[7] E. Cosserat, F. Cosserat, Thérie des Corps Déformables, Herman, Paris, 1909.

[8] V.D. Kupradze, T.G. Gegelia, M.O. Basheleishvili, T.V. Burchuladze, Three Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, Nauka, Moscow, 1976 (in Russian). English translation: North Holland Series in Applied Mathematics and Mechanics **25**, North Holland Publishing Company, Amsterdam, New York, Oxford, 1979.

[9] T.V. Burchuladze, T.G. Gegelia, Development of Potential Methods in the Theory of Elasticity, Metsniereba, Tbilisi, 1985 (in Russian).

[10] D. Natroshvili, L. Giorgashvili, I.G. Stratis, Mathematical problems of the theory of elasticity of chiral materials, Appl. Math. Inform. Mech. 8 (1) (2003) 47–103.

[11] D. Natroshvili, I.G. Stratis, Mathematical problems of the theory of elasticity of chiral materials for Lipschitz domains, Math. Methods Appl. Sci. 29 (4) (2006) 445–478.

[12] D. Natroshvili, L. Giorgashvili, S Zazashvili, Steady state oscillation problems of the theory of elasticity of chiral materials, J. Integral Equations Appl. 17 (1, Spring) (2005) 19–69.

[13] D. Natroshvili, L. Giorgashvili, S. Zazashvili, Mathematical problems of thermoelasticity for hemitropic solids, Mem. Differential Equations Math. Phys. 48 (2009) 97–174.

[14] G. Fichera, Existence Theorems in Elasticity, in: Handb. der Physik, Bd. 6/2, Springer Verlag, Heidelberg, 1973.

[15] T. Buchukuri, O. Chkadua, D. Natroshvili, Mathematical problems of generalized thermo-electro-magneto-elasticity theory, Mem. Differential Equations Math. Phys. 68 (2016) 1–165.

[16] D. Natroshvili, Investigation of Boundary Value and Initial Boundary Value Problems of the Mathematical Theory of Anisotropic Elasticity and Thermoelasticity By Potential Method, (Doctor thesis), Tbilisi Institute of Mathematics (Georgian Academy of Sciences), Tbilisi, 1984, p. 325.

[17] T. Buchukuri, O. Chkadua, R. Duduchava, D. Natroshvili, Interface crack problems for metallic-piezoelectric composite structures, Mem. Differential Equations Math. Phys. 55 (2012) 1–150.

[18] Natroshvili D., Boundary integral equation method in the steady state oscillation problems for anisotropic bodies, Math. Methods Appl. Sci. 20 (2) (1997) 95–119.
[19] D. Natroshvili, A. Djagmaidze, M. Svanadze, Problems of the Linear Theory of Elastic Mixtures, Tbilisi University, Tbilisi, 1986.
[20] D. Natroshvili, O. Chkadua, E. Shargorodsky, Mixed boundary value problems of the anisotropic elasticity, Proc. I. Vekua Inst. Appl. Math. Tbilisi State University 39 (1990) 133–181 (in Russian).
[21] I.N. Vekua, On metaharmonic functions, Tr. Tbilisskogo Mat. Inst. 12 (1943) 105–174 (in Russian).

Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute 171 (2017) 379–380

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

A non-dense subspace in $\mathcal{M}_q^p$ with $1 < q < p < \infty$

Yoshihiro Sawano

Department of Mathematical Science and Information Science, Tokyo Metropolitan University, 1-1 Minami-Ohsawa, Hachioji, Tokyo, 192-0397, Japan

Received 25 January 2017; received in revised form 27 April 2017; accepted 12 May 2017
Available online 30 May 2017

Abstract

The Morrey space $\mathcal{M}_{\tilde{q}}^p$ is disproved dense in $\mathcal{M}_q^p$ if $1 < q < \tilde{q} < p$.

Keywords: Morrey spaces; Inclusion

The Morrey space $\mathcal{M}_q^p$ with $1 \leq q \leq p < \infty$ collects all measurable functions f for which $\|f\|_{\mathcal{M}_q^p} \equiv \sup_Q |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q)}$ is finite, where Q moves over all cubes having sides parallel to coordinate axis in $\mathbb{R}^n$. Recently, more and more attention has been paid to closed subspaces in the Morrey space $\mathcal{M}_q^p$ with $1 < q < p < \infty$ [1,2]. In this connection, we aim here to show the following:

Theorem 1. *Let $1 < q < \tilde{q} < p$. Then $\mathcal{M}_{\tilde{q}}^p$ is not dense in $\mathcal{M}_q^p$.*

Let $E \equiv \{y + (R-1)(a_1 + Ra_2 + \cdots) : \{a_j\}_{j=1}^\infty \in \{0,1\}^n \cap \ell^1(\mathbb{N}), y \in [0,1]^n\}$, where $R > 2$ solves $R^{\frac{n}{p}-\frac{n}{q}} 2^{\frac{n}{q}} = 1$. Note that $E_j \equiv E \cap [0, R^j]^n$ is made up of 2^{jn} cubes of volume 1. According to [3], the indicator function χ of E belongs to $\mathcal{M}_q^p$. We prove that χ is not in the closure of $\mathcal{M}_{\tilde{q}}^p$ by showing $f \notin \mathcal{M}_{\tilde{q}}^p$ if $f \in \mathcal{M}_q^p$ satisfies $\|2\chi - f\|_{\mathcal{M}_q^p} < 1$. Indeed, if K is one of the connected components, then $\|f\|_{L^{\tilde{q}}(K)} \geq \|f\|_{L^q(K)} > 1$ since $1 > \|2\chi - f\|_{\mathcal{M}_q^p} \geq \|2 - f\|_{L^q(K)} \geq 2 - \|f\|_{L^q(K)}$. Thus, $\|f\|_{\mathcal{M}_{\tilde{q}}^p} \geq |[0, R^j]^n|^{\frac{1}{p}-\frac{1}{\tilde{q}}} \|f\|_{L^{\tilde{q}}([0,R^j]^n)} \geq R^{\frac{jn}{p}-\frac{jn}{\tilde{q}}} |E_j|^{\frac{1}{\tilde{q}}} = 2^{\frac{jn}{\tilde{q}}-\frac{jn}{q}} R^{\frac{jn}{q}-\frac{jn}{\tilde{q}}}$ for all $j \in \mathbb{N}$. Hence, $f \notin \mathcal{M}_{\tilde{q}}^p$, since this is valid for all $j \in \mathbb{N}$.

Acknowledgments

The author is supported by Grant-in-Aid for Scientific Research (C) (16K05209) and is thankful to the anonymous reviewers for their careful reading of this paper.

E-mail address: yoshihiro-sawano@celery.ocn.ne.jp.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.05.001

References

[1] L. Caso, R. D'Ambrosio, S. Monsurrò, Some remarks on spaces of Morrey type, Abstr. Appl. Anal. (2010) Art. ID 242079, 22 pp.
[2] Y. Sawano, H. Tanaka, The Fatou property of block spaces, J. Math. Sci. Univ. Tokyo 22 (2015) 663–683.
[3] Y. Sawano, S. Sugano, H. Tanaka, Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces, Trans. Amer. Math. Soc. 363 (12) (2011) 6481–6503.

Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 171 (2017) 381–388
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

The existence of solution for equilibrium problems in Hadamard manifolds

Salahuddin

Department of Mathematics, Jazan University, Jazan, Saudi Arabia

Received 19 December 2016; received in revised form 10 February 2017; accepted 19 April 2017
Available online 8 May 2017

Abstract

In this work, we consider iterative methods for solving a class of *equilibrium problems in Hadamard Manifolds* by using the auxiliary principle techniques. We also discuss the convergence of sequences generated by the algorithms.

Keywords: Equilibrium problems; Variational inequalities; Iterative methods; Algorithms; Existence theory; Hadamard manifolds

1. Introduction

Riemannian manifolds constitute a broad and fruitful framework for the development of different fields. Actually in the last decades concepts and techniques which fit in Euclidean spaces have been extended to this nonlinear framework. Most of the extended methods however require the Riemannian manifold to have non-positive sectional curvature. This is an important property which enjoyed by a large class of Riemannian manifolds and it is strong enough to imply light topological restriction and rigidity phenomena [1–3]. Particularly, Hadamard manifolds which are complete simply connected and finite dimensional Riemannian manifolds of non-positive sectional curvature, have been turned out to be a suitable setting for diverse disciplines. Hadamard manifolds are examples of hyperbolic spaces and geodesic spaces more precisely, a Busemann nonpositive curvature space and a $CAT(0)$ spaces, *see* [4–9].

Equilibrium problem theory provides us with a unified, natural, novel and general framework to study a wide class of problems, which arises in finance, economics, network analysis, transportation and optimization. This theory had applications across all disciplines of pure and applied sciences. Equilibrium problems include variational inequalities and related problems, *see* [10–14]. Very recently, much attention has been given to study the variational inequalities, variational inclusions, complementarity problems, equilibrium problems and related optimization problems on the Riemannian manifold and Hadamard manifold. Several idea and method from the Euclidean space have been extended and generalized to this nonlinear system. Hadamard manifolds are examples of hyperbolic spaces and geodesics, *see* [3–6,9,15–18].

E-mail address: salahuddin12@mailcity.com.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.04.001
2346-8092/

In this paper, we used the auxiliary principle techniques to suggest and analyze an iterative method for solving the *equilibrium problems* on Hadamard manifolds. We also discuss the convergence of sequences generated by the algorithms.

2. Preliminaries

Let M be a simply connected m-dimensional manifold. Given $x \in M$, the tangent space of M at x is denoted by T_xM and the tangent bundle of M by $TM = \bigcup_{x\in M} T_xM$ which is naturally a manifold. A vector field A on M is a mapping of M into TM which associates to each point $x \in M$, a vector $A(x) \in T_xM$. We always assume that M can be endowed with a Riemannian metric to become a Riemannian manifold. We denote by $\langle\cdot,\cdot\rangle$ the scalar product on T_xM with the associated norm $\|\cdot\|_x$, where the subscript x will be omitted. Given a piecewise smooth curve $\gamma : [a, b] \longrightarrow M$ joining x to y (that is, $\gamma(a) = x$ and $\gamma(b) = y$), by using the metric we can define the length of γ as $L(\gamma) = \int_a^b \|\gamma'(t)\|dt$. Then for any $x, y \in M$, the Riemannian distance $d(x, y)$ which includes the original topology on M is defined by minimizing this length over the set of all such curves joining x and y. Let $\triangle$ be the Levi-Civita connection with $(M, \langle\cdot,\cdot\rangle)$. Let γ be a piecewise smooth curve in M. A vector field A is said to be parallel along γ if $\triangle_{\gamma'}A = 0$. If γ' itself is parallel along γ, we say that γ is a geodesic and in this case $\|\gamma'\|$ is a constant when $\|\gamma'\| = 1$, γ is said to be normalized. A geodesic $\gamma_{x,y}$ joining x to y in M is said to be minimal if its length equal to $d(x, y)$. A Riemannian manifold is complete if for any $x \in M$, all geodesics emanating from x are defined for all $t \in R$. By the Hopf–Rinow Theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover (M, d) is a complete metric space and bounded closed subsets are compact.

Let M be complete, then exponential map $exp_x : T_xM \longrightarrow M$ at x is defined by $exp_x v = \gamma_v(1, x)$ for each $v \in T_xM$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at x with velocity v (*i.e.*, $\gamma(0) = x$ and $\gamma'(0) = v$). Then $exp_x tv = \gamma_v(t, x)$ for each real number t. A complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard manifold. Throughout this paper, we always assume that M is an m-dimensional Hadamard manifold. The geodesic triangle $\triangle(x_1, x_2, x_3)$ of a Riemannian manifold is a set consisting of three points x_1, x_2, x_3 and three minimal geodesic joining these points.

Lemma 2.1 ([16]). *Let $x \in M$. Then $exp_x : T_xM \longrightarrow M$ is a diffeomorphism and for any two points $x, y \in M$ there exists a unique normalized geodesic $\gamma_{x,y}$ joining x to y, which is minimal.*

Lemma 2.2 ([5]). *Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3 \pmod 3$, $\gamma_i : [0, \ell_i] \longrightarrow M$ as the geodesic joining x_i to x_{i+1} and set $\alpha_i = L(\gamma_i'(0), -\gamma_{i-1}'(\ell_{i-1}))$, the triangle between the vectors $\gamma_i'(0)$ and $-\gamma_{i-1}'(\ell_{i-1})$, and $\ell_i = L(\gamma_i)$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \le \pi, \tag{2.1}$$

$$\ell_i^2 + \ell_{i+1}^2 - 2\ell_i\ell_{i+1}\cos\alpha_{i+1} \le \ell_{i-1}^2. \tag{2.2}$$

In terms of the distance and the exponential map, the inequality (2.2) can be rewritten as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\rangle \le d^2(x_{i-1}, x_i), \tag{2.3}$$

since

$$\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\rangle = d(x_i, x_{i+1})d(x_{i+1}, x_{i+2})\cos\alpha_{i+1}.$$

Lemma 2.3 ([19]). *Let $\triangle(x, y, z)$ be a geodesic triangle in a Hadamard manifold M. Then there exist $x', y', z' \in \mathbb{R}^2$ such that*

$$d(x, y) = \|x' - y'\|, \quad d(y, z) = \|y' - z'\|, \quad d(z, x) = \|z' - x'\|.$$

The $\triangle(x', y', z')$ is called the comparison triangle of the geodesic triangle $\triangle(x, y, z)$ which is unique up to isometry of M.

From the law of cosine of inequality (2.3), we have the following inequality:

$$\langle \exp_x^{-1} y,\ \exp_x^{-1} z\rangle + \langle \exp_y^{-1} x,\ \exp_y^{-1} z\rangle \ge d^2(x, y),\ \forall x, y, z \in M. \tag{2.4}$$

Lemma 2.4 (*[15]*). *Let $\triangle(x, y, z)$ be a geodesic triangle in a Hadamard manifold M and $\triangle(x', y', z')$ be its comparison triangle.*

(i) *Let α, β, γ (resp. α', β', γ') be the angles of $\triangle(x, y, z)$(resp. $\triangle(x', y', z')$) at the vertices x, y, z (resp. x', y', z'). Then*

$$\alpha' \geq \alpha, \quad \beta' \geq \beta, \quad \gamma' \geq \gamma. \tag{2.5}$$

(ii) *Given any point q belonging to the geodesic which join x to y, its comparison point is the point q' in the interval $[x', y']$ such that $d(q, x) = \|q' - x'\|$ and $d(q, y) = \|q' - y'\|$. Then*

$$d(z, q) \leq \|z' - q'\|. \tag{2.6}$$

Lemma 2.5 (*[15]*). *For all $x, y, z \in M$ and $q \in M$ with $d(x, q) = d(y, q) = d(x, y)/2$, one has*

$$d^2(z, q) \leq \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y). \tag{2.7}$$

Lemma 2.6 (*[16]*). *Let $x_0 \in M$ and $\{x_n\}$ be a sequence in M such that $x_n \longrightarrow x_0$. Then the following assertions hold:*

(i) *For any $y \in M$*

$$\exp_{x_n}^{-1} y \longrightarrow \exp_{x_0}^{-1} y \quad \textit{and} \quad \exp_y^{-1} x_n \longrightarrow \exp_y^{-1} x_0.$$

(ii) *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \longrightarrow v_0$, then $v_0 \in T_{x_0}M$.*

(iii) *Given sequences $\{u_n\}$ and $\{v_n\}$ satisfying $u_n, v_n \in T_{x_n}M$, if $u_n \longrightarrow u_0$ and $v_n \longrightarrow v_0$ with $u_0, v_0 \in T_{x_0}M$, then*

$$\langle u_n \, , \, v_n \rangle \to \langle u_0 \, , \, v_0 \rangle.$$

A subset $K \subseteq M$ is said to be convex if for any two points $x, y \in K$, the geodesic joining x and y is contained in K, that is, if $\gamma : [a, b] \longrightarrow M$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then

$$\gamma((1 - t)a + tb) \in K, \ \forall t \in [0, 1].$$

A real valued function f defined on K is said to be convex if for any geodesic γ of M, the composition function $f \circ \gamma : \mathbb{R} \longrightarrow \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(ta + (1 - t)b) \leq t(f \circ \gamma)(a) + (1 - t)(f \circ \gamma)(b), \forall a, b \in \mathbb{R}, t \in [0, 1].$$

The subdifferential of a function $f : M \longrightarrow \mathbb{R}$ is a set valued mapping $\partial f : M \longrightarrow 2^{TM}$, defined as

$$\partial f(x) = \{u \in T_x M : \langle u, \ \exp_x^{-1} y\rangle \leq f(y) - f(x), \ \forall y \in M\}, x \in M,$$

and its elements are called subgradients. The subdifferential $\partial f(x)$ at a point $x \in M$ is a closed and convex (possibly empty) set. Let $D(\partial f)$ denote the domain of ∂f defined by

$$D(\partial f) = \{x \in M : \partial f(x) \neq \emptyset\}.$$

Lemma 2.7 (*[15]*). *Let M be a Hadamard manifold and $f : M \longrightarrow \mathbb{R}$ convex. Then for any $x \in M$, the subdifferential $\partial f(x)$ of f at x is nonempty. That is $D(\partial f) = M$.*

Definition 2.8 (*[20]*). The bifunction $\varphi : K \times K \longrightarrow \mathbb{R} \bigcup \{+\infty\}$ is called skew-symmetric if and only if

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) - \varphi(v, v) \geq 0, \quad \forall u, v \in K.$$

Clearly, if the skew-symmetric (bifunction $\varphi(\cdot, \cdot)$) is bilinear, then

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) = \varphi(u - v, u - v) \geq 0, \quad \forall u, v \in K.$$

For a given nonlinear continuous trifunction $F : K \times K \times K \longrightarrow \mathbb{R}$, a single valued mapping $T : K \longrightarrow TM$ and a continuous bifunction $\varphi : K \times K \longrightarrow \mathbb{R} \bigcup \{+\infty\}$, we consider a problem of finding $u \in K$ such that

$$F(u, Tu, v) + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K \tag{2.8}$$

called the *equilibrium problems* on Hadamard manifolds.

We note that if $T \equiv 0$ is a zero operator and $F(\cdot,\cdot,\cdot) = F(\cdot,\cdot)$, then (2.8) reduces to finding $u \in K$ such that

$$F(u, v) + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K. \tag{2.9}$$

Again if $\varphi(u, u) \equiv \varphi(u) \equiv 0$, then (2.9) reduces to *equilibrium problems* on Hadamard manifolds for finding $u \in K$ such that

$$F(u, v) \geq 0, \quad \forall v \in K \tag{2.10}$$

studied by Noor and Noor [21].
If $M \equiv \mathbb{R}^n$, (2.10) is called equilibrium problem, *see* [11].

Again we note that if $F(u, Tu, v) = \langle Tu, \exp_u^{-1} v\rangle$, where $T : K \longrightarrow TM$ is a single valued vector field, then problem (2.8) is equivalent to finding $u \in K$ such that

$$\langle Tu, \exp_u^{-1} v\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \ \forall v \in K \tag{2.11}$$

called *variational inclusions* in Hadamard manifolds.
Again we note that if $\varphi(u, u) = \varphi(u)$, then problem (2.11) is equivalent to finding $u \in K$ such that

$$\langle Tu, \exp_u^{-1} v\rangle + \varphi(v) - \varphi(u) \geq 0, \ \forall v \in K \tag{2.12}$$

called *variational inclusions* in Hadamard manifolds.
Again we note that if $\varphi(u) \equiv 0$, then problem (2.12) is equivalent to finding $u \in K$ such that

$$\langle Tu, \exp_u^{-1} v\rangle \geq 0, \ \forall v \in K \tag{2.13}$$

called the *variational inequalities* on Hadamard manifolds. Nemeth [22] and Tang et al. [18] studied the variational inequalities on Hadamard manifolds from different points of view.

Definition 2.9. A trifunction $F : K \times K \times K \longrightarrow \mathbb{R}$, with respect to the operator $T : K \longrightarrow TM$, is said to be

(i) jointly pseudomonotone, if

$$F(u, Tu, v) + \varphi(v, u) - \varphi(u, u) \geq 0$$

implies

$$- F(v, Tv, u) + \varphi(v, u) - \varphi(u, u) \geq 0, \forall u, v \in K.$$

(ii) partially relaxed strongly joint monotone if there exists a constant $\alpha > 0$ such that

$$F(u, Tu, v) + F(v, Tv, u) \leq \alpha d^2(z, u), \forall u, v, z \in K.$$

We note that if $z = u$, then a partially relaxed strongly joint monotonicity reduces to

$$F(u, Tu, v) + F(v, Tv, u) \leq 0, \ \forall u, v \in K,$$

which is known as the joint monotonicity of F.

3. Main results

We used the auxiliary principle techniques of Glowinski et al. [23] to suggest and analyze some iterative methods for solving the *equilibrium problems* (2.8).

For given $u \in K$ satisfying (2.8), consider the following problem of finding $w \in K$ such that

$$\rho F(w, Tw, v) + \langle \exp_u^{-1} w, \ \exp_w^{-1} v\rangle \geq \rho\varphi(u, u) - \rho\varphi(v, u), \ \forall v \in K, \tag{3.1}$$

where $\rho > 0$ is a constant. Inequality (3.1) is called *auxiliary equilibrium problems* on Hadamard manifolds. We note that if $w = u$, then w is a solution of (2.8). This simple observation enables us to suggest the following iterative methods for solving (2.8).

