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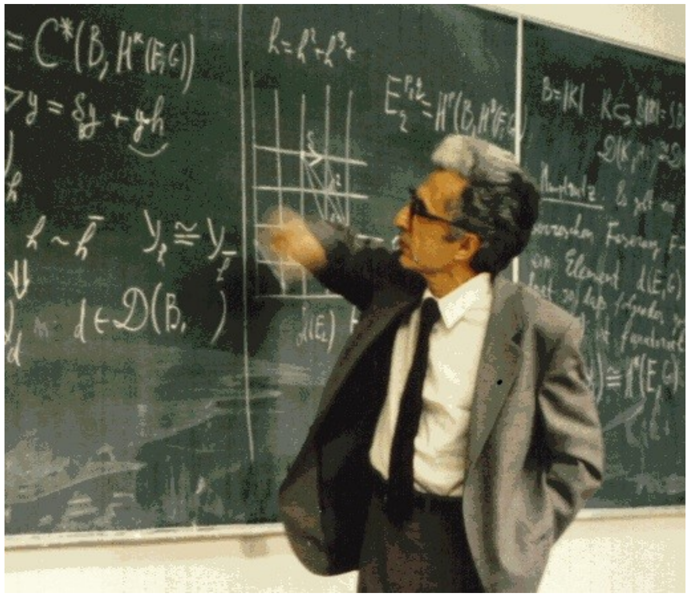

NODAR BERIKASHVILI'S 90TH BIRTHDAY ANNIVERSARY

This year we celebrate the 90th anniversary of birthday of Nodar Berikashvili, widely recognized famous Georgian mathematician, one of the founders of the Georgian topological school.

He was born in 1929. Upon graduation from the Tbilisi State University in 1952, he continued his education in a post-graduate course (PhD student) under the supervision of academician George Chogoshvili. Nodar Berikashvili's dissertation (PhD thesis) attracted attention of theworld famous scientist, topologist Peter Alexandrov, who invited young Nodar Berikashvili to work at Steklov Mathematical Institute in Moscow. Nodar Berikashvili defended his PhD degree at the same Institute and worked there till 1959. Of interest is the fact that on the sites of Steklov Institute and Moscow State University the name of Nodar Berikashvili is listed among prominent scientists.

In 1959, he comes back to Georgia and starts his work at A. Razmadze Mathematical Institute of the Georgian Academy of Sciences, at the Department of Geometry and Topology founded by academician George Chogoshvili. In 1971, he defended his Doctor's degree at Steklov Institute, and in 2001 he was elected the member of Georgian Academy of Sciences.

Besides his scientific activity, Nodar Berikashvili was deeply involved in the pedagogical activity, he was lecturing at Tbilisi State University and was supervisor of PhD students at A. Razmadze Mathematical Institute.

Below we present a brief survey of Nodar Berikashvili's scientific works.

The first cycle of his works is dedicated to the homology theory of general spaces. Namely, N. Berikashvili continued the approach suggested by his supervisors G. Chogoshvili and P. S. Alexandrov, the duality theory, which in Berikashvili's works achieved its complete form. Towards this end, Berikashvili has constructed some new homology theories and developed axiomatic theory for limits of spectra. All these works place N. Berikashvili among the founders of the homology theory of general spaces.

The second cycle is concerned with the index theory for singular integral equations, the field, traditional for Georgian mathematics. In 1963, using topological methods, N. Berikashvili has proved the index formula for an arbitrary 2-dimensional manifold, proven earlier by Wolpert for 2-spheres. Approximately, at the same time, the famous Atiah-Singer theorem has been proven, which generalized these results.

The next cycle of N. Berikashvili's works is an important contribution to one of the most powerful tools of algebraic topology, the Lere-Serre's spectral sequences. N. Berikashvili's theory of predifferentials strengthened essentially this method and enriched it by a new computational potential.

The predifferential theory is based on the new homotopy invariant, i.e., on the functor D, which uses not only cocycle information from the cochain complex of a space (as the homology functor does), but also the so-called twisting cochains, thus this functor is more informative, than the homology functor is. For example, it perceives Hopf's invariant. For a large class of spectral sequences including the spectral sequences of fibrations and coverings, N. Berikashvili has constructed the so-called "predifferential", an element of the functor D, which determines all differentials and extensions in the corresponding spectral sequence, thus it completely reconstructs the limit.

This method demonstrates the connections between various methods of investigation of the homology theory of fibrations: the method of spectral sequences of Lere-Serre, the Hirsch method and Brown's theory of twisted tensor products. Moreover, the predifferential theory develops and generalizes these methods. The Hirsch method is developed in the sense that it negates the restriction about the freeness of homology, N. Berikashvili replaces the nonfree homology by its free resolution,the novelty for that time, besides, the Hirsch method has been extended to the spectral sequences of coverings. Brown's method is developed in the sense that there was introduced the organization in a set of twisting cochains allowing one to choose a twisting cochain in a more simple form for calculation of homology of a fibration.

In the 70-80th of the past century, the predifferential theory achieved essential refinement and was developed as a powerful tool for modeling of spaces and fibrations. N. Berikashvili has constructed new versions of the functor D which determine the multiplicative structure and high rank Steenrod operations.

At present, using his methods, N. Berikashvili is engaged in the central problem of Algebraic Topology, in the problem of homotopy classification, particularly, in the obstruction theory for the section of a fibration. He has developed the complete form of the second obstruction problem. His high order obstruction functors seem to be a perspective tool for this important and extremely complicated problem.

His 90th birthday Professor Nodar Berikashvili meets full of energy and plans. We wish him further success in his scientific and pedagogical activity.

T. Kadeishvili and S. Saneblidze

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PARTICULARITIES OF INTEGRATING THE PERIODIC FUNCTIONS IN THE PRESENCE OF THE TIMESCALE GRADIENTS AND TURBULENCE ISSUES

A. APTSIAURI AND G. APTSIAURI

Abstract. The paper demonstrates that with a strictly periodic nature of fluctuation of scalar parameters at a fixed point of a space, in the presence of timescale gradients at this point, the gradients of these parameters are not periodic functions, but they are complex, almost periodic in the form of a combination of regular and low-frequency (or periodic) fluctuations. This result indicates the link between the discrete spectrum of turbulence and the frequency gradients, as well as the need for considering a more general mode with regard for the timescale effect.

Introduction

In our studies, we are aimed at the fact that the main cause of turbulence problem is disregard of the timescales in integrating the Navier-Stokes equations. In particular, the work [1] has demonstrated that consideration of timescale gradients, when integrating the differential equations for conservation of mass and energy, allows one to get two scalar equations for the turbulent processes.

In this section, we show that taking into account the specifics of integration of periodic processes is even more important, since this offers rich possibilities to solve the turbulence problem.

To this end, in this section, we want to engage the reader's interest in the specifics of integration within a single cycle and within a large (or infinite) time interval. In fact, we consider the simplest problem which, to some extent, corresponds to the mean value theorem based on the Liouville-Arnold theorem.

Strictly periodic functions. Consider a function F, which changes periodically, with an interval τ_o, for an infinitely long period of time $\tau_\infty \to \infty$ (Figure 1). Assuming that all the cyclic processes are identical, we can say that there is a **strictly periodic process.** Moreover, there may also exist such cyclical fluctuations differing in form, during which the mean values of the parameter, in different cycles, are identical. In this case, we will have **processes that are strictly periodic by a mean value.** In this paper, the oscillatory processes of both natures will be referred to as **strictly periodic.**

Mean value of a function F, within a cycle, can be determined by integrating

$$\overline{F}_\tau = \frac{1}{\tau_O}\int\limits_o^{\tau_O} F dt.$$

Naturally, if the process is strictly periodic, then the average value of this parameter will not be changed during the integration within several cycles.

$$\overline{F_\tau} = \frac{1}{\tau_O}\int\limits_o^{\tau_O} F dt = \frac{1}{2\tau_O}\int\limits_0^{2\tau_O} F dt = \cdots = \frac{1}{N\tau_o}\int\limits_O^{N\tau_O} F dt.$$

In this connection, averaging over the cycle, the integration limits should be observed exactly, since for low deviations from this condition, the average value of the parameter will be determined with a significant error.

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On the other hand, the average value of this parameter can also be determined within an interval of an infinitely long period of time [2]:

$$\overline{F}_\infty = \frac{1}{\tau_\infty}\int\limits_o^{\tau_\infty} F dt.$$

It is not difficult to show that the accuracy of determining the mean value increases with increasing τ_∞. Naturally, for the strictly periodic functions, the average values of the function within a cycle and a large interval are equal. At the same time, unlike averaging over a cycle, when averaging over a large interval, the period of integration τ_∞ can be chosen in an arbitrary way, without observing the multiplicity condition.

The almost periodic or complex periodic functions. If the average value of the parameter changes insignificantly in each subsequent cycle, then it can be said that the function is not strictly periodic, almost periodic or complex.

In order to determine the nature of this function, it is necessary to compare the average within a cycle value with the mean value within a large time interval.

Naturally, if the mean value of the function in each cycle increases (or decreases) monotonely, then in an infinitely large interval, the difference over time $\delta(t) = \delta(\tau_\infty)$ will increase infinitely, and we can say that in a large interval, the function is not periodic.

If the difference between the mean values has some limiting value $\delta(x, y, z)$, which is not a function of the averaging period, then we can say that we have a complex function, which is the sum of a strictly periodic function and some low-frequency periodic function.

So, we have:

A) For the nonperiodic functions $\overline{F_\infty} = \overline{F_\tau} + \delta(\tau_\infty)$,

B) For the strictly periodic functions $\overline{F_\infty} = \overline{F_\tau} = \overline{F}$,

C) For the complex periodic functions $\overline{F_\infty} = \overline{F_\tau} + \delta(x, y, z)$, (1)

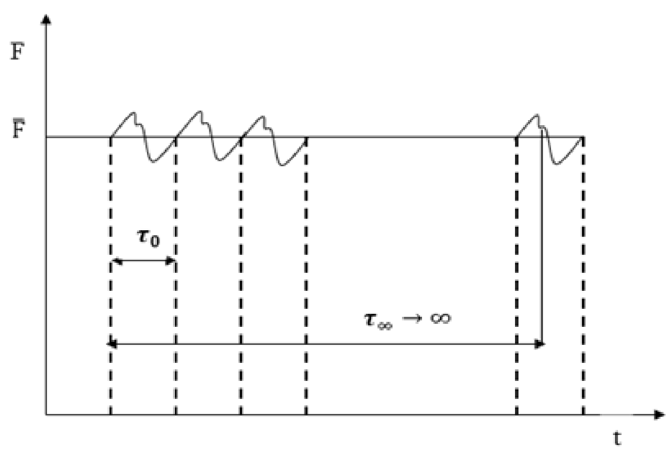

FIGURE 1. A strictly periodic function.

Thus, when we have the strictly periodic functions, the average values with respect to a cycle and to a long period are identical for them. On the other hand, there are various requirements to the limits of integration. As we will see below, this difference provides very interesting information necessary to solve the turbulence problem.

Let us consider, in a continuous medium space, a function ∇F from a strictly periodic function F, for which the fair condition is $\overline{F_\infty} = \overline{F_\tau} = \overline{F}$, and let us show that in the presence of the timescale gradients, ∇F is a complex periodic function.

Let us determine the average value of a function ∇F within a large period of time which can be set as the same everywhere inside the space $\tau_\infty = \text{const}$. Accordingly, at any point in this space, the average value with respect to a large time period of a function ∇F can be determined from the equation

$$\overline{\nabla F}_\infty = \frac{1}{\tau_\infty}\int\limits_0^{\tau_\infty}(\nabla F)dt = \frac{1}{\tau_\infty}\nabla\int\limits_0^{\tau_\infty}F dt = \frac{1}{\tau_\infty}\nabla(\tau_\infty \overline{F_\infty}). \tag{2}$$

In this connection, if the limit of integration is the same for the entire space, τ_∞ can be taken outside the differential sign. Therefore, we have

$$\overline{\nabla F}_\infty = \nabla \overline{F}_\infty = \nabla \overline{F}. \tag{3}$$

Thus, when averaging the Navier-Stokes equation [3–6], the averaging sign is introduced inside the sign ∇, it is good to bear in mind that the averaging takes place within an infinitely long period of time. Of course, this is true, but, at the same time, the cycle integration gives also the quite legitimate, different results, and this difference bears serious information for the problem of turbulence.

Let us determine the mean value of a function F with respect to a cycle

$$\overline{\nabla F}_\tau = \frac{1}{\tau_o}\int\limits_0^{\tau_0}(\nabla F)dt = \frac{1}{\tau_0}\nabla\int\limits_0^{\tau_o}F dt = \frac{1}{\tau_o}\nabla(\tau_o \overline{F}),$$

or

$$\overline{\nabla F}_\tau = \nabla\overline{F} + \frac{\overline{F}}{\tau_o}\nabla\tau_o = \nabla\overline{F} + \overline{F}\,\mathrm{grad}(\ln\tau_0). \tag{4}$$

Comparison of (2) and (4) leads to the conclusion

$$\overline{\nabla F}_\tau = \overline{\nabla F}_\infty + \overline{F}\,\mathrm{grad}(\ln\tau_0).$$

Accordingly, in the general case, when $\overline{F}\,\mathrm{grad}(\ln\tau_0) \neq 0$, we have

$$\overline{\nabla F}_\tau \neq \overline{\nabla F}_\infty.$$

Thus, **if at a fixed point in space the periodic function F varies strictly periodically, and at this point there are the frequency gradients, then the function ∇F is, generally, nonperiodic, or complex periodic**.

On the basis of the law of conservation of the integral flows of mass and energy, the work [1] demonstrated that the fair conditions look as follows:

$$V\,\mathrm{grad}(\tau_0) = 0, \tag{5}$$

$$e_{\sum}\,\mathrm{grad}(\tau_0) = 0. \tag{6}$$

Accordingly, for the vectors of the flows of mass $g = \rho W$ and energy $e_{\sum}$, we have

$$\overline{\nabla\rho W}_\tau = \overline{\nabla\rho W}_\infty,$$

$$\overline{\nabla e_{\sum}}_\tau = \overline{\nabla e_{\sum}}_\infty.$$

This means that the functions $\nabla\rho W$ and $\nabla e_{\sum}$ are strictly periodic.

Moreover, in the general case, for the parameters of the turbulent flow of continuous medium,

$$\overline{F}\,\mathrm{grad}(\tau_0) \neq 0.$$

The obtained result gives very interesting information for the theory of turbulence.

Result 1. Let us consider an arbitrary scalar parameter S (pressure, density, temperature, etc.) to be a periodic function. If the fluctuation of a scalar parameter is strictly periodic (or strictly periodic with respect to a mean value) and the condition (3) is satisfied for it, then in this case, the following expressions

$$\overline{\nabla S}_\infty = \nabla\overline{S},$$
$$\overline{\nabla S}_\tau = \nabla\overline{S} + \overline{S}\,\mathrm{grad}(\tau_0) \tag{7}$$

are fair.

In the case of the timescale gradients, the last term in (7) cannot be zero. Therefore, we have an instance

$$\overline{\nabla S}_\infty = \overline{\nabla S}_\tau + \delta(x, y, z),$$

corresponding to condition (1). The obtained expression allows us for conclude **that for a periodic nature of fluctuation of the scalar parameters at a fixed point of the turbulent flow of a continuous medium, in the presence of the timescale gradients at this point, the scalar parameter gradients are the complex periodic functions, which indicates the appearance of the low-frequency variations of gradients under the influence of the timescale gradients.**

Result 2. If the pressure, as a scalar parameter, varies strictly periodically at different points of the flow, then the pressure gradient, in the general case, may be of the nature of a complex periodic function.

Moreover, the pressure gradient is the main factor of the velocity change. Accordingly, the low-frequency variations of the pressure gradients are able to generate the additional velocity changes or the accompanying fluctuations that distort the reality of the regular periodic pulsations.

For this reason, **the velocity vector in a turbulent flow should be sought as the sum of a stationary function and the independent pulsations of at least two types (traditional and low-frequency, or the slow ones).**

$$W = V + \upsilon + u, \tag{8}$$

where υ reflects the regular pulsations, but u corresponds to periodic fluctuations, or to the low-frequency pulsations, which manifest themselves during a large time interval. These low-frequency fluctuations cannot be correlated with the regular pulsations, but they can create an additional tensor $\tau(\overline{u, u})$.

Let us consider the Navier-Stockes equation

$$\frac{\partial \rho W}{\partial t} + \nabla I_{\sum} = 0, \tag{9}$$
$$I_{\sum} = \rho\tau(W, W) + PI + (2/3)\mu\,\mathrm{div}(W)I - 2\mu D(W) = 0.$$

Integration of (9) within a large time interval, taking into account (8) results in

$$\rho(V\nabla)V = -\,\mathrm{div}[\rho\tau(\overline{\upsilon, \upsilon}) + \rho\tau(\overline{u, u}) + PI + (2/3)\mu\,\mathrm{div}(V)I - 2\mu D(V)]. \tag{10}$$

The obtained expression differs from the traditional Reynolds equation by the presence of an additional tensor of the accompanying pulsations $\tau(\overline{u, u})$.

And now, we can integrate this equation within a single cycle. The probability of the occurrence of low-frequency fluctuations within a single pulsation is low. Therefore, the flow velocity in a short interval τ_0 can be considered to be a sum of the mean velocity and regular pulsations

$$W = V + \upsilon.$$

In such conditions, integration of (9) results in

$$\nabla\overline{I_{\sum\tau}} = -\frac{\overline{I_{\sum\tau}}}{\tau_o}\,\mathrm{grad}\,\tau_0, \tag{11}$$
$$\overline{I_{\sum\tau}} = \rho\tau(V, V) + \rho\tau(\overline{\upsilon, \upsilon}) + PI + \overline{\sigma_\tau},$$

where the mean value of the strain tensor has the form

$$\overline{\sigma_\tau} = (2/3)\mu\,\mathrm{div}(V)I - 2\mu\overline{D(V)}_\tau. \tag{12}$$

When determining the last term of the equation (12), we take into account the fact that the time-average value of the strain tensor from the vector F is determined from the equation:

$$\overline{D(F)} = (1 - \ln \tau_o) D(\overline{F}) + D(\overline{F} \ln \tau_o).$$

At the same time, the following expression

$$D(C\overline{F}) = CD(F) + \frac{1}{2}[\tau(F, \operatorname{grad} C) + \tau(\operatorname{grad} C, F)]$$

is true. Therefore, we have

$$\overline{D(V)} = (1 - \ln \tau_o) D(V) + D(V \ln \tau_o) = D(V) - \frac{1}{2}[\tau(V, A) + \tau(A, V)].$$

In the response to the latter, we have

$$\overline{I_{\sum_\tau}} = \rho\tau(V, V) + \rho\tau(\overline{v, v}) + PI + (2/3)\mu \operatorname{div}(V) I - 2\mu D(V) + \mu\tau(V, A) + \mu\tau(A, V). \tag{13}$$

Putting (13) into (11), the equation of motion becomes

$$\begin{aligned}\rho(V\nabla)V = &- \operatorname{div}[\rho\tau(\overline{v, v}) + PI + (2/3)\mu \operatorname{div}(V) I - 2\mu D(V) + \mu\tau(V, A) + \mu\tau(A, V)] \\ &+ [\rho\tau(V, V) + \rho\tau(\overline{v, v}) + PI + (2/3)\mu \operatorname{div}(V) I - 2\mu D(V) + \mu\tau(V, A) + \mu\tau(A, V)]A.\end{aligned} \tag{14}$$

By comparing (14) and (10), for a tensor of the accompanying pulsations, we obtain the following equation:

$$\begin{aligned}\operatorname{div}[\rho\tau(\overline{u, u})] = \operatorname{div}\{\mu[\tau(V, A) + \tau(A, V)]\} - [\rho\tau(\overline{v, v}) + PI \\ + (2/3)\mu \operatorname{div}(V) I - 2\mu D(V)]A - \mu A^2 V.\end{aligned}$$

As we can see, the influence of this tensor depends on the timescale gradient, and it disappears in the absence of these gradients. To determine the vector $A = -\operatorname{grad} \ln \tau_0$, we already have two scalar equations – (5) and (6).
Thus, taking into account the time gradients makes it necessary to apply a more general model. Moreover, as we will see later, this insignificant complication simplifies the solution of this very complex problem of turbulence.

Conclusion

The presence of the timescale gradients is the reason for the appearance of additional low-frequency oscillations or fluctuations in the velocity of the turbulent flow of a continuous medium, highlighting the need to consider a more general model that not only takes into account the mentioned features of the flow, but also provides the additional equations for solving the turbulence problem.

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SOLUTION FOR SYSTEM OF IMPLICIT ORDERED VARIATIONAL INCLUSIONS

I. ARGYROS[1] AND SALAHUDDIN[2*]

Abstract. The purpose of this paper is to present an existence theorem for a new class of *system of implicit ordered variational inclusions* in real ordered Banach spaces. Using the concept of resolvent operator, we prove the convergence of sequences generated by an algorithm.

1. Brief Prehistory

Generalized nonlinear ordered variational inclusions have wide applications in many fields including, for example, mathematical physics, optimization and control theory, mathematical programming, economics and engineering sciences. Recently, nonlinear mappings, fixed point theory and their applications have been extensively studied in ordered Banach spaces. In 2008, H.G. Li [6] introduced the generalized nonlinear ordered variational inequalities, studied an approximation algorithm and an approximate solution for a class of generalized nonlinear ordered variational inequalities in ordered Banach spaces. In 2009, by using the B-restricted accretive method of the mapping A with constants α_1, α_2, Li [7] studied a new class of general nonlinear ordered variational equations and established an existence theorem and an approximation algorithm of solutions for this kind of generalized nonlinear ordered variational equations in ordered Banach spaces.

Motivated and inspired by the recent research works [1–5,9,13], in this paper, we consider a *system of implicit ordered variational inclusions* in real ordered Banach spaces. We design an iterative algorithm based on the resolvent operator for solving a *system of implicit ordered variational inclusions.* We prove an existence, as well as a convergence theorem for our problem.

2. Prelude

Definition 2.1. Let $C(\neq \emptyset)$ be a closed, convex subset of X. C is said to be a pointed cone if
  (i) for $x \in C$ and $\lambda > 0$, $\lambda x \in C$;
  (ii) if x and $-x \in C$, then $x = \theta$,
where θ is a zero vector in X.

Definition 2.2 ([4]). C is called a normal cone if and only if there exists a constant $\lambda_C > 0$ such that $\theta \le x \le y$ implies $\|x\| \le \lambda_C \|y\|$, where λ_C is called the normal constant of C.

Definition 2.3 ([12]). For arbitrary elements $x, y \in X$, $x \le y$ if and only if $x - y \in C$, then the relation $\le$ is a partial ordered relation in X. The real Banach space X with the ordered relation $\le$ defined by C is called a real ordered Banach space.

Throughout this paper, we assume X to be a real ordered Banach space with norm $\|\cdot\|$, an order pair $\langle\cdot,\cdot\rangle$ and partial ordered relation $\le$ defined by the normal cone C with a normal constant λ_C. Let $CB(X)$ be the family of all nonempty closed and bounded subsets of X, and $\mathfrak{D}$ be the Hausdorff metric defined on $CB(X)$ by

$$\mathfrak{D}(A\,,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\},$$

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where $A, B \in CB(X), d(x, B) = \inf_{y \in B} d(x, y)$.

Definition 2.4 ([13]). For arbitrary elements $x, y \in X$, if $x \leq y$ or $y \leq x$, then x and y are called comparable and this is denoted by $x \propto y$.

Lemma 2.5 ([13]). Let X be an ordered Banach space. For arbitrary $x, y \in X$, $lub\{x, y\}$ and $glb\{x, y\}$ express the least upper bound of the set $\{x, y\}$ and the greatest lower bound of the set $\{x, y\}$ on the partial ordered relation $\leq$, *respectively.* Suppose $glb\{x, y\}$ and $lub\{x, y\}$ exist. Some binary operators can be defined as follows:

- $x \vee y = lub\{x, y\}$;
- $x \wedge y = glb\{x, y\}$;
- $x \oplus y = (x - y) \vee (y - x)$.

$\vee$, $\wedge$ and $\oplus$ are called OR, AND and XOR operation, *respectively.* For arbitrary $x, y, w \in X$, the following relations hold:

(i) if $x \leq y$, then $x \vee y = y,\ x \wedge y = x$;
(ii) if x and y are comparable, then $\theta \leq x \oplus y$;
(iii) $(x + w) \vee (y + w)$ exists and $(x + w) \vee (y + w) = (x \vee y) + w$;
(iv) $(x + w) \wedge (y + w)$ exists and $(x + w) \wedge (y + w) = (x \wedge y) + w$;
(v) $(x \wedge y) = (x + y) - (x \vee y)$;
(vi) if $\lambda \geq 0$, then $\lambda(x \vee y) = \lambda x \vee \lambda y$;
(vii) if $\lambda \leq 0$, then $\lambda(x \wedge y) = \lambda x \vee \lambda y$;
(viii) $x \wedge y = -(-x \vee -y)$ and $(-x) \wedge (x) \leq \theta \leq (-x) \vee x$;
(ix) if $x \leq y$ and $s \leq t$ then $x + s \leq y + t$;
(x) if $\theta \leq x$ and $x \neq \theta$, and $\alpha > 0$ then $\theta \leq \alpha x$ and $\alpha x \neq \theta$;
(xi) if X is an ordered Banach space, and if for any $x, y \in X$, either $x \vee y$ and $x \wedge y$ exist, then X is a Banach lattice.

Definition 2.6 ([8]). Let $A : X \longrightarrow X$ be a single-valued mapping.

(i) A is said to be comparison mapping if for each $x, y \in X,\ x \propto y$, then $A(x) \propto A(y), x \propto A(x)$ and $y \propto A(y)$;
(ii) A is said to be strongly comparison mapping if A is a comparison mapping and $A(x) \propto A(y)$, if and only if $x \propto y$;
(iii) A is said to be β-ordered compression mapping if it is a comparison mapping and there exists a constant $0 < \beta < 1$ such that
$$A(x) \oplus A(y) \leq \beta(x \oplus y);$$
(iv) A is said to be γ-order non-extended mapping if there exists a constant $\gamma > 0$ such that
$$\gamma(x \oplus y) \leq A(x) \oplus A(y), \forall x, y \in X.$$

Lemma 2.7 ([4]). If x and y are comparable, then $lub\{x, y\}$ and $glb\{x, y\}$ exist,
$$x - y \propto y - x, \text{ and } \theta \leq (x - y) \vee (y - x).$$

Lemma 2.8 ([4]). If for any natural number $n, x \propto y_n$ and $y_n \longrightarrow y (n \longrightarrow \infty)$, then $x \propto y$.

Lemma 2.9 ([4]). Let C be a normal cone with a normal constant λ_C in X, then for each $x, y \in X$, we have the relations:

(i) $\|\theta \oplus \theta\| = \|\theta\| = \theta$;
(ii) $\|x \wedge y\| \leq \|x\| \wedge \|y\| \leq \|x\| + \|y\|$;
(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_C \|x \oplus y\|$;
(iv) if $x \propto y$, then $\|x \oplus y\| = \|x - y\|$.

Lemma 2.10 ([8,9]). Let $\leq$ be a partial order relation defined by the cone C with a normal constant λ_C in X in Definition 2.3. Then the following relations are satisfied:

(i) $x \oplus y = y \oplus x,\ x \oplus x = \theta$;
(ii) $\theta \leq x \oplus \theta$;

(iii) $(x \oplus \theta) - (y \oplus \theta) \leq (x - y) \oplus \theta$;
(iv) if $x \propto \theta$, then $-x \oplus \theta \leq x \leq x \oplus \theta$;
(v) if $x \propto y$, then $(x \oplus \theta) \oplus (y \oplus \theta) \leq (x \oplus y) \oplus \theta$;
(vi) allow λ to be real, then $(\lambda x) \oplus (\lambda y) =\mid \lambda \mid (x \oplus y)$;
(vii) if x, y and w are comparable, then $(x \oplus y) \leq (x \oplus w) + (w \oplus y)$;
(viii) if x, y, r, w are comparable, then

$$(x \wedge y) \oplus (r \wedge w) \leq ((x \oplus r) \vee (y \oplus w)) \wedge ((x \oplus w) \vee (y \oplus r));$$

(ix) let $(x + y) \vee (s + t)$ exist, and if $x \propto s, t$, and $y \propto s, t$, then

$$(x + y) \oplus (s + t) \leq (x \oplus s + y \oplus t) \wedge (x \oplus t + y \oplus s).$$

Definition 2.11 ([11])**.** Allow $A : X \longrightarrow X$ and $M : X \longrightarrow CB(X)$ to be the mappings.

(i) M is called a weak-comparison mapping, if given comparable $x, y \in X$ and $t_x \in M(x)$, there exists $t_y \in M(y)$ such that t_x and t_y are comparable.
(ii) M is called an α-weak-non-ordinary difference mapping associated with A, if it is a weak comparison and there exists $\alpha > 0$, and $t_x \in M(A(x))$ and $t_y \in M(A(y))$ such that

$$(t_x \oplus t_y) \oplus \alpha(A(x) \oplus A(y)) = \theta.$$

(iii) M is called a λ-order different weak-comparison mapping associated with A if for the given comparable $x, y \in X$, there exist $\lambda > 0$, and $t_x \in M(A(x)), t_y \in M(A(y))$ such that

$$\lambda(t_x - t_y) \propto x - y.$$

(iv) M (a weak-comparison map) is called an ordered (α_A, λ)-weak-ANODM mapping, if it is α-weak-non-ordinary difference mapping and λ-order different weak-comparison mapping associated with A, and $(A + \lambda M)(X) = X$, for $\alpha, \lambda > 0$.

Definition 2.12 ([11])**.** Let $M : X \longrightarrow CB(X)$ be γ-order non-extended mapping and α-non-ordinary difference mapping with respect to a mapping $A : X \longrightarrow X$. The resolvent operator $R^M_{A,\lambda} : X \longrightarrow X$ associated with both A and M is defined by

$$R^M_{A,\lambda}(x) = (A + \lambda M)^{-1}(x), \text{ for all } x \in X, \tag{2.1}$$

where $\gamma, \alpha, \lambda > 0$ are the constants.

Definition 2.13 ([6])**.** A bi-mapping $B : X \times X \longrightarrow X$ is called (α_1, α_2)-restricted-accretive if it is comparison and there exist constants $0 \leq \alpha_1, \alpha_2 \leq 1$ such that

$$B(x_1, y_1) \oplus B(x_2, y_2) \leq \alpha_1(B(x_1) \oplus B(x_2)) + \alpha_2(B(y_1) \oplus B(y_2)), \text{ for all } x_1, x_2, y_1, y_2 \in X.$$

Lemma 2.14 ([11])**.** Let $M : X \longrightarrow CB(X)$ be γ-order non-extended and α-weak non-ordinary difference mapping associated with a mapping $A : X \longrightarrow X$, and $\alpha\gamma \neq 1$, then $M_\theta = \{\theta \oplus x \mid x \in M\}$ is α-weak non-ordinary difference mapping associated with A and the resolvent operator $R^{M_\theta}_{A,\lambda} = (A + \lambda M_\theta)^{-1}$ of $(A + \lambda M_\theta)$ is a single valued for $\alpha, \lambda > 0$, *i.e.*, $R^{M_\theta}_{A,\lambda} : X \longrightarrow X$ of M_θ holds.

Lemma 2.15 ([11])**.** Let $A : X \longrightarrow X$ be a mapping and $M : X \longrightarrow CB(X)$ be (α_A, λ)-weak-ANODD set-valued and strongly comparison mapping associated with $R^M_{A,\lambda}$. Then the resolvent operator $R^M_{A,\lambda} : X \longrightarrow X$ is a comparison mapping.

Lemma 2.16 ([11])**.** Let $A : X \longrightarrow X$ be a mapping and $M : X \longrightarrow CB(X)$ be ordered (α_A, λ)-weak-ANODD and γ-ordered non-extended mapping associated with $R^M_{A,\lambda}$, for $\alpha_A > \frac{1}{\lambda}$. Then the following relation

$$R^M_{A,\lambda}(x) \oplus R^M_{A,\lambda}(y) \leq \frac{1}{\gamma(\alpha_A \lambda - 1)}(x \oplus y), \text{ for all } x, y \in X. \tag{2.2}$$

holds.

3. Formulation of the Problem

Let X be a real Banach space and C be a normal cone having the normal constant λ_C. Suppose $f_i, g_i : X \longrightarrow X$ $(i = 1, 2)$ and $Q_i : X \times X \longrightarrow X$ $(i = 1, 2)$ are single-valued mappings. Assume that $T_1, T_2, F_1, F_2 : X \longrightarrow CB(X)$ and $M, N : X \times X \longrightarrow CB(X)$ are set-valued mappings. Now we look at the problem: for some $(w_1, w_2) \in X \times X$ and $\rho_1, \rho_2 > 0$, find $x, y \in X$, $u \in T_1(x)$, $v \in T_2(y)$, $p \in F_1(x)$, $q \in F_2(y)$ such that

$$\begin{aligned} &w_1 \in Q_1(f_1(x), v) + \rho_1 M(g_1(x), q), \\ &w_2 \in Q_2(u, f_2(y)) + \rho_2 N(p, g_2(y)). \end{aligned} \tag{3.1}$$

Problem (3.1) is called *a system of implicit ordered variational inclusions.*

Special Cases:

(1) If T_1, T_2, F_1, F_2 are single-valued mappings, then (3.1) reduces to the problem of finding some $(w_1, w_2) \in X \times X$, $\rho_1, \rho_2 > 0$, and $x, y \in X$ such that

$$\begin{aligned} &w_1 \in Q_1(f_1(x), T_2(y)) + \rho_1 M(g_1(x), F_2(y)), \\ &w_2 \in Q_2(T_1(x), f_2(y)) + \rho_2 N(F_1(x), g_2(y)), \end{aligned} \tag{3.2}$$

called *a system of generalized ordered variational inclusions.*

(2) If T_1, T_2, F_1, F_2 are identity mappings, then (3.2) reduces to the problem of finding some $(w_1, w_2) \in X \times X$, $\rho_1, \rho_2 > 0$, and $x, y \in X$ such that

$$\begin{aligned} &w_1 \in Q_1(f_1(x), y) + \rho_1 M(g_1(x), y), \\ &w_2 \in Q_2(x, f_2(y)) + \rho_2 N(x, g_2(y)), \end{aligned} \tag{3.3}$$

called *a system of general ordered variational inclusions.*

(3) If $g_1 = f_2 = I$ (the identity mapping on X), M and N are single-valued mappings and $M(g_1(x), y) = M(x, y)$, then (3.3) reduces to the problem of finding some $w_1, w_2 \in X$, and $x, y \in X$ such that

$$\begin{aligned} &w_1 \in Q_1(f_1(x), y) + \rho_1 M(y, x), \\ &w_2 \in Q_2(x, y) + \rho_2 N(x, g_2(y)), \end{aligned} \tag{3.4}$$

a variant form studied in [9].

(4) If $w_2 = 0$, $Q_2 = f_2 = N = g_2 = 0$, then problem (3.4) is to find $x, y \in X$ such that

$$w_1 \in Q_1(f_1(x), y) + \rho_1 M(y, x), \tag{3.5}$$

a variant form of generalized variational inclusions.

(5) If $\rho_1 = 1$, $w_1 = 0$, then problem (3.5) reduces to finding $x, y \in X$ such that

$$0 \in Q_1(f_1(x), y) + M(y, x), \tag{3.6}$$

considered and studied in [14].

(6) If $\rho_1 = \rho$, $w_1 = w$, $Q_1(f_1(x), y) = f(x)$ and $M(y, x) = M(x)$, then problem (3.5) becomes that of finding $x \in X$ such that

$$w \in f(x) + \rho M(x). \tag{3.7}$$

Problem (3.7) was studied in [11].

(7) If $f = 0$ is a zero mapping, then problem (3.7) reduces to finding $x \in X$ such that

$$w \in \rho M(x). \tag{3.8}$$

Problem (3.9) was initiated and studied in [10].

Now, we mention the fixed point formulation of (3.1).

Lemma 3.1. Let $x, y \in X$, $u \in T(x) \in CB(X)$, $v \in T(y) \in CB(X)$, $p \in F_1(x) \in CB(X)$, $q \in F_2(y) \in CB(X)$ be a solution of (3.1) if and only if $x, y \in X$, $u \in T_1(x) \in CB(X)$, $v \in T_2(y) \in CB(X)$, $p \in F_1(x) \in CB(X)$, $q \in F_2(y) \in CB(X)$ fulfill the following relations:

$$\begin{aligned} x &= R_{A,\lambda}^{M(g_1(\cdot),q)}\Big[A(x) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x), v))\Big], \\ y &= R_{A,\lambda}^{N(p,g_2(\cdot))}\Big[A(y) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u, f_2(y)))\Big]. \end{aligned} \tag{3.9}$$

Proof. The proof follows from the definition of the resolvent operator (2.1). □

4. Main Results

In this section, we present existence results for the *system of implicit ordered variational inclusions* under some suitable conditions. Let us also discuss the convergence of sequences suggested by an iterative algorithm.

Theorem 4.1. Let C be a normal cone having a normal constant λ_C in a real ordered Banach space X. Let $A, f_i, g_i : X \longrightarrow X$ be single-valued mappings such that A is a λ_A-compression, f_i is λ_{f_i}-compression and g_i is comparison mappings for $i = 1, 2$. Let $Q_i : X \times X \longrightarrow X$ $(i = 1, 2)$ be single-valued mappings such that Q_1 is an (α_1, α_2)-restricted-accretive mapping with respect to f_1, and Q_2 is (β_1, β_2)-restricted accretive mapping with respect to f_2. Suppose that $T_i, F_i : X \longrightarrow CB(X)$ $(i = 1, 2)$ be the $\mathfrak{D}$-Lipschitz continuous mappings with respect to the constants $\varrho_i, \sigma_i > 0$. Suppose $M, N : X \times X \longrightarrow CB(X)$ are the mappings such that M is (α_A, λ)-weak-ANODD and N is $(\alpha_{A'}, \lambda)$-weak-ANODD set-valued mappings.
In addition, if $x_i \propto y_i$, $u_i \propto v_i$, $p_i \propto q_i$, $R^M_{A,\lambda}(x_i) \propto R^M_{A,\lambda}(y_i)$, $R^N_{A,\lambda_2}(x_i) \propto R^N_{A,\lambda}(y_i)$ $(i = 1, 2)$ and for all $\lambda_i, \delta_i > 0$ $(i = 1, 2)$, the following condition

$$\begin{aligned} R_{A,\lambda}^{M(g_1(\cdot),q_1)}(x_1) \oplus R_{A,\lambda}^{M(g_1(\cdot),q_2)}(x_1) &\le \delta_2(q_1 \oplus q_2), \\ R_{A,\lambda}^{N(p_1,g_2(\cdot))}(y_1) \oplus R_{A,\lambda}^{N(p_2,g_2(\cdot))}(y_1) &\le \delta_1(p_1 \oplus p_2), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\frac{\lambda_C}{\rho_1\rho_2}[\rho_2\mu_1\lambda\alpha_1\lambda_{f_1} + \rho_1\mu_2\lambda\beta_1\varrho_1] < 1 - \lambda_C(\mu_1\lambda_A + \delta_1\sigma_1), \\ &\frac{\lambda_C}{\rho_1\rho_2}[\rho_1\mu_2\lambda\beta_2\lambda_{f_2} + \rho_2\mu_1\lambda\alpha_2\varrho_2] < 1 - \lambda_C(\mu_2\lambda_A + \delta_2\sigma_2) \end{aligned} \tag{4.2}$$

are satisfied. Then (3.1) grants a solution (x, y, u, v, p, q).

Proof. From Lemma 2.16, we know that the resolvent operators $R^M_{A,\lambda}(\cdot)$ and $R^N_{A,\lambda}(\cdot)$ are Lipschitz continuous with the constants $\mu_1 = \frac{1}{\gamma_1(\alpha_A\lambda - 1)}$ and $\mu_2 = \frac{1}{\gamma_2(\alpha_{A'}\lambda - 1)}$, *respectively.*
Now, define a mapping $P : X \times X \longrightarrow X \times X$ by

$$P(x, y) = (G(x_i, y_i), S(x_i, y_i)),\ \forall (x, y) \in X \times X,\ (i = 1, 2) \tag{4.3}$$

where $G, S : X \times X \longrightarrow X$ are the mappings defined as

$$G(x_i, y_i) = R_{A,\lambda}^{M(g_1(\cdot),q_i)}\Big[A(x_i) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_i), v_i))\Big], \tag{4.4}$$

and

$$S(x_i, y_i) = R_{A,\lambda}^{N(p_i,g_2(\cdot))}\Big[A(y_i) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_i, f_2(y_i)))\Big]. \tag{4.5}$$

For any $x_i, y_i \in X$ and $x_i \propto y_j$, $u_i \propto v_j$, $p_i \propto q_j$ $(i, j = 1, 2)$. By using (4.4), Definition 2.6, Definition 2.13 and Lemmas 2.16 and 2.10, we have

$$\begin{aligned} 0 &\le G(x_1, y_1) \oplus G(x_2, y_2) \\ &= R_{A,\lambda}^{M(g_1(\cdot),q_1)}\Big[A(x_1) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_1), v_1))\Big] \oplus R_{A,\lambda}^{M(g_1(\cdot),q_2)}\Big[A(x_2) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_2), v_2))\Big] \end{aligned}$$

$$
\begin{aligned}
&\le R_{A,\lambda}^{M(g_1(\cdot),q_1)}\Big[A(x_1)+\frac{\lambda}{\rho_1}(w_1-Q_1(f_1(x_1),v_1))\Big]\oplus R_{A,\lambda}^{M(g_1(\cdot),q_1)}\Big[A(x_2)+\frac{\lambda}{\rho_1}(w_1-Q_1(f_1(x_2),v_2))\Big]\\
&\oplus R_{A,\lambda}^{M(g_1(\cdot),q_1)}\Big[A(x_2)+\frac{\lambda}{\rho_1}(w_1-Q_1(f_1(x_2),v_2))\Big]\oplus R_{A,\lambda}^{M(g_1(\cdot),q_2)}\Big[A(x_2)+\frac{\lambda}{\rho_1}(w_1-Q_1(f_1(x_2),v_2))\Big]\\
&\le \mu_1\Big[A(x_1)\oplus A(x_2)+\frac{\lambda}{\rho_1}(Q_1(f_1(x_1),v_1)\oplus Q_1(f_1(x_2),v_2))\Big]\oplus\delta_2(q_1\oplus q_2)\\
&\le \mu_1\Big[A(x_1)\oplus A(x_2)+\frac{\lambda}{\rho_1}(\alpha_1(f_1(x_1)\oplus f_1(x_2))+\alpha_2(v_1\oplus v_2))\Big]\oplus\delta_2(q_1\oplus q_2)\\
&\le \mu_1\Big[\lambda_A(x_1\oplus x_2)+\frac{\lambda}{\rho_1}(\alpha_1\lambda_{f_1}(x_1\oplus x_2)+\alpha_2(v_1\oplus v_2))\Big]\oplus\delta_2(q_1\oplus q_2)\\
&\le \mu_1\Big[\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)(x_1\oplus x_2)+\frac{\lambda\alpha_2}{\rho_1}(v_1\oplus v_2)\Big]\oplus\delta_2(q_1\oplus q_2).
\end{aligned}\tag{4.6}
$$

From Definition 2.2 and Lemma 2.9, we have

$$
\begin{aligned}
&\|G(x_1,y_1)\oplus G(x_2,y_2)\|=\|G(x_1,y_1)-G(x_2,y_2)\|\\
&\le\lambda_C\Big\|\mu_1\Big[\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)(x_1\oplus x_2)+\frac{\lambda\alpha_2}{\rho_1}(v_1\oplus v_2)\Big]\oplus\delta_2(q_1\oplus q_2)\Big\|\\
&\le\lambda_C\Big\{\mu_1\Big\|\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)(x_1\oplus x_2)\Big\|+\mu_1\Big\|\frac{\lambda\alpha_2}{\rho_1}(v_1\oplus v_2)\|+\delta_2\|q_1\oplus q_2\Big\|\Big\}\\
&\le\lambda_C\Big\{\mu_1\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)\|x_1-x_2\|+\mu_1\frac{\lambda\alpha_2}{\rho_1}\|v_1-v_2\|+\delta_2\|q_1-q_2\|\Big\}\\
&\le\lambda_C\Big\{\mu_1\Big(\lambda_A+\frac{\lambda\alpha_1\lambda_{f_1}}{\rho_1}\Big)\|x_1-x_2\|+\frac{\mu_1\lambda\alpha_2}{\rho_1}\mathfrak{D}(T_2(y_1),T_2(y_2))+\delta_2\mathfrak{D}(F_2(y_1),F_2(y_2))\Big\}\\
&\le\lambda_C\Big\{\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_1-x_2\|+\frac{\mu_1\lambda\alpha_2\varrho_2}{\rho_1}\|y_1-y_2\|+\delta_2\sigma_2\|y_1-y_2\|\Big\}\\
&\le\lambda_C\Big\{\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_1-x_2\|+\frac{\mu_1\lambda\alpha_2\varrho_2+\delta_2\sigma_2\rho_1}{\rho_1}\|y_1-y_2\|\Big\}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\|G(x_1,y_1)-G(x_2,y_2)\|\le{}&\lambda_C\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_1-x_2\|\\
&+\lambda_C\frac{(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)}{\rho_1}\|y_1-y_2\|.
\end{aligned}\tag{4.7}
$$

Again,

$$
\begin{aligned}
0&\le S(x_1,y_1)\oplus S(x_2,y_2)\\
&=R_{A,\lambda}^{N(p_1,g_2(\cdot))}\Big[A(y_1)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_1,f_2(y_1)))\Big]\oplus R_{A,\lambda}^{N(p_2,g_2(\cdot))}\Big[A(y_2)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_2,f_2(y_2)))\Big]\\
&\le R_{A,\lambda}^{N(p_1,g_2(\cdot))}\Big[A(y_1)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_1,f_2(y_1)))\Big]\oplus R_{A,\lambda}^{N(p_1,g_2(\cdot))}\Big[A(y_2)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_2,f_2(y_2)))\Big]\\
&\oplus R_{A,\lambda}^{N(p_1,g_2(\cdot))}\Big[A(y_2)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_2,f_2(y_2)))\Big]\oplus R_{A,\lambda}^{N(p_2,g_2(\cdot))}\Big[A(y_2)+\frac{\lambda}{\rho_2}(w_2-Q_2(u_2,f_2(y_2)))\Big]\\
&\le\mu_2\Big[A(y_1)\oplus A(y_2)+\frac{\lambda}{\rho_2}(Q_2(u_1,f_2(y_1))\oplus Q_2(u_2,f_2(y_2)))\Big]\oplus\delta_1(p_1\oplus p_2)\\
&\le\mu_2\Big[A(y_1)\oplus A(y_2)+\frac{\lambda}{\rho_2}(\beta_1(u_1\oplus u_2)+\beta_2\lambda_{f_2}(y_1\oplus y_2))\Big]\oplus\delta_1(p_1\oplus p_2)\\
&\le\mu_2\Big[\lambda_A(y_1\oplus y_2)+\frac{\lambda}{\rho_2}(\beta_1(u_1\oplus u_2)+\beta_2\lambda_{f_2}(y_1\oplus y_2))\Big]\oplus\delta_1(p_1\oplus p_2).
\end{aligned}\tag{4.8}
$$

From Definition 2.2 and Lemma 2.9, we have

$$
\begin{aligned}
&\|S(x_1,y_1)\oplus S(x_2,y_2)\| = \|S(x_1,y_1)-S(x_2,y_2)\|\\
&\leq \lambda_C\Big\|\mu_2\lambda_A(y_1\oplus y_2)+\frac{\mu_2\lambda\beta_1}{\rho_2}(u_1\oplus u_2)+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}(y_1\oplus y_2)+\delta_1(p_1\oplus p_2)\Big\|\\
&\leq \lambda_C\Big[\mu_2\lambda_A\|y_1-y_2\|+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}\|y_1-y_2\|+\frac{\mu_2\lambda\beta_1}{\rho_2}|u_1-u_2\|+\delta_1\|p_1-p_2\|\Big]\\
&\leq \lambda_C\Big[\Big(\mu_2\lambda_A+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}\Big)\|y_1-y_2\|+\frac{\mu_2\lambda\beta_1}{\rho_2}\mathfrak{D}(T_1(x_1),T_1(x_2))+\delta_1\mathfrak{D}(F_1(x_1),F_1(x_2))\Big]\\
&\leq \lambda_C\Big[\Big(\mu_2\lambda_A+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}\Big)\|y_1-y_2\|+\frac{\mu_2\lambda\beta_1\varrho_1}{\rho_2}\|x_1-x_2\|+\delta_1\sigma_1\|x_1-x_2\|\Big]\\
&\leq \lambda_C\Big(\mu_2\lambda_A+\frac{\mu_2\lambda\beta_2\lambda_{f_2}}{\rho_2}\Big)\|y_1-y_2\|+\lambda_C\Big(\frac{\mu_2\lambda\beta_1\varrho_1}{\rho_2}+\delta_1\sigma_1\Big)\|x_1-x_2\|.
\end{aligned}
$$

That is,

$$
\|S(x_1,y_1)-S(x_2,y_2)\| \leq \frac{\lambda_C\mu_2(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})}{\rho_2}\|y_1-y_2\|+\frac{\lambda_C(\mu_2\lambda\beta_1\varrho_1+\rho_2\delta_1\sigma_1)}{\rho_2}\|x_1-x_2\|. \tag{4.9}
$$

From (4.7) and (4.9), we have

$$
\begin{aligned}
&\|G(x_1,y_1)-G(x_2,y_2)\|+\|S(x_1,y_1)-S(x_2,y_2)\| \leq \lambda_C\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_1-x_2\|\\
&+\lambda_C\frac{(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)}{\rho_1}\|y_1-y_2\|+\lambda_C\frac{\mu_2(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})}{\rho_2}\|y_1-y_2\|\\
&+\lambda_C\frac{(\mu_2\lambda\beta_1\varrho_1+\delta_1\sigma_1\rho_2)}{\rho_2}\|x_1-x_2\|\\
&\leq \lambda_C\Big[\frac{\mu_1(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})}{\rho_1}+\frac{(\mu_2\lambda\beta_1\varrho_1+\delta_1\sigma_1\rho_2)}{\rho_2}\Big]\|x_1-x_2\|\\
&+\lambda_C\Big[\frac{\mu_2(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})}{\rho_2}+\frac{(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)}{\rho_1}\Big]\|y_1-y_2\|\\
&\leq \frac{\lambda_C}{\rho_1\rho_2}[\mu_1\rho_2(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})+\rho_1(\mu_2\lambda\beta_1\varrho_1+\delta_1\sigma_1\rho_2)]\|x_1-x_2\|\\
&+\frac{\lambda_C}{\rho_1\rho_2}[\mu_2\rho_1(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})+\rho_2(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)]\|y_1-y_2\|\\
&\leq \Omega_1\|x_1-x_2\|+\Omega_2\|y_1-y_2\|\\
&\leq \max\{\Omega_1,\Omega_2\}(\|x_1-x_2\|+\|y_1-y_2\|),
\end{aligned} \tag{4.10}
$$

where

$$
\Omega_1=\frac{\lambda_C}{\rho_1\rho_2}\Big[\mu_1\rho_2(\lambda_A\rho_1+\lambda\alpha_1\lambda_{f_1})+\rho_1(\mu_2\lambda\beta_1\varrho_1+\delta_1\sigma_1\rho_2)\Big]
$$

and

$$
\Omega_2=\frac{\lambda_C}{\rho_1\rho_2}\Big[\mu_2\rho_1(\lambda_A\rho_2+\lambda\beta_2\lambda_{f_2})+\rho_2(\mu_1\lambda\alpha_2\varrho_2+\rho_1\delta_2\sigma_2)\Big].
$$

Now, we define $\|(x,y)\|_*$ on $X\times X$ by

$$
\|(x,y)\|_* = \|x\|+\|y\|,\ \forall (x,y)\in X\times X. \tag{4.11}
$$

One can easily show that $(X\times X,\|\cdot\|)$ is a Banach space. Hence from (4.3), (4.10) and (4.11), we have

$$
\|P(x_1,y_1)-P(x_2,y_2)\|_* \leq \max\{\Omega_1,\Omega_2\}(\|x_1-x_2\|+\|y_1-y_2\|). \tag{4.12}
$$

By (4.2), we know that $\max\{\Omega_1,\Omega_2\}<1$. It follows from (4.12) that P is a contraction mapping. Hence there exists unique $(x,y)\in X\times X$ such that

$$
P(x,y)=(x,y).
$$

This leads to

$$x = R_{A,\lambda}^{M(g_1(\cdot),q)}\Big[A(x) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x), v))\Big],$$

and

$$y = R_{A,\lambda}^{N(p,g_2(\cdot))}\Big[A(y) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u, f_2(y)))\Big].$$

It is determined by Lemma 3.1 that (x, y, u, v, p, q) is a solution of (3.1). $\square$

Now, we suggest an iterative scheme for problem (3.1).

Algorithm 4.2. Let C be a normal cone with a normal constant λ_C in a real ordered Banach space X. Assume that $f_i, g_i : X \longrightarrow X$ and $Q_i : X \times X \longrightarrow X$ are single-valued mappings for $i = 1, 2$. Let $M, N : X \times X \longrightarrow CB(X)$ and $T_i, F_i : X \longrightarrow CB(X)(i = 1, 2)$ be the set-valued mappings.
For any given $x_0, y_0 \in X$, $u_0 \in T_1(x_0)$, $v_0 \in T_2(y_0)$, $p_0 \in F_1(x_0)$, $q_0 \in F_2(y_0)$, let

$$x_1 = (1-\pi)x_0 + \pi R_{A,\lambda}^{M(g_1(\cdot),q_0)}\Big[A(x_0) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_0), v_0))\Big],$$

$$y_1 = (1-\pi)y_0 + \pi R_{A,\lambda}^{N(p_0,g_2(\cdot))}\Big[A(y_0) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_0, f_2(y_0)))\Big],$$

there exist $u_1 \in T_1(x_1) \in CB(X)$, $v_1 \in T_2(y_1) \in CB(X)$, $p_1 \in F_1(x_1) \in CB(X)$, $q_1 \in F_2(y_1) \in CB(X)$, and assume that $x_0 \propto x_1$, $y_0 \propto y_1$, $u_0 \propto u_1$, $v_0 \propto v_1$, $p_0 \propto p_1$, $q_0 \propto q_1$ such that

$$\|u_1 \oplus u_0\| = \|u_1 - u_0\| \le (1+1)\mathfrak{D}(T_1(x_1), T_1(x_0));$$
$$\|v_1 \oplus v_0\| = \|v_1 - v_0\| \le (1+1)\mathfrak{D}(T_2(y_1), T_2(y_0));$$
$$\|p_1 \oplus p_0\| = \|p_1 - p_0\| \le (1+1)\mathfrak{D}(F_1(x_1), F_1(x_0));$$
$$\|q_1 \oplus q_0\| = \|q_1 - q_0\| \le (1+1)\mathfrak{D}(F_2(y_1), F_2(y_0)).$$

Continuing in this way, we can define iterative sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{p_n\}, \{q_n\}$ with the supposition that $x_{n+1} \propto x_n, y_{n+1} \propto y_n, u_{n+1} \propto u_n, v_{n+1} \propto v_n, p_{n+1} \propto p_n, q_{n+1} \propto q_n$, for all $n \in \mathbb{R}$. We have the following iterative schemes:

$$x_{n+1} = (1-\pi)x_n + \pi R_{A,\lambda}^{M(g_1(\cdot),q_n)}\Big[A(x_n) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_n), v_n))\Big], \tag{4.13}$$

$$y_{n+1} = (1-\pi)y_n + \pi R_{A,\lambda}^{N(p_n,g_2(\cdot))}\Big[A(y_n) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_n, f_2(y_n)))\Big], \tag{4.14}$$

with

$$u_{n+1} \in T_1(x_{n+1}), \|u_{n+1} \oplus u_n\| = \|u_{n+1} - u_n\| \le \Big(1 + \frac{1}{n+1}\Big)\mathfrak{D}(T_1(x_{n+1}), T_1(x_n));$$
$$v_{n+1} \in T_2(y_{n+1}), \|v_{n+1} \oplus v_n\| = \|v_{n+1} - v_n\| \le \Big(1 + \frac{1}{n+1}\Big)\mathfrak{D}(T_2(y_{n+1}), T_2(y_n));$$
$$p_{n+1} \in F_1(x_{n+1}), \|p_{n+1} \oplus p_n\| = \|p_{n+1} - p_n\| \le \Big(1 + \frac{1}{n+1}\Big)\mathfrak{D}(F_1(x_{n+1}), F_1(x_n));$$
$$q_{n+1} \in F_2(x_{n+1}), \|q_{n+1} \oplus q_n\| = \|q_{n+1} - q_n\| \le \Big(1 + \frac{1}{n+1}\Big)\mathfrak{D}(F_2(y_{n+1}), F_2(y_n)); \tag{4.15}$$

for $n = 0, 1, 2, 3, 4, \ldots$, where $0 \le \pi < 1$ and $\lambda, \rho > 0$ are the constants.

Theorem 4.3. Allow $X, C, M, N, f_i, g_i, Q_i, T_i, F_i$ $(i = 1, 2)$ to be as in Theorem 4.1. Then the sequences $\{(x_n, y_n, u_n, v_n, p_n, q_n)\}$ formulated by Algorithm 4.2, converge strongly to $\{(x, y, u, v, p, q)\}$ of (3.1).

Proof. From Algorithm 4.2, (4.1) and Lemma 2.10, we get

$$\begin{aligned}0 &\le x_{n+1} \oplus x_n \\ &= (1-\pi)x_n + \pi R_{A,\lambda}^{M(g_1(\cdot),q_n)}\Big[A(x_n) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_n), v_n))\Big]\end{aligned}$$

$$
\begin{aligned}
&\oplus (1-\pi)x_{n-1} + \pi R_{A,\lambda}^{M(g_1(\cdot),q_{n-1})}\Big[A(x_{n-1}) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_{n-1}), v_{n-1}))\Big] \\
&= (1-\pi)(x_n \oplus x_{n-1}) + \pi\Big[R_{A,\lambda}^{M(g_1(\cdot),q_n)}\Big[A(x_n) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_n), v_n))\Big] \\
&\oplus R_{A,\lambda}^{M(g_1(\cdot),q_{n-1})}\Big[A(x_{n-1}) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_{n-1}), v_{n-1}))\Big]\Big]. \qquad (4.16)
\end{aligned}
$$

By using the same argument as in Theorem 4.1, for (4.7), we have

$$
\begin{aligned}
&\|x_{n+1} \oplus x_n\| = \|x_{n+1} - x_n\| \\
&\leq (1-\pi)\|x_n - x_{n-1}\| + \pi\Big[\lambda_C \frac{\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1})}{\rho_1}\|x_n - x_{n-1}\| \\
&+ \lambda_C \frac{(1+\frac{1}{n+1})(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2)}{\rho_1}\|y_n - y_{n-1}\|\Big] \\
&\leq \Big[(1-\pi) + \pi\frac{\lambda_C\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1})}{\rho_1}\Big]\|x_n - x_{n-1}\| \\
&+ \pi\Big[\frac{(1+\frac{1}{n+1})\lambda_C(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2)}{\rho_1}\Big]\|y_n - y_{n-1}\|. \qquad (4.17)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&0 \leq y_{n+1} \oplus y_n \\
&= \Big((1-\pi)y_n + \pi R_{A,\lambda}^{N(p_n,g_2(\cdot))}\Big[A(y_n) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_n, f_2(y_n)))\Big]\Big) \\
&\oplus \Big((1-\pi)y_{n-1} + \pi R_{A,\lambda}^{N(p_{n-1},g_2(\cdot))}\Big[A(y_{n-1}) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_{n-1}, f_2(y_{n-1})))\Big]\Big) \\
&= (1-\pi)(y_n \oplus y_{n-1}) + \pi\Big(R_{A,\lambda}^{N(p_n,g_2(\cdot))}\Big[A(y_n) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_n, f_2(y_n)))\Big] \\
&\oplus R_{A,\lambda}^{N(p_{n-1},g_2(\cdot))}\Big[A(y_{n-1}) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_{n-1}, f_2(y_{n-1})))\Big]\Big). \qquad (4.18)
\end{aligned}
$$

Importing the same logic as in Theorem 4.1 for (4.9), we have

$$
\begin{aligned}
\|y_{n+1} \oplus y_{n-1}\| = \|y_{n+1} - y_{n-1}\| \leq &\Big[(1-\pi) + \pi\frac{\lambda_C\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})}{\rho_2}\Big]\|y_n - y_{n-1}\| \\
&+ \pi\frac{\lambda_C\big(1+\frac{1}{n+1}\big)(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)}{\rho_2}\|x_n - x_{n-1}\|. \qquad (4.19)
\end{aligned}
$$

From (4.17) and (4.19), we have

$$
\begin{aligned}
&\|x_{n+1} - x_{n-1}\| + \|y_{n+1} - y_{n-1}\| \leq \Big[(1-\pi) + \frac{\pi\lambda_C\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1})}{\rho_1}\Big]\|x_n - x_{n-1}\| \\
&+ \Big[\frac{\pi\big(1+\frac{1}{n+1}\big)\lambda_C(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2)}{\rho_1}\Big]\|y_n - y_{n-1}\| \\
&+ \Big[(1-\pi) + \pi\frac{\lambda_C\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})}{\rho_2}\Big]\|y_n - y_{n-1}\| \\
&+ \pi\frac{\lambda_C\big(1+\frac{1}{n+1}\big)(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)}{\rho_2}\|x_n - x_{n-1}\| \\
&\leq (1-\pi)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) \\
&+ \pi\Big[\frac{\lambda_C\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1})}{\rho_1} + \frac{\lambda_C\big(1+\frac{1}{n+1}\big)(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)}{\rho_2}\Big]\|x_n - x_{n-1}\| \\
&+ \pi\Big[\frac{\lambda_C\big(1+\frac{1}{n+1}\big)(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2)}{\rho_1} + \frac{\lambda_C\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})}{\rho_2}\Big]\|y_n - y_{n-1}\|
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\pi)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) + \pi(\Omega_1^n\|x_n - x_{n-1}\| + \Omega_2^n\|y_n - y_{n-1}\|)\\
&= (1-\pi)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) + \pi\Omega^n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \qquad (4.20)
\end{aligned}
$$

where $\Omega^n = \max\{\Omega_1^n, \Omega_2^n\}$ and

$$
\begin{aligned}
\Omega_1^n &= \frac{\lambda_C}{\rho_1\rho_2}\Big[\mu_1\rho_2(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1}) + \Big(1+\frac{1}{n+1}\Big)\rho_1(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)\Big],\\
\Omega_2^n &= \frac{\lambda_C}{\rho_1\rho_2}\Big[\Big(1+\frac{1}{n+1}\Big)\rho_2(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2) + \rho_1\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})\Big].
\end{aligned}
$$

Let $\Omega_1^n \longrightarrow \Omega_1$ and $\Omega_2^n \longrightarrow \Omega_2$ whenever $n \longrightarrow \infty$, therefore $\Omega^n \longrightarrow \Omega$ as $n \longrightarrow \infty$. Then condition (4.2) implies $\Omega < 1$ and so $\Omega_n < 1$ for sufficiently large n. By (4.20), for sufficient n, we have

$$
\begin{aligned}
\|x_{n+1} - x_{n-1}\| + \|y_{n+1} - y_{n-1}\| &\le (1-\pi)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)\\
&\quad + \pi\Omega(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)\\
&\le (1-\pi+\pi\Omega)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|)\\
&\le \varsigma(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|). \qquad (4.21)
\end{aligned}
$$

where $\varsigma = 1 - \pi + \pi\Omega$ and $\Omega = \max\{\Omega_1, \Omega_2\}$, and

$$
\begin{aligned}
\Omega_1 &= \frac{\lambda_C}{\rho_1\rho_2}[\rho_2\mu_1(\lambda_A\rho_1 + \lambda\alpha_1\lambda_{f_1}) + \rho_1(\mu_2\lambda\beta_1\varrho_1 + \delta_1\sigma_1\rho_2)],\\
\Omega_2 &= \frac{\lambda_C}{\rho_1\rho_2}[\rho_2(\mu_1\lambda\alpha_2\varrho_2 + \rho_1\delta_2\sigma_2) + \rho_1\mu_2(\lambda_A\rho_2 + \lambda\beta_2\lambda_{f_2})].
\end{aligned}
$$

By (4.2), we have $\Omega < 1$. So there exists $\Omega^0 < 1$ such that for sufficiently large n, $\Omega^n < \Omega^0$ and

$$
\|x_{n+1} - x_{n-1}\| + \|y_{n+1} - y_{n-1}\| \le \varsigma^0(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \qquad (4.22)
$$

where $\varsigma^0 = 1 - \pi + \pi\Omega^0 < 1$.
It follow that $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete space, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. From (4.22), $\{y_n\}$ is also a Cauchy sequence in X and $y_n \longrightarrow y$ as $n \longrightarrow \infty$. Condition (4.15) and the $\mathfrak{D}$-Lipschitz continuity of T_1, T_2, F_1, F_2 imply that $\{u_n\}, \{v_n\}, \{p_n\}$ and $\{q_n\}$ are all the Cauchy sequences. Let $u_n \longrightarrow u$, $v_n \longrightarrow v$, $p_n \longrightarrow p$ and $q_n \longrightarrow q$, *respectively*. By (4.15), we have

$$
\begin{aligned}
d(u, T_1(u)) &\le \|u - u_n\| + d(u_n, T(u))\\
&\le \|u - u_n\| + \mathfrak{D}(T_1(u_n), T_1(u))\\
&\le \|u - u_n\| + \varrho_1\|u_n - u\| \longrightarrow 0, \text{ as } n \longrightarrow \infty, \qquad (4.23)
\end{aligned}
$$

and so $u \in T_1(x)$. Similarly, we can show that $v \in T_2(y), p \in F_1(x)$ and $q \in F_2(y)$. By (4.15), we have

$$
\begin{aligned}
x_{n+1} &= (1-\pi)x_n + \pi R_{A,\lambda}^{M(g_1(\cdot),q_n)}\Big[A(x_n) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x_n), v_n))\Big],\\
y_{n+1} &= (1-\pi)y_n + \pi R_{A,\lambda}^{N(p_n,g_2(\cdot))}\Big[A(y_n) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u_n, f_2(y_n)))\Big].
\end{aligned}
$$

By Lemma 2.16 and the assumptions in Theorem 4.1, letting $n \longrightarrow \infty$ in the above equations, we can obtain

$$
\begin{aligned}
x &= (1-\pi)x + \pi R_{A,\lambda}^{M(g_1(\cdot),q)}\Big[A(x) + \frac{\lambda}{\rho_1}(w_1 - Q_1(f_1(x), v))\Big],\\
y &= (1-\pi)y + \pi R_{A,\lambda}^{N(p,g_2(\cdot))}\Big[A(y) + \frac{\lambda}{\rho_2}(w_2 - Q_2(u, f_2(y)))\Big].
\end{aligned}
$$

By Lemma 3.1, $\{(x, y, u, v, p, q)\}$ is a solution of system (3.1). This completes the proof. □

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PROJECTIVITY AND UNIFICATION IN LOCALLY FINITE VARIETIES OF MONADIC MV-ALGEBRAS

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Abstract. A duality between the category of finite monadic MV-algebras and a category of labelled finite Boolean spaces is given. A characterization of projectivity in some locally finite varieties of monadic MV-algebras is provided. Finally, we show that the unification type of these varieties is unitary.

1. Introduction

The finitely-valued propositional calculi, which have been described by Lukasiewicz and Tarski in [15], are extended to the corresponding predicate calculi. The predicate Łukasiewicz (infinitely-valued) logic QL is defined in the following standard way. The existential (universal) quantifier is interpreted as a supremum (infimum) in a complete MV-algebra. Then the valid formulas of predicate calculus are defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by Rutledge in [16]. Scarpellini in [17] has proved that the set of valid formulas is not recursively enumerable. We also refer the reader to papers [10, 18, 19] concerning the Łukasiewicz predicate calculus.

Monadic MV-algebras were introduced and studied by Rutledge in [16] as an algebraic model for the predicate calculus QL of Łukasiewicz infinite-valued logic, in which there occurs only a single individual variable. Rutledge followed P.R. Halmos' study of monadic Boolean algebras. In view of the incompleteness of the predicate calculus, the result of Rutledge in [16], showing the completeness of the monadic predicate calculus, has been of great interest.

Let L denote a first-order language based on $\cdot, +, \to, \neg$ (intended as propositional connectives) and let L_m denote a propositional language based on the propositional connectives $\cdot$, $+$, $\to$, $\neg$, $\exists$ (where $\exists$ denotes a unary propositional connective). Let $Form(L)$ and $Form(L_m)$ be the set of all formulas of L and L_m, respectively. We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation $\Psi : Form(L_m) \to Form(L)$ by putting:

- $\Psi(p) = p^*(x)$ if p is a propositional variable;
- $\Psi(\neg\alpha) = \neg\Psi(\alpha)$;
- $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \to$;
- $\Psi(\exists\alpha) = \exists x \Psi(\alpha)$.

