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Original article

q-deformation of the square white noise Lie algebra

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Abstract

For $q \in (0, 1)$, the q-deformation of the square white noise Lie algebra is introduced using the q-calculus. A representation of this Lie algebra is given, using the q-derivative (or Jackson derivative) and the multiplication operator. The free square white noise Lie algebra is defined. Moreover, its representation on the Hardy space is given.

Keywords: q-derivative; q-square white noise Lie algebra; Free square white noise Lie algebra

1. Introduction

In mathematics, the first order white noise over $\mathbb{R}$ can be described as the current algebra of the 1-mode CCR algebra $[a, a^+] = \mathbf{1}$ over the algebra $C := \{step\ functions\ on\ \mathbb{R}^d\}$. Similarly the second order white noise can be described as the current algebra of the $\mathfrak{sl}_2(\mathbb{R})$-Lie algebra

$$[a^2, a^{+2}] = c + 4a^+a, \quad [a^2, a^+a] = 2a^+a$$

based on the same algebra C as above. The combined 1-st and 2-nd order Boson white noise can be defined as the current algebra over the Lie algebra, with generators: $a, a^+, \mathbf{1}, a^+a, a^2, a^{+2}$ also called the Schrodinger algebra, based on the same algebra C as above.

In the following commutation relations [1]

$$[a_s^2, a_t^{+2}] = 4\delta(t-s)a_s^+a_t + 2\delta(t-s)^2$$

the term $\delta(t-s)^2$ shows that a_t^{+2} and a_t^2 are not well defined even as operator valued distributions. The following formula, due to Ivanov, for the square of the delta function $\delta^2(t) = c\delta(t)$, c is arbitrary constant, was used by Accardi,

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Lu and Volovich to realize the program for the second powers of white noise. See [1]. Using this, one can obtain the renormalized commutation relation:

$$[a_s^2, a_t^{+2}] = 4\delta(t-s)a_s^+ a_t + 2c\delta(t-s). \tag{1}$$

Moreover

$$[a_s^2, a_t^+ a_t] = 2\delta(t-s)a_t^2. \tag{2}$$

The relations (1), (2) are then taken as the definition of the renormalized square of white noise (RSWN)-Lie algebra. Recalling that $\mathfrak{sl}_2(\mathbb{R})$ is the-Lie algebra with three generators B^-, B^+, M and relations

$$[B^-, B^+] = M, \quad [M, B^\pm] = \mp 2B^\pm, \quad (B^-)^* = B^+, \quad M^* = M$$

one concludes that the RSWN-Lie algebra is isomorphic to a current algebra, over $\mathbb{R}$, of a central extension of $\mathfrak{sl}_2(\mathbb{R})$ (see [1]).

Without using the renormalization conditions, in the papers [2–10], it was given many representations of this and other Lie algebras on nuclear algebras of entire functions. More precisely, they introduce a new product of two test functions and based on the space of entire functions with θ-exponential growth of minimal type, they define a new family of infinite dimensional analytical operators using the holomorphic derivative and its adjoint. Also, they introduce a new operator obtained from the quantum white noise derivatives which satisfies new important commutation relations generalizing those of the renormalized power white noise Lie algebra without using renormalization conditions.

In the above introduction (for $q = 1$), we have described the renormalization problem obtained from the above commutation relations, moreover the square white noise Lie algebra was introduced (see [1]). On the other hand, in recent years the q-deformation of the Heisenberg commutation relation has drawn attention. In the paper [11], the purpose was to understand the probability distribution of a non-commutative random variable $a + a^*$, where a is a bounded operator on some Hilbert space satisfying

$$aa^* - qa^*a = \mathbf{1}, \tag{3}$$

for some $q \in [-1, 1)$. The calculation is inspired by the case, $q = 0$, where a and a^* turn out to be the left and right shift on $l^2(\mathbf{N})$. In this case a and a^* can be quite nicely represented as operators on the Hardy class $\mathcal{H}^2$ of all analytic functions on the unit disk with L^2 limits toward the boundary.

Subsequently, they find a measure μ_q, $q \in [0, 1)$, on the complex plane that replaces the Lebesgue measure on the unit circle in the above: μ_q is concentrated on a family of concentric circles, the largest of which has the radius $\frac{1}{\sqrt{1-q}}$. Their representation space (see [11]) will be $\mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$, the completion of the analytic functions on $\mathfrak{D}_q = \{z \in \mathbb{C} \mid z|^2 < \frac{1}{(1-q)}\}$ with respect to the inner product defined by μ_q. In this space annihilation operator a is represented by a q difference operator D_q. As q tends to 1, μ_q will tend to the Gauss measure on $\mathbb{C}$ and D_q becomes differentiation. So, it is natural to ask what is the q-deformation of the square white noise Lie algebra.

In this paper, we introduce the q-square white noise Lie algebra and we give its representation without any renormalization conditions. Also, the free square white noise Lie algebra is defined. Moreover, its representation on the Hardy space is given.

The paper is organized as follows. In Section 2, we briefly recall well-known results on q-calculus, Jackson derivative (or q-derivative) and useful representations. In Section 3, we introduce the free square white noise Lie algebra and we give its representation on the Hardy space $\mathcal{H}^2$. In Section 4, we define the q-square white noise Lie algebra. Moreover, we give its representation on the space $\mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$.

2. Preliminaries

We recall some basic notations of the language of q-calculus (see [11–15]). The natural number n has the following q deformation:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, \ \ with \ [0]_q = 0.$$

Occasionally we shall write $[\infty]_q$ for the limit of these numbers: $\frac{1}{(1-q)}$. The q factorials and q binomial coefficients are defined naturally as

$$[n]_q! := [1]_q \cdot [2]_q \cdots [n]_q \ \ with \ [0]_q := 1.$$

Recall that from [11], for $q \in (0, 1)$ relation (3) admits, up to unitary equivalence, a unique non-trivial bounded irreducible representation given on the canonical basis $\{e_n | n \in \mathbb{N}\}$ of $l^2(\mathbb{N})$ by:

(i) $a^* e_n = e_{n+1}$
(ii) $a e_n = [n]_q e_{n-1}$
(iii) $\langle e_n, e_m \rangle = \delta_{n,m} [n]_q!$

For $q \in (0, 1)$ and analytic $f : \mathbb{C} \to \mathbb{C}$ define operators Z and D_q as follows (see [11,14,15])

$$(Zf)(z) := zf(z),$$

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, z \neq 0 \\ f'(0). \end{cases}$$

The operator D_q has the following properties :

(i) $\lim_{q \uparrow 1} (D_q f)(z) = f'(z)$,
(ii) $D_q(z^n) = [n]_q z^{n-1}$,
(iii) $D_q(f(z)g(z)) = (D_q f)(z)g(z) + f(qz)(D_q g)(z)$,
(iv) $D_q\left(\frac{f(z)}{g(z)}\right) = \frac{(D_q f)(z)g(z) - f(z)(D_q g)(z)}{g(z)g(qz)}$.

It is well known [11] that the operators D_q and Z give a bounded representation of (3), i.e., D_q and Z satisfy

$$D_q Z - qZD_q = \mathbf{1}.$$

With respect to the measure (see [11])

$$\mu_q(dz) = (q; q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q; q)_k} \lambda_{r_k}(dz), \;\; 0 < q < 1 \; and \; r_k = \frac{q^{\frac{k}{2}}}{\sqrt{1-q}}$$

where λ_{r_k} is the normalized Lebesgue measure on the circle with radius r_k, they define the inner product

$$\langle f, g \rangle_{\mu_q} := \int_{\mathbb{C}} \overline{f(z)} g(z) \mu_q(dz)$$

for all $f, g \in \mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$. Note that $\mu_q \to \mu_0$ when $q \to 0$, where μ_0 is the normalized Lebesgue measure on the unit circle and that, in the limit $q \uparrow 1$, μ_q tends to the Gauss measure on the complex plane.

The identification $a = D_q$ and $a^* = Z$ determine a representation of (3) on $\mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$. In particular, with $e_n := z^n$, $(i), (ii)$ *and* (iii) are satisfied, and therefore $D_q^* = Z$. For more details see Ref. [11].

3. Free square white noise Lie algebra

Relation (3) is reduced to $aa^* = 1$ when $q \to 0$ and this obviously admits more than one representation: any isometry a^* suffices. By a representation with $q \to 0$, we shall simply mean one satisfying (i), (ii) and (iii).

The calculation is inspired by the case, $q = 0$, where a and a^* turn out to be the left and right shift on $l^2(\mathbf{N})$. In this case a and a^* can be quite nicely represented as operators on the Hardy class $\mathcal{H}^2$ of all analytic functions on the unit disk with L^2 limits toward the boundary via the equivalence $l^2(\mathbf{N}) \to \mathcal{H}^2$ given by

$$(\xi_n)_{n \in \mathbf{N}} \mapsto \sum_{n=0}^{\infty} \xi_n z^n, \mid z \mid < 1.$$

Under this equivalence a^* and a change into multiplication by z and the operator

$$(Df)(z) := \frac{f(z) - f(0)}{z}.$$

Definition 3.1. The free square white noise Lie algebra is by definition the Lie algebra spanned by the operators A, B and C such that

$$[A, C] = 2A$$
$$[B, C] = -2B$$
$$[A, B] = 0.$$

Let Δ^-, Δ^+ and N given by $\Delta^- := D^2$, $\Delta^+ := Z^2$ and N is the classical number operator which verifies $N(z^n) := nz^n$. Then, we have the following

Theorem 3.1 (*Representation of Free Square White Noise Lie Algebra*). *We have*

$$[\Delta^-, N] = 2\Delta^- \tag{4}$$

$$[\Delta^+, N] = -2\Delta^+ \tag{5}$$

$$[\Delta^-, \Delta^+] = 0 \tag{6}$$

on the Hardy space $\mathcal{H}^2$.

Proof. Applying Δ^- to $N(z^n)$, we get

$$\begin{aligned}\Delta^- N(z^n) &= \Delta^-(nz^n)\\ &= n\Delta^-(z^n).\end{aligned}$$

But, we have

$$\begin{aligned}\Delta^-(z^n) &= D^2(z^n)\\ &= \lim_{q\to 0}[n]_q[n-1]_q z^{n-2}\\ &= z^{n-2}.\end{aligned}$$

Then, we get

$$\Delta^- N(z^n) = nz^{n-2}.$$

Then, on the other hand, we obtain

$$\begin{aligned}N\Delta^-(z^n) &= N\big(z^{n-2}\big)\\ &= N\big(z^{n-2}\big)\\ &= (n-2)z^{n-2}\end{aligned}$$

from which we get

$$\begin{aligned}[\Delta^-, N]z^n &= \big(n-(n-2)\big)z^{n-2}\\ &= 2z^{n-2}\\ &= 2\Delta^-(z^n).\end{aligned}$$

This proves (4).

Now, applying Δ^+ to $N(z^n)$, we get

$$\begin{aligned}\Delta^+ N(z^n) &= \Delta^+(nz^n)\\ &= n\Delta^+(z^n)\\ &= nz^{n+2}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}N\Delta^+(z^n) &= N(z^{n+2})\\ &= (n+2)z^{n+2}\end{aligned}$$

then, we get

$$\begin{aligned}[\Delta^+, N]z^n &= \big(n-(n+2)\big)z^{n+2}\\ &= -2z^{n+2}\\ &= -2\Delta^+(z^n).\end{aligned}$$

This proves (5).

Finally, we have

$$\begin{aligned}\Delta^-\big(\Delta^+(z^n)\big) &= \Delta^-(z^{n+2})\\ &= z^n\end{aligned}$$

and

$$\begin{aligned}\Delta^+\big(\Delta^-(z^n)\big) &= \Delta^+\big(z^{n-2}\big)\\ &= z^n\end{aligned}$$

then, we get

$$[\Delta^-, \Delta^+]z^n = 0,$$

which completes the proof. □

Remark 1. Theorem 3.1 is not a simple case (when $q = 0$) of Theorem 4.1 (see below), because of the representations, i.e., $\mathcal{H}^2$ is not a particular case of $\mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$.

4. q-square white noise Lie algebra

Let $0 < q < 1$. Then, one can define the q-square white noise Lie algebra as follows:

Definition 4.1. The q-square white noise Lie algebra is by definition the Lie algebra spanned by the operators A, B and C such that

$$\begin{aligned}&[A, C] = 2A\\ &[B, C] = -2B\\ &[A, B] = \frac{-[2]_q}{1-q}\big(q^{2C-\mathbf{1}} - q^C - q^{C-\mathbf{1}}\big).\end{aligned}$$

Theorem 4.1 (*Representation of q-square White Noise Lie Algebra)*). *Let $0 < q < 1$. Then, we have*

$$[\Delta_q^-, N] = 2\Delta_q^- \tag{7}$$

$$[\Delta^+, N] = -2\Delta^+ \tag{8}$$

$$[\Delta_q^-, \Delta^+] = -[2]_q\Big(\frac{q^{2N-I} - q^N - q^{N-I}}{1-q}\Big) \tag{9}$$

on $\mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$, where Δ_q^- is given by $\Delta_q^- := D_q^2$.

Proof. Let $0 < q < 1$. Then, we have

$$\begin{aligned}\Delta_q^- N(z^n) &= \Delta_q^-(nz^n)\\ &= n\Delta_q^-(z^n)\\ &= n[n]_q[n-1]_q z^{n-2}.\end{aligned}$$

On the other hand

$$\begin{aligned}N\Delta_q^-(z^n) &= N\big([n]_q[n-1]_q z^{n-2}\big)\\ &= (n-2)[n]_q[n-1]_q z^{n-2}\end{aligned}$$

from which we get

$$\begin{aligned}[\Delta_q^-, N]z^n &= [n]_q[n-1]_q\big(n-(n-2)\big)z^{n-2}\\ &= 2[n]_q[n-1]_q z^{n-2}\\ &= 2\Delta_q^-(z^n).\end{aligned}$$

This proves (7). The proof of (8) is the same as (5).

We have

$$\begin{aligned}\Delta_q^-\big(\Delta^+(z^n)\big) &= \Delta_q^-(z^{n+2})\\ &= [n+2]_q[n+1]_q z^n\end{aligned}$$

and

$$\begin{aligned}\Delta^+\big(\Delta_q^-(z^n)\big) &= \Delta^+\big([n]_q[n-1]_q z^{n-2}\big)\\ &= [n]_q[n-1]_q z^n\end{aligned}$$

then, we get

$$[\Delta_q^-, \Delta^+]z^n = \big([n+2]_q[n+1]_q - [n]_q[n-1]_q\big)z^n.$$

But we know that

$$\begin{aligned}[n+2]_q[n+1]_q - [n]_q[n-1]_q &= \frac{(1-q^{n+2})(1-q^{n+1})}{(1-q)^2} - \frac{(1-q^n)(1-q^{n-1})}{(1-q)^2}\\ &= \frac{1-q^{n+1}-q^{n+2}+q^{2n+3}-(1-q^n-q^{n-1}+q^{2n-1})}{(1-q)^2}\\ &= \frac{q^2(q^{2n-1}-q^n-q^{n-1})-(q^{2n-1}-q^n-q^{n-1})}{(1-q)^2}\\ &= \frac{(q^2-1)(q^{2n-1}-q^n-q^{n-1})}{(1-q)^2}\\ &= -[2]_q\Big(\frac{q^{2n-1}-q^n-q^{n-1}}{1-q}\Big). \qquad (10)\end{aligned}$$

This completes the proof. □

Remark 2. Using (10), one can get

$$\frac{-[2]_q}{1-q}(q^{2n-1}-q^n-q^{n-1}) \to 4n+1$$

when $q \to 1$, for which the relation (9) gives

$$[\Delta_q^-, \Delta^+] \to 4N + 2I$$

this shows that the q-square white noise Lie algebra gives the square white noise Lie algebra when $q \to 1$.

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Original article

Several inequalities For log-convex functions

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Abstract

In this paper, we recall Ostrowski's inequality, Hadamard's inequality and the definition of log-convex functions. We also mention an useful integral identity in the first part of our study. The second part of our study includes new results. We prove new generalizations for log-convex functions. Several new Ostrowski type inequalities have been established and some special cases have been given by choosing $h = 0$ or $x = \frac{a+b}{2}$.

Keywords: log-convex functions; Ostrowski inequality; Hermite–Hadamard inequality; Hölder inequality

1. Introduction

Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, such that $f' \in L[a, b]$ where $a, b \in I$ with $a < b$. If $\left|f'(x)\right| \le M$, then the following inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \le \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]. \tag{1.1}$$

This inequality is well known in the literature as the Ostrowski inequality.

The following inequality is well known in the literature as the Hermite–Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

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In [1], Pečarić et al. mentioned log-convex functions as follows:

A function $f : I \to [0, \infty)$ is said to be log-convex or multiplicatively convex if $\log f$ is convex, or, equivalently, for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality

$$f(tx + (1-t)y) \le [f(x)]^t [f(y)]^{1-t}.$$

Example 1. The function $f(x) = \frac{1}{x}$, $x \in (0, \infty)$ is log-convex on $(0, \infty)$. The function $f(x) = x^x$, $x > 0$ or $f(x) = e^x + 1$, $x \in \mathbb{R}$, etc.

Many different extensions, generalizations and improvements related to log-convex functions can be found in [1–9]. In order to prove our main results we use the following equality from [10] that is mentioned in [11]:

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b), then this equality holds*

$$\int_a^b f(t)\,dt = (b-a)(1-h)f(x) - (b-a)(1-h)\left(x - \frac{a+b}{2}\right) f'(x) + h\frac{b-a}{2}(f(a)+f(b)) - \frac{h^2(b-a)^2}{8}\left(f'(b) - f'(a)\right) + \int_a^b K(x,t) f''(t)dt$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ and $h \in [0, 1]$. Here $K : [a, b]^2 \to \mathbb{R}$

$$K(x,t) = \begin{cases} \dfrac{1}{2}\left[t - \left(a + h\dfrac{b-a}{2}\right)\right]^2, & \text{if } t \in [a, x] \\ \dfrac{1}{2}\left[t - \left(b - h\dfrac{b-a}{2}\right)\right]^2, & \text{if } t \in (x, b]. \end{cases}$$

The main purpose of this paper is to give some new integral inequalities of Ostrowski type for logarithmically convex functions by using the above lemma.

2. Main results

Let us start our first result:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b). If $\left|f''\right|$ is* log-*convex, the following inequality holds for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$:*

$$\left|\int_a^b f(t)\,dt - (b-a)(1-h)f(x) + (b-a)(1-h)\left(x - \frac{a+b}{2}\right) f'(x) - h\frac{b-a}{2}(f(a)+f(b)) + \frac{h^2(b-a)^2}{8}\left(f'(b) - f'(a)\right)\right|$$

$$\le \frac{1}{2}\left[\left(\frac{|f''(a)|^x}{|f''(x)|^a}\right)^{\frac{1}{x-a}} \tau_1 + \left(\frac{|f''(x)|^b}{|f''(b)|^x}\right)^{\frac{1}{b-x}} \tau_2\right]$$

where

$$\tau_1 = \left[x - \left(a + h\frac{b-a}{2}\right)\right]^2 \frac{T^x}{\ln T} - h^2\frac{(b-a)^2}{4}\frac{T^a}{\ln T} - \frac{2}{(\ln T)^2}\left[x - \left(a + h\frac{b-a}{2}\right)\right] T^x + h\frac{b-a}{(\ln T)^2} T^a + \frac{2}{(\ln T)^3}\left[T^x - T^a\right],$$

$$\tau_2 = h^2\frac{(b-a)^2}{4}\frac{M^b}{\ln M} - \left[x-\left(b-h\frac{b-a}{2}\right)\right]^2\frac{M^x}{\ln M} - h\frac{b-a}{(\ln M)^2}M^b + \frac{2}{(\ln M)^2}\left[x-\left(b-h\frac{b-a}{2}\right)\right]M^x - \frac{2}{(\ln M)^3}\left[M^x - M^b\right].$$

Here $T = \left(\frac{|f''(x)|}{|f''(a)|}\right)^{\frac{1}{x-a}}$ *and* $M = \left(\frac{|f''(b)|}{|f''(x)|}\right)^{\frac{1}{b-x}}$. *And also* $T, M \neq 1$.

Proof. From Lemma 1, and using the property of the modulus and log-convexity of $|f''|$ we can write

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - (b-a)(1-h)f(x) + (b-a)(1-h)\left(x-\frac{a+b}{2}\right)f'(x)\right.\\
&\left.\quad - h\frac{b-a}{2}(f(a)+f(b)) + \frac{h^2(b-a)^2}{8}\left(f'(b)-f'(a)\right)\right|\\
&\le \int_a^b |K(x,t)|\left|f''(t)\right| dt\\
&= \int_a^x \frac{1}{2}\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2\left|f''\left(\frac{t-a}{x-a}x+\frac{x-t}{x-a}a\right)\right| dt\\
&\quad + \int_x^b \frac{1}{2}\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2\left|f''\left(\frac{t-x}{b-x}b+\frac{b-t}{b-x}x\right)\right| dt\\
&\le \int_a^x \frac{1}{2}\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2\left[\left|f''(x)\right|^{\frac{t-a}{x-a}}\left|f''(a)\right|^{\frac{x-t}{x-a}}\right] dt\\
&\quad + \int_x^b \frac{1}{2}\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2\left[\left|f''(b)\right|^{\frac{t-x}{b-x}}\left|f''(x)\right|^{\frac{b-t}{b-x}}\right] dt\\
&= \frac{1}{2}\left(\frac{\left|f''(a)\right|^x}{|f''(x)|^a}\right)^{\frac{1}{x-a}}\int_a^x\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2 T^t dt\\
&\quad + \frac{1}{2}\left(\frac{\left|f''(x)\right|^b}{|f''(b)|^x}\right)^{\frac{1}{b-x}}\int_x^b\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2 M^t dt.
\end{aligned}$$

By computing the above integrals, we get the desired result. □

Corollary 1. *Under the assumptions of Theorem* 1, *if we choose* $h = 0$, *the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x) - \left(x-\frac{a+b}{2}\right)f'(x)\right|\\
&\le \frac{1}{2(b-a)}\left(\frac{\left|f''(a)\right|^x}{|f''(x)|^a}\right)^{\frac{1}{x-a}}\\
&\quad\times\left[(x-a)^2\frac{T^x}{\ln T} - 2(x-a)\frac{T^x}{(\ln T)^2} + \frac{2}{(\ln T)^3}(T^x - T^a)\right]\\
&\quad + \frac{1}{2(b-a)}\left(\frac{\left|f''(x)\right|^b}{|f''(b)|^x}\right)^{\frac{1}{b-x}}\\
&\quad\times\left[2(x-b)\frac{M^x}{(\ln M)^2} - (x-b)^2\frac{M^x}{\ln M} + \frac{2}{(\ln M)^3}(M^b - M^x)\right]
\end{aligned}$$

Corollary 2. *Under the assumptions of Corollary 1, if we choose $x = \frac{a+b}{2}$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{1}{2(b-a)}\left(\frac{\left|f''(a)\right|^{\frac{a+b}{2}}}{\left|f''\left(\frac{a+b}{2}\right)\right|^{a}}\right)^{\frac{2}{b-a}} \\
&\quad\times\left[\left(\frac{b-a}{2}\right)^2\frac{K_1^{\frac{a+b}{2}}}{\ln K_1} - (b-a)\frac{K_1^{\frac{a+b}{2}}}{(\ln K_1)^2} + \frac{2}{(\ln K_1)^3}\left(K_1^{\frac{a+b}{2}} - K_1^{a}\right)\right] \\
&\quad+\frac{1}{2(b-a)}\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^{b}}{\left|f''(b)\right|^{\frac{a+b}{2}}}\right)^{\frac{2}{b-a}} \\
&\quad\times\left[(b-a)\frac{M_1^{\frac{a+b}{2}}}{(\ln M_1)^2} - \left(\frac{b-a}{2}\right)^2\frac{M_1^{\frac{a+b}{2}}}{\ln M_1} + \frac{2}{(\ln M_1)^3}\left(M_1^{b} - M_1^{\frac{a+b}{2}}\right)\right]
\end{aligned}$$

where $K_1 = \left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|}{\left|f''(a)\right|}\right)^{\frac{2}{b-a}}$ and $M_1 = \left(\frac{\left|f''(b)\right|}{\left|f''\left(\frac{a+b}{2}\right)\right|}\right)^{\frac{2}{b-a}}$.

Corollary 3. *Under the assumptions of Theorem 1, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{b-a}{8}\left(f'(b)-f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{2(b-a)}\left(\frac{\left|f''(a)\right|^{x}}{\left|f''(x)\right|^{a}}\right)^{\frac{1}{x-a}}\tau_3 + \frac{1}{2(b-a)}\left(\frac{\left|f''(x)\right|^{b}}{\left|f''(b)\right|^{x}}\right)^{\frac{1}{b-x}}\tau_4
\end{aligned}$$

where

$$\begin{aligned}
\tau_3 &= \left(x-\frac{a+b}{2}\right)^2\frac{T^x}{\ln T} - \frac{(b-a)^2}{4}\frac{T^a}{\ln T} \\
&\quad -2\frac{T^x}{(\ln T)^2}\left(x-\frac{a+b}{2}\right) + \frac{b-a}{(\ln T)^2}T^a + \frac{2}{(\ln T)^3}(T^x - T^a), \\
\tau_4 &= \frac{(b-a)^2}{4}\frac{M^b}{\ln M} - \left(x-\frac{a+b}{2}\right)^2\frac{M^x}{\ln M} \\
&\quad -\frac{b-a}{(\ln M)^2}M^b + 2\frac{M^x}{(\ln M)^2}\left(x-\frac{a+b}{2}\right) + \frac{2}{(\ln M)^3}(M^b - M^x)
\end{aligned}$$

and T, M are defined as in Theorem 1.

Corollary 4. *Under the assumptions of Corollary 3, if we choose $x = \frac{a+b}{2}$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{b-a}{8}\left(f'(b)-f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{2(b-a)}\left(\frac{\left|f''(a)\right|^{\frac{a+b}{2}}}{\left|f''\left(\frac{a+b}{2}\right)\right|^{a}}\right)^{\frac{2}{b-a}} \\
&\quad\times\left[-\frac{(b-a)^2}{4}\frac{K_1^{a}}{\ln K_1} + \frac{b-a}{(\ln K_1)^2}K_1^{a} + \frac{2}{(\ln K_1)^3}\left(K_1^{\frac{a+b}{2}} - K_1^{a}\right)\right]
\end{aligned}$$

$$+\frac{1}{2(b-a)}\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^{b}}{|f''(b)|^{\frac{a+b}{2}}}\right)^{\frac{2}{b-a}}$$
$$\times\left[\frac{(b-a)^2}{4}\frac{M_1^b}{\ln M_1}-\frac{b-a}{(\ln M_1)^2}M_1^b+\frac{2}{(\ln K_1)^3}(M_1^b-M_1^{\frac{a+b}{2}})\right]$$

where K_1 and M_1 are defined as in Corollary 2.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b). If $\left|f''\right|^q$ is $\log-$convex, the following inequality holds for all $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$*

$$\left|\int_a^b f(t)\,dt-(b-a)(1-h)f(x)+(b-a)(1-h)\left(x-\frac{a+b}{2}\right)f'(x)\right.$$
$$\left.-h\frac{b-a}{2}(f(a)+f(b))+\frac{h^2(b-a)^2}{8}\left(f'(b)-f'(a)\right)\right|$$
$$\le\frac{1}{2(b-a)}\left(\frac{\left|f''(a)\right|^{x}}{|f''(x)|^{a}}\right)^{\frac{1}{x-a}}\left(\frac{T^{qx}-T^{qa}}{q\ln T}\right)^{\frac{1}{q}}$$
$$\times\left(\frac{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{2p+1}-\left(h\frac{b-a}{2}\right)^{2p+1}}{2p+1}\right)^{\frac{1}{p}}$$
$$+\frac{1}{2(b-a)}\left(\frac{\left|f''(x)\right|^{b}}{|f''(b)|^{x}}\right)^{\frac{1}{b-x}}\left(\frac{M^{qb}-M^{qx}}{q\ln M}\right)^{\frac{1}{q}}$$
$$\times\left(\frac{\left(h\frac{b-a}{2}\right)^{2p+1}-\left(x-\left(b-h\frac{b-a}{2}\right)\right)^{2p+1}}{2p+1}\right)^{\frac{1}{p}}$$

where $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and T, M are defined as in Theorem 1.

Proof. From Lemma 1 and using the property of the modulus, Hölder inequality and $\log-$convexity of $\left|f''\right|^q$ we can write

$$\left|\int_a^b f(t)\,dt-(b-a)(1-h)f(x)+(b-a)(1-h)\left(x-\frac{a+b}{2}\right)f'(x)\right.$$
$$\left.-h\frac{b-a}{2}(f(a)+f(b))+\frac{h^2(b-a)^2}{8}\left(f'(b)-f'(a)\right)\right|$$
$$\le\int_a^b|K(x,t)|\left|f''(t)\right|dt$$
$$\le\frac{1}{2}\left(\int_a^x\left[t-\left(a+h\frac{b-a}{2}\right)\right]^{2p}dt\right)^{\frac{1}{p}}\left(\int_a^x\left|f''\left(\frac{t-a}{x-a}x+\frac{x-t}{x-a}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{1}{2}\left(\int_x^b\left[t-\left(b-h\frac{b-a}{2}\right)\right]^{2p}\right)^{\frac{1}{p}}\left(\int_x^b\left|f''\left(\frac{t-x}{b-x}b+\frac{b-t}{b-x}x\right)^q\right|dt\right)^{\frac{1}{q}}$$
$$\le\frac{1}{2}\left(\int_a^x\left[t-\left(a+h\frac{b-a}{2}\right)\right]^{2p}dt\right)^{\frac{1}{p}}\left(\int_a^x\left|f''(x)\right|^{\left(\frac{t-a}{x-a}\right)q}\left|f''(a)\right|^{\left(\frac{x-t}{x-a}\right)q}dt\right)^{\frac{1}{q}}$$
$$+\frac{1}{2}\left(\int_x^b\left[t-\left(b-h\frac{b-a}{2}\right)\right]^{2p}\right)^{\frac{1}{p}}\left(\int_x^b\left|f''(b)\right|^{\left(\frac{t-x}{b-x}\right)q}\left|f''(x)\right|^{\left(\frac{b-t}{b-x}\right)q}dt\right)^{\frac{1}{q}}.$$

By computing the above integrals, we get the desired result. □

Corollary 5. *Under the assumptions of Theorem 2, if we choose $h = 0$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x) - \left(x - \frac{a+b}{2}\right) f'(x)\right| \\
&\leq \frac{1}{2(b-a)}\left(\frac{(x-a)^{2p+1}}{2p+1}\right)^{\frac{1}{p}} \left(\frac{\left|f''(a)\right|^x}{\left|f''(x)\right|^a}\right)^{\frac{1}{x-a}} \left(\frac{T^{qx} - T^{qa}}{q\ln T}\right)^{\frac{1}{q}} \\
&\quad + \frac{1}{2(b-a)}\left(\frac{(b-x)^{2p+1}}{2p+1}\right)^{\frac{1}{p}} \left(\frac{\left|f''(x)\right|^b}{\left|f''(b)\right|^x}\right)^{\frac{1}{b-x}} \left(\frac{M^{qb} - M^{qx}}{q\ln M}\right)^{\frac{1}{q}}
\end{aligned}$$

where T, M are defined as in Theorem 1.

Corollary 6. *Under the assumptions of Corollary 6, if we choose $x = \frac{a+b}{2}$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{1}{2^{3+\frac{1}{p}}}\left(\frac{(b-a)^{2p+1}}{2p+1}\right)^{\frac{1}{p}} \left[\left(\frac{\left|f''(a)\right|^{\frac{a+b}{2}}}{\left|f''\left(\frac{a+b}{2}\right)\right|^a}\right)^{\frac{2}{b-a}} \left(\frac{K_1^{q\frac{a+b}{2}} - K_1^{qa}}{q\ln K_1}\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^b}{\left|f''(b)\right|^{\frac{a+b}{2}}}\right)^{\frac{2}{b-a}} \left(\frac{M_1^{qb} - M_1^{q\frac{a+b}{2}}}{q\ln K_1}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where K_1 and M_1 are defined as in Corollary 2.

Remark 1. Many applications can be given based on our results to the special means and to numerical integration, we omit the details.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

Authors Contributions

MAA and AOA carried out the design of the study and performed the analysis. MEO participated in its design and coordination. All authors read and approved the final manuscript.

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Original Article

Classes of pseudo BL-algebras with right Boolean lifting property

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Abstract

In this paper, we define the right Boolean lifting property (left Boolean lifting property) RBLP (LBLP) for pseudo BL-algebra to be the property that all Boolean elements can be lifted modulo every right filter (left filter) and next we study the behavior of RBLP (LBLP) with respect to direct products of pseudo BL-algebra. We introduce some conditions, which turn out to be a strengthening and a weakling of RBLP (LBLP) respectively and which open new ways of approaching the study of the RBLP (LBLP) in pseudo BL-algebras.

Keywords: Lifting property; Boolean center; (prime, maximal) filter; (maximal, local, hyper Archimedean, quasi-local, semi local) pseudo BL-algebra

1. Introduction

In 1998, Hajek introduced BL-algebra; an algebraic semantics of basic fuzzy logic which is generated by continuous t-norms on the interval [0, 1] and their residuals [1]. Then Georgescu introduced pseudo BL-algebra as a non-commutative extension of BL-algebra [2]. The idea of pseudo BL-algebra originates not only in logic and algebra, but also in algebraic properties that come from the syntax of certain non-classical propositional logics and intuitionistic logic. A lifting property for Boolean elements appears in the study of maximal MV-algebras and maximal BL-algebra. The left lifting property for Boolean elements modulo radical plays an essential part in the structure theorem for maximal pseudo BL-algebra. Extending previous works, Georgescu and Muresan studied Boolean lifting property for arbitrary residuated lattice [1]. The results of this study were similar to idempotent elements in the rings. In [3] we studied pseudo BL-algebra which satisfies the left (right) lifting property of Boolean elements modulo every left filter, a property that we have called LBLP (RBLP) for abbreviation. Also it shows that each Boolean algebra infused a pseudo BL-algebra with LBLP (RBLP), that hyper Archimedean have LBLP (RBLP).

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It turns out that the algebras at pseudo BL-algebra with LBLP (RBLP) are exactly the quasi-local pseudo BL-algebras. The target of this article is to study and introduce two conditions which share some properties with the RBLP (LBLP), (Propositions 5.12, 5.14 and 5.15) and appear to also differ by some properties from the RBLP (LBLP). Moreover, it shows that a finite direct product pseudo BL-algebra has RBLP iff each pseudo BL-algebra in the products has RBLP (LBLP) and this holds for individual filter, as well. Weaker results hold for arbitrary direct product of pseudo BL-algebra until mentioned otherwise, let $(A_i)_{i\in I}$ be a non-empty family of pseudo BL-algebras and $A = \prod_{i\in I} A_i$, since pseudo BL-algebra form an equational class, it follows that A becomes a pseudo BL-algebra with the operations defined canonically that is componentwise. Also, clearly, all elements are idempotent in A iff in A_i, for each $i \in I$. Throughout this section, unless mentioned otherwise, A will be an arbitrary pseudo BL-algebra. These are the main sources that inspire the research on pseudo BL-algebra. Section 2 shows theorems that satisfy the semi local condition and consists of previously known concepts about pseudo BL-algebra which are necessary in the next sections. In Section 3, we define the RBLP (LBLP) for pseudo BL-algebra and characterization of the RBLP (LBLP). In Section 4, we analyze the RBLP (LBLP) in direct products of pseudo BL-algebra. In Section 5, we set the RBLP (LBLP) in relation to two new arithmetic conditions. In Section 6, we establish relationships between the classes of the local, semi local, maximal, quasi-local, pseudo BL-algebras with RBLP (LBLP) and we obtain representation theorems for semilocal and maximal pseudo BL-algebras with RBLP (LBLP).

2. Preliminaries

In this section, we recall some basic definitions and results related to pseudo BL-algebra, all of them will be used in the paper. We shall denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^*$ the set of nonzero natural numbers.

Definition 2.1 ([4]). A pseudo BL-algebra is an algebra $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ of type (2, 2, 2, 2, 2, 0, 0) satisfying the following

(PSBL_1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;
(PSBL_2) $(A, \odot, 1)$ is a monoid;
(PSBL$_3$) $a \odot b \le c$ iff $a \le b \to c$ iff $b \le a \rightsquigarrow c$, for all $a, b, c \in A$;
(PSBL_4) $a \wedge b = (a \to b) \odot a = a \odot (a \rightsquigarrow b)$;
(PSBL_5) $(a \to b) \vee (b \to a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$, for all $a, b \in A$.

Example 2.2. Let $a, b, c, d \in \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers. We put definition

$$(a, b) \le (c, d) \Longleftrightarrow a < c \text{ or } (a = c \text{ and } b \le d).$$

For any $a, b \in \mathbb{R} \times \mathbb{R}$, we define operations $\vee$ and $\wedge$ as follows: $a \vee b = max\{a, b\}$ and $a \wedge b = min\{a, b\}$. Let $A = \{(\frac{1}{2}, b) \in \mathbb{R}^2 : b \ge 0\} \cup \{(a, b) \in \mathbb{R}^2 : \frac{1}{2} < a < 1, b \in \mathbb{R}\} \cup \left\{(1, b) \in \mathbb{R}^2 : b \le 0\right\}$. For $(a, b), (c, d) \in A$, we put:

$$(a, b) \odot (c, d) = (\frac{1}{2}, 0) \vee (ac, bc + d),$$
$$(a, b) \to (c, d) = (\frac{1}{2}, 0) \vee [(\frac{c}{a}, \frac{d - b}{a}) \wedge (1, 0)],$$
$$(a, b) \rightsquigarrow (c, d) = (\frac{1}{2}, 0) \vee [(\frac{c}{a}, \frac{ad - bc}{a}) \wedge (1, 0)].$$

Then $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, (\frac{1}{2}, 0), (1, 0))$ is a pseudo BL-algebra.

Proposition 2.3 ([3]). *If A is a pseudo BL-algebra and $a, b, c \in A$, then*

(psbl-c_1) $a \le b$ *iff* $a \to b = 1$ *iff* $a \rightsquigarrow b = 1$*;*
(psbl-c_2) $a \rightsquigarrow a = a \to a = 1$*;*
(psbl-c_3) $1 \rightsquigarrow a = 1 \to a = a$*;*
(psbl-c_4) $b \le a \rightsquigarrow b$ *and* $b \le a \to b$*;*

(psbl-c_5) $a \odot b \leq a \wedge b$ *and* $a \odot b \leq a, b$;
(psbl-c_6) $a \leq b$ *implies* $b^\sim \leq a^\sim$ *and* $b^- \leq a^-$;
(psbl-c_7) $(a \odot b)^- = a \to b^-$, $(a \odot b)^\sim = b \rightsquigarrow a^\sim$;
(psbl-c_8) $(a \wedge b)^\sim = a^\sim \vee b^\sim$, $(a \vee b)^\sim = a^\sim \wedge b^\sim$;
(psbl-c_9) $(a \wedge b)^- = a^- \vee b^-$, $(a \vee b)^- = a^- \wedge b^-$;
(psbl-c_{10}) $\tilde{1} = \overline{1} = 0$, $\tilde{0} = \overline{0} = 1$;
(psbl-c_{11}) $a \odot a^\sim = a^- \odot a = 0$;
(psbl-c_{12}) $b \leq a^\sim$ *iff* $a \odot b = 0$;
(psbl-c_{13}) $b \leq a^-$ *iff* $b \odot a = 0$;
(psbl-c_{14}) $a \leq a^- \rightsquigarrow b$, $a \leq a^\sim \to b$;
(psbl-c_{15}) $a \to a^\sim = a \rightsquigarrow a^-$.

Lemma 2.4 (*[5]*). *For all $a \in A$ and $n \in \mathbb{N}^*$, $(\overline{a})^n \leq (a^n)^-$, $(\tilde{a})^n \leq (a^n)^\sim$.*

Definition 2.5 (*[6]*). A non empty subset $F \subseteq A$ is called a filter of A, if the following conditions are satisfied
(F_1) If $a, b \in F$, then $a \odot b \in F$;
(F_2) if $a \in F$, $b \in A$, $a \leq b$, then $b \in F$.

The set of all filters of A is denoted by $F(A)$. Clearly if $\odot = \wedge$, then $F(A)$ coincides with the set of filters of the bounded lattice reduct of A.

Proposition 2.6 (*[6]*).
(i) *If $a, b \in A$, and $a \leq b$, then $[b) \subseteq [a)$;*
(ii) *If $a, b \in A$, then $[a \vee b) = [a) \cap [b)$;*
(iii) *For every $a, b \in A$, $[a) \vee [b) = [a \wedge b) = [a \odot b)$.*

Lemma 2.7 (*[7]*). *Let $M \in \mathrm{Max}(A)$. $a \notin M$ iff there exists $m \geq 1$, such that $(a^m)^\sim = (a^m)^- = 1$.*

Definition 2.8 (*[8]*). The intersection of all of maximal filters of A is called the radical of A and will be denoted by $\mathrm{Rad}(A)$.

It is obvious that $\mathrm{Rad}(A)$ is filter of A clearly $\mathrm{Rad}(A) = A$ iff A is trivial, and $\mathrm{Rad}(A)$ is proper filter of A, iff A is non-trivial. An element $a \in A$ is said to be dense iff $\tilde{a} = \overline{a} = 0$. The set of the dense elements of A is denoted by $D(A)$, that is $D(A) = \{a \in A \mid \tilde{a} = \overline{a} = 0\}$, clearly $D(A) \neq \varnothing$ since $\overline{1} = \tilde{1} = 0$.

The set of the complemented elements of the bounded lattice reduct of A is called Boolean center of A and is denoted by $B(A)$. Clearly $0, 1 \in B(A)$. The elements of $B(A)$ are called Boolean elements of A. It is known that $B(A)$ is a Boolean algebra, with the operation induced by those of A together with the complementation operation given by the negation in A. Also it is straightforward that $B(A)$ is a subalgebra of the pseudo BL-algebra. Here are some more properties of the Boolean center of A pseudo BL-algebra.

Remark 2.9. Consider the pseudo BL-algebra $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, (\frac{1}{2}, 0), (1, 0))$ in Example 2.2. Then

$$D(A) = \{(1, 0)\},\ B(A) = \{(1, 0), (\frac{1}{2}, 0)\},\ \mathrm{Max}(A) = [(1, 0)),\ \mathrm{Rad}(A) = [(1, 0)).$$

Lemma 2.10 (*[3]*). $\mathrm{Rad}(A) = \{a \in A \mid (\forall n \in \mathbb{N}), (\exists k_n \in \mathbb{N}^*)$ *s.t.* $\left((a^n)^-\right)^{k_n} = ((a^n)^\sim)^{k_n} = 0\}$.

Corollary 2.11 (*[3]*). *Any element $a \in \mathrm{Rad}(A)$ has $a^\sim, a^- \in N(A)$.*

Proposition 2.12 (*[3]*). *Let A be a pseudo BL-algebra. Then*
(i) $B(A) \cap \mathrm{Rad}(A) = \{1\}$;
(ii) $D(A)$ *is a filter of* A *and* $D(A) \subseteq \mathrm{Rad}(A)$;
(iii) $B(A) \cap D(A) = \{1\}$.

Remark 2.13 (*[9]*).
(i) $e \in B(A)$ has unique complemented, equal to $\tilde{e} = \overline{e}$, and $(\tilde{e})^{-} = (\overline{e})^{\sim} = e$;
(ii) $\tilde{e}, \overline{e} = 0$ iff $e = 1$.

Proposition 2.14 (*[10]*). *If A is a pseudo BL-algebra, then for $e \in A$, the following conditions are equivalent*
(i) $e \in B(A)$;
(ii) $e \odot e = e$ *and* $(\tilde{e})^{-} = (\overline{e})^{\sim} = e$;
(iii) $\tilde{e} \vee e = 1$ *and* $\overline{e} \vee e = 1$.

Lemma 2.15 (*[11]*). *If $e \in B(A)$, $a \in A$ then*
(i) $e \odot a = e \wedge a = a \odot e$;
(ii) $e \wedge \check{e} = 0 = e \wedge \overline{e}$;
(iii) $e \rightsquigarrow a = e \to a$.

Definition 2.16 (*[6]*). A is said to be local iff it has exactly one maximal filter.

Definition above is equivalent to the fact that $\mathrm{Rad}(A)$ is a maximal filter of A, that is A is local iff $\mathrm{Rad}(A) \in \mathrm{Max}(A)$ $[\mathrm{Max}(A) = \{\mathrm{Rad}(A)\}]$.

Proposition 2.17 (*[6]*). *The following conditions are equivalent*
(i) A *is local;*
(ii) $A \setminus N(A)$ *is local;*
(iii) $A \setminus N(A)$ *is the only maximal filter of* A;
(iv) $\mathrm{Rad}(A) = A \setminus N(A)$;
(v) $A = N(A) \cup \mathrm{Rad}(A)$.

A is said to be semilocal iff $\mathrm{Max}(A)$ is finite. Semilocal pseudo BL-algebras include the trivial pseudo BL-algebra, local pseudo BL-algebra, finite BL-algebra, finite direct product of local or other semi local pseudo BL-algebra. The pseudo BL-algebra A is said to be simple iff it has exactly two filters. that is iff A is non-trivial and $\mathcal{F}(A) = \{1, A\}$, A is simple iff $\{1\}$ is a maximal filter of A iff $\{1\}$ is the unique maximal filter of A iff A is local and $\mathrm{Rad}(A) = \{1\}$. An element $a \in A$ is said to be Archimedean iff $a^n \in B(A)$ for some $n \in \mathbb{N}^*$, equivalent with $a \vee (a^n)^{-} = 1$ or $a \vee (a^n)^{\sim} = 1$. A pseudo BL-algebra is called hyper Archimedean iff all elements are Archimedean. Clearly, if $B(A) = A$ that is if underlying bounded lattice of A is a Boolean algebra, then A is a hyper Archimedean pseudo BL-algebra.

Lemma 2.18 (*[3]*).
(i) *Any maximal pseudo BL-algebra is semi local;*
(ii) *If A is local, then* $B(A) = \{0, 1\}$ *and* $A = N(A) \cup \{a \in A \mid \tilde{a}, \overline{a} \in N(A)\}$;
(iii) *A finite direct product of local pseudo BL-algebra has RBLP.*

Proposition 2.19 (*[12]*). *The following are equivalent*
(i) A *is semi local and has RBLP;*
(ii) A *is semi local and* $\mathrm{Rad}(A)$ *has RBLP;*
(iii) *There exist* $n \in \mathbb{N}^*$ *and a complete set* $\{e_1, \ldots, e_n\} \subseteq B(A)$ *such that* $[e_1), \ldots, [e_n)$ *are local pseudo BL-algebra;*
(iv) A *is isomorphic to a finite direct product of local pseudo BL-algebra.*

Remark 2.20. In Example 2.2, A is a local, hence $A = N(A) \cup \{a \in A \mid \tilde{a}, \overline{a} \in N(A)\}$.

Definition 2.21 (*[3]*). A is said to be quasi-local iff for all $a \in A$ there exist $e \in B(A)$ and $n \in \mathbb{N}^*$, such that $a^n \odot e = 0$ and $\overline{e} \odot (\tilde{a})^n = 0$, $(\overline{a})^n \odot \tilde{e} = 0$.

Proposition 2.22 (*[3]*). *The following conditions are equivalent*

(i) *A is quasi-local;*

(ii) *A has RBLP.*

Let us consider a filter F of A. Define two binary relations on A by:

$\equiv L_{(F)}$: $a \equiv L_{(F)} b$ iff $(a \to b \wedge b \to a) \in F$.

$\equiv R_{(F)}$: $a \equiv R_{(F)} b$ iff $(a \rightsquigarrow b) \wedge (b \rightsquigarrow a) \in F$.

For a given filter F, the relations $\equiv L_{(F)}$ and $\equiv R_{(F)}$ are equivalence relations on A, moreover we have $F = \{a \in A, a \equiv L_{(F)} 1\} = \{a \in A, a \equiv R_{(F)} 1\}$.

We shall denote by $A/L_{(F)}$ ($A/R_{(F)}$, respectively) the quotient set associated with $\equiv L_{(F)}$ ($\equiv R_{(F)}$, respectively). $a/L_{(F)}$ ($a/R_{(F)}$, respectively) will denote the equivalence class of $a \in A$ with respect to $\equiv L_{(F)}$($\equiv R_{(F)}$, respectively).

We shall denote the quotient set $A/\equiv (\bmod L(F))$, simply by $A/L(F)$, and its elements by $a/L(F)$) with $a \in A$, so $A/L(F) = \{a/L(F) \mid a \in A\}$, where, for every $a \in A$, $a/L(F) = \{b \in A \mid a \equiv (\bmod L(F))\}$. Also we shall denote by $P_{L(F)}$: $A \to A/L(F)$ the canonical surjection, $P_{L(F)}(a) = a/L(F)$ for all $a \in A$, and for every $X \subseteq A$ we shall denote $P_{L(F)}(X) = X/L(F) = \{a/L(F) \mid a \in A\}$. In particular x, $1 \in L([a))$ iff $(x \to 1) \wedge (1 \to x) \in [a)$ iff $x \wedge 1 \in [a)$ iff $x \in [a)$.

Lemma 2.23 (*[3]*). *For every $a \in A$, the following conditions are equivalent*

(i) *there exists $e \in B(A)$ such that $e \leftrightsquigarrow a \in \left[a \vee \tilde{a}\right)$;*

(ii) *there exists $e \in B(A)$ such that $e \in [a)$ and $\tilde{e} \in [\tilde{a})$.*

Remark 2.24 (*[3]*). For every $a \in A$, the following conditions are equivalent

(i) there exists $e \in B(A)$ such that $e \longleftrightarrow a \in [a \vee \overline{a})$;

(ii) there exists $e \in B(A)$ such that $e \in [a)$ and $\overline{e} \in [\overline{a})$.

Notation 2.25. We shall denote

$$S(A) = \left\{a \in A \mid (\exists\, e \in B(A))\, e \longleftrightarrow a \in \left[a \vee a^-\right),\; e \leftrightsquigarrow a \in [a \vee \tilde{a})\right\}.$$

Remark 2.26. According to Remark 2.24 we have

$$S(A) = \{a \in A \mid (\exists\, e \in B(A)) \text{ such that } e \in [a) \text{ and } \overline{e} \in [\overline{a}) \text{ or } \tilde{e} \in [\tilde{a})\}.$$

Lemma 2.27 (*[13]*).

(i) $B(A) = \{a \in A \mid a \vee \overline{a} = 1\}$;

(ii) $B(A)/F = \left\{a/F \mid a \in A, a \vee a^- = 1\right\}$;

(iii) $B(A/F) = \left\{a/F \mid a \in A \;\; a \vee a^- \in F\right\}$;

(iv) $B(A)/F \subseteq B(A/F)$.

Proposition 2.28 (*[3]*). *The following statements are equivalent*

(i) *A has RBLP;*

(ii) *For all $a \in A$, there exists $e \in B(A)$ such that $e \longleftrightarrow a \in [a \vee \overline{a})$;*

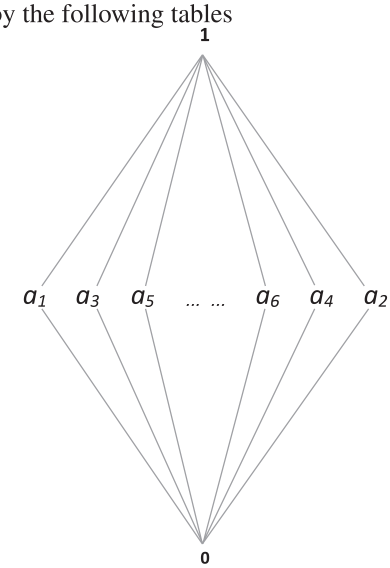
(iii) *For all $a \in A$, there exists $e \in B(A)$ such that $e \leftrightsquigarrow a \in [a \vee \tilde{a})$;*

(iv) *For all $a \in A$, there exists $e \in B(A)$ such that $e \in [a)$ and $\overline{e} \in [\overline{a})$;*

(v) *For all* $a \in A$*, there exists* $e \in B(A)$ *such that* $e \in [a)$ *and* $\tilde{e} \in [\tilde{a})$*;*
(vi) $S(A) = A$.

Remark 2.29. Clearly in Example 2.2, $S(A) = A$, hence A has RBLP.

Example 2.30 (*[3]*)**.** Consider $A = \{0, a_1, a_2, a_3, \ldots, 1\}$, with A bounded lattice structure given by the Hasse diagram below, the implication and $\odot$ given by the following tables



$\rightsquigarrow$	0	a_1	a_2	a_3	a_4	$\ldots$	1
0	1	1	1	1	1	$\ldots$	1
a_1	a_4	1	a_3	1	a_4	$\ldots$	1
a_2	a_3	a_4	1	1	1	$\ldots$	1
a_3	a_2	a_3	a_4	1	a_4	$\ldots$	1
a_4	a_1	a_1	a_3	a_3	1	$\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	a_1	a_2	a_3	a_4	$\ldots$	1

$\rightarrow$	0	a_1	a_2	a_3	a_4	$\ldots$	1
0	1	1	1	1	1	$\ldots$	1
a_1	a_4	1	a_4	1	a_4	$\ldots$	1
a_2	a_3	a_3	1	1	1	$\ldots$	1
a_3	a_2	a_3	a_4	1	a_4	$\ldots$	1
a_4	a_1	a_1	a_3	a_3	1	$\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	a_1	a_2	a_3	a_4	$\ldots$	1

$4\odot$	0	a_1	a_2	a_3	a_4	$\ldots$	1
0	0	0	0	0	0	$\ldots$	0
a_1	0	a_1	0	a_1	0	$\ldots$	a_1
a_2	0	0	0	0	a_2	$\ldots$	a_2
a_3	0	a_1	0	a_1	a_2	$\ldots$	a_3
a_4	0	0	a_2	a_2	a_4	$\ldots$	a_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	a_1	a_2	a_3	a_4	$\ldots$	1

then $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$, is a pseudo BL-algebra.

3. Right (left) Boolean lifting property

Throughout this section unless mentioned otherwise A will be an arbitrary pseudo BL-algebra and F will be an arbitrary filter of A. The canonical morphism $P_{R(F)} : A \to A/R(F)$ induces a Boolean morphism $B(P_{R(F)}) : B(A) \to B(A/R(F))$. The range of this Boolean morphism is $B(P_{R(F)})(B(A)) = P_{R(F)}(B(A)) = B(A)/R(F)$.

Definition 3.1. We say that a Boolean element $f \in B(A/R(F))$ can be right lifted iff there exists a Boolean element $e \in B(A)$ such that $e/R(F) = f$. In other words, $f \in B(A/R(F))$ can be right lifted iff $f \in B(A)/R(F)$.

We say that the equivalence relation $R(F)$ has the right Boolean lifting property (RBLP) iff all Boolean elements of $A/R(F)$ can be right lifted.

Remark 3.2.

(i) For every filter F of A, $R(F)$ has RBLP iff $B(P_{R(F)})$ is surjective.

(ii) For any linearly ordered pseudo BL-algebra A, obviously $B(A) = \{0, 1\}$.

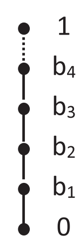
Proof. (i) $R(F)$ has RBLP iff Boolean morphism $B(P_{R(F)}) : B\,(A) \longrightarrow B(A/R(F))$ is surjective. In other words, $B(P_{R(F)})\,(B\,(A)) = B(A/R(F))$ iff $B(A)/R(F) = B(A/R(F))$.

(ii) Let $\tilde{a}$ be a complement of a that means $a \le \tilde{a}$ or $a \ge \tilde{a}$ (since $\tilde{a}, a \in A$, and A is linearly ordered pseudo BL-algebra). Then $a \vee \tilde{a} = 1$, $a \wedge \tilde{a} = 0$, thus $a = 0$ or $a = 1$.

We say that pseudo BL-algebra A has the right Boolean lifting property (RBLP) iff all of its right equivalence relations have RBLP.

For any equivalence relation $R(F)$ of A, the pseudo BL-algebra $A/R(F)$ is also linearly ordered, hence $B(A/R(F)) = \{0/R(F), 1/R(F)\}$ hence $R(F)$ has RBLP.

Example 3.3. Let us consider the chain $A = \{0, b_1, b_2, \ldots, 1\}$, $0 < b_1 < b_2 < \cdots < 1$, organized as a lattice by $a \vee b = max\{a, b\}$, $a \wedge b = min\{a, b\}$, and as in the following tables



$\rightsquigarrow$	0	b_1	b_2	b_3	b_4	$\ldots$	1
0	1	1	1	1	1	$\ldots$	1
b_1	b_3	1	1	1	1	$\ldots$	1
b_2	b_3	b_3	1	1	1	$\ldots$	1
b_3	0	b_3	b_3	1	1	$\ldots$	1
b_4	0	b_1	b_2	b_3	1	$\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	b_1	b_2	b_3	b_4	$\ldots$	1

$\rightarrow$	0	b_1	b_2	b_3	b_4	$\ldots$	1
0	1	1	1	1	1	$\ldots$	1
b_1	b_3	1	1	1	1	$\ldots$	1
b_2	b_3	b_3	1	1	1	$\ldots$	1
b_3	b_3	b_3	b_3	1	1	$\ldots$	1
b_4	0	b_1	b_2	b_3	1	$\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	b_1	b_2	b_3	b_4	$\ldots$	1

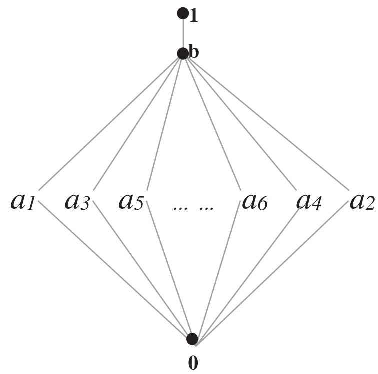
$\odot$	0	b_1	b_2	b_3	b_4	$\ldots$	1
0	0	0	0	0	0	$\ldots$	0
b_1	0	0	0	0	b_1	$\ldots$	b_1
b_2	0	0	0	0	b_2	$\ldots$	b_2
b_3	0	0	0	0	b_3	$\ldots$	b_3
b_4	0	b_1	b_2	b_3	b_4	$\ldots$	b_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	b_1	b_2	b_3	b_4	$\ldots$	1

then $(A\,, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$, is a linearly ordered pseudo BL-algebra.

By Remark 3.2 we have $B(A)/R(F) = B\,(A/R(F))$, for any equivalence relation $R(F)$ of A, hence Boolean elements 0, 1 can be right lifted.

Also in Example 2.2, Boolean elements $(\frac{1}{2}, 0)$, $(1, 0))$ can be right lifted and for any equivalence relation $R(F)$ of A, the pseudo BL-algebra $A/R(F)$ is also linearly ordered, hence $B(A/R(F)) = \left\{(\frac{1}{2}, 0)/R(F), (1, 0)/R(F)\right\}$, thus $R(F)$ has RBLP, therefore A has RBLP.

Example 3.4 (*[3]*)**.** Consider pseudo BL-algebra $A = \{0, a_1, a_2, b, a_3, \ldots, 1\}$, with A bounded lattice structure given by the Hasse diagram below, the implication and $\odot$ given by the following tables



$\odot$	0	a_1	a_2	b	a_3	…	1
0	0	0	0	0	0	…	0
a_1	0	a_1	0	a_1	0	…	a_1
a_2	0	0	0	0	a_2	…	a_2
b	0	a_1	0	a_1	a_2	…	b
a_3	0	0	a_2	a_2	a_3	…	a_3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	0	a_1	a_2	b	a_3	…	1

$\rightarrow$	0	a_1	a_2	b	a_3	…	1
0	1	1	1	1	1	…	1
a_1	a_3	1	a_3	1	a_3	…	1
a_2	b	b	1	1	1	…	1
b	a_2	b	a_3	1	a_3	…	1
a_3	a_1	a_1	b	b	1	…	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	0	a_1	a_2	b	a_3	…	1

$\rightsquigarrow$	0	a_1	a_2	b	a_3	…	1
0	1	1	1	1	1	…	1
a_1	a_3	1	b	1	a_3	…	1
a_2	b	a_3	1	1	1	…	1
b	a_2	b	a_3	1	a_3	…	1
a_3	a_1	a_1	b	b	1	…	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	0	a_1	a_2	b	a_3	…	1

Example 3.5. Consider the pseudo BL-algebra $A = \{0, a_1, a_2, b, a_3, \ldots, 1\}$ in Example 3.4. Then $B(A) = \{0, 1\}$, let us consider the filter $R([b)) = \{b, 1\}$. The element $a_3 \notin B(A)$, $\tilde{a}_3 = a_3 \rightsquigarrow 0 = a_1$. Thus $a_1 \vee a_3 = 1 \in R([b))$ (by Lemma 2.27 (iii)) we have $a_3/R([b)) \in B(A/R([b)))$. And $a_3 \notin R([b)) = \{b, 1\}$. Thus $a_3/R([b)) \neq b/R([b))$ and $0/R([b)) \neq 1/R([b))$, $a_3 \leftrightsquigarrow 0 = (a_3 \rightsquigarrow 0) \wedge (0 \rightsquigarrow a_3) = a_1 \wedge 1 = a_1 \notin R([b))$, $a_3 \leftrightsquigarrow b = (a_3 \rightsquigarrow b) \wedge (b \rightsquigarrow a_3) = b \wedge a_3 = a_3 \notin R([b))$, $a_3 \leftrightsquigarrow a_1 = (a_3 \rightsquigarrow a_1) \wedge (a_1 \rightsquigarrow a_3) = 1 \wedge a_1 = a_1 \notin R([b))$. Hence $a_3/R([b)) \neq a_1/R([b))$, $a_3/R([b)) \neq b/R([b))$, $a_3/R([b)) \neq a_i/R([b))$, $i \in \mathbb{N}$, therefore $a_3/R([b)) = \{a_3\}$. Summarizing the above we have $a_3/R([b)) \in B(A/R([b)))$, while $a_3 \notin B(A)$ and $a_3/R([b)) = \{a_3\}$ hence $a_3/R([b)) \notin B(A)/R([b))$ therefore $B(A/R([b))) \neq B(A)/R([b))$ which means that $R([b))$ does not have RBLP. Hence A does not have RBLP, notice that the maximal filters of A are $R([a_i))$, $i \in \mathbb{N}$, hence $\mathrm{Rad}(A) = \cap R([a_i)) = R([b))$, $i \in \mathbb{N}$, thus $\mathrm{Rad}(A)$ does not have RBLP.

Proposition 3.6. *For every filter F of A, the following conditions are equivalent*
(i) *$B(P_{R(F)})$ is injective;*
(ii) *$B(A) \cap R(F) = \{1\}$.*

Proof. We have $P_{R(F)} : A \rightarrow A/R(F)$, therefore $B(P_{R(F)}) : B(A) \rightarrow B(A/R(F))$.

(ii) $\rightarrow$ (i) Assume that $B(A) \cap R(F) = \{1\}$, $a, b \in B(A)$ such that $B(P_{R(F)})(a) = B(P_{R(F)})(b)$, that is $P_{R(F)}(a) = P_{R(F)}(b)$ which means that $a/R(F) = b/R(F)$ iff $a \longleftrightarrow b \in R(F)$.

(i) $\rightarrow$ (ii) $1 \in B(A)$, $1 \in R(F)$, then $\{1\} \subseteq B(A) \cap R(F)$. Assume that $a \in B(A) \cap R(F)$, thus $a \in R(F)$ hence $a/R(F) = 1/R(F)$, therefore $P_{R(F)}(a) = P_{R(F)}(1)$, then $B_{(P_{R(F)})}(a) = B_{(P_{R(F)})}(1)$ so $a = 1$, hence $B(A) \cap R(F) \subseteq \{1\}$.

Corollary 3.7. *If $B(A) = \{0, 1\}$, then for every proper filter F of A, $B(P_{R(F)})$ is injective.*

Proof. Since $1 \in R(F)$, therefore $B(A) \cap R(F) = \{1\}$, then $B(P_{R(F)})$ is injective.

Remark 3.8. If $(F_i)_{i\in I}$ is a non empty family of filters of A, then $B\big(P_{\cap \boldsymbol{R(F_i)}}\big)_{i\in I} = \cap B\big(P_{\boldsymbol{R(F_i)}}\big)_{i\in I}$.

Proof. We have $B(P_{R(F_i)}) : B(A) \to B\ (A/R(F_i))$, hence $B\big(P_{\cap \boldsymbol{R(F_i)}}\big)_{i\in I} = \cap B\big(P_{\boldsymbol{R(F_i)}}\big)_{i\in I}$ iff $B\big(P_{\cap \boldsymbol{R(F_i)}}\big)_{i\in I}(e) = \cap B\big(P_{\boldsymbol{R(F_i)}}\big)_{i\in I}(e)$, for all $e \in B(A)$ iff $e/\cap R(F_i) = \cap e/R(F_i)$ iff $\{a \in A|(a \rightsquigarrow e) \wedge (e \rightsquigarrow a) \in \cap F_i\}$ iff $\{a \in A|(a \rightsquigarrow e) \wedge (e \rightsquigarrow a) \in F_i\}$, for all $i \in I$, iff $\cap\{a \in A|(a \rightsquigarrow e) \wedge (e \rightsquigarrow a) \in F_i\}$.

Remark 3.9. In Example 2.2, $B(P_{R(F)})$ is injective, for every proper filter F of A, since $B(A) \cap R(F) = \{(1, 0)\}$.

Corollary 3.10. *If $(F_i)_{i\in I}$ is a non empty family of filters of A such that $B(P_{R(F_i)})$, is injective for every $i \in I$, then $B\big(P_{\cap \boldsymbol{R(F_i)}}\big)_{i\in I}$ is injective.*

Proof. $B(P_{R(F_i)})$ is injective by Proposition 3.6, we have $B(A) \cap R(F_i) = \{1\}$ $(\forall i)$, then $B(A) \cap (\cap R(F_i)) = \{1\}$ (for all $i \in I$) hence $B\big(P_{\cap \boldsymbol{R(F_i)}}\big)_{i\in I}$ is injective.

Corollary 3.11.

(i) *Any filter F of A such that $F \subseteq \mathrm{Rad}(A)$, then $B(P_{R(F)})$ injective;*

(ii) *$B(P_{D(A)})$ is injective.*

Proof. (i) We have $R(F) \subseteq \mathrm{Rad}(A)$ therefore $R(F) \cap B(A) \subseteq B(A) \cap \mathrm{Rad}(A) = \{1\}$ then $R(F) \cap B(A) = \{1\}$ hence $(P_{R(F)})$ is injective.

(ii) By Proposition 2.12, we have $D(A) \subseteq \mathrm{Rad}(A)$, thus $D(A) \cap B(A) \subseteq \mathrm{Rad}(A) \cap B(A) = \{1\}$ hence $B(A) \cap D(A) = \{1\}$, therefore $B(P_{D(A)})$ is injective.

Remark 3.12. If $B(A) = \{0, 1\}$, then, according to Remark 2.26 and Proposition 2.3 (psbl-c_{10}), $S(A)$ formed of the element $a \in A$ which satisfy one of the following conditions. $0 \in L([a])$ and $\overline{0} = 1 \in L([\overline{a}])$, that is $a^n = 0$ for some $n \in \mathbb{N}^*$. $1 \in L([a])$ and $\overline{1} = 0 \in L([\overline{a}])$, that is $(\overline{a})^n = 0$ for some $n \in \mathbb{N}^*$. Thus, $S(A)$ contains exactly the element $a \in A$ such that $a, \overline{a}$ are nilpotent that is $S(A) = N(A) \cup \{a \in A \mid \overline{a}, \tilde{a} \in N\,(A)\}$.

In order to prove the main result, we state and prove some lemma and proposition.

4. Right (left) Boolean Lifting property and direct products of pseudo BL-algebra

In this section, we shall prove that a finite direct product pseudo BL-algebra has RBLP (LBLP) iff each pseudo BL-algebra in the products has RBLP (LBLP). Let $(A_i)_{i\in I}$ be a non-empty family of pseudo BL-algebras and $A = \prod_{i\in I} A_i$, clearly, all of elements of A are idempotent, iff in A_i, for each $i \in I$. It is straightforward, from Definition 2.8 that $\mathrm{Rad}(A) \subseteq \prod_{i\in I}\mathrm{Rad}(A)$ and if I is finite, say $I = \overline{1, n}$ with $n \in \mathbb{N}^*$, then the converse inclusion holds as well, so that $\mathrm{Rad}(A) = \prod_{i\in I}\mathrm{Rad}(A_i)$. We denote, for each $i \in I$, the canonical projection by $P_{r_i} : A \to A_i$, which is obviously, a surjective pseudo BL-algebra morphism. Hence for every filter F of A, $P_{r_i}(R\,(F))$ is a right filter of A_i, then clearly $R(F) = \prod_{i\in I} R(F_i)$ is a filter of A with $P_{r_i}(R\,(F)) = R(F_i)$ for all $i \in I$. Generally, main purpose of this section is to study the behavior of RBLP (LBLP) with respect to direct products of pseudo BL-algebra.

Lemma 4.1. *Let $(A_i)_{i\in I}$ be a non-empty family of pseudo BL-algebras and $A = \prod_{i\in I} A_i$. Then $S(A) = S(\prod_{i\in I} A_i) \subseteq \prod_{i\in I} S(A_i)$.*

Proof. By using the fact that $B(A) = B(\prod_{i\in I} A_i) = \prod_{i\in I} B(A_i)$, along with Remark 2.26 for every $a \in S(A)$, then there exists $e \in B(A)$ such that $e \in [a)$ and $\overline{e} \in [\overline{a})$, $\tilde{e} \in [\tilde{a})$. But then $a = (a_i)_{i\in I}$ and $e = (e_i)_{i\in I}$ with $a_i \in A_i$, $e_i \in B(A_i)$ for each $i \in I$ immediately that, for all $i \in I$, we have $e_i \in [a_i)$ and $\overline{e_i} \in [\overline{a_i})$, $\widetilde{e_i} \in [\widetilde{a_i})$ which mean that $a_i \in S(A_i)$, thus $S(A) = S(\prod_{i\in I} A_i) \subseteq \prod_{i\in I} S\,(A_i)$.