Algorithm 3.1. For given $u_0 \in K$, compute an approximate solution $u_{n+1} \in K$ by iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v) + \langle \exp_{u_n}^{-1} u_{n+1}, \ \exp_{u_{n+1}}^{-1} v\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}), \ \forall v \in K. \tag{3.2}$$

Algorithm 3.1 is called the proximal point algorithm for solving *equilibrium problems* on Hadamard manifolds.

If K is convex set in $\mathbb{R}^n$, then Algorithm 3.1 collapses to:

Algorithm 3.2. For given $u_0 \in K$, compute an approximate solution $u_{n+1} \in K$ by iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v) + \langle u_{n+1} - u_n,\ v - u_{n+1}\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}),\ \forall v \in K. \tag{3.3}$$

Algorithm 3.2 is called the proximal point algorithm for solving the *equilibrium problems*.

If $F(u, Tu, v) = \langle Tu, \exp_u^{-1} v\rangle$, where T is a single valued vector field $T : K \longrightarrow TM$, then Algorithm 3.1 reduces to the following proximal point method for solving the variational inclusions.

Algorithm 3.3. For given $u_0 \in K$, compute an approximate solution $u_{n+1} \in K$ by iterative scheme

$$\langle \rho Tu_n + (\exp_{u_n}^{-1} u_{n+1}), \exp_{u_{n+1}}^{-1} v\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}),\ \forall v \in K. \tag{3.4}$$

For $M = \mathbb{R}^n$, Algorithm 3.3 reduces to:

Algorithm 3.4. For given $u_0 \in K$, compute an approximate solution $u_{n+1} \in K$ by iterative scheme

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1}\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}),\ \forall v \in K. \tag{3.5}$$

We note that if $\varphi(u, u) \equiv \varphi(u) \equiv 0$, then Algorithm 3.4 becomes:

Algorithm 3.5. For given $u_0 \in K$, compute an approximate solution $u_{n+1} \in K$ by iterative scheme

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0,\ \forall v \in K, \tag{3.6}$$

which can be written in the following equivalent form:

Algorithm 3.6. For given $u_0 \in K$, compute an approximate solution $u_{n+1} \in K$ by iterative scheme

$$u_{n+1} = P_K[u_n - \rho Tu_n],\quad n = 0, 1, 2, \ldots, \tag{3.7}$$

which is known as a projection method.

In a similar way, one can obtain several iterative methods for solving the variational inclusions and variational inequalities on Hadamard manifolds.

Now, we consider the convergence of Algorithm 3.1 for solving the *variational inclusions* on Hadamard manifolds, which is a motivation of our next results.

Theorem 3.7. *Let $u \in K$ be a solution of* (2.8) *and let u_n be the approximate solution obtained from Algorithm* 3.1. *If $F(\cdot, \cdot, \cdot)$ is the jointly pseudomonotone and the bifunction $\varphi(\cdot, \cdot)$ is skew-symmetric, then*

$$d^2(u_{n+1}, u) \leq d^2(u_n, u) - d^2(u_{n+1}, u_n). \tag{3.8}$$

Proof. Let $u \in K$ be a solution of (2.8), then

$$F(u, Tu, v) \geq \varphi(u, u) - \varphi(v, u), \forall v \in K. \tag{3.9}$$

Now take $v = u_{n+1}$ in (3.9) we have

$$F(u, Tu, u_{n+1}) \geq \varphi(u, u) - \varphi(u_{n+1}, u), \tag{3.10}$$

which implies that

$$-F(u_{n+1}, Tu_{n+1}, u) \geq \varphi(u, u) - \varphi(u_{n+1}, u), \tag{3.11}$$

since $F(\cdot, \cdot, \cdot)$ is a pseudomonotone operator.

Taking $v = u$ in (3.2) we get

$$\rho F(u_{n+1}, Tu_{n+1}, u) + \langle \exp_{u_n}^{-1} u_{n+1},\ \exp_{u_{n+1}}^{-1} u\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(u, u_{n+1}) \tag{3.12}$$

which can be written as

$$\begin{aligned}\langle \exp^{-1}_{u_{n+1}} u_n,\ \exp^{-1}_{u_{n+1}} u\rangle &\geq -\rho F(u_{n+1}, Tu_{n+1}, u) + \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\}\\ &\geq \rho\{\varphi(u, u) - \varphi(u, u_{n+1}) - \varphi(u_{n+1}, u) + \varphi(u_{n+1}, u_{n+1})\}\\ &\geq 0, \end{aligned} \tag{3.13}$$

where we have used (3.11) and fact that the bifunction $\varphi(\cdot, \cdot)$ is skew-symmetric. Now from the geodesic triangle $\triangle(u_n, u_{n+1}, u)$, we have

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) - 2\langle \exp^{-1}_{u_{n+1}} u_n, \exp^{-1}_{u_{n+1}} u\rangle \leq d^2(u_n, u). \tag{3.14}$$

Combining (3.13) and (3.14), we have

$$d^2(u, u_{n+1}) \leq d^2(u_n, u) - d^2(u_{n+1}, u_n), \tag{3.15}$$

the required results (3.8). □

Theorem 3.8. *Let $u \in K$ be a solution of* (2.8) *and let u_{n+1} be the approximate solution obtained from Algorithm* 3.1. *If $\rho < \frac{1}{2\gamma}$, then the sequence $\{u_n\}$ given by Algorithm* 3.1 *converges to a solution u of* (2.8)*, i.e.,*

$$\lim_{n\longrightarrow\infty} u_{n+1} = u.$$

Proof. Let $u \in K$ be a solution of (2.8). Then from (3.8) it follows that the sequence $\{u_n\}$ is monotonically decreasing and bounded. Furthermore, we have

$$\sum_{n=0}^{\infty} d^2(u_{n+1}, u_n) \leq d^2(u_0, u),$$

which implies that

$$\lim_{n\longrightarrow\infty} d(u_{n+1}, u_n) = 0. \tag{3.16}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$. Then there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ converging to $\hat{u}$. Replacing u_{n+1} by u_{n_i} in (3.2), taking the limit $n_i \longrightarrow \infty$ and using (3.16) we have

$$F(\hat{u}, T\hat{u}, v) \geq \varphi(\hat{u}, \hat{u}) - \varphi(v, \hat{u}), \forall v \in K, \tag{3.17}$$

which implies that $\hat{u}$ solves the *equilibrium problems* on Hadamard manifolds (2.8) and

$$d^2(u_{n+1}, \hat{u}) \leq d^2(u_n, \hat{u}).$$

Thus, it follows from the above inequality that $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\longrightarrow\infty} u_n = \hat{u}$$

is a solution of (2.8), the required results. □

It is well known that to implement the proximal methods, one has to calculate the approximate solution implicitly, which is itself a different problem. To overcome this drawback we suggest another iterative method, the convergence of the sequence requires only the partially relaxed strong monotonicity, which is a weaker condition than the cocoercivity.

For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\rho F(u, Tu, v) + \langle \exp^{-1}_{u} w,\ \exp^{-1}_{w} v\rangle \geq \rho\{\varphi(w, w) - \varphi(v, w)\}, \forall v \in K, \tag{3.18}$$

which is also called the *auxiliary uniformly equilibrium problems* on Hadamard manifolds. Note that the problems (3.1) and (3.18) are quite different. If $w = u$, then clearly w is a solution of *equilibrium problems* on Hadamard manifolds (2.8). This fact enables us to suggest and analyze the following iterative methods for solving (2.8).

Algorithm 3.9. For given $u_0 \in K$, compute an approximate solution $u_{n+1} \in K$ by the iterative scheme

$$\rho F(u_n, Tu_n, v) + \langle \exp^{-1}_{u_n} u_{n+1},\ \exp^{-1}_{u_{n+1}} v\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}),\ \forall v \in K. \tag{3.19}$$

Theorem 3.10. *Let the trifunction $F(\cdot,\cdot,\cdot)$ be the partially relaxed strongly jointly monotone with the constant $\alpha > 0$ and the bifunction $\varphi(\cdot,\cdot)$ be skew-symmetric, If u_{n+1} is the approximate solution obtained from Algorithm* 3.9 *and $u \in K$ is a solution of* (2.8)*, then*

$$d^2(u_{n+1}, u) \le d^2(u_n, u) - (1 - 2\rho\alpha)d^2(u_n, u_{n+1}). \tag{3.20}$$

Proof. Let $u \in K$ be a solution of (2.8), then

$$F(u, Tu, v) \ge \varphi(u, u) - \varphi(v, u), \ \forall v \in K. \tag{3.21}$$

Now take $v = u_{n+1}$ in (3.21) we have

$$F(u, Tu, u_{n+1}) \ge \varphi(u, u) - \varphi(u_{n+1}, u), \ \forall v \in K. \tag{3.22}$$

Taking $v = u$ in (3.19), we have

$$\rho F(u_n, Tu_n, u) + \langle \exp_{u_n}^{-1} u_{n+1}, \ \exp_{u_{n+1}}^{-1} u\rangle \ge \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\},$$

which implies that

$$\langle \exp_{u_{n+1}}^{-1} u_n, \ \exp_{u_{n+1}}^{-1} u\rangle \ge -\rho F(u_n, Tu_n, u) + \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\}. \tag{3.23}$$

From (3.22) and (3.23), we have

$$\begin{aligned}\langle \exp_{u_{n+1}}^{-1} u_n, \ \exp_{u_{n+1}}^{-1} u\rangle &\ge -\rho\{F(u_n, Tu_n, u) + F(u, Tu, u_{n+1})\} + \rho\{\varphi(u, u) \\ &\quad - \varphi(u, u_{n+1}) - \varphi(u_{n+1}, u) + \varphi(u_{n+1}, u_{n+1})\} \\ &\ge -\rho\alpha d^2(u_{n+1}, u_n),\end{aligned} \tag{3.24}$$

where we have used the fact that the trifunction $F(\cdot,\cdot,\cdot)$ is partially relaxed strongly jointly monotone with a constant $\alpha > 0$ and bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric. For the geodesic triangle $\triangle(u_n, u_{n+1}, u)$, the inequality (3.24) can be written as

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) - 2\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} u\rangle \le d^2(u_n, u). \tag{3.25}$$

Combining (3.24) and (3.25), we have

$$d^2(u, u_{n+1}) \le d^2(u, u_n) - (1 - 2\rho\alpha)d^2(u_n, u_{n+1}), \tag{3.26}$$

the required results. □

4. Conclusion

The auxiliary principle technique is used to suggest and analyze proximal methods for solving the *equilibrium problems* on Hadamard manifolds. It is shown that the convergence analysis of this method requires the joint pseudomonotonicity and also partially relaxed strongly joint monotonicity.

References

[1] J. Jost, Non-positive Curvature, in: Geometric and Analytic Aspects, in: Lectures Math. ETH Zurich, Birkhauser, Verlag, Basel, 1997.

[2] T. Rapcsak, Sectional curvature in nonlinear optimization, J. Global Optim. 40 (2008) 375–388.

[3] C. Udriste, Convex Functions and Optimization Methods on Riemannian Manifolds, Kluwer Acad. Pub, Dordrecht, 1994.

[4] D. Azagra, J. Ferrera, F. Lopez Mesa, Nonsmooth analysis and Hamiltonian-Jocobi equilibrium on Riemannian Manifolds, J. Funct. Anal. 220 (2005) 304–361.

[5] M.P. DoCarmo, Riemannian Geometry, Birkhauser, Boston, 1992.

[6] O.P. Ferrera, P.R. Oliveira, Proximal point algorithms on Riemannian manifolds, Optimization 51 (2) (2002) 257–270.

[7] C. Li, G. Lopez, V. Martin-Marquez, W. Wang, Resolvent of set valued monotone vector field in Hadamard manifolds, Set-Valued Var. Anal. 19 (3) (2011) 361–383.

[8] M.A. Noor, S. Zainab, Y. Yao, Implicit methods for equilibriums on Hadamard manifolds, J. Appl. Math. 2012 (2012).

[9] Salahuddin, Regularized equilibrium problems on Hadamard manifolds, Nonlinear Anal. Forum 21 (2) (2016) 99–101.

[10] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123–145.

[11] F. Giannessi, A. Maugeri, Variational Inequalities and Network Equilibrium Problems, Kluwer Acad. Pub, Dordrecht, 2000.

[12] F. Giannessi, A. Maugeri, P.M. Pardalos, Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models, Kluwer Acad. Pub, Dordrecht, 2001.
[13] Salahuddin, Descent techniques for generalized equilibrium problems, Trans. Math. Program. Appl. 3 (2) (2015) 30–39.
[14] A.H. Siddiqi, M.K. Ahmad, Salahuddin, Existence results for generalized nonlinear variational inclusions, Appl. Math. Lett. 18 (8) (2005) 859–864.
[15] V. Colao, G. Lopez, G. Marino, V. Martin-Marquez, Equilibrium problems in Hadamard manifolds, J. Math. Anal. Appl. 388 (2012) 61–77.
[16] T. Sakai, Riemannian Geometry, Am. Math. Soc. Providence, 1996, p. 149.
[17] Salahuddin, Regularized equilibrium problems in Banach spaces, Korean J. Math. 24 (1) (2016) 51–63.
[18] G. Tang, L.W. Zhan, N. J. Huang, The proximal point algorithm for pseudomonotone variational inequalities in Hadamard manifolds. Optim. Lett. 2012, http://dx.doi.org/10.1007/811590-12-0459-7.
[19] M. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, New York, 1999.
[20] X.P. Ding, Auxiliary principle and algorithm for mixed equilibrium problems and bilevel mixed equilibrium problems in Banach spaces, J. Optim. Theory Appl. http://dx.doi.org/10.1007/s10957-010-9651-z.
[21] M.A. Noor, K.I. Noor, Some algorithms for equilibrium problems on Hadamard manifolds, J. Inequal. Appl. 2012 (2012) 230.
[22] S.Z. Nemeth, Variational inequalities on Hadamard manifolds, Nonlinear Anal. 52 (5) (2003) 1491–1498.
[23] R. Glowinski, J.L. Lions, R. Tremolieres, Numerical Analysis of Variational Inequalities, North Holland, Amsterdam, Holland, 1981.

Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 171 (2017) 389–395
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

The loop cohomology of a space with the polynomial cohomology algebra

Samson Saneblidze

A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University 6, Tamarashvili st., Tbilisi 0177, GA, United States

Received 8 February 2017; received in revised form 5 July 2017; accepted 18 July 2017
Available online 3 August 2017

To Jim Stasheff on the occasion of his 80th birthday

Abstract

Given a simply connected space X with polynomial cohomology $H^*(X;\mathbb{Z}_2)$, we calculate the loop cohomology algebra $H^*(\Omega X;\mathbb{Z}_2)$ by means of the action of the Steenrod cohomology operation Sq_1 on $H^*(X;\mathbb{Z}_2)$. This calculation uses an explicit construction of the minimal Hirsch filtered model of the cochain algebra $C^*(X;\mathbb{Z}_2)$. As a consequence we obtain that $H^*(\Omega X;\mathbb{Z}_2)$ is the exterior algebra if and only if Sq_1 is multiplicatively decomposable on $H^*(X;\mathbb{Z}_2)$. The last statement in fact contains a converse of a theorem of A. Borel (Switzer, 1975, Theorem 15.60).

Keywords: Loop space; Polynomial cohomology; Hirsch algebra; Multiplicative resolution; Steenrod operation

1. Introduction

Let X denote a simply connected topological space. The cohomology $H^*(X)$ is considered with coefficients $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ unless otherwise specified explicitly. A. Borel gave a condition for $H^*(X)$ to be polynomial in terms of *a simple system of generators* of the loop space cohomology $H^*(\Omega X)$ that are transgressive [1, Theorem 15.60] , [2, p. 88] (see also [3]). This was one of the first nice applications of Leray–Serre spectral sequences [4], and led in particular to calculations of the cohomology of the Eilenberg–MacLane spaces (see [3]). For the converse direction, that is to determine $H^*(\Omega X)$ as an algebra for a given X with $H^*(X)$ polynomial, the first step is the existence of an additive isomorphism $H^*(\Omega X) \approx H^*(BH^*(X))$ where $BH^*(X)$ denotes the bar construction of $H^*(X)$ (cf. [5]). The module $BH^*(X)$ with the shuffle product is a graded differential algebra, but we get no algebra isomorphism above (cf. [6]). In general, a correct product on $BH^*(X)$ is induced by higher order operations on the cochain complex $C^*(X)$ (see below), but when $H^*(X)$ is polynomial we show that these operations reduce to the $\smile_1$-product on $C^*(X)$.

E-mail address: sane@rmi.ge.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.07.002
2346-8092/

Consequently, the multiplicative structure of $BH^*(X)$ is determined by the Steenrod cohomology operation Sq_1 on $H^*(X)$. This reduction is beyond a spectral sequence argument.

In this paper we completely calculate the algebra $H^*(\Omega X)$ for $H^*(X)$ polynomial by means of Sq_1 on $H^*(X)$ (Theorem 1) and then establish the criterion for $H^*(\Omega X)$ to be exterior (Corollary 1). Namely, given $H^*(X) = H(C^*(X), d)$ with the $\smile_1$-product on $C^*(X)$, let

$$Sq_1 : H^n(X) \to H^{2n-1}(X) \quad [c] \to [c \smile_1 c], \quad c \in C^n(X), dc = 0.$$

Let now $H^*(X) = \mathbb{Z}_2[y_1, \dots, y_k, \dots]$ with $\mathcal{Y} = \{y_k\}$ to be a set of polynomial generators. Define a subset $\mathcal{S} \subseteq \mathcal{Y}$ as

$$\mathcal{S} = \{z_s \in \mathcal{Y} \mid z_s \notin \operatorname{Im} Sq_1 \bmod H^+ \cdot H^+\}.$$

Thus $\mathcal{S} = \mathcal{Y}$ if and only if $Sq_1(y_k) \in H^+ \cdot H^+$ for all k. Let $0 \leq \nu_i < \infty$ be the smallest integer such that $Sq_1^{(\nu_i+1)}(y_i) \in H^+ \cdot H^+$, where $Sq_1^{(m)}$ denotes the m-fold composition $Sq_1 \circ \cdots \circ Sq_1$. The integer ν_i is referred to as the *weak* $\smile_1$*-height* of y_i; when the finite integer ν_i does not exist, we say that y_i has the infinite weak $\smile_1$-height $\nu_i = \infty$. (This notion is motivated by the fact that Sq_1 induces a binary $\smile_1$-product on $(H^*(X), 0)$; cf. Remark 1(a).)

Let $\sigma : H^*(X) \to H^{*-1}(\Omega X)$ be the suspension homomorphism.

Theorem 1. *Let X be a simply connected space with $H^*(X) = \mathbb{Z}_2[y_1, \dots, y_k, \dots]$ and ν_k to be the weak $\smile_1$-height of y_k. Then the algebra $H^*(\Omega X)$ is multiplicatively generated by the elements $\bar{z}_s = \sigma z_s$ satisfying only the relations $\bar{z}_s^{m_s} = 0$ for $m_s = 2^{\nu_s+1}$ and $\bar{z}_{s_1}^{m_1} + \cdots + \bar{z}_{s_r}^{m_r} = 0$ for $Sq^{(n_1)}(z_{s_1}) + \cdots + Sq^{(n_r)}(z_{s_r}) \in H^+ \cdot H^+$, $m_i = 2^{n_i+1}$, $n_i \leq \nu_i$, $r \geq 2$, $z_{s_i} \in \mathcal{S}$.*

Corollary 1. *$H^*(\Omega X) = \Lambda(\bar{y}_1, \dots, \bar{y}_k, \dots)$ is the exterior algebra if and only if y_k is of zero weak $\smile_1$-height, i.e., $Sq_1(y_k) \in H^+ \cdot H^+$ for all k.*

When $\mathcal{Y}$ is chosen such that y_i is uniquely determined by the equality $Sq_1(y_i) = y_k \bmod H^+ \cdot H^+$, we get

Corollary 2. *$H^*(\Omega X) = \mathbb{Z}_2[\bar{z}_1, \dots, \bar{z}_s, \dots]$ is the polynomial algebra if and only if z_s is of the infinite weak $\smile_1$-height for all s.*

Our method of proving the theorem consists of using the *filtered Hirsch* model $(RH^*, d + h) \to C^*(X)$ of X [7] (see Section 2). Note that the underlying differential (bi)graded algebra (RH^*, d) is a non-commutative version of Tate–Jozefiak resolution of the commutative algebra H^* [8,9], while h is a perturbation of d similar to [10]. Furthermore, the tensor algebra $RH^* = T(V)$ is endowed with higher order operations $E = \{E_{p,q}\}$ that extend $\smile_1$-product measuring the non-commutativity of the product on RH^*; and there also is a binary operation $\cup_2$ on RH^* measuring the non-commutativity of the $\smile_1$-product. In general, by means of $(RH^*, d + h)$ one can recognize the cohomology $H(BC^*(X))$ of the bar construction $BC^*(X)$ as an algebra. The case of polynomial H^* is distinguished because of H^* has no multiplicative relations unless that of the commutativity; furthermore, we can equivalently take a small multiplicative resolution $R_\tau H^* = T(V_\tau)$ in which the Hirsch algebra structure is completely determined by commutative and associative $\smile_1$-product on V_τ. This allows an explicit calculation of the algebra $H(BC^*(X))$, and, consequently, of the loop space cohomology $H^*(\Omega X)$ in question.

Obviously the hypothesis of Corollary 1 is satisfied for an evenly graded polynomial algebra $H^*(X)$. Note that our method can be in fact applied to an evenly graded polynomial algebra $H^*(X; \Bbbk)$ for any coefficient ring $\Bbbk$ to establish that $H^*(\Omega X; \Bbbk)$ is exterior. Though, this fact can be also deduced from the Eilenberg–Moore spectral sequence (see, for example, [3]; for further references of spaces with polynomial cohomology rings see also [11,12]).

