ISSN 2346-8092

ა. რაზმაძის მათემატიკის ინსტიტუტის შრომები

ივანე ჯავახიშვილის სახელობის თბილისის სახელმწიფო უნივერსიტეტი

ტომი 173, N3, 2019

Transactions of A. Razmadze Mathematical Institute is a continuation of Travaux de L' Institut Mathematique de Tbilisi, Vol. 1–15 (1937–1947), Trudy Tbilisskogo Matematicheskogo Instituta, Vol. 16–99 (1948–1989), Proceedings of A. Razmadze Mathematical Institute, Vol. 100–169 (1990–2015), Transactions of A. Razmadze Mathematical Institute (published by Elsevier), Vol. 170–172 (2016–2018).

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Short Communications

Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 1–6

THE ASYMPTOTIC BEHAVIOR OF PRECISE LOWER ESTIMATE OF RECONSTRUCTION OF A LINEAR ORDER ON A FINITE SET

SHALVA BERIASHVILI

Abstract. In the present paper we consider reconstructions of a linear order on a finite set and give the extremal lower estimate of those reconstructions. The asymptotic behavior of such estimate is studied.

The study of discrete point-systems is one of the most important directions in modern mathematics and the methodology of studies of such point-systems is quite diverse. In particular, it uses the methods and principles of combinatorial set theory, mathematical analysis, algebra, graph theory, etc.

The present article is devoted to a concrete topic of discrete mathematics and describes some extremal cases connected with finite linearly ordered point sets. Discrete linearly ordered point-systems can be met in various fields of pure and applied mathematics. One can indicate several such directions in contemporary mathematics, for instance, discrete and computational geometry, classical number theory, combinatorics (finite or infinite), the theory of convex sets, discrete optimization, etc. The investigation of the combinatorial structure of various discrete and finite point-systems in Euclidean spaces is a rather attractive and important topic.

Properties of various discrete point systems are considered in many works (see, t.g., [2–11].)

Throughout this article, we use the following standard notation:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{R}^m$ is the m-dimensional Euclidean space, where $m \geq 1$;

$(X, \leq)$ is a linearly ordered set with $\mathrm{card}(X) = n$, where n is a natural number.

For our further purpose, we need to formulate one important result, which is a particular case of the so-called Master's Theorem. The mentioned universal theorem plays an important role in investigation of various combinatorial problems and questions. Let us formulate a weak version of the Master's Theorem.

Lemma 1. *Let $f : \mathbf{N} \to \mathbf{N}$ be an increasing (in general, not strongly increasing) function such that the inequality*

$$f(2n) \leq 2f(n) + an + b$$

holds true for two fixed real numbers $a \geq 0$ and $b \geq 0$ and for all $n \in \mathbf{N}$. Then there is an upper estimate of f in the form

$$f(n) = O(\log_2(n)).$$

In other words, there exists a real constant $d > 0$ such that

$$f(n) \leq d \cdot \log_2(n)$$

for all natural numbers $n > 1$.

The proof of Lemma 1 can be found in many works, textbooks and monographs (see, e.g., [2, 5, 7]).

Suppose that a nonempty finite linearly ordered set $(X, \preceq)$ is given with

$$\mathrm{card}(X) = n.$$

Take any two-element subset $\{x, y\}$ of X, where $x \neq y$, and compare x and y with respect to "$\preceq$". In out further considerations, such a comparing will be called an elementary operation. Since $\preceq$

2010 *Mathematics Subject Classification.* 05C30, 52C99.

Key words and phrases. Linearly ordered set; Elementary operation; Stirling's formula.

trivially induces the linear ordering on $\{x, y\}$, we have the disjunction $x \preceq y \vee y \preceq x$. Moreover, since $x \neq y$, we can write

$$x \prec y \vee y \prec x.$$

Suppose now that for every two-element subset $\{x, y\}$ of X, we are able to specify, by using exactly one elementary operation, which of the two relations $x \prec y$ and $y \prec x$ is valid. Briefly speaking, we are in the situation where full information on the induced orderings

$$\preceq_{\{x;y\}} \quad (x \in X, y \in X, x \neq y)$$

is available. For our future purpose, several simple auxiliary propositions will be helpful.

Recall that any pair of the form (V, E), where V is a set and E is some subset of the family of all two-element parts of V, is called a graph (see, e.g., [6, 11]).

Lemma 2. *If a finite graph (V, E) is such that*

$$\text{card}(V) = n, \qquad \text{card}(E) < n - 1,$$

then this graph is not connected.

The above assertion immediately follows from the fact that any nonempty finite connected graph (V, E) contains a subtree (V, E') such that

$$\text{card}(E') = \text{card}(V) - 1.$$

Lemma 3. *Let $(L, \leq)$ be a linearly ordered set and let X and Y be any two nonempty disjoint finite subsets of L such that*

$$X = \{x_1, x_2, \ldots, x_m\}, \quad Y = \{y_1, y_2, \ldots, y_n\},$$
$$x_1 < x_2 < x_3 < \cdots < x_m, \quad y_1 < y_2 < y_3 < \cdots < y_n.$$

Consider the set $Z = X \cup Y$. Then $m + n - 1$ elementary operations are sufficient for describing the ordering on Z induced by $\leq$.

The proof of this lemma can be done by induction on the sum $m + n$.

Lemma 4. *Let $(X, \preceq)$ be a linearly ordered set with* $\text{card}(X) = n$, *where $n \geq 2$. Then no test of $n-2$ (or less) pairs of elements from X can give full description of the ordering $\preceq$.*

Proof. Suppose otherwise. Then there are $n - 2$ (or less) elementary operations which allow us to reconstruct the given linear ordering $\preceq$.

Consider the graph (V, E) where $V = X$ and E consists of all those two-element parts of V which were used in the process of making the above-mentioned elementary operations. Our assumption means that $\text{card}(E) \leq n - 2$.

Then Lemma 2 says that the graph (V, E) is not connected. i.e., we have a representation

$$X = V = V_1 \cup V_2,$$

where V_1 and V_2 are nonempty disjoint sets and no edge of (V, E) has one vertex in V_1 and the order vertex in V_2.

Let us denote

$$\preceq' = (\preceq \cap (V_1 \times V_1)) \cup (\preceq \cap (V_2 \times V_2)).$$

Obviously $\preceq'$ is a partial ordering on X. Taking into account this circumstance, we can readily define two distinct linear orderings on X which both satisfy the list of the carried out elementary operations.

Namely, the first linear ordering is such that it extends $\preceq'$ and all elements from V_1 are strictly less than all those from V_2, and the second linear ordering is such that it also extends $\preceq'$, but all elements from V_2 are strictly less than all elements from V_1. In other words, we see that the test suggested by the made $n - 2$ elementary operations could not reconstruct the given linear ordering $\preceq$. □

In connection with the last lemma, there naturally arise the questions: how many two-element subsets of X should be taken for total reconstruction of the linear ordering $\preceq$ on X?

Equivalently, one may ask: how many elementary operations are sufficient for the total description of the linear orderings $\preceq$ on X?

By using Lemma 1 and Lemma 3, one can deduce the next well-known statement.

Theorem 1. *The minimal number of those two-element subsets of a nonempty finite linearly ordered set $(L, \leq)$ with $\text{card}(L) = n$ that suffices to reconstruct the given ordering $\leq$ is estimated from the above by $O(n \cdot \log_2(n))$.*

Equivalently, $O(n \cdot \log_2(n))$ elementary operations are enough to reconstruct the linear ordering $\leq$ on L.

Example. Let $(L, \preceq)$ be an arbitrary linearly ordered set. Recall that for every set $\{x, y\} \subset L$, where $x \neq y$, the elementary operation corresponding to $\{x, y\}$ allows one to recognize the induced ordering on $\{x, y\}$ or, in other words, provides information which of the two relations $x \prec y$ and $y \prec x$ is valid.

Let now $X_1, X_2, X_3, \ldots, X_m$ be some subsets of a linearly ordered set $(L, \preceq)$. Consider the Cartesian product $X_1 \times X_2 \times X_3 \times \times \times X_m$ and equip it with the so-called lexicographical ordering $\leq$. In particular, if $m = 2$, then we have

$$(x_1, y_1) < (x_2, y_2) \Leftrightarrow ((x_1 \prec x_2 \vee (x_1 = x_2 \& y_1 \prec y_2)),$$

where (x_1, y_1) and (x_2, y_2) are any two distinct pairs from $X_1 \times X_2$.

Let n be a nonzero natural number and Z be the subsets of the Cartesian product $X_1 \times X_2 \times X_3 \times \times \times X_m$ with $\text{card}(Z) = n$. Using Theorem 1, one can demonstrate that $O(n \cdot \log_2(n))$ elementary operations, each of which is applied either to a two-element subset of X_1, or to a two-element subset of $X_2, \ldots$, or to a two-element subset of X_m, are sufficient to reconstruct the lexicographical ordering on Z .

For more detailed information about Theorem 1, its generalizations and applications in the discrete and combinatorial geometry, see [3–5, 8, 9, 13].

Theorem 2. *Let $(X, \leq)$ be a nonempty finite linearly ordered set with $\text{card}(X) = n$. The probability that exactly $n - 1$ elementary operations suffice to reconstruct the given ordered $\leq$, is equal to*

$$p = \frac{1}{\binom{\binom{n}{2}}{n-1}}.$$

Proof. First of all, let us find a number of all elementary operations on $(X, \leq)$. In fact, we wish to find a number of all two-element subsets of the given linearly ordered set $(X, \leq)$. It is well known that, the number of all two-element subsets of a finite set with $\text{card}(X) = n$ is $\binom{n}{2}$.

At the second step we calculate the number of all possible $(n-1)$-subsets of $\binom{n}{2}$, which is obviously equal to

$$\binom{\binom{n}{2}}{n-1}.$$

Let us prove that exactly one set of two-element subsets enables us to reconstruct the given ordering.

Let us consider two-element subsets $\{x_i, x_j\}$, which allow us to reconstruct the given linear ordering

$$x_1 < x_2 < \cdots < x_n.$$

Notice that to reconstruct given ordering, at least one pair must exist which contains x_1, otherwise reconstruction will be not uniquely determined.

Let us look at those two-element subsets that contain the element x_1. Enumerate all elements of X in the pairs with x_1 as follows:

$$x_{i_1}, x_{i_2}, \ldots, x_{i_k}$$

and consider the set of all such two-element subsets which contain x_1

$$\{x_1, x_{i_1}\}, \{x_1, x_{i_2}\}, \dots, \{x_1, x_{i_k}\}.$$

We fixed the ordering

$$x_2 < x_3 < \dots < x_n$$

and by induction $n-2$ elementary operations are enough to reconstruct the above mentioned ordering.

Suppose that $k \geq 2$. Then $n-k-1$ elementary operations are needed to reconstruct the ordering of the remaining elements $x_2, x_3, \dots, x_n$.

It is clear that if $k \geq 2$, then

$$n - k - 1 < n - 2.$$

But this contradicts the inductive assumption.

Therefore, $k = 1$ and x_1 is in the pair with just x_{i_1}.

Now let us prove that

$$x_{i_1} = x_2.$$

Suppose that $x_{i_1} \neq x_2$ and $x_{i_1} = x_k$ $(k \neq 2)$. In such a case we get another ordering of the given set $(X, \leq)$.

For example,

$$x_2 < x_3 < \dots < x_{k-1} < x_1 < x_k < \dots < x_n$$

or

$$x_2 < x_3 < \dots < x_1 < x_{k-1} < x_k < \dots < x_n.$$

It is clear that such an ordering is not the given one. Consequently, x_{i_1} should be x_2 in order for a linear ordering to be uniquely determined. □

We thus obtain

$$p = \frac{1}{\binom{\binom{n}{2}}{n-1}}.$$

By using the well-known combinatorial formulas, we get

$$\binom{\binom{n}{2}}{n-1} = \frac{\left(\frac{n(n-1)}{2}\right)!}{(n-1)!\left(\frac{(n-1)(n-2)}{2}\right)!}.$$

A precise estimate of $n!$ that is of importance both for numerical calculations and for theoretical analysis is Stirlinng's formula (see, e.g., [1]):

$$n! \sim n^n e^{-n} \sqrt{2\pi n}.$$

This formula implies that

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1.$$

For more details about this formula and its applications see, e.g., [12].

Applying Stirling's approximation in our case, we get

$$\frac{\left(\frac{n(n-1)}{2}\right)!}{(n-1)!\left(\frac{(n-1)(n-2)}{2}\right)!} \sim \frac{\left(\frac{n(n-1)}{2e}\right)^{\frac{n(n-1)}{2}} \sqrt{\frac{2\pi n(n-1)}{2}}}{\left(\frac{n-1}{e}\right)^{n-1} \sqrt{2\pi(n-1)} \left(\frac{(n-1)(n-2)}{2e}\right)^{\frac{(n-1)(n-2)}{2}} \sqrt{\pi(n-1)(n-2)}}.$$

Since

$$\frac{e^{-\frac{(n-1)n}{2}}}{e^{-(n-1)} e^{-\frac{(n-2)(n-1)}{2}}} = 1,$$

we have

$$\binom{\binom{n}{2}}{n-1} \sim \frac{\left(\frac{n(n-1)}{2}\right)^{\frac{n(n-1)}{2}}}{(n-1)^{n-1}\left(\frac{(n-1)(n-2)}{2}\right)^{\frac{(n-1)(n-2)}{2}}}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \left[\frac{\left(\frac{n(n-1)}{2}\right)^{\frac{n}{2}}}{(n-1)\left(\frac{(n-1)(n-2)}{2}\right)^{\frac{(n-2)}{2}}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \left[\frac{(n(n-1))^{\frac{n}{2}}2^{-\frac{n}{2}}}{(n-1)((n-1)(n-2))^{\frac{(n-2)}{2}}2^{-\frac{n-2}{2}}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \frac{1}{2^{n-1}}\left[\frac{n^{\frac{n}{2}}(n-1)^{\frac{n}{2}}}{(n-1)(n-1)^{\frac{(n-2)}{2}}(n-2)^{\frac{(n-2)}{2}}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \frac{1}{2^{n-1}}\left[\frac{n^{\frac{n}{2}}}{(n-2)^{\frac{n}{2}}(n-2)^{-1}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \frac{1}{2^{n-1}}\left[\frac{n-2}{\left(\frac{n-2}{n}\right)^{\frac{n}{2}}}\right]^{n-1}\sqrt{\frac{n}{2\pi(n-2)(n-1)}}$$

$$= \frac{1}{2^{n-1}}(n-2)^{n-1}e^{n-1}(n-2)^{-\frac{1}{2}}\sqrt{\frac{n}{2\pi(n-1)}}$$

$$= \frac{1}{2^{n-1}}(n-2)^{n-\frac{3}{2}}e^{n-1}\sqrt{\frac{n}{2\pi(n-1)}} \sim n^n\left(\frac{e}{2}\right)^n\sqrt{\frac{1}{2\pi}}.$$

Remark. The number $n^n(\frac{e}{2})^n\sqrt{\frac{1}{2\pi}}$ is much bigger than n^n for sufficiently large natural numbers n. Since the probability p in Theorem 2 is asymptotically equal to $\frac{1}{n^n(\frac{e}{2})^n\sqrt{\frac{1}{2\pi}}}$, we conclude that even for $n = 15$, this probability is almost zero.

In case $n = 15$, the 15^{15} is an extremely big number. In particular, about 10^{11} many stars are in the Milky Way.

References

1. V. Chelidze, E. Tsitlanadze, *Course of Mathematical Analysis, Part II*, Publishing House of Tbilisi State University, Tbilisi, 1975. (in Georgian).
2. T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein, *Introduction to Algorithms.* Third edition. MIT Press, Cambridge, MA, 2009.
3. S. L. Devadoss, J. O'Rourke, *Discrete and Computational Geometry.* Princeton University Press, Princeton, NJ, 2011.
4. H. Edelsbrunner, *Algorithms in Combinatorial Geometry.* EATCS Monographs on Theoretical Computer Science, 10. Springer-Verlag, Berlin, 1987.
5. M. T. Goodrich, R. Tamassia, *Algorithm Design: Foundation, Analysis and Internet Examples.* John Wiley & Sons, 2006.
6. F. Harary, *Graph Theory.* Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.
7. A. B. Kharazishvili, *Introduction to Combinatorial Geometry.* (in Russian) Tbilis. Gos. Univ., Tbilisi, 1985.
8. A. B. Kharazishvili, *Elements of Combinatorial Geometry.* Georgian National Academy of Sciences, Tbilisi, 2016.
9. J. Matoušek, *Lectures on Discrete Geometry.* Graduate Texts in Mathematics, 212. Springer-Verlag, New York, 2002.
10. W. Moser, J. Pach, Recent developments in combinatorial geometry. *New trends in discrete and computational geometry*, 281–302, Algorithms Combin., 10, *Springer, Berlin*, 1993.

11. O. Ore, *Theory of Graphs*. Amer. Math. Soc., Providence, R. I. 1962.
12. Ch. Pisot, M. Zamansky, *Mathematiques Generales*. Algebre-Analyse, Dunod Paris, 1966.
13. F. P. Preparata, M. I. Shamos, *Computational Geometry: An Introduction*. Springer-Verlag, New York, 1985.

(Received 30.04.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 7–29

CERTAIN FRACTIONAL INTEGRAL AND FRACTIONAL DERIVATIVE FORMULAE WITH THEIR IMAGE FORMULAE INVOLVING GENERALIZED MULTI-INDEX MITTAG-LEFFLER FUNCTION

MEHAR CHAND[1], HAMED DAEI KASMAEI[2], AND MEHMET SENOL[3]

Abstract. The main objective of this paper is to establish some image formulas by applying the Riemann–Liouville fractional derivative and integral operators to the product of generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(.)$. Some more image formulas are derived by applying integral transforms. The results obtained here are quite general in nature and capable of yielding a very large number of known and (presumably) new results.

1. Introduction

In fractional calculus several important functions known as special functions are presented via improper integrals or series. Among these vital functions, the Bessel function is widely used in physical sciences and engineering by many authors (see [9, 13, 15, 14, 1, 3, 2, 4, 7, 6, 8]). In recent years, a remarkably sizable amount of research works involving generalizations of Mittag–Leffler function is presented by several researchers.

For our present study we start with recalling the previous work. The Mittag–Leffler function is given as (see Marichev [23]): In this section we recall some known facts about Mittag–Leffler function and its generalizations, and also about the Riemann–Liouville fractional integral and a derivative operator.

Let us begin with few notions and facts related to the Mittag–Leffler function. In this presentation we follow mainly the review article [12] (see also [11]).

The Mittag-Lefler function $E_\alpha(z)$ with $\alpha > 0$ is named in honour of the great Swedish mathematician G.M. Mittag–Leffler who introduced it in the early of this century in a sequence of five notes and defined in the form of series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{1.1}$$

It was noted by Mittag–Leffler himself that for all α; $\Re(\alpha) > 0$ the series in (1.1) converges in the whole complex plane (and thus is an entire function of a complex variable z). For special values of parameter α the function $E_\alpha(z)$ coincides with some elementary and special functions. In particular, $E_1(z) = \exp(z)$. Hence, sometimes, the Mittag–Leffler function is called a generalized exponential. Anyway, the asymptotic behavior at infinity of this function differs of that for exponential function, namely, for all α, $0 < \Re(\alpha) < 2$, $\alpha \neq 1$ there exists an angle of exponential growth, and an angle at which the function is bounded.

First generalization of the function $E_\alpha(z)$ was mentioned by Wiman [34],

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

For each $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$; $E_{\alpha,\beta}(z)$ is an entire function. The function $E_{\alpha,\beta}(z)$ reduces to the classical Mittag–Leffler function if we choose $\beta = 1$.

2010 *Mathematics Subject Classification.* 26A33, 33C45, 33C60, 33C70.

Key words and phrases. Pochhemmer symbol; Fractional calculus; Fractional derivative; Fractional integration; Mittag–Leffler function; Beta transform; Laplace transform; Whittaker transform.

Further generalization of the function $E_\alpha(z)$ was proposed by Prabhakar [25]:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha,\beta \in \mathbb{C}$, $\Re(\alpha) > 0$ and $(\gamma)_k$ is the Pochhammer symbol:

$$(\gamma)_k := \begin{cases} 1, & k=0, \\ \gamma(\gamma+1)\dots(\gamma+k-1), & k \in \mathbb{N}. \end{cases}$$

Extended exposition on the theory and applications of this function is given in [22]. Evidently, the function $E^{\gamma}_{\alpha,\beta}(z)$ is related to the classical Mittag–Leffler function $E_\alpha(z)$ and two-parametric Mittag–Leffler function $E_{\alpha,\beta}(z)$:

$$E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z); \quad E^{1}_{\alpha,1}(z) = E_{\alpha}(z).$$

Another generalization of two-parametric Mittag–Leffler function is the so-called four-parametric function.

$$E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k+\beta_1)\,\Gamma(\alpha_1 k+\beta_1)}.$$

For positive $\alpha_1 > 0$; $\alpha_2 > 0$ and real $\beta_1;\beta_2 \in \mathbb{R}$ it was introduced by Djrbashian [10]. It is not hard to see that the convergence conditions for this function can be extended to all $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{C}$; $\Re(\alpha_1) > 0$, $\Re(\alpha_2) > 0$. Besides, $E_{\alpha,\beta;0,1}(z) = E_{\alpha,\beta}(z)$.

Generalizing the four-parametric Mittag–Leffler function, Al-Bassam and Luchko [5] introduced the Mittag–Leffler type function

$$E_{(\alpha_j,\beta_j)_m}(z) = E\big((\alpha_j,\beta_j)^m_{j=1}; z\big) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{j=1}^{m}\Gamma(\alpha_j k+\beta_j)}. \tag{1.2}$$

with $2m$ real parameters $\alpha_j > 0$; $\beta_j \in \mathbb{R}$ $(j = 1,\dots,m)$ and with complex $z \in \mathbb{C}$. In [5], an explicit solution to the Cauchy type problem for a fractional differential equation is given in terms of (1.2). The theory of this class of functions was developed in a series of articles by Kiryakova et al. [18, 19, 20, 22, 21] (see also [16]).

Generalization of the Prabhakar type function was done by Shukla and Prajapati [29]:

$$E^{\gamma,q}_{(\alpha,\beta)}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{\Gamma(\alpha k+\beta)}, \quad (n \in \mathbb{N}). \tag{1.3}$$

under the following assumptions on parameters: $q \in (0,1) \cup \mathbb{N}$ and $\min\{\Re(\beta);\Re(\gamma)\}$. In [33], it is shown the existence of the function (1.3) for a wider set of parameters:

$$\big\{\alpha,\beta,\gamma \in \mathbb{C}; \Re(\alpha) > \max\{0;\Re(q-1)\}; \Re(q) > 0\big\}.$$

The definition (1.3) was combined with (1.2) in [27] (see also [28]). As a result, there appeared the following definition of generalized multi-index Mittag–Leffler function

$$E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}}{\prod_{j=1}^{m}\Gamma(\alpha_j k+\beta_j)} \frac{(z)^k}{k!},$$

where $m \in \mathbb{N}$, $\alpha_j,\beta_j,\gamma,q,z \in \mathbb{C}$ $(j = 1,\dots,m)$ such that $\sum_{j=1}^{m}\Re(\alpha_j) > \max\{0;\Re(q)-1\}$; $\Re(q) > 0$.

The results given by Kiryakova [17], Miller and Ross [24], Srivastava et. al., [32] can be referred for some basic results on fractional calculus. The Fox-Wright function ${}_p\Psi_q$ is defined as (see, for details, Srivastava and Karlsson 1985, [31])

$${}_p\Psi_q[z] = {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p);\\(b_1,\beta_1),\dots,(b_q,\beta_q);\end{matrix}\; z\right] = {}_p\Psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p};\\(b_j,\beta_j)_{1,q};\end{matrix}\; z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}, \tag{1.4}$$

where the coefficients $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q \in \mathbb{R}^+$ such that

$$1 + \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0.$$

2. Fractional Derivative and Integral Operators

The right-sided Riemann–Liouville fractional integral operator I^{σ}_{a+} and the left-sided Riemann–Liouville fractional integral operator I^{σ}_{a-} and the corresponding Riemann–Liouville fractional derivative operator D^{σ}_{a+} and D^{σ}_{a-} are given as follows [26].

Lemma 1 (Riemann–Liouville fractional integral operators [26])**.** *Let $\Psi = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The left-sided Riemann–Liouville fractional integral operators I^{σ}_{a+} and the right-sided Riemann–Liouville fractional integral operators I^{σ}_{b-} of order $\sigma \in \mathbb{C}$ are defined as*

$$\left(I^{\sigma}_{a+} f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_a^x \frac{f(t)}{(x-t)^{1-\sigma}} dt \quad (x > a;\ \Re(\sigma) > 0), \tag{2.1}$$

$$\left(I^{\sigma}_{b-} f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_x^b \frac{f(t)}{(t-x)^{1-\sigma}} dt \quad (x < b;\ \Re(\sigma) > 0).$$

Lemma 2 (Riemann–Liouville fractional derivative operators [26])**.** *Let $\Psi = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional derivative operators D^{σ}_{a+} and D^{σ}_{b-} of order $\sigma \in \mathbb{C}$ are defined as*

$$\left(D^{\sigma}_{a+} f\right)(x) = \frac{d^n}{dx^n} \left(I^{n-\sigma}_{a+} f\right)(x), \quad \big(\Re(\sigma) \geq 0;\ \ n = 1 + [\Re(\sigma)]\big), \tag{2.2}$$

$$\left(D^{\sigma}_{b-} f\right)(x) = (-1)^n \frac{d^n}{dx^n} \left(I^{n-\sigma}_{b-} f\right)(x), \quad \big(\Re(\sigma) \geq 0;\ \ n = 1 + [\Re(\sigma)]\big),$$

where the function is locally integrable, $\Re(\sigma)$ denotes real part of the complex number and $[\Re(\sigma)]$ means the greatest integer in $\Re(\sigma)$. Also, the following n^{th} order derivative of x^{α} is defined as:

$$\frac{d^n}{dx^n}(x^{\alpha}) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)} x^{\alpha-n}, \quad \Re(\alpha) > 0. \tag{2.3}$$

For our present work, the following result is also required:

$$\int_b^a (a-t)^{\beta-1}(t-b)^{\alpha-1} dt = (a-b)^{\alpha+\beta-1} B(\alpha, \beta) \quad \big(\Re(\alpha) > 0;\ \ \Re(\beta) > 0;\ \ b < a\big). \tag{2.4}$$

3. Fractional Integral and Derivative Formulae Involving Generalized Multi-index Mittag–Leffler Function

In this section, we derive the formulae by using the Riemann–Liouville fractional integral and derivative operator involving generalized multi-index Mittag–Leffler function.

Theorem 1. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\Big(I^{\sigma}_{a+} (t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_{ij}, \beta_{ij})_m} \left(\xi (t-a)^{\mu}\right)\Big)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1) \ldots \Gamma(\gamma_r)}$$

$$\times {}_{r+1}\Psi_{mr+r} \left[\begin{matrix} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r) \\ (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma+\rho+1, \mu r), (-1, 1)_{1,r-1} \end{matrix} \,\middle|\, (\xi (x-a)^{\mu})^r \right] \tag{3.1}$$

holds.

Proof. Let the left-hand side of equation (3.1) be denoted by $\mathcal{I}$. Applying (1.4) and using the definition in equation (2.1) and interchanging the order of integration and summation, we have

$$\mathcal{I}=\frac{1}{\Gamma(\sigma)}\prod_{i=1}^{r}\left\{\sum_{k=0}^{\infty}\frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m}\Gamma(k\alpha_{ij}+\beta_{ij})}\frac{\xi^k}{k!}\right\}\int\limits_a^x(x-t)^{\sigma-1}(t-a)^{\rho+\mu kr}dt, \tag{3.2}$$

applying the result (2.4), the above equation (3.2) reduces to

$$\mathcal{I}=\frac{1}{\Gamma(\sigma)}\prod_{i=1}^{r}\left\{\sum_{k=0}^{\infty}\frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m}\Gamma(k\alpha_{ij}+\beta_{ij})}\frac{\xi^k}{k!}\right\}(x-a)^{\sigma+\rho+\mu kr}B(\sigma,\rho+\mu kr+1). \tag{3.3}$$

After simplification, the above equation (3.3) reduces to

$$\mathcal{I}=(x-a)^{\sigma+\rho}\prod_{i=1}^{r}\frac{1}{\Gamma(\gamma_i)}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_i+kq_i)}{\prod_{j=1}^{m}\Gamma(k\alpha_{ij}+\beta_{ij})}\frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)}\frac{(\xi(x-a)^{\mu})^{kr}}{k!}\right\}, \tag{3.4}$$

the above equation (3.4) can be written as

$$\mathcal{I}=\frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+kq_1)\ldots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$
$$\left.\times\frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}}\frac{(\xi(x-a)^{\mu})^{kr}}{k!}\right\}, \tag{3.5}$$

interpret the above equation (3.5), in the view of (1.4), we have the required result (3.1) □

Theorem 2. *Let* $m,r\in\mathbb{N}$, $\alpha_{ij},\beta_{ij},\gamma_i,q_i,\rho,\xi,\mu\in\mathbb{C}$ $(i=1,\ldots,r;\ j=1,\ldots,m)$ *such that* $\sum_{j=1}^{m}\Re(\alpha_{ij})>\max\{0;\Re(q_i)-1\}$; $\Re(q_i)>0$ *and* $\Re(\sigma)>0$. *For* $b>x$, *the following integral formula*

$$\left(I_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i}\left(\xi(b-t)^{\mu}\right)\right)(x)=\frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times{}_{r+1}\Psi_{mr+r}\left[\begin{matrix}(\gamma_1,q_1),\ldots,(\gamma_r,q_r),(\rho+1,\mu r)\\(\beta_{1j},\alpha_{1j})_{1,m},\ldots,(\beta_{rj},\alpha_{rj})_{1,m},(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{matrix}\middle|(\xi(b-x)^{\mu})^{r}\right] \tag{3.6}$$

holds.

Proof. The proof of Theorem 2 is similar to that of Theorem 1, therefore we omit the details. □

Theorem 3. *Let* $m,r\in\mathbb{N}$, $\alpha_{ij},\beta_{ij},\gamma_i,q_i,\rho,\xi,\mu\in\mathbb{C}$ $(i=1,\ldots,r;\ j=1,\ldots,m)$ *such that* $\sum_{j=1}^{m}\Re(\alpha_{ij})>\max\{0;\Re(q_i)-1\}$; $\Re(q_i)>0$ *and* $\Re(\sigma)\geq 0$. *For* $x>a$, *the following integral formula*

$$\left(D_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i}\left(\xi(t-a)^{\mu}\right)\right)(x)=\frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times{}_{r+1}\Psi_{mr+r}\left[\begin{matrix}(\gamma_1,q_1),\ldots,(\gamma_r,q_r),(\rho+1,\mu r)\\(\beta_{1j},\alpha_{1j})_{1,m},\ldots,(\beta_{rj},\alpha_{rj})_{1,m},(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{matrix}\middle|(\xi(x-a)^{\mu})^{r}\right] \tag{3.7}$$

holds.

Proof. Let the left-hand side of equation (3.7) be denoted by $\mathcal{D}$, then interchanging the order of differentiation and summation, we have

$$\mathcal{D}=\prod_{i=1}^{r}\left\{\sum_{k=0}^{\infty}\frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m}\Gamma(k\alpha_{ij}+\beta_{ij})}\frac{\xi^k}{k!}\right\}\left(D_{a+}^{\sigma}(t-a)^{\rho+\mu kr}\right)(x), \tag{3.8}$$

now using the result given in equation (2.2) and further applying the result (2.1), the above equation (3.8) reduces to the following form

$$\mathcal{D} = \frac{1}{\Gamma(n-\sigma)} \prod_{i=1}^{r} \left\{ \sum_{k=0}^{\infty} \frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m} \Gamma(k\alpha_{ij}+\beta_{ij})} \frac{\xi^k}{k!} \right\} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\sigma-1}(t-a)^{\rho+\mu kr} dt, \tag{3.9}$$

substituting the result (2.4) into the above equation (3.9) and after simplification, we get

$$\mathcal{D} = \prod_{i=1}^{r} \left\{ \sum_{k=0}^{\infty} \frac{(\gamma_i)_{kq_i}}{\prod_{j=1}^{m} \Gamma(k\alpha_{ij}+\beta_{ij})} \frac{\xi^k}{k!} \right\} \frac{\Gamma(\rho+\mu kr+1)}{\Gamma(k-\sigma+\rho+\mu kr+1)} \frac{d^n}{dx^n} (x-a)^{\rho-\sigma+n+\mu kr}, \tag{3.10}$$

using the result given in equation (2.3) into the above equation (3.10), we have

$$\mathcal{I} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1+kq_1)\ldots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m} \Gamma(k\alpha_{1j}+\beta_{1j}) \cdots \prod_{j=1}^{m} \Gamma(k\alpha_{rj}+\beta_{rj})} \right.$$
$$\left. \times \frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\rho-\sigma+\mu kr+1)(\Gamma(k-1))^{r-1}} \frac{(\xi(x-a)^{\mu})^{kr}}{k!} \right\}, \tag{3.11}$$

further, interpret the above equation (3.11) with the view of (1.4), we obtain the required result (3.7). □

Theorem 4. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ such that $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) \geq 0$. For $b > x$, the following integral formula*

$$\left(D_{b-}^{\sigma} (b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i} (\xi(b-t)^{\mu}) \right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{mr+r} \left[\begin{matrix} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r) \\ (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\rho-\sigma+1, \mu r), (-1,1)_{1,r-1} \end{matrix} \middle| (\xi(b-x)^{\mu})^r \right] \tag{3.12}$$

holds.

Proof. The proof of Theorem 4 is similar to that of Theorem 3, therefore we omit the details. □

3.1. **Special cases of fractional integral and derivative formulae.** Choose $m = 1$, the generalized multi-index Mittag–Leffler function $E_{(\alpha_j,\beta_j)_m}^{\gamma,q}(.)$ reduces to the generalized Mittag–Leffler function $E_{(\alpha,\beta)}^{\gamma,q}(.)$ defined by Shukla and Prajapati. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form

Corollary 1. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I_{a+}^{\sigma} (t-a)^{\rho} \prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}^{\gamma_i,q_i} (\xi(t-a)^{\mu}) \right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r} \left[\begin{matrix} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\sigma+\rho+1, \mu r), (-1,1)_{1,r-1} \end{matrix} \middle| (\xi(x-a)^{\mu})^r \right]$$

holds.

Corollary 2. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I_{b-}^{\sigma} (b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}^{\gamma_i,q_i} (\xi(b-t)^{\mu}) \right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r} \left[\begin{matrix} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\sigma+\rho+1, \mu r), (-1,1)_{1,r-1} \end{matrix} \middle| (\xi(b-x)^{\mu})^r \right]$$

holds.

Corollary 3. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(D_{a+}^{\sigma}(t-a)^{\rho} \prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}^{\gamma_i,q_i}\left(\xi(t-a)^{\mu}\right)\right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\rho-\sigma+1, \mu r), (-1, 1)_{1,r-1} \end{array} \middle| (\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 4. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(D_{b-}^{\sigma}(b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}^{\gamma_i,q_i}\left(\xi(b-t)^{\mu}\right)\right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\rho-\sigma+1, \mu r), (-1, 1)_{1,r-1} \end{array} \middle| (\xi(b-x)^{\mu})^{r}\right]$$

holds.

Choosing $q = m = 1$, the generalized multi-index Mittag–Leffler function $E_{(\alpha_j,\beta_j)_m}^{\gamma,q}(.)$ reduces to the generalized Mittag–Leffler function $E_{(\alpha,\beta)}^{\gamma}(.)$ defined by Prabhakar. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 5. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I_{a+}^{\sigma}(t-a)^{\rho} \prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}^{\gamma_i}\left(\xi(t-a)^{\mu}\right)\right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, 1), \ldots, (\gamma_r, 1), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\sigma+\rho+1, \mu r), (-1, 1)_{1,r-1} \end{array} \middle| (\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 6. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I_{b-}^{\sigma}(b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}^{\gamma_i}\left(\xi(b-t)^{\mu}\right)\right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, 1), \ldots, (\gamma_r, 1), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\sigma+\rho+1, \mu r), (-1, 1)_{1,r-1} \end{array} \middle| (\xi(b-x)^{\mu})^{r}\right]$$

holds.

Corollary 7. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(D_{a+}^{\sigma}(t-a)^{\rho} \prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}^{\gamma_i}\left(\xi(t-a)^{\mu}\right)\right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}$$
$$\times {}_{r+1}\Psi_{2r}\left[\begin{array}{l} (\gamma_1, 1), \ldots, (\gamma_r, 1), (\rho+1, \mu r) \\ (\beta_1, \alpha_1), \ldots, (\beta_r, \alpha_r), (\rho-\sigma+1, \mu r), (-1, 1)_{1,r-1} \end{array} \middle| (\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 8. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \dots, r)$ such that $\Re(\alpha_i) > 0$; $\Re(q_i) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\Big(D^{\sigma}_{b-}(b-t)^{\rho}\prod_{i=1}^{r} E^{\gamma_i}_{(\alpha_i,\beta_i)}\left(\xi(b-t)^{\mu}\right)\Big)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\dots\Gamma(\gamma_r)}$$

$$\times_{r+1}\Psi_{2r}\left[\begin{array}{l}(\gamma_1,1),\dots,(\gamma_r,1),(\rho+1,\mu r)\\(\beta_1,\alpha_1),\dots,(\beta_r,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Choosing $\gamma = q = m = 1$, the generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(.)$ reduces to the generalized Mittag–Leffler function $E_{(\alpha,\beta)}(.)$ defined by Prabhakar. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 9. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \dots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\Big(I^{\sigma}_{a+}(t-a)^{\rho}\prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}\left(\xi(t-a)^{\mu}\right)\Big)(x)$$

$$= (x-a)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(\beta_1,\alpha_1),\dots,(\beta_r,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 10. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \dots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\Big(I^{\sigma}_{b-}(b-t)^{\rho}\prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}\left(\xi(b-t)^{\mu}\right)\Big)(x)$$

$$= (b-x)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(\beta_1,\alpha_1),\dots,(\beta_r,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Corollary 11. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \dots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\Big(D^{\sigma}_{a+}(t-a)^{\rho}\prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}\left(\xi(t-a)^{\mu}\right)\Big)(x)$$

$$= (x-a)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(\beta_1,\alpha_1),\dots,(\beta_r,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 12. *Let $r \in \mathbb{N}$, $\alpha_i, \beta_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \dots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\Big(D^{\sigma}_{b-}(b-t)^{\rho}\prod_{i=1}^{r} E_{(\alpha_i,\beta_i)}\left(\xi(b-t)^{\mu}\right)\Big)(x)$$

$$= (b-x)^{\sigma+\rho}{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(\beta_1,\alpha_1),\dots,(\beta_r,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Choosing $\beta_j = \gamma = q = m = 1$, the generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(.)$ reduces to the Mittag–Leffler function $E_{\alpha}(.)$ defined by the great Swedish mathematician G.M. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 13. *Let $r \in \mathbb{N}$, $\alpha_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\Big(I_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{\alpha_i}\left(\xi(t-a)^{\mu}\right)\Big)(x)$$
$$= (x-a)^{\sigma+\rho}\,{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(1,\alpha_1),\ldots,(1,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\Bigg|(\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 14. *Let $r \in \mathbb{N}$, $\alpha_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\Big(I_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{\alpha_i}\left(\xi(b-t)^{\mu}\right)\Big)(x)$$
$$= (b-x)^{\sigma+\rho}\,{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(1,\alpha_1),\ldots,(1,\alpha_r),(\sigma+\rho+1,\mu r),(-1,1)_{1,r-1}\end{array}\Bigg|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Corollary 15. *Let $r \in \mathbb{N}$, $\alpha_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\Big(D_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{\alpha_i}\left(\xi(t-a)^{\mu}\right)\Big)(x)$$
$$= (x-a)^{\sigma+\rho}\,{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(1,\alpha_1),\ldots,(1,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\Bigg|(\xi(x-a)^{\mu})^{r}\right]$$

holds.

Corollary 16. *Let $r \in \mathbb{N}$ $\alpha_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r)$ such that $\Re(\alpha_i) > 0$; and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\Big(D_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{\alpha_i}\left(\xi(b-t)^{\mu}\right)\Big)(x)$$
$$= (b-x)^{\sigma+\rho}\,{}_1\Psi_{2r}\left[\begin{array}{l}(\rho+1,\mu r)\\(1,\alpha_1),\ldots,(1,\alpha_r),(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{array}\Bigg|(\xi(b-x)^{\mu})^{r}\right]$$

holds.

Choosing $r = m = 1$, the generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(.)$ reduces to the Mittag–Leffler function $E^{\gamma,q}_{(\alpha,\beta)}(.)$ defined by the great Swedish mathematician G.M. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 17. *Let $\alpha, \beta, \gamma, q, \rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\alpha) > \max\{0; \Re(q) - 1\}$; $\Re(q) > 0$ and $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\Big(I_{a+}^{\sigma}(t-a)^{\rho}E^{\gamma,q}_{(\alpha,\beta)}\left(\xi(t-a)^{\mu}\right)\Big)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{array}{l}(\gamma,q),(\rho+1,\mu)\\(\beta,\alpha),(\sigma+\rho+1,\mu)\end{array}\Bigg|\xi(x-a)^{\mu}\right]$$

holds.

Corollary 18. *Let $\alpha, \beta, \gamma, q, \rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\alpha) > \max\{0; \Re(q) - 1\}$; $\Re(q) > 0$ and $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\Big(I_{b-}^{\sigma}(b-t)^{\rho}E^{\gamma,q}_{(\alpha,\beta)}\left(\xi(b-t)^{\mu}\right)\Big)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{array}{l}(\gamma,q),(\rho+1,\mu)\\(\beta,\alpha),(\sigma+\rho+1,\mu)\end{array}\Bigg|\xi(b-x)^{\mu}\right].$$

holds.

Corollary 19. *Let $\alpha, \beta, \gamma, q, \rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\alpha) > \max\{0; \Re(q) - 1\}$; $\Re(q) > 0$ and $\Re(\sigma) \geq 0$. For $x > a$, the following integral formula*

$$\left(D^{\sigma}_{a+}(t-a)^{\rho} E^{\gamma,q}_{(\alpha,\beta)}\left(\xi(t-a)^{\mu}\right)\right)(x) = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma)}\, {}_2\Psi_2 \left[\begin{array}{c} (\gamma, q), (\rho+1, \mu) \\ (\beta, \alpha), (\rho-\sigma+1, \mu) \end{array} \middle| \xi(x-a)^{\mu}\right]$$

holds.

Corollary 20. *Let $\alpha, \beta, \gamma, q, \rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\alpha) > \max\{0; \Re(q) - 1\}; \Re(q) > 0$ and $\Re(\sigma) \geq 0$. For $b > x$, the following integral formula*

$$\left(D^{\sigma}_{b-}(b-t)^{\rho} E^{\gamma,q}_{(\alpha,\beta)}\left(\xi(b-t)^{\mu}\right)\right)(x) = \frac{(b-x)^{\sigma+\rho}}{\Gamma(\gamma)}\, {}_2\Psi_2 \left[\begin{array}{c} (\gamma, q), (\rho+1, \mu) \\ (\beta, \alpha), (\rho-\sigma+1, \mu) \end{array} \middle| \xi(b-x)^{\mu}\right]$$

holds.

Choosing $r = m = \gamma = q = \alpha = \beta = 1$, the generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(.)$ reduces to the Mittag–Leffler function $E^{1,1}_{(1,1)_1}(.) = E^{1,1}_{(1,1)}(.) = \exp(.)$ defined by the great Swedish mathematician G.M. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 21. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I^{\sigma}_{a+}(t-a)^{\rho} \exp\left(\xi(t-a)^{\mu}\right)\right)(x) = (x-a)^{\sigma+\rho}\, {}_1\Psi_1 \left[\begin{array}{c} (\rho+1, \mu) \\ (\sigma+\rho+1, \mu) \end{array} \middle| \xi(x-a)^{\mu}\right]$$

holds.

Corollary 22. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I^{\sigma}_{b-}(b-t)^{\rho} \exp\left(\xi(b-t)^{\mu}\right)\right)(x) = (b-x)^{\sigma+\rho}\, {}_1\Psi_1 \left[\begin{array}{c} (\rho+1, \mu) \\ (\sigma+\rho+1, \mu) \end{array} \middle| \xi(b-x)^{\mu}\right]$$

holds.

Corollary 23. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $x > a$, the following integral formula*

$$\left(D^{\sigma}_{a+}(t-a)^{\rho} \exp\left(\xi(t-a)^{\mu}\right)\right)(x) = (x-a)^{\sigma+\rho}\, {}_1\Psi_1 \left[\begin{array}{c} (\rho+1, \mu) \\ (\rho-\sigma+1, \mu) \end{array} \middle| \xi(x-a)^{\mu}\right]$$

holds.

Corollary 24. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $b > x$, the following integral formula*

$$\left(D^{\sigma}_{b-}(b-t)^{\rho} \exp\left(\xi(b-t)^{\mu}\right)\right)(x) = (b-x)^{\sigma+\rho}\, {}_1\Psi_1 \left[\begin{array}{c} (\rho+1, \mu) \\ (\rho-\sigma+1, \mu) \end{array} \middle| \xi(b-x)^{\mu}\right]$$

holds.

Choosing $r = m = \gamma = q = \beta = 1; \alpha = 2$, the generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(.)$ reduces to the Mittag–Leffler function $E^{1,1}_{(2,1)_1}(.) = E^{1,1}_{(2,1)}(.) = \cosh\sqrt{(.)}$ defined by the great Swedish mathematician G.M. Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 25. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $x > a$, the following integral formula*

$$\left(I^{\sigma}_{a+}(t-a)^{\rho} \cosh\sqrt{(\xi(t-a)^{\mu})}\right)(x) = (x-a)^{\sigma+\rho}\, {}_1\Psi_2 \left[\begin{array}{c} (\rho+1, \mu) \\ (1, 2), (\sigma+\rho+1, \mu) \end{array} \middle| \xi(x-a)^{\mu}\right]$$

holds.

Corollary 26. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $b > x$, the following integral formula*

$$\left(I^{\sigma}_{b-}(b-t)^{\rho} \cosh\sqrt{(\xi(b-t)^{\mu})}\right)(x) = (b-x)^{\sigma+\rho}\, {}_1\Psi_2 \left[\begin{array}{c} (\rho+1, \mu) \\ (1, 2), (\sigma+\rho+1, \mu) \end{array} \middle| \xi(b-x)^{\mu}\right]$$

holds.

Corollary 27. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $x > a$, the following integral formula*

$$\left(D^{\sigma}_{a+}(t-a)^{\rho} \cosh \sqrt{(\xi(t-a)^{\mu})}\right)(x) = (x-a)^{\sigma+\rho}\, {}_1\Psi_2 \left[\begin{array}{l} (\rho+1, \mu) \\ (1,2), (\rho-\sigma+1, \mu) \end{array} \middle| \xi(x-a)^{\mu} \right]$$

holds.

Corollary 28. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $b > x$, the following integral formula*

$$\left(D^{\sigma}_{b-}(b-t)^{\rho} \cosh \sqrt{(\xi(b-t)^{\mu})}\right)(x) = (b-x)^{\sigma+\rho}\, {}_1\Psi_2 \left[\begin{array}{l} (\rho+1, \mu) \\ (1,2), (\rho-\sigma+1, \mu) \end{array} \middle| \xi(b-x)^{\mu} \right]$$

holds.

Choosing $r = m = \gamma = q = \beta = 1$; $\alpha = 0$, the generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(z)$ reduces to the Mittag–Leffler function $E^{1,1}_{(0,1)_1}(z) = E^{1,1}_{(0,1)}(z) = (1-z)^{-1}$ (where $|z| < 1$). Using this concept, the results established in equations (3.1), (3.6), (3.7) and (3.12) are reduced to the following form.

Corollary 29. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $|\xi(t-a)^{\mu}| < 1$, the following integral formula*

$$\left(I^{\sigma}_{a+} \frac{(t-a)^{\rho}}{1-\xi(t-a)^{\mu}}\right)(x) = (x-a)^{\sigma+\rho}\, {}_1\Psi_1 \left[\begin{array}{l} (\rho+1, \mu) \\ (\sigma+\rho+1, \mu) \end{array} \middle| \xi(x-a)^{\mu} \right]$$

holds.

Corollary 30. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) > 0$. For $|\xi(b-t)^{\mu}| < 1$, the following integral formula*

$$\left(I^{\sigma}_{b-} \frac{(b-t)^{\rho}}{1-\xi(b-t)^{\mu}}\right)(x) = (b-x)^{\sigma+\rho}\, {}_1\Psi_1 \left[\begin{array}{l} (\rho+1, \mu) \\ (\sigma+\rho+1, \mu) \end{array} \middle| \xi(b-x)^{\mu} \right].$$

holds.

Corollary 31. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $|\xi(t-a)^{\mu}| < 1$, the following integral formula*

$$\left(D^{\sigma}_{a+} \frac{(t-a)^{\rho}}{1-\xi(t-a)^{\mu}}\right)(x) = (x-a)^{\sigma+\rho}\, {}_1\Psi_1 \left[\begin{array}{l} (\rho+1, \mu) \\ (\rho-\sigma+1, \mu) \end{array} \middle| \xi(x-a)^{\mu} \right]$$

holds.

Corollary 32. *Let $\rho, \xi, \mu \in \mathbb{C}$ such that $\Re(\sigma) \geq 0$. For $|\xi(b-t)^{\mu}| < 1$, the following integral formula*

$$\left(D^{\sigma}_{b-} \frac{(b-t)^{\rho}}{1-\xi(b-t)^{\mu}}\right)(x) = (b-x)^{\sigma+\rho}\, {}_1\Psi_1 \left[\begin{array}{l} (\rho+1, \mu) \\ (\rho-\sigma+1, \mu) \end{array} \middle| \xi(b-x)^{\mu} \right]$$

holds.

4. Numerical Results and Graphical Interpretation

In this section, the numerical results of the formulae established in equations (3.1), (3.6), (3.7) and (3.12) are presented in Tables 1, 2, 3 and 4, respectively. The graphs of the formulae are plotted in Figures 1–7. All these numerical values are selected for $r = m = 2$. The product of Mittag–Leffler function for these values reduces to the form

$$\prod_{i=1}^{2} E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_2}(.) = E^{\gamma_1,q_1}_{(\alpha_{11},\beta_{11};\alpha_{12},\beta_{12})}(.)\, E^{\gamma_2,q_2}_{(\alpha_{21},\beta_{21};\alpha_{22},\beta_{22})}(.)$$

We select the values of parameters $\gamma_1 = 0.2$, $\gamma_2 = 0.3$, $q_1 = 0.05$, $q_2 = 0.06$; $\alpha_{11} = 0.3$, $\beta_{11} = 0.5$; $\alpha_{12} = 0.4$, $\beta_{12} = 0.6$; $\alpha_{21} = 0.5$, $\beta_{21} = 0.7$; $\alpha_{22} = 0.6$, $\beta_{22} = 0.8$; $\nu = 0.1$; $\xi = 0.5$; $\mu = 0.2$; in all Figures 1–7 and Tables 1–4, which are fixed for our investigation for generalized multi-index Mittag–Leffler function. It is also noted that imaginary part of complex numerical values of fractional integrals and fractional derivatives arguments are ignored in each case. The generalized multi-index Mittag–Leffler function $E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_m}(.)$ given in equation (1.4) is in summation form and all the results established in equations (3.1), (3.6), (3.7) and (3.12) involve the generalized multi-index Mittag–Leffler

TABLE 1. Numerical values of equation (3.1)

x	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$
0.00	7.38	-1.90	-12.39	-19.64
0.50	5.95	-1.21	-9.08	-14.33
1.00	4.68	-0.69	-6.40	-10.08
1.50	3.56	-0.32	-4.31	-6.77
2.00	2.59	-0.08	-2.72	-4.28
2.50	1.78	0.05	-1.58	-2.49
3.00	1.13	0.10	-0.81	-1.28
3.50	0.63	0.10	-0.34	-0.55
4.00	0.27	0.06	-0.10	-0.17
4.50	0.07	0.02	-0.01	-0.02
5.00	0.00	0.00	0.00	0.00
5.50	0.11	0.08	0.05	0.04
6.00	0.50	0.41	0.33	0.26
6.50	1.25	1.10	0.95	0.82
7.00	2.41	2.24	2.05	1.86
7.50	4.01	3.89	3.73	3.53
8.00	6.10	6.14	6.10	5.98
8.50	8.70	19.03	9.24	9.35
9.00	11.85	12.62	13.27	13.78
9.50	15.58	16.98	18.27	19.41
10.00	19.91	22.15	24.33	26.40

TABLE 2. Numerical values of the equation (3.6)

x	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$
0.00	19.91	22.15	24.33	26.40
0.50	15.58	16.98	18.27	19.41
1.00	11.85	12.62	13.27	13.78
1.50	8.70	9.03	9.24	9.35
2.00	6.10	6.14	6.10	5.98
2.50	4.01	3.89	3.73	3.53
3.00	2.41	2.24	2.05	1.86
3.50	1.25	1.10	0.95	0.82
4.00	0.50	0.41	0.33	0.26
4.50	0.11	0.08	0.05	0.04
5.00	0.00	0.00	0.00	0.00
5.50	0.07	0.02	-0.01	-0.02
6.00	0.27	0.06	-0.10	-0.17
6.50	0.63	0.10	-0.34	-0.55
7.00	1.13	0.10	-0.81	-1.28
7.50	1.78	0.05	-1.58	-2.49
8.00	2.59	-0.08	-2.72	-4.28
8.50	3.56	-0.32	-4.31	-6.77
9.00	4.68	-0.69	-6.40	-10.08
9.50	5.95	-1.21	-9.08	-14.33
10.00	7.38	-1.90	-12.39	-19.64

function. To establish the graphs and data-base of the results, we take a sum of the first 500 terms of the summation involved in each result.

In Figure 1, we have opted the parametric value as $\rho = .01$; $a = b = .5$ and $\sigma = 0.01 : 0.02 : 0.07$. $\rho = 1.8$; $a = b = 2$ and $\sigma = 0.1 : 0.2 : 0.7$ are opted for Figure 2. $\rho = 2.5$; $a = b = 2$ and $\sigma = 0.1 : 0.2 : 0.7$ are opted for Figure 3. Tables 1–2 are established on the basis of parametric values as those of Figure 4. Figures 5-7 are plotted for the values of the parameters as those of the values of the parameters of the Figures 1–3, which depict that the graphs of formulae (3.1) and (3.6) are reflected identically in each figure, respectively.

5. Image Formulas Associated With Integral Transform

In this section, we establish certain theorems involving the results obtained in previous section associated with the integral transforms like Beta transform, Laplace transform and Whittaker transform.

5.1. **Beta transform.** The Beta transform of $f(z)$ is defined as [30]

$$B\{f(z) : \alpha, \beta\} = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} f(z)dz,$$

Theorem 5. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}\,(i = 1, \ldots, r; j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}; \Re(q_i) > 0$ *and* $\Re(\sigma) > 0$. *For* $x > a$, *the following integral formula*

$$\begin{aligned} &B\left\{\left(I^{\sigma}_{a+}(t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_m}\left[\xi\big(z(t-a)\big)^{\mu}\right]\right)(x) : \alpha, \beta\right\} \\ &= \frac{\Gamma(\beta)(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\, {}_{r+2}\Psi_{(m+1)r+1}\left[\begin{array}{c} A_1 \\ B_1 \end{array}\middle| (\xi(x-a)^{\mu})^{r}\right], \end{aligned} \tag{5.1}$$

$$\begin{aligned} A_1 &= (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (\alpha, \mu r), \\ B_1 &= (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma+\rho+1, \mu r), (\alpha+\beta, \mu r), (-1, 1)_{1,r-1} \end{aligned} \tag{5.2}$$

holds.

Proof. For convenience, we denote the left-hand side of the result (5.1) by $\mathcal{B}$, then using the definition of beta transform, we have

$$\mathcal{B} = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\left(I^{\sigma}_{a+}(t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_m}\left[\xi\big(z(t-a)\big)^{\mu}\right]\right)(x)dz, \tag{5.3}$$

further applying the result from equation (3.5) to the above equation (5.3), then interchanging the order of integration and summation, we have

$$\begin{aligned} \mathcal{B} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)} &\left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1 + kq_1)\ldots\Gamma(\gamma_r + kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})} \right. \\ &\left. \times \frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}} \frac{(\xi(x-a)^{\mu})^{kr}}{k!} \right\} \int_0^1 z^{\alpha+\mu kr-1}(1-z)^{\beta-1}dz, \end{aligned}$$

applying the definition of beta transform and after little simplification, we have

$$\begin{aligned} \mathcal{B} = \frac{\Gamma(\beta)(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)} &\left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1 + kq_1)\ldots\Gamma(\gamma_r + kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})} \right. \\ &\left. \times \frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}} \frac{\Gamma(\alpha+\mu kr)}{\Gamma(\alpha+\beta+\mu kr)} \frac{(\xi(x-a)^{\mu})^{kr}}{k!} \right\}, \end{aligned}$$

now interpreting in view of (1.4), we have the required result (5.1). □

Theorem 6. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) > 0$. *For $b > x$, the following integral formula*

$$B\left\{\left(I_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i}\left[\xi(z(b-t))^{\mu}\right]\right)(x):\alpha,\beta\right\}$$
$$=\frac{\Gamma(\beta)(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\,{}_{r+2}\Psi_{(m+1)r+1}\left[\begin{array}{c}A_1\\B_1\end{array}\middle|(\xi(b-x)^{\mu})^{r}\right]$$

holds, where A_1 and B_1 are defined in equation (5.2).

Theorem 7. *Let $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For $x > a$, the following integral formula*

$$B\left\{\left(D_{a+}^{\sigma}(t-a)^{\rho}\prod_{i=1}^{r}E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i}\left[\xi(z(t-a))^{\mu}\right]\right)(x):\alpha,\beta\right\}$$
$$=\frac{\Gamma(\beta)(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\,{}_{r+2}\Psi_{(m+1)r+1}\left[\begin{array}{c}A_2\\B_2\end{array}\middle|(\xi(x-a)^{\mu})^{r}\right],$$
$$\begin{aligned}A_2 &= (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (\alpha, \mu r),\\ B_2 &= (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\rho-\sigma+1, \mu r), (\alpha+\beta, \mu r), (-1, 1)_{1,r-1}\end{aligned} \tag{5.4}$$

holds.

TABLE 3. Numerical values of the equation (3.7)

x	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$
0.00	8.84	-4.71	-36.63	-82.17
0.50	7.13	-3.17	-26.86	-59.87
1.00	5.60	-1.98	-18.98	-42.04
1.50	4.26	-1.11	-12.80	-28.17
2.00	3.11	-0.51	-8.11	-17.75
2.50	2.14	-0.14	-4.73	-10.29
3.00	1.35	0.05	-2.43	-5.29
3.50	0.75	0.11	-1.03	-2.25
4.00	0.33	0.08	-0.30	-0.68
4.50	0.08	0.03	-0.04	-0.09
5.00	0.00	0.00	0.00	0.00
5.50	0.13	0.14	0.14	0.14
6.00	0.61	0.73	0.86	0.99
6.50	1.52	1.98	2.53	3.18
7.00	2.93	4.04	5.49	7.31
7.50	4.89	7.06	10.02	13.99
8.00	7.44	11.15	16.43	23.80
8.50	10.63	16.43	24.99	37.36
9.00	14.48	23.01	35.97	55.26
9.50	19.05	31.00	49.64	78.12
10.00	24.35	40.49	66.25	106.53

TABLE 4. Numerical values of the equation (3.12)

x	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$
0.00	24.35	40.49	66.25	106.53
0.50	19.05	31.00	49.64	78.12
1.00	14.48	23.01	35.97	55.26
1.50	10.63	16.43	24.99	37.36
2.00	7.44	11.15	16.43	23.80
2.50	4.89	7.06	10.02	13.99
3.00	2.93	4.04	5.49	7.31
3.50	1.52	1.98	2.53	3.18
4.00	0.61	0.73	0.86	0.99
4.50	0.13	0.14	0.14	0.14
5.00	0.00	0.00	0.00	0.00
5.50	0.08	0.03	-0.04	-0.09
6.00	0.33	0.08	-0.30	-0.68
6.50	0.75	0.11	-1.03	-2.25
7.00	1.35	0.05	-2.43	-5.29
7.50	2.14	-0.14	-4.73	-10.29
8.00	3.11	-0.51	-8.11	-17.75
8.50	4.26	-1.11	-12.80	-28.17
9.00	5.60	-1.98	-18.98	-42.04
9.50	7.13	-3.17	-26.86	-59.87
10.00	8.84	-4.71	-36.63	-82.17

Theorem 8. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r; j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For* $b > x$, *the following integral formula*

$$B\left\{\left(D_{b-}^{\sigma}(b-t)^{\rho}\prod_{i=1}^{r}E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i,q_i}\left[\xi(z(b-t))^{\mu}\right]\right)(x):\alpha,\beta\right\}$$
$$= \frac{\Gamma(\beta)(b-x)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\,{}_{r+2}\Psi_{(m+1)r+1}\left[\begin{array}{c} A_2 \\ B_2 \end{array}\middle| (\xi(b-x)^{\mu})^{r}\right]$$

holds, where A_2 *and* B_2 *are defined in equation* (5.4).

Proof. The proof of Theorems 6, 7 and 8 is the same as that of Theorem 5, therefore we omit the details. □

5.2. **Laplace transform.** The Laplace transform of $f(z)$ is defined as [30]

$$L\{f(z)\} = \int_0^{\infty} e^{-sz} f(z)dz$$

Theorem 9. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$*;* $\Re(q_i) > 0$ *and* $\Re(\sigma) > 0$*. For* $x > a$*, the following integral formula*

$$L\Bigg\{z^{l-1}\Bigg(I^{\sigma}_{a+}(t-a)^{\rho}\prod_{i=1}^{r}E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_m}\Big[\xi(z(t-a))^{\mu}\Big]\Bigg)(x)\Bigg\}$$
$$= \frac{(x-a)^{\sigma+\rho}}{s^{l}\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\,{}_{r+2}\Psi_{mr+r}\left[\begin{array}{c} C_1 \\ D_1 \end{array}\Bigg|\left(\xi\Big(\frac{x-a}{s}\Big)^{\mu}\right)^{r}\right], \tag{5.5}$$

$$C_1 = (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (l, \mu r),$$
$$D_1 = (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma+\rho+1, \mu r), (-1, 1)_{1,r-1}$$

holds.

Proof. For convenience, we denote the left-hand side of the result (5.5) by $\mathcal{L}$, then using the definition of Laplace transform, we have:

$$\mathcal{L} = \int_0^{\infty} e^{-sz}z^{l-1}\Bigg(I^{\sigma}_{a+}(t-a)^{\rho}\prod_{i=1}^{r}E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_m}\Big[\xi(z(t-a))^{\mu}\Big]\Bigg)(x)dz, \tag{5.6}$$

further applying the result from equation (3.5) to the above equation (5.6), interchanging the order of integration and summation, we have

$$\mathcal{L} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\Bigg\{\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+kq_1)\ldots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}$$
$$\times\frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}}\frac{(\xi(x-a)^{\mu})^{kr}}{k!}\Bigg\}\int_0^{\infty}e^{-sz}z^{l+\mu kr-1}dz,$$

applying the definition of Laplace transform, after little simplification, we have

$$\mathcal{L} = \frac{(x-a)^{\sigma+\rho}}{s^{l}\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\Bigg\{\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+kq_1)\ldots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}$$
$$\times\frac{\Gamma(\rho+\mu kr+1)\Gamma(l+\mu kr)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}}\frac{1}{k!}\left(\xi\left(\frac{x-a}{s}\right)^{\mu}\right)^{kr}\Bigg\} \tag{5.7}$$

interpret the above equation (5.7) in the view of (1.4), we can easily arrive at the required result (5.5). ☐

Theorem 10. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$*;* $\Re(q_i) > 0$ *and* $\Re(\sigma) > 0$*. For* $b > x$*, the following integral formula*

$$L\Bigg\{z^{l-1}\Bigg(I^{\sigma}_{b-}(b-t)^{\rho}\prod_{i=1}^{r}E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_m}\Big[\xi(z(b-t))^{\mu}\Big]\Bigg)(x)\Bigg\} = \frac{(b-x)^{\sigma+\rho}}{s^{l}\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\,{}_{r+2}\Psi_{mr+r}$$
$$\times\left[\begin{array}{l} (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (l, \mu r) \\ (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma+\rho+1, \mu r), (-1, 1)_{1,r-1} \end{array}\Bigg|\left(\xi\left(\frac{b-x}{s}\right)^{\mu}\right)^{r}\right]$$

holds.