Lemma 4.2. *Let $n \in \mathbb{N}^*$, $(A_i)_{i=1}^n$ be a family of pseudo BL-algebras and $A = \prod_{i=1}^n A_i$. Then $S(A) = S(\prod_{i=1}^n A_i) = \prod_{i=1}^n S\,(A_i)$.*

Proof. The fact that $S(A) = S(\prod_{i=1}^{n} A_{i.}) \subseteq \prod_{i=1}^{n} S(A_i)$ follows from Lemma 4.1. Now, let $(a_1, \dots, a_n) \in \prod_{i=1}^{n} S(A_i)$ that is $a_i \in S(A_i)$ for each $i \in \overline{1,n}$ which means that, for every $i \in \overline{1,n}$, there exists an $e_i \in B(A_i)$ such that $e_i \in [a_i)$ and $\overline{e_i} \in [\overline{a_i})$, $\tilde{e} \in [\widetilde{a_i})$ that is $a_i^{j_i} \leq e_i$, $(\overline{a_i})^{k_i} \leq \overline{e_i}$, $(\widetilde{a_i})^{e_i} \leq \widetilde{e_i}$ for some $j_i, k_i, e_i \in \mathbb{N}^*$. Let $J = \max\{j_i \mid i \in \overline{1,n}\} \in \mathbb{N}^*$ and $k = \max\{k_i \mid i \in \overline{1,n}\} \in \mathbb{N}^*$, $e_i = \max\{e_i \mid i \in \overline{1,n}\} \in \mathbb{N}^*$. By psbl-$C_5$ it follows that, for all $i \in \overline{1,n}$, $a_i^j \leq e_i$ and $(\overline{a_i})^k \leq \overline{e_i}$, $(\widetilde{a_i})^{e_i} \leq \widetilde{e_i}$. Hence $a^j = (a_1^j, \dots, a_n^j) \leq (e_1, \dots, e_n)$ and $(\overline{a})^k = ((\overline{a_1})^k, \dots, (\overline{a_n})^k) \leq (\overline{e_1}, \dots, \overline{e_n}) = (e_1, \dots, e_n)^-$, $(\tilde{a})^e = ((\widetilde{a_1})^e, \dots, (\widetilde{a_n})^e) \leq (\widetilde{e_1}, \dots, \widetilde{e_n}) = (e_1, e_2 \dots, e_n)^\sim$, let $e = (e_1, \dots, e_n) \in B(A)$. We have shown that $a^j \leq e$ and $(\overline{a})^k \leq \overline{e}$, $(\tilde{a})^e \leq \tilde{e}$, with $j, k, e \in \mathbb{N}^*$, which means that $e \in [a)$ and $\overline{e} \in [\overline{a})$, $\tilde{e} \in [\tilde{a})$, hence $a \in S(A)$. Therefore $\prod_{i=1}^{n} S(A_i) \subseteq S(A)$. Thus $S(A) = S(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} S(A_i)$.

Lemma 4.3. *Let $(A_i)_{i\in I}$ be a non-empty family of pseudo BL-algebras. If all the elements of A are idempotent, and $A = \prod_{i\in I} A_i$, then, $S(A) = S(\prod_{i\in I} A_i) = \prod_{i\in I} S(A_i)$.*

Proof. The fact that $S(A) = S(\prod_{i\in I} A_i) \subseteq \prod_{i\in I} S(A_i)$ follows from Lemma 4.1. Now, let $a = (a_i)_{i\in I} \in \prod_{i\in I} S(A_i)$ that is $a_i \in S(A_i)$ for each $i \in I$, there exists an $e_i \in B(A_i)$ such that $e_i \in [a)$ and $\overline{e_i} \in [\overline{a_i})$, $\widetilde{e_i} \in [\widetilde{a_i})$ that is $a_i \leq e_i$ and $\overline{a_i} \leq \overline{e_i}$, $\widetilde{a_i} \leq \widetilde{e_i}$, since every element of these pseudo BL-algebras is idempotent. Then $a \leq e$ (since $a_i^n \leq a_i \leq e_i$ hence $a^n = (a_1^n, \dots, a_k, \dots) \leq (e_1^n, \dots)$, therefore $a = a^n \leq (e_1, \dots) = e$ and $\overline{a} \leq \overline{e}$, $\tilde{a} \leq \tilde{e}$ where $e = (e_i)_{i\in I} \in \prod_{i\in I} B(A_i) = B(A)$. Hence $e \in [a)$ and $\overline{e} \in [\overline{a})$, $\tilde{e} \in [\tilde{a})$. Therefore $a \in S(A)$. So the inclusion $\prod_{i\in I} S(A_i) \subseteq S(A)$ holds as well, hence $S(A) = S(\prod_{i\in I} A_i) = \prod_{i\in I} S(A_i)$.

Proposition 4.4. *Let $n \in \mathbb{N}^*$, $(A_i)_{i=1}^{n}$ be a family of pseudo BL-algebras and $A = \prod_{i=1}^{n} A_i$. Then the following conditions are equivalent*

(i) *A has RBLP;*

(ii) *For every $i \in \overline{1,n}$, A_i has RBLP.*

Proof. From Lemma 4.2 and Proposition 2.28 we obtain, A has RBLP iff $S(A_i) = A_i$ iff $S(\prod_{i=1}^{n} A_{i.}) = \prod_{i=1}^{n} A_{i.}$ iff $\prod_{i=1}^{n} S(A_i) = \prod_{i=1}^{n} A_{i.}$ iff $S(A_i) = A_i$ for every $i \in \overline{1,n}$, iff A_i has RBLP for every $i \in \overline{1,n}$.

Proposition 4.5. *Let $(A_i)_{i\in I}$ be a non-empty family of pseudo BL-algebras. If all the elements of A are idempotent and $A = \prod_{i\in I} A_i$, then the following are equivalent*

(i) *A has RBLP;*

(ii) *For every $i \in I$, A_i has RBLP.*

Proof. From Lemma 4.3 and Proposition 2.28 we obtain A has RBLP iff $S(A) = A$, $S(\prod_{i=I} A_i) = \prod_{i\in I} A_i$ iff $\prod_{i=I} S(A_i) = \prod_{i=I} A_i$ iff $S(A_i) = A_i$ for every $i \in I$ iff A_i has RBLP for every $i \in I$.

Proposition 4.6. *Let $(A_i)_{i\in I}$ be a non-empty family of pseudo BL-algebras and $A = \prod_{i\in I} A_i$. If A has RBLP, then for every $i \in I$, A_i has RBLP.*

Proof. From Lemma 4.1 and Proposition 2.28 we obtain if A has RBLP, then $S(A) = A$, thus $\prod_{i\in I} A_i = A = S(A) = S(\prod_{i\in I} A_i) \subseteq \prod_{i=I} S(A_i) \subseteq \prod_{i\in I} A_i$, hence $\prod_{i=I} S(A_i) = \prod_{i\in I} A_i$, therefore $S(A_i) = A_i$ for every $i \in I$, so A_i RBLP for every $i \in I$.

Proposition 4.7. *Let $n \in \mathbb{N}^*.(A_i)_{i=1}^{n}$ be a (finite non-empty) family of pseudo BL-algebras. $A = \prod_{i=1}^{n} A_i$ for each $i \in \overline{1,n}$, let $P_{r_i} : A \to A_i$ be the canonical projection F be an arbitrary filter of A and for every $i \in \overline{1,n}$, let us denote by $R(F_i) = P_{r_i}(R(F))$. Then*

(i) *For each $i \in \overline{1,n}$, Boolean morphism $B(P_{r_i}) : B(A) \to B(A_i)$ is surjective;*

(ii) *The pseudo BL-algebra $A/R(F)$ and $\prod_{i=1}^{n} A_i/R(F_i)$ are isomorphic;*

(iii) *F has RBLP (in A) iff F_i has RBLP (in A_i) for every $i \in \overline{1,n}$;*

(iv) *Boolean morphism $B(P_{R(F)})$ is injective iff the Boolean morphism $B(P_{R(F_i)})$ is injective for every $i \in \overline{1,n}$.*

Proof. (i) $A = \prod_{i=1}^{n} A_i$, hence $B(A) = \prod_{i=1}^{n} B(A_i)$, thus for each $i \in \overline{1,n}$. $B(P_{r_i})$ is the canonical projection from the Boolean algebra $B(A)$ to the Boolean algebra $B(A_i)$ which is a surjective Boolean morphism.

(ii) For all $a \in$A and all $i \in \overline{1,n}$, we shall denote $a_i = P_{r_i}(a) \in A_i$ so a= $(a_1,\ldots,a_n)$ for all $a \in A$, define $\Psi : A/R(F) \to \prod_{i=1}^{n} A_i/R(F_i)$, for every $a \in A$, $\Psi(a/R(F)) = (a_1/R(F_1),\ldots,a_n/R(F_n))$ then, since $R(F) = \prod_{i=1}^{n} R(F_i)$ and for every $a, b \in A$, $a \leftrightsquigarrow b = (a_1 \leftrightsquigarrow b_1,\ldots,a_n \leftrightsquigarrow b_n)$ it follows that Ψ is well defined and injective, because, for all $a, b \in A$, these equivalences hold : $a/R(F) = b/R(F)$ iff $a \leftrightsquigarrow b \in R(F)$ iff $(a_1 \leftrightsquigarrow b_1,\ldots,a_n \leftrightsquigarrow b_n) \in R(F_i)$ iff, for each $i \in \overline{1,n}$, $a_i \leftrightsquigarrow b_i \in R(F_i)$ therefore for each $i \in \overline{1,n}$, $a_i/R(F_i) = b_i/R(F_i)$ iff $(a_1/R(F_1),\ldots,a_n/R(F_n)) = (b_1/R(F_1),\ldots,b_n/R(F_n))$.

Clearly Ψ is a surjective morphism of pseudo BL-algebra. Thus Ψ is pseudo algebra isomorphism.

(iii) According to (ii), $\Psi : A/R(F) \longrightarrow \prod_{i=1}^{n} A_i/R(F_i)$ is a pseudo BL-algebra isomorphism, hence $B(\Psi) : B(A/R(F)) \longrightarrow B(\prod_{i=1}^{n} A_i/R(F_i))$ is a Boolean isomorphism. We denote for every $k \in \overline{1,n}$, by $P_{r'_k} : \prod_{i=1}^{n} A_i/R(F_i) \longrightarrow A_k/R(F_k)$ the canonical projection, then for every $i \in \overline{1,n}$. $P_{r'_i}$ o $\Psi : A/R(F) \longrightarrow A_i/R(F_i)$ is a pseudo BL-algebra morphism, hence $B(P_{r'_i}$ o $\Psi) = B(P_{r'_i})$ o $B(\Psi) : A/R(F) \longrightarrow A_i/R(F_i)$ is a Boolean morphism; moreover, it is a surjective Boolean morphism, because $B(\Psi)$ is a Boolean isomorphism and $B(P_{r'_i})$ is a surjective Boolean morphism. Accordingly, (i) is applied to $\prod_{j=1}^{n} A_j/R(F_j)$ and $A_i/R(F_i)$ instead of A and A_i, respectively. In this implication, $R(F)$ has RBLP. Thus $B(P_{R(F\}})$ is a surjective Boolean morphism. Therefore $B(P_{r'_i}$ o $\Psi)$ o $B(P_{R(F)})$ is a surjective Boolean morphism. For any $i \in \overline{1,n}$. Let $i \in \overline{1,n}$, be arbitrary, we have the following commutative diagram in the category of Boolean algebra.

$$\begin{tikzcd}
\mathrm{B(A)} \arrow[r, "\mathrm{B}(P_{R(F)})"] \arrow[d, "\mathrm{B}(P_{r_i})"'] & \mathrm{B}(A/_{R(F)}) \arrow[d, "\mathrm{B}(P_{r'_i}) \text{ o } \mathrm{B}(\Psi)"'] \\
\mathrm{B}(A_i) \arrow[r, "\mathrm{B}(P_{R(F_i)})"] & \mathrm{B}(A_i/_{R(F_i)})
\end{tikzcd}$$

So $B(P_{R(F_i)})$ o $B(P_{r_i}) = B(P_{r'_i}$ o $\Psi)$ o $B(P_{R(F)})$, which is surjective. Hence $B(P_{R(F_i)})$ is surjective, where F_i has RBLP. Conversely, let for all $i \in \overline{1,n}$, F_i has RBLP, that is $B(A_i/R(F_i)) = B(A_i)/R(F_i)$. Let $a \in A$, such that $a/R(F) \in B(A/R(F))$, then keeping all the notations above.

$$\begin{aligned}
B(\Psi)(a/R(F)) &= \Psi(a/R(F)) \\
&= \prod_{i=1}^{n} a_i/R(F_i) \\
&= (a_1/R(F_1), a_2/R(F_2), \ldots, a_n/R(F_n)) \\
&\in \prod_{i=1}^{n} B(A_i/R(F_i)) \\
&= \prod_{i=1}^{n} B(A_i)/R(F_i)
\end{aligned}$$

thus, for all $i \in \overline{1,n}$, $a_i/R(F_i) \in B(A_i)/R(F_i)$ so there exists $e_i \in B(A_i)$ such that $a_i/R(F_i) = e_i/R(F_i)$.

Since Ψ is injective then $a/R(F) = e/R(F)$, where $e = (e_1,\ldots,e_n) \in \prod_{i=1}^{n} B(A_i) = B(A)$, therefore $a/R(F) = e/R(F) \in B(A)/R(F)$, hence $B(A/R(F)) \subseteq B(A)/R(F)$, thus $B\,(A/R(F)) = B(A)/R(F)$ by Lemma 2.27 (iv), which means that $R(F)$ has RBLP.

(iv) Since $B(A) = \prod_{i=1}^{n} B(A_i)$ and $R(F) = \prod_{i=1}^{n} R(F_i)$, by Proposition 3.6 we get that $B(P_{R(F)})$ is injective iff $B(A) \cap R(F) = \{1\}$ iff, for all $i \in \overline{1,n}$, $B(A_i) \cap R(F_i) = \{1\}$ iff for all $i \in \overline{1,n}$, $B(P_{R(F_i)})$ is injective.

Remark 4.8. If $(A_i)_{i \in I}$ is a non-empty family of pseudo BL-algebras. $A = \prod_{i \in I} A_i$, for all $i \in I$, $P_{r_i} : A \to A_i$ is the canonical projection, F is a filter of A, for all $i \in I$, $P_{r_i}(F) = F_i$ which is a filter of A_i and we define: $\Psi : A/R(F) \to \prod_{i \in I} A_i/R(F_i)$ by for all $a = (a_i)_{i \in I} \in A = \prod_{i \in I} A_i$ (which $a_i \in A_i$ for each $i \in I$), $\psi(a/R(F)) = (a_i/R(F_i))_{i \in I}$, then

(i) For each $i \in I$, the Boolean morphism $B(P_{r_i}) : B(A) \to B(A_i)$ is surjective.

(ii) Ψ is a surjective pseudo BL-algebra morphism.
(iii) If for every $i \in I$, F_i has RBLP, then the Boolean morphism $B(\Psi)$ is surjective.
(iv) If for every $i \in I$, $B(P_{R(F_i)})$ is injective, then $B(P_{R(F)})$ is injective.

5. Some results in pseudo BL-algebras with rith (left) Boolean lifting property

Consider
(*) For all $x \in A$, there exist $u \in \mathrm{Rad}(A)$ and $e \in B(A)$ such that $[x) = [u) \vee [e)$.
(**) For all $x \in A$, there exist $u \in A$ and $e \in B(A)$ such that $\overline{u}, \tilde{u} \in N(A)$ and $[x) = [u) \vee [e)$.
Clearly, the trivial pseudo BL-algebra satisfies in condition (*). Take $x = u = e = 0$, then $[0) = [0) \vee [0) = [0 \odot 0) = [0)$.
Also pseudo BL-algebra of Example 2.2, satisfies in conditions (*) and (**).

Remark 5.1. Corollary 2.11, shows that condition (*) implies condition (**), that is if pseudo-BL algebra A satisfies in condition (*), then A satisfies in condition (**).

Remark 5.2. If all of elements of A are idempotent, then conditions (*) and (**) are equivalent in A. That is, A satisfies in condition (*) iff A satisfies in condition (**).

Proof. By Remark 5.1, condition (*) implies condition (**).

Conversely, since $\overline{u}, \tilde{u} \in N(A)$, by Corollary 2.11, $u \in \mathrm{Rad}(A)$, hence (**) implies (*).

Proposition 5.3.
(i) *If A satisfies in condition* (*)*, then A has RBLP;*
(ii) *If A has RBLP, then A satisfies in condition* (**)*;*
(iii) *If all of elements of A are idempotent, then* (*) *iff* (**) *in A, that is, A satisfies in condition* (*) *iff A has RBLP iff A satisfies in condition* (**).

Proof. (i) Assume that A satisfies in condition (*) and let $x \in A$ be arbitrary. Then there exist $u \in \mathrm{Rad}(A)$ and $e \in B(A)$ such that $[x) = [u) \vee [e) = [u \odot e)$, thus $u \odot e \in [x)$ and $x \in [u \odot e)$, that is there exist $m, n \in \mathbb{N}^*$ such that $x^m \le u \odot e$ and $u^n \odot e = u^n \odot e^n = (u \odot e)^n \le x$ (by Proposition 2.14). Since $x^m \le u \odot e \le e$, we have $e \in [x)$. By psbl-C_{11}, psbl-C_{12}, Proposition 2.14 (ii) $u^n \odot e < x$ implies $\overline{x} \le (u^n \odot e)^- = (e \odot u^n)^- = e \to (u^n)^- = e^{\sim -} \to (u^n)^- = \tilde{e} \vee (u^n)^- = \overline{e} \vee (u^n)^- = (u^n)^- \vee \overline{e}$. Since $u \in \mathrm{Rad}(A)$ by Lemma 2.10 there exists $k_n \in \mathbb{N}^*$ such that $((u^n)^-)^{k_n} = ((u^n)^\sim)^{k_n} = 0$ therefore by psbl-C_{12}, psbl-C_{13}, Proposition 2.14, Lemma 2.15 and distributivity of $\odot$ with respect to $\vee$, and the choice of k_n, we have

$$\begin{aligned}(\overline{x})^{k_n} \odot e &= e \odot (\overline{x})^{k_n} \\ &\le e \odot (\overline{e} \vee (u^n)^-)^{k_n} \\ &= e \odot [(\overline{e})^{k_n} \vee ((\overline{e})^{k_n - 1} \odot (u^n)^- \vee \cdots \vee (\overline{e} \odot ((u^n)^-)^{k'_n - 1}) \vee ((\overline{u})^n)^{k_n}] \\ &= e \odot [\overline{e} \vee (\overline{e} \odot (u^n)^-) \vee \cdots \vee (\overline{e} \odot ((u^n)^-)^{k'_n - 1}) \vee 0] \\ &= e \odot \overline{e}[1 \vee (u^n)^- \vee \cdots \vee ((u^n)^-)^{k'_n - 1})] = 0 \odot 1 = 0\end{aligned}$$

hence $(\overline{x})^{K_n} \le \overline{e}$ and $(\tilde{x})^{K_n} \le \tilde{e}$ so obtain $e \in [x)$, $\overline{e} \in [\overline{x})$, $\tilde{e} \in [\tilde{x})$ and $e \in B(A)$. That is A has RBLP (by Proposition 2.28). (ii) Assume that A has RBLP and let $x \in A$ be arbitrary. By Proposition 2.28, it follows that there exists an $e \in B(A)$ such that $e \in [x)$ and $\overline{e} \in [\overline{x})$, $\tilde{e} \in [\tilde{x})$. So there exist $m, n, s \in \mathbb{N}^*$ such that $x^n \le e$ and $(\overline{x})^m \le \overline{e}$, $(\tilde{x})^s \le \tilde{e}$ thus $e \odot (\tilde{x})^s = 0$, $(\overline{x})^m \odot e = 0$ (by psbl-C_{12}, psbl-C_{13}). Let $u = x \vee \tilde{e} = x \vee \overline{e}$. Then, by psbl-$C_8$, psbl-$C_9$, Proposition 2.14. We have $\tilde{u} = (x \vee \overline{e})^\sim = \tilde{x} \wedge e = \tilde{x} \odot e$, $\overline{u} = (x \vee \tilde{e})^- = \overline{x} \wedge e = \overline{x} \odot e$ hence $(\tilde{u})^s = (\tilde{x} \odot e)^s = (\tilde{x})^s \odot (e)^s = (\tilde{x})^s \odot e = e \odot (\tilde{x})^s = 0$ so $\tilde{u} \in N(A)$ and similarly $\overline{u} \in N(A)$. By the distributivity of $\odot$ with respect to $\vee$ and psbl-C_{11},

$$\begin{aligned} u \odot e &= (x \vee \tilde{e}) \odot e \\ &= (x \vee \overline{e}) \odot e \\ &= (x \odot e) \vee (\tilde{e} \odot e) \\ &= (x \odot e) \vee (\overline{e} \odot e) \\ &= (x \odot e) \vee 0 \\ &= x \odot e \end{aligned}$$

$x^n \leq e$ (see above) and $x^n \leq x$ hence $x^n \leq x \wedge e = x \odot e = u \odot e$ thus $x^n \leq u \odot e$ so u $\odot e \in [x)$, thus $[u \odot e) \subseteq [x)$. But by psbl-$C_5 u \odot e = x \odot e \leq x$, hence $x \in [u \odot e)$, thus $[x) \subseteq [x \odot e) = [x) \vee [e)$. We have obtained, $[x) = [u) \vee [e)$, $\tilde{u}, \overline{u} \in N(A)$ and $e \in B(A)$, so A satisfies in condition (**).

(iii) By (i), (ii) and Remark 5.2.

Lemma 5.4.

(i) *If* $A = B(A) \cup \mathrm{Rad}(A) \cup N(A)$*, then A satisfies in condition* (*)*;*

(ii) *If* $A = \mathrm{Rad}(A) \cup H(A)$*, where* $H(A)$ *is the set of the Archimedean elements of A, then A satisfies in condition* (*).

Proof. (i) Clearly, any Boolean element is Archimedean and any nilpotent element is Archimedean, that is $B(A) \subseteq H(A)$ and $N(A) \subseteq H(A)$, and so this first statement actually follows from the second, we shall provide here a separate proof for the first statement of this lemma. So let that $A = B(A) \cup \mathrm{Rad}(A) \cup N(A)$ and consider an arbitrary element $x \in A$. If $x \in B(A)$, then $u = 1 \in \mathrm{Rad}(A)$ and $e = x \in B(A)$, we have $[x) = [1 \wedge x) = [u \wedge e) = [u) \vee [e)$. If $x \in \mathrm{Rad}(A)$, then u = $x \in \mathrm{Rad}(A)$ and $e = 1 \in B(A)$. We have $[x) = [x \wedge 1) = [u \wedge e) = [u) \vee [e)$. Finally, if $x \in N(A)$, then $[x) = [0) = [1 \wedge 0) = [1) \vee [0)$, and we have $1 \in \mathrm{Rad}(A)$ and $0 \in B(A)$. Therefore A satisfies in condition (*). (ii) Assume that $A = \mathrm{Rad}(A) \cup H(A)$, and let x be an arbitrary element of A. Since the case $x \in \mathrm{Rad}(A)$ has been treated above, it remains to treat the case when $x \in H(A)$. Assume that x is an Archimedean element of A, that is $u = x^n \in B(A)$ for some $n \in \mathbb{N}^*$ then $[x) = [x^n) = [u) = [u \wedge 1) = [u) \vee [1)$, and $1 \in \mathrm{Rad}(A)$. Thus A satisfies in condition (*).

Corollary 5.5. *Any hyper Archimedean pseudo BL-algebra satisfies in condition* (*)*, but the converse is not true.*

Proof. Since all elements of A are Archimedean hence, by above, it is satisfied in condition (*). To show that the converse implication is not true, Example 3.5 has RBLP and all of elements of A are idempotent, thus it satisfies in condition (*) (by Proposition 5.3). But it is not hyper Archimedean since all of its elements are idempotent and $B(A) = \{0, 1\}$, which shows that its middle element a is not Archimedean.

Corollary 5.6. *Any Boolean algebra induces a pseudo-BL algebra with the property* (*).

Proof. For all $x \in A = B(A)$ and $e = 1 \in \mathrm{Rad}(A)$, $u = x \in B(A)$ hence $[x) = [x \wedge 1) = [x) \vee [1) = [u) \vee [e)$.

Remark 5.7. If $(A, \vee, \wedge, *, 0, 1)$ is a Boolean algebra, then if we define for every $x, y \in A, x \odot y = x \wedge y$, $x \to y = x^* \vee y$, $x \rightsquigarrow y = (x \wedge y^*)^*$, then $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$, is a pseudo BL-algebra.

Remark 5.8. If $(A, \vee, \wedge, *, 0, 1)$ is a Boolean algebra, then it satisfies in condition (*), that is for all $a \in A = B(A)$ and $e = 1 \in \mathrm{Rad}(A)$, $u = a \in B(A)$ hence $[a) = [a \wedge 1) = [a) \vee [1) = [u) \vee [e)$.

Proposition 5.9. *Any local pseudo BL-algebra satisfies in condition* (*)*, but the converse is not true.*

Proof. Let A be a pseudo BL-algebra. Then by Proposition 2.17, $A = \mathrm{Rad}(A) \cup N(A)$ now by Lemma 5.4, it follows that A satisfies in condition (*). To show that the converse implication is not true, Example 3.5 has RBLP and all the elements of A are idempotent, thus it satisfies in condition (*) (by Proposition 5.3). But it is not local since $R([a_i))_{i \in I}$ are maximal filters of A.

Lemma 5.10. *Any non-trivial linearly ordered pseudo BL-algebra is local, but the converse is not true.*

Proof. Let A be a non-trivial linearly ordered pseudo BL-algebra. Assume by absurdum that A has two distinct maximal filters M and P, then $M \nsubseteq P$. Thus there exists an element $x \in M \setminus P$, so, for every $y \in P$, $y \nleq x$. But since A is a chain this means that every $y \in P$ satisfies $x < y$, hence $P = [y) \subseteq [x) \subseteq M$, thus $P \subseteq M$, and this is a contradiction to the maximality of P. Therefore A has only one maximal filter; that is A is a local pseudo BL-algebra. But the converse is not true. By using the fact that $\{1\}$ is a maximal filter of A iff $\{1\}$ is the unique maximal filter of A iff A is local and $\mathrm{Rad}(A) = \{1\}$. In Example 2.30, $\{1\}$ is the unique maximal filter of A and $\mathrm{Rad}(A) = \{1\}$, hence, A is local but non-trivial linearly ordered pseudo BL-algebra.

Corollary 5.11. *Any linearly ordered pseudo BL-algebra satisfies in condition* (*).

Proposition 5.12.
(i) *A satisfies in condition* (*) *iff* $A/R(F)$ *satisfies in condition* (*) *for every filter* F *of* A;
(ii) *A satisfies in condition* (**) *iff* $A/R(F)$ *satisfies in condition* (**) *for every filter* F *of* A.

Proof. Let A be satisfied in condition (*) and F be a filter of A. If $x \in A$ arbitrary, then there exist $u \in \mathrm{Rad}(A)$ and $e \in B(A)$ such that $[x) = [u) \vee [e) = [u \odot e)$, that is $x \in [u \odot e)$ and $u \odot e \in [x)$ that is $(u \odot e)^n \leq x$ and $x^k \leq u \odot e$ for some $n, k \in \mathbb{N}^*$. Then $u/R(F) \in \mathrm{Rad}(A/R(F))= Rad(A)/R(F)$ and $e/R(F) \in B(A)/R(F) \subseteq B(A/R(F))$ (by Lemma 2.27 (iv)) $(u/R(F) \odot e/R(F))^n \leq x/R(F)$ and $(x/R(F))^k \leq u/R(F) \odot e/R(F)$, hence $x/R(F) \in [u/R(F) \odot e/R(F))$ and $u/R(F) \odot e/R(F) \in [x/R(F))$. That is $[x/R(F)) = [u/R(F) \odot e/R(F)) = [u/R(F)) \vee [e/R(F))$, therefore $A/R(F)$ satisfies in condition (*).

Conversely, taking $R(F) = \{1\}$.

(ii) Same as the proof for (i). But for the direct implication, if $u \in A$ such that $\overline{u}, \tilde{u} \in$ N(A), then $\overline{u}/R(F)$, $\tilde{u}/R(F) \in N(A/R(F))$.

Remark 5.13. By Proposition 5.3 (iii) and Proposition 4.5, if $(A_i)_{i \in I}$ is a non empty family of pseudo BL-algebras and all elements are idempotent, $A = \prod_{i \in I} A_i$, then A satisfies in condition (*) iff A has RBLP iff A satisfies in condition (**) iff each A_i satisfies in condition (*) iff each A_i has RBLP iff each A_i satisfies in condition (**).

Proposition 5.14. *Let* $(A_i)_{i \in I}$ *be a non-empty family of pseudo BL-algebras and* $A = \prod_{i \in I} A_i$. *Then*
(i) *If A satisfies in condition* (*), *then* A_i *satisfies in condition* (*) *for each* $i \in I$;
(ii) *If A satisfies in condition* (**), *then* A_i *satisfies in condition* (**) *for each* $i \in I$.

Proof. (i) Assume that A satisfies in condition (*), $k \in I$ and $x_k \in A_k$, both arbitrary but fixed. For all $i \in I \backslash_{\{k\}}$, let $x_i \in A_i$ be arbitrary. Then $x = (x_i)_{i \in I} \in A$. Since A satisfies in condition (*), it follows that there exist $u = (u_i)_{i \in I} \in \mathrm{Rad}(A)$ ($u_i \in A_i$ for every $i \in I$) and $e = (e_i)_{i \in I} \in B(A) = \prod_{i \in I} B(A_i)$ ($e_i \in A_i$ for every $i \in I$), such that $[x) = [u) \vee [e) = [u \odot e)$, that is $x \in [u \odot e)$ and $u \odot e \in [x)$ so $(u \odot e)^p \leq x$ and $x^q \leq u \odot e$ for some $p, q \in \mathbb{N}^*$. Thus, for every $i \in I$, $(u_i \odot e_i)^p \leq x_i$ and $x_i^q \leq u_i \odot e_i$ hence $x_i \in [u_i \odot e_i)$ and $u_i \odot e_i \in [x_i)$, which means that $[x_i) = [u_i \odot e_i) = [u_i) \vee [e_i)$, then for every $i \in I$, $u_i \in \mathrm{Rad}(A_i)$ and $e_i \in B(A_i)$, hence $u_k \in \mathrm{Rad}(A_k)$ and $e_k \in B(A_k)$, Also $[x_k) = [u_k) \vee [c_k)$. Hence A_k satisfies in condition (*).

(ii) Similar to the proof of (i), once we notice that, if $u = (u_i)_{i \in I} \in \mathrm{Rad}(A)$, by Corollary 2.11, has $\overline{u}, \tilde{u} \in N(A)$, then for all $i \in I$, $\overline{u_i}, \widetilde{u_i} \in N(A_i)$.

Proposition 5.15. *Let* $n \in \mathbb{N}^*$, $(A_i)_{i=1}^n$ *be a family of pseudo BL-algebras and* $A = \prod_{i=1}^n A_i$. *Then*
(i) *A satisfies in condition* (*) *iff* A_i *satisfies in condition* (*) *for each* $i \in \overline{1, n}$;
(ii) *A satisfies in condition* (**) *iff* A_i *satisfies in condition* (**) *for each* $i \in \overline{1, n}$.

Proof. (i) It follows from Proposition 5.14 (i).

Conversely, assume that each of $A_1, \ldots, A_n$ satisfies in condition (*), and let $x \in A$. Then for every $i \in \overline{1, n}$, there exist $u_i \in \mathrm{Rad}(A_i)$ and $e_i \in B(A_i)$, such that $[x_i) = [u_i) \vee [e_i) = [u_i \odot e_i)$, that is $x_i \in [u_i \odot e_i)$ and $u_i \odot e_i \in [x_i)$ so $(u_i \odot e_i)^{mi} \leq x_i$ and $x_i^{ki} \leq u_i \odot e_i$ for some $m_i, k_i \in \mathbb{N}^*$. Let $e = (e_1, \ldots, e_n) \in \prod_{i=1}^n B(A_i) = B(A)$ and

$u = (u_1, \ldots, u_n) \in \prod_{i=1}^{n} \mathrm{Rad}(A_i)$. Let $m = \max\{m_1, \ldots, m_n\} \in \mathbb{N}^*$ and $k = \max\{k_1, \ldots, k_n\} \in \mathbb{N}^*$. Then we get that, for all $i \in \overline{1, n}$, $(u_i \odot e_i)^m \leq x_i$ and $x_i^{ki} \leq u_i \odot e_i$ that is $(u \odot e)^m \leq x$ and $x^k \leq u \odot e$. Hence $x \in [u \odot e)$ and $u \odot e \in [x)$, which means that $[x) = [u \odot e) = [u) \vee [e)$. Therefore A satisfies in condition (*).

(ii) It follows from Proposition 5.14 (ii).

Conversely, the proof goes similar to the one above in (i). Once we notice that, if $\widetilde{u_i}, \overline{u_i} \in N(A_i)$ for all $i \in \mathbb{N}^*$, then $\tilde{u} = (\widetilde{u_1}, \widetilde{u_2}, \ldots, \widetilde{u_n}) \in N(A)$, $\overline{u} = (\overline{u_1}, \ldots, \overline{u_n}) \in N(A)$.

Proposition 5.16. *The following conditions are equivalent*

(i) *A satisfies in condition (*);*

(ii) *$A/R(\mathrm{Rad}\,(A))$ satisfies in condition (*) and $R(\mathrm{Rad}(A))$ has RBLP (in A).*

Proof. (i)$\Longrightarrow$(ii) By Proposition 5.12 (i), and Proposition 5.3 (i), if A satisfies in condition (*), then $A/R(\mathrm{Rad}\,(A))$ satisfies in condition (*) and A has RBLP, hence $R(\mathrm{Rad}(A))$ has RBLP.

(ii)$\Longrightarrow$(i) Assume that $A/R(\mathrm{Rad}\,(A))$ satisfies in condition (*) and $R(\mathrm{Rad}(A))$ has RBLP. And let $x \in A$ be arbitrary since $A/R(\mathrm{Rad}\,(A))$ satisfies in condition (*), it follows that there exists $u \in A$ such that

$$\begin{aligned} u/R(\mathrm{Rad}\,(A)) &\in \mathrm{Rad}(A/R(\mathrm{Rad}\,(A))) \\ &= \{\{1\}/R(\mathrm{Rad}\,(A))\} \end{aligned}$$

and there exists $e \in A$ such that

$$\begin{aligned} e/R(\mathrm{Rad}\,(A)) &\in B(A/R(\mathrm{Rad}\,(A))) \\ &= B(A)/R(\mathrm{Rad}\,(A)). \end{aligned}$$

Because $R(\mathrm{Rad}(A))$ has RBLP, with the property that

$$\begin{aligned} [x/R(\mathrm{Rad}\,(A))] &= [u/R(\mathrm{Rad}\,(A))] \vee [e/R(\mathrm{Rad}\,(A))] \\ &= [u/R(\mathrm{Rad}\,(A)) \odot e/R(\mathrm{Rad}\,(A))]. \end{aligned}$$

But then $u/R(\mathrm{Rad}\,(A)) = 1/R(\mathrm{Rad}\,(A))$ and there exists $f \in B(A)$ such that

$$e/R(\mathrm{Rad}\,(A)) = f/R(\mathrm{Rad}\,(A)).$$

Therefore,

$$\begin{aligned} [x/R(\mathrm{Rad}\,(A))] &= [1/R(\mathrm{Rad}\,(A)) \odot f/R(\mathrm{Rad}\,(A))] \\ &= [f/R(\mathrm{Rad}\,(A))]. \end{aligned}$$

So $x/R(\mathrm{Rad}\,(A)) \in [f/R(\mathrm{Rad}\,(A))]$ and $[f/R(\mathrm{Rad}\,(A))] \in [x/R(\mathrm{Rad}\,(A))]$, that is

$$\begin{aligned} f/R(\mathrm{Rad}\,(A)) &= f^m/R(\mathrm{Rad}\,(A)) \\ &\leq x/R(\mathrm{Rad}\,(A)) \end{aligned}$$

(since $f \in B(A)$ thus $f^m = f$) and $x^n/R(\mathrm{Rad}\,(A)) \leq f/R(\mathrm{Rad}\,(A))$ for some m, $n \in \mathbb{N}^*$, so

$$\begin{aligned} x^n/R(\mathrm{Rad}\,(A)) &\leq f/R(\mathrm{Rad}\,(A)) \\ &= f^n/R(\mathrm{Rad}\,(A)) \\ &\leq x^n/R(\mathrm{Rad}\,(A)) \end{aligned}$$

thus $x^n/R(\mathrm{Rad}\,(A)) = f/R(\mathrm{Rad}\,(A))$ that is $x^n \leftrightsquigarrow f$, $x^n \leftrightarrow f \in R(\mathrm{Rad}(A))$. So $x^n \leftrightarrow f = u$, $x^n \leftrightsquigarrow f = u$ with $u \in R(\mathrm{Rad}(A))$ thus $(x^n \longrightarrow f) \wedge (f \to x^n) = u$, $(x^n \rightsquigarrow f) \wedge (f \rightsquigarrow x^n) = u$ hence $u \leq x^n \to f$; $u \leq f \to x^n$ and $u \leq x^n \rightsquigarrow f$, $u \leq f \rightsquigarrow x^n$, $u \odot x^n \leq f$, $f \odot u \leq x^n \leq x$ and $x^n \odot u \leq f$, $f \odot u \leq x^n$ since $f \odot u \leq x$ implies $[x) \subseteq [f \odot u) = [u \odot f) = [u) \vee [f)$.

This section contains representation theorems for semi local and maximal pseudo BL-algebra with RBLP (LBLP) and a proof for the fact that local pseudo BL-algebra coincides with quasi-local pseudo-BL algebra whose Boolean center is equal to {0, 1}.

6. Pseudo BL-algebras with right (left) Boolean lifting property

Definition 6.1. A subset $\{x_1, \ldots, x_n\} \subseteq A$, with $n \in \mathbb{N}^*$, is said to complete iff $\bigwedge_{i=1}^{n} x_i = 0$ and $x_i \vee x_j = 1$ for all $i, j \in \overline{1, n}$ such that $i \neq j$. Clearly, if A is non-trivial, then any complete subset of A has at least two elements.

Definition 6.2. Pseudo BL-algebra A is semiperfect iff it satisfies the equivalent conditions from Proposition 2.19.

Corollary 6.3. *Any semiperfect pseudo BL-algebra has RBLP.*

Corollary 6.4. *If* $\mathrm{Rad}(A)$ *has RBLP, then the following are equivalent*

(i) *A is semilocal;*

(ii) *A is isomorphic to a finite direct product of local pseudo BL-algebra.*

Corollary 6.5. *The following conditions are equivalent*

(i) *A is a semilocal pseudo MV-algebra (respectively a semilocal pseudo BL-algebra);*

(ii) *A is isomorphic to a finite direct product of local pseudo MV-algebra (respectively pseudo BL-algebra).*

Corollary 6.6.

(i) *If A is semilocal and* $\mathrm{Rad}(A)$ *has RBLP, then A satisfies in condition* (*)*;*

(ii) *Any semi perfect pseudo BL-algebra satisfies in condition* (*).

Proof. By Corollary 6.4, Propositions 5.9, 5.15 and 2.19 are clear.

Remark 6.7. We show that the converse implication (i) is not true. Example 3.4 has RBLP and all of elements of A are idempotent, thus it satisfies in condition (*) (by Proposition 5.3). But it is not semi local since $R([a_i))_{i \in I}$ are maximal filters of A.

Corollary 6.8. *Any pseudo BL-algebra A with RBLP which is not satisfied in condition (*) has Max(A) infinite. Equivalently if a pseudo BL-algebra A is such that* $\mathrm{Rad}(A)$ *has RBLP and A does not satisfy in condition* (*)*, then A has* $\mathrm{Max}(A)$ *infinite.*

Definition 6.9. A is called a maximal pseudo-BL algebra iff, for any index set I, any family $(a_i)_{i \in I} \subseteq A$ and any family $(F_i)_{i \in I} \subseteq F(A)$, if these families have the property that, given any finite subset J of I, there exists $x_j \in A$ such that $x_j/R(F_i) = a_i/R(F_i)$ for all $i \in J$, then there exists $x \in A$ such that $x/R(F_i) = a_i/R(F_i)$ for all $i \in I$.

Example 6.10. Any simple pseudo BL-algebra is maximal.

Proposition 6.11. *The following conditions are equivalent*

(i) *A is maximal and has RBLP;*

(ii) *A is maximal and* $\mathrm{Rad}(A)$ *has RBLP;*

(iii) *A is isomorphic to a finite direct product of local maximal pseudo BL-algebra.*

Proof. (i)$\Longrightarrow$(ii): Trivial.

(ii)$\Longrightarrow$(iii): According to Lemma 2.18(i) if A is maximal and $e \in B(A)$, then $[e)$ is maximal pseudo BL-algebra. Now apply Proposition 2.19.

(iii)$\Longrightarrow$(i): According to Lemma 2.18 (iii) a finite direct product of local pseudo BL-algebra has RBLP.

Corollary 6.12. *If* $\mathrm{Rad}(A)$ *has RBLP, then the following conditions are equivalent*

(i) *A is maximal;*

(ii) *A is isomorphic to finite direct product of local maximal pseudo BL-algebra.*

Proof. (i) $\Longrightarrow$ (ii) Let A be maximal. Since Rad(A) has RBLP so by Proposition 6.11, A is isomorphic to finite direct product of local maximal pseudo BL-algebra.

(ii)$\Longrightarrow$(i) By Proposition 6.11 is clear.

Proposition 6.13. *The following conditions are equivalent*

(i) *A is local;*

(ii) *A is quasi-local and $B(A) = \{0, 1\}$;*

(iii) *A has RBLP and $B(A) = \{0, 1\}$;*

(iv) *A satisfies* (*) *and $B(A) = \{0, 1\}$;*

(v) $A = N(A) \cup \{x \in A | \overline{x}, \tilde{x} \in N(A)\}$ *and* $B(A) = \{0, 1\}$.

Proof. (i) $\Longrightarrow$ (iv): By Proposition 5.9 and Lemma 2.18.

(iv)$\Longrightarrow$(iii): By Proposition 5.3 (i).

(iii)$\Longrightarrow$(v): By Remark 3.12 and Proposition 2.28.

(v)$\Longrightarrow$(iii): By Lemma 5.4.

(iii)$\Leftrightarrow$(ii): By Proposition 2.22.

(ii)$\Longrightarrow$(i): Assume that A is quasi-local and $B(A) = \{0, 1\}$. Let $x, y \in A$, be such that $x \odot y \in N(A)$. That is $[x) \vee [y) = [x \odot y) = A$, then according to Proposition 2.22 it follows that there exist $e, f \in B(A) = \{0, 1\}$ such that $e \vee F = 1$ (thus $e = 1$ or $f = 1$) and $[x \odot e) = [x) \vee [e) = [x \odot F) = [y) \vee [f) = A$. If $e = 1$, then $[x) = [x \odot 1) = A$, thus $x \in N(A)$: if $f = 1$, then $[y) = [y \odot 1) = A$. Hence $y \in N(A)$, by Proposition 2.17 it follows that A is local.

Corollary 6.14.

(i) *If B(A) = {0, 1}, then A satisfies in condition* (*) *iff A has RBLP iff A is local.*

(ii) *If all the elements of A are idempotent and* $B(A) = \{0, 1\}$*, then A satisfies in condition* (*) *iff A has RBLP iff A satisfies in condition* (**) *iff A is local.*

7. Conclusion

In 1998, Hajek presented BL-algebra; an algebraic semantics of basic fuzzy logic. They are generated by continuous t-norms on the interval [0, 1] and their residuals. Then Georgescu introduced pseudo BL-algebra as a non-commutative extension of BL-algebra. The idea of pseudo BL-algebra originates not only in logic and algebra, but also in algebraic properties that come from the syntax of certain non-classical propositional logics, intuitionistic logic. Lifting property for Boolean elements appears in the study of maximal MV-algebras and maximal BL-algebra. The left lifting property for Boolean elements modulo, the radical, plays an essential part in the structure theorem for maximal pseudo BL-algebra.

We studied the Boolean lifting property on pseudo BL-algebras and it was shown that pseudo BL-algebras with LBLP (RBLP) are exactly the quasi-local pseudo BL-algebras. We showed that arbitrary pseudo BL-algebra has this property iff for each arbitrary element x, there exists a Boolean element in the pseudo BL-algebra such that it belongs to left filter. We consider that our results could contribute to the Boolean lifting theory on pseudo BL-algebras and more some pseudo BL-algebra such as maximal, semi-perfect and local that equal conditions or left Boolean lifting property to them is established, are explained.

We studied the behavior of RBLP (LBLP) with respect to direct products of pseudo BL-algebra. Moreover, we showed that a finite direct product pseudo BL-algebra has RBLP iff each pseudo BL-algebra in the products has RBLP (LBLP) and this holds for individual filter, as well. Weaker results hold for arbitrary direct product of pseudo BL-algebra, so surjective mapping roles are studied in the direct product of pseudo BL-algebras.

In our next research, we are going to consider the notions of Congruence Boolean lifting property, and other lifting properties in particular classes of pseudo BL-algebras.

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Original article

Some aspects of picture fuzzy set

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Abstract

Picture fuzzy set (PFS) is a recently developed tool to deal with uncertainty which is a direct extension of intuitionistic fuzzy set (IFS) that can model uncertainty in such situations involving more answers of these types: yes, abstain, no. In this paper, (α, δ, β)-cut and strong (α, δ, β)-cut of PFS have been defined and decomposition theorems of PFS are proved. Later on extension principle for PFS has been defined and studied some of its properties. Finally, picture fuzzy arithmetic based on extension principle has been performed with examples.

Keywords: Fuzzy set; Fuzzy arithmetic; Extension principle; Picture fuzzy set

1. Introduction

Fuzzy set theory developed by Zadeh [1], plays an important role in decision making under uncertain environment. Various direct/indirect extensions of fuzzy set have been made and successfully applied in most of the problems of real world situation. An important generalization of fuzzy set theory is the theory of intuitionistic fuzzy set (IFS), introduced by Atanassov [2] ascribing a membership degree and a non-membership degree separately in such a way that sum of the two degrees must not exceed one. It is observed that fuzzy sets are IFSs but converse is not necessarily correct. Later IFS has been applied in different areas by various researchers. It is seen that one of the important concept of neutrality degree is lacking in IFS theory. Concept of neutrality degree can be seen in situations when we face human opinions involving more answers of type: yes, abstain, no, refusal. For example, in a democratic election station, the council issues 500 voting papers for a candidate. The voting results are divided into four groups accompanied with the number of papers namely "vote for" (300), "abstain" (64), "vote against" (115) and "refusal of voting" (21). Group "abstain" means that the voting paper is a white paper rejecting both "agree" and "disagree" for the candidate but still takes the vote. Group "refusal of voting" is either invalid voting papers or bypassing the vote. On the other hand, in medical diagnosis degree of neutrality can be considered. E.g., there may not have effect of the symptoms

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temperature, headache on the diseases stomach and chest problems. Similarly, the symptoms stomach pain and chest pain have neutral effect on the diseases viral fever, malaria, typhoid etc. In this regards, Cuong and Kreinovich [3] introduced Picture fuzzy set (PFS) which is a direct extension of fuzzy set and Intuitionistic fuzzy set by incorporating the concept of positive, negative and neutral membership degree of an element. Cuong [4] studied some properties of PFSs and suggested distance measures between PFSs. Phong and co-authors [5] studied some compositions of picture fuzzy relations. Cuong and Hai [6] investigated main fuzzy logic operators: negations, conjunctions, disjunctions and implications on picture fuzzy sets and also constructed main operations for fuzzy inference processes in picture fuzzy systems. Cuong and co-workers [7] presented properties of an involutive picture negator and some corresponding De Morgan fuzzy triples on picture fuzzy sets, Viet and co-authors [8] presented picture fuzzy inference system based on membership graph, Singh [9] studied correlation coefficients of PFSs. Cuong and colleagues [10] investigated the classification of representable picture t-norms and picture t-conorms operators for picture fuzzy sets., Son [11] proposed a new distance measure between PFSs and applied in fuzzy clustering, Son [12] extended basic distance measures in PFSs and examined some of its properties. Son, Viet and Hai [13] proposed fuzzy inference system on PFSs. Peng and Dai [14] proposed an algorithm for PFS and applied in decision making based on new distance measure, Wei [15] presented some process to measure similarity between PFS, Garg [16] studied some picture fuzzy aggregation operations and their applications to multicriteria decision making.

In this paper, an attempt has been made to define (α, δ, β)-cut and strong (α, δ, β)-cut of PFS, height of PFS, level set of PFS and special picture fuzzy set study etc. Based on (α, δ, β)-cut and strong (α, δ, β)-cut of PFS decomposition theorems of PFS will be proved. Then, some properties of (α, δ, β)-cut of PFS will be studied. Further, extension principle for PFS will be defined and some of its properties and finally picture fuzzy arithmetic based on extension principle will be carried out.

2. Preliminaries

In this section some basic concept of fuzzy set, intuitionistic fuzzy set and picture fuzzy set has been reviewed.

2.1. *Fuzzy set [1]*

Fuzzy set is a set in which every element has degree of membership of belonging in it. Mathematically, let X be a universal set. Then the fuzzy subset A of X is defined by its membership function

$$\mu_A : X \to [0, 1]$$

which assign a real number $\mu_A(x)$ in the interval [0, 1], to each element $x \in A$, where the value of $\mu_A(x)$ at x shows the grade of membership of x in A.

2.2. *Intuitionistic fuzzy set [2]*

A Intuitionistic fuzzy set A on a universe of discourse X is of the form

$$A = \{x, \mu_A(x),\ \nu_A(x) : x \in X\},$$

where $\mu_A(x) \in [0, 1]$ is called the "degree of membership of x in A", $\nu_A(x) \in [0, 1]$ is called the "degree of non-membership of x in A", and where $\mu_A(x)$ and $\nu_A(x)$ satisfy the following condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

The amount $\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x))$ is called hesitancy of x which is reflection of lack of commitment or uncertainty associated with the membership or non-membership or both in A.

2.3. *Picture fuzzy set [3]*

A Picture Fuzzy Set (PFS) A on a universe X is an object of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x))\,|x \in X\}$$

where $\mu_A(x) \in [0, 1]$ is called the degree of positive membership (PM) of x in A, $\eta_A(x) \in [0, 1]$ is called the degree of neutral membership (NeuM) of x in A, $\nu_A(x) \in [0, 1]$ is called the degree of negative membership (NM) of x in A.

$\mu_A(x), \eta_A(x), \nu_A(x)$ must satisfy the condition $\mu_A(x) + \eta_A(x) + \nu_A(x) \le 1 \; \forall \, x \in X$.

Then $\forall x \in X,\ 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ is called the degree of refusal membership of "x" in A.

2.4. (α, δ, β)-Cut of picture fuzzy set

Let A be a picture fuzzy set of a universe set X. Then (α, δ, β)-cut of A is a crisp subset $C_{\alpha,\delta,\beta}(A)$ of the IFS A is given by

$C_{\alpha,\delta,\beta}(A) = \{x : x \in X \text{ such that } \mu_A(x) \ge \alpha, \eta_A(x) \le \delta, \nu_A(x) \le \beta\}$,

where $\alpha, \delta, \beta \in [0, 1]$ with $\alpha + \delta + \beta \le 1$.

That is, ${}^{\alpha}A_+ = \{x \in X : \mu_A(x) \ge \alpha\}$, ${}^{\delta}A_\pm = \{x \in X : \eta_A(x) \le \delta\}$ and ${}^{\beta}A_- = \{x \in X : \nu_A(x) \le \beta\}$ are α, δ and β-cut of PF, NeuM & NM respectively of a PFS A, where A_+, $A_\pm$ & A_- indicates positive membership function (PM), Neutral membership function (NeuM) & negative membership function (NM) respectively.

2.5. Strong (α, δ, β)-Cut of picture fuzzy set

Let A be a picture fuzzy set of a universe set X. Then, strong (α, δ, β)-cut of A is a crisp subset ${}^{+}C_{\alpha,\delta,\beta}(A)$ of the IFS A is given by

${}^{+}C_{\alpha,\delta,\beta}(A) = \{x : x \in X \text{ such that } \mu_A(x) > \alpha, \eta_A(x) < \delta, \nu_A(x) < \beta\}$,

where $\alpha, \delta, \beta \in [0,1]$ with $\alpha + \delta + \beta \le 1$.

That is, ${}^{\alpha+}A_+ = \{x \in X : \mu_A(x) > \alpha\}$, ${}^{\delta+}A_\pm = \{x \in X : \eta_A(x) < \delta\}$ and ${}^{\beta+}A_- = \{x \in X : \nu_A(x) < \beta\}$ are strong α, δ and β -cut of PF, NeuM & NM respectively of a PFS A.

2.6. Height of a PFS

Let A be a PFS then height for PM is defined as

$$Hgt(A_+) = Sup\{\mu_A(x)\}$$

height for NeuM is defined as

$$Hgt(A_\pm) = Inf\{\eta_A(x)\}$$

and height for NM is defined as

$$Hgt(A_-) = Inf\{\nu_A(x)\}.$$

2.7. Level Set of PFS

Let A be a PFS then level set for PM is defined as

$$\Lambda(A_+) = \{\alpha : \mu_A(x) = \alpha,\ \alpha \in [0, w_1]\}$$

level set for NeuM is defined as

$$\Lambda(A_\pm) = \{\delta : \eta_A(x) = \delta,\ \delta \in [w_2, 1]\}$$

and level set for NM is defined as

$$\Lambda(A_-) = \{\beta : \nu_A(x) = \beta,\ \beta \in [w_3, 1]\}$$

where w_1, w_2 and w_3 are heights of PM, NeuM and NM of a PFS A.

2.8. Special Picture Fuzzy Set

Let A be a PFS defined on the universe of discourse X.
Then, special Picture fuzzy set with respect to (α, δ, β)-cut is defined as

$$A = (\alpha, \delta, \gamma)C_{(\alpha,\delta,\gamma)}(A).$$

That is, $\alpha A_+ = \alpha.\,{}^{\alpha}A_+$, $\delta A_\pm = \delta.\,{}^{\delta}A_\pm$ and $\gamma A_- = \gamma.\,{}^{\gamma}A_-$ are special picture fuzzy sets of PM, NeuM and NM respectively.

If $x \in {}^{\alpha}A_+$ then $\alpha A_+(x) = \alpha.\,{}^{\alpha}A_+(x) = \alpha$, otherwise $\alpha A_+(x) = \alpha.\,{}^{\alpha}A_+(x) = 0$.
But, if $x \in {}^{\delta}A_\pm$ then $\delta A_\pm(x) = \delta.\,{}^{\delta}A_\pm(x) = \delta$ while if $x \notin {}^{\delta}A_\pm$ then $\delta A_\pm(x) = \delta.\,{}^{\delta}A_\pm(x) = 1$.
Similarly, if $x \in {}^{\gamma}A_-$ then $\gamma A_-(x) = \gamma.{}^{\gamma}A_-(x) = \gamma$ and if $x \notin {}^{\gamma}A_-$ then $\gamma A_-(x) = \gamma.\,{}^{\gamma}A_-(x) = 1$.
In the similar fashion, special picture fuzzy set with respect to strong (α, δ, β)-cut can be defined.

3. Proposition [3]

If $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ and $B = \{(x, \mu_B(x), \eta_B(x), \nu_B(x)) \mid x \in X\}$ be any two PFS of a set X then

(1) $A \subseteq B$ iff $\forall x \in X$, $\mu_A(x) \le \mu_B(x)$, $\eta_A(x) \ge \eta_B(x)$ and $\nu_A(x) \ge \nu_B(x)$.
(2) $A = B$ iff $\forall x \in X$, $\mu_A(x) = \mu_B(x)$, $\eta_A(x) = \eta_B(x)$ and $\nu_A(x) = \nu_B(x)$.
(3) $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x))) \mid x \in X\}$
(4) $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x))) \mid x \in X\}$.

4. Decomposition theorem for PFS

In this section, decomposition theorems for PFS have been discussed.

Theorem 4.1 (*First Decomposition Theorem*).
Let X be a universe of discourse. For any PFS $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ in X,

$$A = \bigcup_{\substack{\alpha\in[0,w_1],\\ \delta\in[w_2,1],\\ \gamma\in[w_3,1]}} (\alpha, \delta, \gamma)C_{(\alpha,\delta,\gamma)}(A)$$
$$= \begin{cases} \bigcup\limits_{\alpha\in[0,w_1]} \alpha A_+ & \text{for PM} \\ \bigcap\limits_{\alpha\in[w_2,1]} \delta A_\pm & \text{for NeuM} \\ \bigcap\limits_{\alpha\in[w_3,1]} \gamma A_- & \text{for NM} \end{cases}$$

where A_+, $A_\pm$&A_- indicates positive membership function (PM), Neutral membership function (NeuF) & negative membership function (NF) respectively; $\alpha A_+ = \alpha.\,{}^{\alpha}A_+$, $\delta A_\pm = \delta.\,{}^{\delta}A_\pm$ and $\gamma A_- = \gamma.\,{}^{\gamma}A_-$ are special picture fuzzy sets; $\cup$ and $\cap$ are standard fuzzy union and intersection respectively. Also, where w_1, w_2 and w_3 are heights of PM, NeuM and NM of the PFS A.

Proof. For PM, let for each $x \in X$, $\mu_A(x) = a$ where $a \in [0, w_1]$ which indicates degree of belonging in A.
Then,

$$\begin{aligned} \bigcup_{\alpha\in[0,w_1]} \alpha A_+(x) &= \mathop{Sup}_{\alpha\in[0,w_1]} \alpha^{\alpha} A_+(x) \\ &= \max\left[\mathop{Sup}_{\alpha\in[0,a]} \alpha^{\alpha} A_+(x),\ \mathop{Sup}_{\alpha\in(a,w_1]} \alpha^{\alpha} A_+(x)\right]. \end{aligned} \tag{4.1}$$

If $\alpha \in [0, a]$ then $\alpha \le a = \mu_A(x)$

i.e., $\alpha \in {}^{\alpha}A_{+}$ then $\alpha^{\alpha}A_{+} = \alpha$.
If $\alpha \in (a, w_1]$ then $\alpha > a = \mu_A(x)$
i.e., $\alpha \notin {}^{\alpha}A_{+}$ then $\alpha^{\alpha}A_{+} = 0$.
Hence from (4.1), we have

$$\begin{aligned}\bigcup_{\alpha\in[0,w_1]} \alpha A_{+} &= \max[\underset{\alpha\in[0,a]}{Sup}\ \alpha, 0]\\ &= a\\ &= \mu_A(x).\end{aligned}$$

For NeuM, let for each $x \in X$, $\eta_A(x) = b$ where $b \in [w_2, 1]$ which indicates degree of neutrality in A.

$$\begin{aligned}\bigcap_{\delta\in[w_2,1]} \delta A_{\pm} &= \underset{\delta\in[w_2,1]}{Inf}\ \delta^{\delta}A_{\pm}\\ &= \min[\underset{\delta\in[w_2,b)}{Inf}\ \delta^{\delta}A_{\pm}, \underset{\delta\in[b,1]}{Inf}\ \delta^{\delta}A_{\pm}].\end{aligned} \tag{4.2}$$

If $\delta \in [w_2, b)$ then $\delta < b = \eta_A(x)$.
i.e., $\delta \notin {}^{\delta}A_{\pm}$ then $\delta^{\delta}A_{\pm} = 1$.
If $\delta \in [b, 1]$ then $\delta \geq b = \eta_A(x)$
i.e., $\delta \in {}^{\delta}A_{\pm}$ then $\delta^{\delta}A_{\pm} = \delta$.
Hence from (4.2), we have

$$\begin{aligned}\bigcap_{\delta\in[w_2,1]} \delta A_{\pm} &= \min[1, \underset{\delta\in[b,1]}{Inf}\delta]\\ &= b\\ &= \eta_A(x), \forall x \in X.\end{aligned}$$

For NM, let for each $x \in X$, $\nu_A(x) = b$ where $c \in [w_3, 1]$ which indicates degree of non-belonging in A.

$$\begin{aligned}\bigcap_{\gamma\in[w_3,1]} \gamma A_{-} &= \underset{\gamma\in[w_3,1]}{Inf}\ \gamma^{\gamma}A_{-}\\ &= \min[\underset{\gamma\in[w_3,c)}{Inf}\ \gamma^{\gamma}A_{-}, \underset{\gamma\in[c,1]}{Inf}\ \gamma^{\gamma}A_{-}]\end{aligned} \tag{4.3}$$

If $\gamma \in [w_3, c)$ then $\gamma < c = \nu_A(x)$
i.e., $\gamma \notin {}^{\gamma}A_{-}$ then $\gamma^{\gamma}A_{-} = 1$.
If $\gamma \in [c, 1]$ then $\alpha \geq c = \nu_A(x)$
i.e., $\gamma \in {}^{\gamma}A_{-}$ then $\gamma^{\gamma}A_{-} = \gamma$.
Hence from (4.3), we have

$$\begin{aligned}\bigcap_{\gamma\in[w_3,1]} \gamma A_{-} &= \min[1, \underset{\gamma\in[c,1]}{Inf}\gamma]\\ &= c\\ &= \nu_A(x), \forall x \in X.\end{aligned}$$

Thus, $A = \bigcup_{\substack{\alpha\in[0,w_1],\\ \delta\in[w_2,1],\\ \gamma\in[w_3,1]}} (\alpha, \delta, \gamma)C_{(\alpha,\delta,\gamma)}(A)$.

Theorem 4.2 (*Second Decomposition Theorem*).

Let X be a universe of discourse. For any PFS $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ in X,

$$A = \bigcup_{\substack{\alpha\in[0,w_1],\\ \delta\in[w_2,1],\\ \gamma\in[w_3,1]}} (\alpha,\delta,\gamma)^{[+]} C_{(\alpha,\delta,\gamma)}(A)$$

$$= \begin{cases} \bigcup_{\alpha\in[0,w_1]} (\alpha+)A_+ & \text{for MP} \\ \bigcap_{\alpha\in[w_2,1]} (\delta+)A_\pm & \text{for NeuM} \\ \bigcap_{\alpha\in[w_3,1]} (\gamma+)A_- & \text{for NM} \end{cases}$$

where A_+, $A_\pm$&A_- indicates positive membership function (PM), Neutral membership function (NeuF) & negative membership function (NF) respectively; $(\alpha+)A_+ = \alpha.\ {}^{\alpha+}A_+$, $(\delta+)A_\pm = \delta.\ {}^{\delta+}A_\pm$ and $(\gamma+)A_- = \gamma.\ {}^{\gamma+}A_-$ are special picture fuzzy sets; $\cup$ and $\cap$ are standard fuzzy union and intersection respectively. Also, where w_1, w_2 and w_3 are heights of PM, NeuM and NM of a PFS A.

Proof. For PM, let for each $x \in X$, $\mu_A(x) = a$ where $a \in [0, w_1]$ which indicates degree of belonging in A.

Then,

$$\begin{aligned} \bigcup_{\alpha\in[0,w_1]} (\alpha+)A_+ &= \operatorname*{Sup}_{\alpha\in[0,w_1]} \alpha^{\alpha+}A_+ \\ &= \max\left[\operatorname*{Sup}_{\alpha\in[0,a)} \alpha^{\alpha+}A_+,\ \operatorname*{Sup}_{\alpha\in[a,w_1]} \alpha^{\alpha+}A_+\right]. \end{aligned} \tag{4.4}$$

If $\alpha \in [0, a)$ then $\alpha < a = \mu_A(x)$

i.e., $\alpha \in {}^{\alpha+}A_+$ then $\alpha^{\alpha+}A_+ = \alpha$.

If $\alpha \in [a, w_1]$ then $\alpha \geq a = \mu_A(x)$

i.e., $\alpha \notin {}^{\alpha+}A_+$ then $\alpha^{\alpha+}A_+ = 0$.

Hence from (4.4), we have

$$\begin{aligned} \bigcup_{\alpha\in[0,w_1]} (\alpha+)A_+ &= \max[\operatorname*{Sup}_{\alpha\in[0,a)} \alpha, 0] \\ &= a \\ &= \mu_A(x), \forall x \in X. \end{aligned}$$

For NeuM, let for each $x \in X$, $\eta_A(x) = b$ where $b \in [w_2, 1]$ which indicates degree of neutrality in A.

$$\begin{aligned} \bigcap_{\delta\in[w_2,1]} (\delta+)A_\pm &= \operatorname*{Inf}_{\delta\in[w_2,1]} \delta^{\delta+}A_\pm \\ &= \min\left[\operatorname*{Inf}_{\delta\in[w_2,b]} \delta^{\delta+}A_\pm,\ \operatorname*{Inf}_{\delta\in(b,1]} \delta^{\delta+}A_\pm\right]. \end{aligned} \tag{4.5}$$

If $\delta \in [w_2, b]$ then $\delta \leq b = \eta_A(x)$.

i.e., $\delta \notin {}^{\delta+}A_\pm$ then $\delta^{\delta+}A_\pm = 1$.

If $\delta \in (b, 1]$ then $\delta > b = \eta_A(x)$

i.e., $\delta \in {}^{\delta+}A_\pm$ then $\delta^{\delta+}A_\pm = \delta$.

Hence from (4.5), we have

$$\begin{aligned} \bigcap_{\delta\in[w_2,1]} (\delta+)A_\pm &= \min[1, \operatorname*{Inf}_{\delta\in[b,1]} \delta] \\ &= b \\ &= \eta_A(x), \forall x \in X. \end{aligned}$$

For NM, let for each $x \in X$, $\nu_A(x) = b$ where $c \in [w_3, 1]$ which indicates degree of non-belonging in A.

$$\begin{aligned}\bigcap_{\gamma\in[w_3,1]}(\gamma+)A_- &= \mathop{Inf}_{\gamma\in[w_3,1]} \gamma^{\gamma+}A_- \\ &= \min\left[\mathop{Inf}_{\gamma\in[w_3,c]} \gamma^{\gamma+}A_-, \mathop{Inf}_{\gamma\in(c,1]} \gamma^{\gamma+}A_-\right].\end{aligned} \tag{4.6}$$

If $\gamma \in [w_3, c]$ then $\gamma \le c = \nu_A(x)$
i.e., $\gamma \notin {}^{\gamma+}A_-$ then $\gamma^{\gamma+}A_- = 1$.
If $\gamma \in (c, 1]$ then $\alpha > c = \nu_A(x)$
i.e., $\gamma \in {}^{\gamma+}A_-$ then $\gamma^{\gamma+}A_- = \gamma$.
Hence from (4.6), we have

$$\begin{aligned}&\bigcap_{\gamma\in[w_3,1]}(\gamma+)A_- = \min[1, \mathop{Inf}_{\gamma\in[c,1]}\gamma] \\ &= c \\ &= \nu_A(x), \forall x \in X.\end{aligned}$$

$$A = \bigcup_{\substack{\alpha\in[0,w_1],\\ \delta\in[w_2,1],\\ \gamma\in[w_3,1]}} (\alpha, \delta, \gamma)^{[+]}C_{(\alpha,\delta,\gamma)}(A)$$

Theorem 4.3 (*Third Decomposition Theorem*).
Let X be a universe of discourse. For any PFS $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \,|x \in X\}$ in X ,

$$\begin{aligned}A &= \cup_{\substack{\alpha\in\Lambda(A_+),\\ \delta\in\Lambda(A_\pm),\\ \gamma\in\Lambda(A_-)}} (\alpha, \delta, \gamma)C_{(\alpha,\delta,\gamma)}(A) \\ &= \begin{cases}\bigcup_{\alpha\in\Lambda(A_+)} \alpha A_+ & \text{for PM}\\ \bigcap_{\alpha\in\Lambda(A_\pm)} \delta A_\pm & \text{for NeuM}\\ \cap_{\alpha\in\Lambda(A_-)} \gamma A_- & \text{for NM}\end{cases}\end{aligned}$$

where $A_+, A_\pm \& A_-$ indicates positive membership function (PM), Neutral membership function (NeuF) & non-membership function (NF) respectively; $\alpha A_+ = \alpha.\, {}^{\alpha}A_+, \delta A_\pm = \delta.\, {}^{\delta}A_\pm$ and $\gamma A_- = \gamma.\, {}^{\gamma}A_-$ are special picture fuzzy sets; $\cup$ and $\cap$ are standard fuzzy union and intersection respectively and $\Lambda(A_+)$, $\Lambda(A_\pm)$ and $\Lambda(A_-)$ are the level sets of PM, NeuM and NM of A.
Proof is straight forward as above.

5. Properties of picture fuzzy set

In this section, some properties of (α, δ, β)-Cut of picture fuzzy set (PFS).

Theorem 5.1. *If A and B be two PFS' of a universe set X, then the following holds*

(I) $C_{\alpha,\delta,\beta}(A) \subseteq C_{??,??,??}(A)$ *if* $\alpha \ge ??, \delta \le ??, \beta \le ??$
(II) $C_{1-\delta-\beta,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,1-\alpha-\beta,\beta}(A)$
(III) $A \subseteq B$ *implies* $C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(B)$
(IV) $C_{\alpha,\delta,\beta}(A \cap B) = C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B)$
(V) $C_{\alpha,\delta,\beta}(A \cup B) \supseteq C_{\alpha,\delta,\beta}(A) \cup C_{\alpha,\delta,\beta}(B)$
(VI) $C_{\alpha,\delta,\beta}(\cap A_i) = \cap C_{\alpha,\delta,\beta}(A_i)$
(VII) $C_{1,0,0}(A) = X$.

Proof.

(I) $C_{\alpha,\delta,\beta}(A) \subseteq C_{??,??,??}(A)$ if $\alpha \geq ??$, $\delta \leq ??$, $\beta \leq ??$.

Let $x \in C_{\alpha,\delta,\beta}(A)$
$\Rightarrow \mu_A(x) \geq \alpha,\ \eta_A(x) \leq \delta,\ \nu_A(x) \leq \beta$
Since, we have $\alpha \geq ??$, $\delta \leq ??$, $\beta \leq ??$
$\Rightarrow \mu_A(x) \geq \alpha \geq ??,\ \eta_A(x) \leq \delta \leq ??,\ \nu_A(x) \leq \beta \leq ??$
$\Rightarrow \mu_A(x) \geq ??,\ \eta_A(x) \leq ??,\ \nu_A(x) \leq ??$
$\Rightarrow x \in C_{??,\theta,??}(A)$
$\Rightarrow C_{\alpha,\delta,\beta}(A) \subseteq C_{??,??}(A)$.

(II) $C_{1-\delta-\beta,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,1-\alpha-\beta,\beta}(A)$

Since $\alpha + \delta + \beta \leq 1$ implies that $1 - \delta - \beta \geq \alpha$ and $\delta \leq \delta$, $\beta \leq \beta$.

$$\text{Therefore, from } (I) \text{ we get } C_{1-\delta-\beta,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A) \tag{5.1}$$

Again $\alpha \geq \alpha$, $\delta \leq 1 - \alpha - \beta$ and $\beta \leq \beta$

$$\text{Therefore, from } (I) \text{ we get } C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,1-\alpha-\beta,\beta}(A). \tag{5.2}$$

From (5.1) and (5.2) we get

$C_{1-\delta-\beta,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,1-\alpha-\beta,\beta}(A)$

(III) $A \subseteq B$ implies $\mathrm{C}_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(\mathbf{B})$

Let $x \in C_{\alpha,\delta,\beta}(A)$
$\Rightarrow \mu_A(x) \geq \alpha,\ \eta_A(x) \leq \delta,\ \nu_A(x) \leq \beta$
As $B \supseteq A$
$\Rightarrow \mu_B(x) \geq \mu_A(x) \geq \alpha,\ \eta_B(x) \leq \eta_A(x) \leq \delta,\ \nu_B(x) \leq \nu_A(x) \leq \beta$
$\Rightarrow \mu_B(x) \geq \alpha,\ \eta_B(x) \leq \delta,\ \nu_B(x) \leq \beta$
$\Rightarrow x \in C_{\alpha,\delta,\beta}(B)$
$\Rightarrow C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(B)$

(IV) $C_{\alpha,\delta,\beta}(A \cap B) = C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B)$

We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$
Therefore, from (III)
$C_{\alpha,\delta,\beta}(A \cap B) \subseteq C_{\alpha,\delta,\beta}(A)$ and $C_{\alpha,\delta,\beta}(A \cap B) \subseteq C_{\alpha,\delta,\beta}(B)$

$$\Rightarrow C_{\alpha,\delta,\beta}(A \cap B) \subseteq C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B). \tag{5.3}$$

Now, let

$x \in C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B)$
$\Rightarrow x \in C_{\alpha,\delta,\beta}(A)$ and $x \in C_{\alpha,\delta,\beta}(B)$
$\Rightarrow \mu_A(x) \geq \alpha;\ \mu_B(x) \geq \alpha \Rightarrow \mu_A(x) \wedge \mu_B(x) \geq \alpha \Rightarrow (\mu_A \cap \mu_B)(x) \geq \alpha$
$\eta_A(x) \leq \delta;\ \eta_B(x) \leq \delta \Rightarrow \eta_A(x) \vee \eta_B(x) \leq \delta \Rightarrow (\eta_A \cap \eta_B)(x) \leq \delta$
$\nu_A(x) \leq \beta;\ \nu_B(x) \leq \beta \Rightarrow \nu_A(x) \vee \nu_B(x) \leq \beta \Rightarrow (\nu_A \cap \nu_B)(x) \leq \beta$
$\Rightarrow x \in C_{\alpha,\delta,\beta}(A \cap B)$

$$\Rightarrow C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B) \subseteq C_{\alpha,\delta,\beta}(A \cap B). \tag{5.4}$$

From (5.3) and (5.4) we have

$C_{\alpha,\delta,\beta}(A \cap B) = C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B)$.