I wish to thank Jim Stasheff for helpful comments and suggestions. I am also indebted to the referee for a number of helpful comments to improve the exposition.

2. Hirsch resolutions of polynomial algebras

We adopt the notations and terminology of [7] and briefly recall some facts. A Hirsch algebra $(A, d_A, \{E_{p,q}\})$ is an associative dga (A, d_A) equipped with multilinear maps

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A, \;\; p, q \geq 0, \;\; p + q > 0,$$

satisfying the following conditions:

(i) $\deg E_{p,q} = 1 - p - q$;
(ii) $E_{1,0} = Id = E_{0,1}$ and $E_{p>1,0} = 0 = E_{0,q>1}$;
(iii) The homomorphism $E : BA \otimes BA \to A$ defined by

$$E\left([\bar{a}_1|\cdots|\bar{a}_p] \otimes [\bar{b}_1|\cdots|\bar{b}_q]\right) = E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q)$$

is a twisting cochain in the dga $(Hom(BA \otimes BA, A), \nabla, \smile)$, i.e., $\nabla E = -E \smile E$.

A morphism $f : A \to B$ between two Hirsch algebras is a dga map f that commutes with $E_{p,q}$ for all p, q. Condition (iii) implies that $\mu_E : BA \otimes BA \to BA$ is a chain map; thus BA is a dg bialgebra; in particular, $\mu_{E_{10}+E_{01}}$ is the shuffle product on BA.

For a topological space X, there are operations $E = \{E_{p,q}\}$ on the cochain complex $C^*(X)$ making it into a Hirsch algebra. Note that in the simplicial case one can choose $E_{p,q} = 0$ for $q \geq 2$.

A dga (A^*, d) is multialgebra if it is bigraded $A^n = \underset{n=i+j}{\oplus} A^{i,j}$, $i \leq 0$, $j \geq 0$, and $d = d^0 + d^1 + \cdots + d^n + \cdots$ with $d^n : A^{p,q} \to A^{p+n,q-n+1}$. A dga A is bigraded via $A^{0,*} = A^*$ and $A^{i,*} = 0$ for $i \neq 0$; consequently, A is a multialgebra. A multialgebra A is homological if $d^0 = 0$ (hence $d^1 d^1 = 0$) and

$$H^i(\cdots \xrightarrow{d^1} A^{i,*} \xrightarrow{d^1} A^{i+1,*} \xrightarrow{d^1} \cdots \xrightarrow{d^1} A^{0,*}) = 0, \quad i < 0.$$

For a homological multialgebra the sum $d^2 + d^3 + \cdots + d^n + \cdots$ is called a perturbation of d^1. Furthermore, d^1 is denoted by d, d^r is denoted by h^r, and the sum $h^2 + h^3 + \cdots + h^n + \cdots$ is denoted by h. We sometimes denote $d + h$ by d_h.

A multialgebra is quasi-free if it is a tensor algebra over a bigraded $\Bbbk$-module. Given $m \geq 2$, the map $h^m|_{A^{-m,*}} : A^{-m,*} \to A^{0,*}$ is referred to as the transgressive component of h and is denoted by h^{tr}. A multialgebra A with a Hirsch algebra structure

$$E_{p,q} : \otimes_{r=1}^{p} A^{i_r,k_r} \bigotimes \otimes_{n=1}^{q} A^{j_k,\ell_n} \longrightarrow A^{s-p-q+1\,,\,t}$$

with $(s,t) = \left(i_{(p)} + j_{(q)}\,,\, k_{(p)} + \ell_{(q)}\right)$, $p, q \geq 1$, is called Hirsch multialgebra. A multialgebra is quasi-free if it is a tensor algebra over a bigraded $\Bbbk$-module. A quasi-free Hirsch homological multialgebra $(A, d+h, \{E_{p,q}\})$ is a filtered Hirsch algebra if it has the following additional properties:

(i) In $A = T(V)$ a decomposition

$$V^{*,*} = \mathcal{E}^{*,*} \oplus U^{*,*}$$

is fixed where $\mathcal{E}^{*,*} = \underset{p,q \geq 1}{\oplus} \mathcal{E}^{<0,*}_{p,q}$ is distinguished by an isomorphism of modules

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \xrightarrow{\approx} \mathcal{E}_{p,q} \subset V, \quad p, q \geq 1;$$

(ii) The restriction of the perturbation h to $\mathcal{E}$ has no transgressive components h^{tr}, i.e., $h^{tr}|_{\mathcal{E}} = 0$.

An important example of a filtered Hirsch algebra is $A = (R^*H^*, d, \{E_{p,q}\})$, an absolute Hirsch resolution of a graded commutative algebra H^*. In particular, $R^*H^* = T(V)$ with

$$V = \bigoplus\nolimits_{j,m \geq 0} V^{-j,m},$$

where $V^{-j,m} \subset R^{-j}H^m$. The total degree of $R^{-j}H^m$ is the sum $-j+m$, d is of bidegree $(1,0)$ and $\rho : (R^*H^*, d) \to H^*$ is a map of bigraded algebras inducing an isomorphism $\rho^* : H^*(RH, d) \xrightarrow{\approx} H^*$ where H^* is bigraded via $H^{0,*} = H^*$ and $H^{<0,*} = 0$.

Given a Hirsch algebra $(A, d_A, \{E_{p,q}\})$, a submodule $J \subset A$ is a *Hirsch ideal* of A if it is an ideal with $E_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}) \in J$ whenever $a_i \in J$ for some i.

Let $\rho_a : (R^*_a H^*, d) \to H^*$ be an absolute Hirsch resolution and $J \subset R^*_a H^*$ be a Hirsch ideal such that $d : J \to J$ and the quotient map $g : R^*_a H^* \to R^*_a H^*/J$ is a homology isomorphism. A Hirsch resolution of H^* is the Hirsch algebra $R^*H^* = R^*_a H^*/J$ with a map $\rho : R^*H^* \to H^*$ such that $\rho_a = \rho \circ g$. Thus an absolute Hirsch resolution is a Hirsch resolution by taking $J = 0$.

Given a Hirsch algebra $(A, d_A, \{E_{p,q}\})$ with $H^* = H^*(A, d_A)$, there is a filtered Hirsch model

$$f : (R^*H^*, d_h) \to (A, d_A),$$

where R^*H^* denotes an absolute Hirsch resolution. There is a (commutative) binary operation $a\cup_2 b$ on R^*H^* satisfying for basis elements $a, b \in R^*H^*$ the equality

$$d(a\cup_2 b) = \begin{cases} a \cup_2 da + a \smile_1 a, & a = b, \\ a \smile_1 da + da \smile_1 a, & da = b, \\ da \cup_2 b + a\cup_2 db + a \smile_1 b + b \smile_1 a, & \text{otherwise.} \end{cases}$$

(Thus, the first two cases differ $\cup_2$ from the Steenrod $\smile_2$-operation.) In $U \subset V$ we distinguish a submodule $\mathcal{T}^{\leq -2,*} \subset U$ defined by

$$\mathcal{T}^{\leq -2,*} = \{a\cup_2 b \in R^*H^* \mid a\cup_2 b \in U\}.$$

For the sake of minimality of U one can express certain elements $a\cup_2 b \in R^*H^*$ in terms of the $\smile$ and $E_{p,q}$ operations. For example, $da\cup_2 da := a\smile_1 da + a \cdot a$, because $d(a \smile_1 da + a \cdot a) = da \smile_1 da$.

When $H^* = \mathbb{Z}_2[y_1, \ldots, y_k, \ldots]$ is polynomial, the module V is much simplified at the cost of U. Namely,

$$V^{*,*} = \mathcal{E}^{<0,*} \oplus U^{*,*} = \mathcal{E}^{<0,*} \oplus \mathcal{T}^{\leq -2,*} \oplus V^{0,*}.$$

In particular, we have that R^0H^* is a graded subalgebra in R^*H^* and $\operatorname{Ker}\rho \cap R^0H^*$ is an ideal in R^0H^*. Denoting the elements of $\mathcal{V}^{0,*}$ by x_k, i.e., $\rho x_k = y_k$, this ideal is generated by expressions of the form $x_i x_j + x_j x_i$ for $i \neq j$; thus, we get

$$V^{-1,*} = \mathcal{E}^{-1,*} = \langle x_i \smile_1 x_j \mid x_k \in \mathcal{V}^{0,*}\rangle \ \text{ with}$$
$$d(x_i\smile_1 x_j) = d(x_j\smile_1 x_i) = x_i x_j + x_j x_i \text{ for } i \neq j \text{ and } d(x_i\smile_1 x_i) = 0,$$

while

$$\mathcal{T}^{-2,*} = \langle x_i\cup_2 x_j\, (= x_j\cup_2 x_i) \mid x_k \in \mathcal{V}^{0,*}\rangle \text{ with } d(x_i\cup_2 x_j) =$$
$$x_i\smile_1 x_j + x_j \smile_1 x_i \text{ for } i \neq j, \text{ and } d(x_i\cup_2 x_i) = x_i\smile_1 x_i.$$

Here, we can minimize further both an absolute Hirsch resolution R^*H^* and a small Hirsch resolution $R^*_\varsigma H^*$ in [7] to obtain a *minimal* Hirsch resolution $R^*_\tau H^*$; moreover, we give an explicit construction of $R^*_\tau H^*$ below. Namely, set

$$R^*_\tau H^* = R^*H^*/J_\tau$$

where $J_\tau \subset R^*H^*$ is a Hirsch ideal generated by

$$\Big\{E_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}),\ dE_{p,q}(a_1, \ldots, a_p; a_{p+1}, \ldots, a_{p+q}),\ a\cup_2 b,\ d(a\cup_2 b) \mid$$
$$p + q \geq 3,\ a \neq b \text{ in } \mathcal{V}\Big\}$$

with

$$\begin{array}{llllll} a_1, \ldots, a_p \in R^*H^*, & a_{p+1} \in V, & \text{for} & p \geq 1 & \text{and} & q = 1, \\ a_1, \ldots, a_{p+q} \in R^*H^*, & & \text{for} & p \geq 1 & \text{and} & q > 1. \end{array}$$

Because of $d : J_\tau \to J_\tau$, we get a Hirsch algebra map $g_\tau : (R^*H^*, d) \to (R^*_\tau H^*, d)$. Let $\rho_\tau : R^*_\tau H^* \to H^*$ denote a map of bigraded algebras so that the resolution map $\rho : R^*H^* \to H^*$ factors as

$$\rho : (R^*H^*, d) \xrightarrow{g_\tau} (R^*_\tau H^*, d) \xrightarrow{\rho_\tau} H^*.$$

By definition we have $h : \mathcal{E} \to \mathcal{E}$; furthermore, because of the transgressive component h^{tr} of h annihilates $a\cup_2 b$ for $a \neq b$ in $\mathcal{V}$ (cf. [7, Proposition 5]), we get $h : J_\tau \to J_\tau$, too. Thus g_τ extends to a quasi-isomorphism of Hirsch algebras

$$g_\tau : (R^*H^*, d_h) \to (R^*_\tau H^*, d_h),$$

and, hence, A and $R^*_\tau H^*$ are connected via the diagram

$$(A, d_A) \xleftarrow{f} (R^*H^*, d_h) \xrightarrow{g_\tau} (R^*_\tau H^*, d_h).$$

The Hirsch algebra $(R^*_\tau H^*, d_h)$ can be described immediately. Namely, $R^*_\tau H^* = T(V^{*,*}_\tau)$ with $V^{*,*}_\tau = \langle \mathcal{V}^{*,*}_\tau \rangle$,

$$\mathcal{V}_\tau = \Big\{ x_i,\ x_j{}^{\cup_2 q},\ b_{i_1} \smile_1 \cdots \smile_1 b_{i_n} \mid b_{i_r} \in \{x_i, x_j{}^{\cup_2 q}\},\ q = 2^m,\ m \geq 1,\ n \geq 2, \\ x_k \in \mathcal{V}^{0,*}_\tau,\ x^{\cup_2 q} := x \cup_2 \cdots \cup_2 x \Big\}.$$

The $\smile_1$-product is commutative and associative on V_τ and extended on $R^*_\tau H^*$ by the (left) Hirsch formula

$$c \smile_1 ab = (c \smile_1 a)b + a(c \smile_1 b), \qquad a, b, c \in R^*_\tau H^*,$$

and the (right) generalized Hirsch formula

$$ab \smile_1 c = \begin{cases} a(b \smile_1 c) + (a \smile_1 c)b, & a, b \in R^*_\tau H^* \quad \text{and} \quad c \in \{x_i, x_j{}^{\cup_2 q}\}, \\ & \qquad\qquad\qquad\qquad q = 2^m, m \geq 1, \\ a(b \smile_1 c) + (a \smile_1 c)b & \\ +(a \smile_1 c_1)(b \smile_1 c_2) & \\ +(a \smile_1 c_2)(b \smile_1 c_1), & a, b \in R^*_\tau H^* \quad \text{and} \quad c = c_1 \smile_1 c_2 \in V_\tau. \end{cases}$$

The differential d on $R^*_\tau H^*$ is defined by

$$dx_k = 0, \quad d(a \smile_1 b) = da \smile_1 b + a \smile_1 db + ab + ba \quad \text{and} \quad d(a \cup_2 a) = a \smile_1 a,$$

while the perturbation h by

$$hx_k = 0, \quad h(a \smile_1 b) = ha \smile_1 b + a \smile_1 hb$$

and

$$h(x_k \cup_2 x_k) = h^{tr}(x_k \cup_2 x_k) = b_k \quad \text{with} \quad b_k \in R^0_\tau H^* \quad \text{defined by} \quad \rho_\tau b_k = Sq_1(y_k).$$

Note that the value of h on $x_j{}^{\cup_2 2^m}$ for $m > 1$ may be non-zero (see Remark 1(b)). In particular, denoting

$$b_{k,1} := b_k,\ \ b_{k,j+1} := h(b_{k,j} \cup_2 b_{k,j}),\ j \geq 1,$$

and

$$c_0 = x_k \cup_2 x_k,\ \ c_j = x_k^{\smile_1 2^j} \smile_1 c_{j-1} + c_{j-1} \smile_1 b_{k,j} + b_{k,j} \cup_2 b_{k,j},\ j \geq 1,$$

one gets

$$d_h(c_{m-1}) = x_k^{\smile_1 2^m} + b_{k,m} \bmod R_\tau H^+ \cdot R_\tau H^+, m \geq 1, \text{ with } \rho_\tau b_{k,m} = Sq_1^{(m)}(y_k). \tag{2.1}$$

To ensure that $\rho_\tau : (R^*_\tau H^*, d) \to H^*$ is a multiplicative resolution of H^*, it suffices to verify the following.

Proposition 1. *The chain complex $(R^*_\tau H^*, d)$ is acyclic in the negative resolution degrees, i.e., $H^{i,*}(R^i_\tau H^*, d) = 0,\ i < 0$.*

Proof. First observe that as a cochain complex $\operatorname{Ker} \rho_\tau$ can be decomposed via $(\operatorname{Ker} \rho_\tau, d) = (A, d) \oplus (B, d)$ in which $(A, d) = \oplus(A(n), d),\ n \geq 2,\ A(n)$ has a basis consisting of all monomials formed by the $\smile$ and $\smile_1$ products evaluated on generators $x_{i_1}, \ldots, x_{i_n} \in V^{0,*}_\tau$ with distinct x_i's and B has a basis consisting of the other monomials in $\operatorname{Ker} \rho_\tau$. In particular, $(A(n), d)$ can be identified with the cellular chains of the permutohedron P_n (cf. [13]); thus A is acyclic and a chain contracting homotopy $s_A : A \to A$ can be chosen. To see that B is also acyclic, define a map $s_B : B \to B$ of degree -1 as follows. For $ba, ac, bac \in B$ with $a \in A$, let $s_B(ba) = bs_A(a), s_B(ac) = s_A(a)c, s_B(bac) = bs_A(a)c$; otherwise, for $b \smile_1 b$ and $b \smile_1 b \smile_1 c$ with $b, c \in V_\tau$, let $s_B(b \smile_1 b) = b \cup_2 b$ and $s_B(b \smile_1 b \smile_1 c) = b \cup_2 b \smile_1 c$, and then for a monomial $u = u_1 \cdots u_m \in B$, set

$$s_B(u) = \begin{cases} u_1 \cdots u_{i-1} \cdot s_B(u_i) \cdot u_{i+1} \cdots u_m, & u_i \in \{b \smile_1 b,\ b \smile_1 b \smile_1 c\} \text{ and} \\ & u_j \notin \{b \smile_1 b,\ b \smile_1 b \smile_1 c\}, 1 \leq j < i, \\ 0, & \text{otherwise.} \end{cases}$$

Then for each element $b \in B$ there is an integer $n(b) \geq 1$ such that $n(b)$th-iteration of the operator $s_B d + ds_B + Id : B \to B$ evaluated on b is zero, i.e., $(s_B d + ds_B + Id)^{(n(b))}(b) = 0$ as desired. $\square$

3. Proof of Theorem 1

Given the Hirsch algebra $(C^*(X), d_C, \{E_{p,q}\})$, there is an algebra isomorphism [14,15]

$$H^*(\Omega X) \approx H(BC^*(X), d_{BC}, \mu_E).$$

(We assume $C^*(X) = C^*(\mathrm{Sing}^1 X)/C^{>0}(\mathrm{Sing}\ x)$, in which $\mathrm{Sing}^1 X \subset \mathrm{Sing}\, X$ is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard n-simplex Δ^n to the base point x of X.)

Proposition 2. *A morphism $g : A \to A'$ of Hirsch algebras induces a Hopf dga map of the bar constructions*

$$Bg : BA \to BA'$$

and if g is a homology isomorphism, so is Bg.

Proof. The proof is standard by using a spectral sequence comparison argument. □

Denote $\bar{V}_\tau = s^{-1}(V_\tau^{>0}) \oplus \mathbb{Z}_2$ and define the differential $\bar{d}_h := \bar{d} + \bar{h}$ on $\bar{V}_\tau$ by the restriction of $d + h$ to V_τ to obtain the cochain complex $(\bar{V}_\tau, \bar{d}_h)$. Let $\psi : B(R_\tau H) \to \overline{R_\tau H} \to \bar{V}_\tau$ be the standard projection of cochain complexes. We introduce a product on $\bar{V}_\tau$ so that ψ becomes a map of dga's. Namely, for $\bar{a}, \bar{b} \in \bar{V}_\tau$ define

$$\bar{a}\bar{b} = \overline{a \smile_1 b} \quad \text{with} \quad \bar{a}1 = 1\bar{a} = \bar{a}.$$

Then we get the following sequence of algebra isomorphisms

$$H\big(BC^*(X), d_{BC}, \mu_E\big) \underset{\approx}{\overset{Bf^*}{\longleftarrow}} H\big(B(RH^*), d_{B(RH)}, \mu_E\big) \underset{\approx}{\overset{Bg_\tau^*}{\longrightarrow}} H\big(B(R_\tau H^*), d_{B(R_\tau H)}, \mu_{E_\tau}\big) \underset{\approx}{\overset{\psi^*}{\longrightarrow}} H\left(\bar{V}_\tau, \bar{d}_h\right),$$

where the first two isomorphisms are by Proposition 2, while the third isomorphism (additively) is a consequence of a general fact about tensor algebras [16] (see also [5]). Thus the calculation of the algebra $H^*(\Omega X)$ reduces to that of $H^*(\bar{V}_\tau, \bar{d}_h)$. In particular, $[\bar{x}_k] = \sigma(y_k) \in H^*(\Omega X)$. We have that $\bar{h}$ may be non-trivial only on a basis element of the form

$$s^{-1}(x_k{}^{\cup_2 q}) \ \text{and}\ s^{-1}(x_k{}^{\cup_2 q} \smile_1 a), \ \text{some}\ a \in V_\tau,\ q = 2^m,\ m \geq 1.$$

By definition $\bar{x}_k^q = s^{-1}(x_k^{\smile_1 q})$, $q = 2^m$, and taking into account (2.1), the cohomology algebra $H^*(\bar{V}_\tau, \bar{d}_h)$ is as desired.

Remark 1. (a) Refer to Example 4 from [7] and recall that there is a canonical Hirsch algebra structure $Sq = \{Sq_{p,q}\}$ on $H^*(X)$ determined by Sq_1. The isomorphism $H^*(\Omega X) \approx H^*(BH^*(X))$ from the introduction converts into an algebra one when $BH^*(X)$ is endowed with the product μ_{Sq}. Details are left to the interested reader.

(b) In $(\bar{V}_\tau, \bar{d}_h)$ the transgressive terms $\bar{h}^{tr} s^{-1}(x_i^{\cup_2 q})$ detect the Symmetric Massey products $\langle \sigma(y_i) \rangle^q \in H^*(\Omega X)$ for $q = 2^m$, $y_i \in H^*(X)$, or, in general, Stasheff's A_∞-algebra structure on $H^*(\Omega X)$ (cf. [17]). A question arises what else other than the action of Sq_1 on $H^*(X)$ is needed to calculate this structure.

Acknowledgments

This research described in this publication was made possible in part by the grant GNF/ST06/3-007 of the Georgian National Science Foundation.