Through this translation Ψ, we can identify the formulas of L_m with monadic formulas of L containing the variable x. Moreover, it is routine to check that $\Psi(MLPC) \subseteq QL$, where $MLPC$ is the monadic Lukasiewicz propositional calculus [8].

For a detailed consideration of Łukasiewicz predicate calculus we refer to [1, 2, 14, 15].

2. Preliminaries on Monadic MV-algebras

The characterization of monadic MV-algebras as pairs of MV-algebras, where one of them is a special kind of subalgebra (m is a relatively complete subalgebra), is given in [3, 8]. The MV-algebras were introduced by Chang in [4] as an algebraic model for infinitely-valued Łukasiewicz logic.

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An MV-algebra is an algebra $(A, \oplus, \odot, ^*, 0, 1)$, where $(A, \oplus, 0)$ is an abelian monoid, and the following identities hold for all $x, y \in A$: $x \oplus 1 = 1$, $x^{**} = x$, $0^* = 1$, $x \oplus x^* = 1$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, $x \odot y = (x^* \oplus y^*)^*$.

Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \quad \text{iff} \quad x^* \oplus y = 1.$$

$(A, \leq, 0, 1)$ is a bounded distributive lattice. Moreover, in any MV-algebra, the property

$$x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y holds.$$

The unit interval of real numbers $[0, 1]$ endowed with the operations $x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), x^* = 1 - x$, becomes an MV-algebra. It is well known that the MV-algebra $S = ([0, 1], \oplus, \odot, ^*, 0, 1)$ generates the variety **MV** of all MV-algebras, i. e., $\mathcal{V}(S) = \mathbf{MV}$.

Let $\mathbb{Q}$ denote a set of rational numbers; then $[0, 1] \cap \mathbb{Q}$ is an MV-subalgebra of $[0, 1]$.

Moreover, for $(0 \neq)n \in \omega$, we denote by S_n the subalgebra of $[0, 1]$ whose domain is

$$A_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

For any positive integer n, an algebra $(A, \oplus, \odot, ^*, \exists, 0, 1)$ is said to be a monadic MV-algebra (MMV-algebra, for short) if $(A, \oplus, \odot, ^*, 0, 1)$ is an MV-algebra and, in addition, $\exists$ is a unary function and the following identities hold:

E1: $x \leq \exists x$,
E2: $\exists(x \vee y) = \exists x \vee \exists y$,
E3: $\exists(\exists x)^* = (\exists x)^*$,
E4: $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,
E5: $\exists(x \odot x) = \exists x \odot \exists x$,
E6: $\exists(x \oplus x) = \exists x \oplus \exists x$ hold.

Sometimes we shall denote a monadic MV-algebra $(A, \oplus, \odot, ^*, \exists, 0, 1)$ by $(A, \exists)$, for brevity. We can define a unary operation $\forall x = (\exists x^*)^*$, corresponding to the universal quantifier.

Let A_1 and A_2 be any MMV-algebras. A mapping $h : A_1 \to A_2$ is an MMV-homomorphism if h is an MV-homomorphism, and for every $x \in A_1$, $h(\exists x) = \exists h(x)$. Denote by **MMV** the variety and the category of MMV-algebras and MMV-homomorphisms.

From the variety of monadic MV-algebras **MMV** we select the subvariety $\mathbf{K_n}$ for $0 \neq n \in \omega$, which is defined by the following equation [8]:

$$(K_n) \quad x^n = x^{n+1},$$

that is, $\mathbf{K_n} = \mathbf{MMV} + (K_n)$. The main object of our interest are the varieties $\mathbf{K_n}$, which are locally finite, see [8].

A subalgebra A_0 of an MV-algebra A is said to be *relatively complete*, if for every $a \in A$ the set $\{b \in A_0 : a \leq b\}$ has a least element.

A subalgebra A_0 of an MV-algebra A is said to be *m-relatively complete* [8], if A_0 is relatively complete and two additional conditions

(#): $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \odot a \Rightarrow v \geq a \& v \odot v \leq x)$,
(##): $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq a \& v \oplus v \leq x)$ hold.

By [8], there exists a one-to-one correspondence between

(1) the monadic MV-algebras $(A, \exists)$;
(2) the pairs (A, A_0), where A_0 is an m-relatively complete subalgebra of A.

In fact, A_0 and $\exists$ can be uniquely recovered from each other in the following way: A_0 is the range of $\exists$, and $\exists a = inf\{b \in A_0 : a \leq b\}$.

3. Monadic Operators on Finite MV-algebras

In this section, we recall the characterization of all monadic operators over an arbitrary finite MV-algebra given in [3]. In other words, given any finite MV-algebra, we characterize the set of monadic operators which make it an MMV-algebra.

Suppose that A is a finite MV-algebra. Then $A \cong S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$, where the $n_i \geq 1$. Let $\Pi = \{K_1, K_2, \ldots, K_m\}$ be a partition of $\{1, 2, \ldots, k\}$. We shall say that Π is homogeneous if $i, j \in K_l$ implies $S_{n_i} = S_{n_j}$. Given such a Π, each K_i has associated a unique S_{n_j}, which we shall denote by A_i. We clearly have

$$A \cong A_1^{K_1} \times \cdots \times A_m^{K_m}. \tag{1}$$

Since each K_i is finite, there is a monadic operator $\exists_i$ defined on $A_i^{K_i}$ such that $(A_i^{K_i}, \exists_i)$ is an MMV-algebra with $\exists_i(A_i^{K_i}) = A_i$. Setting $\exists = \exists_1 \times \cdots \times \exists_m$ and acting pointwise, we obtain a monadic operator $\exists$ on A, that is, $(A, \exists)$ is an MMV-algebra.

If a $K_i \in \Pi$ has at least two members, then the determined monadic operator will not be trivial, that is, it will not be the identity operator. From this we can see that a given homogeneous partition may give up to $2^m - 1$ non-trivial MMV-algebras.

If we say that $n_1 = n_2 = \cdots = n_k = n$, so $A = S_n^k$, then every partition of $\{1, 2, \ldots, k\}$ will be homogeneous. The question arises as to whether or not every monadic operator on $A \cong S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ is obtained from some homogeneous partition of $\{1, 2, \ldots, k\}$.

Let $(A, \exists)$ be a finite MMV-algebra. Then by [16], $(A, \exists)$ is a subdirect of the product of MMV-algebras $(A_i, \exists_i)$, where $\exists_i A_i$ is totally ordered. Moreover, by [8], each $(A_i, \exists_i)$ is a direct power of $\exists_i A_i$, that is, $(\exists_i A_i)^{K_i}$ for some finite set K_i.

From this we obtain that $(A, \exists)$ is a subdirect product of MMV-algebras, $(A, \exists) \hookrightarrow \prod_{i=1}^{m} ((\exists_i A_i)^{K_i}, \exists_i)$ for some integer m.

4. Labelled Boolean Spaces

A *Boolean space (or Stone space)* is a compact, zero-dimensional and Hausdorff topological space. Boolean spaces form a category whose objects are the Boolean spaces and morphisms are the continuous maps. When a Boolean space is finite, then the topology of the Boolean space is discrete. It is well known that there exists a categorical duality between the category of Boolean algebras **Bool** and the category of Boolean spaces **BS**. Then the category of finite Boolean algebras $\mathbf{Bool}_{fin}$ is dually equivalent to the category of finite Boolean spaces $\mathbf{BS}_{fin}$.

The functors establishing the duality between $\mathbf{Bool}_{fin}$ and $\mathbf{BS}_{fin}$ are as follows. The functor $\mathfrak{C} : \mathbf{Bool}_{fin} \to \mathbf{BS}_{fin}$ sends every finite Boolean algebra B to the set of all ultrafilters of B. The functor $\mathfrak{B} : \mathbf{BS}_{fin} \to \mathbf{Bool}_{fin}$ sends every object $T \in \mathbf{BS}_{fin}$ to the powerset of T.

We now define another category, the category of *labelled Boolean spaces* $\mathbf{BS}^L_{fin}$. Let $X \in \mathbf{BS}_{fin}$ and $\lambda : X \to \omega$. The the set $X_\lambda = \{(x, \lambda(x)) : x \in X\}$ is said to be a *labelled Boolean space*. The map $f : X_\lambda \to Y_{\lambda'}$ is said to be a λ*-map* if for every x, we have $f((x, \lambda(x))) = (f(x), \lambda'(f(x)))$, where $\lambda'(f(x))$ divides $\lambda(x)$. Denote this category by $\mathbf{BS}^L_{fin}$.

Let A be any finite MV-algebra. Then A contains a greatest Boolean subalgebra $B(A) \subseteq A$. The set of ultrafilters of $B(A)$ and the set of MV-ultrafilters of A have the same cardinality. In fact, if $F \subseteq A$ is an MV-ultrafilter of A, then $F \cap B(A)$ is an ultrafilter of $B(A)$. Conversely, if $F \subseteq B(A)$ is an ultrafilter of $B(A)$, then the MV-filter $[F)_A$ generated by F in the MV-algebra A is an MV-ultrafilter of A. So, we have one-to-one correspondence between the set of ultrafilters of $B(A)$ and the set of MV-ultrafilters of A. So, we can identify the corresponding elements of the set of ultrafilters of $B(A)$ and the set of MV-ultrafilters of A. We observe that two different finite MV-algebras A_1 and A_2 may have isomorphic Boolean subalgebras $B(A_1)$ and $B(A_2)$. For example, $B(S_1^2) \cong B(S_1 \times S_2)$ and, so, $\mathfrak{C}(B(A_1)) \cong \mathfrak{C}(B(A_2))$.

Let A be a finite MV-algebra. Label the elements of $\mathfrak{C}(B(A))$ as follows: $\lambda(F) = k$ if $A/[F)_A \cong S_k$. Then let

$$\mathfrak{C}_\lambda(B(A)) = \{(F, \lambda(F)) : F \in \mathfrak{C}(B(A))\}$$

be the resulting labelled Boolean space. We observe that if $A_1 \ncong A_2$, then $\mathfrak{E}_\lambda(B(A_1)) \ncong \mathfrak{E}_\lambda(B(A_2))$. From this observation we can define the functor $\mathfrak{E}_\mathfrak{L}$ from the set $\mathbf{MV}_{fin}$ of finite MV-algebras to the labelled Boolean spaces $\mathbf{BS}^L_{fin}$ in the following way:

$$\mathfrak{E}_\mathfrak{L}(A) = \{(F, \lambda(F)) : \lambda(F) = k \in \omega, F \in \mathfrak{E}(B(A)), A/[F)_A \cong S_k\}.$$

Now let X_λ be a labelled Boolean space. We define the functor $\mathfrak{L}$ from $\mathbf{BS}^L_{fin}$ to $\mathbf{MV}_{fin}$ as follows:

$$\mathfrak{L}(X_\lambda) = \prod_{x \in X} S_{\lambda(x)}.$$

It is easy to verify that $\mathfrak{L}(\mathfrak{E}_\mathfrak{L}(A)) \cong A$ and $\mathfrak{E}_\mathfrak{L}(\mathfrak{L}(X_\lambda)) \cong X_\lambda$. So, we arrive to

Theorem 1. *The category of finite MV-algebras $\boldsymbol{MV}_{fin}$ is dually equivalent to the category of labelled Boolean spaces $\boldsymbol{BS}^L_{fin}$.*

Any subalgebra of a finite Boolean algebra is relatively complete. If a Boolean algebra B_1 embeds into a Boolean algebra B_2, then to this embedding we can associate a surjective map $f : \mathfrak{E}(B_2) \to \mathfrak{E}(B_1)$. The surjective map defines a corresponding partition $E(= Kerf)$. Conversely, any partition E on the Boolean space defines a corresponding subalgebra. Namely, if X is a Boolean space, then the Boolean algebra of all subsets of the set X is the Boolean algebra corresponding to the Boolean space X. Then the set of all E-saturated subsets[1] forms a Boolean subalgebra of the given Boolean algebra.

We are interested in m-relatively complete subalgebras of a finite MV-algebra A. Note that not every subalgebra of a finite MV-algebra A is m-relatively complete.

A partition E of a labelled Boolean space is said to be *correct*, if for any set $U \in E$ and any two elements $x, y \in U$ we have $\lambda(x) = \lambda(y)$. Note that every correct partition is a homogeneous partition in the sense defined above. So, we have

Theorem 2. *Let A be a finite MV-algebra and X_λ be the labelled Boolean space corresponding to it. Then every correct partition of X_λ defines a subalgebra of A which is m-relatively complete, or equivalently, a monadic operator on A.*

Proof. Any correct partition of X_λ defines a decomposition $A = A_1^{K_1} \times \cdots \times A_m^{K_m}$, where $A_1, \ldots, A_m$ are finite MV-chains. From this decomposition, a monadic operator on A can be obtained as that described after equation (1). □

Now we define a category $\mathbf{BS}^{LM}_{fin}$ of monadic labelled Boolean spaces, the objects of which are the pairs (X_λ, E), where X_λ is a labelled Boolean space and E is an equivalence relation which is a correct partition of X_λ.

Let $(A, \exists)$ be a finite monadic MV-algebra. Then $X_\lambda = \mathfrak{E}_\mathfrak{L}(A)$ is a labelled Boolean space. On X_λ there is a homogeneous (correct) partition E corresponding to the monadic operator $\exists$ (see [3]).

Conversely, suppose we have a labelled Boolean space X_λ and a homogeneous (correct) partition E. Let $E(x) = \{y \in X : \textit{there is } U \in E \textit{ such that } x \in U \wedge y \in U\}$.

Then this partition E defines a monadic operator $\exists$ on $A = \mathfrak{L}(X_\lambda)$.

Now define a morphism $f : (X_\lambda, E) \to (X_{\lambda'}, E')$ (similarly to the monadic Boolean algebras) to be a λ-map $f : X_\lambda \to X_{\lambda'}$ which satisfies the following condition: $f(E(x)) = E'(f(x))$, being the condition of strong isotonicity. So, we arrived at

Theorem 3. *The category of monadic labelled Boolean spaces $\boldsymbol{BS}^{LM}_{fin}$ with strongly isotone λ-maps is dually equivalent to the category of finite monadic MV-algebras.*

Remark 4. Let us note that a duality between the category of multisets and the category of finite MV-algebras is established in [5]. The duality established in this section is a particular case of the one given in [5], but represented in another way. We also mention the related paper [7] (especially, Theorem 1.5).

[1]A subset of X is E-saturated if it coincides with the union of E-equivalence classes.

5. Projective Monadic MV-algebras

Now we come back to the subvariety $\mathbf{K_n}$ $(\mathbf{MMV} + (K_n))$ for $1 \leq n \in \omega$.

There is a unique monadic operator $\exists$ on S_n^k , which corresponds to an m-relatively complete totally ordered MV-subalgebra, converting the algebra S_n^k into a simple monadic MV-algebra [8]. This subalgebra coincides with the greatest diagonal subalgebra, i.e., $d(S_n^k) = \{(x, \dots, x) \in S_n^k : x \in S_n\}$. Denote this monadic MV-algebra by $(S_n^k, \exists_d)$. In this case, the "diagonal" monadic operator $\exists_d$ is defined as follows:

$$\exists_d(x_1, \dots, x_k) = (x_j, \dots, x_j),$$

where $x_j = \max(x_1, \dots, x_k)$. The operator $\forall_d$ is defined dually:

$$\forall_d(x_1, \dots, x_k) = (x_i, \dots, x_i),$$

where $x_i = \min(x_1, \dots, x_k)$.

Notice that $\mathbf{K_n}$ is generated by $(S_p^k, \exists_d)$, $p = 1, \dots, n$ and $k \in \omega$. Moreover, $\mathbf{K_n}$ is locally finite and there exists a maximal $k \in \omega$ depending on p and m such that $(S_p^k, \exists_d)$ is m-generated. There exists also a maximal positive number $r(k, p, m)$ depending on k p and m such that $(S_p^k, \exists_d)^{r(k,p,m)}$ is m-generated.

We emphasize that for every m there is a finite number of simple m-generated monadic MV-algebras from $\mathbf{K_n}$.

Observe that, since the variety $\mathbf{K_n}$ is locally finite, the free object in m generators, denoted by $F_{\mathbf{K_n}}(m)$, is finite, and the labelled Boolean space $X_\lambda(m)$ of $F_{\mathbf{K_n}}(m)$ is a finite cardinal sum of one-element labelled points. So, $F_{\mathbf{K_n}}(m)$ is a finite product of simple monadic MV-algebras, where one of the factors coincides with $(S_1^1, \exists_d)$. Therefore we can represent $F_{\mathbf{K_n}}(m)$ as $(S_1^1, \exists_d) \times \prod_{i \in I} A_i$ for some finite set I, where A_i is a simple m-generated monadic MV-algebra from $\mathbf{K_n}$.

Recall now that a *projective* object of a variety is an object which is a retract of a free object. We will give a characterisation of projective finitely generated MMV-algebras and give two proofs of the assertion - algebraic and in dual category.

Theorem 5. *An m-generated MMV-algebra A from $\mathbf{K_n}$ is projective, iff A is isomorphic to $(S_1^1, \exists_d) \times A'$ for some finite MMV-algebra A'.*

Proof. Firstly, we give an algebraic proof. Let A have the form $A' \times (S_1^1, \exists_d)$. Since the m-generated free MMV-algebra in $\mathbf{K_n}$ is a finite product of subdirectly irreducible simple MMV-algebras, we find that any homomorphism of $F_{\mathbf{K_n}}(m)$ is a projection on the factors. Let us suppose that A (in its representation as product) has k factors. Let us permute the factors of $F_{\mathbf{K_n}}(m)$ in such a way that the first k factors are isomorphic to the first k factors of A. So, A is a homomorphic image of $F_{\mathbf{K_n}}(m)$, which is an isomorphic copy of A. Let this homomorphism be a projection $\pi : F_{\mathbf{K_n}}(m) \to A$. So, $\pi(x_1, \dots, x_k, \dots, x_q) = (x_1, \dots, x_k)$ and let us suppose that $x_1 \in S_1^1$.

Let $\overline{\pi}$ be the projection whose image gives the rest part of the product $(S_1^1, \exists_d) \times \prod_{i \in I} A_i$. Then $(S_1^1, \exists_d)$ is a subalgebra of every non-trivial MMV-algebra. So, $\overline{\pi}(F_{\mathbf{K_n}})$ contains a subalgebra which is isomorphic to $(S_1^1, \exists_d)$. In other words, we have an embedding $\varepsilon : A \to F_{\mathbf{K_n}}(m)$ such that $\varepsilon(x_1, \dots, x_k) = (x_1, \dots, x_k, x_1, \dots, x_1)$. Therefore A is a subalgebra of $F_{\mathbf{K_n}}(m)$ such that $\pi\varepsilon = Id_A$. It means that A is a retract of $F_{\mathbf{K_n}}(m)$.

Conversely, if A does not have the form $A' \times (S_1^1, \exists_d)$, then A cannot be embedded into $F_{\mathbf{K_n}}(m)$. From here we conclude the proof of the theorem.

Now we give another proof of this theorem using duality. Let X_λ be the labelled Boolean space of the MMV-algebra A and $Y_{\lambda'}$ the labelled Boolean space of $F_{\mathbf{K_n}}(m)$. We have to show that X_λ is a retract of $Y_{\lambda'}$. Since A has the form $(S_1^1, \exists_d) \times A'$, we find that X_λ has the form of cardinal sum $(x, 1) \sqcup \coprod_{j=1}^{k-1} (x, i_j)$, i. e., X_λ contains the labelled point $(x, 1)$. Since A is a homomorphic image of $F_{\mathbf{K_n}}$, we find that there exists an injective λ-map $f : X_\lambda \to Y_{\lambda'}$. Notice that for every $(S_i, \exists)$ there exists an embedding of $(S_1^1, \exists_d)$ into $(S_i, \exists)$. In the dual picture we have a λ-map from U_λ into $V_{\lambda'}$, where $U_\lambda = \mathfrak{C}_{\mathfrak{L}}((S_i, \exists))$ and $V_{\lambda'} = \mathfrak{C}_{\mathfrak{L}}((S_1^1, \exists_d))$, since $\lambda(x)$ divides $\lambda'(y)$.

Now we construct a λ-map $h : Y_{\lambda'} \to X_\lambda$ in the following way: let $hf((x,i)) = (x,i)$ and for every $(y,j) \in Y_{\lambda'} - f(X_\lambda)$ $h((y,j)) = (x,1) \in X_\lambda$. It is clear that $hf = Id_{X_\lambda}$. Therefore, X_λ is a retract of $Y_{\lambda'}$. It means that A is a retract of $F_{\mathbf{K_n}}(m)$.

Conversely, if A does not have the form $A' \times (S_1^1, \exists_d)$, then X_λ does not contain a point with label 1, i. e., a point $(x,1)$. But $Y_{\lambda'}$ contains points of such kind. In this case, there is no any λ-map from $Y_{\lambda'}$ to X_λ sending this point, because this point must be sent to the point labelled by 1. So, X_λ will not be a retract of $Y_{\lambda'}$. □

Corollary 6. *Any subalgebra of the m-generated free algebra $F_{\mathbf{K_n}}(m)$ is projective.*

Proof. The proof immediately follows from the fact that any subalgebra of the free m-generated algebra $F_{\mathbf{K_n}}(m)$ contains as a factor the algebra which is isomorphic to $(S_1^1, \exists_d)$. □

Consider the variety of MV-algebras $\mathbf{V_n}$, which is generated by $\{S_1, \ldots, S_n\}$. Let us observe that

$$A = \prod_{p=1}^{n} (S_p^1, \exists)^{r(1,p,m)}$$

is an algebra with a trivial monadic operator $\exists$ (i. e. $\exists x = x$) which is isomorphic as an MV-algebra to the m-generated free MV-algebra $F_{\mathbf{V_n}}(m)$, by Lemma 2.2 in [6], and Theorem 1 in [9]. Hence we write $A = (F_{\mathbf{V_n}}(m), \exists)$.

Since $\prod_{p=1}^{n} (S_p^1, \exists)^{r(1,p)}$ contains as a factor an algebra isomorphic to $(S_1^1, \exists_d)$, by Theorem 5 it holds.

Theorem 7. *The MMV-algebra $A = (F_{\mathbf{V_n}}(m), \exists)$ is projective.*

6. Unification Problem

Let E be an equational theory. The E-unification problem is formulated as follows: given two terms s, t, to find a unifier for them, that is, a uniform replacement of the variables occurring in s and t by other terms that makes s and t equal modulo E. For detailed information on unification problems we refer to [11, 12].

Let us be more precise. Let Φ be a set of functional symbols and let V be a set of variables. Let $T_V(\Phi)$ be the term algebra built from Φ and V, and $T_V(\Phi_m)$ be the term algebra of m-variable terms. Let E be a set of equations $p(x) = q(x)$, where $p(x), q(x) \in T_V(\Phi_m)$.

Let $\mathbf{V}$ be the variety of algebras over Φ, axiomatized by the equations in E.

A *unification problem modulo E* is a finite set of pairs

$$\mathcal{E} = \{(s_j, t_j) : s_j, t_j \in T_V(\Phi_m), j \in J\}$$

for some finite set J. A *solution to* (or *unifier for*) $\mathcal{E}$ is a substitution σ (i.e., an endomorphism of the term algebra $T_V(\Phi_m)$) such that the equality $\sigma(s_j) = \sigma(t_j)$ holds in every algebra of the variety $\mathbf{V}$. The problem $\mathcal{E}$ is *solvable* (or *unifiable*) if it admits at least one unifier.

Let $(X, \preceq)$ be a quasi-ordered set (i. e., a reflexive and transitive relation). A *μ-set* for $(X, \preceq)$ (see [12]) is a subset $M \subseteq X$ such that: (1) every $x \in X$ is less than, or equal to some $m \in M$; (2) all elements of M are mutually $\preceq$-incomparable.

There might be no μ-set for $(X, \preceq)$ (in this case we say that $(X, \preceq)$ has *type* 0), or there might be many of them, due to the lack of antisymmetry. However, all μ-sets for $(X, \preceq)$, if any, must have the same cardinality. We say that $(X, \preceq)$ has *type* $1, \omega, \infty$, iff it has a μ-set of cardinality 1, of finite (greater than 1) cardinality or of infinite cardinality, respectively.

Substitutions are compared by instantiation in the following way: we say that $\sigma : T_V(\Phi_m) \to T_V(\Phi_m)$ is *more general, than* $\tau : T_V(\Phi_m) \to T_V(\Phi_m)$ (written as $\tau \preceq \sigma$), iff there is a substitution $\eta : T_V(\Phi_m) \to T_V(\Phi_m)$ such that for all $x \in V_m$, we have $E \vdash \eta(\sigma(x)) = \tau(x)$. The relation $\preceq$ is a quasi-order.

Let $U_E(\mathcal{E})$ be the set of unifiers for the unification problem $\mathcal{E}$; then $(U_E(\mathcal{E}), \preceq)$ is a quasi-ordered set.

We say that an equational theory E has:

1. Unification type 1, iff for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type 1;
2. Unification type ω, iff for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type ω;
3. Unification type ∞, iff for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type 1, or ω, or ∞, and there is a solvable unification problem $\mathcal{E}$ such that $U_E(\mathcal{E})$ has type ∞;
4. Unification type nullary, if none of the preceding cases applies.

An algebra A is called *finitely presented* if A is finitely generated, with the generators $a_1, \ldots, a_m \in A$, and there exist a finite number of equations $P_1(x_1, \ldots, x_m) = Q_1(x_1, \ldots, x_m), \ldots, P_n(x_1, \ldots, x_m) = Q_n(x_1, \ldots, x_m)$ holding in A on the generators $a_1, \ldots, a_m \in A$ such that if there exists an m-generated algebra B, with generators $b_1, \ldots, b_m \in B$, such that the equations $P_1(x_1, \ldots, x_m) = Q_1(x_1, \ldots, x_m)$, $\ldots, P_n(x_1, \ldots, x_m) = Q_n(x_1, \ldots, x_m)$ hold in B on the generators $b_1, \ldots, b_m \in B$, then there exists a homomorphism $h : A \to B$ sending a_i to b_i.

If $\mathbf{V}$ is a variety of algebras and Ω is a finite set of m-ary $\mathbf{V}$-equations, then we denote by $F_{\mathbf{V}}(m, \Omega)$ the object, free over $\mathbf{V}$ with respect to the conditions Ω on the generators (see [13]). If $\Omega = \emptyset$, then $F_{\mathbf{V}}(m, \Omega) = F_{\mathbf{V}}(m)$. $F_{\mathbf{V}}(m, \Omega)$ is a finitely presented algebra.

Now we will give a characterization of finitely presented MMV-algebras.

A filter F of an algebra $(A, \exists) \in \mathbf{MMV}$ is called a *monadic filter* (which is dual to an ideal, see [16]) if for every $a \in A$ we have $a \in F \Rightarrow \forall a \in F$.

For any set $X \subseteq A$, let $[X)$ denote the monadic filter generated by X. It is easy to check that $[X) = \{a \in A : a \geq \forall x_1 \odot \ldots \odot \forall x_n : x_1, \ldots, x_n \in X\}$.

Theorem 8. *Let p be an m-ary term. Then there is a principal monadic filter F such that $F_{\mathbf{MMV}}(m, p = 1) \cong F_{\mathbf{MMV}}(m)/F$.*

Proof. Let

$$F = \{x : x \in F_{\mathbf{MMV}}(m) \text{ and } x \geq \forall p^n(g_1, \ldots, g_m) \text{ for some } n \in \omega\},$$

where $g_1, \ldots, g_m$ are free generators of $F_{\mathbf{MMV}}(m)$. Then $g_1/F, \ldots, g_m/F$ are generators of $F_{\mathbf{MMV}}(m)/F$.

Let A be an MMV-algebra generated by $\{a_1, \ldots, a_m\}$ such that $p(a_1, \ldots, a_m) = 1$, and let $f : F_{\mathbf{MMV}}(m) \to A$ be a homomorphism such that $f(g_i) = a_i$, $i = 1, \ldots, m$. Then $\forall p^n(g_1, \ldots, g_m) \in f^{-1}(1)$ for every $n \in \omega$ and therefore $F \subseteq f^{-1}(1)$. By the homomorphism theorem, there is a homomorphism $f' : F_{\mathbf{MMV}}(m)/F \to A$ such $\pi_F f' = f$. It should be clear that f' is the needed homomorphism extending the map $g_i/F \mapsto a_i$. □

From this theorem it follows that if an algebra A is finitely presented, then there exists a principal monadic filter F of the free algebra $F_{\mathbf{MMV}}(m)$ such that $A \cong F_{\mathbf{MMV}}(m)/F$.

Theorem 9. *Let $u \in F_{\mathbf{MMV}}(m)$ such that $\forall u^n \neq 0$ for any $n \in \omega$. Then $F = \{x : x \geq \forall u^n,\ n \in \omega\}$ is a proper principal monadic filter in $F_{\mathbf{MMV}}(m)$ such that $F_{\mathbf{MMV}}(m)/F \cong F_{\mathbf{MMV}}(m, p = 1)$ for some m-ary term p.*

Proof. Let F be a monadic filter satisfying the condition of the theorem. Then $u = p(g_1, \ldots, g_m)$ for some term p, where $g_1, \ldots, g_m$ are free generators of $F_{\mathbf{MMV}}(m)$. We find that $F_{\mathbf{MMV}}(m)/F$ is generated by $g_1/F, \ldots, g_m/F$, and that $p(g_1/F, \ldots, g_m/F) = p(g_1, \ldots, g_m)/F = 1_{F(m)/F}$. The rest can be verified as in the proof of Theorem 8. □

Combining the two theorems, we arrive at

Theorem 10. *An m-generated MMV-algebra A is finitely presented, iff there exists a principal monadic filter F of $F_{\mathbf{MMV}}(m)$ such that $F_{\mathbf{MMV}}(m)/F \cong A$.*

Now we follow Ghilardi [11], who has introduced the relevant definitions for E-unification from an algebraic point of view. Let E be an equational theory. By an *algebraic unification problem* we mean a finitely presented algebra A of the variety associated to E. A solution for it (also called a unifier for A) is a pair given by a projective algebra P and a homomorphism $u : A \to P$. The set of unifiers for A is denoted by $U_E(A)$. A is said to be *unifiable* or solvable, iff $U_E(A)$ is not empty. Given another algebraic unifier $w : A \to Q$, we say that u is more general, than w, written $w \preceq u$, if there is a homomorphism $g : P \to Q$ such that $w = gu$.

The set of all algebraic unifiers $U_E(A)$ of a finitely presented algebra A forms a quasi-ordered set with the quasi-ordering $\preceq$.

The algebraic unification type of an algebraically unifiable finitely presented algebra A in the variety $\mathbf{V}$ is now defined exactly as in the symbolic case, using the quasi-ordering set $(U_E(A), \preceq)$.

Theorem 11. *The unification type of the equational class* $\mathbf{K_n}$ *is* 1*, i. e., unitary.*

Proof. The proof immediately follows from Theorem 5. Indeed, any finitely presented MMV-algebra in the variety $\mathbf{K_n}$ is finite. The finitely presented projective algebras are those of the kind $(S_1^1, \exists_d) \times A'$ (Theorem 5). Let the MMV-algebra A be unifiable. It means that there is a homomorphism from A into a projective algebra, say $B \times (S_1^1, \exists_d)$, hence also a homomorphism $h : A \to (S_1^1, \exists_d)$. Then A is a retract of $A \times (S_1^1, \exists_d)$ which is projective. Indeed, we can take the homomorphism $\varepsilon = (Id_A, h) : A \to (A \times (S_1^1, \exists_d))$ (i. e., $\varepsilon(a) = (a, h(a))$) and the projection $\pi : A \times (S_1^1, \exists_d) \to A$ so that $h\varepsilon = Id_A$ (i. e., identity homomorphisms are most general unifications). It is obvious that any projective algebra is unifiable. Thereby we have shown that an MMV-algebra A is unifiable, iff it is projective. $\square$

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ON NEW VERSIONS OF THE LINDEBERG–FELLER'S LIMIT THEOREM

SH. FORMANOV AND A. SIROJITDINOV

Abstract. It is well known that the classical Lindeberg condition is sufficient for the validity of the central limit theorem. It will be also a necessary if the summands satisfy the condition of infinite smallness (Feller's theorem). The limit theorems for the distributions of sums of independent random variables which do not use the condition of infinite smallness were called non-classical.

The exact bounds for the Lindeberg, Rotar characteristics using the difference of distribution of sum of independent random variables and a standard normal distribution are established. These results improve Feller's theorem.

Introduction

Let $X_{n1}, X_{n2}, \dots, X_{nn}$, $n = 1, 2, \dots$ be an array of independent random variables (r.v.'s). Assume that

$$EX_{nj} = 0, \quad EX_{nj}^2 = \sigma_{nj}^2, \quad j = 1, 2, \dots,$$

$$S_n = X_{n1} + \cdots + X_{nn}, \quad \sum_{j=1}^{n} \sigma_{nj}^2 = 1.$$

Set

$$F_n(x) = P\left(S_n < x\right), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du,$$

$$\Delta_n = \sup_x \left|F_n(x) - \Phi(x)\right|.$$

It is well-known that the following condition (Feller's characteristic)

$$\max_{1 \le j \le n} \sigma_{nj} \to 0, \quad n \to \infty \tag{F}$$

is called uniform of infinite smallness condition of a sequence of independent r.v.'s. $\{X_{nj},\, j \ge 1\}$. We say that this sequence satisfies Lindeberg condition if for any $\varepsilon > 0$

$$L_n(\varepsilon) = \sum_{j=1}^{n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon)) \to 0, \quad n \to \infty. \tag{L}$$

Here, $I(A)$ denotes an indicator of the event A.

It is well-known that under the condition L,

$$\Delta_n \to 0, \quad n \to \infty,$$

which means a central limit theorem (CLT). The Lindeberg–Feller's theorem improves the above theorem and can be represented in the form of the following implication:

$$(F)\&(CLT) \Leftrightarrow (L),$$

i.e., under the condition (F), Lindeberg's condition is necessary one for CLT.

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1. Estimation of Numerical Characteristics Used in CLT

Following V. M. Zolotarev [8, ch. 5, §5.2], we call the limit theorems non-classical in which the condition (F) is not used. The first non-classical variants of CLT were proved by Zolotarev in 1967 and Rotar in 1975 [6].

In [3], [5], the following estimates of $L_n(\varepsilon)$ $(\varepsilon > 0)$ were obtained.

Theorem A. *There exists an absolute constant $C > 0$ such that for any $\varepsilon > 0$,*

$$\left(1 - e^{-\varepsilon^2/4}\right) \sum_{j=1}^{n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon)) \le C\left(\Delta_n + \sum_{j=1}^{n} \sigma_{nj}^4\right). \tag{1}$$

Note. It is obvious that under the condition (F) and $\sum\limits_{j=1}^{n} \sigma_{nj}^2 = 1$,

$$\sum_{j=1}^{n} \sigma_{nj}^4 \le \max_j \sigma_{nj}^2 \to 0, \quad n \to \infty.$$

Thus (1) implies that if the sequence of independent r.v.'s $\{X_{nj},\, j \ge 1\}$ satisfies CLT (i.e. $\Delta_n \to 0$, $n \to \infty$), then the Lindeberg condition

$$\sum_{j=1}^{n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon)) \to 0$$

holds for any $\varepsilon > 0$ by $n \to \infty$.

Set

$$F_{nj}(x) = P(X_{nj} < x),$$

$\Phi_{nj}(x)$ is a distribution function of normal r.v. with parameters $(0, \sigma_{nj}^2)$ $(j = 1, 2, \dots)$ and for any $\varepsilon > 0$,

$$R_n(\varepsilon) = \sum_{j=1}^{\infty} \int\limits_{|x|>\varepsilon} |x||F_{nj}(x) - \Phi_{nj}(x)|dx.$$

Theorem B (V. I. Rotar [6])**.** *The condition*

$$R_n(\varepsilon) \to 0, \quad n \to \infty, \tag{2}$$

for any $\varepsilon > 0$ is sufficient and necessary for CLT.

The above Theorem B is a nonclassical version of CLT and it generalizes Lindeberg–Feller's theorem. Indeed, in Theorem B we do not use the condition (F). The proof of the necessity of condition (2) is based on the following statement (note that a proof of the necessity of condition (2) given in [6] is rather complicated and it uses the properties of probabilistic metrics).

The following theorem holds.

Theorem 1. *For some $C = C(\varepsilon)$, the following estimation*

$$R_n(\varepsilon) \le C\left(L_n(\varepsilon) + \sum_{j=1}^{n} \sigma_{nj}^{2s}\right) \tag{3}$$

for any $\varepsilon > 0$ and $s \ge 2$, is true.

Proof. In [3], the inequality

$$R_n(\varepsilon) \le \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon} x^2 dF_{nj}(x) + \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon} x^2 d\Phi_{nj}(x) = L_n(\varepsilon) + \Phi_n(\varepsilon) \tag{4}$$

is proved. Further, it is not hard to prove that

$$\Phi_n(\varepsilon)=\sum_{j=1}^{n}\int\limits_{|x|>\varepsilon/\sigma_{nj}}\sigma_{nj}^2x^2d\Phi(x)\le\varepsilon^2\sum_{j=1}^{n}\int\limits_{|x|>\varepsilon/\sigma_{nj}}\left(\frac{\sigma_{nj}^2x^2}{\varepsilon^2}\right)^s d\Phi(x)$$
$$\le\varepsilon^{-2(s-1)}\left(\sum_{j=1}^{n}\sigma_{nj}^{2s}\right)\int\limits_{|x|>\varepsilon/\max\limits_{1\le j\le n}\sigma_{nj}}x^{2s}d\Phi(x).$$

Hence, for any $s\ge 2$,

$$\Phi_n(\varepsilon)\le C(\varepsilon)\left(\sum_{j=1}^{n}\sigma_{nj}^{2s}\right). \tag{5}$$

Now, the fairness of estimation (3) and Theorem 1 follows from relations (4) and (5).

It can easily be checked that the above-proven Theorem 1 has the following corollaries.

1) We say that Rotar's condition holds if

$$R_n(\varepsilon)\to 0,\quad \forall\varepsilon>0,\quad n\to\infty. \tag{R}$$

It is easy to prove that the following estimation (6)

$$\max_{1\le j\le n}\sigma_{nj}^2\le\varepsilon^2+\max_{1\le j\le n}\int\limits_{|x|>\varepsilon}x^2dF_{nj}\left(x\right)\le\varepsilon^2+\sum_{j=1}^{n}\int\limits_{|x|>\varepsilon}x^2dF_{nj}\left(x\right)=\varepsilon^2+L_n\left(\varepsilon\right) \tag{6}$$

is true. Directly from the estimation (3), for $s=2$, we have

$$R_n(\varepsilon)\le C(\varepsilon)\Big(L_n(\varepsilon)+\max_{1\le j\le n}\sigma_{nj}^2\Big). \tag{7}$$

Thus from relations (6) and (7) it follows that if $L_n(\varepsilon)\to 0$, $\forall\varepsilon>0$, $n\to\infty$, then $R_n(\varepsilon)\to 0$, $\forall\varepsilon>0$, $n\to\infty$. Therefore the implication $(L)\Rightarrow(R)$ is true. In turn, from the last it follows that Theorem 1 generalizes classic version of the Lindeberg-Feller limit theorem.

2) Since the integration domain in the expression $\Phi_n(\varepsilon)$ contains in the domain $\left\{x:\frac{\left(\max\limits_{1\le j\le n}\sigma_{nj}\right)^2x^2}{\varepsilon^2}>1\right\}$, inequality (5) can be written as

$$\Phi_n(\varepsilon)\le C(\varepsilon)\left(\max_{1\le j\le n}\sigma_{nj}\right)^{2s}.$$

Hence, estimation (3) can be rewritten as

$$R_n(\varepsilon)\le C(\varepsilon)\left[L_n(\varepsilon)+\Big(\max_{1\le j\le n}\sigma_{nj}\Big)^{2s}\right],\quad s\ge 2.$$

3) In view of Theorem A, estimation (3) can be rewritten as

$$R_n(\varepsilon)\le C(\varepsilon)\left(\Delta_n+\sum_{j=1}^{n}\sigma_{nj}^4\right).$$

Thus the following implication

$$(F)\&(CLT)\Rightarrow(R)$$

holds. From the above corollaries 1)–3) we have that the characteristic $R_n(\varepsilon)$ is thinner then the Lindeberg characteristic. For example, in the case of the equality of distributions $F_{nj}\equiv\Phi_{nj}$, $j=1,2,\dots,n$, it is obvious that the value $R_n(\varepsilon)$ vanishes trivially, but at the same time, $L_n(\varepsilon)>0$, $\forall\varepsilon>0$. It should be noted that if the condition F holds, then these conditions are equivalent, i.e.,

$$(F)\&(R)\Leftrightarrow(L).$$

The last limit relation is proved in the book of A. N. Shiryaev [7]. □

2. Ibragimov-Osipov-Essen Characteristic and Some Relations of Equivalence

We put

$$M_n(\alpha)=\sum_{j=1}^{n}\int\limits_{|x|\le 1}|x|^{2+\alpha}\,dF_{nj}(x)+\sum_{j=1}^{n}\int\limits_{|x|>1}x^2dF_{nj}(x)=m_n(\alpha)+L_n,\quad \alpha>0.$$

This numerical characteristic is the first encountered in [1] and [2] for $\alpha=1$. The appearance of this characteristic is due to the impossibility of estimating the remainder term in the CLT by the single Lindeberg characteristic $L_n(\cdot)$ (see [1]). In [2], it is shown that the value $M_n(1)$ can be used in estimating the rate of convergence in CLT. It should be noted that there are the cases where $m_n(1)=o\left(L_n(\cdot)\right)$ or $L_n(\cdot)=o\left(m_n(1)\right)$ as $n\to\infty$. Therefore in the expression $M_n(\cdot)$ it is impossible to confine ourselves to one of the two terms.

We present some asymptotic properties of $M_n(\alpha)$, as $n\to\infty$.

Lemma 1. *If for some $\alpha=\alpha_0>0$,*

$$m_n(\alpha_0)=\sum_{j=1}^{n}\int\limits_{|x|\le 1}|x|^{2+\alpha_0}\,dF_{nj}(x)\to 0,\quad n\to\infty,$$

then $m_n(\alpha)\to 0$, $n\to\infty$ for any $\alpha>0$.

Proof. Let $\alpha<\alpha_0$. Then for any $0<\varepsilon\le 1$,

$$\begin{aligned}m_n(\alpha)&=\sum_{j=1}^{n}\int\limits_{|x|\le\varepsilon}|x|^{2+\alpha}\,dF_{nj}(x)+\sum_{j=1}^{n}\int\limits_{\varepsilon<|x|\le 1}|x|^{2+\alpha}\,dF_{nj}(x)\\&\le\varepsilon^{\alpha}+\sum_{j=1}^{n}\int\limits_{\varepsilon<|x|\le 1}|x|^{2+\alpha_0}\,|x|^{\alpha-\alpha_0}\,dF_{nj}(x)\\&\le\varepsilon^{\alpha}+\left(\sum_{j=1}^{n}\int\limits_{|x|\le 1}|x|^{2+\alpha_0}\,dF_{nj}(x)\right)\cdot\varepsilon^{-(\alpha_0-\alpha)}=\varepsilon^{\alpha}+\frac{m_n(\alpha_0)}{\varepsilon^{\alpha_0-\alpha}}.\end{aligned}$$

Hence for any $0<\varepsilon\le 1$,

$$\lim_{n\to\infty}\sup\, m_n(\alpha)\le\varepsilon^{\alpha},\quad \alpha<\alpha_0.$$

Since $0<\varepsilon\le 1$ is arbitrary, from the last relation we get the proof of the relation

$$\{m_n(\alpha_0)\to 0\}\Rightarrow\{m_n(\alpha)\to 0,\ \alpha<\alpha_0\}.$$

Now let $\alpha_0<\alpha$. Then swapping α and α_0 in the previous reasoning, we obtain the proof of the following relation:

$$\{m_n,\alpha_0\to 0\}\Rightarrow\{m_n(\alpha)\to 0,\alpha>\alpha_0\}.\qquad\square$$

Lemma 2. *If for some $\alpha=\alpha_0$, the relation $M_n(\alpha_0)\to 0$, $n\to\infty$ is true, then the Feller conditions (F) hold.*

Proof. Indeed, for $0<\varepsilon\le 1$, we have

$$\begin{aligned}\max_{1\le j\le n}\sigma_{nj}^2&\le\varepsilon^2+\max_{1\le j\le n}\Bigg[\int\limits_{\varepsilon<|x|\le 1}x^2dF_{nj}(x)+\int\limits_{|x|>1}x^2dF_{nj}(x)\Bigg]\\&\le\varepsilon^2+\sum_{j=1}^{n}\int\limits_{\varepsilon<|x|\le 1}x^2dF_{nj}(x)+\sum_{j=1}^{n}\int\limits_{|x|>1}x^2dF_{nj}(x)\\&\le\varepsilon^2+\varepsilon^{-\alpha}\cdot\sum_{j=1}^{n}\int\limits_{|x|\le 1}|x|^{2+\alpha}\,dF_{nj}(x)+L_n\le\varepsilon^2+\varepsilon^{-\alpha}m_n(\alpha)+o(1),\ n\to\infty.\end{aligned}$$

Now, by applying Lemma 1, we obtain

$$\lim_{n\to\infty} \sup \max_{1\le j\le n} \sigma_{nj}^2 \le \varepsilon^2, \quad 0<\varepsilon\le 1. \qquad \square$$

Further, we say that the condition (M_α) holds, if for some $\alpha>0$ the value $M_n(\alpha)\to 0,\ n\to\infty$. Then assertion of Lemma 2 can be written as the implication $(M_\alpha)\Rightarrow(F)$.

Theorem 2. *Let for some $\alpha=\alpha_0>0$,*

$$M_n(\alpha_0)=m_n(\alpha_0)+L_n\to 0,\ \ n\to\infty. \tag{M}$$

Then for any α, the following implication

$$\{M_n(\alpha_0)\to 0,\ n\to\infty\}\Leftrightarrow\{L_n(\varepsilon)\to 0,\ \forall\varepsilon>0,\ n\to\infty\}$$

is true.

Remark. In view of Lemma 1, Theorem 2 can be written as an implication of the equivalence

$$(M)\Leftrightarrow(L) \tag{8}$$

Proof of Theorem 2. For simplicity, in the condition M we put $\alpha=1$, i.e.,

$$M_n=M_n(1)=m_n(1)+L_n=\sum_{j=1}^n\int\limits_{|x|\le 1}|x|^3\,dF_{nj}(x)+\sum_{j=1}^n\int\limits_{|x|>1}x^2dF_{nj}(x)\to 0,\ \ n\to\infty.$$

Note that by Lemma 1 this does not limit the generality in the following reasoning.

Let the condition M holds, i.e., for $n\to\infty$, $M_n\to 0$. Then the following relations are clear: for $0<\varepsilon\le 1$,

$$\begin{aligned}L_n(\varepsilon)&=\sum_{j=1}^n\int\limits_{\varepsilon<|x|\le 1}x^2dF_{nj}(x)+L_n(1)\le\frac{1}{\varepsilon}\sum_{j=1}^n\int\limits_{\varepsilon<|x|\le 1}|x|^3\,dF_{nj}(x)+L_n\\&\le\frac{m_n(1)}{\varepsilon}+o(1)\to 0,\quad n\to\infty.\end{aligned} \tag{9}$$

For $\varepsilon\ge 1$, by monotonicity of $L_n(\varepsilon)$, we obtain

$$L_n(\varepsilon)\le L_n(1)=L_n\to 0,\quad n\to\infty. \tag{10}$$

Relations (9) and (10) prove the justice of the implication $(M)\Rightarrow(L)$.

Now, let the Lindeberg condition L hold. Then for $0<\varepsilon\le 1$,

$$L_n=L_n(1)\le L_n(\varepsilon)\to 0,\quad n\to\infty. \tag{11}$$

In addition, for any $0<\varepsilon\le 1$,

$$m_n(1)=\sum_{j=1}^n\int\limits_{|x|\le 1}|x|^3\,dF_{nj}(x)\le\varepsilon\sum_{j=1}^n\int\limits_{|x|\le\varepsilon}x^2dF_{nj}(x)+\sum_{j=1}^n\int\limits_{\varepsilon<|x|\le 1}x^2dF_{nj}(x)\le\varepsilon+L_n(\varepsilon).$$

Thus, from the last estimation it follows that

$$\lim_{n\to\infty}\sup m_n(1)\le\varepsilon,\quad m_n(1)\to 0,\quad n\to\infty. \tag{12}$$

From relations (11), (12), we conclude that the implication $(L)\Rightarrow(M)$ holds. Hence, the equivalence relation (8) is proved, and thus we obtain the proof of Theorem 2. $\square$

3. The Classical Version of Analogue of the Lindeberg-Feller Theorem

The classical version of the Lindeberg-Feller limit theorem can be represented as equivalence of the implications

$$(F)\&(CLT) \Leftrightarrow (L).$$

We give a theorem in which conditions M are used instead of L.

Theorem 3. *In order for the sequence of series of independent random variables* $\{X_{nj}, \quad 1 \le j \le n\}$ *to satisfy the Feller condition* F *and obey* (CLT)*, it is necessary and sufficient that condition* M *be fulfilled.*

Below, we give the proof of Theorem 3 with the direct use of the condition M.

Proof of Theorem 3. We introduce the notions

$$f_{nj}(t) = \int_{-\infty}^{\infty} e^{itx} dF_{nj}(x), \quad g_{nj}(t) = \int_{-\infty}^{\infty} e^{itx} d\Phi_{nj}(x) = e^{-\sigma_{nj}^2 t^2/2},$$

$$f_n(t) = \prod_{j=1}^{n} f_{nj}(t), \quad \prod_{j=1}^{n} g_{nj}(t) = g_n(t) = e^{-t^2/2}.$$

To prove the validity of the CLT, it suffices to make sure that for any $T > 0$,

$$\sup_{|t| \le T} \left| f_n(t) - e^{-t^2/2} \right| \to 0, \quad n \to \infty. \tag{13}$$

Note first that for all complex numbers satisfying the inequalities $|a_k| \le 1$, $|b_k| \le 1$, $k = 1, 2, \dots$, the inequality

$$\left| \prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k \right| \le \sum_{k=1}^{n} |a_k - b_k| \tag{14}$$

holds. By the last inequality (14), we obtain

$$\left| \prod_{j=1}^{n} f_{nj}(t) - \prod_{k=1}^{n} g_{nj}(t) \right| \le \sum_{j=1}^{n} |f_{nj}(t) - g_{nj}(t)| . \tag{15}$$

Therefore, by (15), we can conclude that relation (13) will be proved if it states that

$$d_n(t) = \sum_{j=1}^{n} |f_{nj}(t) - g_{nj}(t)| \to 0, \quad n \to \infty, \tag{16}$$

for any $t \in \mathrm{R}$.

Further, we use the following equalities:

$$\int_{-\infty}^{\infty} x dF_{nj}(x) = \int_{-\infty}^{\infty} x d\Phi_{nj}(x) = 0,$$

$$\int_{-\infty}^{\infty} x^2 dF_{nj}(x) = \int_{-\infty}^{\infty} x^2 d\Phi_{nj}(x) = \sigma_{nj}^2, \quad j = 1, 2, \dots .$$

By these equalities, for $j = 1, 2, \dots$, we can write

$$f_{nj}(t) - g_{nj}(t) = \int_{-\infty}^{\infty} \left[e^{itx} - 1 - itx - \frac{(itx)^2}{2} \right] d\left(F_{nj}(x) - \Phi_{nj}(x)\right). \tag{17}$$

After integrating by parts in the last integral and by virtue of

$$x^2 \left[1 - F_{nj}(x) + F_{nj}(-x)\right] \to 0, \quad x^2 \left[1 - \Phi_{nj}(x) + \Phi_{nj}(-x)\right] \to 0,$$

for $x \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \left[e^{itx} - 1 - itx - \frac{(itx)^2}{2} \right] d\left(F_{nj}(x) - \Phi_{nj}(x)\right)$$
$$= -it \int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx\right) \left(F_{nj}(x) - \Phi_{nj}(x)\right) dx. \tag{18}$$

Now, by (15), (17) and (18), for any $\varepsilon > 0$, we have

$$d_n(t) \le \sum_{j=1}^{n} \left| t \int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx\right) \left(F_{nj}(x) - \Phi_{nj}(x)\right) dx \right|$$
$$\le \frac{|t|^3}{2} \varepsilon \sum_{j=1}^{n} \int_{|x| \le \varepsilon} |x| \left|F_{nj}(x) - \Phi_{nj}(x)\right| dx + 2t^2 \sum_{j=1}^{n} \int_{|x| > \varepsilon} |x| \left|F_{nj}(x) - \Phi_{nj}(x)\right|. \tag{19}$$

By direct integration by parts, it is easy to verify that for any random variable X with the distribution function $F(x)$, the following equality

$$\mathrm{E}\,|X|^n = \int_{-\infty}^{\infty} |x|^n \, dF(x) = n \int_{0}^{\infty} x^{n-1} \left(1 - F(x) + F(-x)\right) dx. \tag{20}$$

holds. Based on formula (20) with $n = 2$, we can verify the following estimate:

$$\sum_{j=1}^{n} \int_{|x| \le \varepsilon} |x| \left|F_{nj}(x) - \Phi_{nj}(x)\right| dx \le 2 \sum_{j=1}^{n} \sigma_{nj}^2 = 2.$$

Hence, inequality (19) can be written as

$$d_n(t) \le |t|^3 \cdot \varepsilon + 2t^2 \cdot R_n(\varepsilon).$$

Now, by Theorem 2, it follows that

$$R_n(\varepsilon) = O\left(L_n(\varepsilon) + \sum_{j=1}^{n} \sigma_{nj}^{2s} \right), \quad s \ge 2.$$

By the last expression, we obtain

$$\overline{\lim_{n\to\infty}} \sup_{|t| \le T} d_n(t) \le T^3 \cdot \varepsilon.$$

Therefore, finally, we have the relation

$$\overline{\lim_{n\to\infty}} \sup_{|t| \le T} \left| f_n(t) - e^{-t^2/2} \right| \le T^3 \cdot \varepsilon. \tag{21}$$

Since $\varepsilon > 0$ is arbitrary, by relation (21) we obtain that for any $T > 0$, relation (13) holds, which proves the sufficiency of the condition M for fulfilling CLT. □

Necessity. By Theorem 2, conditions M and L are equivalent (i.e., $(M) \Leftrightarrow (L)$). Therefore, the necessity of condition M for the validity of CLT follows from Theorem A above.

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ADD-IN FOR SOLVERS OF UNCONSTRAINED MINIMIZATION TO ELIMINATE LOWER BOUNDS OF VARIABLES BY TRANSFORMATION

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Abstract. The paper considers the issue of implementation of a software add-in, which transforms minimization problems with constraints only on lower bounds of variables to unconstrained ones. The add-in eliminates lower bounds of variables through their transformation, without creation of new auxiliary variables. The add-in can be included in any algorithm of unconstrained minimization, which has certain design. At the same time, any number of suitable transformations of variables can be included into the add-in.

The add-in is implemented in C++ and consists of three components: small collection of transformations of variables; small collection of minimization test-problems with constraints only on lower bounds of variables; stopping conditions of minimization algorithms and program-driver.

In the numerical experiments, mainly CG-DESCENT-C-6.8 and l-bfgs were used. Results show that empowering unconstrained minimization algorithms by such add-in is a promising direction, especially, for symmetric matrix games.

Materials of experiments are uploaded at GitHub: https://github.com/kobage.

1. Introduction

Let us consider the minimization problem with constraints only on lower bounds of the variables:

$$\min_{x\in\mathbb{R}^n} f(x) \text{ subject to } x_i \geq l_i, \quad i \in \mathcal{I}, \tag{1}$$

where $f:\mathbb{R}^n \to \mathbb{R}$ is a smooth function, $\mathcal{I}$ is a subset of $1,2,\dots,n$. $\forall i \in \mathcal{I}$, $l_i \in \mathbb{R}$ is a lower bound of x_i variable.

Problems of practical interest are often modelled in the form (1) (see [3]). Symmetric matrix games belong to this set of problems. Usually, such kind of constraints arise when variables represent certain physical quantities.

Problem (1) contains the constraints only on lower bounds of variables and represents particular case of bound constrained or box-constrained optimization problems. Because of the relevance of box-constrained optimization, investigation of their numerical algorithms also is a relevant issue.

From the point of view of citation, L-BFGS-B (see. [10]), based on the gradient projection method, is the most remarkable approach. On the other hand, this method is neither unique, nor the fastest (see [6]), and there are another alternatives. In the presented paper, one of the oldest approaches is promoted. This approach is the most natural from the point of view of mathematics and implements transformations of variables: problem (1) will be transformed into an unconstrained one by transforming the variables. After this, any unconstrained minimization algorithm can be used.

The idea of application of unconstrained minimization algorithms in constrained optimization via transformation of variables was proposed in 1966 by M.J. Box (see [1]). After that, we can mention several papers that attracted attention, certain kind of constraints are named after M.J. Box, but in practical usage this approach did not become competitive. Among the possible reasons we can emphasize the following:

- In [1], there is no emphasis on a particular type of restrictions, therefore it is difficult to systematize the approach and create a software application based on it;

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- [1] emphasizes the possibilities of linearization of problems or maximum simplification, often by transforming to problems of linear programming. The process is accompanied by significant number of auxiliary variables.

Our approach is effective because we are trying to research an issue that is in a certain sense the inverse: elimination of bounds in problems of linear programming (in the form of symmetric matrix games) by transforming variables without creation of auxiliary variables, and transfer to unconstrained minimization problem at the expense of a simple kind of non-linearity.

From the theoretical point of view, transforming variables means the transformation of the problem itself and not the elaboration of a new method of solving. We are trying to implement practically the same idea in the form of the software add-in that can transform problems of the form (1) into unconstrained problems. As a result, after transformation it appears possible to apply powerful unconstrained minimization solvers. Though, to reach the compatibility of the add-in with an algorithm, the latter should satisfy some design requirements. In the call of the algorithm, the following parameters should be passed: the stopping condition and names of one or more implementations (value, gradient, both) of the objective function. Parameters might have the form of a function or a function object. The existence of other parameters causes no difficulties. The algorithms implemented by us get a line search algorithm as one more parameter. Obviously, after transformation and solving of the problem, it is necessary to return to the original variables.

Issues related to the representation of the unconstrained minimization problem in the form, compatible to algorithm, are considered in Section 2.

The structure of the add-in and some related issues are considered in Section 3. Add-in, created in C++, consists of three components: transformations of variables which are formed as objects of the certain class and are piled in container; several test-problems with constraints only on lower bounds of variables, which are also formed as certain objects and gathered in a corresponding container; stopping conditions of minimization algorithms and program-driver.

The program-driver is responsible for choosing of the test-problem and transformation (of variables) in the interactive mode, for calling the unconstrained minimization algorithm and evaluation of the run-time, returning to the original variables and printing the results.

The add-in contains transformations of variables of the following four types: quadratic, quartic, exponential and sinusoidal. The quadratic transformation, for example, means that problem (1) is transformed to the unconstrained one via the transformation

$$x_i = l_i + t_i^2, \quad \forall i \in \mathcal{I}.$$

Adding other transformations to the add-in is a simple job.

The test-problems collection consists of 10 problems. Eight of them are taken from CUTest (see [11]). Two tests correspond to symmetric matrix games; in both cases a game is represented in the form of the minimization problem constrained only with lower bounds of the variables. In one case cubic function (with respect to original variables) is used and the other case- quadratic function.

To benchmark the add-in, two efficient algorithms of unconstrained optimization, l-bfgs (see [9]) and CG-DESCENT (see [7]), are used. CG-DESCENT-C-6.8 is taken from [8] and configured for Visual Studio (see [12]). l-bfgs is implemented by us and its design meets fully the requirements that arise when integrating the add-in into an unconstrained minimization algorithm. This implementation is relatively simple. The only modern technic is the line search procedure borrowed from [8]. But its design is object-oriented and partly generic. As a result, implementation is effective both for unconstrained minimization problems and for problems of the form (1).

The results of numerical experiments are considered in Section 4. MINOS with AMPL (see [13]) interface was used to debug the add-in and test-problems taken from CUTest. GUROBI (see [14]) with C++ interface was used to debug tests related to symmetric matrix games and to evaluate performance of the add-in. Results of experiments show that the use of unconstrained optimization algorithms for symmetric matrix games, even without pretreatment (taking into account specificity of the problem), gives impressive results. Add-in, integrated into l-bfgs, for problems constructed by randomly chosen numbers from [-1, 1] interval in most cases runs as fast as GUROBI. In some cases, when numbers are uniformly distributed, the results are even better. Using of CG-DESCENT-C-6.8 gives similar results.

The last Section 5 evaluates results, makes conclusions and considers the possible directions of future research.

Results of experiments based on codes uploaded at https://github.com/kobage are highlighting two main issues. Using unconstrained minimization algorithms via add-ins for problems, constrained only with lower bounds of variables, presented here, or similar add-ins are promising directions for future research. In the case of improvement and development of the presented here or similar add-ins, the future progress in the sphere of unconstrained minimization will have a direct impact on the progress in the area of symmetric matrix games and linear programming.

2. Some Aspects of Coding of Unconstrained Minimization Problems

Let us consider the problem of minimization of a smooth functional

$$\min f(x) \quad x \in \mathbb{R}^n,$$

with some numerical algorithm, which uses the first order derivatives. Consequently, the data needed to algorithm should be produced in the form of several functions or function objects.

For calling l-bfgs, two functions are needed. One of them takes the following parameters: the dimension of the problem and two vectors (for variable x and its gradient) by reference. It returns the value of the function at x and writes components of the gradient in the vector-parameter. This function is heavily used by l-bfgs and, therefore, it is highly recommended that its implementation takes into account each aspect related to performance speed (for example: reduce to minimum the number of calls of standard functions for simple actions, like the exponentiation, etc.). The second function is for one-time use. It sets initial values of the variable. Adequately, it receives the variable and the dimension as parameters.

To get faster code, the algorithm should distinguish the following cases: only the gradient is used; only the value is used; both are used. Consequently, it is necessary to use two more multiple-use functions. One of them will return only the function's value but will not touch the gradient. The other will not return anything, but calculates and stores the gradient. CG-DESCENT algorithm acts in this way.

In our implementations, the special structure is made for test problems. The structure of objects is storing certain information about the objective function: name, dimension, and names of the above-mentioned two functions (calculating gradient and value and initialising variables). The data of test problems are stored in containers of the corresponding type.

The container of test functions of unconstrained minimization is not a part of the add-in. Neither codes of objective functions of the test problems. They are intended to test unconstrained minimization algorithms and are enclosed to both directories (with prefix LB denoting the lower bound) uploaded at GitHub. Unconstrained minimization algorithms implemented by us have some characteristics in common. They have similar design (to respond to certain requirements) and have the same collections of test-problems. These 44 tests are chosen from the 145, which were used in [7]. But they were used with higher dimensions. The-above mentioned GitHub web-page contains some material, uploaded earlier. The Test-problems collection is among them. The recently uploaded collection of test-problems is essentially refreshed. Implementations of many functions are improved, noticed drawbacks are corrected.

It is worth to note that the performance of the CG-DESCENT-C-6.8 became more efficient on the improved test-problems, than performance of the l-bfgs. This is related to the fact that l-bfgs needs less iterations, than CG-DESCENT. Because of the same reason, l-bfgs is more effective on symmetric matrix games, where calculations of values and gradients are time-consuming.

3. The Structure of the add-in and Issue of its Integration into Unconstrained Optimization Algorithms

The minimization problem with the constraints only on lower bounds of variables contains more data, than unconstrained one, so it has different representation.

After transformations of variables, it is necessary to collect data of the transformed problem. The main challenge lies in efficient implementation of derivatives of the composite function. After solving the transformed problem, it is necessary to return to the original variables.

In mathematics, transformation of variables is a natural and simple task. In order for its equivalent in software also to be natural and simple, it is necessary to elaborate some implementation related separate details:

- for the given test-problem, replacing one transformation of variables with another should be simply feasible;
- adding new transformation of variables into the add-in should be simply feasible;
- adding new test-problems into the add-in should be simply feasible; the structure of test-problems should not be complicated;
- test-problems with different content have different stopping conditions. Therefore, minimization algorithms should allow changing of stopping conditions and accept them as parameters.

The Add-in is implemented in C++ and consists of three components: transformations of variables, which are presented as objects of certain class and are gathered in a container; several minimization test-problems with constraints only on lower bounds of variables, also presented as certain objects and gathered in the corresponding container; stopping conditions of algorithms and program-driver. The program-driver provides choosing of a test-problem and a transformation in the interactive mode, calling unconstrained minimization algorithm, evaluation of the performance time, returning to the original variables and printing the results.

The add-in contains transformations of four types: quadratic: $x_i = l_i + t_i^2$, $\forall i \in \mathcal{I}$; quartic: $x_i = l_i + t_i^4$, $\forall i \in \mathcal{I}$; exponential: $x_i = l_i + e^{t_i}$, $\forall i \in \mathcal{I}$; sinusoidal: $x_i = l_i + 1 + \sin(t_i)$, $\forall i \in \mathcal{I}$. The exponential transformation is not completely correct, because x_i never becomes equal to l_i.

Sinusoidal transformation is correct for test-problems, corresponding to symmetric matrix games, because variables represents probabilities in this case. Besides these four types, it is very simple to add another transformations into the add-in. It should be noted that we are using uniform transformations, the same for each coordinate. Quadratic and sinusoidal transformations, which are the most efficient in numerical experiments, are not invertible. Theoretically, this fact may lead to the emergence of false critical points (at which derivative of the composite function is equal to zero). Practically, this fact causes no problems - because of rounding errors, the possibility of coinciding of a double precision real number and zero is extremely small. The software equivalent of the transformation of variables is an aggregation of three functions: a transformation, its inverse (or its branch) and its derivative

```
double quadraticSubstitution(double v){
        return v*v;
}
double quadraticSubstitutionPrime(double v){
        return 2 * v;
}
double inverseQuadraticSubstitution(double v){
        return sqrt(v);
}
```

In order to store the transformation, we make such a structure that its objects represent software equivalent of transformations:

```
struct lb_substiturtion
{
    double(*substitution) (double);
    double(*inverseSubstitution)(double);
    double(*substitutionPrime)(double);
    string substitutionName;
    void show(void)
    {
         std::cout << "Substitution:  " << substitutionName << std::endl;
```

```
    }
};
```

In the following vector

$$\text{vector} < \text{lb\_substiturtion} > \text{lb\_substiturtionsVector};$$

transformations are added in the standard way. After variables transformation, the objective function becomes a composite function. In C++, implementation of the objective function (or function object) should have access to the transformation. Let us consider one possible approach, simple and yet flexible.

When working with a collection of test-problems, the same objective function may appear both in unconstrained and in constrained problems. In this case, it is convenient to obtain the representation of the constrained problem based on the unconstrained one. Let us consider the functional "hatflda": $hatflda\,(x) = (x_1 - 1)^2 + \sum_{i=2}^{4} \left(x_{i-1} - \sqrt{x_i}\right)^2$. All four variables are constrained with lower bounds: lBounds[i] = 0.0000001. We store the representation of this function with the prototype

```
double hatflda
(
    double *x,
    double *g,
    const int k
);
```

Its implementation is trivial, and we omit it. This representation is suited for the unconstrained minimization. But on its base we construct a composite function which is connected to the transformation via the function pointers:

```
double hatfldaComposite
(
    double *x,
    double *g,
    const int k
)
{
    for (int i = 0; i < k; i++)
       y[i] = lBounds[i] + substitutionName(x[i]);

    double fx = hatflda(y, g, n);
    for (int i = 1; i < k; i++)
           g[i] *= substitutionPrimeName(x[i]);
    return fx;
}
```

Here, substitutionName($x[i]$) is a pointer to the transformation, and substitutionPrimeName($x[i]$) is a pointer to its derivative. Of course, these addresses should be initialized before minimization algorithm will be run. For quadratic transformation, for example, this can be done with following assignments:

substitutionName = quadraticSubstitution; substitutionPrimeName = quadraticSubstitutionPrime;

The other possibilities also existed. For example, to attach test-problem, corresponding to symmetric matrix games, to CG-DESCENT-C-6.8, we used less elegant approach.

The representation of minimization problem (1) needs some additional code in the function, initializing the starting iterate. This portion of the code makes lower bounds and indices of variables with constraints. One more one-time function provides returning to the original variables.

The described add-in was initially debugged with the Modified Heavy Ball algorithm (see [5]) due to the following simple factor: the code of this algorithm consists of 20 lines which makes experiments using it extremely simple. At the same time, it is fast enough. In the debugging and development process, we settled on the design of the algorithms l-bfgs_LB and MHB_LB (see materials uploaded to GitHub). The add-in is fully adapted to these two algorithms. For CG-DESCENT-C-6.8, only the part, corresponding to symmetric matrix games, is adapted. At first, this emphasizes the importance of symmetric games for us. Secondly, we tried to interfere with the code of CG-DESCENT-C-6.8 as little as possible.