Theorem 11. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For* $x > a$, *the following integral formula*

$$L\left\{z^{l-1}\left(D^{\sigma}_{a+}(t-a)^{\rho}\prod_{i=1}^{r}E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_m}\left[\xi(z(t-a))^{\mu}\right]\right)(x)\right\} = \frac{(x-a)^{\sigma+\rho}}{s^l\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\,{}_{r+2}\Psi_{mr+r}$$
$$\times\left[\begin{matrix}(\gamma_1,q_1),\ldots,(\gamma_r,q_r),(\rho+1,\mu r),(l,\mu r)\\ (\beta_{1j},\alpha_{1j})_{1,m},\ldots,(\beta_{rj},\alpha_{rj})_{1,m},(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{matrix}\;\middle|\;\left(\xi\left(\frac{x-a}{s}\right)^{\mu}\right)^{r}\right]$$

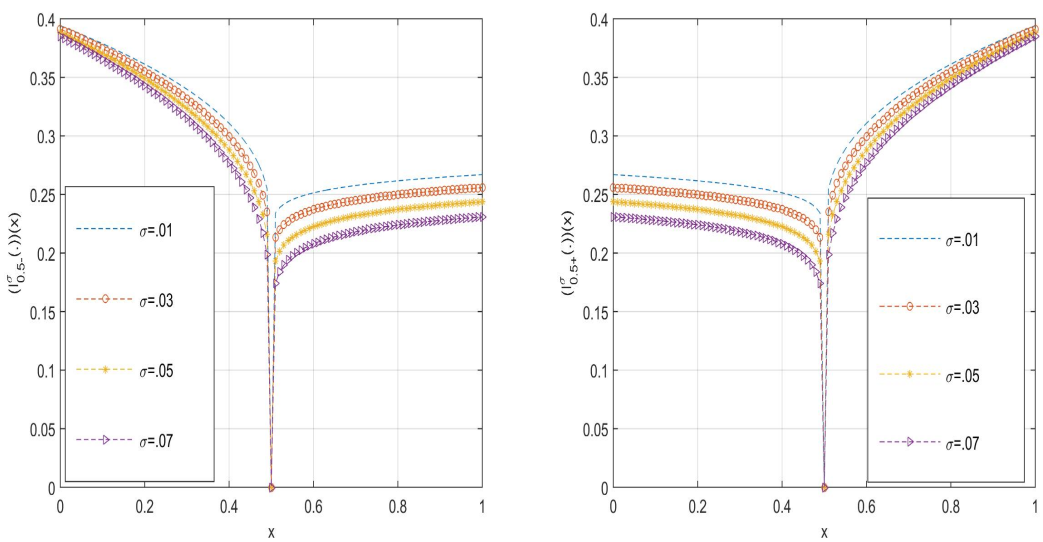
holds.

Theorem 12. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For* $b > x$, *the following integral formula*

$$L\left\{z^{l-1}\left(D^{\sigma}_{b+}(b-t)^{\rho}\prod_{i=1}^{r}E^{\gamma_i,q_i}_{(\alpha_{ij},\beta_{ij})_m}\left[\xi(z(b-t))^{\mu}\right]\right)(x)\right\} = \frac{(b-x)^{\sigma+\rho}}{s^l\Gamma(\gamma_1)\ldots\Gamma(\gamma_r)}\,{}_{r+2}\Psi_{mr+r}$$
$$\times\left[\begin{matrix}(\gamma_1,q_1),\ldots,(\gamma_r,q_r),(\rho+1,\mu r),(l,\mu r)\\ (\beta_{1j},\alpha_{1j})_{1,m},\ldots,(\beta_{rj},\alpha_{rj})_{1,m},(\rho-\sigma+1,\mu r),(-1,1)_{1,r-1}\end{matrix}\;\middle|\;\left(\xi\left(\frac{b-x}{s}\right)^{\mu}\right)^{r}\right]$$

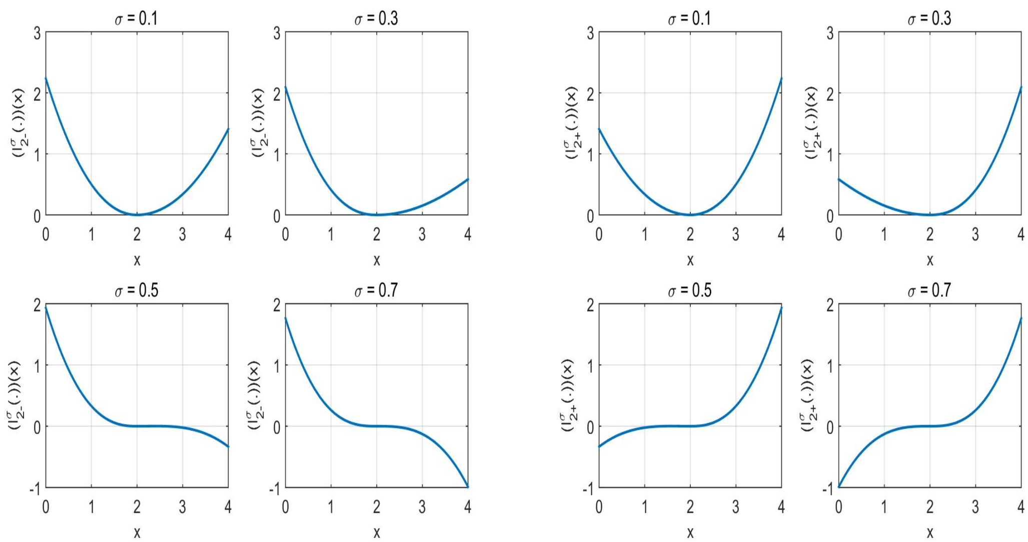
holds.

Proof. The proof of Theorems 10, 11 and 12 is the same as that of Theorem 9, therefore we omit the details. ☐



(a) Plot of Equation (3.6) (b) Plot of Equation (3.1)

FIGURE 1. Graph of fractional integral formulae for $\rho = .01$; $a = b = .5$



(a) Plot of Equation (3.6)

(b) Plot of Equation (3.1)

FIGURE 2. Graph of fractional integral formulae for $\rho = 1.8$; $a = b = 2$

5.3. **Whittaker transform.**

Theorem 13. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}; \Re(q_i) > 0$ *and* $\Re(\sigma) > 0$. *For* $x > a$, *the following integral formula*

$$\int_0^\infty z^{\zeta-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(I^{\sigma}_{a+}(t-a)^{\rho} \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_{ij}, \beta_{ij})_m} \left[\xi (z(t-a))^{\mu} \right] \right)(x) \right\}$$

$$= \frac{(x-a)^{\sigma+\rho}}{\eta^{\zeta} \Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} {}_{r+3}\Psi_{(m+1)r+1} \left[\begin{array}{c} A \\ B \end{array} \middle| \left(\xi \left(\frac{x-a}{\eta} \right)^{\mu} \right)^{r} \right] \tag{5.8}$$

holds, where

$$\begin{aligned} A &= (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (1/2 + \omega + \zeta, \mu r), (1/2 - \omega + \zeta, \mu r) \\ B &= (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\sigma + \rho + 1, \mu r), (1/2 - \tau + \zeta, \mu r), (-1, 1)_{1,r-1}. \end{aligned} \tag{5.9}$$

Proof. For convenience, we denote the left-hand side of the result (5.8) by $\mathcal{W}$, further applying the result from equation (3.5) to the above equation, interchanging the order of integration and summation, we have

$$\mathcal{W} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1 + kq_1) \ldots \Gamma(\gamma_r + kq_r)}{\prod_{j=1}^{m} \Gamma(k\alpha_{1j} + \beta_{1j}) \cdots \prod_{j=1}^{m} \Gamma(k\alpha_{rj} + \beta_{rj})} \right.$$

$$\left. \times \frac{\Gamma(\rho + \mu kr + 1)}{\Gamma(\sigma + \rho + \mu kr + 1)(\Gamma(k-1))^{r-1}} \frac{(\xi(x-a)^{\mu})^{kr}}{k!} \right\} \int_0^\infty z^{\zeta + \mu kr - 1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) dz,$$

by substituting $\eta z = t$, we have

$$\mathcal{W} = \frac{(x-a)^{\sigma+\rho}}{\Gamma(\gamma_1)\dots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+kq_1)\dots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$

$$\left.\times\frac{\Gamma(\rho+\mu kr+1)}{\Gamma(\sigma+\rho+\mu kr+1)(\Gamma(k-1))^{r-1}}\frac{(\xi(x-a)^{\mu})^{kr}}{k!}\right\}\frac{1}{\eta^{\zeta+\mu kr}}\int_0^{\infty} t^{\zeta+\mu kr-1}e^{-t/2}W_{\tau,\omega}(t)dt.$$

Now using the integral formula for Whittaker function

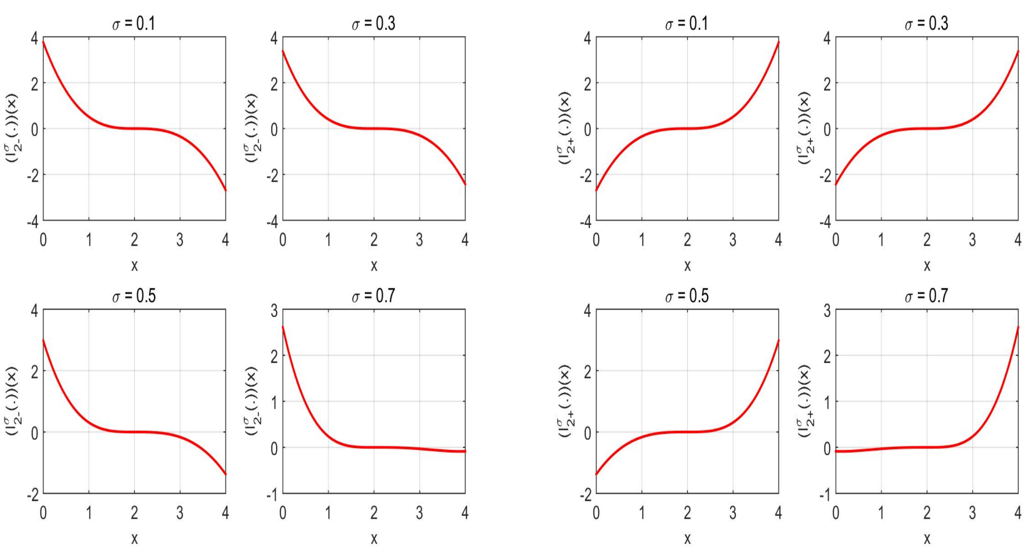
$$\int_0^{\infty} t^{\alpha-1}e^{-t/2}W_{\tau,\omega}(t)dt = \frac{\Gamma\left(1/2+\omega+\alpha\right)\Gamma\left(1/2-\omega+\alpha\right)}{\Gamma\left(1/2-\tau+\alpha\right)}\quad\left(\Re(\alpha\pm\omega)>\frac{-1}{2}\right)$$

and after little simplification, we have

$$\mathcal{W} = \frac{(x-a)^{\sigma+\rho}}{\eta^{\zeta}\Gamma(\gamma_1)\dots\Gamma(\gamma_r)}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\gamma_1+kq_1)\dots\Gamma(\gamma_r+kq_r)}{\prod_{j=1}^{m}\Gamma(k\alpha_{1j}+\beta_{1j})\cdots\prod_{j=1}^{m}\Gamma(k\alpha_{rj}+\beta_{rj})}\right.$$

$$\left.\times\frac{\Gamma(\rho+\mu kr+1)\Gamma\left(1/2+\omega+\zeta+\mu kr\right)\Gamma\left(1/2-\omega+\zeta+\mu kr\right)}{\Gamma(\sigma+\rho+\mu kr+1)\Gamma\left(1/2-\tau+\zeta+\mu kr\right)(\Gamma(k-1))^{r-1}}\frac{1}{k!}\left(\xi\left(\frac{x-a}{\eta}\right)^{\mu}\right)^{kr}\right\} \tag{5.10}$$

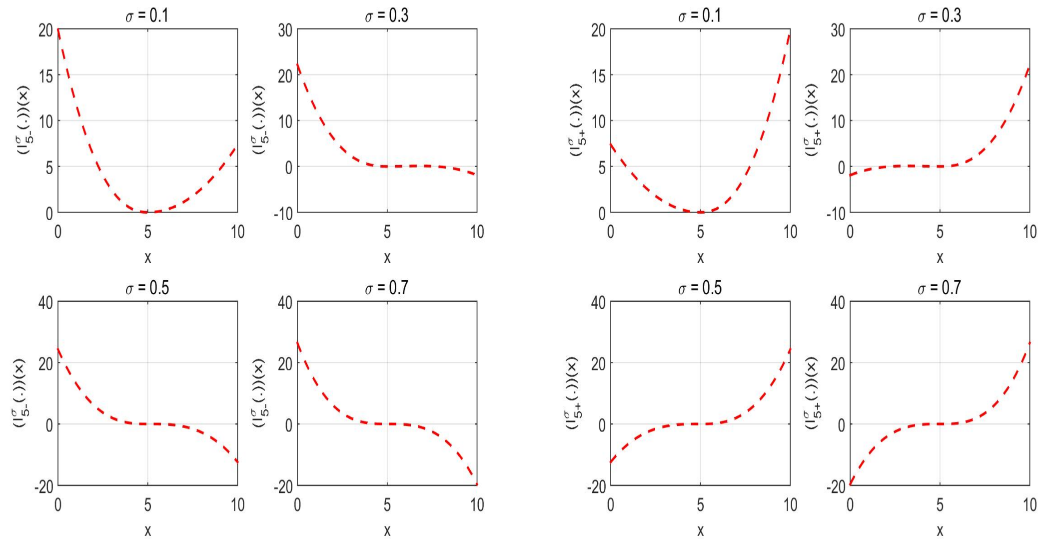
interpret the above equation (5.10) in the view of (1.4), we can easily arrive at the required result (5.5). □



(a) Plot of Equation (3.6)

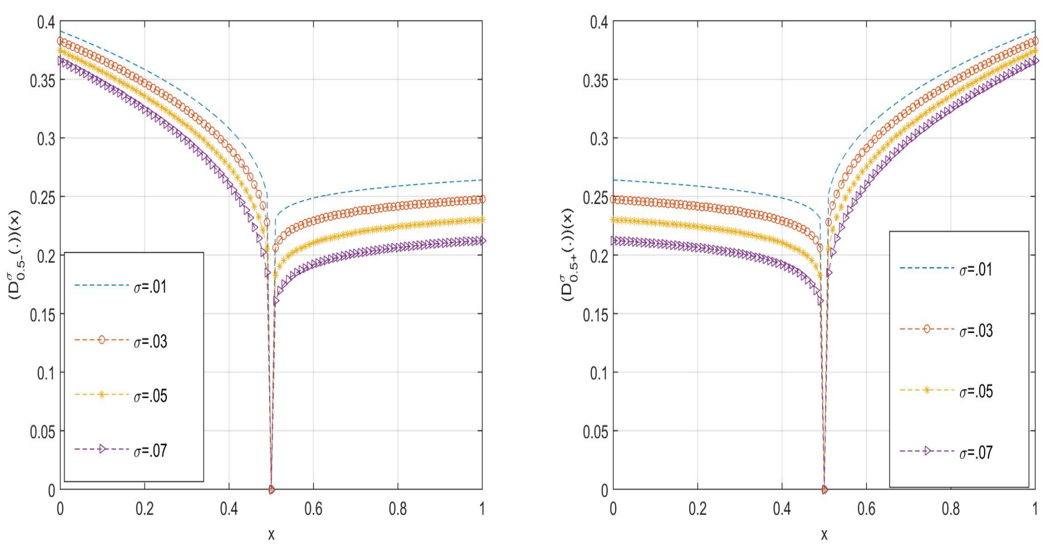
(b) Plot of Equation (3.1)

FIGURE 3. Graph of fractional integral formulae for $\rho = 2.5$; $a = b = 2$



(a) Plot of Equation (3.6) (b) Plot of Equation (3.1)

FIGURE 4. Graph of fractional integral formulae for $\rho = 1.9$; $a = b = 5$



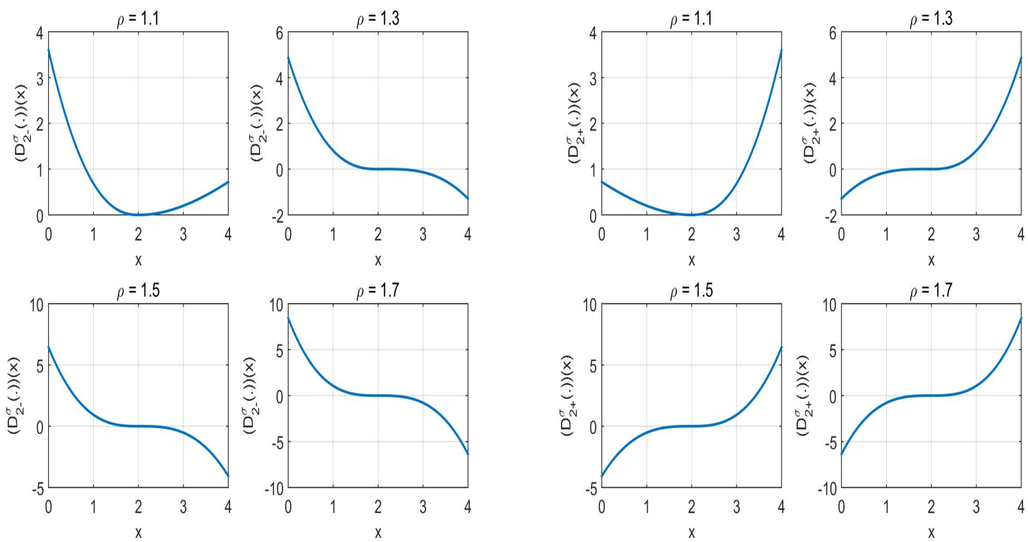
(a) Plot of Equation (3.12) (b) Plot of Equation (3.7)

FIGURE 5. Graph of fractional derivative formulae for $\rho = .01$; $a = b = 0.5$

Theorem 14. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) > 0$. *For* $b > x$, *the following integral formula*

$$\int_0^\infty z^{\zeta-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(I_{b-}^{\sigma} (b-t)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i, q_i} \left[\xi (z(b-t))^{\mu} \right] \right)(x) \right\}$$
$$= \frac{(b-x)^{\sigma+\rho}}{\eta^{\zeta} \Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} \, {}_{r+3}\Psi_{(m+1)r+1} \left[\begin{array}{c} A \\ B \end{array} \middle| \left(\xi \left(\frac{b-x}{\eta} \right)^{\mu} \right)^{r} \right]$$

holds, where A *and* B *are defined in equation* (5.9).



(a) Plot of Equation (3.12)

(b) Plot of Equation (3.7)

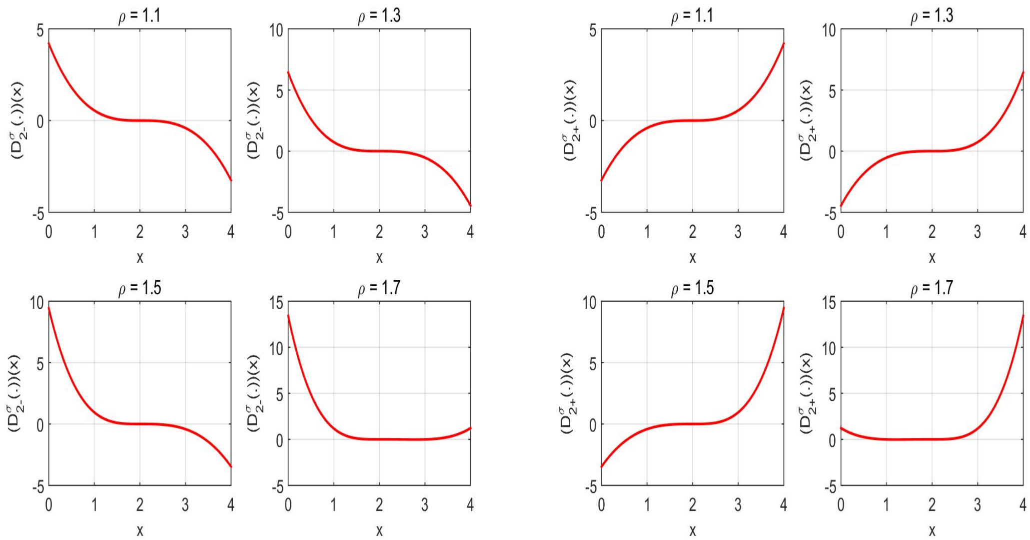
FIGURE 6. Graph of fractional derivative formulae for $\sigma = 1$; $a = b = 2$

Theorem 15. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \ldots, r;\ j = 1, \ldots, m)$ *such that* $\sum_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For* $x > a$, *the following integral formula*

$$\int_0^\infty z^{\zeta-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(D_{a+}^{\sigma} (t-a)^{\rho} \prod_{i=1}^{r} E_{(\alpha_{ij},\beta_{ij})_m}^{\gamma_i, q_i} \left[\xi (z(t-a))^{\mu} \right] \right)(x) \right\}$$
$$= \frac{(x-a)^{\sigma+\rho}}{\eta^{\zeta} \Gamma(\gamma_1) \ldots \Gamma(\gamma_r)} \, {}_{r+3}\Psi_{(m+1)r+1} \left[\begin{array}{c} C \\ D \end{array} \middle| \left(\xi \left(\frac{x-a}{\eta} \right)^{\mu} \right)^{r} \right]$$

holds, where

$$\begin{aligned} C &= (\gamma_1, q_1), \ldots, (\gamma_r, q_r), (\rho+1, \mu r), (1/2+\omega+\zeta, \mu r), (1/2-\omega+\zeta, \mu r) \\ D &= (\beta_{1j}, \alpha_{1j})_{1,m}, \ldots, (\beta_{rj}, \alpha_{rj})_{1,m}, (\rho-\sigma+1, \mu r), (1/2-\tau+\zeta, \mu r), (-1, 1)_{1,r-1} \end{aligned} \tag{5.11}$$



(a) Plot of Equation (3.12) (b) Plot of Equation (3.7)

FIGURE 7. Graph of fractional derivative formulae for $\sigma = 1.5$; $a = b = 2$

Theorem 16. *Let* $m, r \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij}, \gamma_i, q_i, \rho, \xi, \mu \in \mathbb{C}$ $(i = 1, \dots, r;\ j = 1, \dots, m)$ *such that* $\sum\limits_{j=1}^{m} \Re(\alpha_{ij}) > \max\{0; \Re(q_i) - 1\}$; $\Re(q_i) > 0$ *and* $\Re(\sigma) \geq 0$. *For* $b > x$, *the following integral formula*

$$\int\limits_0^\infty z^{\zeta-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(D^\sigma_{b+} (b-t)^\rho \prod_{i=1}^{r} E^{\gamma_i, q_i}_{(\alpha_{ij}, \beta_{ij})_m} \left[\xi (z(b-t))^\mu \right] \right)(x) \right\}$$
$$= \frac{(b-x)^{\sigma+\rho}}{\eta^\zeta \Gamma(\gamma_1) \dots \Gamma(\gamma_r)} {}_{r+3}\Psi_{(m+1)r+1} \left[\begin{array}{c} C \\ D \end{array} \middle| \left(\xi \left(\frac{b-x}{\eta} \right)^\mu \right)^r \right]$$

holds, where C *and* D *are defined in equation* (5.11).

Proof. The proof of Theorems 14, 15 and 16 is the same as that of Theorem 13, therefore we omit the details. □

6. CONCLUSION

In this paper, we have established some image formulas by applying the Riemann–Liouville fractional derivative and integral operators on the product of generalized multi-index Mittag–Leffler function $E^{\gamma,q}_{(\alpha_j,\beta_j)_m}(.)$. Then, some more image formulas are derived by employing the integral transform. To study the nature of these formulae, numerical results and their graphs are plotted for different values of the parameters involved in our main results, which can be simply interpreted and observed. For the numerical results and the graphs, the author has chosen $r = m = 2$. Further, for more investigation of these formulas the reader can choose any value of r and m.

REFERENCES

1. P. Agarwal, M. Chand, G. Singh, Certain fractional kinetic equations involving the product of generalized k-Bessel function. *Alexandria Engineering Journal* **55** (2016), no. 4, 3053–3059.

2. P. Agarwal, S. Jain, M. Chand, S. K. Dwivedi, S. Kumar, Bessel functions associated with Saigo-Maeda fractional derivative operators. *J. Fract. Calc. Appl.* **5** (2014), no. 2, 96–106.
3. P. Agarwal, S. K. Ntouyas, S. Jain, M. Chand, G. Singh, Fractional kinetic equations involving generalized k-Bessel function via Sumudu transform. *Alexandria Engineering Journal* **57** (2017), 1–6. http://dx.doi.org/10.1016/j.aej.2017.03.046
4. P. Agarwal, S. V. Rogosin, E. T. Karimov, M. Chand, Generalized fractional integral operators and the multivariable H-function. *J. Inequal. Appl.* **2015**, 2015:350, 17 pp. DOI 10.1186/s13660-015-0878-y
5. M. A. Al-Bassam, Y. F. Luchko, On generalized fractional calculus and it application to the solution of integro-differential equations. *J. Fract. Calc.***7** (1995), 69–88.
6. M. Chand, P. Agarwal, Z. Hammouch, Certain sequences involving product of k-Bessel function. *Int. J. Appl. Comput. Math.* **4** (2018), no. 4, Art. 101, 9 pp. https://doi.org/10.1007/s40819-018-0532-8
7. M. Chand, P. Agarwal, S. Jain, G. Wang, K. S. Nisar, Image formulas and graphical interpretation of fractional derivatives of R-function and G-function. *Advanced Studies in Contemporary Mathematics* **26** (2016), no. 4, 633–652.
8. M. Chand, J. C. Prajapati, E. Bonyah, Fractional integrals and solution of fractional kinetic equations involving k-Mittag-Leffler function. *Trans. A. Razmadze Math. Inst.* **171** (2017), no. 2, 144–166. http://dx.doi.org/10.1016/j.trmi.2017.03.003
9. J. Choi, P. Agarwal, S. Mathur, S. D. Purohit, Certain new integral formulas involving the generalized Bessel functions. *Bull. Korean Math. Soc.* **51** (2014), no. 4, 995–1003.
10. M. M. Dzherbashian, On integral transforms generated by the generalized Mittag–Leffler function. *Izv. Akad. Nauk Armjan. SSR* **13** (1960), no. 3, 21–63.
11. R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order. *Fractals and fractional calculus in continuum mechanics (Udine, 1996)*, 223–276, CISM Courses and Lect., 378, *Springer, Vienna*, 1997.
12. H. J. Haubold, A. M. Mathai, R. K. Saxena, Mittag–Leffler functions and their applications. *J. Appl. Math.* 2011, Art. ID 298628, 51 pp.
13. N. U. Khan, M. Ghayasuddin, A. Khan Waseem, Zia Sarvat, Certain unified integral involving generalized Bessel–Maitland function. *South East Asian J. Math. Math. Sci.* **11** (2015), no. 2, 27–35.
14. N. U. Khan, T. Kashmin, Some integrals for the generalized Bessel Maitland functions. *Electron. J. Math. Anal. Appl.* **4** (2016), no. 2, 139–149.
15. N. U. Khan, S. W. Khan, M. Ghayasuddin, Some new results associated with the Bessel-Struve kernel function. *Acta Univ. Apulensis Math. Inform.* no. 48, (2016), 89–101.
16. A. A. Kilbas, Multi-parametric Mittag–Leffler functions and their extension. *Fractional Calculus* & *Applied Analysis* **16** (2015), no. 2, 378–404.
17. V. Kiryakova, All the special functions are fractional differintegrals of elementary functions. *J. Phys A* **30** (1977), 5085–5103.
18. V. Kiryakova, Multiindex Mittag–Leffler functions, related Gelfond-Leontiev operators and Laplace type integral transforms. *Fractional Calculus* & *Applied Analysis* **2** (1999), no. 4, 445–462.
19. V. Kiryakova, Multiple (multiindex) Mittag–Leffler functions and relations to generalized fractional calculus. *Journal of Computational and Applied Mathematics* **118** (2000), 241–259.
20. V. Kiryakova, Some special functions related to fractional calculus and fractional (noninteger) order control systems and equations. *Facta Universitatis (Sci. J. of University of Nis*, Series: Automatic Control and Robotics **7** (2008), no. 1, 79–98.
21. V. Kiryakova, The multi-index Mittag–Leffler function as an important class of special functions of fractional calculus. *Comp. Math. Appl.* **59** (2010), no. 5, 1885–1895.
22. V. Kiryakova, The special functions of fractional calculus as generalized fractional calculus operators of some basic functions. *Comput. Math. Appl.* **59** (2010), no. 3, 1128–1141.
23. O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions.* Theory and algorithmic tables. Edited by F. D. Gakhov. Translated from the Russian by L. W. Longdon. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; John Wiley & Sons, Inc., New York, 1983.
24. K. S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations. *Wiley, New York*, 1933.
25. T. R. Prabhakar, A singular integral equation with a generalized Mittag Leffler function in the kernel. *Yokohama Math. J.* **19** (1971), 7–15.
26. S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives Theory and Applications. *Gordan and Breach Science Publishers, Switzerland*, 1993.
27. R. K. Saxena, K. Nishimoto, N-Fractional Calculus of Generalized Mittag–Leffler functions. *J. Fract. Calc.* **37** (2010), 43–52.
28. R. K. Saxena, T. K. Pogoany, J. Ram, J. Daiya, Dirichlet Averages of Generalized Multiindex Mittag–Leffler functions. *Armenian Journal of Mathematics* **3** (2010), no. 4, 174–187.
29. A. K. Shukla, J. C. Prajapati, On a generalization of Mittag–Leffler function and its properties. *J. Math. Anal. Appl.* **336** (2007), 797–811.
30. I. N. Sneddon, *The Use of Integral Transforms.* Tata McGraw-Hill, New Delhi, 1979.

31. H. M. Srivastava, P. W. Karlsson, Multiple Gaussian Hypergeometric Series. *Halsted Press (Ellis Horwood Limited, Chichester, Wiley, New York, Chichester, Brisbane and Toronto*, 1985.
32. H. M. Srivastava, S. D. Lin, P. Y. Wang, Some fractional-calculus results for the H-function associated with a class of Feynman integrals. *Russ J. Math Phys* **13** (2006), 94–100.
33. H. M. Srivastava, Ẑ. Tomovski, Fractional calculus with an integral operator containing generalized Mittag–Leffler function in the kernel. *Appl. Math. Comput.* **211** 2009, no. 1, 198–210.
34. A. Wiman, Über den Fundamentalsatz in der Teorie der Funktionen $E^{a}(x)$. (German) *Acta Math.* **29** (1905), no. 1, 191–201.

(Received 30.04.2018)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 31–36

ON A PARAMETRIZATION OF NON-COMPACT WAVELET MATRICES BY WIENER-HOPF FACTORIZATION

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Abstract. A complete parametrization (one-to-one and onto mapping) of a certain class of non-compact wavelet matrices is introduced in terms of coordinates of infinite-dimensional Euclidian space. The developed method relies on Wiener-Hopf factorization of corresponding unitary matrix functions.

1. Introduction

Let $l^2(\mathbb{Z})$ be the standard Hilbert space of two-sided sequences of complex numbers. A matrix $\mathcal{A}$ with m rows and infinitely many columns

$$\mathcal{A}=\begin{pmatrix} \cdots & a^1_{-1} & a^1_0 & a^1_1 & a^1_2 & \cdots \\ \cdots & a^2_{-1} & a^2_0 & a^2_1 & a^2_2 & \cdots \\ & \vdots & \vdots & & & \\ \cdots & a^m_{-1} & a^m_0 & a^m_1 & a^m_2 & \cdots \end{pmatrix}, \quad a^i_j\in\mathbb{C}, \tag{1}$$

where the rows belong to $l^2(\mathbb{Z})$, is called a wavelet matrix (of rank m) if its rows satisfy the so called *shifted orthogonality condition* [4]:

$$\sum_{k=-\infty}^{\infty} a^i_{k+mj}\,\overline{a^r_{k+ms}}=\delta_{ir}\delta_{js} \quad \text{for all} \quad 1\le i,r\le m;\;\; j,s\in\mathbb{Z} \tag{2}$$

(δ stands for the Kronecker delta). Such matrices are a generalization of ordinary $m\times m$ unitary matrices and they play the crucial role in the theory of wavelets [6] and multirate filter banks [7]. Note that if $\mathcal{A}$ is a wavelet matrix and $\mathcal{A}'$ is obtained by shifting some of its rows by a multiple of m, then $\mathcal{A}'$ is a wavelet matrix as well.

In the *polyphase representation* [8] of matrix $\mathcal{A}$,

$$\mathbf{A}(z)=\sum_{k=-\infty}^{\infty} A_k z^k\,, \tag{3}$$

where $\mathcal{A}=(\dots\ A_{-1}\ A_0\ A_1\ A_2\ \dots)$ is the partition of $\mathcal{A}$ into $m\times m$ blocks $A_k=(a^i_{km+j})$, $1\le i\le m$, $0\le j\le m-1$, condition (2) is equivalent to

$$\mathbf{A}(z)\widetilde{\mathbf{A}}(z)=I_m, \tag{4}$$

where $\widetilde{\mathbf{A}}(z)=\sum_{k=-\infty}^{\infty}A_k^* z^{-k}$ is the *adjoint* of $\mathbf{A}(z)$ ($A^*:=\overline{A}^T$ is the Hermitian conjugate, and I_m stands for the $m\times m$ unit matrix). This is easy to see as (2) can be written in the block matrix form $\sum_{k=-\infty}^{\infty}A_kA^*_{l+k}=\delta_{l0}I_m$.

On the other hand, if series (3) is convergent a.e. on $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$, condition (4) means that $\mathbf{A}$ is a unitary matrix function on the unit circle, i.e.,

$$\mathbf{A}(z)\big(\mathbf{A}(z)\big)^*=I_m \;\text{ for }\; z\in\mathbb{T}. \tag{5}$$

2010 *Mathematics Subject Classification.* 42C40, 47A68.
Key words and phrases. Wavelet matrices; Unitary matrix functions; Wiener-Hopf factorization.

Therefore, wavelet matrices are closely related with unitary matrix functions. There is a natural one-to-one correspondence between them and we will rely on this connection throughout the paper.

Our notion of a wavelet matrix is somewhat different from the standard one. Namely, the *linear condition* $\mathbf{A}(1)\,\mathbf{e} = \sqrt{m}\,\mathbf{e}_1$, where $\mathbf{e} = (1,1,\dots,1)^T$ and $\mathbf{e}_1 = (1,0,\dots,0)^T$, must be satisfied in the usual definition (see [6, Eq. 4.9]) in order the corresponding orthogonal basis of $L^2(\mathbb{R})$ can be constructed by means of $\mathbf{A}$ (see [6, Ch-s 4, 5]). In our consideration, the linear condition is irrelevant. Furthermore, since the structure of coefficients of unitary matrix functions $\mathbf{A}(z)$ and $\mathbf{A}(z)\cdot U$, where U is a constant unitary matrix, are closely related, we introduce the equivalent classes of wavelet matrices as follows:

$$\mathcal{A} \sim \mathcal{A}' \iff A_j = A'_j U \text{ for some constant unitary matrix } U \text{ and every } j \in \mathbb{Z}. \tag{6}$$

We get a unique representative with a corresponding linear condition in each class in this way.

If the number of non-zero columns in (1) is finite, then the wavelet matrix $\mathcal{A}$ is called compact. Otherwise, it is non-compact.

For a compact wavelet matrix

$$\mathbf{A}(z) = \sum_{k=0}^{N} A_k z^k\,, \tag{7}$$

in order to avoid a chaotic rearrangement of the rows of $\mathcal{A}$, we assume that not only $A_0 \neq 0$ and $A_N \neq 0$ (N is called the *order* of (7) in this case) but also

$$\det \mathbf{A}(z) = cz^N. \tag{8}$$

Since it follows from (5) that $\det \mathbf{A}(z)$ is a monomial for compact wavelet matrices, it has necessarily form (8) and the power of z is called the *degree* of (7). It is proved in [1] that the degree of (7) is N if and only if $\operatorname{rank} A_0 = m-1$ (see Lemma 1 therein). This is the maximal possible value for the rank of A_0 and such situation is naturally called nonsingular.

In [1], a complete parametrization (one-to-one and onto mapping) of compact wavelet matrices of rank m and of order and degree N, with a minor restriction that the last row of A_N is not all zeros (this set is denoted by $\mathcal{CWM}_1[m,N,N]$), is proposed in terms of coordinates in the Euclidian space $\mathbb{C}^{(m-1)N}$. Namely, we have

$$\mathcal{CWM}_1[m,N,N] \longleftrightarrow \underbrace{\mathcal{P}_N^- \times \mathcal{P}_N^- \times \cdots \times \mathcal{P}_N^-}_{m-1} \cong \underbrace{\mathbb{C}^N \times \mathbb{C}^N \times \cdots \times \mathbb{C}^N}_{m-1} \tag{9}$$

in the following sence: For each $\mathbf{A} \in \mathcal{CWM}_1[m,N,N]$ there exists a unique Laurent matrix polynomial $F(z)$ of the form

$$F(z) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1^-(z) & \zeta_2^-(z) & \zeta_3^-(z) & \cdots & \zeta_{m-1}^-(z) & 1 \end{pmatrix}, \tag{10}$$

where $\zeta_i^-(z) \in \mathcal{P}_N^-$, $j = 1,2,\dots,m-1$, such that

$$F(z)U(z) \in \mathcal{P}_N^+(m \times m),$$

where

$$U(z) = \operatorname{diag}[1,\dots,1,z^{-N}]\mathbf{A}(z) \tag{11}$$

(the last row of $\mathcal{A}$ is shifted to the left by mN), and

$$\mathcal{P}_N^+ := \left\{ \sum_{k=0}^{N} c_k z^k : c_k \in \mathbb{C},\ k = 0,\dots,N \right\}; \quad \mathcal{P}_N^- := \left\{ \sum_{k=1}^{N} c_k z^{-k} : c_k \in \mathbb{C},\ k = 1,\dots,N \right\}.$$

In other words

$$U(z) = U_-(z)U_+(z),$$

where

$$U_-(z) = F^{-1}(z) \text{ and } U_+(z) = F(z)U(z),$$

is the (right) Wiener-Hopf factorization of U. Note that F^{-1} can be obtained from F if we replace each ζ_i^- in (10) by $-\zeta_i^-$.

It readily follows from (11) and properties of $\mathbf{A}$ that the unitary Laurent matrix polynomial U has the following properties:

$$\det U(z) = \text{Const}, \quad \text{and} \quad \sum_{j=1}^{m} |u_{mj}(0)| > 0.$$

In the present paper, we are going to extend parametrization (9) to a certain class of non-compact wavelet matrices by letting $N \to \infty$ in the above formulations. To this end, we introduce some additional definitions.

Let $L_p^+ = H_p$, where $0 < p \le \infty$, be the Hardy space of analytic functions (we usually identify analytic functions in the unit disk and their boundary values on $\mathbb{T}$) and $L_p^- := \{f : \overline{f} \in L_p^+\}$ be the corresponding set of anti-analytic functions. Denote also

$$L^\pm := \bigcap_{0<p<\infty} L_p^\pm.$$

Obviously, both of the sets L^+ and L^- are closed under multiplication:

$$f, g \in L^\pm \Longrightarrow fg \in L^\pm. \tag{12}$$

Let $\mathcal{WM}^\pm[m]$ be the set of equivalent classes (see (6)) of wavelet matrices (1) with $a_j^i = 0$ for $i = 1, 2, \ldots, m-1$ and $j < 0$ or $i = m$ and $j \ge m$ (i.e., the entries in the first $m-1$ rows in the polyphase representation (3) are from L_∞^+ and the entries in the last row are from L_∞^-) such that

$$\det \mathbf{A}(z) = \text{Const} \quad \text{for a.a.} \quad z \in \mathbb{T}, \tag{13}$$

and the analytic functions $f_j(z) := \widetilde{\mathbf{A}}_{m,j}(z) = \sum_{k=0}^{\infty} \overline{a_{j-1-mk}^m} z^k$, $j = 1, 2, \ldots, m$ (the adjoints of the entries in the last row of $\mathbf{A}(z)$) are not simultaneously equal to 0 in the space of maximal ideals of H_∞, i.e.,

$$\sum_{j=0}^{m} |f_j(z)| > \delta, \quad |z| < 1, \quad \text{for some} \quad \delta > 0;$$

and let $\mathcal{P}_\infty^-$ be the projection of L_∞ on the set of anti-analytic functions vanishing at the infinity, i.e.,

$$\mathcal{P}_\infty^- := \left\{ \sum_{k=-\infty}^{-1} c_k t^k : \text{ there exist } f \in L_\infty \text{ such that } \hat{f}(k) = c_k \text{ for } k < 0 \right\} \subset L^-,$$

where $\hat{f}(k)$ stands for the k-th Fourier coefficient of f. Then we have a ono-to-one and onto mapping similar to (9):

$$\mathcal{WM}^\pm[m] \longleftrightarrow \underbrace{\mathcal{P}_\infty^- \times \mathcal{P}_\infty^- \times \cdots \times \mathcal{P}_\infty^-}_{m-1},$$

which is the claim of the following

Theorem 1. *Let $\mathcal{A} = \mathbf{A}(z) \in \mathcal{WM}^\pm[m]$. Then there exists a unique matrix function $F(z)$ of the form* (10), *where*

$$\zeta_i^- \in \mathcal{P}_\infty^-, \tag{14}$$

$j = 1, 2, \ldots, m-1$, *such that*

$$F(z)\mathbf{A}(z) \in L^+(m \times m). \tag{15}$$

Conversely, for each matrix function (10), (14) *there exists a unique $\mathbf{A}(z) \in \mathcal{WM}^\pm[m]$ such that* (15) *holds.*

The inclusion (15) means again that the representation

$$\mathbf{A}(z)=\mathbf{A}_-(z)\mathbf{A}_+(z),$$

where

$$\mathbf{A}_-(z)=F^{-1}(z) \ \text{ and } \ \mathbf{A}_+(z)=F(z)\mathbf{A}(z),$$

is the (right) Wiener-Hopf factorization of $\mathbf{A}(z)$.

2. Proof of Theorem 1

Proof of Theorem 1 is based on the technique developed in [2].

Since $\mathbf{A}(z)\in L_\infty(m\times m)$ is a unitary matrix function, we have

$$\mathbf{A}^{-1}(z)=\mathbf{A}^*(z) \ \text{ a.e. on } \mathbb{T}. \tag{16}$$

Because of the Carleson Corona Theorem (see, e.g. [5]) there exist functions $g_1,g_2,\dots,g_m$ from H_∞ such that

$$\sum_{j=1}^m f_j(z)g_j(z)=1 \quad \text{for} \quad |z|<1. \tag{17}$$

Let $\mathbf{B}\in L_\infty^+(m\times m)$ be the matrix function $\mathbf{A}$ with its last row replaced by $(g_1,g_2,\dots,g_m)$. Then, since the last column of $\mathbf{A}$ is $(f_1,f_2,\dots,f_m)^T$ and (16), (17) hold, we have

$$\mathbf{B}\mathbf{A}^*=\begin{pmatrix}1&0&\cdots&0&0\\0&1&\cdots&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots\\0&0&\cdots&1&0\\ \zeta_1&\zeta_2&\cdots&\zeta_{m-1}&1\end{pmatrix}=:\Phi\in L_\infty(m\times m),$$

where $\zeta_i=\sum_{k=1}^m g_k\widetilde{\mathbf{A}}_{ik}$. Thus, it follows from (16) that

$$\Phi\mathbf{A}=\mathbf{B}. \tag{18}$$

Let

$$\zeta_i=\zeta_i^+ +\zeta_i^-, \ \text{ where } \ \zeta^\pm\in\mathcal{P}_\infty^\pm, \ \ i=1,2,\dots,m-1 \tag{19}$$

(the definition of $\mathcal{P}_\infty^+$ and the inclusion $\mathcal{P}_\infty^+\subset L^+$ are obvious). Then

$$\Phi=\Phi^+\Phi^-, \tag{20}$$

where $\Phi^\pm\in\mathcal{P}^\pm$ is the matrix Φ whith its last row replaced by $(\zeta_1^\pm,\zeta_2^\pm,\dots,\zeta_{m-1}^\pm,1)$. The equations (18) and (20) imply that

$$\Phi^-\mathbf{A}=(\Phi^+)^{-1}\mathbf{B}\in L^+(m\times m), \tag{21}$$

which proves (15) if we observe that $F(z)=\Phi^-(z)$ and $(\Phi^+)^{-1}$ is the matrix Φ^+ whith its last row replaced by $(-\zeta_1^+,-\zeta_2^+,\dots,-\zeta_{m-1}^+,1)$.

Let us now prove the uniqueness of F.

Assume

$$F_i(z)\mathbf{A}(z)=\Phi_i^+(z)\in L^+(m\times m), \quad i=1,2, \tag{22}$$

are two representations of type (10), (14), where $F_1=F$ and F_2 is the matrix F with its last row replaced by $(\zeta_1',\zeta_2',\dots,\zeta_{m-1}',1)$.

Since $\Phi_i^+\in L^+(m\times m)\Longrightarrow \det\Phi_i^+\in L^+$ (see (12)) and $\det\Phi_i^+(z)=C$ a.e. on $\mathbb{T}$ (see (13), (22)), it follows that $\det\Phi_i^+(z)=C$ for $|z|<1$. Therefore $(\Phi_i^+(z))^{-1}\in L^+(m\times m)$ because of Cramer's formula.

Equations in (22) imply that

$$\mathcal{P}_\infty^-(m\times m)\ni F_2^{-1}(z)F_1(z)=(\Phi_2^+(z))^{-1}\Phi_1^+(z)\in L^+(m\times m).$$

Hence the matrix function $F_2^{-1}F_1$ is constant, while it has form (10) whith its last row replaced by $(\zeta_1^- -\zeta_1',\zeta_2^- -\zeta_2',\dots,\zeta_{m-1}^- -\zeta_{m-1}',1)$. Consequently

$$\zeta_i^-=\zeta_i' \quad \text{for} \quad i=1,2,\dots,m-1.$$

Let us now show the converse part of Theorem 1. The essential part of the claim is proved in [3, Lemma 4]: For each matrix of form (10), where $\zeta_i^- \in L_2^-$, $i = 1, 2, \dots, m-1$, there exists a unique (up to a constant right factor) unitary matrix function

$$U(t) = \begin{pmatrix} u_{11}^+(t) & u_{12}^+(t) & \cdots & u_{1,m-1}^+(t) & u_{1m}^+(t) \\ u_{21}^+(t) & u_{22}^+(t) & \cdots & u_{2,m-1}^+(t) & u_{2m}^+(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}^+(t) & u_{m-1,2}^+(t) & \cdots & u_{m-1,m-1}^+(t) & u_{m-1,m}^+(t) \\ \overline{u_{m1}^+(t)} & \overline{u_{m2}^+(t)} & \cdots & \overline{u_{m,m-1}^+(t)} & \overline{u_{mm}^+(t)} \end{pmatrix}, \quad u_{ij}^+ \in L_\infty^+,$$

with constant determinant

$$\det U(t) = \text{Const} \quad \text{for a.a.} \quad t \in \mathbb{T}, \tag{23}$$

such that

$$F(t)U(t) \in L_2^+(m \times m).$$

It remains to prove that if (14) holds, then

$$\sum_{j=0}^{m} |u_{mj}^+(z)| > \delta, \quad |z| < 1, \text{ for some } \quad \delta > 0, \tag{24}$$

and

$$F(t)U(t) \in L^+(m \times m). \tag{25}$$

We obtain both relations simultaneously.

Since (14) holds, there exist bounded functions $\zeta_i \in L_\infty$ such that (19) holds. Let $\Phi^\pm$ be defined as in (20). Then $\Phi^+ F = \Phi^+ \Phi^- = \Phi$ is bounded and therefore

$$\Phi^+ F U =: \Psi^+ \in L_\infty^+(m \times m). \tag{26}$$

Hence

$$FU = (\Phi^+)^{-1} \Psi^+ \in L^+(m \times m)$$

and (25) holds.

To show (24), let us first observe that $\det \Psi^+(z) = \text{Const}$ for $|z| < 1$ since $\det \Psi^+ \in H_\infty$ and it is constant a.e. on the boundary (see (20), (23), and (26)). Therefore

$$\sum_{j=1}^{m} \Psi_{mi}^+(z) \operatorname{Cof}(\Psi_{mi}^+)(z) = C, \tag{27}$$

where Cof stands for the cofactor. However, the first $m-1$ rows of U and Ψ^+ coincide. So that

$$\operatorname{Cof}(\Psi_{mi}^+) = \operatorname{Cof}(U_{mi}), \ \ j = 1, 2, \dots, m. \tag{28}$$

In addition, since U is unitary, i.e., $U^{-1} = U^*$, the formula for the inverse matrix implies that

$$u_{mj}^+ = \frac{1}{C} \operatorname{Cof}(U_{mj}). \tag{29}$$

Therefore, substituting (28) and (29) in (27), we get

$$\sum_{j=1}^{m} \Psi_{mi}^+(z) u_{mj}^+(z) = 1,$$

and, because of boundedness of the functions Ψ_{mi}^+ (see (26)), relation (24) holds.

3. Open Problems

For compact wavelet matrices, it is proved in [1] that the entries ζ_i^- of the matrix (10) in Theorem 1 can be computed by the formula

$$\zeta_i^-(z) = \mathbb{P}_N^-\big(\widetilde{\mathbf{A}}_{ij}(z)/\mathbf{A}_{mj}(z)\big), \quad \text{if } \mathbf{A}_{mj}(0) \neq 0, \tag{30}$$

where $\mathbb{P}_N^-$ is the projection of a (formal) Fourier series $\sum_{k=-N}^{\infty} c_k t^k$ on $\mathcal{P}_N^-$ (see [1, Eq. (25)]). To describe the conditions under which we can let $N \to \infty$ in equation (30) and to determine in which sense the limit exists is an interesting problem. It is related to the computation of partial indices of Wiener-Hopf factorization for a certain class of matrix functions which is the subject of a forthcoming paper.

4. Acknowledgement

The first two authors were supported by the Shota Rustaveli National Science Foundation of Georgia (Project No. DI-18-118). The third author was also supported in part by Faculty Research funding from the Division of Science and Mathematics, New York University Abu Dhabi.

References

1. L. Ephremidze, E. Lagvilava, On compact wavelet matrices of rank m and of order and degree N. *J. Fourier Anal. Appl.* **20** (2014), no. 2, 401–420.
2. L. Ephremidze, G. Janashia, E. Lagvilava, On the factorization of unitary matrix-functions. *Proc. A. Razmadze Math. Inst.* **116** (1998), 101–106.
3. L. Ephremidze, G. Janashia, E. Lagvilava, On approximate spectral factorization of matrix functions. *J. Fourier Anal. Appl.* **17** (2011), no. 5, 976–990.
4. J. Kautsky, R. Turcajová, Pollen product factorization and construction of higher multiplicity wavelets. *Linear Algebra Appl.* **222** (1995), 241–260.
5. Paul Koosis, *Introduction to H_p Spaces. With an Appendix on Wolff's Proof of the Corona Theorem.* London Mathematical Society Lecture Note Series, 40. Cambridge University Press, Cambridge-New York, 1980.
6. H. L. Resnikoff, R. O. Wells, *Wavelet Analysis.* The scalable structure of information. Springer-Verlag, New York, 1998.
7. P. L. Vaidyanathan, *Multirate Systems and Filter Banks Prentice-Hall.* Englewood Cliffs, NJ 1993.
8. M. Vetterli, J. Kovačević, *Wavelets and Subband Coding.* Prentice Hall PTR, New Jersey, 1995.

(Received 17.07.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 37–44

A NOTE ON THE BILINEAR FRACTIONAL INTEGRAL OPERATOR ACTING ON MORREY SPACES

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Abstract. The boundedness of the bilinear fractional integral operator is investigated. This bilinear fractional integral operator goes back to Kenig and Stein. The paper is oriented to the boundedness of the operator on products of Morrey spaces. This paper uses some averaging techniques to control the Morrey norm. Compared to the earlier work by He and Yan, one feels that the technique can be applied to other function spaces. Among others, the averaging operator will reduce the matters to the existing results.

1. Introduction

Recently, He and Yan investigated fractional integral operators of Grafakos type acting on Morrey spaces [8]. In this paper, by using some known results, we propose to simplify their proofs. Let $0 < q \le p < \infty$. Define the *Morrey norm* $\|\cdot\|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q)} \ : \ Q \text{ is a dyadic cube in } \mathbb{R}^n\right\}$$

for a measurable function f.

The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all the measurable functions f for which $\|f\|_{\mathcal{M}^p_q}$ is finite. We recall the definition of the dyadic cubes precisely in Section 2. A simple geometric observation shows that

$$\|f\|_{\mathcal{M}^p_q} \sim \sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q)} \ : \ Q \text{ is a cube in } \mathbb{R}^n\right\}$$

for any measurable function f. Here let us content ourselves with the intuitive understanding that p serves as the global integrability, as is hinted by the dilation mapping $f \mapsto f(t\cdot)$, and that q serves as the local integrability. We handle the following bilinear operator defined in [5, 13].

Definition 1.1. The *bilinear fractional integral operator of Grafakos type* $\mathcal{J}_\alpha$, $0 < \alpha < n$ is given by

$$\mathcal{J}_\alpha[f_1, f_2](x) \equiv \int_{\mathbb{R}^n} \frac{f_1(x+y)\, f_2(x-y)}{|y|^{n-\alpha}}\, dy \quad (x \in \mathbb{R}^n),$$

where f_1, f_2 are non-negative integrable functions defined in $\mathbb{R}^n$.

The operator $\mathcal{I}_\alpha[f_1, f_2]$, $0 < \alpha < 2n$, defined by

$$\mathcal{I}_\alpha[f_1, f_2](x) \equiv \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{f_1(y_1)\, f_2(y_2)}{\big(|x-y_1| + |x-y_2|\big)^{2n-\alpha}}\, dy \quad (x \in \mathbb{R}^n)$$

for non-negative integrable functions f_1 and f_2 defined in $\mathbb{R}^n$, is a contrast to $\mathcal{J}_\alpha[f_1, f_2]$. These two operators with $0 < \alpha < n$ pass the fractional integral operator I_α in the bilinear case, where I_α is the fractional integral operator

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \quad (x \in \mathbb{R}^n)$$

for a nonnegative measurable function $f : \mathbb{R}^n \to [0, \infty]$.

2010 *Mathematics Subject Classification.* Primary 42B35, Secondary 42B25.

Key words and phrases. Morrey spaces; Bilinear fractional integral operators.

Here and below we assume that the functions are non-negative to ignore the issue of the convergence of the integral definining $\mathcal{J}_\alpha[f_1, f_2](x)$.

The operator $\mathcal{I}_\alpha[f_1, f_2]$ acting on Morrey spaces is investigated by many authors in many settings such as the generalized Morrey spaces [1], the weighted setting [6, 10], the case equipped with the rough kernel [9, 20] and the non-doubling setting [11, 21]. See also [3, 22] for the case of commutators generated by $\mathcal{I}_\alpha$ and other functions. However we do not find so much about the action of the operator $\mathcal{J}_\alpha$ on Morrey spaces. Works [4, 8] considered the boundedness property of $\mathcal{J}_\alpha$. Our aim here is to prove the following estimate

Theorem 1.2. *Let*

$$0 < \alpha < n, \quad 1 < q_1 \le p_1 < \infty, \quad 1 < q_2 \le p_2 < \infty, \quad 1 \le t \le s < \infty.$$

Define p and q by

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Assume that

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}, \quad s < \min(q_1, q_2).$$

Then for all $f_1 \in \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{M}^{p_2}_{q_2}(\mathbb{R}^n)$,

$$\|\mathcal{J}_\alpha[f_1, f_2]\|_{\mathcal{M}^s_t} \lesssim \|f_1\|_{\mathcal{M}^{p_1}_{q_1}} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}}.$$

As is pointed out in [8], the assumption $\frac{q}{p} = \frac{t}{s}$ is essential.

Theorem 1.2 partially extends the following result by Kenig and Stein [13, Theorem 2]

Proposition 1.3. *Let $0 < \alpha < n$ and $1 < p_1, p_2 < \infty$. Assume that $\dfrac{1}{p_1} + \dfrac{1}{p_2} > \dfrac{\alpha}{n}$, so we can define $s > 0$ by $\dfrac{1}{s} = \dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{\alpha}{n}$. Then for all $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$,*

$$\|\mathcal{J}_\alpha[f_1, f_2]\|_{L^s} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.$$

In the first half of [8], He and Yan proved the boundedness of the operator and used the Hölder inequality under the assumption

$$\frac{q_1}{p_1} = \frac{q_2}{p_2}, \quad \frac{1}{\max\left(q_1', \frac{\alpha}{n}p_1\right)} + \frac{1}{\max\left(q_2', \frac{\alpha}{n}p_2\right)} > 1, \tag{1.1}$$

so

$$\frac{q}{p} = \frac{q_1}{p_1} = \frac{q_2}{p_2}$$

and there exists $u \in (1, \infty)$ such that

$$\frac{\alpha}{n}p_1 < u < \left(\frac{\alpha}{n}p_2\right)', \quad (q_2)' < u < q_1.$$

Define s_1, s_2, t_1 and t_2 by

$$\frac{u}{s_1} = \frac{u}{p_1} - \frac{\alpha}{n}, \quad \frac{u'}{s_2} = \frac{u'}{p_2} - \frac{\alpha}{n}, \quad \frac{t_1}{s_1} = \frac{q_1}{p_1}, \quad \frac{t_2}{s_2} = \frac{q_2}{p_2},$$

so $1 < t_1 \le s_1 < \infty$ and $1 < t_2 \le s_2 < \infty$. Then

$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}, \quad \frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2},$$

since

$$\frac{q}{p} = \frac{q_1}{p_1} = \frac{q_2}{p_2}.$$

Meanwhile, by the Hölder inequality we have

$$\mathcal{J}_\alpha[f_1, f_2](x) \le \left(\int_{\mathbb{R}^n} \frac{|f_1(x+y)|^u}{|y|^{n-\alpha}} dy\right)^{\frac{1}{u}} \left(\int_{\mathbb{R}^n} \frac{|f_2(x-y)|^{u'}}{|y|^{n-\alpha}} dy\right)^{\frac{1}{u'}}$$

for any $1 < u < \infty$. Consequently, by the Hölder inequality once again, we obtain

$$\left\|\mathcal{J}_\alpha[f_1,f_2]\right\|_{\mathcal{M}^s_t} \le \left\|I^{(u)}_\alpha f_1\right\|_{\mathcal{M}^{s_1}_{t_1}} \left\|I^{(u')}_\alpha f_2\right\|_{\mathcal{M}^{s_2}_{t_2}},$$

where $I^{(v)}_\alpha f \equiv (I_\alpha[|f|^v])^{\frac{1}{v}}$. If we use the Adams theorem, asserting that $I^{(v)}_\alpha$ maps $\mathcal{M}^P_Q(\mathbb{R}^n)$ boundedly to $\mathcal{M}^S_T(\mathbb{R}^n)$ whenever $v < Q \le P < \infty$, $v < T \le S < \infty$, $\frac{v}{S} = \frac{v}{P} - \frac{\alpha}{n}$ and $\frac{P}{Q} = \frac{S}{T}$, we obtain

$$\left\|\mathcal{J}_\alpha[f_1,f_2]\right\|_{\mathcal{M}^s_t} \lesssim \left\|f_1\right\|_{\mathcal{M}^{p_1}_{q_1}} \left\|f_2\right\|_{\mathcal{M}^{p_2}_{q_2}}.$$

Thus Theorem 1.2 is significant when (1.1) fails. See [4, Theorem 2.2] for the bilinear fractional integral operator of Kenig–Stein type equipped with the rough kernel.

The operator $\mathcal{J}_\alpha$ has a lot to do with the bilinear Hilbert transform defined by

$$\mathcal{H}[f_1,f_2](x) = \lim_{\varepsilon\downarrow 0} \int\limits_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{f_1(x+y)\, f_2(x-y)}{y}\, dy \quad (x \in \mathbb{R}),$$

where f_1 and f_2 are the locally integrable functions. One of the important problems in harmonic analysis is to investigate the boundedness property of the bilinear Hilbert transform. A conjecture of Calderón in 1964 concerned possible extensions of $\mathcal{H}$ to a bounded bilinear operator on products of Lebesgue spaces. A remarkable fact is that $\mathcal{H}$ maps $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ boundedly if $1 < p_1 \le \infty$, $1 < p_2 \le \infty$, $\dfrac{2}{3} < p < \infty$ and $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$ [16, 17]. To understand the boundedness property of this operator, we consider its counterpart to fractional integral operators.

After the authors wrote this article, we were aware that Theorem 1.2 is an unweighted version of [8, Theorem 4.6]. However, our proof differs from theirs in that we use an inequality for norms, while He and Yan used a weighted local estimate. It seems that our results can be extended to Orlicz spaces by using [19, Theorem 7.5]. The details are left for the future works.

2. Preliminaries

For a measurable function f defined on $\mathbb{R}^n$, define a function Mf by

$$Mf(x) \equiv \sup_{B\in\mathcal{B}} \frac{\chi_B(x)}{|B|} \int\limits_B \left|f(y)\right| dy \quad (x \in \mathbb{R}^n). \tag{2.1}$$

The mapping $M : f \mapsto Mf$ is called the *Hardy–Littlewood maximal operator*. It is known that the Hardy–Littlewood maximal operator is bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$ if $1 < q \le p < \infty$ [2]. A dyadic cube is a set of the form Q_{jk} for some $j \in \mathbb{Z}, k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$. The set of all dyadic cubes is denoted by $\mathcal{D}$; $\mathcal{D} \equiv \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$. For $j \in \mathbb{Z}$, the set of dyadic cubes of the j-th generation is given by

$$\mathcal{D}_j = \mathcal{D}_j(\mathbb{R}^n) \equiv \left\{Q_{jk} : k \in \mathbb{Z}^n\right\} = \left\{Q \in \mathcal{D} : \ell(Q) = 2^{-j}\right\}.$$

The following lemma can be located as a standard estimate to handle this bilinear fractional integral operator.

Lemma 2.1. *Let $f_1, f_2 \ge 0$ be measurable functions. Then we have*

$$\mathcal{J}_\alpha[f_1,f_2](x) \lesssim \sum_{l=-\infty}^{\infty} \sum_{Q\in\mathcal{D}_l} 2^{l(n-\alpha)} \chi_Q(x) \int\limits_{B(2^{-l})} f_1(x+y)\, f_2(x-y)\, dy \quad (x \in \mathbb{R}^n).$$

Proof. We will follow the idea used in [13, Theorem 2]. See also [18, Theorem 3.2] and [14, 15]. We decompose

$$\begin{aligned}
\mathcal{J}_\alpha[f_1,f_2](x) &= \int_{\mathbb{R}^n} \frac{f_1(x+y)\, f_2(x-y)}{|y|^{n-\alpha}}\, dy \\
&= \sum_{l=-\infty}^{\infty} \int_{B(2^{-l})\setminus B(2^{-l-1})} \frac{f_1(x+y) f_2(x-y)}{|y|^{n-\alpha}}\, dy \\
&\sim \sum_{l=-\infty}^{\infty} 2^{l(n-\alpha)} \int_{B(2^{-l})\setminus B(2^{-l-1})} f_1(x+y) f_2(x-y)\, dy \\
&\le \sum_{l=-\infty}^{\infty} 2^{l(n-\alpha)} \int_{B(2^{-l})} f_1(x+y) f_2(x-y)\, dy.
\end{aligned}$$

Observe that for each $l \in \mathbb{N}$, there uniquely exists a dyadic cube $Q \in \mathcal{D}_l$ such that $x \in Q$. Thus, we obtain the desired result. □

We now recall the averaging technique.

Lemma 2.2. *Suppose that the parameters p, q, s, t satisfy*

$$1 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad q < t, \quad p < s$$

or

$$1 = q \le p < \infty, \quad 1 = t \le s < \infty, \quad p < s.$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathcal{M}^s_t(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill

$$\operatorname{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_q} < \infty. \tag{2.2}$$

Then $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^q_{\rm loc}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}^p_q} \lesssim_{p,q,s,t} \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q}. \tag{2.3}$$

Proof. This estimate is essentially obtained in [12] if $q > 1$ and in [7] if $q = 1$. Although we distinguished these cases in these papers, we can combine them, since the case of $q = 1$ can almost be emerged into the case of $q > 1$.