References

[1] R.M. Switzer, Algebraic Topology –Homotopy and Homology, Springer-Verlag, 1975.
[2] R.E. Mosher, M.C. Tangora, Cohomology Operations and Applications in Homotopy Theory, Harper and Row Publ, 1968.
[3] J. McCleary, A User'S Guide To Spectral Sequences, second ed., Cambridge University Press, 2001.
[4] J.-P. Serre, Homologie singulière des espaces fibrés. Applications, Ann. of Math. 54 (1951) 425–505.
[5] D. Husemoller, J.C. Moore, J. Stasheff, Differential homological algebra and homogeneous spaces, J. Pure Appl. Algebra 5 (1974) 113–185.
[6] A. Prouté, Un contre-example à la géométricité du shuffle-coproduit de la cobar-construction, C. R. Acad. Sci. Paris I 298 (1984) 31–34.
[7] S. Saneblidze, Filtered Hirsch algebras, Trans. R.M.I. 170 (2016) 114–136.
[8] J. Tate, Homology of noetherian rings and local rings, Illinois J. Math. 1 (1957) 14–27.
[9] J.T. Jozefiak, Tate resolutions for commutative graded algebras over a local ring, Fund. Math. 74 (1972) 209–231.

[10] S. Halperin, J. Stasheff, Obstructions to homotopy equivalences, Adv. Math. 32 (1979) 233–279.
[11] D. Notbohm, in: I.M. James (Ed.), Classifying Spaces of Compact Lie Groups and Finite Loop Spaces, in: Handbook of Algebraic Topology, North-Holland, 1995 (Chapter 21).
[12] K.K.S. Andersen, J. Grodal, The steenrod problem of realizing polynomial cohomology rings, J. Topol. 1 (2008) 460–747.
[13] S. Saneblidze, On the homotopy classification of maps, J. Homotopy Relat. Struct. 4 (2009) 347–357.
[14] H.J. Baues, The cobar construction as a Hopf algebra, Invent. Math. 132 (1998) 467–489.
[15] T. Kadeishvili, S. Saneblidze, A cubical model of a fibration, J. Pure Appl. Algebra 196 (2005) 203–228.
[16] Y. Felix, S. Halperin, J.-C. Thomas, Adams' cobar equivalence, Trans. Amer. Math. Soc. 329 (1992) 531–549.
[17] T. Kadeishvili, On the homology theory of fibre spaces, Russian Math. Surveys 35 (1980) 131–138.

Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute 171 (2017) 396–410

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

Integral equations of the third kind for the case of piecewise monotone coefficients

D. Shulaia*

I. Vekua Institute of Applied Mathematics, Tbilisi State University, University St.2, 0186 Tbilisi, Georgia
Georgian Technical University, Georgia

Received 3 May 2017; accepted 15 May 2017
Available online 7 June 2017

Abstract

We examine the third kind integral equations in Hölder class. The coefficients of the equations are piecewise strictly monotone functions having simple zeros. By singular integral equations theory, for solvability of considered equations, we give the necessary and sufficient conditions. Finding a solution is reduced to solving a regular integral equation of second kind.

Keywords: Piecewise; Eigenvalues; Holomorphic; Singular operator

1. Introduction

The linear integral equation

$$A(x)\varphi(x) + \int_a^b K(x, y)\varphi(y)dy = f(x), \quad x \in]a, b[\tag{1}$$

where $A(x)$ has at least one zero is commonly called an equation of the third kind. Such equations acquire more and more significance in applied problems of mathematical physics. In particular, in kinetic theory, in transport theory, etc. (see [1]) and investigations in this area are of great interest. After the early works of Hilbert and also Picard there appeared a lot papers on equations of the third kind (see e.g. [2–5]). In this paper we present a method for solving Eq. (1) when the coefficient $A(x) \in C^1([a, b])$ is a piecewise strictly monotone function having simple zeros in]a, b[. Moreover, we assume that $A'(x) \in \mathbf{H}$ on $[a, b]$ the kernel $K \in \mathbf{H}$, on $[a, b] \times [a, b]$ and a free term $f \in \mathbf{H}^*$ (Muskhelishvili's class) [6]. Therefore we look for solutions $\varphi \in \mathbf{H}^*$ of this class to be more appropriate in certain applications. Our investigation is based on the spectral expansion ideas by Fridrichs [7] and Hilbert–Schmidt approach for the second kind self-adjoint equations. Methods of the theory singular integral operators are the basic methods for

* Correspondence to: I. Vekua Institute of Applied Mathematics, Tbilisi State University, University St.2, 0186 Tbilisi, Georgia.
E-mail address: dazshul@yahoo.com.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.05.002
2346-8092/

investigating [6]. We have applied this theory often to the similar type problems [8–18]. This paper is structured as follows: First, in Section 2, using the initial equation, we introduce integral operators and their corresponding integral equations which depend on the auxiliary parameter. Some of their properties, which will play an important role in further considerations, are investigated. The singular operator, which is connected to the introduced equation, is defined and its properties are studied in Section 3. In Section 4 the problem of reduction of the singular integral operator is studied. In Section 5, the Hilbert–Schmidt type expansion theorem assertion of an arbitrary function from $\mathbf{H}^*$ can be represented through the singular operator and the eigenfunctions depending on the parameter operator. In Section 6, analogous Hilbert–Schmidt theorems are proved depending on the parameter integral equation and main result is given for the initial equation. Without loss of generality, we assume that, $b > a$ in Eq. (1). This paper is in some sense a continuation of [8].

2. Preliminaries

Let $K(x, y)$ satisfy Hölder conditions on $[a, b] \times [a, b]$. Further assume that the function $g(z, x)$ defined in $(\mathbf{C} \setminus [m_A, M_A]) \times [a, b]$ where $m_A = \min A(x), \quad M_A = \max A(x), x \in [a, b]$ is holomorphic with respect to z and belongs to H with respect to x: Moreover $g(z, x)$ has boundary values

$$g^+(\zeta, x) = \lim_{z\to\zeta} g(z, x), \quad Rez > 0$$

and

$$g^-(\zeta, x) = \lim_{z\to\zeta} g(z, x), \quad Rez < 0, \quad \zeta \in [m_A, M_A].$$

Denote by $\mathbf{\Omega}$ operator $\mathbf{\Omega}: \quad g(z, x) \to (\mathbf{\Omega} g)(z, x)$,

$$(\mathbf{\Omega}_z g(z, \cdot))(z, x) := g(z, x) + \int_a^b \frac{K(x, y)}{A(y) - z} g(z, y)dy, \qquad x \in [a, b] \tag{2}$$

where z is an arbitrary complex number, $A(x) \in \mathbf{C}^1([a, b])$ is the piecewise strictly monotone real-valued function having simple zeros in $]a, b[$ and $A'(x) \in \mathbf{H}$. This operator operating on any function $g(z, x)$ piecewise holomorphic with respect to z with the cut on $[m_A, M_A]$ and satisfying the Holder condition with respect to x, will define with the cut on $[m_A, M_A]$ a piecewise holomorphic function.

Let $\zeta = A(x)$ be the piecewise strictly monotone function, we are able to partition the interval $]a, b[$ into subintervals $]c_{i-1}, c_i[$, $i = \overline{1, n}$, $c_0 = a$, $c_n = b$, such that in these subintervals the function $\zeta = A(x)$ will be strictly monotone and moreover $A'(c_i) = 0$, $i = \overline{1, n-1}$. Let $A_i^{-1}(\zeta)$ be an inverse function of $A(x)$ in subinterval $]c_{i-1}, c_i[$ i.e. $A_i^{-1}(A(x)) = x$ for $x \in [c_{i-1}, c_i]$, $i = \overline{1, n}$ and $A_i^{-1}(\zeta) \in [c_{i-1}, c_i]$.

Now, recall some properties of the Cauchy type integrals. In order to find the boundary values $(\Omega g)^{\pm}(\zeta, x)$ we apply the formulas analogous to formulas for the Cauchy type integrals [19].

Let

$$\Psi(z) = \frac{1}{2\pi i} \int_L \frac{P(\tau, z)}{Q(\tau, z)} d\tau$$

when P and Q are analytic functions with respect to z for all $\tau \in L$

1. P satisfies the Hölder condition with respect to τ
2. Q is the differentiability with respect to τ and $Q'_\tau(\tau, z) \in \mathbf{H}$
3. In the points when $Q(\tau, z) = 0$ we have $Q'_\tau(\tau, z) \neq 0$ and $Q'_z(\tau, z) \neq 0$.

Let $\varsigma = \psi(\tau)$ be solution of the equation $Q(\tau, z) = 0$ and $\tau = \omega(\varsigma)$ is its inverse, then write formulas

$$\Psi^{\pm}(w) = \pm\frac{1}{2} \frac{P(t, w)}{Q'_\tau(t, w)} + \frac{1}{2\pi i} \int_L \frac{P(\tau, w)}{Q(\tau, w)} d\tau$$

when $t = \omega(w)$.

Consequently, according to these formulas we can find the boundary values of Ω as

$$(\boldsymbol{\Omega}_\zeta^\pm g^\pm(\zeta,\cdot))(\zeta,x) = g^\pm(\zeta,x) + \int_a^b \frac{K(x,y)}{A(y)-\zeta} g^\pm(\zeta,y)dy$$
$$\pm\pi\mathbf{i}\sum_{i=1}^n \chi_i(\zeta)K(x,A_i^{-1}(\zeta))(A_i^{-1}(\zeta))' g^\pm(\zeta,A_i^{-1}(\zeta)) \tag{3}$$
$$\zeta\in]m_A,M_A[\setminus\{A(c_i)|i=\overline{0,n}\}\equiv E$$

where $\chi_i(\zeta)$ is characteristic function of $[A(c_{i-1}),A(c_i)]$, $\quad i=\overline{1,n}$.

Also, write the formula

$$\int_L \frac{d\tau}{Q(\tau,w)}\int_L \frac{P(\tau,\sigma)}{\omega(\tau,\sigma)}d\sigma = -\pi^2\frac{P(t,t)}{Q'_\tau(t,w)\omega'_\tau(t,t)} + \int_L d\sigma\int_L \frac{P(\tau,\sigma)}{Q(\tau,w)\omega(\tau,\sigma)}d\tau.$$

Here the function $\omega(\tau,\sigma)$ is continuous, differentiable with respect to τ, satisfying Hölder condition with respect to τ and σ, $\quad \omega(t,t)=0$, but $\omega'_\tau(t,t)\neq 0$.

Let $\aleph$ be a set of values of the z for which the equation

$$\boldsymbol{\Omega}_z g = 0 \tag{4}$$

has non-zero continuous solution. Such values are called eigenvalues of $\boldsymbol{\Omega}_z$. Because this operator's kernel is piecewise analytic in z, vanishing when $z\to\infty$, by Tamarkin's theorem [20] it follows that the set $\aleph$ is almost countable in a plane z with the cut on $[m_A,M_A]$. Note that set of eigenvalues of operators in the form (2) is finite when $K\in\mathbf{H}$ (see e.g. [21,22]).

Obviously: (i_1) Let g be the solution of Eq. (4), then

$$\tau_{z_k}(x) = \frac{g(z_k,x)}{A(x)-z_k} \qquad z_k\in\aleph$$

is the solution of the following equation

$$(A(x)-z)\tau_z(x) + \int_a^b K(x,y)\tau_z(y)dy = 0, \qquad x\in[a,b] \tag{5}$$

as $z=z_k$ and also vice versa.

(i_2) Let z be for $\boldsymbol{\Omega}_z$ the eigenvalue of the multiplicity r, then also z is the eigenvalue of the multiplicity r for

$$(\Omega_z^* q(z,\cdot))(z,x) := q(z,x) + \int_a^b \frac{K(y,x)}{A(y)-z}q(z,y)dy, \qquad x\in[a,b]$$

and also vice versa.

(i_3)

$$\int_a^b \tau_z^*(x)\tau_{z'}(x)dx = 0 \qquad z\neq z' \tag{6}$$

where $\tau_z^*(x)$ be the solution of equation

$$(A(x)-z)\tau_z^*(x) + \int_a^b K(y,x)\tau_z^*(y)dy = 0, \qquad x\in[a,b]. \tag{7}$$

Usually, $\tau_{z_k}(x)$ and $\tau_{z_k}^*(x)$, are called the eigenfunctions of $K(x,y)$ and $K(y,x)$ respectively, corresponding to the eigenvalue $z_k\in\aleph$.

Remark 2.1.

The functions and operators determined by the kernel $K(y,x)$ just in same way as by kernel $K(x,y)$ will be furthermore provided with superscript $*$.

Let

$$\omega(t,x) = \sum_{i=1}^n \vartheta_i(t)\vartheta_i(x), \qquad t,x\in[a,b]$$

where $\vartheta_i,\ i = \overline{1, n-1}$ are the characteristic functions of $[c_{i-1}, c_i[$ respectively and ϑ_n is the characteristic function of $[c_{n-1}, c_n]$. Now we have already introduced special integral equation

$$M(t,x) + \int_a^b \tilde{K}(t,x,y)M(t,y)dy = |A'(t)|K(x,t), \quad t,x \in [a,b] \tag{8}$$

where

$$\tilde{K}(t,x,y) = \sum_{i=1}^{n} \frac{K(x,y) - \chi_i(A(t))K(x,t^{(i)})}{A(y) - A(t)}\omega(t^{(i)}, y),$$

$t^{(i)} = A_i^{-1}(A(t))$ and t is the parameter.

The kernel $\tilde{K}$ of this equation does not belong to that type which as a rule is usually called regular. But, this equation can be reduced to a Fredholm equation (cf. [6, Chapter 14, Section 111]) and therefore to Eq. (8) that are applicable in all Fredholm theorems.

Theorem 2.2. *Let for the some value of parameter $t = t_1 \in [a,b]$, the homogeneous integral equation*

$$M_0(t,x) + \int_a^b \tilde{K}(t,x,y)M_0(t,y)dy = 0, \quad x \in [a,b] \tag{9}$$

have only a trivial solution. Then $z_1 = A(t_1) \notin \aleph$.

Proof. The proof is completely analogous to that of Theorem 1 in [11].

Consequently, if Eq. (9) admits only a trivial solution, then Eq. (8) will have the unique solution which satisfies condition **H** uniformly over t. This solution will also satisfy the condition **H** over t on any closed parts of $]a,b[\setminus\{c_i | i = \overline{1, n-1}\}$ uniformly with respect to x.

In order to eliminate additional arguments, in the sequel we shall assume that;

(j_1) $\aleph$ is the finite set.

(j_2) Eq. (9) and

$$M_0^*(t,x) + \int_a^b \tilde{K}^*(t,x,y)M_0^*(t,y)dy = 0, \quad t,x \in [a,b] \tag{10}$$

where

$$\tilde{K}^*(t,x,y) = \sum_{i=1}^{n} \frac{K(y,x) - \chi_i(A(t))K(t^{(i)},x)}{A(y) - A(t)}\omega(t^{(i)}, y)$$

have only trivial solution for any value of $t \in [a,b]$.

Note that for the sufficiently wide class of the kernels (see e.g. [18]) such conditions are fulfilled.

Consequently, we assume that both Eq. (8) and

$$M^*(t,x) + \int_a^b \tilde{K}^*(t,x,y)M^*(t,y)dy = |A'(t)|K^*(t,x), \quad t,x \in [a,b] \tag{11}$$

will have a unique solution satisfying the condition **H** uniformly with respect to t. These solutions will also satisfy the condition **H** with respect to t on any closed parts of $]a,b[\setminus\{c_i | i = \overline{1, n-1}\}$ uniformly with respect to x.

Before proceeding further with our investigation we will also need

Lemma 2.3. *There holds the equality*

$$\begin{aligned} &\int_a^b \frac{M^*(t_0,x)}{A(t_0) - A(x)} \int_a^b \frac{M(t,x)}{A(t) - A(x)} u(t)dtdx \\ &= \int_a^b \frac{u(t)}{A(t_0) - A(t)} \left(\int_a^b \frac{M^*(t_0,x)M(t,x)}{A(x) - A(t_0)}dx - \int_a^b \frac{M^*(t_0,x)M(t,x)}{A(x) - A(t)}dx \right) dt \\ &\quad + \pi^2 \sum_{i,j=1}^{n} \chi_j(A(t_0))Q^*(t_0, t_0^{(j)})\chi_i(A(t_0))Q(t_0^{(i)}, t_0^{(j)})u(t_0^{(i)}), \quad t_0 \in]a,b[\setminus\{c_i | i = \overline{1,n}\} \end{aligned} \tag{12}$$

where $u \in H^*$, and $Q(t,x)$, $\quad Q^*(t,x)$ defined from the following equations

$$Q(t,x) + \int_a^b \tilde{K}(t,x,y)Q(t,y)dy = K(x,t), \quad x,t \in [a,b]$$

and

$$Q^*(t,x) + \int_a^b \tilde{K}^*(t,x,y)Q^*(t,y)dy = K(t,x), \quad x,t \in [a,b]$$

respectively.

Proof. Indeed, we can write

$$\int_a^b \frac{M^*(t_0,x)}{A(t_0)-A(x)} \int_a^b \frac{M(t,x)}{A(t)-A(x)} u(t)dtdx$$
$$= \sum_{i,j=1}^n \int_{c_{i-1}}^{c_i} \int_{c_{j-1}}^{c_j} \frac{M^*(t_0,x)}{A(t_0)-A(x)} \frac{M(t,x)}{A(t)-A(x)} u(t)dtdx.$$

Then, having applied the analogous formula of the Poincare–Bertrand [19, Chapter 1, Section 9.3] to the every component of this sum we obtain

$$\int_{c_{i-1}}^{c_i} \int_{c_{j-1}}^{c_j} \frac{M^*(t_0,x)}{A(t_0)-A(x)} \frac{M(t,x)}{A(t)-A(x)} u(t)dtdx = \int_{c_{j-1}}^{c_j} \int_{c_{i-1}}^{c_i} \frac{M^*(t_0,x)}{A(t_0)-A(x)} \frac{M(t,x)}{A(t)-A(x)} u(t)dxdt$$
$$+ \pi^2 \chi_j(A(t_0)) Q^*(t_0, t_0^{(j)}) \chi_i(A(t_0)) Q(t_0^{(i)}, t_0^{(j)}) u(t_0^{(i)})$$

and the result follows.

It is seen that

$$M(t,x) = |A'(t)|Q(t,x) \;\; and \;\; M^*(t,x) = |A'(t)|Q^*(t,x).$$

3. Singular integral operator and its fundamental properties

The concept of complete kernel is necessary in the study of second-order linear equations depending on a parameter. In the theory of Hilbert and Schmidt for the second kind integral equations a set of eigenfunctions is assumed to be complete [23], and also an important role is played by one property of eigenfunctions, which corresponds here to the equality

$$(A(x)-z)\tau_{z_k}(x) + \int_a^b K(x,y)\tau_{z_k}(y)dy = (z_k - z)\tau_{z_k}(x). \tag{13}$$

But, a set of the eigenfunctions is not the complete system here. Therefore, based on the spectral expansion theory we consider the following integral operator

$$(\mathbf{L}u(\cdot))(x) := |A'(x)|u(x) + \sum_{i=1}^n \alpha_i(x)u(x^{(i)}) + \int_a^b \frac{M(t,x)}{A(t)-A(x)} u(t)dt \tag{14}$$

where

$$\alpha_i(x) = \int_a^b \frac{\chi_i(A(x))(x^{(i)})' M(x^{(i)},y)}{A(y)-A(x)} \omega(x,y)dy.$$

The singular integral operator $\mathbf{L}$ transforms the arbitrary function $u(t) \in \mathbf{H}^*$, $\quad t \in]a,b[$, into the new function $v(t) \in \mathbf{H}^*$, $\quad t \in]a,b[\setminus\{c_i | i = \overline{1,n}\}$ and if $u(t') = 0$, $\quad t' \in \{c_i | i = \overline{1,n-1}\}$, then $v(t)$ also will satisfy $\mathbf{H}$ condition in a neighborhood of the t' (cf. [6, Chapter 1, Section 20]).

Theorem 3.1. *The following equality is true*

$$(A(x)-z)(\mathbf{L}u(\cdot))(x) + \int_a^b K(x,y)(\mathbf{L}u(\cdot))(y)dy = (\mathbf{L}(A(\cdot)-z)u(\cdot))(x).$$

Proof. From (14) and (8) by simple calculation we have

$$\int_a^b K(x,y)(\mathbf{L}u(\cdot))(y)dy = \int_a^b M(t,x)u(t)dt.$$

Using (8) we deduce

$$(\mathbf{L}(A(\cdot)-A(x))u(\cdot))(x) = \int_a^b M(t,x)u(t)dt$$

and consequently the result yields.

The latter result gives us a motive to more thoroughly study the singular operator $\mathbf{L}$. Before beginning our systematic investigation we shall prove the

Lemma 3.2. *The algebraic equations system*

$$|A'(t^{(i)})|X_i^0 + \sum_{j=1}^{n} \alpha_j(t^{(i)})X_j^0 = 0, \quad i = \overline{1,n} \tag{15}$$

for all $t \in]a,b[\setminus\{c_i | i = \overline{1,n}\}$ *admits only the trivial solution.*

Proof. Let us assume the contrary. Suppose that for some $t = t_0 \in]a,b[\setminus\{c_i | i = \overline{1,n}\}$ this system has a non-zero solution. Then, from (15) and (8),

$$\tilde{M}(t_0,x) = \sum_{i=1}^{n} \chi_i(A(t_0))(t_0^{(i)})' M(t_0^{(i)},x)X_i^0(t_0)$$

will be the non-zero solution of Eq. (4) when $z = A(t_0) \in \aleph$, which contradicts with Theorem 2.2 .

Now, we are able to prove the main result concerning the operator $\mathbf{L}$.