Integration of the add-in into the algorithm requires weak support from the algorithm which is expressed in the possibility of variation of the stopping condition. In the ideal case, a stopping condition should be one of the parameters of the algorithm. The above-mentioned is arranged in algorithms l-bfgs_LB and MHB_LB. This possibility is not defined explicitly in CG-DESCENT-C-6.8, but its structure allows users to consider implicitly the stopping condition as a parameter, using function pointers.

4. Numerical Experiments

In the numerical experiments, two groups of test-problems are used. One group consists of 8 test-problems taken from CUTest. It mainly is used to debug the add-in. MINOS with AMPL interface was used to debug the add-in, to check correctness of results and to evaluate performance characteristics of the add-in on these test-problems. Three different transformations are applicable to each test-problem (to some problems, sinusoidal transformation is applicable as well). Therefore, the number of test-problems is, really, greater. The add-in works reliably. Few exceptions take place in the case of exponential transformation which could be expected.

GUROBI with C++ interface was used to debug the add-in on symmetric matrix games and evaluate performance.

Results of experiments show that application of unconstrained optimization algorithms through the add-in for the problems, constrained only with lower bounds of variables, is fully competitive. In particular, this concerns the theme of matrix games. Formally, only two such test-problems are used. But each of them can be considered with 4 transformations of variables. Moreover, 3 different scenarios can be used to generate random data:

```
std::uniform_real_distribution<double>
std::normal_distribution<double>
std::cauchy_distribution<double>
```

and during generating the objects of distributions, we are able to vary ranges of random numbers. For example, std::cauchy_distribution $<$ double $>$ urdi$(-1, 1)$;

All this allows us to get a fairly complete picture of the perspective of the usage of unconstrained minimization algorithms to solve symmetric matrix games.

Results of experiments show that using integrated into l-bfgs add-in with quadratic and sinusoidal transformations in most cases solves symmetric matrix games, constructed with random numbers from [-1,1], as fast as GUROBI solves problems of linear programming correspondig to the same matrix games. In some cases, when numbers are uniformly distributed, the results are even better. Using of CG-DESCENT-C-6.8 gives similar results. The advantage of GUROBI appears and increases with increasing range of random numbers.

The main goal of this paper is to present the results of developing the add-in and evaluating its performance. Symmetric matrix games represent one tool in this process. The results of numerical experiments demonstrate that matrix games and, more generally, linear programming should be among the key topics of our future research. Because of this, let us shortly consider the point of representation of symmetric matrix games as problems of unconstrained minimization.

Consider a symmetric matrix game, i.e., a game with an $n \times n$ skew matrix $A = (a_{i\,j})$:

$$a_{i\,j} = -a_{j\,i}, \quad \forall i, j \in \{1, 2, \dots, n\}\,.$$

The price of such a game is equal to zero (see [2]), hence solving the game is equivalent to solving the following system of inequalities in $\mathbb{R}^n$:

$$\begin{cases} A_1 x \leq 0, \ldots, A_n x \leq 0, \\ x_1 + \cdots + x_n = 1, \\ x_1 \geq 0, \ldots, x_n \geq 0, \end{cases} \tag{2}$$

where the i-th row is denoted by A_i and $x = (x_1, \ldots, x_n)^T$ is the unknown vector. This problem always has a not necessarily unique solution. If $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$ is the solution of (2), then for some indices, say i, $A_i \tilde{x} = 0$ is fulfilled.

The main problem in numerical solution of symmetric game lies in the third row of (2). To avoid combinatorial approach, the transformation of variables can be used. From the above-mentioned four transformations, the quadratic and sinusoidal transformations are equally effective. For simplicity, let us consider the quadratic transformation

$$x_i = t_i^2, \quad i \in \{1, \ldots, n\}.$$

Now, (2) is equivalent to each of the following two unconstrained minimization problems:

$$\sum_{i=1}^{n} (A_i t \circ t < 0)?0 : (A_i t \circ t)^3 + \left| \left(\sum_{i=1}^{n} t_i^2 - 1 \right)^3 \right| \to \min, \tag{3}$$

which represents test-problem -9 of the add-in,

$$\sum_{i=1}^{n} (A_i t \circ t < 0)?0 : (A_i t \circ t)^2 + \left(\sum_{i=1}^{n} t_i^2 - 1 \right)^2 \to \min, \tag{4}$$

which represents test-problem-10. In (3) and (4), for arbitrarily taken $t = (t_1, \ldots, t_n)^T$ and $s = (s_1, \ldots, s_n)^T$ vectors "$\circ$" operator denotes their Hadamard product $s \circ t \equiv (s_1 t_1, \ldots, s_n t_n)^T$, whilst operator "?" has the following format:

$$\text{(condition) ? (result 1) : (result 2)} \tag{5}$$

if the condition is true, then (5) equals (result 1), or (result 2) if the condition is false.

For $m \times n$ real matrix $A = (a_{i\,j})$, when $m \neq n$, the expression $Ax \circ x$ uniquely means the same as $A(x \circ x)$, because vectors Ax and x have different dimensions and thus their Hadamard product is not defined. Consequently, we are using the denotation $Ax \circ x$ and consider that priority of the Hadamard product is higher, than priority of matrix-vector product.

Let us note that (3) and (4) do not represent penalty functions. Because of variables' transformation, these functions are not convex. Nevertheless, they are smooth enough for minimization algorithms to work efficiently (via LB-add-in). Note also that in case of high dimensions, (3) and (4) are equally effective.

5. Conclusions

The structure of the LB-add-in is simple and flexible. It is easily joinable to unconstrained minimization algorithms; it does not matter if an algorithm has special design or it is unaware of the add-in (as in the case of CG-DESCENT-C-6.8). The add-in itself (excluding transformations and collection of test-problems) is fairly compact and in case of necessity, its implementation in other high level programming languages or developing its parallel versions is a very realistic challenge.

The add-ins presented here are aimed at problems, constrained only with lower bounds of variables. But it can be extended to box-constrained problems, as well. Suitable transformations are described in [1] and technical aspects are developed in the presented paper. Much harder part consists of benchmarking the add-in, which transforms box-constrained problems into unconstrained ones. In order to carry out a complete study, vast infrastructure (algorithms, benchmarking environment, test-problems collection) is needed (see [6]).

Intensive experiments were carried out on symmetric matrix games using the add-in. Interest in matrix games is reinforced by the fact that the canonical form of linear programming is easily reducible to symmetric matrix games. Based on the experiments, we can conclude that if we further develop the approach presented here, the future progress in the sphere of unconstrained minimization will have a direct influence on the progress in the area of symmetric matrix games and linear programming.

Experiments have also showed some weaknesses in the usage of unconstrained minimization algorithms to symmetric matrix games in case of absence of specific-based improvements. It is possible to use various heuristics as well as other themes that are obligatory in design of high-quality solvers. There is a wide choice of solvers in linear programming and among them GUROBI is one of the best. If we make comparisons using some open-source solver, the relative results of the add-in would be (see [4]) much better.

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LOCAL VARIATION FORMULAS OF SOLUTIONS FOR THE NONLINEAR CONTROLLED DIFFERENTIAL EQUATION WITH THE DISCONTINUOUS INITIAL CONDITION AND WITH DELAY IN THE PHASE COORDINATES AND CONTROLS

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Abstract. In the present work, local variation formulas of solutions are given, in which the effects of the discontinuous initial condition and perturbations of delays containing in the phase coordinates and controls are revealed.

1. Introduction

As is known, the real processes contain information about their behavior in the past and are described by the delay differential equation [3], [6], [4]. Linear representation of the main part of the increment of a solution with respect to perturbations of the initial data of a differential equation is called the variation formula of a solution (variation formula). In this paper, the essential novelty is that here the local variation formula is given when there occur simultaneously perturbations of the initial moment and delays both in the phase coordinates and in controls.

The term "variation formula of solution" has been introduced by R. V. Gamkrelidze and proved in [2] for the ordinary differential equation. The effects of perturbation of the initial moment and the discontinuous initial condition in the variation formulas were for the first time revealed by T. A. Tadumadze in [8] for the delay differential equation.

The variation formula plays a basic role in proving the necessary conditions of optimality [2], [5], [9] and in the sensitivity analysis of mathematical models [6] . Moreover, the variation formula allows one to construct an approximate solution of the perturbed equation.

In the present work, for the controlled delay differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t), u(t-\theta_1), \dots, u(t-\theta_k))$$

with the discontinuous initial condition the variation formulas are given. The discontinuity of the initial condition means that the values of the initial function and the trajectory, in general, do not coincide at the initial moment.

The variation formulas for various classes of controlled delay differential equations without perturbations of delay occurring in controls are derived in [1], [7], [11], [10], [12].

2. Formulation of the Main Results

Let $I = [a, b]$ be a finite interval and $0 < h_{i1} < h_{i2}, i = \overline{1, s}$; let $0 < q_{i1} < q_{i2}$, $i = \overline{1, k}$ be the given numbers; suppose that $O \subset \mathbb{R}^n$ and $U_0 \subset \mathbb{R}^r$ are open sets. Let the n-dimensional function $f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k)$, $(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in I \times O^{1+s} \times U_0^{1+k}$ satisfy the following conditions:

a) for almost all fixed $t \in I$, the function $f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k)$ is continuously differentiable with respect to $(x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in O^{1+s} \times U_0^{1+k}$;

b) for each fixed $(x, x_1, \dots, x_s, u, u_1, \dots, u_k) \in O^{1+s} \times U_0^{1+k}$, the functions

$$f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_s), f_x(t, \cdot), f_{x_i}(t, \cdot), \quad i = \overline{1, s}$$

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and

$$f_u(t,\cdot), f_{u_i}(t,\cdot), \quad i=\overline{1,k}$$

are measurable on I;

c) for any compacts $K \subset O$ and $U \subset U_0$, there exists a function $m_{K,U}(t) \in L_1(I,[0,\infty))$ such that for any $(x,x_1,\dots,x_s,u,u_1,\dots,u_k) \in K^{1+s} \times U^{1+k}$ and for almost all $t \in I$, we have

$$|f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_k)| + |f_x(t,\cdot)| + \sum_{i=1}^{s} |f_{x_i}(t,\cdot)|$$
$$+|f_u(t,\cdot)| + \sum_{i=1}^{k} |f_{u_i}(t,\cdot)| \le m_{K,U}(t).$$

Let Φ and Ω be sets of continuously differentiable functions $\varphi : I_1 = [\tau,b] \to O$ and $u : [\theta,b] \to U_0$, respectively, where $\tau = a - \max\{h_{12},\dots,h_{s2}\}$ and $\theta = a - \max\{q_{12},\dots,q_{k2}\}$.

To each element

$$\mu = (t_0,\tau_1,\dots,\tau_s,\theta_1,\dots,\theta_k,x_0,\varphi,u) \in \Lambda = [a,b) \times [h_{11},h_{12}] \times \cdots \times [h_{s1},h_{s2}]$$
$$\times [q_{11},q_{12}] \times \cdots \times [q_{k1},q_{k2}] \times O \times \Phi \times \Omega$$

we assign the controlled delay differential equation

$$\dot{x}(t) = f(t,x(t),x(t-\tau_1),\dots,x(t-\tau_s),u(t),u(t-\theta_1),\dots,u(t-\theta_k)) \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \quad t \in [\tau,t_0), \quad x(t_0) = x_0. \tag{2}$$

Condition (2) is called discontinuous because, in general, $x(t_0) \ne \varphi(t_0)$.

Definition 1. Let $\mu = (t_0,\tau_1,\dots,\tau_s,\theta_1,\dots,\theta_k,x_0,\varphi,u) \in \Lambda$. A function $x(t) = x(t;\mu) \in O$, $t \in [\tau,t_1]$, $t_1 \in (t_0,b]$ is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element μ and defined on the interval $[\tau,t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let $\mu_0 = (t_{00},\tau_{10},\dots,\tau_{s0},\theta_{10},\dots,\theta_{k0},x_{00},\varphi_0,u_0) \in \Lambda$ be a fixed element and let $x_0(t)$ be a solution corresponding to the element μ_0 and defined on the interval $[\tau,t_{10}]$, where

$$t_{00},t_{10} \in (a,b), \quad t_{00} < t_{10}; \quad \tau_{i0} \in (h_{i1},h_{i2}), i=\overline{1,s}; \quad \theta_{i0} \in (q_{i1},q_{i2}), \quad i=\overline{1,k}.$$

Thus, $x_0(t)$ is the solution of the equation

$$\dot{x}(t) = f(t,x(t),x(t-\tau_{10}),\dots,x(t-\tau_{s0}),u_0(t),u_0(t-\theta_{10}),\dots,u_0(t-\theta_{k0})), \quad t \in [t_{00},t_{10}]$$

with the initial condition

$$x(t) = \varphi_0(t), \quad t \in [\tau,t_{00}), \quad x(t_{00}) = x_{00}.$$

Let us introduce the set of variations:

$$V = \Big\{\delta\mu = (\delta t_0,\delta\tau_1,\dots,\delta\tau_s,\delta\theta_1,\dots,\delta\theta_k,\delta x_0,\delta\varphi,\delta u) : \delta t_0 \in (a,b) - t_{00},$$
$$\delta\tau_i \in (h_{i1},h_{i2}) - \tau_{i0}, \quad i=\overline{1,s}; \quad \delta\theta_i \in (q_{i1},q_{i2}) - \theta_{i0}, \quad i=\overline{1,k};$$
$$\delta x_0 \in O - x_{00}, |\delta\tau_i| \le \alpha, \quad i=\overline{1,s}; \quad |\delta\theta_i| \le \alpha, \quad i=\overline{1,k}, \quad |\delta x_0| \le \alpha,$$
$$\delta\varphi = \sum_{i=1}^{m} \lambda_i \delta\varphi_i, \quad \delta u = \sum_{i=1}^{m} \lambda_i \delta u_i, \quad \delta\varphi_i \in \Phi - \varphi_0, \quad \delta u_i \in \Omega - u_0, \quad |\lambda_i| \le \alpha, \quad i=\overline{1,m}\Big\},$$

where $(a,b) - t_{00} := \{\delta t_0 = t_0 - t_{00} : t_0 \in (a,b)\}$ and $\alpha > 0$ is a fixed number.

There exist the numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1) \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$, and a solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\tau,t_{10}+\delta_1] \subset I_1$ corresponds to it (see [9, Theorem 1.4, p. 17]).

We note that $x(t;\mu_0+\varepsilon\delta\mu)$ is the solution of the perturbed equation

$$\dot{x}(t) = f(t,x(t),x(t-\tau_{10}-\varepsilon\delta\tau_1),\dots,x(t-\tau_{s0}-\varepsilon\delta\tau_s),u_0(t)+\varepsilon\delta u(t),u_0(t-\theta_{10}-\varepsilon\delta\theta_1)$$

$$+\varepsilon\delta u(t-\theta_{10}-\varepsilon\delta\theta_1),\dots,u_0(t-\theta_{k0}-\varepsilon\delta\theta_k)+\varepsilon\delta u(t-\theta_{k0}-\varepsilon\delta\theta_k))$$

with the perturbed initial condition

$$x(t)=\varphi_0(t)+\varepsilon\delta\varphi(t),t\in[\tau,t_{00}+\varepsilon\delta t_0),x(t_{00}+\varepsilon\delta t_0)=x_{00}+\varepsilon\delta x_0.$$

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\tau,t_{10}+\delta_1]\subset I_1$. Therefore, in the sequel, the solution $x_0(t)$ is assumed to be defined on the interval $[\tau,t_{10}+\delta_1]$.

For arbitrary $(t,\varepsilon,\delta\mu)\in[\tau,t_{10}+\delta_1]\times(0,\varepsilon_1)\times V$ we define the increment of the solution $x_0(t)=x(t;\mu_0)$:

$$\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t).$$

Theorem 1. *Let the following conditions hold:*

1) $\tau_{s0}>\cdots>\tau_{10}$ and $t_{00}+\tau_{s0}<t_{10}$;

2) *the function* $f(w,u,u_1,\dots,u_k)$, *where* $w=(t,x,x_1,\dots,x_s)$, *is bounded on* $I\times O^{1+s}\times U_0^{1+k}$;

3) *there exists the finite limit*

$$\lim_{w\to w_0}f_0(w)=f_0^-,w\in(a,t_{00}]\times O^{1+s},$$

where $f_0(w)=f(w,u_0(t),u_0(t-\theta_{10}),\dots,u_0(t-\theta_{k0}))$, $w_0=(t_{00},x_{00},\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0}))$;

4) *there exist the finite limits*

$$\lim_{(w_{1i},w_{2i})\to(w_{1i}^0,w_{2i}^0)}[f_0(w_{1i})-f_0(w_{2i})]=f_i,\quad i=\overline{1,s},$$

where $w_{1i},w_{2i}\in(a,b)\times O^{1+s},i=\overline{1,s}$,

$$w_{1i}^0=\Big(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),\dots,x_0(t_{00}+\tau_{i0}-\tau_{i-10}),$$
$$x_{00},x_0(t_{00}+\tau_{i0}-\tau_{i+10}),\dots,x_0(t_{00}+\tau_{i0}-\tau_{s0})\Big),$$
$$w_{2i}^0=\Big(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),\dots,x_0(t_{00}+\tau_{i0}-\tau_{i-10}),$$
$$\varphi_0(t_{00}),x_0(t_{00}+\tau_{i0}-\tau_{i+10}),\dots,x_0(t_{00}+\tau_{i0}-\tau_{s0})\Big);$$

Then there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$, *with* $t_{10}-\delta_2>t_{00}+\tau_{s0}$ *such that for arbitrary*

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^-,$$

where $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$, *we have*

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x_-(t;\delta\mu)+o(t;\varepsilon\delta\mu), \tag{3}$$

where $\delta x_-(t;\delta\mu)$ *has the form*

$$\delta x_-(t;\delta\mu)=-Y(t_{00};t)f_0^-\delta t_0+\beta(t;\delta\mu), \tag{4}$$

$$\beta(t;\delta\mu)=Y(t_{00};t)\delta x_0-\Big[\sum_{i=1}^s Y(t_{00}+\tau_{i0};t)f_i\Big]\delta t_0-\sum_{i=1}^s\Big[Y(t_{00}+\tau_{i0};t)f_i$$
$$+\int_{t_{00}}^t Y(\xi;t)f_{0x_i}[\xi]\Big(\chi_i(\xi)\dot\varphi_0(\xi-\tau_{i0})+(1-\chi_i(\xi))\dot x_0(\xi-\tau_{i0})\Big)d\xi\Big]\delta\tau_i$$
$$+\sum_{i=1}^s\int_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{0x_i}[\xi]\delta\varphi(\xi)d\xi-\sum_{i=1}^k\Big[\int_{t_{00}}^t Y(\xi;t)f_{0u_i}[\xi]\dot u_0(\xi-\theta_{i0})d\xi\Big]\delta\theta_i$$
$$+\int_{t_{00}}^t Y(\xi;t)\Big[f_{0u}[\xi]\delta u(\xi)+\sum_{i=1}^k f_{0u_i}[\xi]\delta u(\xi-\theta_{i0})\Big]d\xi, \tag{5}$$

where $\chi_i(\xi)$ is the characteristic function of the interval $[t_{00}, t_{00}+\tau_{i0}]$; furthermore, $Y(s;t)$ is the $n\times n$-matrix function satisfying the equation

$$Y_\xi(\xi;t) = -Y(\xi;t)f_{0x}[\xi] - \sum_{i=1}^{s} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi+\tau_{i0}], \xi\in[t_{00},t]$$

and the condition

$$Y(\xi;t) = \begin{cases} H, & \text{for} \quad \xi = t, \\ \Theta, & \text{for} \quad \xi > t. \end{cases} \tag{6}$$

Here, $f_{0x}[t] = f_{0x}(t, x_0(t), x_0(t-\tau_{10}), \dots, x_0(t-\tau_{s0}))$, H is the identity matrix and Θ is the zero matrix;

$$\lim_{\varepsilon\to 0}\frac{o(t;\varepsilon\delta\mu)}{\varepsilon} = 0 \quad \text{uniformly for} \quad (t,\delta\mu)\in[t_{10}-\delta_2, t_{10}+\delta_2]\times V^-.$$

Some comments. Theorem 1 corresponds to the case where variation at the point t_{00} is performed on the left. The function $\delta x(t;\delta\mu)$ is called the first variation of the solution $x_0(t)$, $t\in[t_{10}-\delta_2, t_{10}+\delta_2]$ and expression (4) is called the local variation formula.

The expression

$$-\Big[Y(t_{00};t)f_0^- + \sum_{i=1}^{s} Y(t_{00}+\tau_{i0};t)f_i\Big]\delta t_0$$

in formula (4) is the effect of the discontinuous initial condition (2) and perturbation of the initial moment t_{00}.

The addend

$$-\sum_{i=1}^{s}\Big[Y(t_{00}+\tau_{i0};t)f_i + \int_{t_{00}}^{t} Y(\xi;t)f_{0x_i}[\xi][\chi_i(\xi)\dot\varphi_0(\xi-\tau_{i0}) + (1-\chi_i(\xi))\dot x_0(\xi-\tau_{i0})d\xi\Big]\delta\tau_i$$

in formula (4) is the effect of perturbations of the delays τ_{i0}, $i=\overline{1,s}$.

The expression

$$-\sum_{i=1}^{k}\Big[\int_{t_{00}}^{t} Y(\xi;t)f_{0u_i}[\xi]\dot u_0(\xi-\theta_{i0})d\xi\Big]\delta\theta_i$$

in formula (4) is the effect of perturbations of delays θ_{i0}, $i=\overline{1,k}$.

The expression

$$\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi]\delta\varphi(\xi)d\xi$$

in formula (4) is the effect of perturbation of the initial function φ_0.

The expression

$$\int_{t_{00}}^{t} Y(\xi;t)\Big[f_{0u}[\xi]\delta u(\xi) + \sum_{i=1}^{k} f_{0u_i}[\xi]\delta u(\xi-\theta_{i0})\Big]d\xi$$

in formula (4) is the effect of perturbation of the control function u_0.

It is clear that if $\varphi_0(t_{00}) = x_{00}$, then $f_i = 0$, $i=\overline{1,s}$.

It is easy to see that (see (4),(5))

$$\delta x_-(t;\delta\mu) = \delta x_-^{(0)}(t;\delta\mu) - \sum_{i=1}^{s}\delta x^{(i)}(t;\delta\mu), t\in[t_{10}-\delta_2, t_{10}+\delta_2],$$

where

$$\delta x_-^{(0)}(t;\delta\mu) = Y(t_{00};t)\Big[\delta x_0 - f_0^-\delta t_0\Big] + \sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t)f_{0x_i}[\xi]\delta\varphi(\xi)d\xi$$

$$+\int\limits_{t_{00}}^{t} Y(\xi;t)\Big\{-\sum_{i=1}^{s}\Big[f_{0x_i}[\xi]\Big(\chi_i(\xi)\dot{\varphi}_0(\xi-\tau_{i0})+(1-\chi_i(\xi))\dot{x}_0(\xi-\tau_{i0})\Big)\Big]\delta\tau_i$$

$$-\sum_{i=1}^{k} f_{0u_i}[\xi]\dot{u}_0(\xi-\theta_{i0})\delta\theta_i+f_{0u}[\xi]\delta u(\xi)+\sum_{i=1}^{k} f_{0u_i}[\xi]\delta u(\xi-\theta_{i0})\Big\}d\xi$$

and

$$\delta x^{(i)}(t;\delta\mu)=-Y(t_{00}+\tau_{i0};t)f_i\Big(\delta t_0+\delta\tau_i\Big),\quad i=\overline{1,s}.$$

On the basis of the Cauchy formula (see [9, Lemma 2.3, p. 31], the function

$$\delta x_0(t)=\begin{cases}\delta\varphi(t), & t\in[\tau,t_{00}),\\ \delta x_-^{(0)}(t;\delta\mu), & t\in[t_{00},t_{10}]\end{cases}$$

is the solution of the equation in "variations"

$$\dot{\delta}x(t)=f_{0x}[t]\delta x(t)+\sum_{i=1}^{s} f_{0x_i}[t]\delta x(t-\tau_{i0})-\sum_{i=1}^{s}\Big[f_{0x_i}[t]\Big(\chi_i(t)\dot{\varphi}_0(t-\tau_{i0})$$

$$+(1-\chi_i(t))\dot{x}_0(t-\tau_{0i})\Big)\Big]\delta\tau_i-\sum_{i=1}^{k} f_{0u_i}[t]\dot{u}_0(t-\theta_{i0})\delta\theta_i+f_{0u}[t]\delta u(t)+\sum_{i=1}^{k} f_{0u_i}[s]\delta u(t-\theta_{i0})$$

with the discontinuous initial condition

$$\delta x(t)=\delta\varphi(t),t\in[\tau,t_{00}),\delta x(t_{00})=\delta x_0-f_0^-\delta t_0$$

and the function

$$\delta x_i(t)=\begin{cases}0, & t\in[\tau,t_{00}+\tau_{i0}),\\ \delta x^{(i)}(t;\delta\mu), & t\in[t_{00}+\tau_{i0},t_{10}]\end{cases}$$

is the solution of the equation in "variations"

$$\dot{\delta}x(t)=f_{0x}[t]\delta x(t)+\sum_{i=1}^{s} f_{0x_i}[t]\delta x(t-\tau_{i0})$$

with the discontinuous initial condition

$$\delta x(t)=0,t\in[\tau,t_{00}+\tau_{i0}),\delta x(t_{00}+\tau_{i0})=-f_i\Big(\delta t_0+\delta\tau_i\Big).$$

The variation formula allows us to obtain an approximate solution of the perturbed equation in the analytical form. In fact, for a small $\varepsilon>0$, from

$$x(t;\mu_0+\varepsilon\delta\mu)-x_0(t)=\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x_-(t;\delta\mu)+o(t;\varepsilon\delta\mu)$$

(see (3)), it follows that

$$x(t;\mu_0+\varepsilon\delta\mu)\approx x_0(t)+\varepsilon\delta x_-(t;\delta\mu).$$

Theorem 2. *Let conditions* 1), 2) *and* 4) *of Theorem* 1 *hold. Moreover, there exists the finite limit*

$$\lim_{w\to w_0} f_0(w)=f_0^+,w\in[t_{00},b)\times O^{1+s}. \tag{7}$$

Then there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$, with $t_{10}-\delta_2>t_{00}+\tau_{s0}$ such that for arbitrary

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+,$$

where $V^+=\{\delta\mu\in V:\delta t_0\geq 0\}$, we have

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x_+(t;\delta\mu)+o(t;\varepsilon\delta\mu),$$

where $\delta x_+(t;\delta\mu)$ has the form

$$\delta x_+(t;\delta\mu)=-Y(t_{00};t)f_0^+\delta t_0+\beta(t;\delta\mu).$$

Theorem 2 corresponds to the case, where variation at the point t_{00} is performed on the right. Theorems 1 and 2 are proved by the scheme given in [10].

Theorem 3. *Let conditions* 1)–4) *and condition* (7) *hold. Moreover,*

$$f_0^- = f_0^+ := f_0.$$

Then there exist the numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$ *such that for arbitrary*

$$(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V,$$

we have

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu),$$

where $\delta x(t; \delta\mu)$ *has the form*

$$\delta x(t; \delta\mu) = -Y(t_{00}; t) f_0 \delta t_0 + \beta(t; \delta\mu).$$

Theorem 3 corresponds to the case, where variation at the point t_{00} is carried out from double sided and is a corollary to Theorems 1 and 2.

It is clear that if the function $f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k)$ is continuous then

$$f_0 = f(t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), \dots, \varphi_0(t_{00} - \tau_{s0}), u_0(t_{00}), u_0(t_{00} - \theta_{10}), \dots, u_0(t_{00} - \theta_{k0})).$$

Theorem 4. *Let*

$$f(t, x, x_1, \dots, x_s, u, u_1, \dots, u_k) = A(t)x + \sum_{i=1}^{s} B_i(t)x_i + C(t)u + \sum_{i=1}^{k} D_i(t)u_i,$$

where $A(t)$, $B_i(t), i = \overline{1, s}$, $C(t)$ *and* $D_i(t), i = \overline{1, k}$ *are continuous matrix functions. Then there exist the numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$ *such that for arbitrary*

$$(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V,$$

we have

$$\begin{aligned}
\delta x(t; \delta\mu) = {} & Y(t_{00}; t)\Big[\delta x_0 - \Big(A(t_{00})x_{00} + \sum_{i=1}^{s} B_i(t_{00})\varphi_0(t_{00} - \tau_{i0}) + C(t_{00})u_0(t_{00}) \\
& + \sum_{i=1}^{k} D_i(t_{00})u_0(t_{00} - \theta_{i0})\Big)\delta t_0\Big] - \sum_{i=1}^{s} Y(t_{00} + \tau_{i0}; t)B_i(t_{00} + \tau_{i0})\Big(x_{00} - \varphi_0(t_{00})\Big)\delta t_0 \\
& - \sum_{i=1}^{s} \Big[Y(t_{00} + \tau_{i0}; t)B_i(t_{00} + \tau_{i0})\Big(x_{00} - \varphi_0(t_{00})\Big) + \int_{t_{00}}^{t} Y(\xi; t)B_i(\xi)\Big(\chi_i(\xi)\dot{\varphi}_0(\xi - \tau_{i0}) \\
& + (1 - \chi_i(\xi))\dot{x}_0(\xi - \tau_{i0})\Big)d\xi\Big]\delta\tau_i + \sum_{i=1}^{s} \int_{t_{00} - \tau_{i0}}^{t_{00}} Y(\xi + \tau_{i0}; t)B_i(\xi + \tau_{i0})\delta\varphi(\xi)d\xi \\
& - \sum_{i=1}^{k} \Big[\int_{t_{00}}^{t} Y(\xi; t)D(\xi)\dot{u}_0(\xi - \theta_0)d\xi\Big]\delta\theta_i + \int_{t_{00}}^{t} Y(\xi; t)\Big[C(\xi)\delta u(\xi) \\
& + \sum_{i=1}^{k} D_i(\xi)\delta u(\xi - \theta_{i0})\Big]d\xi.
\end{aligned}$$

Here, $Y(\xi; t)$ *is the* $n \times n$*-matrix function satisfying the equation*

$$Y_\xi(\xi; t) = -Y(\xi; t)A(\xi) - \sum_{i=1}^{s} Y(\xi + \tau_{i0}; t)B_i(\xi + \tau_{i0}), \xi \in [t_{00}, t]$$

and condition (6).

Theorem 4 is a simple corollary to Theorem 3.

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SOME COINCIDENCE AND COMMON FIXED POINT RESULTS IN CONE METRIC SPACES OVER BANACH ALGEBRAS VIA WEAK g-φ-CONTRACTIONS

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Abstract. Recently, B. Li and H. Huang [20] introduced the notion of weak φ-contractions on cone metric spaces over Banach algebras. The purpose of this paper is to generalize the main result of Li and Huang [20] by proving some coincidence and common fixed point results in cone metric spaces over Banach algebras via weak g-φ-contractions for weakly compatible mappings. Some examples are presented which verify and illustrate the results proved herein.

1. Introduction

The notion of metric spaces is generalized by several authors in various directions. One such generalization of metric spaces is a cone metric space given by L.G. Huang and X. Zhang [12]. In usual metric spaces, the metric function d is defined from $X \times X$ in the real number system, where X is a nonempty set. When generalizing the metric spaces to a cone metric space, the metric function d is defined from the product $X \times X$ into a Banach space (instead of the real number system). Thus, in cone metric spaces the distance is a vector belongs to the Banach space. Huang and Zhang [12] defined the cone metric spaces and proved some fixed point theorems for various types of contractive mappings and generalized the famous Banach contraction principle [2] in various ways. An example of Huang and Zhang [12] shows that there may be mappings which are contractive in a cone metric space, but fails to be a contraction with usual metric, i.e., the contractive conditions in cone metric spaces are more general, than those in usual metric spaces.

Common fixed point theorems have applications, e.g., in establishing the existence of a common solution for a class of functional equations arising in dynamic programming, in establishing the existence of solution of system of nonlinear integral equations, in establishing the existence of a solution for an implicit integral equation, etc. (see, e.g., [24, 13, 30] and the references therein). G. Jungck [15], introduced a common fixed point theorem for two commutating mappings in such a way that if we take one of them as identity mapping, then we obtain the Banach contraction principle. Although, Jungck's theorem generalizes the Banach contraction principle, but has a drawback that the involved mappings commutate. S. Sessa [28] introduced the notion of weakly commuting mappings and weakened the commutativity of mappings. Further, Jungck [16] introduced the notion of compatible mappings which generalizes the concept of a weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. R.P. Pant [23] introduced R-weakly commuting mappings for proving the common fixed point results. Latter, in [17, 18], Jungck defined weakly compatible mappings which need not to be commuting, but commutes only at their coincidence points. M. Abbas and G. Junck [1] initiated the study of common fixed point theorems in cone metric spaces and proved some common fixed point results for two weakly compatible mappings in cone metric spaces.

Although, some recent papers (see, e.g., [4, 6, 7, 19]) show that the fixed point results in cone metric spaces can be directly derived from their corresponding usual metric versions. To overcome this drawback, Liu and Xu [21] introduced the concept of a cone metric space over the Banach algebra, and proved some fixed point results in such spaces. In the fixed point results of Liu and Xu [21] the scalar control constants of contractive mappings were replaced by a vector control constant. They

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discussed the benefit of taking a vector instead of a scaler as a contractive constant and showed that the fixed point results in this new setting cannot be derived from their usual metric versions. Several authors generalized the results of results obtained by Liu and Xu [21] (see, e.g., [11, 22, 29], etc.). In [3, 26, 27], I.A. Rus and V. Berinde introduced the notion of φ-contraction and also generalized the Banach contraction principle in usual metric spaces. Inspired with these papers, recently, B. Li and H. Huang [20] introduced the notion of weak φ-contractions on cone metric spaces over the Banach algebras and generalized the results of H. Liu and S. Xu [21]. Li and Huang [20] used a vector-valued function as a control function in contractive conditions. They proved some fixed point results for weak φ-contractions and showed that these results on cone metric spaces over the Banach algebra can be applied in finding the solution of elementary system of equations and integral equation.

Inspired by Li and Huang [20], in this paper, we prove some coincidence and common fixed point results in cone metric spaces over the Banach algebras via weak g-φ-contractions for weakly compatible mappings. Our results generalize and unify the results of Huang and Zhang [12], Abbas and Jungck [1] and Li and Huang [20] in cone metric spaces over the Banach algebras. Some example are provided which illustrate the new results.

2. Preliminaries

In this section, we state some known definitions and facts which will be used throughout the paper.

Let $\mathcal{A}$ be a Banach algebra with a unit e and a zero element θ. A nonempty closed subset P of $\mathcal{A}$ is called a cone if the following conditions hold:

(1) $\{\theta, e\} \subset P$;
(2) $\forall \alpha, \beta \in [0, \infty) \Rightarrow \alpha P + \beta P \subseteq P$;
(3) $P^2 = PP \subset P$;
(4) $P \cap (-P) = \{\theta\}$.

A cone P is called a solid cone if $P^\circ \neq \emptyset$, where P° stands for the interior of P.

We can always define a partial ordering $\preceq$ with respect to P by $x \preceq y$, if and only if $y - x \in P$. We shall write $x \ll y$ to indicate that $y - x \in P^\circ$. We shall also write $\|\cdot\|$ as the norm on $\mathcal{A}$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in \mathcal{A}$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.

We always suppose that $\mathcal{A}$ is a Banach algebra with a unit e, P is a solid cone in $\mathcal{A}$, and $\preceq, \ll$ are partial orderings with respect to P.

Definition 2.1 ([26])**.** A function $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison if it satisfies the following two conditions:

(1) φ is monotone nondecreasing, i.e., $0 \leq t_1 \leq t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$;
(2) $\{\varphi^n(t)\}$ $(t > 0)$ converges to 0 as $n \to \infty$.

It is obvious that $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and $\lim_{t \to 0} \varphi(t) = 0$.

Definition 2.2 ([26])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is called a φ-contraction if there exists a comparison $\varphi : [0, \infty) \to [0, \infty)$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad \text{for all} \quad x, y \in X.$$

Rus [26] proved the following fixed point theorem for φ-contractions and generalized the Banach contraction principle.

Theorem 2.3 ([26])**.** *Let (X, d) be a metric space and $T : X \to X$ be a φ-contraction. Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

Definition 2.4 ([21])**.** Let X be a nonempty set and $\mathcal{A}$ be a Banach algebra. A mapping $d : X \times X \to \mathcal{A}$ is called a cone metric if it satisfies:

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$, $d(x, y) = \theta \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y \in X$.

In this case, the pair (X, d) is called a cone metric space over the Banach algebra $\mathcal{A}$.

Definition 2.5 ([5])**.** A sequence $\{u_n\}$ in a Banach algebra $\mathcal{A}$ is said to be a c-sequence if for each $c \gg \theta$, there exists $N \in \mathbb{N}$ such that $u_n \ll c$, for all $n > N$.

Definition 2.6 ([10])**.** Let (X,d) be a cone metric space over a Banach algebra $\mathcal{A}$ and $\{x_n\}$ be a sequence in X. We say that

(i) $\{x_n\}$ converges to $x \in X$ if $\{d(x_n,x)\}$ is a c-sequence and in this case we write $x_n \to x$ as $n \to \infty$;
(ii) $\{x_n\}$ is a Cauchy sequence if $\{d(x_n,x_m)\}$ is a c-sequence for n,m;
(iii) (X,d) is complete if every Cauchy sequence in X is convergent.

It is obvious that the limit of a convergent sequence in a cone metric space (X,d) over a Banach algebra $\mathcal{A}$ is unique.

Lemma 2.7 ([14])**.** *Let $\mathcal{A}$ be a Banach algebra and $u,v,w \in \mathcal{A}$. Then*
(1) *$u \ll w$ if $u \preceq v \ll w$ or $u \ll v \preceq w$;*
(2) *$u = \theta$ if $\theta \preceq u \ll c$ for each $c \gg \theta$.*

Lemma 2.8 ([25])**.** *Let $\mathcal{A}$ be a Banach algebra with its unit e. Then the spectral radius of $u \in \mathcal{A}$ equals to $\rho(u) = \lim\limits_{n\to\infty} \|u^n\|^{\frac{1}{n}}$.*

Lemma 2.9 ([9])**.** *Let P be a cone in a Banach algebra $\mathcal{A}$, $\{u_n\}$ and $\{v_n\}$ be two c-sequences in $\mathcal{A}$, and $\alpha,\beta \in P$ be vectors, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence in $\mathcal{A}$.*

Lemma 2.10 ([8])**.** *Let P be a cone and $k \in P$ with $\rho(k) < 1$. Then $\{k^n\}$ is a c-sequence.*

Definition 2.11 ([20])**.** Let $\mathcal{A}$ be a Banach algebra and P be a cone in $\mathcal{A}$. A mapping $\varphi : P \to P$ is called a weak comparison if the following conditions hold:
(WC1) φ is nondecreasing with respect to $\preceq$, namely, $t_1,t_2 \in P$, $t_1 \preceq t_2 \Rightarrow \varphi(t_1) \preceq \varphi(t_2)$;
(WC2) $\{\varphi^n(t)\}$ $(t \in P)$ is a c-sequence in P;
(WC3) if $\{u_n\}$ is a c-sequence in P, then $\{\varphi(u_n)\}$ is also a c-sequence in P.

Remark 2.12 ([20])**.** By Definition 2.11, we have $\varphi(\theta) = \theta$. Indeed, by (i) of Definition 2.11, we have $\theta \preceq \varphi(\theta) \preceq \varphi^n(\theta)$. Since $\{\varphi^n(\theta)\}$ is a c-sequence, by Lemma 2.7, it may be verified that $\varphi(\theta) = \theta$.

If $\mathcal{A} = \mathbb{R}$ and $P = [0,\infty)$, then the above definition is reduced to the Definition 2.1.

Example 2.13 ([20])**.** Let $\mathcal{A}$ be a Banach algebra, P be a cone in $\mathcal{A}$, and $k \in P$. Take $\varphi(t) = kt$ $(t \in P)$, where $\rho(k) < 1$. Then by Lemma 2.9 and Lemma 2.10, φ is a weak comparison.

Example 2.14 ([20])**.** Let M be a compact set of $\mathbb{R}^n$ and $\mathcal{A} = C(M)$, where $C(M)$ denotes the set of all continuous functions on M. Let $P = \{u \in \mathcal{A} : u(t) \geq 0, t \in M\}$ and define a mapping $\varphi : P \to P$ by $\varphi(u) = \frac{u}{u+1}$. Then φ is a weak comparison.

Definition 2.15 ([20])**.** Let (X,d) be a cone metric space over a Banach algebra $\mathcal{A}$. Let P be a cone and $\varphi : P \to P$ be a weak comparison. Then a mapping $T : X \to X$ is called a weak φ-contraction if

$$d(Tx,Ty) \preceq \varphi(d(x,y)), \quad \text{for all} \quad x,y \in X. \tag{1}$$

Clearly, the above definition generalizes Definition 2.2. The following theorem is the main result of [20].

Theorem 2.16 ([20])**.** *Let (X,d) be a complete cone metric space over a Banach algebra, and $T : X \to X$ be a weak φ-contraction. Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

Definition 2.17 ([1])**.** Let X be a nonempty set and $f,g\colon X \to X$ be two mappings. A point $z \in X$ is called the coincidence point of the pair (f,g) if $fz = gz$. A point $w \in X$ is called the point of coincidence of the pair (f,g) if $fz = gz = w$ for some $z \in X$. The pair (f,g) is called weakly compatible if f and g commute at each coincidence point.

3. Main Results

In this section, we introduce weak g-φ-contractions in the framework of a cone metric space over Banach algebra and obtain some coincidence and common fixed point theorems.

Definition 3.1. Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, P a cone, $\varphi : P \to P$ be a weak comparison and $g \colon X \to X$ be a mapping. Then a mapping $f : X \to X$ is called a weak g-φ-contraction if the following condition

$$d(fx, fy) \preceq \varphi\big(d(gx, gy)\big), \quad \text{for all} \ \ x, y \in X. \tag{2}$$

is satisfied.

Remark 3.2. Take $g = I_X$, the identity mapping of X, a weak g-φ-contraction reduces into weak φ-contraction. Therefore, a weak g-φ-contraction is a generalization of a weak φ-contraction. On the other hand, if we take $\varphi(t) = k$, where $k \in P$ and $\rho(k) < 1$, then we get an improved version of contraction mappings of Abbas and Jungck [1].

In the following theorem, we obtain a coincidence point result for two mappings satisfying the weak g-φ-contractivity condition.

Theorem 3.3. *Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, and $g : X \to X$ be a mapping. Suppose $f \colon X \to X$ is a weak g-φ-contraction and $f(X) \subseteq g(X)$. If $g(X)$ or $f(X)$ is a complete subspace of X, then $f$and g have a point of coincidence.*

Proof. Let $x_0 \in X$ be arbitrary, then $fx_0 \in f(X)$ and since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0 = y_1$ (say). Again, since $fx_1 \in f(X)$ and $f(X) \subseteq g(X)$, there exists $x_2 \in X$ such that $gx_2 = fx_1 = y_2$ (say). epeating this process, we obtain the so-called Jungck sequence defined by $\{y_n\} = \{gx_n\} = \{fx_{n-1}\}$.

If $c \in P^{\circ}$, then by (WC2), the sequence $\{\varphi^n(c)\}$ is a c-sequence, and so, there exists $n_0 \in \mathbb{N}$ such that

$$\varphi^{n_0}(c) \ll c. \tag{3}$$

Now, since f is a weak g-φ-contraction, we obtain

$$d(y_n, y_{n+n_0}) = d\big(fx_{n-1}, fx_{n+n_0-1}\big) \preceq \varphi\big(d(gx_{n-1}, gx_{n+n_0-1})\big) = \varphi\big(d(y_{n-1}, y_{n+n_0-1})\big). \tag{4}$$

Using (WC1), we obtain from the above inequality that

$$\varphi\left(d(y_n, y_{n+n_0})\right) \preceq \varphi\left(\varphi\left(d(y_{n-1}, y_{n+n_0-1})\right)\right) = \varphi^2\big(d(y_{n-1}, y_{n+n_0-1})\big).$$

Replacing n by $n-1$, the above inequality gives $\varphi\big(d(y_{n-1}, y_{n-1+n_0})\big) \preceq \varphi^2\big(d(y_{n-2}, y_{n+n_0-2})\big)$. On using this inequality in (4), we obtain $d(y_n, y_{n+n_0}) \preceq \varphi^2\left(d(y_{n-2}, y_{n+n_0-2})\right)$. Repetition of these arguments yields the following inequality:

$$d(y_n, y_{n+n_0}) \preceq \varphi^n\left(d(y_0, y_{n_0})\right). \tag{5}$$

By (WC2), we have $\{\varphi^n\left(d(y_0, y_{n_0})\right)\}$ is a c-sequence, and so, by Lemma 2.7 and (5) the sequence $\{d(y_n, y_{n+n_0})\}$ is also a c-sequence. Therefore, there exists $n_1 \in \mathbb{N}$ such that

$$d(y_n, y_{n+n_0}) \ll c - \varphi^{n_0}(c), \quad \text{for all} \quad n \geq n_1. \tag{6}$$

We shall show that

$$d(y_n, y_{n+kn_0}) \ll c \ \text{ for all } \ k \in \mathbb{N}, n \geq n_1. \tag{7}$$

To this end, we use mathematical induction.

By (6), the result is true for $k = 1$. Suppose $d(y_n, y_{n+jn_0}) \ll c$, for all $n \geq n_1$ is the induction hypothesis. Then we have

$$\begin{aligned} d\left(y_n, y_{n+(j+1)n_0}\right) &\preceq d\left(y_n, y_{n+n_0}\right) + d\left(y_{n+n_0}, y_{n+(j+1)n_0}\right) \\ &\ll c - \varphi^{n_0}(c) + d\left(fx_{n+n_0-1}, fx_{n+(j+1)n_0-1}\right) \\ &\preceq c - \varphi^{n_0}(c) + \varphi\left(d(gx_{n+n_0-1}, gx_{n+(j+1)n_0-1})\right) \\ &= c - \varphi^{n_0}(c) + \varphi\left(d(y_{n+n_0-1}, y_{n+(j+1)n_0-1})\right). \end{aligned}$$

Repeating the process, we obtain

$$d\left(y_n, y_{n+(j+1)n_0}\right) \ll c - \varphi^{n_0}(c) + \varphi^{n_0}\left(d(y_n, y_{n+jn_0})\right) \preceq c - \varphi^{n_0}(c) + \varphi^{n_0}(c) = c.$$

This completes the inductive proof of (7).

Again, since f is a weak g-φ contraction, one can obtain easily that

$$d(y_n, y_{n+1}) \preceq \varphi^n(d(y_0, y_1)), \text{ for all } n \in \mathbb{N}.$$

Therefore

$$\begin{aligned} &d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+n_0-1}, y_{n+n_0}) \\ \preceq &\varphi^n(d(y_0, y_1)) + \varphi^n(d(y_1, y_2)) + \cdots + \varphi^n(d(y_{n_0-1}, y_{n_0})). \end{aligned}$$

By (WC2), each of the sequences $\{\varphi^n(d(y_i, y_{i+1}))\}$, where $i = 0, 1, \ldots, n_0 - 1$ is a c-sequence, therefore $\{\varphi^n(d(y_0, y_1)) + \varphi^n(d(y_1, y_2)) + \cdots + \varphi^n(d(y_{n_0-1}, y_{n_0}))\}$ is also a c-sequence. Hence, by Lemma 2.7, the sequence $\{d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+n_0-1}, y_{n+n_0})\}$ is also a c-sequence. Thus, for the above $c \gg \theta$, there exists $n_2 \in \mathbb{N}$ such that

$$d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+n_0-1}, y_{n+n_0}) \ll c, \text{ for all } n > n_2.$$

Choose $n_3 = \max\{n_1, n_2\}$ and for $m, n > n_3$, let

$$k_m = \left[\frac{m - n_3}{n_0}\right], \; k_n = \left[\frac{n - n_3}{n_0}\right],$$

where $[\cdot]$ stands for the integer part. Then we have

$$n_3 \le m - k_m n_0 < n_3 + n_0, \; n_3 \le n - k_n n_0 < n_3 + n_0.$$

Therefore, for all $n, m \ge n_3$, we have

$$d(y_m, y_n) \preceq d(y_m, y_{m-k_m n_0}) + d(y_{m-k_m n_0}, y_{n-k_n n_0}) + d(y_{n-k_n n_0}, y_n) \ll 3c.$$

Thus, the sequence $\{y_n\} = \{gx_n\} = \{fx_{n-1}\}$ is a Cauchy sequence.

Suppose that $g(X)$ is complete (the proof for the case where $f(X)$ is complete, is same). Then by the completeness of $g(X)$, there exists $x^* \in X$ such that $y_n \to gx^* = y^*$ (say) as $n \to \infty$.

We shall show that y^* is a point of coincidence of f and g. Then since f is a weak g-φ contraction, we obtain

$$\begin{aligned} d(y^*, fx^*) &\preceq d(y^*, y_{n+1}) + d(y_{n+1}, fx^*) = d(y^*, y_{n+1}) + d(fx_n, fx^*) \\ &\preceq d(y^*, y_{n+1}) + \varphi(d(gx_n, gx^*)) = d(y^*, y_{n+1}) + \varphi(d(y_n, y^*)). \end{aligned}$$

Since $y_n \to gx^* = y^*$ as $n \to \infty$, the sequences $\{d(y^*, y_{n+1})\}$ and $\{d(y_n, y^*)\}$ are c-sequences, and so, by (WC3) and Lemma 2.7, there exists $n_4 \in \mathbb{N}$ such that

$$d(y^*, fx^*) \ll c, \quad \text{for all } \; c \in P^\circ, n \ge n_4.$$

Therefore, by Lemma 2.7, $d(y^*, fx^*) = \theta$, i.e., $fx^* = gx^* = y^*$. It shows that x^* is a coincidence point of f and g, and y^* is the corresponding point of coincidence of f and g. □

In the next theorem, we provide a sufficient condition for the existence of a common fixed point of f and g.

Theorem 3.4. *Suppose that all the conditions of Theorem* 3.3 *are satisfied. In addition, suppose that the mappings f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. The existence of a coincidence point x^* and the corresponding point of coincident y^* follows from Theorem 3.3. Thus, $fx^* = gx^* = y^*$. By weak compatibility of f and g, we have

$$fy^* = fgx^* = gfx^* = gy^*. \tag{8}$$

Since f is a weak g-φ-contraction, we have

$$d(fy^*, y^*) = d(ffx^*, fx^*) \preceq \varphi(d(gfx^*, gx^*)) = \varphi(d(fy^*, y^*))$$

Repetition of this process yields

$$d(fy^*, y^*) \preceq \varphi^n(d(fy^*, y^*)), \text{ for all } n \in \mathbb{N}.$$

By (WC2), the sequence $\{\varphi^n(d(fy^*, y^*))\}$ is a c-sequence, therefore, it follows from Lemma 2.7 and the above inequality that there exists $n_0 \in \mathbb{N}$ such that

$$d(fy^*, y^*) \ll c, \text{ for all } c \in P^\circ, n \geq n_0.$$

The above inequality with Lemma 2.7 yields that $d(fy^*, y^*) = \theta$, i.e., $fy^* = y^*$. Now, by (8), we obtain $fy^* = gy^* = y^*$. Thus, y^* is a common fixed point of f and g.

For the uniqueness of y^*, suppose that z^* is another common fixed point of f and g and $y^* \neq z^*$. Then, since f is a weak g-φ-contraction, we have

$$d(y^*, z^*) = d(fy^*, fz^*) \preceq \varphi(d(gy^*, gz^*)) = \varphi(d(y^*, z^*)).$$

Repetition of this process yields

$$d(y^*, z^*) \preceq \varphi^n(d(y^*, z^*)), \text{ for all } n \in \mathbb{N}.$$

Again, repeating the same arguments as above, we get $d(y^*, z^*) = \theta$, i.e., $y^* = z^*$. This contradiction shows that the common fixed point of f and g is unique. □

In the above theorem, for the existence of a common fixed point of f and g we apply an additional condition on f and g, namely, the condition of a weak compatibility. The following is an interesting example which shows that Theorem 3.3 ensures the existence of a point of coincidence of f and g, but not the existence of common fixed point of f and g, and so, the condition of a weak compatibility in Theorem 3.4 is not superfluous.

Example 3.5. Let $X = \mathbb{R}$, $A = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$ and product defined by $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and the unit $e = (1, 0)$. Define $d\colon X \times X \to \mathcal{A}$ by

$$d(x, y) = (|x - y|, \alpha|x - y|), \text{ for all } x, y \in X$$

where $\alpha > 0$ is a fixed real number. Then (X, d) is a cone metric space over a Banach algebra $\mathcal{A}$.

Define two mappings $f, g\colon X \to X$ by $fx = 1$ for all $x \in X$ and $gx = 1$ if $x \neq 1$ and $g1 = 2$. Then, note that f is a constant mapping, therefore, $d(fx, fy) = \theta$ for all $x, y \in X$. Thus, f is a weak g-φ-contraction. All the conditions of Theorem 3.3 are satisfied, and so, we can conclude the existence of a point of coincidence of f and g. Indeed, all the points of X, except 1, are the coincidence points of f and g and 1 is the corresponding point of coincidence of f and g. On the other hand, one can see that f and g have no common fixed point. Note that the mappings f and g are not weakly compatible. For instance, every $x \neq 1$ is a coincidence point of f and g, but

$$fgx = f1 = 1 \neq gfx = g1 = 2.$$

Therefore, f and g do not commute at their coincidence point, and so, are not weakly compatible.

The following corollary is an improvement of one of the main results of Abbas and Jungck [1] (Theorem 2.1 of Abbas and Jungck [1]) and is a special case of Corollary 2.10 of [8].

Corollary 3.6. *Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$. Suppose that the mappings $f, g\colon X \to X$ satisfy*

$$d(fx, fy) \preceq kd(gx, gy), \quad \text{for all x, y} \in \text{X}$$

where $k \in P$ is such that $\rho(k) < 1$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Taking $\varphi(t) = kt$ in Theorem 3.4, we obtain the required result. □

The following corollary is the main result of Li and Huang [20].

Corollary 3.7. *Let (X, d) be a complete cone metric space over a Banach algebra, and $f : X \to X$ be a weak φ-contraction. Then f has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Proof. Taking $g = I_X$ in Theorem 3.4, we obtain the existence and uniqueness of a fixed point of f. Further, for $g = I_X$, the Jungck sequence $\{y_n\}$ reduces into the iterative sequence $\{f^n x\}$ which converges to the fixed point of f. □

Corollary 3.7 uses the completeness of space X. In the next theorem, we omit the completeness of X by applying a different condition associated with f and the metric d.

Theorem 3.8. *Let (X, d) be a cone metric space over a Banach algebra, and $f : X \to X$ be a weak φ-contraction. Suppose, there exists $x^* \in X$ such that $d(x^*, fx^*) \preceq d(x, fx)$, for all $x \in X$. Then f has a unique fixed point in X and x^* is the fixed point of f.*

Proof. Denote $D(x) = d(x, fx)$ for all $x \in X$. Then by the assumption, we have

$$D(x^*) \preceq D(x), \quad \text{for all } \ x \in X. \tag{9}$$

Since f is a weak φ-contraction, we have

$$D(fx^*) = d(fx^*, ffx^*) \preceq \varphi(d(x^*, fx^*)) = \varphi(D(x^*)).$$

Using (WC1) and (9) and the fact that $f\colon X \to X$ we obtain $\varphi(D(x^*)) \preceq \varphi(D(fx^*))$. Therefore, it follows from the above inequality and Lemma 2.7 that $D(fx^*) \preceq \varphi(D(fx^*))$. Repetition of this process yields

$$D(fx^*) \preceq \varphi^n(D(fx^*)), \text{ for all } n \in \mathbb{N}.$$

Again, using (WC2) and Lemma 2.7, we obtain $D(fx^*) = \theta$, which together with (9) yields $D(x^*) = d(x^*, fx^*) = \theta$ i.e., $fx^* = x^*$. Therefore, x^* is the fixed point of f. The uniqueness of the fixed point follows from the weak φ-contractivity of f and (WC2). □

Example 3.9. Let M be a compact set of $\mathbb{R}^n$ and $\mathcal{A} = C(M)$, where $C(M)$ denotes the set of all continuous functions on M. Let $P = \{u(t) \in \mathcal{A} : u(t) \geq 0, t \in M\}$ and define a mapping $\varphi : P \to P$ by $\varphi(u) = \frac{u}{u+1}$. Then φ is a weak comparison. Let $X = \{u(t) \in \mathcal{A}\colon 0 \leq u(t) \in \mathbb{Q} \text{ for all } t \in M\}$ and define a mapping $d : X \times X \to \mathcal{A}$ by

$$d(u(t), v(t)) = |u(t) - v(t)|, \ t \in M.$$

Then (X, d) is a cone metric space over a Banach algebra. Define a mapping $T : X \to X$ by

$$Tu = \frac{u}{u+1} \text{ for all } u \in X.$$

Clearly, T is a weak φ-contraction. Indeed, if $u, v \in X$, we have

$$d(Tu, Tv) = \left|\frac{u}{u+1} - \frac{v}{v+1}\right| \preceq \frac{|v-u|}{|v-u|+1} = \frac{d(u,v)}{d(u,v)+1} = \varphi(d(x,y)).$$

It is easy to see that (X, d) is not a complete cone metric space, therefore, we cannot apply the results of Li and Huang [20]. On the other hand, there exists a point $x^*(t) = 0(t) = 0 \in X$ such that $d(x^*, Tx^*) \preceq d(u, Tu)$, for all $u \in X$. Therefore, all the conditions of Theorem 3.8 are satisfied and we can conclude the existence and uniqueness of fixed point of T by Theorem 3.8. Indeed, $x^*(t) = 0(t) = 0 \in X$ is the unique fixed point of T.

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THE BEHAVIOR OF SOLUTIONS OF THE DIRICHLET, NEUMANN AND MIXED DIRICHLET NEUMANN PROBLEMS IN THE VICINITY OF SHARP EDGES OF A PIECEWISE SMOOTH BOUNDARY

N. A. TRUBAEV

Abstract. An alternative expression of harmonic function in a three-dimensional case in spherical coordinates is proposed. We consider the behavior of solutions of the Dirichlet, Neumann and mixed Dirichlet-Neumann problems in the vicinity of sharp edges of a piecewise smooth boundary. The conditions of geometry admit only analytical solutions in the vicinity of sharp edges of special type. Some effects of an ideal fluid model in the vicinity of sharp edges of this type are discussed.

1. Introduction

In developing a numerical technique the question of proper modelling of corners in a body has always been a challenge. One can possibly argue, on physical grounds, that a sharp corner in a body is essentially a mathematical artifact. On the other hand, it is well known that variables of interest such as flux may change very rapidly around a rounded corner with a small radius of curvature. Hence for computational efficiency it may be quite advantageous to model a corner as sharp, across which there is a jump in the unit normal and tangential vectors to the boundary of the body. Another type of problem involving mixed boundary values is often encountered in real life. This class of problems involves specifications of incompatible boundary conditions on adjacent segments of the boundary of a body. Such situations may arise irrespective of the local geometry of the boundary of the body being investigated. Hence an ability to model corners effectively and efficiently is very important for numerical techniques for many applications of the Boundary element method (BEM).

A large and growing body of literature exists in this important subject area. It is a difficult task to acknowledge all the contributions in this field in a research article, especially with several excellent reviews published earlier. From a large number of research publications devoted to BEM, very few papers, however, are relevant to the subject of this research. Only publications relevant to the topic are those where singularity of a derivative of solutions is discussed. The reader is referred here to the article by Maz'ya for introduction [16].

The subject has theoretical and practical aspects. It should be pointed out that the majority of researchers consider two-dimensional problems only. Very few manuscripts are devoted to treatments of singularity of a derivative of the solutions in numerical algorithms in three dimensions. Mainly the three-dimensional problems are discussed in a way of a numerical experiment. Let us consider only one representative manuscript written from the sition of numerical algorithms creators [14]; there are numerous sources for additional reading.

The approach to theoretical basis can be gained from the work of Kondrat'ev [7] where the achieved result is the form of a solution of an elliptic equation in the vicinity of irregular points of the boundary (angular or conical points) which consists of a regular function and asymptotic series of solutions of model problems at zero boundary conditions.

This subject receives significant attention since the problem has numerous scientific and technical applications. Most of the cited references and literature on BEM application carry out problem solving without defining a form of the density function of a simple or double layer potential of solutions of the model problems at zero boundary conditions and treating it as an "unknown" function. There is a

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definite need to build the mathematical basis for the applications in order to clarify the methodology and aid for computational problem solving.

The conclusions of Kondrat'ev's work [7] serve as the foundation for the following research publications that mainly use functional analysis [4–20] (for detailed bibliography see [10], [21]). Georgian school of applied mathematics and its founder Kupradze are especially noteworthy [4], [24].

This research formulates and uses somewhat different approach to investigation of derivative's singularity of solutions of problems under consideration. Most authors of the research cited use functional analysis as a research method, while our research is concentrated on the consideration of a harmonic function in comparison with two equivalent forms: the integral form based on the Method of potential and the form with inclusion of trigonometrical functions. The integral form is convenient for proving of the existence and uniqueness of solutions of the considered problems. The trigonometrical functions allow us to use symmetry productively.

The result of this research is formulation of the basis for numerical algorithm as expressions of density functions of a simple or double layer potential for approximation of the terms of Kondrat'ev series having singularity of a derivative in the vicinity of angular, conical points and points of boundary condition's type change in the Dirichlet, Neumann and mixed Dirichlet-Neumann problems. The expression of a harmonic function in a two-dimensional case in the vicinity of these points had been known, but the density function in the expression of the harmonic function by potentials of simple or double layer was unknown.

In a three-dimensional case, the unknown expressions of harmonic function have been obtained, these can be used as an alternative of Legendre functions. The expressions of these functions by potentials of simple and double layer in the vicinity of these points have been proposed.

As any harmonic function satisfying the condition of radiation can be presented by a sum of simple and double layer potentials, the expressions are sufficient for all solutions of problems under consideration.

The results of this research in theoretical aspect are the alternative expressions of harmonic function in a three-dimensional case (3), (4), which have simpler form, than the Legendre functions. Another result is the relation between solutions of the problems with a smooth and piecewise smooth boundary which has been discovered. This relation can be obtained by the conformal mapping by a power function in a two-dimensional case and by two proposed mappings in a three-dimensional case.

Harmonic functions, having singularity of the derivative $r^{\lambda-1}$, $0<\lambda<1$, have in two-dimensional (1), (2) and three-dimensional (3), (4) cases the following expressions:

$$A\,r^{\lambda}\sin(\lambda(\theta+l)), \tag{1}$$

$$B\,r^{\lambda}\cos(\lambda(\theta+l)), \tag{2}$$

$$A\,r^{\lambda}\sin(\lambda\theta)\cos(\beta), \tag{3}$$

$$B\,r^{\lambda}\cos(\lambda\theta), \tag{4}$$

where A, B, λ, l are the constants, r, θ, β are variables. The angles θ, β in (3) are located in two perpendicular planes: if the angle θ is measured from the Ox_1-axis of local orthogonal system of coordinates, the angle β is measured in the plane Ox_2x_3. [1]

Mapping of (1), (2), (3), (4), modificatory the value of the variable λ, maps a harmonic function into a harmonic function. In a two-dimensional case this mapping is conformal by the power function. [2] Below, under the term "conformal mapping" we will mean this case of conformal mappings. In a three-dimensional case we name it "β - mapping". We define two these mappings.

Definition 1. The β_0-mapping: the mapping of a half-space into an infinite wedge with the boundary composed of two half-planes, having intersection in one line. In each plane, perpendicular to the line of intersection, the β_0-mapping corresponds to the conformal mapping in the plane by a power function with center at the point of crossing this plane with this line.

[1]Expressions (1), (2) exist for all values of λ. The proof of the existence of expressions (3), (4) for all values of λ is in Section 3.

[2]The conformal mapping by power function $w=z^k$ is the mapping of a half plane in the angular domain [9].

Definition 2. The β_1-mapping: the mapping of a half-space in the interiority or exteriority of an infinite cone, which is the space with exception of the infinite cone. In each plane, having the axis of the cone, the β_1-mapping corresponds to the conformal mapping in a plane by the power function with center at the cone apex. If the index is not marked below, the β-mapping is the substitution of β_0-mapping or β_1-mapping.

In the domain Θ with a closed smooth boundary $S \in C_1$ the harmonic function $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$ under the condition of radiation:

$$|u| < \frac{c}{\sqrt{r}} \quad \text{as} \quad r \to \infty \quad \text{in a two-dimensional case,} \quad |u| < \frac{c}{r} \quad \text{as} \quad r \to \infty \quad \text{in a three-dimensional case,}$$

where c is a constant, r is a distance from S, can be expressed as a sum of potentials of simple and double layers

$$\delta u(p) = -W_S(p,u) + V_S\left(p, \frac{\partial u}{\partial n}\right), \tag{5}$$

where $\delta = 2$, if $p \in \Theta \backslash S$, and $\delta = 1$, if $p \in S$.

If Θ in (5) is finitesimal (internal problem), it is simply connected. If it is not finitesimal (external problem), Θ is a complement of some simply connected domain with respect to the plane in a two-dimensional case and to the space in a three-dimensional case. The condition of radiation is necessary for external problem only.

The functions V and W, called as a potential of simple and double layer in a two-dimensional case, are defined as

$$V_S(p,\varphi) = -\frac{1}{\pi}\int\limits_S \ln(\overline{r}(p,q))\varphi(q)dS_q, \quad W_S(p,\varphi) = -\frac{1}{\pi}\int\limits_S \frac{\partial}{\partial n_q}\left(\ln(\overline{r}(p,q)\right)\varphi(q)dS_q;$$

the potentials of simple $\overline{V}$ and double $\overline{W}$ layer in a three-dimensional case are defined as [3]

$$\overline{V}_S(p,\varphi) = \frac{1}{2\pi}\int\limits_S \frac{\varphi(q)}{\overline{r}(p,q)}dS_q,$$
$$\overline{W}_S(p,\varphi) = \frac{1}{2\pi}\int\limits_S \frac{\partial}{\partial n_q}\left(\frac{1}{\overline{r}(p,q)}\right)\varphi(q)dS_q,$$

where $\overline{r}$ is a distance from the point q to the point p, n is the normal vector to S, external to Θ, φ is a function of density. If the index is not marked, we mean the potential in S. The notation n_q describes the normal vector n at the point q. In the text below, we omit the first argument of $V, W, \overline{V}, \overline{W}$ if the point of observation p has been specified in the text.

We can find in [26, p. 91] formula (5) for $p \in \Theta \backslash S, \delta = 2$, which was obtained for a smooth in Lyapunov's sense boundary. It is still true for $S \in C_1$. Let us show that the limit of this expression at approach p to $S \in C_1$ for $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$ exists and is equal to (5) for $\delta = 1$.

There are the limiting values for the double layer potential W with differentiable density $\varphi_2 \in C_1(S)$ and potential of simple layer V with continuous density $\varphi_1 \in C_0(S)$ on $S \in C_1$ in two– and in three-dimensional cases:

$$\Big[W(\varphi_2)\Big]^{\pm} = \mp\varphi_2 + W(\varphi_2), \tag{6}$$

$$\Big[V(\varphi_1)\Big]^{\pm} = V(\varphi_1), \tag{7}$$

[3] The underlining is added for distinguishing between the two-dimensional and three-dimensional cases.

where the upper index corresponds to the approach from Θ, and the lower index corresponds to that from the region outside of Θ, which complements Θ with respect to the plane or three-dimensional space. [4]

We can find (5), $p \in S$, taking into account (6), (7), as the limiting value of (5), $p \in \Theta \backslash S$, as $p \to S$ from inside of Θ. [5]

For $\varphi_2 \in C_1(S)$, there are the limiting values of normal derivative for $p \in S(C_1)$ in the two– and in three-dimensional cases:

$$\left[\frac{\partial W(\varphi_2)}{\partial n}\right]^{\pm} = Q(\varphi_2 - \varphi_2(p)), \tag{9}$$

$$\left[\frac{\partial V(\varphi_2)}{\partial n}\right]^{\pm} = \pm\varphi_2(p) + \Gamma(\varphi_2), \tag{10}$$

where

$$Q(\varphi_2 - \varphi_2(p)) = \frac{1}{2\pi}\int\limits_S \Pi(n_p, n_q)(\varphi_2(q) - \varphi_2(p))dS_q,$$

$$\Pi(m_{p_1}, m_{q_1}) = \frac{\partial}{\partial m_{p_1}}\left(\frac{\partial \Upsilon(p_1, q_1)}{\partial m_{q_1}}\right),$$

$$\Gamma(\varphi) = \frac{1}{2\pi}\int\limits_S \frac{\partial \Upsilon(p,q)}{\partial n_p}\varphi_2(q)dS_q,$$

$\Upsilon(p,q) = \dfrac{2}{\ln(\overline{r}(p,q))}$ in the two-dimensional case, $\Upsilon(p,q) = \dfrac{1}{\overline{r}(p,q)}$ in the three-dimensional case; m_{p_1}, m_{q_1} are some unit vectors at the points p_1, q_1.