Let us suppose $q > 1$ for the time being. Let $0 < \eta < \infty$. We will use the *powered Hardy–Littlewood maximal operator* $M^{(\eta)}$ defined by

$$M^{(\eta)} f(x) \equiv \sup_{R>0} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)|^\eta dy \right)^{\frac{1}{\eta}}$$

for a measurable function $f : \mathbb{R}^n \to \mathbb{C}$. If $\eta = 1$, then we write M instead of $M^{(\eta)}$. To prove this, we resort to the duality. For the time being, we assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \ge N$. Let us assume in addition that the a_j's are non-negative. Fix a non-negative function g that is supported on a cube Q such that $\|g\|_{L^{q'}} \le |Q|^{\frac{1}{q'} - \frac{1}{p'}}$. By duality, we will show

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx \lesssim_{p,q,s,t} \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q}, \tag{2.4}$$

to obtain

$$\|f\|_{\mathcal{M}^p_q} \lesssim_{p,q,s,t} \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q}.$$

Assume first that each Q_j contains Q as a proper subset. If we group the j's such that Q_j are identical, we can assume that each Q_j is a j-th parent of Q for each $j \in \mathbb{N}$. Then we have

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx = \sum_{j=1}^\infty \lambda_j \int_Q a_j(x)\, g(x)\, dx \le \sum_{j=1}^\infty \lambda_j \|a_j\|_{L^q(Q)} \|g\|_{L^{q'}(Q)}$$

from $f = \sum_{j=1}^\infty \lambda_j a_j$. By the size condition of a_j and g, we obtain

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx \le \sum_{j=1}^\infty \lambda_j \|a_j\|_{\mathcal{M}^s_t} |Q|^{\frac{1}{q}-\frac{1}{s}} |Q|^{\frac{1}{q'}-\frac{1}{p'}} = \sum_{j=1}^\infty \lambda_j \|a_j\|_{\mathcal{M}^s_t} |Q|^{\frac{1}{p}-\frac{1}{s}}.$$

Note that

$$\left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q} \ge \frac{\|a_{j_0}\|_{\mathcal{M}^s_t}}{|Q_{j_0}|^{\frac{1}{s}}} \|\lambda_{j_0} \chi_{Q_{j_0}}\|_{\mathcal{M}^p_q} = \|a_{j_0}\|_{\mathcal{M}^s_t} |Q_{j_0}|^{\frac{1}{p}-\frac{1}{s}} \lambda_{j_0}$$

for each $j_0 \in \mathbb{N}$. Consequently, it follows from the condition $p < s$ that

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx \le \sum_{j=1}^\infty |Q|^{\frac{1}{p}-\frac{1}{s}} |Q_j|^{\frac{1}{s}-\frac{1}{p}} \cdot \left\| \sum_{k=1}^\infty \lambda_k \frac{\|a_k\|_{\mathcal{M}^s_t}}{|Q_k|^{\frac{1}{s}}} \chi_{Q_k} \right\|_{\mathcal{M}^p_q} \lesssim \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_q}.$$

Conversely, assume that Q contains each Q_j. Then we have

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx = \sum_{j=1}^\infty \lambda_j \int_{Q_j} a_j(x)\, g(x)\, dx \le \sum_{j=1}^\infty \lambda_j \|a_j\|_{L^t(Q_j)} \|g\|_{L^{t'}(Q_j)}.$$

By the condition of a_j, we obtain

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx = \sum_{j=1}^\infty \lambda_j \int_{Q_j} a_j(x)\, g(x)\, dx \le \sum_{j=1}^\infty \lambda_j \|a_j\|_{\mathcal{M}^s_t} |Q_j|^{\frac{1}{t}-\frac{1}{s}} \|g\|_{L^{t'}(Q_j)}.$$

Thus, in terms of the Hardy–Littlewood maximal operator M, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)\, g(x)\, dx &\le \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} |Q_j| \times \inf_{y \in Q_j} M^{(t')} g(y) \\
&\le \int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j}(y) \right) M^{(t')} g(y)\, dy \\
&\le \int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_t}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j}(y) \right) \chi_Q(y) M^{(t')} g(y) dy.
\end{aligned}$$

Hence, we obtain (2.4) by the Hölder inequality, since $\|\chi_Q M^{(t')} g\|_{L^{q'}} \lesssim |Q|^{\frac{1}{p}-\frac{1}{q}}$. Thus the proof for the case of $q > 1$ is complete.

The case of $q = 1$ is a minor modification of the above proof. First, if each Q_j contains Q as a proper subset, the same argument as above works. If each Q contains Q_j, then we can take $g = |Q|^{\frac{1}{p}-1} \chi_Q$ to obtain

$$\int_Q f(x)\, g(x)\, dx \lesssim_{p,s} \left\| \sum_{j=1}^\infty \lambda_j \frac{\|a_j\|_{\mathcal{M}^s_1}}{|Q_j|^{\frac{1}{s}}} \chi_{Q_j} \right\|_{\mathcal{M}^p_1}.$$

We go through the same argument as before, where we will replace $M^{(t')}g$ by 1. Since $\|\chi_Q 1\|_{L^\infty}=1$, we do not have to resort to the boundedness of the maximal operator $M^{(t')}$ as we did in the estimate $\|\chi_Q M^{(t')}g\|_{L^{q'}} \lesssim |Q|^{\frac{1}{p}-\frac{1}{q}}$. So, the proof is complete in this case. □

Lemma 2.3. *Let*

$$0<\alpha<2n, \quad 1<q_j \le p_j<\infty, \quad 0<q\le p<\infty, \quad 0<t\le s<\infty$$

for $j=1,2$. *Assume*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}, \quad \frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}, \quad \frac{1}{s}=\frac{1}{p}-\frac{\alpha}{n}, \quad \frac{q}{p}=\frac{t}{s}.$$

Then

$$|R|^{\frac{1}{s}-\frac{1}{t}} \left\| \sum_{Q\in\mathcal{D}} \frac{\chi_Q}{\ell(Q)^{2n-\alpha}} \int\limits_{(3Q)^2} f_1(y_1)\, f_2(y_2)\, dy_1\, dy_2 \right\|_{L^t(R)} \lesssim \prod_{j=1}^{2} \|f_j\|_{\mathcal{M}^{p_j}_{q_j}} \tag{2.5}$$

for any cube R *and for all non-negative measurable functions* f_1, f_2.

See the proof of [4, Theorem 2.2] for a similar approach.

Proof. Let $L=L(x)$ be a positive number that is specified shortly. We decompose

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}} \frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}} \int\limits_{(3Q)^2} f_1(y_1)\, f_2(y_2)\, dy_1\, dy_2 \\
&\quad= \sum_{Q\in\mathcal{D},\ell(Q)\le L} \frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}} \int\limits_{(3Q)^2} f_1(y_1)\, f_2(y_2)\, dy_1\, dy_2 \\
&\quad\quad+ \sum_{Q\in\mathcal{D},\ell(Q)> L} \frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}} \int\limits_{(3Q)^2} f_1(y_1)\, f_2(y_2)\, dy_1\, dy_2 =: S_1+S_2.
\end{aligned}$$

First, we estimate the quantity S_1:

$$S_1 \lesssim \sum_{Q\in\mathcal{D},\ell(Q)\le L} \chi_Q(x)\, \ell(Q)^\alpha M f_1(x)\, M f_2(x) \sim L^\alpha M f_1(x)\, M f_2(x).$$

Next, we estimate the quantity S_2. By Hölder's inequality,

$$\begin{aligned}
S_2 &\lesssim \sum_{Q\in\mathcal{D},\ell(Q)>L} \frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}} |Q|^{\frac{1}{q_1'}} \|f_1\|_{L^{q_1}(3Q)} \cdot |Q|^{\frac{1}{q_2'}} \|f_1\|_{L^{q_2}(3Q)} \\
&\lesssim \sum_{Q\in\mathcal{D},\ell(Q)>L} \chi_Q(x) |Q|^{-\frac{1}{s}} \|f_1\|_{\mathcal{M}^{p_1}_{q_1}} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}} \sim L^{-\frac{n}{s}} \|f_1\|_{\mathcal{M}^{p_1}_{q_1}} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}}.
\end{aligned}$$

Hence we obtain

$$\sum_{Q\in\mathcal{D}} \frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}} \int\limits_{(3Q)^2} f_1(y_1)\, f_2(y_2)\, dy_1\, dy_2 \lesssim L^\alpha M f_1(x)\, M f_2(x) + L^{-\frac{n}{s}} \|f_1\|_{\mathcal{M}^{p_1}_{q_1}} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}}.$$

In particular, choose the constant $L=L(x)$ to optimize the right-hand side:

$$L = \left(\frac{\|f_1\|_{\mathcal{M}^{p_1}_{q_1}} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}}}{M f_1(x)\, M f_2(x)} \right)^{\frac{p}{n}}.$$

Then we have

$$\sum_{Q\in\mathcal{D}} \frac{\chi_Q(x)}{\ell(Q)^{2n-\alpha}} \int\limits_{(3Q)^2} f_1(y_1)\, f_2(y_2)\, dy_1\, dy_2 \lesssim (M f_1(x)\, M f_2(x))^{\frac{p}{s}} \left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}} \|f_2\|_{\mathcal{M}^{p_2}_{q_2}} \right)^{1-\frac{p}{s}}.$$

Therefore, using Hölder's inequality for Morrey spaces, the $\mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)$-boundedness of M and the $\mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)$-boundedness of M, we have

$$\begin{aligned}
&|R|^{\frac{1}{s}-\frac{1}{t}}\left\|\sum_{Q\in\mathcal{D}}\frac{\chi_Q}{\ell(Q)^{2n-\alpha}}\int\limits_{(3Q)^2} f_1(y_1)\,f_2(y_2)\,dy_1\,dy_2\right\|_{L^t(R)}\\
&\quad\lesssim \left\|(Mf_1\cdot Mf_2)^{\frac{p}{s}}\right\|_{\mathcal{M}^s_t}\left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}\right)^{1-\frac{p}{s}} = \|Mf_1\cdot Mf_2\|_{\mathcal{M}^p_q}^{\frac{p}{s}}\left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}\right)^{1-\frac{p}{s}}\\
&\quad\le \left(\|Mf_1\|_{\mathcal{M}^{p_1}_{q_1}}\|Mf_2\|_{\mathcal{M}^{p_2}_{q_2}}\right)^{\frac{p}{s}}\left(\|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}\right)^{1-\frac{p}{s}} \lesssim \|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}. \qquad\square
\end{aligned}$$

3. Proof of Theorem 1.2

Let $v\in(s,\min(q_1,q_2))$. Let $x\in Q\in\mathcal{D}_l$. By the Minkowski inequality and the Hölder inequality,

$$\begin{aligned}
&\left\|\int\limits_{B(2^{-l})} f_1(\cdot+y)\,f_2(\cdot-y)dy\right\|_{L^v(Q)}\\
&\le \int\limits_{B(2^{-l})}\|f_1(\cdot+y)\,f_2(\cdot-y)\|_{L^v(Q)}dy \le |B(2^{-l})|^{\frac{1}{v'}}\left(\int\limits_{B(2^{-l})}\|f_1(\cdot+y)\,f_2(\cdot-y)\|_{L^v(Q)}{}^v dy\right)^{\frac{1}{v}}\\
&\lesssim |B(2^{-l})|^{\frac{1}{v'}}\|f_1\|_{L^v(Q(x,3\cdot 2^{-l}))}\|f_2\|_{L^v(Q(x,3\cdot 2^{-l}))}\lesssim |B(2^{-l})|^{1+\frac{1}{v}}\inf_{y_1\in Q}M^{(v)}f_1(y_1)\inf_{y_2\in Q}M^{(v)}f_2(y_2).
\end{aligned}$$

Then owing to Theorem 2.2,

$$\begin{aligned}
\|\mathcal{J}_\alpha[f_1,f_2]\|_{\mathcal{M}^s_t} &\lesssim \left\|\sum_{l=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_l}2^{l(n-\alpha)}\frac{\chi_Q}{|Q|^{\frac{1}{v}}}\left\|\int\limits_{B(2^{-l})}f_1(\cdot+y)\,f_2(\cdot-y)\,dy\right\|_{L^v(Q)}\right\|_{\mathcal{M}^s_t}\\
&\lesssim \left\|\sum_{l=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_l}2^{-l\alpha}\frac{\chi_Q}{|Q|}\int\limits_{3Q}M^{(v)}f_1(y_1)\,dy_1\cdot\frac{1}{|Q|}\int\limits_{3Q}M^{(v)}f_2(y_2)\,dy_2\right\|_{\mathcal{M}^s_t}.
\end{aligned}$$

Thus, we are again in the position of using (2.5) to have

$$\|\mathcal{J}_\alpha[f_1,f_2]\|_{\mathcal{M}^s_t}\lesssim \|M^{(v)}f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|M^{(v)}f_2\|_{\mathcal{M}^{p_2}_{q_2}}.$$

Since $v<q_1,q_2$, we are in the position of using the boundedness of M on Morrey spaces obtained by Chiarenza and Frasca [2]. If we use the boundedness of the Hardy–Littlewood maximal operator, then we obtain

$$\|\mathcal{J}_\alpha[f_1,f_2]\|_{\mathcal{M}^s_t}\lesssim \|f_1\|_{\mathcal{M}^{p_1}_{q_1}}\|f_2\|_{\mathcal{M}^{p_2}_{q_2}}.$$

This is the desired result.

To conclude the paper, we remark that Fan and Gao obtained an estimate to control

$$\left\|\int\limits_{B(2^{-l})}f_1(\cdot+y)\,f_2(\cdot-y)\,dy\right\|_{L^v(Q)}$$

in [4, Lemma 2.1].

Acknowledgement

The second-named author was supported by Grant-in-Aid for Scientific Research (C) (16K05209), the Japan Society for the Promotion of Science. The authors are thankful to Dr. Toru Nogayama for his careful reading the paper and his pointing out some typos. The authors are also thankful to Professor Komori–Furuya Yasuo for his kind introduction to [14, 15].

References

1. W. S. Budhi, J. Lindiarti, Boundedness of multilinear generalized fractional integral operators in generalized Morrey space. *Far East J. Math. Sci.* (FJMS) **57** (2011), no. 1, 91–104.
2. F. Chiarenza, M. Frasca, Morrey spaces and Hardy–Littlewood maximal function. *Rend. Mat. Appl.* **7** (1987), no. 3–4, 273–279 (1988).
3. Y. Ding, T. Mei, Boundedness and compactness for the commutators of bilinear operators on Morrey spaces. *Potential Anal.* **42** (2015), no. 3, 717–748.
4. Y. Fan, G. Gao, Some estimates of rough singular bilinear fractional integral. *J. Funct. Spaces Appl.* (2012), Art. ID 406540.
5. L. Grafakos, On multilinear fractional integrals. *Studia Math.* **102** (1992), no. 1, 49–56.
6. V. S. Guliyev, M. N. Omarova, Multilinear singular and fractional integral operators on generalized weighted Morrey spaces. *Azerb. J. Math.* **5** (2015), no. 1, 104–132.
7. V. S. Guliyev, S. G. Hasanov, Y. Sawano, T. Noi, Non-smooth atomic decompositions for generalized Orlicz-Morrey spaces of the third kind. *Acta Appl. Math.* **145** (2016), 133–174.
8. Q. He, D. Yan, Bilinear fractional integral operators on Morrey spaces. arXiv preprint arXiv:1805.01846 (2018).
9. T. Iida, Weighted inequalities on Morrey spaces for linear and multilinear fractional integrals with homogeneous kernels. *Taiwanese J. Math.* **18** (2014), no. 1, 147–185.
10. T. Iida, Various inequalities related to the Adams inequality on weighted Morrey spaces. *Math. Inequal. Appl.* **20** (2017), no. 3, 601–650.
11. T. Iida, E. Sato, Y. Sawano, H. Tanaka, Multilinear fractional integrals on Morrey spaces. *Acta Math. Sin.* (*Engl. Ser.*) **28** (2012), no. 7, 1375–1384.
12. T. Iida, Y. Sawano, H. Tanaka, Atomic decomposition for Morrey spaces. *Z. Anal. Anwend.* **33** (2014), no. 2, 149–170.
13. C. E. Kenig, E. M. Stein, Multilinear estimates and fractional integration. *Math. Res. Lett.* **6** (1999), 1–15.
14. Y. Komori-Furuya, *Weighted Estimates for Bilinear Fractional Integral Operators*. Math. Nachr. 2019.
15. Y. Komori-Furuya, Weighted estimates for bilinear fractional integral operators A necessary and sufficient condition for power weights. *Collect. Math.* 2019. https://doi.org/10.1007/s13348-019-00246-5.
16. M. Lacey, C. Thiele, On Calderón's conjecture. *Ann. of Math. (2)* **149** (1999), no. 2, 475–496.
17. M. Lacey, The bilinear maximal functions map into L^p for $2/3 < p \leq 1$. *Ann. of Math. (2)* **151** (2000), no. 1, 35–57.
18. K. Moen, New weighted estimates for bilinear fractional integral operators. *Trans. Amer. Math. Soc.* **366** (2014), no. 2, 627–646.
19. Y. Sawano, K. P. Ho, D. Yang, S. Yang, Hardy Spaces for Ball Quasi-Banach Function Spaces. *Dissertationes Math.* **525** (2017), 1–102.
20. Y. Shi, Z. Si, X. Tao, Y. Shi, Necessary and sufficient conditions for boundedness of multilinear fractional integrals with rough kernels on Morrey type spaces. *J. Inequal. Appl.* (2016), Paper no. 43, 19 pp.
21. X. X. Tao, T. T. Zheng, Multilinear commutators of fractional integrals over Morrey spaces with non-doubling measures. *NoDEA Nonlinear Differential Equations Appl.* **18** (2011), no. 3, 287–308.
22. Y. Xiao, S. Z. Lu, Olsen-type inequalities for the generalized commutator of multilinear fractional integrals. *Turkish J. Math.* **42** (2018), no. 5, 2348–2370.

(Received 11.03.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 45–78

WEIGHTED BOUNDEDNESS OF THE FRACTIONAL MAXIMAL OPERATOR AND RIESZ POTENTIAL GENERATED BY GEGENBAUER DIFFERENTIAL OPERATOR

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Abstract. In the paper we study the weighted $\left(L_{p,\omega,\lambda}, L_{q,\omega,\lambda}\right)$-boundedness of the fractional maximal operator M_G^α (G is a fractional maximal operator) and the Riesz potential (G is the Riesz potential) generated by the Gegenbauer differential operator

$$G_\lambda \equiv G = \left(x^2-1\right)^{\frac{1}{2}-\lambda} \frac{d}{dx} \left(x^2-1\right)^{\lambda+\frac{1}{2}} \frac{d}{dx}, \ \ x \in (1,\infty), \ \ \lambda \in \left(0, \frac{1}{2}\right).$$

1. Introduction

1.1. As is known the maximal functions, singular integrals and potentials generated by different differential operators are of importance in applications and in different questions of harmonic analysis and therefore their study is very topical. Non-accidentally, there exists extensive literature devoted to difference properties of the afore-mentioned object of harmonic analysis. We cite only those works that relate to the question considered in the paper: D. Adams [1], I. A. Aliev [2], A. P. Calderon [4], R. R. Coifman and C. Fefferman [5], C. Fefferman and E. Stein [8], A. I. Gadjiev [9], V. S. Guliev [11–13], G. H. Hardy and J. E. Littlewood [14], V. M. Kokilashvili, A. Kufner [20], V. M. Kokilashvili and S. Samko [21], I. A. Kipriyanov, M. N. Klyuchancev [16–19], L. N. Lyakhov [25,26], G. Welland [30] and other.

In [12] E. V. Guliyev introduced analogous to Muckenhoupt classes generated by Gegenbauer differential operator and for the fractional maximal function and fractional integrals associated with the Bessel differential operator. He obtained some weighted inequalities, analogous to those given in Propositions 1.5 and 1.6 from [30].

We note that the measure of the homogeneous space satisfies the doubling properties $\mu E(x,2r) \leq c\mu E(x,r)$, where the constant c is independent of x and r, and these properties are essentially used in proving many facts.

1.2. Based on our investigation, we introduce the Gegenbauer differential operator G_λ (see [7]). The shift operator $A^\lambda_{\operatorname{ch} t}$, generated by G_λ is given as follows (see [15]):

$$A^\lambda_{\operatorname{ch} t} f(\operatorname{ch} x) \equiv A_{\operatorname{ch} t}\ f(\operatorname{ch} x) = C_\lambda \int_0^\pi f\left(\operatorname{ch} x \ \operatorname{ch} t - \operatorname{sh} x \ \operatorname{sh} t \ \cos\varphi\right)(\sin\varphi)^{2\lambda-1} d\varphi,$$

where $C_\lambda = \dfrac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} = \left(\int\limits_0^\pi (\sin\varphi)^{2\lambda-1} d\varphi\right)^{-1}$ and possess all properties of the generalized shift operator from B.M. Levitan's monograph [22,23].

Let $H(x,r) = (x-r, x+r) \cap [0,\infty)$, $r \in (0,\infty)$, $x \in \mathbb{R}_+ = [0,\infty)$. In this way,

$$H(x,r) = \begin{cases} (0,\ x+r), & 0 \leq x < r, \\ (x-r,\ x+r), & x \geq r. \end{cases}$$

2010 *Mathematics Subject Classification.* 26A33, 33C45, 33C60, 33C70.

Key words and phrases. Gegenbauer differential operator; G-Fractional maximal operator; G-Riesz potential; Weighted inequalities.

For any set $E \subset \mathbb{R}_+$, $\mu E = |E|_\lambda = \int_E \operatorname{sh}^{2\lambda} t dt$. The maximal functions (G-maximal functions) generated by the Gegenbauer differential operator [15] are defined as follows:

$$M_G\, f(\operatorname{ch} x) = \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_0^r A_{\operatorname{ch} t} \big|f(\operatorname{ch} x)\big| d\mu_\lambda(t),$$

and

$$M_\mu\, f(\operatorname{ch} x) = \sup_{x\in\mathbb{R}_+, r>0} \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} \big|f(\operatorname{ch} t)\big| d\mu_\lambda(t),$$

where

$$d\mu_\lambda(t) = \operatorname{sh}^{2\lambda} t\; dt, \quad \big|H(x,r)\big|_\lambda = \int_{H(x,r)} \operatorname{sh}^{2\lambda} t\; dt, \quad \big|H(0,r)\big|_\lambda = \int_0^r \operatorname{sh}^{2\lambda} t\; dt.$$

The symbol $A \lesssim B$ denotes that there exists a constant C such that $0 < A \le CB$, and C may depend on some parameters. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

In [15], it is proved that

$$M_G\, f(\operatorname{ch} x) \lesssim M_\mu f(\operatorname{ch} x).$$

For any locally integrable function $f(\operatorname{ch} x)$, $x \in \mathbb{R}_+$, we denote the fractional maximal function (G-fractional maximal function) M_G^α generated by the Gegenbauer differential operator as follows:

$$M_G^\alpha\, f(\operatorname{ch} x) = \sup_{r>0} \big|H(0,r)\big|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int_0^r A_{\operatorname{ch} t} \big|f(\operatorname{ch} t)\big| d\mu_\lambda(t), \; 0 \le \alpha < 2\lambda + 1.$$

If $\alpha = 0$, then $M_G^0 f \equiv M_G f$.

We consider the Riezs potential (G-Riezs potential) generated by the Gegenbauer differential operator introduced in [15]

$$I_G^\alpha f(\operatorname{ch} x) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(\operatorname{ch} t) dr \right) A_{\operatorname{ch} t} f(\operatorname{ch} x) d\mu_\lambda(t), \tag{1.1}$$

where

$$h_r(\operatorname{ch} t) = \int_1^\infty e^{-\nu(\nu+2\lambda) r} P_\nu^\lambda(\operatorname{ch} t)(\nu^2 - 1)^{\lambda - \frac{1}{2}} d\nu$$

and $P_\nu^\lambda(\operatorname{ch} t)$ is an eigenfunction of the operator G.

We denote by $L_p(\mathbb{R}_+, G) \equiv L_{p,\lambda}(\mathbb{R}_+)$, $1 \le p \le \infty$, the space of functions measurable on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} \big|f(\operatorname{ch} t)\big|^p d\mu_\lambda(t) \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|f\|_{\infty,\lambda} = \|f\|_\infty = ess \sup_{t\in\mathbb{R}_+} \big|f(\operatorname{ch} t)\big|, \quad p = \infty.$$

We also denote by $WL_{p,\lambda}(\mathbb{R}_+)$ the weak space defined as a set of locally integrable functions on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{WL_{p,\lambda}(\mathbb{R}_+)} = \sup_{t>0} t \Big(\Big| \big\{ x \in \mathbb{R}_+ : |f(\operatorname{ch} x)| > t \big\} \Big|_\lambda \Big)^{\frac{1}{p}}.$$

The main objective of this paper is to obtain the results in [12], [29] and [30] for the operators M_G^α and I_G^α. The paper is organized as follows. In Section 2 we present some auxiliary results. We establish the equivalence of the G-fractional maximal functions M_G^α and M_μ^α in the form $M_G^\alpha\, f(\operatorname{ch} x) \approx M_\mu^\alpha\, f(\operatorname{ch} x)$. We obtain the Calderon–Zygmund decomposition of $\mathbb{R}_+$. In Section 3, we obtain the weighted analogue of the Fefferman-Stein inequality for the maximal function M_μ^α and give the Chebyshev-type inequality. We also introduce the Muckenhoupt-type class and obtain some properties of weights in order to use them in proving the inverse Hölder's inequality. We prove the weighted ($L_{p,w,\lambda}$, $L_{q,w,\lambda}$)- boundedness of the G-fractional maximal operator. In Section 4, we prove the weighted ($L_{p,w,\lambda}$, $L_{q,w,\lambda}$)- boundedness of the G-Riezs potential.

2. Auxiliary Results

In this section we will drive and prove some necessary fact, which we will need in the sequel.

Lemma 2.1 ([15])**.** *For $0<\lambda<\frac{1}{2}$, the following relation*

$$\left|H(0,r)\right|_\lambda \approx \begin{cases} \left(\operatorname{sh}\dfrac{r}{2}\right)^{2\lambda+1}, & 0<r<2,\\ \left(\operatorname{ch}\dfrac{r}{2}\right)^{4\lambda}, & 2\le r<\infty, \end{cases}$$

is valid. Note that this measure does not possess the doubling properties of the homogeneous space, i.e., $\mu E(x,2r)\le c\mu E(x,r)$. In this case, we have

$$\left|H(0,2r)\right|_\lambda \approx \begin{cases} \left(\operatorname{sh}2\cdot\dfrac{r}{2}\right)^{2\lambda+1} \ge 2^{2\lambda+1}\left(\operatorname{sh}\dfrac{r}{2}\right)^{2\lambda+1} \ge c_\lambda\left|H(0,r)\right|_\lambda, & 0<r<2,\\ \left(\operatorname{ch}2\cdot\dfrac{r}{2}\right)^{4\lambda} \ge \left(\operatorname{sh}2\cdot\dfrac{r}{2}\right)^{4\lambda} \ge 2^{4\lambda}\left(\operatorname{sh}\dfrac{r}{2}\right)^{4\lambda} \ge c_\lambda\left|H(0,r)\right|_\lambda, & 2\le r<\infty, \end{cases}$$

where c_λ depends on the parameter $\lambda>0$.

Lemma 2.2 ([15])**.** *Let $0<\lambda<\frac{1}{2}$, $0\le x<\infty$ and $0<r<\infty$. Then for $0<r<2$,*

$$\left|H(x,r)\right|_\lambda \lesssim \begin{cases} \left(\operatorname{sh}\ \dfrac{r}{2}\right)^{2\lambda+1}, & 0\le x<r,\\ \left(\operatorname{sh}\ \dfrac{r}{2}\right)(\operatorname{sh}\ x)^{2\lambda}, & r\le x<\infty, \end{cases}$$

and for $2\le r<\infty$, we have

$$\left|H(x,r)\right|_\lambda \lesssim \begin{cases} (\operatorname{ch}r)^{2\lambda}, & 0\le x<r,\\ (\operatorname{ch}x)^{2\lambda}(\operatorname{ch}r)^{2\lambda}, & 2\le r<\infty. \end{cases}$$

Lemma 2.3. *Let $0\le x<\infty$ and $0<r<\infty$. For any $\gamma>0$, the following relation*

$$\left|H(x,r)\right|_{\frac{\gamma}{2}} \approx \begin{cases} \left(\operatorname{sh}\dfrac{x+r}{2}\right)^{\gamma+1}, & 0<x+r<2,\\ \left(\operatorname{sh}\dfrac{x+r}{2}\right)^{2\gamma}, & 2\le x+r<\infty, \end{cases}$$

holds.

Proof. Let $0\le x<r$. We consider the case where $0<x+r<2$. Let $0<\gamma\le1$. Then

$$\begin{aligned} |H(x,r)|_{\frac{\gamma}{2}} &= \int_0^{x+r}(\operatorname{sh}\ t)^\gamma dt = \int_0^{x+r}(sh\ t)^{\gamma-1}d(\operatorname{ch}\ t) = \int_0^{x+r}\left(\operatorname{ch}^2 t-1\right)^{\frac{\gamma-1}{2}}d(\operatorname{ch}\ t)\\ &= \int_0^{x+r}\frac{(\operatorname{ch}\ t-1)^{\frac{\gamma-1}{2}}}{(\operatorname{ch}\ t+1)^{\frac{1-\gamma}{2}}}d(\operatorname{ch}\ t) \ge (\operatorname{ch}(x+r)+1)^{\frac{\gamma-1}{2}}\int_0^{x+r}(\operatorname{ch}\ t-1)^{\frac{\gamma-1}{2}}d(\operatorname{ch}\ t)\\ &\ge \frac{2}{\gamma+1}(1+\operatorname{ch}\ 2)^{\frac{\gamma-1}{2}}(\operatorname{ch}(x+r)-1)^{\frac{\gamma+1}{2}}\\ &= \frac{2}{\gamma+1}\left(2\operatorname{ch}^2 1\right)^{\frac{\gamma-1}{2}}\left(2\operatorname{sh}^2\frac{x+r}{2}\right)^{\frac{\gamma+1}{2}} = \frac{2^{\gamma+1}}{\gamma+1}(\operatorname{ch}1)^{\gamma-1}\left(\operatorname{sh}\frac{x+r}{2}\right)^{\gamma+1}. \end{aligned} \tag{2.1}$$

Let $\gamma > 1$. Then

$$\int_0^{x+r} (\mathrm{sh}\ t)^\gamma dt = \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} (\mathrm{ch}\ t+1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$\geq 2^{\frac{\gamma-1}{2}} \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$= \frac{2^{\frac{\gamma+1}{2}}}{\gamma+1} (\mathrm{ch}(x+r)-1)^{\frac{\gamma+1}{2}} \frac{2^{\frac{\gamma+1}{2}}}{\gamma+1} \left(2\,\mathrm{sh}^2 \frac{x+r}{2}\right)^{\frac{\gamma+1}{2}} = \frac{2^{\gamma+1}}{\gamma+1} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}. \tag{2.2}$$

On the other hand, for $0 < \gamma \leq 1$ and $0 < x+r < 2$, we have

$$\int_0^{x+r} (\mathrm{sh}\ t)^\gamma dt = \int_0^{x+r} \frac{(\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}}}{(\mathrm{ch}\ t+1)^{\frac{1-\gamma}{2}}} d(\mathrm{ch}\ t) \leq 2^{\frac{\gamma-1}{2}} \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$= \frac{2^{\frac{\gamma+1}{2}}}{\gamma+1} (\mathrm{ch}(x+r)-1)^{\frac{\gamma+1}{2}} = \frac{2^{\gamma+1}}{\gamma+1} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}. \tag{2.3}$$

For $\gamma > 1$,

$$\int_0^{x+r} (\mathrm{sh}\ t)^\gamma dt \int_0^{x+r} (\mathrm{ch}\, t-1)^{\frac{\gamma-1}{2}} (\mathrm{ch}\ t+1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$\leq \frac{2}{\gamma+1} (\mathrm{ch}(x+r)+1)^{\frac{\gamma-1}{2}} (\mathrm{ch}(x+r)-1)^{\frac{\gamma+1}{2}}$$

$$\leq \frac{2}{\gamma+1} (\mathrm{ch}\ 2+1)^{\frac{\gamma-1}{2}} \left(2\,\mathrm{sh}^2 \frac{x+r}{2}\right)^{\frac{\gamma+1}{2}}$$

$$= \frac{2}{\gamma+1} \left(2\,\mathrm{ch}^2 1\right)^{\frac{\gamma-1}{2}} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1} = \frac{2^{\gamma+1}}{\gamma+1} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}. \tag{2.4}$$

Combining (2.1)–(2.4), we obtain

$$\left|H(x,r)\right|_{\frac{\gamma}{2}} \approx \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}, \quad \gamma > 0, \quad 0 < x+r < 2. \tag{2.5}$$

Consider now the case where $2 \leq x+r < \infty$ and $0 \leq x < r$.

Let $0 < \gamma \leq 1$. Then we get

$$\int_0^{x+r} (\mathrm{sh}\ t)^\gamma dt = \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} (\mathrm{ch}\ t+1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$\geq \left(\mathrm{ch}(x+r)+1\right)^{\frac{\gamma-1}{2}} \int_0^{x+r} (\mathrm{ch}\ t-1)^{\frac{\gamma-1}{2}} d(\mathrm{ch}\ t)$$

$$= \frac{2}{\gamma+1} \left(2\,\mathrm{ch}^2 \frac{x+r}{2}\right)^{\frac{\gamma-1}{2}} (\mathrm{ch}(x+r)-1)^{\frac{\gamma+1}{2}} = \frac{2}{\gamma+1} \frac{\left(2\,\mathrm{sh}^2 \frac{x+r}{2}\right)^{\frac{\gamma+1}{2}}}{\left(2\,\mathrm{ch}^2 \frac{x+r}{2}\right)^{\frac{1-\gamma}{2}}}$$

$$= \frac{2^{\gamma+1}}{\gamma+1} \frac{\left(\mathrm{sh}\, \frac{x+r}{2}\right)^{\gamma+1}}{\left(\mathrm{ch}\, \frac{x+r}{2}\right)^{1-\gamma}} \geq \frac{2^{2\gamma}}{\gamma+1} \left(\mathrm{sh}\, \frac{x+r}{2}\right)^{2\gamma}, \tag{2.6}$$

since $\mathrm{ch}\, \frac{t}{2} \leq 2\,\mathrm{sh}\, \frac{t}{2}$ for $t \geq 2$.

On the other hand, for $0 < \gamma \leq 1$,

$$\int_0^{x+r} (\text{sh } t)^\gamma dt = \int_0^{x+r} \left(2\,\text{sh}\,\frac{t}{2}\,\text{ch}\,\frac{t}{2}\right)^\gamma dt = 2^{\gamma+1} \int_0^{x+r} \left(\text{sh}\,\frac{t}{2}\right)^\gamma \left(\text{ch}\,\frac{t}{2}\right)^{\gamma-1} d\left(\text{sh}\,\frac{t}{2}\right)$$

$$\leq 2^{\gamma+1} \int_0^{x+r} \left(\text{sh}\,\frac{t}{2}\right)^{2\gamma-1} d\left(\text{sh}\,\frac{t}{2}\right) = \frac{2^\gamma}{\gamma}\left(\text{sh}\,\frac{x+r}{2}\right)^{2\gamma}. \tag{2.7}$$

Let $\gamma > 1$. Then

$$\int_0^{x+r} (\text{sh } t)^\gamma dt = 2^{\gamma+1} \int_0^{x+r} \left(\text{sh}\,\frac{t}{2}\right)^\gamma \left(\text{ch}\,\frac{t}{2}\right)^{\gamma-1} d\left(\text{sh}\,\frac{t}{2}\right)$$

$$\geq 2^{\gamma+1} \int_0^{x+r} \left(\text{sh}\,\frac{t}{2}\right)^{2\gamma-1} d\left(\text{sh}\,\frac{t}{2}\right) = \frac{2^\gamma}{\gamma}\left(\text{sh}\,\frac{x+r}{2}\right)^{2\gamma}, \tag{2.8}$$

and also

$$\int_0^{x+r} (\text{sh } t)^\gamma dt = 2^{\gamma+1} \int_0^{x+r} \left(\text{sh}\,\frac{t}{2}\right)^\gamma \left(\text{ch}\,\frac{t}{2}\right)^{\gamma-1} d\left(\text{sh}\,\frac{t}{2}\right)$$

$$\leq 2^{\gamma+1} \int_0^{x+r} \left(\text{sh}\,\frac{t}{2}\right)^{2\gamma-1} d\left(\text{sh}\,\frac{t}{2}\right) = \frac{2^\gamma}{\gamma}\left(\text{sh}\,\frac{x+r}{2}\right)^{2\gamma}. \tag{2.9}$$

Combining (2.6)–(2.9), we obtain

$$\left|H(x,r)\right|_{\frac{\gamma}{2}} \approx \left(\text{sh}\,\frac{x+r}{2}\right)^{2\gamma}, \quad \gamma > 0, \quad 2 \leq x+r < \infty. \tag{2.10}$$

Now, from (2.5) and (2.10), for $0 \leq x < r$, we have

$$\left|H(x,r)\right|_{\frac{\gamma}{2}} = \int_0^{x+r} (\text{sh } t)^\gamma dt \approx \begin{cases} \left(\text{sh}\,\dfrac{x+r}{2}\right)^{\gamma+1}, & 0 < x+r < 2, \\ \left(\text{sh}\,\dfrac{x+r}{2}\right)^{2\gamma}, & 2 \leq x+r < \infty. \end{cases} \tag{2.11}$$

Now, let $x \geq r$. Then from (2.3), (2.4), (2.7) and (2.9), for $\gamma > 0$, we get

$$\left|H(x,r)\right|_{\frac{\gamma}{2}} = \int_{x-r}^{x+r} (\text{sh } t)^\gamma dt \leq \int_0^{x+r} (\text{sh } t)^\gamma dt \lesssim \begin{cases} \left(\text{sh}\,\dfrac{x+r}{2}\right)^{\gamma+1}, & 0 < x+r < 2, \\ \left(\text{sh}\,\dfrac{x+r}{2}\right)^{2\gamma}, & 2 \leq x+r < \infty. \end{cases} \tag{2.12}$$

Now, let $\gamma > 0$ and $0 < x + r < 2$. Then

$$\left|H(x,r)\right|_\lambda = \int_{x-r}^{x+r} (\text{sh } t)^\gamma dt \geq \int_{\frac{x+r}{2}}^{x+r} (\text{sh } t)^\gamma dt \geq \frac{x+r}{2}\left(\text{sh}\,\frac{x+r}{2}\right)^\gamma \approx \left(\text{sh}\,\frac{x+r}{2}\right)^{\gamma+1},$$

since $\text{sh}\,t \approx t$ for $0 < t < 1$.

On the other hand, from (2.3) and (2.4), for $\gamma > 0$, we have

$$\left|H(x,r)\right|_\gamma = \int_{x-r}^{x+r} (\text{sh } t)^\gamma dt \leq \int_0^{x+r} (\text{sh } t)^\gamma dt \leq \frac{2^{\gamma+1}}{\gamma+1}\left(\text{sh}\,\frac{x+r}{r}\right)^{\gamma+1}.$$

In this way, by any $\gamma > 0$ and $0 < x + r < 2$,

$$\left|H(x,r)\right|_{\frac{\gamma}{2}} = \int\limits_{x-r}^{x+r} (\text{sh } t)^{\gamma} dt \approx \left(\text{sh } \frac{x+r}{2}\right)^{\gamma+1}. \tag{2.13}$$

Consider now the case where $2 \leq x + r < \infty$. Let $\gamma > 0$, then

$$\begin{aligned}\left|H(x,r)\right|_{\frac{\gamma}{2}} &= \int\limits_{x-r}^{x+r} (\text{sh } t)^{\gamma} dt \geq \int\limits_{\frac{x+r}{2}}^{x+r} (\text{sh } t)^{\gamma} dt = \int\limits_{\frac{x+r}{2}}^{x+r} \frac{(\text{sh } t)^{\gamma}}{\text{ch } t} d(\text{sh } t)\\
&\geq \frac{1}{2} \int\limits_{\frac{x+r}{2}}^{x+r} (\text{sh } t)^{\gamma-1} d(\text{sh } t) = \frac{1}{2\gamma}\left(\text{sh}^{\gamma}(x+r) - \text{sh}^{\gamma}\left(\frac{x+r}{2}\right)\right)\\
&\geq \frac{1}{2\gamma}\left(\text{sh}^{\gamma}(x+r) - \frac{1}{2^{\gamma}}\text{sh}^{\gamma}(x+r)\right) = \frac{1}{2\gamma}\left(1 - \frac{1}{2^{\gamma}}\right)\text{sh}^{\gamma}(x+r)\\
&= \frac{2^{\gamma}-1}{2\gamma}\left(\text{sh}^{\gamma}\frac{x+r}{2}\,\text{ch}^{\gamma}\frac{x+r}{2}\right) \geq \frac{2^{\gamma}-1}{2\gamma}\left(\text{sh } \frac{x+r}{2}\right)^{2\gamma}.\end{aligned} \tag{2.14}$$

On the other hand, we can get from (2.7) and (2.9) for $\gamma > 0$

$$\left|H(x,r)\right|_{\frac{\gamma}{2}} = \int\limits_{x-r}^{x+r} (\text{sh } t)^{\gamma} dt \leq \int\limits_{0}^{x+r} (\text{sh } t)^{\gamma} dt \leq \frac{2^{\gamma}}{\gamma}\left(\text{sh } \frac{x+r}{2}\right)^{2\gamma}. \tag{2.15}$$

Thus from (2.14) and (2.15), for $x \geq r$, we obtain

$$\left|H(x,r)\right|_{\frac{\gamma}{2}} = \int\limits_{x-r}^{x+r} (\text{sh } t)^{\gamma} dt \approx \begin{cases}\left(\text{sh } \dfrac{x+r}{2}\right)^{\gamma+1}, & 0 < x + r < 2,\\ \left(\text{sh } \dfrac{x+r}{2}\right)^{2\gamma}, & 2 \leq x + r < \infty.\end{cases} \tag{2.16}$$

The assertion of Lemma 2.1 follows from (2.11)–(2.16). □

Supposing $\gamma = 2\lambda$ in Lemma 2.3, we obtain

$$\left|H(x,r)\right|_{\lambda} \approx \begin{cases}\left(\text{sh } \dfrac{x+r}{2}\right)^{2\lambda+1}, & 0 < x + r < 2,\\ \left(\text{sh } \dfrac{x+r}{2}\right)^{4\lambda}, & 2 \leq x + r < \infty.\end{cases} \tag{2.17}$$

Since $\text{sh } t \approx \text{ch}\, t$ for $t \geq 1$, then from this for $x = 0$ we, in particular, get Lemma 2.1.

Lemma 2.4. *For a nonnegative function $f(\text{ch}\, x)$, $x \in \mathbb{R}_+$, the following relation*

$$\int\limits_0^r A^{\lambda}_{\text{ch}\, t} f(\text{ch } x)\,\text{sh}^{2\lambda} t\, dt \approx \int\limits_{H(x,r)} f(\text{ch}\, u)\,\text{sh}^{2\lambda} u\; du$$

holds.

Proof. In [15], it is proved that

$$J(x,r) = \int\limits_0^r A^{\lambda}_{\text{ch}\, t} f(\text{ch}\, x)\,\text{sh}^{2\lambda} t\; dt = C_{\lambda} \int\limits_{\text{ch}(x-r)}^{\text{ch}(x+r)} f(z)(z^2-1)^{\lambda-\frac{1}{2}} \int\limits_{\varphi(z,x,r)}^{1} (1-u^2)^{\lambda-1} du\; dz,$$

where

$$\varphi(z,x,r) = \frac{z \text{ ch } x - \text{ch } r}{\sqrt{z^2-1}\,\text{sh } x}, \quad -1 \leq \varphi(z,x,r) \leq 1, \quad C_{\lambda} = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma\left(\lambda\right)\Gamma\left(\frac{1}{2}\right)}.$$

Then

$$A(z,x,r) = C_\lambda \int\limits_{\varphi(z,x,r)}^{1} (1-u^2)^{\lambda-1} du \le C_\lambda \int\limits_{-1}^{1} (1-u^2)^{\lambda-1} du = 1.$$

We now estimate the integral $A(z,x,r)$ from below. Let $-1 \le \varphi(z,x,r) \le 0$. Then

$$A(z,x,r) = C_\lambda \int\limits_{\varphi(z,x,r)}^{1} (1-u^2)^{\lambda-1} du \ge C_\lambda \int\limits_{0}^{1} (1-u^2)^{\lambda-1} du$$

$$\ge 2^{\lambda-1} C_\lambda \int\limits_{0}^{1} (1-u)^{\lambda-1} du = \frac{2^{\lambda-1}}{\lambda} C_\lambda.$$

Now, let $0 \le \varphi(z,x,r) \le 1$, then

$$\begin{aligned}
A(z,x,r) &= C_\lambda \int\limits_{\varphi(z,x,r)}^{1} (1-u)^{\lambda-1}(1+u)^{\lambda-1} du \\
&= C_\lambda \int\limits_{0}^{1-\varphi(z,x,r)} u^{\lambda-1}(2-u)^{\lambda-1} du = C_\lambda \int\limits_{\frac{1}{1-\varphi(z,x,r)}}^{\infty} u^{-\lambda-1}\left(2-\frac{1}{u}\right)^{\lambda-1} du \\
&= C_\lambda \int\limits_{\frac{1}{1-\varphi(z,x,r)}}^{\infty} u^{-2\lambda}(2u-1)^{\lambda-1} du = 2^{2\lambda-1} C_\lambda \int\limits_{\frac{2}{1-\varphi(z,x,r)}}^{\infty} u^{-2\lambda}(u-1)^{\lambda-1} du \\
&= 2^{2\lambda-1} C_\lambda \int\limits_{\frac{1-\varphi(z,x,r)}{1+\varphi(z,x,r)}}^{\infty} (u+1)^{-2\lambda} u^{\lambda-1} du = 2^{2\lambda-1} \cdot C_\lambda \int\limits_{0}^{\frac{1+\varphi(z,x,r)}{1-\varphi(z,x,r)}} (1+u)^{-2\lambda} u^{\lambda-1} du \\
&\ge 2^{2\lambda-1} C_\lambda \int\limits_{0}^{1} (1+u)^{-2\lambda} u^{\lambda-1} du \ge 2^{2\lambda-1} C_\lambda \int\limits_{0}^{1} \frac{u^{\lambda-1}}{(1+u)^{2\lambda}} du \ge \frac{C_\lambda}{2} \int\limits_{0}^{1} u^{\lambda-1} du = \frac{C_\lambda}{2\lambda}\ .
\end{aligned}$$

Consequently,

$$A(z,x,r) = \int\limits_{\varphi(z,x,r)}^{1} (1-u^2)^{\lambda-1} du \approx 1$$

and

$$J(x,r) \approx \int\limits_{\operatorname{ch}(x-r)}^{\operatorname{ch}(x+r)} f(z)(z^2-1)^{\lambda-\frac{1}{2}} dz = \int\limits_{H(x,r)} f(\operatorname{ch} u) \operatorname{sh}^{2\lambda} u \, du. \qquad \square$$

Theorem 2.1. *For $0 \le x < \infty$ and $0 < r < \infty$, the relation*

$$M_G^\alpha f(\operatorname{ch} x) \approx M_\mu^\alpha f(\operatorname{ch} x)$$

is valid.

Proof. First, we prove that

$$M_G^\alpha f(\operatorname{ch} x) \lesssim M_\mu^\alpha f(\operatorname{ch} x).$$

We consider

$$A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x) = C_\lambda \int\limits_{0}^{\pi} \chi_{(0,r)}(x,t)_\varphi (\sin\varphi)^{2\lambda-1} d\varphi,$$

where $(x,t)_\varphi = \operatorname{ch} x \ \operatorname{ch} t - \operatorname{sh} x \ \operatorname{sh} t \cos\varphi$, and $\chi_{(0,r)}$ is the characteristic function on the interval $(0,r)$, i.e.,

$$\chi_{(0,r)}(x,t)_\varphi = \begin{cases} 1, & (x,t)_\varphi \le r, \\ 0, & (x,t)_\varphi > r. \end{cases}$$

Since

$$x - t < \operatorname{ch}(x-t) \le (x,t)_\varphi \le \operatorname{ch}(x+t),$$

we have $x - t > r \Longrightarrow (x,t)_\varphi > r$.

From the inequality $|x-t| > r$ it follows that $A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x) = 0$. This shows that the carrier at the function $A_{\operatorname{ch} t}\chi_{(0,r)}$ belongs to $H(x,r)$.

More generally, $x,t \in \mathbb{R}_+$

$$A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x) = C_\lambda \int_0^\pi \chi_{(0,r)}(x,t)_\varphi (\operatorname{sh}\ \varphi)^{2\lambda-1} d\varphi \le C_\lambda \int_0^\pi (\sin\varphi)^{2\lambda-1} d\varphi = 1.$$

We estimate $A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x)$:

$$A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x) = C_\lambda \int\limits_{\{\varphi\in[0,\pi]:(x,t)_\varphi < r\}} (\sin\varphi)^{2\lambda-1} d\varphi = A(x,t,r).$$

Making the substitution $\cos\varphi = y$, we obtain

$$A(x,t,r) = C_\lambda \int\limits_{\max\left\{-1,\ \frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t}\right\}}^{1} (1-y^2)^{\lambda-1} dy.$$

For any $x,t \in \mathbb{R}_+$, we have

$$\frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t} \ge -1 \Leftrightarrow \operatorname{ch} x\ \operatorname{ch} t + \operatorname{sh} x\ \operatorname{sh} t \ge r \Leftrightarrow \operatorname{ch}(x+t) \ge r.$$

Then in the case for $\operatorname{ch}(x-t) < r < \operatorname{ch}(x+t)$, we obtain

$$\begin{aligned} A(x,t,r) &\lesssim \int\limits_{\frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t}}^{1} (1-y^2)^{\lambda-1} dy \lesssim \int\limits_{\frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t}}^{1} (1-y)^{\lambda-1} dy \\ &\lesssim \left(\frac{r - \operatorname{ch}(x-t)}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda \lesssim \left(\frac{\operatorname{ch}(x+t) - \operatorname{ch}(x-t)}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda = \left(\frac{2\ \operatorname{sh} x\ \operatorname{sh} t}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda = 2^\lambda. \end{aligned} \tag{2.18}$$

On the other hand,

$$A(x,t,r) \lesssim \left(\frac{r - \operatorname{ch}(x-t)}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda \lesssim \left(\frac{\operatorname{ch}\ 2r - 1}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda = \left(\frac{2\ \operatorname{sh}^2 r}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda.$$

From this, for $t \ge x$, we have

$$A(x,t,r) \lesssim \left(\frac{\operatorname{sh}\ r}{\operatorname{sh}\ x}\right)^{2\lambda}. \tag{2.19}$$

We consider the case $0 < t < x$:

$$\begin{aligned} A(x,t,r) &\lesssim \int\limits_{\frac{\operatorname{ch} x\ \operatorname{ch} t - r}{\operatorname{sh} x\ \operatorname{sh} t}}^{1} (1-y^2)^{\lambda-1} dy \lesssim \int\limits_{\frac{\operatorname{ch} x\ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x\ \operatorname{sh} t}}^{1} (1-y)^{\lambda-1} dy \\ &\lesssim \left(1 - \frac{\operatorname{ch} x\ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x\ \operatorname{sh} t}\right)^\lambda \lesssim \left(1 - \left(\frac{\operatorname{ch} x\ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x\ \operatorname{sh} t}\right)^2\right)^\lambda. \end{aligned} \tag{2.20}$$

We find the extremum of the function

$$
\begin{aligned}
f(\operatorname{ch} t) &= 1 - \left(\frac{\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x \ \operatorname{sh} t}\right)^2, \\
(\operatorname{sh} t) f'(\operatorname{ch} t) &= -2\left(\frac{\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x \ \operatorname{sh} t}\right) \frac{\operatorname{sh}^2 t \ \operatorname{ch} x \ \operatorname{sh} x - \operatorname{sh} x \ \operatorname{ch} t \cdot (\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r)}{(\operatorname{sh} x \ \operatorname{sh} t)^2} \\
&= -2\left(\frac{\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} t}\right)\left(\frac{\operatorname{ch} x \ \operatorname{sh}^2 t - \operatorname{ch}^2 t \ \operatorname{ch} x + \operatorname{ch} t \ \operatorname{ch} r)}{\operatorname{sh}^2 x \ \operatorname{sh}^2 t}\right) \\
&= \frac{2(\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r)(\operatorname{ch} x - \operatorname{ch} t \ \operatorname{ch} r)}{\operatorname{sh}^3 t \ \operatorname{sh}^2 x}.
\end{aligned}
$$

Then it follows that

$$
f'(\operatorname{ch} t) = \frac{2(\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r)(\operatorname{ch} x - \operatorname{ch} t \ \operatorname{ch} r)}{\operatorname{sh}^4 t \ \operatorname{sh}^2 x}.
$$

From this, for $x > r$, we have

$$
\begin{aligned}
f_{\max}\left(\frac{\operatorname{ch} x}{\operatorname{ch} r}\right) &= 1 - \left(\frac{\operatorname{ch}^2 x - \operatorname{ch}^2 r}{\sqrt{\operatorname{ch}^2 x - \operatorname{ch}^2 r} \ \operatorname{sh} x}\right)^2 \\
&= 1 - \frac{\operatorname{ch}^2 x - \operatorname{ch}^2 r}{\operatorname{sh}^2 x} = \frac{\operatorname{ch}^2 r - 1}{\operatorname{sh}^2 x} = \left(\frac{\operatorname{sh} r}{\operatorname{sh} x}\right)^2.
\end{aligned}
$$

From this and (2.20), we obtain

$$
A(x,t,r) \lesssim \left(\frac{\operatorname{sh} r}{\operatorname{sh} x}\right)^{2\lambda}, \quad 0 < t < x, \ \ x > r. \tag{2.21}
$$

Thus from (2.18)–(2.21), we get

$$
A(x,t,r) \lesssim \min\left\{1, \left(\frac{\operatorname{sh} r}{\operatorname{sh} x}\right)^{2\lambda}\right\}.
$$

Consequently, for any $t \in H(x,r)$,

$$
A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x) \lesssim \min\left\{1, \ \left(\frac{\operatorname{sh} r}{\operatorname{sh} x}\right)^{2\lambda}\right\}, \quad x > r. \tag{2.22}
$$

We now have

$$
\begin{aligned}
M_G^\alpha f(\operatorname{ch} x) &\le M_{G.1}^\alpha f(\operatorname{ch} x) + M_{G.2}^\alpha f(\operatorname{ch} x) \\
&= \sup_{0 \le x < r < 2} \left|H(0,r)\right|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} \left|f(\operatorname{ch} t)\right| A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} t) d\mu_\lambda(t) \\
&+ \sup_{r \le x < 2} \left|H(0,r)\right|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} \left|f(\operatorname{ch} t)\right| A_{\operatorname{ch} t}\chi_{(0,r)} d\mu_\lambda(t).
\end{aligned} \tag{2.23}
$$

Let $0 \le x < r < 2$. From (2.22), it follows that $A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} x) \le 1$. From Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
M^{\alpha}_{G.1} f(\operatorname{ch} x) &\le \sup_{0\le x<r<2} |H(0,r)|_\lambda^{\frac{\alpha}{2\lambda+1}} \int\limits_{H(x,r)} \big|f(\operatorname{ch} t)\big| d\mu_\lambda(t) \\
&\le \sup_{0\le x<r<2} \frac{|H(0,r)|_\lambda^{\frac{\alpha}{2\lambda+1}-1} |H(x,r)|_\lambda^{\frac{\alpha}{2\lambda+1}-1}}{|H(x,r)|_\lambda^{\frac{\alpha}{2\lambda+1}-1}} \int\limits_{H(x,r)} |f(\operatorname{ch} t)| d\mu_\lambda(t) \\
&\lesssim \sup_{0\le x<r<2} \left(\frac{\operatorname{sh}\frac{r}{2}}{\operatorname{sh}\frac{r}{2}}\right)^{2\lambda+1-\alpha} |H(x,r)|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} \big|f(\operatorname{ch} t)\big| d\mu_\lambda(t) \\
&\lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x).
\end{aligned} \tag{2.24}$$

From Lemmas 2.1 and 2.2, and also (2.21), for $r < x < 2$. we have

$$\begin{aligned}
M^{\alpha}_{G.2} f(\operatorname{ch} x) &\lesssim \sup_{r<x<2} \left(\frac{\operatorname{sh}\frac{r}{2} \operatorname{sh}^{2\lambda} x}{\left(\operatorname{sh}\frac{r}{2}\right)^{2\lambda+1}}\right)^{1-\frac{\alpha}{2\lambda+1}} \left(\frac{\operatorname{sh}\ r}{\operatorname{sh}\ x}\right)^{2\lambda} M^{\alpha}_{\mu} f(\operatorname{ch} x) \\
&\lesssim \sup_{r<x<2} \left(\frac{\operatorname{sh}\ x}{\operatorname{sh}\frac{r}{2}}\right)^{2\lambda-\frac{2\lambda\alpha}{2\lambda+1}} \left(\frac{\operatorname{sh}\ r}{\operatorname{sh}\ x}\right)^{2\lambda} M^{\alpha}_{\mu} f(\operatorname{ch} x) \\
&\lesssim \sup_{r<x<2} \left(\frac{\operatorname{sh}\ \frac{r}{2}}{\operatorname{sh}\ x}\right)^{\frac{2\lambda\alpha}{2\lambda+1}} \left(2\operatorname{ch}\frac{r}{2}\right)^{2\lambda} M^{\alpha}_{\mu} f(\operatorname{ch} x) \\
&\lesssim \left(\frac{\operatorname{sh}\ \frac{r}{2}}{\operatorname{sh}\ r}\right)^{\frac{2\lambda\alpha}{2\lambda+1}} \left(2\operatorname{ch}\frac{r}{2}\right)^{2\lambda} M^{\alpha}_{\mu} f(\operatorname{ch} x) \lesssim \left(2\operatorname{ch}\frac{r}{2}\right)^{2\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)} M^{\alpha}_{\mu} f(\operatorname{ch} x) \\
&\lesssim (2\operatorname{ch} 1)^{2\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)} M^{\alpha}_{\mu} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x).
\end{aligned} \tag{2.25}$$

Taking into account (2.24) and (2.25) in (2.23), we obtain

$$M^{\alpha}_{G} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x), \quad 0 \le x < 2, \quad 0 < r < 2.$$

Now, let $0 \le x < r$ and $2 \le r < \infty$. Then

$$\begin{aligned}
M^{\alpha}_{G} f(\operatorname{ch} x) &\le M^{\alpha}_{G.1} f(\operatorname{ch} x) + M^{\alpha}_{G.2} f(\operatorname{ch} x) \\
&= \sup_{0\le x<r} |H(0,r)|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} |f(\operatorname{ch} t)|\, A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} t) d\mu_\lambda(t) \\
&+ \sup_{x\ge r} |H(0,r)|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} |f(\operatorname{ch} t)|\, A_{\operatorname{ch} t}\chi_{(0,r)}(\operatorname{ch} t) d\mu_\lambda(t).
\end{aligned} \tag{2.26}$$

Using Lemmas 2.1 and 2.2, for $2 \le r < \infty$, we get

$$\begin{aligned}
M^{\alpha}_{G.1} f(\operatorname{ch} x) &\lesssim \left(\frac{|H(0,r)|_\lambda}{|H(x,r)|_\lambda}\right)^{\frac{\alpha}{2\lambda+1}-1} |H(x,r)|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} |f(\operatorname{ch} t)|\, d\mu_\lambda(t) \\
&\lesssim \left(\frac{\operatorname{ch}^{2\lambda} r}{\operatorname{ch}^{4\lambda}\frac{r}{2}}\right)^{1-\frac{\alpha}{2\lambda+1}} M^{\alpha}_{\mu} f(\operatorname{ch} x) = \left(\frac{4^{\lambda}\operatorname{ch}^{2\lambda} r}{(1+\operatorname{ch} r)^{2\lambda}}\right)^{1-\frac{\alpha}{2\lambda+1}} M^{\alpha}_{\mu} f(\operatorname{ch} x) \\
&\lesssim 4^{\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)} M^{\alpha}_{\mu} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x).
\end{aligned}$$

Thus for $0 \le x < r$ and $2 \le r < \infty$,

$$M^{\alpha}_{G.1} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x). \tag{2.27}$$

We consider the cases $r \le x < \infty$ and $2 \le r < \infty$. We investigate the function

$$f(\operatorname{ch} x) = \frac{\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x \ \operatorname{sh} t} = \frac{\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r}{\sqrt{\operatorname{ch}^2 t - 1}\,\operatorname{sh} x}.$$

Putting $u = cht$, we obtain

$$f(u) = \frac{u \ \operatorname{ch} x - \operatorname{ch} r}{\sqrt{u^2 - 1}\,\operatorname{sh} x}.$$

We will now find extremum of the function

$$f'(u) = \frac{\sqrt{u^2-1}\,\operatorname{ch} x \ \operatorname{sh} x - u(u^2-1)^{-\frac{1}{2}}\operatorname{sh} x\,(u \ \operatorname{ch} x - \operatorname{ch} r)}{(u^2-1)\operatorname{sh}^2 x}$$

$$= \frac{(u^2-1)\operatorname{ch} x \ \operatorname{sh} x - u^2 \operatorname{sh} x \ \operatorname{ch} x + u \ \operatorname{ch} r \ \operatorname{sh} x}{(u^2-1)^{\frac{3}{2}}\operatorname{sh}^2 x} = \frac{u \ \operatorname{ch} r - \operatorname{ch} x}{(u^2-1)^{\frac{3}{2}}\operatorname{sh} x} = 0 \ \Leftrightarrow \ u = \frac{\operatorname{ch} x}{\operatorname{ch} r}.$$

By $u = \frac{\operatorname{ch} x}{\operatorname{ch} r}$, the function $f(u)$ has minimum

$$f_{\min}\left(\frac{\operatorname{ch} x}{\operatorname{ch} r}\right) = \frac{\operatorname{ch}^2 x - \operatorname{ch}^2 r}{\sqrt{\operatorname{ch}^2 x - \operatorname{ch}^2 r}\,\operatorname{sh} x}\frac{\sqrt{\operatorname{ch}^2 x - \operatorname{ch}^2 r}}{\operatorname{sh} x} = \frac{\operatorname{ch} x\sqrt{1 - \left(\frac{\operatorname{ch} r}{\operatorname{ch} x}\right)^2}}{\operatorname{sh} x} \sim \frac{\operatorname{sh} x}{\operatorname{ch} x},$$

as $x \to \infty$.

From this, by $x > r \ge 2$, we have

$$A(x,t,r) \lesssim \int\limits_{\frac{\operatorname{ch} x \ \operatorname{ch} t - \operatorname{ch} r}{\operatorname{sh} x \ \operatorname{sh} t}}^{1} (1-y^2)^{\lambda-1} dy \lesssim \int\limits_{\frac{\operatorname{sh} x}{\operatorname{ch} x}}^{1} (1-y)^{\lambda-1} dy$$

$$\lesssim \left(1 - \frac{\operatorname{sh} x}{\operatorname{ch} x}\right)^{\lambda} \lesssim \left(1 - \frac{\operatorname{sh}^2 x}{\operatorname{ch}^2 x}\right)^{\lambda} = (\operatorname{ch} x)^{-2\lambda}.$$

Then by Lemmas 2.1 and 2.2, we obtain

$$M^{\alpha}_{G.2} f(\operatorname{ch} x) \lesssim \sup_{r \ge 2}\left(\frac{\operatorname{ch}^{2\lambda} x \ \operatorname{ch}^{2\lambda} r}{\operatorname{ch}^{4\lambda} \frac{r}{2}}\right)^{1-\frac{\alpha}{2\lambda+1}} (\operatorname{ch} x)^{-2\lambda} M_{\mu} f(\operatorname{ch} x)$$

$$\lesssim (\operatorname{ch} r)^{-\frac{2\lambda\alpha}{2\lambda+1}} M^{\alpha}_{\mu} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x), \ \ 2 \le r < x < \infty. \tag{2.28}$$

Taking into account (2.27) and (2.28) on (2.26), we have

$$M^{\alpha}_{G} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x), \quad 0 \le x < \infty, \ \ 2 \le r < \infty. \tag{2.29}$$

Combining (2.27) and (2.29), we obtain

$$M^{\alpha}_{G} f(\operatorname{ch} x) \lesssim M^{\alpha}_{\mu} f(\operatorname{ch} x), \ \ 0 \le x < \infty, \ \ 0 < r < \infty. \tag{2.30}$$

Now we are going to prove that

$$M^{\alpha}_{\mu} f(\operatorname{ch} x) \lesssim M^{\alpha}_{G} f(\operatorname{ch} x). \tag{2.31}$$

From (2.17), it follows that $\|H(x,r)|_{\lambda} \ge |H(0,r)|_{\lambda}$, then by Lemma 2.4, we have

$$\left|H(x,r)\right|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(x,r)} \left|f(\operatorname{ch} u)\right| d\mu_{\lambda}(u) \lesssim \left|H(0,r)\right|_{\lambda}^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{0}^{r} A_{\operatorname{ch} t} f(\operatorname{ch} x) d\mu_{\lambda}(t),$$

from which we get (2.31).