Theorem 3.3. *Let* $\psi_0 \in \mathbf{H}^*$ *on* $]a,b[$. *For the singular integral equation*

$$\mathbf{L}u = \psi_0 \tag{16}$$

to have a solution in the class $\mathbf{H}^*$, *it is necessary and sufficient that the function* ψ_0 *satisfies the conditions*

$$\int_a^b \psi_0 \tau_{z_k}^* dx = 0, \quad z_k \in \aleph. \tag{17}$$

If are fulfilled these conditions, then the solution is unique.

Proof of the Necessity. Suppose that $u \in \mathbf{H}^*$ satisfies (16) and introduce into consideration the function

$$\Psi(z,x) = \frac{1}{2\pi\mathbf{i}} \int_a^b \frac{M(t,x)}{A(t)-z} u(t)dt \qquad x \in [a,b], \quad z \notin [m_A, M_A].$$

It is seen that this function has the properties:

(p_1) It is piecewise holomorphic with respect to z, in a plane with the cut $[m_A, M_A]$, while for x it satisfies the $\mathbf{H}$ condition.

(p_2) It tends to zero uniformly in x as $z \to \infty$.

(p_3) From the Plemelj formulas

$$\Psi^{\pm}(\zeta,x) = \frac{1}{2\pi\mathbf{i}} \int_a^b \frac{M(t,x)}{A(t)-\zeta} u(t)dt$$
$$\pm \frac{1}{2} \sum_{j=1}^{n} \chi_j(\zeta)(A_j^{-1}(\zeta))' M(A_j^{-1}(\zeta),x)u(A_j^{-1}(\zeta)),$$
$$\zeta \in E, \quad x \in [a,b].$$

Combining (3) with the latter equality we get

$$\begin{aligned}
&(\mathbf{\Omega}^{+}_{\zeta}\Psi^{+}(\zeta,\cdot))(\zeta,x)-(\mathbf{\Omega}^{-}_{\zeta}\Psi^{-}(\zeta,\cdot))(\zeta,x)\\
&\quad=\sum_{j=1}^{n}\Bigg(\chi_j(\zeta)(A_j^{-1}(\zeta))'M(A_j^{-1}(\zeta),x)u(A_j^{-1}(\zeta))\\
&\quad+\int_a^b\frac{K(x,y)}{A(y)-\zeta}\chi_j(\zeta)M(A_j^{-1}(\zeta),x)u(A_j^{-1}(\zeta))dy\\
&\quad+\chi_j(\zeta)K(x,A_j^{-1}(\zeta))\int_a^b\frac{M(t,A_j^{-1}(\zeta))}{A(t)-\zeta}u(t)dt\Bigg)
\end{aligned}$$

where $\zeta\in E$ and $x\in[a,b]$.

Recall that $M(t,x)$ satisfies (8), from (16) we obtain

$$\begin{aligned}
&(\mathbf{\Omega}^{+}_{\zeta}\Psi^{+})(\zeta,x)-(\mathbf{\Omega}^{-}_{\zeta}\Psi^{-})(\zeta,x)\\
&\quad=\sum_{j=1}^{n}\chi_j(\zeta)(A_j^{-1}(\zeta))'K(x,A_j^{-1}(\zeta))\psi_0(A_j^{-1}(\zeta))\\
&\quad\zeta\in E,\quad x\in[a,b].
\end{aligned}$$

Taking into account the Plemelj formula, we can also write

$$(\mathbf{\Omega}_z\Psi)(z,x)=\frac{1}{2\pi\mathbf{i}}\int_{m_A}^{M_A}\sum_{j=1}^{n}\frac{\chi_j(\zeta)(A_j^{-1}(\zeta))'K(x,A_j^{-1}(\zeta))\psi_0(A_j^{-1}(\zeta))}{\zeta-z}d\zeta$$

where $\mathbf{\Omega}_z$ is the operator (2). After transformation we conclude that the function $\Psi(z,x)$ satisfies the following integral equation

$$(\mathbf{\Omega}_z\Psi)(z,x)=\frac{1}{2\pi\mathbf{i}}\int_a^b\frac{K(x,t)}{A(t)-z}\psi_0(t)dt,\quad x\in[a,b]. \tag{18}$$

However the condition of the solubility of the integral equation (18) is that its free term be orthogonal to the eigenfunctions of the kernel $K(y,z)$. That is,

$$\int_a^b\tau^{*}_{z_k}(x)\int_a^b\frac{K(x,t)}{A(t)-z_k}\psi_0(t)dtdx=0,\qquad z_k\in\aleph.$$

By using (7) we immediately come to conditions (17).

Proof of the Sufficiency. Let $\psi_0\in H^{*}$ satisfy conditions (17). From Tamarkin's Theorem [20] we get; there is the unique solution of (18) and for this solution the following properties hold:

(r_1) With respect to z it is the piecewise holomorphic in the plane with a cut $[m_A,M_A]$, while for x satisfies **H** condition

(r_2) It tends to zero uniformly in x as $z\to\infty$

(r_3) It can be written as

$$\Psi(z,x)=\frac{1}{2\pi\mathbf{i}}\int_{m_A}^{M_A}\frac{\tilde{\kappa}(t,x)}{t-z}dt,\quad x\in[a,b],\quad z\notin[m_A,M_A]$$

where $\tilde{\kappa}(t,x)$ is the uniquely determined function.

By the Plemelj formulas for the boundary values the equality is true

$$\Psi^{+}(\zeta,x)+\Psi^{-}(\zeta,x)=\frac{1}{\pi\mathbf{i}}\int_{m_A}^{M_A}\frac{\Psi^{+}(t,x)-\Psi^{-}(t,x)}{t-\zeta}dt.$$

From (3) and (18) we obtain

$$\begin{aligned}&\tilde{\Psi}(\zeta,x)+\int_a^b \frac{K(x,y)}{A(y)-\zeta}\tilde{\Psi}(\zeta,y)dy\\ &\quad+\sum_{j=1}^n \chi_j(\zeta)K(x,A_j^{-1}(\zeta))\left(\int_{m_A}^{M_A}\frac{\tilde{\Psi}(t,A_i^{-1}(\zeta))}{t-\zeta}dt+\psi_0(A_j^{-1}(\zeta))\right)=0\\ &\quad\zeta\in E,\quad x\in[a,b]\end{aligned} \tag{19}$$

where

$$\tilde{\Psi}(\zeta,x)=\Psi^+(\zeta,x)-\Psi^-(\zeta,x).$$

Now consider the nonhomogeneous system of equations:

$$A'(x^{(i)})X_i+\sum_{j=1}^n \alpha_j(x^{(i)})X_j=\tilde{\psi}_0(x^{(i)}),\quad i=\overline{1,n}\quad x\in]a,b[\setminus\{c_k|k=\overline{1,n}\} \tag{20}$$

where $x^{(i)}=A_i^{-1}(A(x))$ and

$$\tilde{\psi}_0(x)=\int_{m_A}^{M_A}\frac{\tilde{\Psi}(t,x)}{t-A(x)}dt+\psi_0(x).$$

Lemma 3.2 shows us that there is a unique solution of (20). Moreover, $X_i(x)$ is a function of variables $x^{(i)}$, i.e. $X_i(x)=u(x^{(i)})$. In addition, $u(c_i)=0$ when $i=\overline{1,n-1}$.

Denote

$$\tilde{M}(\zeta,x)=\tilde{\Psi}(\zeta,x)-\sum_{j=1}^n \chi_j(\zeta)M(A_j^{-1}(\zeta),x)u(A_j^{-1}(\zeta)),\quad \zeta\in]m_A,M_A[.$$

By (8) and (20) and from (19) we conclude that $\tilde{M}(\zeta,x)$ is solution of Eq. (4) when $z=\zeta\in]m_A,M_A[$. On the other hand $\mathbf{\Omega}_z$ has no eigenvalues on $]m_A,M_A[$ and consequently,

$$\tilde{\Psi}(\zeta,x)=\sum_{j=1}^n \chi_j(\zeta)M(A_j^{-1}(\zeta),x)u(A_j^{-1}(\zeta)),\qquad x\in[a,b].$$

By this assertion, from (20), we can write

$$|A'(x)|u(x)+\sum_{j=1}^n \alpha_j(x)u(x^{(j)})=\psi_0(x)-\int_{m_A}^{M_A}\sum_{j=0}^n \frac{\chi_j(\zeta)M(A_j^{-1}(\zeta),x)u(A_j^{-1}(\zeta))}{\zeta-A(x)}d\zeta.$$

Consequently, (16) holds and proof is complete.

Remark 3.4. If $\psi_0\in\mathbf{H}^*$ on $]a,b[\setminus\{c_i|i=\overline{1,n}\}$ then result is also true and $u\in\mathbf{H}^*$ on $]a,b[\setminus\{c_i|i=\overline{1,n}\}$.

4. On the reduction of singular integral operator

Now a goal of ours is to study the singular operator $\mathbf{L}$ more deeply. To do this, we introduce, in the class $\mathbf{H}^*$, the following singular operators:

$$(\mathbf{S}v(\cdot))(x):=|A'(x)|v(x)+\sum_{j=1}^n \beta_j(x)v(x^{(j)})+\int_a^b \frac{M(x,t)}{A(x)-A(t)}v(t)dt,\quad x\in]a,b[\tag{21}$$

where

$$\beta_j(x)=\int_a^b \frac{\chi_j(A(x))M(x,y)}{A(y)-A(x)}\omega(x^{(j)},y)dy,\qquad v\in H^*$$

and operator $\mathbf{S}^*$ which is defined analogously on the $\mathbf{S}$ accordingly mentioned in Remark 2.1 to the rule

$$(\mathbf{S}^* v(\cdot))(x) := |A'(x)|v(x) + \sum_{j=1}^{n} \beta_j^*(x) v(x^{(j)}) + \int_a^b \frac{M^*(x,t)}{A(x)-A(t)} v(t)dt, \quad x \in]a,b[$$

where

$$\beta_j^*(x) = \int_a^b \frac{\chi_j(A(x)) M^*(x,y)}{A(y)-A(x)} \omega(x^{(j)}, y) dy.$$

It can be proved that $\alpha_i(x^{(j)}) = \beta_j(x^{(i)})$. Note that singular operator $\mathbf{S}$ can be rewritten as

$$(\mathbf{S}v(\cdot))(x) := |A'(x)|v(x) + \int_a^b \sum_{j=1}^{n} \frac{v(y) - \chi_j(A(y)) v(x^{(j)})}{A(x)-A(y)} \omega(x^{(j)}, y) M(x,y) dy.$$

Also the singular operator $\mathbf{S}^*$ can be represented similarly.

It is easy to see that for each two functions v and u from $\mathbf{H}^*$

$$\int_a^b u(x)(\mathbf{S}v(\cdot))(x)dx = \int_a^b v(x)(\mathbf{L}u(\cdot))(x)dx.$$

Hence, if there exists u such that (16) is fulfilled, then it is necessary that

$$\int_a^b v(x)\psi_0(x)dx = 0, \tag{22}$$

here v is solution of the equation

$$\mathbf{S}v = 0. \tag{23}$$

Also is true the converse statement.

To this end we formulate some properties of the introduced operators. From definition of $\mathbf{S}$ together with Eq. (8) as an immediate consequence

Lemma 4.1. *The following equality*

$$(\mathbf{S}K(x,\cdot))(t) = M(t,x), \qquad x,t \in [a,b],$$

is true.

A similar lemma is valid for the operator $\mathbf{S}^*$.

Lemma 4.2. *The following equality*

$$(\mathbf{S}^* K(\cdot,x))(t) = M^*(t,x), \qquad x,t \in [a,b],$$

is true.

It can be seen that

Lemma 4.3. *The following equality*

$$(\mathbf{S}^* M(x,\cdot))(t) = (\mathbf{S}M^*(t,\cdot))(x), \qquad x,\ t \in [a,b], \tag{24}$$

is true.

Proof. Really, from (8), by Lemma 4.2 we obtain

$$(\mathbf{S}^* M(x,\cdot))(t) = |A'(x)|M^*(t,x) + \int_a^b \sum_{i=1}^{n} \frac{M^*(t,y) - \chi(x^{(i)}) M^*(t,x^{(i)})}{A(x)-A(y)} \omega(x^{(i)}, y) M(x,y) dy$$
$$= (\mathbf{S}M^*(t,\cdot))(x).$$

Denote

$$\gamma_j(x) = |A'(x^{(j)})|\beta_j^*(x) + |A'(x)|\alpha_j(x) + \sum_{s=1}^{n}\big(\beta_s^*(x)\alpha_j(x^{(s)})$$
$$+ \pi^2 \sum_{s=1}^{n} \chi_s(A(x))Q^*(x, x^{(s)})\chi_j(A(x))Q(x^{(j)}, x^{(s)})\big).$$

We shall prove the following theorem

Theorem 4.4. *Composition* $\mathbf{S}^*\mathbf{L}$ *contains no singularity and there exists the following equality*

$$(\mathbf{S}^*\mathbf{L}u)(x) = (A'(x))^2u(x) + \sum_{j=1}^{n} \gamma_j(x)u(x^{(j)}). \tag{25}$$

Proof. After performing operations which are indicated on left-hand side of (25) and using the identity (12) for the repeated integration, we deduce

$$(\mathbf{S}^*\mathbf{L}u)(x) = (A'(x))^2u(x) + \sum_{j=1}^{n} \gamma_j(x)u(x^{(j)}) \int_a^b \frac{u(t)}{A(x) - A(t)}\big((\mathbf{S}^*M(t, \cdot))(x) - (\mathbf{S}M^*(x, \cdot))(t)\big)dt$$

and Lemma 4.3 completes proof.

Consequently, the operator $\mathbf{S}^*$ reduces to $\mathbf{L}$.

5. Expansion theorem

Now, we have to prove that any function from $\mathbf{H}^*$ can be expressed by eigenfunctions and singular integral operator $\mathbf{L}$. Note that this result plays the same role in the investigation of integral equations of the type (1) as the well known Hilbert–Schmidt expansion Theorem does in the theory of the self-adjoint Fredholm integral equations. To do this we prove some necessary statements.

Taking Lemma 4.3 into account, by substituting into (21) $t = x_0^{(i)},\ x = x_0^{(j)}$ we have

Corollary 5.1. *The equality*

$$|A'(x^{(i)})|M(x^{(j)}, x^{(i)}) + \sum_{s=1}^{n} \beta_s^*(x^{(i)})M(x^{(j)}, x^{(s)})$$
$$= |A'(x^{(j)})|M^*(x^{(i)}, x^{(j)}) + \sum_{s=1}^{n} \beta_s(x^{(j)})M^*(x^{(i)}, x^{(s)})$$
$$i, j = \overline{1, n}$$

is true (here, the x_0 *is replaced by the* x*).*

Now, we examine the following system of equations

$$(A'(x^{(i)}))^2X_i^0 + \sum_{j=1}^{n} \gamma_j(x^{(i)})X_j^0 = 0, \quad i = \overline{1, n}. \tag{26}$$

Before we proceed further, for the study of this system, it is convenient to use the matrix notation. To this end, we introduced the following matrices:

1. $\mathbf{A} = \|a_{ij}(x)\| \quad (i, j = \overline{1, n})$ is square matrix with elements:

$$a_{ij} = \delta_{i,j}|A'(x^{(i)})| + \alpha_j(x^{(i)})$$

where $\delta_{i,j}$ is the Kronecker symbol

2. $\mathbf{A}^* = \|a_{ij}^*(x)\|$ $(i, j = \overline{1, n})$ is the square matrix with the elements:

$$a_{ij}^* = \delta_{i,j}|A'(x^{(i)})| + \beta_i^*(x^{(j)})$$

3. $\mathbf{Q} = \|q_{ij}(x)\| \quad (i, j = \overline{1, n})$ with elements:

$q_{ij} = Q(x^{(j)}, x^{(i)})$

4. $\mathbf{Q}^* = \|q^*_{ij}(x)\| \quad (i, j = \overline{1, n})$ with elements:

$q^*_{ij} = Q^*(x^{(j)}, x^{(i)})$

5. $\mathbf{B} = \|b_{ij}(x)\| \quad (i, j = \overline{1, n})$ with the elements:

$b_{ij} = \delta_{i,j}(A'(x^{(i)}))^2 + \gamma_j(x^{(i)})$.

From Corollary 5.1, after simple calculation we get

$$\mathbf{A}^{*'}\mathbf{Q} = \mathbf{Q}^{*'}\mathbf{A}$$

(here, the sign $'$ denotes transposed).
From this fact it follows

Lemma 5.2. *The matrix* $\mathbf{B}$ *admits the following decomposition*

$$\mathbf{B} = (\mathbf{A}^{*'} + \mathbf{i}\pi\mathbf{Q}^{*'})(\mathbf{A} - \mathbf{i}\pi\mathbf{Q}).$$

From the Lemmas 5.2 and 3.2 *the following result can be derived.*

Lemma 5.3. *The following system of the equations* (26)

$$\boldsymbol{BX = 0}$$

has only zero solution and therefore

$$| \mathbf{B}(x) | := \det \mathbf{B} \neq 0.$$

Denote

$$(\mathbf{T}v(\cdot))(x) := \sum_{i,j=1}^{n} \frac{\vartheta_i(x)}{| \mathbf{B}(x) |} B_{ij}(x)(\mathbf{S}^*(v))(x^{(j)}) \qquad x \in]a, b[\tag{27}$$

where $B_{ij}(x)$ *is algebraic adjunct of* $b_{ji}(x)$ *in* $| \mathbf{B}(x) |$.

The operator $\mathbf{T}$ transforms any function $v(t) \in \mathbf{H}^*$ into a new function $u(t) \in \mathbf{H}^*$ on $]a, b[$. Moreover, $u(c_i) = 0$ when $i = \overline{1, n-1}$. For Theorem 4.4 we get

Theorem 5.4. *The singular operator* $\mathbf{T}$ *regularizes the operator* $\mathbf{L}$ *and the equality*

$$\mathbf{TL}u = u \tag{28}$$

is true.
Now we shall investigate a relationship between the eigenfunctions and the above introduced operators.

Theorem 5.5. *Eigenfunctions of kernel* $K(y, x)$ *satisfy the singular* Eq. (23)*, i.e.*

$$\mathbf{S}\tau^*_{z_k} = 0, \qquad z_k \in \aleph.$$

Proof. Taking (7) into account, from the definition (21) of S we obtain

$$\mathbf{S}\tau^*_{z_k} = |A'(x)|\tau^*_{z_k}(x) + \sum_{i=1}^{n} \beta_i(x)\tau^*_{z_k}(x^{(i)}) + \int_a^b \frac{M(x, t)}{A(t) - A(x)} \int_a^b \frac{K(y, t)\tau^*_{z_k}(y)}{A(t) - z_k} dydt.$$

Because

$$\frac{1}{A(t) - A(x)} \frac{1}{A(t) - z_k} = \left(\frac{1}{A(t) - A(x)} - \frac{1}{A(t) - z_k} \right) \frac{1}{A(x) - z_k},$$

we have that

$$\mathbf{S}\tau^*_{z_k} = |A'(x)|\tau^*_{z_k}(x) + \sum_{i=1}^{n} \beta_i(x)\tau^*_{z_k}(x^{(i)})$$
$$\int_a^b \frac{dt}{A(x)-z_k}\frac{M(x,t)}{A(t)-A(x)}\int_a^b K(y,t)\tau^*_{z_k}(y)dy - \int_a^b \frac{M(x,t)}{A(x)-z_k}\tau^*_{z_k}(t)dt$$

and from (8), we get

$$\int_a^b \frac{\tau^*_{z_k}(t)}{A(x)-z_k}M(x,t)dt = |A'(x)|\tau^*_{z_k}(x) + \sum_{i=1}^{n} \beta_i(x)\tau^*_{z_k}(x^{(i)})\int_a^b \frac{dt}{A(x)-z_k}\frac{M(x,t)}{A(t)-A(x)}\int_a^b K(y,t)\tau^*_{z_k}(y)dy.$$

The proof is complete.

Similarly we have

Theorem 5.6. *The equality*

$$\mathbf{S}^*\tau_{z_k} = 0, \qquad z_k \in \aleph$$

is true.

Corollary 5.7. *The equality*

$$\mathbf{T}\tau_{z_k} = 0, \qquad z_k \in \aleph$$

is true.

Now we shall prove one important property of the eigenfunctions.

Theorem 5.8. *Systems of the eigenfunctions $\{\tau_{z_k}\}$ and $\{\tau^*_{z_k}\}$ represent the biorthogonal system.*

Proof. Owing to equality (6), it remains for us to show that

$$N_{z_k} = \int_a^b \tau_{z_k}\tau^*_{z_k}dx, \quad z_k \in \aleph$$

are different from zero. Let us assume on the contrary that $N_{z_p} = 0$ is true for the some z_p. Then, τ_{z_p} satisfies conditions of Theorem 3.3 and therefore the singular integral equation

$$\mathbf{L}u = \tau_{z_p}$$

admits the unique solution. It follows from Theorem 5.4 and Corollary 5.7, that $u = 0$. Thus we obtain a contradiction and the theorem is proved.

Remark 5.9. This result implies:

(q_1) Solutions of Eq. (23) are only the eigenfunctions $\tau^*_{z_k}$, $z_k \in \aleph$, also their linear combination.

(q_2) The condition (22) also is sufficient for the solvability of (16).

The main result of this section is summarized in the following Hilbert–Schmidt type expansion theorem.

Theorem 5.10. *Let $\psi \in \mathbf{H}^*$, then*

$$\psi = \sum_k d_k\tau_{z_k} + \mathbf{L}u \tag{29}$$

where

$$d_k = \frac{1}{N_{z_k}}\int_a^b \psi\tau^*_{z_k}dx, \quad u = \mathbf{T}\psi.$$

Moreover, d_k and also u are defined uniquely.