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[4]If we use Gauss's theorem (8) we can obtain (6) because of continuity of the integral $W(\varphi_2 - \varphi_2(p_1))$ when point p cross the boundary $S \in C_1$ at the point $p_1 \in S$.

$$\int\limits_S \overline{\Upsilon}(p,q)dS_q = -2, \ \ p \in \Theta\backslash S, \qquad \int\limits_S \overline{\Upsilon}(p,q)dS_q = 0, \ \ p \notin \Theta \cup S, \qquad \int\limits_S \overline{\Upsilon}(p,q)dS_q = -1, \ \ p \in S, \tag{8}$$

where

$$\overline{\Upsilon}(p,q) = \frac{1}{\pi}\frac{\partial}{\partial n_q}\left(\ln\left(\frac{1}{r(p,q)}\right)\right) \text{ in two-dimensional case,}$$

$$\overline{\Upsilon}(p,q) = \frac{1}{2\pi}\frac{\partial}{\partial n_q}\left(\frac{1}{r(p,q)}\right) \text{ in three-dimensional case.}$$

The expression (7) follows from continuity of the integral with weak singularity V.

[5]The second Green's formula [26, p. 90], where one of the functions is Newtonian or logarithmic source and the second is harmonic function $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$, at the point $p_1 \in \Theta\backslash S$ in the vicinity of $p \in S$, $S \in C_1$, by the Gauss's theorem (8) can be represented in the form

$$2\big(u(p_1) - u(p)\big) = -W\big(p_1, u - u(p)\big) + V\left(p_1, \frac{\partial u}{\partial n}\right).$$

When $p_1 \to p$, the left hand-side tends to zero, the right-hand side is continuous for $p \in S$,

$$= -W\big(p, u - u(p)\big) + V\left(p, \frac{\partial u}{\partial n}\right), p_1 = p, p \in S.$$

The equivalent of this equation is

$$u(p) = -W(p, u) + V\left(p, \frac{\partial u}{\partial n}\right), p \in S.$$

$Q(\varphi_2-\varphi_2(p))$ is the operator of the function $\varphi_2-\varphi_2(p)$, $\varphi_2\in C_1(S)$, integral in the operator exists in $p\in S$, $S\in C_1$, in the sense of principal value. [6] Γ is singular integral. [7]

If $u\in C_m(\Theta)\cap C_k(\overline{\Theta})$, $m\geq 2, k\geq 1$, the reasoning is also true right up to $m=\infty, k=\infty$, when u is analytical in some region, which includes Θ.

In the part below we consider the behavior of the potentials on a piecewise smooth boundary next to the angular or conical point and to the point of change of type of the boundary conditions in the function space $u\in L_2^{(1)}(\Theta)$. [8] [9]

Definition 3. If

$$\int\limits_{\Theta}\left(|u|^2+|\mathrm{grad}u|^2\right)d\Theta<\infty, \text{ we say that } u\in L_2^{(1)}(\Theta).$$

We make generalization of (5) for piecewise smooth boundary by using conformal mapping and β - mappings and prove expressions 3), (4).

Comparing the expressions (1), (2) and (3), (4) with (5), in the vicinity of these points the following result can be obtained: geometry of sharp edges of special type allows only analytical solutions in the Dirichlet, Neumann and mixed Dirichlet-Neumann problems. This result has important consequences, and as it will be discussed later, can be used in an ideal fluid model in applications, where this model corresponds to the real physical processes with required degree of accuracy, because the velocity of potential flow in this model is a solution of the Neumann problem.

2. The Potentials $V_L(r^\lambda)$, $W_L(r^\lambda)$ on the Ray L.

Let us consider the integral $W_L(\varphi)$, $\varphi=r^\lambda$, in the two-dimensional case on the ray located along the Ox_1-axis of the left orthogonal coordinate system for positive values of x_1 (L coincides with the right half of the Ox_1)-axis, r is the distance from the point of the ray emergence at the origin of

---

[6]Because of the Gauss's theorem, in the two-dimensional and in three-dimensional cases (8) for $p_1\in\Theta\backslash S$ in the vicinity of $p\in S$ the expression

$$\frac{\partial W(p_1,\varphi_2-\varphi_2(p))}{\partial n_p}=\frac{\partial W(p_1,\varphi_2)}{\partial n_p} \tag{11}$$

is true. Let us split S in two parts: $S=S_R\cup S_a$, where S_R is a part of S inside of the circle in two-dimensional case or sphere in the three-dimensional case with small radius R and center at p , S_a is the remaining part of S. According to the condition $\varphi_2\in C_1(S)$, we can present φ_2 in p by two terms of Taylor series and addition. When $p_1\to p$, because of the subtracting in (11) the first term has no influence on the integral sum. If $S\in C_1$, when R is small, we can replace S_R by a segment of tangent line in the two-dimensional case and by a circle in a tangent plane in the three-dimensional case, and the corresponding integral on S_R of the second term is equal to zero, it exists as singular integral. The integral of the addition on S_R converges as integral with a weak singularity. Therefore in the limiting expression of (11) as $p_1\to p$ the integral exists in the sense of the principal value because there is the limit of the integral sum on S_a when the radius of the circle or sphere tends to zero. It is also true at approaching $p_1\to p$, if $p_1\notin\Theta$. Finally, we get (9), where integral in the right-hand side exists in the sense of the principal value for $\varphi_2\in C_1(S)$, $S\in C_1$.

[7]When R, is small we can replace S_R by a segment of tangent line in the two-dimensional case and by a circle in a tangent plane in the three-dimensional case. The integral Γ on S_R of the second term of Taylor series in p of φ_2 at approach $p_1\to p$, $p_1\in\Theta\backslash S$ converges as the integral with a weak singularity. Integral Γ of addition on S_R and integral Γ of φ_2 on S_a converges. We have to consider the integral Γ of the first term on S_R at approach $p_1\to p$. The first term is a constant, consequently, the integral Γ from it coincides on S_R with the integral in Gauss's theorem (8) with a different sign. Therefore the gap in $p\in S$ of limiting values of the integral from different sides of S is defined by the integral in the Gauss's theorem. Finally, we get (10).

The integral Γ on S_R of φ_2 is equal to zero because the numerator of Γ is equal to zero when the points p and q belong to one line with normal vector n_p in the two-dimensional case or one plane with a normal vector n_p in the three-dimensional case. Consequently, the condition [26, p. 58, (3.20)] of the existence of singular integral is satisfied.

[8]If the following expression of the first Green's formula, where both of the functions are equal to u, exists:

$$\int\limits_S u\frac{\partial u}{\partial n}dS=\int\limits_\Theta\sum_{k=1}^m\left(\frac{\partial u}{\partial x_k}\right)^2 d\Theta$$

($m=2$ in the two-dimensional case, $m=3$ in the three-dimensional case), the function $u\in L_2^{(1)}(\Theta)$. The quadratic form in the right part corresponds to the expression of energy. The condition $u\in L_2^{(1)}(\Theta)$ is equivalency of finiteness of value of energy in applications.

[9]We define the function space as it has been done in [15, pp. 122–130], this is used for applications in Section 5.

coordinates, $0 < \lambda < 1$. Let us show that the limiting values of $W_L(r^\lambda)$ at approach p from region of negative values of x_2 (upper sign) and from the region of positive values of x_2 (lower sign) are equal to (12) when we mean the normal vector $\overrightarrow{n} = (0, 1)$.

$$\left[W_L(r^\lambda)\right]^{\pm} = \mp r^\lambda. \tag{12}$$

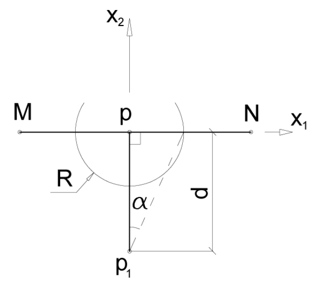

FIGURE 1. Approach point p_1 to the point $p \in]MN[$.

$\diamondsuit$ If we consider potential $W_{[MN]}(\varphi)$, $\varphi \in C_n, n \geq 1$, on the segment of straight line $[MN]$ at the approach point p_1 to the point $p \in]MN[$ in the local system of coordinates (Figure 1), we find: only thefirst term of Taylor series of presentation of φ at p corresponds to the integral having different limiting values in the sides of $[MN]$. Integrals of other terms are continuous when the point p_1 crosses the $[MN]$, they are equal to zero for $p \in]MN[$ when p_1 coincides with p. Let $[MN]$ consist of two parts: $[MN] = S_R \cup S_a$, where S_R is a part of $[MN]$ inside of the circle with small radius R and center at p. The integral of the first term on S_a creates continuous in p function, which is equal to zero at $p \in]MN[$ when p_1 coincides with p. The integral $W_{[MN]}(\varphi) = 0$ for $p \in]MN[$ when p_1 coincides with p because the numerator of $W_{[MN]}$ is equal to zero, it is still true, if we replace $[MN]$ by the ray with the vertex in M. Consequently, we have to consider the integral of the first term on S_R

$$\frac{\varphi(p)}{\pi} \int\limits_{S_R} \frac{d}{x_1^2 + d^2} dx_1 = \operatorname{sign}(d) \frac{\varphi(p)}{\pi} \left(\arctan \frac{x_1}{|d|} \right) \Bigg|_{-R}^{R},$$

when the point of observation p_1 approaches p (Figure 1). The arctangent $\arctan \frac{R}{|d|})$ in the last expression is equal to the value of the angle α (Figure 1) which approaches $\pi/2$ when the point p_1 approaches the point p.

When $\varphi(p) = r^\lambda$, $0 < \lambda < 1$, $p \in [MN]$, r is the distance from the point M, the limiting values of the last expression at the approach point p_1 to the point p is equivalent of (12) if we replace $[MN]$ by the ray with the vertex in M. We can do this because the corresponding integral on the infinite part of the ray converges at p_1 as $p_1 \to p$ and this integral is equal to zero when p_1 coincides with p; in the vertex of the ray this function $\varphi(p) = r^\lambda$ is equal to zero. $\diamondsuit$

Let us consider the function

$$A(r, \Omega) = -r^\lambda \frac{\sin(\lambda(\Omega - \pi))}{\sin(\lambda\pi)} - W_L(r^\lambda)$$

and its derivative

$$\frac{\partial A}{\partial x_1} \tag{13}$$

in the polar coordinate system:

$$x_1 = r\cos(\Omega), x_2 = r\sin(\Omega), \tag{14}$$

the angle Ω is measured anticlockwise from Ox_1, the ray L coincides with the right half of the Ox_1-axis. The harmonic function (13) in the domain of plane with a slit in the half of the $Ox_1, x_1 \geq 0$-axis, has zero values in the boundary: in the half of the $Ox_1, x_1 \geq 0$-axis, and in the infinite boundary. Consequently, (13) is zero function. Therefore the function A is a constant. Since A is equal to zero at the origin of coordinates, this constant is equal to zero. Thus we have found: the harmonic function of the potential $W_L(r^\lambda)$, $0<\lambda<1$ which is defined on the ray L, has the expression

$$W_L(r^\lambda) = -r^\lambda \frac{\sin(\lambda(\Omega-\pi))}{\sin(\lambda\pi)}, \quad 0 \le \Omega \le 2\pi. \tag{15}$$

If we consider a derivative of $W_L(r^\lambda)$ by the angle Ω in the polar coordinate system in continuation of ray L at $\Omega=\pi$, we find that it is equal to [10]

$$-\frac{\lambda r^\lambda}{\sin(\lambda\pi)}. \tag{16}$$

This result justifies (15).

Let us consider simple layer V with density r^λ which is defined on the ray $V_L(r^\lambda)$ in the polar system of coordinates (14). There are the relations between values of derivatives of some function T when $x_2=0$ for $\Omega=0$, $\Omega=2\pi$, $\Omega=\pi$

$$\left.\frac{\partial T}{\partial x_2}\right|_{\Omega=0} = \left.\frac{\partial T}{r\partial\Omega}\right|_{\Omega=0}, \quad \left.\frac{\partial T}{\partial x_2}\right|_{\Omega=2\pi} = \left.\frac{\partial T}{r\partial\Omega}\right|_{\Omega=2\pi}, \quad \left.\frac{\partial T}{\partial x_2}\right|_{\Omega=\pi} = -\left.\frac{\partial T}{r\partial\Omega}\right|_{\Omega=\pi}.$$

Owing to the last expressions, (15) and the relation

$$\Gamma_L(\varphi) = -W_L(\varphi), \tag{17}$$

the function $V_L(r^\lambda)$ has to satisfy the conditions [11]

$$\left.\frac{\partial V_L(r^\lambda)}{r\partial\Omega}\right|_{\Omega=0} = -r^\lambda, \quad \left.\frac{\partial V_L(r^\lambda)}{r\partial\Omega}\right|_{\Omega=2\pi} = r^\lambda, \quad \left.\frac{\partial V_L(r^\lambda)}{r\partial\Omega}\right|_{\Omega=\pi} = 0.$$

Only one two-dimensional harmonic function (2) satisfies the conditions, thus:

$$V_L(r^\lambda) = \frac{1}{\lambda+1}\frac{r^{\lambda+1}}{\sin(\lambda\pi)}\cos((\lambda+1)(\Omega-\pi)) + C, \tag{18}$$

$0<\lambda<1$, $0\le\Omega\le2\pi$, C is a constant.

Let the potential W be defined on two rays with one vertex at the origin of coordinates located under the angles: plus α and minus α from the Ox_1-axis, and have density with an absolute value r^λ on each ray and symmetric normal vectors on the rays which coincide with external normal vectors of the wedge with aperture angle 2α . This potential creates the functions u_w^+ and u_w^-, when the density functions on the rays have symmetric positive values or antisymmetric values: positive values on the one ray and negative values on the other ray, accordingly.

$$u_w^+(r,\Omega) = -\Big(r^\lambda\sin(\lambda(\Omega-\alpha-\pi)) + r^\lambda\sin(\lambda(-\Omega-\alpha+\pi))\Big)/\sin(\lambda\pi),$$
$$u_w^-(r,\Omega) = -\Big(r^\lambda\sin(\lambda(\Omega-\alpha-\pi)) - r^\lambda\sin(\lambda(-\Omega-\alpha+\pi))\Big)/\sin(\lambda\pi).$$

After transformations and substitution $\theta=\Omega-\pi$, we get

$$u_w^+(r,\theta) = -\frac{2\sin(-\lambda\alpha)}{\sin(\lambda\pi)}r^\lambda\cos(\lambda\theta), \tag{19}$$

$$u_w^-(r,\theta) = -\frac{2\cos(-\lambda\alpha)}{\sin(\lambda\pi)}r^\lambda\sin(\lambda\theta), \tag{20}$$

where$-\pi+\alpha\le\theta\le\pi-\alpha$.

---

[10] After substitution $\Omega=\pi$ the value of the derivative by Ω has the expression

$$-\frac{1}{\pi}\int_0^\infty \frac{rb^\lambda}{(r+b)^2}db = -\frac{\lambda r^\lambda}{\sin(\lambda\pi)}.$$

This integral can be calculated by "Wolfram Mathematica 9" (www.wolfram.com):

FullSimplify[Integrate[(r*b^L)/(r + b)^2, {b, 0, Infinity}]]

[11] Despite the fact that the functions $V_L(r^\lambda)$, $W_L(r^\lambda)$ are defined for $0\le\Omega\le2\pi$, the two-dimensional harmonic functions (1) , (2) exist for all values of Ω, thus the derivatives for $\Omega=0,\Omega=2\pi$ can be considered, the values of derivatives of $V_L(r^\lambda)$ for $\Omega=0,\Omega=2\pi$ are the limits as aspiration $\Omega\to0$ and $\Omega\to2\pi$.

If we consider the potential V on the two rays with same combination of density functions and normal vector's directions, we obtain

$$u_v^+(r,\theta) = \frac{1}{\lambda+1}\frac{2\cos\big((\lambda+1)\alpha\big)}{\sin(\lambda\pi)} r^{\lambda+1}\cos\big(\theta(\lambda+1)\big) + C_1, \tag{21}$$

$$u_v^-(r,\theta) = \frac{-1}{\lambda+1}\frac{2\sin\big((\lambda+1)\alpha\big)}{\sin(\lambda\pi)} r^{\lambda+1}\sin\big(\theta(\lambda+1)\big) + C_2, \tag{22}$$

where $-\pi+\alpha \le \theta \le \pi-\alpha$, C_1, C_2 are the constants. [12]

Let us introduce the notation: $\psi_1(r,\theta) = C_\psi u_w^+(R,\theta)$, $\psi_2(r,\theta) = C_\psi u_w^-(r,\theta)$, $\psi_3(r,\theta) = C_\psi u_v^+(r,\theta)$, $\psi_4(r,\theta) = C_\psi u_v^-(r,\theta)$, where $C_\psi = \dfrac{\lambda+1}{4}$.

Expressions (19), (20), (21), (22) were obtained for $0<\lambda<1$. For these values of λ, the integrals $W_L(r^\lambda)$, $V_L(r^\lambda)$ converge at a point for which $r<\infty$.

There are the values $\lambda \ge 1$ for which expressions (19), (20), (21), (22) are equal to zero when the point of observation of the potentials W, V of the two rays belongs to one of the rays. In this case the values of the potential W, V of the two rays are finite for all points where $r<\infty$ despite the potentials $W_L(r^\lambda)$, $V_L(r^\lambda)$ of one ray do not converge at these points, the rays "counterpoise" each other. We can consider expressions (19), (20), (21), (22) having zero values at the points of rays for $\lambda \ge 1$ as a result of conformal mapping by power function with center in the common vertex of the rays from the same expressions for $0<\lambda<1$. The mapping do not change the zero boundary conditions therefore the integrals of the potentials converge for $\lambda \ge 1$ at the points of the rays after the mapping as they have zero values there.

The integrals converge in the described sense only. We do not need to calculate these integrals numerically, since expressions (19), (20), (21), (22) have already been obtained, but we have to take into account the possibility of changing the multiplier function in different ranges of λ. The functions (19), (20), (21), (22) can be written in the form

$$f_w(\lambda)u_w^+(\lambda,r,\theta) = B_w(\lambda,\alpha)r^\lambda\cos(\lambda\theta), \tag{23}$$

$$f_w(\lambda)u_w^-(\lambda,r,\theta) = A_w(\lambda,\alpha)r^\lambda\sin(\lambda\theta). \tag{24}$$

$$f_v(\lambda)u_v^+(\lambda,r,\theta) = B_v(\lambda,\alpha)r^{\lambda+1}\cos(\theta(\lambda+1)), \tag{25}$$

$$f_v(\lambda)u_v^-(\lambda,r,\theta) = A_v(\lambda,\alpha)r^{\lambda+1}\sin(\theta(\lambda+1)), \tag{26}$$

where $f_w(\lambda)=1$, $f_v(\lambda)=\lambda+1$ at $0\le\lambda\le1$. If λ and α are such that (23), (24), (25), (25) are equal to zero on the rays, the functions A_w, B_w, A_v, B_v depend on λ only, as the values of α are determined by the values of λ. Since A_w, B_w, A_v, B_v are periodic, the range $0\le\lambda\le1$ defines these functions for all values of λ. The limiting values for $\lambda\to0$ and $\lambda\to1$ of (23), (24), (25), (26) under the described conditions exist and are equal to zero. Since $u_w^+, u_w^-, u_v^+, u_v^-$ are equal to zero on the rays for $0\le\lambda\le1$ and conformal mapping by power function of the functions created by $u_w^+, u_w^-, u_v^+, u_v^-$ for $0\le\lambda\le1$, have zero values on the rays for all values of λ, the integrals, corresponding to $u_w^+, u_w^-, u_v^+, u_v^-$, converge for all values of λ at the points of the rays. Therefore they converge at other points of the domain at $r<\infty$, $\lambda\ge0$ and at $r\ne0$, $\lambda<0$ for all values of λ. The right– and left-hand sides of expressions (23), (24), (25), (26) are harmonic functions of the forms (1), (2). Consequently, the result of conformal mapping by a power function of the both parts of equalities (23), (24), (25), (26) are harmonic functions of the forms (1), (2), despite the factors $f_w(\lambda), f_v(\lambda)$ are not known for all values of λ.

Any harmonic function has maximal and minimal values in the boundary of the domain where the function is defined. The potentials are equal to zero in the rays and expressions (19), (20), (21), (22)

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[12] Expression (21) with the value of the angle $\alpha = \dfrac{\pi}{2(\lambda+1)}$ is equal to constant C_1. Conformal mapping from constant function by power function of the domain $-\pi+\alpha\le\theta\le\pi-\alpha$ to a domain with different value of α is the same constant function. Consequently, in all expressions (21), equal to the constant, this constant has the same magnitude; the value of C_1 does not depend on λ and α. Below, in the footnote 18it is shown that this is possible if $C_1=0$ only. As $C_1=2C$, $C=0$ in (18), $C_2=0$ in (22).

have infinite values in the infinite boundary $r = \infty$, consequently, the potentials have finite values at the points of the domain where $r < \infty$. If we consider expressions (19), (20), (21), (22) having zero values at the points of the rays for $\lambda \le 0$ as a result of conformal mapping by a power function with center in the common vertex of the rays from the same expressions for $0 < \lambda < 1$ we find that the integrals of the potentials converge at all points of the domain with the exception of the point of singularity: $r = 0$. [13]

3. The Potentials $\overline{V}_{\widehat{S}}(r^\lambda)$, $\overline{W}_{\widehat{S}}(r^\lambda)$ on the Sector $\widehat{S}$.

A solution of three-dimensional Laplace equation in spherical coordinate system:

$$\triangle u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\Omega)}\frac{\partial}{\partial \Omega}\left(\sin(\Omega)\frac{\partial u}{\partial \Omega}\right) + \frac{1}{r^2\sin(\Omega)^2}\frac{\partial^2 u}{\partial \beta^2} = 0 \tag{27}$$

can be considered in the form

$$u = r^\lambda U(\Omega)\cos(\kappa\beta), \tag{28}$$

where κ is integer, $\kappa \ge 0$ [26, p. 319]. Equation (27) has singular points for $r = 0$, $\Omega = 0$, $\Omega = \pi$.

Let us suppose that expression (28), $0 < \lambda < 1$, is an axi-symmetric solution of equation (27), which is equal outside of a cone to the potential $\overline{V}$ located on the cone surface. Because of the axial symmetry, solution (28) has to have the expression ($\kappa = 0$)

$$u = r^\lambda U_v(\Omega) = \int_0^{2\pi} \overline{V}_{\widehat{S}}(\varphi_v)d\beta, \tag{29}$$

where the angle β is measured in the plane, perpendicular to the axis of the cone; the angle Ω is measured from axis of the cone, Ω is equal to a half of aperture angle at the points of cone surface; $\overline{V}_{\widehat{S}}(\varphi_v)$ is the potential $\overline{V}$ located on the sector $\widehat{S}$ of the cone with an infinitely small angle $d\beta$ which corresponds to the sector with an infinitely small angle $d\eta$ in the local coordinate system (Figure 2). In the limiting case, when the cone surface transforms by the β_1-mapping in a plane, perpendicular to the axis of the cone, the angle η coincides with β. [14]

When we consider an integral of some function $A(p,q)$ over a square of sector $\widehat{S}$ in the local system of coordinates (Figure 2):

$$\int_{\widehat{S}} A(p,q)dS_q = \int_r A(p,q)rdrd\eta,$$

we mean: a sector is located on the plane Ox_1x_3, the angle η is measured from Ox_1, the bisector of the sector $d\eta$ is located iin the Ox_1-axis. Element of the surface integral over the square of the sector: $d\widehat{S}_q = rdrd\eta$, where q is the point on the bisector in the Ox_1-axis, r is the distance from the vertex (Figure 2). Consequently, at the point p of the bisector integral $\overline{V}_{\widehat{S}}(\varphi_v)$, $\varphi_v \in C_0(\widehat{S})$, is the integral with a weak singularity.

The first and second derivatives of (29) in direction of perpendicular to a plane, having axis of the cone, are equal to zero in the plane because of axial symmetry of the problem. Therefore $r^\lambda U_v(\Omega)$ is a two-dimensional harmonic function in any plane having the axis of the cone, it has the axis of symmetry in the axis of the cone. Consequently, the β_1-mapping of the cone transforms two-dimensional harmonic function $r^\lambda U_v(\Omega)$ into another two-dimensional harmonic function with same the form of expression (2), but different value of λ in the expression.

Because the conformal mapping by a power function with center at the point $r = 0$ can be done for any expression (2) and can transform the expression in other expression with the same form, but different value of λ, there are special case of the β_1-mapping which transforms the cone surface into a

[13] The right-hand sides of expressions (19), (20), (21), (22) for zero values on the rays include the solution of the Dirichlet or Neumann problem for a wedge under zero boundary conditions [26, pp. 305–310].

[14] The solution of the Dirichlet problem with singularity of derivative $r^{\lambda-1}, 0 < \lambda < 1$, outside of the cone in the vicinity of its vertex: $r = 0$, has to be under the zero boundary conditions. This result is related to the term "Noetherian" [7] and can be illustrated by the conformal mapping as is shown in the text below. Here we do not use this restriction, therefore (29) may be not equal to zero on the cone surface.

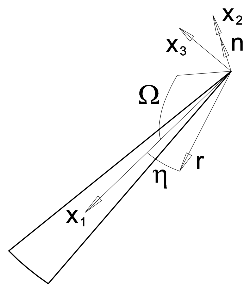

FIGURE 2. Sector $\widehat{S}$ with infinitely small angle $d\eta$.

ray and the corresponding expression (29). In this case, the Newtonian potential $\overline{V}_{\widehat{S}}(\varphi_v)$ on each of the sectors (Figure 2) after transformation will correspond to the ray, and the right-hand side of (29) after the mapping will correspond to 2π such rays. [15] Consequently, the potential of one ray $\overline{V}_{\widehat{S}}(\varphi_v)$ has axial symmetry and expression (28) for $\kappa = 0$ in which the two-dimensional harmonic function $r^\lambda U_v(\Omega)$ is a sum of symmetric functions having expression (2) and a constant. [16]

Let us consider the function

$$\Gamma_{\widehat{S}}(\varphi_v) = -\overline{W}_{\widehat{S}}(\varphi_v), \tag{30}$$

where $\overline{W}_{\widehat{S}}$ is the potential $\overline{W}$ on the sector (2). The function (30) has to be a harmonic function having expression (28) for $\kappa = 1$, because of one plane of symmetry Ox_1x_2. Therefore the potential $\overline{W}_{\widehat{S}}(\varphi_v)$ has the expression

$$fpw\overline{W}_{\widehat{S}}(\varphi_v) = r^\lambda U_w(\Omega)\cos(\beta), \tag{31}$$

where the angle β is measured in plane Ox_2x_3 from the Ox_2-axis (2). The function $r^\lambda U_w(\Omega)$ in the plane Ox_1x_2 is two-dimensional harmonic function because it is the derivative of two-dimensional harmonic function in this plane (30).

Expression (31) has anti-symmetry by the plane Ox_1x_3 and symmetry by the plane Ox_1x_2. Therefore the two-dimensional harmonic function $r^\lambda U_w(\Omega)$ has the expression (1)

$$s400r^\lambda U_w(\Omega) = c_w r^\lambda \sin(\lambda(\Omega - \pi)), \tag{32}$$

where the constant c_w is unknown, $c_w \neq 0$.

Consequently,

$$\left.\frac{\partial \overline{W}_{\widehat{S}}(\varphi_v)}{\partial \Omega}\right|_{\substack{\beta=0\\ \Omega=\pi}} = r^\lambda \left.\frac{\partial U_w(\Omega)}{\partial \Omega}\right|_{\Omega=\pi} = \lambda c_w r^\lambda \cos(\lambda(\Omega-\pi))\Big|_{\Omega=\pi} = \lambda c_w r^\lambda. \tag{33}$$

Let us suppose $\varphi_v = r^\lambda$, $0 < \lambda < 1$ and consider under this guess the following derivative: [17]

$$\left.\frac{\partial \overline{W}_{\widehat{S}}(r^\lambda)}{\partial \Omega}\right|_{\substack{\beta=0\\ \Omega=\pi}} = -\frac{1}{2\pi}\int_0^\infty \frac{r b^{\lambda+1}}{(r+b)^3}\,db = -\frac{1}{4}\lambda(1+\lambda)\frac{r^\lambda}{\sin(\lambda\pi)}. \tag{34}$$

If we compare (34) with (33), we find that our guess is true: $c_w = -\dfrac{1+\lambda}{4\sin(\lambda\pi)}$. Finally, we get

$$\overline{W}_{\widehat{S}}(r^\lambda) = -(\lambda+1)r^\lambda\frac{\sin(\lambda(\Omega-\pi))}{4\sin(\lambda\pi)}\cos(\beta), \quad 0<\lambda<1, \quad 0 \le \Omega \le 2\pi, \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}. \tag{35}$$

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[15] For $0 < \lambda < 1$, after the mapping the value of λ in new expression (29) will be in the same range of values.

[16] The angle $d\eta$ is infinitesimal, the Newtonian potential on the sector (Figure 2) behaves as an (integral) sum of Newtonian sources located on the ray of its bisector. Therefore the potential field is axisymmetric, rotation of the sector around its bisector does not change the field. We can consider one sector as we consider two sectors below (41) with the same result: $\kappa = 0$.

[17] The following integral can be calculated analogously (16), footnote 10.

The two-dimensional harmonic functions $r^\lambda U_v(\Omega)$, $r^\lambda U_w(\Omega)$ of sector's $\widehat{S}$ (2) have relation between each other in the plane Ox_1x_2 (30) which is the same relation of the functions $V_L(r^\lambda)$, $W_L(r^\lambda)$ of the ray L (17 Expressions of the functions $r^\lambda U_w(\Omega)$ (32) and $W_L(r^\lambda)$ (15) are identical to the difference only in coefficient value. Consequently, the functions $r^\lambda U_v(\Omega)$, $V_L(r^\lambda)$ are identical with difference only in a value of this coefficient. Thus we get [18]

$$\overline{V}_{\widehat{S}}(r^\lambda) = \frac{r^{\lambda+1}}{4\sin(\lambda\pi)}\cos\big((\lambda+1)(\Omega-\pi)\big), \quad 0<\lambda<1, \quad 0\le\Omega\le 2\pi, \quad -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}. \tag{36}$$

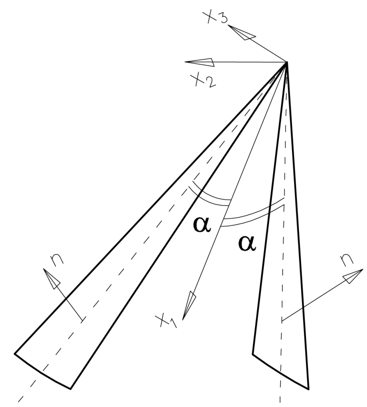


FIGURE 3. Two sectors (see 2) are in the plane Ox_1x_2.

Let us consider two sectors (3) which are the pair of sectors (2). Let us show that the potentials $\overline{V}$, $\overline{W}$, having values of density function in the plane Ox_1x_2 and the direction of normal vectors, same as the density function and the direction of normal vectors on the pair of rays of the expressions (19), (20), (21), (22), define the three-dimensional harmonic functions

$$\overline{u}_w^+(r,\theta,\beta) = -(\lambda+1)\frac{\sin(-\lambda\alpha)}{2\sin(\lambda\pi)}r^\lambda\cos(\lambda\theta), \tag{37}$$

$$\overline{u}_w^-(r,\theta,\beta) = -(\lambda+1)\frac{\cos(-\lambda\alpha)}{2\sin(\lambda\pi)}r^\lambda\sin(\lambda\theta)\cos(\beta), \tag{38}$$

$$\overline{u}_v^+(r,\theta,\beta) = \frac{\cos\big((\lambda+1)\alpha\big)}{2\sin(\lambda\pi)}r^{\lambda+1}\cos\big(\theta(\lambda+1)\big), \tag{39}$$

$$\overline{u}_v^-(r,\theta,\beta) = \frac{-\sin\big((\lambda+1)\alpha\big)}{2\sin(\lambda\pi)}r^{\lambda+1}\sin\big(\theta(\lambda+1)\big)\cos(\beta), \tag{40}$$

where $0<\lambda<1$, $\theta=\Omega-\pi$, $-\pi+\alpha\le\theta\le\pi-\alpha$, $-\pi/2\le\beta<\pi/2$.

Because of the above-described relations of the two-dimensional harmonic functions of the sector $\widehat{S}$ and the ray L in the plane Ox_1x_2 (3), expressions (37), (38), (39), (40) are identical to (19), (20), (21), (22) in this plane with difference only in coefficient value. As each of the functions (37), (38), (39), (40) is a three-dimensional harmonic function having the plane of symmetry Ox_1x_2 that has expression

[18]The potential $\overline{V}_{\widehat{S}}(r^\lambda)$ can be calulated explicitly in the continuation of the sector's bisector $\Omega=\pi$ (Figure 2)

$$\overline{V}_{\widehat{S}}(r^\lambda) = \frac{1}{2\pi}\int_0^\infty \frac{b^{\lambda+1}}{r+b}db = \frac{r^{\lambda+1}}{2\sin(\lambda\pi)}, \quad \text{where} \quad -2<\lambda<-1.$$

Additive constant is equal to zero in this expression therefore additive constant is equal to zero in the source expression (18): $C=0, C_1=0, C_2=0$. See footnotes (12) and the end of Section 2.

The additional conclusion of this result is justification of the supposition at the end of Section 2: the multiplier function in different ranges of λ has different forms as the coefficient of r^λ in the range $-2<\lambda<-1$ is doubled in comparison with the same coefficient in the range $0<\lambda<1$ (36). (Since the cited computer program shows the wrong ranges of λ, the author used [12, p. 360] for integration.)

(28) in which the two-dimensional harmonic function $r^\lambda U(\Omega)$ is already known, we have to find the coefficient κ only. [19] The β_1-mapping of expression (28) transforms the two-dimensional harmonic function $r^\lambda U(\Omega)$ into another two-dimensional harmonic function with same form and different value of λ, the function $\cos(\kappa\beta)$ not changes. Consequently, we can transform a pair of the sectors by the β_1-mapping into one sector in the plane Ox_1x_3 (3). After this mapping we have one sector with a double density function for the potential $\overline{W}$ when values of the density function are anti-symmetrical, consequently, the function $\cos(\kappa\beta)$ in (38) is equal to $\cos(\beta)$ as it is in the expression for one sector (35). By the analogy, the function $\cos(\kappa\beta)$ is equal to 1 in (39) when values of the density function are symmetrical due to (36).

Let us rewrite the functions in the form

$$\begin{aligned}
\overline{u}_w^+(r,\theta,\beta) &= \psi_1(r,\theta)\cos(\kappa_1\beta),\\
\overline{u}_w^-(r,\theta,\beta) &= \psi_2(r,\theta)\cos(\kappa_2\beta),\\
\overline{u}_v^+(r,\theta,\beta) &= \psi_3(r,\theta)\cos(\kappa_3\beta),\\
\overline{u}_v^-(r,\theta,\beta) &= \psi_4(r,\theta)\cos(\kappa_4\beta),
\end{aligned}$$

where $0<\lambda<1$, $-\pi+\alpha\le\theta\le\pi-\alpha$, $-\pi/2\le\beta<\pi/2$, the functions ψ_i are known, $\kappa_2=1$, $\kappa_3=0$, the coefficients κ_1, κ_4 are unknown. Let us find them.

$\diamondsuit$ If the function ξ is created by the integral (β_0 is measured as β)

$$\begin{aligned}
\xi(r,\theta,\beta) &= \int\limits_{-\pi/2}^{\pi/2}\overline{u}_w^+(r,\theta,\beta+\beta_0)d\beta_0 = \int\limits_{-\pi/2}^{\pi/2}\psi_1(r,\theta)\cos(\kappa_1(\beta+\beta_0))d\beta_0\\
&= \psi_1(r,\theta)\frac{1}{\kappa_1}\sin(\kappa_1(\beta+\beta_0))\Big|_{-\pi/2}^{\pi/2}, \qquad (41)
\end{aligned}$$

it has to have the axial symmetry by Ox_1. This symmetry exists if $\kappa_1=0$, or $\kappa_1=2n, n\in N$. For $\alpha>\pi/2$, the function $\overline{u}_w^+$ in the ranges of the angles does not change its sign. This is possible for $\kappa_1=0$ only. The β - mapping does not change the value of κ_1. Hence $\kappa_1=0$. Analogously, we can find: $\kappa_3=0$. The functions $\overline{u}_w^-,\overline{u}_v^-$ have one plane of symmetry, therefore: $\kappa_2=1$, $\kappa_4=1$. $\diamondsuit$

Finally, we get (37), (38), (39), (40).

In expression (28) for $\kappa=0$ (37), (39) $\left.\dfrac{U(\Omega)}{\partial\Omega}\right|_{\theta=0}=0$, for $\kappa=1$ (38), (40) $\left.U(\Omega)\right|_{\theta=0}=0$, consequently, the Laplace equation (27) with a solution in the form (28) does not degenerate in the axis $\theta=0$, $\Omega=\pi$. [20]

The expressions (37), (38), (39), (40) represent the three-dimensional harmonic function for $\lambda\neq n, n\in N$. When $\lambda=n$, $n\in N$, the denominator of the expressions converts to zero, consequently, in this case the expressions exist if they are equal to zero only. We have proved the existence of three-dimensional harmonic functions (3), (4) for $\lambda\neq n$, $n\in N$. [21]

$\diamondsuit$ The expression of the three-dimensional harmonic function (28) having only one plane of symmetry is equal in the plane to the two-dimensional harmonic function $r^\lambda U(\Omega)$ of the form (1), (2): $\kappa=1$; $\beta=0$ in the plane of symmetry. The derivative of expression (28) by r, having values of the derivative of (1), (2) by r in the plane of symmetry, does not change it's form when the value of λ changes, therefore the first summand in (27) does not change it's form, the second summand does not change its form because the derivative of the expression (28) by Ω, having values of derivative of (1),

[19]The first and second derivatives of (28) in the direction of the normal vector of the plane Ox_1x_2 are equal to zero because this plane is the plane of symmetry in the considered problems, therefore the function $r^\lambda U(\Omega)$ is two-dimensional harmonic function in expressions (28) of three-dimensional harmonic functions of the solutions of these problems.

[20]The points on the axis $\theta=\pi$ are not in the domain with the range of the angle θ: $-\pi+\alpha\le\theta\le\pi-\alpha$.

[21]The conformal mapping by a power function of two-dimensional harmonic functions involved in (37), (38), (39), (40) is carried out similarly to the two-dimensional case. See the end of Section 2.

(2) by Ω in the plane of symmetry, does not change its form. The third summand in (27) is equal to

$$\frac{1}{r^2\sin(\Omega)^2}\frac{\partial^2 u}{\partial\beta^2} = -\frac{\cos(\beta)}{r^2\sin(\Omega)^2}\left(r^\lambda U(\Omega)\right)$$

and does not change its form, as well. The conformal mapping by a power function with center at the point $r = 0$ maps one two-dimensional harmonic function (1), (2) into another with the same form and different value of λ. Consequently, from the existence of expression (3) for $\lambda \neq n$, $n \in N$, follows the existence of the three-dimensional harmonic function (3) for all values of λ, because (3) has one plane of symmetry with the relation of two-dimensional harmonic functions of the form $r^\lambda U(\Omega)$ in (28) by the conformal mapping in the plane of symmetry.

We can repeat the logic for the expression of three-dimensional harmonic function (28) having axial symmetry and find an analogous result: the existence of three-dimensional harmonic function (4) for all values of λ follows from the existence of expression (4) for $\lambda \neq n$, $n \in N$. [22] $\diamondsuit$

4. Solutions of Boundary Value Problems in the Vicinity of Angular or Conical Point as Mapping of Solutions for Domain with a Smooth Boundary

4.1. **Two-dimensional problems.** We compare below the integrals of (5) and the integrals of normal derivative of (5) with the same integrals on the boundary of a half plane. At the end of the subsection, these comparisons are used to determin the existence conditions of solutions.

Dirichlet problem

Let us consider the limiting expression (5) in a two-dimensional case. This was proved in Section 1 for $S \in C_1$, $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$ under the condition of radiation. Consequently, this is still true for a regular function u. [23] A regular two-dimensional harmonic function u in the vicinity of $p \in S$ in the polar system of coordinates with center in p has to have expression by a sum of functions (1), (2), $\lambda = n$, $n \in N$, and a constant. If we consider the integrals (19), (20), (21), (22) in (5) in the vicinity of $p \in S$, we find that (5) may present regular u in p when the function u is the sum of functions (1), (2) and a constant. The corresponding integrals (19), (20), (21), (22) on the half plane's boundary converge for $\lambda = n$, $n \in N$, $\alpha = \frac{\pi}{2}$, $|\theta| = \frac{\pi}{2}$ for zero boundary values at the points of the boundary of the half-plane. [24] If we perform conformal mapping by a power function with center in p, the part S_R of the S inside the circle with small radius and center in p transforms in a part of the infinite wedge's boundary, the integrals (19), (20), (21), (22) transform in the expressions for $\lambda \neq n$, $n \in N$, for the infinite wedge.

Because the conformal mapping is not conformal at one point p [9], we have to consider this point separately. The functions (1), (2) have zero values in p which not change after the mapping. Therefore we have to consider the constant function only. At the point p, the mapping of the constant can be calculated by an extended determination of Gauss's theorem and is determined by the value of aperture angle of the wedge. [25] At all other points, the mapping of constant is itself the constant. Consequently, after the mapping we will have expression (5) in which $\delta \neq 1$ at the point $p \in S$.

Neumann problem

The existence of limiting values of the normal derivative of u (5) in $S \in C_1$, $u \in C_2(\Theta) \cap C_1(\overline{\Theta})$ under the condition of radiation has been proved in Section 1. This is still true for a regular function u. If we consider (5) for $p \in \Theta\backslash S$ and limiting expression of normal derivative of u at approach the point p to the point of the normal vector in S, we find that the corresponding integrals of the normal derivative of (19), (20), (21), (22) exist (integrals converge) in the boundary of the half-plane for $\lambda = n$, $n \in N$,

[22] In this case the third summand in 27 is equal to zero.

[23] In the paper the term "regular function" is equivalent of "infinitely differentiable function".

[24] The corresponding integrals (19), (20), (21), (22) are equal to zero at $\alpha = \frac{\pi}{2}$, $|\theta| = \frac{\pi}{2}$ despite the denominator $\sin(\lambda\pi) = 0$.

[25] If $p \in S$ is angular or conical point in (8) the coefficient -1 will be replaced by $-\chi$, $\chi \neq 1$, $0 < \chi < 2$. In two-dimensional case χ is equal to the aperture angle of the wedge divided by π, in three-dimensional case χ is equal to a solid angle in the vertex of cone divided by 2π.

$\alpha = \dfrac{\pi}{2}$, $|\theta| = \dfrac{\pi}{2}$ for zero boundary values at the points of the boundary of the half-plane. It becomes obvious if we recall the relations of the derivatives

$$\frac{\partial}{\partial n}\bigg|_{\theta=\pi/2} = \frac{1}{r}\frac{\partial}{\partial\theta}\bigg|_{\theta=\pi/2}, \qquad \frac{\partial}{\partial n}\bigg|_{\theta=-\pi/2} = -\frac{1}{r}\frac{\partial}{\partial\theta}\bigg|_{\theta=-\pi/2},$$
$$\frac{\partial\sin(\theta)}{\partial\theta} = \cos(\theta), \qquad \frac{\partial\cos(\theta)}{\partial\theta} = -\sin(\theta), \tag{42}$$

owing to which in the expressions of normal derivatives of (19), (19), (20), (21), (22), ($\alpha = \pi/2$). [26] We can obtain conformal mapping by a power function with center at p of the limiting expression of the normal derivative of (5) for $\lambda \neq n$, $n \in N$, when S_R transforms in a part of the infinite wedge's boundary [27] and the expressions (19), (20), (21), (22) correspond to the integrals on infinite boundary of the infinite wedge. [28]

Dirichlet-Neumann mixed problem

Let the boundary S consist of two parts $S = S_u \cup S_t$, we have the Dirichlet boundary conditions on S_u and Neumann boundary conditions on S_t. Let the point p, $p \in S$, be a common point of S_u and S_t; S_R is part of S inside of the circle of small radius with center at p which we replace by a segment of straight line, since $S \in C_1$. If we suppose that the integrals (19), (20), (21), (22) correspond to the integrals of (5) on S_R we find that the solution of the mixed problem exists when the function $u \in L_2^{(1)}(\Theta)$ consists of the functions (1), (2) and has the expression

$$u(r,\theta) = Ar^{\lambda}\sin(\lambda\theta) - Ar^{\lambda}\cos(\lambda\theta), \tag{43}$$
$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad \lambda = \frac{4n+1}{2}, \quad n = 0, 1, 2, 3, dots,$$

where A is a constant; the part S_{Ru} of S_R belongs to S_u ($\theta = \dfrac{\pi}{2}$); the part S_{Rt} of S_R belongs to S_t ($\theta = -\dfrac{\pi}{2}$); $S_R = S_{Ru} \cup S_{Rt}$.

If we consider the integrals of (5) for (43), for points of S_{Rt} and values of the normal derivative of the integrals of (5) for the points of $S_{Ru}$with by using (42), we will find that values of the corresponding integrals of (19), (20), (21), (22) on the both parts of the boundary of the half-plane: on the part with the Dirichlet boundary conditions and on the part with Neumann boundary conditions, have to be equal to zero at the points of the boundary, only under these conditions there is a finite limiting value of normal derivative of u (5) at a point belonging to S_{Ru} for $0 < \lambda < 1$. [29]

Indeed, let the boundary of a half-plane consist of two rays with one vertex at the point p: the ray $\breve{S}_u$ for $\theta = \dfrac{\pi}{2}$ corresponds to the Dirichlet boundary conditions, the ray $\breve{S}_t$ for $\theta = -\dfrac{\pi}{2}$ corresponds to the Neumann boundary conditions. Let us consider the integrals in the right-hand side of (5) on the infinite boundary of the half-plane $W(u), V\left(\dfrac{\partial u}{\partial\Omega}\right)$ (43), at a point of observation belonging to the

[26]The corresponding integrals of norm,(21), (22) the $\cos(\theta)$ “swap” $\sin(\theta)$ and the $\sin(\theta)$ “swap” $\cos(\theta)$ in comparison with the initial expressions all derivative of (19), (20), (21), (22) are equal to zero for $\alpha = \dfrac{\pi}{2}$, $|\theta| = \dfrac{\pi}{2}$ despite the denominator $\sin(\lambda\pi) = 0$.

[27]The integrals (21), (22) of the constant density function create the functions of the form: cr, on the straight line $\widehat{S}$, $S_R \in \widehat{S}$, where c is the constant. These functions are equal to zero at p, $r = 0$, therefore this point has not to be considered separately under the conformal mapping.

[28]At the points of the wedge's boundary in two dimensions and at the points of wedge's boundary or cone's boundary in three dimensions there are expressions similar to (42) for $|\theta| \neq \dfrac{\pi}{2}$. Consequently, we do not need to consider the expressions of limiting values of the normal derivative at these points, since the boundary values of the derivative are determined by $\dfrac{\partial}{\partial\theta}$.

[29]The solution for the value $0 < \lambda < 1$ is most important in applications because it has singularity of the derivative $r^{\lambda-1}$ and belongs to $L_2^{(1)}(\Theta)$ (see footnote 8, [26, pp. 305,309]).

boundary of the half-plane:

$$\Big(-W_{\breve{S}_t}\big(-2Ar^{\lambda}\sin(\lambda\pi/2)\big) + V_{\breve{S}_t}(0) - W_{\breve{S}_u}(0) + V_{\breve{S}_u}\big(2Ar^{\lambda-1}\lambda\sin(\lambda\pi/2)\big)\Big)\Big|_{\theta=-\frac{\pi}{2}} = 0, \tag{44}$$

$$\frac{\partial}{\partial n}\Big(-W_{\breve{S}_t}\big(-2Ar^{\lambda}\sin(\lambda\pi/2)\big) + V_{\breve{S}_t}(0) - W_{\breve{S}_u}(0) + V_{\breve{S}_u}\big(2Ar^{\lambda-1}\lambda\sin(\lambda\pi/2)\big)\Big)\Big|_{\theta=\frac{\pi}{2}} = 0. \tag{45}$$

Equality (44) was obtained by (43) taking into account (42) and expressions (18) and (15); in expression (15), the sign corresponds to the approach to the boundary in direction of the normal vector (upper sign in (12), $\Omega = 2\pi$ in (15)). Equality (45) can be obtained by (44) taking into account (42).

From (45) follows the existence of limiting values of the normal derivative of (5) at the points of S_{Ru} for $0 < \lambda < 1$. Let us show that conditions (43) are exclusive ones for this existence and also existence of solution of the mixed problem. We can rewrite expression of (5) and expression of normal derivative of (5) in the form

$$u + W_{S_t}(u) - V_{S_u}\left(\frac{\partial u}{\partial n}\right) = -W_{S_u}(u) + V_{S_t}\left(\frac{\partial u}{\partial n}\right), \quad p \in S_t, \tag{46}$$

$$\frac{\partial u}{\partial n} + \frac{\partial}{\partial n}\big(W_{S_t}(u)\big) - \frac{\partial}{\partial n}\left(V_{S_u}\left(\frac{\partial u}{\partial n}\right)\right) = -\frac{\partial}{\partial n}\big(W_{S_u}(u)\big) + \frac{\partial}{\partial n}\left(V_{S_t}\left(\frac{\partial u}{\partial n}\right)\right), \quad p \in S_u. \tag{47}$$

We can obtain (47) as a limiting expression of the derivative in direction of normal vector n_{p1} of expression (5) at the point $p \in \Theta\backslash S$ ($\delta = 2$) as $p \to p_1$, $p_1 \in S(C_1)$, taking into account (9), (10). In the mixed problem, we know u on S_u and $\dfrac{\partial u}{\partial n}$ on S_t, thus (46), (47) is a system of resolving equations of the Dirichlet-Neumann mixed problem. In the resolving equation (47), at the points of S_{Ru}, $0 < \lambda < 1$, the divergent integral of u on S_{Rt} has to be "compensated" by the divergent integral of $\dfrac{\partial u}{\partial n_p}$ on S_{Ru}, for the existence of the equation, we can use only "half" of S_R for the integral with unknown density u, because another "half" is "migrated" in the right-hand side of equation (47). The integral of the second "half" of S_R with density $\dfrac{\partial u}{\partial n}$ has to "compensate" the integral of the first "half" with density u, thus the sum of the integrals may be finite in the resolving equation (47), this is possible for (45) only. Consequently, conditions (43) are exclusive for the existence of a solution of the mixed problem having singularity of derivative $r^{\lambda-1}$, $0 < \lambda < 1$. (The transition from the integrals on two rays (44), (45) to the integrals on S_R will be discussed below at the end of the subsection.) Any harmonic function (1), (2) $\lambda \neq n$, $n \in N$, has relation with a regular one through the conformal mapping by a power function, if we mean the harmonic function u has this relation as well we get (43).

Let us denote the boundary line of the half-plane as S_L and mean S_R as a part of the boundary of the half-plane inside of the circle of small radius with center in p, $p \in S_L$, coinciding with the part of S in (5) inside of the circle of small radius, with center at p, $p \in S$. As all three problems: the Dirichlet problem, the Neumann problem and the Dirichlet-Neumann mixed problem have the corresponding integrals (19), (20), (21), (22), having zero values at the points of the half- plane's boundary S_L, the conformal mapping of (5) by a power function with center in p, $p \in S$, corresponds to infinite wedge, having zero values at the points of its boundary.

Integrals (19), (20), (21), (22) on S_L under zero boundary conditions in all three problems have finite values at a point of the half=plane, where $r < \infty$. The parts of integrals (19), (20), (21), (22) on S_R have finite values at all points of the half-plane. [30] Consequently, the parts of the integrals (19), (20), (21), (22) on $S_L\backslash S_R$ have finite values in the vicinity of p, these parts of the integrals are harmonic functions which are infinitely differentiable at a point p_1, $p_1 \notin S_L\backslash S_R$. Consequently, the parts of integrals (19), (20), (21), (22) on S_R in the vicinity of p correspond to a solution of one of three problems for the half-plane under zero boundary conditions with addition of an infinitely differentiable function, after the mapping in the vicinity of p it corresponds to the solution for an

[30] We mean direct calculation of $V_{S_R}(r^{\lambda})$, $W_{S_R}(r^{\lambda})$.

infinite wedge under zero boundary conditions and addition of an infinitely differentiable function. The parts of integrals (5) on $S\backslash S_R$ before and after the mapping create in the vicinity of p an infinitely differentiable function, therefore the integrals of (5) after the mapping in the vicinity of p correspond to the solution for an infinite wedge for zero boundary conditions and addition of an infinitely differentiable function.

As a consequence, the described conditions are necessary for representation (5), $S \in C_1$, of a regular harmonic function near the considered point in the Dirichlet and Neumann problems. Since (5) is the representation of any harmonic function, satisfying the radiation condition, the described conditions are necessary for the existence of solutions of these two boundary problems. [31] In the Dirichlet-Neumann mixed problem the described conditions are necessary for the existence of the representation of limiting values on a part of the boundary, adjacent to a point of change of type of the boundary conditions for $u \in L_2^{(1)}(\Theta)$.

Since a domain with an angular point is connected by a conformal mapping by a power function with a domain with a smooth boundary, the existence conditions for the solutions of boundary problems for domain with an angular point are obtained through this mapping.

If the functions of solutions (5) of the Dirichlet and Neumann problems for the boundary S, $S \in C_1$, are regular or belong to $C_2(\Theta) \cap C_1(\overline{\Theta})$, the results of the mapping belong to $L_2^{(1)}(\Theta)$. The initial function of solution (5) of the Dirichlet-Neumann mixed problem for S, $S \in C_1$, and the result of the mapping together belong to $L_2^{(1)}(\Theta)$.

4.2. **Three-Dimensional Problems.** Since two-dimensional harmonic functions are special case of three-dimensional harmonic functions, all functions (1), (2), (3), (4) for $\lambda = n$, $n \in N$, and a constant are the forms of any regular three-dimensional harmonic function which has expression (5). [32] Because of relations (37), (38), (39), (40) with (19), (20), (21), (22), through the forms of two-dimensional harmonic function (1), (2), we can consider three-dimensional boundary value problems with using (5) and β-mappings as generalization of two-dimensional problems.

Let S_R be a part of S, $S \in C_1$, inside of the sphere of small radius and center at p, $p \in S$. As $S \in C_1$, we can replace S_R by the circle with center at p. If we repeat the reasoning of the previous subsection, we get: the potentials $\overline{W}_{S_R}(u)$, $\overline{V}_{S_R}\left(\dfrac{\partial u}{\partial n_p}\right)$ of (5) in the Dirichlet and Neumann problems correspond to solutions for the half-space under zero boundary conditions and an infinite differentiable function. The β_0-mapping of (5) for solutions of the Dirichlet and Neumann problems in the vicinity of p corresponds to solutions for an infinite wedge under zero boundary conditions and an infinitely differentiable function. The β_1-mapping of (5) for solutions of the Dirichlet and Neumann problems in the vicinity of p corresponds to solutions for an infinite cone under zero boundary conditions and

[31] Integrals (19), (20), (21), (22) of the solutions for a half-plane have density functions corresponding to the density functions of integrals (5) in p, $p \in S(C_1)$: in the Dirichlet and Neumann problems, the density functions correspond to the terms of Taylor's decomposition in p of regular u and regular $\dfrac{\partial u}{\partial n_p}$.

[32] If the density function in three dimensions $\breve{\varphi}$ has constant values in the direction of Ox_3-axis, there is the relation between potentials W and $\overline{W}$ at the point p located at the origin of the local system of coordinates $p(0,0,0)$:

$$\overline{W}(\breve{\varphi}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{L} \frac{n_{1q}x_{1q} + n_{2q}x_{2q}}{(x_{1q}^2 + x_{2q}^2 + x_{3q}^2)^{\frac{3}{2}}}\,\breve{\varphi}(q)dLdx_{3q} = \frac{1}{2\pi}\int\limits_{L} \frac{(n_{1q}x_{1q} + n_{2q}x_{2q})x_{3q}}{(x_{1q}^2 + x_{2q}^2 + x_{3q}^2)^{\frac{1}{2}}(x_{1q}^2 + x_{2q}^2)}\,\breve{\varphi}(q)dL_q\Big|_{x_{3q}=-\infty}^{x_{3q}=\infty}$$
$$= \frac{1}{\pi}\int\limits_{L} \frac{n_{1q}x_{1q} + n_{2q}x_{2q}}{x_{1q}^2 + x_{2q}^2}\,\breve{\varphi}(q)dL_q = W(\breve{\varphi}),$$

where L is a two-dimensional closed line in the plane Ox_1x_2, n is the normal vector of L. There is the same relation between $\dfrac{\partial \overline{V}(\breve{\varphi})}{\partial n_p}$ and $\dfrac{\partial V(\breve{\varphi})}{\partial n_p}$.

The integral of the potential $\overline{V}$ of an infinite straight line does not converges, therefore the potential V is defined through the relation of $\dfrac{\partial \overline{V}(\breve{\varphi})}{\partial x_i}$ and $\dfrac{\partial V(\breve{\varphi})}{\partial x_i}$, $i = 1, 2$, [28, pp. 351-353]. Because these derivatives are equal, the potentials of infinite straight line may have difference in a constant only despite the integral $\overline{V}$ cannot perform the expression of the potential of the line since it is not converges.

an infinitely differentiable function. For the β_0-mapping, the initial regular harmonic function has to be of the form (1), (2), for β_1-mapping it has to be of the form (3), (4).

The conditions of the existence of a solution of the three-dimensional Dirichlet-Neumann mixed problem for the wedge coincide with those of the two-dimensional problem, and the β_0-mapping maps a wedge into the other one with a different aperture angle.

A solution of the three-dimensional mixed Dirichlet-Neumann problem for a cone does not exist, since the sum of nonzero functions (3), (4) having zero values of derivative by θ of the sum in a part of the cone's boundary with nonzero square does not exist.

Indeed, if we consider the equation analogous to the two-dimensional one (47) in three-dimensional case for $0 < \lambda < 1$, we find that the integrals analogous to (47) in the left-hand side must "compensate" each other on S_R, since the integrals diverge separately. Consequently, in the right-hand side the integrals must have zero densities on S_R, in other case, the "compensation" for two three-dimensional sectors (35), (36) at the boundary of a half-space in the left-hand side does not occur analogously to (44), (45) for two rays (15 (18). The sum of nonzero functions (3), (4), equal to zero on S_{R_u}, having zero values of the normal derivative on S_{R_t}, does not exist (42), $S_R = S_{R_u} \cup S_{R_t}$, hence there is no resolving equation in the three-dimensional case for a cone for $0 < \lambda < 1$ analogous to (47). Since (3), (4) for $0 < \lambda < 1$ have relation through the β_1-mapping with (3), (4) for $\lambda \neq n$, $n \in N$, the resolving equation for a cone does not exist for these values of λ. The values $\lambda = n$, $n \in N$, correspond to solutions of the Dirichlet or Neumann problem in the boundary of the half-space, the "compensation" of sectors in the Dirichlet-Neumann mixed problem is impossible. Finally, we get: the resolving equation in the three-dimensional case for a cone, analogous to (47), does not exist.

The rest reasoning of representation of solutions of the Dirichlet, Neumann problems and the Dirichlet-Neumann mixed problem by (5) in three dimensions is analogous to the reasoning for two dimensions.

If three-dimensional functions of solutions (5) of the Dirichlet and Neumann problems for the boundary S, $S \in C_1$, are regular or belong to $C_2(\Theta) \cap C_1(\overline{\Theta})$, the results of the mapping belong to $L_2^{(1)}(\Theta)$. The initial function of solution (5) of the Dirichlet-Neumann mixed problem for S, $S \in C_1$, and the result of the β_0-mapping together belong to $L_2^{(1)}(\Theta)$.

5. Some Effects of the Model of an Ideal Incompressible Fluid

Expression (5) is true for any regular three-dimensional harmonic function under the condition of radiation. As is stated above, a regular three-dimensional harmonic function is a sum of functions (1), (2), (3), (4) and a constant, all these forms of three-dimensional harmonic functions include the two-dimensional harmonic functions (1), (2) as multiplier. The two-dimensional harmonic functions (1), (2) for $\lambda = n$, $n \in N$, have relations with the functions of the same forms for $\lambda \neq n$, $n \in N$, through the conformal mapping, consequently, three-dimensional functions (1), (2), (3), (4) for $\lambda = n$, $n \in N$, have relations with the functions of the same forms for $\lambda \neq n$, $n \in N$, through the β_0-mapping or β_1-mapping. Therefore expression (5) of a regular three-dimensional harmonic function has to have relation with nonregular three-dimensional harmonic function through the β_0-mapping or β_1-mapping. [33] If the local geometry of S in (5) in the vicinity of $p \in S$ is not a wedge or a cone, expression (5) cannot be the expression of nonregular in p three-dimensional harmonic function, thus the geometry of the S admits a regular at p solution of a boundary value problem only. [34]

Let us consider the surface (4). Since the local geometry of the surface (4) at a point belonging to $]DE[$ is not a wedge or a cone, a solution of the boundary value problem in the vicinity of the point is a regular function when the surface (Figure 4) is part of S in (5). The situation is the same in the vicinity of P when surface (5) is a part of S in (5). Consequently, in the framework of the model of an ideal incompressible fluid the sharp edges of surfaces (4), (5) are the alternative of rounded edges. Moreover, if the regular three-dimensional harmolic function u is the function of velocity in some direction which is a derivative in this direction of another regular harmonic function of the potential

[33] The conformal mapping by a power function at the point $r = 0$, is not conformal. See footnote (25).

[34] Despite the fact that the solution of the Dirichlet-Neumann mixed problem (5), $S \in C_1$, is a nonregular function, the β_0-mapping can transform it into a regular function which has no physical sense.

of the potential field, the function u in the vicinity of the points of $]DE[$ (4) and in the vicinity of the point P (5) satisfies the conditions: $u = 0$, $\dfrac{\partial u}{\partial n} = 0$. Indeed, a regular three-dimensional harmonic function u is a sum of functions (1), (2), (3), (4) for $\lambda = n$, $n \in N$, and constant, equal to zero, [35] has expression (5) in which the potentials $\overline{V}_{S_R}\left(\dfrac{\partial u}{\partial n}\right)$, $\overline{W}_{S_R}(u)$ may create a regular harmonic function at these points, if only $S_R \in C_1$, where S_R is a part of S inside of the sphere with of small radius, with center in the considered point. Because at any of the considered points $S_R \notin C_1$ (Figure 4),(Figure 5) and the potentials $\overline{V}_{S\setminus S_R}\left(\dfrac{\partial u}{\partial n}\right)$, $\overline{W}_{S\setminus S_R}(u)$ of (5) create a regular harmonic function at the point, the regularity of u (5) at the point can exist if $u(p) = 0\Big|_{p\in S_R}$, $\dfrac{\partial u(p)}{\partial n} = 0\Big|_{p\in S_R}$ only. [36]

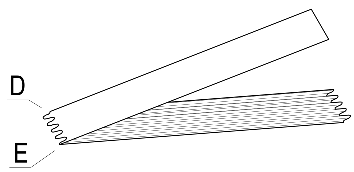


FIGURE 4. Surface with a sharp edge $]DE[$.

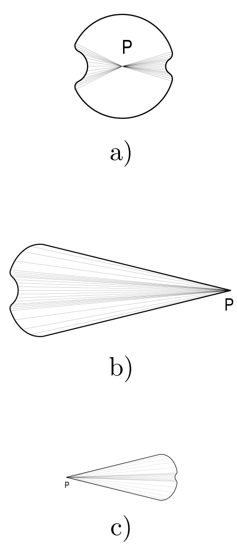


FIGURE 5. Three viewings of surface (a, b, c) with a sharp edge at the point P.

In the ideal fluid model, velocity of the potential flow is a gradient of harmonic function [13]. Despite the fact that this model is the simplest one, it has applications in numerical calculations of

[35]The constant is equal to zero because the function u is a derivative of another regular harmonic function of potential of the potential field which has the same form. The constant differentiation disappears.

[36]We provide regularity of u at the considered point only. The function u, $u \neq 0$, cannot be regular at all poins of $\Theta \cup S$ (5), because in this case the logic can be repeated for all next derivatives of potential of the potential field, since the expression (5) exists for each of them. This is possible if the potential of the potential field is a constant in the local domain of the Θ near the sharp edge, consequently in this case the potential of the potential field is the constant in the whole domain Θ. The boundary conditions for the existence of solutions of the boundary value problems in $L_2^{(1)}(\Theta)$ are discussed in the next section.

the airplane wing [1]. The expressions of harmonic functions by potentials of simple and double layers are used there [28, pp. 146,175]. The problem of impermeability of the wing by the potential flow is the external Neumann problem.

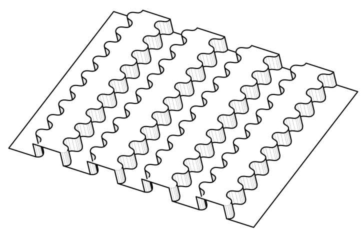


FIGURE 6. Example of a surface, in the vicinity of which the initially appearing vortex in a potential flow is delayed or prevented in the framework of an ideal fluid model.

Let us consider the surface (Figure 6). If it is a part of the boundary of a body in a gas flow or liquid, the surface is in some sense better, than a smooth surface. The velocity in the vicinity of sharp edges is a regular function in the framework of the model, the potentials of expression (5) of the velocity have a zero density function there. Therefore this surface behaves as a body with voids, as a lattice, the breaks of which are on the sharp edges, because in sense of the model the potentials with a zero density function are equivalent to the "absence of boundary" there.

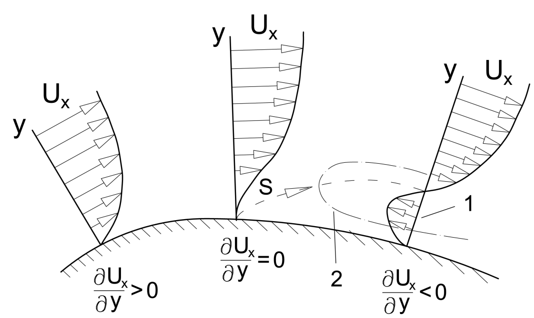


FIGURE 7. Schema of vortex emerging. 1 - zone of backward flow, 2 = vortex, U - velocity, S is a point of vortex formation.

These voids act against the powers, which are created by the initial vortex (Figure 7) [11]. The point S of maximum of the harmonic function of potential of the potential field, in which velocity U_x changes the sign (Figure 7), may be in the boundary of the domain only because of the "principle of maximum", but it cannot be near the voids (Figures 6,8). [37] [38]

Evidently, because of the regularity of solutions in the vicinity of the sharp edges of this type (Figures 4,5), they can be the alternative for rounded edges. It should be pointed out that the region

[37]The flow ceases to be a potential one from the moment when the point S "moves away" from the boundary (Figure 7). In theory, a surface with spaced arrays of sharp edges of this type serves in a potential flow for delaying or preventing the formation of initial vortex.

[38]Likely, special applications described in the patents: US5540406, US20090304511, US8256846, US5171623, US2261558, US5378524, US5289997, US8141936, US4776535 are based on this effect.

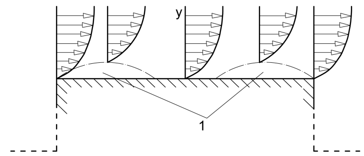
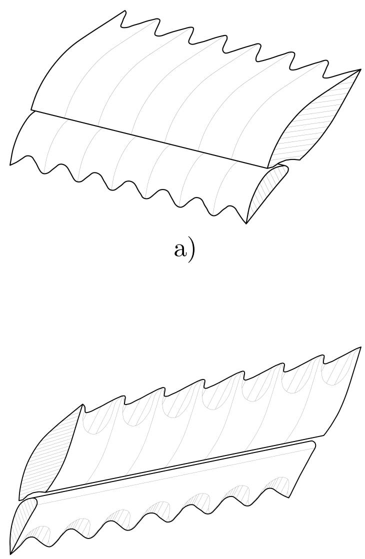


FIGURE 8. Schema of supposed distribution of velocity U_x in a cross-section of a part of the surface (Figure 6) 1 - zone of the zero velocities.

of resistance is smaller, than that of a rounded edge, therefore theoretically the resistance of flow can be decreased in the framework of the model. Let us consider few examples.



a)

b)

FIGURE 9. Prototype an airplane wing (a, b).

The prototype of an airplane wing with sharp edges of type (Figure 4) is shown in (Figure 9). [39] As it has been shown above, in the vicinity of sharp edges of this wing the formation of initial vortex in potential flow is impossible in the framework of the ideal fluid model. Theoretically, the resistance of the flow can be decreased in comparison with rounded edges.