(V) $C_{\alpha,\delta,\beta}(A \cup B) \supseteq C_{\alpha,\delta,\beta}(A) \cup C_{\alpha,\delta,\beta}(\mathbf{B})$

Again, since $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
Hence from (III) we get
$C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A \cup B)$ and $C_{\alpha,\delta,\beta}(B) \subseteq C_{\alpha,\delta,\beta}(A \cup B)$
$\Rightarrow C_{\alpha,\delta,\beta}(A) \cup C_{\alpha,\delta,\beta}(B) \subseteq C_{\alpha,\delta,\beta}(A \cup B)$

(VI) $C_{\alpha,\delta,\beta}(\cap A_i) = \cap C_{\alpha,\delta,\beta}(A_i)$

: Let $x \in C_{\alpha,\delta,\beta}(\cap A_i)$
$\Rightarrow (\cap \mu_{Ai})(x) \geq \alpha, (\cap \eta_{Ai})(x) \leq \delta, (\cap \nu_{Ai})(x) \leq \beta$
$\Rightarrow \wedge \mu_{Ai}(x) \geq \alpha, \vee \eta_{Ai}(x) \leq \delta, \vee \nu_{Ai}(x) \leq \beta$
$\Rightarrow x \in C_{\alpha,\delta,\beta}(A_i)$; for all '$i$'
$\Rightarrow x \in \cap C_{\alpha,\delta,\beta}(A_i)$
$$\Rightarrow C_{\alpha,\delta,\beta}(\cap A_i) \subseteq \cap C_{\alpha,\delta,\beta}(A_i) \tag{5.5}$$

Let $x \in \cap C_{\alpha,\delta,\beta}(A_i)$
$\Rightarrow x \in C_{\alpha,\delta,\beta}(A_i)\ \forall i$.
$\Rightarrow \wedge \mu_{Ai}(x) \geq \alpha, \vee \eta_{Ai}(x) \leq \delta, \vee \nu_{Ai}(x) \leq \beta$
$\Rightarrow (\cap \mu_{Ai})(x) \geq \alpha, (\cap \eta_{Ai})(x) \leq \delta, (\cap \nu_{Ai})(x) \leq \beta$
$\Rightarrow x \in C_{\alpha,\delta,\beta}(\cap A_i)$
$$\Rightarrow \cap C_{\alpha,\delta,\beta}(A_i) \subseteq C_{\alpha,\delta,\beta}(\cap A_i). \tag{5.6}$$

From (5.5) and (5.6) we have

$C_{\alpha,\delta,\beta}(\cap A_i) = \cap C_{\alpha,\delta,\beta}(A_i)$.

(VII) $C_{1,0,0}(A) = X$

$$\begin{aligned} C_{1,0,0}(A) &= \{x : x \in X \text{ such that } \mu_A(x) \geq 1, \eta_A(x) \leq 0, \nu_A(x) \leq 0\} \\ &= X. \end{aligned}$$

6. Extension principle of PFS

Definition 6.1. Let X and Y be two non empty sets and $f : X \to Y$ be a mapping. Let A and B be PFS' of X and Y respectively. Then the image of A under the map f is denoted by $f(A)$ and is defined as

$f(A)(y) = (\mu_{f(A)}(y), \eta_{f(A)}(y), \nu_{f(A)}(y))$, where
$\mu_{f(A)}(y) = \vee\{\mu_A(x) : x \in f^{-1}(y)\}$;
$\eta_{f(A)}(y) = \wedge\{\eta_A(x) : x \in f^{-1}(y)\}$;
$\nu_{f(A)}(y) = \wedge\{\nu_A(x) : x \in f^{-1}(y)\}$;

i.e $f(A)(y) = (\vee\{\mu_A(x) : x \in f^{-1}(y)\}, \wedge\{\eta_A(x) : x \in f^{-1}(y)\}, \wedge\{\nu_A(x) : x \in f^{-1}(y)\})$.

Also the pre-image of B under 'f' is denoted by $f^{-1}(B)$ and is defined as

$f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x), \eta_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x))$

where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x)), \eta_{f^{-1}(B)}(x) = \eta_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$

i.e $f^{-1}(B)(x) = \mu_B(f(x)), \eta_B(f(x)), \nu_B(f(x))$.

Theorem 6.1. *Let $f : X \to Y$ be a mapping. Then the following holds*
(i) $f(C_{\alpha,\delta,\beta}(A)) \subseteq C_{\alpha,\delta,\beta}(f(A))$, for all $A \in \mathrm{PFS}(X)$.
(ii) $f^{-1}(C_{\alpha,\delta,\beta}(B)) = C_{\alpha,\delta,\beta}(f^{-1}(B))$, for all $B \in \mathrm{PFS}(Y)$.

Proof. (i) Let $y \in f(C_{\alpha,\delta,\beta}(A))$ then there exists a $x \in C_{\alpha,\delta,\beta}(A)$ such that

$$f(x) = y \text{ and } \mu_A(x) \geq \alpha,\ \eta_A(x) \leq \delta,\ \nu_A(x) \leq \beta$$
$$\Rightarrow \vee\{\mu_A(x) : x \in f^{-1}(y)\} \geq \alpha,\ \wedge\{\eta_A(x) : x \in f^{-1}(y)\} \leq \delta,\ \wedge\{\nu_A(x) : x \in f^{-1}(y)\} \leq \beta$$
$$\text{i.e } \mu_{f(A)}(y) \geq \alpha,\ \eta_{f(A)}(y) \leq \delta,\ \nu_{f(A)}(y) \leq \beta$$
$$\text{i.e } y \in C_{\alpha,\delta,\beta}(f(A)).$$

Hence $f(C_{\alpha,\delta,\beta}(A)) \subseteq C_{\alpha,\delta,\beta}(f(A))$, for all A $\in IFS(X)$.
(ii) $C_{\alpha,\delta,\beta}(f^{-1}(B))$

$$\begin{aligned}
&= \{x \in X : \mu_{f^{-1}(B)}(x) \geq \alpha,\ \eta_{f^{-1}(B)}(x) \leq \delta,\ \nu_{f^{-1}(B)}(x) \leq \beta\} \\
&= \{x \in X : \mu_B(f(x)) \geq \alpha,\ \eta_B(f(x)) \leq \delta,\ \nu_B(f(x)) \leq \beta\} \\
&= \{x \in X : f(x) \in C_{\alpha,\delta,\beta}(B)\} \\
&= \{x \in X : x \in f^{-1}(C_{\alpha,\delta,\beta}(B))\} \\
&= f^{-1}(C_{\alpha,\delta,\beta}(B))
\end{aligned}$$

Thus, $C_{\alpha,\delta,\beta}(f^{-1}(B)) = f^{-1}(C_{\alpha,\delta,\beta}(B))$

7. Picture fuzzy arithmetic

In this section, picture fuzzy arithmetic operations will be performed based on extension principle with numerical illustrations.

Let A and B be PFSs. Then, $A * B$ (where $* \in (+, -, \cdot, /)$) is defined as

$$A * B = \{z, \mu_{A*B}(z), \eta_{A*B}(z), \nu_{A*B}(z)\}$$

where $\mu_{A*B}(z) = \vee[\mu_A(x) \wedge \mu_B(y)]$, $\eta_{A*B}(z) = \wedge[\eta_A(x) \vee \eta_B(y)]$, $\nu_{A*B}(z) = \wedge[\nu_A(x) \vee \nu_B(y)]$ and $x * y = z$.

Example: Let A and B be two picture fuzzy sets where

$$A = \{(2, 0.4, 0.2, 0.3), (3, 0.7, 0.1, 0.1), (4, 0.6, 0.2, 0.2)\} \text{ and } B = \{(1, 0.6, 0.1, 0.2), (2, 0.5, 0.2, 0.1)\}.$$

7.1. Addition of picture fuzzy sets:

Addition of two picture fuzzy sets A and B can be written as:

$$A + B = \{z, \mu_{A+B}(z), \eta_{A+B}(z), \nu_{A+B}(z)\}$$

where $\mu_{A+B}(z) = \vee[\mu_A(x) \wedge \mu_B(y)]$, $\eta_{A+B}(z) = \wedge[\eta_A(x) \vee \eta_B(y)]$, $\nu_{A+B}(z) = \wedge[\nu_A(x) \vee \nu_B(y)]$ and $x + y = z$.

Then, the addition of the picture fuzzy sets A and B is

$$\begin{aligned}
A + B = {} & \{(2 + 1, \min(0.4, 0.6), \max(0.2, 0.1), \max(0.3, 0.2)), (2 + 2, \min(0.4, 0.5), \max(0.2, 0.2), \\
& \max(0.3, 0.1))(3 + 1, \min(0.7, 0.6), \max(0.1, 0.1), \max(0.1, 0.2)), (3 + 2, \min(0.7, 0.5), \\
& \max(0.1, 0.2), \max(0.1, 0.1))(4 + 1, \min(0.6, 0.6), \max(0.2, 0.1), \max(0.2, 0.2)), \\
& (4 + 2, \min(0.6, 0.5), \max(0.2, 0.2), \max(0.2, 0.1))\} \\
= {} & \{(3, 0.4, 0.2, 0.3), (4, 0.4, 0.2, 0.3), (4, 0.6, 0.1, 0.2), (5, 0.5, 0.2, 0.1), \\
& (5, 0.6, 0.2, 0.2), (6, 0.5, 0.2, 0.2)\} \\
= {} & \{(3, 0.4, 0.2, 0.3), (4, \max(0.4, 0.6), \min(0.2, 0.1), \min(0.3, 0.2)), (5, \max(0.5, 0.6), \min(0.2, 0.2), \\
& \min(0.1, 0.2)), (6, 0.5, 0.2, 0.2)\} \\
= {} & \{(3, 0.4, 0.2, 0.3), (4, 0.6, 0.1, 0.2), (5, 0.6, 0.2, 0.1), (6, 0.5, 0.2, 0.2)\}
\end{aligned}$$

7.2. Subtraction of picture fuzzy sets:

Subtraction of two picture fuzzy sets A and B can be written as:

$$A - B = \{z, \mu_{A-B}(z), \eta_{A-B}(z), \nu_{A-B}(z)\}$$

where $\mu_{A-B}(z) = \vee[\mu_A(x) \wedge \mu_B(y)]$, $\eta_{A-B}(z) = \wedge[\eta_A(x) \vee \eta_B(y)]$, $\nu_{A-B}(z) = \wedge[\nu_A(x) \vee \nu_B(y)]$ and $x - y = z$.
Then, the subtraction of the picture fuzzy sets A and B is

$$\begin{aligned} A - B = \{&(2-1, \min(0.4, 0.6), \max(0.2, 0.1), \max(0.3, 0.2)), (2-2, \min(0.4, 0.5),\\ &\max(0.2, 0.2), \max(0.3, 0.1)), (3-1, \min(0.7, 0.6), \max(0.1, 0.1),\\ &\max(0.1, 0.2)), (3-2, \min(0.7, 0.5), \max(0.1, 0.2), \max(0.1, 0.1)), (4-1, \min(0.6, 0.6),\\ &\max(0.2, 0.1), \max(0.2, 0.2)), (4-2, \min(0.6, 0.5), \max(0.2, 0.2), \max(0.2, 0.1))\}\\ = \{&(1, 0.4, 0.2, 0.3), (0, 0.4, 0.2, 0.3), (2, 0.6, 0.1, 0.2), (1, 0.5, 0.2, 0.1),\\ &(3, 0.6, 0.2, 0.2), (2, 0.5, 0.2, 0.2)\}\\ = \{&(0, 0.4, 0.2, 0.3), (1, \max(0.4, 0.5), \min(0.2, 0.2), \min(0.3, 0.1)), (2, \max(0.6, 0.5),\\ &\min(0.1, 0.2), \min(0.2, 0.2)), (3, 0.6, 0.2, 0.2)\}\\ = \{&(0, 0.4, 0.2, 0.3), (1, 0.5, 0.2, 0.1), (2, 0.6, 0.1, 0.2), (3, 0.6, 0.2, 0.2)\} \end{aligned}$$

7.3. Multiplication of Picture fuzzy sets:

Multiplication of two fuzzy sets A and B can be written as:

$$A \times B = \{z, \mu_{A\times B}(z), \eta_{A\times B}(z), \nu_{A\times B}(z)\}$$

where $\mu_{A\times B}(z) = \vee[\mu_A(x) \wedge \mu_B(y)]$, $\eta_{A\times B}(z) = \wedge[\eta_A(x) \vee \eta_B(y)]$, $\nu_{A\times B}(z) = \wedge[\nu_A(x) \vee \nu_B(y)]$ and $x \times y = z$.
Then, the multiplication of the fuzzy sets A and B is

$$\begin{aligned} A \times B = \{&(1\times 2, \min(0.2, 0.1), \max(0.3, 0.3), \max(0.1, 0.3)), (1\times 3, \min(0.2, 0.2), \max(0.3, 0.4),\\ &\max(0.1, 0.4))(2\times 2, \min(0.1, 0.1), \max(0.3, 0.3), \max(0.4, 0.3)), (2\times 3, \min(0.1, 0.2),\\ &\max(0.3, 0.4), \max(0.4, 0.4)), (3\times 2, \min(0.5, 0.1), \max(0.3, 0.3), \max(0.1, 0.3)),\\ &(3\times 3, \min(0.5, 0.2), \max(0.3, 0.4), \max(0.1, 0.4))\}\\ = \{&(2, 0.1, 0.3, 0.3), (3, 0.2, 0.4, 0.4), (4, 0.1, 0.3, 0.4), (6, 0.1, 0.4, 0.4),\\ &(6, 0.1, 0.3, 0.3), (9, 0.2, 0.4, 0.4)\}\\ = \{&(2, 0.1, 0.3, 0.3), (3, 0.2, 0.4, 0.4), (4, 0.1, 0.3, 0.4), (6, \max(0.1, 0.1), \min(0.3, 0.4),\\ &\min(0.3, 0.4)), (9, 0.2, 0.4, 0.4)\}\\ = \{&(2, 0.1, 0.3, 0.3), (3, 0.2, 0.4, 0.4), (4, 0.1, 0.3, 0.4), (6, 0.1, 0.3, 0.3), (9, 0.2, 0.4, 0.4)\}. \end{aligned}$$

7.4. Division of picture fuzzy sets:

Division of two fuzzy sets A and B can be written as:

$$A/B = \{z, \mu_{A/B}(z), \eta_{A/B}(z), \nu_{A/B}(z)\}$$

where $\mu_{A/B}(z) = \vee[\mu_A(x) \wedge \mu_B(y)]$, $\eta_{A/B}(z) = \wedge[\eta_A(x) \vee \eta_B(y)]$, $\nu_{A/B}(z) = \wedge[\nu_A(x) \vee \nu_B(y)]$ and $x/y = z$.
Then, the division of the fuzzy sets A and B is

$$\begin{aligned} A/B = \{&(2/1, \min(0.1, .6), \max(0.3, 0.2), \max(0.2, 0.2)), (2/2, \min(0.1, 0.7), \max(0.3, 0.1),\\ &\max(0.2, 0.2))(4/1, \min(0.1, 0.6), \max(0.6, 0.2), \max(0.2, 0.2)),\\ &(4/2, \min(0.1, 0.7), \max(0.6, 0.1), \max(0.2, 0.2))\}\\ = \{&(2, 0.1, 0.3, 0.2), (1, 0.1, 0.3, 0.2), (4, 0.1, 0.6, 0.2), (2, 0.1, 0.6, 0.2)\}\\ = \{&(1, 0.1, 0.3, 0.2), (2, \max(0.1, 0.1), \min(0.3, 0.6), \min(0.2, 0.2)), (4, 0.1, 0.6, 0.2)\}\\ = \{&(1, 0.1, 0.3, 0.2), (2, 0.1, 0.3, 0.2), (4, 0.1, 0.6, 0.2)\}. \end{aligned}$$

8. Conclusion

In this piece of work, foremost (α, δ, β)-cut and strong (α, δ, β)-cut of PFS, height of PFS, level set of PFS and special PFS have been defined. Afterwards Decomposition theorems of PFS have been proved. Although some sort of works on PFSs can be seen in literature but Decomposition theorems or other defined important component of PFS have not been defined or proved yet which are integral parts of PFS. Especially, Decomposition theorems will the researches to perform arithmetic operations of continuous PFSs. Later on extension principle for PFS has been defined and studied some of its properties. Finally, picture fuzzy arithmetic based on extension principle has been performed with examples.

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Original Article

Speeding up the convergence of the Polyak's Heavy Ball algorithm

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Abstract

In the presented work, some procedures, usually used in modern algorithms of unconstrained optimization, are added to Polyak's heavy ball method. Namely, periodical restarts, which guarantees monotonic decrease of the objective function along successive iterates, while restarts involve updating of the step size on the base of line search method.

For smooth objective functions, the Heavy Ball (briefly HB) and Modified Heavy Ball (briefly MHB) algorithms are described along with the problem of simplifying the form of used line-search algorithm (without changing its content). MHB and the set of test functions are implemented in C++. The set of test functions contains 44 functions, taken from Cuter/st. Solver CG_DESCENT-C-6.8 was used for MHB benchmarking. Test-functions and other materials, related to benchmarking, are uploaded to GitHub: https://github.com/kobage/.

In case of smooth and convex objective function, the convergence analysis is concentrated on reducing transformations and their orbits. A concept of reducing transformation allows us to investigate algebraic structure of convergent methods.

Keywords: Modified heavy ball algorithm; Nonlinear conjugate gradient methods; Nonlinear programming; C++; Visual studio; AMPL; Knitro; CG_DESCENT-C-6.8

1. Introduction

Motivation. Polyak's heavy ball method or simply HB (see [1]), in contrast to the modern efficient methods (see [2,3]) of solving

$$f(x) \to \min, \quad x \in \mathbb{R}^n,$$

has not been fully studied. At the same time, there exist objective and subjective reasons to develop HB. In general, different methods of local minimization often converge to distinct minimals. HB behaves in other manner when minimization problem has many local minimals. Like effective modern algorithms — the Limited-memory

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BFGS (briefly L-BFGS, [2]) and the limited memory conjugate gradient method (briefly lcg, [3]) HB has modest requirements in terms of the memory usage and is easy to use for sparsely populated data. HB and MHB are easily programmable in a single and multi-threaded environments.

There are publications, explicitly or implicitly based on heavy ball model, having practical interest. But in this research we will not focus on them. Subjective factor to further develop HB is based on its effective physical and geometrical interpretations. For MHB, analogical interpretations are given and the algebraic structure of MHB is considered as well.

The issue of speeding up the convergence. Tremendous difference in performance between HB and, on the other hand, L-BFGS and lcg, is the result of a long-time evolution of these latter ones. Evolution was achieved due to the improvement of the following auxiliary procedures:

1. Periodical restarts;
2. Inexact line search;
3. Preconditioning;
4. Updating search directions.

In [4], the first attempt of speeding up the convergence through implementing periodical restarts every time when at some iterate the objective function ceases to decrease showed a significant increase in speed for some simple tests, and persuaded us in the necessity of further research.

Now, at every restart MHB uses best algorithm of inexact line search (see [5]) in order to determine the length of the step.

The speed of convergence of MHB is evaluated via numerical experiments conducted in C++ using well-known methodology (see [6]). To apply this approach, two sets — a set of test problems and a set of methods must be chosen carefully.

As the set of test functions, 45 functions from 145, used in [3] mainly on high dimensions, were chosen. It is interesting that two of these 45 tests, DQRTIC and QARTC, are identical. The only difference is the number of variables. All 45 tests were used to produce performance profiles (MHB vs lcg). In order to implement and debug functions, corresponding to selected tests, we have essentially used the material in other high-tech languages FORTRAN and AMPL (see [7,8]). To check correctness of implementations, solver *knitro* with the AMPL interface was used (see [9,10]). On the set of test-functions, MHB has been compared with well-known solver CG_DESCENT-C-6.8 (see https://www.math.lsu.edu/~hozhang/SoftArchive/). Test-functions compatible with both MHB and CG_DESCENT-C-6.8, the code of MHB and other materials related to benchmarking, are uploaded to GitHub. MHB does not take into account the possibility of preconditioning. Nevertheless, the results of benchmarking persuade us again in the necessity of next research.

Algebro-limiting properties of the convergence. In Section 3, we study the convergence in case of smooth convex objective function, in terms of reducing transformations and their orbits.

The sequences created from powers of transformations play an important role in functional analysis (e.g. fixed points), differential equations and other fields. Transformation exponentiation, fast exponentiation and studies of orbit properties are also the focus points of the modern computer science (see [11,12]). We introduce a concept of reducing transformation. Iterations of some optimization algorithms, including MHB-algorithm, minimize the objective function along the orbits of reducing transformations, i.e. iterations have form $x_k = T^k x_0$, where T is a reducing transformation specific to algorithm. To facilitate the analysis of these algorithms, we comprehensively study the algebraic structure of the set of these transformations. It should be noted that Theorem 2 is especially useful, since it proves both the convergence of gradients towards zero and the convergence of objective function towards a global minimum along an orbit. We define specific reducing transformations. One of them corresponds to MHB. This allows us to prove the convergence of MHB algorithm.

2. Description of HB, MHB and line search algorithms. Numerical experiments

2.1. HB method

Let the smallness of supremum norm of the gradient be the stopping condition of the algorithm, x_0 be initial iterate, constants $\alpha > 0$ and $0 \leq \beta < 1$ be given. Then the pseudo-code of HB has the form given in Fig. 1.

Calling the swap(u_0, u_1) means that vectors u_0 and u_1 are changing their names: former u_0 after call becomes u_1 and vice versa.

Note two problematic issues related with HB:

hb

1 $g_0 = \left(f'(x_0)\right)^T$
2 if $(|g_0|_\infty < EPS)$ **return**;
3 $x_1 = x_0 - \alpha g_0$;
4 **while**(**true**)
5 $\quad dir = x_1 - x_0$;
6 $\quad$ swap(x_0, x_1);
7 $\quad g_0 = \left(f'(x_0)\right)^T$;
8 $\quad$ if $(|g_0|_\infty < EPS)$ **return**;
9 $\quad x_1 = x_0 - \alpha g_0 + \beta \cdot dir$;

Fig. 1. The heavy ball method.

- The objective function does not decrease monotonically along the iterates.
- General scheme of choosing parameters α and β is not known.

2.2. MHB method

In the modified method, due to restarts, the objective function monotonically decreases along the iterates. Each restart updates the parameter α on the basis of line search method. In HB, β is responsible for amplitude of oscillation decreasing along iterates. In MHB, due to restarts, the objective function monotonically decreases along the iterates, therefore, we have no oscillation and correspondingly no need for existence of parameter β. So we give it a fixed value $\beta = 1$. In more detail, every time when the standard step of HB is done a certain number of times (in our case 10 000 times), or when at some iteration the objective function increases, MHB restarts. Each restart, in order to maintain the high-speed of the convergence, determines the parameter α on the basis of presented in [3] line-search algorithm. In contrast with typical interpretation of HB in case of continuous model, MHB sharply changes its direction along its trajectory in the vicinity of local minimals (turns in the direction of anti-gradient), but with preservation of high-speed of convergence (using line search).

Let us explain meanings of lines, added in MHB.

Lines 6 and 14 are necessary in order to count HB standard steps (line 12). After 10 000 standard steps, MHB is calling line search (Line 5), even if objective function was still decreasing. On one hand, this is necessary to maintain high speed of the convergence. On the other, this is required to guarantee the convergence of the MHB itself. The number 10 000 gives us the understanding of how badly can high dimensionality test problem be conditioned. In case of the lower number of repetitions, MHB does not have enough time to accelerate for some tests. Line 1 of MHB (see Fig. 1) differs from Line 1 of HB (see Fig. 2). To restart, values of the objective function are needed. Therefore, implementation of the objective function calculates both gradient and function values.

The very first call of line search does the job of the Line 3 of HB's pseudo-code. Afterwards, at every restart, line search (Line 5 of MHB) recalculates x_1 along the anti-gradient and gradient in x_1. This is necessary when at the point x_1 (calculated by Lines 12–13) the value of objective function does not decrease. MHB does not guarantee that at x_1, which is calculated by Line 12, function will decrease. That is why we calculate function's value and gradient beforehand at the point x_1 — to be ready to run line search in the case of necessity.

2.3. Line search

In our experiments, we use line search, described in [3], which fully met our expectations. Although, the programming of this variant of the line search procedure is the serious challenge for any researcher. In our opinion, the following form of the line search procedure, which actively uses Literate Programming style (see [13]), maintains all features of its old form but is significantly easier to program. Suppose, parameters of line search have the same default values as in [3]:

$$\delta = 0.1, \quad \sigma = 0.9, \quad \varepsilon = 10^{-6}, \quad \gamma = 0.66, \quad \rho = 5, \quad \psi_0 = 0.01, \quad \psi_2 = 2., \quad \theta = 0.5.$$

mhb

$g_0 = \left(f'(x_0)\right)^T$; $\quad f_0 = f_1 = f(x_0)$;
if $(|g_0|_\infty < \; EPS)$ **return**;
while(**true**)
  if ($f_0 \Leftarrow f_1$ || counter == 10000)
    lineSearch( );
    counter = 0;
  $dir = x_1 - x_0$;
  swap(x_0, x_1);
  swap(g_0, g_1);
  if($|g_0|_\infty < \; EPS$) **return**;
  $f_0 = f_1$;
  $x_1 = x_0 - \alpha g_0 + dir$;
  $f_1 = f(x_1)$; $\quad g_1 = \left(f'(x_1)\right)^T$;
  ++counter;

Fig. 2. The modified method.

line_search( )

< I: Initial guess for b >;
< G: Generate initial $[a, b]$ >;
while(1)
  length$= b - a$;
  $c = \dfrac{a\varphi'(b) - b\varphi'(a)}{\varphi'(b) - \varphi'(a)}$;
  < $U3$: Update a, b, c >;
  < $U2$: Update a, b >;
  if$(b - a > \gamma * length)$
    $c = \dfrac{a+b}{2}$;
    < $U2$: Update a, b >;

Fig. 3. Line search algorithm.

In C++ language, there exist several different ways of sharing data by different parts of the program. Each of them implies an implementation, distinct from others. We do not lose generality by fixing the form of implementation. Instead, we mention only the data, which should be shared to line search at the beginning of its run, and data that should be shared from line search at the end of its work. Data could be shared using pointers or references, or using some kind of shared variables.

In order to *line search* start its work correctly, x_0, f_0 and direction of (guaranteed) descent d from x_0 (in our case — g_0) should be accessible. Line search finishes when it finds the vector $x_0 + t \cdot d$, at which the function $\varphi(t) = f(x_0 + td)$ satisfies the approximate Wolfe condition $(2\delta - 1)\varphi'(0) \ge \varphi'(t) \ge \sigma\varphi'(0)$ and condition $\varphi(t) \le \varphi(0) + \varepsilon$. When it finished, $\alpha = t$, $x_1 = x_0 + \alpha d$, f_1, g_1 (gradient at the x_1) should be accessible from the main program.

Let us represent *line search* procedure as the sequence of several macros (see Fig. 3).

Macros are described in Table 1.

In more detail, pseudo-codes of these macros have the following form:

< T: Termination test in *t* >

Calculate $x_1 = x_0 + t \cdot d$, $\quad \varphi(t) = f(x_1)$ and g_1 at the point t;
Calculate $\varphi'(t)$ at t;
// if the following condition is valid, then $f(x_1)$ and $f'(x_1)$ are accessible from solver

Table 1
Description of macros.

	Macro	Description	Used in
T	$<$ Termination test in t $>$	If at the point t the approximate Wolfe condition $(2\delta - 1)\,\varphi'(0) \geq \varphi'(t) \geq \sigma\varphi'(0)$ and $\varphi(t) \leq \varphi(0) + \varepsilon$ are fulfilled, then algorithm line search terminates and makes: coefficient $\alpha = t$, vector $x_1 = x_0 + \alpha d$, value $f_1 = \varphi(\alpha)$ and gradient g_1 (at x_1)	B, G, U2, U3
I	$<$ Initial guess for b $>$	Determines initial guess for the high endpoint of $[a, b]$	line_search
B	$<$ Bisection on $[a, b]$ $>$	At the beginning, $[a, b]$ satisfies conditions $\varphi'(a) < 0,\ \varphi(a) \leq \varphi(0) + \varepsilon$, $\varphi'(b) < 0,\ \varphi(b) > \varphi(0) + \varepsilon$. Finishes with $[a, b]$ satisfying $\varphi(a) \leq \varphi(0) + \varepsilon$, $\varphi'(a) < 0,\ \varphi'(b) \geq 0$	G, U2, U3
G	$<$ Generate initial $[a, b]$ $>$	Generates an initial $[a, b]$, satisfying the opposite slope condition: $\varphi(a) \leq \varphi(0) + \varepsilon$, $\varphi'(a) < 0, \varphi'(b) \geq 0$	line_search
U3	$<$ Update a, b, c $>$	Starts with secant c and interval $[a, b]$ satisfying the opposite slope condition. If $c \in (a, b)$, then finishes with updated (shrinked) $[a, b]$ satisfying the opposite slope condition, and generates new secant c	line_search
U2	$<$ Update a, b $>$	Starts with secant c and interval $[a, b]$ satisfying the opposite slope condition. If $c \in (a, b)$, then finishes with updated (shrinked) $[a, b]$ satisfying the opposite slope condition	line_search

if $((2\delta - 1)\varphi'(0) \geq \varphi'(t) \geq \sigma\varphi'(0) \quad \&\& \quad \varphi(t) \leq \varphi(0) + \varepsilon)$
    $\alpha = t;$
    ++lineSearchCounter;
    return; // end of **line_search( )**

Remark 1. The main idea behind MHB is to prevent oscillations of the objective function. Therefore, in our case, line search returns vector, located at the anti-gradient direction, but until the local minimal. Consequently, the termination condition of MHB has form:

if $(0 \geq \varphi'(t) \geq \sigma\varphi'(0) \quad \&\& \quad \varphi(t) \leq \varphi(0) + \varepsilon).$

$<$ I: Initial guess for b $>$
  if (lineSearchCounter > 0)
    $b = \psi_2 \cdot \alpha;$
  else
    if $(x_0 \neq 0)$
      $b = (\psi_0 \cdot \|x_0\|_\infty) / \|g_0\|_\infty;$
    else
      if $(f(x_0) \neq 0)$
        $b = (\psi_0 \cdot |f(x_0)|)/(\|g_0\|_2)^2;$
      else
        $b = 1;$

Remark 2. This procedure here is presented in simplified form.

$<$ B: Bisection on $[a, b]$ $>$
  while (1)
    $d = (1 - \theta) \cdot a + \theta \cdot b$
    $<$ Termination test in d $>$
    if $(\varphi'(d) \geq 0)$
      $b = d;$
      break;
    if $(\varphi(d) < \varphi(0) + \varepsilon)$
      $a = d;$

else
$b = d$;

< G: Generate initial $[a, b]$ **>**
$a = 0$
while (1)
< Termination test in b >;
if $(\varphi'(b) \geq 0)$ **break**;
if $(\varphi(b) > \varphi(0) + \varepsilon)$
< Bisection on $[a, b]$ >;
break;
$a = b$
$b = b \cdot \rho$;

< U3: Update *a*, *b*, *c* >
if $(a < c < b)$
< Termination test in c >;
if $(\varphi'(c) < 0 \quad \&\& \quad \varphi(c) > \varphi(0) + \varepsilon)$
$b = c$;
< Bisection on $[a, b]$ >;
else
if $(\varphi'(c) \geq 0)$;
$$c_1 = \frac{c\varphi'(b) - b\varphi'(c)}{\varphi'(b) - \varphi'(c)}$$
$b = c$;
else
$$c_1 = \frac{a\varphi'(c) - c\varphi'(a)}{\varphi'(c) - \varphi'(a)}$$
$a = c$;
$c = c_1$;

Remark 3. Even in case when $a = b$ (and we have one more such case), c_1 will take certain value due to rounding errors. Therefore, it is not worthwhile to complicate U3.

< U2: Update *a*, *b* >
if $(a < c < b)$
< Termination test in c >;
if $(\varphi'(c) < 0 \quad \&\& \quad \varphi(c) > \varphi(0) + \varepsilon)$
$b = c$;
< Bisection on $[a, b]$ >;
else
if $(\varphi'(c) \geq 0)$;
$b = c$;
else
$a = c$;

2.4. Numerical experiments

Testing mathematical software means that different debugged and correctly working products have to be compared. The main methodology in benchmarking optimization software is considered to be an article by E.D. Dolan, J.J. Moré titled "Benchmarking optimization software with performance profiles" (see [6]). In order to apply this approach, two sets — a set of test problems and a set of methods, must be collected. The classical example of such environment

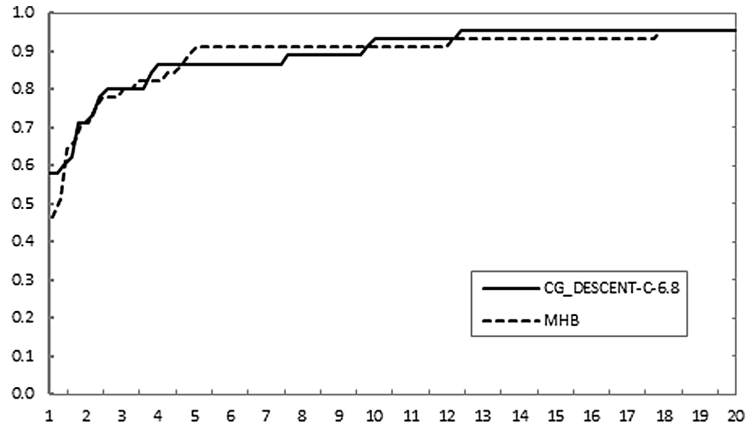

Fig. 4. Performance profiles for CG_ DESCENT-C-6.8 and MHB based on CPU time.

is CUTEst (see [7]), which allows adding tests as well as methods. However, it is FORTRAN-based, that makes it uncomfortable for C++ users. To create test collection on C++, we used existing CUTEst (see [7]) and AMPL (see [8]) test collections. Especially AMPL, because its code is easier compared to SIF (see [14]). AMPL gives us additional opportunity to check correctness of implementation. There is a full collection of test functions on this language, and via NEOS server (see [15]) it is possible to run them and compare obtained results. This approach, containing elements of metaprogramming, is very comfortable and allows us to implement algorithms on IDE and platforms we feel comfortable working with.

For experiments, 45 functions are selected. In performance profiles, all tests are used (in [3], only 79 from 145 were used). On the chosen set of test problems, MHB has been compared with the solver CG_DESCENT-C-6.8. For unconstrained minimization, there exist other efficient algorithms, but because of http://users.clas.ufl.edu/hager/papers/CG/results6.0.txt, there is no need to consider them. MHB, auxiliary algorithms and tests are programmed in C++, using Visual Studio 2015 environment (see [16]). We have debugged CG_DESCENT-C-6.8 in the same environment.

To the files, placed at https://www.math.lsu.edu/~hozhang/SoftArchive/, only drivers and test functions are added. From CG_DESCENT-C-6.8, we only changed file cg_descent.c at line 1677, in order to change termination criterion to the form shown at Line 9 of Fig. 2.

In the line search used in MHB, we have changed procedure of generation of initial interval. In the general case, at the very first run of line search, parameter `lineSearchCounter` takes value 0. At the end of the line search, `lineSearchCounter` increases. At every call except first one, the value of parameter b is calculated based on its previous value. Lines 7–14 (see Fig. 2) are executed sequentially many times and without a call of line search. Therefore, for MHB $< I$: Initial guess for $b >$ is implemented without its first 3 lines. In other words, MHB does not use `lineSearchCounter`.

As was mentioned in the Introduction, we did not pay much attention to code optimization, because the presented variant of MHB is yet far from being complete. In contrast to CG_DESCENT-C-6.8, MHB uses only inner product of vectors from Basic Linear Algebra Subprograms (BLAS). Besides, MHB uses only one realization for each test function, calculating value and gradient. The calculations were executed on a personal computer with the following parameters: Intel(R) Core(TM) 2 Duo CPU T7500 2.20 GHz, 2.00 GB of RAM. For each test function, MHB was run 10 times. Performance profiles were built based on median results. Smallness of the supremum norm of the gradient (less then 10e-6) represented the termination criterion for algorithms. Profile presented on Fig. 4 gives us the first impression about opportunities of the MHB, although the preconditioning techniques are not used here.

At present, it is not clear the importance of preconditioners for MHB. But it is clear for other methods. For example, CG DESCENT is easily programmable without preconditioning and its importance is easily evaluable. Besides, using simplest diagonal preconditioner for MHB in tests where it is usable speeds up convergence tens of times.

3. Convergence analysis of smooth convex objective function

3.1. Reducing transformation and its orbit

Definition 1. Transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a reducing transformation of functional $f : \mathbb{R}^n \to \mathbb{R}$ (or simply a reducing transformation, when functional is unambiguous), if $f(T(x)) \le f(x)$, $\quad \forall x \in \mathbb{R}^n$.

Identity transformation $id : \mathbb{R}^n \to \mathbb{R}^n$, $\quad id(x) \equiv x$, represents a trivial reducer.

Definition 2. Let T be a reducing transformation of a differentiable functional $f : \mathbb{R}^n \to \mathbb{R}$. We say that T has a property to reduce proportionally to the gradient square, or that T is a $g2p$-reducing transformation, if $f(x) - f(T(x)) \ge \beta \left\| f'(x) \right\|^2$ for some positive β and $\forall x \in \mathbb{R}^n$.

Definition 3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an arbitrary transformation and $x \in \mathbb{R}^n$ be an arbitrary vector. In this case the sequence $\left\{T^i(x)\right\}_{i=0}^{\infty}$ is called the orbit of T. Transformation powers are defined according to a recurrent relation:

$$T^i(x) = \begin{cases} x, & i = 0, \\ T\left(T^{i-1}(x)\right), & i > 0. \end{cases}$$

Let us define the conditions under which any orbit is a minimizing sequence.

Theorem 1. *Let the function $f : \mathbb{R}^n \to \mathbb{R}$ be bounded below and differentiable, and T be a $g2p$-reducing transformation. In this case for every $x \in \mathbb{R}^n$ we have:*

$$\lim_{i\to\infty} \left\| f'\left(T^i(x)\right)\right\| = 0. \tag{1}$$

Proof. Let us choose an arbitrary $x \in \mathbb{R}^n$ and denote:

$$x_i = T^i(x), \quad i = 0, 1, 2 \ldots .$$

According to the condition, for some real number f_* and $\forall x \in \mathbb{R}^n$ it is always true that $f(x) \ge f_*$. Applying the definition of a $g2p$-reducing transformation we have:

$$f(x_i) - \beta \left\| f'(x_i)\right\|^2 \ge f(x_{i+1}), \quad i = 0, 1, 2 \ldots .$$

Suppose (1) does not hold. Then there exist some $\eta > 0$ and a subsequence $\left\{x_{i_k}\right\}_{k=0}^{\infty}$ of sequence $\{x_i\}_{i=0}^{\infty}$, such that

$$\left\| f'\left(x_{i_k}\right)\right\| > \eta, \quad k = 0, 1, 2 \ldots . \tag{2}$$

Since the functional f is decreasing for the elements of the initial sequence, it is clear that

$$f\left(x_{i_k}\right) - \beta \left\| f'\left(x_{i_k}\right)\right\|^2 \ge f\left(x_{i_k+1}\right) \ge f\left(x_{i_{k+1}}\right), \quad k = 0, 1, 2 \ldots . \tag{3}$$

Combining (2) and (3) we arrive to:

$$f\left(x_{i_0}\right) - f_* \ge f\left(x_{i_0}\right) - f\left(x_{i_k}\right) = \left(f\left(x_{i_0}\right) - f\left(x_{i_1}\right)\right) + \cdots + \left(f\left(x_{i_{k-1}}\right) - f\left(x_{i_k}\right)\right) \ge \beta k \eta^2$$

which is impossible, since the right-hand side becomes arbitrarily large due to k,while the left-hand side does not depend on this factor. □

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded below and continuously differentiable convex functional, T be a $g2p$-reducing transformation of f and $\{T^i(x_0)\}_{i=0}^{\infty}$ be a bounded orbit for some $x_0 \in \mathbb{R}^n$. Then f has global minimum f_* and*

$$\lim_{i\to\infty} \left\| f'\left(T^i(x_0)\right)\right\| = 0,$$
$$\lim_{i\to\infty} f\left(T^i(x_0)\right) = f_*.$$

Proof. The first estimate stems from the previous theorem. To prove the second, we need to show the existence of global minimal. Let us denote $x_i = T^i x_0$ and take some $\varepsilon > 0$ some. Since the sequence $\{x_i\}_{i=0}^{\infty}$ is bounded, there exists its subsequence $\{x_{i_k}\}_{k=0}^{\infty}$ which converges to some vector $x_* \in \mathbb{R}^n$. f is continuously differentiable, so from $\lim\limits_{k\to\infty} \|f'(x_{i_k})\| = 0$ and $\lim\limits_{k\to\infty} x_{i_k} = x_*$ follows $\|f'(x_*)\| = 0$. Making use of convexity of f, we have

$$f(x) - f(x_*) \geq f'(x_*)(x - x_*) = 0, \quad \forall x \in \mathbb{R}^n.$$

Therefore, $x_* \in \mathbb{R}^n$ is global minimal and $f(x_*) = f_*$.

Because of boundedness of $\{T^i(x_0)\}_{i=0}^{\infty}$, there exists a positive distance $r > 0$, such that

$$\|T^i(x_0) - x_*\| \leq r, \quad \forall i \in \mathbb{N}.$$

Applying the first part of the theorem, for any positive ε we can choose such an index $i_0 \in \mathbb{N}$ that

$$\|f'(T^i(x_0))\| \leq \frac{\varepsilon}{r}, \quad \forall i \geq i_0.$$

Let us take $i \geq i_0$ and consider continuously differentiable convex function depending on one variable:

$$g(t) = f\left((1-t)T^i(x_0) + t x_*\right), \quad t \in \mathbb{R}.$$

Since the function is convex, it is true that $g(1) - g(0) \geq g'(0)$. Since both sides are negative we can rewrite the inequality as: $0 \leq g(0) - g(1) \leq |g'(0)|$, i.e.

$$0 \leq f(T^i(x_0)) - f_* \leq \|f'(T^i(x_0))\| \cdot \|x_* - T^i(x_0)\| \leq \varepsilon, \quad \forall i \geq i_0. \quad \square$$

Note 1. In order to guarantee boundedness of the sequence $\{T^i(x)\}_{i=0}^{\infty}$, it is possible to use various sufficient conditions. For example, one can demand $f(x) \to \infty$ when $\|x\| \to \infty$, which is widespread case in applications. However, checking boundedness is rather convenient during calculations.

As we can see, every bounded orbit of the $g2p$-reducing transformation represents a minimizing sequence, under relatively general conditions. In other words, construction of a $g2p$-reducing transformation is equivalent to defining a minimizing algorithm, since by virtue of Theorem 2, the objective function along the orbit converges towards the minimum. Therefore, it is important for us to easily check whether a particular transformation is indeed $g2p$-decreasing.

Proposition 1. *Let $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ be two reducing transformations. In such case, $T_1 \circ T_2$ is also a reducing transformation.*

Proof. Clearly stems just from the definition: $f(T_1(T_2(x))) \leq f(T_2(x)) \leq f(x), \forall x \in X_0$. $\square$

This relatively simple statement proves that the reducing transformations form a monoid with respect to the composition operation.

Proposition 2. *Let $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ be a reducing transformation, and $T_2 : \mathbb{R}^n \to \mathbb{R}^n$ be a $g2p$-reducing transformation. In this case $T_1 \circ T_2$ is also a $g2p$-reducing transformation.*

Proof. Since T_2 is a $g2p$-reducing transformation, it is true that

$$f(x) - \beta \|f'(x)\|^2 \geq f(T_2(x)) \tag{4}$$

for some positive β and $\forall x \in \mathbb{R}^n$. Then, for the right-hand side of (4) the following holds:

$$f(T_2(x)) \geq f(T_1(T_2(x))),$$

which together with (4) proves that $T_1 \circ T_2$ is a $g2p$-reducing transformation. $\square$

Proposition 2 shows that $g2p$-reducing transformations form the structure of the left-ideal in the monoid of reducing transformations.

Definition 4. Let us assume that T_1, T_2 are two reducing transformations of f. We will say that $T_1 \overset{f}{\leq} T_2$, if $f(T_1(x)) \leq f(T_2(x)), \forall x \in \mathbb{R}^n$, i.e. if the first transformation decreases f uniformly better.

Proposition 3. *Let* $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ *be reducing transformation,* $T_2 : \mathbb{R}^n \to \mathbb{R}^n$ *-* $g2p$*-reducing transformation and let* $T_1 \overset{f}{\leq} T_2$*. In this case* T_1 *is also a* $g2p$*-reducing transformation.*

Proof. The following inequalities

- $f(x) - \beta \left\| f'(x) \right\|^2 \geq f(T_2(x))$, for some positive β and $\forall x \in \mathbb{R}^n$,
- $f(T_1(x)) \leq f(T_2(x))$

together yield the desirable inequality $f(x) - \beta \left\| f'(x) \right\|^2 \geq f(T_1(x))$. □

3.2. Gradient descent with fixed step size

Now, we directly construct only one $g2p$-reducing transformation, corresponding to Gradient descent with fixed step size. Results of previous section allows us indirectly check that reducing transformation, corresponding to MHB algorithm also have $g2p$-property. As a result, we obtain convergence provided by conclusions of Theorems 1 and 2.

Lemma 1. *Let* f *be differentiable, the gradient of* f *be Lipschitz continuous* $\left\| f'(x_0) - f'(y) \right\| \leq L \left\| x_0 - y \right\|$, $\forall x, y \in \mathbb{R}^n$*, and* $0 < s < 2/L$*. Then, the transformation* $T_s : \mathbb{R}^n \to \mathbb{R}^n$ *defined by the rule* $T_s(x) = x - s\left(f'(x)\right)^T$ *is* $g2p$*-reducing with the coefficient* $\beta = s\left(1 - \frac{Ls}{2}\right)$*, i.e.* $f(x) - \beta \left\| f'(x) \right\|^2 \geq f(T_s(x))$, $\forall x \in \mathbb{R}^n$.

Proof. Let us denote $g(x) = \left(f'(x)\right)^T$ and use the formula:

$$f(x+y) = f(x) + \int_0^1 f'(x + \tau y) \cdot y \, d\tau$$

which yields:

$$\begin{aligned}
& f(T_s(x)) = f(x - sg(x)) = f(x) + \int_0^1 \left[f'(x + \tau(-sg(x)))\right] \cdot (-sg(x)) \, d\tau \\
& = (\text{subtract and add} \quad s \cdot f'(x) \cdot g(x)) \\
& = f(x) - s \cdot f'(x) \cdot g(x) + \int_0^1 \left[f'(x + \tau(-s \cdot g(x))) - f'(x)\right] \cdot (-s \cdot g(x)) \, d\tau \\
& \leq f(x) - s \cdot \left\| f'(x) \right\|^2 + \left| \int_0^1 \left[f'(x + \tau(-s \cdot g(x))) - f'(x)\right] \cdot (-s \cdot g(x)) \, d\tau \right| \leq f(x) \\
& - s \cdot \left\| f'(x) \right\|^2 + \int_0^1 \left\| f'(x + \tau(-s \cdot g(x))) - f'(x) \right\| \cdot \left\| -s \cdot g(x) \right\| d\tau \\
& \leq f(x) - s \cdot \left\| f'(x) \right\|^2 + \int_0^1 L \cdot \left\| \tau(-s \cdot g(x)) \right\| \cdot \left\| -s \cdot g(x) \right\| d\tau \\
& \leq f(x) - s \cdot \left\| f'(x) \right\|^2 + \frac{Ls^2}{2} \left\| f'(x) \right\|^2 = f(x) - s\left(1 - \frac{Ls}{2}\right) \left\| f'(x) \right\|^2
\end{aligned}$$

so $f(T_s(x)) \leq f(x) - \beta \left\| f'(x) \right\|^2$. □

In the conditions of Lemma 1, the $g2p$-reducing transformation $T_s : \mathbb{R}^n \to \mathbb{R}^n$ can be referred to as a fixed step reducing transformation due to the way it operates.

Theorem 3. *Let* f *be differentiable and bounded below, the gradient of* f *be Lipschitz continuous with constant* L *and* $0 < s < 2/L$*. Then the derivative of the functional* f *converges to zero along any orbit of the* $g2p$*-decreaser* T_s*. Besides, if* f *is convex functional, then values of the functional* f *converge to the global minimum along any bounded orbit of the* $g2p$*-reducer* T_s*.*

Proof. Conditions of Lemma 1 are following from conditions of Theorem 3. Therefore T_s is $g2p$-reducer. This latter fact together with Theorem 3 conditions guarantees the fulfillment of Theorem 1 conditions. Consequently,

$$\lim_{i\to\infty} \left\| f'\left(T_s{}^i(x)\right)\right\| = 0, \quad \forall x \in \mathbb{R}^n.$$

Suppose now that functional is convex. Gradient's Lipschitz continuity means it's continuity. Therefore conditions of Theorem 2 are fulfilled and the second part of Theorem 3 follows from Theorem 2. □

By the definition, $T_s(x) = x - sg^T$ (g denotes gradient at x). When f satisfies conditions of Theorem 2, then from the same theorem follows that along every sequence $\{x_i\}_0^\infty$ defined by the scheme $x_{i+1} = x_i - sg_i$ $i = 0, 1, 2, \ldots$ (g_i-gradient at x_i, initial iterate x_0 is taken arbitrarily) gradients converge to zero vector. Besides, if f is convex functional and above mentioned sequence is bounded, then values of the functional f converges to its global minimum.

3.3. Specifics of the line search algorithm for convex functions

When line search algorithm is used with MHB for convex functions, pseudo-code from sub-Section 2.3 remains valid, but some macros are subjected to changes. First of all, it concerns to termination test. In case of convex function, line search must find such vector $x_0 - \alpha g_0$, located in the antigradient direction from x_0, at which function $\varphi(t) = f(x_0 - tg_0)$ satisfies the condition $0 \geq \varphi'(t) \geq \sigma\varphi'(0)$. As $x_0 - \alpha g_0$ is located between x_0 and the local minimal in the antigradient direction, the condition $\varphi(\alpha) \leq \varphi(0) + \varepsilon$ is satisfied automatically by virtue of the convexity. So, we omit it and termination test has the form:

< T: Termination test in *t* >
    Calculate $x_1 = x_0 + t \cdot d$, $\varphi(t) = f(x_1)$ and g_1 at the point t;
    Calculate $\varphi'(t)$ at t;
    // if the following condition is valid, then $f(x_1)$ and $f'(x_1)$ are accessible from solver
    if $(0 \geq \varphi'(t) \geq \sigma\varphi'(0))$
        $\alpha = t$;
        ++lineSearchCounter;
        return;   // end of **line_search**( )

In our notations, from $\varphi'(t) < 0$ and $t > 0$ automatically follows $\varphi(t) < \varphi(0) + \varepsilon$, therefore there is no need for bisection. Consequently, the procedures, where it was used, now have the forms:

< G: Generate initial $[a, b]$ **>**
    $a = 0$
    while (1)
        < Termination test in b >;
        if $(\varphi'(b) \geq 0)$ **break**;
        $a = b$
        $b = b \cdot \rho$;

< U3: Update *a*, *b*, *c* >
    if $(a < c < b)$
        < Termination test in c >;
        if $(\varphi'(c) \geq 0)$
            $c_1 = \dfrac{c\varphi'(b) - b\varphi'(c)}{\varphi'(b) - \varphi'(c)}$;
            $b = c$;
        else
            $c_1 = \dfrac{a\varphi'(c) - c\varphi'(a)}{\varphi'(c) - \varphi'(a)}$
            $a = c$;
        $c = c_1$;

< U2: Update a, b >

```
if (a< c < b)
    < Termination test in c >;
    if (φ′(c) ≥ 0)
        b = c;
    else
        a = c;
```

3.4. Gradient descent with inexact line search

Consider gradient descent with variable step-size (g_i denotes gradient at x_i)

$$x_{i+1} = x_i - \alpha_i g_i. \tag{5}$$

Let us define the reducing transformation, one of whose orbits consists of iterates of scheme (5). Let for each $x \in \mathbb{R}^n$ α (depended on x) be that value of the parameter t, at which the line search method terminates:

$$T_\alpha(x) = x - \alpha \cdot g, \quad \forall x \in \mathbb{R}^n.$$

Obviously, T_α is reducing transformation. But, to prove convergence of iterates defined by (5), we need to show that T_α is $g2p$-reducer as well.

Theorem 4. *Let f be differentiable, convex and bounded below, the gradient of f be Lipschitz continuous. Then T_α is the $g2p$-reducer, the derivative of f converges to zero along any orbit of T_α and the values of the functional f converge to the global minimum along any bounded orbit of T_α.*

Proof. Suppose $\forall x_0 \in \mathbb{R}^n$. According to definition, $T_\alpha(x) = x_0 - \alpha g_0$. According to termination criterion of line search, $0 \geq \varphi'(\alpha) \geq \sigma\varphi'(0)$, i.e. $0 \geq -f'(x_0 - \alpha g_0) \cdot g_0 \geq -\sigma g_0^T g_0$. Let us add positive term $g_0^T g_0$ to the right-hand side inequality:

$$f'(x_0) \cdot g_0 - f'(x_0 - \alpha g_0) \cdot g_0 \geq g_0^T g_0 - \sigma g_0^T g_0, \quad \text{i.e.} \quad \left(f'(x_0) - f'(x_0 - \alpha g_0)\right) \cdot g_0 \geq (1-\sigma)\, g_0^T g_0.$$

Left-hand side is positive because right-hand side is positive and we can use the Lipschitz continuity, obtaining $L \cdot \alpha \cdot g_0^T g_0 \geq (1-\sigma)\, g_0^T g_0$. As a result, $\alpha \geq \frac{1-\sigma}{L}$.

As we can see, for $\forall x_0 \in \mathbb{R}^n$ the value of the step-size, at which the line search method terminates, satisfies inequality $\alpha \geq \frac{1}{10 \cdot L}$.

Let us take $s = \frac{1}{10 \cdot L}$. Obviously $s < \frac{2}{L}$, hence by virtue of Lemma 1, T_s is $g2p$-reducer taking into account following facts for $\forall x_0 \in \mathbb{R}^n$:

- $\alpha \geq s$,
- let $\tilde{x}$ be local minimal of the objective function in the anti-gradient direction (from x_0). Because of convexity of the objective function, the restriction of the objective function on $[x_0, \tilde{x}]$ decreases monotonically,

we obtain the following inequality:

$$f(T_\alpha(x_0)) \leq f(T_s(x_0)), \quad \forall x_0 \in \mathbb{R}^n, \quad \text{i.e.} \quad T_\alpha \overset{f}{\leq} T_s.$$

The last result means by virtue Proposition 3 that T_α is g2p-reducing transformation as well. Now, from conditions of Theorem 4 are following conditions of Theorems 1 and 2. Consequently, their conclusions are valid as well. □

By definition of T_α, in the conclusions of the last theorem instead of orbits we can speak about the iterates of gradient descent. In this case we obtain needed information about convergence of the method.

Note 2. For new conjugate gradient method suggested in [5], it is possible to take analogical results with analogical approach — on the basis of the relationship between descent direction d and anti-gradient established by the Theorem 1.1 (see [5]).

3.5. Convergence of MHB

Let us define reducing transformation $T_{hb} : \mathbb{R}^n \to \mathbb{R}^n$ to make conclusions about convergence of MHB. Let us take $x_0 \in \mathbb{R}^n$ arbitrarily. To define $T_{hb}(x_0)$, consider several repetitions of lines 4–14 (Fig. 2) without stopping condition (i.e. without line 10), beginning from counter = 0 and finishing with the last non-zero value of counter (which should be followed by the call of line search, according the pseudo code). Suppose, the last non-zero value of the counter for given x_0 is equal to k, and let x_i denote the vector x_0, which corresponds to counter $= i$ (see Fig. 2). Thus, $x_1 = T_\alpha(x_0)$ and $f(x_0) > f(x_1) > \cdots > f(x_k)$. Now, defining $T_{hb}(x_0) = x_k$, we see that:

$$T_{hb}(x) \le T_\alpha(x), \quad \forall x \in \mathbb{R}^n,$$

which means that T_{hb} is $g2p$-reducing transformation. Correspondingly, Theorems 1 and 2 are valid. However, because orbits of T_{hb} represent subsequences of the iterates of MHB, we can make weaker conclusion compared to the previous cases:

Theorem 5. *Let f be differentiable, convex and bounded below, the gradient of f be Lipschitz continuous. Then:*

- *T_{hb} is the $g2p$-reducer, the derivative of f converges to zero along any orbit of T_{hb} and the values of the functional f converge to the global minimum along any bounded orbit of T_{hb}.*
- *$\forall x_0 \in \mathbb{R}^n$, if $\{x_i\}_{i=0}^{\infty}$ denotes sequence of the iterates of MHB (each consecutive value of vector x_0, not considering line 10), then gradients of f converge to zero on some subsequence of $\{x_i\}_{i=0}^{\infty}$;*
- *if $\{x_i\}_{i=0}^{\infty}$ is bounded, then f converges to the global minimum on this sequence.*

Proof. First part is shown during constructing T_{hb}. The second part is following from the simple fact that the orbit $\left\{T_{hb}^i(x_0)\right\}_{i=0}^{\infty}$ represents subsequence of $\{x_i\}_{i=0}^{\infty}$ and, by virtue of Theorem 1, gradients of the objective function converge to zero vector along the orbit. To prove the remaining part, note that from the boundedness of the $\{x_i\}_{i=0}^{\infty}$ follows boundedness of the orbit $\left\{T_{hb}^i(x_0)\right\}_{i=0}^{\infty}$. $\{f(x_i)\}_{i=0}^{\infty}$ is decreasing sequence, having convergent subsequence (values of f on the orbit) to the global minimum. As a result, $\{f(x_i)\}_{i=0}^{\infty}$ converges to the same limit. □

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Original article

Lyapunov inequalities of nested fractional boundary value problems and applications

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Abstract

In this paper, we study certain classes of nested fractional boundary value problems including both of the Riemann–Liouville and Caputo fractional derivatives. In addition, since we will use the signed-power operators $\phi_\nu z := |z|^{\nu-1}z,\ \nu \in (0, \infty)$ in the governing equations, so our desired boundary value problems possess half-linear nature. Our investigation theoretically reaches so called Lyapunov inequalities of the considered nested fractional boundary value problems, while in viewpoint of applicability using the obtained Lyapunov inequalities we establish some qualitative behavior criteria for nested fractional boundary value problems such as a disconjugacy criterion that will also be used to establish nonexistence results, upper bound estimation for maximum number of zeros of the nontrivial solutions and distance between consecutive zeros of the oscillatory solutions. Also, considering corresponding nested fractional eigenvalue problems we find spreading interval of the eigenvalues.

Keywords: Fractional derivatives and integrals; Nested fractional boundary value problems; Nontrivial solutions; Lyapunov inequalities; Disconjugacy; Nonexistence; Oscillatory solutions; Fractional eigenvalue problems

1. Introduction

The cornerstone of Lyapunov inequalities founded in last decade of the nineteenth century. More precisely, the Russian mathematician A. M. Lyapunov during his investigation about stability of motion considered second order differential equations with periodic coefficients. In this way, Lyapunov presented the following stability criterion.

Theorem 1.1 (*[1], Chapter III, Theorem II*). *Consider the second order differential equation with ω-periodic coefficient*

$$y'' + q(t)y = 0, \quad -\infty < t < \infty. \tag{1.1}$$

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If the function q takes only positive or zero values (without being identically zero), and if further it satisfies the condition

$$\omega \int_0^\omega q(t) \leq 4, \tag{1.2}$$

then roots of the characteristic equation corresponding to (1.1) *will always be complex and their modulus is equal to 1.*

Indeed, making use of the Floquet theory, one may prove that Theorem 1.1 provides a stability tool to the second order periodic differential equation (1.1). The counter inequality corresponding to the inequality (1.2), namely

$$\int_0^\omega q(t) > \frac{4}{\omega}, \tag{1.3}$$

is known as the Lyapunov inequality in the literature. Nowadays we know that Lyapunov inequalities need not to be periodic in the sense of (1.3). P. Hartman in [2], shows this fact by the following theorem.

Theorem 1.2. *Assume that $u(t)$ is a nontrivial solution of the second order boundary value problem*

$$\begin{cases} u'' + q(t)u = 0, & t \in (a, b), \\ u(a) = 0 = u(b). \end{cases} \tag{1.4}$$

where $q \in C[a, b]$. Then the Lyapunov inequality

$$\int_a^b |q(t)| > \frac{4}{b-a}, \tag{1.5}$$

holds.

Over 125 years investigation on Lyapunov inequalities, nowadays we are able to study qualitative behavior of the differential/difference equations of arbitrary order and arbitrary structure via their relevant Lyapunov inequalities. Here we suggest some of the interesting papers and monographs dealing with the Lyapunov inequalities of ordinary differential/difference equations and their applications and cited bibliography as [3–27]. These research works, indicate that not only stability but also disconjugacy, nonexistence, maximum number of zeros of the nontrivial solutions, distance between consecutive zeros of the oscillatory solutions and some spectral properties can be established by means of the Lyapunov inequalities.

Between this variety, if we restrict ourselves into the fractional order differential/difference equations, we shall turn to the late 2013 where R. A. C. Ferreira in [9], for first time in literature considered the fractional order Riemann–Liouville boundary value problem

$$\begin{cases} ({}_aD^\alpha y)(t) + q(t)y(t) = 0, & a < t < a,\ 1 < \alpha \leq 2, \\ y(a) = 0, \quad y(b) = 0, \end{cases} \tag{1.6}$$

where $q \in C[a, b]$. Using Green function technique, the author obtained the first Lyapunov inequality for fractional differential equations corresponding with (1.6) as follows.

$$\int_a^b |q(t)|dt > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.7}$$

As can be observed, the Lyapunov inequality (1.7) generalizes the Lyapunov inequality (1.5), as the boundary value problem (1.6) generalizes the boundary value problem (1.4). Making use of the Lyapunov inequality (1.7), the author proved that in the Riemann–Liouville fractional eigenvalue problem

$$\begin{cases} ({}_aD^\alpha y)(t) + \lambda y(t) = 0, & a < t < a,\ 1 < \alpha \leq 2, \\ y(a) = 0, \quad y(b) = 0, \end{cases}$$

the eigenvalues λ are indeed real zeros of the Mittag-Leffler function $E_\alpha(z)$ provided that

$$|\lambda| > \Gamma(\alpha)\frac{4^{\alpha-1}}{(b-a)^\alpha}.$$

In the recent past years, so many researchers in the community of fractional calculus have studied various fractional order boundary value problems in viewpoint of their Lyapunov inequalities. Generally, these researchers have been concentrated on nonexistence criteria and spreading region of the eigenvalues. Some of the selected papers concerning with the above discussion about Lyapunov inequalities of the fractional boundary value problems are presented here as [7,28], [9–11], [21,29,30]. Getting focus on these papers, it can be seen that the most attention of the authors has been devoted on extracting Lyapunov inequalities from considered fractional boundary value problems. In order to fill this gap and demonstrating more applicability of Lyapunov inequalities of fractional order problems the authors in papers [12–16], studying a wide class of fractional order problems such as continuous and discrete linear and half-linear boundary value problems, impulsive and non-impulsive linear differential/difference systems and boundary value problems over higher dimensional spaces, have obtained corresponding Lyapunov inequalities and then making use of these inequalities qualitative behavior of the mentioned fractional order problems including stability, disconjugacy, nonexistence, upper bound estimation for maximum number of zeros of the nontrivial solutions, distance between consecutive zeros of the oscillatory solutions and spreading regions of the eigenvalues in corresponding fractional eigenvalue problems have estimated.

Here, we state the main problems of this paper. To this aim, first we consider the following nested third order half-linear boundary value problems:

$$\left(\phi_{\alpha_2}\left(\phi_{\alpha_1}x'\right)'\right)' + q(t)\phi_{\alpha_1\alpha_2}(x) = 0, \quad -\infty < a < b < +\infty, \tag{1.8}$$

$$\left(r_2(t)\phi_{\alpha_2}\left(r_1(t)\phi_{\alpha_1}x'\right)'\right)' + q(t)\phi_{\alpha_1\alpha_2}(x) = 0, \quad -\infty < a < b < +\infty, \tag{1.9}$$

where $\phi_\nu(x) := |x|^{\nu-1}x,\ \nu \in (0,\infty), \alpha_1, \alpha_2 > 0$, and $q, r_1, r_2 \in C(\mathbb{R}, \mathbb{R})$ with $r_i(t) > 0,\ i = 1, 2$. The authors in [6], studying the nested differential equations (1.8) and (1.9) subject to the boundary conditions $x(a) = 0$, $x(b) = 0$ and some other appropriate conditions, obtained the following Lyapunov inequalities:

$$\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds > \left(\frac{2}{b-a}\right)^{(\alpha_1+1)\alpha_2}, \quad \xi \in [a, b], \tag{1.10}$$

$$\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds > \frac{2^{(\alpha_1+1)\alpha_2}}{\left(\int_a^b r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1\alpha_2}\left(\int_a^b r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_2}}, \quad \xi \in [a, b], \tag{1.11}$$

respectively, in which $q_\pm(s) := \frac{q(s)\pm|q(s)|}{2}$ denote nonnegative and negative parts of $q(s)$. Essentially using the Lyapunov inequality (1.10), some criteria for maximum number of zeros of the nontrivial solutions, distance between consecutive zeros of the oscillatory solutions and nonexistence of nontrivial solutions of the nested third order differential equation (1.8) have presented. Motivated by the above work, we introduce the main problems of this paper as follows:

$$\begin{cases} {}^c\mathcal{D}^\alpha_{a^+}\left(\phi_{\beta_2}\left(\mathcal{D}^\alpha_{b_-}\left(\phi_{\beta_1}\left({}^c\mathcal{D}^\alpha_{a^+}x\right)\right)\right)\right) - q(t)\phi_{\beta_1\beta_2}x = 0, \quad a \le t \le b, \\ x^{(k)}(a) = 0,\ x^{(k)}(b) = 0,\ k = 0, 1, \dots, n-1, \end{cases} \tag{1.12}$$

$$\begin{cases} {}^c\mathcal{D}^\alpha_{a^+}\left(r_2(t)\phi_{\beta_2}\left(\mathcal{D}^\alpha_{b_-}\left(r_1(t)\phi_{\beta_1}\left({}^c\mathcal{D}^\alpha_{a^+}x\right)\right)\right)\right) - q(t)\phi_{\beta_1\beta_2}x = 0, \quad a \le t \le b, \\ x^{(k)}(a) = 0,\ x^{(k)}(b) = 0,\ k = 0, 1, \dots, n-1, \end{cases} \tag{1.13}$$

where ${}^c\mathcal{D}^\alpha_{a^+}$ and $\mathcal{D}^\alpha_{b_-}$ stand for the left sided Caputo and right sided Riemann–Liouville fractional derivatives of order α with $n-1 < \alpha \le n,\ n \in \mathbb{N}_{\ge 2}$, respectively. Also, $\phi_\nu x := |x|^{\nu-1}x,\ \nu \in (0,\infty), q, r_1, r_2 \in C(\mathbb{R}, \mathbb{R})$ such that $r_i(t) > 0,\ i = 1, 2$ and r_2 is an increasing function on $[a, b]$.

At the end of this section, we state the organization of the paper as follows. Section 2contains those parts of the fractional calculus that will be needed to obtain the main results. In this section we restrict ourselves to the left and right sided Riemann–Liouville and Caputo fractional operators, their interactions and fractional order integration by parts. In Section 3, making use of the presented fractional tools in Section 2, we will obtain main theoretical results of this paper that are Lyapunov inequalities of the nested fractional half-linear boundary value problems (1.12) and (1.13) under some relevant conditions. In Section 4, as the applied aspect of this paper using the obtained Lyapunov inequalities, qualitative behavior of the nested fractional boundary value problems (1.12) and (1.13) such as disconjugacy, nonexistence, upper bound estimation for maximum number of zeros of the nontrivial solutions, distance between consecutive zeros of the oscillatory solutions and spreading region of the eigenvalues in corresponding nested fractional eigenvalue problems will be established.

2. Preliminaries

This section is started by definitions of the left and right sided Riemann–Liouville fractional integration and differentiation operators.

Definition 2.1 (*[31]*)**.** The left and right sided Riemann–Liouville fractional integrals of order $\alpha \geq 0$ for function $f \in L^1[a,b]$ are given by:

$$\mathcal{I}^{\alpha}_{a^+(b_-)} f(t) = \begin{cases} \mathcal{I}^{\alpha}_{a+} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f(s)ds; & \alpha > 0, \\ \mathcal{I}^{\alpha}_{b-} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_t^b (s-t)^{\alpha-1} f(s)ds; & \alpha > 0, \\ f(t) \qquad ; & \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.2 (*[31]*)**.** The left and right sided Riemann–Liouville fractional derivatives of order $\alpha \geq 0$ for function $f \in AC^n(a,b)$ are defined by:

$$\mathcal{D}^{\alpha}_{a^+(b_-)} f(t) = \begin{cases} \mathcal{D}^{\alpha}_{a+} f(t) := \left(\dfrac{d^n}{dt^n}\right) \mathcal{I}^{n-\alpha}_{a^+} f(t) = \dfrac{1}{\Gamma(n-\alpha)} \left(\dfrac{d^n}{dt^n}\right) \displaystyle\int_a^t (t-s)^{n-\alpha-1} f(s)ds; & \alpha > 0, \\ \mathcal{D}^{\alpha}_{b-} f(t) := \left(\dfrac{d^n}{dt^n}\right) \mathcal{I}^{n-\alpha}_{b-} f(t) = \dfrac{(-1)^n}{\Gamma(n-\alpha)} \left(\dfrac{d^n}{dt^n}\right) \displaystyle\int_t^b (s-t)^{n-\alpha-1} f(s)ds; & \alpha > 0, \\ f(t) \qquad ; & \alpha = 0, \end{cases} \tag{2.2}$$

where $n = [\alpha] + 1$.