The assertion of Theorem 2.1 follows from (2.30) and (2.31). □

Theorem 2.2 (Lebesgue differentiation theorem)**.** *Let f be a nonnegative monotone nondecreasing function and let $f \in L^{loc}_{p,\lambda}(\mathbb{R}_+)$, $1 \le p < \infty$. Then*

$$\lim_{r\to 0} \frac{1}{|H(x,r)|_{\lambda}} \int\limits_{H(x,r)} f(\operatorname{ch} y)^p \operatorname{sh}^{2\lambda} y \ dy = f(\operatorname{ch} x)^p$$

for almost every $x \in \mathbb{R}_+$.

Proof. By the locality of the problem, we may assume that $f \in L_{1,\lambda}(\mathbb{R}_+)$. In a general case, one can multiply f by a characteristic function of the interval $[0, r)$ and obtain the required convergence almost everywhere interior of this interval. Then by tending r to infinity, one can obtain it for the whole interval $[0, \infty)$. Suppose for any $r > 0$ and for any $x \in [0, \infty)$,

$$f_r(\operatorname{ch} x) = \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} f(\operatorname{ch} y)^p \operatorname{sh}^{2\lambda} y \, dy$$

and

$$\Omega_f(\operatorname{ch} x) = \Big| \varlimsup_{r \to 0} f_r(\operatorname{ch} x) - \varliminf_{r \to 0} f_r \ (\operatorname{ch} x) \Big|.$$

Then we have

$$\Omega_f(\operatorname{ch} x) \le 2 \sup_{r>0} \big| f_r(\operatorname{ch} x) \big| = 2 M_G \, f(\operatorname{ch} x).$$

First we show that for any $\beta > 0$,

$$\big| x \in \mathbb{R}_+ : \Omega_f(\operatorname{ch} x) > \beta \big|_\lambda = 0. \tag{2.32}$$

In fact, as is known, the set of all continuous functions with compact support in $\mathbb{R}_+$ is dense in $L_{p,\lambda}(\mathbb{R}_+)$ (see [21], Theorem 4.2).

Therefore for any number $\varepsilon > 0$, there exists a continuous function h with a compact support in $\mathbb{R}_+$ such that

$$\big\| f - h \big\|_{L_{p,\lambda}(\mathbb{R}_+)} < \varepsilon.$$

Suppose $g = f - h$, then $g \in L_{p,\lambda}(\mathbb{R}_+)$ and

$$\big\| g \big\|_{L_{p,\lambda}(\mathbb{R}_+)} < \varepsilon.$$

Thus, if $f \in L_{p,\lambda}(\mathbb{R}_+)$, then for any $\varepsilon > 0$, there exist a continuous function h with a compact support and a function $g \in L_{p,\lambda}(\mathbb{R})_+$, with the condition $\big\| g \big\|_{L_{p,\lambda}(\mathbb{R}_+)} < \varepsilon$ such that $f = h + g$. But $\Omega_f \le \Omega_h + \Omega_g$. If g is a continuous function with a compact support on $\mathbb{R}_+$, then g_r converges to g and, consequently, in this case we get $\Omega_g(\operatorname{ch} x) \equiv 0$. Therefore, for any $\beta > 0$ (see [15], Theorem 2.2),

$$\big| x \in \mathbb{R}_+ : \Omega_g(\operatorname{ch} x) > \beta \big|_\lambda \lesssim \frac{1}{\beta} \big\| g \big\|_{L_{1,\lambda}(\mathbb{R}_+)} \lesssim \frac{\varepsilon}{\beta}.$$

Since ε is arbitrarily small, we get (2.32), from which it follows that $\lim\limits_{r \to 0} f_r(\operatorname{ch} x)$ exists for almost everywhere on $\mathbb{R}_+$. Further, we have

$$\lim_{r \to 0} \big\| f_r - f \big\|_{L_{p,\lambda}} = \lim_{r \to 0} \left(\int\limits_{\mathbb{R}_+} \left| \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy - f(\operatorname{ch} x) \right|^p \operatorname{sh}^{2\lambda} x \, dx \right)^{\frac{1}{p}}.$$

By Lemma 2.3, we have $|H(x,r)|_\lambda \geq |H(0,r)|_\lambda$ and therefore, we find

$$\begin{aligned}
\lim_{r\to 0}\big\|f_r - f\big\|_{L_{p,\lambda}} &\lesssim \lim_{r\to 0}\left\{\int\limits_{\mathbb{R}^+}\left[\frac{1}{\big|H(x,r)\big|_\lambda}\int\limits_{x-r}^{x+r}\big|f(\operatorname{ch} y) - f(\operatorname{ch} x)\big|\operatorname{sh}^{2\lambda} y\,dy\right]^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}} \\
&\lesssim \lim_{r\to 0}\left\{\int\limits_{\mathbb{R}^+}\left[\frac{1}{\big|H(0,r)\big|_\lambda}\int\limits_{-r}^{r}\big|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)\big|\operatorname{sh}^{2\lambda}(x-y)dy\right]^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}} \\
&\lesssim \lim_{r\to 0}\left\{\int\limits_{\mathbb{R}^+}\left[\frac{\operatorname{sh}^{2\lambda} r}{\big|H(0,r)\big|_\lambda}\int\limits_{-r}^{r}\big|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)\big|dy\right]^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}} \\
&\lesssim \lim_{r\to 0}\frac{\operatorname{sh}^{2\lambda} r}{|H(0,r)|_\lambda}\int\limits_{-r}^{r}\left\{\int\limits_{\mathbb{R}^+}|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)|^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}} dy \\
&\lesssim \lim_{r\to 0}\frac{\operatorname{sh}^{2\lambda}\frac{r}{2}\operatorname{ch}^{2\lambda}\frac{r}{2}}{(\operatorname{sh}\frac{r}{2})^{2\lambda+1}}\int\limits_{-r}^{r}\left\{\int\limits_{\mathbb{R}^+}|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)|^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}} dy
\end{aligned}$$

(since by Lemma 2.1, $\operatorname{sh} r \approx r$)

$$\lesssim \lim_{r\to 0}\frac{1}{2r}\int\limits_{-r}^{r}\left\{\int\limits_{\mathbb{R}^+}\big|f(\operatorname{ch}(x-y)) - f(\operatorname{ch} x)\big|^p \operatorname{sh}^{2\lambda} x\,dx\right\}^{\frac{1}{p}} dy = 0. \tag{2.33}$$

Further, by the monotonicity of the function f, we have

$$\begin{aligned}
f(\operatorname{ch}(x-y)) = f(\operatorname{ch} x\operatorname{ch} y - \operatorname{sh} x\operatorname{sh} y) &= \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)}\int\limits_0^\pi f(\operatorname{ch} x\operatorname{ch} y - \operatorname{sh} x\operatorname{sh} y)(\sin\varphi)^{2\lambda-1}d\varphi \\
&\leq \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)}\int\limits_0^\pi f\big(\operatorname{ch} x\operatorname{ch} y - \operatorname{sh} x\operatorname{sh} y\cos\varphi\big)(\sin\varphi)^{2\lambda-1}d\varphi = A_{\operatorname{ch} y} f(\operatorname{ch} x).
\end{aligned}$$

Then we have

$$\begin{aligned}
\lim_{r\to 0}\|f_r - f\|_{L_{p,\lambda}} &\lesssim \lim_{r\to 0}\frac{1}{2r}\int\limits_{-r}^{r}\left\{\int\limits_{\mathbb{R}^+}|A_{\operatorname{ch} y} f(\operatorname{ch} x) - f(\operatorname{ch} x)|^p \operatorname{sh}^{2\lambda} x\right\}^{\frac{1}{p}} dy \\
&\lesssim \lim_{r\to 0}\frac{1}{2r}\int\limits_{-r}^{r}\|A_{\operatorname{ch} y} f - f\|_{L_{p,\lambda}}\,dy = 0,
\end{aligned}$$

since $\sup\limits_{0<y\leq r}\big\|A^\lambda_{\operatorname{ch} y} f - f\big\|_{L_{p,\lambda}} = \omega_f(r)$ as $r\to 0$ (see [15], proof of Corollary 2.1).

From (2.33), it follows that there exists a subsequence r_k satisfying $r_k \to 0$ as $k\to\infty$ such that

$$\lim_{k\to\infty} f_{r_k}(\operatorname{ch} x) = f(\operatorname{ch} x)$$

for a.e. $x\in\mathbb{R}_+$. Because $\lim\limits_{r\to 0} f_r(\operatorname{ch} x)$ exists for a.e. $x\in\mathbb{R}_+$, thus

$$\lim_{r\to 0}\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)} f(\operatorname{ch} y)^p \operatorname{sh}^{2\lambda} y\,dy = f(\operatorname{ch} x)^p,$$

which is the desired conclusion. □

Applying the Lebesgue differentiation theorem, we may give a decomposition of $\mathbb{R}_+$, called as Calderon–Zygmund decomposition, which is extremely useful in harmonic analysis.

Theorem 2.3. *Suppose that f is a nonnegative integrable function on $\mathbb{R}_+$. Then for any fixed number $\beta > 0$, there exists a sequence $\{H_j(x_j, r_j)\} = \{H_j\}$ of disjoint intervals such that*

(1) $f(\operatorname{ch} x) \le \beta$, $x \notin \bigcup_j H_j$;

(2) $\left|\bigcup_j H_j\right|_\lambda \le \frac{1}{\beta}\|f\|_{1,\lambda}$;

(3) $\beta < \frac{1}{|H_j|_\lambda} \int_{H_j} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \lesssim 2^{(2\lambda+1)n}\beta$, $n = 1, 2, \dots$.

Proof. Since $f \in L_{1,\lambda}(\mathbb{R}_+)$, we may decompose $\mathbb{R}_+$ into a net of equal intervals (by the Lindelof covering theorem this is possible (see [24])) such that for every H, from the net

$$\frac{1}{|H|_\lambda} \int_H f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \le \beta. \tag{2.34}$$

In fact, for any $\beta > 0$, $\exists \delta = \delta(\beta) > 0$ and for every H_j with measure $|H_j|_\lambda = |H|_\lambda < \delta$, we have

$$\int_{H_j} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy < \beta, \quad j = 1, 2, \dots,$$

where $H_j = (x_j - r, x_j + r)$ and $|H|_\lambda = |H_j|_\lambda = \int_{x_j - r}^{x_j + r} \operatorname{sh}^{2\lambda} y \, dy$, $(j = 1, 2, \dots)$.

First, we prove (3).

Let $H_1 = (x_1 - r,\ x_1 + r)$ be a fixed interval in the net. Then by (2.33), we can write

$$\frac{1}{|H_1|_\lambda} \int_{H_1} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \le \beta. \tag{2.35}$$

We divide the interval H_1 into 2^n equal intervals and let $H_1' = \left(\frac{x_1 - r}{2^n}, \frac{x_1 + r}{2^n}\right)$ be one of those intervals. By (2.17), we have

$$|H_1'|_\lambda = \int_{\frac{x_1 - r}{2^n}}^{\frac{x_1 + r}{2^n}} \operatorname{sh}^{2\lambda} y \, dy \approx \left(\operatorname{sh} \frac{x_1 + r}{2^{n+1}}\right)^{2\lambda+1}, \quad 0 < \frac{x_1 + r}{2^n} < 2.$$

Since for $0 < t < 1$, $\operatorname{sh} t \approx t$, we obtain

$$|H_1'|_\lambda \approx \left(\operatorname{sh} \frac{x_1 + r}{2^{n+1}}\right)^{2\lambda+1} \approx \left(\frac{x_1 + r}{2^{n+1}}\right)^{2\lambda+1} \approx \left(\frac{1}{2^n} \operatorname{sh} \frac{x_1 + r}{2}\right)^{2\lambda+1} = 2^{-(2\lambda+1)n}\left|tH_1\right|_\lambda. \tag{2.36}$$

There exist possibly two cases concerning H_1':

$$\text{(A)} \qquad \frac{1}{|H_1'|_\lambda} \int_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy > \beta,$$

$$\text{(B)} \qquad \frac{1}{|H_1'|_\lambda} \int_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \le \beta.$$

In case (A), from (2.35) and (2.36), we obtain

$$\begin{aligned} \beta &< \frac{1}{|H_1'|_\lambda} \int_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \approx \frac{2^n}{|H_1|_\lambda} \int_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \\ &\lesssim \frac{2^n}{|H_1|_\lambda} \int_{H_1} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \lesssim 2^{(2\lambda+1)n}\beta. \end{aligned}$$

Now, for H_1', we choose a sequence $\{H_j\}$.

We consider case (B). Suppose $H_1' = H_2(x_2 - r,\ x_2 + r)$. Dividing this interval into 2^n equal parts, we obtain

$$\beta < \frac{1}{|H_2'|_\lambda} \int\limits_{H_2'} f(\operatorname{ch} y)\, \operatorname{sh}^{2\lambda} y\, dy \lesssim \frac{2^n}{|H_1'|_\lambda} \int\limits_{H_1'} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le 2^{(2\lambda+1)n}\beta,$$

where for H_2', we choose a sequence $\{H_j\}$. Continuing this process, we obtain a sequence of disjoint intervals $\{H_j\}$ such that

$$\beta < \frac{1}{|H_j|_\lambda} \int\limits_{H_j} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \lesssim 2^{(2\lambda+1)n}\beta, \quad (j = 1, 2, \dots).$$

Proof of (1). Taking into account (2.34), from Theorem 2.2, we have

$$f(\operatorname{ch} x) = \lim_{r\to 0} \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le \beta$$

for a.e. $x \notin \bigcup\limits_j H_j$.

Proof of (2). Passing to the limit by $n \to \infty$ in the inequality

$$\Big| \bigcup_{j=1,2,\dots,n} H_j(x_j, r_j)\Big|_\lambda \le \sum_{j=1}^{n} \big|H_j(x_j, r_j)\big|_\lambda \le \frac{1}{\beta} \sum_{j=1}^{n} \int\limits_{H_j(x_j, 3r_j)} f(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy,$$

which is contained in the proof of Theorem 2.2, from [15], we obtain approval (2). □

Remark 2.1. The Caldeon-Zygmund decomposition stay valid if we replace $\mathbb{R}_+$ by a fixed interval $H_0(x_0, r_0)$ for $f \in L_{p,\lambda}(H_0)$.

3. Weighted $(L_{p,\omega,\lambda}, L_{q,\omega,\lambda})$-Boundedness of the Fractional Maximal Operator Generated by Gegenbauer Differential Operator

In this section, we prove the weighted $(L_{p,\omega,\lambda}, L_{q,\omega,\lambda})$-boundedness of the fractional maximal operator M_G^α (G-fractional maximal operator) generated by the Gegenbauer differential operator.

We need the following theorem.

Theorem (Marcinkiewicz interpolation theorem, [3, n.3.2., p. 43])**.** *Let $(\mathbb{R}_+, \varphi)$ and $(\mathbb{R}_+, \nu)$ be two measure spaces and let the sublinear operator T be both of weak type (p_0, p_0) and of weak type (p_1, p_1) for $1 \le p_0 < p_1 \le \infty$, that is, there exists a constant $C_0 > 0$ such that for any $\alpha > 0$,*

$$\text{(a)} \qquad \nu\Big(\Big\{x \in \mathbb{R}_+ : |Tf(\operatorname{ch} x)| > \alpha\Big\}\Big) \le \Big(\frac{C_0}{\alpha}\big\|f\big\|_{p_0,\varphi}\Big)^{p_0},$$

$$\text{(b)} \qquad \nu\Big(\Big\{x \in \mathbb{R}_+ : |Tf(\operatorname{ch} x)| > \alpha\Big\}\Big) \le \Big(\frac{C_0}{\alpha}\big\|f\big\|_{p_1,\varphi}\Big)^{p_1}, \quad p_1 < \infty.$$

If $p_1 = \infty$, then the weak type and strong type coincide by the definition

$$\big\|Tf\big\|_{\infty,\nu} \lesssim \big\|f\big\|_{\infty,\varphi}.$$

Then T is also of the type (p,p) for all $p_0 < p < p_1$, i.e., for any $f \in L_p(\mathbb{R}_+, \varphi)$,

$$\int\limits_{\mathbb{R}_+} \big|Tf(\operatorname{ch} x)\big|^p d\nu(x) \lesssim \int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p d\varphi(x).$$

Denote by $L_{p,\omega,\lambda}(\mathbb{R}_+, G)$ the set of measurable functions on $\mathbb{R}_+$ with a finite norm

$$\big\|f\big\|_{L_{p,\omega,\lambda}(\mathbb{R}_+,G)} = \Bigg(\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p\, d\omega_\lambda(x)\Bigg)^{\frac{1}{p}} < \infty, \ \ 1 \le p < \infty,$$

where $d\omega_\lambda(x) = \omega(\operatorname{ch} x) d\mu_\lambda(x)$.

The following theorem is a version of the Fefferman-Stein inequality.

Theorem 3.1. *Let $1 \le p < \infty$ and g be a nonnegative function such that $g \in L^{loc}_{1,\omega,\lambda}(\mathbb{R}_+, G)$. Then for any function $f \in L_{p,\omega,\lambda}(\mathbb{R}_+, G)$, the following inequality*

$$\int_{\mathbb{R}_+} (M_\mu f(\operatorname{ch} x))^p \, g(\operatorname{ch} x) d\mu_\lambda(x) \le \int_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p \, M_\mu g(\operatorname{ch} x) d\mu_\lambda(x)$$

is valid.

Proof. Without loss of generality, we may assume that $M_\mu g(\operatorname{ch} x) < \infty$, a.e. $x \in \mathbb{R}_+$ and $M_\mu g(\operatorname{ch} x) > 0$. If we denote $d\nu_\lambda(x) = g(\operatorname{ch} x) d\mu_\lambda(x)$ and $d\varphi_\lambda(x) = M_\mu g(\operatorname{ch} x) d\mu_\lambda(x)$, then by the Marcinkiewicz interpolation theorem for the validity of our assertion it suffices to prove that M_μ is both of type $(L_{\infty,\varphi}, L_{\infty,\nu})$ and of weak type $(L_{1,\varphi,\lambda}, L_{1,\nu,\lambda})$.

Let us first show that M_μ is one of the type $(L_{\infty,\varphi}, L_{\infty,\nu})$. In fact, if $\|f\|_{\infty,\varphi} \le a < \infty$, then

$$\int_{\{x \in \mathbb{R}_+ : |f(\operatorname{ch} x)| > a\}} M_\mu g(\operatorname{ch} x) d\mu_\lambda(x) = \Big|\Big\{x \in \mathbb{R}_+ : \big|f(\operatorname{ch} x)\big| > a\Big\}\Big|_{\mu_\lambda} = 0.$$

Since $M_\mu g(\operatorname{ch} x) > 0$ for any $x \in \mathbb{R}_+$, we get $\Big|\Big\{x \in \mathbb{R}_+ : |f(\operatorname{ch} x)| > a\Big\}\Big|_{\mu_\lambda} = 0$, equivalently, $\big|f(\operatorname{ch} x)\big| \le a$, a.e. $x \in \mathbb{R}_+$. Thus $M_\mu f(\operatorname{ch} x) \le a$, a.e. $x \in \mathbb{R}_+$, and thus it follows that $\|M_\mu f\|_{\infty,\nu_\lambda} \le a$.

Therefore $\big\|M_\mu f\big\|_{\infty,\nu_\lambda} \le \|f\|_{\infty,\varphi_\lambda}$.

Now we can show that M_μ has weak type $(L_{1,\varphi,\lambda}, L_{1,\nu,\lambda})$. For this we need to prove that for any $\alpha > 0$ and $f \in L_{1,\varphi,\lambda}(\mathbb{R}_+)$

$$\int_{\{x \in \mathbb{R}_+ : M_\mu f(\operatorname{ch} x) > \alpha\}} g(\operatorname{ch} x) \, d\mu_\lambda(x) \lesssim \frac{1}{\alpha} \int_{\mathbb{R}_+} f(\operatorname{ch} x) \, M_\mu g(\operatorname{ch} x) d\mu_\lambda(x).$$

By Theorem 2.3 (3), we have

$$\int_{H_i} f(\operatorname{ch} x) M_\mu g(\operatorname{ch} x) d\mu_\lambda(x) \ge \int_{H_i} f(\operatorname{ch} x) \left(\frac{1}{|H_i|_\lambda} \int_{H_i} g(\operatorname{ch} t) d\mu_\lambda(t) \right) d\mu_\lambda(x)$$
$$\approx \alpha \int_{H_i} g(\operatorname{ch} u) d\mu_\lambda(u).$$

Summing over i, we obtain

$$\int_{\mathbb{R}_+} f(\operatorname{ch} x) M_\mu g(\operatorname{ch} x) d\mu_\lambda(x) \ge \alpha \int_{\mathbb{R}_+} g(\operatorname{ch} u) d\mu_\lambda(u)$$
$$\ge \alpha \int_{\{u \in \mathbb{R}_+ : M_\mu f(\operatorname{ch} u) > \alpha\}} g(\operatorname{ch} u) \, d\mu_\lambda(u).$$

Thus M_μ has weak type $(L_{1,\varphi,\lambda}, L_{1,\nu,\lambda})$ and the Fefferman-Stein inequality follows from the Marcinkiewicz interpolation theorem by $p_0 = 1$ and $p_1 = \infty$. □

Theorem 3.2. *The Chebychev type inequality*

$$\Big|\Big\{x \in \mathbb{R}_+ : M_\mu f(\operatorname{ch} x) > \alpha\Big\}\Big|_\omega \le \frac{1}{\alpha} \int_{\mathbb{R}_+} M_\mu f(\operatorname{ch} x) \, d\omega_\lambda(x)$$

is valid for all $\alpha > 0$ and $t > 0$.

Proof. Since

$$M_\mu f(\operatorname{ch} x) \ge \alpha \chi_{\{M_\mu f(\operatorname{ch} x) > \alpha\}}(\operatorname{ch} x),$$

we have

$$\int\limits_{\mathbb{R}_+} M_\mu f(\operatorname{ch} x)\, d\omega_\lambda(x) \geq \alpha \int\limits_{\mathbb{R}_+} \chi_{\{M_\mu f(\operatorname{ch} x)>\alpha\}}(\operatorname{ch} x) d\omega_\lambda(x) = \alpha \Big|\{x \in \mathbb{R}_+ : M_\mu f(\operatorname{ch} x) > \alpha\}\Big|_\omega .$$

Thus our assertion is proved. □

Definition 3.1. The weight function ω belongs to the class $A_p^\lambda(\mathbb{R}_+)$ for $1 < p < \infty$ if

$$\sup_{x\in\mathbb{R}_+, r>0} \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} u) \operatorname{sh}^{2\lambda} u \, du \right) \times \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} u)^{-\frac{1}{p-1}} \operatorname{sh}^{2\lambda} u \, du \right)^{p-1} < \infty \tag{3.1}$$

and ω belongs to $A_1^\lambda(\mathbb{R}_+)$ if there exists a positive constant C such that for any $x \in \mathbb{R}_+$ and $r > 0$

$$M_G\, \omega(\operatorname{ch} x) \leq C\omega(\operatorname{ch} x). \tag{3.2}$$

Remark 3.1. Inequality (3.2) is equivalent to the inequality

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \leq C \operatorname*{ess\,inf}_{y\in H(x,r)} \omega(\operatorname{ch} y). \tag{3.3}$$

Remark 3.2. In inequalities (3.2) and (3.3), for $C \geq 1$, by Hölder's inequalility, we have

$$\begin{aligned} 1 &= \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y)^{\frac{1}{p}} \omega(\operatorname{ch} y)^{-\frac{1}{p}} \operatorname{sh}^{2\lambda} y \, dy \\ &\leq \left\{ \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \right) \right. \\ &\quad \times \left. \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y)^{-\frac{1}{p-1}} \operatorname{sh}^{2\lambda} y \, dy \right)^{p-1} \right\}^{\frac{1}{p}} \leq C^{\frac{1}{p}}. \end{aligned}$$

We show that $\operatorname{sh}^\alpha u \in A_p^\lambda(\mathbb{R}_+)$, $1 < p < \infty$, if and only if $-(2\lambda+1) < \alpha < (2\lambda+1)(p-1)$ and $\operatorname{sh}^\alpha u \in A_1^\lambda(\mathbb{R}_+)$ if and only if $-(2\lambda+1) < \alpha \leq 0$.

By using Lemma 2.3, for $\gamma = 2\lambda - \frac{\alpha}{p-1}$ and (2.17) for $0 < x + r < 2$, we obtain

$$\begin{aligned} \left(\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} (\operatorname{sh} u)^{2\lambda - \frac{\alpha}{p-1}} du \right)^{p-1} &\approx \left(\frac{\left(\operatorname{sh} \frac{x+r}{2}\right)^{2\lambda+1-\frac{\alpha}{p-1}}}{\left(\operatorname{sh} \frac{x+r}{2}\right)^{2\lambda+1}} \right)^{p-1} \\ &= \left(\operatorname{sh} \frac{x+r}{2} \right)^{-\alpha}, \quad \alpha < (2\lambda+1)(p-1), \end{aligned}$$

and also for $\gamma = \alpha + 2\lambda$ and (2.18),

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} (\operatorname{sh} u)^{\alpha+2\lambda} \approx \left(\operatorname{sh} \frac{x+r}{2} \right)^{-\alpha}, \quad \alpha > -2\lambda - 1.$$

Taking into account the relation in (3.1), we obtain that for

$$-(2\lambda+1) < \alpha < (2\lambda+1)(p-1)$$

$\operatorname{sh}^\alpha u \in A_p^\lambda(\mathbb{R}_+)$.

Now, let $2 \le x+r < \infty$. Then assuming $\gamma = 2\lambda - \dfrac{\alpha}{p-1}$ in Lemma 2.3 and using (2.18), we obtain

$$\left(\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)} (\operatorname{sh} u)^{2\lambda-\frac{\alpha}{p-1}}du\right)^{p-1} \approx \left(\operatorname{sh}\frac{x+r}{2}\right)^{-2\alpha}, \quad \alpha < (2\lambda+1)(p-1).$$

and also for $\gamma = \alpha + 2\lambda$,

$$\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)} (\operatorname{sh} u)^{\alpha+2\lambda}du \approx \left(\operatorname{sh}\frac{x+r}{2}\right)^{2\alpha}, \quad -(2\lambda+1) < \alpha \le 0.$$

That is, for $2 \le x+r < \infty$, $-(2\lambda+1) < \alpha < (2\lambda+1)(p-1)$ $(\operatorname{sh} u)^\alpha \in A_p^\lambda(\mathbb{R}_+)$ with $1 < p < \infty$.

Let $p = 1$, then for $0 < x+r < 2$ and $\gamma = \alpha + 2\lambda$, we have

$$\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)} (\operatorname{sh} u)^{\alpha+2\lambda}du \approx \left(\operatorname{sh}\frac{x+r}{2}\right)^{\alpha}, \quad -(2\lambda+1) < \alpha \le 0,$$

and for $2 \le x+r < \infty$ and $\gamma = \alpha + 2\lambda$

$$\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)} (\operatorname{sh} u)^{\alpha+2\lambda}du \approx \left(\operatorname{sh}\frac{x+r}{2}\right)^{\alpha} \lesssim \left(\operatorname{sh}\frac{x+r}{2}\right)^{\alpha}, \quad (2\lambda+1) < \alpha \le 0.$$

Thus, for any $0 < x+r < \infty$,

$$(\operatorname{sh} u)^\alpha \in A_1^\lambda(\mathbb{R}_+), \quad -(2\lambda+1) < \alpha \le 0.$$

We are going to prove some properties of $A_1^\lambda(\mathbb{R}_+)$, which we will need later. Note that in proving these properties and Theorem 3.3, we use the outline from [23].

Proposition 3.1. *If $1 \le p < q < \infty$, then $A_p^\lambda(\mathbb{R}_+) \subsetneqq A_1^\lambda(\mathbb{R}_+)$. In fact, by Hölder's inequality, we have*

$$\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\, dy \le \left(\int\limits_{H(x,r)} \omega^{-\frac{k}{q-1}}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\, dy\right)^{\frac{1}{k}} \left(\int\limits_{H(x,r)} \operatorname{sh}^{2\lambda} y\, dy\right)^{\frac{k-1}{k}}.$$

Supposing here $k = \frac{q-1}{p-1}$, we obtain

$$\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\, dy \le \left(\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\, dy\right)^{\frac{p-1}{q-1}} \left(\int\limits_{H(x,r)} \operatorname{sh}^{2\lambda} y\, dy\right)^{\frac{q-p}{q-1}},$$

whence we have

$$\begin{aligned}
&\left(\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\, dy\right)^{q-1} \\
&\qquad \le \frac{1}{|H(x,r)|_\lambda^{q-1}}\left(\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\, dy\right)^{p-1}\left(\int\limits_{H(x,r)} \operatorname{sh}^{2\lambda} y\, dy\right)^{q-p} \\
&\qquad = \frac{|H(x,r)|_\lambda^{q-p}}{|H(x,r)|_\lambda^{q-1}}\left(\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\, dy\right)^{p-1} \\
&\qquad = \left(\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)} \omega^{-\frac{1}{q-1}}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\, dy\right)^{p-1}.
\end{aligned}$$

If $p=1$, then by (3.3), we have

$$\left(\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)}\omega^{-\frac{1}{q-1}}(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\;dy\right)^{q-1}\le ess\sup_{y\in H(x,r)}\omega^{-1}(\operatorname{ch}y)=\left(\operatorname*{ess\,inf}_{y\in H(x,r)}\omega(\operatorname{ch}y)\right)^{-1}$$
$$\le C\left(\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\;dy\right)^{-1}.$$

Thus, if $\omega\in A_p^\lambda(\mathbb{R}_+)$, then $\omega\in A_q^\lambda(\mathbb{R}_+)$ for $q>p$. On the other hand, $(\operatorname{sh}u)^\alpha\in A_p^\lambda(\mathbb{R}_+)$, if and only if $-(2\lambda+1)<\alpha<(2\lambda+1)(p-1)$, therefore $A_p^\lambda(\mathbb{R}_+)\ne A_q^\lambda(\mathbb{R}_+)$.

Proposition 3.2. *If $\omega\in A_p^\lambda(\mathbb{R}_+)$ $(1\le p<\infty)$, then for any $\alpha\in(0,1)$, there exists $\beta\in(0,1)$ such that for any measurable set $E\subset H$, $|E|_\lambda\le\alpha|H|_\lambda$ and $\omega(E)\le\beta\;\omega\;(H)$, where $\omega(A)=\int_A\omega(\operatorname{ch}x)\operatorname{sh}^{2\lambda}x\;dx$.*

Proof. In fact, let $S=H\backslash E$ and $f(\operatorname{ch}x)=\chi_s(\operatorname{ch}x)$. Then

$$\left(\frac{|S(x,r)|_\lambda}{|H(x,r)|_\lambda}\right)^p\omega(H)\le C\int\limits_{S(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\;dy$$
$$\le C\left(\int\limits_{H(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\;dy-\int\limits_{E(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\;dy\right)=C\left(\omega(H)-\omega(E)\right).$$

Further,

$$\big|E(x,r)\big|_\lambda\le\alpha\big|H(x,r)\big|_\lambda\Leftrightarrow\frac{|E(x,r)|_\lambda}{|H(x,r)|_\lambda}\le\alpha$$
$$\Leftrightarrow-\frac{|E(x,r)|_\lambda}{|H(x,r)|_\lambda}\ge-\alpha\Leftrightarrow1-\alpha\le1-\frac{|E(x,r)|_\lambda}{|H(x,r)|_\lambda}$$
$$\Leftrightarrow(1-\alpha)^p\omega(H)\le\left(1-\frac{|E(x,r)|_\lambda}{|H(x,r)|_\lambda}\right)^p\omega(H)\le C\big(\omega(H)-\omega(E)\big).$$

Taking into account that $C\ge1$, we obtain

$$(1-\alpha)^p\,\omega(H)\le C\,\omega(H)-C\,\omega(E)\Leftrightarrow C\,\omega(E)\le\big(1-(1-\alpha)^p\big)\omega(H)$$
$$\Leftrightarrow\omega(E)\le\frac{C-(1-\alpha)^p}{C}\,\omega(H).$$

Thus, we get our assertion with $\beta=\frac{C-(1-\alpha)^p}{C}$. □

Further, we need the reverse of Hölder's inequality.

Theorem 3.3. *Let $\omega\in A_p^\lambda(\mathbb{R}_+)$, $1\le p<\infty$. Then there exist a constant $C>0$ and $\varepsilon>0$ depending only on p such that for any interval $H(x,r)$, the inequality*

$$\left(\frac{1}{|H(x,r)|_\lambda}\int\limits_{H(x,r)}\omega^{1+\varepsilon}(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\;dy\right)^{\frac{1}{1\varepsilon}}\le\frac{C}{|H(x,r)|_\lambda}\int\limits_{H(x,r)}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\;dy$$

is valid.

Proof. Fix an interval $H_0(x_0,r_0)$. By Remark 3.1, we apply inequality (3) from Theorem 2.3 with respect to H_0 for ω, and the increasing sequence $\{\beta_k\}$, $k=0,1,\dots$, we can write

$$\left\{\frac{\omega(H)}{|H|_\lambda}=\beta_0<\beta_1<\cdots<\beta_k<\cdots\right\}.$$

For each β_k, by property (1), we can get a disjoint sequence $\{H_{k,i}\}$ such that $\omega(\operatorname{ch} x) \le \beta_k$ for $x \notin \Lambda_k = \bigcup_i H_{k,i}$, and by property (3),

$$\beta_k < \frac{1}{|H_{k,i}|_\lambda} \int\limits_{H_{k,i}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \; dy \le 2^{(2\lambda+1)n} \beta_k.$$

Since $\beta_{k+1} > \beta_k$, for every interval $H_{k+1,j}$, it is either equal to $H_{k,i}$ or a subinterval of $H_{k,i}$ for some i, therefore

$$\begin{aligned} |H_{k+1,j}|_\lambda &< \frac{1}{\beta_{k+1}} \int\limits_{H_{k+1,j}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy = \frac{|H_{k,i}|_\lambda}{\beta_{k+1}} \cdot \frac{1}{|H_{k,i}|_\lambda} \int\limits_{H_{k+1,j}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \\ &\le \frac{|H_{k,i}|_\lambda}{\beta_{k+1}} \cdot \frac{1}{|H_{k,i}|_\lambda} \int\limits_{H_{k,j}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \le 2^{(2\lambda+1)n} \cdot \frac{\beta_k}{\beta_{k+1}} \, |H_{k,i}|_\lambda . \end{aligned}$$

From this, we get

$$\left|H_{k,i} \cap \Lambda_{k+1}\right|_\lambda \le 2^{(2\lambda+1)n} \frac{\beta_k}{\beta_{k+1}} \, |H_{k,i}|_\lambda .$$

For fixed $\alpha < 1$ we choose a sequence $\{\beta_k\}$ such that

$$\frac{2^{(2\lambda+1)n} \beta_k}{\beta_{k+1}} = \alpha \Leftrightarrow \beta_k = \left(\frac{2^{(2\lambda+1)n}}{\alpha} \right)^k \beta_0,$$

where $\beta_0 = \left(\frac{\alpha}{2^{(2\lambda+1)n}}\right)^{k+1} \beta_{k+1}$. Thus,

$$\left|H_{k,i} \cap \Lambda_{k+1}\right|_\lambda \le \alpha \, |H_{k,i}|_\lambda .$$

From Property 2 of class $A_p^\lambda(\mathbb{R}_+)$, there exists $\gamma \in (0,1)$ such that

$$\omega\big(H_{k,i} \cap \Lambda_{k+1}\big) \le \gamma \omega \left(H_{k,i}\right).$$

From this, we have

$$\bigcup_i \omega\big(H_{k,i} \cap \Lambda_{k+1}\big) \le \gamma \bigcup_i \omega\left(H_{k,i}\right),$$

that equivalently

$$\omega\left(\Lambda_{k+1}\right) \le \gamma \omega\left(\Lambda_k\right),$$

from which it follows that

$$\omega\left(\Lambda_{k+1}\right) \le \gamma^k \omega\left(\Lambda_0\right).$$

Analogously, we have $|\Lambda_{k+1}|_\lambda \le \alpha \, |\Lambda_k|_\lambda$ and $|\Lambda_{k+1}|_\lambda \le \alpha^k \, |\Lambda_0|_\lambda$. Consequently,

$$\Big| \bigcap_{k=0}^{\infty} \Lambda_k \Big|_\lambda = \lim_{k\to\infty} |\Lambda_k| = 0.$$

Thus,

$$\begin{aligned} \int\limits_H \omega^{1+\varepsilon}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy &= \int\limits_{H \setminus \Lambda_0} \omega^{1+\varepsilon}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy + \sum_{k=0}^{\infty} \int\limits_{\Lambda_k \setminus \Lambda_{k+1}} \omega^{1+\varepsilon}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y \, dy \\ &\le \beta_0^\varepsilon \omega\left(H \setminus \Lambda_0\right) + \sum_{k=0}^{\infty} \beta_{k+1}^\varepsilon \omega\left(\Lambda_k \setminus \Lambda_{k+1}\right) \\ &\le \beta_0^\varepsilon \left(\omega\left(H \setminus \Lambda_0\right) + \sum_{k=0}^{\infty} \left(\frac{2^{(2\lambda+1)n}}{\alpha} \right)^{(k+1)\varepsilon} \gamma^k \omega\left(\Lambda_0\right) \right) \\ &\le \beta_0^\varepsilon \left(\left[\omega\left(H \setminus \Lambda_0\right) + \left(\frac{2^{(2\lambda+1)n}}{\alpha} \right)^\varepsilon \sum_{k=0}^{\infty} \left(\frac{2^{(2\lambda+1)n}}{\alpha} \right)^\varepsilon \gamma \right]^k \omega\left(\Lambda_0\right) \right). \end{aligned}$$

Let $\varepsilon > 0$ be small enough such that $\left(\frac{2^{(2\lambda+1)n}}{\alpha}\right)^{\varepsilon}\gamma < 1$. Then the series converges. Therefore we have

$$\int\limits_{H} \omega^{1+\varepsilon}(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy \le C\beta_0^{\varepsilon}\left(\omega\left(H\backslash\Lambda_0\right)+\omega\left(\Lambda_0\right)\right) = C\beta_0^{\varepsilon}\omega(H)$$

$$= C\frac{\omega^{\varepsilon}(H)}{|H|_{\lambda}^{\varepsilon}}\cdot\omega(H) = C\frac{\omega^{1+\varepsilon}(H)}{|H|_{\lambda}^{1+\varepsilon}}|H|_{\lambda} = C\left(\frac{1}{|H|_{\lambda}}\int\limits_{H}\omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy\right)^{1+\varepsilon}|H|_{\lambda},$$

thus there follows the assertion of theorem. □

Proposition 3.3. *Let $\omega \in A_p^{\lambda}(\mathbb{R}_+)$, $1 < p < \infty$. Then there exists an $\varepsilon > 0$ such that $p-\varepsilon > 1$ and $\omega \in A_{p-\varepsilon}^{\lambda}(\mathbb{R}_+)$.*

Proof. If $\omega \in A_p^{\lambda}(\mathbb{R}_+)$, then by Property 2, $\omega^{-\frac{1}{p-1}} \in A^{\lambda}_{1+\frac{1}{p-1}}(\mathbb{R}_+)$. Applying Theorem 3.3, we obtain

$$\left(\frac{1}{|H(x,r)|_{\lambda}}\int\limits_{H(x,r)}\omega(\operatorname{ch} y)^{\frac{1+\theta}{1-p}}\operatorname{sh}^{2\lambda} y\,dy\right)^{\frac{p-1}{1+\theta}} \le C^{p-1}\left(\frac{1}{|H(x,r)|_{\lambda}}\int\limits_{H(x,r)}\omega(\operatorname{ch} y)^{-\frac{1}{1-p}}\operatorname{sh}^{2\lambda} y\,dy\right)^{p-1},$$

where $\theta > 0$. Multiplying both sides of the inequality by

$$\frac{1}{|H(x,r)|_{\lambda}}\int\limits_{H(x,r)}\omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy,$$

we have

$$\left(\frac{1}{|H(x,r)|_{\lambda}}\int\limits_{H(x,r)}\omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy\right)\times\left(\frac{1}{|H(x,r)|_{\lambda}}\int\limits_{H(x,r)}\omega(\operatorname{ch} y)^{-\frac{1+\theta}{1-p}}\operatorname{sh}^{2\lambda} y\,dy\right)^{\frac{p-1}{1+\theta}}$$

$$\le C^{p-1}\left(\frac{1}{|H(x,r)|_{\lambda}}\int\limits_{H(x,r)}\omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy\right)$$

$$\times\left(\frac{1}{|H(x,r)|_{\lambda}}\int\limits_{H(x,r)}\omega(\operatorname{ch} y)^{-\frac{1}{1-p}}\operatorname{sh}^{2\lambda} y\,dy\right)^{p-1} \le C_1.$$

Suppose $\frac{1+\theta}{p-1} = \frac{1}{q-1} \Leftrightarrow (q-1)(1+\theta) = p-1 \Leftrightarrow p-q = \theta(q-1) > 0 \Leftrightarrow p > q$, then $p > q > 1$ and $\omega \in A_q^{\lambda}(\mathbb{R}_+)$. Thus we get Property 3 with $\varepsilon = p-q$. □

The following theorems are the analogues of the corresponding Theorems 2 and 3 from [29].

Theorem 3.4. *Let $0 \le \alpha < 2\lambda+1$, $1 \le p < \frac{2\lambda+1}{\alpha}$, $\frac{1}{p}-\frac{1}{q} = \frac{\alpha}{2\lambda+1}$, $\beta > 0$, $E_{\beta} = \{x \in \mathbb{R}_+ : M_G^{\alpha} f(\operatorname{ch} x) > \beta\}$ and $V(\operatorname{ch} x)$ is a nonnegative function on $\mathbb{R}_+$ such that for every interval $H \subset \mathbb{R}_+$, the inequality*

$$\left(\frac{1}{|H|_{\lambda}}\int\limits_{H} V(\operatorname{ch} x)^{q}\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}}\left(\frac{1}{|H|_{\lambda}}\int\limits_{H} V(\operatorname{ch} x)^{-p'}\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{p'}} \le K \qquad (3.4)$$

holds with K, independent of H, then there is a C, independent of f such that

$$\left(\int\limits_{E_{\beta}} V(\operatorname{ch} x)^{q}\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}} \le \frac{C}{\beta}\left(\int\limits_{\mathbb{R}_+}|f(\operatorname{ch} x)V(\operatorname{ch} x)|^{p}\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{p}}. \qquad (3.5)$$

Proof. Fix $M>0$ and let $E_{\beta,M}$, an interval of radius M, be the intersection of the set E_β. For each $x\in E_{\beta,M}$, there is an interval H centered at x such that

$$|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H |f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda}x\,dx>\beta. \tag{3.6}$$

By the Lindelof covering theorem (see [24]), there is a sequence $\{H_k\}$ such that $E_{\beta,M}\subset\cup H_k$, then we can write

$$\begin{aligned}\Bigg(\int\limits_{E_{\beta,M}} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda}x\,dx\Bigg)^{\frac{p}{q}} &\le \Bigg(\sum_k\int\limits_{H_k} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda}x\,dx\Bigg)^{\frac{p}{q}}\\ &\le \sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda}x\,dx\Bigg)^{\frac{p}{q}}, \end{aligned}\tag{3.7}$$

so, $\frac{p}{q}\le 1$.

Since interval H_k satisfies (3.6), from (3.7), we have

$$\begin{aligned}\sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}} &\le \sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}}\\ &\times\Bigg(\frac{1}{\beta}|H_k|^{\frac{\alpha}{2\lambda+1}-1}\int\limits_{H_k}\big|f(\operatorname{ch} x)\big|\operatorname{sh}^{2\lambda}x,dx\Bigg)^p\\ &=\sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}}\frac{1}{\beta^p}|H_k|^{1-p-\frac{p}{q}}\\ &\times\Bigg(\int\limits_{H_k}|f(\operatorname{ch} x)|\,V(\operatorname{ch} x)V(\operatorname{ch} x)^{-1}\operatorname{sh}^{2\lambda}x,dx\Bigg)^p.\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}&\Bigg(\int\limits_{H_k}\big|f(\operatorname{ch} x)\big|\,V(\operatorname{ch} x)\,V(\operatorname{ch} x)^{-1}\operatorname{sh}^{2\lambda}x,dx\Bigg)^p\\ &\le\Bigg(\int\limits_{H_k}\big|f(\operatorname{ch} x)\,V(\operatorname{ch} x)\big|^p\operatorname{sh}^{2\lambda}x,dx\Bigg)\Bigg(\int\limits_{H_k}V(\operatorname{ch} x)^{-p'}\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{p'}}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}} &\le \sum_k\Bigg(\frac{1}{|H_k|_\lambda}\int\limits_{H_k} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}}\\ \times\Bigg(\frac{1}{\beta^p}\int\limits_{H_k}\big|f(\operatorname{ch} x)V(\operatorname{ch} x)\big|^p\operatorname{sh}^{2\lambda}x,dx\Bigg)&\Bigg(\frac{1}{|H_k|_\lambda}\int\limits_{H_k}V(\operatorname{ch} x)^{-p'}\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{p'}}.\end{aligned}$$

Taking into account (3.4), we have

$$\sum_k\Bigg(\int\limits_{H_k} V(\operatorname{ch} x)^q\operatorname{sh}^{2\lambda}x,dx\Bigg)^{\frac{p}{q}}\le C\beta^{-p}\Bigg(\int\limits_{H_k}\big|f(\operatorname{ch} x)V(\operatorname{ch} x)\big|^p\operatorname{sh}^{2\lambda}x,dx\Bigg).$$

From this and (3.7), it follows that

$$\left(\int\limits_{E_{\beta,M}} V(\operatorname{ch} x)^q \operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}} \le \frac{C}{\beta}\left(\int\limits_{H_k} |f(\operatorname{ch} x)V(\operatorname{ch} x)|^p \operatorname{sh}^{2\lambda} x dx\right)^{\frac{1}{p}}$$
$$\le \frac{C}{\beta}\left(\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)V(\operatorname{ch} x)|^p \operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{p}}.$$

So, (3.5) follows from the monotone convergence theorem. □

Theorem 3.5. *Let $0<\alpha<2\lambda+1$, $1<p<\frac{2\lambda+1}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$ and $V(\operatorname{ch} x)$ be a nonnegative function on $\mathbb{R}_+$ such that for every interval H, (3.4) holds with K, independent of H. Then there is a constant C, independent of φ such that*

$$\left(\int\limits_{\mathbb{R}_+} \left[M_G^\alpha\varphi(\operatorname{ch} x)V(\operatorname{ch} x)\right]^q \operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}} \le C\left(\int\limits_{\mathbb{R}_+} \left|\varphi(\operatorname{ch} x)V(\operatorname{ch} x)\right|^p \operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{p}}. \tag{3.8}$$

Proof. Suppose $W(\operatorname{ch} x)=V(\operatorname{ch} x)^q$ and note that the condition (3.4) is equivalent to

$$\left(\frac{1}{|H|_\lambda}\int\limits_H W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)\left(\frac{1}{|H|_\lambda}\int\limits_H W(\operatorname{ch} x)^{-\frac{1}{r-1}}\operatorname{sh}^{2\lambda} x, dx\right)^{r-1} \le C,$$

where $r=1+\frac{q}{p'}$. That is, $W(\operatorname{ch} x)$ belongs to class $A_r^\lambda(\mathbb{R}_+)$. Then by Properties 1 and 3, there exists an $\varepsilon>0$ such that $r_2(\varepsilon)<r<r_1(\varepsilon)$ and simultaneously, $W\in A^\lambda_{r_1(\varepsilon)}(\mathbb{R}_+)$ and $W\in A^\lambda_{r_2(\varepsilon)}(\mathbb{R}_+)$. Let $0<\varepsilon<\min(\alpha,2\lambda+1-\alpha)$.

Suppose

$$\frac{1}{p_1}=\frac{1}{p}-\frac{\varepsilon}{2\lambda+1}<\frac{1}{p}<\frac{1}{p}+\frac{\varepsilon}{2\lambda+1}=\frac{1}{p_2}\Longrightarrow p_2<p<p_1,$$
$$\frac{1}{q_1}=\frac{1}{p}-\frac{\alpha+\varepsilon}{2\lambda+1}=\frac{}{p}-\frac{\varepsilon}{2\lambda+1}-\frac{\alpha}{2\lambda+1}=\frac{1}{p_1}-\frac{\alpha}{2\lambda+1},$$
$$\frac{1}{q_2}=\frac{1}{p}-\frac{\alpha-\varepsilon}{2\lambda+1}=\frac{1}{p}+\frac{\varepsilon}{2\lambda+1}-\frac{\alpha}{2\lambda+1}=\frac{1}{p_2}-\frac{\alpha}{2\lambda+1}.$$

From this it follows that simultaneously $\frac{1}{q_1}=\frac{1}{p_1}-\frac{\alpha}{2\lambda+1}$ and $\frac{1}{q_2}=\frac{1}{p_2}-\frac{\alpha}{2\lambda+1}$, and then suppose

$$r_1(\varepsilon)=1+\frac{p_1(2\lambda+1)}{p_1'(2\lambda+1-(\alpha+\varepsilon)p_1)}$$
$$=1+\frac{p_1(2\lambda+1)}{p_1'(2\lambda+1-\alpha p_1)}=r_1=1+\frac{q_1}{p_1'},\quad p_1p_1'=p_1+p_1',$$
$$r_2(\varepsilon)=1+\frac{p_2(2\lambda+1)}{p_2'(2\lambda+1-(\alpha+\varepsilon)p_2)}$$
$$=1+\frac{p_2(2\lambda+1)}{p_2'(2\lambda+1-\alpha p_2)}=r_2=1+\frac{q_2}{p_2'},\quad p_2p_2'=p_2+p_2'.$$

We obtain for $r_2<r<r_1$, from $p_2<p<p_1$ it follows that $p_1'<p'<p_2'$, but then simultaneously $W\in A^\alpha_{1+\frac{q_1}{p_1'}}(\mathbb{R}_+)$ and $W\in A^\alpha_{1+\frac{q_2}{p_2'}}(\mathbb{R}_+)$.

By Theorem 3.4, there exists a constant C such that

$$\left(\int\limits_{E_\beta} W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{p_i}{q_i}} \le C\beta^{-p_i}\int\limits_{\mathbb{R}_+}\left|\varphi(\operatorname{ch} x)\right|^{p_i} W(\operatorname{ch} x)^{\frac{p_i}{q_i}}\operatorname{sh}^{2\lambda} x, dx,\quad i=1,2. \tag{3.9}$$

Now define a sublinear operator T by

$$Tg(\operatorname{ch} x) = M_G^{\alpha}\left[g(\operatorname{ch} x)W(\operatorname{ch} x)^{\frac{\alpha}{2\lambda+1}}\right].$$

Then with $\varphi(\operatorname{ch} x) = g(\operatorname{ch} x)W(\operatorname{ch} x)^{\frac{\alpha}{2\lambda+1}}$, (3.9) can be written in the form

$$\int\limits_{\{x\in E_\beta : Tg(\operatorname{ch} x)>\beta\}} W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx \le C\beta^{-q_i}\left(\int\limits_{\mathbb{R}_+} |g(\operatorname{ch} x)|^{p_i} W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{q_i}{p_i}}, \quad i = 1, 2.$$

From this it follows that the operator T has simultaneously weak type (p_1, q_1) and (p_2, q_2).

$$\left(\int\limits_{\mathbb{R}_+} [Tg(\operatorname{ch} x)]^q W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{q}} \le C\left(\int\limits_{\mathbb{R}_+} [g(\operatorname{ch} x)]^p W(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{p}}.$$

Supposing here $g(\operatorname{ch} x) = \varphi(\operatorname{ch} x)W(\operatorname{ch} x)^{-\frac{\alpha}{2\lambda+1}}$ and $W(\operatorname{ch} x) = V(\operatorname{ch} x)^q$, we obtain the assertion of the theorem. □

4. Main Results

4.1. Weighted $(L_{p,\omega,\lambda},\ L_{q,\omega,\lambda})$ Boundedness Gegenbauer Fractional Maximal Operator. Next two theorems are analogues of works [12] and [30].

Theorem 4.1. *Let $1 < p < \frac{2\lambda+1}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$. Then the next two conditions are equivalent:*

(i) $\exists C > 0$ *such that* $\forall f \in L_{p,\omega,\lambda}(\mathbb{R}_+, G)$ *the following inequality*

$$\left\{\int\limits_{\mathbb{R}_+} \left[M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)\right]^q \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right\}^{\frac{1}{q}} \le C\left(\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x, dx\right)^{\frac{1}{p}} \textit{ is valid,} \tag{4.1}$$

(ii) $\omega \in A_{1+\frac{q}{p'}}(\mathbb{R}_+)$, $p\,p' = p + p'$,

$$\sup_H\left(\frac{1}{|H|_\lambda}\int\limits_H \omega(\operatorname{ch} y)\operatorname{sh}^{2\lambda} y\,dy\right)\left(\frac{1}{|H|_\lambda}\int\limits_H \omega(\operatorname{ch} y)^{-\frac{p'}{q}}\operatorname{sh}^{2\lambda} y\,dy\right)^{\frac{q}{p'}} < \infty. \tag{4.2}$$

Proof. We show that from (4.1), (4.2) we have the following. For every fixing interval $H \subset [0, \infty)$, we can write

$$\begin{aligned} M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x) &= \sup_H\left(|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \left|\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)\right|\operatorname{sh}^{2\lambda} y\,dy\right) \\ &\ge \left(|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \left|\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)\right|\operatorname{sh}^{2\lambda} y\,dy\right)\chi_H(\operatorname{ch} x). \end{aligned}$$

Taking into account (4.1), we obtain

$$\begin{aligned} &\left(\int\limits_H \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{q}}\left(\left|H\right|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \left|\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)\right|\operatorname{sh}^{2\lambda} y\,dy\right) \\ &\qquad \le C\left(\int\limits_H |f(\operatorname{ch} x)|^p \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\right)^{\frac{1}{p}}. \end{aligned}$$

Thus,

$$|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \left|\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)\right|\operatorname{sh}^{2\lambda} y\,dy$$

$$\leq C\Bigg(\int\limits_H \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{-\frac{1}{q}}\Bigg(\int\limits_H |f(\operatorname{ch} x)|^p\,\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{1}{p}}.$$

Supposing $f(\operatorname{ch} x)=\omega(\operatorname{ch} x)^{-\frac{1}{p}\left(1+\frac{p'}{q}\right)}$, we obtain

$$|H|_\lambda^{\frac{\alpha}{2\lambda+1}-1}\int\limits_H \omega(\operatorname{ch} x)^{-\frac{p'}{q}}\operatorname{sh}^{2\lambda} x\,dx\leq C\Bigg(\int\limits_H \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{-\frac{1}{q}}\Bigg(\int\limits_H \omega(\operatorname{ch} x)^{-\frac{p'}{q}}\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{1}{p}}.$$

From this it follows that

$$|H|_\lambda^{\left(\frac{\alpha}{2\lambda+1}-1\right)q}\Bigg(\int\limits_H \omega(\operatorname{ch} x)^{-\frac{p'}{q}}\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{q}{p'}}\leq C^q\Bigg(\int\limits_H \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{-1}.$$

So, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}\Leftrightarrow\frac{1}{q}-\frac{1}{p}+1=1-\frac{\alpha}{2\lambda+1}\Leftrightarrow\frac{1}{q}+\frac{1}{p'}=1-\frac{\alpha}{2\lambda+1}\Leftrightarrow 1+\frac{q}{p'}=\left(1-\frac{\alpha}{2\lambda+1}\right)q$, then (4.2) is provided. We show that from inequality (4.2) follows inequality (4.1). Suppose in (3.8) $\varphi(\operatorname{ch} x)=f(\operatorname{ch} x)\ \omega(\operatorname{ch} x)^{\frac{\alpha}{2\lambda+1}}$ and $V(\operatorname{ch} x)^q=\omega(\operatorname{ch} x)$, we obtain

$$\Bigg(\int\limits_{\mathbb{R}_+}\left[M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)\right]^q\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{1}{q}}\leq C\Bigg(\int\limits_{\mathbb{R}_+}[f(\operatorname{ch} x)]^p\,[\omega(\operatorname{ch} x)]^{\frac{p\alpha}{2\lambda+1}+\frac{p}{q}}\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{1}{p}}$$

$$=C\Bigg(\int\limits_{\mathbb{R}_+}[f(\operatorname{ch} x)]^p\,\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{1}{p}},$$

since $p\left(\frac{\alpha}{2\lambda+1}+\frac{1}{q}\right)=1\Leftrightarrow\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$. □

Theorem 4.2. *Let $q=\frac{2\lambda+1}{2\lambda+1-\alpha}$. Then the next two conditions are equivalent:*

(i)
$$\int\limits_{\left\{x\in\mathbb{R}_+:M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)>\beta\right\}}\leq C\Bigg(\frac{1}{\beta}\int\limits_{\mathbb{R}_+}|f(\operatorname{ch} x)|\,\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^q,$$

where the constant C does not depend on f and $\beta>0$.

(ii) $\omega\in A_1^\lambda(\mathbb{R}_+)$, *i.e.*, $M\omega(\operatorname{ch} x)\leq C\omega(\operatorname{ch} x)$.

Proof. Let $H_1\subset H$. Suppose $f\omega^{\frac{\alpha}{2\lambda+1}}=|H|_\lambda^{\frac{\alpha}{2\lambda+1}}\chi_{H_1}$, where χ_{H_1} is the characteristic function of H_1. From this we have

$$M_\mu^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)=|H|_\lambda^{\frac{\alpha}{2\lambda+1}}\,M_\mu\chi_{H_1}(\operatorname{ch} x).\tag{4.3}$$

But for any $x\in H$,

$$M_G\chi_{H_1}(\operatorname{ch} x)=\sup_{r>0}\frac{|H_1\cap H|_\lambda}{|H|_\lambda}\geq\frac{|H_1|_\lambda}{|H|_\lambda}.\tag{4.4}$$

From (4.3) and (4.4), for any $x\in H$, we have

$$M\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)\geq|H|_\lambda^{\frac{\alpha}{2\lambda+1}}\frac{|H_1|_\lambda}{|H|_\lambda}>\beta>0,$$

from this it follows that

$$H\subset\left\{x\in\mathbb{R}:M\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)>\beta\right\}$$

for every $0<\beta<|H|_\lambda^{\frac{\alpha}{2\lambda+1}}\frac{|H_1|_\lambda}{|H|_\lambda}$.

By (i) and Hölder's inequality, we obtain

$$\beta^q \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le \beta^q \int\limits_{\left\{y\in\mathbb{R}_+ : M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} y)>\beta\right\}} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy$$

$$\le \left(\int\limits_{H_1(x,r_1)} \omega^{1-\frac{\alpha}{2\lambda+1}}(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy\right)^q = C\left(\int\limits_{H_1(x,r_1)} \left[\omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\right]^{\frac{1}{q}} (\operatorname{sh}^{2\lambda} y)^{1-\frac{1}{q}} dy\right)^q$$

$$\le C\left(\int\limits_{H_1(x,r_1)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy\right)\left(\int\limits_{H_1(x,r_1)} \operatorname{sh}^{2\lambda} y\, dy\right)^{q-1}$$

$$= C\, |H_1(x,r_1)|_\lambda^q \left(\frac{1}{|H_1(x,r_1)|_\lambda} \int\limits_{H_1(x,r_1)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy\right).$$

From this it follows that

$$\frac{|H_1(x,r_1)|_\lambda^q}{|H(x,r)|_\lambda^q} |H(x,r)|_\lambda^{q-1} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy$$

$$\le C\, |H_1(x,r_1)|_\lambda^q \left(\frac{1}{|H_1(x,r_1)|_\lambda} \int\limits_{H_1(x,r_1)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy\right),$$

which is equivalent to

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le C \frac{1}{|H_1(x,r_1)|_\lambda} \int\limits_{H_1(x,r_1)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy.$$

Applying the Lebesgue differentiation theorem, we obtain

$$\frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \omega(\operatorname{ch} y) \operatorname{sh}^{2\lambda} y\, dy \le C\omega(\operatorname{ch} x)$$

for a.e. $x \in \mathbb{R}_+$.

Thus, $\omega \in A_1^\lambda(\mathbb{R}_+)$.

Now we show that from (ii) (i) we have the following. Applying Hölder's inequality, we obtain

$$M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x) = \sup_{r>0} \frac{1}{|H(x,r)|_\lambda^{1-\frac{\alpha}{2\lambda+1}}} \int\limits_{H(x,r)} |f(\operatorname{ch} t)|\, \omega(\operatorname{ch} t)^{\frac{\alpha}{2\lambda+1}} \operatorname{sh}^{2\lambda} t\, dt$$

$$= \sup_{r>0} \frac{1}{|H(x,r)|_\lambda^{1-\frac{\alpha}{2\lambda+1}}} \int\limits_{H(x,r)} \left[(f\omega)(\operatorname{ch} t) \operatorname{sh}^{2\lambda} t\right]^{\frac{\alpha}{2\lambda+1}} \left[f(\operatorname{ch} t) \operatorname{sh}^{2\lambda} t\right]^{1-\frac{\alpha}{2\lambda+1}} dt$$

$$\le \sup_{r>0} \frac{1}{|H(x,r)|_\lambda^{1-\frac{\alpha}{2\lambda+1}}} \left(\int\limits_{H(x,r)} f(\operatorname{ch} t)\omega(\operatorname{ch} t) \operatorname{sh}^{2\lambda} t\, dt\right)^{\frac{\alpha}{2\lambda+1}} \left(\int\limits_{H(x,r)} f(\operatorname{ch} t) \operatorname{sh}^{2\lambda} t\, dt\right)^{1-\frac{\alpha}{2\lambda+1}}$$

$$\le \left(M_\mu f(\operatorname{ch} x)\right)^{\frac{1}{q}} \left(\|f\|_{L^\lambda_{1,\omega,\lambda}}\right)^{1-\frac{1}{q}}.$$

From this it follows that

$$M_\mu f(\operatorname{ch} x) \ge M_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)^q \|f\|_{L_{1,\omega,\lambda}}^{1-q}.$$

Then taking into account Theorem 3.2, we obtain

$$\int\limits_{\left\{y\in H(x,r):M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch}y)>\beta\right\}}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy\le\int\limits_{\left\{y\in\mathbb{R}_+:M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch}y)>\beta\right\}}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy$$
$$\le\int\limits_{\left\{y\in\mathbb{R}_+:M_\mu f(\operatorname{ch}y)>\beta^q\|f\|^{1-q}_{L_{1,\omega,\lambda}}\right\}}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy\le C\beta^{-q}\|f\|^{1-q}_{L_{1,\omega,\lambda}}\int\limits_{\mathbb{R}_+}M_\mu f(\operatorname{ch}y)\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy.$$

Using Theorem 3.1, for $p=1$, and also condition (ii), we get

$$\int\limits_{\left\{y\in\mathbb{R}_+:M_G^{\alpha}\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch}y)>\beta\right\}}\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy\le C\beta^{-q}\|f\|^{q-1}_{L_{1,\omega,\lambda}}\int\limits_{\mathbb{R}_+}f(\operatorname{ch}y)M_\mu\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy$$
$$\le C\beta^{-q}\|f\|^{q-1}_{L_{1,\omega,\lambda}}\int\limits_{\mathbb{R}_+}f(\operatorname{ch}y)\omega(\operatorname{ch}y)\operatorname{sh}^{2\lambda}y\,dy=C\left(\frac{1}{\beta}\|f\|_{L_{1,\omega,\lambda}}\right)^q.\qquad\square$$

4.2. Weighted $(L_{p,\omega,\lambda},\ L_{q,\omega,\lambda})$ Boundedness of G-Riesz Potential. In this section we obtain some results for the G-Riesz potential (1.1), which are analogous to the corresponding results obtained in [12] for the B-Riesz potential.