Proof. In view of Theorem 5.8 it is evident that the following function

$$\psi_0 = \psi - \sum_k d_k \tau_{z_k}$$

where

$$d_k = \frac{1}{N_{z_k}} \int_a^b \psi \tau^*_{z_k} dx, \qquad z_k \in \aleph,$$

satisfies the conditions:

$$\int_a^b \psi_0 \tau^*_{z_k} dx = 0, \quad z_k \in \aleph.$$

By Theorem 3.3 this yields (29). The question of uniqueness of the d_k and u is obvious.

6. Main results

Now, from a comparison of the results, obtained in the preceding sections, with the foundations of the Hilbert–Schmidt approach from the theory of Fredholm integral equations of second kind we can solve the equation

$$(A(x) - z)\tilde{\varphi}_z(x) + \int_a^b K(x; y)\tilde{\varphi}_z(y)dy = f(x), \quad x \in]a, b[, \tag{30}$$

(cf. [8]).

Theorem 6.1. *Let $f \in \mathbf{H}^*$ and let $z \notin [m_A, M_A] \cup \aleph$. Then Eq.* (30)) *has one and only one solution $\tilde{\varphi}_z \in \mathbf{H}^*$ expressed by the formula*

$$\tilde{\varphi}_z(x) = \sum_k \frac{\tau_{z_k}(x)}{z_k - z} \frac{1}{N_{z_k}} \int_a^b f(y)\tau^*_{z_k}(y)dy + \left(\mathbf{L}\frac{1}{A(\cdot) - z}(\mathbf{T}f)(\cdot)\right)(x). \tag{31}$$

Proof. Let $\tilde{\varphi}_z \in \mathbf{H}^*$ be the solution of (30). By virtue of Theorem 5.10 this solution can be written in the form

$$\tilde{\varphi}_z = \sum_k \tilde{d}_k \tau_{z_k} + \mathbf{L}\tilde{u}. \tag{32}$$

To find the coefficients $\tilde{d}_k$ and the function $\tilde{u}$, we proceed in the following way. Putting (32) into Eq. (30) and using the relation (13) and Theorem 3.1, we get

$$\sum_k \tilde{d}_k (z_k - z)\tau_{z_k} + \mathbf{L}(A(\cdot) - z)\tilde{u}(\cdot) = f.$$

From this, by Theorem 5.10 we obtain

$$(z_k - z)\tilde{d}_k = \frac{1}{N_{z_k}} \int_a^b f \tau^*_{z_k} dy, \qquad z_k \in \aleph,$$

$$(A(t) - z)\tilde{u}(t) = (\mathbf{T}f)(t), \qquad t \in]a, b[.$$

Now, after replacing in (30) $\tilde{\varphi}_z$ by expression (31), direct calculation gives

$$\sum_k \tau_{z_k} \frac{1}{N_{z_k}} \int_a^b f \tau^*_{z_k} dy + \mathbf{LT}f = f.$$

But from Theorem 5.10 it follows, this last equality holds.

Theorem 6.2. *If $z = z_1 \in \aleph$ is an eigenvalue of the multiplicity r of the kernel K, then the solution of* Eq. (30) *exists only when the conditions*

$$\int_a^b f \tau^*_{z_k} dx = 0, \quad k \le r \tag{33}$$

are fulfilled. Then Eq. (30) *has in the class* $\mathbf{H}^*$ *solutions represented by the formula*

$$\tilde{\varphi}_{z_1}(x) = \sum_{k\le r} \tilde{d}_k \tau_{z_k}(x) + \sum_{k>r} \frac{\tau_{z_k}(x)}{z_k - z_1} \frac{1}{N_{z_k}} \int_a^b f(y)\tau^*_{z_k}(y)dy + \left(\mathbf{L}\frac{1}{A(\cdot) - z_1}(\mathbf{T}f)(\cdot)\right)(x) \tag{34}$$

where $\{\tilde{d}_k\}$ *are arbitrary constants.*

Proof. Assume that $\tilde{\varphi}_{z_1} \in \mathbf{H}^*$ is the solution of Eq. (30). Using the equality

$$(A(x) - z)\tau^*_{z_k}(x) + \int_a^b K(y, x,)\tau^*_{z_k}(y)dy = (z_k - z)\tau^*_{z_k}(x),$$

owing to (30) we get

$$(z_k - z_1)\int_a^b \tilde{\varphi}_{z_1}(x)\tau^*_{z_k}(x)dx = \int_a^b f(x)\tau^*_{z_k}(x)dx.$$

Since $z_k = z_1$ for $k \le r$, we have (33). We are now able to show that the following function

$$\tilde{\varphi}^0_{z_1} = \sum_{k\le r} \tilde{d}_k \tau_{z_k}$$

satisfies Eq. (30). Also, we have to prove that the following function

$$\bar{\varphi}_{z_1}(x) = \sum_{k>r} \frac{\tau_{z_k}(x)}{z_k - z_1} \frac{1}{N_{z_k}} \int_a^b f\tau^*_{z_k}dx + \left(\mathbf{L}\frac{1}{A(\cdot) - z_1}(\mathbf{T}f)(\cdot)\right)(z_1, x)$$

satisfies Eq. (30). Really, just as in Theorem 6.1 we find

$$\sum_{k>r} \varphi_{z_k} \frac{1}{N_{z_k}} \int_a^b f\tau^*_{z_k}dx + \mathbf{LT}f = f.$$

From Theorem 5.10 it follows that there holds latter equality

Theorem 6.3. *Let* $z = A(t_0)$ *where* $t_0 \in]a, b[\setminus\{c_i\}$. *In order that Eq.* (30) *be solvable in the class* $\mathbf{H}^*$*; it is necessary and sufficient that its free term* f *satisfies the conditions*

$$(\mathbf{T}f)(t_0^{(i)}) = 0, \quad i = \overline{1, n-1}. \tag{35}$$

Then the unique solution of Eq. (30) *may be represented by* (31).

Proof. As in Theorem 6.1, if a solution of (30) exists, then

$$A(t) - A(t_0)u(t) = (\mathbf{T}f)(t).$$

Therefore, in this case, when $t = t_0^{(i)}$ we have (35). Besides, if the conditions (35) are fulfilled, we are able to show that the function $\varphi_z \in \mathbf{H}^*$, defined by (31) where $z = A(t_0)$, satisfies Eq. (30).

Corollary 6.4. *Let* $z \notin \aleph$ *and let*

$$f(x) = \sum_k m_k \tau_{z_k}(x) \tag{36}$$

where m_k *is the arbitrary constant. Then Eq.* (30) *has one and only one solution which is expressed by the formula*

$$\tilde{\varphi}_z(x) = \sum_k \frac{m_k}{z_k - z}\tau_{z_k}(x).$$

Corollary 6.5. *Let* $z = z_1$ *be the eigenvalue of the multiplicity* r *and assume that* f *is of the form* (36). *Then* Eq. (30) *admits the solutions if and only if the following conditions* $m_k = 0, \quad k \le r$ *are fulfilled. When these latter conditions* *are fulfilled, then the solution of Eq.* (30) *may be given by the*

$$\tilde{\varphi}_{z_1}(x) = \sum_{k\le r} d_k \tau_{z_k}(x) + \sum_{k>r} \frac{m_k}{z_k - z_1}\tau_{z_k}(x)$$

where $\{d_k\}$ *are the arbitrary constants.*

A comparison of the obtained results, namely Theorems 3.1 and 5.10, with the foundations of the theory of Hilbert–Schmidt leads to a solution of the integral equation (1)

$$A(x)\varphi(x) + \int_a^b K(x, y)\varphi(y)dy = f(x), \quad x \in]a, b[.$$

Let $A(x) \in \mathbf{C}^1$ be the piecewise strictly monotone function having simple zeros x_s on $]a, b[\setminus\{c_i | i = \overline{1, n}\}$, $s = \overline{1, n_0}$, in addition we suppose that $A'(x) \in \mathbf{H}$ and $K(x, y) \in \mathbf{H}$ such that the assumptions (j_1) and (j_2) are fulfilled, $f(x) \in \mathbf{H}^*$.

Main Theorem.

Eq. (1) is solvable in the class H^* if and only if f satisfies the conditions

$$(\mathbf{T}f)(x_s) = 0, \qquad s = \overline{1, n_0}.$$

Provided these conditions are satisfied, Eq. (1) admits one and only one solution $\varphi(x) \in \mathbf{H}^*$ on $]a, b[$, moreover this solution may be written as

$$\varphi(x) = \sum_k \frac{\tau_{z_k}(x)}{z_k N_{z_k}} \int_a^b f(y)\tau^*_{z_k}(y)dy + \left(\mathbf{L}\frac{1}{A(\cdot)}(\mathbf{T}f)(\cdot)\right)(x)$$

where the operators L and T are defined by (14) and (27) respectively.

References

[1] J.H. Ferziger, H.G. Kaper, Mathematical Theory of Transport Processes in Gases, North-Holland Publishing Company, Amsterdam-London, 1972.
[2] N. Sukavanam, A Fredholm-type theory for third-kind linear integral equations, J. Math. Anal. Appl. 100 (1984) 478–485.
[3] E. Schok, Integral equations of the third kind, Studia Math. 81 (1985) 1–11.
[4] G.R. Bart, R.L. Warnock, Linear integral equations of the third kinds, SIAM J. Math. Anal. 4 (4) (1971). http://dx.doi.org/10.1137/0504053.
[5] N.S. Gabbasov, A special version of the collocation method for integral equations of the third kind, Differ. Equ. 41 (12) (2005) 1768–1774.
[6] N. Muskhelishvili, Singular Integral Equations, P.Noordhooff:Groningen, 1953.
[7] K.O. Friedrichs, Uber die Spectralzerlegung Eines Integral operators, Math. Ann. 115 (2) (1938) 249–300 (in German).
[8] D. Shulaia, Linear integral equations of the third kind arising from the neutron transport theory, Math. Methods Appl. Sci. 30 (15) (2007) 1941–1964.
[9] D. Shulaia, N. Sharashidze, A spectral representation of the multigroup transport problem, Transp. Theory Stat. Phys. 33 (2) (2004) 183–202.
[10] D. Shulaia, On one Fredholm integral equation of the third kind, Georgian Math. J. 4 (5) (1997) 461–476.
[11] D. Shulaia, Solution of a linear integral equation of third kind, Georgian Math. J. 9 (1) (2002) 179–196.
[12] D. Shulaia, On one two-dimensional linear integral equation with a coefficient that has zero, J. Math. Sci. 208 (6) (2015) 693–705.
[13] D. Shulaia, P. Ghurtskaia, On one linear integral equation with a coefficient that has zeros, J. Math. Sci. 208 (6) (2015) 706–722.
[14] D. Shulaia, On the expansion of solutions of equations of the linear multivelocity transport theory by eigenfunctions of the characteristic equation, Dokl. Akad. Nauk SSSR 310 (4) (1990) 844–849 (in Russian).
[15] D. Shulaia, Linear equation of the multivelocity transport theory, J. Vichislitelnoi Mat. Mat. Phisiki 23 (5) (1983) 1125–1140 (in Russian).
[16] D. Shulaia, Inverse problem of the linear multivelocity transport theory, Dokl. Akad. Nauk SSSR 270 (1) (1983) 82–87 (in Russian).
[17] D. Shulaia, Completness theorems in linear multivelocity transport theory, Dokl. Akad. Nauk SSSR 259 (2) (1981) 332–335 (in Russian).
[18] T. Germogenova, D. Shulaia, On the characteristic equation of the radiation transport theory, Dokl. Akad. Nauk SSSR 231 (4) (1976) 841–844 (in Russian).
[19] F. Gakhov, Boundary Value Problems, Nauka, Moscow, 1963 (in Russian).
[20] J. Tamarkin, On fredholm integral equations whose kernels are analytical in a parameter, Ann. of Math. 2 (28) (1927) 127–152.
[21] L.D. Faddeev, On friedrichs model in the theory of perturbations of a continuous spectrum, Tr. Mat. Inst. Steklova 73 (1964) 292–313 (in Russian).
[22] I.A. Feldman, On the discrete spectrum of the characteristic equation of the radiation transport theory, Mat. Issled. 10 N 1 (35) (1975) 236–242 (in Russian).
[23] F. Tricomi, Integral Equations, Interscience Publishers, New york, 1957.

Further reading

[1] M. Krasnov, Integral Equations, Nauka, Moscow, 1975 (in Russian).

Available online at www.sciencedirect.com
ScienceDirect

Transactions of A. Razmadze Mathematical Institute 171 (2017) 411–423

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

Application of Sinc-Galerkin method for solving a nonlinear inverse parabolic problem

A. Zakeri*, A.H. Salehi Shayegan, S. Sakaki

Faculty of Mathematics, K.N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran

Received 27 November 2016; received in revised form 27 April 2017; accepted 29 May 2017
Available online 22 June 2017

Abstract

In this paper, using Sinc-Galerkin and Levenberg–Marquardt methods a stable numerical solution is obtained to a nonlinear inverse parabolic problem. Due to this, this problem is reduced to a parameter approximation problem. To approximate unknown parameters, we consider an optimization problem where objective function is minimized by Levenberg–Marquardt method. This objective function is obtained by using Sinc-Galerkin method and the overposed measured data. Finally, some numerical examples are given to demonstrate the accuracy and reliability of the proposed method.

Keywords: Inverse problem; Sinc-Galerkin method; Nonlinear parabolic partial differential equation; Levenberg–Marquardt method

1. Introduction

Sinc methods have been increasingly used for finding a numerical solution of ordinary and partial differential equations [1–3]. The books [4,5] provide overviews of existing methods based on Sinc functions for solving ODEs, PDEs, and integral equations [1]. These methods have also been employed for some inverse problems [6–8].

There are many reasons that why these methods motivated authors to use them. First, the most important benefit of the Sinc methods is good accuracy that they make in the neighborhood of singularities [5,9]. Second, they are typified by exponentially decaying errors and in special cases by optimal convergence rate, even for problems over infinite and semi-infinite domains [5,9]. Finally, due to their rapid convergence rate, these methods do not suffer from the usual instability problems that typically occur in different methods [5,9,10].

* Corresponding author.
E-mail addresses: a.zakeri@kntu.ac.ir (A. Zakeri), ah.salehi@mail.kntu.ac.ir (A.H. Salehi Shayegan), sakakisaman@gmail.com (S. Sakaki).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.05.003
2346-8092/

The main aim of this paper is to use the Sinc-Galerkin method for solving a nonlinear inverse parabolic problem of the form

$$\begin{array}{ll} u_t - u_{xx} + H(u) = F(x,t), & 0 < x < 1,\ t > 0 \\ u(x,0) = \phi(x) & 0 \leqslant x \leqslant 1 \\ u(0,t) = p(t) & t \geqslant 0 \\ u(1,t) = q(t) & t \geqslant 0 \end{array} \tag{1.1}$$

where $H(u)$ is considered as follows

$$H(u) = u^n (n > 1), \sin(u), \cos(u), \exp(\pm u), \sinh(u), \cosh(u),$$

or any analytic function of u that has a power series expansion. In the above problem, $F(x,t)$, $\phi(x)$ and $q(t)$ are known analytic functions in an open interval $0 < x < 1$, $t > 0$ and may be singular in 0 or 1 or both, and the analytic functions $p(t)$ and $u(x,t)$ are unknown. If $p = p(t)$ is given, then the problem (1.1) is called *direct problem* (DP). The existence and uniqueness of DP (1.1) have been widely investigated in [11–14].

To find the pair (u, p), we use the overposed measured data

$$u(x^*, t) = E(t), \quad 0 < x^* < 1. \tag{1.2}$$

Let us denote by the notation $u[x, t; p]$ the solution of the DP (1.1). Then from the additional condition (1.2) it is seen that the nonlinear inverse parabolic problem (1.1) consists of solving the following nonlinear functional equation

$$u[x^*, t; p] = E(t), \quad 0 < x^* < 1. \tag{1.3}$$

In general, instead of solving the functional equation (1.3), an optimization problem is solved, where objective function is minimized by an effective regularization method. This objective function is defined as

$$S(p) = \sum_{i=1}^{I} (u[x^*, t_i; p] - E(t_i))^2. \tag{1.4}$$

In this paper, we attempt to obtain an approximate solution for the unknown function $p(t)$. For this purpose, first let

$$\bar{p}(t) \simeq \sum_{i=1}^{n} p_i Sinc\left(\frac{t - ih}{h}\right),$$

be a linear combination of Sinc functions, where h is the step size of time and p_i's are unknown parameters that should be derived. Then, the Sinc-Galerkin Method is used to obtain the approximate solution $u_{m_x,m_t}[x, t, \bar{p}]$ of the problem (1.1) with $\bar{p}(t)$ instead of $p(t)$. In other words, the problem (1.1) is reduced to a parameter approximation problem. These parameters are determined by minimizing the objective function (1.4) such that $u[x^*, t_i; p]$ is replaced by $u_{m_x,m_t}[x, t, \bar{p}]$. Due to this the Levenberg–Marquardt method is used. This method is a Newton-type method for nonlinear least-squares problem that is treated in many numerical optimization text books, e.g. [15]. The Levenberg–Marquardt method has also been successfully applied to the solution of linear problems that are too ill-conditioned to permit the application of linear algorithms [16,17].

The paper is organized as follows. Section 2 is devoted to the basic formulation of the Sinc function required for our subsequent development. In Section 3, the computational algorithm based on the Sinc-Galerkin method and the Levenberg–Marquardt method is provided and sensitivity matrix is obtained. Finally, in Section 4 some numerical examples are given and shown the efficiency and accuracy of the proposed numerical scheme.

2. Sinc function properties

In this section using the notations of [2,3,5,10], an overview of the basic formulation of the Sinc function is presented.

The Sinc function is defined on the whole real line $-\infty < x < \infty$ by

$$Sinc(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

For $h_x > 0$ and $h_t > 0$, the translated Sinc functions with evenly spaced nodes for space and time variables are given as

$$S(k, h_x)(z) = Sinc\left(\frac{z - kh_x}{h_x}\right), \quad k = 0, \pm 1, \pm 2, \ldots$$
$$S^*(k, h_t)(z) = Sinc\left(\frac{z - kh_t}{h_t}\right), \quad k = 0, \pm 1, \pm 2, \ldots.$$

To construct approximations on the interval $(0, 1)$, which is used in this paper, the eye-shaped domain in the z-plane, $D_E = \left\{z = x + iy : \left|\arg\left(\frac{z}{1-z}\right)\right| < d \leqslant \frac{\pi}{2}\right\}$ is mapped conformally onto the infinite strip $D_S = \left\{w = t + is : \ |s| < d \leqslant \frac{\pi}{2}\right\}$ with

$$w = \varphi(z) = \ln\left(\frac{z}{1 - z}\right).$$

The composition

$$S_j(x) = S(j, h_x)o\varphi(x) = Sinc\left(\frac{\varphi(x) - jh_x}{h_x}\right),$$

defines the basis element on the interval $(0, 1)$, where h_x is the mesh size in D_S for the uniform grids kh_x, $-\infty < k < \infty$. The inverse map of $w = \varphi(z)$ is

$$z = \varphi^{-1}(w) = \psi(w) = \frac{\exp(w)}{\exp(w) + 1}.$$

Thus, the inverse images of the equispaced grids are $x_k = \psi(kh_x)$, $k = 0, \pm 1, \pm 2, \ldots$.

For the temporal space [5], we define the function $\Upsilon(t) = \ln(t)$ which is a conformal mapping from $D_W = \left\{t = r + is : \ |\arg(t)| < d \leqslant \frac{\pi}{2}\right\}$ the wedge-shaped temporal domain onto D_S. The basis element on the interval $(0, \infty)$ are derived from the composite translated Sinc functions

$$S_j{}^*(t) = S^*(j, h_t)o\Upsilon(t) = Sinc\left(\frac{\Upsilon(t) - jh_t}{h_t}\right).$$

The mesh size h_t is the mesh size in D_S for the uniform grids kh_t, $-\infty < k < \infty$. The inverse map $\Upsilon^{-1}(t)$ is $\exp(t)$. Thus, the inverse images of the equispaced grids are $t_k = \exp(kh_t)$, $k = 0, \pm 1, \pm 2, \ldots$.