Replacing the affection position of the nth order derivative $\frac{d^n}{dt^n}$ as follows, gives us the left and right sided Caputo fractional derivatives

$${}^c\mathcal{D}^{\alpha}_{a^+(b_-)} f(t) = \begin{cases} {}^c\mathcal{D}^{\alpha}_{a+} f(t) := \mathcal{I}^{n-\alpha}_{a^+} \left(\dfrac{d^n}{dt^n} f\right)(t) = \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t (t-s)^{n-\alpha-1} \left(\dfrac{d^n}{dt^n} f\right)(s)ds; & \alpha > 0, \\ {}^c\mathcal{D}^{\alpha}_{b-} f(t) := \mathcal{I}^{n-\alpha}_{b-} \left(\dfrac{d^n}{dt^n} f\right)(t) = \dfrac{(-1)^n}{\Gamma(n-\alpha)} \displaystyle\int_t^b (s-t)^{n-\alpha-1} \left(\dfrac{d^n}{dt^n} f\right)(s)ds; & \alpha > 0, \\ f(t) \qquad ; & \alpha = 0. \end{cases} \tag{2.3}$$

We give interaction between the Riemann–Liouville and Caputo fractional derivatives in the next lemma.

Lemma 2.3 (*[31]*)**.** *Assume $x \in AC^n(a,b)$ and $n-1 < \alpha \leq n,\ n \in \mathbb{Z}^+$. Then*

$${}^c\mathcal{D}^{\alpha}_{a+} x(t) = \mathcal{D}^{\alpha}_{a+} x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \tag{2.4}$$

$${}^c\mathcal{D}^{\alpha}_{b-} x(t) = \mathcal{D}^{\alpha}_{b-} x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}. \tag{2.5}$$

Next lemma deals with the inverse rules of the left and right sided Caputo fractional derivatives.

Lemma 2.4 (*[31]*)**.** *Assume $\alpha > 0$.*

(i) *If $x \in L_1(a,b)$, then*

$${}^c\mathcal{D}^{\alpha}_{a+} \mathcal{I}^{\alpha}_{a+} x(t) = x(t), \quad {}^c\mathcal{D}^{\alpha}_{b-} \mathcal{I}^{\alpha}_{b-} x(t) = x(t). \tag{2.6}$$

(ii) *If* $x \in AC^n(a,b)$ *and* $n-1<\alpha\le n,\ n\in\mathbb{Z}^+$, *then*

$$\mathcal{I}^{\alpha}_{a^+}\,{}^c\mathcal{D}^{\alpha}_{a^+}x(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^k, \tag{2.7}$$

$$\mathcal{I}^{\alpha}_{b_-}\,{}^c\mathcal{D}^{\alpha}_{b_-}x(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(b)}{k!}(b-t)^k. \tag{2.8}$$

As will be seen, fractional integration by parts will play a crucial role to obtain Lyapunov inequalities of the nested fractional boundary value problems (1.12) and (1.13). So, this fractional tool is given in the next lemma.

Lemma 2.5 (*[32]*). *(Fractional integration by parts). Assume* $f,g\in AC^n(a,b)$ *and* $n-1<\alpha\le n,\ n\in\mathbb{Z}^+$. *Then*

$$\int_a^b g(t)\left({}^c\mathcal{D}^{\alpha}_{a^+}f\right)(t)dt=\int_a^b f(t)\left(\mathcal{D}^{\alpha}_{b_-}g\right)(t)dt+\sum_{k=0}^{n-1}\left[\mathcal{D}^{\alpha+k-n}_{b_-}g(t).f^{(n-1-k)}(t)\right]_a^b, \tag{2.9}$$

$$\int_a^b g(t)\left({}^c\mathcal{D}^{\alpha}_{b_-}f\right)(t)dt=\int_a^b f(t)\left(\mathcal{D}^{\alpha}_{a^+}g\right)(t)dt+\sum_{k=0}^{n-1}\left[(-1)^{n+k}\mathcal{D}^{\alpha+k-n}_{a^+}g(t).f^{(n-1-k)}(t)\right]_a^b. \tag{2.10}$$

In this position we discuss about half-linearity of the nested fractional boundary value problems (1.12) and (1.13). We say the fractional boundary value problem (1.12) (or (1.13)) is half-linear, since the solution space of the aforementioned differential equations has just one half of properties which characterize linearity, that is homogeneity but not additivity. For more details, we refer to [8].

Remark 2.6. The half-linear signed-power operator $\phi_\nu x:=|x|^{\nu-1}x,\ \nu\in(0,\infty)$ is invertible and $\phi_\nu^{-1}:=\phi_{\nu^{-1}}$.

At the end, we will keep in mind that everywhere needed, by $\|.\|$ we mean the standard max norm $\|x\|:=\max\{|x(t)|:t\in[a,b]\}$.

3. Main results

As explained in Section 1, this section is indeed theoretical body of our investigation. So, making use of the theoretical analysis techniques we shall extract desired Lyapunov inequalities corresponding to the nested fractional boundary value problems (1.12) and (1.13). To this aim, we first recall here boundary value problem (1.12).

$$\begin{cases}{}^c\mathcal{D}^{\alpha}_{a^+}\left(\phi_{\beta_2}\left(\mathcal{D}^{\alpha}_{b_-}\left(\phi_{\beta_1}\left({}^c\mathcal{D}^{\alpha}_{a^+}x\right)\right)\right)\right)-q(t)\phi_{\beta_1\beta_2}x=0, \quad a\le t\le b,\ n-1<\alpha\le n,\ n\in\mathbb{N}_{\ge2},\\ x^{(k)}(a)=0,\ x^{(k)}(b)=0,\ k=0,1,\dots,n-1.\end{cases} \tag{3.1}$$

Here we start with Lyapunov inequality of the nested fractional boundary value problem (3.1).

Theorem 3.1. *Let* $-\infty<a<b<+\infty$. *Assume* $x(t)$ *is a nontrivial solution of the boundary value problem* (3.1). *Suppose that there exists* $\xi\in[a,t],\ t\le b$ *such that*

$$\left(\phi_{\beta_2}\left(\mathcal{D}^{\alpha}_{b_-}\left(\phi_{\beta_1}\left({}^c\mathcal{D}^{\alpha}_{a^+}x\right)\right)\right)\right)^{(k)}(a)=0,\qquad\int_a^{\xi}(t-s)^{\alpha-1}q\phi_{\beta_1\beta_2}x(s)ds=0. \tag{3.2}$$

Then Lyapunov type inequality of the nested fractional boundary value problem (3.1) *is as follows:*

$$\int_a^{\xi}q_-(s)ds+\int_{\xi}^b q^+(s)ds>\frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}}. \tag{3.3}$$

Proof. Without loss of generality, assume $x(t)>0$ on (a,b). Thus there exists $d\in[a,b]$ such that $m:=x(d)=\|x\|$. Accordingly, we get

$$m=\mathcal{I}^{\alpha}_{a^+}\,{}^c\mathcal{D}^{\alpha}_{a^+}x(d)\le\mathcal{I}^{\alpha}_{a^+}\left|{}^c\mathcal{D}^{\alpha}_{a^+}x\right|(d)\le\int_a^d\frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\left|{}^c\mathcal{D}^{\alpha}_{a^+}x(s)\right|ds.$$

On the other hand, using the boundary condition $x(b) = 0$, one has

$$
\begin{aligned}
x(b) = 0 &= \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} {}^c\mathcal{D}_{a^+}^{\alpha} x(s)ds \\
&= \int_a^d \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} {}^c\mathcal{D}_{a^+}^{\alpha} x(s)ds + \int_d^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} {}^c\mathcal{D}_{a^+}^{\alpha} x(s)ds \\
&\geq \int_a^d \frac{(d-s)^{\alpha-1}}{\Gamma(\alpha)} {}^c\mathcal{D}_{a^+}^{\alpha} x(s)ds + \int_d^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} {}^c\mathcal{D}_{a^+}^{\alpha} x(s)ds \\
&= m + \int_d^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} {}^c\mathcal{D}_{a^+}^{\alpha} x(s)ds,
\end{aligned}
$$

that is

$$
\begin{aligned}
m &= -\int_d^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} {}^c\mathcal{D}_{a^+}^{\alpha} x(s)ds \\
&\leq \int_d^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} \left|{}^c\mathcal{D}_{a^+}^{\alpha} x(s)\right| ds.
\end{aligned}
$$

Therefore, we get the following

$$
\begin{cases}
m \leq \displaystyle\int_a^d \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} \left|{}^c\mathcal{D}_{a^+}^{\alpha} x(s)\right| ds, \\
m \leq \displaystyle\int_d^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} \left|{}^c\mathcal{D}_{a^+}^{\alpha} x(s)\right| ds.
\end{cases} \tag{3.4}
$$

Finally, the inequalities (3.4), give us

$$
2m \leq \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b \left|{}^c\mathcal{D}_{a^+}^{\alpha} x(t)\right| dt. \tag{3.5}
$$

Attempting to the next step requires the Hölder inequality that stands for the real valued measurable functions f and g, as follows;

$$
\int_a^b |f(t)g(t)|dt \leq \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q dt\right)^{\frac{1}{q}}, \quad 1 < p < \infty, \; p^{-1} + q^{-1} = 1. \tag{3.6}
$$

Choosing the setting $f(t) = {}^c\mathcal{D}_{a^+}^{\alpha} x(t)$, $g(t) = 1$, $p = \beta_1 + 1$ and $q = 1 + \frac{1}{\beta_1}$, we come to conclusion that

$$
\int_a^b \left|{}^c\mathcal{D}_{a^+}^{\alpha} x(t)\right| dt \leq (b-a)^{\frac{\beta_1}{\beta_1+1}} \left(\int_a^b \left|{}^c\mathcal{D}_{a^+}^{\alpha} x(t)\right|^{\beta_1+1} dt\right)^{\frac{1}{\beta_1+1}}. \tag{3.7}
$$

In viewpoint of (3.5), the inequality (3.7) yields

$$
\frac{(2m\Gamma(\alpha))^{\beta_1+1}}{(b-a)^{(\alpha-1)(\beta_1+1)+\beta_1}} \leq \int_a^b \left|{}^c\mathcal{D}_{a^+}^{\alpha} x(t)\right|^{\beta_1+1} dt = \int_a^b {}^c\mathcal{D}_{a^+}^{\alpha} x(t) \phi_{\beta_1}\left({}^c\mathcal{D}_{a^+}^{\alpha} x(t)\right) dt. \tag{3.8}
$$

Applying the fractional integration by parts formula (2.9) on the right hand side of the inequality (3.8) and then using the boundary conditions $x^{(k)}(a) = 0 = x^{(k)}(b)$, $k = 0, 1, 2, \ldots, n-1$, we reach the following

$$
\frac{(2m\Gamma(\alpha))^{\beta_1+1}}{(b-a)^{(\alpha-1)(\beta_1+1)+\beta_1}} \leq \int_a^b x(t)\mathcal{D}_{b^-}^{\alpha}\left(\phi_{\beta_1}\left({}^c\mathcal{D}_{a^+}^{\alpha} x(t)\right)\right) dt. \tag{3.9}
$$

In order to use the inequality (3.9), we shall recall the governing equation (3.1) as follows;

$$
{}^c\mathcal{D}_{a^+}^{\alpha}\left(\phi_{\beta_2}\left(\mathcal{D}_{b^-}^{\alpha}\left(\phi_{\beta_1}\left({}^c\mathcal{D}_{a^+}^{\alpha} x\right)\right)\right)\right) - q(t)\phi_{\beta_1\beta_2} x = 0.
$$

If we take fractional integration of order α from both sides of the recent equation, so using the inversion rule (2.7) and the assumptions (3.2), we get that

$$\phi_{\beta_2}\left(\mathcal{D}^{\alpha}_{b-}\left(\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{a+}x\right)\right)\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{\xi}^{t}(t-s)^{\alpha-1}q(s)\phi_{\beta_1\beta_2}x(s)ds. \tag{3.10}$$

In the sequel we define

$$q^{*}(s)=\begin{cases}-q_{-}(s), & a\le s\le \xi,\\ q^{+}(s), & \xi\le s\le b.\end{cases} \tag{3.11}$$

The definition (3.11), implies that $-q_{-}(t)\le q(t)\le q^{+}(t)$. So we have

$$\int_{\xi}^{t}q(s)\phi_{\beta_1\beta_2}x(s)ds\le\begin{cases}\displaystyle\int_{\xi}^{t}-q_{-}(s)\phi_{\beta_1\beta_2}x(s)ds, & t<\xi,\\[2ex] \displaystyle\int_{\xi}^{t}q^{+}(s)\phi_{\beta_1\beta_2}x(s)ds, & t\ge\xi.\end{cases} \tag{3.12}$$

Now we use (3.12) to reach

$$\phi_{\beta_2}\left(\mathcal{D}^{\alpha}_{b-}\left(\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{a+}x\right)\right)\right)\le\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_{\xi}^{t}q^{*}(s)\phi_{\beta_1\beta_2}x(s)ds. \tag{3.13}$$

Equivalently, we get that

$$\mathcal{D}^{\alpha}_{b-}\left(\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{a+}x\right)\right)\le\frac{(b-a)^{\frac{\alpha-1}{\beta_2}}}{\Gamma^{\frac{1}{\beta_2}}(\alpha)}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{*}(s)\phi_{\beta_1\beta_2}x(s)ds\right). \tag{3.14}$$

Consequently

$$\mathcal{D}^{\alpha}_{b-}\left(\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{a+}x\right)\right)<\frac{(b-a)^{\frac{\alpha-1}{\beta_2}}}{\Gamma^{\frac{1}{\beta_2}}(\alpha)}m^{\beta_1}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{*}(s)ds\right). \tag{3.15}$$

Here we substitute (3.9) into (3.15) to achieve

$$\begin{aligned}\frac{(2m\Gamma(\alpha))^{\beta_1+1}}{(b-a)^{(\alpha-1)(\beta_1+1)+\beta_1}}&<\frac{(b-a)^{\frac{\alpha-1}{\beta_2}}}{\Gamma^{\frac{1}{\beta_2}}(\alpha)}m^{\beta_1}\int_{a}^{b}x(t)\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{*}(s)ds\right)dt\\&\le\frac{(b-a)^{\frac{\alpha-1}{\beta_2}}}{\Gamma^{\frac{1}{\beta_2}}(\alpha)}m^{\beta_1+1}\int_{a}^{b}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{*}(s)ds\right)dt\\&\le\frac{(b-a)^{\frac{\alpha-1}{\beta_2}}}{\Gamma^{\frac{1}{\beta_2}}(\alpha)}m^{\beta_1+1}\left\{\int_{a}^{\xi}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{*}(s)ds\right)dt+\int_{\xi}^{b}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{*}(s)ds\right)dt\right\}.\end{aligned} \tag{3.16}$$

Applying the definition (3.11) into the inequality (3.16), gives us the following inequality

$$\begin{aligned}\frac{2^{\beta_1+1}\Gamma(\alpha)^{\beta_1+\frac{1}{\beta_2}+1}}{(b-a)^{\frac{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_1\beta_2}{\beta_2}}}&<\int_{a}^{\xi}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{*}(s)ds\right)dt+\int_{\xi}^{b}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{*}(s)ds\right)dt\\&=\int_{a}^{\xi}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}-q_{-}(s)ds\right)dt+\int_{\xi}^{b}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{+}(s)ds\right)dt\\&=\int_{a}^{\xi}\phi_{\beta_2^{-1}}\left(\int_{t}^{\xi}q_{-}(s)ds\right)dt+\int_{\xi}^{b}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t}q^{+}(s)ds\right)dt\\&\le\int_{a}^{\xi}\phi_{\beta_2^{-1}}\left(\int_{a}^{\xi}q_{-}(s)ds\right)dt+\int_{\xi}^{b}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{b}q^{+}(s)ds\right)dt.\end{aligned}$$

Hence,

$$\frac{2^{\beta_1+1}\Gamma(\alpha)^{\beta_1+\frac{1}{\beta_2}+1}}{(b-a)^{\frac{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_1\beta_2}{\beta_2}}} < (\xi-a)\phi_{\beta_2^{-1}}\left(\int_a^\xi q_-(s)ds\right) + (b-\xi)\phi_{\beta_2^{-1}}\left(\int_\xi^b q^+(s)ds\right). \tag{3.17}$$

Now, we are in such position that using the upcoming discussion desired Lyapunov inequality for the nested fractional boundary value problem (3.1) can be concluded. To this aim, the last step of the proof is divided into two cases $\beta_2 \geq 1$ and $0 < \beta_2 < 1$ as follows:

(i) Suppose that $\beta_2 \geq 1$. In this case, positivity of second derivative of ϕ_{β_2} guarantees that ϕ_{β_2} is a concave-up function on $[0,\infty)$ in the sense that for any $x_1, x_2 \in [0,\infty)$ and $t \in [0,1]$, we have

$$\phi_{\beta_2}(tx_1+(1-t)x_2) \leq t\phi_{\beta_2}(x_1)+(1-t)\phi_{\beta_2}(x_2). \tag{3.18}$$

Dividing both sides of (3.17) by $b-a$ and applying ϕ_{β_2} gives us

$$\begin{aligned}\frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}} &= \phi_{\beta_2}\left(\frac{2^{\beta_1+1}\Gamma(\alpha)^{\beta_1+\frac{1}{\beta_2}+1}}{(b-a)^{\frac{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}{\beta_2}}}\right)\\ &< \phi_{\beta_2}\left(\left(\frac{\xi-a}{b-a}\right)\phi_{\beta_2^{-1}}\left(\int_a^\xi q_-(s)ds\right)+\left(\frac{b-\xi}{b-a}\right)\phi_{\beta_2^{-1}}\left(\int_\xi^b q^+(s)ds\right)\right).\end{aligned}$$

Since $\frac{\xi-a}{b-a}+\frac{b-\xi}{b-a}=1$, then taking $t=\frac{\xi-a}{b-a}$ in the (3.18), desired Lyapunov inequality is obtained as follows:

$$\begin{aligned}\frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}} &< \phi_{\beta_2}\Bigg(\left(\frac{\xi-a}{b-a}\right)\Theta_{\beta_2^{-1}}\left(\int_a^\xi q_-(s)ds\right)\\ &\quad+\left(\frac{b-\xi}{b-a}\right)\phi_{\beta_2^{-1}}\left(\int_\xi^b q^+(s)ds\right)\Bigg)\\ &\leq \left(\frac{\xi-a}{b-a}\right)\int_a^\xi q_-(s)ds+\left(\frac{b-\xi}{b-a}\right)\int_\xi^b q^+(s)ds\\ &\leq \int_a^\xi q_-(s)ds+\int_\xi^b q^+(s)ds.\end{aligned}$$

(ii) In second phase of the last step, we assume that $0<\beta_2<1$. Because of negativity of second derivative, ϕ_{β_2} is a concave-down function on $[0,\infty)$. Therefore for any $x_1, x_2 \in [0,\infty)$, we have

$$\phi_{\beta_2}(x_1+x_2) \leq \phi_{\beta_2}x_1+\phi_{\beta_2}x_2. \tag{3.19}$$

In this case according to the inequality (3.16), we get that

$$\frac{2^{\beta_1+1}\Gamma(\alpha)^{\beta_1+\frac{1}{\beta_2}+1}}{(b-a)^{\frac{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_1\beta_2}{\beta_2}}} < (b-a)\left[\phi_{\beta_2^{-1}}\left(\int_a^\xi q_-(s)ds\right)+\phi_{\beta_2^{-1}}\left(\int_\xi^b q^+(s)ds\right)\right].$$

To obtain desired result, it is enough to divide both sides of the recent inequality by $b-a$ and take ϕ_{β_2} from both sides. Now, applying property (3.19) yields the expected Lyapunov inequality

$$\begin{aligned}\frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}} &< \phi_{\beta_2}\left(\phi_{\beta_2^{-1}}\left(\int_a^\xi q_-(s)ds\right)\right)+\phi_{\beta_2}\left(\phi_{\beta_2^{-1}}\left(\int_\xi^b q^+(s)ds\right)\right)\\ &= \int_a^\xi q_-(s)ds+\int_\xi^b q^+(s)ds.\end{aligned}$$

Considering the outcomes of the cases (i) and (ii), shows that the proof is complete. □

It is time to evaluate the nested fractional boundary value problem

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{a+}\left(r_2(t)\phi_{\beta_2}\left(\mathcal{D}^{\alpha}_{b-}\left(r_1(t)\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{a+}x\right)\right)\right)\right) - q(t)\phi_{\beta_1\beta_2}x = 0, \quad a \le t \le b,\ n-1 < \alpha \le n,\ n \in \mathbb{N}_{\ge 2}, \\ x^{(k)}(a) = 0,\ x^{(k)}(b) = 0,\ k = 0, 1, \ldots, n-1. \end{cases} \tag{3.20}$$

The next theorem contains an analytic algorithm that its output is Lyapunov inequality of the nested fractional boundary value problem (3.20).

Theorem 3.2. *Let $-\infty < a < b < +\infty$. Assume $x(t)$ is a nontrivial solution of the nested fractional boundary value problem* (3.20). *Suppose there exists $\xi \in [a, t],\ t \le b$ such that*

$$\left(r_2\phi_{\beta_2}\left(\mathcal{D}^{\alpha}_{b-}\left(r_1\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{a+}x\right)\right)\right)\right)^{(k)}(a) = 0, \qquad \int_a^{\xi}(t-s)^{\alpha-1}\left(q\phi_{\beta_1\beta_2}x\right)(s)ds = 0. \tag{3.21}$$

Then Lyapunov type inequality of the nested fractional boundary value problem (3.20) *is as follows:*

$$\int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q^+(s)ds > \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{\left(\int_a^b (r_1(t))^{\frac{-1}{\beta_1}}dt\right)^{\beta_1\beta_2}\left(\int_a^b (r_2(t))^{\frac{-1}{\beta_2}}dt\right)^{\beta_2}(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)}}. \tag{3.22}$$

Proof. Without loss of generality, suppose that $x(t) > 0$ on (a, b). Then there exists $d \in [a, b]$ such that

$$m := x(d) = \|x\|.$$

Thus, similar to Theorem 3.1, one has

$$2m \le \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b \left|{}^{c}\mathcal{D}^{\alpha}_{a+}x(t)\right|dt = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b (r_1(t))^{-\frac{1}{\beta_1+1}}\left((r_1(t))^{\frac{1}{\beta_1+1}}\left|{}^{c}\mathcal{D}^{\alpha}_{a+}x(t)\right|\right)dt. \tag{3.23}$$

If we apply the Hölder inequality with parameters $p = 1 + \frac{1}{\beta_1}$ and $q = \beta_1 + 1$ on the inequality (3.23), the following inequality is obtained:

$$\begin{aligned} \frac{(2m\Gamma(\alpha))^{\beta_1+1}}{(b-a)^{(\beta_1+1)(\alpha-1)}} &\le \left(\int_a^b (r_1(t))^{-\frac{1}{\beta_1}}dt\right)^{\beta_1}\int_a^b r_1(t)\left|{}^{c}\mathcal{D}^{\alpha}_{a+}x(t)\right|^{\beta_1+1}dt \\ &= \left(\int_a^b (r_1(t))^{-\frac{1}{\beta_1}}dt\right)^{\beta_1}\int_a^b {}^{c}\mathcal{D}^{\alpha}_{a+}x(t)\left(r_1(t)\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{a+}x(t)\right)\right)dt. \end{aligned} \tag{3.24}$$

Here is the position that we shall use the fractional integration by parts on the second integral in the right hand side of (3.24). In this case we have

$$\frac{(2m\Gamma(\alpha))^{\beta_1+1}}{(b-a)^{(\beta_1+1)(\alpha-1)}} \le \left(\int_a^b (r_1(t))^{-\frac{1}{\beta_1}}dt\right)^{\beta_1}\int_a^b x(t)\mathcal{D}^{\alpha}_{b-}\left(r_1(t)\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{a+}x(t)\right)\right)dt. \tag{3.25}$$

Now, considering the governing equation (3.20) and having similar argument that led to the inequality (3.15) in Theorem 3.1, it is easy to obtain the following inequality:

$$\mathcal{D}^{\alpha}_{b-}\left(r_1(t)\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{a+}x(t)\right)\right) \le \frac{m^{\beta_1}(b-a)^{\frac{\alpha-1}{\beta_2}}}{r_2(t)^{\frac{1}{\beta_2}}\Gamma(\alpha)^{\frac{1}{\beta_2}}}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{t} q^*(s)ds\right). \tag{3.26}$$

In this position we need to use increasing nature of the functions $r_2(t)$ and $\phi^{-1}_{\beta_2}(x > 0)$ on $[a, b]$ also the following well known identity

$$(b-a)\int_a^b f(t)g(t)dt \le \left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right),$$

where f and g are increasing and decreasing integrable functions on $[a, b]$, respectively. In this case, substituting (3.25) into (3.26), it follows that

$$\begin{aligned}
&\frac{2^{\beta_1+1}\Gamma(\alpha)^{\frac{\beta_2(\beta_1+1)+1}{\beta_2}}}{\left(\int_a^b (r_1(t))^{\frac{-1}{\beta_1}}dt\right)^{\beta_1}\left(\int_a^b (r_2(t))^{\frac{-1}{\beta_2}}dt\right)(b-a)^{\frac{(\alpha-1)(\beta_2(\beta_1+1)+1)}{\beta_2}}}\\
&\quad< \int_a^{\xi}\phi_{\beta_2^{-1}}\left(\int_a^{\xi} q_-(s)ds\right)dt + \int_{\xi}^{b}\phi_{\beta_2^{-1}}\left(\int_{\xi}^{b} q^+(s)ds\right)dt\\
&\quad\le (\xi-a)\phi_{\beta_2^{-1}}\left(\int_a^{\xi} q_-(s)ds\right) + (b-\xi)\phi_{\beta^{-1}{}_2}\left(\int_{\xi}^{b} q^+(s)ds\right).
\end{aligned} \tag{3.27}$$

Here we are in such a position that the inequality (3.17) was in Theorem 3.1. Therefore, with a similar estimation we conclude that

$$\int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q^+(s)ds > \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{\left(\int_a^b (r_1(t))^{\frac{-1}{\beta_1}}dt\right)^{\beta_1\beta_2}\left(\int_a^b (r_2(t))^{\frac{-1}{\beta_2}}dt\right)^{\beta_2}(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)}}. \tag{3.28}$$

This completes the proof. □

Remark 3.3. Note that setting $r_1(t) = r_2(t) = 1$, reduces the Lyapunov type inequality (3.22) to the Lyapunov type inequality (3.3). So, we deduce that the nested fractional boundary value problem (3.20) generalizes the nested fractional boundary value problem (3.1), as the Lyapunov inequality (3.22) generalizes the Lyapunov inequality (3.3).

Remark 3.4. Let us consider the Lyapunov inequalities (3.3) and (3.22). Assume $n = 2$. If we take $\alpha \to 1$, then these Lyapunov inequalities reduce to the Lyapunov inequalities (1.10) and (1.11), respectively. Now, there is an interesting point regarding to this generalization. If we take $\alpha \to 1$, then in the nested fractional boundary value problem (3.1) we have four boundary conditions $x(a) = x'(a) = 0$ and $x(b) = x'(b) = 0$, while in the nested ordinary boundary value problem (1.9), we have just couple of the boundary condition $x(a) = 0$ and $x(b) = 0$. So, we conclude that both of the conditions

$$\begin{cases} x(a) = 0, \quad x(b) = 0,\\ \left(\phi_{\alpha_1}\left(x'\right)\right)'(\xi) = 0, \quad \xi \in [a, b], \end{cases} \tag{3.29}$$

and

$$\begin{cases} x(a) = x'(a) = 0, \quad x(b) = x'(b) = 0,\\ \phi_{\beta_2}\left(\phi_{\beta_1}\left(x'\right)\right)'(a) = 0,\ \left(\phi_{\beta_2}\left(\phi_{\beta_1}\left(x'\right)\right)'\right)'(a) = 0, \quad \int_a^{\xi} q\phi_{\beta_1\beta_2}x(s)ds = 0, \quad \xi \in [a, b], \end{cases} \tag{3.30}$$

give us the Lyapunov inequality (1.10) corresponding to the third order nested boundary value problem (1.8). Comparing the boundary conditions (3.29) and (3.30), it is clear that if $\alpha \to 1$, then, the Lyapunov inequality (3.3) reduces into the Lyapunov inequality (1.10) subject to stronger conditions. Interpreting this situation one can state that if $\alpha \to 1^-$ i.e. $\alpha \in (0, 1)$, in this case the power function $(t - s)^{\alpha-1}$ is decreasing with respect to the variable t and increasing with respect to the variable s. Thus, this function cannot reach its finite bounds for each $\alpha \in (0, 1)$. Therefore, we choose $n - 1 < \alpha \le n,\ n \in \mathbb{N}_{\ge 2}$ that consequences the stronger conditions (3.30).

4. Applications

This section can be considered as the applied aspect of this paper. Since, in this section relying on the Lyapunov inequalities (3.3) and (3.22), we are going to establish qualitative dynamics of nontrivial solutions of the nested fractional boundary value problems (3.1) and (3.20). In this way, we are interested in the study of disconjugacy, nonexistence, zero count, distance between consecutive zeros of oscillatory solutions and eigenvalue intervals for nested fractional eigenvalue problems corresponding to the boundary value problems (3.1) and (3.20). So, we begin as follows.

► **Disconjugacy**. In order to establish disconjugacy of the nested fractional boundary value problems (3.1) and (3.20), we first define it as follows.

Definition 4.1. The nested fractional boundary value problem (3.1) (or (3.20)), is said to be disconjugate on the interval $[a, b]$, if and only if each of its nontrivial solutions has less then $3n$ zeros in the interval $[a, b]$. Otherwise, we have a conjugate nested fractional boundary value problem.

Theorem 4.2. *Assume there exists* $\xi \in [s_1, s_2],\ a \le s_1 < s_2 \le b$ *such that*

$$\left(\phi_{\beta_2}\left(\mathcal{D}^{\alpha}_{s_2-}\left(\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{s_1+}x\right)\right)\right)\right)^{(k)}(s_1) = 0, \quad \left(\mathcal{I}^{\alpha}_{s_1^+}(q\phi_{\beta_1\beta_2}x)\right)(\xi) = 0.$$

Let $x^{(k)}(s_1) = 0$ *and* $x^{(k)}(s_2) = 0$ *for* $k = 1, 2, 3, \ldots, n-1$. *If*

$$\int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q^+(s)ds \le \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}}, \tag{4.1}$$

then, the nested fractional boundary value problem (3.1) *is disconjugate on* $[a, b]$.

Proof. Suppose on the contrary that the nested fractional boundary value problem (3.1) is conjugate on $[a, b]$. So, in accordance with Definition 4.1, there exists nontrivial solution x with at least $3n,\ n \in \mathbb{N}_2$ zeros in the interval $[a, b]$. Suppose that $s_1,\ s_2 \in [a, b]$ are two distinct zeros of the x. Thus, $x(s_1) = 0 = x(s_2)$. Now, since all of the assumptions of Theorem 3.1 hold, then the Lyapunov inequality

$$\int_{s_1}^{\xi} q_-(s)ds + \int_{\xi}^{s_2} q^+(s)ds > \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(s_2-s_1)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}},$$

is obtained. Therefore, taking $s_1 = a$ and $s_2 = b$,

$$\int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q^+(s)ds > \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}},$$

which contradicts the hypothesis (4.1). Thus, the nested fractional boundary value problem (3.1) is disconjugate on the interval $[a, b]$. □

Similar to the disconjugacy criterion presented in Theorem 4.2, one may present a disconjugacy criterion for the nested fractional boundary value problem (3.20).

Lemma 4.3. *Assume there exists* $\xi \in [s_1, s_2],\ a \le s_1 < s_2 \le b$ *such that*

$$\left(r_2\phi_{\beta_2}\left(\mathcal{D}^{\alpha}_{s_2-}\left(r_1\phi_{\beta_1}\left({}^{c}\mathcal{D}^{\alpha}_{s_1+}x\right)\right)\right)\right)^{(k)}(s_1) = 0, \quad \left(\mathcal{I}^{\alpha}_{s_1^+}(q\phi_{\beta_1\beta_2}x)\right)(\xi) = 0.$$

Let $x^{(k)}(s_1) = 0$ *and* $x^{(k)}(s_2) = 0$ *for* $k = 1, 2, 3, \ldots, n-1$. *If*

$$\int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q^+(s)ds \le \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{\left(\int_a^b (r_1(t))^{\frac{-1}{\beta_1}}dt\right)^{\beta_1\beta_2}\left(\int_a^b (r_2(t))^{\frac{-1}{\beta_2}}dt\right)^{\beta_2}(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)}}, \tag{4.2}$$

then, the nested fractional boundary value problem (3.20) *is disconjugate on* $[a, b]$.

► **Nonexistence**. Here we present a nonexistence criterion for nontrivial solutions of the nested fractional boundary value problems (3.1) and (3.20). As claimed at the beginning, in fact, we will show that the disconjugacy criteria in Theorem 4.2 and Lemma 4.3, are also nonexistence criteria for nontrivial solutions of the boundary value problems (3.1) and (3.20). To this aim, we first state and prove the following theorem.

Theorem 4.4. *Assume that the assumptions of Theorem* 4.2 *hold. Then the nested fractional boundary value problem* (3.1) *has no nontrivial solution on* $[a, b]$.

Proof. Assume on contrary that the nested fractional boundary value problem (3.1) has a nontrivial solution such as x. Thus, there exist consecutive zeros $s_1, s_2 \in [a, b]$ such that $x^{(k)}(s_1) = 0$ and $x^{(k)}(s_2) = 0$ for $k = 0, 1, 2, \dots, n-1$. So, relying on the assumptions of Theorem 4.2 and according to Theorem 3.1 we get the following Lyapunov inequality:

$$\int_{s_1}^{\xi} q_-(s)ds + \int_{\xi}^{s_2} q^+(s)ds > \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(t_2-t_1)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}}.$$

So, using the setting $t_1 = a$ and $t_2 = b$ it follows that

$$\int_{a}^{\xi} q_-(s)ds + \int_{\xi}^{b} q^+(s)ds > \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}},$$

which contradicts the assumption (4.1). Hence, the nested fractional boundary value problem (3.1) has no nontrivial solution. □

A similar criterion for the nonexistence of nontrivial solutions for the nested fractional boundary value problem (3.20) is given as follows.

Lemma 4.5. *Suppose $r_1, r_2, q \in C(\mathbb{R}, \mathbb{R})$ such that $r_1(t), r_2(t) > 0$ and $r_2(t)$ is an increasing function on* $[a, b]$. *If assumptions of Lemma* 4.3 *hold, then the nested fractional boundary value problem* (3.20) *has no nontrivial solution.*

From now on, we concentrate only on the Lyapunov inequality (3.3). Since establishing applicability of the Lyapunov inequality (3.22) in what remains of the expected applications, requires more restrictive assumptions.

► **Zerocount**. Third application of the Lyapunov inequality (3.3), is dealt with the maximum number of zeros of the nontrivial solutions of the nested fractional boundary value problem (3.1). This zero count estimation is given in the following theorem.

Theorem 4.6. *Suppose that $x(t)$ is a nontrivial solution of the nested fractional boundary value problem* (3.1), *with $\beta_2 \geq 1$. Let $\{t_k\}_{k=1}^N$, $N \geq 1$ be an increasing sequence of zeros of $x^{(k)}(t)$, $k = 0, 1, 2, \dots, n-1$ in a compact interval I with length l. Then, an upper bound for maximum number of zeros of $x(t)$ is given by:*

$$\begin{aligned} N < \Bigg(&\frac{l^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)-1}}{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}} \sum_{k=1}^{N-1} \times \\ &\times \max_{\xi_k \in [t_k, t_{k+1}]} \left\{ \int_{t_k}^{\xi_k} q_-(s)ds + \int_{\xi_k}^{t_{k+1}} q^+(s)ds \right\} \Bigg)^{\frac{1}{-((\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)-1)}} + 1. \end{aligned} \tag{4.3}$$

Proof. Considering intervals $[t_k, t_{k+1}] \subset I$ for $k = 1, 2, \dots, N-1$, Theorem 3.1 gives us the following Lyapunov inequality:

$$\max_{\xi_k \in [t_k, t_{k+1}]} \left\{ \int_{t_k}^{\xi_k} q_-(s)ds + \int_{\xi_k}^{t_{k+1}} q^+(s)ds \right\} > \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(t_{2k+1}-t_{2k-1})^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}}.$$

Now, taking the sum on both sides of the recent inequality from 1 to $N-1$, and by means of inequalities

$$\frac{1}{M}\sum_{k=1}^{M} A_k \geq \left(\prod_{k=1}^{M} A_k\right)^{\frac{1}{M}} \geq \left(\frac{1}{M}\sum_{k=1}^{M}\frac{1}{A_k}\right)^{-1}, \quad A_k > 0,\ k = 1, 2, \dots, M,\ (see\ [22]),$$

we come to conclusion that

$$\begin{aligned}
&\sum_{k=1}^{N-1} \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(t_{k+1}-t_k)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}} \\
&= (N-1)\left(\frac{1}{N-1}\sum_{k=1}^{N-1} \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(t_{k+1}-t_k)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}}\right) \\
&\geq (N-1)2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}\left(\frac{1}{N-1}\sum_{k=1}^{N-1}\frac{1}{t_{k+1}-t_k}\right)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)} \\
&\geq \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(N-1)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)-1}}\left(\sum_{k=1}^{N-1} t_{k+1}-t_k\right)^{-((\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1))} \\
&\geq \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(N-1)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)-1}}(t_N-t_1)^{-((\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1))} \\
&\geq \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{((N-1)l)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)-1}}.
\end{aligned}$$

So, we have the following upper bound estimation for N

$$\begin{aligned}
N < &\left(\frac{l^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)-1}}{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}\sum_{k=1}^{N-1}\times\right. \\
&\left.\times \max_{\xi_k\in[t_k,t_{k+1}]}\left\{\int_{t_k}^{\xi_k} q_-(s)ds + \int_{\xi_k}^{t_{k+1}} q^+(s)ds\right\}\right)^{\frac{1}{-((\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)-1)}} + 1,
\end{aligned} \tag{4.4}$$

that completes the proof. $\square$

Remark 4.7. Since $q_-(t), q^+(t) \leq |q(t)|$ for $t \in [a, b]$, so, we get that

$$\int_a^b |q(t)|dt \geq \int_a^\xi q_-(s)ds + \int_\xi^b q^+(s)ds.$$

Therefore, in what follows we will use the Lyapunov inequality:

$$\int_a^b |q(t)|dt > \frac{2^{\beta_2(\beta_1+1)}\Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}}. \tag{4.5}$$

► **Distance between consecutive zeros of oscillatory solutions.** In this position we examine the Lyapunov inequality (4.5) from a different viewpoint. Indeed, the Lyapunov inequality (4.5) will help us to demonstrate that, if given nontrivial solution of the nested fractional boundary value problem (3.1) is also oscillatory, then it is impossible that distance between consecutive zeros of this solution at infinity be finite. Here we state and prove this criterion.

Theorem 4.8. *Assume $x(t)$ be an oscillatory solution of the nested fractional boundary value problem* (3.1)*. Let $\{t_n\}_{n=1}^\infty$ be an increasing sequence of zeros of $x^{(k)}(t)$, $k = 0, 1, 2, \ldots, n-1$ in $[0, \infty)$. If there exist a $\sigma \geq 1$, such that for any positive real constant M, we have*

$$\int_t^{t+M} |q(s)|^\sigma ds \to 0, \quad t \to \infty, \tag{4.6}$$

then, $t_{n+1} - t_n \to \infty$ as $n \to \infty$.

Proof. At the beginning, we remind this point that the Hölder inequality is playing crucial role to obtain desired result. Thus, choosing the parameters $p = \sigma$ and $q = \frac{\sigma}{\sigma-1}$, it follows that

$$\int_t^{t+M} |q(s)|ds \le M^{\frac{\sigma}{\sigma-1}} \left(\int_t^{t+M} |q(s)|^\sigma ds \right)^{\frac{1}{\sigma}} \to 0, \quad t \to \infty.$$

Since our proof is based on the contradiction, suppose on the contrary that there exist a positive real constant M and a subsequence $\{t_{n_k}\}_{k=1}^\infty$ of $\{t_n\}_{n=1}^\infty$ such that $t_{n_k+1} - t_{n_k} \le M$ for all large k. Thus, the assumption (4.6) gives us the following:

$$\int_{t_{n_k}}^{t_{n_k+1}} |q(t)|dt \le \int_{t_{n_k}}^{t_{n_k}+M} |q(t)|dt \to 0 \quad k \to \infty.$$

In the sequel if we apply Theorem 3.1 on the interval $[t_{n_k}, t_{n_k+1}]$, it follows that

$$\int_{t_{n_k}}^{t_{n_k+1}} |q(t)|dt > \frac{2^{\beta_2(\beta_1+1)} \Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(t_{n_k+1} - t_{n_k})^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}}.$$

Equivalently, we have

$$\begin{aligned} 1 &< 2^{-\beta_2(\beta_1+1)} \Gamma(\alpha)^{-\beta_2(\beta_1+1)-1} (t_{n_k+1} - t_{n_k})^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)} \int_{t_{n_k}}^{t_{n_k+1}} |q(t)|dt \\ &\le 2^{-\beta_2(\beta_1+1)} \Gamma(\alpha)^{-\beta_2(\beta_1+1)-1} M^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)} \int_{t_{n_k}}^{t_{n_k+1}} |q(t)|dt \to 0, \;\; k \to \infty. \end{aligned}$$

This algebraic contradiction completes the proof. □

► **Eigenvalue interval.** This is the last examination to show applicability of the Lyapunov inequality (4.5). To this aim, we consider the nested fractional eigenvalue problem

$$\begin{cases} {}^c\mathcal{D}^\alpha_{a^+} \left(\phi_{\beta_2} \left(\mathcal{D}^\alpha_{b_-} \left(\phi_{\beta_1} \left({}^c\mathcal{D}^\alpha_{a^+} x \right) \right) \right) \right) - \lambda \phi_{\beta_1 \beta_2} x = 0, \quad a \le t \le b, \ \lambda \in \mathbb{R}, \ n-1 < \alpha \le n, \ n \in \mathbb{N}_{\ge 2}, \\ x^{(k)}(a) = 0, \ x^{(k)}(b) = 0, \ k = 0, 1, \dots, n-1, \end{cases} \tag{4.7}$$

corresponding to the nested fractional boundary value problem (3.1). Therefore according to the Lyapunov inequality (4.5), we conclude that

$$\int_a^b |\lambda|dt > \frac{2^{\beta_2(\beta_1+1)} \Gamma(\alpha)^{\beta_2(\beta_1+1)+1}}{(b-a)^{(\alpha-1)(\beta_2(\beta_1+1)+1)+\beta_2(\beta_1+1)}}. \tag{4.8}$$

Equivalently,

$$|\lambda| > \frac{1}{2} \left(\frac{2\Gamma(\alpha)}{(b-a)^\alpha} \right)^{\beta_2(\beta_1+1)+1}. \tag{4.9}$$

Hence, it follows that for each

$$\lambda \in \left[-\frac{1}{2} \left(\frac{2\Gamma(\alpha)}{(b-a)^\alpha} \right)^{\beta_2(\beta_1+1)+1}, \frac{1}{2} \left(\frac{2\Gamma(\alpha)}{(b-a)^\alpha} \right)^{\beta_2(\beta_1+1)+1} \right],$$

λ is not an eigenvalue of the nested fractional eigenvalue problem (4.7). Also,

$$\mathcal{LB}_{\text{eigenvalue}} := \frac{1}{2} \left(\frac{2\Gamma(\alpha)}{(b-a)^\alpha} \right)^{\beta_2(\beta_1+1)+1},$$

can be considered as a lower bound for the positive eigenvalues of the eigenvalue problem (4.8).

5. Concluding remarks and an open problem

In this paper, two classes of nested half-linear fractional boundary value problems of the form (3.1) and (3.20) have studied. The main aims of this study can be summarized as follows. First using analytic techniques, the Lyapunov inequalities (3.3) and (3.22) corresponding to the boundary value problems (3.1) and (3.20) are obtained, respectively. Second part of our study has dealt with applicability examination of the Lyapunov inequalities (3.3) and (3.22), that is in five steps we demonstrated that disconjugacy, nonexistence, zero count for nontrivial solutions, distance between consecutive zeros of the oscillatory solutions at infinity and eigenvalue intervals for corresponding nested half-linear fractional eigenvalue problems can be estimated via Lyapunov inequalities.

The end point of our study concerns with an open problem regarding to non-integer linear differential systems. Actually, if we consider the linear planar Hamiltonian systems of the form

$$x' = a(t)x + b(t)u, \quad u' = -c(t)x - a(t)u, \qquad t \in \mathbb{R}, \tag{5.1}$$

the authors in [17], making use of the Leibniz rule obtained the corresponding Lyapunov inequality

$$\int_\alpha^\beta |a(t)|dt + \left\{\int_\alpha^\beta b(t)dt.\int_\alpha^\beta c_+(t)dt\right\}^{\frac{1}{2}} \geq 2, \quad \alpha, \beta \in \mathbb{R},\ \alpha < \beta, \tag{5.2}$$

for some real-valued piece-wise continuous functions a, b and c. Now, let us replace the first order derivatives $\frac{d}{dt}$ with Riemann–Liouville fractional derivative $\mathcal{D}^\alpha$, for $0 < \alpha \leq 1$. So, we have the fractional linear differential system

$$\mathcal{D}^\alpha x = a(t)x + b(t)u, \quad \mathcal{D}^\alpha u = -c(t)x - a(t)u, \qquad t \in \mathbb{R}. \tag{5.3}$$

As we know, none of the Riemann–Liouville based fractional differentiation operators satisfy the first order Leibniz rule in the sense of ordinary differential calculus. Now, we must try to answer this question that can we obtain Lyapunov inequality of the fractional linear differential system (5.3)? and if the answer is yes, what is the desired method to reach the appropriate Lyapunov inequality?

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Original article

Weighted Hardy-type inequalities involving convex function for fractional calculus operators

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Abstract

The aim of this paper is to establish some new weighted Hardy-type inequalities involving convex and monotone convex functions using Hilfer fractional derivative and fractional integral operator with generalized Mittag-Leffler function in its kernel. We also discuss one dimensional cases of our related results. As a special case of our general results we obtain the results of Iqbal et al. (2017). Moreover, the refinement of Hardy-type inequalities for Hilfer fractional derivative is also included.

Keywords: Convex function; Kernel; Hilfer fractional derivatives; Fractional integral

1. Introduction

Fractional calculus deals with the study of fractional order integral and derivative operators calculus and have been of great importance during the last few decades. Oldham and Spanier [1] published their fundamental work in their book in 1974 and Podlubny [2] publication from 1999, which deals principally with fractional differential equations. For further details and literature about the fractional calculus we refer to [3–5] and the references cited therein. Numerous mathematicians obtained new Hardy-type inequalities for different fractional integrals and fractional derivatives. For details we refer to [6–12].

The general theory for the Hardy-type inequalities has attracted a lot of attention during a long time, see e.g. the books [13–15] and the reference therein. One reason is that such results are of special interest for technical sciences.

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Especially actions of kernels operators of type (1.2) and (1.3) are important since the kernel $k(x, y)$ represent unit impulse answers in systems which need not to be time invariant ($f(y)$ and $g(x)$ represent the "insignals" and "outsignals" respectively). Some current knowledge can be found in Section 7.5 of the new 2017 book [15] by Kufner, Persson and Samko, see also the related review article [16]. But still there are many open questions in this area, see e.g. those pointed out in [15, Section 7.5]. In this paper we present some new results concerning Hardy-type inequalities not covered by the literature mentioned above.

The following definitions are presented in [17].

Definition 1.1. Let I be an interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is called convex if

$$\Phi(\lambda x + (1-\lambda)y) \le \lambda\Phi(x) + (1-\lambda)\Phi(y), \tag{1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0, 1]$. The function Φ is strictly convex if inequality (1.1) holds strictly for all distinct points in I and $\lambda \in (0, 1)$.

Definition 1.2. Let $\Phi : I \longrightarrow \mathbb{R}$ be a convex function, then the sub-differential of Φ at x, denoted by $\partial\Phi(x)$, is defined as

$$\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(y) - \Phi(x) - \alpha(y - x) \ge 0,\ y \in I\}.$$

Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $U(f)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x, y) f(y) d\mu_2(y), \tag{1.2}$$

and A_k be an integral operator defined by

$$(A_k f)(x) := \frac{g(x)}{K(x)} = \frac{1}{K(x)} \int_{\Omega_2} k(x, y) f(y) d\mu_2(y), \tag{1.3}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f : \Omega_2 \to \mathbb{R}$ is measurable function and

$$0 < K(x) := \int_{\Omega_2} k(x, y) d\mu_2(y), \qquad x \in \Omega_1. \tag{1.4}$$

The following theorem was given in [18] and [19] (see also [20]).

Theorem 1.3. *Let $0 < p \le q < \infty$, or $-\infty < q \le p < 0$, $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, K be defined on Ω_1 by (1.4) and that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each $y \in \Omega_2$, and that v is defined on Ω_2 by*

$$v(y) := \left(\int_{\Omega_1} u(x)\left(\frac{k(x, y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int I$, then the inequality

$$\begin{aligned} &\left(\int_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)[\Phi\left((A_k f)(x)\right)]^{\frac{q}{p}} d\mu_1(x) \\ &\quad \ge \frac{q}{p}\int_{\Omega_1} \frac{u(x)}{K(x)} \Phi^{\frac{q}{p}-1}((A_k f)(x)) \int_{\Omega_2} k(x, y) r(x, y) d\mu_2(y) d\mu_1(x) \end{aligned} \tag{1.5}$$

holds for all measurable functions $f : \Omega_2 \to I$, where A_k is defined by (1.3) and $r : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a non-negative function defined by

$$r(x, y) = \big|\,|\Phi(f(y)) - \Phi((A_k f)(x))| - |\varphi((A_k f)(x))|\,|f(y) - (A_k f)(x)|\,\big|. \tag{1.6}$$

If Φ is a non-negative concave function, then the order of terms on the left hand side of (1.5) *is reversed. If Φ is a non-negative monotone convex function on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality*

$$\left(\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}((A_k f)(x))\,d\mu_1(x)$$
$$\geq \frac{q}{p}\left|\int_{\Omega_1} \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}((A_k f)(x))\int_{\Omega_2} sgn(f(y)-(A_k f)(x))k(x,y)r_1(x,y)\,d\mu_2(y)\,d\mu_1(x)\right| \tag{1.7}$$

holds for all measurable functions $f : \Omega_2 \to I$, where $A_k f$ is defined by (1.3) *and $r_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a non-negative function defined by*

$$r_1(x,y) = \Phi(f(y)) - \Phi((A_k f)(x)) - \varphi((A_k f)(x)) \cdot (f(y) - (A_k f)(x)). \tag{1.8}$$

If Φ is a non-negative monotone concave function, then the order of terms on the left hand side of (1.7) *is reversed.*

Remark 1.4. For $p = q$, Theorem 1.3 becomes [6, Theorem 2.1] (see also [20, Theorem 4.1]) and convex function Φ need not to be non-negative.

Although the inequalities (1.5) and (1.7) hold for non-negative convex and monotone convex functions some choices of Φ are of our particular interest. Here, we consider the power weight function i.e. the function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^s$. It is a non-negative, convex and monotone function. Obviously, $\varphi(x) = \Phi'(x) = sx^{s-1}$, $x \in \mathbb{R}_+$, so Φ is convex for $s \in \mathbb{R} \setminus [0, 1)$, concave for $s \in (0, 1]$, and affine, that is, both convex and concave for $s = 1$.

Corollary 1.5. *Let $\Omega_1, \Omega_2, \mu_1, \mu_2, u, k, K, p, q$ and v be as in Theorem* 1.3. *Let $s \in \mathbb{R}$ be such that $s \neq 0$, $f : \Omega_2 \to \mathbb{R}$ be a non-negative measurable function (positive for $s < 0$), $A_k f$ be defined by* (1.3) *and*

$$r_{s,k}f(x,y) = \left|\left|f^s(y) - ((A_k f)(x))^s\right| - |s| \cdot ((A_k f)(x))^{s-1}|f(y) - (A_k f)(x)|\right|, \tag{1.9}$$

for $x \in \Omega_1$, $y \in \Omega_2$. If $s \geq 1$ or $s < 0$, then the following inequality

$$\left(\int_{\Omega_2} v(y)f^s(y)d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)A_k^{\frac{qs}{p}} f(x)d\mu_1(x)$$
$$\geq \frac{q}{p}\int_{\Omega_1} \frac{u(x)}{K(x)}((A_k f)(x))^{\frac{(q-p)s}{p}}\int_{\Omega_2} k(x,y)r_{s,k}f(x,y)\,d\mu_2(y)\,d\mu_1(x) \tag{1.10}$$

holds.

Let

$$M_{s,k}f(x,y) = f^s(y) - A_k^{\frac{qs}{p}} f(x) - s \cdot ((A_k f)(x))^{s-1}(f(y) - (A_k f)(x)) \tag{1.11}$$

for $x \in \Omega_1$, $y \in \Omega_2$. If $s \geq 1$ or $s < 0$, then the inequality

$$\left(\int_{\Omega_2} v(y)f^s(y)d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)A_k^{\frac{qs}{p}} f(x)d\mu_1(x)$$
$$\geq \frac{q}{p}\left|\int_{\Omega_1} \frac{u(x)}{K(x)}((A_k f)(x))^{\frac{(q-p)s}{p}}\int_{\Omega_2} sgn(f(y)-(A_k f)(x))k(x,y)M_{s,k}f(x,y)\,d\mu_2(y)\,d\mu_1(x)\right| \tag{1.12}$$

holds. If $s \in (0, 1)$, then relations (1.10) *and* (1.12) *hold with*

$$\int_{\Omega_1} u(x)A_k^{\frac{qs}{p}} f(x)d\mu_1(x) - \left(\int_{\Omega_2} v(y)f^s(y)d\mu_2(y)\right)^{\frac{q}{p}}$$

on their left hand sides.

Result for one dimensional settings, with intervals in $\mathbb{R}$ and Lebesgue measures was given in the following theorem (see [18] and c.f. also [20, Theorem 5.7]).

Theorem 1.6. *Let $0 < b \leq \infty$ and $k : (0,b) \times (0,b) \to \mathbb{R}$, $u : (0,b) \to \mathbb{R}$ be a non-negative measurable functions satisfying*

$$K(x) =: \int_0^x k(x,y)dy, \quad x \in (0,b), \tag{1.13}$$

and

$$w(y) = y\left(\int_y^b \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} u(x)\frac{dx}{x}\right)^{\frac{p}{q}} < \infty, \qquad y \in (0,b). \tag{1.14}$$

If $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ satisfies that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int I$, then the following inequality

$$\begin{aligned}&\left(\int_0^b w(y)\Phi(f(y))\frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b u(x)\Phi^{\frac{q}{p}}((A_k f)(x))\frac{dx}{x}\\ &\geq \frac{q}{p}\int_0^b \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}((A_k f)(x))\int_0^x k(x,y)r(x,y)dy\frac{dx}{x}\end{aligned} \tag{1.15}$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$ with values in I, where r is defined by (1.6). *If Φ is a non-negative monotone convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ satisfies that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int I$, then the following inequality*

$$\begin{aligned}&\left(\int_0^b w(y)\Phi(f(y))\frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b u(x)\Phi^{\frac{q}{p}}((A_k f)(x))\frac{dx}{x}\\ &\geq \frac{q}{p}\left|\int_0^b \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}((A_k f)(x))\int_0^x sgn(f(y)-(A_k f)(x))k(x,y)r_1(x,y)dy\frac{dx}{x}\right|\end{aligned} \tag{1.16}$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$, where r_1 is defined by (1.8) *and $A_k f$ is defined by*

$$(A_k f)(x) := \frac{1}{K(x)}\int_{0+}^x k(x,y)f(y)dy, \quad x \in (0,b). \tag{1.17}$$

If $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, and Φ is a non-negative (monotone) concave function, then (1.15) *and* (1.16) *hold with reverse order integral of their left hand sides.*

The paper is organized in the following way: After this introduction, in Section 2 we give the generalized Hardy-type inequalities involving generalized Mittag-Leffler function appearing in the kernel for convex and monotone convex functions and Hilfer fractional derivative. We give the related inequalities as an application for the power function. We also include the results for the one dimensional settings. In addition to this we construct inequalities in quotient for the generalized fractional integral operator. In Section 3 we derive the results for Hilfer fractional derivative. Results analogous to those in Section 2 given for Hilfer fractional derivative. We present some new inequalities of Hardy-type for Hilfer fractional derivative. Moreover, we deduce in particular the results of [21] and [22] from our general results.

2. Refined Hardy-type inequalities for fractional integral operator with generalized Mittag-Leffler function in its kernel

In this section, we first give the definition of Mittag-Leffler function [23] and fractional integral operator involving generalized Mittag-Leffler function appearing in the kernel [24]. Let $\Re(\alpha)$ be a real part of complex number α.

Definition 2.1. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$; $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0$; $p, q > 0$. Then the generalized Mittag-Leffler function defined in [24] is given by

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{(\delta)_{pn}}, \tag{2.1}$$

where $(\gamma)_n$ represents the Pochhammer symbol, defined by $(\gamma)_n = \gamma(\gamma-1)(\gamma-2)\ldots(\gamma-n+1)$. The function (2.1) represents all the previous generalizations of Mittag-Leffler function by setting

- $p = q = 1$, it reduces to $E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_n}$ defined by Salim in [25].
- $\delta = p = 1$, it represents $E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$, which was introduced by Shukla and Prajapati in [26]. In [27] Srivastava and Tomovski investigated the properties of this function and its existence for a wider set of parameters.
- $\delta = p = q = 1$, the operator (2.1) was defined by Prabhakar in [28] and was denoted as: $E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$.
- $\gamma = \delta = p = q = 1$, it reduces to Wiman's function presented in [29]. Moreover, if $\beta = 1$, then the original Mittag-Leffler function $E_\alpha(z)$ will be the result (see [23]).

We denote

$$\left(e^{\gamma,\delta,q}_{\alpha,\beta,p}\right)(x;\omega) = x^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}\left(\omega x^{\alpha}\right).$$

Definition 2.2. Let $\alpha, \beta, \gamma, \delta, \omega \in \mathbb{C}$; $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0$; $p, q > 0$. For all $f \in L(a, b)$ we introduce an integral operator

$$\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x) = \int_a^x e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-t;\omega) f(t)dt, \tag{2.2}$$

which contains the generalized Mittag-Leffler function (2.1) in its kernel.

Our first main result is given in the following theorem.

Theorem 2.3. *Let* $0 < p \le q < \infty$, *or* $-\infty < q \le p < 0$ *and* $\alpha, \beta, \gamma, \delta, p, q$ *be as in Definition* (2.2) *and let* u *be a weight function defined on* (a, b). *For each* $y \in (a, b)$, $\tilde{v}$ *is defined on* (a, b) *by*

$$\tilde{v}(y) := \left(\int_y^b u(x) \left(\frac{e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty.$$

If Φ *is a non-negative convex function on the interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in Int I$, *then the inequality*

$$\begin{aligned} &\left(\int_a^b \tilde{v}(y) \Phi(f(y))\, dy \right)^{\frac{q}{p}} - \int_a^b u(x) \Phi^{\frac{q}{p}} \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)} \right) dx \\ &\ge \frac{q}{p} \int_a^b \frac{u(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)} \Phi^{\frac{q}{p}-1} \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)} \right) \\ &\times \int_a^x e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\, \tilde{r}(x,y) dy dx \end{aligned} \tag{2.3}$$

holds for all measurable functions $f : (a, b) \to \mathbb{R}$ *and* $\tilde{r} : (a, b) \times (a, b) \to \mathbb{R}$ *is a non-negative function defined by*

$$\begin{aligned} \tilde{r}(x,y) = \Bigg| &\left| \Phi(f(y)) - \Phi\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)} \right) \right| \\ &- \left| \varphi\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)} \right) \right| \left| f(y) - \frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)} \right| \Bigg|. \end{aligned} \tag{2.4}$$

If Φ is a non-negative monotone convex function on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality

$$\begin{aligned}
&\left(\int_a^b \tilde{v}(y)\Phi\left(f(y)\right)dy\right)^{\frac{q}{p}} - \int_a^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)dx \\
&\geq \frac{q}{p}\left|\int_a^b \frac{u(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\Phi^{\frac{q}{p}-1}\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)\right. \\
&\left.\times\int_a^x sgn\left(f(y)-\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\,\tilde{r}_1(x,y)\,dy\,dx\right| \qquad (2.5)
\end{aligned}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$ and $\tilde{r}_1 : (a,b)\times(a,b) \to \mathbb{R}$ is a non-negative function defined by

$$\begin{aligned}
\tilde{r}_1(x,y) = {} & \Phi(f(y)) - \Phi\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right) \\
&- \varphi\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)\cdot\left(f(y)-\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right). \qquad (2.6)
\end{aligned}$$

If Φ is a non-negative (monotone) concave, then the order of terms on the left hand side of inequalities (2.3) *and* (2.5) *is reversed.*

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$\tilde{k}(x,y) = \begin{cases} e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\,, & a \le y \le x\,; \\ 0, & x < y \le b\,, \end{cases} \qquad (2.7)$$

where (see Lemma 3.2 in [30]), and

$$\tilde{K}(x) = \int_a^x e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\,dy = e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)\,,$$

and

$$(A_k f)(x) = \frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}, \qquad (2.8)$$

we get inequalities (2.3) and (2.5). ■

Remark 2.4. Theorem 2.3 generalizes the result of [22], i.e. for $p = q$, Theorem 2.3 becomes [22, Theorem 3.6] and convex function Φ need not to be non-negative.

Particular to our interest, we consider the power function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ defined by $\Phi(x) = x^s$. Obviously, $\varphi(x) = \Phi'(x) = sx^{s-1}$, $x \in \mathbb{R}_+$.

Corollary 2.5. *Let $\tilde{k}$, $\tilde{K}$, p, q and $\tilde{v}$ be as in Theorem* 2.3. *Let $s \in \mathbb{R}$ be such that $s \neq 0$, $f : (a,b) \to \mathbb{R}$ be a non-negative measurable function (positive for $s < 0$) and*

$$\begin{aligned}
\tilde{r}_{s,k}f(x,y) = {} & \left|\left|f^s(y) - \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)^s\right|\right. \\
&\left. - |s|\cdot\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)^{s-1}\left|f(y)-\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right|\right|, \qquad (2.9)
\end{aligned}$$

for $x, y \in (a, b)$. If $s \geq 1$ or $s < 0$, then the inequality

$$\left(\int_a^b \tilde{v}(y) f^s(y) dy\right)^{\frac{q}{p}} - \int_a^b u(x) \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)^{\frac{qs}{p}} dx$$
$$\geq \frac{q}{p} \int_a^b \frac{u(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)} \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)^{\frac{(q-p)s}{p}}$$
$$\times \int_a^x e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\, \tilde{r}_{s,k} f(x,y)\, dy\, dx \tag{2.10}$$

holds.

Let

$$\tilde{M}_{s,k} f(x,y) = f^s(y) - \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)^s -$$
$$s\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)^{s-1} \left(f(y) - \frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right) \tag{2.11}$$

for $x, y \in (a, b)$. If $s \geq 1$ or $s < 0$, then the inequality

$$\left(\int_a^b \tilde{v}(y) f^s(y) dy\right)^{\frac{q}{p}} - \int_a^b u(x) \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)^{\frac{qs}{p}} dx$$
$$\geq \frac{q}{p} \Bigg| \int_a^b \frac{u(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)} \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right)^{\frac{(q-p)s}{p}}$$
$$\times \int_a^x sgn\left(f(y) - \frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x-a;\omega)}\right) e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\, \tilde{M}_{s,k} f(x,y)\, dy\, dx \Bigg| \tag{2.12}$$

holds. If $s \in (0, 1)$, then versions of the inequalities (2.10) *and* (2.12) *hold with reverse order of terms on the left hand sides.*

Here we give the result for one dimensional settings, with intervals in $\mathbb{R}$ and Lebesgue measures in next theorem.

Theorem 2.6. *Let $0 < b \leq \infty$, $\alpha, \beta, \gamma, \delta, p, q$ be as Definition* (2.2) *and u be a weight function. For each $y \in (0, b)$ we let the function $\tilde{w} : (0, b) \to \mathbb{R}$ be defined by*

$$\tilde{w}(y) = y \left(\int_y^b \left(\frac{e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x;\omega)}\right)^{\frac{q}{p}} u(x) \frac{dx}{x}\right)^{\frac{p}{q}}. \tag{2.13}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ satisfies that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int I$, then the following inequality

$$\left(\int_0^b \tilde{w}(y) \Phi(f(y)) \frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b u(x) \Phi^{\frac{q}{p}} \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x;\omega)}\right) \frac{dx}{x}$$
$$\geq \frac{q}{p} \int_0^b \frac{u(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x;\omega)} \Phi^{\frac{q}{p}-1} \left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x;\omega)}\right) \int_0^x e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\, \tilde{r}(x,y) dy \frac{dx}{x} \tag{2.14}$$

holds for all measurable functions $f : (0, b) \to \mathbb{R}$ and $\tilde{r}(x, y)$ is defined by (2.4). *If Φ is a non-negative monotone convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ satisfies that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, then the following inequality*

$$\left(\int_0^b \tilde{w}(y)\Phi(f(y))\frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x;\omega)}\right)\frac{dx}{x}$$
$$\geq \frac{q}{p}\left|\int_0^b \frac{u(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x;\omega)}\Phi^{\frac{q}{p}-1}\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x;\omega)}\right)\right.$$
$$\left.\times\int_0^x sgn\left(f(y) - \frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x;\omega)}\right)e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\,\tilde{r}_1(x,y)dy\frac{dx}{x}\right| \tag{2.15}$$

holds for all measurable functions $f : (0, b) \to \mathbb{R}$ and $\tilde{r}_1(x, y)$ defined by (2.6).

Proof. Applying Theorem 1.6 with $\tilde{k}(x, t)$ given by (2.7) and

$$(A_k f)(x) = \frac{1}{e^{\gamma,\delta,q}_{\alpha,\beta+1,p}(x;\omega)}\int_0^x e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)f(y)dy,$$

then we obtain inequalities (2.14) and (2.15). ■

Remark 2.7. Some special cases of the above results are given below.

- If we take $m = k = 1$ in Theorem 2.3, Corollary 2.5 and in Theorem 2.6, then the inequalities reduce to the case $E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{(\delta)_n}$.
- If we take $\delta = m = 1$ in Theorem 2.3, Corollary 2.5 and in Theorem 2.6, then the inequalities reduce to the case $E^{\gamma,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{kn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}$.
- If we take $\delta = m = k = 1$ in Theorem 2.3, Corollary 2.5 and in Theorem 2.6, then the inequalities reduce to the case $E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}$.
- If we take $\gamma = \delta = m = k = 1$ in Theorem 2.3, Corollary 2.5 and in Theorem 2.6 then the inequalities reduce to Wiman's function. Moreover, if $\beta = 1$, then the original Mittag-Leffler function $E_\alpha(z)$ will be the result.

Next we will present some new generalized Hardy-type inequalities in quotient form. For this if we substitute $k(x, y)$ by $k(x, y)f_2(y)$ and f by $\frac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}, (i = 1, 2)$ are measurable functions in Theorem 1.6 we obtain the following result.

Theorem 2.8. *Let $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures. Let $f_i : \Omega_2 \to \mathbb{R}$ be measurable functions, $g_i \in U(f_i)$, $(i = 1, 2)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$ and let the function $x \mapsto u(x)\left(\frac{k(x,y)}{g_2(x)}\right)^{\frac{q}{p}}$ be integrable on Ω_1. For each $y \in \Omega_2$ define $s = s(y)$ on Ω_2 by*

$$s(y) := f_2(y)\left(\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{g_2(x)}\right)^{\frac{q}{p}}d\mu_1(x)\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, then the inequality

$$\left(\int_{\Omega_2} s(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}\left(\frac{g_1(x)}{g_2(x)}\right)d\mu_1(x)$$
$$\geq \frac{q}{p}\int_{\Omega_1}\frac{u(x)}{g_2(x)}\Phi^{\frac{q}{p}-1}\left(\frac{g_1(x)}{g_2(x)}\right)\int_{\Omega_2} k(x,y)f_2(y)d(x,y)d\mu_2(y)d\mu_1(x) \tag{2.16}$$

holds for all measurable functions $f_i : \Omega_2 \to I, (i = 1, 2)$ such that $\frac{f_1(y)}{f_2(y)} \in I$, for all $y \in \Omega_2$, and $d : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a non-negative function defined by

$$d(x,y) = \left| \left| \Phi\left(\frac{f_1(y)}{f_2(y)}\right) - \Phi\left(\frac{g_1(x)}{g_2(x)}\right) \right| - \left| \varphi\left(\frac{g_1(x)}{g_2(x)}\right) \right| \left| \frac{f_1(y)}{f_2(y)} - \frac{g_1(x)}{g_2(x)} \right| \right|.$$

If Φ is a non-negative monotone convex function on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality

$$\begin{aligned}&\left(\int_{\Omega_2} s(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}\left(\frac{g_1(x)}{g_2(x)}\right) d\mu_1(x) \\ &\geq \frac{q}{p}\left| \int_{\Omega_1} \frac{u(x)}{g_2(x)}\Phi^{\frac{q}{p}-1}\left(\frac{g_1(x)}{g_2(x)}\right) \int_{\Omega_2} sgn\left(\frac{f_1(y)}{f_2(y)} - \frac{g_1(x)}{g_2(x)}\right) k(x,y) f_2(y) d_1(x,y)\, d\mu_2(y)\, d\mu_1(x) \right| \end{aligned} \tag{2.17}$$

holds for all measurable functions $f_i : \Omega_2 \to I, (i = 1, 2)$ such that $\frac{f_1(y)}{f_2(y)} \in I$, for all $y \in \Omega_2$ and $d_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a non-negative function defined by

$$d_1(x,y) = \Phi\left(\frac{f_1(y)}{f_2(y)}\right) - \Phi\left(\frac{g_1(x)}{g_2(x)}\right) - \varphi\left(\frac{g_1(x)}{g_2(x)}\right) \cdot \left(\frac{f_1(y)}{f_2(y)} - \frac{g_1(x)}{g_2(x)}\right).$$

If Φ is a non-negative (monotone) concave function, then the order of terms on the left hand sides of (2.16) *and* (2.17) *are reversed.*

Remark 2.9. For $p = q$ in Theorem 2.8 we get the result in [21, Theorem 1.3].