Lemma 4.1. *Let $0<\alpha<2\lambda+1,\ 1\le p<\frac{\beta}{\alpha}$. Then there is a positive constant C such that for any $r>0$ and $x\in\mathbb{R}_+$, we have*

$$|I_G^{\alpha}f(\operatorname{ch}x)|\le C\Big((\operatorname{sh}r)^{\alpha}M_Gf(\operatorname{ch}x)+(\operatorname{sh}r)^{\alpha-\frac{\beta}{p}}M_G^{\frac{\beta}{p}}f(\operatorname{ch}x)\Big).\tag{4.5}$$

Proof. From (1.1), we have

$$I_G^{\alpha}f(\operatorname{ch}x)=\left(\int\limits_0^r+\int\limits_r^{\infty}\right)\left(\int\limits_0^{\infty}r^{\frac{\alpha}{2}-1}h_r(\operatorname{ch}t)dr\right)A^{\lambda}_{\operatorname{ch}\ t}f(\operatorname{ch}x)\operatorname{sh}^{2\lambda}t\,dt=A_1(x,r)+A_2(x,r).\tag{4.6}$$

We consider $A_1(x,r)$. Let $0<r<2$. Then from Lemma 3.2 and Corollary 3.1 [15], we have

$$|A_1(x,r)|\le\int\limits_0^r A^{\lambda}_{\operatorname{ch}t}|f(\operatorname{ch}x)|(\operatorname{sh}t)^{2\lambda}(\operatorname{sh}t)^{\alpha-2\lambda-1}dt\le\sum_{k=0}^{\infty}\int\limits_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}}\frac{A^{\lambda}_{\operatorname{ch}t}|f(\operatorname{ch}x)|\operatorname{sh}^{2\lambda}t\,dt}{(\operatorname{sh}t)^{2\lambda+1-\alpha}}$$
$$\le\sum_{k=0}^{\infty}\left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\alpha}\left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{-2\lambda-1}\int\limits_0^{\frac{r}{2^k}}A_{\operatorname{ch}t}|f(\operatorname{ch}x)|\operatorname{sh}^{2\lambda}t\,dt$$
$$\lesssim M_Gf(\operatorname{ch}x)\sum_{k=0}^{\infty}\left(\frac{1}{2^{k+1}}\operatorname{sh}r\right)^{\alpha}\lesssim(\operatorname{sh}r)^{\alpha}M_Gf(\operatorname{ch}x)\sum_{k=0}^{\infty}\frac{1}{2^{(k+1)^{\alpha}}}$$
$$\lesssim(\operatorname{sh}r)^{\alpha}M_Gf(\operatorname{ch}x),\tag{4.7}$$

since $\operatorname{sh}\frac{t}{a}\le\frac{1}{a}\operatorname{sh}t$ for $a\ge1$.

Now let $2 \le r < \infty$ and $0 < \alpha < 4\lambda$. Then from the proof of Corollary 3.1 in [15], we have

$$\begin{aligned}
|A_1(x,r)| &\le \int_0^r \frac{A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt}{(\operatorname{ch} t)^{2\lambda+1-\alpha}} \le \int_0^r \frac{A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t\, dt}{(\operatorname{ch} t)^{4\lambda-\alpha}} \\
&\le \int_0^r \frac{A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt}{(\operatorname{sh} t)^{4\lambda-\alpha}} \le \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt}{(\operatorname{sh} t)^{4\lambda-\alpha}} \\
&\le \sum_{k=0}^\infty \Big(\operatorname{sh}\frac{r}{2^{k+1}}\Big)^\alpha \Big(\operatorname{sh}\frac{r}{2^{k+1}}\Big)^{-4\lambda} \int_0^{\frac{r}{2^k}} A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt \\
&\le M_G f(\operatorname{ch} x)\sum_{k=0}^\infty \Big(\operatorname{sh}\frac{r}{2^{k+1}}\Big)^\alpha \le (\operatorname{sh} r)^\alpha M_G f(\operatorname{ch} x), \quad 0<\alpha<4\lambda. \qquad (4.8)
\end{aligned}$$

Now let $4\lambda \le \alpha < 2\lambda+1$. From the proof of Corollary 3.1 in [15], it follows that $\int_0^\infty r^{\frac{\alpha}{2}-1} h_2(\operatorname{ch} t) dr \lesssim 1$, then

$$\begin{aligned}
|A_1(x,r)| &\le \int_0^r \frac{A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t}{(\operatorname{ch} t)^{2\lambda+1-\alpha}} dt \\
&\le \int_0^r A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt = \frac{(\operatorname{sh}\frac{r}{2})^{4\lambda}}{(\operatorname{sh}\frac{r}{2})^{4\lambda}} \int_0^r A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t\, dt \\
&\le \Big(\operatorname{sh}\frac{r}{2}\Big)^{4\lambda} M_G\, f(\operatorname{ch} x) \le (\operatorname{sh} r)^\alpha M_G f(\operatorname{ch} x), \quad 4\lambda<\alpha<2\lambda+1. \qquad (4.9)
\end{aligned}$$

Thus from (4.7)–(4.9), it follows that for every $0 < r < \infty$ and $0 < \alpha < 2\lambda + 1$,

$$|A_1(x,r)| \lesssim (\operatorname{sh} r)^\alpha M_G f(\operatorname{ch} x). \qquad (4.10)$$

We estimate $A_2(x,r)$. Let $0 < r < 2$. Then

$$\begin{aligned}
|A_2(x,r)| &\le \int_r^\infty \frac{A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} dt = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \frac{A^\lambda_{cht}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} dt \\
&\le \sum_{k=0}^\infty \big(\operatorname{sh} 2^k r\big)^{\alpha-2\lambda-1} \int_0^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt \\
&= \sum_{k=0}^\infty \big(\operatorname{sh} 2^k r\big)^{\alpha-\frac{\beta}{p}} \big(\operatorname{sh} 2^k r\big)^{\frac{\beta}{p}-2\lambda-1} \int_0^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t\, dt \\
&\le M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\sum_{k=0}^\infty \big(\operatorname{sh} 2^k r\big)^{\alpha-\frac{\beta}{p}} \le (\operatorname{sh} r)^{\alpha-\frac{\beta}{p}}\, M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\sum_{k=0}^\infty 2^{k\left(\alpha-\frac{\beta}{p}\right)} \\
&\lesssim (\operatorname{sh} r)^{\alpha-\frac{\beta}{p}}\, M_G^{\frac{\beta}{p}} f(\operatorname{ch} x), \qquad (4.11)
\end{aligned}$$

by the condition $\alpha - \frac{\beta}{p} < 0$.

Now let $2 \le r < \infty$. Then for $0 < \alpha < 4\lambda$, we have

$$A_2(x,r) \le \int\limits_r^\infty \frac{A^\lambda_{\operatorname{ch} t}\,|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{4\lambda-\alpha}}dt \le \sum_{k=0}^\infty \int\limits_{2^k r}^{2^{k+1}r} \frac{A^\lambda_{\operatorname{ch} t}\,|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{4\lambda-\alpha}}dt$$

$$\le \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-4\lambda} \int\limits_0^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}\,|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt$$

$$= \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-\frac{\beta}{p}} (\operatorname{sh} 2^k r)^{\frac{\beta}{p}-4\lambda} \int\limits_0^{2^k r} A^\lambda_{\operatorname{ch} t}\,|f(\operatorname{ch} x)|\operatorname{sh} 2\lambda t dt$$

$$\le M_G^{\frac{\beta}{p}} f(\operatorname{ch} x) \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-\frac{\beta}{p}} \le\ (\operatorname{sh} r)^{\alpha-\frac{\beta}{p}} M_G^{\frac{\beta}{p}} f(\operatorname{ch} x). \tag{4.12}$$

We consider the case $4\lambda < \alpha < 2\lambda + 1$. Then

$$|A_2(x,r)| \lesssim \sum_{k=0}^\infty \int\limits_{2^k r}^{2^{k+1}r} \frac{A^\lambda_{\operatorname{ch} t}\,|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}dt \le \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-2\lambda-1} \int\limits_{2^k r}^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}\,|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t\, dt$$

$$= \sum_{k=0}^\infty (\operatorname{sh} 2^k r)^{\alpha-\frac{\beta}{p}} (\operatorname{sh}\ 2^k r)^{\frac{\beta}{p}-4\lambda} (\operatorname{sh}\ 2^k r)^{2\lambda-1} \int\limits_{2^k r}^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}\,|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt$$

$$\lesssim \sum_{k=0}^\infty (\operatorname{sh}\ 2^k r)^{\alpha-\frac{\beta}{p}} (\operatorname{sh}\ 2^k r)^{\frac{\beta}{p}-4\lambda} \int\limits_{2^k r}^{2^{k+1}r} A^\lambda_{\operatorname{ch} t}\,|f(\operatorname{ch} x)|\operatorname{sh}^{2\lambda} t dt$$

$$\lesssim (\operatorname{sh}\ r)^{\alpha-\frac{\beta}{p}} M_G^{\frac{\beta}{p}} f(\operatorname{ch} x). \tag{4.13}$$

From (4.11)–(4.13) it follows that for any $0 < r < \infty$ and $0 < \alpha < 2\lambda + 1$, the inequality

$$A_2(x,r) \lesssim (\operatorname{sh}\ r)^{\alpha-\frac{\beta}{p}} M_G^{\frac{\beta}{p}} f(\operatorname{ch} x) \tag{4.14}$$

is valid. Taking into account (4.10) and (4.14) in (4.6), we obtain the statement of Lemma 4.1. □

Theorem 4.3. *Let $0 < \alpha < \beta \le 2\lambda + 1$, $1 < p < \frac{\beta}{\alpha}$, $1 \le r \le \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta} + \frac{\alpha p}{\beta r}$. Then for any function $f \in L_{p,\lambda}(\mathbb{R}_+)$ and $M_G^{\frac{\beta}{p}} f \in L_{r,\lambda}(\mathbb{R}_+)$ the estimate*

$$\left\| I_G^\alpha f \right\|_{L_{q,\lambda}(\mathbb{R}_+)} \lesssim \left\| M_G^{\frac{\beta}{p}} f \right\|^{\frac{\alpha\beta}{\beta}}_{L_{r,\lambda}(\mathbb{R}_+)} \cdot \left\| f \right\|^{1-\frac{\alpha\beta}{\beta}}_{L_{p,\lambda}(\mathbb{R}_+)}$$

is valid.

Proof. From (4.5), for

$$\operatorname{sh} r = \operatorname{sh} r(\operatorname{ch} x) = \left(\frac{M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)}{M_G f(\operatorname{ch} x)} \right)^{\frac{p}{\beta}},$$

we obtain

$$\left| I_G^\alpha f(\operatorname{ch} x) \right| \lesssim \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x) \right)^{\frac{\alpha p}{\beta}} \left(M_G f(\operatorname{ch} x) \right)^{1-\frac{\alpha p}{\beta}}$$

for each $x \in \mathbb{R}_+$.

Considering both sides of inequality (4.1) to the power of q, integrate by x and using Hölder's inequality, we have

$$\int\limits_{\mathbb{R}_+} \left|I_G^\alpha f(\operatorname{ch} x)\right|^q \operatorname{sh}^{2\lambda} x\, dx \lesssim \int\limits_{\mathbb{R}_+} \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\right)^{\frac{\alpha p q}{\beta}} \left(M_G f(\operatorname{ch} x)\right)^{q-\frac{\alpha p q}{\beta}} \operatorname{sh}^{2\lambda} x\, dx$$

$$\lesssim \left(\int\limits_{\mathbb{R}_+} \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\right)^{\frac{\alpha p q}{\beta} s'} \operatorname{sh}^{2\lambda} x\, dx\right)^{\frac{1}{s'}} \left(\int\limits_{\mathbb{R}_+} \left(M_G f(\operatorname{ch} x)\right)^{\left(q-\frac{\alpha p q}{\beta}\right) s} \operatorname{sh}^{2\lambda} x\, dx\right)^{\frac{1}{s}},$$

where

$$\left(q-\frac{\alpha p q}{\beta}\right)s = p, \quad s' = \frac{s}{s-1} = \frac{\beta r}{\alpha p q}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta} + \frac{\alpha p}{\beta r}.$$

Therefore,

$$\left(\int\limits_{\mathbb{R}_+} |I_G^\alpha f(\operatorname{ch} x)|^q\, d\mu_\lambda(x)\right)^{\frac{1}{q}} \lesssim \left(\int\limits_{\mathbb{R}_+} \left(M_G f(\operatorname{ch} x)\right)^p d\mu_\lambda(x)\right)^{\frac{1}{sq}} \left(\int\limits_{\mathbb{R}_+} \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\right)^r d\mu_\lambda(x)\right)^{\frac{\alpha p}{\beta r}}$$

$$\lesssim \left(\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|^p\, d\mu_\lambda(x)\right)^{\frac{1}{sq}} \left(\int\limits_{\mathbb{R}_+} \left(M_G^{\frac{\beta}{p}} f(\operatorname{ch} x)\right)^r d\mu_\lambda(x)\right)^{\frac{\alpha p}{\beta r}},$$

which is equivalent to

$$\left\|I_G^\alpha f\right\|_{L_{q,\lambda}(\mathbb{R}_+)} \lesssim \left\|f\right\|_{L_{p,\lambda}(\mathbb{R}_+)}^{1-\frac{\alpha p}{\beta}} \cdot \left\|M_G^{\frac{\beta}{p}} f\right\|_{L_{r,\lambda}(\mathbb{R}_+)}^{\frac{\alpha p}{\beta}}.$$

The theorem is proved. □

Lemma 4.2. *Let $0 < \varepsilon < \min(\alpha,\ 2\lambda+1-\alpha)$. Then there is the constant $C_\varepsilon > 0$ such that for any nonnegative function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and for every point $x \in \mathbb{R}_+$, the inequality*

$$I_G^\alpha \varphi(\operatorname{ch} x) \le C_\varepsilon \sqrt{M_G^{\alpha-\varepsilon}\varphi(\operatorname{ch} x) M_G^{\alpha+\varepsilon}\varphi(\operatorname{ch} x)} \tag{4.15}$$

is valid.

Proof. Let r be an arbitrary positive number. Using the scheme of the proof of Lemma 4.1, we have

$$I_G^\alpha \varphi(\operatorname{ch} x) \lesssim \left(\int\limits_0^r + \int\limits_r^\infty\right) A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ (\operatorname{sh} t)^{\alpha-2\lambda-1} \operatorname{sh}^{2\lambda} t\, dt = J_1 + J_2. \tag{4.16}$$

Let $0 < \varepsilon < \alpha$, then

$$J_1 = \int\limits_0^r \frac{A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} dt = \sum_{k=0}^\infty \int\limits_{2^{-k-1}r}^{2^{-k}r} \frac{A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} dt$$

$$\le \sum_{k=0}^\infty \left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\alpha-2\lambda-1} \int\limits_0^{2^{-k}r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\, dt$$

$$\le \sum_{k=0}^\infty \left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\varepsilon} \left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\alpha-2\lambda-1-\varepsilon} \int\limits_0^{2^{-k}r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\, dt$$

$$\le (\operatorname{sh} r)^\varepsilon \sum_{k=0}^\infty 2^{-(k+1)\varepsilon} \left(\operatorname{sh}\frac{r}{2^{k+1}}\right)^{\alpha-2\lambda-1-\varepsilon} \int\limits_0^{2^{-k}r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\, dt$$

$$\le C_\varepsilon (\operatorname{sh} r)^\varepsilon M_G^{\alpha-\varepsilon}\varphi(\operatorname{ch} x). \tag{4.17}$$

Now let $0<\varepsilon<2\lambda+1-\alpha$. Then

$$J_2=\int\limits_r^\infty \frac{A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}dt=\sum_{k=0}^\infty \int\limits_{2^k r}^{2^{k+1}r} \frac{A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}dt$$

$$\le \sum_{k=0}^\infty \left(\operatorname{sh}\ 2^k r\right)^{\alpha-2\lambda-1}\int\limits_0^{2^{k+1}r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t\,dt$$

$$\le \sum_{k=0}^\infty \left(\operatorname{sh}\ 2^k r\right)^{-\varepsilon}\left(\operatorname{sh}\ 2^k r\right)^{\alpha+\varepsilon-2\lambda-1}\int\limits_0^{2^{k+1}r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t\,dt$$

$$\le (\operatorname{sh} r)^{-\varepsilon}\sum_{k=0}^\infty \left(2^{-k\varepsilon}\right)\left(\operatorname{sh}\ 2^k r\right)^{\alpha+\varepsilon-2\lambda-1}\int\limits_0^{2^{k+1}r} A_{\operatorname{ch} t}\varphi(\operatorname{ch} x)\ \operatorname{sh}^{2\lambda} t\,dt$$

$$\le C_\varepsilon(\operatorname{sh} r)^{-\varepsilon}M_G^{\alpha+\varepsilon}\varphi(\operatorname{ch} x). \tag{4.18}$$

Taking into account (4.17) and (4.18) in (4.16), we get that for any $\varepsilon>0$ with $0<\varepsilon<\min(\alpha,2\lambda+1-\alpha)$, there exists $C_\varepsilon>0$ such that for every nonnegative function φ, for any point $x\in\mathbb{R}_+$ and $r>0$, the following inequality

$$I_G^\alpha\varphi(\operatorname{ch} x)\le C_\varepsilon\left((\operatorname{sh} r)^\varepsilon M_G^{\alpha-\varepsilon}\varphi(\operatorname{ch} x)+(\operatorname{sh} r)^{-\varepsilon}M_G^{\alpha+\varepsilon}\varphi(\operatorname{ch} x)\right) \tag{4.19}$$

holds.

Assuming in (4.19)

$$(\operatorname{sh} r)^\varepsilon=\left(\frac{M_G^{\alpha+\varepsilon}\varphi(\operatorname{ch} x)}{M_G^{\alpha-\varepsilon}\varphi(\operatorname{ch} x)}\right)^{\frac{1}{q}},$$

we obtain inequality (4.15). □

Theorem 4.4. *Let $1<p<\frac{2\lambda+1}{\alpha}$ and $\frac1q=\frac1p-\frac{\alpha}{2\lambda+1}$. Then for ensuring the inequality*

$$\left(\int\limits_{\mathbb{R}_+}\left|I_G^\alpha\left(f\omega^\alpha\right)(\operatorname{ch} x)\right|^q\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda}x\,dx\right)^{\frac1q}\lesssim\left(\int\limits_{\mathbb{R}_+}|f(\operatorname{ch} x)|^p\,\omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda}x\,dx\right)^{\frac1p}$$

the necessary and sufficient condition is

$$\omega\in A_\beta^\lambda(\mathbb{R}_+),\quad \beta=1+\frac{q}{p'},\quad pp'=p+p'$$

for any $f\in L_{p,\omega,\lambda}(\mathbb{R}_+)$.

Proof. Sufficiency. Let $\omega\in A_\beta^\lambda(\mathbb{R}_+)$, then $\omega\in A_{\beta-\mu}^\lambda(\mathbb{R}_+)$ for any $\mu>0$ sufficiently small. Therefore, for $0<\varepsilon<\min\left(\alpha,\ 2\lambda+1-\alpha\right)$, we have $\omega\in A_{\beta_1}^\lambda(\mathbb{R}_+)$ with $\beta_1=1+\frac{p(2\lambda+1)}{p'(2\lambda+1-(\alpha+\varepsilon)p)}$ and $\omega\in A_{\beta_2}^\lambda(\mathbb{R}_+)$ with $\beta_2=1+\frac{p(2\lambda+1)}{p'(2\lambda+1-(\alpha-\varepsilon)p)}$. Now, if we take

$$\frac{1}{q_\varepsilon}=\frac1p-\frac{\alpha+\varepsilon}{2\lambda+1},\quad \frac{1}{q_\varepsilon}=\frac1p-\frac{\alpha-\varepsilon}{2\lambda+1},$$

then we find that $\omega\in A_{1+\frac{q_\varepsilon}{p'}}^\lambda(\mathbb{R}_+)$ and $\omega\in A_{1+\frac{\overline{q}_\varepsilon}{p'}}^\lambda(\mathbb{R}_+)$.

In view of $p_1=\frac{2q_\varepsilon}{q}$ and $p_2=\frac{2\overline{q}_\varepsilon}{q}$, we will have

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{q}{2}\left(\frac{1}{q_\varepsilon}+\frac{1}{\overline{q}_\varepsilon}\right)=\frac{q}{2}\left(\frac1p-\frac{\alpha+\varepsilon}{2\lambda+1}+\frac1p-\frac{\alpha-\varepsilon}{2\lambda+1}\right)$$

$$=q\left(\frac1p-\frac{\alpha}{2\lambda+1}\right)=1\Longleftrightarrow\frac1q=\frac1p-\frac{\alpha}{2\lambda+1}.$$

Suppose

$$F_1(\operatorname{ch} x) = \left(M_G^{\alpha+\varepsilon}\left(f\omega^\alpha\right)(\operatorname{ch} x)\right)^{\frac{q}{2}} \omega(\operatorname{ch} x)^{\frac{1}{p_1}}$$

and

$$F_2(\operatorname{ch} x) = \left(M_G^{\alpha-\varepsilon}\left(f\omega^\alpha\right)(\operatorname{ch} x)\right)^{\frac{q}{2}} \omega(\operatorname{ch} x)^{\frac{1}{p_2}}.$$

From (4.12), by Ḧolder's inequality, we have

$$\begin{aligned}
\int\limits_{\mathbb{R}_+} \left|I_G^\alpha\left(f\omega^\alpha\right)(\operatorname{ch} x)\right|^q \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx &\leq C_\varepsilon \int\limits_{\mathbb{R}_+} F_1(\operatorname{ch} x) F_2(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx \\
&\leq C_\varepsilon \Bigg(\int\limits_{\mathbb{R}_+} \left(M_G^{\alpha+\varepsilon}(f\omega^\alpha)(\operatorname{ch} x)\right)^{\frac{qp_1}{2}} \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{1}{p_1}} \\
&\quad\times \Bigg(\int\limits_{\mathbb{R}_+} \left(M_G^{\alpha-\varepsilon}(f\omega^\alpha)(\operatorname{ch} x)\right)^{\frac{qp_2}{2}} \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{1}{p_2}} \\
&= C_\varepsilon \Bigg(\int\limits_{\mathbb{R}_+} \left(M_G^{\alpha+\varepsilon}(f\omega^\alpha)(\operatorname{ch} x)\right)^{q_\varepsilon} \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{1}{p_1}} \\
&\quad\times \Bigg(\int\limits_{\mathbb{R}_+} \left(M_G^{\alpha-\varepsilon}(f\omega^\alpha)(\operatorname{ch} x)\right)^{\overline{q}_\varepsilon} \omega(\operatorname{ch} x)\operatorname{sh}^{2\lambda} x\,dx\Bigg)^{\frac{1}{p_2}}.
\end{aligned}$$

Finally, using Theorem 4.1, we obtain

$$\left\|I_G^\alpha\left(f\omega^\alpha\right)\right\|_{L_{q,\omega,\lambda}(\mathbb{R}_+)} \lesssim \|f\|_{L_{p,\omega,\lambda}(\mathbb{R}_+)}.$$

Necessity. We show that

$$M_G^\alpha\left(f\omega^\alpha\right)(\operatorname{ch} x) \lesssim I_G^\alpha\left(|f|\,\omega^\alpha\right)(\operatorname{ch} x). \tag{4.20}$$

In fact,

$$\begin{aligned}
\int\limits_{H(0,r)} A_{\operatorname{ch} t}\left(f\omega^\alpha\right)(\operatorname{ch} x)\operatorname{sh}^{2\lambda} t\,dt &= \int\limits_0^r A_{\operatorname{ch} t}\left(f\omega^\alpha\right)(\operatorname{ch} x)\operatorname{sh}^{2\lambda} t\,dt \\
&= \int\limits_0^r \frac{A_{\operatorname{ch} t}\left(f\omega^\alpha\right)(\operatorname{ch} x)(\operatorname{sh} t)^{2\lambda+1-\alpha}\operatorname{sh}^{2\lambda} t\,dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} \\
&= \sum_{k=0}^\infty \int\limits_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{\operatorname{ch} t}\left(f\omega^\alpha\right)(\operatorname{ch} x)(\operatorname{sh} t)^{2\lambda+1-\alpha}\operatorname{sh}^{2\lambda} t\,dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} \\
&\leq \sum_{k=0}^\infty \left(\operatorname{sh}\frac{r}{2^k}\right)^{2\lambda+1-\alpha} \int\limits_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{\operatorname{ch} t}\left(|f|\,\omega^\alpha\right)(\operatorname{ch} x)\operatorname{sh}^{2\lambda} t\,dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} \\
&\leq \left(\operatorname{sh}\frac{r}{2}\right)^{2\lambda+1-\alpha} \sum_{k=0}^\infty \frac{1}{2^{(k-1)(2\lambda+1-\alpha)}} \times \int\limits_0^\infty \frac{A_{\operatorname{ch} t}\left(|f|\,\omega^\alpha\right)(\operatorname{ch} x)\operatorname{sh}^{2\lambda} t\,dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}} \\
&\lesssim \left(\operatorname{sh}\frac{r}{2}\right)^{2\lambda+1-\alpha} I_G^\alpha\left(|f|\,\omega^\alpha\right)(\operatorname{ch} x).
\end{aligned} \tag{4.21}$$

Taking into account Lemma 2.1, by $0 < r < 2$ and (4.21), we have

$$M_G^\alpha (f\omega^\alpha)(\operatorname{ch} x) = \sup_{r>0} |H(0,r)|^{\frac{\alpha}{2\lambda+1}-1} \int\limits_{H(0,r)} A_{\operatorname{ch} t}(f\omega^\alpha)(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\, dt$$

$$\lesssim \sup_{r>0} \left(\operatorname{sh}\frac{r}{2}\right)^{(2\lambda+1)\left(\frac{\alpha}{2\lambda+1}-1\right)} \left(\operatorname{sh}\frac{r}{2}\right)^{2\lambda+1-\alpha} I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x) \lesssim I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x). \qquad (4.22)$$

On the other hand,

$$\int\limits_0^r A_{\operatorname{ch} t}(f\omega^\alpha)(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t dt \le \sum_{k=0}^\infty \int\limits_{\left(\frac{r}{2^{k+1}}\right)^{\frac{4\lambda}{2\lambda+1}}}^{\left(\frac{r}{2^k}\right)^{\frac{4\lambda}{2\lambda+1}}} \frac{A_{\operatorname{ch} t}(f\omega^\alpha)(\operatorname{ch} x)(\operatorname{sh} t)^{2\lambda+1-\alpha} \operatorname{sh}^{2\lambda} t\, dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}$$

$$\le \sum_{k=0}^\infty \left(\operatorname{sh}\frac{r}{2^k}\right)^{\frac{4\lambda}{2\lambda+1}(2\lambda+1-\alpha)} \int\limits_0^\infty \frac{A_{\operatorname{ch} t}(|f|\,\omega^\alpha)(\operatorname{ch} x) \operatorname{sh}^{2\lambda} t\, dt}{(\operatorname{sh} t)^{2\lambda+1-\alpha}}$$

$$\le \left(\operatorname{sh}\frac{r}{2}\right)^{\frac{4\lambda}{2\lambda+1}(2\lambda+1-\alpha)} I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x) \sum_{k=0}^\infty 2^{(1-k)\frac{4\lambda}{2\lambda+1}(2\lambda+1-\alpha)}$$

$$\lesssim \left(\operatorname{sh}\frac{r}{2}\right)^{\frac{4\lambda}{2\lambda+1}(2\lambda+1-\alpha)} I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x). \qquad (4.23)$$

Applying Lemma 2.1, for $2 \le r < \infty$ and (4.23), we obtain

$$M_G^\alpha (f\omega^\alpha)(\operatorname{ch} x) \lesssim \sup_{r>0} \left(\operatorname{sh}\frac{r}{2}\right)^{4\lambda\left(\frac{\alpha}{2\lambda+1}-1\right)} \left(\operatorname{sh}\frac{r}{2}\right)^{4\lambda\left(1-\frac{\alpha}{2\lambda+1}\right)} I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x)$$

$$\lesssim I_G^\alpha(|f|\,\omega^\alpha)(\operatorname{ch} x). \qquad (4.24)$$

Inequality (4.20) follows from inequalities (4.22) and (4.24). □

Theorem 4.5. *Let* $q = \frac{2\lambda+1}{2\lambda+1-\alpha}$. *Then the following two conditions are equivalent:*

(i) $$\int\limits_{\left\{x\in\mathbb{R}:\ I_G^\alpha\left(f\omega^{\frac{\alpha}{2\lambda+1}}\right)(\operatorname{ch} x)>\beta\right\}} \omega(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx \le C\left(\frac{1}{\beta}\int\limits_{\mathbb{R}_+} |f(\operatorname{ch} x)|\, \omega(\operatorname{ch} x) \operatorname{sh}^{2\lambda} x\, dx\right)^q$$

with a constant C, independent of f and $\lambda > 0$,
(ii) $\omega \in A_1^\lambda(\mathbb{R}_+)$.

The assertion of the Theorem follows from inequality (4.20) and Theorem 4.2.

Acknowledgements

The authors would like to express their gratitude to the referees for their very valuable comments and suggestions. The research of V. S. Guliyev and E. Ibragimov was partially supported by the grant of 1st Azerbaijan-Russia Joint Grant Competition (Agreement number EIF-BGM-4-RFTF-1/2017-21/01/1).

References

1. D. Adams, A note on Riesz potentials. *Duke Math. J.* **42** (1975), no. 4, 765–778.
2. I. A. Aliev, A. D. Gadjiev, Weighted estimates for multidimensional singular integrals generated by a generalized shift operator. (Russian) *Russian Acad. Sci. Sb. Math.* **77** (1994), no. 1, 37–55.
3. O. V. Besov, V. P. Il'in, S. M. Nikol'skiĭ, *Integral Representations of Functions, and Embedding Theorems*. Second edition. (Russian) Fizmatlit Nauka, Moscow, 1996, 480 pp.
4. A. P. Calderon, Inequalities for the maximal function relative to a metric. *Studia Math.* **57** (1976), no. 3, 297–306.
5. R. R. Coifman, C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* **51** (1974), 241–250.

6. Y. Ding, S. Z. Lu, Weighted norm inequalities for fractional integral operator with rough kernel. Canad. J. Math. 50 (1998), 29–39.
7. L. Durand, P. M. Fishbane, L. M. Simmons, Expansion formulas and addition theorems for Gegenbauer functions. *J. Mathematical Phys.* **17** (1976), no. 11, 1933–1948.
8. C. Fefferman, E. Stein, Some maximal inequalities. *Amer. J. Math.* **93** (1971), 107–115.
9. A. I. Gadziyev, On fractional maximal functions and fractional integrals, generated by Bessel differential operators. *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **20** (2000), no. 4, Math. Mech., 52–63, 265.
10. I. S. Gradshteyn, I. M. Rijik, *Tables of Integrals, Sums, Series and Products.* Nanka, Moscow, 1971.
11. V. S. Guliev, On maximal function and fractional integral, associated with the Bessel differential operator. *Math. Inequal. Appl.* **6** (2003), no. 2, 317–330.
12. E. V. Guliyev, Weighted inequality for fractional maximal functions and fractional integrals, associated with the Laplace-Bessel differential operator. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **26** (2006), no. 1, Math. Mech., 71–80.
13. V. S. Guliyev, E. J. Ibrahimov, S. Ar. Jafarova, Gegenbauer harmonic analysis and approximation of functions on the half line. *Advances in Analysis* **2** (2017), no. 3, 167–195.
14. G. H. Hardy, J. E. Littlewood, Some properties of fractional integrals. I. *Math. Z.* **27** (1928), no. 1, 565–606.
15. E. J. Ibrahimov, A. Akbulut, The Hardy-Littlewood-Sobolev theorem for Riesz potential generated by Gegenbauer operator. *Trans. A. Razmadze Math. Inst.* **170** (2016), no. 2, 166–199.
16. I. A. Kipriyanov, M. N. Ključancev, Estimates of a surface potential which is generated by a generalized shift operator. (Russian) *Dokl. Akad. Nauk SSSR* **188** (1969), 997–1000.
17. I. A. Kipriyanov, M. N. Ključancev, The boundedness of a certain class of singular integral operators. (Russian) *Dokl. Akad. Nauk SSSR* **186** (1969), 1253–1255.
18. I. A. Kipriyanov, M. I. Ključancev, Singular integrals that are generated by a generalized shift operator. II. (Russian) *Sibirsk. Mat. Ž.* **11** (1970) 1060–1083, 1196–1197.
19. M. N. Ključancev, ingular integrals that are generated by a generalized shift operator. I. (Russian) *Sibirsk. Mat. Ž* **11** (1970)m 810–821, 956–957.
20. V. M. Kokilashvili, A. Kufner, Fractional integrals on spaces of homogeneous type. *Comment. Math. Univ. Carolin.* **30** (1989), no. 3, 511–523.
21. V. M. Kokilashvili, S. Samko, Singular integrals in weighted Lebesgue spaces with variable exponent. *Georgian Math. J.* **10** (2003), no. 1, 145–156.
22. B. M. Levitan, Expansion in Fourier series and integrals with Bessel functions. (Russian) *Uspehi Matem. Nauk (N.S.)* **6** (1951). no. 2(42), 102–143.
23. B. M. Levitan, *Theory of Generalized Shift Operators.* (Russian) Izdat. Nauka, Moscow, 1973, 312 pp.
24. L. Lindelöf, *Le Calcul Des Résidus Et Ses Applications à la Théorie des Fonctions.* vol. 8. Gauthier-Villars, 1905.
25. L. N. Lyakhov, Inversion of the B-Riesz potentials. (Russian) *Dokl. Akad. Nauk SSSR* **321** (1991), no. 3. 466–469.
26. L. N. Lyakhov, Spaces of Riesz B-potentials. (Russian) *translated from Dokl. Akad. Nauk* **334** (1994), no. 3, 278–280 *Russian Acad. Sci. Dokl. Math.* **49** (1994), no. 1, 83–87
27. R. A. Macias, C. Segovia, *A Well Behaved Quasi-Distance for Spaces of Homogeneous Type.* vol. 32 of Trabajos de Matemática. Inst. Argentino Mat. 1981.
28. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165** (1972), 207–226.
29. B. Muckenhoupt, R. L. Wheeden, Weighted norm inequalities for fractional integrals. *Trans. Amer. Math. Soc.* **192** (1974), 261–274.
30. G. Welland, Weighted norm inequalities for fractional integrals. *Proc. Amer. Math. Soc.* **51** (1975), 143–148.
31. S. Yu, Y. Ding, D. Yan, *Singular Integrals and Related Topics.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
32. A. Zygmund, *Trigonometric Series.* (Russian) Izdat. Mir, Moscow, 1965 vol. I: 615 pp.; vol. II: 537 pp.

(Received 27.12.2018)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 79–91

SOLUTIONS OF SOME DIOPHANTINE EQUATIONS IN TERMS OF HORADAM SEQUENCE

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Abstract. Let a, b, and P be integers such that $(a,b) \neq (0,0)$. In this study, we give all solutions of the equations $x^2 - Pxy - y^2 = \pm\left(b^2 - Pab - a^2\right)$, $x^2 - (P^2+4)y^2 = \pm 4(b^2 - Pab - a^2)$, $x^2-(P^2+4)y^2 = \pm 4(b^2-Pab-a^2)^2$, $x^2-Pxy+y^2 = b^2-Pab+a^2$, $x^2-(P^2-4)y^2 = 4(b^2-Pab+a^2)$, and $x^2 - (P^2-4)y^2 = 4(b^2 - Pab + a^2)^2$ in terms of the second order recurrence sequences when $|b^2 - Pab \pm a^2|$ is odd prime.

1. Introduction

The second order recurrence sequence $\{W_n\} = \{W_n(a,b;P,Q)\}$ is defined by

$$W_0 = a,\ W_1 = b, \quad \text{and} \quad W_n = PW_{n-1} + QW_{n-2} \quad \text{for} \quad n \geq 2,$$

where a, b, P, and Q are integers with $PQ \neq 0$ and $(a,b) \neq (0,0)$. Particular cases of $\{W_n\}$ are the Lucas sequence of the first kind $\{U_n(P,Q)\} = \{W_n(0,1;P,Q)\}$ and the Lucas sequence of the second kind $\{V_n(P,Q)\} = \{W_n(2,P;P,Q)\}$. Now we define the sequence $\{X_n\} = \{X_n(a,b;P,Q)\}$ by

$$X_0 = 2b - aP, \quad X_1 = bP + 2aQ, \quad \text{and} \quad X_n = PX_{n-1} + QX_{n-2} \quad \text{for} \quad n \geq 2.$$

It is convenient to consider $\{X_n\}$ to be the companion sequence of $\{W_n\}$, in the same way that $\{V_n\}$ is the companion sequence of $\{U_n\}$. Let α and β be the roots of the equation $x^2 - Px - Q = 0$. Then $\alpha = (P + \sqrt{P^2+4Q})/2$ and $\beta = (P - \sqrt{P^2+4Q})/2$. Clearly, $\alpha + \beta = P$, $\alpha - \beta = \sqrt{P^2+4Q}$, and $\alpha\beta = -Q$. Assume that $P^2 + 4Q \neq 0$. Then Binet formulas of $\{W_n\}$ and $\{X_n\}$ are given by

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \quad \text{and} \quad X_n = A\alpha^n + B\beta^n, \tag{1.1}$$

where $A = b - a\beta$ and $B = b - a\alpha$. It can be seen that $AB = b^2 - abP - a^2Q$. Moreover, it can be easily shown that there are the following relations between the terms of the sequences $\{W_n\}$, $\{X_n\}$, $\{U_n\}$, and $\{V_n\}$ given by

$$X_n = W_{n+1} + QW_{n-1} = PW_n + 2QW_{n-1}, \tag{1.2}$$

$$(P^2+4Q)W_n = X_{n+1} + QX_{n-1}, \tag{1.3}$$

$$W_n = bU_n + aQU_{n-1} \quad \text{and} \quad X_n = bV_n + aQV_{n-1} \tag{1.4}$$

for $n \geq 1$. It is well known that the numbers U_n and V_n for negative subscripts are defined as

$$U_{-n} = \frac{-U_n}{(-Q)^n} \text{ and } V_{-n} = \frac{V_n}{(-Q)^n}$$

for $n \geq 1$. By using (1.1) together with (1.4), it is convenient to define the numbers W_n and X_n for negative subscripts by

$$W_{-n} = \frac{A\alpha^{-n} - B\beta^{-n}}{\alpha - \beta} \quad \text{and} \quad X_{-n} = A\alpha^{-n} + B\beta^{-n}.$$

Then it follows that

$$W_{-n} = \frac{-bU_n + aU_{n+1}}{(-Q)^n} \quad \text{and} \quad X_{-n} = \frac{bV_n - aV_{n+1}}{(-Q)^n} \tag{1.5}$$

2010 *Mathematics Subject Classification.* 11B37, 11D09.
Key words and phrases. Second order recurrence sequence; Diophantine equation.

and therefore

$$W_{-n} = bU_{-n} + aQU_{-n-1} \quad \text{and} \quad X_{-n} = bV_{-n} + aQV_{-n-1}.$$

Thus it is seen that identities (1.2), (1.3), and (1.4) hold for all integers n. For more information about the sequence one can consult [2, 10, 11, 13, 15].

In the literature, integer solutions of the equations $x^2 - Pxy - y^2 = 1$, $x^2 - Pxy - y^2 = -1$, $x^2 - (P^2+4)y^2 = 4$, $x^2 - (P^2+4)y^2 = -4$, $x^2 - Pxy + y^2 = 1$, and $x^2 - (P^2-4)y^2 = 4$ are given in terms of the sequences $\{U_n(P, \pm 1)\}$ and $\{V_n(P, \pm 1)\}$(see [4–9, 12, 16]). More clearly, we can state them by

Equations	**Solutions**
$x^2 - Pxy - y^2 = 1$	$(x, y) = \pm(U_n(P, 1), U_{n-1}(P, 1))$ with n odd,
$x^2 - Pxy - y^2 = -1$	$(x, y) = \pm(U_n(P, 1), U_{n-1}(P, 1))$ with n even,
$x^2 - (P^2+4)y^2 = 4$	$(x, y) = \pm(V_n(P, 1), U_n(P, 1))$ with n even,
$x^2 - (P^2+4)y^2 = -4$	$(x, y) = \pm(V_n(P, 1), U_n(P, 1))$ with n odd,
$x^2 - Pxy + y^2 = 1$	$(x, y) = \pm(U_n(P, -1), U_{n-1}(P, -1))$,
$x^2 - (P^2-4)y^2 = 4$	$(x, y) = \pm(V_n(P, -1), U_n(P, -1))$.

Moreover, if $P^2 \pm 4$ is square free, then all integer solutions of the equations $x^2 - Pxy - y^2 = P^2 + 4, x^2 - Pxy - y^2 = -(P^2+4)$, and $x^2 - Pxy + y^2 = -(P^2-4)$ are given in terms of the sequence $\{V_n(P, \pm 1)\}$(see [7]). When $P^2 \pm 4$ is square free, we get

Equations	**Solutions**
$x^2 - Pxy - y^2 = P^2 + 4$	$(x, y) = \pm(V_n(P, 1), V_{n-1}(P, 1))$ with n even,
$x^2 - Pxy - y^2 = -(P^2+4)$	$(x, y) = \pm(V_n, (P, 1), V_{n-1}(P, 1))$ with n odd,
$x^2 - Pxy + y^2 = -(P^2-4)$	$(x, y) = \pm(V_n(P, -1), V_{n-1}(P, -1))$.

In this paper, we give all integer solutions of the equations

$$\begin{aligned} x^2 - Pxy - y^2 =& b^2 - Pab - a^2,\ x^2 - Pxy - y^2 = -(b^2 - Pab - a^2) \\ x^2 - (P^2+4)y^2 =& 4(b^2 - Pab - a^2),\ x^2 - (P^2+4)y^2 = -4(b^2 - Pab - a^2), \\ x^2 - (P^2+4)y^2 =& 4(b^2 - Pab - a^2)^2,\ x^2 - (P^2+4)y^2 = -4(b^2 - Pab - a^2)^2, \\ x^2 - Pxy + y^2 =& b^2 - Pab + a^2,\ x^2 - (P^2-4)y^2 = 4(b^2 - Pab + a^2), \end{aligned}$$

and

$$x^2 - (P^2-4)y^2 = 4(b^2 - Pab + a^2)^2$$

in terms of second order recurrence sequences when $|b^2 - Pab \pm a^2|$ is odd prime. In the second section, we give some identities between the sequence $\{W_n\}$ and its companion sequence $\{X_n\}$. After that, we give our main theorem in the third section.

2. Preliminaries

In this section, we give some identities, theorems, and lemmas, which will be used later. The following identities concerning the sequence $\{W_n\}$ and its companion sequence $\{X_n\}$ hold.

$$X_n^2 - (P^2+4Q)W_n^2 = 4(-Q)^n(b^2 - Pab - Qa^2), \tag{2.1}$$

$$W_{n+1}^2 - PW_{n+1}W_n - QW_n^2 = (-Q)^n(b^2 - Pab - Qa^2), \tag{2.2}$$

$$W_n^2 - PW_{n+1}W_{n-1} = (-Q)^{n-1}(b^2 - Pab - Qa^2), \tag{2.3}$$

$$X_{n+1}^2 - PX_{n+1}X_n - QX_n^2 = -(-Q)^n(P^2+4Q)(b^2 - Pab - Qa^2), \tag{2.4}$$

and

$$X_{n+1}X_{n-1} - X_n^2 = (-Q)^{n-1}(P^2+4Q)(b^2 - Pab - Qa^2). \tag{2.5}$$

One can find the above identities in [2] and [15]. Let

$$W_n^* = bW_n + aQW_{n-1} \quad \text{and} \quad X_n^* = bX_n + aQX_{n-1}. \tag{2.6}$$

Then it can be shown that

$$bW_n - aW_{n+1} = (b^2 - Pab - a^2Q)U_n \quad \text{and} \quad bX_n - aX_{n+1} = (b^2 - Pab - a^2Q)V_n \tag{2.7}$$

$$(X_n^*)^2 - (P^2 + 4Q)(W_n^*)^2 = 4(-Q)^n(b^2 - Pab - Qa^2)^2, \tag{2.8}$$

$$\left(W_{n+1}^*\right)^2 - PW_{n+1}^*W_n^* - Q(W_n^*)^2 = (-Q)^n(b^2 - Pab - Qa^2)^2, \tag{2.9}$$

and

$$\left(X_{n+1}^*\right)^2 - PX_{n+1}^*X_n^* - Q(X_n^*)^2 = -(-Q)^n(P^2 + 4Q)(b^2 - Pab - Qa^2)^2 \tag{2.10}$$

by (2.1), (2.2), (2.3), (2.4), and (2.5).

From now on, we write W_n, X_n, U_n, and V_n instead of $W_n(a, b; P, 1), X_n(a, b; P, 1)$, $U_n(P, 1)$, and $V_n(P, 1)$, respectively. We represent $W_n(a, b; P, -1), X_n(a, b; P, -1)$, $U_n(P, -1)$, and $V_n(P, -1)$ by w_n, x_n, u_n, and v_n, respectively. We write x_n^* and w_n^* instead of $X_n^*(a, b; P, -1)$ and $W_n^*(a, b; P, -1)$, respectively. The following three theorems are given in [7].

Theorem 2.1. *Let u and v be integers. Then $u^2 - (P^2 + 4)v^2 = \pm 4$ if and only if $(u, v) = \mp(V_n, U_n)$ for some $n \in \mathbb{Z}$.*

Theorem 2.2. *Let $P > 3$. Then all integer solutions of the equation $u^2 - (P^2 - 4)v^2 = 4$ are given by $(u, v) = \mp(v_n, u_n)$ with $n \in \mathbb{Z}$.*

Theorem 2.3. *Let $P > 3$. Then the equation $u^2 - (P^2 - 4)v^2 = -4$ has no integer solutions.*

3. Main Theorems

3.1. **Solutions of some Diophantine equations for $Q = 1$.** In this subsection, we will assume that $Q = 1, P \geq 1$, and $\Delta = b^2 - Pab - a^2$ such that $|\Delta| > 2$ and $|\Delta|$ is prime.

Theorem 3.1. *Let x and y be integers. Then $x^2 - (P^2+4)y^2 = \pm 4\Delta$ if and only if $(x, y) = \pm(X_n, W_n)$ or $\pm((-1)^{n-1}X_n, (-1)^nW_n)$ for some $n \in \mathbb{Z}$.*

Proof. If $(x, y) = \pm(X_n, W_n)$ or $\pm((-1)^{n-1}X_n, (-1)^nW_n)$, then it follows that $x^2 - (P^2 + 4)y^2 = \pm 4\Delta$ by (2.1). Now let $x^2 - (P^2 + 4)y^2 = \pm 4\Delta$. Assume that $\Delta | y$. Then $\Delta | x$ and this shows that $\Delta^2 | x^2 - (P^2 + 4)y^2$. Then we get $\Delta^2 | 4\Delta$, but this is impossible, since $|\Delta| > 2$ and $|\Delta|$ is prime. Therefore $\Delta \nmid y$.

It is obvious that $4\Delta = (2b - Pa)^2 - (P^2 + 4)a^2$. Thus

$$\Delta | [(2b - Pa)^2 - (P^2 + 4)a^2] \tag{3.1}$$

and

$$\Delta | [x^2 - (P^2 + 4)y^2]. \tag{3.2}$$

From (3.1) and (3.2), we get

$$\Delta | [a^2\left(x^2 - (P^2 + 4)y^2\right) - y^2\left((2b - Pa)^2 - (P^2 + 4)a^2\right)],$$

i.e.,

$$\Delta | [ax + y(2b - Pa)][ax - y(2b - Pa)].$$

Since $|\Delta|$ is prime, it follows that

$$\Delta | [ax + y(2b - Pa)]$$

or

$$\Delta | [ax - y(2b - Pa)].$$

Also, from (3.1) and (3.2), we get

$$\Delta | [a^2(P^2 + 4)\left(x^2 - (P^2 + 4)y^2\right) + x^2\left((2b - Pa)^2 - (P^2 + 4)a^2\right)],$$

i.e.,

$$\Delta | [(2b - Pa)x - ay(P^2 + 4)][(2b - Pa)x + ay(P^2 + 4)].$$

This implies that

$$\Delta|[(2b-Pa)x-ay(P^2+4)]$$

or

$$\Delta|[(2b-Pa)x+ay(P^2+4)].$$

Hence, we have

$$\Delta|[ax+y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x-ay(P^2+4)] \tag{3.3}$$

or

$$\Delta|[ax-y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x+ay(P^2+4)], \tag{3.4}$$

and

$$\Delta|[ax-y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x-ay(P^2+4)] \tag{3.5}$$

or

$$\Delta|[ax+y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x+ay(P^2+4)]. \tag{3.6}$$

Now assume that (3.3) is satisfied. Then we get

$$\Delta|[x\left(ax+y(2b-Pa)\right)-y\left((2b-Pa)x-ay(P^2+4)\right)],$$

i.e.,

$$\Delta \mid a[x^2+(P^2+4)y^2].$$

This implies that $\Delta|a$ or $\Delta|\left(x^2+(P^2+4)y^2\right)$. Assume that $\Delta|a$. Then $\Delta|b$ since $\Delta=b^2-Pab-a^2$. Thus $\Delta^2|\Delta$ and this shows that $\Delta|1$, but this is impossible. Therefore $\Delta|\left(x^2+(P^2+4)y^2\right)$. Then we see that $\Delta|2(P^2+4)y^2$, since $\Delta|\left(x^2-(P^2+4)y^2\right)$. Hence, $\Delta|2(P^2+4)$, since $\Delta\nmid y$. Then it follows that

$$\Delta|\left[(P^2+4)2ay+(2b-Pa)x-ay(P^2+4)\right],$$

i.e.,

$$\Delta|[(2b-Pa)x+ay(P^2+4)].$$

In this case, (3.3) coincides with (3.6). Similarly, it is seen that (3.4) coincides with (3.5).

Now, let us show that $2|[(2b-Pa)x\pm ay(P^2+4)]$ and $2|[ax\pm y(2b-Pa)]$. It is seen that $x^2\equiv(Py)^2 \pmod 4$ from the equation $x^2-(P^2+4)y^2=\pm4\Delta$. This implies that x and Py have the same parity. Therefore, we see that $2|[(2b-Pa)x\pm ay(P^2+4)]$ and $2|[ax\pm y(2b-Pa)]$.

Consequently, we should examine two cases

$$2\Delta|[(2b-Pa)x-ay(P^2+4)] \quad \text{and} \quad 2\Delta|[ax-y(2b-Pa)] \tag{3.7}$$

and

$$2\Delta|[(2b-Pa)x+ay(P^2+4)] \quad \text{and} \quad 2\Delta|[ax+y(2b-Pa)]. \tag{3.8}$$

Assume that (3.7) is satisfied. Let

$$u=\frac{(2b-Pa)x-ay(P^2+4)}{2\Delta} \quad \text{and} \quad v=\frac{[(2b-Pa)y-ax]}{2\Delta}.$$

Then it follows that

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} 2b-Pa & -a(P^2+4) \\ -a & 2b-Pa \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and so a simple computation shows that

$$\begin{bmatrix} 2b-Pa & a(P^2+4) \\ a & 2b-Pa \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}. \tag{3.9}$$

By using the identities

$$x^2-(P^2+4)y^2=\pm4\Delta \quad \text{and} \quad 4\Delta=(2b-Pa)^2-(P^2+4)a^2,$$

it is seen that

$$u^2-(P^2+4)v^2=\pm4.$$

Thus we have $(u,v)=\mp(V_n,U_n)$ for some $n\in\mathbb{Z}$ by Theorem 2.1. Then $2x=\pm((2b-Pa)\,V_n+a(P^2+4)U_n)$ and $2y=\pm(aV_n+(2b-Pa)\,U_n)$ by (3.9). By using (1.2), (1.3), and (1.4), we get

$$\begin{aligned} x &= \pm\left((2b-Pa)\,V_n+a(P^2+4)U_n\right)/2=\pm\left(2bV_n-PaV_n+aV_{n+1}+aV_{n-1}\right)/2\\ &= \pm\left(bV_n+aV_{n-1}\right)=\pm X_n \end{aligned}$$

and

$$\begin{aligned} y &= \pm\left(aV_n+(2b-Pa)\,U_n\right)/2=\pm\left(aU_{n+1}+aU_{n-1}+2bU_n-PaU_n\right)/2\\ &= \pm\left(bU_n+aU_{n-1}\right)=\pm W_n. \end{aligned}$$

Now assume that (3.8) is satisfied. Let

$$u=\frac{(2b-Pa)x+ay(P^2+4)}{2\Delta}\quad\text{and}\quad v=\frac{[(2b-Pa)y+ax]}{2\Delta}.$$

Then we can see by a simple computation that

$$\begin{bmatrix} 2b-Pa & -a(P^2+4)\\ -a & 2b-Pa \end{bmatrix}\begin{bmatrix} u\\ v\end{bmatrix}=\begin{bmatrix} 2x\\ 2y\end{bmatrix}$$

and

$$u^2-(P^2+4)v^2=\pm 4.$$

Thus we have $(u,v)=\mp(V_m,U_m)$ for some $m\in\mathbb{Z}$ by Theorem 2.1. Similarly, it can be shown that $(x,y)=\pm((-1)^mX_{-m},(-1)^{m+1}W_{-m})$. Taking $n=-m$, it is seen that

$$\begin{aligned} (x,y) &= \pm\left((-1)^{-n}X_n,(-1)^{-n+1}W_n\right)=\pm((-1)^{-n-1}X_n,(-1)^{-n}W_n)\\ &= \pm\left((-1)^{n-1}X_n,(-1)^nW_n\right). \end{aligned}$$

□

From the above theorem and (2.1), the following corollaries can be given.

Corollary 1. *All integer solutions of the equation* $x^2-(P^2+4)y^2=4\Delta$ *are given by* $(x,y)=\pm(X_{2n},W_{2n})$ *or* $\pm(-X_{2n},W_{2n})$ *with* $n\in\mathbb{Z}$.

Corollary 2. *All integer solutions of the equation* $x^2-(P^2+4)y^2=-4\Delta$ *are given by* $(x,y)=\pm(X_{2n-1},W_{2n-1})$ *or* $\pm(X_{2n-1},-W_{2n-1})$ *with* $n\in\mathbb{Z}$.

Theorem 3.2. *Let x and y be integers. Then* $x^2-Pxy-y^2=\pm\Delta$ *if and only if* $(x,y)=\pm(W_{n+1},W_n)$ *or* $\pm\left((-1)^nW_n,(-1)^{n+1}W_{n+1}\right)$ *for some* $n\in\mathbb{Z}$.

Proof. If $(x,y)=\pm(W_{n+1},W_n)$ or $\pm\left((-1)^nW_n,(-1)^{n+1}W_{n+1}\right)$, then it follows that $x^2-Pxy-y^2=\pm\Delta$ by (2.2). Assume that $x^2-Pxy-y^2=\pm\Delta$. Completing the square gives $(2x-Py)^2-(P^2+4)y^2=\pm4\Delta$.This implies that $(2x-Py,y)=\pm(X_n,W_n)$ or $\pm((-1)^{n-1}X_n,(-1)^nW_n)$ for some $n\in\mathbb{Z}$ by Theorem 3.1. If $(2x-Py,y)=\pm(X_n,W_n)$, then we get $(x,y)=\pm(W_{n+1},W_n)$. If $(2x-Py,y)=\pm((-1)^{n-1}X_n,(-1)^nW_n)$, then $(x,y)=\pm\left((-1)^{n-1}W_{n-1},(-1)^nW_n\right)$. □

From the above theorem and (2.2), the following corollaries can be given.

Corollary 3. *All integer solutions of the equation* $x^2-Pxy-y^2=\Delta$ *are given by* $(x,y)=\pm(W_{2n+1},W_{2n})$ *or* $\pm\left(-W_{2n+1},W_{2n+2}\right)$ *with* $n\in\mathbb{Z}$.

Corollary 4. *All integer solutions of the equation* $x^2-Pxy-y^2=-\Delta$ *are given by* $(x,y)=\pm(W_{2n},W_{2n-1})$ *or* $\pm\left(W_{2n},-W_{2n+1}\right)$ *with* $n\in\mathbb{Z}$.

Since $b^2-3ab+a^2=(b-a)^2-(b-a)a-a^2$, we can give the following corollaries.

Corollary 5. *Let* $|b^2-3ab+a^2|$ *be a prime number. Then all integer solutions of the equation* $x^2-xy-y^2=b^2-3ab+a^2$ *are given by* $(x,y)=\pm(W_{2n+1},W_{2n})$ *or* $\pm\left(-W_{2n+1},W_{2n+2}\right)$ *with* $n\in\mathbb{Z}$, *where* $W_n=W_n(a,b-a,1,1)$.

Corollary 6. *Let* $|b^2-3ab+a^2|$ *be a prime number. Then all integer solutions of the equation* $x^2-xy-y^2=-(b^2-3ab+a^2)$ *are given by* $(x,y)=\pm(W_{2n},W_{2n-1})$ *or* $\pm\left(W_{2n},-W_{2n+1}\right)$ *with* $n\in\mathbb{Z}$, *where* $W_n=W_n(a,b-a,1,1)$.

Theorem 3.3. *Let P^2+4 be square free. Then $x^2 - Pxy - y^2 = \pm(P^2+4)\Delta$ for some integers x and y if and only if $(x, y) = \pm(X_{n+1}, X_n)$ or $\pm((-1)^n X_{n-1}, (-1)^{n-1} X_n)$ for some $n \in \mathbb{Z}$.*

Proof. If $(x, y) = \pm(X_{n+1}, X_n)$ or $\pm((-1)^n X_{n-1}, (-1)^{n-1} X_n)$, then it follows that $x^2 - Pxy - y^2 = \pm(P^2+4)\Delta$ by (2.4). Now assume that P^2+4 is square free and $x^2 - Pxy - y^2 = \pm(P^2+4)\Delta$ for some integers x and y. Then $(2x - Py)^2 - (P^2+4)y^2 = \pm 4(P^2+4)\Delta$. Since P^2+4 is square free, it is seen that $(P^2+4)|(2x - Py)$. Therefore, if we take

$$u = \frac{2x - Py}{P^2+4} \quad \text{and} \quad v = y,$$

then we get $v^2 - \left(P^2+4\right) u^2 = \pm 4\Delta$. This implies that $(v, u) = \mp (X_n, W_n)$ or $\pm((-1)^{n-1} X_n, (-1)^n W_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.1. If $(v, u) = \mp (X_n, W_n)$, then it follows that $y = v = \pm X_n$ and

$$\begin{aligned} x &= \left(\left(P^2+4\right) u + Pv\right)/2 = \pm\left(\left(P^2+4\right) W_n + PX_n\right)/2 \\ &= \pm\left(X_{n+1} + X_{n-1} + PX_n\right)/2 \\ &= \pm\, X_{n+1} \end{aligned}$$

by (1.3). Similarly, it can be seen that $(x, y) = \pm((-1)^n X_{n-1}, (-1)^{n-1} X_n)$ if $(v, u) = \pm((-1)^{n-1} X_n, (-1)^n W_n)$. □

We can give the following corollaries from the above theorem and (2.4).

Corollary 7. *Let P^2+4 be square free. Then all integer solutions of the equation $x^2 - Pxy - y^2 = (P^2+4)\Delta$ are given by $(x, y) = \pm(X_{2n+2}, X_{2n+1})$ or $\pm(-X_{2n}, X_{2n+1})$ with $n \in \mathbb{Z}$.*

Corollary 8. *Let P^2+4 be square free. Then all integer solutions of the equation $x^2 - Pxy - y^2 = -(P^2+4)\Delta$ are given by $(x, y) = \pm(X_{2n+1}, X_{2n})$ or $\pm(X_{2n-1}, -X_{2n})$ with $n \in \mathbb{Z}$.*

Theorem 3.4. *Let x and y be integers. Then $x^2 - (P^2+4)y^2 = \pm 4\Delta^2$ if and only if $(x, y) = \pm(X_n^*, W_n^*)$, $\pm((-1)^{n-1} X_n^*, (-1)^n W_n^*)$, or $\pm(\Delta V_n, \Delta U_n)$ for some $n \in \mathbb{Z}$.*

Proof. If $(x, y) = \pm(X_n^*, W_n^*), \pm((-1)^{n-1} X_n^*, (-1)^n W_n^*)$, or $\pm(\Delta V_n, \Delta U_n)$, then it follows that $x^2 - (P^2+4)y^2 = \pm 4\Delta^2$ by (2.1) and (2.8). Let $x^2 - (P^2+4)y^2 = \pm 4\Delta^2$. Now we divide the proof into two cases:

Case I: Assume that $\Delta \nmid y$.

It is obvious that $4\Delta = (2b - Pa)^2 - (P^2+4)a^2$. Thus

$$\Delta | [(2b - Pa)^2 - (P^2+4)a^2] \tag{3.10}$$

and

$$\Delta | [x^2 - (P^2+4)y^2]. \tag{3.11}$$

From (3.10) and (3.11), we get

$$\Delta | [a^2\left(x^2 - (P^2+4)y^2\right) - y^2\left((2b - Pa)^2 - (P^2+4)a^2\right)],$$

i.e.,

$$\Delta | [ax + y(2b - Pa)][ax - y(2b - Pa)].$$

Since $|\Delta|$ is a prime number, it follows that

$$\Delta | [ax + y(2b - Pa)]$$

or

$$\Delta | [ax - y(2b - Pa)].$$

Also, from (3.10) and (3.11), we get

$$\Delta | [a^2(P^2+4)\left(x^2 - (P^2+4)y^2\right) + x^2\left((2b - Pa)^2 - (P^2+4)a^2\right)],$$

i.e.,

$$\Delta | [(2b - Pa)x - ay(P^2+4)][(2b - Pa)x + ay(P^2+4)].$$

This implies that

$$\Delta | [(2b - Pa)x - ay(P^2+4)]$$

or

$$\Delta|[(2b - Pa)x + ay(P^2 + 4)].$$

Hence, we have

$$\Delta|[ax + y(2b - Pa)] \quad \text{and} \quad \Delta|[(2b - Pa)x - ay(P^2 + 4)] \tag{3.12}$$

or

$$\Delta|[ax - y(2b - Pa)] \text{ and } \Delta|[(2b - Pa)x + ay(P^2 + 4)] \tag{3.13}$$

and

$$\Delta|[ax - y(2b - Pa)] \quad \text{and} \quad \Delta|[(2b - Pa)x - ay(P^2 + 4)] \tag{3.14}$$

or

$$\Delta|[ax + y(2b - Pa)] \quad \text{and} \quad \Delta|[(2b - Pa)x + ay(P^2 + 4)]. \tag{3.15}$$

Now assume that (3.12) is satisfied. Then we get

$$\Delta|[x\left(ax + y(2b - Pa)\right) - y\left((2b - Pa)x - ay(P^2 + 4)\right)],$$

i.e.,

$$\Delta \mid a[x^2 + (P^2 + 4)y^2].$$

This implies that $\Delta|a$ or $\Delta|\left(x^2 + (P^2 + 4)y^2\right)$. Assume that $\Delta|a$. Then $\Delta|b$, since $\Delta = b^2 - Pab - a^2$. Thus $\Delta^2|\Delta$ and this shows that $\Delta|1$, which is impossible. Therefore $\Delta|\left(x^2 + (P^2 + 4)y^2\right)$. Then we see that $\Delta|2(P^2 + 4)y^2$ since $\Delta|\left(x^2 - (P^2 + 4)y^2\right)$. Hence $\Delta|2(P^2 + 4)$ since $\Delta \nmid y$. Then it follows that

$$\Delta|\left[(P^2 + 4)2ay + (2b - Pa)x - ay(P^2 + 4)\right],$$

i.e.,

$$\Delta|[(2b - Pa)x + ay(P^2 + 4)].$$

In this case, (3.12) coincides with (3.15). Similarly, it is seen that (3.13) coincides with (3.14).

It can be seen that $2|[(2b - Pa)x \pm ay(P^2 + 4)]$ and $2|[ax \pm y(2b - Pa)]$.