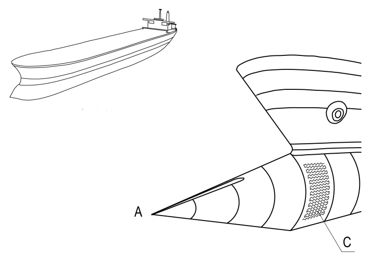


FIGURE 10. Prototype of bulbous bow for a bulk carrier.

[39]In trailing edge of the wing of this form, the Chaplygin-Zhukovskii hypothesis (Kutta condition) is true because of the described effect.

Possible application of the surface (Figure 5) at point A for a leading sharp edge of bulbous bow for a bulk carrier is shown in (Figure 10). Theoretically, the resistance of flow can be decreased in comparison with a rounded edge without appearance of vortexes. In the region marked by letter C, where formation of initial vortex is possible because of the pressure drop, the surface (Figure 6) can be used to prevent or delay the occurrence of this vortex.

6. Boundary Conditions on Piecewise Smooth Boundary for Solutions of Boundary Value Problems in $L_2^{(1)}$

Let $\widetilde{S}$ be a closed piecewise smooth boundary of a simply-connected domain $\widetilde{\Theta}$ in the form of a union of surfaces S_i, $i = 1, \ldots, N$, $S_i \in C_1$. It follows from the above discussion that the form of harmonic function $L_2^{(1)}(\widetilde{\Theta})$ for a solution of the Dirichlet and Neumann problems in polar coordinates in a two-dimensional case and in spherical coordinates in a three-dimensional case at any point of the piecewise smooth boundary $\widetilde{S}$ is determined by representations (1), (2) and (3), (4), respectively, and the existence of conformal mapping by a power function or β-mapping to regular function for which representation (5) exists. For the mixed problem, the form of this function is determined by the existence of the mapping to the solution function of the problem with a smooth boundary at the considered point and the corresponding representation (5).

In the two-dimensional case boundary values of the solution in the vicinity of any point of $\widetilde{S}$ has to be the values of the sum of the functions (1), (2), $\lambda > 0$, and a constant. These functions (1), (2) are the solution of the boundary value problem for an infinite wedge with aperture angle τ under zero boundary conditions, where the angle τ is the internal angle under which the tangent lines to $\widetilde{S}$ intersect at the considered point of $\widetilde{S}$. On a smooth part of the $\widetilde{S}$ $\tau = \pi$. (See 4.1.)

In the three-dimensional case, boundary values of the solution in the vicinity of any point of $\widetilde{S}$ has to be the values of representation by one of the four following variants.

1) The sum of functions of the form (1), (2), $\lambda > 0$, and a constant if a part of the surface $\widetilde{S}$ inside of the sphere of small radius and center at the considered point tends to the wedge surface whose wedge includes the considered point when the radius tends to zero. The functions (1), (2) are the solution of the boundary-value problem for an infinite wedge for zero boundary conditions. (This corresponds to the two-dimensional case.)

2) The sum of functions of the form (3), (4), $\lambda > 0$, and a constant if a part of the surface $\widetilde{S}$ inside of the sphere of small radius and with center at the considered point tends to the surface of the cone whose vertex is at the considered point; any cone generatrix is tangent to $\widetilde{S}$ at the considered point. The functions (3), (4) correspond to the solution for infinite cone at zero boundary conditions. (See 4.2.)

3) The sum of the functions (1), (2), (3), (4), $\lambda > 0$, and a constant if the considered point is located on the smooth part of $\widetilde{S}$. The functions (1), (2) correspond to the solution for infinite wedge with aperture angle π at zero boundary conditions. The functions (3), (4) correspond to the solution for an infinite cone with aperture angle π at zero boundary conditions.

4) A constant if the conditions of each of the variants 1-3 are not fulfilled at the considered point of $\widetilde{S}$. [40] (Examples of the points of variant 4) are the points of the line DE (4) and the point P (5). Another example is the point of vertex of a pyramid.) [41]

[40] If the boundary $\widetilde{S}$ includes the points of variant 4), footnote 36 has to be taken into account. When the solution function is infinitely differentiable at all points of $\widetilde{\Theta} \cup \widetilde{S}$, each of its derivatives is a harmonic function having expression (5). Consequently, all derivatives of the solution in direction of the normal vector to $\widetilde{S}$ in vicinity of a point of the variant 4) are equal to zero, all tangents to $\widetilde{S}$ derivatives are equal to zero there, as well. Therefore the solution function is a constant in the vicinity of the considered point, because the first term of an infinite Taylor's decomposition of the function is not equal to zero only. Thus this solution function is a constant in the whole $\widetilde{\Theta}$.

[41] In all four variants the factor δ in expression (5) is equal to the value of the solid angle at the considered point of $\widetilde{S}$ divided by 2π.

For convergence of results of the algorithm of Boundary element method, when the total number of the boundary elements increases at the decreasing of characteristic size of elements, it suffices to make approximation of functions (1), (2), (3), (4), $0 < \lambda < 2$, and a constant at all points of $\widetilde{S}$. Functions (1), (2), (3), (4) for $0 < \lambda < 1$ may have singularity of derivative only, if this feature is approximated, the numerical results are refined, when the characteristic size of the elements decreases.

Statements of this section can be repeated if domain $\widetilde{\Theta}$ is a complement of some simply connected domain with respect to the plane in a two-dimensional case or to the space in a three-dimensional case and the radiation condition is satisfied.

7. Conclusion

We have obtained expressions (3), (4) of harmonic functions in the three-dimensional case. These representations of harmonic functions in three dimensions could be used as an alternative to the well-known Legendre functions and can be applied to various fields of mathematics and technology with the benefit of a more simple form.

The expressions of the summands with possibly infinite derivative in the solutions of the Dirichlet, Neumann and Dirichlet-Neumann mixed problems in $L_2^{(1)}$ in the vicinity of sharp edges of piecewise smooth boundary by potentials of simple or double layer are proposed. These allow us to exclude the key shortcomings of traditional formulation of the Method of potential (traditional form of the Boundary element method). These expressions will allow us to formulate the method for piecewise smooth boundary and mixed boundary conditions. It opens up opportunities for simulation applications of the Laplace equation and can be generalized to solve other equations, solutions of which have representation in the form of a combination of harmonic functions, for example, for the equations of the theory of elasticity.

The suggested technique will bring about simplicity of numerical algorithms and proximity to the analytical methods which would increase the calculation accuracy. This factor presents a definite benefit over other methods and makes it highly competitive.

The found parallel between solutions for smooth and piecewise smooth boundaries in the vicinity of angular or conical points eliminates the key shortcomings of traditional BEM. This relationship is maintained through the conformal mapping in the two-dimensional case or β-mapping in the three-dimension case. The Taylor series of presentations of density functions of potentials in the expression of solution of the Dirichlet or Neumann problem by a sum of potentials of simple and double layers on a smooth boundary are transformed after the mapping into the series of density functions, the potentials of which present the series of Kondrat'ev solutions of these problems in the vicinity of angular or conical point.

Some types of sharp edges allow only regular solutions, and the suggested technique shows a possible alternative to rounding. This alternative may be used to improve the efficiency of technical embodiment in many areas.

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A BRIEF NOTE ON QUATERNION ANALYSIS

O. DZAGNIDZE

Abstract. This paper presents a brief historical survey of quaternion functions of a quaternion variable.

It has long been known that holomorphic (analytic) functions of one complex variable have numerous applications in solving various important problems of natural sciences.

Because of these applications and an active mathematical interest shown in them there arose the problem whether there exists or not an analogous theory for functions depending on three and more real variables.

This problem for functions with four real variables x_0, x_1, x_2, x_3 was studied for over a decade by William Rowan Hamilton (1805-1865). In 1843, Hamilton introduced into consideration the quaternions $z = x_0 i_0 + x_1 i_1 + x_2 t_2 + x_3 i_3$ with the real unit $i_0 = 1$ and three imaginary units i_1, i_2, i_3, having the properties $i_1^2 = i_2^2 = i_3^2 = -1$, $i_1 i_2 = -i_2 i_1 = i_3$, $i_2 i_3 = -i_3 i_2 = i_1$, $i_3 i_1 = -i_1 i_3 = i_2$.

Therefore the product of quaternions depends on the order of succession of cofactors, i.e. the product does not possess the property of commutativity. That is why two equations

$$q_2 x = q_1 \tag{1}$$

and

$$x q_2 = q_1 \tag{2}$$

are considered for quaternions. The solution of equation (1) is called the left quotient of division of q_1 by q_2 and here we denote it by the symbol $\vdash\!\frac{q_1}{q_2}$ (the numerator is not allowed to move to the left). Analogously, the right quotient of division of q_1 by q_2 is called the solution of equation (2) denoted by the symbol $\frac{q_1}{q_2}\!\dashv$ (the numerator is not allowed to move to the right). For $q_1 = 1$ we see that each $q_2 \neq 0$ has an inverse quaternion $\frac{\overline{q}_2}{|q_2|^2}$, $|q| = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{1/2}$, $\overline{q} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3$. Hence we have

$$\frac{q_1}{q_2}\!\dashv = q_1 \cdot \frac{1}{q_2} \tag{3}$$

and

$$\vdash\!\frac{q_1}{q_2} = \frac{1}{q_2} \cdot q_1. \tag{4}$$

It should be said that in 1837 Janos Bolyai (1802–1860) wrote a treatise on the theory of imaginary values which he sent to Leipzig University to participate in the announced students' competition. Unfortunately the jury adopted a negative decision.

In the middle of the 19th century, Hamilton set the problem: for quaternions functions of a quaternion variable to develop a theory analogous to the theory of holomorphic functions of one complex variable.

As to this problem, we know that several not quite successful attempts were undertaken to solve it. The problem was considered for the first time by A. N. Kryloff (1863–1945) [6] who, using equalities (3) and (4), introduced into consideration the right and the left derivative at the point z for the function f by means of the equalities

$$A(z) = \lim_{h \to 0} [f(z+h) - f(z)] \cdot \frac{1}{h}, \quad B(z) = \lim_{h \to 0} \frac{1}{h} \cdot [f(z+h) - f(z)]$$

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Key words and phrases. Quaternion functions; H-derivative.

respectively. It turned out that [7], $A(z)$ exists only for functions of the form $az+b$, $B(z)$ exists in the case of functions of the form $za+b$ and $A(z)=B(z)$ only for functions of the form $rz+b$ where the quaternions a and b are constant with respect to z, and r is a real number. Hence the derivative method is not suitable for quaternion functions. An analogous opinion exists about the polynomial method [1,5]. This perhaps led G. E. Shilov to make the following statement: "The dream of Hamilton was to create the theory of functions of a quaternion variable. However the hopes set on quaternions did not come true" – G. E. Shilov, Mathematical Analysis-Functions of Several Real Variables, Parts 1 and 2, Nauka, Moscow, 1972, p. 385.

In 1985, D. Solomentsev, the head of the mathematical analysis sector of the Mathematical Synopses Journal, advised me not to get involved in the study of Hamilton's problem because no serious reviewer could be found not in any country of the world.

Following this advice, for several years my research had been focused on problems of the existence of bihedral angular limits of partial derivatives of the spherical Poisson integral and on finding a necessary and sufficient condition for the existence of the total differential of a function of many variables (the results appeared in press).

However later I renewed the work on Hamilton's problem and finished it in a certain sense in 2009. The results obtained by me on the existence of the operation of differentiation for quaternion functions of a quaternion variable were submitted for publication in the form of a paper to an American mathematical journal. Two years after I received from the editor of that journal an enthusiastic letter with the attached reviewer's complimentary report. The editor advised me to get acquainted with the work of two Italian mathematicians and asked to inform him of my opinion. The results of the Italian colleagues turned out to be the newly obtained findings of A. N. Kryloff's results. I sent this information to the editor and soon from the new editor of the same journal I received a refusal to publish my paper.

I was surprised by this fact and told about it to Academician Hverdri Inassaridze who in his student years made a report on quaternion numbers at a session of the circle under the mechanical-and-mathematical department of Tbilisi State University and who advised me to submit my paper to Tbilisi Mathematical Journal of which he was the editor-in-chief. The paper was published [1] and presently I also have the survey paper [5] published in the Journal of Mathematical Sciences.

In these works, for quaternion functions of a quaternion variable z the notion of a H-derivative is introduced (in honor of Hamilton) and the following results are established:

1) formulas $(z^n)'=nz^{n-1}$, $(e^z)'=e^z$, $(\sin z)'=\cos z$, $(\cos z)'=-\sin z$;

2) rules of H-differentiation $(f+\varphi)'=f'+\varphi'$, $(f\cdot\varphi)'=f'\cdot\varphi+f\cdot\varphi'$, $\left(\frac{1}{\varphi}\right)'=-\frac{1}{\varphi}\cdot\varphi'\cdot\frac{1}{\varphi}$, $\left(\frac{f}{\varphi}\!\dashv\right)'=f'\cdot\frac{1}{\varphi}-f\cdot\frac{1}{\varphi}\cdot\varphi'\cdot\frac{1}{\varphi}$, $\left(\vdash\!\frac{f}{\varphi}\right)'=-\frac{1}{\varphi}\cdot\varphi'\cdot\frac{1}{\varphi}\cdot f+\frac{1}{\varphi}\cdot f'$. It should be noted that the equality for $\left(\frac{1}{\varphi}\right)'$ was established by me only after establishing the equality $\frac{1}{z_1}(z_1-z_2)\frac{1}{z_2}=\frac{1}{z_2}(z_1-z_2)\frac{1}{z_1}$ which, after simplification, yields the equality $\frac{1}{z_2}-\frac{1}{z_1}=\frac{1}{z_2}-\frac{1}{z_1}$. In addition to these results:

3) the necessary and sufficient condition for the existence of a H-derivative is obtained;

4) for a $\mathbb{C}^2$-holomorphic in the domain D quaternion function, its integral representation and its expansion into a power series with respect to two complex variables are established;

5) A. Moivre's (1667–1754) formula is obtained by using the imaginary unit I_z with the property $I_z^2=-1$ that depends on the variable z. It should be noted that F. G. Frobenius (1849–1917) established that for functions of three real variables there exists no theory analogous to the theory of holomorphic functions of one complex variable.

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SEMISIMPLICITY SETS FOR CYCLIC ELEMENTS IN SIMPLE LIE ALGEBRAS

A. ELASHVILI[1], M. JIBLADZE[1], AND V. KAC[2]

Abstract. This paper is a continuation of the theory of cyclic elements in semisimple Lie algebras, developed by Elashvili, Kac and Vinberg. We classify semisimple cyclic elements in terms of various nonassociative algebra structures on certain subspaces of the corresponding Lie algebra. The importance of such classification stems from the fact that each such element gives rise to an integrable hierarchy of Hamiltonian PDE of Drinfeld-Sokolov type.

1. Introduction

Let us recall that for an element a of a Lie algebra $\mathfrak{g}$, the *adjoint action operator* $\operatorname{ad} a : \mathfrak{g} \to \mathfrak{g}$ is given by

$$(\operatorname{ad} a)x = [a, x], \qquad x \in \mathfrak{g}.$$

The element a is called *semisimple* if the operator $\operatorname{ad} a$ is diagonalizable, and *nilpotent* if $\operatorname{ad} a$ is a nilpotent operator, i. e. if $(\operatorname{ad} a)^n = 0$ for some n.

Consider a semisimple finite-dimensional Lie algebra $\mathfrak{g}$. To each nilpotent element $e \in \mathfrak{g}$ corresponds a *grading* of $\mathfrak{g}$, i. e. a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{j=-d}^{d} \mathfrak{g}_j, \text{ where } \mathfrak{g}_{\pm d} \neq 0$$

of $\mathfrak{g}$ with the property $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. It is obtained as follows: by the Morozov–Jacobson theorem, there exists an $\mathfrak{sl}(2)$-*triple* $\mathfrak{s} = (e, h, f)$ for e, i. e. another nilpotent element $f \in \mathfrak{g}$ such that $[e, f] = h$ is semisimple and satisfies $[h, e] = 2e$ and $[h, f] = -2f$. One then defines $\mathfrak{g}_j := \{x \in \mathfrak{g} \mid [h, x] = jx\}$. The positive integer d is called the *depth* of the nilpotent element e.

Elements of the form $e+F$ for $F \in \mathfrak{g}_{-d}$ are called *cyclic elements* for e. Classification of semisimple cyclic elements is interesting by (at least) two different reasons. First, such elements can be used to understand the structure of regular elements in Weyl groups [3, 7, 10]. Second, such elements give rise to integrable Hamiltonian hierarchies of partial differential equations [1, 2].

2. Singular Sets of Nilpotent Elements

For a nilpotent e in a simple Lie algebra $\mathfrak{g}$, we call

$$\boldsymbol{\Sigma}_{\mathfrak{g}}(e) = \{F \in \mathfrak{g}_{-d} \mid e + F \text{ is not semisimple}\}$$

the *singular set* of e.

Our task is the description of these sets.

Our work is continuation of [3], where systematic classification of nilpotent elements from the above point of view has been undertaken.

If there exists a semisimple cyclic element $e+F$, then the nilpotent $e \in \mathfrak{g}$ is said to be of *semisimple type.*

If $e + F$ is nilpotent for every $F \in \mathfrak{g}_{-d}$, then e is of *nilpotent type.*

If none of these is true for e, then it is of *mixed type.*

Thus for elements of mixed or nilpotent types, the set $\boldsymbol{\Sigma}_{\mathfrak{g}}(e)$ coincides with the whole of $\mathfrak{g}_{-d}$, and the interesting case for us is that of e of semisimple type.

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In [3] nilpotents of each of these types have been completely described. One of the central notions in that paper are the notion of a *reducing subalgebra* and of *rank* for a nilpotent e.

For an $\mathfrak{sl}(2)$-triple $\mathfrak{s}$ for e as above, let $Z(\mathfrak{s})$ denote the centralizer of $\mathfrak{s}$ under the action of the adjoint group G of $\mathfrak{g}$.

Definition 2.1. A semisimple subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$ is called *reducing* for e, if it is normalized by $\mathfrak{s}$ and moreover $Z(\mathfrak{s})\mathfrak{q}_{-d}$ is Zariski dense in $\mathfrak{g}_{-d}$.

The *rank* $\operatorname{rk} e$ of e is the smallest possible dimension $\dim \mathfrak{q}_{-d}$ for any reducing subalgebras $\mathfrak{q}$ for e.

A nilpotent $e \in \mathfrak{g}$ is called *irreducible* if it does not admit any reducing subalgebras different from $\mathfrak{g}$.

Note that in particular for an irreducible nilpotent e we have $\operatorname{rk} e = \dim \mathfrak{g}_{-d}$.

For us, reducing subalgebras are crucial as they enable us to give description of the singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ in terms of $\mathbf{\Sigma}_{\mathfrak{q}}(e)$, for smallest possible reducing subalgebras $\mathfrak{q}$ for e. When e is of semisimple type, it is irreducible in such $\mathfrak{q}$, so in a sense all kinds of singular sets can be described in terms of singular sets for irreducible nilpotents. It turns out that there are very few cases of irreducible nilpotents. These are as follows ($k \geqslant 1$):

Table 1: Irreducible nilpotents of semisimple type

#	$\mathfrak{g}$	orbit	depth	rank	$Z(\mathfrak{s})\vert\mathfrak{g}_{-d}$
1_k	$\mathfrak{sl}(2k+1)$	$[2k+1]$	$4k$	1	1
2_k	$\mathfrak{sp}(2k)$	$[2k]$	$4k-2$	1	1
3_k	$\mathfrak{so}(2k+1)$	$[2k+1]$	$4k-2$	1	1
4_k	$\mathfrak{so}(4k+4)$	$[2k+3, 2k+1]$	$4k+2$	2	1
5	G_2	G_2	10	1	1
6	F_4	F_4	22	1	1
7	F_4	$\mathrm{F}_4(a_2)$	10	2	π_2
8	E_6	$\mathrm{E}_6(a_1)$	16	1	1
9	E_7	E_7	34	1	1
10	E_7	$\mathrm{E}_7(a_1)$	26	1	1
11	E_7	$\mathrm{E}_7(a_5)$	10	3	π_3
12	E_8	E_8	58	1	1
13	E_8	$\mathrm{E}_8(a_1)$	46	1	1
14	E_8	$\mathrm{E}_8(a_2)$	38	1	1
15	E_8	$\mathrm{E}_8(a_4)$	28	1	1
16	E_8	$\mathrm{E}_8(a_5)$	22	2	π_2
17	E_8	$\mathrm{E}_8(a_6)$	18	2	σ_2
18	E_8	$\mathrm{E}_8(a_7)$	10	4	σ_4

The last column of the table shows that in all these cases the group $Z(\mathfrak{s})|\mathfrak{g}_{-d}$ is finite: here π_n, resp. σ_{n-1} denotes the permutation representation, resp. the $n-1$-dimensional irreducible subrepresentation, of the symmetric group S_n.

Thus for irreducible e, only ranks $\leqslant 4$ occur, and in these cases it turns out that we have

Theorem 2.2. *For an irreducible nilpotent e with $\operatorname{rk} e = r$, the singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ is a union of exactly $\frac{r(r+1)}{2}$ distinct $r-1$-dimensional linear subspaces of $\mathfrak{g}_{-d}$.*

Moreover for any given irreducible e the singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ can be explicitly determined.

Example 2.3. Let $\mathfrak{g}$ be the simple Lie algebra of type D_{2k}, i. e. isomorphic to the algebra $\mathfrak{so}(4k)$ of $4k \times 4k$ skew-symmetric matrices with the Lie bracket given by commutators of matrices. Let e be a nilpotent corresponding to the partition $(2k+1, 2k-1)$, i. e. the one which acts on the standard $4k$-dimensional representation of $\mathfrak{g}$ via the matrix with two Jordan blocks, of sizes $2k+1$ and $2k-1$.

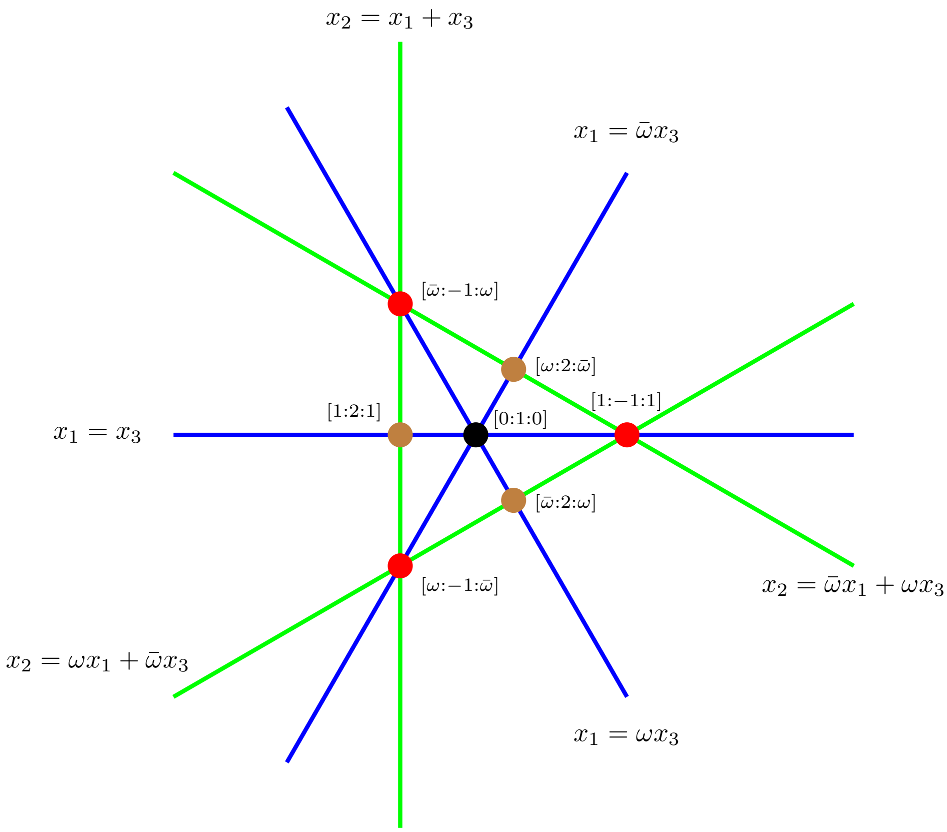
For this e we have $\dim \mathfrak{g}_{-d} = 2$, and in the root vector basis (F_1, F_2) of $\mathfrak{g}_{-d}$ the set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ consists of three lines given by scalar multiples of F_2, $F_1 + F_2$ and $F_1 - F_2$ respectively.

Example 2.4. Let $\mathfrak{g}$ be of type E_7 and

$$e := e_{\substack{100000\\0}} + e_{\substack{000111\\0}} + e_{\substack{000110\\1}} + e_{\substack{001100\\1}} + e_{\substack{001110\\0}} + e_{\substack{011100\\0}} + e_{\substack{011111\\1}}$$

be the nilpotent with label $\mathrm{E}_7(a_5)$. Here a letter e with subscripts denotes a root vector corresponding to a root given by the linear combination of simple roots with coefficients indicated in the subscript.

In this case $\dim \mathfrak{g}_{-d} = 3$ and the image of the singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ in the projective plane is as follows:

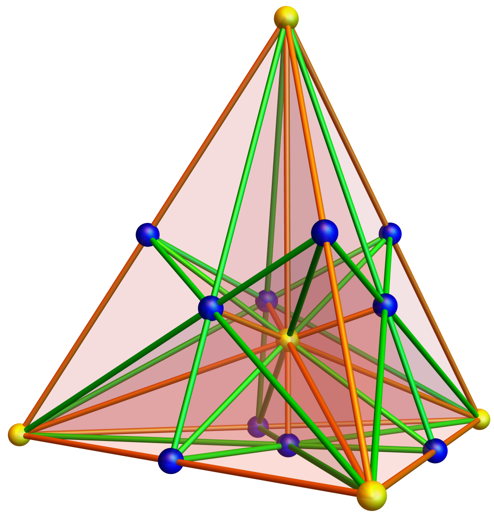


where ω is the third root of 1. That is, a cyclic element

$$c := e + x_1 f_{\substack{123421\\2}} + x_2 f_{\substack{123431\\2}} + x_3 f_{\substack{123432\\2}}$$

fails to be semisimple if and only if the corresponding point $[x_1 : x_2 : x_3]$ of the projective plane lies in the indicated set.

Example 2.5. As a last example, let $\mathfrak{g}$ be of type E_8 and e a nilpotent with label $\mathrm{E}_8(a_7)$. Then $\dim \mathfrak{g}_{-d} = 4$, and $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ is the union of 10 3-dimensional subspaces. For a particular choice of e, these subspaces are given by the equations $x_1 + x_2 + x_3 + x_4 = 0$, $x_i + x_j = x_k + x_4$ and $x_i \pm x_j = 0$, with $\{i, j, k\} = \{1, 2, 3\}$. Projectivization of these subspaces gives a configuration of ten planes in the projective 3-space that looks as follows:



The authors are indebted to Noam Elkies for finding a particularly tractable parametrization of this configuration which helped them to identify it [4].

In the remaining (non-irreducible) cases, the space $\mathfrak{g}_{-d}$ contains a copy of one of the above singular sets $\mathbf{\Sigma}_{\mathfrak{q}}(e)$ for some reducing subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$, in which e becomes irreducible, while the whole singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ in $\mathfrak{g}_{-d}$ can be described in terms of the image of $\mathbf{\Sigma}_{\mathfrak{q}}(e)$ under the action of $Z_{\mathfrak{g}}(\mathfrak{s})$ on $\mathfrak{g}_{-d}$.

3. "Explanation" of Singular Sets by Algebra Structures

It is also possible to understand more conceptually why exactly are the singular sets of the form that we found. For this aim, we equip the space $\mathfrak{g}_{-d}$ with additional structure.

First,

$$(x, y)_e := \left\langle (\operatorname{ad} e)^d x, y \right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the Killing form on $\mathfrak{g}$, defines a nondegenerate $Z(\mathfrak{s})$-invariant symmetric bilinear form on $\mathfrak{g}_{-d}$.

Second, when d is even (which happens precisely if e is not of nilpotent type), the formula

$$x \underset{e}{\star} y := [(\operatorname{ad} e)^{\frac{d}{2}} x, y]$$

defines a bilinear operation on $\mathfrak{g}_{-d}$. This operation is skew-commutative if $d/2$ is even and commutative if $d/2$ is odd. Moreover the above symmetric bilinear form is invariant for this operation, in the sense that

$$(x \underset{e}{\star} y, z)_e = (x, y \underset{e}{\star} z)_e$$

holds for any $x, y, z \in \mathfrak{g}_{-d}$.

It then turns out that

Theorem 3.1. *The singular set $\mathbf{\Sigma}_{\mathfrak{g}}(e)$ is the union of all proper $\underset{e}{\star}$-subalgebras of the algebra $(\mathfrak{g}_{-d}, \underset{e}{\star})$.*

In each of the cases then, the algebra $(\mathfrak{g}_{-d}, \underset{e}{\star})$ can be identified with interesting well known algebras. First, when d is divisible by 4 then it turns out that this algebra is a direct sum of an abelian algebra (with zero multiplication), some simple Lie algebras, and a simple 7-dimensional Maltsev algebra of imaginary octonions under the commutator operation $[x, y] = xy - yx$ [8, 9].

In the case $d = 2$ it has been known for a long time that using the operation $\underset{e}{\star}$ one may obtain all simple Jordan algebras. It turns out that more generally, for even d not divisible by 4 the algebra $(\mathfrak{g}_{-d}, \underset{e}{\star})$ is Jordan provided e is of semisimple type and not irreducible. Whereas if e is irreducible, then one obtains one of the family of commutative algebras $\mathscr{C}_\lambda(n)$ given by the basis $p_1,\dots,p_n$ with the multiplication table

$$p_i^2 = p_i, \qquad p_ip_j = \lambda(p_i+p_j),\ i\neq j.$$

These algebras are known in the literature related to representations of sporadic finite simple groups, to Hessian algebras appearing in differential geometry, and to the theory of vertex algebras [5, 6].

For example, in 2.3 we get the algebra $\mathscr{C}_{1-k}(2)$, in 2.4 it is $\mathscr{C}_{-\frac{1}{3}}(3)$, and in 2.5 it is $\mathscr{C}_{-\frac{1}{3}}(4)$.

The algebras $\mathscr{C}_\lambda(n)$ are mostly not Jordan, but satisfy quartic identities

$$\langle a,b,c\rangle\, d - \langle a,d,c\rangle\, b = (ab)(cd) - (ad)(bc)$$

and

$$\langle a,bd,c\rangle + \langle b,cd,a\rangle + \langle c,ad,b\rangle = 0,$$

where $\langle x,y,z\rangle$ denotes the associator $(xy)z - x(yz)$. The latter identity can be also written in the form

$$[\mathrm{L}_a,\mathrm{L}_b]\,\mathrm{L}_c + [\mathrm{L}_b,\mathrm{L}_c]\,\mathrm{L}_a + [\mathrm{L}_c,\mathrm{L}_a]\,\mathrm{L}_b = 0,$$

where L_x denotes the multiplication operator, $\mathrm{L}_x(y)=xy$. Note the close resemblance with the Jordan identity, which is equivalent to

$$\langle ab,d,c\rangle + \langle bc,d,a\rangle + \langle ca,d,b\rangle = 0,$$

or in terms of the multiplication operators,

$$[\mathrm{L}_{ab},\mathrm{L}_c] + [\mathrm{L}_{bc},\mathrm{L}_a] + [\mathrm{L}_{ca},\mathrm{L}_b] = 0.$$

These algebras then give an explanation of the particular form of singular sets $\mathbf{\Sigma}_{\mathfrak{g}}(e)$, in view of 3.1. Indeed, all algebras $\mathscr{C}_\lambda(n)$ that occur in our case contain exactly 2^n-1 nonzero idempotents, and all of their subalgebras are spanned by linearly independent subsets of idempotents. For example, the algebra $\mathscr{C}_{1-k}(2)$ from 2.3 has exactly three nonzero idempotents, and its subalgebras are the one-dimensional ones spanned by one of them, which explains why in 2.2 exactly three 1-dimensional subspaces occur when $\dim\mathfrak{g}_{-d}=2$. Similarly, when $\dim\mathfrak{g}_{-d}=3$, every maximal proper subalgebra of $\mathscr{C}_{-\frac{1}{3}}(3)$ is spanned by two linearly independent idempotents and contains three of them, which gives six distinct subalgebras, while for $\dim\mathfrak{g}_{-d}=4$, any three linearly independent idempotents of $\mathscr{C}_{-\frac{1}{3}}(4)$ span a 3-dimensional subalgebra that contains seven of the idempotents, which gives ten 3-dimensional subalgebras in total, according to the projectivized picture in 2.5 where points correspond to 1-dimensional subalgebras, lines to 2-dimensional subalgebras and planes to 3-dimensional subalgebras.

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PUBLIC KEY EXCHANGE USING CROSSED MODULES OF GROUPS

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Abstract. A notion of one-way crossed module of groups is introduced and a new general public key exchange protocol based on one-way crossed modules of groups is described.

1. Introduction

There are many attempts to use modern algebraic structures in different cryptographic constructions (see [1, 3, 8–12, 16] and related references therein). Some typical cryptosystems based on non-commutative groups (semigroups, algebras) described in recent papers are the best understood as various generalizations of the classical Diffie–Hellman scheme [5]. In the Diffie–Hellman case the underlying algorithmic problem is the famous discrete logarithm problem for the platform group. Other group based schemes the underlying algorithmic problems could be different such as the conjugacy search problem, the membership search problem, the decomposition and factorization problems and so on.

In this note we propose a new general public key exchange protocol based on one-way crossed modules of groups, that is, one-way functions having an additional algebraic structure. Here, we only give some well-known examples of crossed modules of groups, whose computing algorithms have the cryptographic nature, which will be demonstrated in the forthcoming paper suggesting candidates of one-way crossed modules.

1.1. **Basic notions and notations.** We proceed with recalling some basic notions and notations. Let G and H be the groups. By *an action* of H on G we mean a left action written by ${}^h g$, for $h \in H$ and $g \in G$. It is always understood that a group H acts on its normal subgroup H' by conjugation: ${}^h h' = hh'h^{-1}$, for $h \in H$, $h' \in H'$. Moreover, given a group G, it is always understood that the group of automorphisms $\mathrm{Aut}(G)$ acts on G in the following natural way:

$$ {}^{\phi} g = \phi(g), \quad \text{for } \phi \in \mathrm{Aut}(G),\ g \in G. $$

Given an action of a group H on a group G, *the semi-direct product* of G and H is defined to be the group $G \rtimes H$ with the underlying set $G \times H$ and the multiplication

$$ (g, h)(g', h') = (g\ {}^h g', hh'), \quad \text{for all } g, g' \in G,\ h, h' \in H. $$

In particular, the product of elements in $G \rtimes \mathrm{Aut}(G)$ is given by

$$ (g, \phi)(g', \psi) = (g\phi(g'), \phi\psi), \quad \text{for all } g, g' \in G,\ \phi, \psi \in \mathrm{Aut}(G). $$

2. One-way Crossed Modules of Groups

The general concept of a crossed module takes its origin in the work of Whitehead in the late 40s of the past century [15]. The crossed modules of groups are algebraic models for the path-connected CW-spaces whose homotopy groups are trivial in dimensions > 2. Since their introduction, crossed modules played an important role in the homotopy theory.

Definition 2.1. A *crossed module of groups* is a group homomorphism $\mu \colon G \to P$ together with an action of P on G such that

$$ \mu({}^p g) = p\mu(g)p^{-1}, \quad {}^{\mu(g)} g' = gg'g^{-1}, $$

for all $g, g' \in G$ and $p \in P$, denoted by (G, P, μ).

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Now we give examples of crossed modules of groups which are useful in cryptographic sense for constructing candidates of one-way crossed modules.

Examples 2.2. i) A normal inclusion $N \trianglelefteq G$ is a crossed module, where G acts on its normal subgroup N by conjugation.

ii) Let G be a group. There is a homomorphism

$$\alpha : G \to \mathrm{Aut}(G)$$

given by $\alpha(g) = \phi_g$, where ϕ_g is the inner automorphism defined by $g \in G$, i.e., $\phi_g(g') = gg'g^{-1}$. It is easy to check that α together with the action of $\mathrm{Aut}(G)$ on G described above, is a crossed module.

iii) Let G and H be groups acting on each other compatibly, that is, the following conditions:

$$^{(^g h)}(g') = {}^g({}^h({}^{g^{-1}}g'))\ , \qquad {}^{(^h g)}(h') = {}^h({}^g({}^{h^{-1}}h'))\ ,$$

hold for all $g, g' \in G$ and $h, h' \in H$. Let us denote by $G \bowtie H$ the quotient group $(G \rtimes H)/L$, where L is the normal subgroup of the semi-direct product $G \rtimes H$ generated (as a group) by all elements of the form $(g^h g^{-1}, h^g h^{-1})$, $g \in G$, $h \in H$. As a consequence of the compatibility conditions, the canonical maps

$$\mu : G \to G \bowtie H \quad \text{and} \quad \nu : H \to G \bowtie H$$

are the crossed modules (see [6]).

Now we introduce our main (cryptographic) notion, which will in the sequel be the subject of our investigation and applications in a key exchange protocol.

Definition 2.3. A crossed module of groups $\mu : G \to P$ is called a *one-way crossed module* if it is a one-way function (OWF) by its cryptographic nature, i. e., intuitively saying, a function that is easy to compute, but computationally hard to invert.

Having in mind that the existence of OWFs is not proven yet, we may follow the common practice for the proposition of new candidates of OWFs for cryptographic applications. Consequently, the following two necessary (but not sufficient) conditions may be required for the maps to be candidates of OWFs:

(1) The computation of direct value of the map is computationally easy;

(2) A certain hard problem without a known polynomial time algorithm is reducible, in polynomial time, to inverting problem of the map.

General one-wayness assumption. In this subsection we address the general question of one-wayness of a crossed module.

By the Peiffer identity in Definition 2.1, knowing $\mu(x)$ for some $x \in G$, one can compute

$$^{\mu(x)}g = xgx^{-1} \quad \text{for any} \quad g \in G.$$

So, we arrive at the computation problem which we call the *total conjugacy search problem* (TCSP). It has the form:

> given a group G, a map $f : G \to G$ given by some polynomial algorithm and the information that $f(g) = {}^x g$ for some x and any $g \in G$, find at least one particular element x with this property.

This problem is related to the known problem, called the k-simultaneous conjugacy search problem (k-SCSP) for a fixed $k \in \mathbb{N}$:

> given a group G and two k-tuples of elements in G, $(g_1, \dots, g_k), (h_1, \dots, h_k)$, and the information that ${}^x g_i = h_i$ for all $1 \le i \le n$ and some $x \in G$, find at least one particular element x with this property.

Namely, we have the following theorem.

Theorem 2.4. *Let G be a finitely generated group with k generators $g_1, \dots, g_k$. Then TCSP is polynomial-time equivalent to the k-SCSP for generators, i.e. to the problem:*

given a k-tuple of elements in G, $(h_1, \dots, h_k)$, and information that ${}^x g_i = h_i$ for all $1 \leq i \leq n$ and some $x \in G$, find at least one particular element x with this property.

We note that simultaneous conjugacy decision problem turns out to be quite hard in many groups, and even unsolvable in some cases. This problem was studied for various classes of groups (see [7,13,14]). It is shown that the solvability of the conjugacy problem does not imply the solvability of simultaneous conjugacy problem [4]. More recently, in [2], the examples of finitely presented groups are constructed, where the ordinary conjugacy problem is solvable, but the k-simultaneous conjugacy problem is unsolvable for every $k \geq 2$.

We suppose that the simultaneous conjugacy search problem for generators will have at least the same hardness, than the k-simultaneous conjugacy search problem. We also note that if there is an algorithm for solving the simultaneous conjugacy search problem for generators which calculates some $g'' \in G$ such that

$$^{\mu(g)}g' = g''g'g''^{-1} \quad \text{for any} g' \in G,$$

there is no guarantee that $g = g''$. Indeed, it means that $g^{-1}g'' \in Z(G)$ (center of G). Hence if the center of the group is large enough, it will be unfeasible to recover g from g'' by simple multiplications on center elements of the group G.

3. Key Exchange Protocol

In this section, we give a new general key exchange protocol using the idea of one-wayness of crossed modules. Then choosing a platform crossed module of groups, we give practical instances of these protocols.

Protocol. Let (G, P, μ) be a one-way crossed module of groups. The group G is considered to be a set of possible public keys, i.e., elements $g, g' \in G$ are chosen and made public. Both Alice and Bob are going to work with the crossed module (G, P, μ) if they wish to create a shared key. Then, for creating a shared key, Alice and Bob can proceed as follows:

1. Alice selects at random a private key $m \in \mathbb{N}$. Then she computes the element
$$a = g^m \cdot g' \cdot g^{-m},$$
applies the one-way crossed module μ to the element a and sends $\mu(a)$ to Bob.
2. Bob selects at random a private key $n \in \mathbb{N}$. Then he computes the element
$$b = g^n \cdot g' \cdot g^{-n},$$
applies the one-way crossed module μ to the element b and sends $\mu(b)$ to Alice.
3. Alice computes her key as follows:
$$K_A = (g^m) \cdot \left({}^{\mu(b)} \left(g^{-m} \cdot g' \cdot g^m \right) \right) \cdot (g^{-m}).$$
Bob computes his key as follows:
$$K_B = (g^n) \cdot \left({}^{\mu(a)} \left(g^{-n} \cdot g' \cdot g^n \right) \right) \cdot (g^{-n}).$$
Using the crossed module structure, namely, the Peiffer identity of Definition 2.1, we have the following equalities:
$$\begin{aligned} K_A &= (g^m) \cdot \left(g^n \cdot g' \cdot g^{-n} \right) g^{-m} \cdot g' \cdot g^m \cdot \left(g^n \cdot g'^{-1} \cdot g^{-n} \right) \cdot (g^{-m}) \\ &= (g^n) \cdot \left(g^m \cdot g' \cdot g^{-m} \right) g^{-n} \cdot g' \cdot g^n \cdot \left(g^m \cdot g'^{-1} \cdot g^{-m} \right) \cdot (g^{-n}) = K_B. \end{aligned}$$
So, Alice and Bob have the shared key $K = K_A = K_B$.

Remark 3.1. Note that the publicly known elements g and g' of G should be chosen in such a way that $gg' \neq g'g$. Otherwise, the scheme will become trivial. Moreover, as in the Habeeb-Kahrobaei-Shpilrain key exchange protocol [8,11] and in contrast to the "standard" Diffie–Hellman key exchange, the correctness here is based on the equality $g^m \cdot g^n = g^n \cdot g^m = g^{m+n}$, rather than on the equality $(g^m)^n = (g^n)^m = g^{mn}$.

Security assumptions. Since the shared secrete key in the Protocol is

$$K = g^{m+n} \cdot g' \cdot g^{-(m+n)} \cdot g' \cdot g^{m+n} \cdot g'^{-1} \cdot g^{-(m+n)},$$

our security assumption is that it is computationally hard to retrieve the key K from the quadruple $(g, g', \mu(g^m \cdot g' \cdot g^{-m}), \mu(g^n \cdot g' \cdot g^{-n}))$. If the adversary chooses a "direct" attack by trying to recover the private keys a, she will have to invert one-way function μ on $\mu(a)$, then to solve the conjugacy problem for $g^m \cdot g' \cdot g^{-m}$ and, finally, to solve the discrete logarithm problem for g^m.

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BERIKASHVILI'S FUNCTOR D: GENERALIZATION AND APPLICATION

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Abstract. We present the notion of Berikashvili's functor D, its generalization for the $A(\infty)$ case and the corresponding algebraic model of a fibre bundle. It is a pleasure to dedicate this paper to my teacher Nodar Berikashvili on his 90th anniversary.

1. Twisting Cochains for DG-algebras

1.1. **Browns twisting cochains.** Let $(K, d_K : K_* \to K_{*-1}, \nabla_K : K \to K \otimes K)$ be a DG-coalgebra and $(A, d_A : A_* \to A_{*-1}, \mu_A : A \otimes A \to A)$ be a DG-algebra. Then the *cochain complex* $C^*(K, A) = Hom(K, A)$ with a differential $\delta\alpha = \alpha d_K + d_A\alpha$ and multiplication $\alpha \smile \beta = \mu_A(\alpha \otimes \beta)\nabla_K$ is a DG-algebra.

A *Brown's twisting cochain* [3] is a homomorphism $\phi : K_* \to A_{*-1}$, i.e., $deg\ \phi = -1$, satisfying $\delta\phi = \phi \smile \phi$.

Twisted tensor product. Let $(P, d_P, \nu : A \otimes P \to P)$ be a DG A-module. Then any twisting cochain $\phi : K \to A$ determines a homomorphism $d_\phi : K \otimes P \to K \otimes P$ by $d_\phi(k \otimes p) = d_K k \otimes p + k \otimes d_P m + (k \otimes p) \cap \phi$ where $(k \otimes p) \cap \phi = (id_K \otimes \nu)(id_K \otimes \phi \otimes id_P)(\nabla_K \otimes id_P)(k \otimes p)$. The Brown's condition $d\phi = \phi \smile \phi$ implies that $d_\phi d_\phi = 0$. The obtained chain complex $(K \otimes M,\ d_\phi)$ is called *twisted tensor product* and is denoted as $K \otimes_\phi M$.

Using these notions, Edgar Brown constructed an algebraic model of a fibre bundle (see below).

For a morphism of DG-algebras $f : A \to A'$, a morphism of modules $g : P \to P'$, $g(a \cdot p) = f(a) \cdot g(p)$ and a twisting cochain $\phi : K \to A$ the map $id_K \otimes g : K \otimes_\phi P \to K \otimes_{f\phi} P'$ is a chain map.

1.2. **Berikashvili's equivalence of twisting cochains.** Two twisting cochains $\phi, \psi : K \to A$ are equivalent (Berikashvili [2]) if there exists $c : K \to A,\ deg\ c = 0$, such that $\psi = \phi + \delta c + \psi \smile c + c \smile \phi$, notation $\phi \sim_c \psi$. This equivalence allows one to *perturb* twisting cochains.

Essential applications of Berikashvili's equivalnce give the following.

Theorem 1. *If $\phi \sim_c \psi$, then $K \otimes_\phi P \xrightarrow{F_c} K \otimes_\psi P$ given by $F_c(k \otimes p) = (k \otimes p) \cap c$ is an isomorphism of DG-comodules.*

Berikashvili's functor D. Let $Tw(K, A) = \{\phi : K \to A,\ \delta\phi = \phi \circ \phi\}$ be the set of all twisting cochains. *Berikasvili's functor* $D(K, A)$ is defined as the factorset $D(K, A) = \frac{Tw(K,A)}{\sim}$.

The following property of D plays an essential role in some constructions.

Theorem 2 (Berikashvili [2]). *Let (K, d_K, ∇_K) be a DG-colagebra with free K_is and (A, d_A, μ_A) be a connected DG-algebra. If $f : A \to A'$ is a weak equivalence of connected DG-algebras (i.e., homology isomorphism), then $D(f) : D(K, A) \to D(K, A')$ is a bijection.*

1.3. **Bar interpretation.** The notions of twisting cochain and their equivalence have useful interpretation in terms of Adams's bar construction [1].

Twisting Cochains and the Bar Construction. Any twisting cochain $\phi : K \to A$ induces a map of DG-coalgebras $f_\phi : K \to B(A)$ given by $f_\phi = \sum_i (\phi \otimes\ \cdots\ \otimes \phi)\nabla^i_K$.

Bar interpretation of equivalence of twisting Cochains. In the category of DG-coalgebras there is the following notion of homotopy: two DG-coalgebra maps $f, g : (K, d_K, \nabla_K) \to (K', d_{K'}, \nabla_{K'})$ are

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homotopic, if there exists chain homotopy $D : K \to K'$, $d_{K'}D + Dd_k = f - g$, which, in addition, is a $f - g$-coderivation, that is, $\nabla_{K'}D = (f \otimes D + D \otimes g)\nabla_K$.

If $\phi \sim_c \psi$, then f_ϕ and f_ψ are homotopic by $D(c) : K \to BA$ given by $D(c) = \sum_{i,j}(\psi \otimes \cdots (j - times) \cdots \otimes \psi \otimes c \otimes \phi \otimes \cdots \otimes \phi)\nabla^i_K$.

Bar interpretation of a functor D. Asigning to a twisting cohain $\phi : K \to A$ the DG-coalgebra map $f_\phi : K \to BA$ and having in mind that $\phi \sim_c \psi$ implies $f_\phi \sim_{D(c)} f_\psi$, we obtain a bijection $D(K, A) \leftrightarrow [K, BA]$ where $[K, BA]$ denotes the set of chain homotopy classes in the category of DG-coalgebras.

We remark here that the theorem 2 means that for a weak equivalence of DG-algebras $A \to A'$ the induced map $[K, BA] \to [K, BA']$ is a bijection.

2. Twisting Cochains for $A(\infty)$-algebras

Here we are going to step from the DG-algebra (A, d, μ) to an $A(\infty)$-algebra $(M, \{m_i\})$, this notion was introduced by James Stasheff in [9].

2.1. Category of $A(\infty)$-algebras.

Definition 1 (Stasheff [9])**.** *An $A(\infty)$ algebra $(A, \{m_i\})$ is a graded module A equipped with a sequence of operations $\{m_i : A^{\otimes i} \to A,\ i = 1, 2, 3, 4, \dots\}$ which satisfies the following conditions: deg $m_i = 2-i$ and*

$$\sum_{i+j=n+1} \sum_{k=0}^{n-j} m_{n-j+1}(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes a_{k+j+1} \otimes \cdots \otimes a_n) = 0.$$

Stasheff's defining condition for $n = 1$ gives $m_1m_1 = 0$, i.e., m_1 is a differential, for $n = 2$, m_1 is a derivation with respect to the multiplication m_2, and for $n = 3$, m_2 is homotopy associative, and the appropriate homotopy is m_3. So, $(A, \{m_i\})$ is a *strong homotopy associative* (sha) algebra.

Bar interpretation. The Stasheff's condition guarantees that the coderivation $d_m : B(A) \to B(A)$ given by

$$\begin{gathered} d_m(a_1 \otimes \cdots \otimes a_n) \\ = \sum_{k=0}^{n} \sum_{j=1}^{n-k} a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes a_{k+j+1} \otimes \cdots \otimes a_n \end{gathered}$$

satisfies $d_md_m = 0$: the Stasheff's condition is the projection of this equality on the cogenerating module A. So, the bar construction $(B(A, \{m_i\}), d_m)$ with this perturbed differential is a DG-coalgebra.

Particular case. The notion of an $A(\infty)$ algebra generalizes the notion of DG-algebra: an $A(\infty)$-algebra of type $(A, \{m_1, m_2, m_3 = 0, m_4 = 0, \dots\})$ is a DG-algebra with the differential m_1 and associative multiplication m_2.

Morphism of A_∞-algebras. This notion was introduced in [5]. A morphism of $A(\infty)$-algebras $(A, \{m_i\}) \to (A', \{m'_i\})$ is defined as a sequence of homomorphisms $\{f_i : A^{\otimes i} \to A',\ i = 1, 2, \dots\}$, which satisfy the following conditions: deg $f_i = 1 - i$ and

$$\begin{gathered} \sum_{i+j=n+1} \sum_{k=0}^{n-j} f_i(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n) \\ = \sum_{t=1}^{n} \sum_{k_1+\cdots+k_t=n} m'_t(f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes f_{k_2}(a_{k_1+1} \otimes \cdots \otimes a_{k_1+k_2}) \otimes \\ \cdots \otimes f_{k_t}(a_{k_1+\cdots+k_{t-1}+1} \otimes \cdots \otimes a_n)). \end{gathered}$$

Particulary, for $n = 1$, this gives $f_1m_1 = m'_1f_1$, that is, $f_1 : (A, m_1) \to (A', m'_1)$ is a chain map. We call $\{f_i\}$ *a weak equivalence* if f_1 induces isomorphism of homologies.

Bar interpretation. A morphism $\{f_i\}$ defines the DG-coalgebra map of the bar constructions $B(\{f_i\}) : B(A, \{m_i\}) \to B(A', \{m'_i\})$, given by

$$\begin{gathered} B(\{f_i\})(a_1 \otimes \cdots \otimes a_n) \\ = \sum_{t=1}^{n} \sum_{k_1+\cdots+k_t=n} f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes f_{k_t}(a_{k_1+\cdots+k_{t-1}+1} \otimes \cdots \otimes a_n). \end{gathered}$$

Particularly, a morphism of A_∞-algebras

$$\{f_1, f_2 = 0, f_3 = 0, \dots\} :$$

$$(A, \{m_1, m_2, m_3 = 0, m_4 = 0, \dots\}) \to (A', \{m'_1, m'_2, m'_3 = 0, m'_4 = 0, \dots\})$$

is an ordinary map of DG-algebras.

2.2. **Category of $A(\infty)$-modules.** This definition from [5] generalizes the notion of a DG-module over a DG-algebra. An $A(\infty)$-module over an $A(\infty)$-algebra $(A,\{m_i\})$ is a graded module P equipped with a sequence of "actions" $\{p_i : P\otimes A^{\otimes i}\to A,\ i=0,1,2,3,\dots\}$ satisfying the conditions: $\deg p_i = 1-i$ and, for $a_k\in A,\ x\in P$,

$$\begin{array}{l}\sum_{i=0}^{n} p_{n-i}(p_i(x\otimes a_1\otimes\cdots\otimes a_i)\otimes a_{i+1}\otimes\cdots\otimes a_n)+\sum_{k=0}^{n}\sum_{i=1}^{n-k}\\ p_{n-i+1}(x\otimes a_1\otimes\cdots\otimes a_k\otimes m_i(a_{k+1},\otimes\cdots\otimes a_{k+i})\otimes a_{k+i+1}\otimes\cdots\otimes a_n)=0.\end{array}$$

Bar interpretation. This structure induces $d_p : P\otimes B(A)\to P\otimes B(A)$ by

$$\begin{array}{l}d_p(x\otimes a_1\otimes\cdots\otimes a_n)=\sum_{i=0}^{n} p_i(x\otimes a_1\otimes\cdots\otimes a_i)\otimes a_{i+1}\otimes\cdots\otimes a_n\\ +\sum_{k=0}^{n}\sum_{i=1}^{n-k} x\otimes a_1\otimes\cdots\otimes a_k\otimes m_i(a_{k+1}\otimes\cdots\otimes a_{k+i})\otimes a_{k+i+1}\otimes\cdots\otimes a_n\end{array}$$

which satisfies $d_pd_p=0$, thus $(P\otimes B(A),d_p)$ is a DG-comodule over DG-coalgebra $(B(A,\{m_i\}),d_m)$.
Particular cases. (1) An $A(\infty)$-module $(P,\{p_1,p_2,0,0,\dots\})$ over an $A(\infty)$-algebra $(A,\{m_1,m_2,0,0,\dots\})$ is a DG-module over DG-algebra (A,m_1,m_2) with a differential $p_0 : P\to P$ and strictly associative action $p_1 : P\otimes A\to A$. (2) An $A(\infty)$-algebra $(A,\{m_i\})$ is an $A(\infty)$-module over itself with structure maps $p_n(x\otimes a_1\otimes\cdots\otimes a_n)=m_{n+1}(x\otimes a_1\otimes\cdots\otimes a_n)$.
Morphism of $A(\infty)$-modules. Let $(P,\{p_i\})$ be an $A(\infty)$-module over an $A(\infty)$-algebra $(A,\{m_i\})$, and let $(P',\{p'_i\})$ be an $A(\infty)$-module over an $A(\infty)$-algebra $(A',\{m'_i\})$. A morphism of the couples

$$(\{g_i\},\{f_i\}) : ((P,\{p_i\}),(A,\{m_i\}))\to((P',\{p'_i\}),(A',\{m'_i\}))$$

is defined in [5] as: a morphism of $A(\infty)$-algebras $\{f_i\} : (A,\{m_i\})\to(A',\{m'_i\})$ and a sequence of homomorphisms $\{g_i : P\otimes A^{\otimes i}\to P',\ i=0,1,2,3,\dots\}$ such that $deg\ g_i=-i$ and

$$\begin{array}{c}\sum_{k=0}^{n}\sum_{j=1}^{n-k} g_{n-j+1}(x\otimes a_1\otimes\cdots\otimes a_k\otimes m_j(a_{k+1}\otimes\cdots\otimes a_{k+j})\cdots\otimes a_n)\\ +\sum_{k=0}^{n} g_{n-k}(p_k(x\otimes a_1\otimes\cdots\otimes a_k)\otimes a_{k+1}\otimes\cdots\otimes a_n)\\ =\sum_{t=1}^{n+1}\sum_{k_1+\cdots+k_t=n+1} p_t(g_{k_1}(x\otimes a_1\otimes\cdots\otimes a_{k_1})\otimes f_{k_2}(a_{k_1+1}\otimes\cdots\otimes a_{k_1+k_2})\\ \otimes f_{k_3}(a_{k_1+k_2+1}\otimes\cdots\otimes a_{k_1+k_2+k_3})\otimes\cdots\otimes f_{k_t}(a_{k+k_1+\cdots+k_{t-1}+1}\otimes\cdots\otimes a_n)).\end{array}$$

Bar interpretation. Such a morphism induces the chain map $G : (P\otimes BA,d_p)\to(P'\otimes BA',d_{p'})$ by

$$\begin{array}{l}G(x\otimes a_1\otimes\cdots\otimes a_n)=\sum_{t=1}^{n+1}\sum_{k_1+\cdots+k_t=n+1} g_{k_1}(x\otimes a_1\otimes\cdots\otimes a_{k_1})\\ \otimes f_{k_2}(a_{k_1+1}\otimes\cdots\otimes a_{k_1+k_2})\otimes\cdots\otimes f_{k_t}(a_{k+k_1+\cdots+k_{t-1}+1}\otimes\cdots\otimes a_n).\end{array}$$

2.3. **A_∞-twisting Cochains.** Now we have to replace in the definition of a twisting cochain a DG-algebra (A,d_A,μ) by an A_∞-algebra $(A,\{m_i\})$, see [6], [7].

An $A(\infty)$-*twisting cochain* we define as a homomorphism $\phi : K\to A$ of degree -1 satisfying the condition $\sum_{k=1}^{\infty} m_k(\phi\otimes\ \cdots\ \otimes\phi)\nabla_K^k=\phi d_K$.

The set of all $A(\infty)$-twisting cochains $Tw_\infty(K,A)$ is a bifunctor: for a morphism of DG-coalgebras $h : K'\to K$, the composition $\phi\circ h$ belongs to $Tw_\infty(K,A)$, similarly, for a morphism of A_∞-algebras $f=\{f_i\} : (A,\{m_i\})\to(A',\{m'_i\})$, the composition $f(\phi)=\sum f_i(\phi\otimes\cdots\otimes\phi)\nabla^i$ belongs to $Tw_\infty(K,A')$.
Bar Interpretation. An A_∞-twisting cochain $\phi : K\to A$ induces the DG-coalgebra morphism $f_\phi : K\to B(A)$ by $f_\phi=\sum_i(\phi\otimes\ \cdots\ \otimes\phi)\nabla_K^i$. Conversely, any DG-coalgebra map $f : K\to B(A)$ is f_ϕ for the $A(\infty)$-twisting cochain $\phi=p\circ f : K\to B(A)\to A$. So, $Mor_{dgcoalg}(K,B(A))\leftrightarrow T_\infty(K,A)$.
Equivalence of A_∞-twisting Cochains [6]. Two A_∞-twisting cochains $\phi,\psi : K\to A$ are equivalent if there exists $c : K\to A,\ \ deg\ c=0$, such that

$$\psi-\phi=cd_K+\sum_{k,j} m_k(\psi\otimes\ \cdots\ (j)\ \cdots\ \otimes\psi\otimes c\otimes\phi\otimes\ \cdots\ \otimes\phi)\nabla^k,$$

notation $\phi\sim_c\psi$.
Bar interpretation. If $\phi\sim_c\psi$, then f_ϕ and f_ψ are homotopic in the category of DG-coalgebras: chain homotopy $D_\infty(c) : K\to B(A)$ is given by $D_\infty(c)=\sum_{i,j}(\psi\otimes\ \cdots\ (j-times)\ \cdots\ \otimes\psi\otimes c\otimes\phi\otimes\ \cdots\ \otimes\phi)\nabla_K^i$.

FunctorD_∞**.** By $D_\infty(K,A)$ we denote the factorset $D_\infty(K,A)=\frac{T_\infty(K,A)}{\sim}$. Thus we have a bijection $[K,B(A)]\leftrightarrow D_\infty(K,A)$.

Suppose $f=\{f_i\}:(A,\{m_i\})\to(A',\{m'_i\})$ is a morphism of A_∞-algebras and $\phi:K\to A$ is an A_∞-twisting cochain.

From the bar construction interpretation it follows that $f(\phi):K\to A'$ given by $f(\phi)=\sum_i f_i(\phi\otimes\cdots\otimes\phi)\nabla_K^i$ is an A_∞-twisting cochain, too. Moreover, if $\phi\sim_c\psi$, then $f(\phi)\sim_{c'}f(\psi)$ with $c':K\to A'$ given by

$$c'=\sum_{i,j}f_i(\psi\otimes\ \cdots\ (j-times)\ \cdots\ \otimes\ \psi\otimes c\otimes\phi\ \otimes\ \cdots\ \otimes\phi)\nabla_K^i,$$

thus we have a map $D_\infty(f):D_\infty(K,A)\to D_\infty(K,A')$.

The following theorem is an analogue of Berikashvilis's theorem 2 for A_∞-algebras proved in [6].

Theorem 3. *Let (K,d_K,∇_K) be a DG-colagebra and $(A,\{m_i\})$ be a connected DG-algebra. If $f=\{f_i\}:(A,\{m_i\})\to(A',\{m'_i\})$ is a weak equivalence of A_∞-algebras, then $D_\infty(f):D_\infty(K,A)\to D_\infty(K,A')$ is a bijection, consequently, $[K,B(A)]\leftrightarrow[K,B(A)']$.*

2.4. **Twisted tensor product, the A_∞-case.** Let (K,d_K,∇_K) be a DG-coalgebra, $(A,\{m_i\})$ be an A_∞-algebra, $(P,\{p_i\})$ be an A_∞-module over an $(A,\{m_i\})$, and $\phi:K\to A$ be an A_∞-twisting cochain. It defines on the tensor product $K\otimes P$ a differential $\partial_\phi:K\otimes P\to K\otimes P$ given by

$$\partial_\phi=d\otimes id_P+\sum_{i=1}^{\infty}(\hat{id}\otimes p_i)(id_K\otimes\phi\otimes...\otimes\phi\otimes id_P)(\Delta^i\otimes id_P)$$

which turns $K\otimes_\phi P=(K\otimes P,\partial_\phi)$ into a differential comodule over (K,d).

Particular case. If A is an A_∞-algebra of the form $(A,\{m_1,m_2,0,0,...\})$, and P is an A_∞-module of the form $(P,\{p_1,p_2,0,0,...\})$, then ϕ is the usual twisting cochain, and $K\otimes_\phi P$ coincides with the usual twisted tensor product.

As in the DG-algebra case, equivalent A_∞-twisting cochais $\phi\sim_c\psi$ produce isomorphic twisted tensor products.

Theorem 4. *If $\phi\sim_c\psi$, then $K\otimes_\phi P\xrightarrow{F_c}K\otimes_\psi P$ given by $F_c(k\otimes p)=(k\otimes p)\cap c$ is an isomorphism of DG-comodules.*

Functoriality. For a morphism of couples

$$\{g_i\},\{f_i\}:((P,\{p_i\}),(A,\{m_i\}))\to((P',\{p'_i\}),(A',\{m'_i\}))$$

and an A_∞-twisting cochain $\phi:K\to A$ there exists the chain map $K\otimes_\phi P\to K\otimes_{f(\phi)}P'$.

3. Minimality Theorems

Minimal A_∞-algebras. An A_∞-algebra $(M,\{m_i\})$ we call *minimal* if $m_1=0$, in this case (M,m_2) is *strictly* associative graded algebra. Suppose $f:(M,\{m_i\})\to(M',\{m'_i\})$ is a weak equivalence of minimal A_∞-algebras, then $f_1:(M,m_1=0)\to(M',m'_1=0)$, which by definition should induce isomorphism of homology, is automatically an isomorphism. It is not hard to check that in this case $\{f_i\}$ is an isomorphism of A_∞-algebras, thus each *weak equivalence of minimal A_∞-algebras is an isomorphism.* This fact motivates the word *minimal* in this notion: the Sullivan's minimal model has similar property - a weak equivalence of minimal DG algebras is an isomorphism.

Let us present here the minimality theorem from [5].

Theorem 5. *For a DG algebra (A,d,μ) its homology $H(A)$ (all $H_i(X)$-s are assumed to be free) can be equipped with a sequence of multi-operations*

$$m_i:H(A)^{\otimes i}\to H(A),\ i=1,2,3,\dots\ ;\ m_1=0,\ m_2=\mu^*$$

turning $(H(A),\{m_i\})$ into a minimal *A_∞-algebra for which $m_2=\mu^*$ and there exists a weak equivalence of A_∞-algebras*

$$f=\{f_i\}:(H(A),\{m_i\})\to(A,\{m_1=d,m_2=\mu,m_3=0,m_4=0,\dots\}).$$

This structure is unique up to an isomorphism in the category of A_∞-algebras.

Analogous results hold for the modules: if $(P, d_P, \nu_P : A \otimes P \to P)$ is a DG-module over A, then there exist on $H(P, d_P)$ a structure of A_∞-module $(H(P), \{p_i\})$ over $(H(A), \{m_i\})$ and a morhism of couples

$$(\{g_i\}, \{f_i\}) : ((H(P), \{p_i\}), (H(A), \{m_i\})) \to (P, A)$$

such that $p_1 = 0,\ p_2 = \nu^*$ and $g_1^* = id_{H(P)} : H(P) \to H(P)$.

Furthermore, by (3), there is a bijection $D_\infty(f) : D_\infty(K, A) \to D_\infty(K, A')$, this implies the following

Theorem 6. *For a twisting cochain $\phi : K \to A$, there exists an A_∞-cochain $\psi : K \to (H(A), \{m_i\})$ such that $\phi \sim f(\psi)$, consequently, there exists a chain map inducing an isomorphism in the homologies $K \otimes_\psi H(P) \to K \otimes_{f(\psi)} P \overset{\approx}{\to} K \otimes_\phi P$.*

4. Application: A_∞-model of a Fibre Bundle

The minimality theorem (5) and the theorem (6) about the lifting of twisting cochains allow one to construct an effective model of a fibre bundle. Actually, this model and higher operations $\{m_i\}$ and $\{p_i\}$ were constructed in [4]. Later, we have recognized that they form Stasheff's A_∞ structures, and the model in these terms was presented in [5]. Similar model was also presented in [8]. **Topological level.** Let $\xi = (X, p, B, G)$ be a principal G-fibration. If F is a G-space, then the action $G \times F \to F$ determines the *associated fibre bundle* $\xi(F) = (E, p, B, F, G)$ with fiber F. Thus, ξ and the action $G \times F \to F$ on the topological level determine E.

Chain level. Let $K = C_*(B),\ A = C_*(G),\ P = C_*(F)$. The classical result of E. Brown [3] states that the principal fibration ξ determines a twisting cochain $\phi : K = C_*(B) \to A = C_*(G)$ and the action on chain level $C_*(G) \otimes C_*(F) \to C_*(F)$ defines the twisted tensor product $K \otimes_\phi P = C_*(B) \otimes_\phi C_*(F)$ which gives homology modules of the total space $H_*(E)$. Thus, ξ and the action on the chain level $C_*(G) \otimes C_*(F) \to C_*(F)$ determine $H_*(E)$.

The twisting cochain ψ is not uniquely determined and it can be perturbed by the above equivalence relations for computational reasons.

Homology level. Nodar Berikashvili stated the problem to lift the previous "chain level" model of associated fibration to "homology level", i.e., to construct "twisted differential" on $C_*(B) \otimes H_*(F)$. Investigation has shown that the principal fibration ξ and the action of Pontriagin's ring $H_*(G)$ on $H_*(F)$, that is, the pairing $H_*(G) \otimes H_*(F) \to H_*(F)$ *do not determine* $H_*(E)$. But by the minimality theorem it appeared that $H_*(G)$ carries not only Pontriagin's product $H_*(G) \otimes H_*(G) \to H_*(G)$, but also a richer algebraic structure, namely, the structure of minimal A_∞-algebra $(H_*(G), \{m_i\})$, furthermore, the action $G \times F \to F$ induces not only the pairing $H_*(G) \otimes H_*(F) \to H_*(F)$, but also the structure of a minimal A_∞-module $(H_*(F), \{p_i\})$, and all these operations allow one to define correct differential on $C_*(B) \otimes H_*(F)$: according to the theorem (6), there is a weak equivalence, a homology isomorphism

$$C_*(B) \otimes_\psi H_*(F) = K \otimes_\psi H(P) \to K \otimes_\phi P = C_*(B) \otimes_\phi C_*(F) \sim C_*(E).$$

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WAVE PROPAGATION THROUGH A SQUARE LATTICE WITH SOURCES ON LINE SEGMENTS

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Abstract. We study the problems related to the propagation of time harmonic waves through a two-dimensional square lattice with sources on line segments. The discrete Helmholtz equation with the wave number $k \in (0, 2\sqrt{2})\backslash\{2\}$ and input data prescribed on finite rows/columns of lattice sites is investigated without passing to the complex wave number. Similarly to the continuum theory, we use the notion of radiating solution. The unique solvability result and the Green's representation formula are obtained with the help of difference potentials. Finally, we propose a method for numerical calculation. Efficiency of our approach is demonstrated in examples related to the propagation problems in the left-handed 2D inductor-capacitor metamaterial.