Our next result reads;

Theorem 2.10. *Let $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, $\alpha, \beta, \gamma, \delta, p, q$ be as in Definition* (2.2) *and let u be a weight function defined on (a, b). For each $y \in (a, b)$, define a function*

$$\tilde{s}(y) := f_2(y)\left(\int_y^b u(x)\left(\frac{e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2\right)(x)}\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int I$, then the inequality

$$\begin{aligned}&\left(\int_a^b \tilde{s}(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) dy\right)^{\frac{q}{p}} - \int_a^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2\right)(x)}\right) dx \\ &\geq \frac{q}{p}\int_a^b \frac{u(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2\right)(x)}\Phi^{\frac{q}{p}-1}\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2\right)(x)}\right) \\ &\quad\times \int_a^x e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\, f_2(y)\tilde{d}(x,y) dy dx \end{aligned} \tag{2.18}$$

holds for all measurable functions $f_i : \Omega_2 \to I, (i = 1, 2)$ such that $\frac{f_1(y)}{f_2(y)} \in I$, for all $y \in (a, b)$, and $\tilde{d} : (a, b) \times (a, b) \to \mathbb{R}$ is a non-negative function defined by

$$\begin{aligned}\tilde{d}(x,y) = &\left| \left| \Phi\left(\frac{f_1(y)}{f_2(y)}\right) - \Phi\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2\right)(x)}\right) \right| \right. \\ &\left. - \left| \varphi\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2\right)(x)}\right) \right| \cdot \left| \frac{f_1(y)}{f_2(y)} - \frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2\right)(x)} \right| \right|.\end{aligned}$$

If Φ is a non-negative monotone convex function on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality

$$\left(\int_a^b \tilde{s}(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)dy\right)^{\frac{q}{p}} - \int_a^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2\right)(x)}\right)dx$$
$$\geq \frac{q}{p}\left|\int_a^b \frac{u(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2\right)(x)}\Phi^{\frac{q}{p}-1}\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2\right)(x)}\right)\right.$$
$$\left.\times\int_a^x sgn\left(\frac{f_1(y)}{f_2(y)} - \frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2\right)(x)}\right)e^{\gamma,\delta,q}_{\alpha,\beta,p}(x-y;\omega)\, f_2(y)\tilde{d}_1(x,y)\,dy\,dx\right| \tag{2.19}$$

holds for all measurable functions $f_i : \Omega_2 \to I, (i = 1, 2)$ such that $\frac{f_1(y)}{f_2(y)} \in I$, for all $y \in (a, b)$ and $\tilde{d}_1 : (a, b) \times (a, b) \to \mathbb{R}$ is a non-negative function defined by

$$\tilde{d}_1(x,y) = \Phi\left(\frac{f_1(y)}{f_2(y)}\right) - \Phi\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2\right)(x)}\right)$$
$$- \varphi\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2\right)(x)}\right)\cdot\left(\frac{f_1(y)}{f_2(y)} - \frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2\right)(x)}\right).$$

If Φ is a non-negative (monotone) concave function, then the order of terms on the left hand sides of (2.18) *and* (2.19) *are reversed.*

Proof. Applying Theorem 2.8 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, $g_1(x) = (\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_1)(x)$, $g_2(x) = (\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2)(x)$ and $k(x, y) = e^{\gamma,\delta,q}_{\alpha,\beta,p}(x - y; \omega)$, we obtain inequalities (2.18) and (2.19). ■

Remark 2.11. Since the right hand sides of the inequalities (2.18) and (2.19) are non-negative, therefore we obtain the following inequality

$$\int_a^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_1\right)(x)}{\left(\mathbf{E}^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2\right)(x)}\right)dx \leq \left(\int_a^b v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)dy\right)^{\frac{q}{p}}. \tag{2.20}$$

Particularly for $p = q$, we obtain the inequality given in [22, Theorem 3.4].

Remark 2.12. If Φ is strictly convex on I and $\frac{f_1(x)}{f_2(x)}$ is non-constant, then the inequality given in (2.20) is strict.

3. Hardy-type inequalities for Hilfer fractional derivative operator

Let $x > a > 0$. By $L^1(a, x)$ we denote the space of all Lebesgue integrable functions on the interval (a, x). For any $f \in L^1(a, x)$ the Riemann–Liouville fractional integral of f of order ν is defined by

$$(I^{\nu}_{a_+}f)(s) = \frac{1}{\Gamma(\alpha)}\int_a^s (x-y)^{\nu-1}f(t)dt = (f * K_\nu)(s), \quad s \in [a, x], \quad (\nu > 0), \tag{3.1}$$

where $K_\nu(s) = \frac{s^{\nu-1}}{\Gamma(\nu)}$. The integral on the right side of (3.1) exists for almost $s \in [a, x]$ and $I^{\nu}_{a+}f \in L^1(a, x)$. The Riemann–Liouville fractional derivative of $f \in L^1(a, x)$ of order ν is defined by

$$(D^{\nu}_{a_+}f)(s) = \left(\frac{d}{dx}\right)^n (I^{n-\nu}_{a_+}f)(x), (\nu > 0, n = [\nu + 1]).$$

By $C^m[a, x]$ we denote the space of all functions on $[a, x]$ which have continuous derivatives up to order m, and $AC[a, x]$ is the space of all absolutely continuous functions on $[a, x]$. By $AC^m[a, x]$ we denote the space of all functions $f \in C^m[a,x]$ with $f^{(m-1)} \in AC[a,x]$. By $L_\infty(a,x)$ we denote the space of all measurable functions essentially bounded on $[a,x]$. Let $\mu > 0, m = [\mu] + 1$ and $f \in AC^m[a,b]$. The Caputo derivative of order $\mu > 0$ is defined as

$$({}^C D^{\mu}_{a+} f)(x) = \left(I^{m-\mu}_{a+} \frac{d^m}{dx^m} f \right)(x) = \frac{1}{\Gamma(m-\mu)} \int_a^x (x-s)^{m-\mu-1} \frac{d^m}{dx^m} f(s) ds.$$

Let us recall the definition of Hilfer fractional derivative presented in [31].

Definition 3.1. Let $f \in L^1[a,b]$, $f * K_{(1-\nu)(1-\mu)} \in AC^1[a,b]$. The fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $0 < \mu < 1$ and type $0 < \nu \leq 1$ with respect to $x \in [a,b]$ is defined by

$$\left(D^{\mu,\nu}_{a+} f\right)(x) := I^{\nu(1-\mu)}_{a+} \frac{d}{dx} \left(I^{(1-\nu)(1-\mu)}_{a+} f(x) \right), \tag{3.2}$$

whenever the right hand side exists. The derivative (3.2) is usually called Hilfer fractional derivative.

The more general integral representation of Eq. (3.2) given in [32] is defined by:

Let $f \in L^1[a,b]$, $f * K_{(1-\nu)(n-\mu)} \in AC^n[a,b]$, $n-1 < \mu < n, 0 < \nu \leq 1, n \in \mathbb{N}$. Then

$$\left(D^{\mu,\nu}_{a+} f\right)(x) = \left(I^{\nu(n-\mu)}_{a+} \frac{d^n}{dx^n} \left(I^{(1-\nu)(n-\mu)}_{a+} f(x) \right) \right), \tag{3.3}$$

which coincide with (3.2) for $n = 1$.

Specially for $\nu = 0$, $D^{\mu,0}_{a+} f = D^{\mu}_{a+} f$ is a Riemann–Liouville fractional derivative of order μ, and for $\nu = 1$ it is a Caputo fractional derivative $D^{\mu,1}_{a+} f = {}^C D^{\mu}_{a+} f$ of order μ. Applying the properties of Riemann–Liouville integral the relation (3.3) can be rewritten in the form:

$$\begin{aligned} \left(D^{\mu,\nu}_{a+} f\right)(x) &= \left(I^{\nu(n-\mu)}_{a+} \left(\left(D^{n-(1-\nu)(n-\mu)}_{a+} f \right)(x) \right) \right) \\ &= \frac{1}{\Gamma(\nu(n-\mu))} \int_a^x (x-y)^{\nu(n-\mu)-1} \left(\left(D^{\mu+\nu(n-\mu)}_{a+} f \right)(t) \right) dt. \end{aligned} \tag{3.4}$$

Our first result of this section is given in next theorem.

Theorem 3.2. *Let $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, $f \in L^1[a,b]$ and the fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $n-1 < \mu < n$ and type $0 < \nu \leq 1$, and let u be a weight function on (a,b). For each $y \in (a,b)$, $\hat{v} = \hat{v}(y)$ is defined on (a,b) by*

$$\hat{v}(y) := (\nu(n-\mu)) \left(\int_y^b u(x) \left(\frac{(x-y)^{\nu(n-\mu)-1}}{(x-a)^{\nu(n-\mu)}} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty. \tag{3.5}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int I$, then the inequality

$$\begin{aligned} &\left(\int_a^b \hat{v}(y) \Phi \left(\left(D^{\mu+\nu(n-\mu)}_{a+} f \right)(y) \right) dy \right)^{\frac{q}{p}} \\ &\quad - \int_a^b u(x) \Phi^{\frac{q}{p}} \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D^{\mu,\nu}_{a+} f\right)(x) \right) dx \\ &\geq (\nu(n-\mu)) \frac{q}{p} \int_a^b \frac{u(x)}{(x-a)^{\nu(n-\mu)}} \Phi^{\frac{q}{p}-1} \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D^{\mu,\nu}_{a+} f\right)(x) \right) \\ &\quad \times \int_a^x (x-y)^{\nu(n-\mu)-1} \hat{r}(x,y) dy dx \end{aligned} \tag{3.6}$$

holds for all measurable functions $D_{a+}^{\mu+\nu(n-\mu)} f : (a,b) \to \mathbb{R}$ *and* $\hat{r} : (a,b) \times (a,b) \to \mathbb{R}$ *is a non-negative function defined by*

$$\begin{aligned}\hat{r}(x,y) = \Bigg| &\left|\Phi\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(y) - \Phi\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right)\right| \\ &- \left|\varphi\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right)\right|\left|f(y) - \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right|\Bigg|. \qquad (3.7)\end{aligned}$$

If Φ *is a non-negative monotone convex function on the interval* $I \subseteq \mathbb{R}$, *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in Int\, I$, *then the inequality*

$$\begin{aligned}&\left(\int_a^b \hat{v}(y)\Phi\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(y)\right)dy\right)^{\frac{q}{p}} \\ &\quad - \int_a^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right)dx \\ &\geq (\nu(n-\mu))\frac{q}{p}\Bigg|\int_a^b \frac{u(x)}{(x-a)^{\nu(n-\mu)}}\Phi^{\frac{q}{p}-1}\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right) \\ &\quad \times \int_a^x sgn\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(y) - \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right) \\ &\quad \times (x-y)^{\nu(n-\mu)-1}\hat{r}_1(x,y)\,dy\,dx\Bigg| \qquad (3.8)\end{aligned}$$

holds for all measurable functions $D_{a+}^{\mu+\nu(n-\mu)} f : (a,b) \to \mathbb{R}$ *and* $\hat{r}_1 : (a,b) \times (a,b) \to \mathbb{R}$ *is a non-negative function defined by*

$$\begin{aligned}\hat{r}_1(x,y) = &\ \Phi\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(y)\right) - \Phi\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right) \\ &- \varphi\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right) \\ &\times \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(y) - \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x)\right). \qquad (3.9)\end{aligned}$$

If Φ *is a non-negative (monotone) concave function, then the order of terms on the left hand sides of* (3.6) *and* (3.8) *are reversed.*

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$\hat{k}(x,y) = \begin{cases} \dfrac{(x-y)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}, & a \leq y \leq x; \\ 0, & x < y \leq b, \end{cases} \qquad (3.10)$$

we get

$$\hat{K}(x) = \frac{(x-a)^{\nu(n-\mu)}}{\Gamma(\nu(n-\mu)+1)} \qquad (3.11)$$

and the integral operator $A_k f(x)$ takes the form

$$(A_k f)(x) = \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu} f\right)(x) \qquad (3.12)$$

and $\hat{v}$ as in (3.5), we get inequalities (3.6) and (3.8). ■

Remark 3.3. For $p = q$, Theorem 1.3 becomes [22, Theorem 3.6] and the convex function Φ need not to be non-negative.

Especially for the power function we obtain the next corollary.

Corollary 3.4. *Let $u, \hat{k}, \hat{K}$ and $\hat{v}$ be as in Theorem* 3.2. *Let $s \in \mathbb{R}$ be such that $s \neq 0$, $D_{a+}^{\mu+\nu(n-\mu)} f : (a,b) \to \mathbb{R}$ be a non-negative measurable function (positive for $s < 0$) and*

$$h_{s,k} f(x,y) = \left| \left| \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right)^s - \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right)^s \right| \right.$$
$$\left. - |s| \cdot \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right)^{s-1} \left| f(y) - \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right| \right|, \quad (3.13)$$

for $x, y \in (a,b)$. If $s \geq 1$ or $s < 0$, then the inequality

$$\left(\int_a^b \hat{v}(y) \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right)^s dy \right)^{\frac{q}{p}}$$
$$- \int_a^b u(x) \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right)^{\frac{qs}{p}} dx$$
$$\geq (\nu(n-\mu)) \frac{q}{p} \int_a^b \frac{u(x)}{(x-a)^{\nu(n-\mu)}} \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right)^{\frac{(q-p)s}{p}}$$
$$\times \int_a^x (x-y)^{\nu(n-\mu)-1} h_{s,k} f(x,y) \, dy \, dx \quad (3.14)$$

holds.

Let

$$N_{s,k} f(x,y) = \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right)^s - \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right)^s$$
$$- s \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right)^{s-1}$$
$$\times \left(\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right) - \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right) \quad (3.15)$$

for $x, y \in (a,b)$. If $s \geq 1$ or $s < 0$, then the inequality

$$\left(\int_a^b v(y) \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right)^s dy \right)^{\frac{q}{p}}$$
$$- \int_a^b u(x) \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right)^{\frac{qs}{p}} dx$$
$$\geq (\nu(n-\mu)) \frac{q}{p} \left| \int_a^b \frac{u(x)}{(x-a)^{\nu(n-\mu)}} \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right)^{\frac{(q-p)s}{p}} \right.$$
$$\times \int_a^x sgn \left(\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right) - \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right)$$
$$\left. \times (x-y)^{\nu(n-\mu)-1} N_{s,k} f(x,y) \, dy \, dx \right| \quad (3.16)$$

holds. If $s \in (0,1)$, then inequalities corresponding to (3.14) *and* (3.16) *hold with reverse order of terms on the left hand sides.*

Next we give the results for one dimensional settings involving Hilfer fractional derivative.

Theorem 3.5. *Let $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, $f \in L^1[a,b]$ and the fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $n-1 < \mu < n$ and type $0 < \nu \leq 1$, and let u be a weight function. For each $y \in (0,b)$ let the function*

$\hat{w}:(0,b)\to\mathbb{R}$ be defined by

$$\hat{w}(y)=y\left(\nu\left(n-\mu\right)\right)\left(\int_y^b\left(\frac{(x-y)^{\nu(n-\mu)-1}}{(x-a)^{\nu(n-\mu)}}\right)^{\frac{q}{p}}u(x)\frac{dx}{x}\right)^{\frac{p}{q}}. \tag{3.17}$$

If Φ is a non-negative convex function on the interval $I\subseteq\mathbb{R}$ and $\varphi:I\to\mathbb{R}$ satisfies that $\varphi(x)\in\partial\Phi(x)$ for all $x\in IntI$, then the following inequality

$$\begin{aligned}&\left(\int_0^b\hat{w}(y)\Phi\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(y)\right)\frac{dy}{y}\right)^{\frac{q}{p}}\\&\quad-\int_0^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\Gamma\left(\nu\left(n-\mu\right)+1\right)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu}f\right)(x)\right)\frac{dx}{x}\\&\geq\left(\nu\left(n-\mu\right)\right)\frac{q}{p}\int_0^b\frac{u(x)}{(x-a)^{\nu(n-\mu)}}\Phi^{\frac{q}{p}-1}\left(\frac{\Gamma\left(\nu\left(n-\mu\right)+1\right)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu}f\right)(x)\right)\\&\quad\times\int_0^x(x-y)^{\nu(n-\mu)-1}\hat{r}(x,y)dy\frac{dx}{x}\end{aligned}\tag{3.18}$$

holds for all measurable functions $D_{a+}^{\mu+\nu(n-\mu)}f:(0,b)\to\mathbb{R}$ and $\hat{r}(x,y)$ is defined by (3.7)*. If Φ is a non-negative monotone convex function on the interval $I\subseteq\mathbb{R}$ and $\varphi:I\to\mathbb{R}$ satisfies that $\varphi(x)\in\partial\Phi(x)$ for all $x\in IntI$, then the following inequality*

$$\begin{aligned}&\left(\int_0^b\hat{w}(y)\Phi\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(y)\right)\frac{dy}{y}\right)^{\frac{q}{p}}\\&\quad-\int_0^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\Gamma\left(\nu\left(n-\mu\right)+1\right)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu}f\right)(x)\right)\frac{dx}{x}\\&\geq\left(\nu\left(n-\mu\right)\right)\frac{q}{p}\left|\int_0^b\frac{u(x)}{(x-a)^{\nu(n-\mu)}}\Phi^{\frac{q}{p}-1}\left(\frac{\Gamma\left(\nu\left(n-\mu\right)+1\right)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu}f\right)(x)\right)\right.\\&\quad\times\int_0^x sgn\left(\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(y)\right)-\frac{\Gamma\left(\nu\left(n-\mu\right)+1\right)}{(x-a)^{\nu(n-\mu)}}\left(D_{a+}^{\mu,\nu}f\right)(x)\right)\\&\quad\left.\times(x-y)^{\nu(n-\mu)-1}\hat{r}_1(x,y)dy\frac{dx}{x}\right|\end{aligned}\tag{3.19}$$

holds for all measurable functions $D_{a+}^{\mu+\nu(n-\mu)}f:(0,b)\to\mathbb{R}$ and $\hat{r}_1(x,y)$ is defined by (3.9)*.*

Our next result reads:

Theorem 3.6. *Let $0<p\leq q<\infty$, or $-\infty<q\leq p<0$, $\alpha,\beta,\gamma,\delta,p,q$ be as in Definition* (2.2) *and let u be a weight function defined on (a,b). For each $y\in(a,b)$, define a function*

$$\hat{s}(y):=\frac{(D_{a+}^{\mu+\nu(n-\mu)}f_2)(y)}{(\nu\left(n-\mu\right))}\left(\int_y^b u(x)\left(\frac{(x-y)^{\nu(n-\mu)-1}}{\left(D_{a+}^{\mu,\nu}f_2\right)(x)}\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}}<\infty.$$

If Φ is a non-negative convex function on the interval $I\subseteq\mathbb{R}$ and $\varphi:I\to\mathbb{R}$ is any function, such that $\varphi(x)\in\partial\Phi(x)$ for all $x\in IntI$, then the inequality

$$\begin{aligned}&\left(\int_a^b\hat{s}(y)\Phi\left(\frac{(D_{a+}^{\mu+\nu(n-\mu)}f_1)(y)}{(D_{a+}^{\mu+\nu(n-\mu)}f_2)(y)}\right)dy\right)^{\frac{q}{p}}-\int_a^b u(x)\Phi\left(\frac{\left(D_{a+}^{\mu,\nu}f_1\right)(x)}{\left(D_{a+}^{\mu,\nu}f_2\right)(x)}\right)^{\frac{q}{p}}dx\\&\geq\frac{q}{p}\int_a^b\frac{u(x)}{\left(D_{a+}^{\mu,\nu}f_2\right)(x)}\Phi^{\frac{q}{p}-1}\left(\frac{\left(D_{a+}^{\mu,\nu}f_1\right)(x)}{\left(D_{a+}^{\mu,\nu}f_2\right)(x)}\right)\\&\quad\times\int_a^x\frac{(x-y)^{\nu(n-\mu)-1}}{\Gamma\left(\nu\left(n-\mu\right)\right)}(D_{a+}^{\mu+\nu(n-\mu)}f_2)(y)\hat{d}(x,y)dydx\end{aligned}\tag{3.20}$$

holds for all measurable functions $D_{a_+}^{\mu+\nu(n-\mu)} f_i : (a,b) \to \mathbb{R}$, $(i = 1,2)$ *such that* $\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)} \in I$ *for all* $y \in (a,b)$, *and* $\hat{d} : (a,b)\times(a,b) \to \mathbb{R}$ *is a non-negative function defined by*

$$\hat{d}(x,y) = \left|\left|\Phi\left(\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)}\right) - \Phi\left(\frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right)\right| - \left|\varphi\left(\frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right)\right|\left|\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)} - \frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right|\right|. \tag{3.21}$$

If Φ *is a non-negative monotone convex function on the interval* $I \subseteq \mathbb{R}$, *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in Int\, I$, *then the inequality*

$$\left(\int_a^b \hat{s}(y)\Phi\left(\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)}\right) dy\right)^{\frac{q}{p}} - \int_a^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right) dx$$
$$\geq \frac{q}{p}\left|\int_a^b \frac{u(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)} \Phi^{\frac{q}{p}-1}\left(\frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right)\int_a^x sgn\left(\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)} - \frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right)\right.$$
$$\left.\times \frac{(x-y)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)\hat{d}_1(x,y)\, dy\, dx\right| \tag{3.22}$$

holds for all measurable functions $D_{a_+}^{\mu+\nu(n-\mu)} f_i : (a,b) \to \mathbb{R}$, $(i = 1,2)$ *such that* $\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)} \in I$, *for all* $y \in (a,b)$ *and* $\hat{d}_1 : (a,b)\times(a,b) \to \mathbb{R}$ *is a non-negative function defined by*

$$\hat{d}_1(x,y) = \Big[\Phi\left(\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)}\right) - \Phi\left(\frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right) - \varphi\left(\frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right)\cdot\left(\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)} - \frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right)\Big]. \tag{3.23}$$

If Φ *is a non-negative (monotone) concave function, then the order of terms on the left hand sides of* (3.20) *and* (3.22) *are reversed.*

Proof. Applying Theorem 2.8 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $g_1(x) = \left(D_{a_+}^{\mu,\nu} f_1\right)(x)$, $g_2(x) = \left(D_{a_+}^{\mu,\nu} f_2\right)(x)$, $f_1(y) = (D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)$, and $f_2(y) = (D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)$, $k(x,y)$ defined by (3.10), we obtain inequalities (3.20) and (3.22). ■

Remark 3.7. Since the right hand sides of the inequalities (3.20) and (3.22) are non-negative, we obtain the following inequality

$$\int_a^b u(x)\Phi^{\frac{q}{p}}\left(\frac{\left(D_{a_+}^{\mu,\nu} f_1\right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2\right)(x)}\right) dx \leq \left(\int_a^b v(y)\Phi\left(\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)}\right) dy\right)^{\frac{q}{p}}. \tag{3.24}$$

Particularly for $p = q$, we obtain the inequality given in [22, Theorem 3.4].

Remark 3.8. If Φ is strictly convex on I and $\frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y)}{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)}$ is non-constant, then the inequality given in (3.24) is strict.

Now we present some Hardy-type inequalities for Hilfer fractional derivative. We continue our analysis about improvements by taking the non-negative difference of the left hand side and the right hand side of the inequalities given in (1.10) and (1.12) as:

$$\rho(s) = \left(\int_a^b v(y) f^s(y) d\mu_2(y) \right)^{\frac{q}{p}} - \int_a^b u(x)((A_k f)(x))^{\frac{sq}{p}} d\mu_1(x)$$
$$- \frac{q}{p} \int_a^b \frac{u(x)}{K(x)} ((A_k f)(x))^{s(\frac{q}{p}-1)} \int_a^b k(x,y) r_{p,k} f(x,y) dy dx, \tag{3.25}$$

where $r_{p,k} f(x, y)$ is defined by (1.9) and

$$\pi(s) = \left(\int_a^b v(y) f^s(y) dy \right)^{\frac{q}{p}} - \int_a^b u(x) \left((A_k f)(x)\right)^{\frac{sq}{p}} dx$$
$$- \frac{q}{p} \left| \int_a^b \frac{u(x)}{K(x)} \Phi^{\frac{q}{p}-1} ((A_k f)(x)) \int_a^b sgn(f(y) - (A_k f)(x)) k(x,y) M_{p,k} f(x,y) \, dy \, dx \right|, \tag{3.26}$$

where $M_{p,k} f(x, y)$ is defined by (1.11).

Theorem 3.9. *Let* $0 < p \le q < \infty$, $s \ge 1$, $\nu(n-\mu) \ge 1 - \frac{p}{q}$, $f \in L^1[a,b]$ *and the fractional derivative operator* $D_{a+}^{\mu,\nu}$ *of order* $n-1 < \mu < n$ *and type* $0 < \nu \le 1$. *Then for non-negative functions* $D_{a_+}^{\mu+\nu(n-\mu)} f$ *and* $D_{a+}^{\mu,\nu} f$ *the following inequality holds true:*

$$0 \le \rho(s) \le H(s) - M(s) \le H(s),$$

where

$$\rho(s) = \frac{(\nu\,(n-\mu))^{\frac{q}{p}}}{(\nu\,(n-\mu)-1)\frac{q}{p}+1} \left(\int_a^b (b-y)^{\nu(n-\mu)-1+\frac{p}{q}} ((D_{a_+}^{\mu+\nu(n-\mu)} f)(y))^s dy \right)^{\frac{q}{p}}$$
$$- (\Gamma(\nu\,(n-\mu)+1))^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{(\nu(n-\mu))q(1-s)}{p}} \left((D_{a+}^{\mu,\nu} f)(x)\right)^{\frac{sq}{p}} dx - M(s),$$

$$M(s) = \frac{q(\nu\,(n-\mu))(\Gamma(\nu\,(n-\mu)+1))^{s(\frac{q}{p}-1)}}{p} \int_a^b (x-a)^{\frac{(\nu(n-\mu))(q-p)(1-s)}{p}} \left((D_{a+}^{\mu,\nu} f)(x)\right)^{s(\frac{q}{p}-1)}$$
$$\times \int_a^x h_{p,k} f(x,y) (x-y)^{\nu(n-\mu)-1} dy\, dx,$$

where $h_{p,k} f(x, y)$ *is defined by* (3.13) *and*

$$H(s) = (b-a)^{(\nu(n-\mu))\frac{q}{p}(1-s)}$$
$$\times \left[\frac{(\nu\,(n-\mu))^{\frac{q}{p}} (b-a)^{\frac{q((\nu(n-\mu))s-1)+p}{p}}}{(\nu\,(n-\mu)-1)\frac{q}{p}+1} \left(\int_a^b (D_{a_+}^{\mu+\nu(n-\mu)} f(y))^s dy \right)^{\frac{q}{p}} \right.$$
$$\left. - (\Gamma(\nu\,(n-\mu)+1))^{\frac{sq}{p}} \int_a^b (D_{a+}^{\mu,\nu} f(x))^{\frac{sq}{p}} dx \right]. \tag{3.27}$$

Moreover,

$$0 \le \pi(s) \le H(s) - B(s) \le H(s),$$

where

$$\pi(s) = \frac{(\nu\,(n-\mu))^{\frac{q}{p}}}{(\nu\,(n-\mu)-1)\frac{q}{p}+1} \left(\int_a^b (b-y)^{\nu(n-\mu)-1+\frac{p}{q}} ((D_{a_+}^{\mu+\nu(n-\mu)} f)(y))^s dy \right)^{\frac{q}{p}}$$
$$- (\Gamma(\nu\,(n-\mu)+1))^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{(\nu(n-\mu))q(1-s)}{p}} \left((D_{a+}^{\mu,\nu} f)(x)\right)^{\frac{sq}{p}} dx - B(s),$$

$$B(s) = \frac{q(\nu\,(n-\mu))(\Gamma(\nu\,(n-\mu)+1))^{s(\frac{q}{p}-1)}}{p}$$

$$\times \left| \int_a^b \int_a^x sgn\left((D_{a+}^{\mu+\nu(n-\mu)} f)(y) - \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} (D_{a+}^{\mu,\nu} f)(x) \right) \right.$$
$$\left. \times (x-a)^{\frac{(\nu(n-\mu))(q-p)(1-s)}{p}} \left((D_{a+}^{\mu,\nu} f)(x) \right)^{s(\frac{q}{p}-1)} (x-y)^{\nu(n-\mu)-1} N_{p,k} f(x,y) dy\, dx \right|$$

and $N_{p,k} f(x,y)$ is defined by (3.15) *and $H(s)$ is defined by* (3.27).

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_2(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \dfrac{(x-y)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}, & a < y \le x; \\ 0, & x < y \le b, \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\nu(n-\mu)}}{\Gamma(\nu(n-\mu)+1)}$ and $(A_k f)(x) = \frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} (D_{a+}^{\mu,\nu} f)(x)$. Replace f by $D_{a+}^{\mu+\nu(n-\mu)} f$. For the particular weight function $u(x) = (x-a)^{\frac{(\nu(n-\mu))q}{p}}$, $x \in (a,b)$ we get $v(y) = ((\nu(n-\mu))(b-y)^{\nu(n-\mu)-1+\frac{p}{q}})/(((\nu(n-\mu)-1)\frac{q}{p}+1)^{\frac{p}{q}})$ and then (3.25) takes the form

$$\rho(s) = \frac{(\nu(n-\mu))^{\frac{q}{p}}}{(\nu(n-\mu)-1)\frac{q}{p}+1} \left(\int_a^b (b-y)^{\nu(n-\mu)-1+\frac{p}{q}} ((D_{a+}^{\mu+\nu(n-\mu)} f)(y))^s dy \right)^{\frac{q}{p}}$$
$$- (\Gamma(\nu(n-\mu)+1))^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{(\nu(n-\mu))q(1-s)}{p}} \left((D_{a+}^{\mu,\nu} f)(x) \right)^{\frac{sq}{p}} dx - M(s).$$

Since $\frac{(\nu(n-\mu))q}{p}(1-s) \le 0$ and $M(s) \ge 0$, we obtain that

$$\begin{aligned} \rho(s) &\le \frac{(\nu(n-\mu))^{\frac{q}{p}} (b-a)^{(\nu(n-\mu)-1)\frac{q}{p}+1}}{(\nu(n-\mu)-1)\frac{q}{p}+1} \left(\int_a^b ((D_{a+}^{\mu+\nu(n-\mu)} f)(y))^s dy \right)^{\frac{q}{p}} \\ &\quad - (b-a)^{\frac{(\nu(n-\mu))q}{p}(1-s)} (\Gamma(\nu(n-\mu)+1))^{\frac{sq}{p}} \int_a^b \left((D_{a+}^{\mu,\nu} f)(x) \right)^{\frac{sq}{p}} dx - M(s) \\ &= H(s) - M(s) \\ &\le H(s). \end{aligned}$$

Moreover, (3.26) takes the form

$$\pi(s) = \frac{(\nu(n-\mu))^{\frac{q}{p}}}{(\nu(n-\mu)-1)\frac{q}{p}+1} \left(\int_a^b (b-y)^{\nu(n-\mu)-1+\frac{p}{q}} ((D_{a+}^{\mu+\nu(n-\mu)} f)(y))^s dy \right)^{\frac{q}{p}}$$
$$- (\Gamma(\nu(n-\mu)+1))^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{(\nu(n-\mu))q(1-s)}{p}} \left((D_{a+}^{\mu,\nu} f)(x) \right)^{\frac{sq}{p}} dx - B(s).$$

Since $\frac{(\nu(n-\mu))q}{p}(1-s) \le 0$ and $B(s) \ge 0$, we obtain that

$$\begin{aligned} \pi(s) &\le \frac{(\nu(n-\mu))^{\frac{q}{p}} (b-a)^{(\nu(n-\mu)-1)\frac{q}{p}+1}}{(\nu(n-\mu)-1)\frac{q}{p}+1} \left(\int_a^b ((D_{a+}^{\mu+\nu(n-\mu)} f)(y))^s dy \right)^{\frac{q}{p}} \\ &\quad - (b-a)^{\frac{(\nu(n-\mu))q}{p}(1-s)} (\Gamma(\nu(n-\mu)+1))^{\frac{sq}{p}} \int_a^b \left((D_{a+}^{\mu,\nu} f)(x) \right)^{\frac{sq}{p}} dx - B(s) \\ &= H(s) - B(s) \\ &\le H(s). \end{aligned}$$

The proof is complete. ■

Remark 3.10. Since $H(s) > 0$ in Theorem 3.9, then after an elementary calculation we get the inequality given in [30, Remark 2.6].

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Original article

Aitken type methods with high efficiency

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Abstract

In this paper, we study the iterative method of Aitken type for solving the non-linear equations, in which the interpolation nodes are controlled by variant of Newton method or by a general method of order p. By combining such methods with a generalized secant method, it is shown that the order of convergence can be increased to as high as desired and also in the limiting case efficiency of the method is 2. Several numerical examples are provided in support of the theoretical results.

Keywords: Non-linear equations; Newton method; Aitken type method; Generalized secant method

1. Introduction

Non-linear equations are encountered in all branch of science and engineering. It is hardly possible to solve such equations analytically and therefore iterative methods are employed. For a given non-linear equation

$$f(x) = 0,$$

a very well known method widely used is the Newton method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which is quadratically convergent. There have been several ways by which the order of convergence can be increased. Recently, in [1], Păvăloiu and Cătinaş obtained and studied the following Aitken method:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(y_n)}{f'(y_n)},$$

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$$x_{n+1} = z_n - \frac{f(z_n)}{[y_n, z_n; f]}. \tag{1.1}$$

Along with other considerations, it was proved in [1] that the method (1.1) is of order 6 with efficiency index 1.431 which is higher than the Newton method or the standard Aitken method.

Recently, McDougall and Wotherspoon in [2] gave the following modification of Newton's method:

$$\begin{aligned} x_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_{n-1} + x_{n-1}^*]\right)} \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_n + x_n^*]\right)}. \end{aligned} \tag{1.2}$$

They proved that the above method (1.2) has order of convergence $1+\sqrt{2} \approx 2.4$ and requires two functions evaluation per iteration so that the efficiency becomes 1.5537. In this paper, to begin with, we propose the following method in which the Newton iterates in (1.1) are replaced by (1.2):

$$\begin{aligned} y_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_{n-1} + y_{n-1}^*]\right)} \\ y_n &= x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_n + y_n^*]\right)} \\ z_n^* &= y_n - \frac{f(y_n)}{f'\left(\frac{1}{2}[y_{n-1} + z_{n-1}^*]\right)} \\ z_n &= y_n - \frac{f(y_n)}{f'\left(\frac{1}{2}[y_n + z_n^*]\right)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{[y_n, z_n; f]}. \end{aligned} \tag{1.3}$$

We prove that the order of convergence of the method (1.3) is 6.76137 and efficiency is 1.4655 higher than the method (1.1). This is done in Section 2.

Also in Section 2, we study a method more general than (1.1) or (1.2). We replace, in (1.1), the Newton iterates by the iterates of any arbitrary method. Let $\phi(x)$ be an iterative function such that the method

$$x_{n+1} = \phi(x_n)$$

is of order p. We propose the following generalized Aitken-type method:

$$\begin{aligned} y_n &= \phi(x_n), \\ z_n &= \phi(y_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{[y_n, z_n; f]}. \end{aligned} \tag{1.4}$$

We prove, in Section 2, that the method (1.4) is of order $p^2 + p$. This strategy would enable to produce an iterative method of any desired order. We demonstrate it with the help of certain examples.

There have been several methods which are based on approximations of integrals. In this direction Weerakoon and Fernando [3] obtained the following third order method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}. \end{aligned} \tag{1.5}$$

Several authors have obtained similar methods, see, e.g., [3–5]. It is noted that the method (1.5) will not proceed if at any iterate $f'(x_n) = 0$. To overcome this problem, on the lines of Wu [6], in [7], the following modification of (1.5) was proposed and studied:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}. \tag{1.6}$$

We develop, in Section 3, a method more general than (1.6) and study its convergence. This method is further generalized to Aitken-type method. In Section 4, we propose and study the Aitken-type methods which are based on power means.

Next, note that the last iterate in the method (1.1) is, in fact, the secant iterate which uses the previously calculated nodes y_n and z_n. In [8] and [9], those authors generalized the secant method which involves arbitrary number of previously calculated nodes. We exploit this generalized secant method in Section 5. In fact, we replace in (1.1), the secant iterate by generalized secant iterate. We show that as the number of iterate increases, not only the order but also the efficiency of the corresponding method increases. Moreover, in the limiting case as the number of iterates increases to infinity, the efficiency tends to 2. Finally, in Section 6, some numerical examples are provided based on the methods developed in this paper.

2. Order of convergence of general Aitken method

We begin with the convergence analysis of the method (1.3).

Theorem 2.1. *Let f be a sufficiently differentiable function in the neighbourhood of α which is a simple zero of f. If x_0 is sufficiently close to α, the order of convergence of the method* (1.3) *is* 6.76137 *with efficiency index* 1.4655.

Proof. Let e_n, d_n, and θ_n be the errors in, respectively, x_n, y_n, and z_n. Using Taylor series, we have

$$f(x_n) = f'(\alpha)[e_n + C_2e_n^2 + C_3e_n^3 + O(e_n^4)] \tag{2.1}$$

and

$$f'(x_n) = f'(\alpha)[1 + 2C_2e_n + 3C_3e_n^2 + 4C_4e_n^3 + O(e_n^4)] \tag{2.2}$$

where $C_n = \frac{1}{n!}\frac{f^n(\alpha)}{f'(\alpha)}$. Also,

$$\frac{1}{2}[x_{n-1} + y^*_{n-1}] = \frac{e_{n-1} + d^*_{n-1}}{2} + \alpha \tag{2.3}$$

where d^*_n is the error in y^*_n. Using (2.2) and (2.3), we get

$$\begin{aligned} f'\left(\frac{1}{2}[x_{n-1} + y^*_{n-1}]\right) = f'(\alpha)[1 &+ 2C_2\left(\frac{e_{n-1} + d^*_{n-1}}{2}\right) \\ &+ 3C_3\left(\frac{e_{n-1} + d^*_{n-1}}{2}\right)^2 + O(e_n^3)] \end{aligned}$$

which on using (2.1) gives

$$\begin{aligned} \frac{f(x_n)}{f'\left(\frac{1}{2}[x_{n-1} + y^*_{n-1}]\right)} &= e_n - C_2e_{n-1}e_n - C_2d^*_{n-1}e_n + C_2e_n^2 \\ &\quad - C_2^2e_{n-1}e_n^2 - C_2d^*_{n-1}e_n^2 + C_3e_n^3. \end{aligned}$$

Consequently, (1.3) gives

$$\begin{aligned} d^*_n &= e_n - (e_n - C_2e_{n-1}e_n - C_2d^*_{n-1}e_n + C_2e_n^2 \\ &\quad - C_2^2e_{n-1}e_n^2 - C_2d^*_{n-1}e_n^2 + C_3e_n^3) \\ &\approx C_2e_{n-1}e_n. \end{aligned} \tag{2.4}$$

Again by using Taylor series and (2.4), we get

$$d_n \approx C_2d^*_ne_n = C_2^2e_{n-1}e_n^2. \tag{2.5}$$

Similarly, if θ^*_n is the error in z_n then by (2.5), we obtain

$$\theta^*_n \approx C_2d_{n-1}d_n = C_2^5e_{n-2}e_{n-1}^3e_n^2. \tag{2.6}$$

Now, (2.5) and (2.6), give

$$\theta_n \approx \theta_n^* d_n = C_2^8 e_{n-2} e_{n-1}^4 e_n^4. \tag{2.7}$$

By Taylor series expansion, we have from (1.3)

$$e_{n+1} \approx \theta_n d_n$$

which by using (2.5) and (2.7) gives

$$e_{n+1} = C_2^{11} e_{n-2} e_{n-1}^5 e_n^6. \tag{2.8}$$

Now, by definition, if the method (1.3) has order of convergence p, then there exists a constant $A > 0$ such that the following holds

$$e_{n+1} = A e_n^p, \tag{2.9}$$

which implies that

$$e_n = A e_{n-1}^p \tag{2.10}$$

or

$$e_{n-1} = A e_{n-2}^p. \tag{2.11}$$

By using (2.9), (2.10) and (2.11) in (2.8), we get

$$A e_n^p = B (e_n^{1/p})^{1/p} e_n^{5/p} e_n^6,$$

where $B = C_2^{11} A^{-\frac{1}{p}-\frac{1}{p^2}}$. On equating the powers of e_n of both R.H.S and L.H.S, we have

$$p^3 - 6p^2 - 5p - 1 = 0. \tag{2.12}$$

On solving (2.12), we get that its positive root is $p = 6.76137$ which is the order of convergence of the method (1.3). Since the method (1.3) requires 5 functions evaluation per iteration, the efficiency follows. □

Next, we study the convergence analysis of the general Aitken method (1.4).

Theorem 2.2. *Let f be a sufficiently differentiable function in a neighbourhood of α which is a simple root of $f(x) = 0$. If $\phi(x)$ is an iterative function such that the method*

$$x_{n+1} = \phi(x_n) \tag{2.13}$$

has order of convergence p, then the method (1.4) *has order of convergence $p^2 + p$.*

Proof. Let e_n, d_n, θ_n denote the errors involved in the iterates x_n, y_n, z_n, respectively. Since the method (2.13) is of order p, the error equations for the iterates y_n and z_n in (1.4) are given by

$$d_n = P e_n^p + O(e_n^{p+1}) \tag{2.14}$$

$$\theta_n = Q d_n^p + O(d_n^{p+1}), \tag{2.15}$$

where P and Q are certain constants. Now, using Taylor's expansion, we have

$$\begin{aligned} f(z_n) = f(\alpha + \theta_n) &= f(\alpha) + f'(\alpha)\theta_n + \frac{1}{2!} f''(\alpha)\theta_n^2 + \frac{1}{3!} f'''(\alpha)\theta_n^3 + O(e_n^4) \\ &= f'(\alpha)\left[\theta_n + C_2\theta_n^2 + C_3\theta_n^3 + O(\theta_n^4)\right], \end{aligned} \tag{2.16}$$

where $C_n = \frac{1}{n!}\frac{f^n(\alpha)}{f'(\alpha)}$ and therefore

$$\begin{aligned}[y_n, z_n; f] &= \frac{f(y_n) - f(z_n)}{y_n - z_n} \\ &= \frac{f'(\alpha)(d_n - \theta_n)\left[1 + C_2(d_n + \theta_n) + C_3(d_n^2 + \theta_n^2 + d_n\theta_n) + O(e_n^4)\right]}{(d_n - \theta_n)} \\ &= f'(\alpha)\left[1 + C_2(d_n + \theta_n) + C_3(d_n^2 + \theta_n^2 + d_n\theta_n) + O(e_n^4)\right].\end{aligned}$$

Consequently, by (2.16) we get

$$\begin{aligned}\frac{f(z_n)}{[y_n, z_n; f]} &= \frac{f'(\alpha)\left[\theta_n + C_2\theta_n^2 + C_3\theta_n^3 + O(\theta_n^4)\right]}{f'(\alpha)\left[1 + C_2(d_n + \theta_n) + C_3(d_n^2 + \theta_n^2 + d_n\theta_n) + O(e_n^4)\right]} \\ &= \left(\theta_n + C_2\theta_n^2 + C_3\theta_n^3\right)\left(1 - C_2(d_n + \theta_n) - C_3(d_n^2 + \theta_n^2 + d_n\theta_n)\right) \\ &= \theta_n - C_2 d_n\theta_n - (C_2^2 + C_3)\theta_n^2 d_n - C_3 d_n^2\theta_n - C_2^2\theta_n^3,\end{aligned}$$

using which the error equation of the iterate x_{n+1} in (1.4) is obtained as

$$\begin{aligned}e_{n+1} &= C_2 d_n\theta_n + (C_2^2 + C_3)\theta_n^2 d_n + C_3 d_n^2\theta_n + C_2^2\theta_n^3 \\ &\approx C_2 d_n\theta_n,\end{aligned}$$

so that (2.14) and (2.15) give

$$\begin{aligned}e_{n+1} &= C_2(P{e_n}^p)(Q{d_n}^p) \\ &= C_2(P{e_n}^p)(Q(P{e_n}^p)^p) \\ &= C_2 P^{p+1} Q e_n^{p^2+p}\end{aligned} \tag{2.17}$$

and the assertion follows. □

Below, we apply Theorem 2.2 and obtain certain higher order methods.

Example 2.3. If we consider

$$\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)},$$

then $x_{n+1} = \phi(x_n)$ is the Newton method which is of order 2, i.e., in this case $p = 2$. For this ϕ, the method (1.4) becomes (1.1) obtained by Păvăloiu and Cătinaş [1]. The order 6 of the method is confirmed by Theorem 2.2.

Remark 2.4. If we consider

$$\phi(x_n) = x_n - \frac{f(x_n)}{f'(1/2[x_n + x_n^*])},$$

where

$$x_n^* = x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_{n-1} + x_{n-1}^*]\right)}$$

then the method

$$x_{n+1} = \phi(x_n)$$

is the method (1.2) given by McDougall and Wotherspoon [2] having order of convergence $p = 2.414$. Consequently, in view of Theorem 2.2 the order of convergence of the Aitken type method (1.3) is 8.16. This demonstrate the effectiveness of Theorem 2.2.

Example 2.5. In [10], Wang considered the method

$$x_{n+1} = x_n - \frac{f(x_n)}{(1-\beta)f'(x_n) + \beta f'(x_n - \frac{f(x_n)}{2\beta f'(x_n)})}, \tag{2.18}$$

where $\beta \neq 0$. is an arbitrary real number. The method (2.18) was proved to be of order 3 for any $\beta \neq 0$. Note that for $\beta = 1$ and $\beta = 1/2$, (2.18) becomes the methods, respectively, corresponding to the mid point rule and the trapezoidal rule, see [10]. Thus, if we consider

$$\phi(x_n) = x_n - \frac{f(x_n)}{(1-\beta)f'(x_n) + \beta f'(x_n - \frac{f(x_n)}{2\beta f'(x_n)})},$$

then the method (1.4) becomes the following:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{(1-\beta)f'(x_n) + \beta f'\left(x_n - \frac{f(x_n)}{2\beta f'(x_n)}\right)}, \\ z_n &= y_n - \frac{f(y_n)}{(1-\beta)f'(y_n) + \beta f'\left(y_n - \frac{f(y_n)}{2\beta f'(y_n)}\right)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{[y_n, z_n; f]}. \end{aligned} \tag{2.19}$$

In view of Theorem 2.2, the method (2.19) is of order 12.

Remark 2.6. Let $\phi(x)$ is an iterative function such that the method (2.13) has order of convergence p and requires m functions evaluation per iteration. Then the efficiency index of (2.13) is p^1/m. Moreover, with this ϕ, in view of Theorem 2.2, the efficiency index of the method (1.4) is $(p^2 + p)^{1/(2m+1)}$. Obviously for any $p \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$,

$$(p^2 + p)^{1/(2m+1)} > p^1/m.$$

This demonstrates that Aitken-type method (1.4) increases the efficiency of any given method.

3. Methods based on approximation of integrals

In [3], Weerakoon and Fernando used Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt$$

and approximated the indefinite integral by trapezoidal rule as

$$\int_{x_n}^{x} f'(t)dt \approx \frac{x - x_n}{2}[f'(x_n) + f(x)]$$

and obtained the following method

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{2f(x_n)}{[f'(x_n) + f'(y_n)]}. \end{aligned} \tag{3.1}$$

Similarly, by approximating the indefinite integral by mid point rule as

$$\int_{x_n}^{x} f'(t)dt \approx (x - x_n)f'(\frac{x_n + x}{2})$$

(see [11]), one can obtain the following method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{2f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(y_n)}. \end{aligned} \tag{3.2}$$

Recall that the method (2.18) of Wang [10] contains both the methods (3.1) and (3.2). It is noted that if, in the methods (3.1), (3.2) or (2.18), $f'(x_n)$ becomes zero at any iterate, then the method cannot proceed further. To overcome this problem, method (3.1) can be redefined as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)},$$
$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}. \tag{3.3}$$

Similarly, method (3.2) can be redefined as

$$y_n = x_n - \frac{f(x_n)}{2[f'(x_n) - \lambda f(x_n)]}, \quad \lambda \in \mathbb{R}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(y_n)}. \tag{3.4}$$

This type of modification has been used in [7]. We use the similar modification in the method (2.18) and propose the following method:

$$y_n = x_n - \frac{f(x_n)}{2\beta[f'(x_n) - \lambda f(x_n)]},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{(1-\beta)f'(x_n) + \beta f'(y_n)}. \tag{3.5}$$

Note that for $\beta = 1$, the method (3.5) becomes (3.4) and for $\beta = \frac{1}{2}$, it becomes (3.3). Below, we prove the convergence of the method (3.5).

Theorem 3.1. *Let f be a sufficiently differentiable function in the neighbourhood of α which is a simple zero of f. If x_0 is sufficiently close to α, the order of convergence of the method* (3.5) *is three.*

Proof. Using the Taylor expansion, it is standard to have

$$\begin{aligned} f(x_n) &= f(\alpha + e_n) \\ &= f'(\alpha)\left[e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\right], \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} f'(x_n) &= f'(\alpha + e_n) \\ &= f'(\alpha)\left[1 + 2C_2 e_n + 3C_3 e_n^2 + O(e_n^3)\right], \end{aligned} \tag{3.7}$$

where $C_n = \frac{1}{n!}\frac{f^n(\alpha)}{f'(\alpha)}$. From (3.6) and (3.7) we get,

$$f'(x_n) - \lambda f(x_n) = f'(\alpha)\left[1 + (2C_2 - \lambda)e_n + (3C_3 - \lambda C_2)e_n^2 + O(e_n^3)\right] \tag{3.8}$$

which gives that

$$\begin{aligned} \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)} &= \frac{[e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)]}{[1 + (2C_2 - \lambda)e_n + (3C_3 - \lambda C_2)e_n^2 + O(e_n^3)]} \\ &= [e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)] \\ &\quad \times [1 - (2C_2 - \lambda)e_n - (3C_3 - \lambda C_2)e_n^2 + (2C_2 - \lambda)^2 e_n^2] \\ &= e_n + (\lambda - C_2)e_n^2 + (2C_2(1+\lambda) - 2C_3 - 2C_2^2 - \lambda)e_n^3. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{f(x_n)}{2\beta(f'(x_n) - \lambda f(x_n))} &= (1/2\beta)e_n + (1/2\beta)(\lambda - C_2)e_n^2 \\ &\quad + (1/2\beta)(2C_2(1+\lambda) - 2C_3 - 2C_2^2 - \lambda)e_n^3. \end{aligned} \tag{3.9}$$

Let d_n be the error involved in y_n, so that (3.5) and (3.9) give

$$d_n = (1 - 1/2\beta)e_n - (1/2\beta)(\lambda - C_2)e_n^2 - (1/2\beta)[2C_2(1+\lambda) - 2C_3 - 2C_2^2 - \lambda]e_n^3 + O(e_n^4).$$

Again by using Taylor expansion we have,

$$\begin{aligned} f'(y_n) &= f'(\alpha) + f''(\alpha)[(1 - 1/2\beta)e_n - (1/2\beta)(\lambda - C_2)e_n^2 \\ &\quad - (1/2\beta)(2C_2(1+\lambda) - 2C_3 - 2C_2^2 - \lambda)e_n^3] \\ &= f'(\alpha)[1 + 2C_2(1 - 1/2\beta)e_n - (C_2/\beta)(\lambda - C_2)e_n^2 \\ &\quad - (C_2/\beta)(2C_2(1+\lambda) - 2C_3 - 2C_2^2 - \lambda)e_n^3]. \end{aligned} \tag{3.10}$$

Now, by (3.7) and (3.10) we have

$$(1-\beta)f'(x_n) + \beta f(y_n) = f'(\alpha)\left[1 + C_2 e_n + (3C_3(1-\beta) - C_2\lambda + C_2^2)e_n^2 + O(e_n^4)\right]. \tag{3.11}$$

Using (3.9) and (3.11) in (3.8) we have,

$$e_{n+1} = (3C_3\beta - 2C_3 + C_2\lambda + C_2^2)e_n^3 + O(e_n^4), \tag{3.12}$$

which proves that the method (3.5) is of order three. □

Example 3.2. As in Example 2.5, if we consider

$$\phi(x_n) = x_n - \frac{f(x_n)}{(1-\beta)f'(x_n) + \beta f'\left(x_n - \frac{f(x_n)}{2\beta[f'(x_n) - \lambda f(x_n)]}\right)}$$

then the method (1.4) reads as:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{(1-\beta)f'(x_n) + \beta f'\left(x_n - \frac{f(x_n)}{2\beta[f'(x_n) - \lambda f(x_n)]}\right)}, \\ z_n &= y_n - \frac{f(y_n)}{(1-\beta)f'(y_n) + \beta f'\left(y_n - \frac{f(y_n)}{2\beta[f'(y_n) - \lambda f(y_n)]}\right)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{[y_n, z_n; f]}, \end{aligned} \tag{3.13}$$

and in view of Theorems 2.2 and 3.1, the method (3.13) is of order 12.

We can rewrite the trapezoidal Newton's method (3.1) of Weerakoon and Fernando [3] as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{[f'(x_n) + f'(y_n)]/2}. \end{aligned} \tag{3.14}$$

This can be seen as obtained by using arithmetic mean of $f'(x_n)$ and $f'(y_n)$. In [5], Özban, instead of arithmetic mean, used harmonic mean which leads to the following method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)\left(f'(x_n) + f'(y_n)\right)}{2f'(x_n)f'(y_n)}. \end{aligned} \tag{3.15}$$

Özban, in [5] proved that this method is of order 3.

We propose the following variant of (3.15):

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)\left(f'(x_n) + f'(y_n)\right)}{2f'(x_n)f'(y_n)}. \end{aligned} \tag{3.16}$$

The order of convergence of the method (3.16) is given in the following:

Theorem 3.3. *Let f be sufficiently differentiable function in the neighbourhood of α which is a simple zero of f. If x_0 is sufficiently close to α, then the method* (3.16) *has order of convergence three.*

Proof. It can be shown, similar to the proof of Theorem 3.1, that the error equation of (3.16) satisfies

$$e_{n+1} = \left((1/2)C_3 - \lambda C_2 - 2C_2^2\right) e_n^3 + O(e_n^4)$$

and the assertion follows. □

Remark 3.4. Using the iterates of the method (3.15), an Aitken-type method can be proposed which, in view of Theorems 2.2 and 3.3 will be of order 12.

4. Methods based on power means

Let p be a finite real number. For two non-negative real numbers a and b, their p-power mean, denoted by m_p is defined as

$$m_p = \left(\frac{a^p + b^p}{2}\right)^{1/p}.$$

For different values of p, in particular for $p = 1, -1, 2, -2$ and $1/2$, we get several cases of well-known means as given below:

$$m_1 = \left(\frac{a+b}{2}\right), m_{-}1 = \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1}, m_2 = \left(\frac{a^2+b^2}{2}\right)^{1/2}$$

$$m_{-}2 = \left(\frac{a^{-2}+b^{-2}}{2}\right)^{1/-2}, m_{1/2} = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2$$

which, respectively, are called arithmetic mean, harmonic mean, root mean square, inverse root inverse-square mean and square-mean root of a and b. We also include the case $p = 0$. In this case, $m_0(a, b) = \lim_{p \to 0} m_p(a, b) = \sqrt{ab}$, which is standard geometric mean of a and b.

In [12], those authors replaced the arithmetic mean in (3.1) by the power mean and proved the third order convergence of their general method which now covers several other means as well. We adopt the same strategy for the method (3.2) and obtain a new class of methods as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{x_n^p + y_n^p}{2}\right)^{1/p}}. \end{aligned} \tag{4.1}$$

For the convergence of the method (4.1), we prove the following:

Theorem 4.1. *Let f be a sufficiently differentiable function in the neighbourhood of α which is a simple root of $f(x) = 0$. Then, the order of convergence of the method* (4.1) *is 3.*

Proof. Let e_n, d_n be the errors in, respectively, x_n, y_n, i.e.,

$$\begin{aligned} x_n &= e_n + \alpha, \\ y_n &= d_n + \alpha. \end{aligned} \tag{4.2}$$

By expanding $f(x_n)$ and $f'(x_n)$ using Taylor series about α, we have

$$f(x_n) = f'(\alpha)[e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)] \tag{4.3}$$

and

$$f'(x_n) = f'(\alpha)[1 + 2C_2 e_n + 3C_3 e_n^2 + 4C_4 e_n^3 + O(e_n^4)]. \tag{4.4}$$

By using (4.3) and (4.4) in (4.1), we get

$$d_n = C_2 e_n^2 + (2C_3 - 2C_2^2)e_n^3 + O(e_n^4). \tag{4.5}$$

Now, from (4.2) and (4.5), we have

$$\begin{aligned} x_n^p &= (e_n + \alpha)^p \\ &= \alpha^p \left(1 + p\frac{e_n}{\alpha} + \frac{1}{2}p(p-1)\frac{e_n^2}{\alpha^2}\right) \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} y_n^p &= (d_n + \alpha)^p \\ &= \alpha^p \left(1 + \frac{p}{\alpha}\left[C_2 e_n^2 + (2C_3 - 2C_2^2)e_n^3\right]\right). \end{aligned} \tag{4.7}$$

Therefore, (4.6) and (4.7) give

$$\begin{aligned} \left(\frac{x_n^p + y_n^p}{2}\right)^{1/p} &= \frac{1}{2^{1/p}}\left[\alpha^p\left(1 + p\frac{e_n}{\alpha} + \frac{1}{2}p(p-1)\frac{e_n^2}{\alpha^2}\right) + \alpha^p\left(1 + \frac{p}{\alpha}C_2 e_n^2\right)\right]^{1/p} \\ &= \alpha\left[1 + \frac{1}{2\alpha}e_n + \frac{1}{p}\left(\frac{1}{4\alpha^2}p(p-1) + \frac{p}{2\alpha}C_2 + \frac{1}{2}(\frac{1}{p}-1)\frac{p^2}{4\alpha^2}\right)e_n^2\right] \\ &= \left[\alpha + \frac{e_n}{2} + (\frac{1}{8\alpha}(p-1) + \frac{C_2}{2})e_n^2\right]. \end{aligned} \tag{4.8}$$

Again expanding $f'\left(\frac{x_n^p + y_n^p}{2}\right)^{1/p}$ by Taylor series and using (4.4), we get

$$\begin{aligned} f'\left(\frac{x_n^p + y_n^p}{2}\right)^{1/p} &= f'(\alpha)\left[1 + 2C_2\left(\frac{e_n}{2} + (\frac{1}{8\alpha}(p-1) + \frac{C_2}{2})e_n^2\right) + 3C_3(\frac{e_n^2}{4})\right] \\ &= f'(\alpha)\left[1 + C_2 e_n + (2C_2(\frac{1}{8\alpha}(p-1) + \frac{C_2}{2} + \frac{3}{4}C_3)e_n^2)\right]. \end{aligned}$$

Consequently, (4.1) gives

$$\begin{aligned} e_{n+1} &= e_n - \left[e_n + C_2 e_n^2 + C_3 e_n^3\right]\left[1 - C_2 e_n - \left(2C_2(\frac{1}{8\alpha}(p-1) + \frac{C_2}{2})\right) + \frac{3}{4}C_3 - C_2^2 e_n^2\right] \\ &= \left[2C_2\left(\frac{1}{8\alpha}(p-1) + \frac{C_2}{2}\right) - \frac{1}{4}C_3\right] + O(e_n^4) \\ &= \left[\frac{1}{4\alpha}C_2(p-1) + C_2^2 - \frac{1}{4}C_3\right]e_n^3 + O(e_n^4). \end{aligned}$$

Hence, the method (4.1) has the order of convergence 3. □

Example 4.2. As done in Section 2, if we consider

$$\phi(x_n) = x_n - \frac{f(x_n)}{f'\left(\frac{x_n^p + (x_n - \frac{f(x_n)}{f'(x_n)})^p}{2}\right)^{1/p}},$$

then the following Aitken-type method can be considered:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'\left(\frac{x_n^p + (x_n - \frac{f(x_n)}{f'(x_n)})^p}{2}\right)^{1/p}}, \\ z_n &= y_n - \frac{f(y_n)}{f'\left(\frac{y_n^p + (y_n - \frac{f(y_n)}{f'(y_n)})^p}{2}\right)^{1/p}}, \end{aligned}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{[y_n, z_n; f]}. \tag{4.9}$$

In the light of Theorem 2.2, we can easily see that the method (4.9) is of order 12.

5. Increasing the efficiency

Recall that the standard Secant method reads as

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})},$$

which, in terms of divided difference, can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_{n-1}, x_n; f]} \tag{5.1}$$

secant method is a two point method with memory which has order of convergence 1.618. It requires only one function evaluation per iteration and as a result has efficiency 1.618. In (1.1) or (1.4), the strategy was to use method (5.1) once the two nodes are calculated from other methods.

Very recently in [8,9] Kogan et al. used the Newton divided difference formula

$$f(x) = f(x_n) + [x_{n-1}, x_n; f](x - x_n) + \cdots + [x_0, x_n; f]\prod_{j=1}^{n}(x - x_j) + R_n,$$

where

$$R_n = f(x, x_n, \ldots, x_0)\prod_{j=1}^{n}(x - x_j)$$

and generalized the secant method (5.1) as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_{n-1}, x_n; f] + \sum_{i=2}^{k}[x_{n-i}, x_n; f]\prod_{j=1}^{i-1}(x_n - x_{n-j})}, \quad n = k, k+1, \ldots \tag{5.2}$$

where $k \geq 1$ is an arbitrary fixed integer and the initial k approximations $x_0, x_1, \ldots . . x_k$ are known. Obviously for $k = 2$, (5.2) becomes (5.1).

Remark 5.1. For the later use, let us mention that (see [9]) based on $k+1$ initial approximations $x_0, x_1, \ldots x_k$, the error equation corresponding to the method (5.2) is given by

$$e_{n+1} = C_k \prod_{j=0}^{k} e_{n-j} + O(\prod e_{n-j}). \tag{5.3}$$

In the light of above discussion, we propose a multipoint Aitken method of the type (1.1) as follows:

$$x_n^{(0)} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_n^{(1)} = x_n^{(0)} - \frac{f(x_n^{(1)})}{f'(x_n^{(1)})}$$

$$x_n^{(2)} = x_n^{(1)} - \frac{f(x_n^{(1)})}{[x_n^{(0)}, x_n^{(1)}; f]}$$

$$\vdots$$

$$x_{n+1} = x_n^{(k)} - \frac{f(x_n^{(k)})}{[x_n^{(k-1)}, x_n^{(k)}; f] + \sum_{i=2}^{k}[x_n^{(k-i)}, x_n^{(k)}; f]\prod_{j=1}^{i-1}(x_n^{(k)} - x_n^{(k-j)})}, \quad k = 1, 2, 3, \ldots \tag{5.4}$$

with the initial approximation x_0. Clearly, for $k = 1$, the method (5.4) becomes (1.1). We shall prove that as k increases, not only the order of convergence but also the efficiency of (5.4) increases. Precisely, we prove the following:

Theorem 5.2. *Let f be a sufficiently differentiable function in a neighbourhood of α which is a simple root of $f(x) = 0$. Let O(k) and EI(k) denote, respectively, the order of convergence and efficiency index of* (5.4) *for* $k = 1, 2, 3, \ldots$ *Then*

(a) $O(k) = 6 \times 2^{k-1}$
(b) $EI(k) = (6 \times 2^{k-2})^{\frac{1}{k+4}}$
(c) *EI(k) is strictly increasing*
(d) $EI(k) \to 2 \quad as \quad k \to \infty$.

Proof. We only prove (a) and (b). It is straightforward to verify (c) and (d).

(a) Let e_n, e_{n+1} denote the errors in the iterates x_n, x_{n+1}. For the intermediate steps, let $e_{n,k}$ denote the errors in $x_n^{(k)}$, $k = 0, 1, 2, 3, \ldots$ Since $x_n^{(0)}$ and $x_n^{(1)}$ are Newton iterates, it is standard that the corresponding errors are given by

$$e_{n,0} \approx A_1 e_n^2 \tag{5.5}$$

$$e_{n,1} \approx A_2 e_{n,0}^2 \approx A_1^2 A_2 e_n^4, \tag{5.6}$$

where A_1 and A_2 are appropriate constants. For the intermediate steps, for $k = 1, 2, 3, \ldots,$ the corresponding error equations, in view of (5.3), are given by

$$e_{n,k+1} \approx C_{k+1} \prod_{j=1}^{k+1} e_{n,k+1-j} \tag{5.7}$$

and once $k = 1, 2, 3, \ldots$ is fixed, we shall write

$$e_{n+1} = e_{n,k+1}.$$

We shall prove by induction that the order of convergence of the method (5.4) is $6 \times 2^{k-1}$ for $k = 1, 2, 3, \ldots$.

For $k = 1$, (5.7) becomes

$$e_{n,2} \approx C_2 e_{n,1}.e_{n,0}$$

which by using (5.5) and (5.6) gives

$$e_{n,3} \approx C_2 A_1^3 A_2 e_n^6$$

i.e.,

$$e_{n+1} \approx C_2 A_1^3 A_2 e_n^6.$$

Therefore, the assertion holds for $k = 1$. Assume that it holds for k, i.e.,

$$e_{n+1} = e_{n,k+1} \approx D_k e_n^{6\times 2^{k-1}}, \tag{5.8}$$

where D_k is some constant. Note that by (5.7),

$$e_{n,k+1} \approx C_{k+1}.e_{n,k}.e_{n,k-1}.e_{n,k-2} \ldots e_{n,1}.e_{n,0}. \tag{5.9}$$

For k replaced by $k + 1$, (5.7) gives

$$e_{n,k+2} \approx C_{k+2} \prod_{j=1}^{k+2} e_{n,k+2-j}$$

which using (5.8) and (5.9) gives

$$\begin{aligned}
e_{n,k+2} &\approx C_{k+2}.e_{n,k+1}.e_{n,k}.e_{n,k-1}.e_{n,k-2}\ldots e_{n,1}.e_{n,0}\\
&\approx \frac{C_{k+2}}{C_k+1}e^2_{n,k+1}\\
&\approx \frac{C_{k+2}}{C_k+1}D_k e_n^{6\times 2^{k-1}\times 2}\\
&= \frac{C_{k+2}}{C_k+1}D_k e_n^{6\times 2^{k}}
\end{aligned}$$

and the assertion follows.

(b) In the method (5.4), first two steps are Newton's iterates that require two functions evaluation each per iterations. Thus for these two steps, a total of 4 function evaluations per iteration are required. After third step onwards, the method requires only one function evaluation per iteration since it uses the previously calculated values. Thus for $k = 1, 2, 3, \ldots$ a total of $k+4$ functions need to be evaluated per iteration. Combining this information with the order of the method, the result follows. □

The method (5.4) can be modified to give rise a more general method by replacing the Newton iterates by any arbitrary method as was done in (1.4). In this way, the order of convergence of corresponding method will, of course, depend upon the chosen method, the efficiency still approaches to 2 as $k \to \infty$. We construct the method as follows:

Let $\phi(x)$ be an iterative function such that the method

$$x_{n+1} = \phi(x_n)$$

is of order p. we propose the following method:

$$\begin{aligned}
x_n^{(0)} &= \phi(x_n)\\
x_n^{(1)} &= \phi(x_n^{(0)})\\
x_n^{(2)} &= x_n^{(1)} - \frac{f(x_n^{(1)})}{[x_n^{(0)}, x_n^{(1)}; f]}\\
&\vdots\\
x_{n+1} &= x_n^{(k)} - \frac{f(x_n^{(k)})}{[x_n^{(k-1)}, x_n^{(k)}; f] + \sum_{i=2}^{k}[x_n^{(k-i)}, x_n^{(k)}; f]\prod_{j=1}^{i-1}(x_n^{(k)} - x_n^{(k-j)})}, \quad k = 1, 2, 3, \ldots.
\end{aligned} \tag{5.10}$$

We prove the following theorem:

Theorem 5.3. *Let f be a sufficiently differentiable function in a neighbourhood of α which is a simple zero of f. If $\phi(x) = 0$ is an iterative function such that the method*

$$x_{n+1} = \phi(x_n) \tag{5.11}$$

has order of convergence p.Let O(k) and EI(k) denote, respectively, the order of convergence and efficiency index of (5.10) *for $k = 1, 2, 3, \ldots$. Then*

(a) *O(k)=$(p^2 + p) \times 2^{k-1}$*
(b) *EI(k)=$[(p^2 + p) \times 2^{k-2}]^{\frac{1}{k+4}}$*
(c) *EI(k) is strictly increasing*
(d) *EI(k)$\to 2$ as $k \to \infty$.*

Proof. It can be shown, similar to the proof of Theorems 5.3 and 2.2 that the error equation of (5.10) satisfies

$$e_{n,k} = \frac{C_{k+1}}{C_k}D_k e_n^{(p^2+p)\times 2^{(k-1)}}$$

and the assertion follows. □

Table 1

n	x_n	$f(x_n)$	$f(x_n) - f(x_{n-1})$	$x_n - x_{n-1}$
1	1.4072530700	−0.0068699926	−0.7149434109	0.4072530702
2	1.4044916480	$-2.4202861940 \times 10^{-13}$	0.0068699926	−0.0027614220
3	1.4044916480	$3.3306690740 \times 10^{-16}$	$2.4236168630 \times 10^{-13}$	$-9.7699626170 \times 10^{-14}$
4	1.4044916480	$3.3306690740 \times 10^{-16}$	0.0000000000	0.0000000000
5	1.4044916480	$3.3306690740 \times 10^{-16}$	0.0000000000	0.0000000000

Table 2

n	x_n	$f(x_n)$	$f(x_n) - f(x_{n-1})$	$x_n - x_{n-1}$
1	0.9023469634	0.0192246037	−2.1544019144	−0.5976530365
2	0.8951770213	−0.0000777429	−0.0193023466	−0.0071699421
3	0.8952060449	$-1.2703803564 \times 10^{-9}$	0.0000777416	0.0000290235
4	0.8952060453	$1.1102230246 \times 10^{-16}$	$1.2703804674 \times 10^{-9}$	$4.7426629290 \times 10^{-10}$
5	0.8952060453	$1.1102230246 \times 10^{-16}$	0.0000000000	0.0000000000
6	0.8952060453	$1.1102230246 \times 10^{-16}$	0.0000000000	0.0000000000

Table 3

n	x_n	$f(x_n)$	$f(x_n) - f(x_{n-1})$	$x_n - x_{n-1}$
1	0.8952058992	$-3.9138720608 \times 10^{-7}$	−2.1736269095	−0.6047941007
2	0.8952060453	$1.1102230246 \times 10^{-16}$	$3.9138720620 \times 10^{-7}$	$1.4611511678 \times 10^{-7}$
3	Division by 0	Division by 0	Division by 0	Division by 0
4				

Table 4

n	x_n	$f(x_n)$	$f(x_n) - f(x_{n-1})$	$x_n - x_{n-1}$
1	2.4788706270	29.8113615300	−23.1886384700	−0.5211293734
2	2.0864736370	16.4966853500	−13.3146761700	−0.3923969897
3	1.8034714550	8.8758448120	−7.6208405420	−0.2830021820
4	1.6064524160	4.4685121220	−4.4073326900	−0.1970190390
5	1.4449994650	1.3692865880	−3.0992255340	−0.1614529510
6	1.5745885450	3.8212395230	2.4519529350	0.1295890801
7	1.3921047750	0.4496602138	−3.3715793090	−0.1824837697
8	1.3465447930	−0.3057365137	−0.7553967275	−0.0455599821
9	1.3621951380	−0.0500415762	0.2556949375	0.0156503445
10	1.3651247990	−0.0017373659	0.0483042103	0.0029296608
11	1.3652298790	$-2.2150939110 \times 10^{-6}$	0.0017351508	0.0001050808
12	1.3652300130	$-3.6077807410 \times 10^{-12}$	$2.2150903030 \times 10^{-6}$	$1.3413897640 \times 10^{-7}$
13	1.3652300130	0.0000000000	$3.6077807410 \times 10^{-12}$	$2.1849189120 \times 10^{-13}$
14	1.3652300130	0.0000000000	0.0000000000	0.0000000000
15	1.3652300130	0.0000000000	0.0000000000	0.0000000000

6. Examples

In this section, we provide numerical examples to demonstrate the order of convergence of the methods proposed in previous sections.

Example 6.1. We consider the equation

$$f(x) = \sin^2 x - x^2 + 1$$

and implement the method (2.19) on this function. Table 1, shows the corresponding iterate. Here, we take the initial value as $x_0 = 1.0$ and $\beta = 1.0$. The method (2.19) is of order 12, which is demonstrated in Table 1.