Consequently, we should examine two cases

$$2\Delta|[(2b - Pa)x - ay(P^2 + 4)] \quad \text{and} \quad 2\Delta|[ax - y(2b - Pa)] \tag{3.16}$$

and

$$2\Delta|[(2b - Pa)x + ay(P^2 + 4)] \quad \text{and} \quad 2\Delta|[ax + y(2b - Pa)]. \tag{3.17}$$

Assume that (3.16) is satisfied. Let

$$u = \frac{(2b - Pa)x - ay(P^2 + 4)}{2\Delta} \quad \text{and} \quad v = \frac{[(2b - Pa)y - ax]}{2\Delta}.$$

Then it follows that

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} 2b - Pa & -a(P^2 + 4) \\ -a & 2b - Pa \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and so a simple computation shows that

$$\begin{bmatrix} 2b - Pa & a(P^2 + 4) \\ a & 2b - Pa \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}. \tag{3.18}$$

By using the equalties

$$x^2 - (P^2 + 4)y^2 = \pm 4\Delta^2 \quad \text{and} \quad 4\Delta = (2b - Pa)^2 - (P^2 + 4)a^2,$$

it is seen that

$$u^2 - (P^2 + 4)v^2 = \pm 4\Delta.$$

Thus we have $(u, v) = \pm(X_n, W_n)$ or $\pm((-1)^{n-1}X_n, (-1)^n W_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.1. If $(u, v) = \pm(X_n, W_n)$, then $2x = \pm((2b - Pa)\,X_n + a(P^2 + 4)W_n)$ and $2y = \pm(aX_n + (2b - Pa)\,W_n)$ by (3.18). By using (1.2), (1.3), and (2.6), we get

$$\begin{aligned} x &= \pm\left((2b - Pa)\,X_n + a(P^2 + 4)W_n\right)/2 = \pm\left(2bX_n - PaX_n + aX_{n+1} + aX_{n-1}\right)/2 \\ &= \pm\left(bX_n + aX_{n-1}\right) = \pm X_n^* \end{aligned}$$

and

$$\begin{aligned} y &= \pm\left(aX_n + (2b - Pa)\, W_n\right)/2 = \pm\left(aW_{n+1} + aW_{n-1} + 2bW_n - PaW_n\right)/2 \\ &= \pm\left(bW_n + aW_{n-1}\right) = \pm W_n^*. \end{aligned}$$

Assume that $(u, v) = \pm\left((-1)^{n-1}X_n, (-1)^n W_n\right)$. Then from (3.18) and (2.7), we get

$$\begin{aligned} y &= \pm\left(a(-1)^{n-1}X_n + (2b - Pa)\,(-1)^n W_n\right)/2 \\ &= \pm\,(-1)^n\left(-aW_{n+1} - aW_{n-1} + 2bW_n - PaW_n\right)/2 \\ &= \pm\,(-1)^n\left(bW_n - aW_{n+1}\right) = \pm(-1)^n \Delta U_n. \end{aligned}$$

However, this is impossible since $\Delta \nmid y$.

Now assume that (3.17) is satisfied. Let

$$u = \frac{(2b - Pa)x + ay(P^2 + 4)}{2\Delta} \quad \text{and} \quad v = \frac{[(2b - Pa)y + ax]}{2\Delta}.$$

Then we can see by a simple computation that

$$\begin{bmatrix} 2b - Pa & -a(P^2 + 4) \\ -a & 2b - Pa \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

and

$$u^2 - (P^2 + 4)v^2 = \pm 4\Delta.$$

Thus we have $(u, v) = \pm(X_n, W_n)$ or $\pm((-1)^{n-1}X_n, (-1)^n W_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.1. Similarly, it can be shown that $(x, y) = \pm((-1)^{n-1}X_n^*, (-1)^n W_n^*)$.

Case II. Assume that $\Delta | y$. Then $\Delta | x$ and therefore

$$(x/\Delta)^2 - (P^2 + 4)(y/\Delta)^2 = \pm 4.$$

Thus we get $(x, y) = \pm(\Delta V_n, \Delta U_n)$ for some integer n by Theorem 2.1. □

Now, we can give the following results by using (2.8) and Theorem 3.4.

Corollary 9. *All integer solutions of the equation* $x^2 - (P^2 + 4)y^2 = 4\Delta^2$ *are given by* $(x, y) = \pm(X_{2n}^*, W_{2n}^*)$, $\pm(-X_{2n}^*, W_{2n}^*)$, *or* $\pm(\Delta V_{2n}, \Delta U_{2n})$ *with* $n \in \mathbb{Z}$.

Corollary 10. *All integer solutions of the equation* $x^2 - (P^2 + 4)y^2 = -4\Delta^2$ *are given by* $(x, y) = \pm(X_{2n+1}^*, W_{2n+1}^*)$, $\pm(X_{2n+1}^*, -W_{2n+1}^*)$, *or* $\pm(\Delta V_{2n+1}, \Delta U_{2n+1})$ *with* $n \in \mathbb{Z}$.

Theorem 3.5. *Let x and y be integers. Then* $x^2 - Pxy - y^2 = \pm\Delta^2$ *if and only if* $(x, y) = \pm(W_{n+1}^*, W_n^*), \pm((-1)^{n-1}W_{n-1}^*, (-1)^n W_n^*)$, *or* $\pm(\Delta U_{n+1}, \Delta U_n)$ *for some* $n \in \mathbb{Z}$.

Proof. If $(x, y) = \pm(W_{n+1}^*, W_n^*), \pm((-1)^{n-1}W_{n-1}^*, (-1)^n W_n^*)$, or $\pm(\Delta U_{n+1}, \Delta U_n)$, then it follows that $x^2 - Pxy - y^2 = \pm\Delta^2$ by (2.2) and (2.9). Assume that $x^2 - Pxy - y^2 = \pm\Delta^2$ for some integers x and y. Then

$$(2x - Py)^2 - (P^2 + 4)y^2 = \pm 4\Delta^2.$$

Taking

$$u = 2x - Py \quad \text{and} \quad v = y, \tag{3.19}$$

we get

$$u^2 - (P^2 + 4)v^2 = \pm 4\Delta^2.$$

Hence, $(u, v) = \pm\,(X_n^*, W_n^*)$, $\pm((-1)^{n-1}X_n^*, (-1)^n W_n^*)$, or $\pm(\Delta V_n, \Delta U_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.4. If $(u, v) = \pm\,(X_n^*, W_n^*)$, then we get $(x, y) = \pm(W_{n+1}^*, W_n^*)$ by (3.19) and (1.2). If $(u, v) = \pm((-1)^{n-1}X_n^*, (-1)^n W_n^*)$, then it is seen that $(x, y) = \pm((-1)^{n-1}W_{n-1}^*, (-1)^n W_n^*)$. If $(u, v) = \pm(\Delta V_n, \Delta U_n)$, it can be shown that $(x, y) = \pm(\Delta U_{n+1}, \Delta U_n)$. □

From (2.9) and Theorem 3.5, we have the following immediate corollaries.

Corollary 11. *All integer solutions of the equation* $x^2 - Pxy - y^2 = \Delta^2$ *are given by* $(x, y) = \pm(W_{2n+1}^*, W_{2n}^*), \pm(-W_{2n-1}^*, W_{2n}^*)$, *or* $\pm(\Delta U_{2n+1}, \Delta U_{2n})$ *with* $n \in \mathbb{Z}$.

Corollary 12. *All integer solutions of the equation* $x^2 - Pxy - y^2 = -\Delta^2$ *are given by* $(x,y) = \pm(W^*_{2n+2}, W^*_{2n+1}), \pm(W^*_{2n}, -W^*_{2n+1})$*, or* $\pm(\Delta U_{2n+2}, \Delta U_{2n+1})$ *with* $n \in \mathbb{Z}$*.*

Theorem 3.6. *Let* P^2+4 *be square free. Then* $x^2 - Pxy - y^2 = \pm(P^2+4)\Delta^2$ *for some integers* x *and* y *if and only if* $(x,y) = \pm(X^*_{n+1}, X^*_n)$*,* $\pm((-1)^n X^*_{n-1}, (-1)^{n-1} X^*_n)$*, or* $\pm(\Delta V_{n+1}, \Delta V_n)$ *for some* $n \in \mathbb{Z}$*.*

Proof. If $(x,y) = \pm(X^*_{n+1}, X^*_n), \pm((-1)^n X^*_{n-1}, (-1)^{n-1} X^*_n)$, or $\pm(\Delta V_{n+1}, \Delta V_n)$, then it follows that $x^2 - Pxy - y^2 = \pm(P^2+4)\Delta^2$ by (2.4) and (2.10). Assume that P^2+4 is square free, and $x^2 - Pxy - y^2 = \pm(P^2+4)\Delta^2$ for some integers x and y. Then

$$(2x - Py)^2 - (P^2+4)y^2 = \pm 4(P^2+4)\Delta^2.$$

Since P^2+4 is square free, we get $(P^2+4)|(2x - Py)$. Let

$$u = \frac{2x - Py}{P^2+4} \quad \text{and} \quad v = y. \tag{3.20}$$

Then it can be seen that

$$v^2 - \left(P^2+4\right) u^2 = \pm 4\Delta^2.$$

This implies that $(v,u) = \pm\ (X^*_n, W^*_n)$, $\pm((-1)^{n-1} X^*_n, (-1)^n W^*_n)$, or $\pm(\Delta V_n, \Delta U_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.4. The result follows from (1.3). □

We can give the following results from (2.5) and the above theorem.

Corollary 13. *Let* P^2+4 *be square free. Then all integer solutions of the equation* $x^2 - Pxy - y^2 = (P^2+4)\Delta^2$ *are given by* $(x,y) = \pm(X^*_{2n}, X^*_{2n-1})$*,* $\pm(-X^*_{2n}, X^*_{2n+1})$*, or* $\pm(\Delta V_{2n}, \Delta V_{2n-1})$ *with* $n \in \mathbb{Z}$*.*

Corollary 14. *Let* P^2+4 *be square free. Then all integer solutions of the equation* $x^2 - Pxy - y^2 = -(P^2+4)\Delta^2$ *are given by* $(x,y) = \pm(X^*_{2n+1}, X^*_{2n})$*,* $\pm(X^*_{2n-1}, -X^*_{2n})$*, or* $\pm(\Delta V_{2n+1}, \Delta V_{2n})$ *with* $n \in \mathbb{Z}$*.*

3.2. **Solutions of some Diophantine equations for** $Q = -1$**.** In this subsection, we will assume that $P > 3$, $Q = -1$, and $\Delta = b^2 - Pab + a^2$ such that $|\Delta| > 2$ and $|\Delta|$ is prime.

Theorem 3.7. *All integer solutions of the equation* $x^2 - (P^2-4)y^2 = 4\Delta$ *are given by* $(x,y) = \pm(x_n, w_n)$ *or* $\pm(-x_n, w_n)$ *with* $n \in \mathbb{Z}$*.*

Proof. If $(x,y) = \pm(x_n, w_n)$ or $\pm(-x_n, w_n)$, it follows that $x^2 - (P^2-4)y^2 = 4\Delta$ by (2.1). Now let $x^2 - (P^2-4)y^2 = 4\Delta$ for some integers x and y. It can be shown that $\Delta \nmid y$.

It is obvious that $4\Delta = (2b - Pa)^2 - (P^2-4)a^2$. Thus

$$\Delta|[(2b - Pa)^2 - (P^2-4)a^2] \tag{3.21}$$

and

$$\Delta|[x^2 - (P^2-4)y^2]. \tag{3.22}$$

From (3.21) and (3.22), we get

$$\Delta|[a^2\left(x^2 - (P^2-4)y^2\right) - y^2\left((2b - Pa)^2 - (P^2-4)a^2\right)],$$

i.e.,

$$\Delta|[ax + y(2b - Pa)][ax - y(2b - Pa)].$$

Since $|\Delta|$ is prime, it follows that

$$\Delta|[ax + y(2b - Pa)]$$

or

$$\Delta|[ax - y(2b - Pa)].$$

Also, from (3.21) and (3.22), we get

$$\Delta|[a^2(P^2-4)\left(x^2 - (P^2-4)y^2\right) + x^2\left((2b - Pa)^2 - (P^2-4)a^2\right)],$$

i.e.,

$$\Delta|[(2b - Pa)x - ay(P^2-4)][(2b - Pa)x + ay(P^2-4)].$$

This implies that

$$\Delta|[(2b - Pa)x - ay(P^2-4)]$$

or

$$\Delta|[(2b-Pa)x+ay(P^2-4)].$$

Hence, we have

$$\Delta|[ax+y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x-ay(P^2-4)] \tag{3.23}$$

or

$$\Delta|[ax-y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x+ay(P^2-4)] \tag{3.24}$$

and

$$\Delta|[ax-y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x-ay(P^2-4)] \tag{3.25}$$

or

$$\Delta|[ax+y(2b-Pa)] \text{ and } \Delta|[(2b-Pa)x+ay(P^2-4)]. \tag{3.26}$$

Now assume that (3.23) is satisfied. Then we get

$$\Delta|[x\,(ax+y(2b-Pa))-y\left((2b-Pa)x-ay(P^2-4)\right)],$$

i.e.,

$$\Delta \mid a[x^2+(P^2-4)y^2].$$

This implies that $\Delta|a$ or $\Delta|\left(x^2+(P^2-4)y^2\right)$. Assume that $\Delta|a$. Then $\Delta|b$, since $\Delta=b^2-Pab+a^2$. Thus $\Delta^2|\Delta$ and this shows that $\Delta|1$, which is impossible. Therefore $\Delta|\left(x^2+(P^2-4)y^2\right)$. Then we see that $\Delta|2(P^2-4)y^2$ since $\Delta|\left(x^2-(P^2-4)y^2\right)$. Hence, $\Delta|2(P^2-4)$, since $\Delta\nmid y$. Then it follows that

$$\Delta|\left[(P^2-4)2ay+(2b-Pa)x-ay(P^2-4)\right],$$

i.e.,

$$\Delta|[(2b-Pa)x+ay(P^2-4)].$$

In this case, (3.23) coincides with (3.26). Similarly, it is seen that (3.24) coincides with (3.25).

Now, let us show that $2|[(2b-Pa)x\pm ay(P^2-4)]$ and $2|[ax\pm y(2b-Pa)]$. It is seen that $x^2\equiv(Py)^2$ (mod4) from the equation $x^2-(P^2-4)y^2=4\Delta$. This implies that x and Py have the same parity. Therefore, we see that $2|[(2b-Pa)x\pm ay(P^2-4)]$ and $2|[ax\pm y(2b-Pa)]$.

Consequently, we should examine two cases

$$2\Delta|[(2b-Pa)x-ay(P^2-4)] \quad\text{and}\quad 2\Delta|[ax-y(2b-Pa)] \tag{3.27}$$

and

$$2\Delta|[(2b-Pa)x+ay(P^2-4)] \quad\text{and}\quad 2\Delta|[ax+y(2b-Pa)]. \tag{3.28}$$

Assume that (3.27) is satisfied. Let

$$u=\frac{(2b-Pa)x-ay(P^2-4)}{2\Delta} \quad\text{and}\quad v=\frac{[(2b-Pa)y-ax]}{2\Delta}.$$

Then it follows that

$$\begin{bmatrix} u \\ v \end{bmatrix}=\frac{1}{2\Delta}\begin{bmatrix} 2b-Pa & -a(P^2-4) \\ -a & 2b-Pa \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and so a simple computation shows that

$$\begin{bmatrix} 2b-Pa & a(P^2-4) \\ a & 2b-Pa \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} 2x \\ 2y \end{bmatrix}. \tag{3.29}$$

Since $x^2-(P^2-4)y^2=4\Delta$, using the equalty

$$4\Delta=(2b-Pa)^2-(P^2-4)a^2,$$

it is seen that

$$u^2-(P^2-4)v^2=4.$$

Thus we have $(u,v)=\mp(v_n,u_n)$ for some $n\in\mathbb{Z}$ by Theorem 2.2. Then $2x=\pm((2b-Pa)\,v_n+a(P^2-4)u_n)$ and $2y=\pm(av_n+(2b-Pa)\,u_n)$ by (3.29). By using (1.2), (1.3), and (1.4), we get

$$\begin{aligned} x &= \pm\left((2b-Pa)\,v_n+a(P^2-4)u_n\right)/2=\pm\left(2bv_n-Pav_n+av_{n+1}-av_{n-1}\right)/2 \\ &= \pm\left(bv_n-av_{n-1}\right)=\pm x_n \end{aligned}$$

and

$$\begin{aligned} y &= \pm\left(av_n + (2b - Pa)\,u_n\right)/2 = \pm\left(au_{n+1} - au_{n-1} + 2bu_n - Pau_n\right)/2 \\ &= \pm\left(bu_n - au_{n-1}\right) = \pm w_n. \end{aligned}$$

Now assume that (3.28) is satisfied. Let

$$u = \frac{(2b - Pa)x + ay(P^2 - 4)}{2\Delta} \quad \text{and} \quad v = \frac{[(2b - Pa)y + ax]}{2\Delta}.$$

Then we can see by a simple computation that

$$\begin{bmatrix} 2b - Pa & -a(P^2 - 4) \\ -a & 2b - Pa \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

and

$$u^2 - (P^2 - 4)v^2 = 4,$$

since $x^2 - (P^2 - 4)y^2 = 4\Delta$. Thus we have $(u, v) = \mp(v_n, u_n)$ for some $n \in \mathbb{Z}$ by Theorem 2.2. Similarly, it can be shown that $(x, y) = \pm(x_{-n}, -w_{-n})$. □

Since the equation $x^2 - (P^2 - 4)y^2 = -4$ has no integer solutions by Theorem 2.3, using the same argument in the proof of the above theorem, we can give the following theorem.

Theorem 3.8. *The equation $x^2 - (P^2 - 4)y^2 = -4\Delta$ has no integer solutions.*

Corollary 15. *The equation $x^2 - Pxy + y^2 = -\Delta$ has no integer solutions.*

Proof. Assume that $x^2 - Pxy + y^2 = -\Delta$ for some integers x and y. Completing the square gives $(2x - Py)^2 - (P^2 - 4)y^2 = -4\Delta$, which is impossible by Theorem 3.8. □

We can give the following corollaries from Theorem 3.8.

Corollary 16. *The equation $x^2 - (P^2 - 4)y^2 = -\Delta$ has no integer solutions.*

Corollary 17. *Let P be odd. Then the equation $x^2 - (P^2 - 4)y^2 = -16\Delta$ has no integer solutions.*

Proof. Assume that P is odd and $x^2 - (P^2 - 4)y^2 = -16\Delta$ for some integers x and y. Then it is seen that x and y are even and this implies that $(x/2)^2 - (P^2 - 4)(y/2)^2 = -4\Delta$, which is impossible by Theorem 3.8. □

Corollary 18. *Let P be odd. Then the equation $x^2 - Pxy + y^2 = -4\Delta$ has no integer solutions.*

Proof. Since $x^2 - Pxy + y^2 = -4\Delta$ if and only if $(2x - Py)^2 - (P^2 - 4)y^2 = -16\Delta$, the proof follows. □

Theorem 3.9. *All integer solutions of the equation $x^2 - Pxy + y^2 = \Delta$ are given by $(x, y) = \pm(w_{n+1}, w_n)$ with $n \in \mathbb{Z}$.*

Proof. If $(x, y) = \pm(w_{n+1}, w_n)$, then it follows that $x^2 - Pxy + y^2 = \Delta$ by (2.2). Assume that $x^2 - Pxy + y^2 = \Delta$. Completing the square gives $(2x - Py)^2 - (P^2 - 4)y^2 = 4\Delta$. This implies that $(2x - Py, y) = \pm(x_n, w_n)$ or $\pm(x_n, -w_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.7. Hence, $(x, y) = \pm(w_{n+1}, w_n)$ or $\pm(w_{n-1}, w_n)$. Since the role of x and y is symmetric, the proof follows. □

Theorem 3.10. *Let $P^2 - 4$ be square free. Then all integer solutions of the equation $x^2 - Pxy + y^2 = -(P^2 - 4)\Delta$ are given by $(x, y) = \pm(x_{n+1}, x_n)$ with $n \in \mathbb{Z}$.*

Proof. If $(x, y) = \pm(x_{n+1}, x_n)$, then it follows that $x^2 - Pxy + y^2 = -(P^2 - 4)\Delta$ by (2.4). Now assume that $P^2 - 4$ is square free and $x^2 - Pxy + y^2 = -(P^2 - 4)\Delta$ for some integers x and y. Then $(2x - Py)^2 - (P^2 - 4)y^2 = -4(P^2 - 4)\Delta$. Since $P^2 - 4$ is square free, it is seen that $(P^2 - 4)|(2x - Py)$. Therefore, taking

$$u = \frac{2x - Py}{P^2 - 4} \quad \text{and} \quad v = y,$$

we get $v^2 - \left(P^2 - 4\right)u^2 = 4\Delta$. This implies that $(v, u) = \pm(x_n, w_n)$ or $\pm(-x_n, w_n)$ for some $n \in \mathbb{Z}$ by Theorem 3.7. Hence, the proof follows from (1.2) and (1.3). □

From Theorem 3.8, we can give the following corollary.

Corollary 19. *Let P^2-4 be square free. Then the equation $x^2-Pxy+y^2=(P^2-4)\Delta$ has no integer solutions.*

Since the proof of the following theorem is similar to that of Theorem 3.4, we omit it.

Theorem 3.11. *All integer solutions of the equation $x^2-(P^2-4)y^2=4\Delta^2$ are given by $(x,y)=\pm(x_n^*,w_n^*),\pm(-x_n^*,w_n^*)$, or $\pm(\Delta v_n,\Delta u_n)$ with $n\in\mathbb{Z}$.*

Corollary 20. *All integer solutions of the equation $x^2-Pxy+y^2=\Delta^2$ are given by $(x,y)=\pm(w_{n+1}^*,w_n^*)$ or $\pm(\Delta u_{n+1},\Delta u_n)$ with $n\in\mathbb{Z}$.*

Theorem 3.12. *The equation $x^2-(P^2-4)y^2=-4\Delta^2$ has no integer solutions.*

Proof. If we follow the way as in the proof of Theorem 3.4, then we have the equation $u^2-(P^2-4)v^2=-4\Delta$, where $u=\left[(2b-Pa)x+ay(P^2-4)\right]/2\Delta$ and $v=\left[(2b-Pa)y+ax\right]/2\Delta$ or $u=\left[(2b-Pa)x-ay(P^2-4)\right]/2\Delta$ and $v=\left[(2b-Pa)y-ax\right]/2\Delta$. Since the equation $u^2-(P^2-4)v^2=-4\Delta$ is impossible by Theorem 3.8, the equation $x^2-(P^2-4)y^2=-4\Delta^2$ has no integer solutions. □

Corollary 21. *Let P^2-4 be square free. Then all integer solutions of the equation $x^2-Pxy+y^2=-(P^2-4)\Delta^2$ are given by $(x,y)=\pm(x_{n+1}^*,x_n^*)$ or $\pm(\Delta v_{n+1},\Delta v_n)$ with $n\in\mathbb{Z}$.*

Corollary 22. *The equation $x^2-Pxy+y^2=-\Delta^2$ has no integer solutions.*

Corollary 23. *The equation $x^2-(P^2-4)y^2=-\Delta^2$ has no integer solutions.*

Corollary 24. *Let P be odd. Then the equation $x^2-(P^2-4)y^2=-16\Delta^2$ has no integer solutions.*

Corollary 25. *Let P be odd. Then the equation $x^2-Pxy+y=-4\Delta^2$ has no integer solutions.*

Corollary 26. *Suppose that $|b^2-ba-a^2|$ is prime. Then all integer solutions of the equation $x^2-3xy+y^2=b^2-ba-a^2$ are given by $(x,y)=\pm(W_{2n+2},W_{2n})$ with $n\in\mathbb{Z}$, where $W_n=W_n(a,b;1,1)$.*

Proof. Suppose that $(x,y)=\pm(W_{2n+2},W_{2n})$. Then it is easy to see that

$$\begin{aligned}W_{2n+2}^2-3W_{2n+2}W_{2n}+W_{2n}^2 =&(W_{2n+2}-W_{2n})^2-(W_{2n+2}-W_{2n})W_{2n}-W_{2n}^2\\ =&W_{2n+1}^2-W_{2n+1}W_{2n}-W_{2n}^2=b^2-ba-a^2\end{aligned}$$

by (2.2). Now suppose that $x^2-3xy+y^2=b^2-ba-a^2$ for some integers x and y. Then $(x-y)^2-y(x-y)-y^2=b^2-ba-a^2$ and therefore $(x-y,y)=\pm(W_{2n+1},W_{2n})$ or $\pm\left(-W_{2n+1},W_{2n+2}\right)$ for some $n\in\mathbb{Z}$ by Corollary 3, where $W_n=W_n(a,b;1,1)$. If $(x-y,y)=\pm(W_{2n+1},W_{2n})$, then $(x,y)=\pm(W_{2n+2},W_{2n})$. If $(x-y,y)=\pm\left(-W_{2n+1},W_{2n+2}\right)$, then $(x,y)=\pm(W_{2n+2},W_{2n})$. Since the role of x and y is symmetric, the proof follows. □

The following corollary can be proved in a similar way.

Corollary 27. *Suppose that $|b^2-ba-a^2|$ is prime. Then all integer solutions of the equation $x^2-3xy+y^2=-(b^2-ba-a^2)$ are given by $(x,y)=\pm(W_{2n+1},W_{2n-1})$ with $n\in\mathbb{Z}$, where $W_n=W_n(a,b;1,1)$.*

Corollary 28. *Suppose that $|b^2-3ba+a^2|$ is prime. Then all integer solutions of the equation $x^2-3xy+y^2=b^2-3ba+a^2$ are given by $(x,y)=\pm(W_{2n+2},W_{2n})$ with $n\in\mathbb{Z}$, where $W_n=W_n(a,b-a;1,1)$.*

Proof. Suppose that $(x,y)=\pm(W_{2n+2},W_{2n})$ with $W_n=W_n(a,b-a;1,1)$. Then it can be seen that

$$\begin{aligned}W_{2n+2}^2-3W_{2n+2}W_{2n}+W_{2n}^2 =&(W_{2n+2}-W_{2n})^2-(W_{2n+2}-W_{2n})W_{2n}-W_{2n}^2\\ =&W_{2n+1}^2-W_{2n+1}W_{2n}-W_{2n}^2\\ =&(b-a)^2-(b-a)a-a^2\\ =&b^2-3ba+a^2\end{aligned}$$

by (2.2). Now suppose that $x^2-3xy+y^2=b^2-3ba+a^2$ for some integers x and y. Then $(x-y)^2-y(x-y)-y^2=(b-a)^2-a(b-a)-a^2$ and therefore $(x-y,y)=\pm(W_{2n+1},W_{2n})$ or $(x-y,y)=\pm(-W_{2n+1},W_{2n+2})$ for some $n\in\mathbb{Z}$ by Corollary 3, where $W_n=W_n(a,b-a;1,1)$. Let $(x-y,y)=\pm(W_{2n+1},W_{2n})$. Then $y=\pm W_{2n}$ and $x-y=\pm W_{2n+1}$, which implies that $x=\pm(W_{2n+1}+W_{2n})=\pm W_{2n+2}$. Let $(x-y,y)=\pm(-W_{2n+1},W_{2n+2})$. Then $y=\pm W_{2n+2}$ and $x-y=\pm(-W_{2n+1})$. Thus $x=\pm(-W_{2n+1}+W_{2n+2})=\pm W_{2n}$. This completes the proof. □

Corollary 29. *Suppose that $|b^2-3ba+a^2|$ is prime. Then all integer solutions of the equation $x^2-3xy+y^2=-(b^2-3ba+a^2)$ are given by $(x,y)=\pm(W_{2n+1},W_{2n-1})$ with $n\in\mathbb{Z}$, where $W_n=W_n(a,b-a;1,1)$.*

By using Corollaries 1 and 2, we can give the following corollaries.

Corollary 30. *Suppose that $|b^2-3ba+a^2|$ is prime. Then all integer solutions of the equation $x^2-5y^2=4(b^2-3ba+a^2)$ are given by $(x,y)=\pm(X_{2n},W_{2n})$ or $\pm(-X_{2n},W_{2n})$ with $n\in\mathbb{Z}$, where $W_n=W_n(a,b-a;1,1)$ and $X_n=X_n(a,b-a;1,1)$.*

Corollary 31. *Suppose that $|b^2-3ba+a^2|$ is prime. Then all integer solutions of the equation $x^2-5y^2=-4(b^2-3ba+a^2)$ are given by $(x,y)=\pm(X_{2n-1},W_{2n-1})$ or $\pm(X_{2n-1},-W_{2n-1})$ with $n\in\mathbb{Z}$, where $W_n=W_n(a,b-a;1,1)$ and $X_n=X_n(a,b-a;1,1)$.*

References

1. G. E. Bergum, Addenda to geometry of a generalized Simson's formula. *Fibonacci Quart.* **22** (1984), no.1, 22–28.
2. A. F. Horadam, Basic properties of a certain generalized sequence of numbers. *Fibonacci Quart.* **3** (1965), 161–176.
3. A. F. Horadam, Geometry of a generalized Simson's formula. *Fibonacci Quart.* **20** (1982), 164–68.
4. M. E. H. Ismail, One parameter generalizations of the Fibonacci and Lucas numbers. *Fibonacci Quart.* **46/47** (2008/09), no. 2, 167–180.
5. J. P. Jones, Representation of solutions of Pell equations using Lucas sequences. *Acta Acad. Paedagog. Agriensis Sect. Math. (N.S.)* **30** (2003), 75–86.
6. R. Keskin, Solutions of some quadratic Diophantine equations. *Comput. Math. Appl.* **60** (2010), no. 8, 2225–2230.
7. R. Keskin, B. Demirtürk, Solutions of some Diophantine equations using generalized Fibonacci and Lucas sequences. *Ars Combin.* **111** (2013), 161–179.
8. C. Kimberling, Fibonacci hyperbolas. *Fibonacci Quart.* **28** (1990), no. 1, 22–27.
9. W. L. McDaniel, Diophantine representation of Lucas sequences. *Fibonacci Quart.* **33** (1995), no. 1, 59–63.
10. R. S. Melham, A. G. Shannon, Some congruence properties of generalized second-order integer sequences. *Fibonacci Quart.* **32** (1994), no. 5, 424–428.
11. R. S. Melham, Certain classes of finite sums that involve generalized Fibonacci and Lucas numbers. *Fibonacci Quart.* **42** (2004), no. 1, 47–54.
12. R. Melham, Conics which characterize certain Lucas sequences. *Fibonacci Quart.* **35** (1997), no. 3, 248–251.
13. A. G. Shannon, A. F. Horadam, Special recurrence relations associated with the $\{W_n(a,b;p,q)\}$. *Fibonacci Quart.* **17** (1979), no. 4, 294–299.
14. Z. Şiar, R. Keskin, Some new identities concerning generalized Fibonacci and Lucas numbers. *Hacet. J. Math. Stat.* **42** (2013), no. 3, 211–222.
15. Z. Şiar, R. Keskin, Some new identities concerning the Horadam sequence and its companion sequence. *Commun. Korean Math. Soc.* **34** (2019), no. 1, 1–16.
16. S. Zhwei, Singlefold Diophantine representation of the sequence $U_0=0,U_1=1$ and $U_{n+2}=mU_{n+1}+U_n$, Pure and Applied Logic. (Zhang Jinwen ed.), *Beijing Univ. Press, Beijing*, (1992), 97–101.

(Received 07.05.2018)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 93–102

CONSTRUCTION OF A KERNEL DENSITY ESTIMATOR OF ROSENBLATT-PARZEN TYPE BY CONDITIONALLY INDEPENDENT OBSERVATIONS

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Abstract. On the probabilistic space (Ω, F, P), we consider the conditionally independent sequence $\{X_i\}_{i\geq 1}$ controlled by the sequence $\{\xi_i\}_{i\geq 1}$. The members of $\{\xi_i\}_{i\geq 1}$ are independent, identically distributed random variables $\xi_i = b_1 I_{(\xi_1=b_1)} + b_2 I_{(\xi_1=b_2)} + \cdots + b_r I_{(\xi_1=b_r)}$. The elements of the sequence $\{X_i\}_{i\geq 1}$ are the observations of some random variable X. Conditional distributions $\mathcal{P}_{X_i|\xi_i=b_i}$, $i=\overline{1,r}$, have unknown densities $f_i(x)$, $i=\overline{1,r}$, respectively. Using observations $\{X_i\}_{i\geq 1}$, a kernel density estimator $\bar{f}(x) = \sum\limits_{i=1}^{r} p_i f_i(x)$ of Rosenblatt-Parzen type is constructed, where $p_i = P(\xi_1 = b_i)$. The accuracy of approximation of the constructed estimator to the unknown function $\bar{f}(x)$ is established.

Distribution density estimators are intensively studied by many authors. In this paper, a nonparametric density estimator is constructed by dependent observations. The class of conditionally independent observations is considered. The nonparametric density estimators which have so far been considered are constructed by independent samples.

Below we present some definitions and auxiliary facts for nonparametric estimates of a distribution density which were constructed by independent observations.

Let the values X_i, $x_i \in R$, i=1,2,..., be independent observations of some random value X_i with unknown density $g(x)$. Various methods are available for obtaining estimators of $g(x)$. In the works of M. Rosenblatt and E. Parzen (see [8,9]) the estimators of$g_n^*(x)$ obtained by the kernel $k(x)$

$$g_n^*(x, h_n) = \frac{1}{nh_n} \sum_{i=1}^{n} k\left(\frac{(x - X_i)}{h_n}\right)$$

were considered, where $\{h_n\}_{n\geq 1}$ is the sequence of positive numbers such that

$$\lim_{n\to\infty} h_n = 0; \quad \lim_{n\to\infty} nh_n = \infty,$$

and the kernel $k(x)$ is some Lebesgue-integrable Borel function. In [5,7,10], the results of [9] were generalized by modifying the conditions on $k(x)$ and using the observations of vectors $X_i \in R^m$ $(m > 1)$.

Along with estimators of Rosenblatt-Parzen type, projection type estimators were also considered (see [2,7]) using the spectral decomposition of the kernel $k(x)$ with respect to the orthonormal basis of functions. Applying smoothing functions, L. Devroye and L. Györfi (see [3]) constructed the adaptive kernel estimators for densities with a finite number of discontinuity points. As a divergence measure of the constructed estimators of $g(x)$ some authors considered various characteristics in terms of metrics L_1, ([3,6]); L_2 ([7,9]) and so on.

In [7], E. Nadaraya obtained the sufficient conditions for the uniform convergence of the estimator

$$\hat{g}_n(x, a_n) = \frac{a_n}{n} \sum_{i=1}^{n} k\left(a_n(x - X_i)\right)$$

2010 *Mathematics Subject Classification.* 62G05, 62G07.
Key words and phrases. Conditionally independent sequence; Kernel estimate; Conditional density.

to $g(x)$ with probability 1. The divergence measure between $g(x)$ and $\hat{g}_n(x, a_n)$ is the value

$$E\int\limits_{-\infty}^{\infty}\left[\hat{g}_n(x, a_n) - g(x)\right]^2 dx,$$

where $\{a_n\}_{n\geq 1}$ is the sequence of positive numbers such that

$$\lim_{n\to\infty} a_n = \infty, \quad a_n = o(n). \tag{1}$$

Definition 1. Denote by H_s ($s \geq 2$; s is an even number) a set of functions $k(x)$ satisfying the conditions

$$k(-x) = k(x), \quad \int\limits_{-\infty}^{\infty} k(x)\,dx = 1, \quad \sup|k(x)| \leq A < \infty,$$

$$\int\limits_{-\infty}^{\infty} x^i k(x)\,dx = 0, \quad i = 1, 2, \dots, s-1; \quad \int\limits_{-\infty}^{\infty} x^s k(x)\,dx \neq 0, \quad \int\limits_{-\infty}^{\infty} x^s |k(x)|\,dx < \infty. \tag{2}$$

Definition 2. Denote by W_s a set of functions $\varphi(x)$ having derivatives up to the s-th order ($s \geq 2$) inclusive, and note that $\varphi^{(s)}(x)$ is a continuous bounded function from the class $L_2(-\infty, \infty)$.

Lemma (see [7])**.** *If the variables X_i, $x_i \in R$, $i = 1, 2, \dots,$ are independent observations of some random variable X with unknown density $g(x)$, $g(x) \in W_s \cap L_2(-\infty, \infty)$, $k(x) \in H_s \cap L_2(-\infty, \infty)$ and*

$$\hat{g}_n(x, a_n) = \frac{a_n}{n}\sum_{i=1}^{n} k(a_n(x - X_i)),$$

then for $n \to \infty$, the equalities

$$\int\limits_{-\infty}^{\infty} D\hat{g}_n(x, a_n)\,dx = \frac{a_n}{n}\int\limits_{-\infty}^{\infty} k^2(x)\,dx + O\left(\frac{a_n}{n}\right), \tag{3}$$

$$\int\limits_{-\infty}^{\infty}\left[E\hat{g}_n(x, a_n) - g(x)\right]^2 dx = a_n^{-2s}\frac{\alpha^2}{(s!)^2}\int\limits_{-\infty}^{\infty}\left[g^{(s)}(x)\right]^2 dx + O\left(a_n^{-2s}\right) \tag{4}$$

hold, where $\{a_n\}_{n\geq 1}$ is the sequence (1)*, and*

$$\alpha = \int\limits_{-\infty}^{\infty} x^s k(x)\,dx.$$

Let us present our result.

In practice we encounter the situation when at random moments of time the distribution of the observed variable X changes depending on the conditions (of the controlling sequence $\{\xi_i\}_{i\geq 1}$). This brings about changes of the densities of observations $\{X_i\}_{i\geq 1}$. For example, in stock-exchange transactions the price of some commodities changes depending on a season, though this price is fixed at the auction. As a result, the flow of revenues due to such transactions also changes, and so on.

In such situations, to estimate the density X it is appropriate to consider dependent observations. Here we consider the class of conditionally independent observations.

On the probability space (Ω, F, P), let us consider the two-component stationary (in the narrow sense) sequence of random variables

$$\{\xi_i, X_i\}_{i\geq 1}, \tag{5}$$

where $\xi_i : \Omega \to \Xi$, $X_i : \Omega \to R^m$ and Ξ is some space.

Definition 3. The sequence $\{X_i\}_{i\geq 1}$ from (4) is called a conditionally independent sequence (see [1]) controlled by the sequence $\{\xi_i\}_{i\geq 1}$ if for any natural n and the fixed trajectory $\bar{\xi}_{1n} = (\xi_1, \xi_2, \ldots, \xi_n)$, the values $X_1, X_2, \ldots, X_n$ become independent and for all natural numbers i, k, n, $j_1, j_2, \ldots, j_k$, $(2 \leq k \leq n;\ i \leq n;\ 1 \leq j_1 < j_2 <, \cdots, < j_k \leq n)$ the equalities

$$\mathcal{P}_{(X_{j_1}, X_{j_2}, \ldots, X_{j_k})|\xi_{1n}} = \mathcal{P}_{X_{j_1}|\xi_{j_1}} \times \mathcal{P}_{X_{j_2}|\xi_{j_2}} \times \cdots \times \mathcal{P}_{X_{j_k}|\xi_{j_k}},$$
$$\mathcal{P}_{X_i|\xi_{1n}} = \mathcal{P}_{X_i|\xi_i},$$

are fulfilled, where $\mathcal{P}_{X|Y}$ is the conditional distribution of the variable X under the condition Y.

Consider the sequence (5). Let ξ_i, $i = 1, 2, \ldots,$ be independent, identically distributed random variables and let

$$\Xi = \{b_1, b_2, \ldots, b_r\}; \quad P(\xi_1 = b_i) = p_i, \quad i = \overline{1, r}, \quad p_1 + p_2 + \cdots + p_r = 1,$$

$\{X_i\}_{i\geq 1}$ is the conditionally independent sequence whose elements are observations of the variable X. It is assumed that the conditional distributions $\mathcal{P}_{X_1|\xi_1 = b_i}$, $i = \overline{1, r}$, have unknown densities $f_i(x)$, $i = \overline{1, r}$, respectively. The sum $\hat{f}_n(x, a_n) = \frac{a_n}{n} \sum\limits_{j=1}^{n} k(a_n(x - X_j))$ constructed by conditionally independent observations is considered as the density estimator $\bar{f}(x) = \sum\limits_{i=1}^{r} p_i f_i(x)$, while the estimator accuracy is established by the expression $u(a_n) = E \int\limits_{-\infty}^{\infty} \left[\hat{f}_n(x, a_n) - \bar{f}(x)\right]^2 dx$.

On the fixed trajectory $\bar{\xi}_{1n} = (\xi_1, \xi_2, \ldots, \xi_n)$ of the sequence $\{\xi_i\}_{i\geq 1}$, we denote by $\nu_n(1), \nu_n(2), \ldots,$ $\nu_n(r)$ the frequencies for which the first n members of the sequence adopt the values $b_1, b_2, \ldots, b_r$.

Theorem. *Let us consider the sequence* (5). *The elements of the controlling sequence* $\{\xi_i\}_{i\geq 1}(\xi_i : \Omega \to \{b_1, b_2, \ldots, b_r\})$ *are independent, identically distributed values* $\xi_i = \sum\limits_{i=1}^{r} b_i I_{(\xi_1 = b_i)}$. *Assume that for every function* $\Psi : \Xi \to R^1$, *for which* $E\Psi(\xi_i) < \infty$ *as* $n \to \infty$, *we have the convergence*

$$\frac{1}{n} \sum_{j=1}^{n} \Psi(\xi_i) \to E\Psi(\xi_1) \quad a.s. \tag{6}$$

The elements of the conditionally independent sequence $\{X_i\}_{i\geq 1}$ are observations of the variable X. The conditional distributions $\mathcal{P}_{X_1|\xi_1 = b_i}$, $i = \overline{1, r}$, have unknown densities $f_i(x)$, $i = \overline{1, r}$, respectively. Assume $f_i(x) \in W_s \cap L_2(-\infty, \infty)$ and $k(x) \in H_s \cap L_2(-\infty, \infty)$. If for the frequencies $\nu_n(i)$, $i = \overline{1, r}$, the inequalities

$$D\left(\frac{\nu_n(i)}{n}\right) \leq \frac{c_i}{\sqrt{n}}, \quad i = \overline{1, r} \tag{7}$$

are fulfilled, then for any natural n the estimator of the density $\bar{f}(x) = \sum\limits_{i=1}^{r} p_i f_i(x)$ is the sum

$$\hat{f}_n(x, a_n) = \frac{a_n}{n} \sum_{j=1}^{n} k(a_n(x - X_j)) \tag{8}$$

and the following asymptotic equality

$$u(a_n) \leq \left(\sum_{i=1}^{r} M_i\right)^2 + \frac{a_n}{n} \int\limits_{-\infty}^{\infty} k^2(x)\,dx + \left(\sum_{i=1}^{r} (C_i n^{-1/2} + {p_i}^2)\right) O\Big(\frac{a_n}{n}\Big)$$

is valid, where

$$M_i = T_i^{1/2} + \left(C_i n^{-1/2} \int\limits_{-\infty}^{\infty} f_i^2\left(x\right)dx \right)^{\frac{1}{2}}$$

$$T_i = (a_n^{-2s}\frac{\alpha^2}{(s!)^2}\int\limits_{-\infty}^{\infty}[f_i^{(s)}\left(x\right)]^2 dx + O(a_n^{-2s}))(C_i n^{-1/2} + p_i^2) \quad i = \overline{1,r}$$

and

$$\alpha = \int\limits_{-\infty}^{\infty} x^s k\left(x\right)dx.$$

Proof. The proof of the theorem is based on the decomposition of $\hat{f}_n\left(x, a_n\right)$ (the trajectory $\overline{\xi}_{1n}$is assumed to be fixed) into independent sums of random variables. For the fixed trajectory $\overline{\xi}_{1n}$ we enumerate individually the moments of time at which the first n members of the sequence $\{\xi_i\}_{i\geq 1}$ take the value b_i, $i = \overline{1,r}$, respectively,

$$\tau_0(i) = 0, \quad \tau_m(i) = \min\{j|\tau_{m-1} < j \leq n; \xi_i = b_i\}; \quad m = \overline{1, \nu_n\left(i\right)}, \quad i = \overline{1,r}.$$

We obtain the sequence of indices

$$\tau_1(i), \tau_2(i), \ldots, \tau_{\nu_n(i)}(i) \quad i = \overline{1,r}$$

for which the equalities

$$\xi_{\tau_m(i)} = b_i \quad m = \overline{1, \nu_n\left(i\right)}, \quad i = \overline{1,r}$$

are valid.

When the trajectory $\overline{\xi}_{1n}$ is fixed, the sum (8) can be decomposed as follows

$$\hat{f}_n\left(x, a_n\right) = \sum_{i=1}^{r} \frac{\nu_n\left(i\right)}{n} \hat{f}_{in}\left(x, a_n\right),$$

where

$$\hat{f}_{in}\left(x, a_n\right) = \frac{a_n}{\nu_n\left(i\right)} \sum_{m=1}^{\nu_n(i)} k\left(a_n\left(x - X_{\tau_{m(i)}}\right)\right) \quad i = \overline{1,r}.$$

Naturally, if $\nu_n\left(i\right) = 0$, then the summand $\hat{f}_{in}\left(x, a_n\right)$, $i = \overline{1,r}$, does not exist. Let us prove the finiteness of $E\hat{f}_n\left(x, a_n\right)$ and $D\hat{f}_n\left(x, a_n\right)$. On the fixed trajectory $\overline{\xi}_{1n}$, we represent $E\hat{f}_n\left(x, a_n\right)$ as a conditional mathematical expectation

$$E\hat{f}_n\left(x, a_n\right) = E\{E(\hat{f}_n\left(x, a_n\right)|\xi_{1n})\} = E\bigg\{E\bigg(\sum_{i=1}^{r}\frac{\nu_n\left(i\right)}{n}\hat{f}_{in}\left(x, a_n\right)|\xi_{1n}\bigg)\bigg\}.$$

In proving the theorem, we take into account that $\nu_n\left(i\right)$, $i = \overline{1,r}$, are measurable functions with respect to the σ-algebra which is generated when the probability space Ω is partitioned as a result of fixing the trajectory $\overline{\xi}_{1n}$. Therefore these functions can be taken outside the sign of mathematical expectation. In the above equality and in the sequel we keep in mind the fact that the following equality

$$E\frac{\nu_n\left(i\right)}{n} = p_i$$

is fulfilled by virtue of conditions (6), and applying condition (7), we obtain the estimator

$$E\bigg(\frac{\nu_n\left(i\right)}{n}\bigg)^2 = D\left(\frac{\nu_n\left(i\right)}{n}\right) + \bigg(E\frac{\nu_n\left(i\right)}{n}\bigg)^2 \leq n^{-1/2}c_i + p_i^2. \tag{9}$$

Hence, using conditions (2), after replacing the variable under the integration sign, we see that the following chain of equalities

$$E\hat{f}_n(x,a_n)=\sum_{i=1}^{r}E\Big\{\frac{\nu_n(i)}{n}E\Big(\frac{a_n}{\nu_n(i)}\sum_{m=1}^{\nu_n(i)}k\left(a_n\left(x-X_{\tau_{m(i)}}\right)\right)|\xi_{1n}\Big)\Big\}$$
$$=\sum_{i=1}^{r}E\Big\{\frac{\nu_n(i)}{n}E\Big(\frac{a_n}{\nu_n(i)}\nu_n(i)\,k\left(a_n\left(x-X_{\tau_{m(i)}}\right)\right)|\xi_{\tau_{m(i)}}\Big)\Big\}$$
$$=\sum_{i=1}^{r}a_n\int_{-\infty}^{\infty}k(a_n(x-u))f_i(u)du E\frac{\nu_n(i)}{n}=\sum_{i=1}^{r}p_i\int_{-\infty}^{\infty}k(t)f_i\Big(\frac{t}{a_n}+x\Big)dt$$

is valid.

Since$f_i(x)$ is the density and $|k(t)|$ is bounded by the infinite constant A, we conclude that $E\hat{f}_n(x,a_n)$ is finite.

On the fixed trajectory $\overline{\xi}_{1n}$, the sums $\hat{f}_{in}(x,a_n)$, $i=\overline{1,r}$, and their constituent summands too, are independent and the following equalities

$$D\hat{f}_n(x,a_n)=E\{E([\hat{f}_n(x,a_n)-E\hat{f}_n(x,a_n)]^2|\xi_{1n})\}$$
$$=E\Big\{E\Big(\Big[\sum_{i=1}^{r}\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)-E\sum_{i=1}^{r}\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)\Big]^2|\xi_{1n}\Big)\Big\}$$
$$=E\Big\{E\Big(\Big[\sum_{i=1}^{r}\frac{\nu_n(i)}{n}(\hat{f}_{in}(x,a_n)-E\hat{f}_{in}(x,a_n))\Big]^2|\xi_{1n}\Big)\Big\}$$
$$=E\Big\{E\Big(\Big[\sum_{i=1}^{r}\Big(\frac{\nu_n(i)}{n}\Big)^2\hat{f}_{in}(x,a_n)-E\hat{f}_{in}(x,a_n)\Big]^2|\xi_{1n}\Big)\Big\}$$
$$=\sum_{i=1}^{r}\Big\{E\Big(\frac{\nu_n(i)}{n}\Big)^2E([\hat{f}_{in}(x,a_n)-E\hat{f}_{in}(x,a_n)]^2|\xi_{1n})\Big\}$$
$$=\sum_{i=1}^{r}E\Big\{\Big(\frac{\nu_n(i)}{n}\Big)^2E\Big[\sum_{j=1}^{\nu_n(i)}\frac{a_n}{\nu_n(i)}\left(k\left(a_n\left(x-X_{\tau_{j(i)}}\right)\right)-Ek\left(a_n\left(x-X_{\tau_{j(i)}}\right)\right)\right)\Big]^2|\xi_{1n}\Big\}$$
$$=\sum_{i=1}^{r}E\Big\{\Big(\frac{\nu_n(i)}{n}\Big)^2\Big(\frac{a_n}{\nu_n(i)}\Big)^2E\Big(\sum_{j=1}^{\nu_n(i)}[k\left(a_n\left(x-X_{\tau_{j(i)}}\right)\right)-Ek\left(a_n\left(x-X_{\tau_{j(i)}}\right)\right)]^2|\xi_{1n}\Big)\Big\}$$
$$=\sum_{i=1}^{r}E\Big(\frac{\nu_n(i)}{n}\Big)^2\Big(\frac{a_n}{\nu_n(i)}\Big)^2\nu_n(i)\int_{-\infty}^{+\infty}\Big[k(a_n(x-u))-\int_{-\infty}^{+\infty}k(a_n(x-y))f_i(y)dy\Big]^2f_i(u)\,du$$

are true.

Using equality (9), we obtain for $D\hat{f}_n(x,a_n)$ the following expression:

$$D\hat{f}_n(x,a_n)=\frac{a_n}{n}\sum_{i=1}^{2}p_i\int_{-\infty}^{\infty}[k(t)-\int_{-\infty}^{\infty}k(a_n(x-y))f_i(y)dy]^2f_i\Big(\frac{t}{a_n}+x\Big)dt.$$

Let us estimate $u(a_n)$. We apply Fubini's theorem and divide $u(a_n)$ into two parts

$$u(a_n)=\int_{-\infty}^{\infty}E\Big[\hat{f}_n(x,a_n)-\bar{f}(x)\Big]^2dx$$
$$=\int_{-\infty}^{\infty}E\Big[\hat{f}_n(x,a_n)-E\hat{f}_n(x,a_n)\Big]^2dx+\int_{-\infty}^{\infty}E\Big[E\hat{f}_n(x,a_n)-\overline{f}(x)\Big]^2dx=I_1+I_2. \quad (10)$$

To estimate I_1, we again apply Fubini's theorem and obtain

$$I_1=\int\limits_{-\infty}^{\infty}E\Big[\hat{f}_n(x,a_n)-E\hat{f}_n(x,a_n)\Big]^2dx=\int\limits_{-\infty}^{\infty}E\{E([\hat{f}_n(x,a_n)-E\hat{f}_n(x,a_n)]^2|\xi_{1n})\}dx$$

$$=E\int\limits_{-\infty}^{+\infty}E\bigg(\Big[\sum_{i=1}^{r}\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)-E\sum_{i=1}^{r}\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)\Big]^2|\xi_{1n}\bigg)dx$$

$$=E\bigg\{\int\limits_{-\infty}^{+\infty}E\bigg(\sum_{i=1}^{r}\Big(\frac{\nu_n(i)}{n}\Big)^2[\hat{f}_{in}(x,a_n)-E\hat{f}_{in}(x,a_n)]^2|\xi_{1n}\bigg)dx\bigg\}$$

$$=E\bigg\{\int\limits_{-\infty}^{+\infty}\sum_{i=1}^{r}\Big(\frac{\nu_n(i)}{n}\Big)^2E([\hat{f}_{in}(x,a_n)-E\hat{f}_{in}(x,a_n)]^2|\xi_{1n})dx\bigg\}.$$

Using equality (3) from the Lemma, we have

$$I_1=\sum_{i=1}^{r}E\bigg\{\Big(\frac{\nu_n(i)}{n}\Big)^2\bigg(\frac{a_n}{\nu_n(i)}\int\limits_{-\infty}^{+\infty}k(x)dx+o\Big(\frac{a_n}{n}\Big)\bigg)\bigg\}$$

$$=\sum_{i=1}^{r}E\bigg\{\Big(\frac{\nu_n(i)}{n}\Big)^2\frac{a_n}{\nu_n(i)}\int\limits_{-\infty}^{+\infty}k^2(x)dx\bigg\}+o\Big(\frac{a_n}{n}\Big)\sum_{i=1}^{r}E\Big(\frac{\nu_n(i)}{n}\Big)^2$$

$$=\frac{a_n}{n}\int\limits_{-\infty}^{+\infty}k^2(x)dx\sum_{i=1}^{r}E\Big(\frac{\nu_n(i)}{n}\Big)+o\Big(\frac{a_n}{n}\Big)\sum_{i=1}^{r}\bigg[D\Big(\frac{\nu_n(i)}{n}\Big)+\Big(E\frac{\nu_n(i)}{n}\Big)^2\bigg].$$

By applying inequality (9), we complete the estimation of I_1,

$$I_1\le\frac{a_n}{n}\int\limits_{-\infty}^{+\infty}k^2(x)dx\sum_{i=1}^{r}p_i+o\Big(\frac{a_n}{n}\Big)\sum_{i=1}^{r}(C_in^{-1/2}+{p_i}^2)$$

$$=\frac{a_n}{n}\int\limits_{-\infty}^{+\infty}k^2(x)dx+o\Big(\frac{a_n}{n}\Big)\sum_{i=1}^{r}(C_in^{-1/2}+{p_i}^2).\tag{11}$$

Applying Fubini's theorem once more and decomposing I_2 into two sums, we have

$$I_2=\int\limits_{-\infty}^{\infty}E\Big[E\hat{f}_n(x,a_n)-\overline{f}(x)\Big]^2dx=E\int\limits_{-\infty}^{\infty}\Big[E\hat{f}_n(x,a_n)-\overline{f}(x)\Big]^2dx$$

$$=E\bigg\{E\bigg(\int\limits_{-\infty}^{\infty}\Big[E\hat{f}_n(x,a_n)-\overline{f}(x)\Big]^2dx|\xi_{1n}\bigg)\bigg\}$$

$$=E\bigg\{E\bigg(\int\limits_{-\infty}^{\infty}\bigg[E\bigg(\sum_{i=1}^{r}\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)\bigg)-\sum_{i=1}^{r}p_if_i(x)\bigg]^2dx|\xi_{1n}\bigg)\bigg\}$$

$$=E\bigg\{E\bigg(\int\limits_{-\infty}^{\infty}\bigg[\sum_{i=1}^{r}\Big(E\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)-p_if_i(x)\Big)\bigg]^2dx|\xi_{1n}\bigg)\bigg\}$$

$$=E\bigg\{E\bigg(\int\limits_{-\infty}^{\infty}\bigg(\sum_{i=1}^{r}\Big(E\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)-p_if_i(x)\Big)^2$$

$$+2\sum_{\substack{i,j=1\\ i<j}}^{r}\Big(E\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)-p_if_i(x)\Big)\Big)\Big(E\frac{\nu_n(j)}{n}\hat{f}_{jn}(x,a_n)-p_jf_j(x)\Big)\Big)dx|\xi_{1n}\Big)\Big\}$$

$$=E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\sum_{i=1}^{r}E\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)-p_if_i(x))^2dx|\xi_{1n}\Big)\Big\}$$

$$+E\Big\{E\Big(\int\limits_{-\infty}^{\infty}2\sum_{\substack{i,j=1\\ i<j}}^{r}\Big(E\frac{\nu_n(i)}{n}\hat{f}_{in}(x,a_n)-p_if_i(x)\Big)\Big(E\frac{\nu_n(j)}{n}\hat{f}_{jn}(x,a_n)-p_jf_j(x)\Big)dx|\xi_{1n}\Big)\Big\}$$

$$=\sum_{i=1}^{r}E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n(i)}{n}E\hat{f}_{in}(x,a_n)-p_if_i(x)\Big]^2dx|\xi_{1n}\Big)\Big\}$$

$$+2\sum_{\substack{i,j=1\\ i<j}}^{r}E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n(i)}{n}E\hat{f}_{in}(x,a_n)-p_if_i(x)\Big]\Big[\frac{\nu_n(j)}{n}E\hat{f}_{jn}(x,a_n)-p_jf_j(x)\Big]dx|\xi_{1n}\Big)\Big\}=I_{21}+I_{22}.$$

We decompose the sum I_{21} into three parts

$$I_{21}=\sum_{i=1}^{r}E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n(i)}{n}E\hat{f}_{in}(x,a_n)-\frac{\nu_n(i)}{n}f_i(x)+\frac{\nu_n(i)}{n}f_i(x)-p_if_i(x)\Big]^2dx|\xi_{1n}\Big)\Big\}$$

$$=\sum_{i=1}^{r}E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\Big(\frac{\nu_n(i)}{n}\Big)^2[E\hat{f}_{in}(x,a_n)-f_i(x)]^2dx|\xi_{1n}\Big)\Big\}$$

$$+\sum_{i=1}^{r}E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\Big(\frac{\nu_n(i)}{n}-p_i\Big)^2f_i^2(x)dx|\xi_{1n}\Big)\Big\}$$

$$+2\sum_{i=1}^{r}E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\frac{\nu_n(i)}{n}\Big(\frac{\nu_n(i)}{n}-p_i\Big)[E\hat{f}_{in}(x,a_n)-f_i(x)]f_i(x)dx|\xi_{1n}\Big)\Big\}=A_1+A_2+A_3.$$

Using equality (4) from the Lemma and the estimator (9), we obtain

$$A_1=\sum_{i=1}^{r}E\Big\{\Big(\frac{\nu_n(i)}{n}\Big)^2E\Big(\int\limits_{-\infty}^{\infty}[E\hat{f}_{in}(x,a_n)-f_i(x)]^2dx|\xi_{1n}\Big)\Big\}$$

$$=\sum_{i=1}^{r}E\Big\{\Big(\frac{\nu_n(i)}{n}\Big)^2\Big(a_n^{-2s}\frac{\alpha^2}{(s!)^2}\int\limits_{-\infty}^{\infty}[f_i^{(s)}(x)]^2dx+o(a_n^{-2s})\Big)\Big\}$$

$$=\sum_{i=1}^{r}\Big(a_n^{-2s}\frac{\alpha^2}{(s!)^2}\int\limits_{-\infty}^{\infty}[f_i^{(s)}(x)]^2dx+o(a_n^{-2s})\Big)(C_in^{-1/2}+p_i^2)=\sum_{i=1}^{r}T_i,$$

where

$$T_i=\Big(a_n^{-2s}\frac{\alpha^2}{(s!)^2}\int\limits_{-\infty}^{\infty}[f_i^{(s)}(x)]^2dx+o(a_n^{-2s})\Big)(C_in^{-1/2}+p_i^2).$$

It is not difficult to derive the estimator for the sum A_2,

$$A_2=\sum_{i=1}^{r}E\Big\{\Big(\frac{\nu_n(i)}{n}-p_i\Big)^2E\int\limits_{-\infty}^{\infty}f_i^2(x)dx\Big\}\le\sum_{i=1}^{r}C_in^{-1/2}\int\limits_{-\infty}^{\infty}f_i^2(x)dx.$$

We use the fact that the values $\frac{\nu_n(i)}{n}-p_i$, $i=\overline{1,r}$, as well as $\frac{\nu_n(i)}{n}$, are measurable with respect to the σ-algebra generated by the fixed trajectory $\overline{\xi}_{1n}$. By Hölder's inequality, for the summand A_3, we obtain the following estimator

$$A_3 = 2\sum_{i=1}^{r} E\Big\{\frac{\nu_n\,(i)}{n}\Big(\frac{\nu_n\,(i)}{n}-p_i\Big)E\Big(\int\limits_{-\infty}^{\infty}[E\hat{f}_{in}\,(x,a_n)-f_i(x)]f_i(x)dx|\xi_{1n}\Big)\Big\}$$

$$\leq 2\sum_{i=1}^{r} E\left\{\frac{\nu_n\,(i)}{n}\Big(\frac{\nu_n\,(i)}{n}-p_i\Big)E\left(\sqrt{\int\limits_{-\infty}^{\infty}[E\hat{f}_{in}\,(x,a_n)-f_i(x)]^2dx}\sqrt{\int\limits_{-\infty}^{\infty}f_i{}^2(x)dx|\xi_{1n}}\right)\right\}$$

$$\leq 2\sum_{i=1}^{r}\sqrt{a_n^{-2s}\frac{\alpha^2}{(s!)^2}\int\limits_{-\infty}^{\infty}\big[f_i^{(s)}\,(x)\big]^2dx+o(a_n^{-2s})}\sqrt{\int\limits_{-\infty}^{\infty}f_i{}^2(x)dx}\times E\Big\{\frac{\nu_n\,(i)}{n}\Big(\frac{\nu_n\,(i)}{n}-p_i\Big)\Big\}$$

$$\leq 2\sum_{i=1}^{r}\sqrt{a_n^{-2s}\frac{\alpha^2}{(s!)^2}\int\limits_{-\infty}^{\infty}\big[f_i^{(s)}\,(x)\big]^2dx+o(a_n^{-2s})}\sqrt{\int\limits_{-\infty}^{\infty}f_i{}^2(x)dx}\times\sqrt{E\Big(\frac{\nu_n\,(i)}{n}\Big)^2}\sqrt{E\Big(\frac{\nu_n\,(i)}{n}-p_i\Big)^2}$$

$$\leq 2\sum_{i=1}^{r}\sqrt{a_n^{-2s}\frac{\alpha^2}{(s!)^2}\int\limits_{-\infty}^{\infty}\big[f_i^{(s)}\,(x)\big]^2dx+o(a_n^{-2s})}\sqrt{\int\limits_{-\infty}^{\infty}f_i{}^2(x)dx}\cdot\sqrt{C_in^{-1/2}}\sqrt{C_in^{-1/2}+p_i^2}$$

$$\equiv 2\sum_{i=1}^{r}\sqrt{T_i}\cdot\sqrt{C_in^{-1/2}\int\limits_{-\infty}^{\infty}f_i{}^2(x)dx}.$$

The summation of the estimators A_1 A_2 and A_3 gives

$$I_{21}\leq\sum_{i=1}^{r}\Big(\sqrt{T_i}+\sqrt{C_in^{-1/2}\int\limits_{-\infty}^{\infty}f_i{}^2(x)dx}\Big)^2\equiv\sum_{i=1}^{r}M_i^2, \tag{12}$$

where

$$M_i=\sqrt{T_i}+\sqrt{C_in^{-1/2}\int\limits_{-\infty}^{\infty}f_i{}^2(x)dx}.$$

Consider the sum I_{22}

$$I_{22}=2\sum_{\substack{i,j=1\\ i<j}}^{r}E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n\,(i)}{n}E\hat{f}_{in}\,(x,a_n)-p_if_i(x)\Big]\Big[\frac{\nu_n\,(j)}{n}E\hat{f}_{jn}\,(x,a_n)-p_jf_j(x)\Big]dx|\xi_{1n}\Big)\Big\}$$

$$=2\sum_{\substack{i,j=1\\ i<j}}^{r}B_{ij}.$$

The summands of this sum are estimated in the same manner as above. Let us estimate one of them by applying Fubini's and Hölder's theorems:

$$B_{ij}=E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n\,(i)}{n}E\hat{f}_{in}\,(x,a_n)-p_if_i(x)\Big]\Big[\frac{\nu_n\,(j)}{n}E\hat{f}_{jn}\,(x,a_n)-p_jf_j(x)\Big]dx|\xi_{1n}\Big)\Big\}$$

$$\leq E\left\{E\sqrt{\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n(i)}{n}E\hat{f}_{in}(x,a_n)-p_if_i(x)\Big]^2dx}\sqrt{\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n(j)}{n}E\hat{f}_{jn}(x,a_n)-p_jf_j(x)\Big]^2dx|\xi_{1n}}\right\}$$

$$\leq E\left\{\sqrt{E\Big(\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n(i)}{n}E\hat{f}_{in}(x,a_n)-p_if_i(x)\Big]^2dx|\xi_{1n}\Big)}\right.$$

$$\left.\times\sqrt{E\Big(\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n(j)}{n}E\hat{f}_{jn}(x,a_n)-p_jf_j(x)\Big]^2dx|\xi_{1n}\Big)}\right\}$$

$$\leq\sqrt{E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n(i)}{n}E\hat{f}_{in}(x,a_n)-p_if_i(x)\Big]^2dx|\xi_{1n}\Big)\Big\}}$$

$$\times\sqrt{E\Big\{E\Big(\int\limits_{-\infty}^{\infty}\Big[\frac{\nu_n(j)}{n}E\hat{f}_{jn}(x,a_n)-p_jf_j(x)\Big]^2dx|\xi_{1n}\Big)\Big\}}.$$

Each of the obtained two multipliers is estimated like a summand of the sum I_{21}. Thus we obtain the estimate B_{ij}

$$B_{ij}\leq M_iM_j.$$

Hence an estimator of the sum I_{22} has the form

$$I_{22}\leq 2\sum_{\substack{i,j=1\\ i<j}}^{r}M_iM_j. \tag{13}$$

Thus, in view of the decomposition (10) and the derived estimators (11), (12) and (13), the theorem is proved. □

Note that the proposed method enables one to construct density estimators for other types of dependence of observations too, for example, when the controlling sequence $\{\xi_i\}_{i\geq1}$ is a Markov chain, i.e., $\{X_i\}_{i\geq1}$ are observations with the chain dependence (see [4]).