1. Introduction

Consider two-dimensional passive propagation media that can be used for signal processing and filtering. Assume that at the fine scale these media consist of a lattice of repeated single type cells. As an example of such propagation media, we can take a host microstrip line network periodically loaded with series capacitors and shunt inductors as is shown in Figure 1. This type of inductor-capacitor lattice is referred to a negative-refractive-index transmission-line (NRI-TL) metamaterial [6], or simply, left-handed 2D metamaterial. Suppose that monochromatic inputs are applied to finite rows/columns of lattice sites. Assume that the number of unit cells in this slab is large enough to make it prohibitively expensive to solve numerically for the voltage/current at every cell in the lattice until the system reaches steady state. As a simplifying strategy, it can be anticipated that the limiting case, when the lattice is effectively infinite, is more amenable to analysis and provides a good approximation of the steady-state output at an exterior boundary.

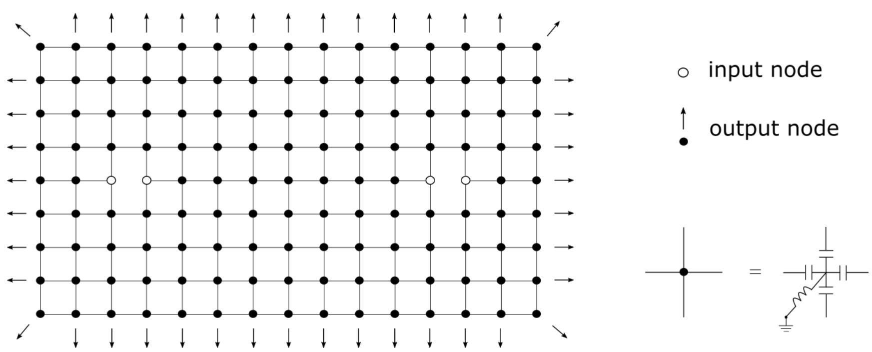

Figure 1. Left-handed 2D inductor-capacitor metamaterial with sources on two line segments. A host transmission-line is loaded periodically with series capacitors and shunt inductors.

The present paper examines the effect of finite sources on line segments in an infinite square lattice. Mathematical modeling of the propagation problem under consideration leads us to study an exterior problem for a discrete 2D Helmholtz equation with Dirichlet boundary conditions. Note that the same

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category of problems can be originated by composite right/left-handed and dual-composite right/left-handed lattices [2, 3, 5, 6, 11, 14] and mass spring lattices (cf., e.g., [18, 19]).

It is well known that for the negative discrete Laplacian the spectrum is (absolutely continuous) $[0, 8]$, but there is an exceptional set $\{0, 4, 8\}$ in $[0,\ 8]$, where the limiting absorption principle fails [15]. Therefore for "admissible" wave numbers $k \in (0, 2\sqrt{2})\backslash\{2\}$, one can study the problem as the limit $k + \iota 0$ of the complex wave number. This method is applied by Sharma in [16–19], where diffraction problems on a square lattice are investigated by a finite crack and a rigid constraint. In this paper, we use the results obtained in [9] and carry out our investigation without passing to the complex wave number. We use the radiation conditions and asymptotic estimates from Shaban et al. [15], a Rellich-Vekua type theorem from Isozaki el al. [8], and asymptotic estimates of the lattice Green's function derived by Martin [12]. Further, the unique solvability result and the Green's representation formula are obtained with the help of difference potentials. For the numerical calculation, we apply the method developed by Berciu et al. [1] which allows us to calculate the lattice Green's functions without the need to perform integrals and appears to be much more effective than the recurrence relations due to Morita [13].

2. Basic Notations and Formulation of the Problem

Following the customary notation in mathematics, let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of integers, positive integers, real numbers, and complex numbers, respectively. We denote by $e_1 = (1, 0)$, $e_2 = (0, 1)$ the standard base of the square lattice $\mathbb{Z}^2$ $(= \mathbb{Z} \times \mathbb{Z})$.

For any point $x = (x_1, x_2) \in \mathbb{Z}^2$, we define the 4-neighborhood F_x^0 as the set of points $\{(x_1 - 1, x_2), (x_1 + 1, x_2), (x_1, x_2 - 1), (x_1, x_2 + 1)\}$ and the neighborhood F_x as $F_x^0 \bigcup \{x\}$. We say that $R \subset \mathbb{Z}^2$ is a region if there exist disjoint nonempty subsets $\mathring{R}$ and ∂R of R such that

(a) $R = \mathring{R} \cup \partial R$,
(b) if $x \in \mathring{R}$ then $F_x \subset R$,
(c) if $x \in \partial R$ then there is at least one point $y \in F_x^0$ such that $y \in \mathring{R}$.

Clearly, the subsets $\mathring{R}$ and ∂R are not defined uniquely by R, but henceforth, for a given region R in $\mathbb{Z}^2$ it will always be assumed that $\mathring{R}$ and ∂R are also given and fixed. Next, we say that x is an interior (boundary) point of R if $x \in \mathring{R}$ ($x \in \partial R$). Further, a region $R \subset \mathbb{Z}^2$ is said to be connected if for any $y, z \in R$ there exists a sequence $x^{(1)}, \ldots, x^{(n)} \in R$ with $x^{(1)} = y$ and $x^{(n)} = z$, such that for all $0 \leq i \leq n - 1$, $|x^{(i)} - x^{(i+1)}| = 1$. By definition, a region R with one interior point x is connected and coincides with F_x. Denote by S_N a region defined as a discrete square $([-N, N]^2 \cap \mathbb{Z}^2)\backslash\{(N, N), (-N, N), (-N, -N), (N, -N)\}$, $N \in \mathbb{N}$, where $\mathring{S}_N := [-N + 1, N - 1]^2 \cap \mathbb{Z}^2$ and $\partial S_N := S_N \backslash \mathring{S}_N$ will be fixed throughout the paper.

A boundary point $y \in \partial R$ is said to be

$$\begin{aligned} &\text{a left point if} && y + e_1 \in \mathring{R}, \\ &\text{a right point if} && y - e_1 \in \mathring{R}, \\ &\text{a top point if} && y - e_2 \in \mathring{R}, \\ &\text{a bottom point if} && y + e_2 \in \mathring{R}. \end{aligned}$$

The union of all left (right, top and bottom) points we denote by ∂R_l (∂R_r, ∂R_t and ∂R_b, respectively) and call it a side of the boundary ∂R. Note that a boundary point y can simultaneously be a left, right, top and bottom point. Thus, ∂R_l, ∂R_r, ∂R_t and ∂R_b may overlap each other. Clearly, ∂R is the union of its four sides, $\partial R = \partial R_l \bigcup \partial R_r \bigcup \partial R_t \bigcup \partial R_b$.

Let Γ_j, $j = 1, \ldots, n$, $n \in \mathbb{N}$, be a finite row or column of lattice sites and consider a region $\mathring{\Omega} = \mathbb{Z}^2 \backslash \partial\Omega$, where $\partial\Omega = \cup_{j=1}^n \Gamma_j$. From now on, we assume that Γ_j are located so that $\mathring{\Omega}$ is connected and satisfies the cone condition, cf. [8, 9]. Finally, we emphasize that $\Omega = \mathring{\Omega} \cup \partial\Omega$ and $\mathbb{Z}^2$ coincide as the sets.

Given the problem and assumptions described in Introduction, we suppose that there is an inductor connecting each node $x \in \mathring{\Omega}$ to a common ground plane, and there is a capacitor connecting each node

$x \in \mathring{\Omega}$ to its four nearest neighbors $(x_1 \pm 1, x_2 \pm 1)$ (cf., Figure 1). Assume that all inductances equal to L and all capacitances equal to C, where both L and C are positive constants. Then, Kirchhoff's laws of voltage and current (while suppressing the explicit dependence on time t) imply the following second-order equation for the voltage $U(x)$ across the inductor at the node x:

$$LC\frac{d^2}{dt^2}(\Delta_d U(x)) = U(x). \tag{1}$$

Here, Δ_d denotes the discrete Laplacian defined as follows:

$$\Delta_d U(x) = \sum_{i=1}^{2}(U(x+e_i) + U(x-e_i)) - 4U(x). \tag{2}$$

We specify that (1) holds for all $x \in \mathring{\Omega}$ and along the boundary $\partial\Omega$ we have the time-dependent boundary condition

$$U(y) = f(y)e^{-\iota\omega t}. \tag{3}$$

Here, ι denotes the imaginary unit, and $f : \partial\Omega \to \mathbb{C}$ is a given function. We assume that at time $t = 0$, $U(x)$ and all its derivatives are zero for all $x \in \mathring{\Omega}$. Then, as t increases, the boundary term causes wave to propagate into the lattice, and the system approaches steady state. At this point, the solution is given by $U(x) = u(x)e^{-\iota\omega t}$. Substituting this expression into (1) and (3), for the discrete Helmholtz equation in Ω, we obtain the following problem:

$$(\Delta_d + k^2)u(x) = 0, \qquad \text{in } \mathring{\Omega}, \tag{4a}$$

$$u(y) = f(y), \quad \text{on } \partial\Omega, \tag{4b}$$

where k and ω are related through the relation $k^2 = (\omega^2 LC)^{-1}$. It is well known that (1) admits plane wave solutions $U(x) = Ae^{\iota(-\xi_1 x_1 - \xi_2 x_2 - \omega t)}$, where $A \in \mathbb{C}$ is a constant, as long as the following dispersion relation

$$\omega^2 = \frac{1}{4LC(\sin^2\frac{\xi_1}{2} + \sin^2\frac{\xi_2}{2})},$$

is satisfied, where $(\xi_1, \xi_2) \in [-\pi, \pi]^2$ known as the first Brillouin zone [4]. Thus we have

$$k^2 = 4\left(\sin^2\frac{\xi_1}{2} + \sin^2\frac{\xi_2}{2}\right) \in [0, 8].$$

Clearly, other values of k are also a subject of investigation, but this case is rather straightforward and will not be considered here.

Recall that for the negative discrete Laplacian, the spectrum is (absolutely continuous) $[0, 8]$, but there is an exceptional set $\{0, 4, 8\}$ in $[0, 8]$, where the limiting absorption principle fails (cf., [15]). Consequently, we assume that $k \in (0, 2\sqrt{2})\backslash\{2\}$.

Thus, we are interested in studying the problem of the existence and uniqueness of a function $u : \Omega \to \mathbb{C}$ such that $u(x)$ satisfies the discrete Helmholtz equation (4a) with $k \in (0, 2\sqrt{2})\backslash\{2\}$ and the boundary condition (4b). From now on, we will refer to this problem as Problem $\mathcal{P}$.

3. Green's Representation Formula and Uniqueness Result

In this section we mainly recall the results from [9]. Let R be a region in $\mathbb{Z}^2$. As it was already mentioned above, $y \in \partial R$ may be a point of intersection of several sides of ∂R. However, in our arguments presented below, it will always be clear which side is needed to be considered. Under this condition, we define the discrete derivative in the outward normal direction

$$Tu(y) = u(y) - u(y - \nu_y), \quad y \in \partial R,$$

where ν_y is $-e_1$ (e_1, e_2 or $-e_2$) if y is an element of ∂R_l (∂R_r, ∂R_t or ∂R_b).

Let R be a finite region. Then we have a discrete analogues of the Green's first and second identities

$$\sum_{x \in \mathring{R}}(\nabla_d^+ u(x) \cdot \nabla_d^+ v(x) + \nabla_d^- u(x) \cdot \nabla_d^- v(x) + u(x)\Delta_d v(x)) = \sum_{y \in \partial R} u(y)Tv(y), \tag{5}$$

and

$$\sum_{x\in\mathring{R}}(u(x)\Delta_d v(x) - v(x)\Delta_d u(x)) = \sum_{y\in\partial R}(u(y)Tv(y) - v(y)Tu(y)), \tag{6}$$

respectively. Here, ∇_d^+ and ∇_d^- are defined as follows:

$$\nabla_d^+ u(x) := \begin{pmatrix} u(x+e_1) - u(x) \\ u(x+e_2) - u(x) \end{pmatrix},$$

and

$$\nabla_d^- u(x) := \begin{pmatrix} u(x-e_1) - u(x) \\ u(x-e_2) - u(x) \end{pmatrix}.$$

The next step in deriving the Green's representation formula is the introduction of the Green's function. Denote by $\mathcal{G}(x-y)$ the Green's function for (4a) centered at the point x and evaluated at y. Then $\mathcal{G}(x-y)$ satisfies

$$(\Delta_d + k^2)\mathcal{G}(x-y) = \delta_{x,y}, \tag{7}$$

where $\delta_{x,y}$ is the Kronecker delta. The lattice Green's function $\mathcal{G}$ is quite well known (cf., e.g., [7, 10, 12, 20]) and can be written in the following form:

$$\mathcal{G}(x) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{-\iota(x_1\xi_1 + x_2\xi_2)}}{\sigma(\xi_1,\xi_2;k)}\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{8}$$

or, equivalently, as

$$\mathcal{G}(x) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\frac{\cos(x_1\xi_1)\cos(x_2\xi_2)}{\sigma(\xi;k)}\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{9}$$

where

$$\begin{aligned}
\sigma(\xi;k) &= k^2 - 4 + 2\cos\xi_1 + 2\cos\xi_2 \\
&= k^2 - 4\sin^2\frac{\xi_1}{2} - 4\sin^2\frac{\xi_2}{2} \\
&= k^2 - 8 + 4\cos^2\frac{\xi_1}{2} + 4\cos^2\frac{\xi_2}{2} \\
&= k^2 - 4 + 4\cos\frac{\xi_1+\xi_2}{2}\cos\frac{\xi_1-\xi_2}{2}, \quad \xi = (\xi_1,\xi_2).
\end{aligned} \tag{10}$$

Notice that if $k^2 \in \mathbb{C}\backslash[0,8]$, then $\sigma \neq 0$ and, consequently, $\mathcal{G}(x)$ in (8) is well defined. In this case, $\mathcal{G}(x)$ decays exponentially when $|x| \to \infty$. For the cases $0 < k^2 < 4$ and $4 < k^2 < 8$, the expression (8) is understood as follows: we replace k^2 by $k^2 + \iota\varepsilon$ with $0 < \varepsilon \ll 1$, and let $\varepsilon \to 0$ at the end of the calculation. We use Koster's method to evaluate the integrals, cf., [12]. Finally, notice that $\mathcal{G}(x-y) = \mathcal{G}(y-x)$.

Theorem 1. *Let R be a finite region. Then for a given function $u : \mathring{R} \to \mathbb{C}$ and any point $x \in \mathring{R}$, we have a discrete Green's representation formula*

$$u(x) = \sum_{y\in\partial R}\big(u(y)T\mathcal{G}(x-y) - \mathcal{G}(x-y)Tu(y)\big) + \sum_{y\in\mathring{R}}\mathcal{G}(x-y)(\Delta_d + k^2)u(y).$$

In particular, if u is a solution to the discrete Helmholtz equation

$$(\Delta_d + k^2)u(x) = 0 \quad \text{in } \mathring{R},$$

then

$$u(x) = \sum_{y\in\partial R}\big(u(y)T\mathcal{G}(x-y) - \mathcal{G}(x-y)Tu(y)\big). \tag{11}$$

Further, we need to apply the notion of a radiation condition for the discrete Helmholtz operators. We emphasize that we are dealing with the case $k \in (0, 2\sqrt{2})\backslash\{2\}$, and, therefore, it is natural to require an extra condition at infinity (cf., [15]). We say that $u : \Omega \to \mathbb{C}$ satisfies the radiation condition at infinity, if

$$\begin{cases} u(x) = O(|x|^{-\frac{1}{2}}) \\ u(x+e_j) = -e^{\iota \xi_j^*(\alpha,k)} u(x) + O(|x|^{-\frac{3}{2}}), \quad j = 1, 2, \end{cases} \tag{12}$$

with the remaining term decaying uniformly in all directions $x/|x|$, where $x \in \Omega$ is characterized as $x_1 = |x|\cos\alpha$, $x_2 = |x|\sin\alpha$, $0 \le \alpha < 2\pi$. Here, $\xi_j^*(\alpha, k)$ is the jth coordinate of the point $\xi^*(\alpha, k)$ defined as follows:

$$\begin{aligned} \xi_1^*(\alpha,k) &= 2\,\mathrm{sgn}\big((\pi/2-\alpha)(3\pi/2-\alpha)\big) \arcsin\left(\frac{k}{2}\cos\theta^*\right), \\ \xi_2^*(\alpha,k) &= 2\,\mathrm{sgn}\big(\pi-\alpha\big) \arcsin\left(\frac{k}{2}\sin\theta^*\right), \end{aligned} \tag{13}$$

for $0 < k^2 < 4$. Here,

$$\theta^* = \theta^*(\alpha,k) := \begin{cases} \arctan\sqrt{-\lambda + \sqrt{\lambda^2 + \tan^2\alpha}}, & \text{if} \quad \alpha \neq \frac{\pi}{2}, \frac{3\pi}{2}, \\ \frac{\pi}{2}, & \text{if} \quad \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \end{cases} \tag{14}$$

where

$$\lambda = \lambda(\alpha,k) := \frac{2(1-\tan^2\alpha)}{4-k^2}.$$

In the second case $4 < k^2 < 8$, we have

$$\begin{aligned} \xi_1^*(\alpha,k) &= 2\,\mathrm{sgn}\big((\pi/2-\alpha)(3\pi/2-\alpha)\big) \arccos\left(\frac{\sqrt{8-k^2}}{2}\cos\theta^*\right), \\ \xi_2^*(\alpha,k) &= 2\,\mathrm{sgn}\big(\pi-\alpha\big) \arccos\left(\frac{\sqrt{8-k^2}}{2}\sin\theta^*\right), \end{aligned} \tag{15}$$

where θ^* is defined as in (14) with

$$\lambda = \lambda(\alpha,k) := \frac{2(1-\tan^2\alpha)}{k^2-4}.$$

Definition 2. Let $k \in (0, 2\sqrt{2})\backslash\{2\}$. A solution u to the discrete Helmholtz equation (4a) is called radiating if it satisfies the radiation condition (12).

Notice that the second condition in (12) can be written in the following forms

$$u(x) - u(x+e_j) = \left(1 + e^{\iota\xi_j^*(\alpha,k)}\right) u(x) + O(|x|^{-\frac{3}{2}}), \quad |x| \to \infty, \tag{16}$$

and

$$u(x+e_j) - u(x) = \left(1 + e^{-\iota\xi_j^*(\alpha,k)}\right) u(x+e_j) + O(|x|^{-\frac{3}{2}}), \quad |x| \to \infty. \tag{17}$$

Notice also that $u(x) = O(|x|^{-\frac{1}{2}})$ implies

$$\frac{1}{N} \sum_{x \in S_{2N}\backslash S_N} |u(x)|^2 < const < \infty$$

for any integer $N \ge N_0$.

For the fixed point $x \in \mathbb{Z}^2$ and any point $y \in \partial S_N$, the radiation conditions (12) imply

$$\sum_{y \in \partial S_N} \big(u(y) T\mathcal{G}(x-y) - \mathcal{G}(x-y) Tu(y)\big) \to 0, \quad N \to \infty.$$

Indeed, for instance, since $\alpha(y-x)$ tends to $\alpha=\alpha(y)$ as $|y|\to\infty$, therefore for a sufficiently large N, we have

$$\begin{aligned}
u(y)\big(\mathcal{G}(x-y)-\mathcal{G}(x-(y+e_1))\big)&-\mathcal{G}(x-y)(u(y)-u(y+e_1))\\
&=u(y)\left[(e^{\iota\xi_j^*(\alpha,k)}+1)\mathcal{G}(x-y)+O(N^{-\frac{3}{2}})\right]\\
&\quad-\mathcal{G}(x-y)\left[(e^{\iota\xi_j^*(\alpha,k)}+1)u(y)+O(N^{-\frac{3}{2}})\right]\\
&=u(y)\cdot O(N^{-\frac{3}{2}})+\mathcal{G}(x-y)\cdot O(N^{-\frac{3}{2}})=O(N^{-2}).
\end{aligned}$$

Consequently, Theorem 1 applied for $\Omega\cap S_N$, where $N\in\mathbb{N}$ is sufficiently large, and then passing to the limit $N\to\infty$, yields the following Green's formula for a radiating solution u to the discrete Helmholtz equation (4a)

$$u(x)=\sum_{y\in\partial\Omega}(u(y)T\mathcal{G}(x-y)-\mathcal{G}(x-y)Tu(y))=\sum_{j=1}^{n}\sum_{y\in\Gamma_j}(u(y)T\mathcal{G}(x-y)-\mathcal{G}(x-y)Tu(y)). \tag{18}$$

Using the results obtained in [12, 15] and (18), we can conclude that every radiating solution u to the discrete Helmholtz equation (4a) has the following asymptotic expansion:

$$u(x)=-\frac{e^{\iota\mu(\alpha,k)|x|}}{|x|^{\frac{1}{2}}}\left\{u_\infty(\hat{x})+O\left(\frac{1}{|x|}\right)\right\},\quad |x|\to\infty, \tag{19}$$

where $\mu(\alpha,k):=\xi^*(\alpha,k)\cdot\hat{x}$, $\hat{x}:=x/|x|$, and $\xi^*(\alpha,k)=(\xi_1^*(\alpha,k),\xi_2^*(\alpha,k))$ (cf., (13) and (15)). Here, the function $u_\infty(\hat{x})$, known as the far field pattern of u, can be expressed with the help of formula (13) from [15].

Finally, let us formulate the following uniqueness theorem.

Theorem 3 (cf. [9])**.** *The problem $\mathcal{P}$ has at most one radiating solution.*

4. Difference Potentials and Existence of a Solution

For any function $\varphi:\partial R\to\mathbb{C}$, we define difference single-layer and double-layer potentials as follows:

$$V\varphi(x)=\sum_{y\in\partial R}\mathcal{G}(x-y)\varphi(y),\quad \text{for all } x\in\mathbb{Z}^2, \tag{20}$$

and

$$W\varphi(x)=\sum_{y\in\partial R}\big(T\mathcal{G}(x-y)+\delta_{x,y}\big)\varphi(y),\quad \text{for all } x\in\mathbb{Z}^2, \tag{21}$$

respectively. Since, $\delta_{x,y}=0$ for every $x\in\mathring{R}$ and $y\in\partial R$, the equation (18) can be written as

$$u(x)=Wu(x)-V(Tu)(x),\quad x\in\mathring{\Omega}.$$

The role of the summand $\delta_{x,y}$ is clarified by the following result.

Lemma 4 (cf. [9])**.** *For every $x\in\mathring{R}$, we have*

$$(\Delta_d+k^2)V\varphi(x)=0,\quad\text{and}\quad(\Delta_d+k^2)W\varphi(x)=0.$$

As a consequence of Lemma 4,

$$V\varphi(x)=\sum_{y\in\partial\Omega}\mathcal{G}(x-y)\varphi(y),\quad x\in\mathring{\Omega},$$

and

$$W\varphi(x)=\sum_{y\in\partial\Omega}\left(T\mathcal{G}(x-y)+\delta_{x,y}\right)\varphi(y),\quad x\in\mathring{\Omega},$$

are radiating solutions to the equation (4a) for any function $\varphi:\partial\Omega\to\mathbb{C}$.

Now we are ready to find a solution to Problem $\mathcal{P}$. In order to reduce a number of numerical computations, we do the following: if y_i is a point of intersection of several sides, we choose and fix only one side of the boundary. Let m be a number of points of $\partial\Omega$. Then $\partial\Omega$ can be represented as a sequence of points $y_1, y_2, ..., y_m$, such that $y_i = y_j$, if and only if $i = j$, for all $1 \le i, j \le m$. Thus, in (20) we will have only one summand related to the boundary point y_i, and we denote the corresponding difference potentials by $\widetilde{V}$. Further, from the given function f on $\partial\Omega$, we form a vector $F = (f_1, \dots, f_m)^\top$, $f_i := f(y_i)$. Similarly, for an unknown function φ, we write $\Phi = (\varphi_1, \dots, \varphi_m)^\top$, $\varphi_i := \varphi(y_i)$.

We look for a solution to Problem $\mathcal{P}$ in the form

$$u(x) = \widetilde{V}\varphi(x) = \sum_{i=1}^{m} \mathcal{G}(x - y_i)\varphi_i, \quad x \in \mathring{\Omega}. \tag{22}$$

As in the proof of Lemma 4, we can easily show that u is a radiating solution to Eq. (4a), and we only need to satisfy the boundary conditions (4b). Then (4b) implies the following linear system of boundary equations

$$\mathcal{H}\Phi = F, \tag{23}$$

where

$$\mathcal{H} = \begin{pmatrix} \mathcal{G}(y_1 - y_1) & \mathcal{G}(y_1 - y_2) & \mathcal{G}(y_1 - y_3) & \dots & \mathcal{G}(y_1 - y_m) \\ \mathcal{G}(y_2 - y_1) & \mathcal{G}(y_2 - y_2) & \mathcal{G}(y_2 - y_3) & \dots & \mathcal{G}(y_2 - y_m) \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{G}(y_m - y_1) & \mathcal{G}(y_m - y_2) & \mathcal{G}(y_m - y_3) & \dots & \mathcal{G}(y_m - y_m) \end{pmatrix}.$$

Lemma 5. *The linear system of boundary equations* (23) *is uniquely solvable.*

Proof. It is sufficient to proof that the homogeneous system

$$\mathcal{H}\Phi = 0 \tag{24}$$

has only the trivial solution. Let $\Phi^* = (\varphi_1^*, \dots, \varphi_m^*)^\top$ be a nonzero solution to (24). Then

$$u(x) = \widetilde{V}\varphi^*(x)$$

is a radiating solution to the homogeneous Problem $\mathcal{P}$. Therefore, due to Theorem 3, we have $u \equiv 0$ in Ω. Since $\Omega = \mathbb{Z}^2$, at any boundary point $y_i \in \partial\Omega$, we have

$$0 = (\Delta_d + k^2)u(y_i) = (\Delta_d + k^2)\widetilde{V}\varphi^*(y_i) = \sum_{i=1}^{m} \delta_{y_i,y}\varphi^*(y) = \varphi^*(y_i) = \varphi_i^*,$$

for all $1 \le i \le m$. □

Due to a direct combination of the results obtained above, we have now the main conclusions of the present work.

Theorem 6. *Problem $\mathcal{P}$ has a unique radiating solution which is represented as* (22)*, where Φ is a unique solution to the system of linear equations* (23)*.*

5. Numerical Results and Illustrative Examples

The main task in numerically evaluating (23) is to compute the lattice Green's function. For this purpose, we apply the method developed in [1]. Using the eightfold symmetry, we need only to compute the lattice Green's function $\mathcal{G}(i, j)$ with $i \ge j \ge 0$. Following [1], let us introduce the vectors $\mathcal{V}_{2p} = (\mathcal{G}(2p, 0), \mathcal{G}(2p-1, 1), \dots, \mathcal{G}(p, p))^\top$, and $\mathcal{V}_{2p+1} = (\mathcal{G}(2p+1, 0), \mathcal{G}(2p, 1), \dots, \mathcal{G}(p+1, p))^\top$ that collect all distinct Green's functions $\mathcal{G}(i, j)$ with "Manhattan distances" $|i| + |j|$ of $2p$ and $2p + 1$, respectively. For any Manhattan distance larger than 1, the equation

$$(\Delta_d + k^2)\mathcal{G}(x) = \delta_{x,0}$$

can be written in the matrix form $\mathcal{V}_n = \alpha_n(k)\mathcal{V}_{n-1} + \beta_n(k)\mathcal{V}_{n+1}$, where $\alpha_n(k)$ and $\beta_n(k)$ are easy to identify sparse matrices (cf., "Appendix A"). Notice that only the dimensions of these matrices depend on n, their elements are just multiples of $\frac{1}{4-k^2}$ and do not depend on n. It is shown in [1] that for any $n \ge 1$, we have

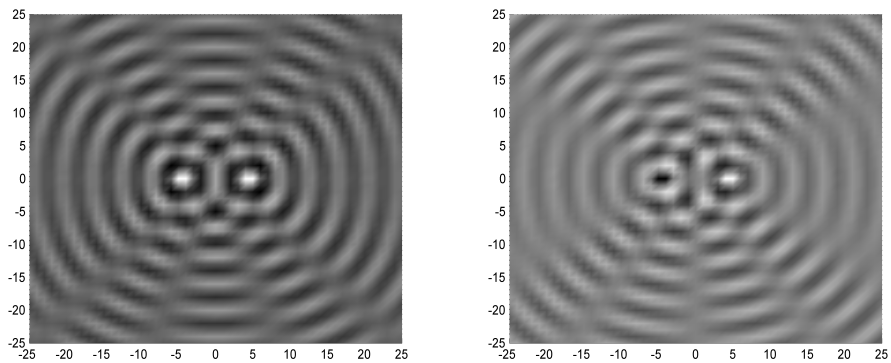
$$\mathcal{V}_n = A_n(k)\mathcal{V}_{n-1},$$

where the matrices $A_n(k)$ are defined by the following recurrence formula:

$$A_n(k) = [1 - \beta_n(k) A_{n+1}]^{-1} \alpha_n(k).$$

They can be computed starting from the sufficiently large N with $A_{N+1}(k) = 0$. However, we may have a better "initial guess" than $A_{N+1}(k) = 0$ (cf., "Appendix A").

Since $A_n(k)$ are known, we have $\mathcal{V}_n = A_n(k) \dots A_1(k)\mathcal{V}_0$, where $\mathcal{V}_0 = \mathcal{G}(0,0)$. In particular, $\mathcal{V}_1 = \mathcal{G}(1,0) = A_1(k)\mathcal{G}(0,0)$ which, together with $(k^2-4)\mathcal{G}(0,0) + 4\mathcal{G}(1,0) = 1$, gives $\mathcal{G}(0,0) = 1/[k^2 - 4 - 4A_1(k)]$. This completes the calculation of Green's function by using elementary operations and no integrals. Notice also one more important advantage of this method. The $A_n(k)$ matrices are calculated coming down from asymptotically large Manhattan distances. As they are propagated towards smaller Manhattan distances, it will definitely give us the physical solution.



(A) The density plot of $\mathrm{Re}\, u$. Symmetric mode.

(B) The density plot of $\mathrm{Re}\, u$. Skew-symmetric mode.

FIGURE 2. In the density plots, darker shade represents the lower values.

Finally, we demonstrate our theoretical and numerical approaches to Problem $\mathcal{P}$, when $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{(-5,0), (-4,0)\}$ and $\Gamma_2 = \{(4,0), (5,0)\}$, cf. Figure 1. We consider two examples. In the first case of the symmetric mode, we take for simplicity $f(y) = 1$ on both line segments, while in the second case of the skew-symmetric mode, we take $f(y) = -1$ on Γ_1 and $f(y) = 1$ on Γ_2.

The vector $\Phi = (\varphi_1, \dots, \varphi_4)^\top$ is a unique solution to (23), where

$$\mathcal{H} = \begin{pmatrix} \mathcal{G}(0,0) & \mathcal{G}(1,0) & \mathcal{G}(9,0) & \mathcal{G}(10,0) \\ \mathcal{G}(1,0) & \mathcal{G}(0,0) & \mathcal{G}(8,0) & \mathcal{G}(9,0) \\ \mathcal{G}(9,0) & \mathcal{G}(8,0) & \mathcal{G}(0,0) & \mathcal{G}(1,0) \\ \mathcal{G}(10,0) & \mathcal{G}(9,0) & \mathcal{G}(1,0) & \mathcal{G}(0,0) \end{pmatrix}$$

is a symmetric matrix. In order to solve the obtained system of linear equations (23) and then find the solution u, we have developed MATLAB code that uses the efficient method described above to compute Green's functions. As a technical aside, these data were obtained in several minutes on a regular desktop. Some results of numerical evaluations are plotted in Figure 2, where (a) and (b) show the density plots of $\Re\mathrm{e}\, u$ for the first and second examples, respectively, when $k = \sqrt{2}$. Some key features of numerical solution can be immediately observed. Namely, the symmetry of $\Re\mathrm{e}\, u$ and the interference of waves.

Appendix A: Sparse Matrices

The sparse matrices $\alpha_n(k)$ and $\beta_n(k)$ are defined as follows: if $n = 2p$, then $\alpha_{2p}(k)$ is a $(p+1) \times p$ matrix such that $\alpha_{2p}(k)|_{i,i} = \frac{1}{4-k^2}$, $i = \overline{1,p}$, $\alpha_{2p}(k)|_{i,i-1} = \frac{1}{4-k^2}$, $i = \overline{2,p}$, while $\alpha_{2p}(k)|_{p+1,p} = \frac{2}{4-k^2}$, and all other matrix elements are zero. The $\beta_{2p}(k)$ is a $(p+1) \times (p+1)$ matrix such that $\beta_{2p}(k)|_{i,i} =$

$\frac{1}{4-k^2}$, $i = \overline{1,p}$, $\beta_{2p}(k)|_{i,i+1} = \frac{1}{4-k^2}$, $i = \overline{2,p}$, while $\beta_{2p}(k)|_{p+1,p+1} = \beta_{2p}(k)|_{1,2}\frac{2}{4-k^2}$, and all other matrix elements are zero.

If $n = 2p+1$, then $\alpha_{2p+1}(k)$ is a $(p+1) \times (p+1)$ matrix such that $\alpha_{2p+1}(k)|_{i,i} = \frac{1}{4-k^2}$, $i = \overline{1,p+1}$, $\alpha_{2p+1}(k)|_{i,i-1} = \frac{1}{4-k^2}$, $i = \overline{2,p}$, and all other matrix elements are zero. The $\beta_{2p+1}(k)$ is a $(p+1) \times (p+2)$ matrix such that $\beta_{2p+1}(k)|_{i,i} = \frac{1}{4-k^2}$, $i = \overline{1,p+1}$, $\beta_{2p+1}(k)|_{i,i+1} = \frac{1}{4-k^2}$, $i = \overline{2,p+1}$, while $\beta_{2p+1}(k)|_{1,2} = \frac{2}{4-k^2}$, and all other matrix elements are zero.

Let N be a sufficiently large number with $N+1 = 2p$, $p \in \mathbb{Z}$ and let $\lambda = \lambda(k)$ be a root of the polynomial $2\lambda^2 + (k^2-4)\lambda + 2 = 0$ such that $|\lambda| < 1$. Then $A_{N+1}(k)$ is a $(p+1) \times p$ matrix, where $A_{N+1}(k)|_{i,i} = A_{N+1}(k)|_{i,i-1} = \frac{1}{4-k^2-2\lambda}$, $i = \overline{2,p}$, while $A_{N+1}(k)|_{1,1} = \frac{1}{4-k^2-3\lambda}$, $A_{N+1}(k)|_{p+1,p} = \frac{2}{4-k^2-2\lambda}$, and all other matrix elements are zero.

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THE PROBLEMS OF A PUNCH IN THE LINEAR THEORY OF VISCO-ELASTICITY

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Abstract. The problem of pressure of a rigid punch upon a viscous half-plane is considered. As is known, building and composition materials possess the property of visco-elasticity and its affect is reflected in the Hook's law. Unlike the elastic bond, the stresses for visco-elastic bodies are proportional to deformations and to their time derivatives. Investigations of different possible forms of visco-elastic correlations can be found in [1–5, 8–10].

The goal of the present work is to extend the well-known Kolosov–Muskhelishvili's method elaborated for the problem of pressure of a rigid punch in the case of the classical theory of plane elasticity to the theory of linear visco-elasticity based of the Kelvin–Vogt model [9].

1. Introduction

One of the models of the linear theory of visco-elasticity is the Kelvin–Vogt model which is characterized by the fact that stresses in the Hook's law are proportional both to deformations and to time derivatives, where the former describes the Hook's law and the latter the Newton law of viscosity.

Following the Kelvin–Vogt model [9], the Hook's law for visco-elasic bodies has the form

$$\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{xx}, \\ Y_y &= \lambda\theta + 2\mu e_{yy} + \lambda^*\dot{\theta} + 2\mu^*\dot{e}_{yy}, \\ X_y &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu^*\left(\frac{\partial \dot{v}}{\partial x} + \frac{\partial \dot{u}}{\partial y}\right), \end{aligned} \tag{1}$$

where $\vartheta = e_{xx} + e_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, X_x, Y_y, X_y, u, v, e_{xx}, e_{yy}, e_{xy} are the functions of variables x, y, t. Under t we mean the time parameter and the points in the expressions $\dot{\theta}, \dots, \dot{u}$ denote derivatives in time t; λ, μ and λ^*, μ^* are, respectively, elastic and visco-elastic constants.

In what follows, the use will be made of the known Kolosov–Muskhelishvili's formulas which may be referred to any solid bodies. The above-mentioned formulas are of the form [6]

$$\begin{aligned} X_x + Y_y &= 4\operatorname{Re}[\Phi(z,t)] = 4\operatorname{Re}[\varphi'(z,t)], \\ Y_y - X_x + 2iX_y &= 2[\overline{z}\varphi''(z,t) + \psi'(z,t)] = 2[\overline{z}\Phi'(z,t) + \Psi(z,t)], \end{aligned} \tag{2}$$

where $\Phi(z,t) = \varphi'(z,t)$; $\Psi(z,t) = \psi'(z,t)$. From (2) we have the formula

$$Y_y - iX_y = \Phi(z,t) + \overline{\Phi(z,t)} + z\overline{\Phi'(z,t)} + \overline{\Psi(z,t)}, \tag{3}$$

which will frequently be used in the sequel.

The principle vector (X, Y) of external forces applied to the boundary is assumed to be finite and the stresses and rotation vanish at infinity, hence for large $|z|$, we have

$$\Phi(z,t) = -\frac{X+iY}{2\pi z} + o\left(\frac{1}{z}\right); \quad \Psi(z,t) = \frac{X-iY}{2\pi z} + o\left(\frac{1}{z}\right).$$

From relations (1), in view of (2), for the function $\vartheta(z,t) = e_{xx} + e_{yy}$ we get the differential equation

$$\dot{\vartheta}(z,t) + k\vartheta(z,t) = \frac{2}{\lambda^* + \mu^*}\operatorname{Re}[\varphi'(z,t)], \quad \left(k = \frac{\lambda+\mu}{\lambda^*+\mu^*}\right),$$

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a solution of which under zero initial conditions (i.e., for $\vartheta(z;0)=0$) has the form

$$\vartheta(z,t)=\frac{2}{\lambda^*+\mu^*}\int\limits_0^t \operatorname{Re}[\varphi'(z,\tau)]e^{k(\tau-t)}d\tau. \tag{4}$$

Analogously, from (1) and (2), for the function $\gamma(z,t)=e_{xx}-e_{yy}$ we obtain the differential equation

$$\dot\gamma(z,t)+m\gamma(z,t)=-\frac{1}{\mu^*}\operatorname{Re}[\overline{z}\varphi''(z,t)+\psi'(z,t)], \quad \left(m=\frac{\mu}{\mu^*}\right),$$

a solution of which under zero initial conditions has the form

$$\gamma(z,t)=-\frac{1}{\mu^*}\int\limits_0^t \operatorname{Re}[\overline{z}\varphi''(z,\tau)+\psi'(z,\tau)]e^{m(\tau-t)}d\tau. \tag{5}$$

Thus in view of (4) and (5), with respect to e_{xx} and e_{yy}, we have a system which solution is represented as follows:

$$\begin{aligned} e_{xx}=&\frac{1}{2\mu^*}\int\limits_0^t \operatorname{Re}\Big[\text{æ}^*\varphi'(z,\tau)e^{k(\tau-t)}-\left(\overline{z}\varphi''(z,\tau)+\psi'(z,\tau)\right)e^{m(\tau-t)}\Big]d\tau,\\ e_{yy}=&\frac{1}{2\mu^*}\int\limits_0^t \operatorname{Re}\Big[\text{æ}^*\varphi'(z,\tau)e^{k(\tau-t)}+\left(\overline{z}\varphi''(z,\tau)+\psi'(z,\tau)\right)e^{m(\tau-t)}\Big]d\tau, \end{aligned} \tag{6}$$

where

$$\text{æ}^*=\frac{2\mu^*}{\lambda^*+\mu^*}.$$

Taking into account equalities $dx=dz$, $dx=d\overline{z}$, $dy=-idz$, $dy=id\overline{z}$, and integrating (6), we obtain the formula

$$2\mu^*(u+iv)=\int\limits_0^t \Big[\text{æ}^*\varphi(z,\tau)e^{k(\tau-t)}+\Big(\varphi(z,\tau)-z\overline{\varphi'(z,\tau)}-\overline{\psi(z,\tau)}\Big)e^{m(\tau-t)}\Big]d\tau+2\mu^*(u_0+iv_0), \tag{7}$$

where $u_0=u(z,0)$, $v_0=v(z,0)$.

Formula (7) is an analogue of Kolosov–Muskhelishvili's formula for the second basic problem of the plane theory of elasticity (see [6]) in the case of a visco-elastic isotropic body.

From formula (7), differentiating with respect to x, we get

$$\begin{aligned} 2\mu^*v'(x,y,t)=&\operatorname{Im}\Big[\int\limits_0^t \text{æ}^*e^{k(\tau-t)}\Phi(z,\tau)d\tau\Big]\\ &+\operatorname{Im}\Big[\int\limits_0^t e^{m(\tau-t)}\Big(\Phi(z,\tau)-\overline{\Phi(z,\tau)}-z\overline{\Phi'(z,\tau)}-\overline{\Psi(z,\tau)}\Big)d\tau\Big]+2\mu^*v_0'(x,y,0). \end{aligned} \tag{8}$$

Statement of the Problem. Let a visco-elastic body occupy the lower half-plane S^-. By L we denote the boundary of that domain (i.e., the Ox-axis) and assume that a segment $L'=[-1;1]$ enters in contact with a punch having a given base shape and the punch is pressed into the half-plane with a given force directed vertically downward. Assume also that the punch displacement is translational in a normal direction with respect to the boundary, in the absence of friction. Under the given assumptions, tangential stress is zero and the boundary conditions have the form

$$\begin{gathered} X_y^-(x,0,t)=0, \quad x\in L; \quad Y_y^-(x,0,t)=0, \quad x\in L''=L-L';\\ v^-(x,0,t)=f(x,t), \quad x\in L', \end{gathered} \tag{9}$$

where $f(x,0)=f(x)$ is the given function defining the shape of the punch base before it is pressed into the half-plane.

In the sequel, the expression $v^-(x,0,t)$ will be written as $v^-(x,0,t) = v^-(x,t)$ and so we will do for other similar expressions.

Assume that external forces acting on the punch have a resultant

$$X = 0, \qquad Y = -N_0 = -\int\limits_{-1}^{1} N(x,t)dx,$$

where $N(x,t)$ is a normal stress at the point $x \in L'$.

Our problem is to define elastic equilibrium of the domain S^- and normal stress $P(x,t)$ acting under the punch.

Solution of the Problem. Passing in (8) to the limit as $z \to x \in L'$ $(z \in S^-)$ and taking into account (3) and (9), we have

$$\operatorname{Im}\left[æ^* e^{-kt}\int\limits_0^t \Phi^-(x,\tau)e^{k\tau}d\tau + 2e^{-mt}\int\limits_0^t \Phi^-(x,\tau)e^{m\tau}d\tau\right] = 2\mu^* v'^-(x,t) - 2\mu^* v_0'^-(x,0). \tag{10}$$

Following N. I. Muskhelishvili (see [6]), we extend the function $\Phi(z,t)$ to the upper half-plane (S^+) in such a way that its values continue analytically the values of $\Phi(z,t)$ in the lower half-plane through the unloaded sections (i.e., on the section L'').

In our case, proceeding from the boundary conditions and formula (3), we define $\Phi(z,t)$ in S^+ as follows:

$$\Phi(z,t) = -\Phi_*(z,t) - z\Phi'_*(z,t) - \Psi_*(z,t), \qquad z \in S^+, \tag{11}$$

where $\Phi_*(z,t) = \overline{\Phi(\overline{z},t)}$; $\Psi_*(z,t) = \overline{\Psi(\overline{z},t)},$. From (11), we have

$$\Phi_*(z,t) = -\Phi(z,t) - z\Phi'(z,t) - \Psi(z,t).$$

The obtained in a such a way piecewise-holomorphic function we again denote by $\Phi(z,t)$. Then for finding the function $\Psi(z,t)$ by means of $\Phi(z,t)$, we obtain the following correlation

$$\Psi(z,t) = -\Phi(z,t) - \Phi_*(z,t) - z\Phi'(z,t),$$

and thus the stress and displacement components are expressed by one piecewise-holomorphic function $\Phi(z,t)$. Substituting the obtained value of $\Psi(z,t)$ into (3), we find that

$$Y_y - iX_y = \Phi(z,t) - \Phi(\overline{z},t) + (z-\overline{z})\overline{\Phi'(z,t)}.$$

On the basis of the above formula, we have

$$Y_y^-(x,t) - i\,X_y^-(x,t) = \Phi^-(x,t) - \Phi^+(x,t), \quad x \in L', \tag{12}$$

or passing to the complex-conjugate value and taking into account the equalities $\overline{\Phi^-(x,t)} = \Phi_*^+(x,t)$; $\overline{\Phi^+(x,t)} = \Phi_*^-(x,t)$, we obtain

$$Y_y^-(x,t) + i\,X_y^-(x,t) = \Phi_*^+(x,t) - \Phi_*^-(x,t). \tag{13}$$

Subtracting (18) and (13), in view of the fact that $X_y^-(x,t) = 0$, $x \in L$, we obtain

$$\Phi^-(z,t) + \Phi_*^-(z,t) = \Phi^+(z,t) + \Phi_*^+(z,t).$$

This implies that the function $\Phi(z,t) + \Phi_*(z,t)$ is holomorphic on the whole plane, and since it vanishes at infinity, we have the equality

$$\Phi_*(z,t) = -\Phi(z,t). \tag{14}$$

We get back now to equality (10). On the basis of (14), formula (10) can be written in the form

$$e^{-kt}\int\limits_0^t æ^* e^{k\tau}\Omega(x,\tau)d\tau + 2e^{-mt}\int\limits_0^t e^{m\tau}\Omega(x,\tau)d\tau = 4i\mu^* f_1(x,t), \tag{15}$$

where

$$\Omega(x,t) = \Phi^+(x,t) + \Phi^-; \quad f_1(x,t) = 4i\mu^*[f'(x,t) - f'(x)]. \tag{16}$$

Differentiating (15) with respect to t, we obtain

$$-ke^{-kt}\int\limits_0^t æ^* e^{k\tau}\Omega(x,\tau)d\tau - 2me^{-mt}\int\limits_0^t e^{m\tau}\Omega(x,\tau)d\tau + (æ^*+2)\Omega(x,t) = \dot{f}_1(x,t). \tag{17}$$

Multiplying (15) by m and summing with (17), we get

$$æ^*(m-k)\int\limits_0^t e^{k\tau}\Omega(x,\tau)d\tau + (æ^*+2)e^{kt}\Omega(x,t) = \dot{f}_2(x,t),$$

where

$$f_2(x,t) = e^{kt}[f_1(x,t) + mf_1(x,t)] \tag{18}$$

from which after differentiation with respect to t, we obtain the following equation:

$$\dot{\Omega}(x,t) + n\,\Omega(x,t) = -\frac{\dot{f}_2(x,t)}{æ^*+2}e^{-kt}, \tag{19}$$

where

$$n = \frac{\mathrm{m}æ^*+2k}{æ^*+2}.$$

Substituting $t=0$ into (17), we have

$$\Omega(x,0) = \frac{\dot{f}_1(x,0)}{æ^*+2}. \tag{20}$$

A solution of differential equation (19) under the initial condition (20) takes the form

$$\Omega(x,t) = e^{-nt}\left[\Omega(x,0) + \int\limits_0^t \frac{e^{(n-k)\tau}\dot{f}_2(x,\tau)}{æ^*+2}d\tau\right].$$

Thus, on the basis of (16), for the function $\Phi(z,t)$ we obtain the boundary value problem of linear conjugation

$$\Phi^+(x,t) + \Phi^-(x,t) = F(x,t), \tag{21}$$

where

$$F(x,t) = e^{-nt}\left[\Omega(0,\omega) + \frac{1}{æ^*+2}\int\limits_0^t e^{(n-k)\tau}\dot{f}_2(x,\tau)d\tau\right]. \tag{22}$$

The vanishing at infinity solution of problem (21) of the class h_0 (i.e., unbounded at the ends of the segment L') has the form (see [7])

$$\Phi(z,t) = -\frac{1}{2\pi\chi_0(z)}\int\limits_{-1}^{1}\frac{\chi_0^+(\sigma)F(\sigma,t)}{\sigma-z}d\sigma + \frac{C_0}{\chi_0(z)},$$

where $\chi_0(z) = \sqrt{(1-z)(1+z)} = -i\sqrt{(z-1)(z+1)}$, $\chi_0^+(\sigma)$ is the positive value of the function $\chi_0(z)$ on the left-hand side (i.e., from S^+) of the segment L'.

Taking into account behaviour of the function $\Phi(z,t)\chi_0(z)$ at infinity, for the constant C_0 we obtain the formula

$$C_0 = \frac{N_0}{2\pi}.$$

For the normal stress $P(x,t)$ under the punch, on the basis of (18) and (14), we get

$$P(x,t) = 2\,\mathrm{Re}\,\Phi^+(x,t),$$

or taking into account that from (22) follows

$$\mathrm{Re}\,\Phi^+(x,t) = \frac{1}{2\pi i}\frac{1}{\chi_0^+(x)}\int\limits_{-1}^{1}\frac{\chi_0^+(\sigma)F(\sigma,t)}{\sigma-x}d\sigma + \frac{N_0}{2\pi\chi_0^+(x)},$$

we will have

$$P(x,t) = -\frac{1}{\pi i \chi_0^+(x)} \int_{-1}^{1} \frac{\chi_0^+(\sigma)F(\sigma,t)}{\sigma - x} + \frac{N_0}{\pi \chi_0^+(x)},$$

where $F(\sigma,t)$ is defined by formula (22).

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MAXIMAL AND CALDERÓN–ZYGMUND OPERATORS IN WEIGHTED GRAND VARIABLE EXPONENT LEBESGUE SPACES

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Abstract. The boundedness of maximal and Calderón–Zygmund operators is established in weighted grand variable exponent Lebesgue spaces with power-type weights. The same problem for commutators of Calderón–Zygmund operators is also studied. These spaces unify two non-standard function spaces, namely, grand Lebesgue and variable exponent Lebesgue spaces. The spaces and operators are defined, generally speaking, on quasi-metric measure spaces with doubling measure. Exponents of spaces satisfy log-Hölder continuity condition.

1. Introduction

In this note, the boundedeness of maximal and singular integral operators is derived in the weighted grand variable exponent Lebesgue spaces (GVELS, for short) with power weights. The same problem is studied also for commutators of singular integrals. The operators and spaces are defined on quasi-metric measure spaces with doubling measure. The spaces of functions under consideration $L_w^{p(\cdot),\theta}$ are non-reflexive, non-separable and non-rearrangement invariant. Generally speaking, GVELS unifies two non-standard function spaces: variable exponent Lebesgue spaces and grand Lebesgue spaces. The unweighted GVELSs were introduced in [11] (see also [13, Chapter 14]), where also the mapping properties of operators of Harmonic Analysis were established. Later, in [6], the authors introduced new scale of GVELSs and studied integral operators boundedness in those spaces. In [7], the boundedness of the operator

$$Mf(x) := \sup_{r>0} M_r f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)\cap\Omega} |f(y)|dy$$

in $L_w^{p(\cdot),\theta}(\Omega)$ with $w(x) = |x - x_0|^\gamma$ was proved. In particular, it was proved the following statement:

Theorem 1.1 ([7]). *Let Ω be a bounded open set in $\mathbb{R}^n$, $1 < p_-(\Omega) \le p_+(\Omega) < \infty$, $\theta > 0$, $x_0 \in \Omega$, and suppose that γ is a constant satisfying the condition $-n < \gamma < n(p(x_0) - 1)$. Then the operator M is bounded in $L_w^{p(\cdot),\theta}(\Omega)$ with $w(x) = |x - x_0|^\gamma$.*

In the recent years it was realized that the classical function spaces are no longer appropriate spaces when we attempt to solve a number of contemporary problems arising naturally in: the non-linear elasticity theory, fluid mechanics, mathematical modelling of various physical phenomena, solvability problems of non-linear partial differential equations. It thus became necessary to introduce and study the spaces mentioned above from various viewpoints. One of such spaces is the variable exponent Lebesgue space (we refer, e.g., to the monographs [4], [3], [12] for the properties of variable exponent function spaces and integral operators in these spaces).

In this paper, we deal with the weighted grand variable exponent Lebesgue space $L_w^{p(\cdot),\theta}(X)$, with power-type weight $w(x) = d(x_0, x)^\beta$. This space for $\beta = 0$ has been introduced in [11] (see also [13, Ch. 14]). The norm in $L_w^{p(\cdot),\theta}(X)$ entrains those ones of the two variants of the Lebesgue spaces: variable exponent and grand Lebesgue spaces. This space is a Banach space. In [11], the authors also derived the boundedness of some operators of Harmonic Analysis in $L^{p(\cdot),\theta}$ defined on quasi-metric measure spaces with doubling measure.

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The grand Lebesgue spaces were introduced in the 90s of the past century by T. Iwaniec and C. Sbordone [10]. In the subsequent years, quite a number of problems of Harmonic Analysis and the theory of non-linear differential equations were studied in these spaces (see, e.g., the monographs [12], [13] and references cited therein).

2. Preliminaries

Let (X,d,μ) be a quasi-metric measure spaces. A quasi-metric d is a function $d: X\times X\to[0,\infty)$ which satisfies the following conditions:

(a) $d(x,y)=0$ if and only if $x=y$.

(b) For all $x,y\in X$, $d(x,y)=d(y,x)$.

(c) There is a constant $\mathcal{K}>0$ such that $d(x,y)\le\mathcal{K}(d(x,z)+d(z,y))$ for all $x,y,z\in X$.

Let

$$d_X:=\operatorname{diam}(X):=\sup\{d(x,y):x,y\in X\}$$

be the diameter of X. We denote by $B(x,r)$ the ball with center x and radius r, i.e., $B(x,r):=\{y\in X: d(x,y)<r\}$.

We assume that $\mu(X)<\infty$.

We assume also that the following condition holds for (X,d,μ): there are positive constants n, C_1 and C_2 such that for all $x\in X$ and $0<r<d_X$ the inequality

$$C_1r^n\le\mu\big(B(x,r)\big)\le C_2r^n. \tag{1}$$

It is easy to check that condition (1) guarantees that (X,d,μ) is a space of homogeneous type (SHT, for short), i.e., measure μ satisfies the doubling condition ($\mu\in DC(X)$): there is a constant $D_\mu>0$ such that

$$\mu B(x,2r)\le D_\mu\mu B(x,r)$$

for every $x\in X$ and $r>0$.

As examples of SHT are regular curves (see the definition below), domains in $\mathbb{R}^n$ with the so-called $\mathcal{A}$ condition, nilpotent Lie groups with Haar measure (homogeneous groups), etc. (see, e.g., [2], [5]).

We denote by $P(X)$ a family of all real-valued μ- measurable functions p on X such that

$$1<p_-\le p_+<\infty,$$

where $p_-:=p_-(X):=\inf_X p(\cdot)$, $p_+:=p_+(X):=\sup_X p(\cdot)$.

Let w be a weight function on X, i.e., w be $\mu-$ a.e. locally integrable function on X. The Lebesgue space with a variable exponent $p(\cdot)$ with a weight function w denoted by $L^{p(\cdot)}_w(X)$ (or by $L^{p(x)}_w(X)$) is the class of all μ- measurable functions f on X for which

$$S_{p,w}(f):=\int\limits_X|f(x)|^{p(x)}w(x)d\mu(x)<\infty.$$

The norm in $L^{p(\cdot)}_w(X)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}_w(X)}=\inf\Big\{\lambda>0: S_{p,w}(f/\lambda)\le1\Big\}.$$

If $\beta=0$, then we denote $L^{p(\cdot)}_\beta(X)$ by $L^{p(\cdot)}(X)$.

The space $L^{p(\cdot),\theta}_w(X)$ is a Banach space. The closure of $L^{p(\cdot)}_w(X)$ in $L^{p(\cdot),\theta}_w(X)$ consists of those $f\in L^{p(\cdot),\theta}_w(X)$ for which $\lim\limits_{c\to0}\varepsilon^{\frac{\theta}{p_--c}}\|f(\cdot)\|_{L^{p(\cdot)-c}_w(X)}=0$.

Let $\theta>0$. For a weight function w, we denote (see [11] in an unweighted case) by $L^{p(\cdot),\theta}_w(X)$ the class of all measurable functions $f: X\mapsto\mathbb{R}$ for which the norm

$$\|f\|_{L^{p(\cdot),\theta}_w(X)}:=\sup_{0<c<p_--1}c^{\frac{\theta}{p_--c}}\|f\|_{L^{p(x)-c}_w(X)}$$

is finite. In this definition, c is a constant.

If $w(x)=d(x_0,x)^\beta$, where x_0 is a fixed point in X, then we use the symbol $L^{p(\cdot),\theta}_\beta(X)$ for $L^{p(\cdot),\theta}_w(X)$.

Let $\mathcal{P}^{\log}_{\mu}(X)$ be a class of those exponents p belonging to $P(X)$ that satisfy the following log-Hölder condition: there exists a positive constant a such that for all $x, y \in X$ with $d(x,y) \le 1/2$,

$$|p(x) - p(y)| \le \frac{a}{-\ln\big(d(x,y)\big)}.$$

3. Main Results

We denote by M the maximal operator defined on X:

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int\limits_B |f(y)| d\mu(y),$$

where f is a locally integrable function on X.

Our result regarding the maximal operator reads as follows:

Theorem 3.1. *Let $p \in P(X) \cap \mathcal{P}^{\log}_{\mu}(X)$, $\theta > 0$ and let x_0 be a point in X. Suppose that $-n < \beta < n(p(x_0) - 1)$, where n is from* (1). *Then the Hardy–Littlewood maximal operator M is bounded in $L^{p(\cdot),\theta}_{\beta}(X)$.*

Let K be the Calderón-Zygmund operator defined on a SHT, i.e., K satisfies the following conditions (see, e.g., [1], [2]):

(i) K is linear and bounded in $L^{p_c}(X)$ for every $p_c \in (1, \infty)$;

(ii) there is a measurable function $k : X \times X \mapsto \mathbb{R}$ such that for every $f \in D(X)$,

$$Kf(x) = \int\limits_X k(x,y) f(y) d\mu(y),$$

for a.e. $x \notin \operatorname{supp} f$, where $D(X)$ is the class of bounded functions with compact supports defined on X.

(iii) the kernels k and k^* (here, $k^*(x,y) := k(y,x)$) satisfy the following pointwise Hörmander's condition: there are positive constants C, β and $A > 1$ such that

$$|k(x_0, y) - k(x,y)| \le C \frac{d(x_0,x)^{\beta - n}}{d(x_0,y)^{\beta}}$$

holds for every $x_0 \in X$, $r > 0$, $x \in B(x_0, r)$, $y \in X \setminus B(x_0, Ay)$;

(iv) there is a positive constant C such that for all $x, y \in X$,

$$|k(x,y)| \le \frac{C}{d(x,y)^n}.$$

We say that a weight w belongs to the Muckenhoupt class $A_{p_c}(X)$, where p_c is a constant such that $1 < p_c < \infty$, if

$$[w]_{A_{p_c}(X)} := \sup_B \bigg(\frac{1}{\mu B} \int\limits_B w d\mu \bigg) \bigg(\frac{1}{\mu B} \int\limits_B w^{1-(p_c)'} d\mu \bigg)^{p_c - 1} < \infty, \quad (p_c)' = \frac{p_c}{p_c - 1},$$

where the supremum is taken with respect to all balls B in X.

The operator K (see, e.g., [14] and references therein) is bounded in $L^{p_c}_w(X)$ for $1 < p_c < \infty$ and $w \in A_{p_c}(X)$. Moreover, the following estimate

$$\|K\|_{L^{p_c}_w(X)} \le c_0([w]_{A_{p_c}(X)})$$

holds, where $c_0([w]_{A_{p_c}(X)})$ is a constant depending on $[w]_{A_{p_c}(X)}$ so that the mapping $x \mapsto c_0(x)$ is non-decreasing.

Theorem 3.2. *Let the conditions of Theorem* 3.1 *be satisfied. Then there is a positive constant c depending only on p such that the following inequality*

$$\|Kf\|_{L^{p(\cdot),\theta}_{\beta}(X)} \le c \|f\|_{L^{p(\cdot),\theta}_{\beta}(X)}, \quad f \in D(X),$$

holds.

Let K be a Calderón-Zygmund operator. By K_b^m we denote a higher order commutator on X. For a function b on X,

$$K_b^1 f = bKf - K(bf)$$

is well defined for $b \in BMO(X)$ and $f \in D(X)$ (see [14]), where $BMO(X)$ is the well-known class of functions b defined on X for which

$$\|b\|_{BMO} = \sup_{\substack{x\in X\\ 0<r<d_X}} \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| d\mu(y) < \infty.$$

Further, let m be an integer $m \geq 2$. Then by the definition,

$$K_b^m f(x) = \int\limits_X (b(x) - b(y))^m k(x,y) f(y) d\mu(y).$$

Theorem 3.3. *Let the conditions of Theorem* 3.1 *be satisfied and let $b \in BMO(X)$. Then there is a positive constant c such that for all $f \in D(X)$, the inequality*

$$\|K_b^m f\|_{L_\beta^{p(\cdot),\theta}(X)} \leq c \|f\|_{L_\beta^{p(\cdot),\theta}(X)}$$

holds.

As an application, we can obtain weighted estimates for maximal and singular integral operators defined on a rectifiable curve Γ.

Let

$$\Gamma(z,r) := \Gamma \cap D(z,r), \quad z \in \Gamma, \quad r > 0,$$

where $D(z,r)$ is a disc on the complex plane with center z and radius r.

We say that the curve Γ is regular (Carleson) if there is a positive constant C such that for all $z \in \Gamma$ and $r > 0$,

$$\nu\big(\Gamma(z,r)\big) \leq Cr,$$

where ν is the arc-length measure on γ.

This condition guarantees that Γ with the Euclidean distance and ν is a SHT, where the balls are $\Gamma(z,r)$.

Observe that

$$C_1 r \leq \nu\big(\Gamma(z,r)\big) \leq C_2 r, \quad 0 < r < d_\Gamma,$$

for some positive constants C_1 and C_2.

Suppose that by M_Γ and K_Γ are denoted the Hardy-Littlewood maximal operator and the Cauchy singular integrals on Γ defined, respectively, as follows:

$$M_\Gamma f(z) = \sup_{r>0} \frac{1}{\Gamma(z,r)} \int\limits_{\Gamma(z,r)} |f(t)| d\nu(t), \quad z \in \Gamma, \quad f \in L_{\mathrm{loc}}(\Gamma),$$

$$S_\Gamma f(z) = (p.v.) \int\limits_\Gamma \frac{f(t))}{t-z} d\nu(t), \quad z \in \Gamma, \quad f \in D(\Gamma).$$

As a consequence of Theorems 3.1–3.3, we have

Theorem 3.4. *Let $p \in P(\Gamma) \cap \mathcal{P}_\mu^{\log}(\Gamma)$, $\theta > 0$ and let z_0 be a point on Γ. Suppose that Γ is a regular curve and $-1 < \beta < p(z_0) - 1$. Then the Hardy–Littlewood maximal operator M_Γ is bounded in $L_\beta^{p(\cdot),\theta}(\Gamma)$.*

Theorem 3.5. *Let the conditions of Theorem* 3.4 *be satisfied. Then there is a positive constant c depending only on p such that the following inequality*

$$\|S_\Gamma f\|_{L_\beta^{p(\cdot),\theta}(\Gamma)} \leq c \|f\|_{L_\beta^{p(\cdot),\theta}(\Gamma)}, \quad f \in D(\Gamma),$$

holds.

Let us denote
$$S^m_{\Gamma,b} f(z) = (p.v.) \int\limits_{\Gamma} \frac{(b(z)-b(t))^m}{z-t} f(t) d\nu(t), \quad z \in \Gamma.$$

Theorem 3.6. *Let the conditions of Theorem* 3.4 *be satisfied and let $b \in BMO(\Gamma)$. Then there is a positive constant c such that for all $f \in D(\Gamma)$, the inequality*
$$\|S^m_{\Gamma,b} f\|_{L^{p(\cdot),\theta}_{\beta}(\Gamma)} \le c \|f\|_{L^{p(\cdot),\theta}_{\beta}(\Gamma)}$$
is fulfilled.

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ANGULAR TRIGONOMETRIC APPROXIMATION IN THE FRAMEWORK OF NEW SCALE OF FUNCTION SPACES

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Abstract. In this note we announce our results on approximation by angle of functions of two variables in weighted Lebesgue spaces with mixed norms. The related problems of multidimensional Fourier Analysis are explored as well.

The investigation of trigonometric approximation by angle in classical Lebesgue spaces L^p $(1 < p < \infty)$ was initiated by M. K. Potapov. To this problem are devoted the series of his papers (see, for example, [6–8] and the survay paper [9]). Recently these results were extended to the L^p $(1 < p < \infty)$ spaces with Muckenhoupt weights [1, 2]. The main goal of present paper is to study the similar problem in weighted Lebesgue spaces with mixed norms.

1. In the sequel we denote by $\mathbb{T}^2$ the torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, where $\mathbb{T}$ is the circle $\{e^{i\varphi},\ \varphi \in [0, 2\pi]\}$. The function $v : \mathbb{T} \longrightarrow \mathbb{R}^1$ is called a weight if v is a measurable on $\mathbb{T}$, positive almost everywhere and integrable. For a Borel measure $e \subset \mathbb{T}$ we define the absolute continuous measure

$$ve = \int\limits_e v(x)dx.$$

A weight function v is said to be of Muckenhoupt class $A_p(\mathbb{T})$ if

$$\sup \bigg(\frac{1}{|I|}\int\limits_I v(x)dx\bigg)\bigg(\frac{1}{|I|}\int\limits_I v^{1-p'}(x)dx\bigg)^{p-1} < \infty, \quad p' = \frac{p}{p-1},$$

where the supremum is taken over all intervals $I \subset \mathbb{T}$.

In the sequel we consider the set of the measurable functions $f(x, y) : \mathbb{T}^2 \longrightarrow \mathbb{R}^1$ such that $f(x, y)$ is 2π-periodic with respect to each variable x and y.

Definition 1. Let $1 < p_i,\ s_i < \infty$ $(i = 1, 2)$, v and w be the weight functions defined on $\mathbb{T}$.

By $L^{p_1}_v$ $(L^{p_2}_w)$ $(\mathbb{T}^2)$, denote the set of measurable functions $f : \mathbb{T}^2 \longrightarrow \mathbb{R}$ for which the norm

$$\|f\|_{L^{p_1}_v(L^{p_2}_w)(\mathbb{T}^2)} = \Big\|\|f(x, \cdot)\|_{L^{p_2}_w}\Big\|_{L^{p_1}_v}$$

is finite.

The space $L^{p_1}_v(L^{p_2}_w)$ is a Banach function space. Introduction and the study of properties of this space was initiated in [3].

In the sequel always we assume that $v \in A_{p_1}(\mathbb{T})$ and $w \in A_{p_2}(\mathbb{T})$.

To avoid an inconvience of notation in this section we set $X := L^{p_1}_v(L^{p_2}_w)$.

The definition of the fractional modulus of smoothness in our case is similar as in [2], only the norm of Lebesgue spaces is changed by the norm in $L^{p_1}_v(L^{p_2}_w)(\mathbb{T}^2)$.

The mixed modulus of smoothness in $L^{p_1 s_1}_v(L^{p_2 s_2}_w)$ is denoted by $\Omega_r(f, \delta_1, \delta_2)$. We have

$$\Omega_r(f, \delta_1, \delta_2)_X \le c\|f\|_X$$

with a constant c nondepending on f, δ_1 and δ_2.

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By $P_{m,0}$ (respectively, $P_{0,n}$) is denoted the set of all trigonometric polynomial of degree m (at most n) with respect to the variable x (variable y). Also $P_{m,n}$ is defined as the set of all trigonometric polynomial of degree at most m with respect to variable x and of degree at most n with respect to variable y.