Example 6.2. Consider the equation

$$f(x) = x^2 \sin x - \cos x.$$

We implement the methods (3.5) and (3.13) on this equation. The corresponding iterates have been tabulated in Tables 2 and 3, respectively, where the initial value is taken to be $x_0 = 1.5$, $\lambda = 1.5$ and $\beta = 0.5$. The method (3.5) is of order 3 where as the method (3.13) is of order 12. Tables 2 and 3 show the higher rate of convergence of the method (3.13).

Example 6.3. We consider the equation

$$f(x) = x^3 + 4x^2 - 10$$

and implement the method (3.16) on this function. Table 4, shows the corresponding iterate. Here, we take the initial value as $x_0 = 3.0$ and $\lambda = -1.5$. The method (3.16) is of order 3.

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Original article

Study residuated lattice via some elements

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Abstract

In this paper, the notions of distributive, standard and neutral elements in residuated lattices were introduced and relationships between them were investigated. Also we study the sets of distributive, standard and neutral elements in residuated lattices. Then we show that under some conditions, the sets of distributive, standard and neutral elements in residuated lattices become a MTL-algebra. Finally, special elements of type 2 in residuated lattices were introduced.

MSC: 03B47; 03G25

Keywords: Residuated lattice; MTL-algebra; BL-algebra; Distributive element; Standard element; Neutral element

1. Introduction and preliminaries

The concept of residuated lattices was introduced by M. Ward and R. P. Dilworth [1] as a generalization of the structure of t he set of ideals of a ring. These algebras are a common structure among algebras associated with logical systems. The residuated lattices have interesting algebraic and logical properties [2–5]. The main example of residuated lattices related to logic is BL-algebras. A basic logic algebra (BL-algebra for short) is an important class of logical algebras introduced by H'ajek [6] in order to provide an algebraic proof of the completeness of "Basic Logic" (BL for short). MV-algebras introduced by Chang [7] in 1958 are the most known classes of BL-algebras.

The concepts of distributive, standard and neutral elements introduced in lattices by O. Ore [8], G. Gratzer [9] and G. Birkhoff [10], respectively and have been extended to trellises by S. B. Rai in [11].

We decide to generalize this concepts to residuated lattices. In this paper, we introduce the notions of distributive, standard and neutral elements in residuated lattices and verify relationships between them. Also we study the sets of $Dis(L)$, $St(L)$ and $Neu(L)$. Then, we study relationships between distributive, standard and neutral elements with

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some other special elements, likeness dense, boolean, node and regular elements in residuated lattices. Finally, we study image of distributive, standard and neutral elements under a homomorphism.

In this section, we recall some definitions and results about residuated lattices which are used in the sequel.

Definition 1.1 (*[6]*)**.** A residuated lattice is an algebra $(L, \vee, \wedge, \odot, \longrightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that:
(RL1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
(RL2) $(L, \odot, 1)$ is an abelian monoid,
(RL3) $x \odot z \leq y$ if and only if $z \leq x \to y$, for all $x, y, z \in L$.

Definition 1.2 (*[6,12,13]*)**.** Let L be a residuated lattice and $a \in L$,
(a) a is called idempotent iff $a^2 = a$,
(b) a is called nilpotent iff there exists a natural number n, such that $a^n = 0$,
(c) a is called dense iff $a^* = 0$, where $x^* = x \to 0$,
(d) a is called regular iff $a^{**} = a$ and $x^* \odot (x^* \to x) = 0$,
(e) a is called boolean iff there exists $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$,
(f) a is called node iff for every filter F of L, $[a) \subseteq F$ or $F \subseteq [a)$.

Theorem 1.3 (*[6]*)**.** *In any residuated lattice* $(L, \vee, \wedge, \odot, \to, 0, 1)$ *the following properties are valid:*
(1) $1 \to x = x$,
(2) $x \to x = 1$,
(3) $x \odot y \leq x, y$, *so* $x \odot y \leq x \wedge y$,
(4) $y \leq x \to y$,
(5) $x \leq y \Leftrightarrow x \to y = 1$,
(6) if $x \to y = y \to x = 1$, *then* $x = y$,
(7) $x \to 1 = 1$,
(8) $0 \to x = 1$,
(9) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
for all $x, y, z, \in L$.

Definition 1.4 (*[6]*)**.** A residuated lattice L is called MTL-algebra if the following property is valid:
(BL5) $(x \to y) \vee (y \to x) = 1$, for all $x, y \in L$.

Definition 1.5 (*[6]*)**.** A MTL-algebra L is called BL-algebra if the following property is valid:
(BL4) $x \odot (x \to y) = x \wedge y$, for all $x, y \in L$.

Definition 1.6 (*[6]*)**.** A nonempty subset F of residuated lattice L is called a filter of L if F satisfies the following conditions:
$(F1)$ if $x \in F$, $x \leq y$ and $y \in L$, then $y \in F$,
$(F2)$ $x \odot y \in F$ for every $x, y \in F$ that is, F is a subsemigroup of L.

Definition 1.7 (*[14]*)**.** An element a of a lattice L is called
(1) distributive, if $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$,
(2) standard, if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$,
(3) neutral, if $(a \wedge x) \vee (a \wedge y) \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \wedge (x \vee y)$,
for all $x, y \in L$.

The concepts of dually distributive and dually standard elements are obtained by dualizing (1) and (2) respectively. The notion of a neutral element is self-dual.

2. On distributive elements in residuated lattices

From now on $L = (L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice unless otherwise specified.

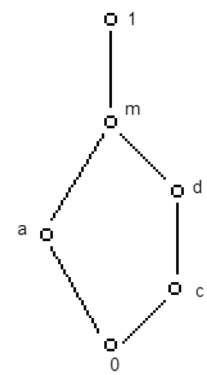
Definition 2.1. An element a of L is said to be distributive if

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$$

for all $x, y \in L$.

We denote the set of all distributive elements of L with $Dis(L)$.
Note: In every L, $\{0, 1\} \subseteq Dis(L)$.

Example 2.2. (a) Let $L = \{0, a, c, d, m, 1\}$, with $0 < a < m < 1$, $0 < c < d < m < 1$, but a incomparable with c, d. For all $x, y \in L$, define $\odot$ and $\rightarrow$ as follows:

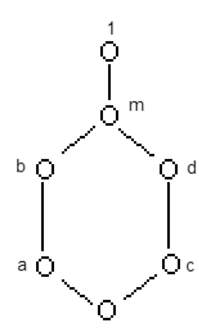


$\odot$	0	a	c	d	m	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
c	0	0	c	c	c	c
d	0	0	c	c	c	d
m	0	a	c	c	m	m
1	0	a	c	d	m	1

$\rightarrow$	0	a	c	d	m	1
0	1	1	1	1	1	1
a	d	1	d	d	1	1
c	a	a	1	1	1	1
d	a	a	m	1	1	1
m	0	a	d	d	1	1
1	0	a	c	d	m	1

then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice. $\{0, a, d, m, 1\}$ is the set of all distributive elements of L and c is not a distributive element of L, because $c \vee (a \wedge d) \neq (c \vee a) \wedge (c \vee d)$.

(b) Let $L = \{0, a, b, c, d, m, 1\}$, with $0 < a < b < m < 1$, $0 < c < d < m < 1$ and elements $\{a, c\}$ and $\{b, d\}$ are pairwise incomparable. For all $x, y \in L$, define $\odot$ and $\rightarrow$ as follows:



$\odot$	0	a	b	c	d	m	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
m	0	a	a	c	c	m	m
1	0	a	b	c	d	m	1

$\rightarrow$	0	a	b	c	d	m	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	m	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	m	1	1	1
m	0	b	b	d	d	1	1
1	0	a	b	c	d	m	1

then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice. $\{0, b, d, m, 1\}$ is the set of all distributive elements of L, but a, c are not distributive elements of L, because $c \vee (a \wedge d) \neq (c \vee a) \wedge (c \vee d)$ and $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$.

Definition 2.3. An element a of L is said to be distributive type 2, if

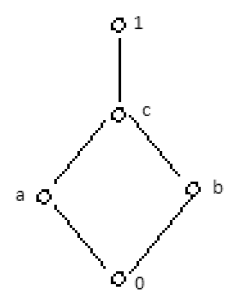
$$a \wedge (x \odot y) = (a \wedge x) \odot (a \wedge y)$$

for all $x, y \in L$.

Let $Dis2(L)$ denote the set of all distributive type 2 elements of L.
Note: In every L, $0, 1 \in Dis2(L)$.

Example 2.4. (a) In Example 2.2, (a), $\{0, a, c, 1\}$ is the set of all distributive type 2 elements of L, but m, d are not distributive type 2 elements of L, because $m \wedge (d \odot 1) \neq (m \wedge d) \odot (m \wedge 1)$ and $d \wedge (d \odot 1) \neq (d \wedge d) \odot (d \wedge 1)$.

(b) Let $L = \{0, a, b, c, 1\}$, with $0 < a, b < c < 1$, but a, b are incomparable. We define $\odot$ and $\to$ as follows:

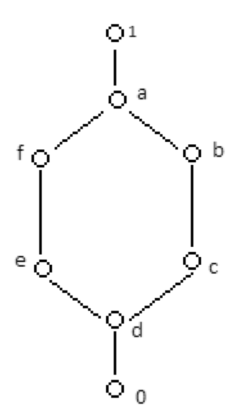


$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a *non*-linearly ordered residuated lattice and $Dis2(L) = \{0, a, b, c, 1\}$.

(c) Let $L = \{0, a, b, c, d, e, f, 1\}$ with $0 < c < d < b < a < 1$, $0 < d < e < f < a < 1$, and elements $\{b, f\}$ and $\{c, e\}$ are pairwise incomparable. We define $\odot$ and $\to$ as follows:



$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	1	1	1	f
c	f	1	1	1	f	f	0	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	0	a	b	c	d	e	f	1

then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a *non*-linearly ordered residuated lattice. We get $Dis2(L) = \{0, 1\}$ but $a, b, c, d, e, f \notin Dis2(L)$.

With compare Examples 2.2, (a) and 2.4, (a), we get $Dis(L) \nsubseteq Dis2(L)$ and $Dis2(L) \nsubseteq Dis(L)$.

Proposition 2.5. *Define $\varphi : L \to [a)$ by $x \mapsto a \vee x$. If φ is a homomorphism, then a becomes a distributive element of L.*

Proof.

$$\begin{aligned} a \vee (x \wedge y) &= \varphi(x \wedge y) \\ &= \varphi(x) \wedge \varphi(y) \\ &= (a \vee x) \wedge (a \vee y). \qquad \square \end{aligned}$$

Proposition 2.6. *Define $\varphi : L \to [a)$ by $x \mapsto a \wedge x$. If φ is a homomorphism, then a is a distributive type 2 element of L.*

Proof.

$$\begin{aligned} a \wedge (x \odot y) &= \varphi(x \odot y) \\ &= \varphi(x) \odot \varphi(y) \\ &= (a \wedge x) \odot (a \wedge y). \end{aligned}$$

So $a \in Dis2(L)$. $\square$

Example 2.7. (a) In Example 2.2, (b), b is a distributive element, but the map $\varphi : L \to [b)$ by $x \mapsto b \vee x$ is not a homomorphism, because

$$m = \varphi(a \to c) \neq \varphi(a) \to \varphi(c) = 1.$$

(b) In Example 2.4 (b), a is a distributive type 2 element, but the map $\varphi : L \to [a)$ by $x \mapsto a \wedge x$ is not a homomorphism, because

$$a = \varphi(b \to c) \neq \varphi(b) \to \varphi(c) = 1.$$

Theorem 2.8. *Let a be a distributive element of L. Then the map $\varphi : L \to [a)$ by $x \mapsto a \vee x$, for all $x \in L$, is onto $[a)$.*

Theorem 2.9. *Let $\equiv_{\alpha_a}$ be a congruence relation on L defined as follows*

$$x \equiv_{\alpha_a} y \text{ if and only if } a \vee x = a \vee y.$$

Then a becomes a distributive element of L.

Proof. Since $a \vee x = a \vee (a \vee x)$ and $a \vee y = a \vee (a \vee y)$, so $x \equiv_{\alpha_a} a \vee x$ and $y \equiv_{\alpha_a} a \vee y$, thus $x \wedge y \equiv_{\alpha_a} (a \vee x) \wedge (a \vee y)$. Therefore $a \vee (x \wedge y) = a \vee ((a \vee x) \wedge (a \vee y)) = (a \vee x) \wedge (a \vee y)$. $\square$

Proposition 2.10. *Let a be a boolean element of L. Then $a \in Dis^{\delta}(L)$.*

Proof. Let $a \in B(L)$. Then $a \wedge x = a \odot x$, for every $x \in L$. Thus by Theorem 1.3, (9),

$$\begin{aligned} a \wedge (x \vee y) &= a \odot (x \vee y) \\ &= (a \odot x) \vee (a \odot y) \\ &= (a \wedge x) \vee (a \wedge y). \qquad \square \end{aligned}$$

3. On standard elements in residuated lattices

Definition 3.1. An element a of a residuated lattice L is said to be standard element if

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$$

for all $x, y \in L$.

We denote the set of all standard elements of L with $St(L)$.
Note: In every L, $\{0, 1\} \subseteq St(L)$.

Example 3.2. (a) In Example 2.2, (a), $\{0, d, m, 1\}$ is the set of all standard elements of L but a, c are not standard elements of L, because $d \wedge (a \vee c) \neq (d \wedge a) \vee (d \wedge c)$ and $d \wedge (c \vee a) \neq (d \wedge c) \vee (d \wedge a)$.

(b) In Example 2.2, (b), $\{0, m, 1\}$ is the set of all standard elements of L and the elements of a, b, c, d are not standard elements of L, because $a \wedge (d \vee c) \neq (a \wedge d) \vee (a \wedge c)$ and $b \wedge (a \vee c) \neq (b \wedge a) \vee (b \wedge c)$.

Definition 3.3. An element a of L is said to be standard type 2, if

$$x \odot (a \wedge y) = (x \odot a) \wedge (x \odot y)$$

for all $x, y \in L$.

Let $St2(L)$ denote the set of all standard type 2 elements of L.
Note: In every L, $0, 1 \in St2(L)$.

Example 3.4. (a) In Example 2.2, (a), $\{0, a, c, d, m, 1\}$ is the set of all standard type 2 elements of L.
(b) In Example 2.4, (b), $St2(L) = \{0, a, b, c, 1\}$.
(c) In Example 2.4, (c), $e \notin St2(L)$ because $a \odot (e \wedge c) \neq (a \odot e) \wedge (a \wedge c)$.

Problem. Whether in every L, $St(L) \subseteq St2(L)$

Theorem 3.5. *Let a be a distributive element of L and*

$$a \vee x = a \vee y \text{ and } a \wedge x = a \wedge y \text{ imply that } x = y$$

for all $x, y \in L$. Then $a \in St(L)$.

Proof. Let $x, y \in L$. Define

$$b = x \wedge (a \vee y),$$

$$c = (x \wedge a) \vee (x \wedge y).$$

In order to show $b = c$, it will be sufficient to prove that

$$a \vee b = a \vee c,$$

$$a \wedge b = a \wedge c.$$

To prove that first, we compute, using the fact that $a \in Dis(L)$

$$\begin{aligned} a \vee b &= a \vee ((x \wedge (a \vee y))) \\ &= (a \vee x) \wedge (a \vee y) \\ &= a \vee (x \wedge y) \\ &= a \vee (x \wedge a) \vee (x \wedge y) = a \vee c. \end{aligned}$$

To prove that second,

$$\begin{aligned} a \wedge x &\le a \wedge c \\ &\le a \wedge b \\ &= a \wedge x \wedge (a \vee y) = a \wedge x, \end{aligned}$$

and hence $a \wedge c = a \wedge b$. $\square$

Theorem 3.6. *Let $\equiv_{\alpha_a}$ is defined on L as $x \equiv_{\alpha_a} y$ if $(x \wedge y) \vee a_1 = x \vee y$ for some $a_1 \le a$, be a congruence relation. Then $a \in St(L)$.*

Proof. We can show that $a \in Dis(L)$ just as in Theorem 2.9. Now let

$$a \vee x = a \vee y,$$

$$a \wedge x = a \wedge y.$$

Since $y \equiv_{\alpha_a} a \vee y$, meeting both sides with x and using $a \vee y = a \vee x$, we obtain that

$$x \wedge y \equiv_{\alpha_a} x \wedge (a \vee y) = x \wedge (a \vee x) = x.$$

Thus $x = (x \wedge y) \vee a_1$ for some $a_1 \le a$. Also $a_1 \le x$, hence $a_1 \le a \wedge x = a \wedge y$, and so $a_1 \le x \wedge y$. We conclude that $x = x \wedge y$. Similarly, $y = x \wedge y$, and so $x = y$. $\square$

Theorem 3.7. *Let $a \in St(L)$. Then $a \in Dis(L)$.*

Proof. Let $a \in St(L)$. So for all $x, y \in L$,

$$\begin{aligned} (a \vee x) \wedge (a \vee y) &= ((a \vee x) \wedge a) \vee ((a \vee x) \wedge y) \\ &= a \vee ((a \vee x) \wedge y) \\ &= a \vee ((y \wedge a) \vee (y \wedge x)) \\ &= (a \vee (y \wedge a)) \vee (y \wedge x) \\ &= a \vee (y \wedge x). \qquad \square \end{aligned}$$

Proposition 3.8. *(a) Let $a, b \in Dis2(L)$. Then $a \wedge b \in Dis2(L)$.*
(b) Let $a, b \in St2(L)$. Then $a \wedge b \in St2(L)$.

Proof. (a) Let $a, b \in Dis2(L)$. Then

$$\begin{aligned} (a \wedge b) \wedge (x \odot y) &= a \wedge (b \wedge (x \odot y)) \\ &= a \wedge ((b \wedge x) \odot (b \wedge y)) \\ &= (a \wedge (b \wedge x)) \odot (a \wedge (b \wedge y)) \\ &= ((a \wedge b) \wedge x) \odot ((a \wedge b) \wedge y). \end{aligned}$$

So $a \wedge b \in Dis2(L)$.
(b) Let $a, b \in St2(L)$. Then

$$\begin{aligned} x \odot ((a \wedge b) \wedge y) &= x \odot (a \wedge (b \wedge y)) \\ &= (x \odot a) \wedge (x \odot (b \wedge y)) \\ &= (x \odot a) \wedge (x \odot b) \wedge (x \odot y) \\ &= (x \odot (a \wedge b)) \wedge (x \odot y). \end{aligned}$$

So $a \wedge b \in St2(L)$. $\square$

Theorem 3.9. *Let L be a BL-algebra. Then $St2(L) = L$.*

Theorem 3.10. *Let L be a residuated lattice. Then*
(a) $Dis2(L)$ is a meet-subsemi lattice of L.
(b) $St2(L)$ is a meet-subsemi of L.

4. On neutral elements in residuated lattices

Definition 4.1. An element a of a residuated lattice L is said to be neutral if

$$(a \wedge x) \vee (a \wedge y) \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \wedge (x \vee y)$$

for all $x, y \in L$.

Let $Neu(L)$ denote the set of all neutral elements of L.
Note: In every L, $\{0, 1\} \subseteq Neu(L)$.

Example 4.2. (a) In Example 2.2, (a), $\{0, d, m, 1\}$ is the set of all neutral elements of L, but a, c are not neutral elements of L.
(b) In Example 2.2, (b), $\{0, m, 1\}$ is the set of all neutral elements of L but a, b, c, d are not neutral elements of L.

Definition 4.3. An element a of L is said to be neutral type 2, if

$$(a \odot x) \wedge (a \odot y) \wedge (x \odot y) = (a \wedge x) \odot (a \wedge y) \odot (x \wedge y)$$

for all $x, y \in L$.

Let $Neu2(L)$ denote the set of all neutral type 2 elements of L.
Note: In every L, $0 \in Neu2(L)$.

Example 4.4. In Example 2.2, (a), $\{0, a, c, m\}$ is the set of all neutral type 2 elements of L, but d, 1 are not neutral type 2 elements of L, because

$$(d \odot 1) \wedge (d \odot 1) \wedge (1 \odot 1) \neq (d \wedge 1) \odot (d \wedge 1) \odot (1 \wedge 1)$$

and

$$(1 \odot 1) \wedge (1 \odot d) \wedge (1 \odot d) \neq (1 \wedge 1) \odot (1 \wedge d) \odot (1 \wedge d).$$

Example 4.5. (a) In Example 2.2, (a), $d \in St2(L)$ but $d \notin Dis2(L)$. So in general $St2(L) \nsubseteq Dis2(L)$.
(b) In Example 2.2, (a), $m \in Neu2(L)$ but $m \notin Dis2(L)$. So in general $Neu2(L) \nsubseteq Dis2(L)$.
(c) In Example 2.2, (a), $1 \in Dis2(L)$ but $1 \notin Neu2(L)$. So in general $Dis2(L) \nsubseteq Neu2(L)$.

With compare Examples 4.2, (a) and 4.4, (a), we get $Neu(L) \nsubseteq Neu2(L)$ and $Neu2(L) \nsubseteq Neu(L)$.

Theorem 4.6. *Let L be a residuated lattice and $a \in L$. Then the following conditions are equivalent:*
(a) $a \in Neu(L)$,
(b) $a \in Dis(L)$, $a \in Dis^{\delta}(L)$ and

$$a \vee x = a \vee y,$$

$$a \wedge x = a \wedge y,$$

imply that $x = y$ for every $x, y \in L$.

Proof. $(a) \to (b)$ Let $a \in Neu(L)$. Then for $x \geq a$

$$(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = a \vee (x \wedge y)$$

and

$$(a \vee x) \wedge (x \vee y) \wedge (y \vee a) = x \wedge (a \vee y),$$

so

$$a \vee (x \wedge y) = x \wedge (a \vee y). \tag{4.1}$$

To show that $a \in Dis(L)$, consider

$$\begin{aligned} a \vee (x \wedge y) &= a \vee ((a \wedge x) \vee (x \wedge y) \vee (y \wedge a)) \\ &= a \vee ((a \vee x) \wedge ((x \vee y) \wedge (y \vee a))) \end{aligned}$$

(apply (4.1) to a, $a \vee x$, $(x \vee y) \wedge (y \vee a)$)

$$= (a \vee x) \wedge (a \vee ((x \vee y) \wedge (y \vee a)))$$

(apply (4.1) to a, $y \vee a$, $x \vee y$)

$$\begin{aligned} &= (a \vee x) \wedge (y \vee a) \wedge (a \vee x \vee y) \\ &= (a \vee x) \wedge (a \vee y), \end{aligned}$$

as claimed. By duality, we get that $a \in Dis^{\delta}(L)$.

Finally, let $a \vee x = a \vee y$ and $a \wedge x = a \wedge y$. Then

$$\begin{aligned} x &= x \wedge (a \vee x) \wedge (a \vee y) \wedge (x \vee y) \\ &= x \wedge ((a \wedge x) \vee (x \wedge y) \vee (a \wedge y)) \\ &= x \wedge ((a \wedge x) \vee (x \wedge y)) \\ &= (a \wedge x) \vee (x \wedge y) \\ &= (a \wedge x) \vee (a \wedge y) \vee (x \wedge y). \end{aligned}$$

Since the right-hand side is symmetric in x and y, we conclude that $x = y$.

$(b) \to (a)$ since $a \in Dis(L)$

$$(a \vee x) \wedge (x \vee y) \wedge (y \vee a) = [a \vee (x \wedge y)] \wedge (x \vee y)$$

by (b) and Theorem 2.9, $a \in St(L)$

$$\begin{aligned} &= [a \wedge (x \vee y)] \vee [(x \wedge y) \wedge (x \vee y)] \\ &= [a \wedge (x \vee y)] \vee (x \wedge y) \end{aligned}$$

since $a \in Dis^{\delta}(L)$

$$= [(a \wedge x) \vee (a \wedge y)] \vee (x \wedge y).$$

That is, $a \in Neu(L)$. □

Theorem 4.7. *Let L be a residuated lattice. Then $Neu(L) \subseteq St(L) \subseteq Dis(L)$.*

Proof. By Theorems 3.5, 3.7 and 4.6. □

Theorem 4.8. *Let L be a residuated lattice. Then*
a) $Dis(L)$ is a join-subsemi lattice of L.
(b) $St(L)$ is a sublattice of L.
(c) $Neu(L)$ is a sublattice of L.

Proof. (a) Let $a, b \in Dis(L)$. Consider

$$\begin{aligned} (a \vee b) \vee (x \wedge y) &= a \vee b \vee (x \wedge y) \\ &= a \vee ((b \vee x) \wedge (b \vee y)) \\ &= (a \vee b \vee x) \wedge (a \vee b \vee y), \end{aligned}$$

so $a \vee b \in Dis(L)$.

(b) Let $a, b \in St(L)$. Consider

$$\begin{aligned} x \wedge (a \vee b \vee y) &= (x \wedge a) \vee (x \wedge (b \vee y)) \\ &= (x \wedge a) \vee (x \wedge b) \vee (x \wedge y) \\ &= (x \wedge (a \vee b)) \vee (x \wedge y), \end{aligned}$$

proving that $a \vee b \in St(L)$. Now we verify the formula

$$\alpha_a \wedge \alpha_b = \alpha_{a \wedge b}$$

where α_a, α_b and $\alpha_{a\wedge b}$ are the relation described by Theorem 2.9. Since $\alpha_a \wedge \alpha_b \leq \alpha_{a\wedge b}$ is trivial, let $x \equiv_{\alpha_a \wedge \alpha_b} y$. Then $x \equiv_{\alpha_a} y$, so $(x \wedge y) \vee a_1 = x \vee y$ for some $a_1 \leq a$. Also we have $x \equiv_{\alpha_b} y$, therefore

$$a_1 = a_1 \wedge (x \vee y) \equiv_{\alpha_b} a_1 \wedge x \wedge y.$$

Thus $a_1 = (a_1 \wedge x \wedge y) \vee b_1$ for some $b_1 \leq b$. Now

$$(x \wedge y) \vee b_1 = (x \wedge y) \vee (a_1 \wedge x \wedge y) \vee b_1 = (x \wedge y) \vee a_1 = x \vee y,$$

since $b_1 \leq b$ and $b_1 \leq a_1 \leq a$, we obtain that $b_1 \leq a \wedge b$, this verify that $x \equiv_{\alpha_{a\wedge b}} y$.

This formula shows that if $a, b \in St(L)$, then the relation $\alpha_{a\wedge b}$ of Theorem 2.9, is a congruence relation, hence $a \wedge b \in St(L)$ by Theorem 2.9. □

Example 4.9. In Example 2.2, (a),
(a) $d, m \in Dis(L)$ but $d \odot m = c \notin Dis(L)$.
(b) $d, m \in St(L)$ but $d \odot m = c \notin St(L)$.
(c) $d, m \in Neu(L)$ but $d \odot m = c \notin Neu(L)$.

Theorem 4.10. *Let L be a MTL-algebra. Then*

$$Neu(L) = St(L) = Dis(L) = L.$$

Corollary 4.11. *Let L be a BL-algebra. Then*

$$Neu(L) = St(L) = Dis(L) = L.$$

Corollary 4.12. *Let L be a MTL-algebra. Then $Neu(L)$, $St(L)$ and $Dis(L)$ are MTL-algebra.*

Example 4.13. In Example 2.4, (b), $Neu(L) = St(L) = Dis(L) = L$, but L is not a MTL-algebra. So the converse of Corollary 4.12 is not valid, in general.

In the following, we study the relationship between distributive, standard, neutral element and other types of elements in residuated lattices.

Example 4.14. (a) In Example 2.2, (b), $m \in Dis(L), St(L), Neu(L)$ but $m \notin Reg(L)$. Then every distributive, standard and neutral element is not a regular element.

(b) Every regular element is not a standard or neutral element. In Example 2.2, (b), $d \in Reg(L)$ but $d \notin St(L), Neu(L)$.

(c) In Example 2.4, (c), $a \in Dis(L), St(L), Neu(L)$ but $a \notin Idem(L)$. So every distributive, standard and neutral element is not a idempotent element.

(d) Every idempotent element is not a distributive or standard or neutral element. In Example 2.2, (b), $c \in Idem(L)$ but $c \notin Dis(L), St(L), Neu(L)$.

(e) Every distributive, standard and neutral element is not a nilpotent element. In Example 2.4, (c), $a \in Dis(L), St(L), Neu(L)$ but $a \notin Nil(L)$.

(f) Every nilpotent element is not a distributive or standard or neutral element. In Example 2.4, (c), $e \in Nil(L)$ but $e \notin Dis(L), St(L), Neu(L)$.

(g) Every distributive, standard and neutral element is not a element of MV-center of L. In Example 2.2, (b), $m \in Dis(L), St(L), Neu(L)$ but $m \notin MV(L)$.

(h) Every element of MV-center of L is not a distributive or standard or neutral element. In Example 2.4, (c), $c \in MV(L)$ but $c \notin Dis(L), St(L), Neu(L)$.

(i) Every distributive, standard and neutral element is not a dense element. In Example 2.2, (b), $0 \in Dis(L), St(L), Neu(L)$ but $0 \notin D(L)$.

(j) Every distributive, standard and neutral element is not a boolean element. In Example 2.2, (b), $m \in Dis(L), St(L), Neu(L)$ but $m \notin B(L)$.

(k) Every distributive, standard and neutral element is not a node element. In Example 2.4, (b), $a \in Dis(L), St(L), Neu(L)$ but $a \notin Nod(L)$.

Example 4.15. If F is a filter of L, then $Dis(L), St(L), Neu(L)$ are not necessary a filter of L. In Example 2.4, (c), consider $F = \{a, b, c, 1\}$, then F is a filter of L, but $Dis(F) = \{a, b, 1\}$ and $St(F) = Neu(F) = \{a, 1\}$ are not filter of L.

Theorem 4.16. *Let f be an epimorphism on L.*
(a) If $a \in Dis(L)$, then $f(a) \in Dis(L)$,
(b) if $a \in St(L)$, then $f(a) \in St(L)$,
(c) if $a \in Neu(L)$, then $f(a) \in Neu(L)$.

Proof. Let f be an epimorphism on L and $a \in Dis(L)$. Since f is onto, so for every $x, y \in L$, there exist $x', y' \in L$ such that $f(x') = x,\ f(y') = y$. So (a)

$$\begin{aligned} f(a) \vee (x \wedge y) &= f(a) \vee (f(x') \wedge f(y')) \\ &= f(a \vee (x' \wedge y')) \\ &= f((a \vee x') \wedge (a \vee y')) \\ &= f(a \vee x') \wedge f(a \vee y') \\ &= (f(a) \vee f(x')) \wedge (f(a) \vee f(y')) \\ &= (f(a) \vee x) \wedge (f(a) \vee y). \end{aligned}$$

(b)

$$\begin{aligned} x \wedge (f(a) \vee y) &= f(x') \wedge (f(a) \vee f(y')) \\ &= f(x' \wedge (a \vee y')) \\ &= f((x' \wedge a) \vee (x' \wedge y')) \\ &= f(x' \wedge a) \vee f(x' \wedge y') \\ &= (f(x') \wedge f(a)) \vee (f(x') \wedge f(y')) \\ &= (x \wedge f(a)) \vee (x \wedge y). \end{aligned}$$

(c)

$$\begin{aligned} (f(a) \wedge x) \vee (f(a) \wedge y) \vee (x \wedge y) &= (f(a) \wedge f(x')) \vee (f(a) \wedge f(y')) \vee (f(x') \wedge f(y')) \\ &= f(a \wedge x') \vee (a \wedge y') \vee (x' \wedge y') \\ &= f((a \vee x') \wedge (a \vee y') \wedge (x' \vee y')) \\ &= (f(a) \vee f(x')) \wedge (f(a) \vee f(y')) \wedge (f(x') \vee f(y')) \\ &= (f(a) \vee x) \wedge (f(a) \vee y) \wedge (x \vee y). \qquad \square \end{aligned}$$

In the following, we discuss some different cases of distributive, standard and neutral elements such that, use $\odot$ and $\rightarrow$ instead of $\wedge$ and $\vee$.

Example 4.17. (a) In Example 2.2, (b), we have $b, d, m \in Dis(L)$, but

$$m \rightarrow (b \vee d) \neq (m \rightarrow b) \vee (m \rightarrow d)$$

and

$b \odot (d \to m) \neq (b \odot d) \to (b \odot m)$.

(b) In Example 2.2, (b), we have $b, d \in Dis(L)$, but

$b \vee (a \odot d) \neq (b \vee a) \odot (b \vee d)$.

(c) In Example 2.2, (a, b), if $x \in Dis(L)$, then

$x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$

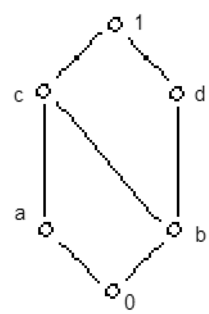
and

$x \to (y \to z) = (x \to y) \to (x \to z)$.

(d) In Example 2.2, (b), we have $d, m \in St(L)$, but

$m \to (b \vee d) \neq (m \to b) \vee (m \to d)$

(e) Let $L = \{0, a, b, c, d, 1\}$, with $0 < a, b < c < 1$, $0 < b < d < 1$, but a, b and respective c, d are incomparable. We define $\odot$ and $\to$ as follows:



$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice. $d \in St(L)$, but

$c \wedge (d \odot b) \neq (c \wedge d) \odot (c \wedge b)$

.

(f) In Example 2.2, (a), we have $d, m, 1 \in Neu(L)$, but

$(1 \to d) \vee (1 \to m) \vee (d \to m) \neq (1 \vee d) \to (1 \vee m) \to (d \vee m)$.

(g) In Example 2.2, (a), we have $d, m, 1 \in Neu(L)$, but

$(d \odot m) \to (d \odot 1) \to (m \odot 1) \neq (d \to m) \odot (d \to 1) \odot (m \to 1)$.

(h) In Example 2.2, (a), we have $d \in Neu(L)$, but

$(d \to c) \wedge (d \to m) \wedge (c \to m) \neq (d \wedge c) \to (d \wedge m) \to (c \wedge m)$.

Example 4.18. In Example 2.2,

(a) $a, c \in Dis2(L)$ but $a \vee c = m \notin Dis2(L)$.
(b) $a, c \in Dis2(L)$ but $a \to c = d \notin Dis2(L)$.
(c) $a, c \in Neu2(L)$ but $a \to c = d \notin Neu2(L)$.

Theorem 4.19. *Let f be an epimorphism on L.*
(a) *If $a \in Dis2(L)$, then $f(a) \in Dis2(L)$,*
(b) *if $a \in St2(L)$, then $f(a) \in St2(L)$,*
(c) *if $a \in Neu2(L)$, then $f(a) \in Neu2(L)$.*

5. Conclusion and future research

Residuation is a fundamental concept of ordered structures and categories. The origin of residuated lattices is in Mathematical Logic without contraction.

In this paper, we introduced the notions of distributive, standard and neutral elements in residuated lattices and verified relationships between them. Also we studied the sets of $Dis(L)$, $St(L)$ and $Neu(L)$ and proved that for MTL-algebra L, $Neu(L)$, $St(L)$ and $Dis(L)$ are MTL-algebra. Finally, we introduced and studied special elements of type 2 in residuated lattices.

Some important issues for future work are:
(i) developing the properties of distributive, standard and neutral elements in residuated lattices,
(ii) developing the properties of distributive, standard and neutral type 2 elements in residuated lattices,
(iii) finding useful results on other structures,
(iv) constructing the related logical properties of such structures.

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Original article

Stationary distribution and extinction of a three-species food chain stochastic model

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Abstract

In this paper, we investigate long-time behaviour of a stochastic three-species food chain model. By Markov semigroups theory, we prove that the densities of this model can converge to an invariant density or can converge weakly to a singular measure in L^1 under appropriate conditions. Further, several sufficient conditions for the extinction of the three species were obtained. Finally, numerical simulations are carried out to illustrate our theoretical results.

Keywords: Food chain; Stationary distribution; Markov semigroups; Extinction

1. Introduction

The dynamical relationship of three species predator–prey systems has long been one of the hot topics in mathematical biology. To clarify the short-term or long-term behaviour of ecosystems, it is essential to understand the interacting dynamics of three species food chain models. Since 1970s, there have been some interesting results on investigating the dynamics of three species predator–prey systems [1–6]. In particular, Krikorian [4] considered the Volterra predator–prey model in the three species case and to say as much as possible about global properties of its solution. Zhou [3] investigated the existence and global stability of the positive periodic solutions of delayed discrete food chains with omnivory. Hsu [5] considered a three species Lotka–Volterra food web model with omnivory which is defined as feeding on more than one trophic level. A famous three-species food chain model [4] can be expressed

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as follows:

$$\begin{cases} \dot{x}_1(t) = x_1(t)[a_1 - b_{11}x_1(t) - b_{12}x_2(t)], \\ \dot{x}_2(t) = x_2(t)[-a_2 + b_{21}x_1(t) - b_{22}x_2(t) - b_{23}x_3(t)], \\ \dot{x}_3(t) = x_3(t)[-a_3 + b_{32}x_2(t) - b_{33}x_3(t)], \end{cases} \tag{1}$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the densities of prey, predator and top-predator population at time t respectively. The parameters a_1, a_2 and a_3 are positive constants that stand for the intrinsic growth rate of the species $x_1(t)$, the death rate of the species $x_2(t)$, and the death rate of the species $x_3(t)$, respectively. The coefficients b_{11}, b_{22}, b_{33} are the intra-specific competition in the resource, b_{12}, b_{23} represent the rate of consumption and b_{21}, b_{32} represent the contribution of prey to the growth of predator.

Actually, the growth of populations in the natural world is always affected by environmental stochastic perturbations which should be taken into consideration in the process of mathematical modelling. In the literatures, many authors have studied population systems affected by white noise [7–16]. Especially, Takeuchi [9] investigated the evolution of a system composed of two predator–prey deterministic systems described by Lotka–Volterra equations in random environment. Ji [11] considered a predator–prey model with modified Leslie–Gower and Holling-type II schemes with stochastic perturbation. Nguyen [16] investigated a stochastic ratio-dependent predator–prey model. Inspired by the above literatures, in this paper, we consider the effect of white noise on the three-species food chain. From model (1) one can derive the following model with stochastic perturbations:

$$\begin{cases} dx_1(t) = x_1(t)\big[\big(a_1 - b_{11}x_1(t) - b_{12}x_2(t)\big)dt + \sigma_1 dB(t)\big], \\ dx_2(t) = x_2(t)\big[\big(-a_2 + b_{21}x_1(t) - b_{22}x_2(t) - b_{23}x_3(t)\big)dt + \sigma_2 dB(t)\big], \\ dx_3(t) = x_3(t)\big[\big(-a_3 + b_{32}x_2(t) - b_{33}x_3(t)\big)dt + \sigma_3 dB(t)\big], \end{cases} \tag{2}$$

where $B(t)$ are white noises, and σ_i is a positive constant representing the intensity of the white noise. We always assume that σ_i is not all equal. The existence, uniqueness and non-extinction property of the solution of system (2) have been studied in [15]. We replace model (2) by a slightly simpler one. Let $u_1 = \ln x_1, u_2 = \ln x_2, u_3 = \ln x_3$. Then, by Itô's formula, the random variables u_1, u_2, u_3 satisfy

$$\begin{cases} du_1(t) = \big(c_1 - b_{11}e^{u_1} - b_{12}e^{u_2}\big)dt + \sigma_1 dB(t), \\ du_2(t) = \big(-c_2 + b_{21}e^{u_1} - b_{22}e^{u_2} - b_{23}e^{u_3}\big)dt + \sigma_2 dB(t), \\ du_3(t) = \big(-c_3 + b_{32}e^{u_2} - b_{33}e^{u_3}\big)dt + \sigma_3 dB(t) \end{cases} \tag{3}$$

where $c_1 = a_1 - \sigma_1^2/2, c_2 = a_2 + \sigma_2^2/2, c_3 = a_3 + \sigma_3^2/2$.

The aim of this paper is to study the long-time behaviour of the solutions. The long-time behaviour of system (3) depends on the constants $b_{11}, b_{12}, b_{21}, b_{22}, b_{23}, b_{32}, b_{33}, c_1, c_2, c_3$. The study reveals that the other dynamic scenarios of system (3) are characterized by those parameters. The main results are listed as follows:

• Under some conditions (see Theorem 3.1), we show that the density of the distribution of system (3) converge to a stationary density;

• If $c_1 < 0$, then $\lim_{t\to\infty} u_i(t) = -\infty,\ a.e.\ \ i = 1, 2, 3$ (see Theorem 3.10);

• If $c_1 > 0, b_{11}c_2 > b_{21}c_1$, then $\lim_{t\to\infty} u_i(t) = -\infty,\ a.e.\ \ i = 2, 3$, and the distribution of the process $u_1(t)$ converges weakly to the measure which has the density $f_*(x) = Cexp\{2c_1x/\sigma_1^2 - (2b_{11}/\sigma_1^2)e^x\}$ (see Theorem 3.9);

• If $c_1 > 0, b_{11}c_2 < b_{21}c_1, b_{11}b_{22}c_3 + b_{11}b_{32}c_2 > b_{21}b_{32}c_1$, then $\lim_{t\to\infty} u_3(t) = -\infty$, $a.e$, and there exists a unique density $\overline{U}_*(x, y)$ which is a stationary solution of the first two equations of system (3) (see Theorem 3.8).

In this paper we focus on the existence of stationary distribution of system (3). Since the Fokker–Planck equation corresponding to system (3) is of degenerate type, the approach used in [17] to obtain the existence of stationary distribution is invalid for system (3). Our new approach comes from Markov semigroup theory which was used in [18–21].

The rest of the paper is organized as follows: In Section 2, we present some auxiliary results concerning Markov semigroups. In Section 3, we formulate the main results and its proof. Finally, we illustrate some results through an example in Section 4.

2. Preliminaries

Throughout the paper we will need the following notations: $\mathbb{R}^3_+ := \{x = (x_1, x_2, x_3)' : x_i > 0, i = 1, 2, 3\}$; Id: the identity matrix; Prob$\{\cdot\}$: probability of an event; C: constant; L^p: p-th integral function space; C^∞: Infinitely differentiable; $I_{[a,b]}$: indicator function.

In this section, we provide some auxiliary definitions and results concerning Markov semigroups and asymptotic properties to prove our main results. Let $X = \mathbb{R}^3_+$, Σ be the σ-algebra of Borel subsets of X, and m be the Lebesgue measure on (X, Σ), the triple (X, Σ, m) be a σ-finite measure space. Denote by $D = D(X, \Sigma, m)$ the subset of the space $L^1 = L^1(X, \Sigma, m)$ which contains all *densities*, i.e.,

$$D = \{g \in L^1 : g \geq 0, \|g\| = 1\},$$

where $\|\cdot\|$ stands for the norm in L^1. A linear mapping $P : L^1 \to L^1$ is called a *Markov operator* if $P(D) \subset D$.

If $k : X \times X \to [0, \infty)$ is a measurable function such that

$$\int_X k(\mathbf{x}, \mathbf{y}) m(d\mathbf{x}) = 1 \tag{4}$$

for almost all $\mathbf{y} \in X$, then

$$Pg(\mathbf{x}) = \int_X k(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) m(d\mathbf{y}) \tag{5}$$

is an *integral* Markov operator. The function k is called a *kernel* of the Markov operator P.

If a family $\{P(t)\}_{t\geq 0}$ of Markov operators satisfies the following conditions:

(a) $P(0) = \mathrm{Id}$,

(b) $P(s + t) = P(s)P(t)$ for $s, t \geq 0$,

(c) for each $g \in L^1$ the function $t \mapsto P(t)g$ is continuous with respect to the L^1 norm, then $\{P(t)\}_{t\geq 0}$ is called a *Markov semigroup*.

A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called *integral*, if for each $t > 0$, the operator $P(t)$ is an integral Markov operator, that is, there exists a measurable function $k : (0, \infty) \times X \times X \to [0, \infty)$ such that

$$P(t)g(\mathbf{x}) = \int_X k(t, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) m(d\mathbf{y}) \tag{6}$$

for every density g.

Next we provide two definitions concerning the asymptotic behaviour of a Markov semigroup. A density g_* is called invariant under the Markov semigroup $\{P(t)\}_{t\geq 0}$ if $P(t)g_* = g_*$ for each $t > 0$. The Markov semigroup $\{P(t)\}_{t\geq 0}$ is called *asymptotically stable* if there is an invariant density g_* such that

$$\lim_{t\to\infty} \|P(t)g - g_*\| = 0 \tag{7}$$

for $g \in D$. If the semigroup $\{P(t)\}_{t\geq 0}$ is generated by some differential equations, then the asymptotic stability means that all solutions of the equation starting from a density converge to the invariant density.

A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called *sweeping* with respect to a set $A \in \Sigma$ if for every $g \in D$

$$\lim_{t\to\infty} \int_A P(t)g(\mathbf{x}) m(d\mathbf{x}) = 0. \tag{8}$$

We need some results concerning asymptotic stability and sweeping which can be found in [18, Corollary 1].

Lemma 2.1. *Let $\{P(t)\}_{t\geq 0}$ be an integral Markov semigroup with a continuous kernel $k(t, \mathbf{x}, \mathbf{y})$, $t > 0$, which satisfies* (4) *for all $\mathbf{y} \in X$. Assume that for every $g \in D$ we have*

$$\int_0^\infty P(t)g(\mathbf{x}) dt > 0.$$

Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.

The property that a Markov semigroup $\{P(t)\}_{t\geq 0}$ is *asymptotically stable* or *sweeping* from a sufficiently large family of sets (from all compact sets) is called the *Foguel alternative*.

For any $A \in \Sigma$, we denote the transition probability function by $\mathcal{P}(t, x, y, z, A)$ for the diffusion process $(u_1(t), u_2(t), u_3(t))$, i.e., $\mathcal{P}(t, x, y, z, A) = \text{Prob}\{(u_1(t), u_2(t), u_3(t)) \in A\}$ with the initial condition $(u_1(0), u_2(0), u_3(0)) = (x, y, z)$. Assume that $(u_1(t), u_2(t), u_3(t))$ is a solution of system (3) such that the distribution of $(u_1(0), u_2(0), u_3(0))$ is absolutely continuous and has the density $v(x, y, z)$. Then $(u_1(t), u_2(t), u_3(t))$ has also the density $U(t, x, y, z)$ and U satisfies the Fokker–Planck equation:

$$\begin{aligned}\frac{\partial U}{\partial t} = {} & \frac{1}{2}\sigma_1^2\frac{\partial^2 U}{\partial x^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 U}{\partial y^2} + \frac{1}{2}\sigma_3^2\frac{\partial^2 U}{\partial z^2} + \sigma_1\sigma_2\frac{\partial^2 U}{\partial x \partial y} + \sigma_2\sigma_3\frac{\partial^2 U}{\partial y \partial z} + \sigma_1\sigma_3\frac{\partial^2 U}{\partial x \partial z} \\ & - \frac{\partial(f_1 U)}{\partial x} - \frac{\partial(f_2 U)}{\partial y} - \frac{\partial(f_3 U)}{\partial z}\end{aligned} \tag{9}$$

where $f_1(x, y, z) = c_1 - b_{11}e^x - b_{12}e^y$, $f_2(x, y, z) = -c_2 + b_{21}e^x - b_{22}e^y - b_{23}e^z$, $f_3(x, y, z) = -c_3 + b_{32}e^y - b_{33}e^z$.

Next we introduce a Markov semigroup connected with (9). Let $P(t)V(x, y, z) = U(x, y, z)$ for $V \in D$. Since the operator $P(t)$ is a contraction on D, it can be extended to a contraction on L^1. Thus the operators $\{P(t)\}_{t\geq 0}$ form a Markov semigroup, i.e.,

$$\begin{aligned}\mathcal{A}V = {} & \frac{1}{2}\sigma_1^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial y^2} + \frac{1}{2}\sigma_3^2\frac{\partial^2 V}{\partial z^2} + \sigma_1\sigma_2\frac{\partial^2 V}{\partial x \partial y} + \sigma_2\sigma_3\frac{\partial^2 V}{\partial y \partial z} + \sigma_1\sigma_3\frac{\partial^2 V}{\partial x \partial z} \\ & - \frac{\partial(f_1 V)}{\partial x} - \frac{\partial(f_2 V)}{\partial y} - \frac{\partial(f_3 V)}{\partial z}.\end{aligned}$$

The adjoint operator of $\mathcal{A}$ is of the form

$$\begin{aligned}\mathcal{A}^*V = {} & \frac{1}{2}\sigma_1^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 V}{\partial y^2} + \frac{1}{2}\sigma_3^2\frac{\partial^2 V}{\partial z^2} + \sigma_1\sigma_2\frac{\partial^2 V}{\partial x \partial y} + \sigma_2\sigma_3\frac{\partial^2 V}{\partial y \partial z} + \sigma_1\sigma_3\frac{\partial^2 V}{\partial x \partial z} \\ & + f_1\frac{\partial(V)}{\partial x} + f_2\frac{\partial(V)}{\partial y} + f_3\frac{\partial(V)}{\partial z}.\end{aligned} \tag{10}$$

3. Main results

In this section we present the main conclusions of the paper.

Theorem 3.1. *Let $(u_1(t), u_2(t), u_3(t))$ be a solution of system (3), then for every $t > 0$ the distribution of $(u_1(t), u_2(t), u_3(t))$ has a density $U(t, x, y, z)$. If $\Xi > 0$, then there exist a unique density $U_*(x, y, z)$ which is a stationary solution of system (3) and*

$$\lim_{t\to\infty} \iiint_X |U(t, x, y, z) - U_*(x, y, z)|dxdydz = 0, \tag{11}$$

where

$$\Xi = b_{21}c_1 - b_{12}c_2 - \frac{b_{12}b_{23}c_3}{b_{32}} - \frac{\Delta_1^2}{4b_{11}b_{21}} - \frac{\Delta_2^2}{4b_{12}b_{21}} - \frac{b_{32}\Delta_3^2}{4b_{12}b_{23}},$$

$$\Delta_1 = b_{21}(c_1 + \frac{\sigma_1^2}{2}) + b_{11}b_{21} - b_{12}b_{21}, \quad \Delta_2 = b_{12}(-c_2 + \frac{\sigma_2^2}{2}) + b_{12}b_{21} + b_{12}b_{22} - b_{12}b_{23},$$

$$\Delta_3 = \frac{b_{12}b_{23}}{b_{32}}(-c_3 + \frac{\sigma_3^2}{2}) + b_{12}b_{23} + \frac{b_{12}b_{23}b_{33}}{b_{32}}.$$

Remark 1. Similar to the literature [18], the support of invariant density u_* depends on the coefficients $\sigma_1, \sigma_2, \sigma_3, b_{11}, b_{22}$. If $b_{12}\sigma_2 > b_{22}\sigma_1$ or $\sigma_2 < \sigma_1$, then $U_* > 0$, a.e. If $b_{12}\sigma_2 \leq b_{22}\sigma_1$, $\sigma_2 \geq \sigma_1$, and $\sigma_3 \leq \sigma_1$, then

$$\text{supp}U_* = E(M_1, M_2) = \{(x, y, z) : y \leq \frac{\sigma_2}{\sigma_1}x + M_1, z \leq \frac{\sigma_3}{\sigma_2}y + M_2\},$$

where M_1 is the smallest number such that $f_1\sigma_2 - f_2\sigma_1 \geq 0$ for all $(x, y, z) \notin E(M_1, M_2)$, M_2 is the smallest number such that $f_2\sigma_3 - f_3\sigma_2 \geq 0$ for all $(x, y, z) \notin E(M_1, M_2)$. By the *support* of a measurable function $f = (f_1, f_2, f_3)$ we simply mean the set

$$\text{supp}f = \{(x, y, z) \in X : f(x, y, z) \neq 0\}.$$

The proof of Theorem 3.1 is based on the theory of integral Markov semigroups. The strategy of the proof is the similar as the literatures [18,19].

Lemma 3.2. *For each point* $(x_0, y_0, z_0) \in X$ *and* $t > 0$, *the transition probability function* $\mathcal{P}(t, x_0, y_0, z_0, A)$ *has a continuous density* $k(t, x, y, z; x_0, y_0, z_0)$ *with respect to the Lebesgue measure.*

Proof. In the proof of this lemma, we use the Hörmander theorem [22] on the existence of smooth densities of the transition probability function for degenerate diffusion process. If $a(x)$ and $b(x)$ are vector fields on R^d, then the Lie bracket $[a, b]$ is a vector field given by

$$[a,b]_j(x) = \sum_{k=1}^{d}\left(a_k \frac{\partial b_j}{\partial x_k}(x) - b_k \frac{\partial a_j}{\partial x_k}(x)\right).$$

Let

$$a_0(\xi,\eta,\vartheta) = \begin{pmatrix} a_1 - \dfrac{\sigma_1^2}{2} - b_{11}e^{\xi} - b_{12}e^{\eta} \\ -a_2 - \dfrac{\sigma_2^2}{2} + b_{21}e^{\xi} - b_{22}e^{\eta} - b_{23}e^{\vartheta} \\ -a_3 - \dfrac{\sigma_3^2}{2} + b_{32}e^{\eta} - b_{33}e^{\vartheta} \end{pmatrix} \quad \text{and} \quad a_1(\xi,\eta,\vartheta) = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}.$$

Then, by direct calculating, the Lie bracket $[a_0, a_1]$ is a vector field given by

$$a_2(\xi,\eta,\vartheta) = [a_0,a_1] = \begin{pmatrix} b_{11}\sigma_1 e^{\xi} + b_{12}\sigma_2 e^{\eta} \\ -b_{21}\sigma_1 e^{\xi} + b_{22}\sigma_2 e^{\eta} + b_{23}\sigma_3 e^{\vartheta} \\ -b_{32}\sigma_2 e^{\eta} + b_{33}\sigma_3 e^{\vartheta} \end{pmatrix}$$

and

$$a_3(\xi,\eta,\vartheta) = [a_1,a_2] = \begin{pmatrix} b_{11}\sigma_1^2 e^{\xi} + b_{12}\sigma_2^2 e^{\eta} \\ -b_{21}\sigma_1^2 e^{\xi} + b_{22}\sigma_2^2 e^{\eta} + b_{23}\sigma_3^2 e^{\vartheta} \\ -b_{32}\sigma_2^2 e^{\eta} + b_{33}\sigma_3^2 e^{\vartheta} \end{pmatrix}.$$

Consequently, a_1, a_2, a_3 are linearly independent on X. Thus for every $(u_1, u_2, u_3) \in X$, vector a_1, a_2, a_3 span the space X. In view of Hörmander Theorem [22], this is, the Lie algebra generated by vector fields $\{a_1, a_2, a_3\}$ is 3 dimensional on X, the transition probability function $\mathcal{P}(t, x_0, y_0, z_0, A)$ has a continuous density $k(t, x, y, z; x_0, y_0, z_0)$, $k \in C^{\infty}((0,\infty) \times X \times X)$. □

Next we briefly describe the method based on support theorems [23] which allows us to check where the kernel k is positive. Fixing a point $(x_0, y_0, z_0) \in X$ and a function $\phi \in L^2([0, T]; R)$, consider the following system of integral equations:

$$\begin{aligned} x_\phi(t) &= x_0 + \int_0^t (\sigma_1\phi + f_1(x_\phi(s), y_\phi(s), z_\phi(s)))ds, \\ y_\phi(t) &= y_0 + \int_0^t (\sigma_2\phi + f_2(x_\phi(s), y_\phi(s), z_\phi(s)))ds, \\ z_\phi(t) &= z_0 + \int_0^t (\sigma_3\phi + f_3(x_\phi(s), y_\phi(s), z_\phi(s)))ds, \end{aligned} \tag{12}$$

where $f_1(x, y, z)$, $f_2(x, y, z)$, $f_3(x, y, z)$ are defined as (9).

Let $D_{x_0,y_0,z_0;\phi}$ be the Fréchet derivative of the function $h \to \mathbf{x}_{\phi+h}(T)$ from $L^2([0, T]; R)$ to X, where $\mathbf{x}_{\phi+h} = (x_{\phi+h}, y_{\phi+h}, z_{\phi+h})^T$. For some $\phi \in L^2([0, T]; R)$, the derivative $D_{x_0,y_0,z_0;\phi}$ has rank 3, then

$$k(T, x, y, z; x_0, y_0, z_0) > 0 \quad \text{for} \quad x = x_\phi(T), y = y_\phi(T), z = z_\phi(T).$$

Let $\psi(t) = \mathbf{f}'(x_\phi(t), y_\phi(t), z_\phi(t))$, where $\mathbf{f}'$ is the Jacobian of $\mathbf{f} = (f_1, f_2, f_3)^T$. And let $Q(t, t_0)$, $0 \le t_0 \le t \le T$, be a matrix function such that $Q(t_0, t_0) = \mathrm{Id}$ and $\frac{\partial Q(t,t_0)}{\partial t} = \psi(t)Q(t, t_0)$ and $\mathbf{v} = (\sigma_1, \sigma_2, \sigma_3)^T$. Then

$$D_{x_0,y_0,z_0;\phi} h = \int_0^T Q(T, s)\mathbf{v}h(s)ds. \tag{13}$$

Lemma 3.3. *Let $E = X$ when $b_{12}\sigma_2 > b_{22}\sigma_1$ or $\sigma_2 < \sigma_1, \sigma_3 < \sigma_1$, and $E = E(M_1, M_2)$ when $b_{12}\sigma_2 \le b_{22}\sigma_1$ and $\sigma_3 \le \sigma_1 \le \sigma_2$, for each $(x_0, y_0, z_0) \in X$ and $(x, y, z) \in X$, there exists $T > 0$ such that $k(T, x, y, z; x_0, y_0, z_0) > 0$.*

Proof. We check that the rank of $D_{x_0,y_0,z_0;\phi}$ is 3. Let $\varepsilon \in (0, T)$ and $h = \mathbf{1}_{[T-\varepsilon,T]}$. Since

$$Q(T, s) = \mathrm{Id} + \psi(T)(s - T) + \frac{1}{2}\psi^2(T)(s - T)^2 + o((s - T)^2),$$

we obtain

$$D_{x_0,y_0,z_0;\phi} = \varepsilon\mathbf{v} - \frac{1}{2}\varepsilon^2\psi(T)\mathbf{v} + \frac{1}{6}\varepsilon^3\psi^2(T)\mathbf{v} + o(\varepsilon)^3, \quad \mathbf{v} = (\sigma_1, \sigma_2, \sigma_3)^T.$$

Compute

$$\psi(T)\mathbf{v} = \begin{pmatrix} -b_{11}e^x & -b_{12}e^y & 0 \\ b_{21}e^x & -b_{22}e^y & -b_{23}e^z \\ 0 & b_{32}e^y & -b_{33}e^z \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} -b_{11}\sigma_1 e^x - b_{12}\sigma_2 e^y \\ b_{21}\sigma_1 e^x - b_{22}\sigma_2 e^y - b_{23}\sigma_3 e^z \\ b_{32}\sigma_2 e^y - b_{33}\sigma_3 e^z \end{pmatrix},$$

$$\begin{aligned} \psi^2(T)\mathbf{v} &= \begin{pmatrix} -b_{11}\sigma_1 e^x & -b_{12}\sigma_2 e^y & 0 \\ b_{21}\sigma_1 e^x & -b_{22}\sigma_2 e^y & -b_{23}\sigma_3 e^z \\ 0 & b_{32}\sigma_2 e^y & -b_{33}\sigma_3 e^z \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \\ &= \begin{pmatrix} -b_{11}\sigma_1^2 e^x - b_{12}\sigma_2^2 e^y \\ b_{21}\sigma_1^2 e^x - b_{22}\sigma_2^2 e^y - b_{23}\sigma_3^2 e^z \\ b_{32}\sigma_2^2 e^y - b_{33}\sigma_3^2 e^z \end{pmatrix}. \end{aligned}$$

Therefore it follows that $\mathbf{v}$, $\psi(T)\mathbf{v}$, $\psi^2(T)\mathbf{v}$ are linearly independent and derivative $D_{x_0,y_0,z_0;\phi}$ has rank 3 if $\sigma_i (i = 1, 2, 3)$ is not all equal.

Next, we prove that for any two points $(x_0, y_0, z_0) \in X$ and $(x, y, z) \in X$, there exists a control function ϕ and $T > 0$ such that $x_\phi(0) = x_0$, $y_\phi(0) = y_0$, $z_\phi(0) = z_0$, $x_\phi(T) = x$, $y_\phi(T) = y$, $z_\phi(T) = z$. The system (12) can be replaced by the following differential equations:

$$\begin{aligned} x'_\phi(t) &= \sigma_1\phi + f_1(x_\phi, y_\phi, z_\phi), \\ y'_\phi(t) &= \sigma_2\phi + f_2(x_\phi, y_\phi, z_\phi), \\ z'_\phi(t) &= \sigma_3\phi + f_3(x_\phi, y_\phi, z_\phi), \end{aligned} \tag{14}$$

where $f_1(x, y, z) = c_1 - b_{11}e^x - b_{12}e^y$, $f_2(x, y, z) = -c_2 + b_{21}e^x - b_{22}e^y - b_{23}e^z$, $f_3(x, y, z) = -c_3 + b_{32}e^y - b_{33}e^z$.

Let $\omega_\phi = y_\phi - \frac{\sigma_2}{\sigma_1}x_\phi$, $\nu_\phi = z_\phi - \frac{\sigma_3}{\sigma_1}x_\phi$, then (14) become

$$\begin{aligned} x'_\phi(t) &= \sigma_1\phi + g_1(x_\phi, \omega_\phi, \nu_\phi), \\ \omega'_\phi(t) &= g_2(x_\phi, \omega_\phi, \nu_\phi), \\ \nu'_\phi(t) &= g_3(x_\phi, \omega_\phi, \nu_\phi), \end{aligned} \tag{15}$$

where

$$\begin{aligned} g_1(x_\phi, \omega_\phi, \nu_\phi) &= c_1 - b_{11}e^x - b_{12}e^{\frac{\sigma_2}{\sigma_1}x}e^\omega, \\ g_2(x_\phi, \omega_\phi, \nu_\phi) &= (\frac{\sigma_2 b_{11}}{\sigma_1} + b_{21})e^x + (\frac{\sigma_2 b_{12}}{\sigma_1} - b_{22})e^{\frac{\sigma_2}{\sigma_1}x}e^\omega - b_{23}e^{\frac{\sigma_3}{\sigma_1}x}e^\nu - (\frac{\sigma_2 c_1}{\sigma_1} + c_2), \\ g_3(x_\phi, \omega_\phi, \nu_\phi) &= \frac{\sigma_3 b_{11}}{\sigma_1}e^x + (\frac{\sigma_3 b_{12}}{\sigma_1} + b_{32})e^{\frac{\sigma_2}{\sigma_1}x}e^\omega - b_{33}e^{\frac{\sigma_3}{\sigma_1}x}e^\nu - (\frac{\sigma_3 c_1}{\sigma_1} + c_3). \end{aligned}$$

For the convenience, let $r_1 = \frac{\sigma_2}{\sigma_1}, \alpha_1 = \frac{\sigma_2 b_{11}}{\sigma_1} + b_{21} > 0, \alpha_2 = \frac{\sigma_2 b_{12}}{\sigma_1} - b_{22}, \alpha_3 = \frac{\sigma_2 c_1}{\sigma_1} + c_2 > 0, r_2 = \frac{\sigma_3}{\sigma_1}, \beta_1 = \frac{\sigma_3 b_{11}}{\sigma_1} > 0, \beta_2 = \frac{\sigma_3 b_{12}}{\sigma_1} + b_{32} > 0, \beta_3 = \frac{\sigma_3 c_1}{\sigma_1} + c_3 > 0$, we obtain

$$g_1(x_\phi, \omega_\phi, \nu_\phi) = c_1 - b_{11}e^x - b_{12}e^{r_1 x}e^\omega,$$
$$g_2(x_\phi, \omega_\phi, \nu_\phi) = \alpha_1 e^x + \alpha_2 e^{r_1 x}e^\omega - b_{23}e^{r_2 x}e^\nu - \alpha_3,$$
$$g_3(x_\phi, \omega_\phi, \nu_\phi) = \beta_1 e^x + \beta_2 e^{r_1 x}e^\omega - b_{33}e^{r_2 x}e^\nu - \beta_3.$$

We take five steps to complete the rest of the proof:

(1) For each fixed ν, fix $\omega_0, \omega_1 \in R$ and $\omega_1 < \omega_0$, there exists $x_0 \in R$ such that

$$g_2(x_0, \omega, \nu) \le -\frac{\alpha_3}{2}, \tag{16}$$

or

$$g_3(x_0, \omega, \nu) \le -\frac{\beta_3}{2}, \tag{17}$$

for $\omega \in [\omega_1, \omega_0]$. Consider (15) with $x_\phi \equiv x_0$ and $\omega_\phi(0) = \omega_0$, from the second of (15) and (16) or (17) it follows that there exist function ϕ, ω_ϕ satisfying system (15) and $\omega'_\phi \le -\frac{\alpha_3}{2}, \nu'_\phi \le -\frac{\beta_3}{2}$. Therefore, we find ϕ and $T > 0$ such that $\omega_\phi(T) = \omega_1$. On the other hand, for each fixed ω, fix $\nu_0, \nu_1 \in R$ and $\nu_1 < \nu_0$, then there exists $x_0 \in R$ such that $g_2(x_0, \omega, \nu) \le -\frac{\alpha_3}{2}$ or $g_3(x_0, \omega, \nu) \le -\frac{\beta_3}{2}$. Analogously, Consider (15) with $x_\phi \equiv x_0$ and $\nu_\phi(0) = \nu_0$, we also find ϕ and $T > 0$ such that $\nu_\phi(T) = \nu_1$.

(2) Now we assume that $\alpha_2 \ge 0, \beta_2 \ge 0$ or $0 < r_1 < 1, 0 < r_2 < 1$. For each fixed ν, every $\omega_0, \omega_1 \in R$ and $\omega_0 < \omega_1$ there exists $x_0 \in R$ such that $g_2(x_0, \omega, \nu) \ge 1$ or $g_3(x_0, \omega, \nu) \ge 1$ for $\omega \in [\omega_0, \omega_1]$. Then we find a control function ϕ such that $x_\phi \equiv x_0, \omega_\phi(0) = \omega_0, \omega_\phi(T) = \omega_1$ for some $T > 0$. On the other hand, for any ω, fix $\nu_0, \nu_1 \in R$ and $\nu_1 < \nu_0$, there exists x_0 such that $g_2(x_0, \omega, \nu) \ge 1$ or $g_3(x_0, \omega, \nu) \ge 1$ for $\nu \in [\nu_0, \nu_1]$. Then we also find a control function ϕ such that $x_\phi \equiv x_0, \nu_\phi(0) = \nu_0, \nu_\phi(T) = \nu_1$ for some $T > 0$.

(3) Consider the case $\alpha_2 \le 0, \beta_2 \ge 0$ and $r_1 \ge 1, 0 < r_2 < 1$. For each fixed ν, every $\varepsilon > 0$, there exists a $\delta_1 > 0$ having the following property, if $\omega_1 - \delta \le \omega_0 < \omega_1 \le M_1 - \varepsilon$ then there exists x_0 such that $g_2(x_0, \omega, \nu) \ge \delta_1$ for $\omega \in [\omega_0, \omega_1]$. For every $\omega_0, \omega_1 \in R$ and $\omega_0 < \omega_1$ there exists $x_0 \in R$ such that $g_3(x_0, \omega, \nu) \ge 1$ for $\omega \in [\omega_0, \omega_1]$. Then we find a control function ϕ such that $x_\phi \equiv x_0, \omega_\phi(0) = \omega_0, \omega_\phi(T) = \omega_1$ for some $T > 0$. On the other hand, for each fixed ω, every $\varepsilon > 0$, there exists a $\delta_2 > 0$ having the following property. if $\nu_1 - \delta \le \nu_0 < \nu_1 \le M_2 - \varepsilon$ then there exists x_0 such that $g_2(x_0, \omega, \nu) \ge \delta$ for $\nu \in [\nu_0, \nu_1]$. For every $\nu_0, \nu_1 \in R$ and $\nu_0 < \nu_1$, there exists $x_0 \in R$ such that $g_3(x_0, \omega, \nu) \ge 1$ for $\nu \in [\nu_0, \nu_1]$. Then we find a control function ϕ such that $x_\phi \equiv x_0, \nu_\phi(0) = \nu_0, \nu_\phi(T) = \nu_1$ for some $T > 0$.

(4) Fix $x_0 \in R, L > 0, A_0, A_1 > A_0$ and $\varepsilon > 0$ such that $\varepsilon < \frac{L}{4}$ and $\varepsilon < \frac{A_1 - A_0}{4}$, Let

$$M = \max\left\{|g_1(x, \omega, \nu)| + |g_2(x, \omega, \nu)| + |g_3(x, \omega, \nu)| : x \in [x_0, x_0 + L], \omega \in [A_0, A_1], \nu \in [A_0, A_1]\right\},$$

and $t_0 \in \varepsilon m^{-1}, \phi \equiv \frac{3ML}{4\sigma\varepsilon}$. Then for every $\nu_0 \in [A_0 + \varepsilon, A_1 - \varepsilon]$, the solution of system (14) with $x_\phi(0) = x_0, \omega_\phi(0) = \omega_0, \nu_\phi(0) = \nu_0$ has the following properties:

$$\begin{aligned} &x_\phi(t_0) \in (x_0 + L/2, x_0 + L), \\ &\omega_\phi(t) \in [\omega_0 - \varepsilon, \omega_0 + \varepsilon], \quad \text{for } t \le t_0, \\ &\nu_\phi(t) \in [\nu_0 - \varepsilon, \nu_0 + \varepsilon], \quad \text{for } t \le t_0, \end{aligned} \tag{18}$$

From (18) it follows that for $(x_1, \omega_1, \nu_1) \in (x_0, x_0 + L/2] \times [A_0 + 2\varepsilon, A_0 - 2\varepsilon] \times [A_0 + 2\varepsilon, A_0 - 2\varepsilon]$ there exists $\omega_0 \in [\omega_1 - \varepsilon, \omega_1 + \varepsilon], \nu_0 \in [\nu_1 - \varepsilon, \nu_1 + \varepsilon]$ and $T \in (0, t_0)$ such that $x_\phi(T) = x_1, \omega_\phi(T) = \omega_1, \nu_\phi(T) = \nu_1$. The same proof works for $x_1 \in (x_0 - L/2, x_0]$.

(5) Let $E = X$ when $\alpha_2 \ge 0, \beta_2 \ge 0$ or $0 < r_1 < 1, 0 < r_2 < 1$, and $E = E(M_1, M_2)$ when $\alpha_2 \le 0, \beta_2 \ge 0$ and $r_1 \ge 1, 0 < r_2 < 1$. Then from (1)–(4) it follows that for any two points $(x_0, y_0, z_0) \in E$ and $(x, y, z) \in E$ there exist a control function ϕ and $T > 0$ such that $x_\phi(0) = x_0, y_\phi(0) = y_0, z_\phi(0) = z_0, x_\phi(T) = x, y_\phi(T) = y, z_\phi(T) = z$. This completes the proof. □

Remark 2. Lemma 3.3 shows that the density of the transition function is positive on X.