References

1. I. V. Bokuchava, Z. A. Kvatadze, T. L. Shervashidze, On limit theorems for random vectors controlled by a Markov chain. *Probability theory and mathematical statistics, vol. I (Vilnius, 1985)*, 231–250, *VNU Sci. Press, Utrecht*, 1987.
2. N. N. Čencov, A bound for an unknown distribution density in terms of the observations. (Russian) *Dokl. Akad. Nauk SSSR* **147** (1962), 45–48.
3. L. Devroye, L. Györfi, *Nonparametric Density Estimation.* The L_1 view. Wiley Series in Probability and Mathematical Statistics: Tracts on Probability and Statistics. John Wiley & Sons, Inc., New York, 1985.
4. Z. Kvatadze, T. Shervashidze, On limit theorems for conditionally independent random variables controlled by a finite Markov chain. *Probability theory and mathematical statistics (Kyoto, 1986)*, 250–258, Lecture Notes in Math., 1299, *Springer, Berlin*, 1988.
5. G. M. Mania, *Statistical Estimation of a Probability Distribution Function.* (Russian) Izdat. Tbilis. Univ., Tbilisi, 1974.
6. R. M. Mnacakanov, E. B. Khmaladze, L_1-convergence of statistical kernel estimates of probability density-functions. *Dokl. Akad. Nauk SSSR* **258** (1981), no. 5, 1052–1055.
7. E. A. Nadaraya, *Nonparametric Estimation of a Probability Density Function and a Regression Curve.* Tbilisi univ. publishing house, Tbilisi, 1983.
8. E. Parzen, On the estimation of a probability density function and mode. *Ann. Math. Statist.* **33** (1962), no. 3, 1065–1076.
9. M. Rosenblatt, Remarks on some nonparametric estimates of a density function. *Ann. Math. Statist.* **27** (1956), 832–837.

10. G. S. Watson, M. R. Leadbetter, On the estimation of the probability density. *I. Ann. Math. Statist.* **34** (1963), 480–491.

(Received 23.01.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 103–130

INVESTIGATION OF NONCLASSICAL TRANSMISSION PROBLEMS OF THE THERMO-ELECTRO-MAGNETO ELASTICITY THEORY FOR COMPOSED BODIES BY THE INTEGRAL EQUATION METHOD

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Abstract. We investigate multi-field problems for complex elastic anisotropic structures when in different adjacent components of the composed body different refined models of elasticity theory are considered. In particular, we analyse the case when we have the generalized thermo-electro-magneto elasticity model (GTEME model) in one region of the composed body and the generalized thermo-elasticity model (GTE model) in the other adjacent region. This type of mechanical problem is described mathematically by systems of partial differential equations with appropriate transmission and boundary conditions. In the GTEME model part we have six-dimensional unknown physical field (three components of the displacement vector, electric potential function, magnetic potential function, and temperature distribution function), while in the GTE model part we have four- dimensional unknown physical field (three components of the displacement vector and temperature distribution function). The diversity in dimensions of the interacting physical fields are taken into consideration in mathematical formulation and analysis of the corresponding boundary-transmission problems. We apply the potential method and the theory of pseudodifferential equations and prove the uniqueness and existence theorems of solutions to different type boundary-transmission problems in appropriate Sobolev spaces.

1. Introduction

Modern industrial and technological processes apply widely, on the one hand, composite materials with complex microstructure and, on the other hand, complex composed structures consisting of materials having essentially different physical properties (for example, piezoelectric, piezomagnetic, hemitropic materials, two- and multi-component mixtures, nano-materials, bio-materials, and solid structures constructed by composition of these materials, such as, e.g., Smart Materials and other meta-materials). Therefore the investigation and analysis of mathematical models describing the mechanical, thermal, electric, magnetic and other physical properties of such materials are of crucial importance for both fundamental research and practical applications.

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields. For example, piezoelectric materials (electro-elastic coupling) have been used as ultrasonic transducers and micro-actuators; pyroelectric materials (thermal-electric coupling) have been applied in thermal imaging devices; piezomagnetic materials (elastic-magnetic coupling) are pursued for health monitoring of civil structures (see [9, 12, 13, 15, 24–32, 39, 45–47, 50, 51, 53, 55], and the references therein).

Although natural materials rarely show full coupling between elastic, electric, magnetic, and thermal fields, some artificial materials do. In [54], it was reported that the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite had the electro-magnetic effect not existing in either constituent. Other examples of similar complex coupling can be found in [3–6, 16–18, 34–36, 40, 41, 48, 56]. For more detailed historical and bibliographic data see [1, 7, 49].

In the present paper, we investigate multi-field problems for complex elastic anisotropic structures when in different adjacent components of the composed body different refined models of elasticity

2010 *Mathematics Subject Classification.* 31B10, 35B65, 35C15, 35D30, 35J47, 35J57, 35S05, 47G10, 47G30, 47G40, 74E10, 74F05, 74F15, 74G30, 74G40, 74G55.

Key words and phrases. Thermo-electro-magneto-elasticity; Thermo-elasticity; Green-Lindsay's model; Boundary value problem; Transmission problem; Potential method; Pseudodifferential equations.

theory are considered. In particular, we analyse the case when we have the generalized thermo-electro-magneto elasticity model associated with Green-Lindsay's model (GTEME model) in one region of the composed body and the generalized thermo-elasticity model (GTE model) in the other adjacent region. The essential feature of the generalized models under consideration is that the heat propagation has a finite speed (see [2, 10, 11, 14, 19, 20, 49]). This type of mechanical problem is described mathematically by systems of second order partial differential equations with appropriate transmission and boundary conditions. In the GTEME model part we have six-dimensional unknown physical field (three components of the displacement vector, electric potential function, magnetic potential function, and temperature distribution function), while in the GTE model part we have four- dimensional unknown physical field (three components of the displacement vector and temperature distribution function). Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations.

Note that the basic and mixed initial-boundary value problems for the GTEME theory are investigated in the monograph [7]. The transmission problems for composed elastic structures are also studied when in all adjacent regions of piecewise homogeneous composite bodies the same type GTEME model is considered with different material constants.

As we have already mentioned, the main goal of the present paper is investigation of the transmission problems when in different parts of adjacent regions of piecewise homogeneous composite bodies different models (in particular, GTEME and GTE models) are considered. The diversity in dimensions of the interacting physical fields essentially complicates mathematical formulation and analysis of the corresponding boundary-transmission problems. We apply the potential method and the theory of pseudodifferential equations and prove the uniqueness and existence theorems of solutions to different type basic boundary-transmission problems in appropriate Sobolev spaces. Properties of the layered potentials associated with the matrix differential operators of the GTEME and GTE models and the boundary operators generated by them are studied in [7, 21, 22, 42], and for the readers convenience, some results needed in our analysis are briefly presented in Appendix.

2. Basic Field Equation and Formulation of Boundary Transmission Problems

First we present the pseudo-oscillation equations of the GTEME and GTE models with corresponding Green's identities and afterwards we formulate the transmission problems. The pseudo-oscillation equations considered in the paper are obtained from the corresponding equations of dynamics by the Laplace transform and they contain a complex parameter $\tau = \sigma + i\omega$. Here we investigate the boundary-transmission problems for pseudo-oscillation equations. Solutions to the original dynamical initial-boundary-transmission problems can be then reconstructed by the inverse Laplace transform with respect to the parameter τ from solutions to the pseudo-oscillation problems. The detailed derivations of pseudo-oscillation equations from the dynamical constitutive relations can be found in [7] and [21]. In our analysis we will use essentially the results obtained in the monograph [7] and develop the potential method to complex boundary-transmission problems for anisotropic composed multilayered elastic structures.

2.1. Field equations of the GTEME model and Green's formulas.

The basic linear system of pseudo-oscillation equations for the thermo-electro-magneto-elasticity theory associated with Green-Lindsay's model for homogeneous solids in matrix form reads as [7]

$$A(\partial_x, \tau)\, U(x, \tau) = \Phi(x, \tau)\,,$$

where $U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top := (u, \varphi, \psi, \vartheta)^\top$ is the sought for complex-valued vector function, $\Phi = (\Phi_1, \dots, \Phi_6)^\top$ is a given vector-function, and $A(\partial_x, \tau)$ is a matrix differential operator

$$A(\partial_x,\tau)=[A_{pq}(\partial_x,\tau)]_{6\times 6}:$$

$$:=\begin{bmatrix} [c_{rjkl}\partial_j\partial_l-\varrho\tau^2\delta_{rk}]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -(1+\nu_0\tau)\,p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -(1+\nu_0\tau)m_j\partial_j \\ [-\tau\lambda_{kl}\partial_l]_{1\times 3} & \tau p_l\partial_l & \tau m_l\partial_l & \eta_{jl}\partial_j\partial_l-\tau^2h_0-\tau d_0 \end{bmatrix}_{6\times 6}. \tag{2.1}$$

The superscript $(\cdot)^\top$ denotes transposition operation, $\tau=\sigma+i\omega$ is a complex parameter, the summation over the repeated indices is meant from 1 to 3; $\partial=\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$. The components of the vector function U have the following physical sense: the first three components correspond to the elastic displacement vector $u=(u_1,u_2,u_3)^\top$, the forth and fifth ones, φ and ψ are, respectively, the electric and magnetic potentials, and the sixth component ϑ stands for the temperature distribution; c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are the constants characterizing the relation between thermodynamic processes and electromagnetic effects, λ_{rj} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, ϱ denotes the mass density, ν_0 and h_0 are two relaxation times, d_0 is a constitutive coefficient. These constants satisfy the symmetry conditions:

$$\begin{gathered} c_{rjkl}=c_{jrkl}=c_{klrj},\quad e_{klj}=e_{kjl},\quad q_{klj}=q_{kjl}, \\ \varkappa_{kj}=\varkappa_{jk},\quad \lambda_{kj}=\lambda_{jk},\quad \mu_{kj}=\mu_{jk},\quad a_{kj}=a_{jk},\quad \eta_{kj}=\eta_{jk},\quad r,j,k,l=1,2,3. \end{gathered} \tag{2.2}$$

From physical considerations it follows that (see, e.g., [2, 14, 33, 44, 49]):

$$\begin{gathered} c_{rjkl}\,\xi_{rj}\,\xi_{kl}\geq\delta_0\,\xi_{kl}\,\xi_{kl},\quad \varkappa_{kj}\,\xi_k\,\xi_j\geq\delta_1\,|\xi|^2,\quad \mu_{kj}\,\xi_k\,\xi_j\geq\delta_2\,|\xi|^2,\quad \eta_{kj}\,\xi_k\,\xi_j\geq\delta_3\,|\xi|^2, \\ \text{for all }\ \xi_{kj}=\xi_{jk}\in\mathbb{R}\text{ and for all }\ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3, \end{gathered} \tag{2.3}$$

$$\nu_0>0,\quad h_0>0,\quad d_0\nu_0-h_0>0, \tag{2.4}$$

where δ_0, δ_1, δ_2, and δ_3 are positive constants depending on material parameters.

Due to the symmetry conditions (2.2), with the help of (2.3) one can easily derive the inequalities

$$\begin{gathered} c_{rjkl}\,\zeta_{rj}\,\overline{\zeta_{kl}}\geq\delta_0\,\zeta_{kl}\,\overline{\zeta_{kl}},\quad \varkappa_{kj}\,\zeta_k\,\overline{\zeta_j}\geq\delta_1\,|\zeta|^2,\quad \mu_{kj}\,\zeta_k\,\overline{\zeta_j}\geq\delta_2\,|\zeta|^2,\quad \eta_{kj}\,\zeta_k\,\overline{\zeta_j}\geq\delta_3\,|\,\zeta|^2, \\ \text{for all }\ \zeta_{kj}=\zeta_{jk}\in\mathbb{C}\text{ and for all }\ \zeta=(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3, \end{gathered} \tag{2.5}$$

where the over bar denotes complex conjugation. The positive definiteness of the potential energy and the laws of thermodynamics imply that the following 8×8 matrix

$$M=[M_{kj}]_{8\times 8}:=\begin{bmatrix} [\varkappa_{jl}]_{3\times 3} & [a_{jl}]_{3\times 3} & [p_j]_{3\times 1} & [\nu_0 p_j]_{3\times 1} \\ [a_{jl}]_{3\times 3} & [\mu_{jl}]_{3\times 3} & [m_j]_{3\times 1} & [\nu_0 m_j]_{3\times 1} \\ [p_j]_{1\times 3} & [m_j]_{1\times 3} & d_0 & h_0 \\ [\nu_0 p_j]_{1\times 3} & [\nu_0 m_j]_{1\times 3} & h_0 & \nu_0 h_0 \end{bmatrix}_{8\times 8} \tag{2.6}$$

is positive definite. Moreover, it follows that the matrices

$$\Lambda^{(1)}:=\begin{bmatrix} [\varkappa_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} \end{bmatrix}_{6\times 6},\qquad \Lambda^{(2)}:=\begin{bmatrix} d_0 & h_0 \\ h_0 & \nu_0 h_0 \end{bmatrix}_{2\times 2} \tag{2.7}$$

are positive definite as well, i.e.,

$$\begin{gathered} \varkappa_{kj}\,\zeta'_k\,\overline{\zeta'_j}+a_{kj}\,(\zeta'_k\,\overline{\zeta''_j}+\overline{\zeta'_k}\,\zeta''_j)+\mu_{kj}\,\zeta''_k\,\overline{\zeta''_j}\geq\kappa_1\,(|\zeta'|^2+|\zeta''|^2)\qquad\forall\zeta',\zeta''\in\mathbb{C}^3, \\ d_0\,|z_1|^2+h_0\,(z_1\,\overline{z_2}+\overline{z_1}\,z_2)+\nu_0 h_0\,|z_2|^2\geq\kappa_2\,(|z_1|^2+|z_2|^2)\qquad\forall z_1,z_2\in\mathbb{C}, \end{gathered}$$

with some positive constants κ_1 and κ_2 depending on the material parameters involved in matrices (2.7).

Further, let us introduce the generalized stress operator $\mathcal{T}(\partial_x,n,\tau)$ associated with the pseudo-oscillation operator $A(\partial_x,\tau)$,

$$\mathcal{T}(\partial_x,n,\tau)=\big[\,\mathcal{T}_{pq}(\partial_x,n,\tau)\,\big]_{6\times 6}:$$

$$:=\left[\begin{array}{cccc} [c_{rjkl}\,n_j\partial_l]_{3\times 3} & [e_{lrj}\,n_j\partial_l]_{3\times 1} & [q_{lrj}\,n_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}n_j]_{3\times 1} \\ [-e_{jkl}\,n_j\partial_l]_{1\times 3} & \varkappa_{jl}\,n_j\partial_l & a_{jl}\,n_j\partial_l & -(1+\nu_0\tau)p_jn_j \\ [-q_{jkl}\,n_j\partial_l]_{1\times 3} & a_{jl}\,n_j\partial_l & \mu_{jl}\,n_j\partial_l & -(1+\nu_0\tau)m_jn_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{array}\right]_{6\times 6}. \tag{2.8}$$

For a six vector $U=(u,\varphi,\psi,\vartheta)^\top$ we can calculate the so-called generalized stress vector $\mathcal{T}U$,

$$\begin{aligned}\mathcal{T}(\partial_x,n,\tau)\,U(x,\tau)=&\big(\sigma_{1j}(x,\tau)n_j(x),\ \sigma_{2j}(x,\tau)n_j(x),\ \sigma_{3j}(x,\tau)n_j(x),\\ &-D_j(x,\tau)n_j(x),\ -B_j(x,\tau)n_j(x),\ -T_0^{-1}q_j(x,\tau)n_j(x)\big)^\top.\end{aligned} \tag{2.9}$$

Due to (2.9), the components of the stress vector have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the forth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector, respectively, with opposite sign, and finally, the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector; here $n=(n_1,n_2,n_3)$ stands for the unit normal vector to the corresponding surface element, σ_{ij} are the components of the mechanical stress tensor, T_0 is the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, $D=(D_1,D_2,D_3)^\top$ is the electric displacement vector and $B=(B_1,B_2,B_3)^\top$ is the magnetic induction vector.

Recall that $E=(E_1,E_2,E_3)^\top=-\operatorname{grad}\varphi$ and $H=(H_1,H_2,H_3)^\top=-\operatorname{grad}\psi$ are electric and magnetic fields, respectively, σ_{ij} are the components of the mechanical stress tensor, $\varepsilon_{kj}=2^{-1}(\partial_k u_j+\partial_j u_k)$ are the components of the mechanical strain tensor, $q=(q_1,q_2,q_3)^\top$ is the heat flux vector, and the corresponding constitutive equations read as

$$\begin{aligned}\sigma_{rj}(x,\tau)&=c_{rjkl}\varepsilon_{kl}(x,\tau)+e_{lrj}\partial_l\varphi(x,\tau)+q_{lrj}\partial_l\psi(x,\tau)-(1+\nu_0\tau)\lambda_{rj}\vartheta(x,\tau),\\ D_j(x,\tau)&=e_{jkl}\,\varepsilon_{kl}(x,\tau)-\varkappa_{jl}\,\partial_l\varphi(x,\tau)-a_{jl}\,\partial_l\psi(x,\tau)+(1+\nu_0\tau)p_j\vartheta(x,\tau),\\ B_j(x,\tau)&=q_{jkl}\,\varepsilon_{kl}(x,\tau)-a_{jl}\,\partial_l\varphi(x,\tau)-\mu_{jl}\partial_l\psi(x,\tau)+(1+\nu_0\tau)m_j\vartheta(x,\tau),\\ q_j(x,\tau)&=-T_0\,\eta_{jl}\partial_l\vartheta(x,\tau)\,.\end{aligned}$$

Let $\Omega=\Omega^+$ be a bounded domain of $\mathbb{R}^3$ with a sufficiently smooth boundary $S=\partial\Omega$ and $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega}$. For simplicity, in what follows, we assume that $S\in C^\infty$, if not otherwise stated.

By $C^k(\overline{\Omega})$ we denote the subspace of functions from $C^k(\Omega)$ whose derivatives up to the order k are continuously extendable to S from $\Omega^\pm$; $C^{k,\alpha}(\overline{\Omega^\pm})$ denotes the subspace of functions from $C^k(\overline{\Omega^\pm})$ whose kth order derivatives are Hölder continuous in $\Omega^\pm$ with exponent $\alpha\in(0,1]$. By L_p, $L_{p,\mathrm{loc}}$, $L_{p,\mathrm{comp}}$, W^r_p, $W^r_{p,\mathrm{loc}}$, $W^r_{p,\mathrm{comp}}$, H^s_p, and $B^s_{p,q}$ (with $r\geq 0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\leq q\leq\infty$) we denote the well-known Lebesgue, Sobolev-Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [37, 52]). Recall that $H^r_2=W^r_2=B^r_{2,2}$, $H^s_2=B^s_{2,2}$, $W^t_p=B^t_{p,p}$, and $H^k_p=W^k_p$, for any $r\geq 0$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

For arbitrary vector functions

$$U=(u_1,u_2,u_3,\varphi,\psi,\vartheta)^\top\in\big[C^2(\overline{\Omega})\big]^6 \text{ and } U'=(u_1',u_2',u_3',\varphi',\psi',\vartheta')^\top\in\big[C^2(\overline{\Omega})\big]^6,$$

the following first Green identity

$$\int\limits_{\Omega}\Big[A(\partial_x,\tau)\,U\cdot U'+\mathcal{E}_\tau(U,\overline{U'})\Big]dx=\int\limits_{\partial\Omega}\{\mathcal{T}(\partial_x,n,\tau)U\}^+\cdot\{U'\}^+dS \tag{2.10}$$

holds, where the central dot denotes the scalar product of two vectors in the complex vector space $\mathbb{C}^N$, i.e., $a\cdot b\equiv(a,b):=\sum_{j=1}^N a_j\,\overline{b_j}$ for $a,b\in\mathbb{C}^N$, the symbols $\{\cdot\}^\pm$ denote the one sided limits (the trace operators) on $\partial\Omega^\pm$ from $\Omega^\pm$, the operators $A(\partial_x,\tau)$ and $\mathcal{T}(\partial_x,n,\tau)$ are given by (2.1) and (2.8), respectively, and

$$
\begin{aligned}
&\mathcal{E}_\tau(U,\overline{U'}) := c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u'_r} + \varrho\,\tau^2\,u_r\,\overline{u'_r} + e_{lrj}\,(\partial_l\varphi\,\overline{\partial_j u'_r} - \partial_j u_r\,\overline{\partial_l\varphi'})\\
&+ q_{lrj}\,(\partial_l\psi\,\overline{\partial_j u'_r} - \partial_j u_r\,\overline{\partial_l\psi'}) + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi'} + a_{jl}\,(\partial_l\varphi\,\overline{\partial_j\psi'} + \partial_j\psi\,\overline{\partial_l\varphi'})\\
&+ \mu_{jl}\partial_l\psi\overline{\partial_j\psi'} + \lambda_{kj}[\tau\overline{\vartheta'}\partial_j u_k - (1+\nu_0\tau)\vartheta\overline{\partial_j u'_k}] - p_l[\tau\overline{\vartheta'}\partial_l\varphi + (1+\nu_0\tau)\vartheta\overline{\partial_l\varphi'}]\\
&- m_l\,[\tau\,\overline{\vartheta'}\,\partial_l\psi + (1+\nu_0\tau)\vartheta\,\overline{\partial_l\psi'}] + \eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta'} + \tau\,(h_0\tau + d_0)\,\vartheta\,\overline{\vartheta'}\,. \qquad (2.11)
\end{aligned}
$$

Note that Green's formula (2.10) by a standard limiting procedure can be generalized to the Lipschitz domains and to vector functions $U \in [W_2^1(\Omega)]^6$ with $A(\partial_x,\tau)U \in [L_2(\Omega)]^6$ and $U' \in [W_2^1(\Omega)]^6$. Using Green's first identity, we can correctly determine a *generalized trace of the stress vector* $\{\mathcal{T}(\partial_x,n,\tau)U\}^+ \in [H_2^{-1/2}(\partial\Omega)]^6$ for a function $U \in [W_2^1(\Omega)]^6$ with $A(\partial_x,\tau)U \in [L_2(\Omega)]^6$ by the following relation (cf. [7,38,43])

$$
\big\langle\,\{\mathcal{T}(\partial_x,n,\tau)U\}^+\,,\,\{U'\}^+\big\rangle_{\partial\Omega} := \int\limits_{\Omega} [\,A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\,]\,dx, \qquad (2.12)
$$

where $U' \in [W_2^1(\Omega)]^6$ is an arbitrary vector function. Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denotes the duality pairing of $[H_2^{-1/2}(\partial\Omega)]^6$ with $[H_2^{1/2}(\partial\Omega)]^6$ which extends the usual L_2 inner product for the complex-valued vector functions,

$$
\langle\,f\,,\,g\,\rangle_{\partial\Omega} = \int\limits_{\partial\Omega}\sum_{j=1}^{6} f_j(x)\,\overline{g_j(x)}\,dS \;\;\text{for}\;\; f,g\in[L_2(\partial\Omega)]^6.
$$

2.2. **Field equations of the GTE model and Green's formulas.** The basic linear system of pseudo-oscillation equations for the thermo-elasticity theory associated with Green-Lindsay's model for homogeneous solids in matrix form reads as (see [7,8,21–23])

$$
A(\partial_x,\tau)\,U(x,\tau) = \Phi(x,\tau)\,,
$$

where $U = (u_1,u_2,u_3,\vartheta)^\top := (u,\vartheta)^\top$ is a complex valued unknown vector function with $u=(u_1,u_2,u_3)^\top$ being the elastic displacement vector and ϑ the temperature distribution, $\Phi = (\Phi_1,\Phi_2,\Phi_3,\Phi_4)^\top$ is a given vector function,

$$
A(\partial_x,\tau) = [A_{pq}(\partial_x,\tau)]_{4\times 4} := \begin{bmatrix} [c_{rjkl}\partial_j\partial_l - \varrho\tau^2\delta_{rk}]_{3\times 3} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3\times 1}\\ [-\tau\lambda_{kl}\partial_l]_{1\times 3} & \eta_{jl}\partial_j\partial_l - \tau^2 h_0 - \tau d_0\end{bmatrix}_{4\times 4}. \qquad (2.13)
$$

The corresponding constitutive relations are

$$
\begin{aligned}
\sigma_{rj}(x,\tau) &= c_{rjkl}\varepsilon_{kl}(x,\tau) - (1+\nu_0\tau)\lambda_{rj}\vartheta(x,\tau),\\
q_j(x,\tau) &= -\,T_0\,\eta_{jl}\partial_l\vartheta(x,\tau)\,.
\end{aligned}
$$

The stress operator in the theory of thermo-elasticity has the form

$$
\mathcal{T}(\partial_x,n,\tau) = [\,\mathcal{T}_{pq}(\partial_x,n,\tau)\,]_{4\times 4} := \begin{bmatrix} [c_{rjkl}\,n_j\partial_l]_{3\times 3} & [-(1+\nu_0\tau)\lambda_{rj}n_j]_{3\times 1}\\ [0]_{1\times 3} & \eta_{jl}n_j\partial_l\end{bmatrix}_{4\times 4} \qquad (2.14)
$$

and the corresponding generalized thermo-stress vector is written as

$$
\mathcal{T}(\partial_x,n,\tau)\,U(x,\tau) = \big(\sigma_{1j}(x,\tau)n_j(x),\;\sigma_{2j}(x,\tau)n_j(x),\sigma_{3j}(x,\tau)n_j(x),-T_0^{-1}q_j(x,\tau)n_j(x)\big)^\top,
$$

where the first three components correspond to the mechanical stress vector in the theory of generalized thermo-elasticity and the fourth component is $(-T_0^{-1})$ times the normal component of the heat flux vector; here again, $n=(n_1,n_2,n_3)$ stands for the unit normal vector to the corresponding surface element.

For arbitrary vector functions $U = (u_1,u_2,u_3,\vartheta)^\top \in \big[C^2(\overline{\Omega})\big]^4$ and $U' = (u'_1,u'_2,u'_3,\vartheta')^\top \in \big[C^2(\overline{\Omega})\big]^4$, we have Green's first identity for the thermo-elasticity case

$$
\int\limits_{\Omega}\Big[A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\Big]dx = \int\limits_{\partial\Omega}\{\mathcal{T}(\partial_x,n,\tau)U\}^+\cdot\{U'\}^+dS\,, \qquad (2.15)
$$

where

$$\mathcal{E}_\tau(U,\overline{U'}) := c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u'_r} + \varrho\,\tau^2\,u_r\,\overline{u'_r} + \lambda_{kj}[\tau\overline{\vartheta'}\partial_j u_k - (1+\nu_0\tau)\vartheta\overline{\partial_j u'_k}]$$
$$+\,\eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta'} + \tau\,(h_0\tau+d_0)\,\vartheta\,\overline{\vartheta'}\,. \tag{2.16}$$

By a standard limiting procedure, Green's first formula (2.15) can be extended to the Lipschitz domains and to the vector functions $U \in \big[W_2^1(\Omega)\big]^4$ and $U' \in \big[W_2^1(\Omega)\big]^4$ possessing the property $A(\partial_x,\tau)\,U \in \big[L_2(\Omega)\big]^4$,

$$\int\limits_{\Omega}\big[A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\big]dx = \big\langle\{\mathcal{T}(\partial_x,n,\tau)U\}^+\,,\,\{U'\}^+\big\rangle_S\,.$$

Green's first formula holds also for the exterior unbounded domain Ω^- in the class of functions decaying at infinity.

Definition 2.1. We say that a vector function $U=(u_1,u_2,u_3,\vartheta)^\top \in [W^1_{2,\,\mathrm{loc}}(\Omega^-)]^4$ belongs to the class $\mathbf{Z}_\tau(\Omega^-)$ if the components of U satisfy the following decay conditions at infinity:

$$\begin{aligned} &u_k(x)=\mathcal{O}(|x|^{-2}), && \partial_j u_k(x)=\mathcal{O}(|x|^{-2}),\\ &\vartheta(x)=\mathcal{O}(|x|^{-2}), && \partial_j\vartheta(x)=\mathcal{O}(|x|^{-2}),\quad k,j=1,2,3. \end{aligned} \tag{2.17}$$

Evidently, $\mathbf{Z}_\tau(\Omega^-)\subset [W_2^1(\Omega^-)]^4$.

For arbitrary vector functions $U=(u_1,u_2,u_3,\vartheta)^\top \in [C^1(\overline{\Omega^-})]^4\cap[C^2(\Omega^-)]^4\cap\mathbf{Z}_\tau(\Omega^-)$ and $U'=(u'_1,u'_2,u'_3,\vartheta')^\top \in [C^1(\overline{\Omega^-})]^4\cap\mathbf{Z}_\tau(\Omega^-)$ with $A(\partial_x,\tau)\,U\in[L_{2,\mathrm{comp}}(\Omega^-)]^4$ the following Green's first identity for the exterior domain Ω^-

$$\int\limits_{\Omega^-}\Big[A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\Big]dx = -\int\limits_{\partial\Omega^-}\{\mathcal{T}(\partial_x,n,\tau)U\}^-\cdot\{U'\}^- dS \tag{2.18}$$

holds true. By a standard limiting procedure, Green's formula (2.18) can be extended to the Lipschitz domains and to the vector functions $U\in\big[W^1_{2,\,\mathrm{loc}}(\Omega^-)\big]^4$ and $U'\in\big[W^1_{2,\,\mathrm{loc}}(\Omega^-)\big]^4$ satisfying the decay conditions at infinity (2.17) and possessing the property $A(\partial_x,\tau)\,U\in\big[L_{2,\,\mathrm{comp}}(\Omega^-)\big]^4$,

$$\int\limits_{\Omega^-}\big[A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\big]dx = -\big\langle\{\mathcal{T}(\partial_x,n,\tau)U\}^-\,,\,\{U'\}^-\big\rangle_S\,, \tag{2.19}$$

where $A(\partial_x,\tau)\,U$ is compactly supported and $\{\mathcal{T}(\partial_x,n,\tau)U\}^-\in[H_2^{-1/2}(S)]^4$ is the generalized trace of the stress vector on the boundary surface $S=\partial\Omega^-$. Note that since the operator $A(\partial_x,\tau)$ is strongly elliptic and $A(\partial_x,\tau)\,U$ has a compact support, therefore, actually, U is an analytic vector function of the real variables (x_1,x_2,x_3) in the vicinity of infinity (in the domain $\Omega^-\setminus\operatorname{supp}A(\partial_x,\tau)\,U$) and conditions (2.17) can be understood in the usual classical pointwise sense. Hence, the improper integral over Ω^- in formula (2.19) is convergent and well defined.

2.3. **Formulation of the basic transmission problems.** Here we formulate the basic transmission problems in the classical pointwise sense and in the weak sense, when the whole space $\mathbb{R}^3$ is divided into two simply connected regions $\mathbb{R}^3=\overline{\Omega^{(1)}}\cup\overline{\Omega^{(2)}}$, where $\Omega^{(1)}$ is a bounded domain with a smooth boundary S and $\Omega^{(2)}$ is its unbounded complement, $\Omega^{(2)}=\mathbb{R}^3\setminus\overline{\Omega^{(1)}}$. We assume that the region $\Omega^{(1)}$ is filled up with a material subject to the termo-electro-magneto-elasticity model, while the region $\Omega^{(2)}$ is filled up with a material subject to the thermo-elasticity model. The thermo-mechanical and electro-magnetic characteristics, material constants, differential and boundary operators associated with the domain $\Omega^{(\beta)}$ for $\beta=1,2$, we equip with the superscript (β).

In what follows, we assume that all unknown vector functions and the given vector functions depend on the complex parameter τ, however, we will not show explicitly this dependence and drop the argument τ in the case of functions, but we will keep the argument in the differential and stress operators.

Definition 2.2. A vector $U^{(1)}=(U_1^{(1)},\dots,U_6^{(1)})^\top$ is called regular in the domain $\Omega^{(1)}$ if

$$U^{(1)}\in[C^1(\overline{\Omega^{(1)}})]^6\cap[C^2(\Omega^{(1)})]^6.$$

Similarly, a vector $U^{(2)}=(U_1^{(2)},\dots,U_4^{(2)})^\top$ is called regular in the domain $\Omega^{(2)}$ if

$$U^{(2)}\in[C^1(\overline{\Omega^{(2)}})]^4\cap[C^2(\Omega^{(2)})]^4\cap\mathbf{Z}_\tau(\Omega^{(2)})\,.$$

The basic transmission problem $(\mathbf{TD})_\tau$: *Find regular solutions* $U^{(1)}=(U_1^{(1)},\dots,U_6^{(1)})^\top$ *and* $U^{(2)}=(U_1^{(2)},\dots,U_4^{(2)})^\top$ *to the equations*

$$A^{(1)}(\partial_x,\tau)\,U^{(1)}(x)=\Phi^{(1)}(x)\,,\quad x\in\Omega^{(1)},\tag{2.20}$$

$$A^{(2)}(\partial_x,\tau)\,U^{(2)}(x)=\Phi^{(2)}(x)\,,\quad x\in\Omega^{(2)},\tag{2.21}$$

satisfying on the interface S the following transmission conditions:

$$\{U_j^{(1)}(x)\}^+-\{U_j^{(2)}(x)\}^-=f_j(x),\quad j=1,2,3,\quad x\in S,\tag{2.22}$$

$$\{U_6^{(1)}(x)\}^+-\{U_4^{(2)}(x)\}^-=f_6(x),\quad x\in S,\tag{2.23}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}_j^+-\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}_j^-=F_j(x),\ \ j=1,2,3,\ \ x\in S,\tag{2.24}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}_6^+-\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}_4^-=F_6(x),\quad x\in S,\tag{2.25}$$

and the Dirichlet type boundary conditions

$$\{U_j^{(1)}(x)\}^+=f_j(x),\quad j=4,5,\quad x\in S.\tag{2.26}$$

The basic transmission problem $(\mathbf{TN})_\tau$: *Find regular solutions* $U^{(1)}$ *and* $U^{(2)}$ *to equations* (2.20)–(2.21) *satisfying transmission conditions* (2.22)–(2.25) *and the Neumann type boundary conditions*

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}_j^+=F_j(x),\quad j=4,5,\quad x\in S.\tag{2.27}$$

Here, the differential operators $A^{(\beta)}(\partial_x,\tau)$ and generalized stress operators $\mathcal{T}^{(\beta)}(\partial_x,n,\tau)$, $\beta=1,2$, are defined by (2.1), (2.8) *and* (2.13), (2.14), *respectively. The data of the problems satisfy the following inclusions:*

$$\Phi^{(1)}=(\Phi_1^{(1)},\dots,\Phi_6^{(1)})^\top\in[C(\Omega^{(1)})]^6,$$
$$\Phi^{(2)}=(\Phi_1^{(2)},\dots,\Phi_4^{(2)})^\top\in[C(\Omega^{(2)})]^4,\quad \operatorname{supp}\Phi^{(2)}\ \textit{is compact},$$
$$f_j\in C^1(S),\quad F_j\in C(S),\quad j=1,2,\dots,6.$$

Note that the transmission conditions relate one-sided limits (traces) of similar fields: equations (2.22) relate the components of the displacement vectors $u^{(1)}$ and $u^{(2)}$, equation (2.23) relates temperature functions $U_6^{(1)}=\vartheta^{(1)}$ and $U_4^{(2)}=\vartheta^{(2)}$, equation (2.24) relates components of the mechanical stress vectors, and finally, equation (2.25) relates normal components of the heat flux vectors. Further, equation (2.26) describes the Dirichlet conditions for the electric and magnetic potentials, while equation (2.27) corresponds to the Neumann type boundary conditions for the prescribed normal components of the electric displacement and magnetic induction vectors.

Remark 2.3. Note that Green's formulas can be extended to the general Sobolev $W_p^1(\Omega)$ spaces with arbitrary $p>1$. For example, if $U^{(1)}\in[W_p^1(\Omega^{(1)})]^6$ with $A^{(1)}(\partial_x,\tau)U^{(1)}\in\big[L_p(\Omega^{(1)})\big]^6$ and $U'\in[W^1_{p'}(\Omega^{(1)})]^6$ with $1/p+1/p'=1$, then formula (2.12) holds true due to the inclusion $\{U'\}^+\in[B^{1-1/p'}_{p',p'}(\partial\Omega^{(1)})]^6=[B^{-1/p}_{p',p'}(\partial\Omega^{(1)})]^6$ and defines the generalized trace of the stress vector $\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\}^+\in[B^{-1/p}_{p,p}(\partial\Omega^{(1)})]^6$ on $\partial\Omega^{(1)}$.

Similarly, if vector functions $U^{(2)}\in[W_p^1(\Omega^{(2)})]^4$ and $U'\in[W^1_{p'}(\Omega^{(2)})]^4$ satisfy the decay conditions at infinity (2.17), and $A^{(2)}(\partial_x,\tau)\,U^{(2)}\in\big[L_{p,\,\mathrm{comp}}(\Omega^{(2)})\big]^4$, then formula (2.19) holds true due to the inclusion $\{U'\}^-\in[B^{-1/p}_{p',p'}(\partial\Omega^{(2)})]^4$ and defines the generalized trace of the stress vector $\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\}^-\in[B^{-1/p}_{p,p}(\partial\Omega^{(2)})]^4$ on $\partial\Omega^{(2)}$.

In these cases, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_S$ denotes the duality pairing of the space $[B^{-1/p}_{p,p}(S)]^m$ with the space $[B^{1/p}_{p',p'}(S)]^m$ for $m=6$ and $m=4$, respectively.

These generalized Green's formulas give us possibility to formulate the transmission problems in a week sense.

Weak formulation of the basic transmission problems $(\mathbf{TD})_\tau$ and $(\mathbf{TN})_\tau$:
Find vector functions

$$U^{(1)}\in[W^1_p(\Omega^{(1)})]^6 \quad \textit{and} \quad U^{(2)}\in[W^1_{p,\,\mathrm{loc}}(\Omega^{(2)})]^4\cap\mathbf{Z}_\tau(\Omega^{(2)}), \quad p>1,$$

satisfying the differential equations (2.20) *and* (2.21) *in the distributional sense, transmission conditions* (2.22)–(2.23) *and the Dirichlet type boundary condition* (2.26) *in the usual trace sense, transmission conditions* (2.24)–(2.25) *and the Neumann type boundary condition* (2.27) *in the generalized functional sense defined by Green's formulas* (2.12) *and* (2.19).
In the case of weak setting, we assume that

$$\begin{aligned}&\Phi^{(1)}\in[L_p(\Omega^{(1)})]^6, \quad \Phi^{(2)}\in[L_{p,\,\mathrm{comp}}(\Omega^{(2)})]^4,\\ &f_j\in B^{1-\frac1p}_{p,p}(S), \quad F_j\in B^{-\frac1p}_{p,p}(S), \quad j=1,2,\dots,6.\end{aligned}$$

Recall that for $p=2$ we have $B^{\pm\frac12}_{2,2}(S)=H^{\pm\frac12}_2(S)$.

2.4. **Formulation of the boundary-transmission problems for layered composite structures.** Let us now consider a bounded elastic composite structure $\overline{\Omega^{(1)}}\cup\overline{\Omega^{(2)}}$ with the interface $S^{(1)}$ and the exterior boundary $S^{(2)}$, assuming that in the region $\Omega^{(1)}$, we have again the GTEME model and in the region $\Omega^{(2)}$ the GTE model. Evidently, $\partial\Omega^{(1)}=S^{(1)}$ and $\partial\Omega^{(2)}=S^{(1)}\cup S^{(2)}$. For $x\in S^{(\beta)}$, by $n(x)$ we denote again the outward unit normal vector to the surfaces $S^{(\beta)}$, $\beta=1,2$.

The boundary-transmission problems in the case under consideration are formulated as follows.

We are looking for regular solutions $U^{(1)}=(U^{(1)}_1,\dots,U^{(1)}_6)^\top\in[C^1(\overline{\Omega^{(1)}})]^6\cap[C^2(\Omega^{(1)})]^6$ and $U^{(2)}=(U^{(2)}_1,\dots,U^{(2)}_4)^\top\in[C^1(\overline{\Omega^{(2)}})]^4\cap[C^2(\Omega^{(2)})]^4$ to the corresponding differential equations (2.20) and (2.21), respectively, satisfying the boundary-transmission conditions formulated in the problems $(\mathrm{TD})_\tau$ or $(\mathrm{TN})_\tau$ on the interface $S^{(1)}$ and one of the following boundary conditions on the exterior boundary $S^{(2)}$:
(D) Dirichlet boundary condition

$$\{U^{(2)}(x)\}^+=f^*(x), \quad x\in S^{(2)};$$

(N) Neumann boundary condition

$$\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}^+=F^*(x), \quad x\in S^{(2)};$$

(M) Mixed type boundary conditions

$$\begin{aligned}&\{U^{(2)}(x)\}^+=f^{(D)}(x), \quad x\in S^{(2)}_D,\\ &\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}^+=F^{(N)}(x), \quad x\in S^{(2)}_N,\end{aligned}$$

where $S^{(2)}_D$ and $S^{(2)}_N$ are non-overlapping open submanifolds, $S^{(2)}=\overline{S^{(2)}_D}\cup\overline{S^{(2)}_N}$, $S^{(2)}_D\cap S^{(2)}_N=\varnothing$; $f^*=(f^*_1,\dots,f^*_4)^\top$, $F^*=(F^*_1,\dots,F^*_4)^\top$, $f^{(D)}=(f^{(D)}_1,\dots,f^{(D)}_4)^\top$, and $F^{(N)}=(F^{(N)}_1,\dots,F^{(N)}_4)^\top$ are the given vector functions from the appropriate continuous function spaces.

In the case of weak setting of the problems we look for solutions $U^{(1)}\in[W^1_p(\Omega^{(1)})]^6$ and $U^{(2)}\in[W^1_p(\Omega^{(2)})]^4$ of the differential equations (2.20) and (2.21) in the distributional sense, and satisfying the above-listed boundary and transmission conditions in the trace sense for the Dirichlet data and in the generalized functional trace sense for the Neumann data. In this case, the data of the problem satisfy the inclusions:

$$\begin{aligned}&\Phi^{(1)}\in[L_p(\Omega^{(1)})]^6, && \Phi^{(2)}\in[L_p(\Omega^{(2)})]^4,\\ &f_j\in B^{1-\frac1p}_{p,p}(S^{(1)}), && F_j\in B^{-\frac1p}_{p,p}(S^{(1)}), \quad j=1,2,\dots,6,\\ &f^*_j\in B^{1-\frac1p}_{p,p}(S^{(2)}), && F^*_j\in B^{-\frac1p}_{p,p}(S^{(2)}), \quad j=1,2,\dots,4,\\ &f^{(D)}_j\in B^{1-\frac1p}_{p,p}(S^{(2)}_D), && F^{(N)}_j\in B^{-\frac1p}_{p,p}(S^{(2)}_N), \quad j=1,2,\dots,4.\end{aligned}$$

In what follows,we refer the above problems as $(\mathrm{DTD})_\tau$, $(\mathrm{NTD})_\tau$, $(\mathrm{MTD})_\tau$, $(\mathrm{DTN})_\tau$, $(\mathrm{NTN})_\tau$, and $(\mathrm{MTN})_\tau$, where the first letter indicates the type of boundary conditions on the exterior boundary $S^{(2)}$, while the next two letters indicate the type of boundary-transmission conditions on the interface $S^{(1)}$.

3. Uniqueness Theorems

Due to the linearity of the above-formulated problems, we consider the corresponding homogeneous problems and prove the uniqueness theorems for weak solutions implying the uniqueness for regular solutions as well. In what follows, we assume that the time relaxation parameters $\nu_0^{(1)}$ and $\nu_0^{(2)}$ involved in the equations of the GTEME and GTE models are the same,

$$\nu_0^{(1)} = \nu_0^{(2)} =: \nu_0.$$

Theorem 3.1. *Let the interface surface S be the Lipschitz one and $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$. The basic homogeneous transmission problem $(TD)_\tau$ has only the trivial weak solution for $p = 2$, while the general weak solution to the homogeneous transmission problem $(TN)_\tau$ reads as a pair of vectors $U^{(1)} = (0,0,0,b_1,b_2,0)^\top$ and $U^{(2)} = (0,0,0,0)^\top$, where b_1 and b_2 are arbitrary complex constants.*

Proof. Let a pair of vector functions

$$(U^{(1)},U^{(2)}) \in [W_2^1(\Omega^{(1)})]^6 \times \Big([W_2^1(\Omega^{(2)})]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$$

be a weak solution to one of the homogeneous transmission problems listed in the theorem. For arbitrary vector functions $U' = (u_1',u_2',u_3',\varphi',\psi',\vartheta')^\top \in [W_2^1(\Omega^{(1)})]^6$ and $V' = (v_1',v_2',v_3',\theta')^\top \in [W_2^1(\Omega^{(2)})]^4$, from Green's formulas (2.12) and (2.19), we have

$$\int\limits_{\Omega^{(1)}} \mathcal{E}_\tau^{(1)}(U^{(1)},\overline{U'})\,dx = \big\langle \{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\}^+ \cdot \{U'\}^+ \big\rangle_S\,, \tag{3.1}$$

$$\int\limits_{\Omega^{(2)}} \mathcal{E}_\tau^{(2)}(U^{(2)},\overline{V'})\,dx = -\big\langle \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\}^- \cdot \{V'\}^- \big\rangle_S\,, \tag{3.2}$$

where $\mathcal{E}_\tau^{(1)}(\cdot,\cdot)$ and $\mathcal{E}_\tau^{(2)}(\cdot,\cdot)$ are defined by the relations (2.11) and (2.16) respectively, with the material constants associated with the regions $\Omega^{(1)}$ and $\Omega^{(2)}$,

$$\begin{aligned}
\mathcal{E}_\tau^{(1)}(U^{(1)},\overline{U'}) &= c_{rjkl}^{(1)}\,\partial_l u_k^{(1)}\,\overline{\partial_j u_r'} + \varrho^{(1)}\,\tau^2\,u_r^{(1)}\,\overline{u_r'} + e_{lrj}^{(1)}\big(\partial_l\varphi^{(1)}\,\overline{\partial_j u_r'} - \partial_j u_r^{(1)}\,\overline{\partial_l\varphi'}\,\big)\\
&\quad + q_{lrj}^{(1)}\big(\partial_l\psi^{(1)}\overline{\partial_j u_r'} - \partial_j u_r^{(1)}\overline{\partial_l\psi'}\big) + \varkappa_{jl}^{(1)}\,\partial_l\varphi^{(1)}\overline{\partial_j\varphi'} + a_{jl}^{(1)}\big(\partial_l\varphi^{(1)}\overline{\partial_j\psi'} + \partial_j\psi^{(1)}\overline{\partial_l\varphi'}\,\big)\\
&\quad + \mu_{jl}^{(1)}\,\partial_l\psi^{(1)}\,\overline{\partial_j\psi'} + \lambda_{kj}^{(1)}\,\big[\tau\,\partial_j u_k^{(1)}\,\overline{\vartheta'} - (1+\nu_0\tau)\,\vartheta^{(1)}\,\overline{\partial_j u_k'}\,\big]\\
&\quad - p_l^{(1)}\big[\tau\partial_l\varphi^{(1)}\overline{\vartheta'} + (1+\nu_0\tau)\vartheta^{(1)}\overline{\partial_l\varphi'}\,\big] - m_l^{(1)}\big[\tau\partial_l\psi^{(1)}\overline{\vartheta'} + (1+\nu_0\tau)\vartheta^{(1)}\overline{\partial_l\psi'}\,\big]\\
&\quad + \eta_{jl}^{(1)}\,\partial_l\vartheta^{(1)}\,\overline{\partial_j\vartheta'} + \tau\,\big(h_0^{(1)}\tau + d_0^{(1)}\big)\,\vartheta^{(1)}\,\overline{\vartheta'}\,,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\mathcal{E}_\tau^{(2)}(U^{(2)},\overline{V'}) &= c_{rjkl}^{(2)}\,\partial_l u_k^{(2)}\,\overline{\partial_j v_r'} + \varrho^{(2)}\,\tau^2\,u_r^{(2)}\,\overline{v_r'}\\
&\quad + \lambda_{kj}^{(2)}\,\big[\tau\,\partial_j u_k^{(2)}\,\overline{\theta'} - (1+\nu_0\tau)\,\vartheta^{(2)}\,\overline{\partial_j v_k'}\,\big] + \eta_{jl}^{(2)}\,\partial_l\vartheta^{(2)}\,\overline{\partial_j\theta'}\\
&\quad + \tau\,\big(h_0^{(2)}\tau + d_0^{(2)}\big)\,\vartheta^{(2)}\,\overline{\theta'}\,.
\end{aligned} \tag{3.4}$$

If in Green's formulas (3.1) and (3.2) we substitute successively the vectors

$$\big(u_1^{(1)},u_2^{(1)},u_3^{(1)},0,0,0\big)^\top,\ \ \big(0,0,0,\varphi^{(1)},0,0\big)^\top,\ \ \big(0,0,0,0,\psi^{(1)},0\big)^\top,\\ \Big(0,0,0,0,0,\frac{1+\nu_0\tau}{\overline{\tau}}\vartheta^{(1)}\Big)^\top$$

and

$$\big(u_1^{(2)},u_2^{(2)},u_3^{(2)},0\big)^\top,\ \ \Big(0,0,0,\frac{1+\nu_0\tau}{\overline{\tau}}\vartheta^{(2)}\Big)^\top$$

in the place of the vectors U' and V' respectively, we get the following relations:

$$\int\limits_{\Omega^{(1)}} \big[c^{(1)}_{rjkl}\partial_l u^{(1)}_k \overline{\partial_j u^{(1)}_r} + \varrho^{(1)}\tau^2 u^{(1)}_r \overline{u^{(1)}_r} + e^{(1)}_{lrj}\,\partial_l\varphi^{(1)}\,\overline{\partial_j u^{(1)}_r} + q^{(1)}_{lrj}\,\partial_l\psi^{(1)}\,\overline{\partial_j u^{(1)}_r}$$

$$-(1+\nu_0\tau)\lambda^{(1)}_{kj}\vartheta^{(1)}\,\overline{\partial_j u^{(1)}_k}\,\big]dx = \int\limits_S \{[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}]_r\}^+ \cdot \{u^{(1)}_r\}^+ dS, \tag{3.5}$$

$$\int\limits_{\Omega^{(1)}} \big[-e^{(1)}_{lrj}\,\partial_j u^{(1)}_r\,\overline{\partial_l\varphi^{(1)}} + \varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + a^{(1)}_{jl}\,\partial_j\psi^{(1)}\,\overline{\partial_l\varphi^{(1)}}$$

$$-(1+\nu_0\tau)p^{(1)}_l\,\vartheta^{(1)}\,\overline{\partial_l\varphi^{(1)}}\,\big]\,dx = \int\limits_S \{[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}]_4\}^+ \cdot \{\varphi^{(1)}\}^+ dS, \tag{3.6}$$

$$\int\limits_{\Omega^{(1)}} \big[-q^{(1)}_{lrj}\,\partial_j u^{(1)}_r\,\overline{\partial_l\psi^{(1)}} + a^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\psi^{(1)}} + \mu^{(1)}_{jl}\,\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}$$

$$-(1+\nu_0\tau)m^{(1)}_l\,\vartheta^{(1)}\,\overline{\partial_l\psi^{(1)}}\,\big]\,dx = \int\limits_S \{[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}]_5\}^+ \cdot \{\psi^{(1)}\}^+ dS, \tag{3.7}$$

$$\int\limits_{\Omega^{(1)}} \big\{(1+\nu_0\overline{\tau})\big[\lambda^{(1)}_{kj}\overline{\vartheta^{(1)}}\partial_j u^{(1)}_k - p^{(1)}_l\overline{\vartheta^{(1)}}\partial_l\varphi^{(1)} - m^{(1)}_l\overline{\vartheta^{(1)}}\partial_l\psi^{(1)} + (h^{(1)}_0\tau + d^{(1)}_0)|\vartheta^{(1)}|^2\big]$$

$$+\frac{1+\nu_0\overline{\tau}}{\tau}\eta^{(1)}_{jl}\partial_l\vartheta^{(1)}\overline{\partial_j\vartheta^{(1)}}\big\}dx = \frac{1+\nu_0\overline{\tau}}{\tau}\int\limits_S \{[\mathcal{T}^{(1)}(\partial,n,\tau)U^{(1)}]_6\}^+ \cdot \{\vartheta^{(1)}\}^+ dS, \tag{3.8}$$

$$\int\limits_{\Omega^{(2)}} \big[c^{(2)}_{rjkl}\partial_l u^{(2)}_k \overline{\partial_j u^{(2)}_r} + \varrho^{(2)}\tau^2 u^{(2)}_r \overline{u^{(2)}_r} - (1+\nu_0\tau)\lambda^{(2)}_{kj}\vartheta^{(2)}\,\overline{\partial_j u^{(2)}_k}\,\big]dx$$

$$= -\int\limits_S \{[\mathcal{T}^{(2)}(\partial,n,\tau)U^{(2)}]_r\}^- \cdot \{u^{(2)}_r\}^- dS, \tag{3.9}$$

$$\int\limits_{\Omega^{(2)}} \big\{(1+\nu_0\,\overline{\tau})\big[\lambda^{(2)}_{kj}\overline{\vartheta^{(2)}}\partial_j u^{(2)}_k + (h^{(2)}_0\tau + d^{(2)}_0)|\vartheta^{(2)}|^2\big] + \frac{1+\nu_0\overline{\tau}}{\tau}\eta^{(2)}_{jl}\partial_l\vartheta^{(2)}\overline{\partial_j\vartheta^{(2)}}\big\}dx$$

$$= -\frac{1+\nu_0\,\overline{\tau}}{\tau}\int\limits_S \{[\mathcal{T}^{(2)}(\partial,n,\tau)U^{(2)}]_4\}^- \cdot \{\vartheta^{(2)}\}^- dS. \tag{3.10}$$

Now, if we add termwise equation (3.5), the complex conjugate of equations (3.6)–(3.8), equation (3.9), and the complex conjugate of equation (3.10), and take into account symmetry properties (2.2) of coefficients for both models and the homogeneous transmission and boundary conditions, we arrive at the relation

$$\int\limits_{\Omega^{(1)}} \Big\{ c^{(1)}_{rjkl}\,\partial_l u^{(1)}_k\,\overline{\partial_j u^{(1)}_r} + \varrho^{(1)}\,\tau^2|u^{(1)}|^2 + \varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}}$$

$$+ a^{(1)}_{jl}\big(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + \partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}}\big) + \mu^{(1)}_{jl}\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}$$

$$- 2\,\mathrm{Re}\big[p^{(1)}_l(1+\nu_0\tau)\vartheta^{(1)}\,\overline{\partial_l\varphi^{(1)}}\,\big] - 2\,\mathrm{Re}\big[m^{(1)}_l(1+\nu_0\tau)\vartheta^{(1)}\,\overline{\partial_l\psi^{(1)}}\,\big]$$

$$+ (1+\nu_0\tau)(h^{(1)}_0\overline{\tau} + d^{(1)}_0)\,|\vartheta^{(1)}|^2 + \frac{1+\nu_0\tau}{\overline{\tau}}\,\eta^{(1)}_{jl}\,\partial_l\vartheta^{(1)}\,\overline{\partial_j\vartheta^{(1)}}\,\Big\}\,dx$$

$$+ \int\limits_{\Omega^{(2)}} \Big\{ c^{(2)}_{rjkl}\,\partial_l u^{(2)}_k\,\overline{\partial_j u^{(2)}_r} + \varrho^{(2)}\,\tau^2|u^{(2)}|^2 + (1+\nu_0\tau)(h^{(2)}_0\overline{\tau} + d^{(2)}_0)\,|\vartheta^{(2)}|^2$$

$$+ \frac{1+\nu_0\tau}{\overline{\tau}}\,\eta^{(2)}_{jl}\,\partial_l\vartheta^{(2)}\,\overline{\partial_j\vartheta^{(2)}}\,\Big\}\,dx = 0. \tag{3.11}$$

Due to the relations (2.5) and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.7), the following relations

$$c^{(\beta)}_{rjkl}\,\partial_r u^{(\beta)}_j\,\overline{\partial_k u^{(\beta)}_l}\geq\lambda_0\,\varepsilon^{(\beta)}_{kj}\,\varepsilon^{(\beta)}_{kj},\quad \eta^{(\beta)}_{jl}\,\partial_l\vartheta^{(\beta)}\,\overline{\partial_j\vartheta^{(\beta)}}\geq\lambda_0\,|\nabla\vartheta^{(\beta)}|^2,\quad \beta=1,2,$$
$$[\varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}}+a^{(1)}_{jl}\,(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}}+\partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}})+\mu^{(1)}_{jl}\,\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}]$$
$$\geq\lambda_0(|\nabla\varphi^{(1)}|^2+|\nabla\psi^{(1)}|^2), \tag{3.12}$$

hold true, where λ_0 is a positive constant and $\nabla=(\partial_1,\partial_2,\partial_3)$. Using the equalities

$$\tau^2=\sigma^2-\omega^2+2i\sigma\omega,\qquad \frac{1+\nu_0\tau}{\overline{\tau}}=\frac{\sigma+\nu_0(\sigma^2-\omega^2)}{|\tau|^2}+i\,\frac{\omega(1+2\sigma\nu_0)}{|\tau|^2},$$
$$(1+\nu_0\tau)(h^{(\beta)}_0\overline{\tau}+d^{(\beta)}_0)=d^{(\beta)}_0+\nu_0h^{(\beta)}_0|\tau|^2+(h^{(\beta)}_0+\nu_0d^{(\beta)}_0)\sigma+i\,\omega(\nu_0d^{(\beta)}_0-h^{(\beta)}_0)$$

and separating the imaginary part of relation (3.11), we find

$$\omega\,\Bigg(\sum_{\beta=1}^{2}\int\limits_{\Omega^{(\beta)}}\Big\{2\,\varrho^{(\beta)}\,\sigma\,|u^{(\beta)}|^2+(\nu_0d^{(\beta)}_0-h^{(\beta)}_0)|\vartheta^{(\beta)}|^2$$
$$+\frac{1+2\sigma\nu_0}{|\tau|^2}\,\eta^{(\beta)}_{jl}\,\partial_l\vartheta^{(\beta)}\,\overline{\partial_j\vartheta^{(\beta)}}\Big\}\,dx\Bigg)=0.$$

By inequalities (2.4) and since $\sigma>\sigma_{{}_0}\geqslant 0$, we conclude $u^{(\beta)}_j=0$ and $\vartheta^{(\beta)}=0$ in $\Omega^{(\beta)}$, $\beta=1,2$, for $\omega\neq0$. Then from (3.11) we have

$$\int\limits_{\Omega^{(1)}}\big[\varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}}+a^{(1)}_{jl}\,(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}}+\partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}})$$
$$+\mu^{(1)}_{jl}\,\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}\,\big]\,dx=0,$$

whence, in view of the last inequality in (3.12), we find $\partial_l\varphi^{(1)}=0$, $\partial_l\psi^{(1)}=0$, $l=1,2,3$, in $\Omega^{(1)}$. Thus, if $\omega\neq0$, then

$$\begin{gathered}u^{(1)}=0,\quad \varphi^{(1)}=b_1=\text{const},\quad \psi^{(1)}=b_2=\text{const},\quad \vartheta^{(1)}=0\text{ in }\Omega^{(1)},\\ u^{(2)}=0,\quad \vartheta^{(2)}=0\text{ in }\Omega^{(2)}.\end{gathered}\tag{3.13}$$

If $\omega=0$, then $\tau=\sigma>0$ and (3.11) can be rewritten in the form

$$\int\limits_{\Omega^{(1)}}\Big\{c^{(1)}_{rjkl}\partial_l u^{(1)}_k\,\overline{\partial_j u^{(1)}_r}+\varrho^{(1)}\,\sigma^2|u^{(1)}|^2+\frac{1+\nu_0\sigma}{\sigma}\,\eta^{(1)}_{jl}\,\partial_l\vartheta^{(1)}\,\overline{\partial_j\vartheta^{(1)}}\Big\}\,dx$$
$$+\int\limits_{\Omega^{(1)}}\Big\{\varkappa^{(1)}_{jl}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}}+a^{(1)}_{jl}(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}}+\partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}})+\mu^{(1)}_{jl}\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}}$$
$$-2p^{(1)}_l(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta^{(1)}\,\overline{\partial_l\varphi^{(1)}}\,\big]-2m^{(1)}_l(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta^{(1)}\,\overline{\partial_l\psi^{(1)}}\,\big]$$
$$+(1+\nu_0\sigma)(h^{(1)}_0\sigma+d^{(1)}_0)\,|\vartheta^{(1)}|^2\Big\}\,dx$$
$$+\int\limits_{\Omega^{(2)}}\Big\{c^{(2)}_{rjkl}\partial_l u^{(2)}_k\,\overline{\partial_j u^{(2)}_r}+\varrho^{(2)}\,\sigma^2|u^{(2)}|^2+\frac{1+\nu_0\sigma}{\sigma}\,\eta^{(2)}_{jl}\,\partial_l\vartheta^{(2)}\,\overline{\partial_j\vartheta^{(2)}}\Big\}\,dx$$
$$+\int\limits_{\Omega^{(2)}}\Big\{(1+\nu_0\sigma)(h^{(2)}_0\sigma+d^{(2)}_0)\,|\vartheta^{(2)}|^2\Big\}\,dx=0. \tag{3.14}$$

The integrands in the first and third integrals are nonnegative. Let us show that the integrand in the second integral is nonnegative, as well. Introducing the following notation

$$\zeta_j:=\partial_j\varphi^{(1)},\quad \zeta_{j+3}:=\partial_j\psi^{(1)},\quad \zeta_7:=-\vartheta^{(1)},\quad \zeta_8:=-\sigma\vartheta^{(1)},\quad j=1,2,3,$$

and

$$\Theta := (\zeta_1, \zeta_2, \dots, \zeta_8)^\top,$$

we deduce the relation

$$\begin{aligned}
&\varkappa_{jl}^{(1)}\,\partial_l\varphi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + a_{jl}^{(1)}(\partial_l\psi^{(1)}\,\overline{\partial_j\varphi^{(1)}} + \partial_j\varphi^{(1)}\,\overline{\partial_l\psi^{(1)}})\\
&\quad + \mu_{jl}^{(1)}\partial_l\psi^{(1)}\,\overline{\partial_j\psi^{(1)}} - 2p_l^{(1)}(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta^{(1)}\,\overline{\partial_l\varphi^{(1)}}\,\big]\\
&\quad - 2m_l^{(1)}(1+\nu_0\sigma)\,\mathrm{Re}\big[\vartheta^{(1)}\,\overline{\partial_l\psi^{(1)}}\,\big] + (1+\nu_0\sigma)(h_0^{(1)}\sigma + d_0^{(1)})\,|\vartheta^{(1)}|^2\\
&= [\varkappa_{jl}^{(1)}\,\partial_l\varphi^{(1)} + a_{jl}^{(1)}\,\partial_l\psi^{(1)} + p_j^{(1)}(-\vartheta^{(1)}) + \nu_0 p_j^{(1)}(-\sigma\vartheta^{(1)})]\,\overline{\partial_j\varphi^{(1)}}\\
&\quad + [a_{jl}^{(1)}\,\partial_l\varphi^{(1)} + \mu_{jl}^{(1)}\,\partial_l\psi^{(1)} + m_j^{(1)}(-\vartheta^{(1)}) + \nu_0 m_j^{(1)}(-\sigma\vartheta^{(1)})]\,\overline{\partial_j\psi^{(1)}}\\
&\quad + [p_l^{(1)}\partial_l\varphi^{(1)} + m_l^{(1)}\,\partial_l\psi^{(1)} + d_0^{(1)}(-\vartheta^{(1)}) + h_0^{(1)}(-\sigma\vartheta^{(1)})](-\overline{\vartheta^{(1)}})\\
&\quad + [\nu_0 p_l^{(1)}\partial_l\varphi^{(1)} + \nu_0 m_l^{(1)}\,\partial_l\psi^{(1)} + h_0^{(1)}(-\vartheta^{(1)}) + \nu_0 h_0^{(1)}(-\sigma\vartheta^{(1)})](-\sigma\overline{\vartheta^{(1)}})\\
&\quad + \sigma(d_0^{(1)}\nu_0 - h_0^{(1)})\,|\vartheta^{(1)}|^2\\
&= [\varkappa_{jl}^{(1)}\,\zeta_l + a_{jl}^{(1)}\,\zeta_{l+3} + p_j^{(1)}\zeta_7 + \nu_0 p_j^{(1)}\zeta_8)]\,\overline{\zeta}_j\\
&\quad + [a_{jl}^{(1)}\,\zeta_l + \mu_{jl}^{(1)}\,\zeta_{l+3} + m_j^{(1)}\zeta_7 + \nu_0 m_j^{(1)}\zeta_8]\,\overline{\zeta}_{j+3}\\
&\quad + [p_l^{(1)}\zeta_l + m_l^{(1)}\,\zeta_{l+3} + d_0^{(1)}\zeta_7 + h_0^{(1)}\zeta_8]\,\overline{\zeta}_7\\
&\quad + [\nu_0 p_l^{(1)}\zeta_l + \nu_0 m_l^{(1)}\,\zeta_{l+3} + h_0^{(1)}\zeta_7 + \nu_0 h_0^{(1)}\zeta_8]\,\overline{\zeta}_8\\
&\quad + \sigma(d_0^{(1)}\nu_0 - h_0^{(1)})\,|\vartheta^{(1)}|^2\\
&= \sum_{p,q=1}^{8} M_{pq}\zeta_q\,\overline{\zeta}_p + \sigma(d_0^{(1)}\nu_0 - h_0^{(1)})\,|\vartheta^{(1)}|^2\\
&= M\,\Theta\cdot\Theta + \sigma(d_0^{(1)}\nu_0 - h_0^{(1)})\,|\vartheta^{(1)}|^2 \geq C_0\,|\Theta|^2,
\end{aligned}$$

with some positive constant C_0, due to the positive definiteness of the matrix M defined by (2.6) and the inequality $\sigma(d_0^{(1)}\nu_0 - h_0^{(1)}) > 0$. Taking into account inequalities (2.4) and $\sigma > 0$, it can be easily checked that $(1+\nu_0\sigma)(h_0^{(2)}\sigma + d_0^{(2)})$ is positive, hence the integrand in the forth integral in (3.14) is also nonnegative. Therefore, from (3.14) we see that the relations (3.13) hold true for $\omega = 0$, as well.