3. Sinc-Galerkin solution of the nonlinear inverse parabolic problem

3.1. The direct problem

An approximate solution of DP (1.1) is considered by

$$\hat{u}_{m_x,m_t}(x, t) = \sum_{i=-M_x-1}^{N_x+1} \sum_{j=-M_t-1}^{N_t} u_{i,j} X_i(x) \Theta_j(t), \tag{3.1}$$

where $m_x = M_x + N_x + 1$, $m_t = M_t + N_t + 1$,

$$X_i(x) = \begin{cases} 1 - x & i = -M_x - 1 \\ S(i, h_x)o\varphi(x) & -M_x \leqslant i \leqslant N_x \\ x & i = N_x + 1 \end{cases}$$

and

$$\Theta_j(t) = \begin{cases} \dfrac{t + 1}{t^2 + 1} & j = -M_t - 1 \\ S^*(j, h_t)o\Upsilon(t) & -M_t \leqslant j \leqslant N_t. \end{cases}$$

Using the boundary and initial conditions in DP (1.1), we have

$$\hat{u}_{m_x,m_t}(0,t) = \sum_{i=-M_t-1}^{N_t} u_{-M_x-1,j}\,\Theta_j(t) = p(t),$$

$$\hat{u}_{m_x,m_t}(1,t) = \sum_{i=-M_t-1}^{N_t} u_{N_x+1,j}\,\Theta_j(t) = q(t),$$

$$\hat{u}_{m_x,m_t}(x,0) = \sum_{i=-M_x-1}^{N_x+1} u_{i,-M_t-1} X_i(x) = \phi(x).$$

Thus, we can write the approximate solution (3.1) based on Sinc basis functions as

$$\hat{u}_{m_x,m_t}(x,t) = \sum_{i=-M_x}^{N_x} \sum_{j=-M_t}^{N_t} u_{i,j} S_i(x) S_j^*(t) + p^*(t) X_{-M_x-1}(x) + q^*(t) X_{N_x+1}(x) + \phi(x)\Theta_{-M_t-1}(t),$$

where

$$p^*(t) = p(t) - \phi(0)\Theta_{-M_t-1}(t),$$

and

$$q^*(t) = q(t) - \phi(1)\Theta_{-M_t-1}(t).$$

The unknown coefficients $u_{i,j}$, $i = -M_x, \ldots, N_x, \quad j = -M_t, \ldots, N_t$ are determined by orthogonalizing the residual with respect to the functions $S_{k,l}$, i.e.,

$$\left(L\hat{u}_{m_x,m_t} - F, S_{k,\ell}\right) = 0, \quad -M_x \leqslant k \leqslant N_x, \quad -M_t \leqslant l \leqslant N_t,$$

where $Lu \equiv u_t - u_{xx} + H(u)$ and

$$S_{k,l} = S_k(x) S_l^*(t) = \left(S(k,h_x) o \varphi(x)\right)\left(S(l,h_t) o \Upsilon(t)\right).$$

The weighted inner product here is defined by

$$(f,g) = \int_0^\infty \int_0^1 f(x,t) g(x,t) w(x) \tau(t)\, dx\, dt,$$

where $w(x)\tau(t)$ is a product weight function. This orthogonalization may be written

$$\left(Lu_{m_x,m_t} - F^*, S_{k,\ell}\right) = 0, \quad -M_x \leqslant k \leqslant N_x, \quad -M_t \leqslant l \leqslant N_t,$$

in which the homogeneous part of the approximate solution is given by

$$u_{m_x,m_t}(x,t) = \sum_{i=-M_x}^{N_x} \sum_{j=-M_t}^{N_t} u_{i,j} S_i(x) S_j^*(t), \tag{3.2}$$

and

$$\begin{aligned} F^*(x,t) = F(x,t) &- \frac{\partial}{\partial t}\left(p^*(t) X_{-M_x-1}(x) + q^*(t) X_{N_x+1}(x) + \phi(x)\Theta_{-M_t-1}(t)\right) \\ &+ \frac{\partial^2}{\partial x^2}\left(p^*(t) X_{-M_x-1}(x) + q^*(t) X_{N_x+1}(x) + \phi(x)\Theta_{-M_t-1}(t)\right) \\ &- H\left(p^*(t) X_{-M_x-1}(x) + q^*(t) X_{N_x+1}(x) + \phi(x)\Theta_{-M_t-1}(t)\right), \end{aligned}$$

for more details, one can refer to [4, Page 244].

An alternative approach is to analyze instead

$$\left(\left(u_{m_x,m_t}\right)_t, S_{k,l}\right) - \left(\left(u_{m_x,m_t}\right)_{xx}, S_{k,l}\right) + \left(H\left(u_{m_x,m_t}\right), S_{k,l}\right) = \left(F^*, S_{k,l}\right). \tag{3.3}$$

The method of approximating the integrals in (3.3) begins by integrating by parts to transfer all derivatives from u_{m_x,m_t} to $S_{k,l}$. Thus, we have

$$\begin{aligned}\left(\left(u_{m_x,m_t}\right)_t, S_{k,l}\right) &= \int_0^\infty \int_0^1 \frac{\partial}{\partial t}\left(u_{m_x,m_t}(x,t)\right) S_k(x)w(x)S_l^*(t)\tau(t)dxdt,\\ &= BT_1 - \int_0^\infty \int_0^1 u_{m_x,m_t}(x,t)\frac{\partial}{\partial t}\left(S_l^*(t)\tau(t)\right) S_k(x)w(x)dxdt,\end{aligned}$$

and

$$\begin{aligned}\left(\left(u_{m_x,m_t}\right)_{xx}, S_{k,l}\right) &= \int_0^\infty \int_0^1 \frac{\partial^2}{\partial x^2}\left(u_{m_x,m_t}(x,t)\right) S_k(x)w(x)S_l^*(t)\tau(t)dxdt,\\ &= BT_2 + \int_0^\infty \int_0^1 u_{m_x,m_t}(x,t)\frac{\partial^2}{\partial x^2}\left(S_k(x)w(x)\right) S_l^*(t)\tau(t)dxdt,\end{aligned}$$

where

$$BT_1 = \int_0^1 S_k(x)w(x)\left(u_{m_x,m_t}(x,t)S_l^*(t)\tau(t)\right)\Big|_0^\infty dx,$$

and

$$\begin{aligned}BT_2 = &\int_0^\infty S_l^*(t)\tau(t)\left(\frac{\partial}{\partial x}\left(u_{m_x,m_t}(x,t)\right) S_k(x)w(x)\right)\Big|_0^1 dt\\ &- \int_0^\infty S_l^*(t)\tau(t)\left(u_{m_x,m_t}(x,t)\frac{\partial}{\partial x}\left(S_k(x)w(x)\right)\right)\Big|_0^1 dt.\end{aligned}$$

So, we can write

$$\begin{aligned}&\int_0^\infty \int_0^1 u_{m_x,m_t}(x,t)\left(-\frac{\partial}{\partial t}\left(S_l^*(t)\tau(t)\right) S_k(x)w(x) + \frac{\partial^2}{\partial x^2}\left(S_k(x)w(x)\right) S_l^*(t)\tau(t)\right) dxdt\\ &\quad + BT + \int_0^\infty \int_0^1 H\left(u_{m_x,m_t}(x,t)\right) S_k(x)S_l^*(t)w(x)\tau(t)dxdt\\ &= \int_0^\infty \int_0^1 F^*(x,t)S_k(x)S_l^*(t)w(x)\tau(t)dxdt,\end{aligned}$$

where $BT = BT_1 + BT_2$. The weight functions $w(x)$ and $\tau(t)$ are defined in the following forms

$$w(x) = \frac{1}{\sqrt{\varphi'(x)}}, \qquad \tau(t) = \sqrt{\Upsilon'(t)}.$$

These weight functions cause $BT = 0$.

For approximating the above double integrals, we use the Sinc quadrature rule that is given in the following theorem.

Theorem 3.1 *([18,19]). For each fixed t, let $G(z,t) \in B(D_E)$ and $h > 0$. Let φ and Υ be one-to-one conformal maps of the domains D_E and D_W onto D_S, respectively. Let $z_i = \varphi^{-1}(ih_z)$, $t_j = \Upsilon^{-1}(jh_t)$ and $\Gamma_z = \varphi^{-1}(\mathbb{R})$, $\Gamma_t = \Upsilon^{-1}(\mathbb{R})$. Assume there are positive constants α_z, β_z, and $C_z(t)$ such that*

$$\left|\frac{G(z,t)}{\varphi'(z)}\right| \leqslant C_z(t). \begin{cases}\exp(-\alpha_z\,|\varphi(z)|), & z \in \Gamma_a^{(z)},\\ \exp(-\beta_z\,|\varphi(z)|), & z \in \Gamma_b^{(z)},\end{cases}$$

where $\Gamma_a^{(z)} \equiv \{z \in \Gamma_z :\ \varphi(z) = u \in (-\infty, 0)\}$, $\Gamma_b^{(z)} \equiv \{z \in \Gamma_z :\ \varphi(z) = u \in [0,\infty)\}$. Also for each fixed z, let $G(z,t) \in B(D_W)$ and assume there are positive constants α_t, β_t, and $C_t(z)$ such that

$$\left|\frac{G(z,t)}{\Upsilon'(z)}\right| \leqslant C_t(z). \begin{cases}\exp(-\alpha_t\,|\Upsilon(t)|), & t \in \Gamma_a^{(t)},\\ \exp(-\beta_t\,|\Upsilon(t)|), & t \in \Gamma_b^{(t)},\end{cases}$$

where $\Gamma_a^{(t)} \equiv \{t \in \Gamma_t : \Upsilon(t) = u \in (-\infty, 0)\}$, $\Gamma_b^{(t)} \equiv \{t \in \Gamma_t : \Upsilon(t) = u \in [0, \infty)\}$. *Then the Sinc trapezoidal quadrature rule is*

$$\int_{\Gamma_t}\int_{\Gamma_z} G(z,t)dzdt = h_z h_t \sum_{i=-M_z}^{N_z}\sum_{j=-M_t}^{N_t} \frac{G(z_i,t_j)}{\varphi'(z_i)\Upsilon'(t_j)} + O(\exp(-\alpha_z M_z h_z)) + O(\exp(-\beta_z N_z h_z))$$
$$+\ O(\exp(-2\pi d/h_z)) + O(\exp(-\alpha_t M_t h_t)) + O(\exp(-\beta_t N_t h_t)) + O(\exp(-2\pi d/h_t)).$$

Hence, make the selections

$$N_z = \left[\left|\frac{\alpha_z}{\beta_z}M_z + 1\right|\right], \qquad M_t = \left[\left|\frac{\alpha_z}{\alpha_t}M_z + 1\right|\right], \qquad N_t = \left[\left|\frac{\alpha_z}{\beta_t}M_z + 1\right|\right]$$

where $h \equiv h_z = h_t$, *and*

$$h = \sqrt{2\pi d/(\alpha_z M_z)},$$

and the exponential order of the Sinc trapezoidal quadrature rule is $O(\exp(-\sqrt{2\pi d \alpha_z M_z}))$.

Applying the Sinc quadrature rule that is defined in Theorem 3.1 yields

$$-\int_0^\infty\int_0^1 u_{m_x,m_t}(x,t)S_k(x)w(x)\frac{\partial}{\partial t}\left(S_l^*(t)\tau(t)\right)dxdt$$
$$\simeq -h_x h_t \sum_{p=-M_x}^{N_x}\sum_{q=-M_t}^{N_t} \frac{u_{m_x,m_t}(x_p,t_q)S_k(x_p)w(x_p)\frac{\partial}{\partial t}(S_l^*(t)\tau(t))\,\big|_{t=t_q}}{\varphi'(x_p)\Upsilon'(t_q)}$$
$$\simeq -h_x h_t \sum_{p=-M_x}^{N_x}\sum_{q=-M_t}^{N_t} \frac{u_{p,q}S_k(x_p)w(x_p)\frac{\partial}{\partial t}(S_l^*(t)\tau(t))\big|_{t=t_q}}{\varphi'(x_p)\Upsilon'(t_q)},$$

where $x_p = \varphi^{-1}(ph_x)$ and $t_q = \Upsilon^{-1}(qh_t)$. Also, we have

$$\int_0^\infty\int_0^1 u_{m_x,m_t}(x,t)\frac{\partial^2}{\partial x^2}\left(S_k(x)w(x)\right)S_l^*(t)\tau(t)dxdt$$
$$\simeq h_x h_t \sum_{p=-M_x}^{N_x}\sum_{q=-M_t}^{N_t} \frac{u_{m_x,m_t}(x_p,t_q)S_l^*(t_q)\tau(t_q)\frac{\partial^2}{\partial x^2}(S_k(x)w(x))\,\big|_{x=x_p}}{\varphi'(x_p)\Upsilon'(t_q)}$$
$$\simeq h_x h_t \sum_{p=-M_x}^{N_x}\sum_{q=-M_t}^{N_t} \frac{u_{p,q}S_l^*(t_q)\tau(t_q)\frac{\partial^2}{\partial x^2}(S_k(x)w(x))\,\big|_{x=x_p}}{\varphi'(x_p)\Upsilon'(t_q)},$$

$$\int_0^\infty\int_0^1 H\left(u_{m_x,m_t}(x,t)\right)S_k(x)S_l^*(t)w(x)\tau(t)dxdt$$
$$\simeq h_x h_t \sum_{p=-M_x}^{N_x}\sum_{q=-M_t}^{N_t} \frac{H(u_{m_x,m_t}(x_p,t_q))S_k(x_p)S_l^*(t_q)w(x_p)\tau(t_q)}{\varphi'(x_p)\Upsilon'(t_q)}$$
$$\simeq h_x h_t \sum_{p=-M_x}^{N_x}\sum_{q=-M_t}^{N_t} \frac{H(u_{p,q})S_k(x_p)S_l^*(t_q)w(x_p)\tau(t_q)}{\varphi'(x_p)\Upsilon'(t_q)},$$

and

$$\int_0^\infty\int_0^1 F^*(x,t)S_k(x)S_l^*(t)w(x)\tau(t)dxdt \simeq h_x h_t \sum_{p=-M_x}^{N_x}\sum_{q=-M_t}^{N_t} \frac{F^*(x_p,t_q)S_k(x_p)S_l^*(t_q)w(x_p)\tau(t_q)}{\varphi'(x_p)\Upsilon'(t_q)}.$$

The Sinc-Galerkin method actually requires the evaluated derivatives of Sinc basis functions $S(i,h_x)o\varphi(x)$ and $S^*(j,h_t)o\Upsilon(t)$ at the Sinc nodes $x = x_k$ and $t = t_k$, respectively. The p-th derivative of $S(i,h_x)o\varphi(x)$, with respect to

φ, evaluated at the nodal point x_k is denoted by

$$\delta^{(p)}_{i,k} \equiv h^p \frac{d^p}{d\varphi^p}\left[S(i,h)o\varphi(x)\right]\Big|_{x=x_k}. \tag{3.4}$$

Theorem 3.2 (*[4,5]*)**.** *Let φ be a conformal one-to-one map of the simply connected domain D_E onto D_S then*

$$\delta^{(0)}_{i,k} = \left[S(i,h)o\varphi(x)\right]\Big|_{x=x_k} = \begin{cases} 1, & k=i \\ 0, & k\neq i, \end{cases}$$

$$\delta^{(1)}_{i,k} = h\frac{d}{d\varphi}\left[S(i,h)o\varphi(x)\right]\Big|_{x=x_k} = \begin{cases} 0, & k=i \\ \dfrac{(-1)^{(k-i)}}{(k-i)}, & k\neq i \end{cases}$$

and

$$\delta^{(2)}_{i,k} = h^2\frac{d^2}{d\varphi^2}\left[S(i,h)o\varphi(x)\right]\Big|_{x=x_k} = \begin{cases} \dfrac{-\pi^2}{3}, & k=i \\ \dfrac{-2(-1)^{(k-i)}}{(k-i)^2}, & k\neq i. \end{cases}$$

Proof. See [4,5]. □

We note that, the similar formula as (3.4) and similar theorem as above for $S^*(j,h_t)o\Upsilon(t)$ are satisfied.
Define the $m\times m$ matrices $I^{(p)}_m$ for $0\le p\le 2$ by

$$I^{(0)}_m = \left[\delta^{(0)}_{i,k}\right] = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix} = I,$$

$$I^{(1)}_m = \left[\delta^{(1)}_{i,k}\right] = \begin{pmatrix} 0 & -1 & \frac{1}{2} & \dots & \frac{(-1)^{m-1}}{m-1} \\ -1 & & & & \vdots \\ \frac{1}{2} & & \ddots & & \frac{1}{2} \\ \vdots & & & & -1 \\ \frac{(-1)^{m-1}}{m-1} & \dots & -\frac{1}{2} & 1 & 0 \end{pmatrix},$$

and

$$I^{(2)}_m = \left[\delta^{(2)}_{i,k}\right] = \begin{pmatrix} \frac{-\pi^2}{3} & 2 & \frac{-2}{2^2} & \dots & \frac{-2(-1)^{m-1}}{(m-1)^2} \\ 2 & & & & \vdots \\ \frac{-2}{2^2} & & \ddots & & \frac{-2}{2^2} \\ \vdots & & & & 2 \\ \frac{-2(-1)^{m-1}}{(m-1)^2} & \dots & \frac{-2}{2^2} & 2 & \frac{-\pi^2}{3}, \end{pmatrix}.$$

The above matrices are the $m\times m$ Toeplitz matrices (see [3]). Then the discrete system can be represented in the following matrix form

$$A_x V + V B_t^T + H = G, \tag{3.5}$$

where

$$V = D(w)\bar{U}\,D^*\left(\frac{\tau}{\sqrt{\Upsilon'}}\right),$$

$$H = D(w)\bar{H}\, D^*\left(\frac{\tau}{\sqrt{\Upsilon'}}\right),$$
$$G = D(w)\bar{F}\, D^*\left(\frac{\tau}{\sqrt{\Upsilon'}}\right),$$
$$A_x = D\left(\varphi'(x)\right)\left[-\frac{1}{h_x^2}I_{m_x}^{(2)} + D\left(\frac{-1}{\left(\varphi'_x\right)^{\frac{3}{2}}}\left(\frac{1}{\sqrt{\varphi'(x)}}\right)''\right)\right]D\left(\varphi'(x)\right),$$
$$B_t = D^*\left(\sqrt{\Upsilon'(t)}\right)\left[-\frac{1}{h_t}I_{m_t}^{(1)} - D^*\left(\frac{-\tau'}{\tau\,\Upsilon'(t)}\right)\right]D^*\left(\sqrt{\Upsilon'(t)}\right),$$

and

$$D(g) = \begin{pmatrix} g(x_{-M_x}) & & 0 \\ & \ddots & \\ 0 & & g(x_{N_x}) \end{pmatrix}_{m_x\times m_x},$$
$$D^*(g) = \begin{pmatrix} g(t_{-M_t}) & & 0 \\ & \ddots & \\ 0 & & g(t_{N_t}) \end{pmatrix}_{m_t\times m_t},$$
$$\bar{U} = \left(u_{i,j}\right)_{m_x\times m_t},$$
$$\bar{H} = \left(H(u_{i,j})\right)_{m_x\times m_t},$$
$$\bar{F} = \left(F^*(x_i, t_j)\right)_{m_x\times m_t}.$$

Now, we have a nonlinear system of $m_x \times m_t$ equations of the $m_x \times m_t$ unknown coefficients $u_{i,j}$. These coefficients are obtained by using Newton's method or many other different methods such as, conjugate gradient method, genetic algorithms, Steffensen's methods and so on [4,20,21].

3.2. *The nonlinear inverse parabolic problem*

In this subsection, to find the unknown function $p(t)$ of the problem (1.1), a computational algorithm is provided.

Algorithm: Identification of the unknown function $p(t)$

Step 1. Put

$$\bar{p}(t) \simeq \sum_{i=1}^{n} p_i\, Sinc\left(\frac{t-ih}{h}\right),$$

be an approximation of the unknown function $p(t)$, where h is the step size of time and p_i's are unknown parameters.

Step 2. Using the Sinc-Galerkin solution (3.2), obtain an approximate solution for $u[x, t, \bar{p}]$. In this case, when we solve the nonlinear system of Eqs. (3.5), the unknown coefficients $u_{i,j}$ are obtained according to p_i's. In other words, we have

$$u_{m_x,m_t}[x, t, \bar{p}] = \sum_{i=-M_x}^{N_x}\sum_{j=-M_t}^{N_t} u_{i,j}(p_1, p_2, \ldots, p_n)S_i(x)S_j^*(t).$$

Step 3. Obtain the n unknown parameters p_i, based on the minimization of the least squares norm

$$S(p) = \sum_{i=1}^{I}(u_{m_x,m_t}[x^*, t_i; \bar{p}] - E(t_i))^2. \tag{3.6}$$

Since, the obtained system of algebraic equations is ill-conditioned, therefore the Levenberg–Marquardt method according to step 4 is used.

Step 4. **Levenberg–Marquardt regularization** [17]. Suppose that,

$$U_{m_x,m_t}(p) = [U_1, U_2, \ldots, U_I]^T$$
$$E = [E_1, E_2, \ldots, E_I]^T,$$

and $p = [p_1, p_2, \ldots, p_n]^T$, where $E_i = E(t_i)$ and $U_i = u_{m_x,m_t}[x^*, t_i; \bar{p}]$, $i = 1, 2, \ldots, I$. Then the matrix form of the functional (3.6) is given by

$$S(p) = \left[E - U_{m_x,m_t}(p)\right]^T \left[E - U_{m_x,m_t}(p)\right],$$

in which

$$\left[E - U_{m_x,m_t}(p)\right]^T \equiv [E_1 - U_1, E_2 - U_2, \ldots, E_I - U_I].$$

The superscript T denotes the transpose and I is the total number of measurements. To minimize the least squares norm, the derivatives of $S(p)$ with respect to each unknown parameters $\{p_i\}_{i=1}^{i=n}$ are equated to zero. That is

$$\frac{\partial S(p)}{\partial p_1} = \frac{\partial S(p)}{\partial p_2} = \cdots = \frac{\partial S(p)}{\partial p_n} = 0,$$

or in matrix form

$$\nabla S(p) = 2\left[-\frac{\partial U^T_{m_x,m_t}(p)}{\partial p}\right]\left[E - U_{m_x,m_t}(p)\right] = 0,$$

where

$$\frac{\partial U^T_{m_x,m_t}(p)}{\partial p} = \begin{bmatrix} \frac{\partial}{\partial p_1} \\ \frac{\partial}{\partial p_2} \\ \vdots \\ \frac{\partial}{\partial p_n} \end{bmatrix} \begin{bmatrix} U_1 & U_2 & \ldots & U_I \end{bmatrix}.$$

Hence, the sensitivity matrix can be written in the form [17]

$$J(p) = \left[\frac{\partial U^T_{m_x,m_t}(p)}{\partial p}\right]^T = \begin{bmatrix} \frac{\partial U_1}{\partial p_1} & \frac{\partial U_1}{\partial p_2} & \frac{\partial U_1}{\partial p_3} & \ldots & \frac{\partial U_1}{\partial p_n} \\ \frac{\partial U_2}{\partial p_1} & \frac{\partial U_2}{\partial p_2} & \frac{\partial U_2}{\partial p_3} & \ldots & \frac{\partial U_2}{\partial p_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial U_I}{\partial p_1} & \frac{\partial U_I}{\partial p_2} & \frac{\partial U_I}{\partial p_3} & \ldots & \frac{\partial U_I}{\partial p_n} \end{bmatrix} \tag{3.7}$$

Remark 3.3. We note that [17], when the sensitivity coefficients $(J(P))_{i,j} = \frac{\partial U_i}{\partial p_j}$ are small, we have $\left|(J(P))^T J(P)\right| \approx 0$ and the inverse problem is ill-conditioned. It can also be shown that $\left|(J(P))^T J(P)\right|$ is null if any column of $J(p)$ can be expressed as a linear combination of the other columns [17].