Lemma 3.4. *Assume that $b_{12}\sigma_2 \le b_{22}\sigma_1, \sigma_3 < \sigma_1 < \sigma_2$, and let $E = E(M_1, M_2)$. Then for every density f we have*

$$\lim_{t\to\infty} \iiint_E P(t)f(x, y, z)dxdydz = 1, \tag{19}$$

Proof. Similar to the proof Lemma 3.3, let $\bar{u}_2(t) = u_2 - \frac{\sigma_2}{\sigma_1}u_1, \bar{u}_3(t) = u_3 - \frac{\sigma_3}{\sigma_1}u_1$, then system (3) can be replaced by

$$\begin{cases} du_1(t) = \sigma_1 dB(t) + g_1(u_1, \bar{u}_2, \bar{u}_3)dt, \\ d\bar{u}_2(t) = g_2(u_1, \bar{u}_2, \bar{u}_3)dt, \\ d\bar{u}_3(t) = g_2(u_1, \bar{u}_2, \bar{u}_3)dt, \end{cases} \tag{20}$$

where the functions g_1, g_2 and g_3 are defined in (15). Since for each $\varepsilon_1, \varepsilon_2 > 0$ we have

$$\sup\{g_2(x, y, z) : y \ge M_1 + \varepsilon_1, z \ge M_2 + \varepsilon_1, x \in R\} < 0, \tag{21}$$

$$\sup\{g_3(x, y, z) : y \ge M_1 + \varepsilon_2, z \ge M_2 + \varepsilon_2, x \in R\} < 0, \tag{22}$$

we obtain $\limsup_{t\to\infty}\bar{u}_2(t) \le M_1, \limsup_{t\to\infty}\bar{u}_3(t) \le M_2$. We check that for almost every w there exists $t_0 = t_0(w)$ such that $\bar{u}_2(w) < M_1, \bar{u}_3(w) < M_2$ for $t \ge t_0$. If $\sigma_3 < \sigma_1 < \sigma_2$ then there exists $C_1, C_2 \in R$ such that $g_2(C_1, M_1, M_2) = 0, g_3(C_2, M_1, M_2) = 0$. Let $C_0 = \max\{C_1, C_2\}$ and fix $\kappa > 0, \tau > 0$ and $\iota > 0$. Consider the solution of system (20) such that $u_1(0) = C_0 + 2\kappa, \bar{u}_2(0) = M_1 + \tau, \bar{u}_3(0) = M_2 + \iota$. Let

$$A_{\kappa,\tau,\iota} = [C_0, C_0 + \kappa] \times [M_1, M_1 + \tau] \times [M_2, M_2 + \iota],$$

$$B_{\kappa,\tau,\iota} = [C_0, C_0 + 2\kappa] \times [M_1, M_1 + \tau] \times [M_2, M_2 + \iota],$$

Then there exist $\varepsilon > 0, L > 0$ such that $g_2(x, y, z) < -\varepsilon, g_3(x, y, z) < -\varepsilon$ for $x \ge C_0 + \kappa, (y, z) \in [M_1, M_1 + \tau] \times [M_2, M_2 + \iota]$, and $|g_1(x, y, z)| \le L$ for $(x, y, z) \in B_{\kappa,\tau,\iota}$.

Let $\bar{u}_1(t)$ be a solution of the equation $d\bar{u}_1(t) = \sigma dB(t) - Ldt$ with the initial condition $\bar{u}_1(0) = C_0 + 2\kappa$. Then $\bar{u}_1(t) \le u_1(t)$ and $\bar{u}_2(t) < M_1 + \tau - \varepsilon t, \bar{u}_3(t) < M_2 + \tau - \varepsilon t$ as long as $(u_1(t), \bar{u}_2(t), \bar{u}_3(t)) \in B_{\kappa,\tau,\iota} \setminus A_{\kappa,\tau,\iota}$. Let $t = \tau/\varepsilon$ and $\Omega_{\tau,\iota} = \{w : \bar{u}_1(s, w) \ge C_0 + \kappa, \text{ for } s \le t\}$, then $\lim_{\tau,\iota\to 0}\text{Prob}(\Omega_{\tau,\iota}) = 1$ and $\bar{u}_2(t, w) < 0, \bar{u}_3(t, w) < 0$ for $w \in \Omega_{\tau,\iota}$.

Now let $u_1(t), \bar{u}_2(t), \bar{u}_3(t)$ be any solution of system (20). Then from what has already been proved and the Markov property it follows that if $\inf_{t>0}\bar{u}_2(t, w) \ge M_1, \inf_{t>0}\bar{u}_3(t, w) \ge M_2$, then $\limsup_{t\to\infty}u_1(t, w) \le C_0$. Analogously, we check that if $\inf_{t>0}\bar{u}_2(t, w) \ge M_1, \inf_{t>0}\bar{u}_3(t, w) \ge M_2$, then $\liminf_{t\to\infty}u_1(t, w) \ge C_0$. Thus, $\inf_{t>0}\bar{u}_2(t, w) \ge M_1, \inf_{t>0}\bar{u}_3(t, w) \ge M_2$, then $\lim_{t\to\infty}u_1(t, w) = C_0$.

Assume that $\lim_{t\to\infty}u_1(t, w) = C_0$ with probability $> p_0 > 0$. Set $\gamma = g_1(C_0, M_1, M_2)$, then for every $\varepsilon > 0$ there exist $t_0 > 0$ and a set Ω' such that $\text{Prob}(\Omega') > p_0, |u_1(t, w) - C_0| < \varepsilon$ and

$$\sigma dB(t) + (\gamma - \varepsilon)dt \le du_1(t) \le \sigma dB(t) + (\gamma + \varepsilon)dt \tag{23}$$

for $w \in \Omega'$ and $t \ge t_0$. Then $\text{Prob}\{w \in \Omega' : |u_1(t_0 + 1) - C_0| < \varepsilon\} \le o(\varepsilon)$ which contradicts assumption that $p_0 > 0$. Consequently, for almost every w there exists $t_0 = t_0(w)$ such that $\bar{u}_2(w) < M_1, \bar{u}_3(w) < M_2$ for $t \ge t_0$ and (19) holds. □

Remark 3. From Lemmas 3.3 and 3.4, we know if the Fokker–Planck (9) has a stationary solution U_*, then $\text{supp}U_* = E(M_1, M_2)$.

Lemma 3.5. *The semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable or is sweeping with respect to compact sets.*

Proof. From Lemma 3.2 it follows that $\{P(t)\}_{t\ge 0}$ is an integral Markov semigroup with a continuous kernel $k(t, x, y, z)$ for $t > 0$. Let $E = X$ when $b_{12}\sigma_2 \ge b_{22}\sigma_1$ or $\sigma_2 < \sigma_1, \sigma_3 < \sigma_1$, and $E = clE(M_1, M_2)$ when $b_{12}\sigma_2 < b_{22}\sigma_1$ and $\sigma_2 < \sigma_1 < \sigma_2$. Then according to Lemma 3.3, for every $f \in D$ we have

$$\int_0^\infty P(t)f dt > 0 \quad a.e. \quad \text{on } E. \tag{24}$$

So from Lemma 2.1 it follows immediately that the semigroup $\{P(t)_{t\ge 0}\}$ is asymptotically stable or is sweeping with respect to compact sets, the desired result follows. □

Lemma 3.6. *If $\Xi > 0$, then the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable, where*

$$\Xi = b_{21}c_1 - b_{12}c_2 - \frac{b_{12}b_{23}c_3}{b_{32}} - \frac{\Delta_1^2}{4b_{11}b_{21}} - \frac{\Delta_2^2}{4b_{12}b_{21}} - \frac{b_{32}\Delta_3^2}{4b_{12}b_{23}},$$
$$\Delta_1 = b_{21}(c_1 + \frac{\sigma_1^2}{2}) + b_{11}b_{21} - b_{12}b_{21}, \quad \Delta_2 = b_{12}(-c_2 + \frac{\sigma_2^2}{2}) + b_{12}b_{21} + b_{12}b_{22} - b_{12}b_{23},$$
$$\Delta_3 = \frac{b_{12}b_{23}}{b_{32}}(-c_3 + \frac{\sigma_3^2}{2}) + b_{12}b_{23} + \frac{b_{12}b_{23}b_{33}}{b_{32}}.$$

Proof. According to Lemma 3.5, the semigroup $\{P(t)\}_{t\geq 0}$ satisfies the Foguel alternative. In order to exclude sweeping it is sufficient to construct a non-negative C^2-function V and a closed set $O \in \Sigma$ such that

$$\sup_{(u_1,u_2,u_3)\in X\setminus O} \mathcal{A}^*V < 0. \tag{25}$$

Such a function is called a Khasminskiǐ function. By using similar arguments to those in [24], the existence of a Khasminskiǐ function implies that the semigroup is not sweeping from the set O, which completes the proof. Let

$$V(u_1, u_2, u_3) = b_{21}(e^{u_1} - 1 - u_1) + b_{12}(e^{u_2} - 1 - u_2) + \frac{b_{12}b_{23}}{b_{32}}(e^{u_3} - 1 - u_3). \tag{26}$$

Then

$$\begin{aligned}
\mathcal{A}^*V &= b_{21}(e^{u_1} - 1)(c_1 - b_{11}e^{u_1} - b_{12}e^{u_2}) + \frac{1}{2}\sigma_1^2 b_{21}e^{u_1} \\
&\quad + b_{12}(e^{u_2} - 1)(-c_2 + b_{21}e^{u_1} - b_{22}e^{u_2} - b_{23}e^{u_3}) + \frac{1}{2}\sigma_2^2 b_{12}e^{u_2} \\
&\quad + \frac{b_{12}b_{23}}{b_{32}}(e^{u_3} - 1)(-c_3 + b_{32}e^{u_2} - b_{33}e^{u_3}) + \frac{\sigma_3^2 b_{12}b_{23}}{2b_{32}} \\
&= -b_{11}b_{21}e^{2u_1} - b_{12}b_{21}e^{2u_2} - \frac{b_{12}b_{23}}{b_{32}}e^{2u_3}, \\
&\quad + b_{21}(c_1 + \frac{\sigma_1^2}{2})e^{u_1} + b_{12}(-c_2 + \frac{\sigma_2^2}{2})e^{u_2} + \frac{b_{12}b_{23}}{b_{32}}(-c_3 + \frac{\sigma_3^2}{2})e^{u_3} \\
&\quad - b_{21}c_1 + b_{12}c_2 + \frac{b_{12}b_{23}}{b_{32}}c_3 + (b_{11}b_{21} - b_{12}b_{21})e^{u_1} \\
&\quad + (b_{12}b_{21} + b_{12}b_{22} - b_{12}b_{23})e^{u_2} + (b_{12}b_{23} + \frac{b_{12}b_{23}b_{33}}{b_{32}})e^{u_3} \\
&= -b_{11}b_{21}e^{2u_1} - b_{12}b_{21}e^{2u_2} - \frac{b_{12}b_{23}}{b_{32}}e^{2u_3}, \\
&\quad - b_{21}c_1 + b_{12}c_2 + \frac{b_{12}b_{23}}{b_{32}}c_3 + \Delta_1 e^{u_1} + \Delta_2 e^{u_2} + \Delta_3 e^{u_3}. \\
&= -b_{11}b_{21}(e^{u_1} - \frac{\Delta_1}{2b_{11}b_{21}})^2 - b_{12}b_{21}(e^{u_2} - \frac{\Delta_2}{2b_{12}b_{21}})^2 - \frac{b_{12}b_{23}}{b_{32}}(e^{u_3} - \frac{b_{32}\Delta_3}{2b_{12}b_{23}})^2 - \Xi,
\end{aligned}$$

where $\Xi = b_{21}c_1 - b_{12}c_2 - \frac{b_{12}b_{23}c_3}{b_{32}} - \frac{\Delta_1^2}{4b_{11}b_{21}} - \frac{\Delta_2^2}{4b_{12}b_{21}} - \frac{b_{32}\Delta_3^2}{4b_{12}b_{23}}$.

Condition $\Xi > 0$ implies that there exists a closed set $O \in \Sigma$ such that

$$\sup_{(u_1,u_2,u_3)\in X\setminus O} \mathcal{A}^*V \leq -\Xi < 0. \quad \square$$

So far, Theorem 3.1 has been proved completely. Next several properties of population extinction are given.

Lemma 3.7 (*See [15,11]*)**.** *Let $f \in C([0,\infty)\times\Omega,(0,\infty))$ and $F \in C([0,\infty)\times\Omega, R)$. If there exist positive constants λ_0, λ such that*

$$\log f(t) \le \lambda t - \lambda_0 \int_0^t f(s)ds + F(t), t \ge 0 a.s.,$$

and $\lim_{t\to\infty}(F(t)/t) = 0 a.s.$, then

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t f(s)ds \le \frac{\lambda}{\lambda_0}, a.s.$$

Theorem 3.8. *If $c_1 > 0$, $b_{11}c_2 < b_{21}c_1$ and $b_{11}b_{22}c_3 + b_{11}b_{32}c_2 > b_{21}b_{32}c_1$ then $\lim_{t\to\infty}u_3(t) = -\infty$, $a.e.$ and there exists a unique density $\overline{U}_*(x,y)$ which is a stationary solution of the first two equations of system* (3) *and*

$$\lim_{t\to\infty} \iint_{R_+^2} |\overline{U}(t,x,y) - \overline{U}_*(x,y)|dxdy = 0.$$

Proof. Note that

$$du_1(t) \le (c_1 - b_{11}e^{u_1})dt + \sigma_1 dB(t),$$

then by the comparison principle [17], we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t e^{u_1(s)}ds \le \frac{c_1}{b_{11}}, \quad a.s. \tag{27}$$

Integrating the both sides of the second equation of system (3), we can obtain

$$\begin{aligned}\frac{u_2(t)-u_2(0)}{t} &= -c_2 + \frac{b_{21}}{t}\int_0^t e^{u_1(s)}ds - \frac{b_{22}}{t}\int_0^t e^{u_2(s)}ds - \frac{b_{23}}{t}\int_0^t e^{u_3(s)}ds + \frac{\sigma_2 B(t)}{t}\\ &\le -c_2 + \frac{b_{21}c_1}{b_{11}} - \frac{b_{22}}{t}\int_0^t e^{u_2(s)}ds + \frac{\sigma_2 B(t)}{t}.\end{aligned}$$

That is

$$u_2(t) \le (-c_2 + \frac{b_{21}c_1}{b_{11}})t - b_{22}\int_0^t e^{u_2(s)}ds + \sigma_2 B(t) + u_2(0).$$

By Lemma 3.7, we derive directly

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t e^{u_2(s)}ds \le \frac{b_{21}c_1 - b_{11}c_2}{b_{11}b_{22}}, \quad a.s. \tag{28}$$

By integrating the both sides of the third equation of the system (3), we can obtain

$$\begin{aligned}\frac{u_3(t)-u_3(0)}{t} &= -c_3 + \frac{b_{32}}{t}\int_0^t e^{u_2(s)}ds - \frac{b_{33}}{t}\int_0^t e^{u_3(s)}ds + \frac{\sigma_3 B(t)}{t}\\ &\le -c_3 + \frac{b_{32}}{t}\int_0^t e^{u_2(s)}ds + \frac{\sigma_3 B(t)}{t}.\end{aligned}$$

Combining (28) and $\lim_{t\to\infty}\frac{B(t)}{t} = 0$, we have

$$\limsup_{t\to\infty} \frac{u_3(t)}{t} \le \frac{b_{32}(b_{21}c_1 - b_{11}c_2)}{b_{11}b_{22}} < 0, \quad a.s. \tag{29}$$

Therefore

$$\lim_{t\to\infty} u_3(t) = -\infty, \quad a.s.$$

When $\lim_{t\to\infty}u_3(t) = -\infty$, system (3) becomes

$$\begin{cases} du_1(t) = \big(a_1 - \frac{\sigma_1^2}{2} - b_{11}e^{u_1} - b_{12}e^{u_2}\big)dt + \sigma_1 dB(t),\\ du_2(t) = \big(-a_2 - \frac{\sigma_2^2}{2} + b_{21}e^{u_1} - b_{22}e^{u_2}\big)dt + \sigma_2 dB(t).\end{cases} \tag{30}$$

For system (30), there exists a unique density $\overline{U}_*(x, y)$ which is a stationary solution of system (30). Details are shown in [18]. The result is confirmed. □

In the following proof we use the property of the solutions of a one-dimensional stochastic equation [26, p. 162]. Consider the following stochastic equation:

$$dx_t = \sigma(x_t)dB_t + b(x_t)dt.$$

Let

$$s(x) = \int_0^x \exp\{-\int_0^y \frac{2b(x_t)}{\sigma^2(r)}dr\}dy.$$

If $s(-\infty) > -\infty$ and $s(\infty) = \infty$ then $\lim_{t\to\infty} x_t = -\infty$.

Theorem 3.9. *If $c_1 > 0$ and $b_{11}c_2 > b_{21}c_1$ then $\lim_{t\to\infty} u_i(t) = -\infty$, $a.e.$ $i = 2, 3$. and the distribution of the process $u_1(t)$ converges weakly to the measure which has the density $f_*(x) = Cexp\{2c_1x/\sigma_1^2 - (2b_{11}/\sigma_1^2)e^x\}$.*

Proof. By Lemma 7 of the literature [18], we know that the distribution of the process $u_1(t)$ converges to the measure with the density f_* and $\lim_{t\to\infty} u_2(t) = -\infty$. Since $\lim_{t\to\infty} u_2(t) = -\infty$ we have

$$du_3(t) \le \sigma_3 dB(t) - c_3 dt. \tag{31}$$

Since $c_3 = a_3 + \sigma_3^2/2 > 0$, from (31) it follows that $\lim_{t\to\infty} u_3(t) = -\infty$. □

Theorem 3.10. *If $c_1 < 0$ then $\lim_{t\to\infty} u_i(t) = -\infty$, $a.e.$ $i = 1, 2, 3$.*

Proof. Similar to lemma 6 in [18], it is omitted in this paper. □

4. Numerical simulations

In this section we will introduce one example and some figures to illustrate our main theorems. For numerical simulations of the system (3), we use the Milstein method mentioned in [25]. In this way, system (3) can be rewritten as the following discretized equations:

$$\begin{cases} x_{1,k+1} = x_{1,k} + x_{1,k}\big(a_1 - b_{11}x_{1,k} - b_{12}x_{2,k}\big)\Delta t + \sigma_1 x_{1,k}\sqrt{\Delta t}\xi_k + \dfrac{1}{2}\sigma_1^2 x_{1,k}(\xi_k^2 - 1)\Delta t, \\ x_{2,k+1} = x_{2,k} + x_{2,k}\big(-a_2 + b_{21}x_{1,k} - b_{22}x_{2,k} - b_{23}x_{3,k}\big)\Delta t + \sigma_2 x_{2,k}\sqrt{\Delta t}\xi_k + \dfrac{1}{2}\sigma_2^2 x_{2,k}(\xi_k^2 - 1)\Delta t, \\ x_{3,k+1} = x_{3,k} + x_{3,k}\big(-a_3 + b_{32}x_{2,k} - b_{33}x_{3,k}\big)\Delta t + \sigma_3 x_{3,k}\sqrt{\Delta t}\xi_k + \dfrac{1}{2}\sigma_3^2 x_{3,k}(\xi_k^2 - 1)\Delta t, \end{cases} \tag{32}$$

where $\xi_k, k = 1, 2, \ldots, n$ are independent Gaussian random variables $N(0, 1)$. The main goal of this section is to further investigate the correctness of the theoretical results.

For system (2), using the numerical simulation method given out above and the help of Matlab software, we choose the initial value $(x_1(0), x_2(0), x_3(0)) = (1, 0.8, 0.6)$, time step $\Delta t = 0.01$ and illustrate our main conclusions through the following example and figures.

Example 4.1. Assume that parameter Settings for system (2) are as follows:
Case 1.

$$\begin{aligned} &a_1 = 0.8, b_{11} = 0.5, b_{12} = 0.6, \sigma_1 = 0.15, \\ &a_2 = 0.3, b_{21} = 0.9, b_{22} = 0.4, b_{23} = 0.8, \sigma_2 = 0.1, \\ &a_3 = 0.2, b_{32} = 0.8, b_{33} = 0.3, \sigma_3 = 0.2. \end{aligned}$$

It is easy to know that

$$\Delta_1 = b_{21}a_1 + b_{11}b_{21} - b_{12}b_{21} = 0.6300,$$

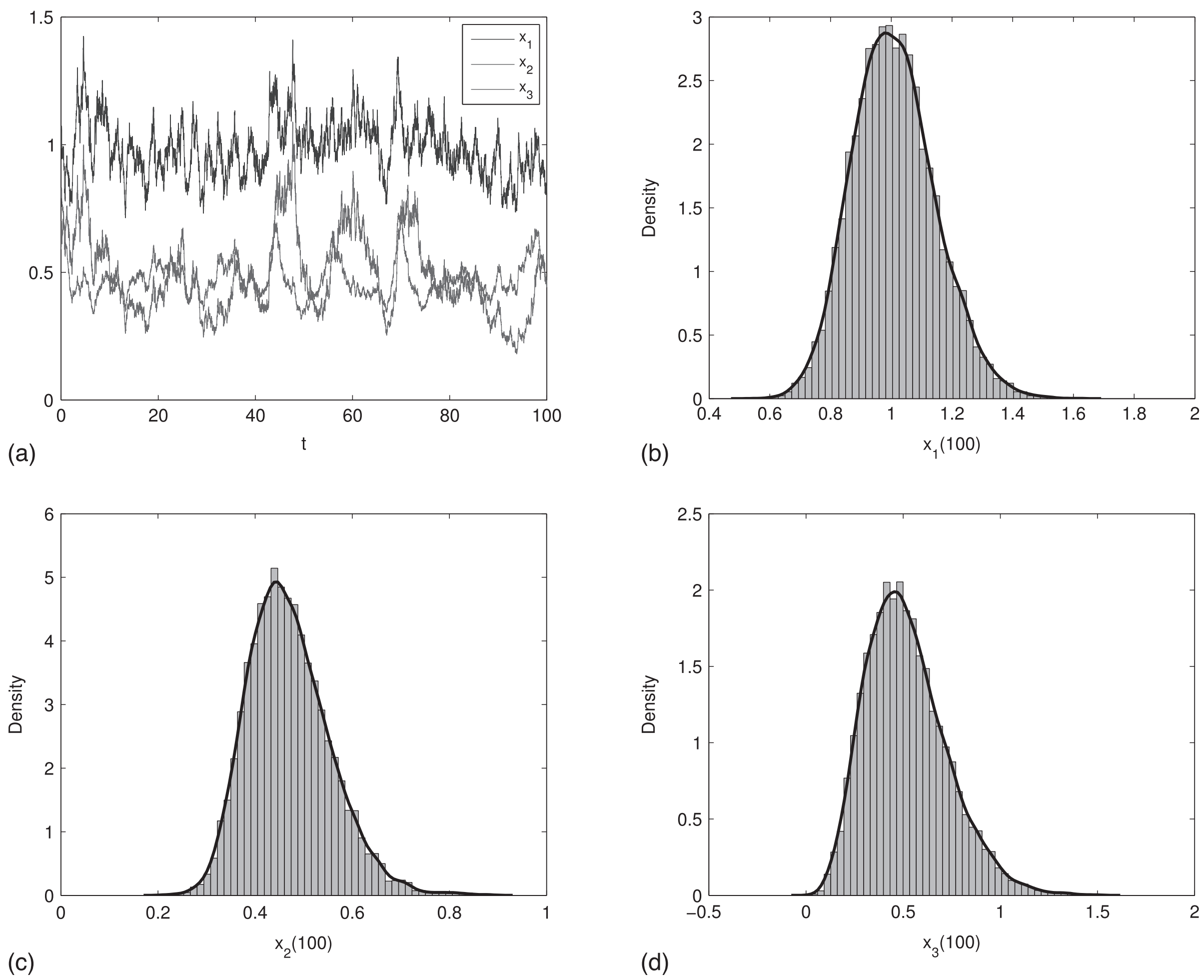

Fig. 1. The paths and Histograms of three-species for the stochastic model (1.3) with $(x_1(0), x_2(0), x_3(0)) = (1, 0.8, 0.6)$, other parameters are given in case 1. (a) The paths of $x_i(t)$; (b) Histogram of the probability density function for $x_1(t)$, the curve is probability density function of $x_1(t)$; (c) Histogram of the probability density function for $x_2(t)$, the curve is probability density function of $x_2(t)$; (d) Histogram of the probability density function for $x_3(t)$, the curve is probability density function of $x_3(t)$.

$$\begin{aligned}
&\Delta_2 = b_{12}(-a_2) + b_{12}b_{21} + b_{12}b_{22} - b_{12}b_{23} = 0.1200,\\
&\Delta_3 = \frac{b_{12}b_{23}}{b_{32}}(-a_3) + b_{12}b_{23} + \frac{b_{12}b_{23}b_{33}}{b_{32}} = 0.5400,\\
&\Xi = b_{21}c_1 - b_{12}c_2 - \frac{b_{12}b_{23}c_3}{b_{32}} - \frac{\Delta_1^2}{4b_{11}b_{21}} - \frac{\Delta_2^2}{4b_{12}b_{21}} - \frac{b_{32}\Delta_3^2}{4b_{12}b_{23}} = 0.0462 > 0.
\end{aligned}$$

Then it follows from Theorem 3.1 that there exists a unique density $U_*(x, y, z)$ which is a stationary solution of system (3). We use Fig. 1 to illustrate Case 1.

Case 2.

$$\begin{aligned}
&a_1 = 0.8, b_{11} = 0.2, b_{12} = 0.6, \sigma_1 = 0.2,\\
&a_2 = 0.8, b_{21} = 0.6, b_{22} = 0.8, b_{23} = 0.8, \sigma_2 = 0.2,\\
&a_3 = 0.5, b_{32} = 0.8, b_{33} = 0.1, \sigma_3 = 0.3.
\end{aligned}$$

It is clear that,

$$c_1 = a_1 - \frac{1}{2}\sigma_1^2 = 0.7800 > 0,$$

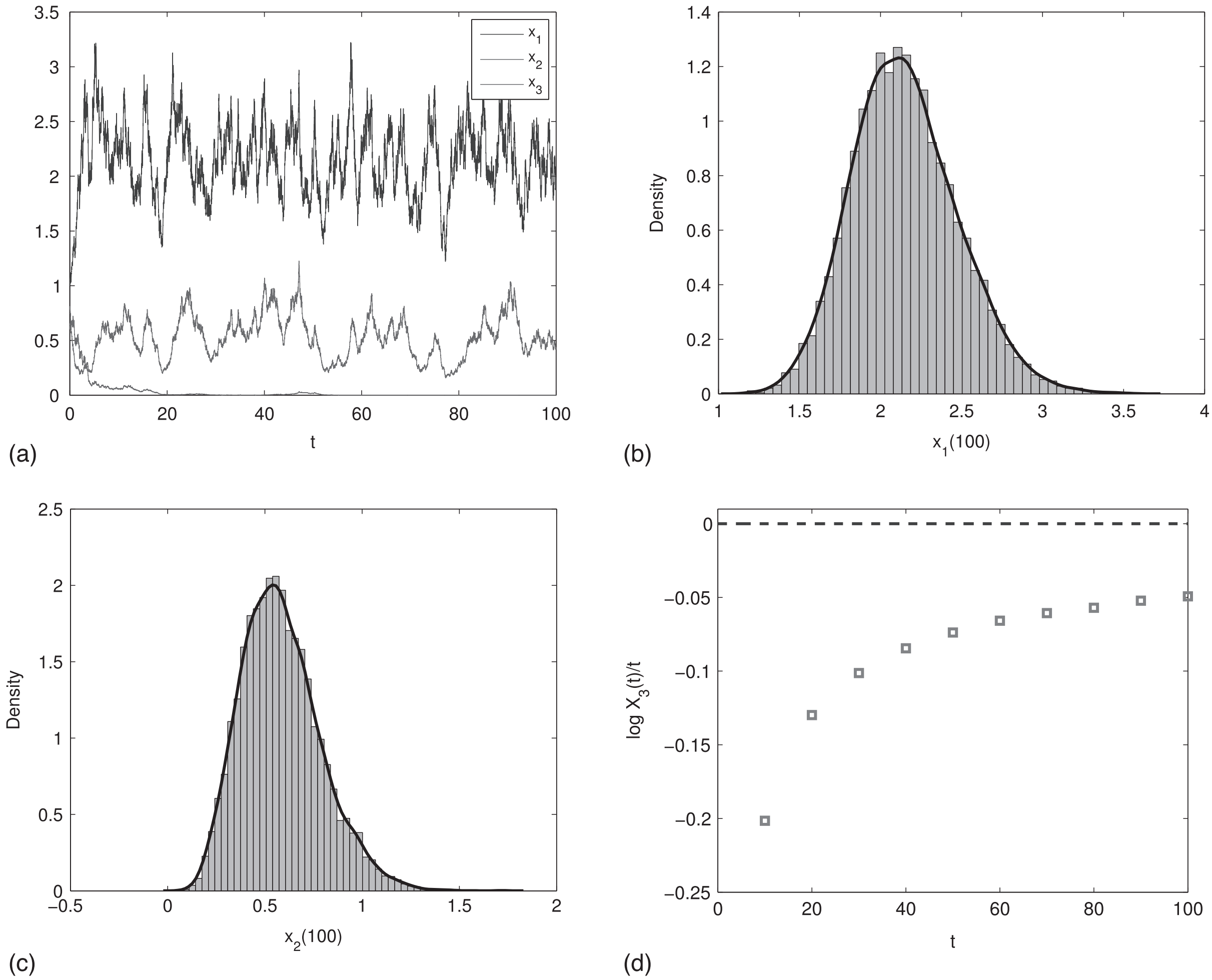

Fig. 2. The paths and Histograms of three-species for the stochastic model (1.3) with $(x_1(0), x_2(0), x_3(0)) = (1, 0.8, 0.6)$, other parameters are given in case 2. (a) The paths of $x_i(t)$; (b) Histogram of the probability density function for $x_1(t)$, the curve is probability density function of $x_1(t)$; (c) Histogram of the probability density function for $x_2(t)$, the curve is probability density function of $x_2(t)$; (d) The mean value of $\log x_3/t$ from 10 000 simulations at $t = 10, 20, \ldots 100$.

$$b_{21}c_1 - b_{11}c_2 = 0.3040 > 0,$$
$$b_{21}b_{32}c_1 - b_{11}b_{22}c_3 - b_{11}b_{32}c_3 = -0.1560 < 0.$$

Then it follows from Theorem 3.8 that $\lim_{t\to\infty} u_3(t) = -\infty,\ a.e.$ and there exists a unique density $\overline{U}_*(x, y)$ which is a stationary solution of the degenerate equations of system (3). We give Fig. 2 to illustrate Case 2.

5. Conclusion

Asymptotic behaviour of stochastic population system have recently been studied by many authors. However, to the best of our knowledge, there are rare results about the stationary distribution of a stochastic food chain model using Markov semi-groups. In this paper, we develop and analyse a stochastic food chain model, which takes white noise into account. We first prove that the distributions of the solutions of system (3) are absolutely continuous. Further, we prove that the densities can converge in L^1 to an invariant density or can converge weakly to a singular measure. Secondly, several sufficient conditions for the extinction of the three populations were obtained. Moreover, Some interesting questions deserve further investigation, such as incorporating intervention strategies into the system. We leave this for future consideration.

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Original article

The uniqueness theorem for cohomologies on the category of polyhedral pairs

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Abstract

Let X be a topological space and $F = \{F_\alpha\}$ be a direct system of all compact subsets F_α of X, directed by inclusions. For any homology theory H_* the groups $\{H_*(F_\alpha) \mid F_\alpha \subset X\}$ constitute a direct system, and the maps $H_*(F_\alpha) \to H_*(X)$ define a homomorphism $i_* : \varinjlim H_*(F_\alpha) \to H_*(X)$.

As is known (Theorem 4.4.6, Spanier, 1966), for the singular homology, the homomorphism i_* is an isomorphism

$$i_* : \varinjlim H_*^s(F_\alpha) \xrightarrow{\sim} H_*^s(X). \tag{1}$$

Using the isomorphism (1), it is proved that for the homologies having compact support H there is the uniqueness theorem on the category of polyhedral pairs (Theorem 4.8.14, Spanier, 1966).

Since the singular homology theory is a homology theory with compact supports, the uniqueness theorem connects all homology theories having compact supports with the singular homology theory.

Let H^* be a cohomology theory. The groups $\{H^*(F_\alpha) \mid F_\alpha \subset X\}$ constitute an inverse system, and the maps $H^*(X) \to H^*(F_\alpha)$ define a homomorphism

$$i^* : H^*(X) \to \varprojlim H^*(F_\alpha).$$

Since the homology functor does not commute with inverse limits, it is not true that the singular cohomology of a space is isomorphic to the inverse limit of the singular cohomology of its compact subsets (that is, there is no general cohomology analogue of Theorem 4.4.6, Spanier, 1966).

In the present work, it will be shown that there is such connection for a singular cohomology. Namely, there exists a finite exact sequence

$$\begin{aligned} 0 &\longrightarrow \varprojlim{}^{(2n-3)} H_s^1(F_\alpha, G) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(1)} H_s^{n-1}(F_\alpha, G) \longrightarrow H_s^n(X, G) \\ &\longrightarrow \varprojlim H_s^n(F_\alpha, G) \longrightarrow \varprojlim{}^{(2)} H_s^{n-1}(F_\alpha, G) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2n-2)} H_s^1(F_\alpha, G) \longrightarrow 0. \end{aligned} \tag{2}$$

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The terms the Alexander cohomology with compact supports and the singular cohomology with compact supports used in the works (Spanier, 1966; Mdzinarishvili, 1984) do not refer to our problem. Therefore, cohomology theory, in particular the singular cohomology, for which there is a finite exact sequence (2), is called a cohomology with partially compact supports.

In the present work, using a finite exact sequence (2), it is proved the uniqueness theorem for a cohomology having partially compact supports on the category of polyhedral pairs. Hence, the uniqueness theorem connects all cohomology theories with partially compact supports with the singular cohomology theory.

Keywords: Singular cohomology; A finite exact sequence; The category of polyhedral pairs

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Let $C_* = \{C_n\}$ be a chain complex of abelian groups C_n,

$$C_* = C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} \cdots \longleftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} \cdots . \tag{3}$$

We denote $Z_n = \operatorname{Ker} \partial_n$, $B_n = \operatorname{Im} \partial_{n+1}$, $H_n = Z_n/B_n = H_n(C_*)$.

Let $\operatorname{Hom}(-, G)$ be the contravariant functor, where G is an abelian group. Using the chain complex C_* from (3) and the functor $\operatorname{Hom}(-, G)$, we have a cochain complex $C^* = \operatorname{Hom}(C_*, G)$, where $C^n = \operatorname{Hom}(C_n, G)$ and $\delta^n : C^{n-1} \to C^n$. Denote also $Z^n = \operatorname{Ker} \delta^{n+1}$, $B^n = \operatorname{Im} \delta^n$, $H^n = Z^n/B^n = H^n(C^*)$.

Lemma 1. *If C_* is a free chain complex, then there is an exact sequence*

$$0 \longrightarrow \operatorname{Hom}(B_{n-1}, G) \longrightarrow Z^n \longrightarrow \operatorname{Hom}(H_n, G) \longrightarrow 0. \tag{4}$$

Proof. Since C_* is a free chain complex, Z_n and B_n are free abelian groups for $n \in \mathbb{Z}$. Consider the exact sequences

$$0 \longrightarrow Z_n \xrightarrow{i_n} C_n \xrightarrow{j_n} B_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_n \longrightarrow Z_n \xrightarrow{t_m} H_n \longrightarrow 0.$$

Using the above sequences, the functor $\operatorname{Hom}(-, G)$, and also Theorems 3.3.2, 3.3.5 and Lemma 1.5.4 [1], we have, respectively, the exact sequences

$$0 \longrightarrow \operatorname{Hom}(B_{n-1}, G) \longrightarrow \operatorname{Hom}(C_n, G) \longrightarrow \operatorname{Hom}(Z_n, G) \longrightarrow 0, \tag{5}$$

$$0 \longrightarrow \operatorname{Hom}(H_n, G) \longrightarrow \operatorname{Hom}(Z_n, G) \longrightarrow \operatorname{Hom}(B_n, G) \longrightarrow \operatorname{Ext}(H_n, G) \longrightarrow 0. \tag{6}$$

The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B_n \arrow[r, "k_n"] & Z_n \arrow[r, "i_n"] & C_n \\
 & & C_{n+1} \arrow[ul, "j_{n+1}"] \arrow[ur, "\partial_{n+1}"'] &
\end{tikzcd} \tag{7}$$

induces a commutative diagram

$$\begin{tikzcd}
 & & \vdots \arrow[d] & 0 \arrow[d] & \\
 & & \mathrm{Hom}(C_{n-1}, G) \arrow[d, "\delta^n"] & \mathrm{Hom}(H_n, G) \arrow[d, "t^n"] & \\
0 \arrow[r] & \mathrm{Hom}(B_{n-1}, G) \arrow[r, "j^n"] & \mathrm{Hom}(C_n, G) \arrow[r, "i^n"] \arrow[d, "\delta^{n+1}"] & \mathrm{Hom}(Z_n, G) \arrow[r] \arrow[d, "k^n"] & 0 \\
 & & \mathrm{Hom}(C_{n+1}, G) \arrow[d] & \mathrm{Hom}(B_n, G) \arrow[l, "j^{n+1}"'] & 0 \arrow[l] \\
 & & \vdots & &
\end{tikzcd} \tag{8}$$

Since j^{n+1} is a monomorphism, the composition

$$Z^n = \mathrm{Ker}\,\delta^{n+1} \xrightarrow{i^n} \mathrm{Hom}(Z_n, G) \xrightarrow{k^n} \mathrm{Hom}(B_n, G)$$

is a trivial map. Since $\mathrm{Im}\,t^n = \mathrm{Ker}\,k^n$, there is $i^n Z^n \subset \mathrm{Im}\,t^n$. Let $\varphi \in \mathrm{Hom}(H_n, G)$. As far as i^n is an epimorphism, there exists $\psi \in \mathrm{Hom}(C_n, G)$ such that $i^n\psi = t^n\varphi$. Since $k^n t^n \varphi = 0$, j^{n+1} is a monomorphism and $j^{n+1}k^n i^n = \delta^{n+1}$, we have $\delta^{n+1}\psi = j^{n+1}k^n i^n \psi = j^{n+1}k^n t^n \varphi = 0$ and $\psi \in Z^n$. Hence there is an epimorphism

$$i^n : \mathrm{Ker}\,\delta^{n+1} = Z^n \longrightarrow \mathrm{Hom}(H_n, G) \longrightarrow 0.$$

We can show that there is the exact sequence (4). Since the homomorphism j^n in the diagram (8) is a monomorphism, $\delta^{n+1} = j^{n+1}k^n i^n$ and $i^n j^n = 0$, it follows that $j^n\,\mathrm{Hom}(B_{n-1}, G) \subset Z^n$.

Let $\varphi \in Z^n$ and $i^n\varphi = 0$. Since t^n is a monomorphism, $i^n\varphi = 0$ and there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}(B_{n-1}, G) \xrightarrow{j^n} \mathrm{Hom}(C_n, G) \xrightarrow{i^n} \mathrm{Hom}(Z_n, G) \longrightarrow 0,$$

there exists $\psi \in \mathrm{Hom}(B_{n-1}, G)$ such that $j^n\psi = \varphi$. □

Lemma 2. *If C_* is a free chain complex, then there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(B_{n-1}, G) \arrow[r] \arrow[d] & Z^n \arrow[r] \arrow[d] & \mathrm{Hom}(H_n, G) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}, G) \arrow[r] \arrow[d] & H^n \arrow[r] \arrow[d] & \mathrm{Hom}(H_n, G) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{9}$$

with exact rows.

Proof. The diagram (7) generates a commutative diagram

$$\begin{tikzcd}
 & & \vdots \arrow[d] & 0 \arrow[d] & \\
0 & \mathrm{Hom}(Z_{n-1},G) \arrow[l] \arrow[d, "k^{n-1}"] & \mathrm{Hom}(C_{n-1},G) \arrow[l, "i^{n-1}"'] \arrow[d, "\delta^n"] & \mathrm{Hom}(H_n,G) \arrow[d, "\iota^n"] & \\
0 \arrow[r] & \mathrm{Hom}(B_{n-1},G) \arrow[r, "j^n"] \arrow[d, "s^{n-1}"] & \mathrm{Hom}(C_n,G) \arrow[r, "i^n"] \arrow[d, "\delta^{n+1}"] & \mathrm{Hom}(Z_n,G) \arrow[r] \arrow[d, "k^n"] & 0 \\
 & \mathrm{Ext}(H_{n-1},G) \arrow[d] & \mathrm{Hom}(C_{n+1}G) \arrow[d] & \mathrm{Hom}(B_n,G) \arrow[l, "j^{n+1}"'] & 0 \arrow[l] \\
 & 0 & \vdots & &
\end{tikzcd} \tag{10}$$

Since $H^n = Z^n/B^n = \mathrm{Ker}\,\delta^{n+1}/\,\mathrm{Im}\,\delta^n$, by Lemma 1 there is an epimorphism

$i^n : \mathrm{Ker}\,\delta^{n+1} \to \mathrm{Hom}(H_n, G) \to 0.$

Since i^n is an epimorphism and j^{n+1} is a monomorphism, there is $\mathrm{Ker}\,\delta^{n+1} = \mathrm{Ker}\,k^n i^n$.

Since i^{n-1} is an epimorphism and j^n is a monomorphism, there is $\mathrm{Im}\,\delta^n = \mathrm{Im}\,j^n k^{n-1}$. Hence $\mathrm{Ker}\,\delta^{n+1}/\,\mathrm{Im}\,\delta^n = \mathrm{Ker}\,k^n i^n/\,\mathrm{Im}\,j^n k^{n-1}$ and $i^n\delta^n = i^n j^n k^{n-1} i^{n-1} = 0$. Therefore there exists a homomorphism in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B^n \arrow[r] & Z^n \arrow[r] \arrow[d] & H^n \arrow[r] \arrow[dl, dashed, "\alpha"] & 0 \\
 & & \mathrm{Hom}(H_n,G) & &
\end{tikzcd}$$

and α is an epimorphism. There is $\mathrm{Ker}\,\alpha = \mathrm{Im}\,j^n/\,\mathrm{Im}\,j^n k^{n-1}$. Since j^n is a monomorphism, we have

$$\mathrm{Ker}\,\alpha = \mathrm{Im}\,j^n/\,\mathrm{Im}\,j^n k^{n-1} \approx \mathrm{Hom}(B_{n-1}, G)/\,\mathrm{Im}\,k^{n-1} \approx \mathrm{Ext}(H_{n-1}, G). \tag{11}$$

Using the commutative diagram (10) and an isomorphism (11), we have the commutative diagram (9). □

Let $\{C_*^\alpha\}_{\alpha\in\Omega}$ be a direct system, where $\Omega = \{\alpha\}$ is a set of indexes of chain complexes C_*^α, where $C_*^\alpha = \{C_n^\alpha\}$, C_n^α is an abelian group.

Theorem 1 (*[2]*)**.** *If $\{C_*^\alpha\}$ is a direct system of free chain complexes, then for $n \in \mathbb{Z}$ and $i \geq 2$ there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(i)}\,\mathrm{Ext}(H_{n-1}^\alpha, G) \longrightarrow \varprojlim{}^{(i)} H_\alpha^n \longrightarrow \varprojlim{}^{(i)}\,\mathrm{Hom}(H_n^\alpha, G) \longrightarrow 0, \tag{12}$$

and this sequence is split.

Proof. Since for each $\alpha \in \Omega$ there is a free chain complex C_*^α, by Lemma 2, there is the commutative diagram (9) with exact rows. By Theorem 5.4.1 [3] the diagram (9) induces a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \varprojlim^{(i)}\mathrm{Hom}(B_{n-1}^\alpha,G) \arrow[r] \arrow[d] & \varprojlim^{(i)} Z_\alpha^n \arrow[r] \arrow[d] & \varprojlim^{(i)}\mathrm{Hom}(H_n^\alpha,G) \arrow[r] \arrow[d, equal] & \varprojlim^{(i+1)}\mathrm{Hom}(B_{n-1}^\alpha,G) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \varprojlim^{(i)}\mathrm{Ext}(H_{n-1}^\alpha,G) \arrow[r] & \varprojlim^{(i)} H_\alpha^n \arrow[r] & \varprojlim^{(i)}\mathrm{Hom}(H_n^\alpha,G) \arrow[r] & \varprojlim^{(i+1)}\mathrm{Ext}(H_{n-1}^\alpha,G) \arrow[r] & \cdots
\end{tikzcd} \tag{13}$$

In [4], for the direct system $\{A_\alpha\}$ of abelian groups A_α, there is an exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)}\,\mathrm{Hom}(A_\alpha, G) \longrightarrow \mathrm{Ext}\left(\varinjlim A_\alpha, G\right) \longrightarrow \varprojlim\,\mathrm{Ext}(A_\alpha, G) \longrightarrow \varprojlim{}^{(2)}\,\mathrm{Hom}(A_\alpha, G) \longrightarrow 0, \tag{14}$$

and for $i \geq 1$, there is an isomorphism

$$\varprojlim^{(i)} \operatorname{Ext}(A_\alpha, G) \approx \varprojlim^{(i+2)} \operatorname{Hom}(A_\alpha, G). \tag{15}$$

If we consider a direct system $\{B^\alpha_{n-1}\}$ of free abelian groups, then from (14) and (15) for $i \geq 2$ one has

$$\varprojlim^{(i)} \operatorname{Hom}(B^\alpha_{n-1}, G) = 0. \tag{16}$$

Using the commutative diagram (13), the exact sequence (14) and the equality (16), we have the following:
(a) an isomorphism $\varprojlim^{(i)} Z^n_\alpha \approx \varprojlim^{(i)} \operatorname{Hom}(H^\alpha_n, G)$ for $i \geq 2$;
(b) an epimorphism $\varprojlim^{(i)} H^n_\alpha \longrightarrow \varprojlim^{(i)} \operatorname{Hom}(H^\alpha_n, G)$ for $i \geq 1$;
(c) a monomorphism $\varprojlim^{(i)} \operatorname{Ext}(H^\alpha_{n-1}, G) \longrightarrow \varprojlim^{(i)} H^\alpha_n$ for $i \geq 2$;
(d) a trivial homomorphism $\varprojlim^{(i)} \operatorname{Hom}(H^\alpha_n, G) \longrightarrow \varprojlim^{(i+1)} \operatorname{Ext}(H^\alpha_{n-1}, G)$ for $i \geq 1$.
Using (a)–(d) for $i \geq 2$, we have an exact sequence

$$0 \longrightarrow \varprojlim^{(i)} \operatorname{Ext}(H^\alpha_{n-1}, G) \longrightarrow \varprojlim^{(i)} H^n_\alpha \longrightarrow \varprojlim^{(i)} \operatorname{Hom}(H^\alpha_n, G) \longrightarrow 0. \tag{17}$$

For $i \geq 2$, define a homomorphism

$$\varprojlim^{(i)} \operatorname{Hom}(H^\alpha_n, G) \xrightarrow{\sim} \varprojlim^{(i)} Z^n_\alpha \longrightarrow \varprojlim^{(i)} H^n_\alpha.$$

We obtain that a composition

$$\varprojlim^{(i)} \operatorname{Hom}(H^\alpha_n, G) \xrightarrow{\sim} \varprojlim^{(i)} Z^n_\alpha \longrightarrow \varprojlim^{(i)} H^n_\alpha \longrightarrow \varprojlim^{(i)} \operatorname{Hom}(H^\alpha_n, G)$$

is the identity map. □

Let $C_* = \{C^\alpha_*\}$ be a direct system of chain complexes C^α_* and let C_* be a chain complex.

Definition 1. A direct system $\underline{C}_*$ of chain complexes C^α_* is said to be an association with a chain complex C_*, if there exists a homomorphism $\underline{C}_* \to C_*$ such that for all $n \in \mathbb{Z}$ there is an isomorphism

$$\varinjlim H^\alpha_n \xrightarrow{\sim} H_n(C_*),$$

where $H^\alpha_n = H_n(C^\alpha_*)$.

Theorem 2. *If a direct system $\underline{C}_* = \{C^\alpha_*\}$ of free chain complexes C^α_* is an association with a free chain complex C_*, then there is an exact sequence*

$$\begin{aligned} \cdots \longrightarrow \varprojlim^{(2i+1)} H^{n-(i+1)}_\alpha \longrightarrow \cdots \longrightarrow \varprojlim^{(1)} H^{n-1}_\alpha \longrightarrow H^n(C^*) \\ \longrightarrow \varprojlim H^n_\alpha \longrightarrow \varprojlim^{(2)} H^{n-1}_\alpha \longrightarrow \cdots \longrightarrow \varprojlim^{(2i)} H^{n-i}_\alpha \longrightarrow \cdots, \end{aligned} \tag{18}$$

where $H^n(C^*) = H^n(\operatorname{Hom}(C_*, G))$, $H^n_\alpha = H^n(\operatorname{Hom}(C^\alpha_*, G))$.

Proof. Using property (d) from the proof of Theorem 1 and the exact sequence (13), we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \varprojlim \operatorname{Ext}(H^\alpha_{n-1}, G) \longrightarrow \varprojlim H^n_\alpha \longrightarrow \varprojlim \operatorname{Hom}(H^\alpha_n, G) \\ \longrightarrow \varprojlim^{(1)} \operatorname{Ext}(H^\alpha_{n-1}, G) \longrightarrow \varprojlim^{(1)} H^n_\alpha \longrightarrow \varprojlim^{(1)} \operatorname{Hom}(H^\alpha_n, G) \longrightarrow 0. \end{aligned} \tag{19}$$

Since a chain complex C_* is free, a homomorphism $\overline{C}_* \to C_*$ induces a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \operatorname{Ker}\psi \arrow[r] \arrow[d] & \operatorname{Ker}\varphi^n \arrow[d] & & \\
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(C_*), G) \arrow[r] \arrow[d, "\psi"] & H^n(C^*) \arrow[r] \arrow[d, "\phi^n"] & \operatorname{Hom}(H_n(C_*), G) \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & \varprojlim \operatorname{Ext}(H^\alpha_{n-1}, G) \arrow[r] \arrow[d] & \varprojlim H^n_\alpha \arrow[r] \arrow[d] & \varprojlim \operatorname{Hom}(H^\alpha_n, G) \arrow[r] & \cdots \\
 & \operatorname{Coker}\psi \arrow[r] \arrow[d] & \operatorname{Coker}\phi^n \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{20}$$

Since a system $\underline{C}_*$ is an association with C_*, we have $\operatorname{Hom}(H_n(C_*), G) \approx \operatorname{Hom}(\varinjlim H^\alpha_n, G) \approx \varprojlim \operatorname{Hom}(H^\alpha_n, G)$ and hence λ is an isomorphism. Since the diagram (20) is commutative and λ is an isomorphism, we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \operatorname{Ker}\psi \arrow[r, "\sim"] \arrow[d] & \operatorname{Ker}\phi^n \arrow[d] & & \\
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(C_*), G) \arrow[r] \arrow[d, "\psi"] & H^n(C^*) \arrow[r] \arrow[d, "\phi^n"] & \operatorname{Hom}(H_n(C_*), G) \arrow[r] \arrow[d, "\lambda", "\approx"'] & 0 \\
0 \arrow[r] & \varprojlim \operatorname{Ext}(H^\alpha_{n-1}, G) \arrow[r] \arrow[d] & \varprojlim H^n_\alpha \arrow[r] \arrow[d] & \varprojlim \operatorname{Hom}(H^\alpha_n, G) \arrow[r] & 0 \\
 & \operatorname{Coker}\psi \arrow[r, "\sim"] \arrow[d] & \operatorname{Coker}\phi^n \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{21}$$

Using the exact sequence (14), an isomorphism λ and the diagram (21), we have the exact sequences

$$0 \longrightarrow \varprojlim^{(1)} \operatorname{Ext}(H^\alpha_{n-1}, G) \longrightarrow \varprojlim^{(1)} H^n_\alpha \longrightarrow \varprojlim^{(1)} \operatorname{Hom}(H^\alpha_n, G) \longrightarrow 0 \tag{22}$$

and

$$0 \longrightarrow \varprojlim^{(1)} \operatorname{Hom}(H^\alpha_{n-1}, G) \longrightarrow H^n(C^*) \longrightarrow \varprojlim H^n_\alpha \longrightarrow \varprojlim^{(2)} \operatorname{Hom}(H^\alpha_{n-1}, G) \longrightarrow 0. \tag{23}$$

Using the exact sequences (12), (22) and (23) for $i \geq 1$, the isomorphism (15) and Theorem 1, there is the exact sequence

$$\begin{tikzcd}
 & & \vdots \arrow[d] & & \\
0 \arrow[r] & \varprojlim^{(3)} \mathrm{Ext}(H^{\alpha}_{n-3}, G) \arrow[r] & \varprojlim^{(3)} H^{n-2}_{\alpha} \arrow[r] \arrow[d] & \varprojlim^{(3)} \mathrm{Hom}(H^{\alpha}_{n-2}, G) \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\
0 & \varprojlim^{(1)} \mathrm{Hom}(H^{\alpha}_{n-1}, G) \arrow[l] \arrow[d, equal] & \varprojlim^{(1)} H^{n-1}_{\alpha} \arrow[l] \arrow[d] & \varprojlim^{(1)} \mathrm{Ext}(H^{\alpha}_{n-2}, G) \arrow[l] & 0 \arrow[l] \\
0 \arrow[r] & \varprojlim^{(1)} \mathrm{Hom}(H^{\alpha}_{n-1}, G) \arrow[r] & H^{n}(C^{*}) \arrow[d] & & \\
 & & \varprojlim H^{n}_{\alpha} \arrow[r] \arrow[d] & \varprojlim^{(2)} \mathrm{Hom}(H^{\alpha}_{n-1}, G) \arrow[r] \arrow[d, equal] & 0 \\
0 & \varprojlim^{(2)} \mathrm{Ext}(H^{\alpha}_{n-2}, G) \arrow[l] \arrow[d, phantom, "\wr"] & \varprojlim^{(2)} H^{n-1}_{\alpha} \arrow[l] \arrow[d] & \varprojlim^{(2)} \mathrm{Hom}(H^{\alpha}_{n-1}, G) \arrow[l] & 0 \arrow[l] \\
0 \arrow[r] & \varprojlim^{(4)} \mathrm{Hom}(H^{\alpha}_{n-2}, G) \arrow[r] & \varprojlim^{(4)} H^{n-2}_{\alpha} \arrow[r] \arrow[d] & \varprojlim^{(4)} \mathrm{Ext}(H^{\alpha}_{n-2}, G) \arrow[r] & 0 \\
 & & \vdots & &
\end{tikzcd} \quad \square$$

Let X be a topological space. A singular q-simplex of space X is a continuous map $\Delta_q \to X$, where Δ_q is a standard q-simplex, $q \geq 0$ [5]. Denote by $S_q(X)$ the free abelian group generated by the set of all singular q-simplexes of space X. For $q < 0$, there are no singular q-simplexes. Denote by $\partial_q : S_q X \to S_{q-1} X$ a homomorphism, defined by the formula $\partial_q(\sigma) = \sum_{j=0}^{q} (-1)^j (\sigma_q e^i_q)$, where $e^i_q : \Delta_{q-1} \to \Delta_q$.

The sequence

$$SX = \cdots \longleftarrow S_{q-1} X \xleftarrow{\partial_q} S_q X \xleftarrow{\partial_{q+1}} S_{q+1} X \longleftarrow \cdots$$

is a free chain complex. Its homology group, denoted by $H^s_* X$, is a gradet group $\{H^s_q X = H_q(SX)\}$, and called the singular homology group of X.

If $f : X \to Y$ is a continuous map and $\sigma : \Delta_q \to X$ is a singular q-simplex, then a composition $f\sigma : \Delta_q \to Y$ is a singular q-simplex of space Y. Hence there is a homomorphism $S_q f : S_q X \to S_q Y$, $(S_q f)(\sigma) = f\sigma$. A sequence of maps $S_q f : S_q X \to S_q Y$, $q \in \mathbb{Z}$, is a chain map and is denoted by $Sf : SX \to SY$. A chain map Sf induces a homomorphism $f_* : H^s_* X \to H^s_* Y$.

Since Δ_q is compact, every singular $\sigma : \Delta_q \to X$ maps Δ_q into some compact subset of X. Hence, if $\{F_\alpha\}$ is the collection of compact subsets of X directed by inclusion, then

$$SX = \varinjlim SF_\alpha. \tag{24}$$

By Theorem 4.1.7 [6], we have the following result.

Theorem 3 (*[6]*)**.** *The singular homology group of a space is isomorphic to the direct limit of the singular homology groups of its compact subsets,*

$$H_*^s X \approx \varinjlim H_*^s F_\alpha.$$

Since all conditions of Theorem 2 are satisfied, for a singular cohomology we have the following theorem.

Theorem 4. *For a singular cohomology of any topological space X there is a finite exact sequence*

$$0 \longrightarrow \varprojlim{}^{(2n-3)} H_s^1 F_\alpha \longrightarrow \cdots \longrightarrow \varprojlim{}^{(1)} H_s^{n-1} F_\alpha \longrightarrow H_s^n X$$
$$\longrightarrow \varprojlim H_s^n F_\alpha \longrightarrow \varprojlim{}^{(2)} H_s^{n-1} F_\alpha \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2n-2)} H_s^1 F_\alpha \longrightarrow 0. \tag{25}$$

Proof. Since there is an isomorphism (24), using Theorem 2, we have an exact sequence

$$\cdots \longrightarrow \varprojlim{}^{(2i+1)} H_s^{n-(i+1)} F_\alpha \longrightarrow \cdots \longrightarrow \varprojlim{}^{(1)} H_s^{n-1} F_\alpha \longrightarrow H_s^n X$$
$$\longrightarrow \varprojlim H_s^n F_\alpha \longrightarrow \varprojlim{}^{(2)} H_s^{n-1} F_\alpha \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2i)} H_s^{n-i} F_\alpha \longrightarrow \cdots \tag{26}$$

(1) $n = 0$. For $q < 0$ and any topological space Y, there is $H_q Y = 0$ and using the universal coefficients formula for a singular cohomology, we have

$$H_s^0 Y \approx \mathrm{Hom}(H_0^s Y, G), \tag{27}$$
$$H_s^q Y = 0, \quad q < 0. \tag{28}$$

Therefore from the exact sequence (26) it follows that there is an isomorphism

$$H_s^0 X \approx \varprojlim H_s^0 F_\alpha.$$

(2) $n = 1$. Using the equality (28) and the exact sequence (26), there are

$$\varprojlim{}^{(3)} H_s^{-1} F_\alpha = 0, \quad \varprojlim{}^{(4)} H_s^{-1} F_\alpha = 0$$

and a finite exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)} H_s^0 F_\alpha \longrightarrow H_s^1 X \longrightarrow \varprojlim H_s^1 F_\alpha \longrightarrow \varprojlim{}^{(2)} H_s^0 F_\alpha \longrightarrow 0.$$

Using the isomorphisms $H_s^0 F_\alpha \approx \mathrm{Hom}(H_0^s F_\alpha, G)$, $\varprojlim{}^{(2)} H_s^0 F_\alpha \approx \varprojlim{}^{(2)} \mathrm{Hom}(H_0^s F_\alpha, G)$, the exact sequence (14) and the equality $\varprojlim \mathrm{Ext}(H_0^s F_\alpha, G) = 0$, since $H_0^s F_\alpha$ is a free abelian group, we have the equality

$$\varprojlim{}^{(2)} H_s^0 F_\alpha = 0.$$

Therefore there is an exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)} H_s^0 F_\alpha \longrightarrow H_s^1 X \longrightarrow \varprojlim H_s^1 F_\alpha \longrightarrow 0.$$

(3) $n = 2$. Using the equality (28) and the exact sequence (26), there are

$$\varprojlim{}^{(5)} H_s^{-1} F_\alpha = 0, \quad \varprojlim{}^{(6)} H_s^{-1} F_\alpha = 0.$$

Using the isomorphism (27), there is

$$\varprojlim{}^{(3)} H_s^0 F_\alpha \approx \varprojlim{}^{(3)} \mathrm{Hom}(H_0^s F_\alpha, G). \tag{29}$$

Using the isomorphisms (15) and (29), there is an isomorphism

$$\varprojlim{}^{(3)} H_s^0 F_\alpha \approx \varprojlim{}^{(1)} \mathrm{Ext}(H_0^s F_\alpha, G). \tag{30}$$

By Corollary 4.8.4 [6], Lemma 1.5.4 and Theorem 3.3.5 [1], there is $\mathrm{Ext}(H_0^s F_\alpha, G) = 0$. Hence, using an isomorphism (30), we have

$$\varprojlim{}^{(3)} H_s^0 F_\alpha = 0, \quad \varprojlim{}^{(4)} H_s^0 F_\alpha = 0.$$

Therefore there is a finite exact sequence

$$0 \longrightarrow \varprojlim^{(1)} H_s^1 F_\alpha \longrightarrow H_s^2 X \longrightarrow \varprojlim H_s^2 F_\alpha \longrightarrow \varprojlim^{(2)} H_s^1 F_\alpha \longrightarrow 0.$$

(4) $n \geq 3$. Using the exact sequence (26) and the equality (28), there are

$$\varprojlim^{(2n+1)} H_s^{-1} F_\alpha = 0, \quad \varprojlim^{(2n+2)} H_s^{-1} F_\alpha = 0.$$

We have the isomorphism

$$\varprojlim^{(2n-1)} H_s^0 F_\alpha \approx \varprojlim^{(2n-1)} \operatorname{Hom}(H_0^s F_\alpha, G) \approx \varprojlim^{(2n-3)} \operatorname{Ext}(H_0^s F_\alpha, G).$$

Since $H_0^s F_\alpha$ is a free abelian group, there is $\operatorname{Ext}(H_0^s F_\alpha, G) = 0$. Therefore $\varprojlim^{(2n-1)} H_s^0 F_\alpha = 0$. Analogously,

$$\varprojlim^{(2n)} H_s^0 F_\alpha \approx \varprojlim^{(2n)} \operatorname{Hom}(H_0^s F_\alpha, G) \approx \varprojlim^{(2n-2)} \operatorname{Ext}(H_0^s F_\alpha, G) = 0.$$

Hence there is an exact sequence (25). □

Definition 2. A cohomology theory H for which there is a finite exact sequence

$$0 \longrightarrow \varprojlim^{(2n-3)} H^1 F_\alpha \longrightarrow \cdots \longrightarrow \varprojlim^{(1)} H^{n-1} F_\alpha \longrightarrow H^n X$$
$$\longrightarrow \varprojlim H^n F_\alpha \longrightarrow \varprojlim^{(2)} H^{n-1} F_\alpha \longrightarrow \cdots \longrightarrow \varprojlim^{(2n-2)} H^1 F_\alpha \longrightarrow 0,$$

where $\{F_\alpha\}$ is a direct system of all compact subsets F_α of X directed by the inclusion, is called a cohomology theory with partially compact supports.

By Theorem 4, the singular cohomology theory is a cohomology theory with partially compact supports.

Corollary 1. *If X is a polyhedron and $\{F_\alpha\}$ is a system of compact subspaces of X, then for a singular cohomology there is a short exact sequence*

$$0 \longrightarrow \varprojlim^{(1)} H_s^{n-1} F_\alpha \longrightarrow H_s^n(X, G) \longrightarrow \varprojlim H_s^n F_\alpha \longrightarrow 0, \tag{31}$$

where $H_s^* F_\alpha = H_s^*(F_\alpha, G)$.

Proof. For any polyhedral space X, the compact polyhedrals X_λ contained in it are cofinal in the family of all compact subspaces F_α. Therefore, for $i \geq 0$, there is an isomorphism

$$\varprojlim^{(i)} H_s^* X_\lambda \approx \varprojlim^{(i)} H_s^* F_\alpha. \tag{32}$$

Since $H_*^s X_\lambda$ is finitely generated, using Corollary 1.5 [4], there is the equalities

$$\text{a}') \quad \varprojlim^{(i)} \operatorname{Hom}(H_*^s X_\lambda, G) = 0, \quad i \geq 2, \tag{33}$$

$$\text{b}') \quad \varprojlim^{(i)} \operatorname{Ext}(H_*^s X_\lambda, G) = 0, \quad i \geq 1. \tag{34}$$

Using Theorem 1, the isomorphism (32) and the equalities (33), (34), we have

$$\varprojlim^{(i)} H_s^* F_\alpha = 0 \quad \text{for} \quad i \geq 2. \tag{35}$$

From the finite exact sequence (25) and the equality (35) follows the short exact sequence (31). □

Theorem 5. *Let h be a homomorphism from cohomology H to cohomology H', that is, an isomorphism for one-point spaces. If H and H' have partially compact supports, h is an isomorphism for any polyhedral pair.*

Proof. The uniqueness theorem is valid for cohomology theories (that is, a homomorphism from one cohomology theory to another, which is an isomorphism for one-point spaces, is an isomorphism for compact polyhedral pairs).

A homomorphism h induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim^{(1)} H^{n-1} X_\lambda \arrow[r] \arrow[d, "\approx"'] & H^n(X) \arrow[r] \arrow[d, "h^n"] & \varprojlim H^n X_\lambda \arrow[r] \arrow[d, "\approx"] & 0 \\ 0 \arrow[r] & \varprojlim^{(1)} H'^{n-1} X_\lambda \arrow[r] & H'^n \arrow[r] & \varprojlim H'^n X_\lambda \arrow[r] & 0. \end{tikzcd} \tag{36}$$

Then for all $n \in \mathbb{Z}$, there is an isomorphism

$$h^n : H^n(X) \xrightarrow{\sim} H'^n(X). \tag{37}$$

For any polyhedral pairs (X, A), there is a commutative diagram with exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^q(X) \arrow[r] \arrow[d, "\approx"] & H^q(A) \arrow[r] \arrow[d, "\approx"] & H^{q+1}(X, A) \arrow[r] \arrow[d, "h"] & H^{q+1}(X) \arrow[r] \arrow[d, "\approx"] & H^{q+1}(A) \arrow[r] \arrow[d, "\approx"] & \cdots \\ \cdots \arrow[r] & H'^q(X) \arrow[r] & H'^q(A) \arrow[r] & H'^{q+1}(X, A) \arrow[r] & H'^{q+1}(X) \arrow[r] & H'^{q+1}(A) \arrow[r] & \cdots \end{tikzcd}$$

Using the five lemma, we have an isomorphism

$$h : H^*(X, A) \xrightarrow{\sim} H'^*(X, A). \quad \square$$

Corollary 2. *If X is a manifold, then there is a finite exact sequence*

$$\begin{aligned} &0 \longrightarrow \varprojlim^{(2n-3)} H_s^1 F_\alpha \longrightarrow \cdots \longrightarrow \varprojlim^{(1)} H_s^{q-1} F_\alpha \longrightarrow \overline{H}^q X \\ &\longrightarrow \varprojlim H_s^q F_\alpha \longrightarrow \varprojlim^{(2)} H_s^{q-1} F_\alpha \longrightarrow \cdots \longrightarrow \varprojlim^{(2n-2)} H_s^1 F_\alpha \longrightarrow 0, \end{aligned} \tag{38}$$

where $\overline{H}^q X = \overline{H}^q(X, G)$ is the Alexander–Spanier cohomology of X, $H_s^p F_\alpha = H_s^p(F_\alpha, G)$, F_α is a compact subset of X.

In particular, if a manifold X has a triangulation, then there is an exact sequence

$$0 \longrightarrow \varprojlim^{(1)} H_s^{q-1} F_\alpha \longrightarrow \overline{H}^q(X) \longrightarrow \varprojlim H_s^q F_\alpha \longrightarrow 0. \tag{39}$$

Proof. If X is a manifold, by Corollary 6.8.7 [6], there is an isomorphism

$$\overline{H}^*(X, G) \approx H_s^*(X, G), \tag{40}$$

where $\overline{H}^*(X, G)$ is the Alexander–Spanier cohomology. Using the isomorphism (40) and Theorem 4, we have a finite exact sequence (38). In particular, if manifold X has a triangulation, then, by Corollary 1 and the isomorphism (40), we have the exact sequence (39). $\square$

Corollary 3. *If A is an arbitrary closed subset of the sphere S^{n+1}, then for $0 < q < n$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim^{(1)} H_{q+1}^s(U) \longrightarrow \overline{H}_q(A) \longrightarrow \varprojlim H_q^s(U) \longrightarrow 0, \tag{41}$$

where $\overline{H}_q(A)$ is the Steenrod homology of A, $U = \{U\}$ is an inverse system of open subsets U such that $A \subset U \subset S^{n+1}$.

Proof. Since A is a compact subset of S^{n+1}, $S^{n+1} \setminus A$ has a triangulation. Using Corollary 2, we have a short exact sequence

$$0 \longrightarrow \varprojlim^{(1)} H_s^{q-1}(S^{n+1} \setminus U) \longrightarrow \overline{H}^q(S^{n+1} \setminus A) \longrightarrow \varprojlim H_s^q(S^{n+1} \setminus U) \longrightarrow 0, \tag{42}$$

where $\overline{H}^q(S^{n+1} \setminus A) = \overline{H}^q(S^{n+1} \setminus A, G)$ is the Alexander–Spanier cohomology of $S^{n+1} \setminus A$. By Corollary 6.8.8 [6], there is an isomorphism $\check{H}^*(S^{n+1} \setminus A, G) \approx \overline{H}^*(S^{n+1} \setminus A, G)$, where $\check{H}^*$ is the Čech cohomology.

Using the Steenrod duality theorem and the Alexander–Pontryagin duality theorem [7,8] and the isomorphism 4.8.4 [6], we have the isomorphisms

$$\check{H}^{n-q}(S^{n+1} \setminus A, G) \approx \overline{H}_q(A, G) \tag{43}$$

and

$$H_s^{n-q}(F) \approx \check{H}^{n-q}(F) \approx H_q^s(S^{n+1} \setminus F), \tag{44}$$

where F is a compact polyhedral subset of $S^{n+1} \setminus A$.

Therefore, using the exact sequence (42), the isomorphisms (37), (43), (44), Corollary 1.1.12 [8], we have a short exact sequence (41). $\square$

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Original article

On the solutions of quasi-static and steady vibrations equations in the theory of viscoelasticity for materials with double porosity

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Abstract

In the present paper the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity is considered. Some basic results on the solutions of the quasi-static and steady vibrations equations are obtained. Indeed, the fundamental solutions of the systems of equations of quasi-static and steady vibrations are constructed by elementary functions and their basic properties are established. Green's formulae and the integral representation of regular solution in the considered theory are obtained. Finally, a wide class of the internal boundary value problems of quasi-static and steady vibrations is formulated and on the basis of Green's formulae the uniqueness theorems for classical solutions of these problems are proved.

Keywords: Viscoelasticity; Double porosity; Fundamental solution; Uniqueness theorems; Quasi-static; Steady vibrations

1. Introduction

Poroelasticity is a well-developed theory for the interaction of fluid and solid phases of porous medium. The mathematical models of single- and multi-porosity media have found applications in many branches of civil and geotechnical engineering, biomechanics and technology (see e.g. Bai at al. [1], Cowin [2] and Vafai [3]).

The first theory of consolidation for elastic materials with single porosity is presented by Biot [4]. This theory is developed for double porosity elastic solid by Wilson and Aifantis [5]. More general models of double porosity materials are introduced by Ieşan and Quintanilla [6], Khalili et al. [7], Masters et al. [8], Gelet et al. [9] and studied by Ciarletta et al. [10], Straughan [11], Gentile and Straughan [12], Ieşan [13], Tsagareli and Svanadze [14]. An extensive review of works on the single- and multi-porosity elasticity and thermoelasticity is given in de Boer [15], Straughan [16–18].