The best partial trigonometric approximation orders are defined as

$$E_{m,0}(f)_X = \inf\{\|f-T\|_X : \ T \in P_{m,0}\},$$

Analogously,

$$E_{0,m}(f)_X = \inf\{\|f-G\|_X : \ G \in P_{0,n}\}.$$

Then the best angular approximation order is defined by the equality

$$E_{m,n}(f)_X = \inf\{\|f-T-G\|_X : \ \mathbb{T} \in P_{m,0}, \ G \in P_{0,n}\}.$$

The following assertions are true:

Theorem 1. *Let $1 < p_i, s_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. Let $r > 0$. For $f \in X$ the following inequality*

$$E_{m,n}(f)_X \leq c_1 \Omega_r \left(f, \frac{1}{m}, \frac{1}{n}\right)_X$$

holds with a constant c_1 nondepending on f, m and n.

Theorem 2. *Let $1 < p_i, s_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$, $f \in L_v^{p_1 s_1}(L_w^{p_2 s_2})(\mathbb{T}^2)$ and $r > 0$. Then*

$$\Omega_r \left(f, \frac{1}{m}, \frac{1}{n}\right)_X \leq \frac{c}{n^{2r}} \sum_{i=0}^{m} \sum_{j=0}^{n} (i+1)^{2r-1} (j+1)^{2r-1} E_{ij}(f)_X.$$

In what follows we discuss some tools, contributing to the proving of aforementioned assertions.

Let $\sigma_{mn}^{\alpha,\beta}(f,x,y)$ $(\alpha > 0, \ \beta > 0)$ be the Cesáro means of double Fourier trigonometric series of $f \in L_v^{p_1 s_1}(L_w^{p_2 s_2})(\mathbb{T}^2)$.

Theorem 3. *Let $1 < p_i, s_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. Then*

$$\|\sigma_{mn}^{\alpha,\beta}(f)\|_X \leq c\|f\|_X$$

and

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \|\sigma_{mn}^{\alpha,\beta}(f) - f\|_X = 0.$$

For the partial sums of double Fourier trigonometric series we have

$$\|S_{mn}(f)\|_X \leq c\|f\|_X$$

with a constant c nondepending on $m, n \in \mathbb{N}$ and $f \in X$.

Further for $f \in X$

$$\lim_{n\to\infty} \|S_{n,n} - f\| = 0.$$

In the sequel under derivatives we assume the derivatives in Weyl's sense.

Theorem 4 (Bernstein type inequalities)**.** *Let $1 < p_i, s_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. Assume that $\alpha, \beta > 0$.*

Let $T_1 \in P_{m,0}$, $T_2 \in P_{0,n}$ and $T_3 \in P_{mn}$. Then

$$\left\| \frac{\partial^\alpha}{\partial x^\alpha} T_1 \right\|_X \leq c_1 m \|T_1\|_X$$

$$\left\| \frac{\partial^\beta}{\partial y^\beta} T_2 \right\|_X \leq c_2 m \|T_2\|_X$$

and

$$\left\| \frac{\partial^\beta}{\partial x^\alpha \partial y^\beta} T_3 \right\|_X \leq c_3 mn \|T_3\|_X,$$

where the constants c_1, c_2 and c_3 are nondepending on m, n and polynomial.

For one and two weighted Bernstein inequalities in Lebesgue spaces we refer to [5], Chapter 6.

Definition 2. Let $f \in L_v^{p_1 s_1}(L_w^{p_2 s_2})(\mathbb{T}^2)$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. Assume that $r > 0$. The mixed K-functional is defined as

$$K(f, \delta, \varepsilon, p_1, p_2, s_1, s_2, v, w, 2r) :$$

$$= \inf_{h_1 h_2, h} \left\{ \|f - h_1 - h_2 - h\|_X + \delta^{2r} \left\| \frac{\partial^{2r} h_1}{\partial x^{2r}} \right\|_X + \varepsilon^{2r} \left\| \frac{\partial^{2r} h_2}{\partial y^{2r}} \right\|_X + \delta^{2r} \varepsilon^{2r} \left\| \frac{\partial^{4r} h}{\partial x^{2r} \partial y^{2r}} \right\|_X \right\},$$

where the infimum is taken from all h_1, h_2, h such that $h_1 \in W_X^{2r,0}$, $h_2 \in W_X^{0,2r}$, $h \in W_X^{4r}$.

Here the following notations are used:

$$W_X^{2r,0} = \left\{ h_1 : \frac{\partial^{2r} h_1}{\partial x^{2r}} \in X \right\},$$

$$W_X^{0,2r} = \left\{ h_2 : \frac{\partial^{2r} h_2}{\partial y^{2r}} \in X \right\}$$

and

$$W_X^{4r} = \left\{ h : \frac{\partial^{4} h}{\partial x^{2r} \partial y^{2r}} \in X \right\}.$$

The following statement is true

Theorem 5. *Let $f \in L_v^{p_1 s_1}(L_w^{p_2 s_2})$, $1 < p_i, s_i < \infty$, $v \in A_{p_1}(\mathbb{T})$, $w \in A_{p_2}(\mathbb{T})$. Then the following equivalence*

$$\Omega_r(f, \delta_1, \delta_2)_X \approx K(f, \delta_1, \delta_2, p_1, p_2, s_1, s_2, v, w, 2r)_X$$

holds with equivalence constants nondepending on f, δ_1 and δ_2.

It should be noted that mixed K-functionals were explored in [4] and [10]. This notion turn out to be very usefule in approximation and interpolation theory.

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ON DESCENT COHOMOLOGY

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Abstract. The zeroth and first descent cohomology sets for a (co)monad on arbitrary base category with coefficients in a (co)algebra are introduced and their basic properties are studied. These sets generalize those for a coring with coefficients in a comodule. It is shown that under this generalization, essential properties and relationships are preserved.

1. Introduction

The aim of this paper is to introduce and investigate low-dimensional descent cohomology sets of comonads with coefficients in coalgebras over the comonad. These include not only the non-abelian descent cohomology for Hopf modules with coefficients in comodule algebras in the sense of P. Nuss and M. Wambst ([23], [24]), and hence the classical non-abelian group cohomology of Serre [25], but also their generalization by T. Brzeziński to comodules over corings ([4]).

Recall that P. Nuss and M. Wambst in [23] and [24] introduced the zeroth and first descent cohomology sets for Hopf modules with coefficients in comodule algebras, and showed that

- their cohomology generalizes the non-abelian group cohomology of Serre [25], and
- the first descent cohomology pointed set classifies twisted forms of Hopf modules and Hopf torsors.

Based on the descent theory for corings (see, [6], [9]) and the fact that an arbitrary Hopf module can be considered as a special case of an entwining module and hence a comodule over an appropriate coring, T. Brzeziński [4] gave a coring approach to the descent cohomology theory. In particular, he introduced the zeroth and first descent cohomology pointed sets for a coring with values in a comodule over the coring and showed that the first descent cohomology pointed sets still classify twisted forms and suitably defined torsors. Brzeziński's definition of these sets involves a k-algebra A (k being a commutative ring with unit), an A-coring $\mathcal{C}$ and a (right) $\mathcal{C}$-comodule (M, ϱ). Given these data, the *zeroth descent cohomology set of* $\mathcal{C}$ *with coefficients in* (M, ϱ) (which is in fact a group) is defined as the group of $\mathcal{C}$-comodule automorphisms of (M, ϱ), while the *first descent cohomology set of* $\mathcal{C}$ *with coefficients in* M as the set of equivalence classes of $\mathcal{C}$-comodule structures on M, where two $\mathcal{C}$-comodule structures are equivalent if they are isomorphic as $\mathcal{C}$-comodules. Since an A-coring can be defined as an A-bimodule $\mathcal{C}$ such that the endofunctor $\mathbb{G}_{\mathcal{C}} = - \otimes_{\mathsf{A}} \mathcal{C}$ on the category of right A-modules $\mathbf{M}_{\mathsf{A}}$ is a comonad, and since right $\mathcal{C}$-comodules are the same as $\mathbb{G}_{\mathcal{C}}$-coalgebras, the concepts of the zeroth and first descent cohomology sets of an A-coring $\mathcal{C}$ with coefficients in a $\mathcal{C}$-comodule can obviously be formulated in pure categorical terms. One may then ask whether the results of [23], [24] and [4] are valid in other categories than the category of (co)modules over a (co)ring. The motivation and main purpose of this paper is to show that this is indeed the case. We demonstrate in particular that several aspects of descent cohomology sets for corings can be generalized in the context of (co)monads on general categories in such a way that their essential properties and relationships with (appropriately generalized to this context) twisted forms and torsors are maintained. We should point out that our method of obtaining this generalization do not use sophisticated machinery of (co)ring and (co)module theory (which is not applicable to our situation due to the great generality of the context we are working in). Our proofs are, in fact, based only on two elementary results concerning pseudo-pullbacks. The first result states that pseudo-pullbacks preserve equivalences of categories, while the second one states that the comparison functor from the pullback to the pseudo-pullback of

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any functor along a functor that lifts isomorphism uniquely, is an equivalence of categories. For the convenience of the reader, we have recalled these results in Section 2.

The outline of this paper is as follows. After recalling in Section 2 some notions and aspects of the theory of (co)monads and (pseudo-)pullbacks, we obtain some categorical results that will be needed for proving our results in the next sections.

In Section 3, we introduce the zeroth and first descent cohomology (pointed) sets of a comonad with coefficients in a coalgebra and study their elementary properties. We close the section by giving two examples of calculating these pointed sets for some comonads.

In Section 4, we introduce twisted forms of an object w.r.t. functors and show how to describe the first descent cohomology sets using them.

Section 5 is concerned with the description of the first cohomology pointed set in terms of subobjects of a certain object.

In Section 6, the first cohomology sets are related with isomorphism classes of suitable defined torsors.

Finally, in the last section, we formally dualize the notions of descent cohomology sets of comonads and define descent cohomology sets of monads. As an application, we calculate descent cohomology sets for some monads.

We refer to S. MacLane [15] and F. Borceux [2,3] for terminology and general results on (co)monads and on (pseudo-)pullbacks, and to T. Brzezinski and R. Wisbauer [5] for coring and comodule theory.

2. Preliminaries

This section introduces the categorical preliminaries necessary for the other sections.

2.1. Monads and comonads. We write $\eta, \varepsilon \colon F \dashv U \colon \mathscr{B} \to \mathscr{A}$ to denote that $F : \mathscr{A} \to \mathscr{B}$ and $U : \mathscr{B} \to \mathscr{A}$ are the functors, where F is left adjoint to U with unit $\eta : 1 \to UF$ and counit $\varepsilon : FU \to 1$.

For a monad $\mathbb{T} = (T, m, e)$ on a category $\mathscr{A}$, we write

- $\mathscr{A}_{\mathbb{T}}$ for the Eilenberg-Moore category of $\mathbb{T}$-algebras;
- $U_{\mathbb{T}} : \mathscr{A}_{\mathbb{T}} \to \mathscr{A},\ (a, h) \to a$, for the forgetful functor;
- $F_{\mathbb{T}} : \mathscr{A} \to \mathscr{A}_{\mathbb{T}},\ a \to (T(a), m_a)$, for the free $\mathbb{T}$-algebra functor, and
- $\eta_{\mathbb{T}}, \varepsilon_{\mathbb{T}} : F_{\mathbb{T}} \dashv U_{\mathbb{T}} : \mathscr{A}_{\mathbb{T}} \to \mathscr{A}$ for the corresponding forgetful-free adjunction, in which $\eta_{\mathbb{T}} = \eta$ and $(\varepsilon_{\mathbb{T}})_{(a,h)} = h$ for each $\mathbb{T}$-algebra (a, h).

Dually, for a comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on $\mathscr{A}$, we write

- $\mathscr{A}^{\mathbb{G}}$ for the category of the Eilenberg-Moore category of $\mathbb{G}$-coalgebras;
- $U^{\mathbb{G}} : \mathscr{A}^{\mathbb{G}} \to \mathscr{A},\ (a, \theta) \to a$, for the forgetful functor;
- $F^{\mathbb{G}} : \mathscr{A} \to \mathscr{A}^{\mathbb{G}},\ a \to (G(a), \delta_a)$, for the cofree $\mathbb{G}$-coalgebra functor, and
- $\eta^{\mathbb{G}}, \varepsilon^{\mathbb{G}} : U^{\mathbb{G}} \dashv F^{\mathbb{G}} : \mathscr{A} \to \mathscr{A}^{\mathbb{G}}$ for the forgetful-cofree adjunction, in which $\varepsilon^{\mathbb{G}} = \varepsilon$ and $(\eta^{\mathbb{G}})_{(a,\theta)} = \theta$ for each $\mathbb{G}$-coalgebra (a, θ).

It is well known (e.g., [15]) that any adjunction $\eta, \varepsilon \colon F \dashv U \colon \mathscr{B} \to \mathscr{A}$ generates a monad $\mathbb{T} = (T, m, e)$ on $\mathscr{A}$, where $T = UF$, $m = U\varepsilon F$, $e = \eta$, and a comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on $\mathscr{B}$, where $\mathbb{G} = FU,\ \delta = F\eta U$.

Let $\eta, \varepsilon : F \dashv U \colon \mathscr{B} \to \mathscr{A}$ be an adjunction and $\mathbb{T}$ and $\mathbb{G}$ be the associated monad and comonad on $\mathscr{A}$ and $\mathscr{B}$, respectively. Then one has the comparison functors $K_{\mathbb{T}} : \mathscr{B} \to \mathscr{A}_{\mathbb{T}}$ and $K^{\mathbb{G}} : \mathscr{A} \to \mathscr{B}^{\mathbb{G}}$ and a diagram of categories and functors

$$\begin{tikzcd}
\mathscr{B}^{\mathbb{G}} \arrow[r, shift right, "U^{\mathbb{G}}"'] & \mathscr{B} \arrow[l, shift right, "F^{\mathbb{G}}"'] \arrow[r, "K_{\mathbb{T}}"] \arrow[d, shift left, "U"] & \mathscr{A}^{\mathbb{T}} \arrow[dl, shift left, "U_{\mathbb{T}}"] \\
& \mathscr{A} \arrow[ul, "K^{\mathbb{G}}"] \arrow[u, shift left, "F"] \arrow[ur, shift left, "F_{\mathbb{T}}"] &
\end{tikzcd}$$

where the functors $K_{\mathbb{T}} : \mathscr{B} \to \mathscr{A}_{\mathbb{T}}$ and $K^{\mathbb{G}} : \mathscr{A} \to \mathscr{B}^{\mathbb{G}}$ are defined by

$$K_{\mathbb{T}}(b) = (U(b), U(\varepsilon_b)) \text{ and } K_{\mathbb{T}}(f) = U(f) : (U(b), U(\varepsilon_b)) \to (U(b'), U(\varepsilon_{b'}))$$

and

$$K^{\mathbb{G}}(a) = (F(a), F(\eta_a)) \text{ and } K^{\mathbb{G}}(g) = F(g) : (F(a), F(\eta_a)) \to (F(a'), F(\eta_{a'})).$$

Thus

$$K_{\mathbb{T}}F = F_{\mathbb{T}}, U_{\mathbb{T}}K_{\mathbb{T}} = U, U^{\mathbb{G}}K^{\mathbb{G}} = F \text{ and } K^{\mathbb{G}}U = F^{\mathbb{G}}.$$

One says that $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is a *monadic* (resp. *premonadic*) adjunction if $K_{\mathbb{T}}$ is an equivalence of categories (resp. full and faithful). Dually, one says that the adjunction $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is *comonadic* (resp. *precomonadic*) if $K^{\mathbb{G}}$ is an equivalence of categories (resp. full and faithful).

When $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is (co)monadic, we write $L_{\mathbb{T}}$ (resp. $R^{\mathbb{G}}$) for the adjoint inverse of $K_{\mathbb{T}}$ (resp. $K^{\mathbb{G}}$) and write $\underline{\eta} : 1 \to K_{\mathbb{T}}L_{\mathbb{T}}$ and $\underline{\varepsilon} : L_{\mathbb{T}}K_{\mathbb{T}} \to 1$ (resp. $\overline{\eta} : 1 \to R^{\mathbb{G}}K^{\mathbb{G}}$ and $\overline{\varepsilon} : K^{\mathbb{G}}R^{\mathbb{G}} \to 1$) for the unit and counit of the adjunction $L_{\mathbb{T}} \dashv K_{\mathbb{T}}$ (resp. $K^{\mathbb{G}} \dashv R^{\mathbb{G}}$).

2.2. Pullbacks and pseudo-pullbacks. We begin with recalling (for example, from [15]) that the *comma category* $(F_1 \downarrow F_2)$ of the functors

$$F_1 : \mathscr{A}_1 \to \mathscr{A} \text{ and } F_2 : \mathscr{A}_2 \to \mathscr{A},$$

is the category whose objects are the triples (a_1, f, a_2), where a_1 is an object of $\mathscr{A}_1$, a_2 one of $\mathscr{A}_2$, and $f : F_1(a_1) \to F_2(a_2)$ is a morphism in $\mathscr{A}$, and whose morphisms $(a_1, f, a_2) \to (a_1', f', a_2')$ are pairs (α_1, α_2), where $\alpha_1 : a_1 \to a_1'$ is a morphism in $\mathscr{A}_1$ and $\alpha_2 : a_2 \to a_2'$ is a morphism in $\mathscr{A}_2$ such that the diagram

$$\begin{tikzcd} F_1(a_1) \arrow[r, "f"] \arrow[d, "F_1(\alpha_1)"'] & F_2(a_2) \arrow[d, "F_2(\alpha_2)"] \\ F_1(a_1') \arrow[r, "f'"'] & F_2(a_2') \end{tikzcd}$$

commutes. Composition and identities in $(F_1 \downarrow F_2)$ are inherited from $\mathscr{A}$, $\mathscr{A}_1$ and $\mathscr{A}_2$. It follows that a morphism (α_1, α_2) is an isomorphism in $(F_1 \downarrow F_2)$ if and only if the morphisms α_1 and α_2 are both isomorphisms.

The comma category $(F_1 \downarrow F_2)$ is equipped with the obvious projections

$$P_1 : (F_1 \downarrow F_2) \to \mathscr{A}_1, \;\; P_2 : (F_1 \downarrow F_2) \to \mathscr{A}_2$$

and a natural transformation $\omega : F_1P_1 \to F_2P_2$ defined by $\omega_{(a_1,\, f,\, a_2)} = f$. Then the square

$$\begin{tikzcd} (F_1 \downarrow F_2) \arrow[r, "P_2"] \arrow[d, "P_1"'] & \mathscr{A}_2 \arrow[d, "F_2"] \\ \mathscr{A}_1 \arrow[r, "F_1"'] \arrow[ur, phantom, "\omega"] & \mathscr{A} \end{tikzcd}$$

is universal among such squares in the sense that given any other such square

$$\begin{tikzcd} \mathscr{B} \arrow[r, "Q"] \arrow[d, "P"'] & \mathscr{A}_2 \arrow[d, "F_2"] \\ \mathscr{A}_1 \arrow[r, "F_1"'] \arrow[ur, phantom, "\varpi"] & \mathscr{A} \end{tikzcd}$$

where $\varpi : F_1P \to F_2Q$ is a natural transformation, then there is a unique functor $F : \mathscr{B} \to (F_1 \downarrow F_2)$ such that $P_1F = P$, $P_2F = Q$ and $\omega F = \varpi$.

We write $\mathscr{W}(F_1, F_2)$ for the unique functor $(F_1 \downarrow F_2) \to \mathscr{A}_1 \times \mathscr{A}_2$ making the diagram

$$\begin{tikzcd}
 & \mathscr{A}_1 & \\
(F_1 \downarrow F_2) \arrow[ur, "P_1"] \arrow[rr, "\mathscr{W}(F_1,F_2)"] \arrow[dr, "P_2"'] & & \mathscr{A}_1 \times \mathscr{A}_2 \arrow[ul, "p_{\mathscr{A}_1}"'] \arrow[dl, "p_{\mathscr{A}_2}"] \\
 & \mathscr{A}_2 &
\end{tikzcd}$$

where $p_{\mathscr{A}_1}$ and $p_{\mathscr{A}_2}$ are the projections, commute. Comma categories are also sometimes called *lax pullbacks.*

Given functors $F_1 : \mathscr{A}_1 \to \mathscr{A}$ and $F_2 : \mathscr{A}_2 \to \mathscr{A}$, their *pullback* $\mathsf{P}(F_1, F_2)$ (resp., *pseudo-pullback* $\mathsf{Ps}(F_1, F_2)$) is the full subcategory of $(F_1 \downarrow F_2)$ consisting of those (a_1, f, a_2) for which f is an identity morphism (resp., an isomorphism). The restrictions of P_1 and P_2 on $\mathsf{P}(F_1, F_2)$ (resp., $\mathsf{Ps}(F_1, F_2)$) are denoted again by P_1 and P_2, respectively. Then $\omega : F_1P_1 \to F_2P_2$ is an identity (resp., invertible) natural transformation and $\mathsf{P}(F_1, F_2)$ (resp., $\mathsf{Ps}(F_1, F_2)$) is universal among those diagrams

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "Q"] \arrow[d, "P"'] & \mathscr{A}_2 \arrow[d, "F_2"] \\
\mathscr{A}_1 \arrow[r, "F_1"'] \arrow[ur, phantom, "\Uparrow\varpi"] & \mathscr{A}
\end{tikzcd}$$

in which ω is an identity (resp., invertible) natural transformation.

2.3. Comparing pullbacks and pseudo-pullbacks. Given functors $F_1 : \mathscr{A}_1 \to \mathscr{A}$ and $F_2 : \mathscr{A}_2 \to \mathscr{A}$, we write K_{P} for the functor $\mathsf{P}(F_1, F_2) \to \mathsf{Ps}(F_1, F_2)$ induced by the universal property of the pseudo-pullback $\mathsf{Ps}(F_1, F_2)$ and the defining diagram for $\mathsf{P}(F_1, F_2)$:

$$\begin{tikzcd}
\mathsf{P}(F_1, F_2) \arrow[r, "P_2"] \arrow[d, "P_1"'] & \mathscr{A}_2 \arrow[d, "F_2"] \\
\mathscr{A}_1 \arrow[r, "F_1"'] \arrow[ur, phantom, "\Uparrow ="] & \mathscr{A}
\end{tikzcd}$$

K_{P} is called the *canonical comparison functor.* It takes (a_1, a_2) to $(a_1, 1, a_2)$, where 1 is the identity morphism of $F_1(a_1) = F_2(a_2)$ in $\mathscr{A}$, and takes (α_1, α_2) to (α_1, α_2).

While clearly fully faithful, K_{P} need not be an equivalence of categories, in general. The following proposition provides a sufficient condition for K_{P} to be an equivalence. Recall (for example, from [1]) that a functor $F : \mathscr{B} \to \mathscr{A}$ *lifts isomorphisms uniquely* if for any isomorphism $f : F(b) \to a$ in $\mathscr{A}$, there exists a unique isomorphism $g : b \to b'$ in $\mathscr{B}$ such that $F(b') = a$ and $F(g) = f$.

2.4. Proposition. ([12]) *Given functors $F_1 : \mathscr{A}_1 \to \mathscr{A}$ and $F_2 : \mathscr{A}_2 \to \mathscr{A}$, the comparison fun ctor*

$$\mathrm{K}_{\mathsf{P}} : \mathsf{P}(F_1, F_2) \to \mathsf{Ps}(F_1, F_2)$$

is an equivalence of categories provided F_1 lifts isomorphism uniquely. When this is the case, an adjoint inverse for K_{P} is the functor sending (a_1, f, a_2) to (a_1', a_2), where a_1' is the unique object of $\mathscr{A}_1$ for which there is an isomorphism $g : a_1 \to a_1'$ in $\mathscr{A}_1$ with $F_1(g) = f$.

Note that $(g, 1)$ is an isomorphism from (a_1, f, a_2) to $\mathrm{K}_{\mathsf{P}}(a_1', a_2) = (a_1', 1, a_2)$.

Recall (for example, from [1]) that if $\mathbb{G}$ is a comonad on a category $\mathscr{A}$, then the forgetful functor $U^{\mathbb{G}} : \mathscr{A}^{\mathbb{G}} \to \mathscr{A}$ lifts isomorphisms as follows. If $(a,\theta) \in \mathscr{A}^{\mathbb{G}}$ is such that there exists an isomorphism $f : U^{\mathbb{G}}(a,\theta) = a \to a'$ in $\mathscr{A}$, then the pair (a',θ'), where θ' is the composite

$$a' \xrightarrow{f^{-1}} a \xrightarrow{\theta} G(a) \xrightarrow{G(f)} G(a'),$$

is a $\mathbb{G}$-coalgebra and f is an isomorphism from (a,θ) to (a',θ'). Quite obviously, $U^{\mathbb{G}}(f : (a,\theta) \to (a',\theta')) = f$. Therefore, as a special case of Proposition 2.4, we have

2.5. Proposition. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{A}$. Then for any functor $F : \mathscr{B} \to \mathscr{A}$, the comparison functor*

$$\mathrm{K}_{\mathsf{P}} : \mathsf{P}(U^{\mathbb{G}}, F) \to \mathsf{Ps}(U^{\mathbb{G}}, F)$$
$$\mathrm{K}_{\mathsf{P}}((a,\theta), b) = ((a,\theta), 1, b), \;\; \mathrm{K}_{\mathsf{P}}(\alpha_1,\alpha_2) = (\alpha_1,\alpha_2)$$

is an equivalence of categories whose adjoint inverse is the functor

$$\overline{\mathrm{K}}_{\mathsf{P}} : \mathsf{Ps}(U^{\mathbb{G}}, F) \to \mathsf{P}(U^{\mathbb{G}}, F)$$
$$\overline{\mathrm{K}}_{\mathsf{P}}((a,\theta), f : U^{\mathbb{G}}(a,\theta) = a \approx F(b), b) = ((F(b), G(f)\cdot\theta\cdot f^{-1}), b).$$

2.6. Functoriality of pseudo-pullbacks. The category $\mathsf{Ps}(F_1,F_2)$ depends functorially both on F_1 and on F_2. The dependence on F_1 is as follows. Given any pair of functors $H : \mathscr{A}_1 \to \mathscr{C}$ and $F : \mathscr{C} \to \mathscr{A}$ and any invertible natural transformation $\omega : FH \approx F_1$, it follows from the universal property of pseudo-pullback that the assignments

$$(a_1, f, a_2) \longmapsto (H(a_1), f\cdot\omega_{a_1}, a_2)$$

and

$$(\alpha_1,\alpha_2) \longmapsto (H(\alpha_1), \alpha_2)$$

define a functor

$$\mathsf{Ps}(\omega, F_2) : \mathsf{Ps}(F_1,F_2) \to \mathsf{Ps}(F,F_2).$$

The situation may be pictured by the following diagram:

$$\begin{tikzcd}
\mathsf{Ps}(F_1,F_2) \arrow[rr, "P_2"] \arrow[dr, "\mathsf{Ps}(\omega,F_2)"] \arrow[dd, "P_1"'] & & \mathscr{A}_2 \arrow[dd, "F_2"] \\
& \mathsf{Ps}(F,F_2) \arrow[ur, bend right, "P_2"'] \arrow[dd, "P_1"] & \\
\mathscr{A}_1 \arrow[rr, dashed, "F_1"] \arrow[dr, "H"'] & \arrow[r, phantom, "\Uparrow\omega"] & \mathscr{A} \\
& \mathscr{C} \arrow[ur, bend right, "F"'] &
\end{tikzcd} \tag{2.1}$$

in which all the rectangles and the top triangle commute.

When H is an equivalence of categories, i.e. when there exists a functor $H' : \mathscr{C} \to \mathscr{A}_1$ with natural isomorphisms $\sigma : HH' \approx 1$ and $\varsigma : 1 \approx H'H$, then the composite

$$\omega^* : F_1H' \xrightarrow{\omega^{-1}H'} FHH' \xrightarrow{F\sigma} F$$

is an isomorphism and the induced functor

$$\mathsf{Ps}(\omega^*, F_2) : \mathsf{Ps}(F,F_2) \to \mathsf{Ps}(F_1,F_2),$$

that takes an object $(b,g,a) \in \mathsf{Ps}(F,F_2)$ to $(H'(b), g\cdot F(\sigma_b)\cdot(\omega_{H'(b)})^{-1}, a)$, is an adjoint inverse of the functor $\mathsf{Ps}(\omega,F_2)$.

In particular, in the case where $\omega = 1_{F_1}$,

$$\mathsf{Ps}(1_{F_1}, F_2)(a_1, f, a_2) = (H(a_1), f, a_2)$$

and

$$\mathsf{Ps}((1_{F_1})^*, F_2)(b, g, a) = (H'(b), g \cdot F(\sigma_b), a).$$

Suppose now that the adjunction $\eta, \varepsilon : F \dashv U$ is comonadic. Then $\overline{\eta}, \overline{\varepsilon} : K^{\mathbb{G}} \dashv R^{\mathbb{G}}$ is an adjoint equivalence and considering the diagrams

$$\begin{tikzcd} \mathscr{B}^{\mathbb{G}} \arrow[drr, bend left, "U^{\mathbb{G}}"] & & \mathscr{X} \arrow[d, "H"] \\ \mathscr{A} \arrow[u, "K^{\mathbb{G}}"] \arrow[rr, "F"'] \arrow[urr, phantom, "\Downarrow ="] & & \mathscr{B} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{B}^{\mathbb{G}} \arrow[d, "R^{\mathbb{G}}"'] \arrow[drr, bend left, "U^{\mathbb{G}}"] & & \mathscr{X} \arrow[d, "H"] \\ \mathscr{A} \arrow[rr, "F"'] \arrow[urr, phantom, "\Uparrow (1_F)^* = U^{\mathbb{G}}\overline{\varepsilon}"] & & \mathscr{B}, \end{tikzcd}$$

where $H : \mathscr{X} \to \mathscr{B}$ is an arbitrary functor, from the facts above follows

2.7. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction and $H : \mathscr{X} \to \mathscr{B}$ be an arbitrary functor. Then the functor*

$$\mathsf{Ps}(1_F, H) : \mathsf{Ps}(F, H) \to \mathsf{Ps}(U^{\mathbb{G}}, H)$$
$$(a, f : F(a) \simeq H(x), x) \longrightarrow (K^{\mathbb{G}}(a), f, x) = ((F(a), F(\eta_a)), f, x)$$

is an equivalence of categories. Its adjoint inverse is the functor

$$\mathsf{Ps}(U^{\mathbb{G}}\overline{\varepsilon}, H) : \mathsf{Ps}(U^{\mathbb{G}}, H) \to \mathsf{Ps}(F, H)$$
$$((b', \theta_{b'}), g : b' \simeq H(x), x) \longrightarrow (R^{\mathbb{G}}(b', \theta_{b'}), g \cdot U^{\mathbb{G}}(\overline{\varepsilon}_{(b', \theta_{b'})}), x).$$

Combining Propositions 2.5 and 2.7, we have

2.8. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction. Then for any functor $H : \mathscr{X} \to \mathscr{B}$, the composite*

$$\mathsf{Ps}(F, H) \xrightarrow{\mathsf{Ps}(1_F, H)} \mathsf{Ps}(U^{\mathbb{G}}, H) \xrightarrow{\overline{K}_{\mathsf{P}}} \mathsf{P}(U^{\mathbb{G}}, H)$$
$$(a, f : F(a) \simeq H(x), x) \to ((H(x), G(f) \cdot F(\eta_a) \cdot f^{-1}), x)$$

is an equivalence of categories. Its adjoint inverse takes $((b, \theta), x) \in \mathsf{P}(U^{\mathbb{G}}, H)$ to $(R^{\mathbb{G}}(b, \theta), U^{\mathbb{G}}(\overline{\varepsilon}_{(b,\theta)}), x) \in \mathsf{Ps}(F, H)$.

The following special case will be the most important for us.

Given a category $\mathscr{X}$ and object x of $\mathscr{X}$, we write $\langle x \rangle$ the full subcategory of $\mathscr{X}$ generated by the object x; this means that $\langle x \rangle$ has only one object x and $\langle x \rangle(x, x) = \mathscr{X}(x, x)$. The canonical inclusion $\langle x \rangle \to \mathscr{X}$ will be denoted by $\iota_{\langle x \rangle}$.

Fixing now an object $b \in \mathscr{B}$ and applying Proposition 2.8 to the case where $H = \iota_{\langle b \rangle}$, we obtain

2.9. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction and $b \in \mathscr{B}$. Then the functor*

$$P_{\langle b \rangle} : \mathsf{Ps}(F, \iota_{\langle b \rangle}) \to \mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle})$$
$$(a, f, b) \to (b, G(f) \cdot F(\eta_a) \cdot f^{-1})$$

is an equivalence of categories. Its adjoint inverse takes $(b, \theta) \in \mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle})$ to $(R^{\mathbb{G}}(b, \theta), U^{\mathbb{G}}(\overline{\varepsilon}_{(b,\theta)}), b) \in \mathsf{Ps}(F, \iota_{\langle b \rangle})$.

For a category $\mathscr{X}$, we write $\pi_0(\mathscr{X})$ for the collection of the isomorphism classes of objects of $\mathscr{X}$. For any $x \in \mathscr{X}$, $[x]$ denotes the class of x. Clearly, for any functor $H : \mathscr{X} \to \mathscr{Y}$, the assignment $[x] \to [H(x)]$ yields a map $\pi_0(S) : \pi_0(\mathscr{X}) \to \pi_0(\mathscr{Y})$.

Proposition 2.9 at once yields the following

2.10. Corollary. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction and let b be an arbitrary fixed object of $\mathscr{B}$. Then the map*

$$\pi_0(P_{\langle b\rangle}) : \pi_0(\mathsf{Ps}(F, \iota_{\langle b\rangle})) \to \pi_0(\mathsf{P}(U^{\mathbb{G}}, i_{\langle b\rangle}))$$
$$[(a, f, b)] \to [(b, G(f) \cdot F(\eta_a) \cdot f^{-1})]$$

is a bijection with inverse

$$[(b, \theta] \longrightarrow [(R^{\mathbb{G}}(b, \theta), U^{\mathbb{G}}(\overline{\varepsilon}_{(b,\theta)}), b)].$$

3. Descent Cohomology Sets of Comonads

In this section, we introduce the zeroth and first descent cohomology (pointed) sets of a comonad and study their elementary properties. As is mentioned in the introduction, these sets should generalize those for a coring with values in a comodule introduced in [4]. To achieve this, we first recall the definitions from [4] and then present them in a form which makes it quite obvious how to transform them to general categories.

Let A be an algebra over a commutative base ring k, $\mathcal{C}$ be an A-coring. Given a right $\mathcal{C}$-comodule (M, ϱ), the *first descent cohomology set of $\mathcal{C}$ with coefficients in M* is defined as the set of equivalence classes of $\mathcal{C}$-comodule structures on M, where two $\mathcal{C}$-comodule structures are equivalent if they are isomorphic as $\mathcal{C}$-comodules. Since M comes already equipped with the right coaction ϱ, the set of equivalence classes of $\mathcal{C}$-comodule structures on M is a pointed set, with the distinguished point given by the equivalence class of (M, ϱ). The *zeroth descent cohomology group of $\mathcal{C}$ with coefficients in* (M, ϱ) is defined as the group $\mathsf{Aut}_{\mathbf{M}^{\mathcal{C}}}(M, \varrho)$ of $\mathcal{C}$-comodule automorphisms of (M, ϱ).

Since the assignment

$$(\mathcal{C}, \delta, \varepsilon) \longmapsto \mathbb{G}_{\mathcal{C}} = (- \otimes_{\mathsf{A}} \mathcal{C}, - \otimes_{\mathsf{A}} \delta, - \otimes_{\mathsf{A}} \varepsilon)$$

yields a bijective correspondence between A-corings and colimit preserving comonads on $\mathbf{M}_{\mathsf{A}}$ (e.g., [5]), and since the category $\mathbf{M}^{\mathcal{C}}$ of (right) $\mathcal{C}$-comodules can be identified with the Eilenberg-Moore category $\mathbf{M}^{\mathbb{G}_{\mathcal{C}}}$ of $\mathbb{G}_{\mathcal{C}}$-colagebras, it is easy to see that the basic structures of [4] can be defined for arbitrary categories $\mathscr{B}$, replacing $\mathbf{M}_{\mathsf{A}}$, and any comonad $\mathbb{G} : \mathscr{A} \to \mathscr{A}$, replacing $- \otimes_{\mathsf{A}} \mathcal{C} : \mathbf{M}_{\mathsf{A}} \to \mathbf{M}_{\mathsf{A}}$, and any $\mathbb{G}$-coalgebra (b, θ), replacing (M, ϱ). This leads to the following definitions.

3.1. Descent cohomology sets of comonads. Given a comonad $\mathbb{G}$ on a category $\mathscr{B}$ and an object $b \in \mathscr{B}$, the *first descent cohomology set of* $\mathbb{G}$ *with values in* b, denoted $\mathsf{Desc}^1(\mathbb{G}, b)$, is the set of equivalence classes of $\mathbb{G}$-coalgebra structures $\theta : b \to G(b)$ on b, where two $\mathbb{G}$-coalgebra structures $\theta_1 : b \to G(b)$ and $\theta_2 : b \to G(b)$ are equivalent if they are isomorphic as the objects of the category $\mathscr{B}^{\mathbb{G}}$, i.e., there exists an isomorphism $f : b \to b$ in $\mathscr{B}$ making the diagram

$$\begin{tikzcd} b \arrow[r, "\theta_1"] \arrow[d, "f"'] & G(b) \arrow[d, "G(f)"] \\ b \arrow[r, "\theta_2"'] & G(b) \end{tikzcd}$$

commute. When b already carries a $\mathbb{G}$-coalgebra structure $\theta : b \to G(b)$, this structure makes $\mathsf{Desc}^1(\mathbb{G}, b)$ a pointed set, with the distinguished point given by the equivalence class of (b, θ). We shall indicate this by writing $\mathsf{Desc}^1(\mathbb{G}, (b, \theta))$ rather than $\mathsf{Desc}^1(\mathbb{G}, b)$. Moreover, in this special case, the 0-*descent cohomology group* $\mathsf{Desc}^0(\mathbb{G}, (b, \theta))$ *of* $\mathbb{G}$ *with coefficients in* (b, θ) can also be defined as the group of all automorphisms of (b, θ) in $\mathscr{B}^{\mathbb{G}}$:

$$\mathsf{Desc}^0(\mathbb{G}, (b, \theta)) := \mathsf{Aut}_{\mathscr{B}^{\mathbb{G}}}(b, \theta).$$

Given a comonad $\mathbb{G}$ on a category $\mathscr{B}$ and an object $b \in \mathscr{B}$, we write $\mathrm{D{\small ESC}}\,(\mathbb{G}, b)$ for the category whose objects are the $\mathbb{G}$-coalgebras with underlying object b, and whose morphisms are those of $\mathscr{A}^{\mathbb{G}}$. It is easy to see that $\pi_0(\mathrm{D{\small ESC}}\,(\mathbb{G}, b)) = \mathsf{Desc}^1(\mathbb{G}, b)$.

3.2. Proposition. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{B}$ and $b \in \mathscr{B}$. Then*

$$\mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b\rangle}) = \mathrm{D{\small ESC}}\,(\mathbb{G}, b).$$

Proof. The objects of both categories are precisely those $\mathbb{G}$-coalgebras whose underlying object is b. A morphism from (b,θ) to (b,θ') in $\mathsf{P}(U_{\mathbb{G}},\iota_{\langle b\rangle})$ is a pair (α,β), where $\beta : b \to b$ is a morphism in $\mathscr{B}$ and $\alpha : (b,\theta) \to (b,\theta)$ is a morphism in $\mathscr{B}^{\mathbb{G}}$ such that $U^{\mathbb{G}}(\alpha) = \iota_{\langle a\rangle}(\beta)$. But since $U^{\mathbb{G}}(\alpha) = \alpha$ and $\iota_{\langle b\rangle}(\beta) = \beta$, it follows that the morphisms in $\mathsf{P}(U^{\mathbb{G}},\iota_{\langle b\rangle})$ is just the morphism in $\mathscr{B}^{\mathbb{G}}$. Thus $\mathsf{P}(U^{\mathbb{G}},\iota_{\langle b\rangle}) = \mathrm{D{\small ESC}}\,(\mathbb{G},b)$. $\square$

Since $\pi_0(\mathrm{D{\small ESC}}\,(\mathbb{G},b)) = \mathsf{Desc}^1(\mathbb{G},b)$, we have immediately the following result.

3.3. PROPOSITION. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{B}$ and $b \in \mathscr{B}$. Then*

$$\pi_0(\mathsf{P}(U^{\mathbb{G}},\iota_{\langle b\rangle})) = \mathsf{Desc}^1(\mathbb{G},b).$$

We give now two examples of calculating Desc^0 and Desc^1 for some comonads.

3.4. EXAMPLE. Recall (for example, from [3]) that a monad $\mathbb{G} = (G,\delta,\varepsilon)$ on a category $\mathscr{B}$ is said to be *idempotent* if it satisfies one (hence all) of the following equivalent conditions:

(i) $\delta : G \to GG$ is a natural isomorphism;
(ii) $G\varepsilon$ (or εG) is an isomorphism;
(iii) the structure morphism of every object in $\mathscr{B}^{\mathbb{G}}$ is an isomorphism; i.e., for every $\mathbb{G}$-coalgebra (b,θ), the $\mathbb{G}$-coaction $\theta : b \to G(b)$ is an isomorphism;
(iv) the forgetful functor $U^{\mathbb{G}} : \mathscr{B}^{\mathbb{G}} \to \mathscr{B}$ is full (and faithful).

3.5. PROPOSITION. *Let $\mathbb{G} = (G,\delta,\varepsilon)$ be an idempotent comonad on a category $\mathscr{B}$ and b be an arbitrary object of $\mathscr{A}$. Then*

$$\mathsf{Desc}^1(\mathbb{G},b) = \begin{cases} \{[(b,\varepsilon_b^{-1})]\}, & \textit{if } \varepsilon_b \textit{ is an isomorphism;} \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Moreover, if ε_b is an isomorphism (and hence $(b,\varepsilon_b^{-1}) \in \mathscr{B}^{\mathbb{G}}$), one has

$$\mathsf{Desc}^0(\mathbb{G},(b,\varepsilon_a^{-1})) = \mathsf{Aut}_{\mathscr{B}}(b)$$

Proof. Since the comonad $\mathbb{G} = (G,\delta,\varepsilon)$ is assumed to be idempotent, the following are equivalent for an arbitrary object b of $\mathscr{B}$ (e.g., [7, Lemma 2.8]):

(1) b carries a $\mathbb{G}$-coalgebra structure.
(2) $\varepsilon_b : G(b) \to b$ is a split epimorphism.
(3) $\varepsilon_b : G(b) \to b$ is an isomorphism.

Accordingly, there is at most one $\mathbb{G}$-algebra structure on b. It then follows that the objects of $\mathscr{B}$ admitting a $\mathbb{G}$-coalgebra structure are precisely those objects b for which the morphism $\varepsilon_b : G(b) \to b$ is an isomorphism; the $\mathbb{G}$-coalgebra structure on such an object b is then unique, being $\varepsilon_b^{-1} : b \to G(b)$. This proves the first part of the proposition.

As, by the very definition, $\mathsf{Desc}^0(\mathbb{G},(b,\varepsilon_b^{-1})) = \mathsf{Aut}_{\mathscr{B}^{\mathbb{G}}}(b,\varepsilon_b^{-1})$, the second part of the proposition follows at once from the fact that the forgetful functor $U^{\mathbb{G}} : \mathscr{B}^{\mathbb{G}} \to \mathbb{G}$ is full and faithful. $\square$

3.6. EXAMPLE. Write $\mathbb{1}$ for the category with just one object $*$ and with $\mathbb{1}(*,*) = 1_*$. Then an arbitrary functor $H : \mathbb{1} \to \mathscr{A}$ simply consists in the choice of an object $a := H(*)$ in $\mathscr{A}$. If $a \in \mathscr{A}$, we write $\ulcorner a \urcorner : \mathbb{1} \to \mathscr{A}$ for the corresponding functor with value a.

Let us fix now a category $\mathscr{A}$ and its object $a \in \mathscr{A}$ and consider the comma category $1_{\mathscr{A}} \downarrow \ulcorner a \urcorner$, which is usually denoted by $\mathscr{A} \downarrow a$. Recall from 2.2 that the objects of this category are the pairs (x,f), where $x \in \mathscr{A}$ and f is a morphism $x \to a$ in $\mathscr{A}$, and the morphisms between two objects $(x, f : x \to a)$ and $(y, g : y \to y)$ are morphisms $h : x \to y$ in $\mathscr{A}$ such that $gh = f$.

Now let $\mathscr{A}$ admit pullbacks. Given morphisms $f : x \to a$, $g : x \to b$, $p : a \to c$ and $q : b \to c$ in $\mathscr{A}$ with $pf = qg$, we write $\langle f, g\rangle : x \to a \times_c b$ for the unique morphism making the diagram

$$\begin{tikzcd}
x \arrow[rr, "g"] \arrow[dr, dashed, "\langle f, g\rangle"] \arrow[dd, "f"'] & & b \arrow[dd, "q"] \\
& a \times_c b \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & \\
a \arrow[rr, "p"'] & & c
\end{tikzcd}$$

commute. Note that for any morphism $k : y \to x$ in $\mathscr{A}$, one has

$$\langle f \cdot k, g \cdot k\rangle = \langle f, g\rangle \cdot k. \tag{3.1}$$

Moreover, if $p' : a' \to c$ and $q' : b' \to c$ are arbitrary morphisms, while $k_1 : a \to a'$ and $k_2 : b \to b'$ are morphisms with $p = p' \cdot k_1$ and $q = q' \cdot k_2$, then

$$(k_1 \times_C k_2) \cdot \langle f, g\rangle = \langle k_1 \cdot f, k_1 \cdot g\rangle. \tag{3.2}$$

Now let $p : e \to b$ be a fixed morphism in $\mathscr{A}$. Then the *change-of-base functor* $p^* : \mathscr{A}/b \to \mathscr{A}/e$ assigns to an object $(x, f : x \to b)$ of $\mathscr{A}/b$ the object $(e \times_b x, \pi_1 : e \times_b x \to e)$ of $\mathscr{A}/e$. It is well-known that p^* has a left adjoint $p_! : \mathscr{A}/e \to \mathscr{A}/b$ given by composition with p. The components of the unit η of this adjunction are given as

$$\eta_{(y,\, g:\, y \to e)} = \langle g, 1_g\rangle : (y, g) \to (e \times_b y, \pi_1)$$

and the components of the counit ε as

$$\varepsilon_{(x,\, f:\, x \to b)} = \pi_2 : (e \times_b x, p \cdot \pi_1) \to (x, f).$$

Let $\mathbb{G}_p$ be the comonad on $\mathscr{A}/b$ generated by the adjunction $p_! \dashv p^*$. Then the functor-part of $\mathbb{G}_p$ is the composite $p_!p^*$, so $\mathbb{G}_p$ takes $(x,\ f : x \to b)$ to the object $(e \times_b x, p \cdot \pi_1)$. The counit $p_!p^* \to 1$ of $\mathbb{G}_p$ is ε, while the comultiplication $\delta : p_!p^* \to p_!p^*p_!p^*$ is defined by

$$\delta_{(x,\, f:\, x \to b)} = p_!(\eta_{p^*(x,\, f)}) = \langle \pi_1, 1_{e \times_b x}\rangle : e \times_b x \to e \times_b e \times_b x.$$

A $\mathbb{G}_p$-coalgebra structure on $(x, f : x \to b) \in \mathscr{A}/b$ is a morphism $\theta : x \to e \times_b x$ in $\mathscr{A}$ making the diagrams

$$\begin{tikzcd}
x \arrow[r, "\theta"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & e \times_b x \arrow[d, "p \cdot \pi_1"] \\
b \arrow[r, equal] & b\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "\theta"] \arrow[d, "1_x"'] \arrow[dr, phantom, "(2)"] & e \times_b x \arrow[d, "\pi_2"] \\
x \arrow[r, equal] & x
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
x \arrow[r, "\theta"] \arrow[d, "\theta"'] \arrow[dr, phantom, "(3)"] & e \times_b x \arrow[d, "\langle \pi_1, 1_{e \times_b x}\rangle"] \\
e \times_b x \arrow[r, "e \times_b \theta"'] & e \times_b e \times_b x\,.
\end{tikzcd} \tag{3.3}$$

commute. Note that the commutativity of diagram (3.3)(1) expresses the fact that θ is a morphism from $(x, f : x \to b)$ to $\mathbb{G}_p(x,\ f :\ x \to b) = (e \times_b x, p \cdot \pi_1)$ in $\mathscr{A}/b$.

One easily concludes from the commutativity of (3.3)(2) that $\theta = \langle \overline{\theta}, 1_X\rangle$, where $\overline{\theta} = \pi_1\theta : x \to e$. Then the commutativity of (3.3)(1) means that $p \cdot \overline{\theta} = f$, while diagram (3.3)(3) can be rewritten as

$$\begin{tikzcd}
x \arrow[r, "\langle \overline{\theta}, 1_x\rangle"] \arrow[d, "\langle \overline{\theta}, 1_X\rangle"'] & e \times_b x \arrow[d, "\langle \pi_1, 1_{e \times_b x}\rangle"] \\
e \times_b x \arrow[r, "1_e \times_b \langle \overline{\theta}, 1_x\rangle"'] & e \times_b e \times_b x.
\end{tikzcd}$$

But since

$$\langle \pi_1,\, 1_{e \times_b x}\rangle \cdot \langle \overline{\theta},\, 1_x\rangle \overset{(3.1)}{=} \langle \pi_1 \cdot \langle \overline{\theta},\, 1_x\rangle, 1_{e \times_b x} \cdot \langle \overline{\theta},\, 1_x\rangle\rangle = \langle \overline{\theta}, \langle \overline{\theta},\, 1_x\rangle\rangle$$

and

$$(1_e \times_b \langle \overline{\theta},\, 1_x\rangle) \cdot \langle \overline{\theta},\, 1_x\rangle \overset{(3.2)}{=} \langle 1_e \cdot \overline{\theta}, \langle \overline{\theta},\, 1_x\rangle \cdot 1_x\rangle = \langle \overline{\theta}, \langle \overline{\theta},\, 1_x\rangle\rangle,$$

it follows that diagram (3.3)(3) is always commutative. Thus, to give a $\mathbb{G}_p$-coalgebra structure on $(x, f : x \to b) \in \mathscr{A}/a$ is to give a morphism $\theta : x \to e$ in $\mathscr{A}$ making the diagram

$$\begin{tikzcd} x \arrow[r, "\theta"] \arrow[d, "f"'] & e \arrow[d, "p"] \\ b \arrow[r, equal] & b \end{tikzcd}$$

commute. It is easy then to see that two $\mathbb{G}_p$-coalgebra structures θ and ϑ on $(x, f : x \to b) \in \mathscr{A}/b$ are isomorphic if and only if there is an automorphism $h : f \to f$ in $\mathscr{A}/b$ (i.e., an automorphism $h : x \to x$ in $\mathscr{A}$ with $f = fh$) making the diagram

$$\begin{tikzcd} x \arrow[r, "\theta"] \arrow[d, "h"'] & e \arrow[d, "1_e"] \\ x \arrow[r, "\vartheta"'] & e \end{tikzcd}$$

commute. Putting $f = 1_b : b \to b$ and using the fact that there is only one automorphism of 1_b in $\mathscr{A}/b$, namely 1_b, we obtain

$$\mathsf{Desc}^1(\mathbb{G}_p, 1_b : b \to b) = \{\theta : b \to e \text{ in } \mathscr{A} \text{ with } p \cdot \theta = 1_b\}.$$

4. Twisted Forms

In [10] (see also [11]), we introduced the notion of a twisted form w.r.t. any adjunction and proved that when the adjunction is comonadic, these twisted forms can be described in terms of the induced command coactons. In this section, we introduce twisted forms of an object w.r.t. functors and show how to describe the firs descent cohomology sets by using them.

4.1. Twisted forms w.r.t. functors. Let $H : \mathscr{Y} \to \mathscr{X}$ be a fixed, chosen functor. For any object $x \in \mathscr{X}$, let $\textsc{Twist}(H, x)$ be the category whose objects are the pairs (y, f), where $y \in \mathscr{Y}$ and $f : H(y) \to x$ is an isomorphism in $\mathscr{X}$, and whose morphisms between two objects (y, f) and (y', f') are just morphisms $y \to y'$ in $\mathscr{Y}$. An object $(y, f) \in \textsc{Twist}(H, y)$ is called an *H-twisted form of x*. A straightforward calculation shows that the following propositopn is valid.

4.2. Proposition. *In the situation considered above, the assignments*

$$(y, f) \to (y, f, x)$$

and

$$\left(\alpha\colon \begin{smallmatrix}(y,f)\\ \downarrow \\ (y',f')\end{smallmatrix}\right) \longmapsto \left((\alpha, f' \cdot H(\alpha) \cdot f^{-1})\colon \begin{smallmatrix}(y,f,x)\\ \downarrow \\ (y',f',x)\end{smallmatrix}\right)$$

yield a functor

$$K_x : \textsc{Twist}(H, x) \to \mathsf{Ps}(H, \iota_{\langle x \rangle}),$$

which is an isomorphism of categories and makes the diagram

$$\begin{tikzcd} \textsc{Twist}(H, x) \arrow[rr, "K_x"] \arrow[dr, "U"'] & & \mathsf{Ps}(H, \iota_{\langle x \rangle}) \arrow[dl, "P_1"] \\ & \mathscr{A} & \end{tikzcd}$$

where $U : \textsc{Twist}(H, x) \to \mathscr{A}$ is the evident forgetful functor, commute. The inverse K_x^{-1} of K_x takes (y, f, x) to (y, f) and takes $(\beta : y \to y', \gamma : x \to x)$ to $\beta : y \to y'$.

Let $\mathsf{Twist}_H(x)$ be the collection of isomorphic classes of h-twisted forms of $x \in \mathscr{X}$. Thus, $\pi_0(\textsc{Twist}(H, x)) = \mathsf{Twist}_H(x)$. According to the definition of the category $\textsc{Twist}(H, x)$, two H-twisted forms (y, f) and (y', f') of b are isomorphic if there exists an isomorphism $y \simeq y'$ in $\mathscr{Y}$.

When $x = H(y)$ for some $y \in \mathscr{Y}$, then an H-twisted form of $H(y)$ is called an *H-twisted form of y*. In this case we simply write $\mathsf{Twist}_H(y)$ instead of $\mathsf{Twist}_H(H(y))$. $\mathsf{Twist}_H(y)$ is a pointed set, with a distinguished point given by the class of $(y, 1_{H(y)})$.

As an immediate consequence of Proposition 4.2, we have

4.3. Corollary. *Let $H : \mathscr{Y} \to \mathscr{X}$ be a functor and $x \in \mathscr{X}$. Then*

$$\pi_0(K_x) : \mathsf{Twist}_H(x) \to \pi_0(\mathsf{Ps}(H, \iota_{\langle x \rangle}))$$

is a bijection.

The next result shows that twisted forms can be used to classify the descent cohomology.

4.4. Theorem. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be an adjunction and let $\mathbb{G} = (FU, \varepsilon, F\eta U)$ be the corresponding comonad on the category $\mathscr{B}$. If F is comonadic, then for any $a \in \mathscr{A}$, the assignment*

$$(x, f) \to ((F(a), F(\eta_a)), FU(f) \cdot F(\eta_x) \cdot f^{-1})$$

yields a bijection

$$\omega_a^{\mathbb{G}} : \mathsf{Twist}_F(a) \to \mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a)))$$

of pointed sets [1].

Proof. Since F is assumed to be comonadic, the map

$$\pi_0(P_{\langle F(a) \rangle}) : \pi_0(\mathsf{Ps}(F, \iota_{\langle F(a) \rangle})) \to \pi_0(\mathsf{P}(U^{\mathbb{G}}, \iota_{\langle F(a) \rangle}))$$

is bijective by Corollary 2.10. So, in the light of Proposition 3.2 and Corollary 4.3, the assignment

$$[(x, f)] \to [(F(a), FU(f) \cdot F(\eta_x) \cdot f^{-1})]$$

yields a bijection from $\mathsf{Twist}_F(a)$ to $\mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a)))$. Moreover, it is easy to see that $K_{\langle F(a) \rangle}(a, 1_{F(a)}) = (F(a), F(\eta_a))$. Hence $\pi_0(K_{\langle F(a) \rangle})$ is an isomorphism of pointed sets. This completes the proof. $\square$

Let $\mathscr{Z} \xrightarrow{G} \mathscr{Y} \xrightarrow{H} \mathscr{X}$ be the functors and y be a fixed object of $\mathscr{Y}$. For any G-twisted form (z, f) of y, the pair $(z, H(f))$ is an (HG)-twisted form of $H(y)$. By the very definition of twisted form, (z, f) and (z', f') are equivalent as G-twisted forms of y if and only if $(a, H(f))$ and $(a', H(f'))$ are equivalent as (HG)-twisted forms of $H(y)$. Thus the passage

$$[(z, f)] \to [(a, H(f))]$$

yields a map

$$S_y : \mathsf{Twist}_G(y) \to \mathsf{Twist}_{(HG)}(H(y)).$$

Quite obviously, S_y is injective. Moreover, it is easy to see that when $y = G(z)$ for some $z \in \mathscr{Z}$, then $S_{G(z)}$, which is a map from $\mathsf{Twist}_G(z)(= \mathsf{Twist}_G(G(z)))$ to $(\mathsf{Twist}_{(HG)}(H(G(z)) = \mathsf{Twist}_{(HG)}((HG)(z))$ $=)\mathsf{Twist}_{(HG)}(z)$, is a morphism of pointed sets.

It turns out that in some cases, this map is surjective (and hence bijective). In order to prove this, we need one preliminary result.

4.5. Lemma. *For any adjunction $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$, the diagram*

$$FUFUF \underset{\varepsilon FUF}{\overset{FU\varepsilon F}{\rightrightarrows}} FUF \xrightarrow{\varepsilon F} F \tag{4.1}$$

is a split coequalizer, a splitting being given by $F\eta$ and $FUF\eta$.

Proof. $\varepsilon F \cdot F\eta = 1$ by one of the triangular identities for the adjunction $F \dashv U$, and hence $FU\varepsilon F \cdot FUF\eta = 1$; and the remaining splitting condition follows from the naturality of $\varepsilon : FU \to 1$. $\square$

[1] Recall that $(F(a), F(\eta_a))$ is a $\mathbb{G}$-coalgebra.

4.6. Proposition. *Let $\mathbb{T}$ be the monad on a category $\mathscr{A}$ generated by an adjoint pair $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$. For any $a \in \mathscr{A}$, the map*

$$S_{F(a)} : \mathsf{Twist}_F(a) \to \mathsf{Twist}_{F_{\mathbb{T}}}(a),$$

induced by the composition $\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{K_{\mathbb{T}}} \mathscr{A}_{\mathbb{T}}$, is an isomorphism of pointed sets.

Proof. Since we have already observed that $S_{F(a)}$ is injective, it suffices to verify that $S_{F(a)}$ is surjective. To this end, consider an arbitrary $F_{\mathbb{T}}$-twisted form

$$(x, f : F_{\mathbb{T}}(x) \to F_{\mathbb{T}}(a))$$

of $F_{\mathbb{T}}(a)$. Since f is an (iso)morphism in $\mathscr{A}_{\mathbb{T}}$, it follows that $f : UF(x) \to UF(a)$ is an (iso)morphism in $\mathscr{A}$ and that the diagram

$$\begin{tikzcd} UFUF(x) \arrow[r, "UF(f)"] \arrow[d, "U(\varepsilon_{F(x)})"'] & UFUF(a) \arrow[d, "U(\varepsilon_{F(a)})"] \\ UF(x) \arrow[r, "f"'] & UF(a) \end{tikzcd} \tag{4.2}$$

commutes. Then since $F(f) \cdot \varepsilon_{FUF(x)} = \varepsilon_{FUF(a)} \cdot FUF(f)$ by naturality of ε, the diagram

$$\begin{tikzcd} FUFUF(x) \arrow[r, shift left, "FU(\varepsilon_{F(x)})"] \arrow[r, shift right, "\varepsilon_{FUF(x)}"'] \arrow[d, "FUF(f)"'] & FUF(x) \arrow[r, "\varepsilon_{F(x)}"] \arrow[d, "F(f)"] & F(x) \arrow[d, dashed, "f'"] \\ FUFUF(a) \arrow[r, shift left, "FU(\varepsilon_{F(a)})"] \arrow[r, shift right, "\varepsilon_{FUF(a)}"'] & FUF(a) \arrow[r, "\varepsilon_{F(a)}"'] & F(a) \end{tikzcd} \tag{4.3}$$

is serially commutative. Since, by Lemma 4.5, each row of this diagram is a (split) coequalizer and since f is an isomorphism, it follows that there exists a unique isomorphism $f' : F(x) \to F(a)$ in $\mathscr{B}$ making the right square in Diagram (4.3) commute. Thus, in particular, $(x, f') \in \mathsf{Twist}_F(a)$ and $U(\varepsilon_{F(a)}) \cdot UF(f) = U(f') \cdot (\varepsilon_{F(x)})$. It then follows – since $U(\varepsilon_{F(x)})$ is a (split) epimorphism – from the commutativity of Diagram (4.2) that $U(f') = f$. Thus $K_{\mathbb{T}}(f) = U(f') = f$, and hence $S_{F(a)}([(x, f')]) = [(x, K_{\mathbb{T}}(f'))] = [(x, f]$. Therefore $S_{F(a)}$ is surjective. □

4.7. Theorem. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be an adjunction, $\mathbb{T} = (UF, \eta, U\varepsilon F)$ be the monad on $\mathscr{A}$ generated by the adjunction, and $\mathbb{G}$ (resp., $\mathbb{G}_{\mathbb{T}}$) be the comonad on $\mathscr{B}$ (resp., $\mathscr{A}_{\mathbb{T}}$) corresponding to the adjunction $F \dashv U$ (resp. $F_{\mathbb{T}} \dashv U_{\mathbb{T}}$). Suppose that idempotents split in $\mathscr{A}$ [2]. If F is comonadic, then for any object $a \in \mathscr{A}$,*

$$\mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a))) = \mathsf{Desc}^1(\mathbb{G}_{\mathbb{T}}, (F_{\mathbb{T}}(a), F_{\mathbb{T}}(\eta_a)))$$

as pointed sets.

Proof. Since idempotents splits in $\mathscr{A}$, to say that the functor F is comonadic is to say that the functor $F_{\mathbb{T}}$ is comonadic (see Theorem 3.20 in [16]).The result now follows from Theorem 4.4 and Proposition 4.6. □

5. Subobjects and Descent Cohomology Sets

We continue supposing that $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is an adjunction and that $\mathbb{G}$ is the comoanad on $\mathscr{B}$ generated by the adjunction.

In this section, we study the relationship between the set $\mathsf{Desc}^1(\mathbb{G}, b)$ for a given object $b \in \mathscr{B}$ and the set of some subobjects of $U(b)$. We will show that when the left adjoint functor is F comonadic, the aforementioned relationship takes the form of a bijection.

[2] It is said that idempotents split in $\mathscr{A}$ if whenever $a \in \mathscr{A}$, $e : a \to a$ with $e^2 = e$, then there exist an object $a' \in \mathscr{A}$ and morphisms $p : a \to a'$ and $\iota : a' \to a$ such that $\iota p = e$ and $p\iota = 1_{a'}$

5.1. The category of subobjects associated with adjunctions. Given an object $b \in \mathscr{B}$, write $\textsc{Sub}_F(U(b))$ for the category whose objects are the pairs (a, ι) for which $\iota : a \to U(b)$ is a regular monomorphism[3] in $\mathscr{A}$ and, moreover, its image $\widetilde{\iota} : F(a) \to b$ under the adjunction bijection

$$\alpha_{a,b} : \mathscr{B}(F(a), b) \simeq \mathscr{A}(a, U(b))$$

is an isomorphism.

Clearly, if $(a, \iota) \in \textsc{Sub}_F(U(b))$, then $(a, \widetilde{\iota}) \in \textsc{Twist}(F, b)$. Hence we can define a functor

$$S_b : \textsc{Sub}_F(U(b)) \to \textsc{Twist}(F, b)$$

by $S_b(a, \iota) = (a, \widetilde{\iota})$. The morphism assignment is given by the identity function. In other words, S_b is the identity on morphisms. It can be easily checked that S_b is indeed a functor.

The following result gives a criterion for determining when the functor S_b is an isomorphism of categories.

5.2. Proposition. *In the situation described above, if the adjunction $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is precomonadic, then the functor*

$$S_b : \textsc{Sub}_F(U(b)) \to \textsc{Twist}(F, b)$$

is an isomorphism of categories.

Proof. Suppose that the adjunction $\eta, \varepsilon : F \dashv U$ is precomonadic. Then (see, for example, [13, Theorem 2.4]) η_a is a regular monomorphism for all $a \in \mathscr{A}$.

Now, if $(a, f) \in \textsc{Twist}(F, b)$, then $f : F(a) \to b$ (and hence also $U(f)$) is an isomorphism, and hence $\widetilde{f} = \alpha_{a,b}(f) = U(f) \cdot \eta_a$ is a regular monomorphism. Thus, $(a, \widetilde{f}) \in \textsc{Sub}_F(U(b))$. It follows – since $\widetilde{\widetilde{f}} = f$ – that $S_b(a, \widetilde{f}) = (a, f)$. The result now follows by noting that (as is easily seen) the functor S_b is full and faithful. □

Combining Propositions 4.2 and 5.2, we have

5.3. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a precomonadic adjunction and $b \in \mathscr{B}$. Then the assignments*

$$(a, \iota) \longmapsto (a, \widetilde{\iota}, b)$$

and

$$(\alpha : (a, \iota) \to (a', \iota')) \longmapsto ((\alpha, \iota' \cdot F(f) \cdot \iota^{-1}) : (a, \widetilde{\iota}, b) \to (a', \widetilde{\iota'}, b))$$

yield an isomorphism of categories

$$\textsc{Sub}_F(U(b)) \simeq \mathsf{Ps}(F, \iota_{\langle b \rangle}).$$

Its inverse takes (a, f, b) into $(a, \widetilde{f})$ and $(\alpha, beta)$ into α.

We write $\mathsf{Sub}_{F \dashv U}(b)$ for $\pi_0(\textsc{Sub}_F(U(b)))$. Note that in case $F \dashv U : \mathscr{B} \to \mathscr{A}$ is a precomonadic adjunction and $b = F(a)$ for some $a \in \mathscr{A}$, we find that $\mathsf{Sub}_{F \dashv U}(F(a))$ is a pointed set with a base point of the class $\eta_a : a \to UF(a)$, which is a regular monomorphism because of the precomonadicity of the adjunction (see again [13, Theorem 2.4]).

As an immediate consequence of Proposition 5.3, we have

5.4. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a precomonadic adjunction and $b \in \mathscr{B}$. Then there is a bijection*

$$\mathbf{Sub}_{F \dashv U}(b) \simeq \pi_0(\mathsf{Ps}(F, \iota_{\langle b \rangle})).$$

[3] Recall that regular monomorphisms are morphisms occurring as equalizers of some pairs of parallel morphisms.

Suppose now that $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is a comonadic adjunction. Write $\mathbb{G}$ for the induced comonad on $\mathscr{B}$. Then in the commutative diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "K^{\mathbb{G}}"] \arrow[dr, "F"'] & & \mathscr{B}^{\mathbb{G}} \arrow[dl, "U^{\mathbb{G}}"] \\
\langle b \rangle \arrow[r, "\iota_{\langle b \rangle}"'] & \mathscr{B} &
\end{tikzcd}$$

$K^{\mathbb{G}}$ is an equivalence of categories and thus it induces (see 2.6) an equivalence of categories

$$\mathsf{Ps}(F, \iota_{\langle b \rangle}) \simeq \mathsf{Ps}(U_{\mathbb{G}}, \iota_{\langle b \rangle}).$$

In the light of Theorem 2.9, from Propositions 3.2, 5.2 and 5.4 follows

5.5. Theorem. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction with the corresponding comonad $\mathbb{G}$ on $\mathscr{B}$ and $b \in \mathscr{B}$. Then there are the bijections*

$$\mathsf{Desc}^1(\mathbb{G}, b) \simeq \mathbf{Sub}_{F \dashv U}(b) \simeq \mathsf{Twist}_F(b).$$

Moreover, when $b = F(a)$ for some $a \in \mathscr{A}$, then these bijections are isomorphisms of pointed sets.