Now, the computational algorithm for the Levenberg–Marquardt regularization is provided as follows [17].

Suppose an initial guess for the vector of unknown coefficients p is available. Denote it with $p^{(0)}$.

1. Set μ_0 be an arbitrary regularization parameter (for example $\mu_0 = 0.001$) and $k = 0$.

2. Compute $U_{m_x,m_t}(p^{(0)})$ and $S(p^{(0)})$.

3. Compute the sensitivity matrix J^k defined by (3.7) and then $\Omega^k = diag[(J^k)^T J^k]$, by using the current values of $p^{(k)}$.

4. Solve the following linear system of algebraic equations

$$\left[(J^k)^T J^k + \mu^k \Omega^k\right] \Delta p^k = (J^k)^T \left[E - U_{m_x,m_t}(p^{(k)})\right],$$

in order to compute $\Delta p^k = p^{k+1} - p^k$.

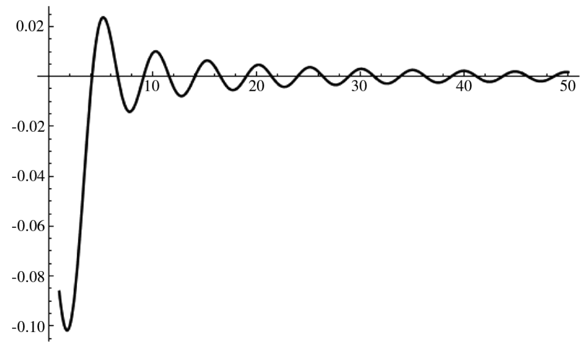

Fig. 1. The approximate solution $\bar{p}(t)$ with $h_x = \frac{\pi}{4}$, $h_t = \frac{\pi}{2}$, $h = \frac{\pi}{4}$, $M_x = N_x = 16$, $M_t = N_t = 4$ and $H(u) = u^2$.

Table 1
The L_1-norm error of the introduced method for $M_x = N_x = M_t = N_t = 4$ and $H(u) = u^2$.

t	1	4.1415	10.4248	19.8496	22.9911	27.7035	32.4159	37.1283	41.8407	48.1239
$\Vert p(t) - \bar{p}(t)\Vert_1$	0.15239	0.20502	0.04755	0.01921	0.00060	0.01034	0.00293	0.00390	0.00456	0.00246

Table 2
The L_1-norm error of the introduced method for $M_x = N_x = 8$, $M_t = N_t = 4$ and $H(u) = u^2$.

t	1	4.1415	10.4248	19.8496	22.9911	27.7035	32.4159	37.1283	41.8407	48.1239
$\Vert p(t) - \bar{p}(t)\Vert_1$	0.17440	0.19443	0.03191	0.00316	0.00788	0.00826	0.01176	0.00322	0.00638	0.00119

5. Compute $p^{k+1} = \Delta p^k + p^k$.
6. If $S(p^{k+1}) \geq S(p^k)$ replace μ^k by $10\mu^k$ and go to 4.
7. If $S(p^{k+1}) < S(p^k)$ accept p^{k+1} and replace μ^k by $0.1\mu^k$.
8. Assume that tol (tolerance) is given. If $\left\Vert p^{k+1} - p^k \right\Vert \leq tol$, then an acceptable approximation is obtained. Otherwise, replace k by $k+1$ and go to 3.

In last section, the application of the proposed approach to solve the problem (1.1) is illustrated by three examples.

4. Numerical results

In this section, to show the validation of the introduced method three numerical examples are given. In these examples, the numerical results are listed with different values of $h_x, h_t, h, M_x, M_t, N_x, N_t$ and for $0 \leq t \leq 50$. Also, in order to solve the obtained nonlinear system of equations in (3.5), we apply Newton's method.

Example 4.1. Consider the nonlinear inverse parabolic problem of the form

$$\begin{array}{ll} u_t - u_{xx} + u^2 = F(x,t), & 0 < x < 1,\ t > 0 \\ u(x,0) = 0 & 0 \leqslant x \leqslant 1 \\ u(0,t) = p(t) & t \geqslant 0 \\ u(1,t) = 0.9te^{-t}\sin(1) & t \geqslant 0 \end{array}$$

where $F(x,t) = -2te^{-t}\cos(x) + e^{-t}(x-0.1)\sin(x)\left(1 + e^{-t}t^2(x-0.1)\sin(x)\right)$ and $p(t)$ is unknown. We note that the exact solutions are $u(x,t) = (x-0.1)te^{-t}\sin(x)$ and $p(t) = 0$. In this example the overposed measured data is considered by $u(0.1,t) = 0$ and $p(t)$ is approximated by

$$p(t) \cong \bar{p}(t) = p_1 Sinc\left(\frac{t-h}{h}\right) + p_2 Sinc\left(\frac{t-2h}{h}\right). \tag{4.1}$$

The numerical results are listed in Table 1 for $h_x = h_t = h = \frac{\pi}{2}$, in Table 2 for $h_x = \frac{\pi}{2\sqrt{2}}$, $h_t = \frac{\pi}{2}$, $h = \frac{\pi}{2\sqrt{2}}$ and in Table 3 for $h_x = \frac{\pi}{4}$, $h_t = \frac{\pi}{2}$, $h = \frac{\pi}{4}$. As we observe, the results show the efficiency and accuracy of the method. Also, Fig. 1 shows the approximate solution $\bar{p}(t)$.

Table 3
The L_1-norm error of the introduced method for $M_x = N_x = 16$, $M_t = N_t = 4$ and $H(u) = u^2$.

t	1	4.1415	10.4248	19.8496	22.9911	27.7035	32.4159	37.1283	41.8407	48.1239
$\Vert p(t) - \bar{p}(t)\Vert_1$	0.08626	0.00097	0.00991	0.00424	0.00381	0.00334	0.00278	0.00218	0.00156	0.00109

Table 4
The L_1-norm error of the introduced method for $M_x = N_x = M_t = N_t = 4$ and $H(u) = \frac{1}{1+u^2}$.

t	1	4.1415	10.4248	19.8496	22.9911	27.7035	32.4159	37.1283	41.8407	48.1239
$\Vert p(t) - \bar{p}(t)\Vert_1$	0.42572	0.70087	0.06571	0.00311	0.07001	0.05146	0.04958	0.04253	0.03651	0.00209

Table 5
The L_1-norm error of the introduced method for $M_x = N_x = 8$, $M_t = N_t = 4$ and $H(u) = \frac{1}{1+u^2}$.

t	1	4.1415	10.4248	19.8496	22.9911	27.7035	32.4159	37.1283	41.8407	48.1239
$\Vert p(t) - \bar{p}(t)\Vert_1$	0.44657	0.77071	0.10003	0.06335	0.05201	0.03154	0.00761	0.03009	0.01810	0.00158

Table 6
The L_1-norm error of the introduced method for $M_x = N_x = 16$, $M_t = N_t = 4$ and $H(u) = \frac{1}{1+u^2}$.

t	1	4.1415	10.4248	19.8496	22.9911	27.7035	32.4159	37.1283	41.8407	48.1239
$\Vert p(t) - \bar{p}(t)\Vert_1$	0.13101	0.08657	0.00116	0.00254	0.00122	0.00045	0.00043	0.00111	0.00108	0.00025

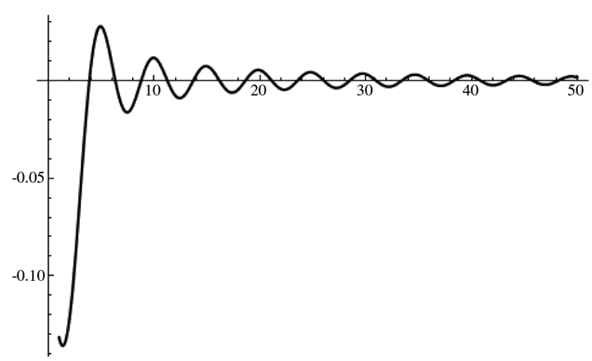

Fig. 2. The approximate solution $\bar{p}(t)$ with $h_x = \frac{\pi}{4}$, $h_t = \frac{\pi}{2}$, $h = \frac{\pi}{4}$, $M_x = N_x = 16$, $M_t = N_t = 4$ and $H(u) = \frac{1}{1+u^2}$.

Example 4.2. For second example, we consider the following problem

$$\begin{array}{ll} u_t - u_{xx} + \dfrac{1}{1+u^2} = F(x,t), & 0 < x < 1,\ t > 0 \\ u(x,0) = 0 & 0 \leqslant x \leqslant 1 \\ u(0,t) = p(t) & t \geqslant 0 \\ u(1,t) = 0.9te^{-t}\sin(1) & t \geqslant 0 \end{array}$$

where

$$F(x,t) = -2e^{-t}t\cos(x) + e^{-t}(x-0.1)\sin(x) + \frac{1}{1 + e^{-2t}t^2(x-0.1)^2\sin(x)^2}$$

and $p(t)$ is unknown. The exact solutions are $u(x,t) = (x-0.1)te^{-t}\sin(x)$ and $p(t) = 0$. Again the overposed measured data is considered by $u(0.1,t) = 0$ and $p(t)$ is approximated by (4.1). The numerical results are listed in Table 4 for $h_x = h_t = h = \frac{\pi}{2}$, in Table 5 for $h_x = \frac{\pi}{2\sqrt{2}}$, $h_t = \frac{\pi}{2}$, $h = \frac{\pi}{2\sqrt{2}}$ and in Table 6 for $h_x = \frac{\pi}{4}$, $h_t = \frac{\pi}{2}$, $h = \frac{\pi}{4}$. As we observe, the results show the validation and accuracy of the method. Also, Fig. 2 shows the approximate solution $\bar{p}(t)$.

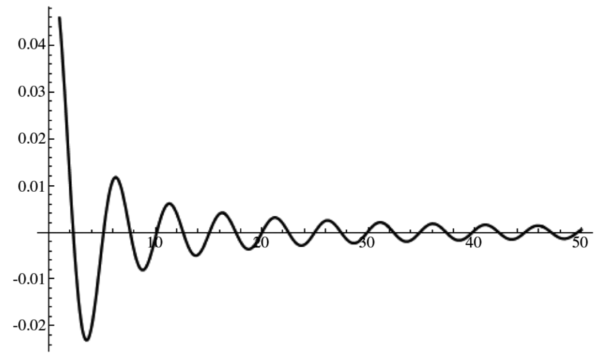

Fig. 3. The approximate solution $\bar{p}(t)$ with $h_x = \frac{\pi}{4}$, $h_t = \frac{\pi}{2}$, $h = \frac{\pi}{4}$, $M_x = N_x = 16$, $M_t = N_t = 4$ and $H(u) = \sin(u)$.

Table 7
The L_1-norm error of the introduced method for $M_x = N_x = M_t = N_t = 4$ and $H(u) = \sin(u)$.

t	1	4.1415	10.4248	19.8496	22.9911	27.7035	32.4159	37.1283	41.8407	48.1239
$\|p(t) - \bar{p}(t)\|_1$	0.14522	0.15112	0.03212	0.01584	0.00431	0.00220	0.00063	0.00047	0.00131	0.00317

Table 8
The L_1-norm error of the introduced method for $M_x = N_x = 8$, $M_t = N_t = 4$ and $H(u) = \sin(u)$.

t	1	4.1415	10.4248	19.8496	22.9911	27.7035	32.4159	37.1283	41.8407	48.1239
$\|p(t) - \bar{p}(t)\|_1$	0.16707	0.16232	0.03149	1.88×10^{-6}	0.00451	0.00826	0.00952	0.00130	0.00622	0.00106

Table 9
The L_1-norm error of the introduced method for $M_x = N_x = 16$, $M_t = N_t = 4$ and $H(u) = \sin(u)$.

t	1	4.1415	10.4248	19.8496	22.9911	27.7035	32.4159	37.1283	41.8407	48.1239
$\|p(t) - \bar{p}(t)\|_1$	0.04484	0.01749	0.00242	0.00075	0.00169	0.00059	0.00011	0.00038	0.00065	0.00109

Example 4.3. Consider the following problem

$$\begin{array}{ll} u_t - u_{xx} + \sin(u) = F(x,t), & 0 < x < 1,\ t > 0 \\ u(x,0) = 0 & 0 \leqslant x \leqslant 1 \\ u(0,t) = p(t) & t \geqslant 0 \\ u(1,t) = 0.9te^{-t}\sin(1) & t \geqslant 0 \end{array}$$

where

$$F(x,t) = -2te^{-t}\cos(x) + e^{-t}(x-0.1)\sin(x) + \sin\left(te^{-t}(x-0.1)\sin(x)\right),$$

and $p(t)$ is unknown. The exact solutions are $u(x,t) = (x-0.1)te^{-t}\sin(x)$ and $p(t) = 0$. The overposed measured data is considered by $u(0.1,t) = 0$ and $p(t)$ is approximated by (4.1). The numerical results are listed in Table 7 for $h_x = h_t = h = \frac{\pi}{2}$, in Table 8 for $h_x = \frac{\pi}{2\sqrt{2}}$, $h_t = \frac{\pi}{2}$, $h = \frac{\pi}{2\sqrt{2}}$ and in Table 9 for $h_x = \frac{\pi}{4}$, $h_t = \frac{\pi}{2}$, $h = \frac{\pi}{4}$. As we observe, the results show the accuracy of the method. Also, Fig. 3 shows the approximate solution $\bar{p}(t)$.

References

[1] J.L. Mueller, T.S. Shores, New Sinc-Galerkin method for convection–diffusion equations with mixed boundary conditions, Comput. Math. Appl. 47 (2004) 803–822.
[2] M. El-Gamel, A.I. Zayed, Sinc-Galerkin method for solving nonlinear boundary-value problems, Comput. Math. Appl. 48 (2004) 1285–1298.
[3] J. Rashidinia, M. Nabati, Sinc-Galerkin and Sinc-Collocation methods in the solution of nonlinear two-point boundary value problems, Comput. Appl. Math. 32 (2013) 315–330.
[4] J. Lund, K. Bowers, Sinc Methods for Quadrature and Differential Equations, SIAM, Philadelphia, 1992.
[5] F. Stenger, Numerical Methods Based on Sine and Analytic Functions, Springer-Verlag, New York, 1993.
[6] J. Lund, C. Vogel, A fully-Galerkin method for the numerical solution of an inverse problem in a parabolic partial differential equation, Inverse Problems 6 (1990) 205–217.

[7] R. Smith, K. Bowers, Sinc-Galerkin estimation of diffusivity in parabolic problems, Inverse Problems 9 (1993) 113–135.
[8] J. Mueller, T. Shores, Uniqueness and numerical recovery of a potential on the real line, Inverse Problems 13 (1997) 289–303.
[9] M. Dehghan, F. Emami-Naeini, The Sinc-collocation and Sinc-Galerkin methods for solving the two-dimensional Schrodinger equation with nonhomogeneous boundary conditions, Appl. Math. Model. 37 (2013) 9379–9397.
[10] A. Saadatmandi, M. Dehghan, The use of Sinc-collocation method for solving multi-point boundary value problems, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 593–601.
[11] R. Agarwal, C. Akrivis, Boundary value problems occurring in plate deflection theory, J. Comput. Appl. Math. 8 (1982) 145–154.
[12] P. Kelevedjiev, Existence of positive solutions to a singular second order boundary value problem, Nonlinear Anal. 50 (2002) 1107–1118.
[13] Y. Liu, H. Yu, Existence and uniqueness of positive solution for singular boundary value problem, Comput. Math. Appl. 50 (2005) 133–143.
[14] M. Badiale, E. Serra, Semilinear Elliptic Equations for Beginners, Springer, 2011.
[15] M. Hanke, The regularizing Levenberg–Marquardt scheme is of optimal order, J. Integral Equations Appl. 22 (2010) 259–283.
[16] C.T. Kelley, Iterative Methods for Optimization, SIAM, Philadelphia, 1999.
[17] M.N. Ozisik, H.R.B. Orlande, Inverse Heat Transfer, Fundamentals and Applications, Taylor Franscis, New York, 2000.
[18] S. Koonprasert, K.L. Bowers, The fully Sinc-Galerkin method for time-dependent boundary conditions, Numer. Methods Partial Differential Equations 20 (2004) 494–526.
[19] A. Shidfar, A. Babaei, The Sinc-Galerkin method for solving an inverse parabolic problem with unknown source term, Numer. Methods Partial Differential Equations 29 (2013) 64–78.
[20] J.M. Ortega, W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.
[21] P.Y. Nie, A null space method for solving system of equations, Appl. Math. Comput. 149 (2004) 215–226.

Guide for authors

Types of papers
Proceedings (Transactions) of A. Razmadze Mathematical Institute focus on significant research articles on both pure and applied mathematics. They should contain original new results with complete proofs. The review papers and short communications, which can be published after Editorial Board's decision, are allowed.

All efforts will be made to process papers efficiently within a minimal amount of time.

Ethics in publishing
For information on Ethics on publishing and Ethical guidelines for journal publication see http://www.elsevier.com/editors/publishing-ethics and http://www.elsevier.com/authors/journal-authors/policies-and-ethics .

Author rights
As an author you (or your employer or institution) have certain rights to reuse your work. For more information see www.elsevier.com/copyright

Contact details for Submission
Authors should submit their manuscript via the Elsevier Editorial System (EES), the online submission, peer-review and editorial system for Proceedings (Transactions) of A. Razmadze Mathematical Institute.

Submission declaration
Submission of an article that described has not be published previously (except in the form of abstract or as part of a published lecture or thesis or as an electronic preprint), that it is not under consideration for publication elsewhere, that its publication is approved by all authors.

Copyright
Upon acceptance of an article, authors will be asked to complete "Journal Publishing Agreement" (for more information see www.elsevier.com/copyright). An e-mail will be sent to the corresponding author confirming receipt of the manuscript together with a "Journal Publishing Agreement" form or a link to the online version of this agreement.

Language
Please write your text in good English (American or British is accepted, but not a mixture of these).

Submission
Our online submission system guides you stepwise through the process of entering your article details and uploading your files. The system converts your article files to a single PDF file used in the peer-review process.

All correspondents, including notification of the Editor's decision and requests for revision, is sent by e-mail.

References
Citation in text
Please ensure that every reference cited in the text is also present in the reference list (and wise versa). Any references cited in the abstract must be given full. Unpublished results and personal communications are not recommended in the reference list, but may be mentioned in the text. Citation of a reference "in press" implies that the item has been accepted for the publication.

Web References
The full URL should be given and the date when the reference was last accepted. Any further information, if known (DOI, author names, dates, reference to a source publication, etc.) should be also given.

LaTeX
It is recommended that each submitted article be prepared in camera-ready form using TeX(plain, LaTeX, AMS-LaTeX) macro page. The type-font for the text is ten point roman with the baselinkship of twelve point. The text area is 190×115 mm excluding page number. The final pagination will be done by the publisher.

Abstracts

The abstract should state briefly the purpose of the research, the principal results and major conclusions. References in the abstract should be avoided, but if essential they must be cited in full, without reference list.

Keywords

Immediately after the abstract, provide a maximum of ten keywords, using American spelling and avoiding general and plural terms and multiple concepts (avoid, for example “and”, or “of”).

Classification codes

Please provide classification codes as per the AMS codes. A full list and information for these can be found at www.ams.orm/msc/ .

Acknowledgements

Collate acknowledgements in a separate section at the end of the article before references and do not, therefore, including them in the title page, as a footnote.

Results

Results should be clear and concise.

Essential title page information

- Title.
  Concise and informative. Avoid abbreviations and formulae where possible.
- Author names and affiliation.
  Please clearly indicate the given name(s) and family name(s) of each author and check that all names are accurately spellt. Present the author’s affiliation addresses below the names. Provide the full postal addresses of each affiliation, including country name and, if available, the e-mail address of each author.
- Corresponding Author.
  Clearly indicate who will handle correspondence at all stages of refereeing and publication.
- Present/permanent address.
  If an author has moved since the work described in the article was done, was visiting at a time, ‘Present address’ (or ‘Permanent address’) may be indicated as a footnote to that author’s name.

Transactions of A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
6 Tamarashvili St., Tbilisi 0177
Georgia

დაიბეჭდა თსუ გამომცემლობის სტამბაში
0179 თბილისი, ი. ჭავჭავაძის გამზ. 1

Published by Tbilisi State University Press
0179 Tbilisi, Ilia Chavchavadze Ave. 1
Tel.: 995(32) 225 14 32; 225 25 27 36
www.press.tsu.ge