Viscoelastic materials play an important role in many branches of engineering (see Brinson and Brinson [19], Gutierrez-Lemini [20], Lakes [21]). Various theories of differential and integral types of viscoelastic materials have

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been proposed by several authors (for details, see Amendola et al. [22], Christensen [23], Fabrizio and Morro [24], Eringen [25] and references therein).

In the last decade there has been interest in formulation of the theories of differential type for elastic materials with microstructures. In this connection, the theories of viscoelasticity and thermoviscoelasticity are presented for binary mixtures by Ieşan [26,27], Ieşan and Nappa [28], Ieşan and Quintanilla [29], Ieşan and Scalia [30]. The mathematical models for Kelvin–Voigt materials with single and double porosity are introduced in [31] and [32], respectively. The basic properties of plane waves are established and some boundary value problems of steady vibrations of the theories of viscoelasticity and thermoviscoelasticity for Kelvin–Voigt materials with double porosity are considered in [33,34].

In this paper the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity is considered. This work is articulated as follows. The next section is based on the governing field equations of quasi-statics and steady vibrations of the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity. In Section 3 the fundamental solutions of the systems of equations of quasi-static and steady vibrations are constructed by elementary functions and their basic properties are established. In Section 4 Green's formulae and the integral representation of regular solution in the considered theory are obtained. Finally, in Section 5 a wide class of the internal boundary value problems is formulated and on the basis of Green's formulae the uniqueness theorems for regular (classical) solutions of these problems are proved.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$, let t denote the time variable, $t \geq 0$. We assume that subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range (1,2,3), and the dot denotes differentiation with respect to t.

In what follows we consider an isotropic and homogeneous viscoelastic Kelvin–Voigt material with double porosity that occupies the region Ω of $\mathbb{R}^3$. Let $\hat{\mathbf{u}}(\mathbf{x}, t)$ be the displacement vector, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$; $\hat{p}_1(\mathbf{x}, t)$ and $\hat{p}_2(\mathbf{x}, t)$ are the pore and fissure fluid pressures, respectively.

The system of dynamical equations in the linear theory of viscoelasticity for Kelvin–Voigt material with double porosity consists of the following equations [32]:

(a) *The equations of motion*

$$t_{lj,j} = \rho(\ddot{\hat{u}}_l - \hat{F}_l), \qquad l = 1, 2, 3, \tag{1}$$

where t_{lj} are the components of the total stress tensor, ρ is the reference mass density, $\rho > 0$, $\hat{\mathbf{F}} = (\hat{F}_1, \hat{F}_2, \hat{F}_3)$ is the body force per unit mass.

(b) *The equations of fluid mass conservation*

$$\begin{aligned} &\operatorname{div} \mathbf{v}^{(1)} + \dot{\zeta}_1 + \beta_1 \dot{e}_{rr} + \gamma(\hat{p}_1 - \hat{p}_2) = 0, \\ &\operatorname{div} \mathbf{v}^{(2)} + \dot{\zeta}_2 + \beta_2 \dot{e}_{rr} - \gamma(\hat{p}_1 - \hat{p}_2) = 0, \end{aligned} \tag{2}$$

where $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are the fluid flux vectors for the pores and fissures, respectively; e_{lj} are the components of the strain tensor,

$$e_{lj} = \frac{1}{2}\left(\hat{u}_{l,j} + \hat{u}_{j,l}\right), \qquad l, j = 1, 2, 3, \tag{3}$$

β_1 and β_2 are the effective stress parameters, γ is the internal transport coefficient (leakage parameter) and corresponds to a fluid transfer rate respecting the intensity of flow between the pores and fissures, $\gamma \geq 0$; ζ_1 and ζ_2 are the increments of fluid (volumetric strain) in the pores and fissures, respectively, and defined by

$$\zeta_1 = \alpha_1 \hat{p}_1 + \alpha_3 \hat{p}_2, \qquad \zeta_1 = \alpha_3 \hat{p}_1 + \alpha_2 \hat{p}_2, \tag{4}$$

α_1 and α_2 measure the compressibilities of the pore and fissure systems, respectively; α_3 is the cross-coupling compressibility for fluid flow at the interface between the two pore systems at a microscopic level (see Khalili et al. [7], Masters et al. [8]).

(c) *The equations of effective stress concept*

$$t_{lj} = t'_{lj} - \left(\beta_1 \hat{p}_1 + \beta_2 \hat{p}_2\right) \delta_{lj}, \qquad l, j = 1, 2, 3, \tag{5}$$

where

$$t'_{lj} = 2\mu e_{lj} + \lambda e_{rr}\delta_{lj} + 2\mu^* \dot{e}_{lj} + \lambda^* \dot{e}_{rr}\delta_{lj}$$

are the components of effective stress tensor, λ, μ, λ^* and μ^* are the constitutive coefficients, δ_{lj} is the Kronecker's delta.

(d) *The Darcy's law for materials with double porosity*

$$\begin{aligned} \mathbf{v}^{(1)} &= -\frac{1}{\hat{\mu}}\left(\hat{k}_1 \nabla \hat{p}_1 + \hat{k}_3 \nabla \hat{p}_2\right) - \rho_1 \mathbf{s}^{(1)}, \\ \mathbf{v}^{(2)} &= -\frac{1}{\hat{\mu}}\left(\hat{k}_3 \nabla \hat{p}_1 + \hat{k}_2 \nabla \hat{p}_2\right) - \rho_2 \mathbf{s}^{(2)}, \end{aligned} \tag{6}$$

where $\hat{\mu}$ is the fluid viscosity, $\hat{k}_1$ and $\hat{k}_2$ are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively; $\hat{k}_3$ is the cross-coupling permeability for fluid flow at the interface between the matrix and fissure phases; ρ_1, $\mathbf{s}^{(1)}$ and ρ_2, $\mathbf{s}^{(2)}$ are the densities of fluid and the external forces (such as gravity) for the pore and fissure phases, respectively.

In the following we assume that $\beta_1^2 + \beta_2^2 > 0$ (the case $\beta_1 = \beta_2 = 0$ is too simple to be considered).

Substituting Eqs. (3)–(6) into (1) and (2), we obtain the following system of equations of motion in the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity expressed in terms of the displacement vector $\hat{\mathbf{u}}$ and pressures $\hat{p}_1$ and $\hat{p}_2$:

$$\begin{aligned} &\mu\, \Delta\hat{\mathbf{u}} + (\lambda + \mu)\, \nabla \mathrm{div}\, \hat{\mathbf{u}} + \mu^* \Delta\dot{\hat{\mathbf{u}}} + (\lambda^* + \mu^*)\, \nabla \mathrm{div}\, \dot{\hat{\mathbf{u}}} \\ &\qquad\qquad -\beta_1 \nabla \hat{p}_1 - \beta_2 \nabla \hat{p}_2 = \rho\,(\ddot{\hat{\mathbf{u}}} - \hat{\mathbf{F}}), \\ &(k_1\Delta - \gamma)\hat{p}_1 + (k_3\Delta + \gamma)\hat{p}_2 - \alpha_1 \dot{\hat{p}}_1 - \alpha_3 \dot{\hat{p}}_2 - \beta_1 \mathrm{div}\, \dot{\hat{\mathbf{u}}} = -\rho_1 \mathrm{div}\, \mathbf{s}^{(1)}, \\ &(k_3\Delta + \gamma)\hat{p}_1 + (k_2\Delta - \gamma)\hat{p}_2 - \alpha_3 \dot{\hat{p}}_1 - \alpha_2 \dot{\hat{p}}_2 - \beta_2 \mathrm{div}\, \dot{\hat{\mathbf{u}}} = -\rho_2 \mathrm{div}\, \mathbf{s}^{(2)}, \end{aligned} \tag{7}$$

where Δ is the Laplacian operator, $k_j = \frac{\hat{k}_j}{\hat{\mu}}$ $(j = 1, 2, 3)$.

If the body force $\hat{\mathbf{F}}$ and the external forces $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ are assumed to be absent, and the displacement vector $\hat{\mathbf{u}}$ and the pressures $\hat{p}_1$ and $\hat{p}_2$ are postulated to have a harmonic time variation, that is,

$$\left\{\hat{\mathbf{u}}, \hat{p}_1, \hat{p}_2\right\}(\mathbf{x}, t) = \mathrm{Re}\left[\{\mathbf{u}, p_1, p_2\}(\mathbf{x})\, e^{-i\omega t}\right],$$

then from the system (7) we obtain the following system of steady vibrations in the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity

$$\begin{aligned} &\mu_1 \Delta\mathbf{u} + (\lambda_1 + \mu_1)\nabla \mathrm{div}\, \mathbf{u} - \beta_1 \nabla p_1 - \beta_2 \nabla p_2 + \rho\omega^2\, \mathbf{u} = \mathbf{0}, \\ &(k_1\Delta + a_1)p_1 + (k_3\Delta + a_3)p_2 + \beta'_1 \mathrm{div}\, \mathbf{u} = 0, \\ &(k_3\Delta + a_3)p_1 + (k_2\Delta + a_2)p_2 + \beta'_2 \mathrm{div}\, \mathbf{u} = 0, \end{aligned} \tag{8}$$

where $\lambda_1 = \lambda - i\omega\lambda^*$, $\mu_1 = \mu - i\omega\mu^*$, $a_j = i\omega\alpha_j - \gamma$, $a_3 = i\omega\alpha_3 + \gamma$, $\beta'_j = i\omega\beta_j$ $(j = 1, 2)$; ω is the oscillation frequency, $\omega > 0$.

Obviously, neglecting inertial effect $(\rho\, \ddot{\hat{\mathbf{u}}})$ in the first equation of (7), from (8) we obtain the following system of homogeneous equations of steady vibrations in the linear quasi-static theory of viscoelasticity for Kelvin–Voigt materials with double porosity:

$$\begin{aligned} &\mu_1 \Delta\mathbf{u} + (\lambda_1 + \mu_1)\nabla \mathrm{div}\, \mathbf{u} - \beta_1 \nabla p_1 - \beta_2 \nabla p_2 = \mathbf{0}, \\ &(k_1\Delta + a_1)p_1 + (k_3\Delta + a_3)p_2 + \beta'_1 \mathrm{div}\, \mathbf{u} = 0, \\ &(k_3\Delta + a_3)p_1 + (k_2\Delta + a_2)p_2 + \beta'_2 \mathrm{div}\, \mathbf{u} = 0. \end{aligned} \tag{9}$$

We introduce the second order matrix differential operators with constant coefficients:

$$\mathbf{A}(\mathbf{D_x}) = \left(A_{lj}(\mathbf{D_x})\right)_{5\times 5}, \qquad \mathbf{B}(\mathbf{D_x}) = \left(B_{lj}(\mathbf{D_x})\right)_{5\times 5},$$

where

$$
\begin{aligned}
&A_{lj}(\mathbf{D_x}) = \mu_1 \Delta\delta_{lj} + (\lambda_1 + \mu_1)\frac{\partial^2}{\partial x_l \partial x_j},\\
&B_{lj}(\mathbf{D_x}) = A_{lj}(\mathbf{D_x}) + \rho\omega^2\delta_{lj},\\
&A_{l;m+3}(\mathbf{D_x}) = B_{l;m+3}(\mathbf{D_x}) = -\beta_m \frac{\partial}{\partial x_l},\\
&A_{m+3;l}(\mathbf{D_x}) = B_{m+3;l}(\mathbf{D_x}) = \beta'_m \frac{\partial}{\partial x_l},\\
&A_{44}(\mathbf{D_x}) = B_{44}(\mathbf{D_x}) = k_1\Delta + a_1,\\
&A_{45}(\mathbf{D_x}) = A_{54}(\mathbf{D_x}) = B_{45}(\mathbf{D_x}) = B_{54}(\mathbf{D_x}) = k_3\Delta + a_3,\\
&A_{55}(\mathbf{D_x}) = B_{55}(\mathbf{D_x}) = k_2\Delta + a_2,\\
&\mathbf{D_x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \qquad m = 1, 2, \qquad l, j = 1, 2, 3.
\end{aligned}
$$

It is easily seen that the systems (9) and (8) can be written as

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{10}$$

and

$$\mathbf{B}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}, \tag{11}$$

respectively, where $\mathbf{U} = (\mathbf{u}, p_1, p_2)$ is a five-component vector function and $\mathbf{x} \in \mathbb{R}^3$.

Obviously, $\mathbf{A}(\mathbf{D_x})$ and $\mathbf{B}(\mathbf{D_x})$ are elliptic differential operators if and only if

$$\mu_1\, \mu_0\, k \neq 0 \tag{12}$$

where $\mu_0 = \lambda_1 + 2\mu_1,\ k = k_1k_2 - k_3^2$.

3. Fundamental solutions

3.1. Fundamental solution of system of equations of quasi-static theory

The fundamental solution of system (9) (the fundamental matrix of operator $\mathbf{A}(\mathbf{D_x})$) is the matrix $\boldsymbol{\Gamma}(\mathbf{x}) = \left(\Gamma_{lj}(\mathbf{x})\right)_{5\times5}$ satisfying condition in the class of generalized functions (for example, see Hörmander [35])

$$\mathbf{A}(\mathbf{D_x})\boldsymbol{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \tag{13}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = \left(\delta_{lj}\right)_{5\times5}$ is the unit matrix, $\mathbf{x} \in \mathbb{R}^3$.

In this section the matrix $\boldsymbol{\Gamma}$ is constructed in terms of elementary functions and some of its basic properties are established.

We consider the system of nonhomogeneous equations

$$
\begin{aligned}
&\mu_1\, \Delta\mathbf{u} + (\lambda_1 + \mu_1)\, \nabla \mathrm{div}\, \mathbf{u} + \beta'_1\, \nabla\, p_1 + \beta'_2\, \nabla\, p_2 = \mathbf{F}',\\
&(k_1\, \Delta + a_1)p_1 + (k_3\, \Delta + a_3)p_2 - \beta_1\, \mathrm{div}\, \mathbf{u} = F_4,\\
&(k_3\, \Delta + a_3)p_1 + (k_2\, \Delta + a_2)p_2 - \beta_2\, \mathrm{div}\, \mathbf{u} = F_5,
\end{aligned} \tag{14}
$$

where $\mathbf{F}' = (F_1, F_2, F_3)$ is a three-component vector function, F_4 and F_5 are scalar functions on $\mathbb{R}^3$. As one may easily verify, the system (14) may be written in the form

$$\mathbf{A}^\top(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{15}$$

where $\mathbf{A}^\top$ is the transpose of matrix $\mathbf{A}$, $\mathbf{F} = (\mathbf{F}', F_4, F_5)$ is a five-component vector function and $\mathbf{x} \in \mathbb{R}^3$.

Applying the operator div to the first equation of (14) from system (14) we obtain

$$\begin{aligned}
&\mu_0 \Delta \operatorname{div} \mathbf{u} + \beta_1' \, \Delta \, p_1 + \beta_2' \, \Delta \, p_2 = \operatorname{div} \mathbf{F}', \\
&(k_1 \Delta + a_1) p_1 + (k_3 \Delta + a_3) p_2 - \beta_1 \operatorname{div} \mathbf{u} = F_4, \\
&(k_3 \Delta + a_3) p_1 + (k_2 \Delta + a_2) p_2 - \beta_2 \operatorname{div} \mathbf{u} = F_5.
\end{aligned} \tag{16}$$

From (16) we have

$$\mathbf{C}(\Delta)\mathbf{V}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x}), \tag{17}$$

where $\mathbf{V} = (\operatorname{div} \mathbf{u}, p_1, p_2)$, $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) = (\operatorname{div} \mathbf{F}', F_4, F_5)$ and

$$\mathbf{C}(\Delta) = \left(C_{lj}(\Delta)\right)_{3\times 3} = \begin{pmatrix} \mu_0 \Delta & \beta_1' \, \Delta & \beta_2' \, \Delta \\ -\beta_1 & k_1 \Delta + a_1 & k_3 \Delta + a_3 \\ -\beta_2 & k_3 \Delta + a_3 & k_2 \Delta + a_2 \end{pmatrix}_{3\times 3}.$$

We introduce the notation

$$\mathbf{C}'(\Delta) = \left(C'_{lj}(\Delta)\right)_{3\times 3} = \begin{pmatrix} \mu_0 & \beta_1' & \beta_2' \\ -\beta_1 & k_1 \Delta + a_1 & k_3 \Delta + a_3 \\ -\beta_2 & k_3 \Delta + a_3 & k_2 \Delta + a_2 \end{pmatrix}_{3\times 3},$$

$$\Lambda_1(\Delta) = \frac{1}{k\mu_0} \det \mathbf{C}'(\Delta).$$

It is easily seen that $\Lambda_1(-\tau) = 0$ is a quadratic equation and there exist two roots τ_1^2 and τ_2^2 (with respect to τ). Then we have

$$\Lambda_1(\Delta) = (\Delta + \tau_1^2)(\Delta + \tau_2^2).$$

The system (17) implies

$$\Lambda_1(\Delta)\,\mathbf{V} = \boldsymbol{\Phi}, \tag{18}$$

where

$$\boldsymbol{\Phi} = (\Phi_1, \Phi_2, \Phi_3), \qquad \Phi_j = \frac{1}{k\mu_0} \sum_{l=1}^{3} C^*_{lj} \, \psi_l, \qquad j = 1, 2, 3 \tag{19}$$

and C^*_{lj} is the cofactor of element C_{lj} of the matrix $\mathbf{C}$.

Now applying the operator $\Lambda_1(\Delta)$ to the first equation of (12) and taking into account (18), we obtain

$$\Lambda_1(\Delta)(\Delta + \tau_3^2)\mathbf{u} = \mathbf{F}'', \tag{20}$$

where $\tau_3^2 = \frac{\rho\omega^2}{\mu_1}$ and

$$\mathbf{F}'' = \frac{1}{\mu_1}\left[\Lambda_1(\Delta)\mathbf{F}' - (\lambda_1 + \mu_1)\nabla \Phi_1 - \beta_1' \nabla \Phi_2 - \beta_2' \nabla \Phi_3\right]. \tag{21}$$

On the basis of (18) and (20) we get

$$\boldsymbol{\Lambda}(\Delta)\mathbf{U}(\mathbf{x}) = \boldsymbol{\Phi}'(\mathbf{x}), \tag{22}$$

where $\boldsymbol{\Phi}' = \left(\mathbf{F}'', \Phi_2, \Phi_3\right)$ is a five-component vector and

$$\boldsymbol{\Lambda}(\Delta) = \left(\Lambda_{lj}(\Delta)\right)_{5\times 5}, \qquad \Lambda_{11}(\Delta) = \Lambda_{22}(\Delta) = \Lambda_{33}(\Delta) = \Lambda_1(\Delta)(\Delta + \tau_3^2),$$

$$\Lambda_{44}(\Delta) = \Lambda_{55}(\Delta) = \Lambda_1(\Delta), \qquad \Lambda_{lj}(\Delta) = 0, \qquad l, j = 1, 2, \ldots, 5, \qquad l \neq j.$$

We introduce the notations

$$
\begin{aligned}
&n_{j1}(\Delta)=-\frac{1}{k\mu_1\mu_0}\left[(\lambda_1+\mu_1)C^*_{j1}(\Delta)+\beta'_1C^*_{j2}(\Delta)+\beta'_2C^*_{j3}(\Delta)\right],\\
&n_{jl}(\Delta)=\frac{1}{k\mu_0}C^*_{jl}(\Delta),\qquad j=1,2,3,\qquad l=2,3.
\end{aligned}
\tag{23}
$$

In view of (19) and (23), from (21) it follows that

$$
\begin{aligned}
&\mathbf{F}''=\left[\frac{1}{\mu_1}\Lambda_1(\Delta)\mathbf{I}+n_{11}(\Delta)\,\nabla\mathrm{div}\right]\mathbf{F}'+n_{21}(\Delta)\,\nabla F_4+n_{31}(\Delta)\,\nabla F_5,\\
&\Phi_m=n_{1m}(\Delta)\,\mathrm{div}\,\mathbf{F}+n_{2m}(\Delta)F_4+n_{3m}(\Delta)F_5,\qquad m=2,3,
\end{aligned}
\tag{24}
$$

where $\mathbf{I}=\left(\delta_{lj}\right)_{3\times 3}$ is the unit matrix.

Thus, from (24) we have

$$
\mathbf{\Phi}'(\mathbf{x})=\mathbf{L}^\top(\mathbf{D_x})\,\mathbf{F}(\mathbf{x}),
\tag{25}
$$

where

$$
\begin{aligned}
&\mathbf{L}(\mathbf{D_x})=\left(L_{lj}(\mathbf{D_x})\right)_{5\times5},\qquad L_{lj}(\mathbf{D_x})=\frac{1}{\mu_1}\Lambda_1(\Delta)\,\delta_{lj}+n_{11}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
&L_{l;m+2}(\mathbf{D_x})=n_{1m}(\Delta)\frac{\partial}{\partial x_l},\qquad L_{m+2;l}(\mathbf{D_x})=n_{m1}(\Delta)\frac{\partial}{\partial x_l},\\
&L_{m+2;r+2}(\mathbf{D_x})=n_{mr}(\Delta),\qquad l,j=1,2,3,\qquad m,r=2,3.
\end{aligned}
\tag{26}
$$

By virtue of (15) and (25), from (22) it follows that $\mathbf{\Lambda U}=\mathbf{L}^\top\mathbf{A}^\top\mathbf{U}$. It is obvious that $\mathbf{L}^\top\mathbf{A}^\top=\mathbf{\Lambda}$ and, hence,

$$
\mathbf{A}(\mathbf{D_x})\mathbf{L}(\mathbf{D_x})=\mathbf{\Lambda}(\Delta).
\tag{27}
$$

We assume that $\tau_1^2\neq\tau_2^2$ and $\tau_j\neq 0\,(j=1,2)$. Let

$$
\begin{aligned}
&\mathbf{Y}(\mathbf{x})=(Y_{lm}(\mathbf{x}))_{5\times5},\qquad Y_{11}(\mathbf{x})=Y_{22}(\mathbf{x})=Y_{33}(\mathbf{x})=\sum_{j=1}^{4}\eta_j\gamma_j(\mathbf{x}),\\
&Y_{44}(\mathbf{x})=Y_{55}(\mathbf{x})=\sum_{j=1}^{2}\eta_{j+4}\gamma_j(\mathbf{x})+\eta_4\gamma_3(\mathbf{x}),\qquad Y_{lm}(\mathbf{x})=0,\\
&l\neq m,\qquad l,m=1,2,\ldots,5,
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
&\gamma_j(\mathbf{x})=-\frac{e^{i\tau_j|\mathbf{x}|}}{4\pi\,|\mathbf{x}|},\qquad \gamma_3(\mathbf{x})=-\frac{1}{4\pi\,|\mathbf{x}|},\qquad \gamma_4(\mathbf{x})=-\frac{|\mathbf{x}|}{8\pi},\\
&\eta_1=\frac{1}{\tau_1^4(\tau_2^2-\tau_1^2)},\qquad \eta_2=\frac{1}{\tau_2^4(\tau_1^2-\tau_2^2)},\qquad \eta_3=-\frac{\tau_1^2+\tau_2^2}{\tau_1^4\tau_2^4},\\
&\eta_4=\frac{1}{\tau_1^2\tau_2^2},\qquad \eta_5=\frac{1}{\tau_1^4(\tau_1^2-\tau_2^2)},\qquad \eta_6=\frac{1}{\tau_2^4(\tau_2^2-\tau_1^2)},\qquad j=1,2.
\end{aligned}
$$

Lemma 1. *The matrix* $\mathbf{Y}$ *is the fundamental solution of operator* $\mathbf{\Lambda}(\Delta)$, *that is,*

$$
\mathbf{\Lambda}(\Delta)\mathbf{Y}(\mathbf{x})=\delta(\mathbf{x})\,\mathbf{J},
\tag{29}
$$

where $\mathbf{x}\in\mathbb{R}^3$.

Lemma 1 is proved by virtue of (28) and equalities

$$
(\Delta+\tau_j^2)\gamma_j(\mathbf{x})=\delta(\mathbf{x}),\qquad \Delta\gamma_3(\mathbf{x})=\delta(\mathbf{x}),\qquad \Delta^2\gamma_4(\mathbf{x})=\delta(\mathbf{x}).
$$

We introduce the matrix

$$
\mathbf{\Gamma}(\mathbf{x})=\mathbf{L}(\mathbf{D_x})\,\mathbf{Y}(\mathbf{x}).
\tag{30}
$$

Using identities (27) and (29) from (30) we get

$$\mathbf{A}\left(\mathbf{D_x}\right)\boldsymbol{\Gamma}(\mathbf{x}) = \mathbf{A}\left(\mathbf{D_x}\right)\mathbf{L}\left(\mathbf{D_x}\right)\mathbf{Y}\left(\mathbf{x}\right) = \boldsymbol{\Lambda}\left(\Delta\right)\mathbf{Y}\left(\mathbf{x}\right) = \delta\left(\mathbf{x}\right)\mathbf{J}.$$

Hence, $\boldsymbol{\Gamma}(\mathbf{x})$ is the solution of (13). We have thereby proved the following

Theorem 1. *If the condition* (12) *is satisfied, then the matrix* $\boldsymbol{\Gamma}\left(\mathbf{x}\right)$ *defined by* (30) *is the fundamental solution of system* (9)*, where the matrices* $\mathbf{L}\left(\mathbf{D_x}\right)$ *and* $\mathbf{Y}\left(\mathbf{x}\right)$ *are given by* (26) *and* (28)*, respectively.*

Remark 1. The matrix $\boldsymbol{\Gamma}\left(\mathbf{x}\right)$ is constructed by harmonic (γ_3), biharmonic (γ_4) and metaharmonic (γ_1 and γ_2) functions.

3.2. Fundamental solution of system of equations of steady vibrations

In a quite similar manner as in the previous subsection we can construct the fundamental solution of system (8).

The fundamental solution of system (8) (the fundamental matrix of operator $\mathbf{B}(\mathbf{D_x})$) is the matrix $\boldsymbol{\Theta}(\mathbf{x}) = \left(\Theta_{lj}(\mathbf{x})\right)_{5\times 5}$ satisfying condition in the class of generalized functions

$$\mathbf{B}(\mathbf{D_x})\boldsymbol{\Theta}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \tag{31}$$

where $\mathbf{x} \in \mathbb{R}^3$.

We introduce the notation:

(1)

$$\hat{\mathbf{C}}(\Delta) = \left(\hat{C}_{lj}(\Delta)\right)_{3\times 3} = \begin{pmatrix} \mu_0\Delta + \rho\omega^2 & \beta_1'\,\Delta & \beta_2'\,\Delta \\ -\beta_1 & k_1\Delta + a_1 & k_3\Delta + a_3 \\ -\beta_2 & k_3\Delta + a_3 & k_2\Delta + a_2 \end{pmatrix}_{3\times 3}.$$

(2)

$$\hat{\Lambda}_1(\Delta) = \frac{1}{k\mu_0}\,\det\hat{\mathbf{C}}\left(\Delta\right) = (\Delta + \xi_1^2)(\Delta + \xi_2^2)(\Delta + \xi_3^2),$$

where ξ_1^2, ξ_2^2 and ξ_3^2 are the roots of equation $\Lambda_1(-\xi) = 0$ (with respect to ξ).

(3)

$$\hat{\boldsymbol{\Lambda}}(\Delta) = \left(\hat{\Lambda}_{lj}\left(\Delta\right)\right)_{5\times 5}, \qquad \hat{\Lambda}_{11}(\Delta) = \hat{\Lambda}_{22}(\Delta) = \hat{\Lambda}_{33}(\Delta) = \hat{\Lambda}_1(\Delta)(\Delta + \xi_4^2),$$

$$\hat{\Lambda}_{44}(\Delta) = \hat{\Lambda}_{55}(\Delta) = \hat{\Lambda}_1(\Delta), \qquad \hat{\Lambda}_{lj}(\Delta) = 0, \qquad l, j = 1, 2, \ldots, 5, \qquad l \neq j$$

where $\xi_4^2 = \tau_3^2$. We assume that $\xi_l^2 \neq \xi_j^2$, where $l, j = 1, 2, 3, 4$ and $l \neq j$.

(4)

$$\hat{n}_{j1}(\Delta) = -\frac{1}{k\mu_1\mu_0}\left[(\lambda_1 + \mu_1)\hat{C}_{j1}^*(\Delta) + i\omega\beta_1\hat{C}_{j2}^*(\Delta) + i\omega\beta_2\hat{C}_{j3}^*(\Delta)\right],$$

$$\hat{n}_{jl}(\Delta) = \frac{1}{k\mu_0}\hat{C}_{jl}^*(\Delta), \qquad j = 1, 2, 3, \qquad l = 2, 3$$

where $\hat{C}_{lj}^*$ is the cofactor of element $\hat{C}_{lj}$ of the matrix $\hat{\mathbf{C}}$.

(5)

$$\begin{aligned} &\hat{\mathbf{L}}\left(\mathbf{D_x}\right) = \left(\hat{L}_{lj}\left(\mathbf{D_x}\right)\right)_{5\times 5}, \qquad \hat{L}_{lj}\left(\mathbf{D_x}\right) = \frac{1}{\mu_1}\Lambda_1(\Delta)\,\delta_{lj} + \hat{n}_{11}(\Delta)\frac{\partial^2}{\partial x_l \partial x_j}, \\ &\hat{L}_{l;m+2}\left(\mathbf{D_x}\right) = \hat{n}_{1m}(\Delta)\frac{\partial}{\partial x_l}, \qquad \hat{L}_{m+2;l}\left(\mathbf{D_x}\right) = \hat{n}_{m1}(\Delta)\frac{\partial}{\partial x_l}, \\ &\hat{L}_{m+2;4}\left(\mathbf{D_x}\right) = \hat{n}_{m2}(\Delta), \qquad \hat{L}_{m+2;5}\left(\mathbf{D_x}\right) = \hat{n}_{m3}(\Delta), \\ &l, j = 1, 2, 3, \qquad m = 2, 3. \end{aligned} \tag{32}$$

(6)

$$
\begin{aligned}
&\hat{\mathbf{Y}}(\mathbf{x}) = \left(\hat{Y}_{lm}(\mathbf{x})\right)_{5\times 5}, \qquad \hat{Y}_{11}(\mathbf{x}) = \hat{Y}_{22}(\mathbf{x}) = \hat{Y}_{33}(\mathbf{x}) = \sum_{j=1}^{4} \eta_{2j}\hat{\gamma}_j(\mathbf{x}),\\
&\hat{Y}_{44}(\mathbf{x}) = \hat{Y}_{55}(\mathbf{x}) = \sum_{j=1}^{3} \eta_{1j}\hat{\gamma}_j(\mathbf{x}), \qquad \hat{Y}_{lm}(\mathbf{x}) = 0,\\
&l \neq m, \qquad l, m = 1, 2, \dots, 5,
\end{aligned}
\tag{33}
$$

where

$$
\hat{\gamma}_j(\mathbf{x}) = -\frac{e^{i\xi_j|\mathbf{x}|}}{4\pi\,|\mathbf{x}|}
\tag{34}
$$

and

$$
\eta_{1m} = \prod_{l=1,\, l\neq m}^{3} (\xi_l^2 - \xi_m^2)^{-1}, \qquad \eta_{2j} = \prod_{l=1,\, l\neq j}^{4} (\xi_l^2 - \xi_j^2)^{-1},
$$
$$
m = 1, 2, 3, \qquad j = 1, 2, 3, 4.
$$

Quite similarly as in the previous subsection we can prove the following identities

$$
\mathbf{B}(\mathbf{D}_\mathbf{x})\hat{\mathbf{L}}(\mathbf{D}_\mathbf{x}) = \hat{\mathbf{\Lambda}}(\Delta). \tag{35}
$$

and

$$
\hat{\mathbf{\Lambda}}(\Delta)\hat{\mathbf{Y}}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{J}, \tag{36}
$$

where $\mathbf{x} \in \mathbb{R}^3$.

We introduce the matrix

$$
\mathbf{\Theta}(\mathbf{x}) = \hat{\mathbf{L}}(\mathbf{D}_\mathbf{x})\,\hat{\mathbf{Y}}(\mathbf{x}). \tag{37}
$$

Using (35) and (36) from (37) we get

$$
\mathbf{B}(\mathbf{D}_\mathbf{x})\,\mathbf{\Theta}(\mathbf{x}) = \mathbf{B}(\mathbf{D}_\mathbf{x})\,\hat{\mathbf{L}}(\mathbf{D}_\mathbf{x})\,\hat{\mathbf{Y}}(\mathbf{x}) = \hat{\mathbf{\Lambda}}(\Delta)\,\hat{\mathbf{Y}}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{J}.
$$

Hence, $\mathbf{\Theta}(\mathbf{x})$ is the solution of (31). We have the following

Theorem 2. *If the condition* (12) *is satisfied, then the matrix* $\mathbf{\Theta}(\mathbf{x})$ *defined by* (37) *is the fundamental solution of system* (8)*, where the matrices* $\hat{\mathbf{L}}(\mathbf{D}_\mathbf{x})$ *and* $\hat{\mathbf{Y}}(\mathbf{x})$ *are given by* (32) *and* (33)*, respectively.*

Remark 2. The fundamental solution of the system (8) is constructed by four metaharmonic functions $\hat{\gamma}_j$, where $j = 1, 2, 3, 4$ (see (34)). Obviously, the matrix $\mathbf{\Gamma}(\mathbf{x})$ is not possible to obtain from $\mathbf{\Theta}(\mathbf{x})$ by replacing $\rho = 0$.

3.3. Basic properties of fundamental solutions

Theorems 1 and 2 lead to the following results.

Theorem 3. *Each column of the matrices* $\mathbf{\Gamma}(\mathbf{x})$ *and* $\mathbf{\Theta}(\mathbf{x})$ *is a solution of homogeneous equations* (11) and (10)*, respectively, at every point* $\mathbf{x} \in \mathbb{R}^3$ *except the origin.*

Theorem 4. *The relations*

$$
\begin{aligned}
&\Gamma_{lj}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \qquad \Gamma_{mq}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right),\\
&\Gamma_{mj}(\mathbf{x}) = O(1), \qquad \Gamma_{jm}(\mathbf{x}) = O(1),\\
&\Theta_{lj}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \qquad \Theta_{mq}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right),\\
&\Theta_{mj}(\mathbf{x}) = O(1), \qquad \Theta_{jm}(\mathbf{x}) = O(1)
\end{aligned}
$$

hold in the neighborhood of the origin, where $l, j = 1, 2, 3,\ m, q = 4, 5.$

Corollary 1. *If condition* (12) *is satisfied, then the fundamental solution of the system*

$$\mu_1 \Delta \mathbf{u} + (\lambda_1 + \mu_1)\nabla \operatorname{div} \mathbf{u} = \mathbf{0},$$

$$k_1 \Delta p_1 + k_3 \Delta p_2 = 0,$$

$$k_3 \Delta p_1 + k_2 \Delta p_2 = 0$$

is the matrix $\mathbf{\Psi}(\mathbf{x}) = \left(\Psi_{lj}(\mathbf{x})\right)_{5\times 5}$, *where*

$$\Psi_{lj}(\mathbf{x}) = \lambda' \frac{\delta_{lj}}{|\mathbf{x}|} + \mu' \frac{x_l x_j}{|\mathbf{x}|^3}, \qquad \Psi_{44}(\mathbf{x}) = \frac{k_2}{k}\gamma_1(\mathbf{x}),$$

$$\Psi_{45}(\mathbf{x}) = \Psi_{54}(\mathbf{x}) = -\frac{k_3}{k}\gamma_1(\mathbf{x}), \qquad \Psi_{55}(\mathbf{x}) = \frac{k_1}{k}\gamma_1(\mathbf{x}),$$

$$\Psi_{lm} = \Psi_{ml} = 0, \qquad \lambda' = -\frac{\lambda_1 + 3\mu_1}{8\pi\mu_1\mu_0}, \qquad \mu' = -\frac{\lambda_1 + \mu_1}{8\pi\mu_1\mu_0},$$

$$l, j = 1, 2, 3, \qquad m = 4, 5.$$

Corollary 2. *The relations*

$$\Psi_{lj}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \qquad \Psi_{mn}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right)$$

hold in the neighborhood of the origin, where $l, j = 1, 2, 3$ *and* $m, n = 4, 5$.

Now we can establish the singular part of the matrices $\mathbf{\Gamma}(\mathbf{x})$ and $\mathbf{\Theta}(\mathbf{x})$ in the neighborhood of the origin.

Theorem 5. *The relations*

$$\Gamma_{lj}(\mathbf{x}) - \Psi_{lj}(\mathbf{x}) = const + O(|\mathbf{x}|),$$

$$\Theta_{lj}(\mathbf{x}) - \Psi_{lj}(\mathbf{x}) = const + O(|\mathbf{x}|)$$

hold in the neighborhood of the origin, where $l, j = 1, 2, \ldots, 5$.

Thus, on the basis of Theorems 4 and 5 and Corollary 2 the matrix $\mathbf{\Psi}(\mathbf{x})$ is the singular part of the fundamental solutions $\mathbf{\Gamma}(\mathbf{x})$ and $\mathbf{\Theta}(\mathbf{x})$ in the neighborhood of the origin.

4. Greens formulae

In this section Green's formulae of the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity are obtained. In what follows we assume that the constitutive coefficients satisfy the conditions

$$\mu^* > 0, \qquad k_1 > 0, \qquad k > 0, \qquad \alpha_1 > 0, \qquad \alpha_1\alpha_2 - \alpha_3^2 > 0 \tag{38}$$

and

$$3\lambda^* + 2\mu^* > 0. \tag{39}$$

Let S be the closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $S \in C^{1,\nu}$, $0 < \nu \le 1$, $\overline{\Omega^+} = \Omega^+ \cup S$. The scalar product of two vectors $\mathbf{U} = (u_1, u_2, \ldots, u_5)$ and $\mathbf{V} = (v_1, v_2, \ldots, v_5)$ is denoted by $\mathbf{U}\cdot\mathbf{V} = \sum_{j=1}^{5} u_j \bar{v}_j$, where $\bar{v}_j$ is the complex conjugate of v_j.

Definition 1. A vector function $\mathbf{U} = (\mathbf{u}, p_1, p_2) = (u_1, u_2, \ldots, u_5)$ is called *regular* in Ω^+ if $u_j \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})$ for $j = 1, 2, \ldots, 5$.

In the sequel, we use the matrix differential operator

$$\mathbf{P}(\mathbf{D_x}, \mathbf{n}) = (P_{lj}(\mathbf{D_x}, \mathbf{n}))_{5\times 5},$$

where

$$
\begin{aligned}
&P_{lj}(\mathbf{D_x},\mathbf{n}) = \mu_1\delta_{lj}\frac{\partial}{\partial\mathbf{n}} + \mu_1 n_j\frac{\partial}{\partial x_l} + \lambda_1 n_l\frac{\partial}{\partial x_j}, \qquad P_{l;m+3}(\mathbf{D_x},\mathbf{n}) = -\beta_m\, n_l,\\
&P_{m+3;l}(\mathbf{D_x},\mathbf{n}) = 0, \qquad P_{44}(\mathbf{D_x},\mathbf{n}) = k_1\frac{\partial}{\partial\mathbf{n}},\\
&P_{45}(\mathbf{D_x},\mathbf{n}) = P_{54}(\mathbf{D_x},\mathbf{n}) = k_3\frac{\partial}{\partial\mathbf{n}}, \qquad P_{55}(\mathbf{D_x},\mathbf{n}) = k_2\frac{\partial}{\partial\mathbf{n}},\\
&m = 1,2, \qquad l,j = 1,2,3,
\end{aligned}
$$

$\mathbf{n} = (n_1, n_2, n_3)$ is the unit vector, $\frac{\partial}{\partial\mathbf{n}}$ is the derivative along the vector $\mathbf{n}$.

Let $\mathbf{U} = (\mathbf{u}, p_1, p_2)$ and $\mathbf{U}' = (\mathbf{u}', p'_1, p'_2)$ be five-component complex vector fields in Ω^+, $\mathbf{u}' = (u'_1, u'_2, u'_3)$.

4.1. Greens formulae in the quasi-static theory

We introduce the notation

$$
\begin{aligned}
&\mathbf{A}^{(0)}(\mathbf{D_x}) = \left(A^{(0)}_{lj}(\mathbf{D_x})\right)_{3\times 3}, \qquad A^{(0)}_{lj}(\mathbf{D_x}) = A_{lj}(\mathbf{D_x}),\\
&\mathbf{P}^{(0)}(\mathbf{D_x},\mathbf{n}) = \left(P^{(0)}_{lj}(\mathbf{D_x},\mathbf{n})\right)_{3\times 3}, \qquad P^{(0)}_{lj}(\mathbf{D_x},\mathbf{n}) = P_{lj}(\mathbf{D_x},\mathbf{n}),\\
&W^{(0)}(\mathbf{u},\mathbf{u}') = \frac{1}{3}(3\lambda_1 + 2\mu_1)\operatorname{div}\mathbf{u}\operatorname{div}\bar{\mathbf{u}}'\\
&+\mu_1\left[\frac{1}{2}\sum_{l,j=1;\, l\neq j}^{3}\left(\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right)\left(\frac{\partial \bar{u}'_j}{\partial x_l} + \frac{\partial \bar{u}'_l}{\partial x_j}\right) + \frac{1}{3}\sum_{l,j=1}^{3}\left(\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial \bar{u}'_l}{\partial x_l} - \frac{\partial \bar{u}'_j}{\partial x_j}\right)\right].
\end{aligned}
\tag{40}
$$

$\mathbf{P}^{(0)}(\mathbf{D_x},\mathbf{n})u$ is the stress vector in the classical theory of viscoelasticity (see Svanadze [36]).

Lemma 2. *Let $\mathbf{U} = (\mathbf{u}, p_1, p_2)$ and $\mathbf{U}' = (\mathbf{u}', p'_1, p'_2)$ be the regular vectors in Ω^+, then*

$$
\begin{aligned}
&\int_{\Omega^+}\left[\mathbf{A}^{(0)}(\mathbf{D_x}) - \beta_1\nabla p_1 - \beta_2\nabla p_2\right]\mathbf{U}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x})\,d\mathbf{x}\\
&\quad+\int_{\Omega^+}\left[W^{(0)}(\mathbf{u},\mathbf{u}') - (\beta_1 p_1 + \beta_2 p_2)\operatorname{div}\bar{\mathbf{u}}'(\mathbf{x})\right]d\mathbf{x}\\
&\quad=\int_S\left[\mathbf{P}^{(0)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{u}(\mathbf{z}) - (\beta_1 p_1 + \beta_2 p_2)\,\mathbf{n}(\mathbf{z})\right]\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S,
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&\int_{\Omega^+}\left[(k_1\Delta + a_1)p_1 + (k_3\Delta + a_3)p_2 + \beta'_1\operatorname{div}\mathbf{u}\right]\bar{p}'_1(\mathbf{x})\,d\mathbf{x}\\
&\quad+\int_{\Omega^+}\left[(k_3\Delta + a_3)p_1 + (k_2\Delta + a_2)p_2 + \beta'_2\operatorname{div}\mathbf{u}\right]\bar{p}'_2(\mathbf{x})\,d\mathbf{x}\\
&\quad+\int_{\Omega^+}\left[(k_1\nabla p_1 + k_3\nabla p_2)\cdot\nabla p'_1 + (k_3\nabla p_1 + k_2\nabla p_2)\cdot\nabla p'_2\right]d\mathbf{x}\\
&\quad-\int_{\Omega^+}\left[a_1 p_1\bar{p}'_1 + a_3\left(p_1\bar{p}'_2 + p_2\bar{p}'_1\right) + a_2 p_2\bar{p}'_2 + \operatorname{div}\mathbf{u}(\beta'_1\bar{p}'_1 + \beta'_2\bar{p}'_2)\right]d\mathbf{x}\\
&=\int_S\left[\left(k_1\frac{\partial p_1}{\partial\mathbf{n}} + k_3\frac{\partial p_2}{\partial\mathbf{n}}\right)\bar{p}'_1 + \left(k_3\frac{\partial p_1}{\partial\mathbf{n}} + k_2\frac{\partial p_2}{\partial\mathbf{n}}\right)\bar{p}'_2\right]\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S.
\end{aligned}
\tag{42}
$$

Proof. By virtue of Green's first formula of the classical theory of elasticity (see, e.g. Kupradze et al. [37])

$$
\int_{\Omega^+}\left[\mathbf{A}^{(0)}(\mathbf{D_x})\mathbf{u}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}) + W^{(0)}(\mathbf{u},\mathbf{u}')\right]d\mathbf{x} = \int_S \mathbf{P}^{(0)}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{u}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S
$$

and identities

$$\int_{\Omega^+}\left[\nabla p_j(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})+p_j(\mathbf{x})\,\mathrm{div}\,\bar{\mathbf{u}}(\mathbf{x})\right]d\mathbf{x}=\int_S p_j(\mathbf{z})\,\mathbf{n}(\mathbf{z})\cdot\mathbf{U}(\mathbf{z})\,d_{\mathbf{z}}S,$$
$$\int_{\Omega^+}\left[\Delta p_j(\mathbf{x})\bar{p}_l(\mathbf{x})+\nabla p_j(\mathbf{x})\cdot\nabla\bar{p}_l(\mathbf{x})\right]d\mathbf{x}=\int_S\frac{\partial p_j(\mathbf{z})}{\partial\mathbf{n}(\mathbf{z})}\,\bar{p}_l(\mathbf{z})\,d_{\mathbf{z}}S,\qquad j=1,2$$

we obtain (41) and (42). □

It is easy to see that Lemma 2 leads to the following result.

Theorem 6. *Let* $\mathbf{U}=(\mathbf{u},p_1,p_2)$ *be regular vector field in* Ω^+ *and* $\mathbf{U}'=(\mathbf{u}',p'_1,p'_2)\in C^1(\Omega^+)$*, then*

$$\int_{\Omega^+}\left[\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x})+W(\mathbf{U},\mathbf{U}')\right]d\mathbf{x}=\int_S\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S,\tag{43}$$

where

$$\begin{aligned}W(\mathbf{U},\mathbf{U}')&=W^{(0)}(\mathbf{u},\mathbf{u}')+k_1\nabla p_1\cdot\nabla p'_1+k_3(\nabla p_1\cdot\nabla p'_2+\nabla p_2\cdot\nabla p'_1)\\&\quad+k_2\nabla p_2\cdot\nabla p'_2-\left[a_1p_1\bar{p}'_1+a_3(p_1\bar{p}'_2+p_2\bar{p}'_1)+a_2p_2\bar{p}'_2\right]\\&\quad-(\beta_1\,p_1+\beta_2\,p_2)\,\mathrm{div}\,\bar{\mathbf{u}}'-\mathrm{div}\,\mathbf{u}(\beta'_1\,\bar{p}'_1+\beta'_2\,\bar{p}'_2).\end{aligned}$$

The formula (43) is Green's first identity in the linear quasi-static theory of viscoelasticity for Kelvin–Voigt materials with double porosity.

The matrix differential operator $\tilde{\mathbf{A}}(\mathbf{D_x})=\left(\tilde{A}_{lj}(\mathbf{D_x})\right)_{5\times5}$ is the associate operator of $\mathbf{A}(\mathbf{D_x})$, where $\tilde{\mathbf{A}}(\mathbf{D_x})=\mathbf{A}^\top(-\mathbf{D_x})$. It is easy to verify that the operator $\tilde{\mathbf{A}}(\mathbf{D_x})$ may be obtained from the operator $\mathbf{A}(\mathbf{D_x})$ by replacing β_j by β'_j $(j=1,2)$ and vice versa. Obviously, the associated system of equations is

$$\begin{aligned}&\mu_1\Delta\tilde{\mathbf{u}}+(\lambda_1+\mu_1)\nabla\mathrm{div}\,\tilde{\mathbf{u}}-\beta'_1\nabla\tilde{p}_1-\beta'_2\nabla\tilde{p}_2=\mathbf{0},\\&(k_1\Delta+a_1)\tilde{p}_1+(k_3\Delta+a_3)\tilde{p}_2+\beta_1\,\mathrm{div}\,\tilde{\mathbf{u}}=0,\\&(k_3\Delta+a_3)\tilde{p}_1+(k_2\,\Delta+a_2)\tilde{p}_2+\beta_2\,\mathrm{div}\,\tilde{\mathbf{u}}=0,\end{aligned}\tag{44}$$

where $\tilde{\mathbf{u}}$ is a three-component vector function, $\tilde{p}_1$ and $\tilde{p}_2$ are functions on Ω^+.

Let $\mathbf{U}=(\mathbf{u},p_1,p_2)=(U_1,U_2,\ldots,U_5)$, the vector $\tilde{\mathbf{U}}_j$ is the jth column of the matrix $\tilde{\mathbf{U}}=(\tilde{U}_{lj})_{5\times5}$, $\tilde{\mathbf{u}}_j=(\tilde{U}_{1j},\tilde{U}_{2j},\tilde{U}_{3j})^\top$, $\tilde{p}_{1j}=\tilde{U}_{4j}$, $\tilde{p}_{2j}=\tilde{U}_{5j}$, $j=1,2,\ldots,5$.

Theorem 7. *If* U *and* $\tilde{U}_j$ $(j=1,2,\ldots,5)$ *are regular vectors in* Ω^+*, then*

$$\begin{aligned}&\int_{\Omega^+}\left\{[\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^\top\mathbf{U}(\mathbf{y})-[\tilde{\mathbf{U}}(\mathbf{y})]^\top\mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y})\right\}d\mathbf{y}\\&\quad=\int_S\{[\tilde{\mathbf{P}}(\mathbf{D_z},\mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^\top\mathbf{U}(\mathbf{z})-[\tilde{\mathbf{U}}(\mathbf{z})]^\top\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\}d_{\mathbf{z}}S,\end{aligned}\tag{45}$$

where

$$\begin{aligned}&\tilde{\mathbf{P}}(\mathbf{D_x},\mathbf{n})=(\tilde{P}_{lj}(\mathbf{D_x},\mathbf{n}))_{5\times5},\qquad\tilde{P}_{lj}(\mathbf{D_x},\mathbf{n})=P_{lj}(\mathbf{D_x},\mathbf{n}),\\&\tilde{P}_{l;m+3}(\mathbf{D_x},\mathbf{n})=-\beta'_m\,n_l,\qquad\tilde{P}_{m+3;l}(\mathbf{D_x},\mathbf{n})=0,\\&\tilde{P}_{m+3;r+3}(\mathbf{D_x},\mathbf{n})=P_{m+3;r+3}(\mathbf{D_x},\mathbf{n}),\qquad l,j=1,2,3,\qquad m,r=1,2.\end{aligned}$$

Theorem 7 is proved by direct calculation. The formula (45) is Green's second identity in the linear quasi-static theory of viscoelasticity for Kelvin–Voigt materials with double porosity.

In what follows we shall use the notation

$$\tilde{\mathbf{\Gamma}}(\mathbf{x})=\mathbf{\Gamma}^\top(-\mathbf{x}).$$

Obviously, the matrix $\tilde{\mathbf{\Gamma}}(\mathbf{x})$ is the fundamental solution of systems (44) (the fundamental matrix of operator $\tilde{\mathbf{A}}(\mathbf{D_x})$).

Theorem 7 leads to the following

Theorem 8. *If U is a regular vector in Ω^+, then*

$$\begin{aligned}\mathbf{U}(\mathbf{x}) = &\int_S \left\{ \left[\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{z}-\mathbf{x})\right]^\top \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x}-\mathbf{z})\mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z})\right\} d_{\mathbf{z}}S \\ &+ \int_{\Omega^+} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y})d\mathbf{y} \quad for \;\; \mathbf{x}\in\Omega^+.\end{aligned} \tag{46}$$

The formula (46) is Green's third identity for integral representation of regular vector (or Somigliana type representation) in the quasi-static linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity.

Obviously, Theorem 8 leads to the following

Corollary 3. *If U is a regular solution of* (10) *in Ω^+, then*

$$\mathbf{U}(\mathbf{x}) = \int_S \left\{ \left[\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{z}-\mathbf{x})\right]^\top \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x}-\mathbf{z})\mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z})\right\} d_{\mathbf{z}}S \tag{47}$$

The formula (47) is the integral representation of regular solution of the homogeneous Eq. (10) in the considered quasi-static theory.

4.2. Green's formulae of equations of steady vibrations

Quite similarly as in the previous subsection we can obtain the Green's formulae for equations of steady vibrations (11).

Theorem 9. *Let $\mathbf{U} = (\mathbf{u}, p_1, p_2)$ be regular vector field in Ω^+ and $\mathbf{U}' = (\mathbf{u}', p_1', p_2') \in C^1(\Omega^+)$, then*

$$\int_{\Omega^+} \left[\mathbf{B}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x}) + \hat{W}(\mathbf{U},\mathbf{U}')\right] d\mathbf{x} = \int_S \mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\, d_{\mathbf{z}}S, \tag{48}$$

where

$$\hat{W}(\mathbf{U},\mathbf{U}') = W^{(0)}(\mathbf{u},\mathbf{u}') - \rho\omega^2\mathbf{u}\cdot\mathbf{u}'.$$

Let $\tilde{\mathbf{B}}(\mathbf{D_x}) = \left(\tilde{B}_{lj}(\mathbf{D_x})\right)_{5\times5}$ be the associated operator of $\mathbf{B}(\mathbf{D_x})$, where $\tilde{\mathbf{B}}(\mathbf{D_x}) = \mathbf{B}^\top(-\mathbf{D_x})$. It is easy to verify that the operator $\tilde{\mathbf{B}}(\mathbf{D_x})$ may be obtained from the operator $\mathbf{B}(\mathbf{D_x})$ by replacing β_j by β_j' $(j = 1, 2)$ and vice versa.

Let $\tilde{\mathbf{U}}_j$ be the jth column of the matrix $\tilde{\mathbf{U}} = (\tilde{U}_{lj})_{5\times5}$, $j = 1, 2, \ldots, 5$, $\mathbf{U} = (\mathbf{u}, p_1, p_2) = (U_1, U_2, \ldots, U_5)$.

Theorem 10. *If U and $\tilde{U}_j$ $(j = 1, 2, \ldots, 5)$ are regular vectors in Ω^+, then*

$$\begin{aligned}&\int_{\Omega^+} \left\{ [\tilde{\mathbf{B}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{B}(\mathbf{D_y})\mathbf{U}(\mathbf{y})\right\} d\mathbf{y} \\ &\quad = \int_S \{[\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z})\} d_{\mathbf{z}}S.\end{aligned} \tag{49}$$

In what follows we shall use the notation

$$\tilde{\mathbf{\Theta}}(\mathbf{x}) = \mathbf{\Theta}^\top(-\mathbf{x}).$$

Obviously, the matrix $\tilde{\mathbf{\Theta}}(\mathbf{x})$ is the fundamental matrix of operator $\tilde{\mathbf{B}}(\mathbf{D_x})$.

Theorem 10 leads to the following

Theorem 11. *If U is a regular vector in Ω^+, then*

$$\begin{aligned}\mathbf{U}(\mathbf{x}) = &\int_S \left\{ \left[\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{\Theta}}(\mathbf{z}-\mathbf{x})\right]^\top \mathbf{U}(\mathbf{z}) - \mathbf{\Theta}(\mathbf{x}-\mathbf{z})\mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z})\right\} d_{\mathbf{z}}S \\ &+ \int_{\Omega^+} \mathbf{\Theta}(\mathbf{x}-\mathbf{y})\mathbf{B}(\mathbf{D_y})\mathbf{U}(\mathbf{y})d\mathbf{y} \quad for \;\; \mathbf{x}\in\Omega^+.\end{aligned} \tag{50}$$

Obviously, Theorem 8 leads to the following

Corollary 4. *If* U *is a regular solution of* (11) *in* Ω^+, *then*

$$\mathbf{U}(\mathbf{x}) = \int_S \left\{ \left[\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n}) \tilde{\mathbf{\Gamma}}(\mathbf{z}-\mathbf{x}) \right]^\top \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x}-\mathbf{z}) \mathbf{P}(\mathbf{D_z}, \mathbf{n}) \mathbf{U}(\mathbf{z}) \right\} d_{\mathbf{z}} S. \tag{51}$$

The formulae (48), (49) and (50) are Green's first, second and third identities in the linear theory of viscoelasticity of steady vibrations for Kelvin–Voigt materials with double porosity. The formula (51) is the integral representation of regular solution of the homogeneous equation (11) in this theory.

5. Uniqueness of solutions of BVPs

In the sequel, we use the matrix differential operators

$$\mathbf{P}^{(1)}(\mathbf{D_x}, \mathbf{n}) = \left(P^{(1)}_{lj}(\mathbf{D_x}, \mathbf{n}) \right)_{3\times 5}, \qquad \mathbf{P}^{(m)}(\mathbf{D_x}, \mathbf{n}) = (P^{(m)}_{1r}(\mathbf{D_x}, \mathbf{n}))_{1\times 2},$$

where

$$P^{(1)}_{lj}(\mathbf{D_x}, \mathbf{n}) = P_{lj}(\mathbf{D_x}, \mathbf{n}), \qquad P^{(m)}_{1r}(\mathbf{D_x}, \mathbf{n}) = P_{m+2;r+3}(\mathbf{D_x}, \mathbf{n}),$$
$$l = 1, 2, 3, \qquad j = 1, 2, \ldots, 5, \qquad m = 2, 3, \qquad r = 1, 2.$$

The basic internal BVPs in the quasi-static theory of viscoelasticity for Kelvin–Voigt materials with double porosity are formulated as follows.

Find a regular (classical) solution $\mathbf{U} = (\mathbf{u}, p_1, p_2)$ to system

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \qquad \text{for} \quad \mathbf{x} \in \Omega^+$$

satisfying the boundary condition

$$\lim_{\Omega^\pm \ni \mathbf{x} \to \ \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the *Problem* $(I)^{(q)}_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^\pm \ni \mathbf{x} \to \ \mathbf{z} \in S} \mathbf{P}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the *Problem* $(II)^{(q)}_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}^{(1)}(\mathbf{z}), \qquad \{\mathbf{P}^{(m)}(\mathbf{D_x}, \mathbf{n})\mathbf{p}(\mathbf{z})\}^+ = f_{m+2}(\mathbf{z})$$

in the *Problem* $(III)^{(q)}_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}^{(1)}(\mathbf{z}), \qquad \{\mathbf{P}^{(2)}(\mathbf{D_x}, \mathbf{n})\mathbf{p}(\mathbf{z})\}^+ = f_4(\mathbf{z}), \qquad \{p_2(\mathbf{z})\}^+ = f_5(\mathbf{z})$$

in the *Problem* $(IV)^{(q)}_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{U}(\mathbf{z})\}^\pm = \mathbf{f}^{(1)}(\mathbf{z}), \qquad \{p_1(\mathbf{z})\}^+ = f_4(\mathbf{z}), \qquad \{\mathbf{P}^{(3)}(\mathbf{D_x}, \mathbf{n})\mathbf{p}(\mathbf{z})\}^+ = f_5(\mathbf{z})$$

in the *Problem* $(V)^{(q)}_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{P}^{(1)}(\mathbf{D_x}, \mathbf{n})\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}^{(1)}(\mathbf{z}), \qquad \{p_1(\mathbf{z})\}^+ = f_4(\mathbf{z}), \qquad \{p_2(\mathbf{z})\}^+ = f_5(\mathbf{z})$$

in the *Problem* $(VI)^{(q)}_{\mathbf{F},\mathbf{f}}$,

$$\{\mathbf{P}^{(1)}(\mathbf{D_x}, \mathbf{n})\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}^{(1)}(\mathbf{z}), \qquad \{p_1(\mathbf{z})\}^+ = f_4(\mathbf{z}),$$
$$\{\mathbf{P}^{(3)}(\mathbf{D_x}, \mathbf{n})\mathbf{p}(\mathbf{z})\}^+ = f_5(\mathbf{z})$$

in the *Problem* $(VII)^{(q)}_{\mathbf{F},\mathbf{f}}$, and

$$\{\mathbf{P}^{(1)}(\mathbf{D_x}, \mathbf{n})\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}^{(1)}(\mathbf{z}), \qquad \{\mathbf{P}^{(2)}(\mathbf{D_x}, \mathbf{n})\mathbf{U}(\mathbf{z})\}^+ = f_4(\mathbf{z}),$$
$$\{p_2(\mathbf{z})\}^+ = f_5(\mathbf{z})$$

in the *Problem* $(VIII)^{(q)}_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}, \mathbf{f} = (\mathbf{f}^{(1)}, f_4, f_5)$ and $\mathbf{f}^{(1)} = (f_1, f_2, f_3)$ are known six- and three-component smooth vector functions, respectively; $\mathbf{p} = (p_1, p_2)$, $\mathbf{n}(\mathbf{z})$ is the external unit normal vector to S at $\mathbf{z}$, $m = 2, 3$.

Quite similarly, the basic internal BVP $(K)^{(s)}_{\mathbf{F},\mathbf{f}}$ of steady vibrations in the theory of viscoelasticity for Kelvin–Voigt materials with double porosity are formulated as follows: find a regular (classical) solution $\mathbf{U} = (\mathbf{u}, p_1, p_2)$ to system

$$\mathbf{B}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \qquad \text{for} \quad \mathbf{x} \in \Omega^+$$

satisfying the boundary condition of BVP $(K)^{(q)}_{\mathbf{F},\mathbf{f}}$, where $K = I, II, \dots, VIII$.

We are now in a position to study the uniqueness of regular solutions of the BVPs $(K)^{(q)}_{\mathbf{F},\mathbf{f}}$ and $(K)^{(s)}_{\mathbf{F},\mathbf{f}}$, where $K = I, II, \dots, VIII$. We have the following results.

Theorem 12. *If the conditions* (38) *and* (39) *are satisfied, then*
(a) the internal BVP $(K)^{(q)}_{\mathbf{F},\mathbf{f}}$ *admits at most one regular solution for* $K = I, III, IV, V$;
(b) any two regular solutions of the internal BVP $(K)^{(q)}_{\mathbf{F},\mathbf{f}}$ *may differ only for an additive vector* $\mathbf{U} = (\mathbf{u}, p_1, p_2)$, *for* $K = II, VI, VII, VIII$, *where* $\mathbf{u}$ *is the rigid displacement vector*

$$\mathbf{U}(\mathbf{x}) = \mathbf{a} + [\mathbf{b} \times \mathbf{x}] \tag{52}$$

and

$$p_1(\mathbf{x}) = p_2(\mathbf{x}) = 0 \tag{53}$$

for $\mathbf{x} \in \Omega^+$, $\mathbf{a}$ *and* $\mathbf{b}$ *are arbitrary three-component vectors and* $[\mathbf{b} \times \mathbf{x}]$ *is the vector product of* $\mathbf{b}$ *and* $\mathbf{x}$.

Proof. Suppose that there are two regular solutions of problem $(K)^{(q)}_{\mathbf{F},\mathbf{f}}$, where $K = I, II, \dots, VIII$. Then their difference $\mathbf{U}$ corresponds to zero data ($\mathbf{F} = \mathbf{f} = \mathbf{0}$), i.e. $\mathbf{U}$ is a regular solution of problem $(K)^{(q)}_{\mathbf{0},\mathbf{0}}$. Hence, the vector $\mathbf{U}$ is a regular (classical) solution to system of homogeneous Eq. (10) in the domain Ω^+ satisfying the homogeneous boundary condition

$$\{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z}) \cdot \mathbf{U}(\mathbf{z})\}^+ = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S. \tag{54}$$

Taking into account (10), (40) and (54) from the identities (41) and (42) (for $\mathbf{U} = \mathbf{U}'$) we obtain

$$\int_{\Omega^+} \left[W^{(0)}(\mathbf{u}, \mathbf{u}) - (\beta_1 p_1 + \beta_2 p_2) \operatorname{div} \bar{\mathbf{u}}(\mathbf{x}) \right] d\mathbf{x} = 0, \tag{55}$$

$$\begin{aligned} &\int_{\Omega^+} \left[k_1 |\nabla p_1|^2 + 2k_3 \mathrm{Re}\,(\nabla p_1 \cdot \nabla p_2) + k_2 |\nabla p_2|^2 \right] d\mathbf{x} \\ &- \int_{\Omega^+} \left[a_1 |p_1|^2 + 2a_3 \mathrm{Re}\,(p_1 \bar{p}_2) + a_2 |p_2|^2 + i\omega \operatorname{div} \mathbf{u} (\beta_1 \bar{p}_1 + \beta_2 \bar{p}_2) \right] d\mathbf{x} = 0, \end{aligned} \tag{56}$$

where

$$\begin{aligned} W^{(0)}(\mathbf{u}, \mathbf{u}) &= \frac{1}{3}(3\lambda_1 + 2\mu_1) |\operatorname{div} \mathbf{u}|^2 \\ &+ \mu_1 \left[\frac{1}{2} \sum_{l,j=1;\, l \neq j}^{3} \left| \frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j} \right|^2 + \frac{1}{3} \sum_{l,j=1}^{3} \left| \frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j} \right|^2 \right]. \end{aligned} \tag{57}$$

Hence, on the basis of (57) the imaginary part of (55) has the following form

$$\omega \int_{\Omega^+} W^{(1)}(\mathbf{u}, \mathbf{u}) d\mathbf{x} + \mathrm{Im} \int_{\Omega^+} (\beta_1\, p_1 + \beta_2\, p_2) \operatorname{div} \bar{\mathbf{u}}\, d\mathbf{x} = 0. \tag{58}$$

where

$$\begin{aligned} W^{(1)}(\mathbf{u}, \mathbf{u}) &= \frac{1}{3}(3\lambda^* + 2\mu^*) |\operatorname{div} \mathbf{u}|^2 \\ &+ \mu^* \left[\frac{1}{2} \sum_{l,j=1;\, l \neq j}^{3} \left| \frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j} \right|^2 + \frac{1}{3} \sum_{l,j=1}^{3} \left| \frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j} \right|^2 \right]. \end{aligned}$$

By conditions (38) and (39) it follows that

$$W^{(1)}(\mathbf{u},\mathbf{u}) \geq 0. \tag{59}$$

On the other hand, by virtue of (38) the real part of (56) can be written as

$$\begin{aligned}&\int_{\Omega^+}\left[k_1|\nabla p_1|^2 + 2k_3\mathrm{Re}\,(\nabla p_1\cdot\nabla p_2) + k_2|\nabla p_2|^2 + \gamma|p_1-p_2|^2\right]d\mathbf{x}\\ &\qquad + \omega\mathrm{Im}\int_{\Omega^+}\mathrm{div}\,\mathbf{u}(\beta_1\bar{p}_1+\beta_2\bar{p}_2)d\mathbf{x} = 0.\end{aligned} \tag{60}$$

Taking into account the identity

$$\mathrm{Im}\int_{\Omega^+}\mathrm{div}\,\mathbf{u}(\beta_1\bar{p}_1+\beta_2\bar{p}_2)d\mathbf{x} = -\mathrm{Im}\int_{\Omega^+}(\beta_1\,p_1+\beta_2\,p_2)\,\mathrm{div}\,\bar{\mathbf{u}}\,d\mathbf{x}$$

from (58) and (60) we have

$$\begin{aligned}&\omega^2\int_{\Omega^+}W^{(1)}(\mathbf{u},\mathbf{u})d\mathbf{x}\\ &+\int_{\Omega^+}\left[k_1|\nabla p_1|^2 + 2k_3\mathrm{Re}\,(\nabla p_1\cdot\nabla p_2) + k_2|\nabla p_2|^2 + \gamma|p_1-p_2|^2\right]d\mathbf{x} = 0.\end{aligned} \tag{61}$$

Obviously, on the basis of (38), (39) and (59) from (61) we get

$$W^{(1)}(\mathbf{u},\mathbf{u}) = 0, \qquad p_1(\mathbf{x}) = p_1' = \mathrm{const}, \qquad p_2(\mathbf{x}) = p_2' = \mathrm{const} \tag{62}$$

for $\mathbf{x}\in\Omega^+$. As in the classical theory of elasticity (see Kupradze et al. [34]) the first equation of (62) implies (52) and $\mathrm{div}\,\mathbf{u}(\mathbf{x}) = \mathbf{0}$. By virtue of (62) the second and third equations of (9) have the following form

$$a_1p_1' + a_3p_2' = 0, \qquad a_3p_1' + a_2p_2' = 0. \tag{63}$$

On account of (38) we have $a_1a_2 - a_3^2 \neq 0$ and the system (63) implies (53). Hence, theorem is proved for $K = II, VI, VII, VIII$.

In addition, if $K = I, III, IV, V$, then on the basis of homogeneous boundary condition $\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0}$ from (52) it follows that $\mathbf{U}(\mathbf{x}) \equiv \mathbf{0}$. Hence, $\mathbf{U}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x}\in\Omega^+$. □

Theorem 13. *If the conditions* (38) *and* (39) *are satisfied, then the internal BVP* $(K)^{(s)}_{\mathbf{F},\mathbf{f}}$ *admits at most one regular solution, where* $K = I, II, \ldots, VIII$.

Proof. Suppose that there are two regular solutions of problem $(K)^{(s)}_{\mathbf{F},\mathbf{f}}$, where $K = I, II, \ldots, VIII$. Then their difference $\mathbf{U}$ corresponds to zero data ($\mathbf{F} = \mathbf{f} = \mathbf{0}$), i.e. $\mathbf{U}$ is a regular solution of problem $(K)^{(s)}_{\mathbf{0},\mathbf{0}}$. Hence, the vector $\mathbf{U}$ is a regular (classical) solution to system of homogeneous equations (11) in the domain Ω^+ satisfying the homogeneous boundary condition (54).

Taking into account (11), (40) and (54) from the identities (41) and (42) (for $\mathbf{U} = \mathbf{U}'$) we obtain (56) and

$$\int_{\Omega^+}\left[W^{(0)}(\mathbf{u},\mathbf{u}) - \rho\omega^2|\mathbf{u}|^2 - (\beta_1p_1+\beta_2p_2)\,\mathrm{div}\,\bar{\mathbf{u}}(\mathbf{x})\right]\,d\mathbf{x} = 0, \tag{64}$$

where $W^{(0)}(\mathbf{u},\mathbf{u})$ is given by (57). Quite similarly, from (56) and (64) we have (52) and (53). It is easy to see that (52) implies $W^{(0)}(\mathbf{u},\mathbf{u}) \equiv 0$ and from (64) we get $\mathbf{U}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x}\in\Omega^+$. □

As in the classical theory of elasticity (see, Knops and Payne [38]) we can prove the uniqueness of regular solutions of the problems $(K)^{(q)}_{\mathbf{F},\mathbf{f}}$ and $(K)^{(s)}_{\mathbf{F},\mathbf{f}}$ under weaker conditions than (39), where $K = I, III, IV, V$. We have the following

Theorem 14. *If the conditions* (38) *and* $\lambda^* + 2\mu^* > 0$ *are satisfied, then the internal BVP* $(K)^{(r)}_{\mathbf{F},\mathbf{f}}$ *admits at most one regular solution, where* $K = I, III, IV, V$ *and* $r = q, s$.

6. Concluding remarks

1. In this paper the following results are obtained:
(i) the fundamental solutions of the systems of equations of quasi-static and steady vibrations in the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity are constructed by elementary functions;
(ii) Green's formulae and the integral representation of regular solution in the considered theory are established;
(iii) the uniqueness theorems for regular solutions of the internal boundary value problems of quasi-static and steady vibrations are proved.

2. On the basis of Theorems 1 to 14 are possible:
(i) to construct the surface (single-layer and double-layer) and volume potentials and to establish their basic properties;
(ii) to prove the existence theorems for the boundary value problems of quasi-statics and steady vibrations in the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity by means of the potential method and the theory of singular integral equations.

3. An extensive review of the works on the potential method in the classical theory of elasticity is given in the book [37] and the review paper [39].

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