6. Torsors

Given a comonad $\mathbb{G}$ on a category $\mathscr{B}$ and an object $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$, a (b, θ)-*torsor* is a triple (x, ϑ, α), where $(x, \vartheta) \in \mathscr{B}^{\mathbb{G}}$ and $\alpha : b \to x$ is an isomorphism in $\mathscr{B}$. Morphism between two (b, θ)-torsors (x, ϑ, α) and $(x', \vartheta', \alpha')$ are morphisms from (x, ϑ) to (x', ϑ') in $\mathscr{B}^{\mathbb{G}}$. The (b, θ)-torsors and their morphisms constitute a category $\textsc{Tors}(b, \theta)$. It is easy to see that this category is just the category $\textsc{Twist}(U^{\mathbb{G}}, (b, \theta))$. Therefore, applying Proposition 4.2, we obtain

6.1. Proposition. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{B}$ and $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$. Then there is an isomorphism of categories*

$$\textsc{Tors}(b, \theta) \simeq \mathsf{Ps}(U^{\mathbb{G}}, \iota_{\langle b \rangle})$$

making the diagram

$$\begin{tikzcd}
\textsc{Tors}(b, \theta) \arrow[rr, "\simeq"] \arrow[dr, "U"'] & & \mathsf{Ps}(U^{\mathbb{G}}, \iota_{\langle b \rangle}) \arrow[dl, "P_1"] \\
& \mathscr{B}^{\mathbb{G}} &
\end{tikzcd}$$

where $U : \textsc{Tors}(b, \theta) \to \mathscr{B}^{\mathbb{G}}$ is the evident forgetful functor, commute.

6.2. The pointed set $\mathsf{Tors}(b, \theta)$. Given a comonad $\mathbb{G}$ on a category $\mathscr{B}$ and an object $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$, denote by $\mathsf{Tors}(b, \theta)$ the set of isomorphic classes of (b, θ)-torsors. It is pointed with distinguished point of the class $(b, \theta, 1_b)$. It is clear that $\mathsf{Tors}(b, \theta) = \pi_0(\textsc{Tors}(b, \theta))$. From Proposition 6.1 follows

6.3. Proposition. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{B}$ and $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$. Then*

$$\pi_0(\mathsf{Ps}(U^{\mathbb{G}}, \iota_{\langle b \rangle})) = \mathsf{Tors}(b, \theta).$$

6.4. Theorem. *For any comonad $\mathbb{G}$ on $\mathscr{B}$ and any $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$, the assignment $(b, \varrho) \longmapsto ((b, \varrho), 1_b)$ yields an isomorphism of pointed sets*

$$\mathsf{Desc}^1(\mathbb{G}, (b, \theta)) \simeq \mathsf{Tors}(b, \theta),$$

whose inverse takes (x, ν, α) to $(b, G(\alpha^{-1}) \cdot \nu \cdot \alpha)$.

Proof. Since

- $\pi_0(\mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle})) = \mathsf{Desc}^1(\mathbb{G}, (b, \theta))$ by Proposition 3.2, and
- $K_{\mathsf{P}} : \mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle}) \to \mathsf{Ps}(U^{\mathbb{G}}, \iota_{\langle b \rangle})$ is an equivalence of categories by Proposition 2.5, and thus $\pi_0(K_{U^{\mathsf{P}}}) : \pi_0(\mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle})) \to \pi_0(\mathsf{Ps}(U^{\mathbb{G}}, \iota_{\langle b \rangle}))$ is an isomorphism of pointed sets,

the result follows from the previous proposition. $\square$

6.5. Galois comodule functors. Given a functor $F : \mathscr{A} \to \mathscr{B}$ and a comonad $\mathbb{H} = (H, \delta, \sigma)$ on $\mathscr{B}$, F is called a *left* $\mathbb{H}$-*comodule* (e.g., [19, Section 3]) if there exists a natural transformation $\kappa_F : F \to HF$ inducing commutativity of the diagrams

$$\begin{tikzcd} F \arrow[r, "\kappa_F"] \arrow[dr, equal] & HF \arrow[d, "\sigma F"] \\ & F, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\kappa_F"] \arrow[d, "\kappa_F"'] & HF \arrow[d, "\delta F"] \\ HF \arrow[r, "H\kappa_F"'] & HHF. \end{tikzcd}$$

Suppose now that F has a right adjoint $U : \mathscr{B} \to \mathscr{A}$, with unit $\eta : 1_{\mathscr{A}} \to UF$ and counit $\varepsilon : FU \to 1_{\mathscr{B}}$. Write $\mathbb{G}$ for the comonad on $\mathscr{B}$ generated by this adjunction. Recall (e.g., from [14]) that there exist bijective correspondences between

(i) functors $K : \mathscr{A} \to \mathscr{B}^{\mathbb{H}}$ with the commutative diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, "K"] \arrow[dr, "F"'] & \mathscr{B}^{\mathbb{H}} \arrow[d, "U^{\mathbb{H}}"] \\ & \mathscr{B}\,; \end{tikzcd} \tag{6.1}$$

(ii) left $\mathbb{H}$-comodule structures $\kappa_F : F \to HF$ on F;

(iii) comonad morphisms $t_K : \mathbb{G} \to \mathbb{H}$.

These bijections are constructed as follows: Given a functor K making Diagram (6.1) commute, then $K(a) = (F(a), \kappa_a)$ for some morphism $\kappa_a : F(a) \to HF(a)$ and the collection $\{\kappa_a,\, a \in \mathscr{A}\}$ constitutes a natural transformation $\kappa_F : F \to HF$ making F a $\mathbb{H}$-comodule. Conversely, if $(F, \kappa_F : F \to HF)$ is a $\mathbb{H}$-module, then $K : \mathscr{A} \to \mathscr{B}^{\mathbb{H}}$ is defined by $K(a) = (F(a), (\kappa_F)_a)$. Next, for any (left) $\mathbb{H}$-comodule structure $\kappa_F : F \to HF$, the composite

$$t_K : FU \xrightarrow{\kappa_F U} HFU \xrightarrow{H\varepsilon} H$$

is a comonad morphism from the comonad $\mathbb{G}$ generated by the adjunction $F \dashv U$ to the comonad $\mathbb{H}$. On the other hand, for any comonad morphism $t : \mathbb{G} \to \mathbb{H}$, the composite

$$\kappa_F : F \xrightarrow{F\eta} FUF \xrightarrow{tF} HF$$

defines a left $\mathbb{H}$-comodule structure on F.

A left $\mathbb{H}$-comodule functor F is said to be $\mathbb{H}$-*Galois* provided t_K is an isomorphism (e.g. [18, Definition 1.3]).

For more details on the Galois comodule functors, see, e.g., [14, 17–21].

Now, let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$, an adjunction with the corresponding comonad $\mathbb{G} = (FU, F\eta U, \varepsilon)$, $\mathbb{H} = (H, \delta, \sigma)$, be a comonad on $\mathscr{B}$ and $K : \mathscr{A} \to \mathscr{B}^{\mathbb{H}}$ be a functor making Diagram (6.1) commute.

6.6. Theorem. *In the situation described above, suppose that F is a comonadic functor and that the comonad morphism $t_K : \mathbb{G} \to \mathbb{H}$ induced by the triangle (6.1), is an isomorphism (i.e., F is an $\mathbb{H}$-Galois comodule functor). Then for any object $a \in \mathscr{A}$, there is an isomorphism of pointed sets*

$$\mathsf{Desc}^1(\mathbb{H}, K(a)) \simeq \mathsf{Twist}_F(a).$$

Proof. Since the functor $F : \mathscr{A} \to \mathscr{B}$ is assumed to be comonadic, it follows from Theorem 4.4 that

$$\mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a))) \simeq \mathsf{Twist}_F(a)$$

as pointed sets. Now, if, in addition, t_K is an isomorphism, then the functor $t_K^* : \mathscr{B}^{\mathbb{G}} \to \mathscr{B}^{\mathbb{H}}$ that takes $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$ to $(b, (t_K)_b \cdot \theta) \in \mathscr{B}^{\mathbb{H}}$ is an isomorphism of categories. Moreover, the diagram

$$\begin{tikzcd} \mathscr{B}^{\mathbb{G}} \arrow[r, "t_K^*"] \arrow[dr, "U^{\mathbb{G}}"'] & \mathscr{B}^{\mathbb{H}} \arrow[d, "U^{\mathbb{H}}"] \\ & \mathscr{B}\,; \end{tikzcd} \tag{6.2}$$

commutes. Then the induced functor $\mathsf{P}(t_K^*, \iota_{\langle b\rangle}) : \mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b\rangle}) \to \mathsf{P}(U^{\mathbb{H}}, \iota_{\langle b\rangle})$ is an isomorphism for all $b \in \mathscr{B}$. It then follows from Proposition 3.3 that the map

$$\pi_0(\mathsf{P}(t_K^*, \iota_{\langle b\rangle})) : \mathsf{Desc}^1(\mathbb{G}, b)) \simeq \mathsf{Desc}^1(\mathbb{H}, b),$$

taking $[(b, \theta)]$ to $[(b, (t_K)_b \cdot \theta)]$, is bijective for all $b \in \mathscr{B}$. In particular,

$$\pi_0(\mathsf{P}(t_K^*, \iota_{\langle F(a)\rangle})) : \mathsf{Desc}^1(\mathbb{G}, (F(a)) \simeq \mathsf{Desc}^1(\mathbb{H}, F(a))$$

is bijective. But since $t_K^*((F(a), F(\eta_a))) = K(a)$ by [17, Lemma 4.3.], it follows that $\pi_0(\mathsf{P}(t_K^*, \iota_{\langle F(a)\rangle}))$ is, in fact, an isomorphism of the pointed sets

$$\mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a))) \simeq \mathsf{Desc}^1(\mathbb{H}, K(a)).$$

Consequently, the pointed sets $\mathsf{Desc}^1(\mathbb{H}, K(a))$ and $\mathsf{Twist}_F(a)$ are isomorphic. $\square$

6.7. Example. Let K be a commutative ring, A be a K-algebra and $\mathcal{C} = (\mathcal{C}, \Delta, \epsilon)$ be an A-coring. Given a right $\mathcal{C}$-comodule (X, θ), we write $\mathsf{Desc}^1(\mathcal{C}, (X, \theta))$ for the pointed set $\mathsf{Desc}^1(\mathbb{G}_{\mathcal{C}}, (X, \theta))$.

Let (Σ, ν) be a fixed, chosen right $\mathcal{C}$-comodule, and consider the K-algebra $B = \mathbf{M}^{\mathcal{C}}((\Sigma, \nu), (\Sigma, \nu))$. Then Σ has a canonical structure of (B, A)-bimodule and one has the adjunction

$$\begin{tikzcd} \mathbf{M}_B \arrow[r, bend left, "-\otimes_B \Sigma"] & \mathbf{M}_A \arrow[l, bend left, "\mathbf{M}_A(\Sigma,-)"] \end{tikzcd}$$

where $- \otimes_B \Sigma$ is left adjoint to $\mathbf{M}_A(\Sigma, -)$. Write $\mathbb{G}_\Sigma$ for the comonad on $\mathbf{M}_A$ generated by this adjunction.

The natural transformation

$$\mathrm{Can} : \mathbf{M}_A(\Sigma, -) \otimes_B \Sigma \xrightarrow{\mathbf{M}_A(\Sigma,-)\otimes_B \nu} \mathbf{M}_A(\Sigma, -) \otimes_B \Sigma \otimes_A \mathcal{C} \xrightarrow{\mathrm{ev}\otimes_A \mathcal{C}} - \otimes_A \mathcal{C},$$

where ev is the evaluation map, is a comonad morphism from the comonad $\mathbb{G}_\Sigma$ to the comonad $\mathbb{G}_{\mathcal{C}}$. (Σ, ν) is called a *Galois comodule* provided Σ_A is finitely generated and projective and the natural transformation Can is an isomorphism. For further details regarding the theory of Galois comodules we refer to [26].

Since Σ is a (B, A)–bimodule, $\Sigma^* = \mathbf{M}_A(\Sigma, A)$ is canonically endowed with a structure of (A, B)–bimodule, and $\Sigma^* \otimes_B \Sigma$ is an A–bimodule in a natural way. Assume that Σ_A is finitely generated and projective with a finite dual basis $\{(e_i^*, e_i)\} \subseteq \Sigma^* \otimes_B \Sigma$. Then A–bimodule $\Sigma^* \otimes_B \Sigma$ is an A-coring with comultiplication and counit defined, respectively, by

$$\Delta(f \otimes_B x) = \sum_i (f \otimes_B e_i \otimes_A e_i^* \otimes_B x) \text{ and } \epsilon(f \otimes_B x) = \mathrm{ev}(f \otimes_B x) = f(x).$$

Then Can may be written as the composite

$$- \otimes_A \Sigma^* \otimes_B \Sigma \xrightarrow{-\otimes_A \Sigma^* \otimes_B \nu} - \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \mathcal{C} \xrightarrow{-\otimes_A \mathrm{ev} \otimes_A \mathcal{C}} - \otimes_A \mathcal{C}.$$

Thus, Can is an isomorphism if and only if the composite

$$\mathrm{Can}_A : \Sigma^* \otimes_B \Sigma \xrightarrow{\Sigma^* \otimes_B \nu} \Sigma^* \otimes_B \Sigma \otimes_A \mathcal{C} \xrightarrow{\mathrm{ev}\otimes_A \mathcal{C}} \mathcal{C}$$

is an isomorphism.

For any $N \in \mathbf{M}_B$, we write $\mathsf{Twist}_B(\Sigma, N)$ for $\mathsf{Twist}_{-\otimes_B \Sigma}(N)$.

By applying Theorem 6.6 to the present situation, we obtain the following slight generalization of [4, Theorem 2.4].

6.8. Theorem. *Let $\mathcal{C}$ be an A-coring, A being an algebra over a commutative ring K, and let (Σ, ν) be a right $\mathcal{C}$-comodule. Write $B = \mathbf{M}^{\mathcal{C}}(\Sigma, \Sigma)$. If the functor*

$$- \otimes_B \Sigma : \mathbf{M}_B \to \mathbf{M}_A$$

is comonadic and Σ is a Galois $\mathcal{C}$-comodule, then for any right B-module N, there exists an isomorphism of pointed sets

$$\mathsf{Desc}^1(\mathcal{C}, N \otimes_B \Sigma) \simeq \mathsf{Twist}_B(\Sigma, N),$$

where $N \otimes_B \Sigma$ is a right $\mathcal{C}$-comodule with the induced coaction

$$N \otimes_B \nu : N \otimes_B \Sigma \to N \otimes_B \Sigma \otimes_B \mathcal{C}.$$

7. Descent Cohomology Sets of Monads

Dualizing the notions of descent cohomology sets of comonads leads to the descent cohomology sets of monads.

Given a monad $\mathbb{T}$ on a category $\mathscr{A}$ and an object $a \in \mathscr{A}$, the *first descent cohomology set of* $\mathbb{T}$ *with values in* a, denoted $\mathsf{Desc}^1(\mathbb{T}, a)$, is the set of equivalence classes of $\mathbb{T}$-algebra structures on a, where two $\mathbb{T}$-algebra structures are equivalent if they are isomorphic as the objects of the category $\mathscr{A}_{\mathbb{T}}$.

When a comes equipped with a $\mathbb{T}$-algebra structure $h : T(a) \to a$, $\mathsf{Desc}^1(\mathbb{T}, a)$ becomes a pointed set with a base point the equivalence class of (a, h), and to indicate this fact, we write $\mathsf{Desc}^1(\mathbb{T}, (a, h))$ in place of $\mathsf{Desc}^1(\mathbb{T}, a)$. Moreover, in this special case, the *zeroth descent cohomology group* $\mathsf{Desc}^0(\mathbb{T}, (a, h))$ *of* $\mathbb{T}$ *with coefficients in* (a, h) is also defined as the group of all automorphisms of (a, h) in $\mathscr{A}_{\mathbb{T}}$. Thus $\mathsf{Desc}^0(\mathbb{T}, (a, h)) = \mathsf{Aut}_{\mathscr{A}_{\mathbb{T}}}(a, h)$.

The rest of this section is devoted to describing descent cohomology sets for some monads.

7.1. Example. Idempotent monads. Dualizing Proposition 3.5 gives the following result for the idempotent monads.

Proposition. *Let $\mathbb{T} = (T, m, e)$ be an idempotent monad on a category $\mathscr{A}$ and a be an arbitrary object of $\mathscr{A}$. Then*

$$\mathsf{Desc}^1(\mathbb{T}, a) = \begin{cases} \{[(a, e_a^{-1})]\}, & \text{if} \quad e_a \text{ is an isomorphism;} \\ \emptyset, & \qquad\text{otherwise.} \end{cases}$$

Moreover, if e_a is an isomorphism (and hence $(a, e_a^{-1}) \in \mathscr{A}_{\mathbb{T}}$), one has

$$\mathsf{Desc}^0(\mathbb{T}, (a, e_a^{-1})) = \mathsf{Aut}_{\mathscr{A}}(a).$$

For the next two examples concerning monads on Set, we recall that a *variety of (finitary, one-sorted) algebras* is a class of algebras determined by finitary operations satisfying suitable identities (with morphisms preserving these operations). It is well known that every such a variety $\mathscr{V}$ is equivalent to $\mathsf{Set}_{\mathbb{T}}$ for some finitary (= filtered colimit preserving) monad $\mathbb{T}$ on Set. Therefore, any finitary variety of algebras $\mathscr{V}$ gives rise to a finitary monad $\mathbb{T}_{\mathscr{V}}$ on Set whose category of algebras is equivalent to the category defined by $\mathscr{V}$.

7.2. Example. Left braces. A *left brace* is an abelian group $(A, +, 0)$ together with a multiplication $\cdot : A \times A \to A$ such that the following identities hold:

- $a \cdot (b + c) = a \cdot b + a \cdot c$;
- $(a \cdot b + a + b) \cdot c = a \cdot (b \cdot c) + a \cdot c + b \cdot c$;
- the map $x \mapsto a \cdot x + x$ is bijective for each $a \in A$.

Left braces and their homomorphisms (additive group homomorphisms that respect multiplication) form a variety denoted LBr.

Proposition. *Let X be a set with cardinality $n = p^2 q$, where $2 < p < q$ are primes. Then*

$$|\mathsf{Desc}^1(\mathbb{T}_{\mathsf{LBr}}, X)| = \begin{cases} 4, & \text{if} \quad p \nmid q-1; \\ p+8, & \text{if} \quad p | q-1, p^2 \nmid q-1; \\ 2p+8, & \text{if} \quad p | q-1. \end{cases}$$

Proof. The result follows from [8, Corollary 3], according to which

$$b(p^2q) = \begin{cases} 4, & \text{if} \quad p \nmid q-1; \\ p+8, & \text{if} \quad p|q-1, p^2 \nmid q-1; \\ 2p+8, & \text{if} \quad p|q-1. \end{cases}$$

for primes $2 < p < q$. Here $b(n)$, n being a positive integer, denotes the number of non-isomorphic left braces of fixed order n. □

7.3. EXAMPLE. Finite Abelian Groups. Let Ab be the variety of abelian groups and $\mathbb{T}_{\mathsf{Ab}}$ be Sthe corresponding finitary monad on Set.

Let N be a positive integer. Recall that a *partition* of N is a non-decreasing sequence $(n_1, n_2, \ldots, n_k)$ of positive integers with $n_1 + n_2 + \cdots + n_k = N$. The *partition function* $\pi(N)$ gives the number of partitions of N.

PROPOSITION. *Let X be a finite set with cardinality $|X| = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$, where $p_1, p_2, \ldots p_k$ are distinct primes. Then*

$$|\mathsf{Desc}^1(\mathbb{T}_{Ab}, X)| = \prod_{i=1}^{k} \pi(n_i).$$

Proof. The result follows from the fact (e.g., [22, p.129]) that there are $\prod_{i=1}^{k} \pi(n_i)$ isomorphism classes of abelian groups of order $\prod_{i=1}^{k} p^{n_i}$. □

For example, if $|X| = 16 = 2^4$, then $|\mathsf{Desc}^1(\mathbb{T}_{\mathsf{Ab}}, X)| = 4$ and

$$\{(\mathbb{Z}_2)^4, \mathbb{Z}_4 \times (\mathbb{Z}_2)^2, \mathbb{Z}_8 \times \mathbb{Z}_2, (\mathbb{Z}_4)^2, \mathbb{Z}_{16}\}$$

is a complete set of the representatives of the isomorphism classes of finite abelian groups of order 16.

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THE WEIGHTED RIGHT FOCAL BOUNDARY VALUE PROBLEM FOR SECOND ORDER SINGULAR IN THE TIME VARIABLE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions are found for the solvability of the boundary value problem

$$u''(t) = f(t, u(\tau(t))),$$

$$\lim_{t\to a} \frac{u(t)}{(t-a)^\alpha} = 0, \quad \lim_{t\to b} u'(t) = 0$$

in the case where the function f has singularities of arbitrary order in the time variable at the point $t = a$ as well as at the points of the interval $]a, b]$.

On a finite open interval $]a, b[$, we consider the differential equation

$$u''(t) = f(t, u(\tau(t))) \tag{1}$$

with the boundary conditions

$$\lim_{t\to a} \frac{u(t)}{(t-a)^\alpha} = 0, \quad \lim_{t\to b} u'(t) = 0, \tag{2}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is a measurable in the first argument and continuous in the second argument function,

$$I \subset]a, b[, \quad \operatorname{mes} I = b - a,$$

and $\alpha \in [0, 1[$.

Suppose

$$f^*(t, r) = \max\{|f(t, x)| : \ |x| \le r\} \ \text{ for } \ t \in I, \ \ r \ge 0.$$

If

$$\int_a^b f^*(t, r)dt < +\infty \quad \text{for } \ r > 0,$$

then problem $(1), (2)$ is said to be regular. If

$$\int_a^b f^*(t, r)dt = +\infty \ \text{ for some } \ r > 0, \tag{3}$$

then this problem is said to be singular in the time variable.

Unimprovable sufficient conditions for the solvability and unique solvability of problem $(1), (2)$ in the case, where $\alpha = 0$, $\tau(t) \equiv t$, and the function f has a singularity of arbitrary order in the time variable at the point $t = a$, are contained in [1, 3–8].

For $\alpha = 0$ and $\tau(t) \not\equiv t$, the singular problem $(1), (2)$ is also studied under the assumption that the function f has a non-integrable singularity in the time variable only at the point $t = a$ (see, [2, 9–12]). Therefore, the papers [2, 9–12] concern only the case where along with (3) the condition

$$\int_t^b f^*(s, r)ds < +\infty \ \text{ for } \ a < t < b, \ \ r \ge 0 \tag{4}$$

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is satisfied.

In contrast to the results of [2,9–12], theorems proven by us on the solvability and unique solvability of problem (1), (2) cover the case where condition (4) is violated , i.e., the case where the function f has a non-integrable singularity in the time variable at the points of the interval $]a,b]$. In particular, it is assumed that there exist points $t_i \in]a,b[$ $(i=1,\dots,n)$ such that for an arbitrarily small $\varepsilon > 0$ and for any $x \neq 0$ and $\lambda > 0$, the conditions

$$\int\limits_{t_i-\varepsilon}^{t_i+\varepsilon} |t-t_i|^\lambda |f(t,x)|dt = +\infty \ \ (i=1,\dots,n), \quad \int\limits_{b-\varepsilon}^{b} (b-t)^\lambda |f(t,x)|dt = +\infty \tag{5}$$

hold.

Introduce the function

$$\chi(t) = \begin{cases} 1 & \text{if } \ t = \tau(t), \\ 0 & \text{if } \ t \neq \tau(t). \end{cases}$$

We investigate the solvability of problem (1), (2) in the case where

$$\int\limits_t^b f^*\big(s, (\tau(s)-a)^\alpha r\big) ds < +\infty \ \text{ for } \ a < t < b, \ \ r \geq 0, \tag{6}$$

and on the set $I \times \mathbb{R}$ the inequality

$$\chi(t) f(t,x)\operatorname{sgn}(x) - (1-\chi(t))|f(t,x)| \geq -g(t)|x| - h(t) \tag{7}$$

is satisfied, where g and $h : I \to [0,+\infty[$ are measurable functions.

When investigating the uniqueness of a solution of problem (1), (2), we assume that the function f on the set $I \times \mathbb{R}$ instead of condition (7) satisfies the one-sided Lipschitz condition

$$\chi(t)\big[f(t,x) - f(t,y)\big]\operatorname{sgn}(x-y) - (1-\chi(t))|f(t,x) - f(t,y)| \geq -g(t)|x-y|. \tag{8}$$

Theorem 1. *If along with* (6) *and* (7) *the conditions*

$$\int\limits_a^b (t-a)^{1-\alpha}(\tau(t)-a)^\alpha g(t)dt < 1 \tag{9}$$

and

$$\int\limits_a^b (t-a)^{1-\alpha} h(t)dt < +\infty \tag{10}$$

hold, then problem (1), (2) *has at least one solution.*

Theorem 2. *If along with* (6) *and* (8) *conditions* (9) *and* (10) *are satisfied, where* $h(t) = |f(t,0)|$, *then problem* (1), (2) *has one and only one solution.*

Remark 1. Inequality (9) in Theorems 1 and 2 cannot be replaced by the inequality

$$\int\limits_a^b (t-a)^{1-\alpha}(\tau(t)-a)^\alpha g(t)dt \leq 1+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be. However, the question of whether it is possible to replace (9) by the nonstrict inequality

$$\int\limits_a^b (t-a)^{1-\alpha}(\tau(t)-a)^\alpha g(t)dt \leq 1$$

remains open.

Example 1. Suppose $\alpha \in]0,1[$, $a < a_0 < b$, m and n are natural numbers, $t_i \in]a,b[\ \ (i = 1,\dots,n)$, $t_{n+1} = b$,

$$\tau(t) = t, \ \ f(t,x) = \exp\Big(\frac{1+|x|}{t-a}\Big)x^{2m-1} + q(t) \ \text{ for } t \in]a,a_0[,\ x \in \mathbb{R},$$

$$\tau(t) = a + (b-a)\exp\bigg(-\sum_{i=1}^{n+1}\frac{1}{|t-t_i|}\bigg), \quad f(t,x) = \ell(t-a)^{\alpha-1}(\tau(t)-a)^{-\alpha}, \quad 0 < \ell(b-a) < 1$$
$$\text{for } \ t \in]a_0,b[\setminus \{t_1,\dots,t_n\}, \ \ x \in \mathbb{R},$$

$q :]a,b[\to \mathbb{R}$ is a measurable function such that

$$\int\limits_a^b (t-a)^{1-\alpha}|q(t)|dt < +\infty. \tag{11}$$

Then by Theorem 2, problem (1), (2) has one and only one solution. On the other hand, in this case the function f satisfies conditions (5) for an arbitrarily small $\varepsilon > 0$ and for any $x \neq 0$ and $\lambda > 0$. It is also evident that the function f has a singularity of arbitrary order in the time variable at the point $t = a$ as well.

The particular case of equation (1) is the linear differential equation

$$u''(t) = p(t)u(\tau(t)) + q(t), \tag{12}$$

where p and $q :]a,b[\to \mathbb{R}$ are measurable functions.

Put

$$p_-(t) = \frac{|p(t)| - p(t)}{2}.$$

From Theorem 2 we have the following statement.

Corollary 1. *If*

$$\int\limits_a^b \chi(t)(t-a)p_-(t)dt + \int\limits_a^b (1-\chi(t))(t-a)^{1-\alpha}(\tau(t)-a)^{\alpha}|p(t)|dt < 1,$$

and the function q satisfies condition (11), *then problem* (12), (2) *has one and only one solution.*

It is easy to see that under the conditions of Corollary 1 the function p may have singularities of arbitrary order at the points of the interval $]a,b]$.

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INTERSECTION AND STRING TOPOLOGY PRODUCTS IN THE FREE LOOP FIBRATION

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Abstract. For an oriented closed triangulated manifold M we derive a cubical cellular structure on M so that $M = |X|$ is the geometric realisation of a cubical set X. We recover the intersection product on the homology $H_*(M)$ by defining the pairing on the cubical chains $C_*(X)$. We construct the permutahedral model $\widehat{\mathbf{\Lambda}}X$ for the free loop space ΛM, and lift the intersection pairing on the permutahedral chains $C_*^{\diamond}(\widehat{\mathbf{\Lambda}}X)$ to recover the string topology product on the free loop homology $H_*(\Lambda M)$ and to establish the compatibility condition with its standard coproduct.

1. Introduction

Let M be an oriented closed triangulated n-manifold, and fix the ground coefficient ring to be a field. A motivation of the paper is to establish relationship between the string topology product of degree $-n$

$$\mu_* : H_*(\Lambda M) \otimes H_*(\Lambda M) \to H_*(\Lambda M)$$

on the homology $H_*(\Lambda M)$ of the free loop space ΛM defined in [1] and the standard coproduct

$$\Delta_* : H_*(\Lambda M) \to H_*(\Lambda M) \otimes H_*(\Lambda M)$$

in fact defined for any topological space Y instead of ΛM. The problem naturally requires first to establish the compatibility condition between the classical intersection product

$$\cap_* : H_*(M) \otimes H_*(M) \to H_*(M)$$

and the coproduct Δ_* on the homology $H_*(M)$. In this way we first define $\cap_*$ as induced by a pairing of degree $-n$

$$\cap_{\#} : C_p(K^{\square}) \otimes C_q(K^{\square}) \to C_{p+q-n}(K^{\square}),$$

where $K^{\square}$ is a cubical subdivision of M canonically derived from a triangulation K of M. Consequently, without using the Poincaré isomorphism $H_i(M) \overset{\approx}{\to} H^{n-i}(M)$, we establish that $\cap_*$ is a map of $H_*(M)$-bicomodules:

Theorem 1. *The following diagrams*

$$\begin{array}{ccc} H_*(M) \otimes H_*(M) & \xleftarrow{\cap_* \otimes 1} & H_*(M) \otimes H_*(M) \otimes H_*(M) \\ {\scriptstyle \Delta_*} \uparrow & & {\scriptstyle (1\otimes T)\circ(\Delta_* \otimes 1)} \uparrow \\ H_*(M) & \xleftarrow{\cap_*} & H_*(M) \otimes H_*(M) \end{array} \tag{1.1}$$

and

$$\begin{array}{ccc} H_*(M) \otimes H_*(M) & \xleftarrow{1 \otimes \cap_*} & H_*(M) \otimes H_*(M) \otimes H_*(M) \\ {\scriptstyle \Delta_*} \uparrow & & \uparrow {\scriptstyle (T\otimes 1)\circ(1 \otimes \Delta_*)} \\ H_*(M) & \xleftarrow{\cap_*} & H_*(M) \otimes H_*(M) \end{array} \tag{1.2}$$

are commutative.

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Furthermore, by replacing simplicial closed necklaces by cubical ones (see Figure 2), we construct a cubical closed necklical set $\widehat{\mathbf{\Lambda}}K^{\square}$ having likewise a canonical permutahedral set structure such that the geometric realization $|\widehat{\mathbf{\Lambda}}K^{\square}|$ is homotopy equivalent to ΛM. The chain pairing $\cap_{\#}$ indices naturally the pairing of permutahedral cellular chains of degree $-n$

$$\mu_{\#}: C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|)\otimes C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|)\to C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|). \tag{1.3}$$

We also define canonical chain maps

$$\nu^l_{\#},\nu^r_{\#}: C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|)\otimes C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|)\to C_*^{\diamond}(|\widehat{\mathbf{\Lambda}}K^{\square}|). \tag{1.4}$$

Then by the isomorphism $H_*(|\widehat{\mathbf{\Lambda}}K^{\square}|)\approx H_*(\Lambda M)$ and a diagonal decomposition of $|\widehat{\mathbf{\Lambda}}K^{\square}|$ by means of an explicit diagonal of permutahedra [5] we immediately get

Theorem 2. *The following diagrams*

$$\begin{array}{ccc} H_*(\Lambda M)\otimes H_*(\Lambda M) & \xleftarrow{\mu_*\otimes\nu^r_*} & H_*(\Lambda M)\otimes H_*(\Lambda M)\otimes H_*(\Lambda M)\otimes H_*(\Lambda M)\\ {\scriptstyle\Delta_*}\uparrow & & {\scriptstyle(1\otimes T\otimes 1)\circ(\Delta_*\otimes\Delta_*)}\uparrow\\ H_*(\Lambda M) & \xleftarrow{\mu_*} & H_*(\Lambda M)\otimes H_*(\Lambda M)\end{array}$$

and

$$\begin{array}{ccc} H_*(\Lambda M)\otimes H_*(\Lambda M) & \xleftarrow{\nu^l_*\otimes\mu_*} & H_*(\Lambda M)\otimes H_*(\Lambda M)\otimes H_*(\Lambda M)\otimes H_*(\Lambda M)\\ {\scriptstyle\Delta_*}\uparrow & & {\scriptstyle(1\otimes T\otimes 1)\circ(\Delta_*\otimes\Delta_*)}\uparrow\\ H_*(\Lambda M) & \xleftarrow{\mu_*} & H_*(\Lambda M)\otimes H_*(\Lambda M)\end{array}$$

are commutative.

Note that the maps ν^r_* and ν^l_* are equivalent to the right and left actions of $H_*(\Omega M)$ on $H_*(\Lambda M)$ whenever the standard map $\iota_*: H_*(\Omega M)\to H_*(\Lambda M)$ is monomorphic. The definition of the map $\cap_{\#}$ uses the pairing (cf. [6])

$$C_p(K)\otimes C_q(K^{\circledast})\to C_{p+q-n}(K'), \tag{1.5}$$

where K is a simplicial subdivision of M, K' is its barycentric subdivision and $K^{\circledast}$ is a block dissection of K' by the barycentric stars $D(\sigma)$ of simplices $\sigma\in K$ involved in the Poincaré isomorphism. The problem of constructing the chain-level intersection pairing gave rise to a number of works. A good reference to the subject is the recent book [2].

The map $\cap_{\#}$ is defined as follows. To each pair $\sigma\supset\tau$ of simplices from K we assign the cubical cell $I(\sigma\supset\tau)$ of dimension $\dim(\sigma)-\dim(\tau)$: Namely, if $\sigma_k\supset\cdots\supset\sigma_1$ denotes a barycentric subdivision simplex formed by subsimplices σ_i of σ_k with $(\sigma_k\supset\cdots\supset\sigma_1)\subset\sigma_k$, then

$$I(\sigma\supset\tau)=\bigcup_{\sigma\supset\sigma_i\supset\tau}\sigma\supset\sigma_r\supset\cdots\supset\sigma_1\supset\tau.$$

Thus $I(\sigma\supset\tau)\subset\sigma$, and the triangulated manifold M with the cubical cellular structure formed by the cubes $I(\sigma\supset\tau)$ for all pair of simplices $\sigma\supset\tau$ is just denoted by $K^{\square}$. Then we define a set map (on the set of the structural cells)

$$\sqcap: K\times K^{\circledast}\longrightarrow K^{\square}$$

for a pair $(\sigma,D(\tau))\in K\times K^{\circledast}$ by

$$\sigma\sqcap D(\tau)=\begin{cases} I(\sigma\supset\tau), & \text{if }\ \sigma\supset\tau,\\ *, & \text{otherwise.}\end{cases}$$

We, obviously, have $\partial(\sigma\sqcap D(\tau))=\partial\sigma\sqcap D(\tau)\bigcup\sigma\sqcap\partial D(\tau)$, and, hence, obtain the induced pairing of degree $-n$

$$\sqcap_{\#}: C_p(K)\otimes C_q(K^{\circledast})\to C_{p+q-n}(K^{\square}).$$

There are the canonical chain maps

$$Sd_{\square}: C_*(K)\to C_*(K^{\square}),\ Sd^{\circledast}_{\square}: C_*(K^{\circledast})\to C_*(K^{\square})\ \text{ and }\ Sd'_{\square}: C_*(K^{\square})\to C_*(K')$$

based on the fact that the chains of a cell is mapped to the chains of a subdivision of the cell (see Figure 1, below).

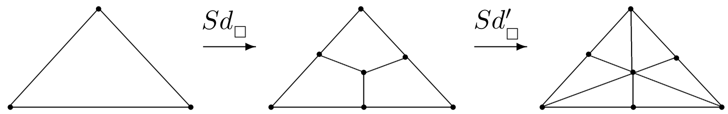

FIGURE 1. The first barycentric cubical and simplicial subdivisions of Δ^2.

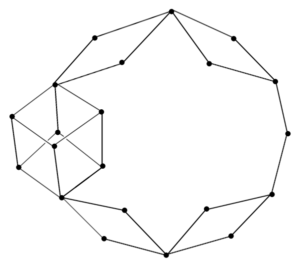

FIGURE 2. A cubical closed necklace $\mathbf{I}^3 \vee I^2 \vee I^2 \vee I^1 \vee I^1 \vee I^2 \vee I^2$ of dimension 7.

These maps have homotopy inverse maps, "cubical displacements"; namely, consider the chain maps

$$\theta_*^{cu} : C_*(K^{\square}) \to C_*(K) \quad \text{and} \quad \theta_*^{st} : C_*(K^{\square}) \to C_*(K^{*}),$$

induced in fact by cellular maps $\theta^{cu} : K^{\square} \to K$ and $\theta^{st} : K^{\square} \to K^{*}$. In particular, the pairing given by (1.6) is then the composition

$$C_p(K) \otimes C_q(K^{*}) \xrightarrow{\sqcap_{\#}} C_{p+q-n}(K^{\square}) \xrightarrow{Sd'_{\square}} C_{p+q-n}(K'), \tag{1.6}$$

while we define $\cap_{\#}$ as the composition

$$\cap_{\#} : C_p(K^{\square}) \otimes C_q(K^{\square}) \xrightarrow{\theta_*^{cu} \times \theta_*^{st}} C_p(K) \otimes C_q(K^{*}) \xrightarrow{\sqcap_{\#}} C_{p+q-n}(K^{\square}).$$

Proof of Theorem 1. The set of structural cells of $K^{\square}$ denote by $\mathcal{I}$, and then θ^{cu}, θ^{st} and $\sqcap$ induce a set map of degree $-n$

$$\cap : \mathcal{I} \times \mathcal{I} \to \mathcal{I}.$$

For the moment denote also by $\Delta_I := \{\Delta_{ij}\}$ the components of the Serre diagonal of a cube. Further, let $\mathcal{I} \curlyvee \mathcal{I} \subset \mathcal{I} \times \mathcal{I}$ be the subset such that for each bouquet $I(\sigma_i \supset \tau_i) \vee I(\sigma_j \supset \tau_j) \in \mathcal{I} \curlyvee \mathcal{I}$ there is the overlap cube $I(\sigma \supset \tau)$ with $\Delta_{ij}(I(\sigma \supset \tau)) = I(\sigma_i \supset \tau_i) \vee I(\sigma_j \supset \tau_j)$.

Then the diagram(s)

$$\begin{array}{ccc} \mathcal{I} \curlyvee \mathcal{I} & \xleftarrow{\cap \times 1} & (\mathcal{I} \curlyvee \mathcal{I}) \times \mathcal{I} \\ {\scriptstyle \Delta_{**}} \uparrow & & \uparrow {\scriptstyle (1 \curlyvee T) \circ (\Delta_{**} \times 1)} \\ \mathcal{I} & \xleftarrow{\cap} & \mathcal{I} \times \mathcal{I} \end{array} \tag{1.7}$$

commutes in a sense that each term of the diagonal fixed in $\mathcal{I} \curlyvee \mathcal{I}$ by one side equals to some term obtained by the other side of the diagram. Consequently, it induces the desired commutativity of (1.1). The commutativity of (1.2) follows from the diagram, similar to (1.7). $\square$

The proof of Theorem (2) requires the construction of the permutahedral model $\widehat{\mathbf{\Lambda}} K^{\square}$ for ΛM. In fact, beside $\widehat{\mathbf{\Lambda}} K^{\square}$ we also construct the cubical necklical set $\widehat{\mathbf{\Omega}} K^{\square}$ modeling the based loops ΩM, and, in general, for a cubical set X we have a quasi-fibration $|\widehat{\mathbf{\Omega}} X| \xrightarrow{\iota} |\widehat{\mathbf{\Lambda}} X| \xrightarrow{\zeta} Y$ modeling the free loop fibration $\Omega Y \to \Lambda Y \to Y$, this time Y is the geometric realisation of the cubical set X, while $\widehat{\mathbf{\Omega}} X$ and $\widehat{\mathbf{\Lambda}} X$ are cubical necklical and closed necklical sets, respectively, hence, both are permutahedral sets.

Evidently, this model uses more complicated polytopes rather than the ones in [4], however, since both ΩY and ΛY are modelling by the same type of sets, it has more symmetries.

We note that certain ideas of interactions of different type polytopes come from the works of N. Berikashvili in the obstruction theory to the section problem of a fibration (see also [3]).

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THE ADHESIVE CONTACT PROBLEMS IN THE PLANE THEORY OF ELASTICITY

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Abstract. The problems of constructing an exact and approximate solutions of system of singular integro-differential equations related to the problems of adhesive interaction between elastic thin finite or infinite nonhomogeneous patch and elastic plate are investigated. For the patch loaded by horizontal and vertical forces the usual model of beam bending in combination with the uniaxial stress state model is valid. Using the methods of theory of analytic functions, integral transformation or orthogonal polynomials the singular integro-differential equations reduced to the different boundary value problems (Karleman type problem with displacements, Riemann problem) of the theory of analytic functions or to the infinite system of linear algebraic equations. The asymptotic analysis of problem is carried out.

Let a finite non-homogeneous patch with modulus of elasticity $E_1(x)$, thickness $h_1(x)$ and Poisson's coefficient ν_1 be attached to the plate (E_2, v_2), which is in the condition of a plane deformation. It is assumed that the horizontal and vertical stresses with intensity $\tau_0(x)$ and $p_0(x)$ acts on the patch along the ox-axis. The patch in the vertical direction bends like a beam (has a finite bending stiffness, model A) or along the horizontal axis the vertical elastic displacements of its points are constant (model B) and besides in the horizontal direction the patch compressed or stretched like rod being in uniaxil stress state. The contact between the plate and patch is realized by a thin glue layer with width h_0 and Lame's constants λ_0, μ_0. The contact condition has the form [1]

$$u_1(x) - u_2(x,0) = k_0\tau(x), \quad v_1 s(x) - v_2(x,0) = m_0 p(x), \quad |x| < 1 \tag{1}$$

where $u_2(x,y)$, $v_2(x,y)$ are displacements of the plate points along the ox-axis, $u_1(x)$, $v_1(x)$ displacements of the patch points along the ox-axis. $k_0 := h_0/\mu_0$, $m_0 : h_0/(\lambda_0 + 2\mu_0)$.

We have to define the law of distribution of tangential and normal contact stresses $\tau(x)$ and $p(x)$ on the line of contact, the asymptotic behavior of these stresses at the end of the patches and the coefficient of stress intensity.

According to the equilibrium equation of patch elements and Hooke's law we have: in model A

$$\begin{gathered}\frac{du_1(x)}{dx} = \frac{1}{E(x)} \int\limits_{-1}^{x} [\tau(t) - \tau_0(t)]dt, \\ \frac{d^2}{dx^2} D(x) \frac{d^2 v_1(x)}{dx^2} = p_0(x) - p(x), \quad |x| < 1\end{gathered} \tag{2}$$

the equilibrium equation of the patch has the form

$$\int\limits_{-1}^{1} [\tau(t) - \tau_0(t)]dt = 0, \quad \int\limits_{-1}^{1} [p(t) - p_0(t)]dt = 0, \quad \int\limits_{-1}^{1} t[p(t) - p_0(t)]dt = 0, \tag{3}$$

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and in model B

$$\frac{du_1(x)}{dx} = \frac{1}{E(x)} \int_{-1}^{x} [\tau(t) - \tau_0(t)]dt, \qquad \frac{dv_1(x)}{dx} = 0, \quad |x| < 1 \tag{2'}$$

where $E(x) = \dfrac{E_1(x)h_1(x)}{1-\nu_1^2}$, $D(x) = \dfrac{E_1(x)h_1^3(x)}{1-\nu_1^2}$. According to known results [2], the horizontal and vertical deformations of the points of the ox axis have the form

$$\frac{du_2(x,0)}{dx} = -ap(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{\tau(t)dt}{t-x}, \quad \frac{dv_2(x,0)}{dx} = -\frac{b}{\pi} \int_{-1}^{1} \frac{p(t)dt}{t-x} - a\tau(x) \tag{4}$$

$$a = \frac{(1+\nu_2)(1-2\nu_2)}{E_2}, \quad b = \frac{2(1-\nu_2^2)}{E_2}.$$

Introducing the notations

$$\varphi(x) = \int_{-1}^{x} [\tau(t) - \tau_0(t)]dt, \quad \psi(x) = \int_{-1}^{x} dt \int_{-1}^{t} [p_0(\tau) - p(\tau)]d\tau$$

from (1), (2) and (4) we obtain the following system of integro-differential equations

$$\frac{\varphi(x)}{E(x)} - \frac{b}{\pi} \int_{-1}^{1} \frac{\varphi'(t)dt}{t-x} - k_0\varphi''(x) - a\psi''(x) = f_1(x), \tag{5}$$

$$\frac{\psi(x)}{D(x)} - \frac{b}{\pi}\frac{d}{dx} \int_{-1}^{1} \frac{\psi''(t)dt}{t-x} + a\varphi''(x) + m_0\psi^{(\mathrm{IV})}(x) = f_2(x), \quad |x| < 1 \tag{6}$$

where

$$f_1(x) = -ap_0(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{\tau_0(t)dt}{t-x} + k_0\tau_0'(x),$$

$$f_2(x) = -a\tau_0'(x) - \frac{b}{\pi}\frac{d}{dx} \int_{-1}^{1} \frac{p_0(t)dt}{t-x} + m_0p_0''(x),$$

or from (1), (2′) and (4) we have

$$\frac{\varphi(x)}{E(x)} - \frac{b}{\pi} \int_{-1}^{1} \frac{\varphi'(t)dt}{t-x} - k_0\varphi''(x) - a\psi''(x) = f_1(x), \tag{5'}$$

$$m_0\psi^{m}(x) - \frac{b}{\pi} \int_{-1}^{1} \frac{\psi''(t)dt}{t-x} + a\varphi'(x) = f_3(x), \quad |x| < 1 \tag{6'}$$

where

$$f_3(x) = -a\tau_0(x) + m_0p_0'(x)$$

and from (3) we have the conditions

$$\varphi(1) = 0, \quad \psi(1) = 0, \quad \psi'(1) = 0. \tag{7}$$

Thus, the above posed boundary contact problem reduced to the system of singular integro-differential equation (5), (6) or ((5′), (6′)) with the condition (7). From the symmetry of the problem, we assume, that $E(x)$, $D(x)$ and $p_0(x)$ are even functions and $\tau_0(x)$ is uneven function, the solutions of equation (5), (6), ((5′), (6′)) under the condition (7) can be sought in the class of even functions, besides

$$\varphi, \varphi' \in H[-1,1], \quad \varphi'' \in H^*(-1,1), \quad \psi, \psi', \psi'' \in H[-1,1], \quad \psi''', \psi^{(\mathrm{IV})} \in H^*(-1,1).$$

We assume that the function $\tau_0(x)$ and $p_0(x)$ is continuous in the Holder's sense, $\tau_0(x)$ has a continuous first order derivative, $p_0(x)$ has a continuous first and second order derivatives on the interval [-1,1]. Under the assumption that

$$E(x) = (1-x^2)^{\gamma} b_0(x), \quad \gamma \geq 0, \quad b_0(x) = b_0(-x), \quad b_0 \in C([-1,1]) \tag{8}$$
$$b_0(x) \geq c_0 = \mathrm{const} > 0$$
$$D(x) = (1-x^2)^{\delta} b_1(x), \quad \delta > 0, \quad b_1(x) = b_1(-x), \quad b_1 \in C([-1,1]), \tag{9}$$
$$b_1(x) \geq c_1 = \mathrm{const} > 0$$

a solutions of problem (5)–(7) will be sought in the class of even function whose derivatives are representable in the form

$$\begin{aligned} \varphi'(x) &= (1-x^2)^{a} g_1(x), \quad \alpha > -1 \\ \psi''(x) &= (1-x^2)^{\beta} g_2(x), \quad \beta > -1 \end{aligned} \tag{10}$$

where

$$g_1(x) = -g_1(-x), \quad g_1 \in C'([-1,1]), \quad g_1(x) \neq 0, \quad x \in [-1,1]$$
$$g_2(x) = g_2(-x), \quad g_2 \in C'([-1,1]), \quad g_2(x) \neq 0, \quad x \in [-1,1].$$

There is valid the following

Theorem. *In condition* (8)*,* (9)*, if the problems* (5)–(7) ((5′)–(7)) *has the solutions in the form* (10)*, then:*

If $\gamma > 2$ and $\delta > 4$, then $\alpha = \gamma - 1$ and $\beta = \delta - 2$,

If $\gamma \leq 2$ and $\delta \leq 4$, then $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 2$.

Remark 1. Numbers and cannot be negative, which corresponds to the physical meaning of the condition (1).

Remark 2. If the inclusion rigidity varies by the law

$$E(x) = (1-x^2)^{n+\frac{1}{2}} b_0(x), \tag{11}$$

where $n \geq 0$ is integer, then from the above asymptotic analysis we obtain:

$$\alpha = n - \frac{1}{2}, \quad \text{for} \quad n = 2, 3, \ldots$$

and $0 < \alpha < 1$, for $n = 0$, $n = 1$ (the same result is obtained for $E(x) = b_0(x) > 0$).

Remark 3. If the inclusion bending rigidity varies by the law

$$D(x) = (1-x^2)^{m+\frac{1}{2}} b_1(x), \tag{12}$$

where $m \geq 0$ is integer, then from the above asymptotic analysis we obtain:

$$\beta = m - \frac{3}{2}, \quad \text{for} \quad m = 4, 5, \ldots$$

and $0 < \beta < 2$, for $m = 0, 1, 2, 3$ (the same result is obtained for $D(x = b_1(x) > 0)$).

Based on the above asymptotic analysis (see. (11), (12)) in cases

$$n = 0; 1, \quad E(x) = b_0(x) > 0,$$
$$m = 0; 1; 2; 3, \quad D(x) = b_1(x) > 0, \quad |x| \leq 1$$

the solution of system of equations (5), (6) we will be sought in the form

$$\varphi'(x)=\sqrt{1-x^2}\sum_{k=1}^{\infty}X_kP_k^{(1/2,1/2\ )}(x),$$

$$\psi''(x)=(1-x^2)^{3/2}\sum_{k=1}^{\infty}\frac{Y_k}{k+1}P_k^{(3/2,3/2)}(x)$$

and for the case $n=2$, $m=4$, the solution of this system will be present in follows

$$\varphi'(x)=(1-x^2)^{3/2}\sum_{k=1}^{\infty}X_kP_k^{(3/2,3/2)}(x),\quad \psi'(x)=(1-x^2)^{5/2}\sum_{k=1}^{\infty}\frac{Y_k}{k+1}P_k^{(3/2,3/2)}(x),$$

where the numbers X_k, Y_k are subject to determination. $P_k^{(\alpha,\beta)}(z)$ are Jacob's orthogonal polynomials, $(k=1,2,\dots)$ [3].

For determination of the numbers X_k, Y_k the pair of infinite systems of linear algebraic equations is obtained and quasi-completely regularity of this system is proved in the class of bounded sequences for any positive values of the parameters k_0, m_0, a, b [4].

In some specific cases the system of equations (5), (6) ((5′), (6′)) is splitting and the integro-differential equations of the following types are obtained:

$$\frac{\varphi(x)}{E_0}+b\int_0^{\infty}\frac{\varphi'(t)dt}{t-x}-k_0\varphi''(x)=0,\quad x>0 \tag{13}$$

$$\varphi(0)=T,\quad \varphi(\infty)=0$$

or

$$b\int_0^{\infty}\frac{p(t)dt}{t-x}=m_0p'(x),\quad x>0, \tag{14}$$

$$\int_0^{\infty}p(t)dt=P_0.$$

The solution of equation (13) or (14) is sought in the class of functions φ, $\varphi'\in H([0,\infty))$, $\varphi''\in H((0,\infty))$, $p\in H([0,\infty)))$, $p'\in H((0,\infty))$, $p(x)=O(x^{-(2+\omega)})$, $x\to\infty$, $\omega>0$.

$E(x)=E_0=\text{const}$, T and P_0 are known constants.

Based to the different boundary value problems of the theory of analytic functions the equations (13), (14) are solved effectively and asymptotic estimates are obtained [5].

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ON CANTOR'S Λ FUNCTIONALS AND THE RECONSTRUCTION OF COEFFICIENTS OF MULTIPLE FUNCTION SERIES

SH. TETUNASHVILI[1,2] AND T. TETUNASHVILI[2,3]

Abstract. In the present paper, the Λ summable single and multiple function series are considered. The notion of a sequence of Cantor's Λ functionals, which represents formulas of reconstruction of coefficients of a single function series and is also a generalization of Fourier formulas for calculation of coefficients of an orthonormal function series is introduced.

A theorem representing a possibility of reconstruction of coefficients of a multiple function series via iterated application of Cantor's Λ functionals is formulated.

Introduction

Let Λ be an infinite matrix, $\Phi = (\varphi_i(t))_{i=0}^{\infty}$ be a system of finite and measurable functions defined on $[0,1]$ and

$$\sum_{i=0}^{\infty} a_i \varphi_i(t) \tag{1}$$

be a series with respect to Φ which is Λ summable to a function $f(t)$.

In the present paper, the problem of reconstruction of coefficients of series (1) in terms of $f(t)$ is considered.

A necessary condition of solvability of the above-mentioned problem is the validity of the following uniqueness theorem:

If a series (1) is summable to zero on $[0,1]$, then every coefficient of this series is equal to zero.

In a particular case, when Λ summability coincides with the convergence of the series and Φ is a trigonometric system, the following Cantor's fundamental theorem [1] concerning the convergence of trigonometric series holds.

Theorem A (Cantor)**.** *If a trigonometric series converges everywhere to zero, then every coefficient of this series is equal to zero.*

The following Vallée-Poussin's theorem [2] generalizing Theorem A holds, as well.

Theorem B (Vallée-Poussin)**.** *If a trigonometric series converges to a finite integrable function f everywhere, except possibly at countably many points, then it is the Fourier series of f.*

(The validity of Theorem A and Theorem B for multiple trigonometric series in Pringsheim's convergence sense is proved in [3]).

It is known that there exists a trigonometric series such that the sum of this series is a Lebesgue nonintegrable function. The following series

$$\sum_{k=2}^{\infty} \frac{\sin 2\pi k t}{\ln k}$$

is an example of the above-mentioned one. So, Vallée-Poussin's theorem and Fourier-Lebesgue's formulas cannot be applied to reconstruct the coefficients of such a series.

On the other hand, the uniqueness of coefficients of an everywhere convergent trigonometric series follows from Cantor's uniqueness theorem.

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The above-mentioned circumstance causes the necessity of a generalization of the Lebesgue integral such that any everywhere convergent trigonometric series is the Fourier series of its sum in the generalized integral sense. This problem has been solved by Denjoy.

The Marcinkiewicz–Zygmund integral and the James integral are also examples of such a generalization of the Lebesgue integral (see [6]).

The presented work consists of two sections.

In Section 1, single series with respect to system Φ which are summable by some Λ-method are considered. For such series, the notion of the sequence of Cantor's Λ functionals is introduced. The notion of a sequence of Cantor's functionals for convergent function series was introduced in [4] and [5].

It should be noted that Fourier formulas for coefficients of a single trigonometric series in every above mentioned generalized integral sense represents a particular example of the notion of the sequence of Cantor's Λ functionals.

In Section 2, the theorem asserting the possibility of reconstruction of coefficients of multiple function series by iterated application of Cantor's Λ functionals is presented.

1. Single Series. Cantor's Λ Functionals

Let $\Phi = (\varphi_i(t))_{i=0}^{\infty}$ be a system of finite and Lebesgue measurable functions defined on $[0,1]$. Also, $\Lambda = \|\lambda_{p,i}\|$ be a matrix of numbers, where $p = 0,1,2,\dots$ and $i = 0,1,2,\dots$.

With respect to Φ, we consider a series

$$\sum_{i=0}^{\infty} a_i \varphi_i(t).$$

Denote by $\sigma_p(t)$ the p-th Λ mean of this series, so,

$$\sigma_p(t) = \sum_{i=0}^{\infty} \lambda_{p,i} a_i \varphi_i(t),$$

which means that for every integer $p \geq 0$, the series

$$\sum_{i=0}^{\infty} \lambda_{p,i} a_i \varphi_i(t)$$

converges for any $t \in [0,1]$.

Definition 1. *We say that a set $A \subset [0,1]$ belongs to a class $U(\Phi,\Lambda)$ if $\lim\limits_{p\to\infty} \sigma_p(t) = 0$, when $t \in A$ implies that $a_i = 0$ for every integer $i \geq 0$.*

Definition 2. *We say that a finite function $f(t)$ belongs to a class $J(A,\Phi,\Lambda)$ if $A \in U(\Phi,\Lambda)$ and there exists a series* (1) *such that*

$$\lim_{p\to\infty} \sigma_p(t) = f(t), \quad \text{when} \quad t \in A. \tag{2}$$

Let us note that if $f \in J(A,\Phi,\Lambda)$, then $A \in U(\Phi,\Lambda)$ implies the uniqueness of coefficients of any series (1) which satisfies (2).

So, for every integer $i \geq 0$, there exists a functional $G_i^{A,\Phi,\Lambda}(\cdot)$ defined on $J(A,\Phi,\Lambda)$ such that

$$a_i = G_i^{A,\Phi,\Lambda}(f).$$

Definition 3. *Let $i \geq 0$ be any fixed and integer number and $G_i^{A,\Phi,\Lambda}(\cdot)$ be a functional defined on $J(A,\Phi,\Lambda)$. We say that $G_i^{A,\Phi,\Lambda}(\cdot)$ is a Cantor's Λ functional if for any series* (1) *satisfying* (2) *the following equality*

$$a_i = G_i^{A,\Phi,\Lambda}(f)$$

holds; we say that the sequence of functionals $\left(G_i^{A,\Phi,\Lambda}(\cdot)\right)_{i=0}^{\infty}$ is a sequence of Cantor's Λ functionals.

The following propositions hold.

Proposition 1. *If $A \in U(\Phi,\Lambda)$, then $J(A,\Phi,\Lambda)$ is a linear space.*

Proposition 2. *If* $\left(G_i^{A,\Phi,\Lambda}(\cdot)\right)_{i=0}^{\infty}$ *is a sequence of Canor's* Λ *functionals defined on* $J(A,\Phi,\Lambda)$*, then* $G_i^{A,\Phi,\Lambda}(\cdot)$ *is a linear functional for every integer* $i \geq 0$.

Thus the sequence of Cantor's Λ functionals represents the general form of formulas to calculate the coefficients of Λ summable any function series in the sense that any such type formula is a particular case of a Cantor's Λ functional.

2. Multiple Series. Coefficients Reconstruction

Let us formulate notation and definitions we use below.

Let $d \geq 2$ be a positive integer. By R^d we denote a d-dimensional Euclidean space and by Z_0^d the set of all points with nonnegative integer coordinates in R^d. The symbols $m = (m_1, \dots, m_d)$ and $n = (n_1, \dots, n_d)$ will stand for the points of Z_0^d. We write $m \to \infty$ if $m_j \to \infty$ for every integer j satisfying $1 \leq j \leq d$ independently of one another. Also, we write $m \geq 0$ if $m_j \geq 0$ $(1 \leq j \leq d)$. By $E_1 \times \cdots \times E_d$ we denote the Cartesian product of sets $E_j \subset [0,1]$, where $1 \leq j \leq d$. We use $x = (x_1, \dots, x_d)$ to denote the points of a unit cube $[0,1]^d$. The linear Lebesgue measure is denoted by μ.

Definition 4. *Let* $\delta \in (0,1]$. *We call a system* $\Phi = \{\varphi_i(t)\}_{i=0}^{\infty}$ *a* δ *linear independence system if any finite part of this system is lineary independent on any set* $A \subset [0,1]$*, where* $\mu A > 1 - \delta$.

If $\delta = 1$*, then* Φ *is called an essentially lineary independence system.*

Let us note that the trigonometric system

$$T = \{1, \cos 2\pi i t, \sin 2\pi i t\}_{i=1}^{\infty}, \quad t \in [0,1]$$

is an essentially lineary independence system.

Let

$$\Phi^{(j)} = \left\{\varphi_{n_j}^{(j)}(x_j)\right\}_{n_j=0}^{\infty}, \quad x_j \in [0,1]$$

for every integer j satisfying inequalities $1 \leq j \leq d$ be a system of measurable and finite functions defined on $[0,1]$.

Let $\overline{\Phi} = \{\Phi_n(x)\}$ be a d-fold system such that

$$\Phi_n(x) = \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j),$$

where $n \in Z_0^d$, $x \in [0,1]^d$ and $x_j \in [0,1]$ and let

$$\Lambda^{(j)} = \left\|\lambda_{m_j,n_j}^{(j)}\right\|, \quad \text{where} \quad m_j = 0,1,2,\dots \ \text{ and } \ n_j = 0,1,2,\dots$$

be a matrix for every integer j satisfying $1 \leq j \leq d$.

Let $\overline{\Lambda} = \left\|\lambda_{m,n}\right\|$ be a d-fold matrix, where $m \in Z_0^d$, $n \in Z_0^d$ and

$$\lambda_{m,n} = \prod_{j=1}^{d} \lambda_{m_j,n_j}^{(j)}.$$

Everywhere below, we mean that for every integer j satisfying $1 \leq j \leq d-1$ a system $\Phi^{(j)} = \left\{\varphi_{n_j}^{(j)}(x_j)\right\}_{n_j=0}^{\infty}$ is a δ_j lineary independence system, where $\delta_j \in (0,1]$ and $\Lambda^{(j)} = \left\|\lambda_{m_j,n_j}\right\|$ is a matrix with finite rows, that is,

$$\lambda_{m_j,n_j} = 0 \ \text{ if } \ n_j \geq \gamma_j(m_j).$$

Consider the d-multiple series with respect to $\overline{\Phi}$:

$$\sum_{n=0}^{\infty} a_n \Phi_n(x) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_1}, \dots, n_d \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j). \tag{3}$$

Denote by $\sigma_m(x)$ the m-th Λ mean of series (3), so

$$\sigma_m(x)=\sum_{n_1=0}^{\gamma_1(m_1)}\cdots\sum_{n_{d-1}=0}^{\gamma_{d-1}(m_{d-1})}\sum_{n_d=0}^{\infty}\prod_{j=1}^{d}\lambda_{m_j,n_j}^{(j)}a_{n_1,\dots,n_d}\prod_{j=1}^{d}\varphi_{n_j}^{(j)}(x_j).$$

Let $E_j\in U\left(\Phi^{(j)},\Lambda^{(j)}\right)$ for every integer j satisfying $1\le j\le d$ and

$$\lim_{m\to\infty}\sigma_m(x)=f(x),\quad if\quad x\in E_1\times E_2\times\cdots\times E_d.$$

Let $n\in Z_0^d$ and j be fixed and $G_{n_j}^{(j)}(f(t))$ stands for $G_{n_j}^{E_j,\Phi^{(j)},\Lambda^{(j)}}(f(t))$, where $t\in E_j$.

Consider

$$G_{n_j}^{(j)}(F(x_j,x_{j+1},\dots,x_d)). \tag{4}$$

In such a case we mean that only $x_j\in E_j$ is a variable and a point $(x_{j+1},\dots,x_d)\in E_{j+1}\times\cdots\times E_d$ is a fixed one, and

$$F(x_j,x_{j+1},\dots,x_d)\in J(E_j,\Phi^{(j)},\Lambda^{(j)}).$$

Thus, (4) depends on $(x_{j+1},\dots,x_d)$ and does not depend on x_j.

The following theorem holds.

Theorem 1. *Let $E_j\in U\left(\Phi^{(j)},\Lambda^{(j)}\right)$ for every integer j satisfying $1\le j\le d$ and $\Phi^{(j)}$ be a δ_j lineary independence system for every integer j satisfying $1\le j\le d-1$, where $\mu E_j>1-\delta_j$ and $\Lambda^{(j)}$ is the matrix with finite rows.*

If $f(x)$ is a finite function defined on $E_1\times\cdots\times E_{d-1}\times E_d$ and $\lim_{m\to\infty}\sigma_m(x)=f(x)$ when $x\in E_1\times\cdots\times E_{d-1}\times E_d$, then

$$a_{n_1,\dots,n_d}=G_{n_d}^{(d)}\left(\cdots\left(G_{n_1}^{(1)}(f(x_1,\dots,x_d))\right)\right),$$

for every $(n_1,\dots,n_d)\in Z_0^d$.

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ON CRITERIA OF CONVERGENCE IN MEASURE OF A SEQUENCE OF FUNCTIONS

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Abstract. It is well known that the Lebesgue and F. Riesz theorems show an interrelation between the convergence in measure and the convergence almost everywhere of a sequence of functions; the first one is a sufficient and the second one is a necessary condition of convergence in measure of a sequence of functions.

In the present paper we formulate a theorem representing a necessary and sufficient condition of convergence in measure of a sequence of functions.

Let $(f_n(x))_{n=1}^{\infty}$ be a sequence of functions defined on a measurable set $E \subset [0,1]$ and μ be the Lebesgue linear measure.

Definition 1. *A sequence of functions $(f_n(x))_{n=1}^{\infty}$ is called convergent in measure to a function $f(x)$, if*

$$\lim_{n\to\infty} \mu\{x \in E : \ |f_n(x) - f(x)| \geq \delta\} = 0$$

for any $\delta > 0$.

The symbol $f_n(x) \xrightarrow{\mu} f(x)$ denotes the convergence in measure of a sequence $(f_n(x))_{n=1}^{\infty}$ to a function $f(x)$.

Lebesgue (see [1, p. 92]) and F. Riesz (see [1, p. 96]) established theorems representing relations between the convergence in measure and the convergence almost everywhere of a sequence of functions.

Namely, the following theorem holds.

Theorem A (Lebesgue)**.** *If $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for a. e. $x \in E$, then $f_n(x) \xrightarrow{\mu} f(x)$.*

It is known that there exists a sequence of functions which is convergent in measure to zero and there exists no point at which this series converges to zero.

However, it should be noted that the following theorem holds.

Theorem B (F. Riesz)**.** *If $f_n(x) \xrightarrow{\mu} f(x)$, then there exists a sequence of natural numbers $n_k \uparrow \infty$ such that*

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \quad \text{for a. e.} \quad x \in E.$$

So, Theorem A is a sufficient condition of the convergence in measure of a sequence of functions and Theorem B is a necessary condition of the convergence in measure of one.

It holds the following theorem which is a necessary and sufficient condition of the convergence in measure of a sequence of functions.

Theorem 1. *$f_n(x) \xrightarrow{\mu} f(x)$ if and only if there exists a sequence of natural numbers $n_k \uparrow \infty$ such that if $(m_k)_{k=1}^{\infty}$ is any sequence of natural numbers such that $n_k \leq m_k < n_{k+1}$, then*

$$\lim_{k\to\infty} f_{m_k}(x) = f(x) \quad \text{a. e.} \quad x \in E.$$

The following Proposition also holds.

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Proposition 1. *Let $(f_n(x))_{n=1}^{\infty}$ be any sequence of measurable functions and $f(x)$ be any measurable function, then the following two conditions are equivalent to each other:*

(i) *there exists a sequence of natural numbers $n_k \uparrow \infty$ such that if $(m_k)_{k=1}^{\infty}$ is any sequence of natural numbers such that $n_k \le m_k < n_{k+1}$, then*

$$\lim_{k\to\infty} f_{m_k}(x) = f(x) \quad a.\ e. \quad x \in E.$$

(ii) *If $(p_k)_{k=1}^{\infty}$ is any sequence of natural numbers, then there exists a sequence of natural numbers $n_k \uparrow \infty$ such that $n_{k+1} - n_k \ge p_k$, and for any sequence of sets $(B_k)_{k=1}^{\infty}$ such that*

$$B_k \subset \{n : \ n_k \le n < n_{k+1}\}, \quad \operatorname{Card} B_k = p_k \ \text{ and } \ B = \bigcup_{k=1}^{\infty} B_k,$$

the following equality

$$\lim_{\substack{n\to\infty \\ n\in B}} f_n(x) = f(x), \quad a.\ e. \quad x \in E$$

holds.

The conjunction of Theorem 1 and Proposition 1 implies the following

Theorem 2. *Let $(f_n(x))_{n=1}^{\infty}$ be a sequence of measurable functions, then the following three conditions are equivalent to each other:*

α) $f_n(x) \xrightarrow{\mu} f(x)$;

β) *there exists a sequence of natural numbers $n_k \uparrow \infty$ such that if $(m_k)_{k=1}^{\infty}$ is any sequence of natural numbers such that $n_k \le m_k < n_{k+1}$, then*

$$\lim_{k\to\infty} f_{m_k}(x) = f(x) \quad a.\ e. \quad x \in E.$$

γ) *If $(p_k)_{k=1}^{\infty}$ is any sequence of natural numbers, then there exists a sequence of natural numbers $n_k \uparrow \infty$ such that $n_{k+1} - n_k \ge p_k$, and for any sequence of sets $(B_k)_{k=1}^{\infty}$ such that*

$$B_k \subset \{n : \ n_k \le n < n_{k+1}\}, \quad \operatorname{Card} B_k = p_k \ \text{ and } \ B = \bigcup_{k=1}^{\infty} B_k$$

the following equality

$$\lim_{\substack{n\to\infty \\ n\in B}} f_n(x) = f(x), \quad a.\ e. \quad x \in E$$

holds.

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