Thus equalities (3.13) hold for arbitrary $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geqslant 0$ and $\omega \in \mathbb{R}$. On the one hand, this implies that the transmission problem $(\mathrm{TD})_\tau$ possesses only the trivial weak solution, since on the interface S due to the homogeneous boundary conditions for $\varphi^{(1)}$ and $\psi^{(1)}$, we have $b_1 = b_2 = 0$. On the other hand, a general weak solution to the transmission problem $(\mathrm{TN})_\tau$ reads as $U^{(1)} = (0,0,0,b_1,b_2,0)^\top$ and $U^{(2)} = (0,0,0,0)^\top$ with arbitrary complex constants b_1 and b_2. This completes the proof. □

Theorem 3.2. *Let the interface surface $S^{(1)}$, the exterior boundary $S^{(2)}$, and the boundary curve $\ell = \partial S_D^{(2)} = \partial S_N^{(2)}$ be Lipschitz continuous. Let $\tau = \sigma + i\omega$ with $\sigma > \sigma_0 \geq 0$ and $\omega \in \mathbb{R}$. Then the homogeneous boundary-transmission problems $(DTD)_\tau$, $(NTD)_\tau$, and $(MTD)_\tau$ have only the trivial weak solutions for p = 2, while the general weak solution to the homogeneous boundary-transmission problems $(DTN)_\tau$, $(NTN)_\tau$, and $(MTN)_\tau$ is a pair of vector functions $U^{(1)} = (0,0,0,b_1,\,b_2,0)^\top$ and $U^{(2)} = (0,0,0,0)^\top$, where b_1 and b_2 are arbitrary complex constants.*

Proof. Let a pair of vectors $(U^{(1)}, U^{(2)})$ be a weak solution to one of the homogeneous transmission problems listed in the theorem. Then for arbitrary vector-functions $U' = (u_1', u_2', u_3', \varphi', \psi', \vartheta')^\top \in [W_2^1(\Omega^{(1)})]^6$ and $V' = (v_1', v_2', v_3', \theta')^\top \in [W_2^1(\Omega^{(2)})]^4$, we have the following Green's formulas:

$$\int\limits_{\Omega^{(1)}} \mathcal{E}_\tau^{(1)}(U^{(1)}, \overline{U'})\,dx = \int\limits_{S^{(1)}} \{\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\}^+ \cdot \{U'\}^+ dS\,,$$

$$\int\limits_{\Omega^{(2)}} \mathcal{E}_\tau^{(2)}(U^{(2)},\overline{V'})\,dx = -\int\limits_{S^{(1)}} \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\}^- \cdot \{V'\}^- dS + \int\limits_{S^{(2)}} \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}\}^+ \cdot \{V'\}^+ dS\,,$$

where $\mathcal{E}_\tau^{(1)}(\cdot\,,\,\cdot)$ and $\mathcal{E}_\tau^{(2)}(\cdot\,,\,\cdot)$ are defined in (3.3) and (3.4).

The arguments used in the proof of Theorem 3.1 extend mutatis mutandis to the present case and we arrive at the relation (3.11), leading to the equalities

$$u^{(1)} = 0, \quad \varphi^{(1)} = b_1 = \text{const}, \quad \psi^{(1)} = b_2 = \text{const}, \quad \vartheta^{(1)} = 0 \text{ in } \Omega^{(1)},$$
$$u^{(2)} = 0, \quad \vartheta^{(2)} = 0 \text{ in } \Omega^{(2)},$$

where b_1 and b_2 are arbitrary complex constants. Therefore the proof of the theorem follows from the homogeneous boundary and transmission conditions. $\square$

4. Existence and Regularity Results

4.1. **Existence results for the basic transmission problem** $(TD)_\tau$**.** Let us consider the basic transmission problem $(\mathrm{TD})_\tau$ in the weak setting sense for the homogeneous differential equations

$$A^{(1)}(\partial_x,\tau)\,U^{(1)}(x,\tau) = 0\,, \quad x \in \Omega^{(1)}, \tag{4.1}$$

$$A^{(2)}(\partial_x,\tau)\,U^{(2)}(x,\tau) = 0\,, \quad x \in \Omega^{(2)}, \tag{4.2}$$

where $A^{(1)}$ and $A^{(2)}$ are defined by (2.1) and (2.13) respectively, and the sought for vectors

$$U^{(1)} = (U_1^{(1)},\dots,U_6^{(1)})^\top \in [W_p^1(\Omega^{(1)})]^6, \quad U^{(2)} = (U_1^{(2)},\dots,U_4^{(2)})^\top \in [W_p^1(\Omega^{(2)})]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)}),$$

satisfy on S (see (2.22)–(2.26)) the following transmission and Dirichlet type boundary conditions:

$$\{U_j^{(1)}(x)\}^+ - \{U_j^{(2)}(x)\}^- = f_j(x), \quad j=1,2,3, \quad x \in S, \tag{4.3}$$

$$\{U_j^{(1)}(x)\}^+ = f_j(x), \quad j=4,5, \quad x \in S, \tag{4.4}$$

$$\{U_6^{(1)}(x)\}^+ - \{U_4^{(2)}(x)\}^- = f_6(x), \quad x \in S, \tag{4.5}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}_j^+ - \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}_j^- = F_j(x), j=1,2,3, \;\; x \in S, \tag{4.6}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}_6^+ - \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}_4^- = F_6(x), \quad x \in S, \tag{4.7}$$

with $p>1$ and the data satisfying the inclusions

$$\begin{aligned} &f_j \in B_{p,p}^{1-\frac{1}{p}}(S), \quad j=1,2,3,4,5,6, \quad F_1,F_2,F_3,F_6 \in B_{p,p}^{-\frac{1}{p}}(S),\\ &S = \partial\Omega^{(1)} = \partial\Omega^{(2)} \;\text{ is a sufficiently smooth surface, say }\; S \in C^\infty. \end{aligned} \tag{4.8}$$

We will investigate the problem by the potential method. For the readers convenience, properties of the layer potentials needed in our analysis are briefly presented in Appendix.

We look for the vectors $U^{(1)}$ and $U^{(2)}$ in the form of single layer potentials associated with the operators $A^{(1)}(\partial_x,\tau)$ and $A^{(2)}(\partial_x,\tau)$ (see Appendix, formulas (5.2) and Corollary 5.4)

$$U^{(1)}(x) = V_S^{(1)}\varphi(x), \quad x \in \Omega^{(1)}, \quad \varphi = (\varphi_1,\dots,\varphi_6)^\top \in \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^6,$$

$$U^{(2)}(x) = V_S^{(2)}\psi(x), \quad x \in \Omega^{(2)}, \quad \psi = (\psi_1,\dots,\psi_4)^\top \in \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^4.$$

The transmission and boundary conditions (4.3)–(4.7) and properties of single layer potentials, presented in Appendix (see Theorems 5.1–5.3) lead then for φ and ψ on S to the following system of integral equations:

$$\sum_{l=1}^{6}[\mathcal{H}_S^{(1)}]_{jl}\,\varphi_l - \sum_{p=1}^{4}[\mathcal{H}_S^{(2)}]_{jp}\,\psi_p = f_j \text{ on } S, \;\; j=1,2,3, \tag{4.9}$$

$$\sum_{l=1}^{6}[\mathcal{H}_S^{(1)}]_{4l}\,\varphi_l = f_4 \text{ on } S, \tag{4.10}$$

$$\sum_{l=1}^{6}[\mathcal{H}_S^{(1)}]_{5l}\,\varphi_l = f_5 \text{ on } S, \tag{4.11}$$

$$\sum_{l=1}^{6}[\mathcal{H}_S^{(1)}]_{6l}\,\varphi_l - \sum_{p=1}^{4}[\mathcal{H}_S^{(2)}]_{4p}\,\psi_p = f_6 \text{ on } S, \tag{4.12}$$

$$\sum_{l=1}^{6}\Big[-\frac{1}{2}\,I_6+\mathcal{K}_S^{(1)}\Big]_{jl}\varphi_l - \sum_{p=1}^{4}\Big[\frac{1}{2}\,I_4+\mathcal{K}_S^{(2)}\Big]_{jp}\psi_p = F_j \text{ on } S, \ \ j=1,2,3, \tag{4.13}$$

$$\sum_{l=1}^{6}\Big[-\frac{1}{2}\,I_6+\mathcal{K}_S^{(1)}\Big]_{6l}\varphi_l - \sum_{p=1}^{4}\Big[\frac{1}{2}\,I_4+\mathcal{K}_S^{(2)}\Big]_{4p}\psi_p = F_6 \text{ on } S, \tag{4.14}$$

where the integral operators $\mathcal{H}_S^{(l)}$ and $\mathcal{K}_S^{(l)}$ are associated with the single layer potentials and are defined by (5.7) and (5.8), respectively.

To prove the unique solvability of the above system, we proceed as follows. Due to the invertibility of the operators (see Theorem 5.3)

$$\mathcal{H}_S^{(1)} : [B_{p,p}^{-\frac{1}{p}}(S)]^6 \to [B_{p,p}^{1-\frac{1}{p}}(S)]^6, \qquad \mathcal{H}_S^{(2)} : [B_{p,p}^{-\frac{1}{p}}(S)]^4 \to [B_{p,p}^{1-\frac{1}{p}}(S)]^4, \tag{4.15}$$

we can introduce new unknown vector functions

$$h=(h_1,\dots,h_6)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^6, \qquad g=(g_1,\dots,g_4)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^4,$$

by the relations $h := \big[\mathcal{H}_S^{(1)}\big]\varphi$ and $g := \big[\mathcal{H}_S^{(2)}\big]\psi$ implying

$$\varphi = \big[\mathcal{H}_S^{(1)}\big]^{-1}h, \quad \psi = \big[\mathcal{H}_S^{(2)}\big]^{-1}g.$$

Evidently, then we have

$$U^{(1)}(x) = V_S^{(1)}\big([\mathcal{H}_S^{(1)}]^{-1}\,h\big)(x) \ \text{ in } \ \Omega^{(1)}, \tag{4.16}$$

$$U^{(2)}(x) = V_S^{(2)}\big([\mathcal{H}_S^{(2)}]^{-1}\,g\big)(x) \ \text{ in } \ \Omega^{(2)}, \tag{4.17}$$

where $[\mathcal{H}_S^{(j)}]^{-1}$ is the inverse to the operator $\mathcal{H}_S^{(j)}$, $j=1,2$, in (4.15) (see Theorem 5.3). Note that these unknown densities are, actually, the traces on S of the sought for vectors

$$h=\big\{U^{(1)}\big\}^+, \quad g=\big\{U^{(2)}\big\}^-. \tag{4.18}$$

By Theorem 5.3, we have

$$\big\{\mathcal{T}^{(1)}V_S^{(1)}\big([\mathcal{H}_S^{(1)}]^{-1}\,h\big)\big\}^+ = \Big(-\frac{1}{2}I_6+\mathcal{K}_S^{(1)}\Big)[\mathcal{H}_S^{(1)}]^{-1}\,h \equiv \mathcal{A}_S^{(1)+}h,$$

$$\big\{\mathcal{T}^{(2)}V_S^{(2)}\big([\mathcal{H}_S^{(2)}]^{-1}\,g\big)\big\}^- = \Big(\frac{1}{2}I_4+\mathcal{K}_S^{(2)}\Big)[\mathcal{H}_S^{(2)}]^{-1}\,g \equiv \mathcal{A}_S^{(2)-}g,$$

with Steklov-Poincaré type operators (see Appendix, formulas (5.12)–(5.13))

$$\mathcal{A}_S^{(1)+} := \Big(-\frac{1}{2}I_6+\mathcal{K}_S^{(1)}\Big)[\mathcal{H}_S^{(1)}]^{-1}, \qquad \mathcal{A}_S^{(2)-} := \Big(\frac{1}{2}I_4+\mathcal{K}_S^{(2)}\Big)[\mathcal{H}_S^{(2)}]^{-1}.$$

System (4.9)–(4.14) can be rewritten then for new unknown vectors h and g as follows

$$h_j - g_j = f_j \quad j=1,2,3, \tag{4.19}$$

$$h_4 = f_4, \tag{4.20}$$

$$h_5 = f_5, \tag{4.21}$$

$$h_6 - g_4 = f_6, \tag{4.22}$$

$$\big[\mathcal{A}_S^{(1)+}h\big]_j - \big[\mathcal{A}_S^{(2)-}g\big]_j = F_j, \quad j=1,2,3, \tag{4.23}$$

$$\big[\mathcal{A}_S^{(1)+}h\big]_6 - \big[\mathcal{A}_S^{(2)-}g\big]_4 = F_6. \tag{4.24}$$

As we see, the unknowns h_4 and h_5 are uniquely defined by equations (4.20) and (4.21) and the above system can be rewritten as

$$h_j - g_j = f_j, \quad j = 1,2,3, \tag{4.25}$$

$$h_6 - g_4 = f_6, \tag{4.26}$$

$$\sum_{l=1,2,3,6}[\mathcal{A}_S^{(1)+}]_{jl}\,h_l - \sum_{l=1}^{4}[\mathcal{A}_S^{(2)-}]_{jl}\,g_l = F_j - [\mathcal{A}_S^{(1)+}]_{j4}f_4 - [\mathcal{A}_S^{(1)+}]_{j5}f_5, \quad j = 1,2,3, \tag{4.27}$$

$$\sum_{l=1,2,3,6}[\mathcal{A}_S^{(1)+}]_{6l}\,h_l - \sum_{l=1}^{4}[\mathcal{A}_S^{(2)-}]_{4l}\,g_l = F_6 - [\mathcal{A}_S^{(1)+}]_{64}f_4 - [\mathcal{A}_S^{(1)+}]_{65}\,f_5, \tag{4.28}$$

$$h_4 = f_4, \tag{4.29}$$

$$h_5 = f_5. \tag{4.30}$$

Further, let

$$\widetilde{h} := (h_1,h_2,h_3,h_6)^\top,$$

$$\widetilde{f} = (f_1,f_2,f_3,f_6)^\top \in [B_{p,p}^{1-\frac{1}{p}}(S)]^4, \quad \widetilde{F} = (\widetilde{F}_1,\widetilde{F}_2,\widetilde{F}_3,\widetilde{F}_6)^\top \in [B_{p,p}^{-\frac{1}{p}}(S)]^4, \tag{4.31}$$

$$\widetilde{F}_j = F_j - [\mathcal{A}_S^{(1)+}]_{j4}f_4 - [\mathcal{A}_S^{(1)+}]_{j5}f_5, \quad j = 1,2,3,6. \tag{4.32}$$

From equations (4.25)–(4.30), we then get

$$\widetilde{h} - g = \widetilde{f}, \tag{4.33}$$

$$\widetilde{\mathcal{A}}_S^{(1)+}\,\widetilde{h} - \mathcal{A}_S^{(2)-}\,g = \widetilde{F}, \tag{4.34}$$

$$h_4 = f_4, \tag{4.35}$$

$$h_5 = f_5, \tag{4.36}$$

where

$$\widetilde{\mathcal{A}}_S^{(1)+} := \begin{bmatrix} [\mathcal{A}_S^{(1)+}]_{11} & [\mathcal{A}_S^{(1)+}]_{12} & [\mathcal{A}_S^{(1)+}]_{13} & [\mathcal{A}_S^{(1)+}]_{16} \\ [\mathcal{A}_S^{(1)+}]_{21} & [\mathcal{A}_S^{(1)+}]_{22} & [\mathcal{A}_S^{(1)+}]_{23} & [\mathcal{A}_S^{(1)+}]_{26} \\ [\mathcal{A}_S^{(1)+}]_{31} & [\mathcal{A}_S^{(1)+}]_{32} & [\mathcal{A}_S^{(1)+}]_{33} & [\mathcal{A}^{(1)+}]_{36} \\ [\mathcal{A}_S^{(1)+}]_{61} & [\mathcal{A}_S^{(1)+}]_{62} & [\mathcal{A}_S^{(1)+}]_{63} & [\mathcal{A}_S^{(1)+}]_{66} \end{bmatrix}_{4\times 4}.$$

Finally, system (4.33)–(4.36) can be equivalently rewritten in more convenient form

$$\widetilde{h} = g + \widetilde{f}, \tag{4.37}$$

$$h_4 = f_4, \tag{4.38}$$

$$h_5 = f_5, \tag{4.39}$$

$$\big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]\,g = \widetilde{F}^*, \tag{4.40}$$

where $\widetilde{F}^* = \widetilde{F} - \widetilde{\mathcal{A}}_S^{(1)+}\widetilde{f} \in [B_{p,p}^{-\frac{1}{p}}(S)]^4$.

Thus, the solvability of system (4.19)–(4.24) is equivalently reduced to the solvability of the matrix pseudodifferential equation (4.40).

Next, we prove the following lemma.

Lemma 4.1. *The operator*

$$\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-} \,:\, [B_{p,p}^{1-\frac{1}{p}}]^4 \to [B_{p,p}^{-\frac{1}{p}}]^4, \quad p > 1, \tag{4.41}$$

is invertible.

Proof. Note that the integral operators $\mathcal{H}_S^{(j)}$ and $\pm\frac{1}{2}I^{(j)} + \mathcal{K}_S^{(j)}$ are elliptic pseudodifferential operators of order -1 and 0, respectively. This implies that $\mathcal{A}_S^{(j)\pm}$, $j = 1,2$, are also elliptic pseudodifferential operators of order $+1$ (see [7]). More detailed analysis shows that the principal homogeneous symbol matrix of the operators $\mathcal{A}_S^{(1)+}$ and $-\mathcal{A}_S^{(2)-}$ are strongly elliptic (see [7, 21]). Therefore, the operator $\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}$ is a strongly elliptic pseudodifferential operator of order +1. Therefore, due to the general theory of pseudodifferential equations on manifolds without boundary, if we show invertibility of operator (4.41) for $p = 2$, then it will imply that of operator (4.41) for all $p > 1$.

Keeping in mind that $B_{2,2}^{\pm\frac{1}{2}}(S) = H_2^{\pm\frac{1}{2}}(S)$, we have to show the invertibility of the operator

$$\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-} \,:\, [H_2^{\frac{1}{2}}(S)]^4 \to [H_2^{-\frac{1}{2}}(S)]^4. \tag{4.42}$$

Using Theorem 5.5 we easily deduce the coercivity inequalities for arbitrary vector function $\widetilde{g} = (\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4)^\top \in \big[H_2^{\frac{1}{2}}(S)\big]^4$,

$$\mathrm{Re}\,\big\langle\, \big(\widetilde{\mathcal{A}}_S^{(1)+} + \widetilde{\mathcal{C}}^{(1)}\big)\widetilde{g}, \widetilde{g}\big\rangle_S \geq C_1 \|\widetilde{g}\|^2_{[H_2^{\frac{1}{2}}(S)]^4},$$
$$\mathrm{Re}\,\big\langle\, \big(-\mathcal{A}_S^{(2)-} + \mathcal{C}^{(2)}\big)\widetilde{g}, \widetilde{g}\big\rangle_S \geq C_2 \|\widetilde{g}\|^2_{[H_2^{\frac{1}{2}}(S)]^4},$$

where $\widetilde{\mathcal{C}}^{(1)} : \big[H_2^{\frac{1}{2}}(S)\big]^4 \to \big[H_2^{-\frac{1}{2}}(S)\big]^4$ and $\mathcal{C}^{(2)} : \big[H_2^{\frac{1}{2}}(S)\big]^4 \to \big[H_2^{-\frac{1}{2}}(S)\big]^4$ are compact operators. Consequently, the following coercivity inequality

$$\mathrm{Re}\,\big\langle\, \big(\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-} + \widetilde{\mathcal{C}}^{(1)} + \mathcal{C}^{(2)}\big)\widetilde{g}, \widetilde{g}\big\rangle_S \geq C \|\widetilde{g}\|^2_{[H_2^{\frac{1}{2}}(S)]^4}, \quad C = \mathrm{const} > 0, \tag{4.43}$$

holds implying that operator (4.42) is Fredholm one with zero index (see, e.g., [38, Ch. 2]).

Further, we show that the null space of operator (4.42) is trivial.

Let $\widetilde{\varphi} = \big(\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{\varphi}_3, \widetilde{\varphi}_4\big)^\top \in \big[H_2^{\frac{1}{2}}(S)\big]^4$ be a solution to the homogeneous equation

$$\big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]\widetilde{\varphi} = 0 \quad \text{on } S \tag{4.44}$$

and construct the vectors

$$U^{(1)}(x) = V_S^{(1)}\big([\mathcal{H}_S^{(1)}]^{-1}\,\varphi\big)(x) \ \text{ in } \ \Omega^{(1)}, \tag{4.45}$$
$$U^{(2)}(x) = V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1}\,\widetilde{\varphi}\big)(x) \ \text{ in } \ \Omega^{(2)}, \tag{4.46}$$

where $\varphi = \big(\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{\varphi}_3, 0, 0, \widetilde{\varphi}_4\big)^\top \in \big[H_2^{\frac{1}{2}}(S)\big]^6$.

By Theorems 5.1–5.3, evidently, $[\mathcal{H}_S^{(1)}]^{-1}\,\varphi \in \big[H_2^{-\frac{1}{2}}(S)\big]^6$, $[\mathcal{H}_S^{(2)}]^{-1}\,\widetilde{\varphi} \in \big[H_2^{-\frac{1}{2}}(S)\big]^4$ and using the mapping properties and the jump relations of the single layer potentials, we deduce:

$$U^{(1)} \in \big[W_2^1(\Omega^{(1)})\big]^6, \quad U^{(2)} \in \big[W_{2,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)}), \tag{4.47}$$
$$\big\{U^{(1)}\big\}^+ = \big\{V_S^{(1)}\big[\mathcal{H}_S^{(1)}\big]^{-1}\,\varphi\big\}^+ = \mathcal{H}_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1}\,\varphi\big) = \varphi, \tag{4.48}$$
$$\big\{U^{(2)}\big\}^- = \big\{V_S^{(2)}\big[\mathcal{H}_S^{(2)}\big]^{-1}\,\widetilde{\varphi}\big\}^- = \mathcal{H}_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1}\,\widetilde{\varphi}\big) = \widetilde{\varphi}, \tag{4.49}$$
$$\big\{\mathcal{T}^{(1)}U^{(1)}\big\}^+ = \big\{\mathcal{T}^{(1)}V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1}\,\varphi\big)\big\}^+ = \big[-\tfrac{1}{2}I_6 + \mathcal{K}_S^{(1)}\big]\big[\mathcal{H}_S^{(1)}\big]^{-1}\,\varphi = \mathcal{A}_S^{(1)+}\varphi, \tag{4.50}$$
$$\big\{\mathcal{T}^{(2)}U^{(2)}\big\}^- = \big\{\mathcal{T}^{(2)}V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1}\,\widetilde{\varphi}\big)\big\}^- = \big[\tfrac{1}{2}I_4 + \mathcal{K}_S^{(2)}\big]\big[\mathcal{H}_S^{(2)}\big]^{-1}\,\widetilde{\varphi} = \mathcal{A}_S^{(2)-}\widetilde{\varphi}. \tag{4.51}$$

Taking into account that $\big[\mathcal{A}_S^{(1)+}\varphi\big]_j = \big[\widetilde{\mathcal{A}}_S^{(1)+}\widetilde{\varphi}\big]_j$ for $j = 1, 2, 3$, and $\big[\mathcal{A}_S^{(1)+}\varphi\big]_6 = \big[\widetilde{\mathcal{A}}_S^{(1)+}\widetilde{\varphi}\big]_4$, from relations (4.44)–(4.51), we conclude that the vectors defined by (4.45)–(4.46) solve the homogeneous differential equations

$$A^{(\beta)}(\partial_x, \tau)\,U^{(\beta)} = 0 \ \text{ in } \ \Omega^{(\beta)}, \quad \beta = 1, 2,$$

and satisfy on S the homogeneous boundary-transmission conditions

$$\{U_j^{(1)}\}^+ - \{U_j^{(2)}\}^- = 0, \quad j = 1, 2, 3,$$
$$\{U_j^{(1)}\}^+ = 0, \quad j = 4, 5,$$
$$\{U_6^{(1)}\}^+ - \{U_4^{(2)}\}^- = 0,$$
$$\{\mathcal{T}^{(1)}U^{(1)}\}_j^+ - \{\mathcal{T}^{(2)}U^{(2)}\}_j^- = 0, \quad j = 1, 2, 3,$$

$$\{\mathcal{T}^{(1)}U^{(1)}\}_6^+ - \{\mathcal{T}^{(2)}U^{(2)}\}_4^- = 0,$$

i.e., the pair $(U^{(1)}, U^{(2)})$ solves the homogeneous basic boundary-transmission problem $(\mathrm{TD})_\tau$. Therefore $U^{(1)} = 0$ in $\Omega^{(1)}$ and $U^{(2)} = 0$ in $\Omega^{(2)}$ by the uniqueness Theorem 3.1. Consequently, $\widetilde{\varphi}_j = 0$, $j = 1,2,3,4$, in view of (4.49) implying that the null space of the operator (4.42) is trivial which completes the proof of the invertibility of the operators (4.42) and (4.41). □

Note that the operator $\mathfrak{M}$ generated by the left hand side expressions of system (4.19)–(4.24) reads as

$$\mathfrak{M} = \|\mathfrak{M}_{kj}\|_{10\times 10} =$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
[\mathcal{A}_S^{(1)+}]_{11} & \cdot & \cdot & \cdot & \cdot & [\mathcal{A}_S^{(1)+}]_{16} & -[\mathcal{A}_S^{(2)-}]_{11} & \cdot & \cdot & -[\mathcal{A}_S^{(2)-}]_{14} \\
[\mathcal{A}_S^{(1)+}]_{21} & \cdot & \cdot & \cdot & \cdot & [\mathcal{A}_S^{(1)+}]_{26} & -[\mathcal{A}_S^{(2)-}]_{21} & \cdot & \cdot & -[\mathcal{A}_S^{(2)-}]_{24} \\
[\mathcal{A}_S^{(1)+}]_{31} & \cdot & \cdot & \cdot & \cdot & [\mathcal{A}_S^{(1)+}]_{36} & -[\mathcal{A}_S^{(2)-}]_{31} & \cdot & \cdot & -[\mathcal{A}_S^{(2)-}]_{34} \\
[\mathcal{A}_S^{(1)+}]_{61} & \cdot & \cdot & \cdot & \cdot & [\mathcal{A}_S^{(1)+}]_{66} & -[\mathcal{A}_S^{(2)-}]_{41} & \cdot & \cdot & -[\mathcal{A}_S^{(2)-}]_{44}
\end{bmatrix}. \tag{4.52}$$

Therefore this system can be rewritten in matrix form as follows

$$\mathfrak{M}\,\Phi = \Psi,$$

where $\mathfrak{M}$ is given by (4.52), $\Phi := (h,g)^\top$ is an unknown vector function, and Ψ is a known vector function, $\Psi := (f_1,\dots,f_6,F_1,F_2,F_3,F_4)^\top$.

The above results imply the following assertions.

Lemma 4.2. *Systems of integral equations* (4.9)–(4.14) *and* (4.19)–(4.24) *are uniquely solvable in the spaces* $\big[B_{p,p}^{-\frac{1}{p}}(S)\big]^6 \times \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^4$ *and* $\big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6 \times \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^4$*, respectively, for arbitrary right-hand side functions satisfying conditions* (4.8).

Proof. Follows from Lemma 4.1 and equivalence of systems (4.9)–(4.14), (4.19)–(4.24), and (4.37)–(4.40). □

Lemma 4.3. *The operator*

$$\mathfrak{M} : \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^{10} \to \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6 \times \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^4$$

is invertible.

Proof. Follows from Lemmas 4.1 and 4.2 and the structure of the operator (4.52). □

From (4.37)–(4.40), for the solution vectors $h \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6$ and $g \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^4$, we derive the following relations:

$$\widetilde{h} = (h_1,h_2,h_3,h_6)^\top = \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\widetilde{F} - \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\mathcal{A}_S^{(2)-}\widetilde{f}, \tag{4.53}$$

$$h_4 = f_4, \tag{4.54}$$

$$h_5 = f_5, \tag{4.55}$$

$$g = (g_1,g_2,g_3,g_4)^\top = \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\widetilde{F} - \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\widetilde{\mathcal{A}}_S^{(1)+}\widetilde{f}. \tag{4.56}$$

Introduce the notation

$$\widetilde{\mathcal{Q}} = \big[\widetilde{\mathcal{Q}}_{kj}\big]_{4\times 4} := \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1},$$

$$\widetilde{\mathcal{R}} = \big[\widetilde{\mathcal{R}}_{kj}\big]_{4\times 4} := \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\mathcal{A}_S^{(2)-},$$

$$\widetilde{\mathcal{M}} = \big[\widetilde{\mathcal{M}}_{kj}\big]_{4\times 4} := \big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]^{-1}\widetilde{\mathcal{A}}_S^{(1)+},$$

and construct the following matrix operators

$$\mathcal{Q} := \begin{bmatrix} \widetilde{\mathcal{Q}}_{11} & \widetilde{\mathcal{Q}}_{12} & \widetilde{\mathcal{Q}}_{13} & 0 & 0 & \widetilde{\mathcal{Q}}_{14} \\ \widetilde{\mathcal{Q}}_{21} & \widetilde{\mathcal{Q}}_{22} & \widetilde{\mathcal{Q}}_{23} & 0 & 0 & \widetilde{\mathcal{Q}}_{24} \\ \widetilde{\mathcal{Q}}_{31} & \widetilde{\mathcal{Q}}_{32} & \widetilde{\mathcal{Q}}_{33} & 0 & 0 & \widetilde{\mathcal{Q}}_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{\mathcal{Q}}_{41} & \widetilde{\mathcal{Q}}_{42} & \widetilde{\mathcal{Q}}_{43} & 0 & 0 & \widetilde{\mathcal{Q}}_{44} \end{bmatrix}_{6\times 6},$$

$$\mathcal{R} := \begin{bmatrix} \widetilde{\mathcal{R}}_{11} & \widetilde{\mathcal{R}}_{12} & \widetilde{\mathcal{R}}_{13} & 0 & 0 & \widetilde{\mathcal{R}}_{14} \\ \widetilde{\mathcal{R}}_{21} & \widetilde{\mathcal{R}}_{22} & \widetilde{\mathcal{R}}_{23} & 0 & 0 & \widetilde{\mathcal{R}}_{24} \\ \widetilde{\mathcal{R}}_{31} & \widetilde{\mathcal{R}}_{32} & \widetilde{\mathcal{R}}_{33} & 0 & 0 & \widetilde{\mathcal{R}}_{34} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \widetilde{\mathcal{R}}_{41} & \widetilde{\mathcal{R}}_{42} & \widetilde{\mathcal{R}}_{43} & 0 & 0 & \widetilde{\mathcal{R}}_{44} \end{bmatrix}_{6\times 6}.$$

The relations (4.53)–(4.56) can be rewritten then in the form

$$h = \mathcal{Q}F - \mathcal{R}f \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^6, \quad g = \widetilde{\mathcal{Q}}\widetilde{F} - \widetilde{\mathcal{M}}\widetilde{f} \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^4,$$

where $F = \big(\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, 0, 0, \widetilde{F}_6\big)^\top$, $f = (f_1, f_2, f_3, f_4, f_5, f_6)^\top$, the vectors $\widetilde{F}$ and $\widetilde{f}$ are defined by (4.31)–(4.32). Consequently, from (4.16) and (4.17), we get the following representation of the solution vectors,

$$U^{(1)} = V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1} h\big) = V_S^{(1)}\big([\mathcal{H}_S^{(1)}]^{-1}\,(\mathcal{Q}F - \mathcal{R}f)\big) \ \text{ in } \ \Omega^{(1)}, \tag{4.57}$$

$$U^{(2)} = V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1} g\big) = V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1}\,\big(\widetilde{\mathcal{Q}}\widetilde{F} - \widetilde{\mathcal{M}}\widetilde{f}\big)\big) \ \text{ in } \ \Omega^{(2)}. \tag{4.58}$$

Finally, let us formulate the following existence result.

Theorem 4.4. *Let conditions* (4.8) *be fulfilled. Then the basic transmission problem* $(TD)_\tau$, (4.1)–(4.7), *is uniquely solvable in the space* $\big[W_p^1(\Omega^{(1)})\big]^6 \times \Big(\big[W_{p,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$ *for* $p > 1$ *and the solution pair of vectors* $(U^{(1)}, U^{(2)})$ *is representable in the form of single layer potentials* (4.57)–(4.58).

Proof. The existence of a solution in the space $\big[W_p^1(\Omega^{(1)})\big]^6 \times \Big(\big[W_{p,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$ for $p > 1$ follows from the representation (4.57)–(4.58) and Lemmas 4.2 and 4.3. For $p = 2$, the solution is unique due to Theorem 3.1. To show the uniqueness for $p \neq 2$, we proceed as follows. Let a pair

$$(U^{(1)}, U^{(2)}) \in \big[W_p^1(\Omega^{(1)})\big]^6 \times \Big(\big[W_{p,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big) \ \text{ for } \ p \neq 2$$

be a solution to the homogeneous basic transmission problem $(\mathrm{TD})_\tau$. Due to Corollary 5.4, $U^{(1)}$ and $U^{(2)}$ are uniquely representable in the form of single layer potentials (4.16) and (4.17), respectively, where the densities h and g are the traces on S of the vectors $U^{(1)}$ and $U^{(2)}$ (see (4.18)). Therefore, in view of the homogenous boundary-transmission conditions on S, with the help of the above-employed arguments we arrive at the homogeneous system of equations on S (cf. (4.37)–(4.40)):

$$\begin{aligned} &\widetilde{h} - g = 0, \\ &h_4 = 0, \\ &h_5 = 0, \\ &\big[\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\big]\, g = 0. \end{aligned}$$

Due to the invertibility of the operator (4.41) (see Lemma 4.1), we deduce that $g = 0$ and $h = 0$ on S implying $U^{(1)} = 0$ in $\Omega^{(1)}$ and $U^{(2)} = 0$ in $\Omega^{(2)}$. This completes the proof. □

Corollary 4.5. *Let S be the Lipschitz one and $p=2$. Then the basic transmission problem (TD)$_\tau$, (4.1)–(4.8) is uniquely solvable in the space $\big[W_2^1(\Omega^{(1)})\big]^6 \times \Big(\big[W_{2,\mathrm{loc}}^1(\Omega^{(2)})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$ and the solution pair of vectors $(U^{(1)}, U^{(2)})$ is representable again in the form (4.57)–(4.58).*

Proof. We seek for a solution again in the form of single layer potentials (4.16) and (4.17). Using the properties of the single layer potentials presented in Theorem 5.2, the problem is then again reduced to system (4.37)–(4.40) and the coercivity inequality (4.43) leads to the invertibility of the operator (4.42) which completes the proof. □

We have the following regularity result.

Corollary 4.6. *Let $S \in C^{m,\alpha'}$ with $0<\alpha<\alpha'\leqslant 1$ and $m\geqslant 2$ being an integer. Further, let*

$$f_j \in C^{k,\alpha}(S),\ \ j=1,\dots,6,\ \ F_1,F_2,F_3,F_6 \in C^{k-1,\alpha}(S),\ \ 1\leqslant k\leqslant m-1.$$

Then the basic transmission problem (TD)$_\tau$, (4.1)–(4.7), is uniquely solvable in the space $\big[C^{k,\alpha}(\overline{\Omega^{(1)}})\big]^6\times \Big(\big[C^{k,\alpha}(\overline{\Omega^{(2)}})\big]^4 \cap \mathbf{Z}_\tau(\Omega^{(2)})\Big)$ and the solution pair of vectors $(U^{(1)}, U^{(2)})$ is representable again in the form (4.57)–(4.58).

Proof. The existence of a unique weak solution representable in the form (4.57)–(4.58) follows from Theorem 4.4. On the other hand, Lemma 4.1 implies that the strongly elliptic pseudodifferential operator

$$\widetilde{\mathcal{A}}_S^{(1)+} - \mathcal{A}_S^{(2)-}\ :\ [C^{k,\alpha}(S)]^4 \to [C^{k-1,\alpha}(S)]^4$$

is invertible. This implies that solution vectors to system (4.37)–(4.40) satisfy the inclusions $h\in[C^{k,\alpha}(S)]^6$ and $g\in[C^{k,\alpha}(S)]^4$. The regularity result then follows from the representation (4.16)–(4.17) (and from (4.57)–(4.58) as well) and the mapping properties of the single layer potentials and the corresponding boundary operators described in Theorem 5.1. □

4.2. **Existence results for the boundary-transmission problem (DTD)$_\tau$.** In this subsection we consider a bounded composite structure $\Omega^{(1)}\cup\Omega^{(2)}$ introduced in Subsection 2.4. Recall that $S^{(1)}$ is the interface between the interior domain $\Omega^{(1)}$ and the exterior domain $\Omega^{(2)}$ and $S^{(2)}$ is the exterior boundary of the composite body. In the region $\Omega^{(1)}$ we have the GTEME model and in the region $\Omega^{(2)}$ the GTE model. Evidently, $\partial\Omega^{(1)}=S^{(1)}$ and $\partial\Omega^{(2)}=S^{(1)}\cup S^{(2)}$, $\overline{\Omega^{(1)}}=\Omega^{(1)}\cup S^{(1)}$, $\overline{\Omega^{(2)}}=\Omega^{(2)}\cup S^{(1)}\cup S^{(2)}$. For simplicity, let us assume that $S^{(1)},S^{(2)}\in C^\infty$.

We will investigate the boundary-transmission problem (DTD)$_\tau$ in the weak setting sense for the homogeneous differential equations

$$A^{(j)}(\partial_x,\tau)\,U^{(j)}(x,\tau)=0\,,\quad x\in\Omega^{(j)},\qquad j=1,2,$$

where the differential operators $A^{(1)}(\partial_x,\tau)$ and $A^{(2)}(\partial_x,\tau)$ are defined by (2.1) and (2.13) respectively, and the sought for vectors

$$U^{(1)}=(U_1^{(1)},\dots,U_6^{(1)})^\top\in[W_p^1(\Omega^{(1)})]^6,\quad U^{(2)}=(U_1^{(2)},\dots,U_4^{(2)})^\top\in[W_p^1(\Omega^{(2)})]^4,$$

satisfy on the interface $S^{(1)}$ (see (2.22)–(2.26)) the following transmission conditions:

$$\{U_j^{(1)}(x)\}^+ - \{U_j^{(2)}(x)\}^- = f_j(x),\quad j=1,2,3,\quad x\in S^{(1)}, \tag{4.59}$$

$$\{U_6^{(1)}(x)\}^+ - \{U_4^{(2)}(x)\}^- = f_6(x),\quad x\in S^{(1)}, \tag{4.60}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}_j^+ - \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}_j^- = F_j(x),\quad j=1,2,3,\ \ x\in S^{(1)}, \tag{4.61}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}_6^+ - \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}_4^- = F_6(x),\quad x\in S^{(1)}, \tag{4.62}$$

$$\{U_j^{(1)}(x)\}^+ = f_j(x),\quad j=4,5,\quad x\in S^{(1)}, \tag{4.63}$$

and on the exterior boundary $S^{(2)}$ the Dirichlet boundary conditions:

$$\{U_j^{(2)}(x)\}^+ = f_j^*(x),\quad j=1,2,3,4,\quad x\in S^{(2)}. \tag{4.64}$$

The data of the problem satisfy the inclusions

$$\begin{aligned} &f_j \in B_{p,p}^{1-\frac{1}{p}}(S^{(1)}), \quad j=1,2,3,4,5,6, \quad F_1, F_2, F_3, F_6 \in B_{p,p}^{-\frac{1}{p}}(S^{(1)}),\\ &f_j^* \in B_{p,p}^{1-\frac{1}{p}}(S^{(2)}), \quad j=1,2,3,4. \end{aligned} \tag{4.65}$$

We look for solutions $U^{(1)}$ and $U^{(2)}$ in the form of a linear combination of single layer potentials associated with the operators $A^{(1)}$ and $A^{(2)}$ and constructed by the corresponding fundamental matrices $\Gamma^{(1)}$ and $\Gamma^{(2)}$ defined by (5.1), respectively:

$$U^{(1)}(x) = V_{S^{(1)}}^{(1)}\varphi^{(1)}(x), \quad x \in \Omega^{(1)}, \tag{4.66}$$

$$U^{(2)}(x) = V_{S^{(1)}}^{(2)}\psi^{(1)}(x) + V_{S^{(2)}}^{(2)}\psi^{(2)}(x), \quad x \in \Omega^{(2)}, \tag{4.67}$$

where

$$\begin{aligned} &\varphi^{(1)} = (\varphi_1^{(1)}, \varphi_2^{(1)}, \varphi_3^{(1)}, \varphi_4^{(1)}, \varphi_5^{(1)}, \varphi_6^{(1)})^\top \in [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^6,\\ &\psi^{(1)} = (\psi_1^{(1)}, \psi_2^{(1)}, \psi_3^{(1)}, \psi_4^{(1)})^\top \in [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^4,\\ &\psi^{(2)} = (\psi_1^{(2)}, \psi_2^{(2)}, \psi_3^{(2)}, \psi_4^{(2)})^\top \in [B_{p,p}^{-\frac{1}{p}}(S^{(2)})]^4, \end{aligned}$$

are unknown density vector functions.

The properties of single layer potentials and the boundary-transmission conditions (4.59)–(4.64) lead to the following system of pseudodifferential equations for $\varphi^{(1)}$, $\psi^{(1)}$, and $\psi^{(2)}$:

$$\sum_{l=1}^{6}[\mathcal{H}_{S^{(1)}}^{(1)}]_{jl}\,\varphi_l^{(1)} - \sum_{p=1}^{4}[\mathcal{H}_{S^{(1)}}^{(2)}]_{jp}\,\psi_p^{(1)} - [\gamma_{S^{(1)}}^{-}\{V_{S^{(2)}}^{(2)}\psi^{(2)}\}]_j = f_j, \ j=1,2,3, \text{ on } S^{(1)}, \tag{4.68}$$

$$\sum_{l=1}^{6}[\mathcal{H}_{S^{(1)}}^{(1)}]_{4l}\,\varphi_l^{(1)} = f_4 \text{ on } S^{(1)}, \tag{4.69}$$

$$\sum_{l=1}^{6}[\mathcal{H}_{S^{(1)}}^{(1)}]_{5l}\,\varphi_l^{(1)} = f_5 \text{ on } S^{(1)}, \tag{4.70}$$

$$\sum_{l=1}^{6}[\mathcal{H}_{S^{(1)}}^{(1)}]_{6l}\,\varphi_l^{(1)} - \sum_{p=1}^{4}[\mathcal{H}_{S^{(1)}}^{(2)}]_{4p}\,\psi_p^{(1)} - [\gamma_{S^{(1)}}^{-}\{V_{S^{(2)}}^{(2)}\psi^{(2)}\}]_4 = f_6 \text{ on } S^{(1)}, \tag{4.71}$$

$$\begin{aligned} \sum_{l=1}^{6}\Big[-\frac{1}{2}\,I_6 + \mathcal{K}_{S^{(1)}}^{(1)}\Big]_{jl}\,\varphi_l^{(1)} - \sum_{p=1}^{4}\Big[\frac{1}{2}\,I_4 + \mathcal{K}_{S^{(1)}}^{(2)}\Big]_{jp}\,\psi_p^{(1)}&\\ - [\gamma_{S^{(1)}}^{-}\{\mathcal{T}^{(2)}V_{S^{(2)}}^{(2)}\psi^{(2)}\}]_j = F_j, \quad j=1,2,3, \text{ on } S^{(1)},& \end{aligned} \tag{4.72}$$

$$\begin{aligned} \sum_{l=1}^{6}\Big[-\frac{1}{2}\,I_6 + \mathcal{K}_{S^{(1)}}^{(1)}\Big]_{6l}\,\varphi_l^{(1)} - \sum_{p=1}^{4}\Big[\frac{1}{2}\,I_4 + \mathcal{K}_{S^{(1)}}^{(2)}\Big]_{4p}\,\psi_p^{(1)}&\\ - [\gamma_{S^{(1)}}^{-}\{\mathcal{T}^{(2)}V_{S^{(2)}}^{(2)}\psi^{(2)}\}]_4 = F_6 \text{ on } S^{(1)},& \end{aligned} \tag{4.73}$$

$$\sum_{p=1}^{4}[\mathcal{H}_{S^{(2)}}^{(2)}]_{jp}\,\psi_p^{(2)} + [\gamma_{S^{(2)}}^{+}\{V_{S^{(1)}}^{(2)}\psi^{(1)}\}]_j = f_j^*, \quad j=1,2,3,4, \text{ on } S^{(2)}, \tag{4.74}$$

where $\gamma_{S^{(j)}}^{\pm}$ denote one-sided traces on $S^{(j)}$, $j=1,2$. The integral operators $\mathcal{H}_{S^{(j)}}^{(l)}$ and $\mathcal{K}_{S^{(j)}}^{(l)}$ are associated with the single layer potentials and are defined by (5.7) and (5.8), respectively.

To prove the unique solvability of the above system, we proceed as follows. Due to the invertibility of the operators

$$\begin{aligned} &\mathcal{H}_{S^{(1)}}^{(1)} : [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^6 \to [B_{p,p}^{1-\frac{1}{p}}(S^{(1)})]^6,\\ &\mathcal{H}_{S^{(j)}}^{(2)} : [B_{p,p}^{-\frac{1}{p}}(S^{(j)})]^4 \to [B_{p,p}^{1-\frac{1}{p}}(S^{(j)})]^4, \ \ j=1,2, \end{aligned}$$

we can introduce new unknown vector functions

$$h^{(1)} = (h_1^{(1)}, \dots, h_6^{(1)})^\top \in [B_{p,p}^{1-\frac{1}{p}}(S^{(1)})]^6,$$
$$g^{(1)} = (g_1^{(1)}, \dots, g_4^{(1)})^\top \in [B_{p,p}^{1-\frac{1}{p}}(S^{(1)})]^4,$$
$$g^{(2)} = (g_1^{(2)}, \dots, g_4^{(2)})^\top \in [B_{p,p}^{1-\frac{1}{p}}(S^{(2)})]^4,$$

by the relations

$$\varphi^{(1)} = \big[\mathcal{H}_{S^{(1)}}^{(1)}\big]^{-1} h^{(1)}, \quad \psi^{(1)} = \big[\mathcal{H}_{S^{(1)}}^{(2)}\big]^{-1} g^{(1)}, \quad \psi^{(2)} = \big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1} g^{(2)}.$$

Then

$$U^{(1)}(x) = V_{S^{(1)}}^{(1)}\big(\big[\mathcal{H}_{S^{(1)}}^{(1)}\big]^{-1} h^{(1)}\big)(x), \quad x \in \Omega^{(1)},$$
$$U^{(2)}(x) = V_{S^{(1)}}^{(2)}\big(\big[\mathcal{H}_{S^{(1)}}^{(2)}\big]^{-1} g^{(1)}\big)(x) + V_{S^{(2)}}^{(2)}\big(\big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1} g^{(2)}\big)(x), \quad x \in \Omega^{(2)},$$

and system (4.68)–(4.74) can be rewritten as follows

$$h_j^{(1)} - g_j^{(1)} - \big[\gamma_{S^{(1)}}^{-}\big\{V_{S^{(2)}}^{(2)}\big(\big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1} g^{(2)}\big)\big\}\big]_j = f_j, \ j = 1,2,3, \ \text{on } S^{(1)}, \tag{4.75}$$

$$h_4^{(1)} = f_4 \text{ on } S^{(1)}, \tag{4.76}$$

$$h_5^{(1)} = f_5 \text{ on } S^{(1)}, \tag{4.77}$$

$$h_6^{(1)} - g_4^{(1)} - \big[\gamma_{S^{(1)}}^{-}\big\{V_{S^{(2)}}^{(2)}\big(\big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1} g^{(2)}\big)\big\}\big]_4 = f_6 \text{ on } S^{(1)}, \tag{4.78}$$

$$\sum_{l=1}^{6} \big[\mathcal{A}_{S^{(1)}}^{(1)+}\big]_{jl} h_l^{(1)} - \sum_{p=1}^{4} \big[\mathcal{A}_{S^{(1)}}^{(2)-}\big]_{jp} g_p^{(1)} - \big[\gamma_{S^{(1)}}^{-}\big\{\mathcal{T}^{(2)} V_{S^{(2)}}^{(2)}\big(\big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1} g^{(2)}\big)\big\}\big]_j = F_j, \tag{4.79}$$
$$j = 1,2,3, \ \text{on } S^{(1)},$$

$$\sum_{l=1}^{6} \big[\mathcal{A}_{S^{(1)}}^{(1)+}\big]_{6l} h_l^{(1)} - \sum_{p=1}^{4} \big[\mathcal{A}_{S^{(1)}}^{(2)-}\big]_{4p} g_p^{(1)} - \big[\gamma_{S^{(1)}}^{-}\big\{\mathcal{T}^{(2)} V_{S^{(2)}}^{(2)}\big(\big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1} g^{(2)}\big)\big\}\big]_4 = F_6 \text{ on } S^{(1)}, \tag{4.80}$$

$$g_j^{(2)} + \big[\gamma_{S^{(2)}}^{+}\big\{V_{S^{(1)}}^{(2)}\big(\big[\mathcal{H}_{S^{(1)}}^{(2)}\big]^{-1} g^{(1)}\big)\big\}\big]_j = f_j^*, \quad j = 1,2,3,4, \ \text{on } S^{(2)}, \tag{4.81}$$

where $\mathcal{A}_{S^{(1)}}^{(1)+}$ and $\mathcal{A}_{S^{(1)}}^{(2)-}$ are the Steklov-Poincaré type operators associated with the interface manifold $S^{(1)}$ (see Appendix, formulas (5.12)–(5.13))

$$\mathcal{A}_{S^{(1)}}^{(1)+} := \Big(-\frac{1}{2} I_6 + \mathcal{K}_{S^{(1)}}^{(1)}\Big)[\mathcal{H}_{S^{(1)}}^{(1)}]^{-1}, \qquad \mathcal{A}_{S^{(1)}}^{(2)-} := \Big(\frac{1}{2} I_4 + \mathcal{K}_{S^{(1)}}^{(2)}\Big)[\mathcal{H}_{S^{(1)}}^{(2)}]^{-1}.$$

Note that the traces of the potentials

$$\gamma_{S^{(1)}}^{-}\big\{V_{S^{(2)}}^{(2)}\big(\big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1} g^{(2)}\big)\big\}, \ \gamma_{S^{(1)}}^{-}\big\{\mathcal{T}^{(2)} V_{S^{(2)}}^{(2)}\big(\big[\mathcal{H}_{S^{(2)}}^{(2)}\big]^{-1} g^{(2)}\big)\big\}, \ \gamma_{S^{(2)}}^{+}\big\{V_{S^{(1)}}^{(2)}\big(\big[\mathcal{H}_{S^{(1)}}^{(2)}\big]^{-1} g^{(1)}\big)\big\}, \tag{4.82}$$

are smoothing operators, since $S^{(1)}$ and $S^{(2)}$ are disjoint surfaces, $S^{(1)} \cap S^{(2)} = \varnothing$.

Therefore the operator $\mathfrak{D}$ generated by the left-hand side expressions in system (4.75)–(4.81) can be written as follows:

$$\mathfrak{D} = \|\mathfrak{D}_{kj}\|_{14\times 14} = \mathfrak{N} + \mathfrak{L}, \tag{4.83}$$

with

$$\mathfrak{N} = \|\mathfrak{N}_{kj}\|_{14\times 14} = \begin{bmatrix} \mathfrak{M} & [0]_{10\times 4} \\ [0]_{4\times 10} & I_4 \end{bmatrix}_{14\times 14},$$

where the operator $\mathfrak{M} = [\mathfrak{M}_{kj}]_{10\times 10}$ is given by (4.52) with $S^{(1)}$ for S, $I_4 = [\delta_{kj}]_{4\times 4}$ is the unit matrix, and $\mathfrak{L} = [\mathfrak{L}_{kj}]_{14\times 14}$ is infinitely smoothing operator generated by the summands of system (4.75)–(4.81) involving operators (4.82).

For the 14-dimensional unknown vector function Φ and for the known vector function Ψ constructed by the transmission and boundary data, we introduce the following notation:

$$\Phi := \big(h^{(1)}, \, g^{(1)}, \, g^{(2)}\big)^\top \in \mathbb{X}_p,$$

$$\Psi := \big(f_1,\, f_2,\, f_3,\, f_4,\, f_5,\, f_6,\; F_1,\, F_2,\, F_3,\, F_6,\; f_1^*,\, f_2^*,\, f_3^*,\, f_4^*\big)^\top \in \mathbb{Y}_p,$$

where

$$\mathbb{X}_p := [B_{p,p}^{1-\frac{1}{p}}(S^{(1)})]^{10} \times [B_{p,p}^{1-\frac{1}{p}}(S^{(2)})]^4,$$
$$\mathbb{Y}_p := [B_{p,p}^{1-\frac{1}{p}}(S^{(1)})]^{6} \times [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^4 \times [B_{p,p}^{1-\frac{1}{p}}(S^{(2)})]^4.$$

The system of equations (4.75)–(4.81) can be then rewritten in the matrix form

$$\mathfrak{D}\Phi = \Psi, \ \text{ i.e., } \big(\mathfrak{N} + \mathfrak{L}\big)\Phi = \Psi.$$

Evidently, we have the following mapping properties

$$\mathfrak{N} : \mathbb{X}_p \to \mathbb{Y}_p, \tag{4.84}$$
$$\mathfrak{L} : \mathbb{X}_p \to \mathbb{Y}_p. \tag{4.85}$$

Now we prove the following

Lemma 4.7. *The operator*

$$\mathfrak{N} + \mathfrak{L} : \mathbb{X}_p \to \mathbb{Y}_p \tag{4.86}$$

is invertible.

Proof. Note that operator (4.85) is compact due to the above mentioned smoothing property of the operator $\mathfrak{L}$. On the other hand, in view of Lemma 4.3 and relation (4.83), we conclude that the operator (4.84) is invertible. Therefore operator (4.86) is the Fredholm one with zero index. Let us show that the null-space of the operator (4.86) is trivial which will complete the proof. To this end, let us assume that $\widetilde{\Phi} = \big(\widetilde{h}^{(1)},\, \widetilde{g}^{(1)},\, \widetilde{g}^{(2)}\big)^\top \in \mathbb{X}_p$ is a solution to the homogeneous equation $\big(\mathfrak{N} + \mathfrak{L}\big)\Phi = 0$. Then in accordance with relations (4.75)–(4.81) the pair of vector functions

$$U^{(1)}(x) = V^{(1)}_{S^{(1)}}\big(\big[\mathcal{H}^{(1)}_{S^{(1)}}\big]^{-1}\widetilde{h}^{(1)}\big)(x), \quad x \in \Omega^{(1)},$$
$$U^{(2)}(x) = V^{(2)}_{S^{(1)}}\big(\big[\mathcal{H}^{(2)}_{S^{(1)}}\big]^{-1}\widetilde{g}^{(1)}\big)(x) + V^{(2)}_{S^{(2)}}\big(\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}\widetilde{g}^{(2)}\big)(x), \quad x \in \Omega^{(2)},$$

solve the homogeneous boundary-transmission problem $(\mathrm{DTD})_\tau$. Therefore $U^{(1)} = 0$ in $\Omega^{(1)}$ and $U^{(2)} = 0$ in $\Omega^{(2)}$ by the uniqueness Theorem 3.2. Using the continuity property of the single layer potentials across the integration surface and the uniqueness theorems for the interior and exterior Dirichlet problems for the operators $A^{(j)}(\partial_x, \tau)$, $j = 1, 2$, we deduce that $V^{(1)}_{S^{(1)}}\big(\big[\mathcal{H}^{(1)}_{S^{(1)}}\big]^{-1}\widetilde{h}^{(1)}\big)(x) = 0$ and $V^{(2)}_{S^{(1)}}\big(\big[\mathcal{H}^{(2)}_{S^{(1)}}\big]^{-1}\widetilde{g}^{(1)}\big)(x) + V^{(2)}_{S^{(2)}}\big(\big[\mathcal{H}^{(2)}_{S^{(2)}}\big]^{-1}\widetilde{g}^{(2)}\big)(x) = 0$ in the whole space $\mathbb{R}^3$ (see, [7, Theorems 2.25 and 2.26]). By the jump relations presented in Theorem 5.3, we finally conclude that $\widetilde{h}^{(1)} = 0$ on $S^{(1)}$, $\widetilde{g}^{(1)} = 0$ on $S^{(1)}$, and $\widetilde{g}^{(2)} = 0$ on $S^{(2)}$, which completes the proof. □

This lemma implies directly the following assertion.

Lemma 4.8. *Let conditions* (4.65) *be satisfied with* $p > 1$. *The systems of pseudodifferential equations* (4.68)–(4.74) *and* (4.75)–(4.81) *be uniquely solvable in appropriate function spaces for arbitrary right-hand side functions.*

We now can prove the existence and regularity theorems of solutions to the problem $(DTD)_\tau$.

Theorem 4.9. *Let conditions* (4.65) *be satisfied with* $p > 1$. *Then the boundary-transmission problem* $(DTD)_\tau$ *is uniquely solvable in the spaces* $[W^1_p(\Omega^{(1)})]^6 \times [W^1_p(\Omega^{(2)})]^4$ *and the solution vectors* $U^{(j)}$, $j = 1, 2$, *are representable in the form of a linear combination of single layer potentials* (4.66)–(4.67), *where the density vectors* $\varphi^{(1)} \in [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^6$, $\psi^{(1)} \in [B_{p,p}^{-\frac{1}{p}}(S^{(1)})]^4$ *and* $\psi^{(2)} \in [B_{p,p}^{-\frac{1}{p}}(S^{(2)})]^4$ *are defined from the uniquely solvable system of pseudodifferential equations* (4.68)–(4.74).

Proof. Is word for word similar to that of Theorem 4.4. □

Corollary 4.10. *Let $S^{(1)}, S^{(2)} \in C^{m,\alpha'}$ with $0<\alpha<\alpha'\leqslant 1$ and $m\geqslant 2$ being an integer. Further, let*

$$f_j \in C^{k,\alpha}(S^{(1)}),\ \ j=1,\dots,6,\ \ F_1,F_2,F_3,F_6\in C^{k-1,\alpha}(S^{(1)}),$$
$$f_j^* \in C^{k,\alpha}(S^{(2)}),\ \ j=1,\dots,4,\ \ 1\leqslant k\leqslant m-1.$$

Then the transmission problem (DTD)$_\tau$ is uniquely solvable in the space $\big[C^{k,\alpha}(\overline{\Omega^{(1)}})\big]^6\times\big[C^{k,\alpha}(\overline{\Omega^{(2)}})\big]^4$ and the solution pair of vectors $(U^{(1)},U^{(2)})$ is representable in the form (4.66)–(4.67).

Proof. Is word for word similar to that of Corollary 4.6. □

5. Appendix

Here we collect some results from references [7, 21], and [22] which are employed in the main text of the present paper.

Fundamental matrices $\Gamma^{(j)}(x,\tau)$ of the operators $A^{(j)}(\partial_x,\tau)$, $j=1,2$, can be constructed with the help of the Fourier transform

$$\Gamma^{(j)}(x,\tau)=\mathcal{F}^{-1}_{\xi\to x}\big[\big(A^{(j)}(-i\,\xi,\tau)\big)^{-1}\big], \tag{5.1}$$

where $\big(A^{(j)}(-i\,\xi,\tau)\big)^{-1}$ is the matrix inverse to $A(-i\,\xi,\tau)$, and $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse distributional Fourier transforms in the space of tempered distributions which for regular summable functions f and g read as follows

$$\mathcal{F}_{x\to\xi}[f]=\int\limits_{\mathbb{R}^3} f(x)\,e^{i\,x\cdot\xi}dx,\qquad \mathcal{F}^{-1}_{\xi\to x}[g]=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} g(\xi)\,e^{-i\,x\cdot\xi}d\xi,$$

where $x=(x_1,x_2,x_3)$ and $\xi=(\xi_1,\xi_2,\xi_3)$.

These fundamental matrices solve the following distributional equations

$$A^{(j)}(\partial_x,\tau)\Gamma^{(j)}(x,\tau)=I^{(j)}\,\delta(x),$$

where $I^{(1)}=I_6$ and $I^{(2)}=I_4$ are 6×6 and 4×4 unit matrices and $\delta(x)$ is Dirac's distribution.

The entries of the matrices $\Gamma^{(1)}(x,\tau)$ and $\Gamma^{(2)}(x,\tau)$ in the vicinity of the origin have the property

$$\Gamma^{(1)}(x,\tau)=\begin{bmatrix}[\mathcal{O}(|x|^{-1})]_{5\times5} & [\mathcal{O}(1)]_{5\times1}\\ [\mathcal{O}(1)]_{1\times5} & \mathcal{O}(|x|^{-1})\end{bmatrix}_{6\times6},$$

$$\Gamma^{(2)}(x,\tau)=\begin{bmatrix}[\mathcal{O}(|x|^{-1})]_{3\times3} & [\mathcal{O}(1)]_{3\times1}\\ [\mathcal{O}(1)]_{1\times3} & \mathcal{O}(|x|^{-1})\end{bmatrix}_{4\times4},$$

while at infinity they have the following asymptotic behaviour

$$\Gamma^{(1)}(x,\tau)=\begin{bmatrix}
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2})\\
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2})\\
\mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-3})
\end{bmatrix}_{6\times6},$$

$$\Gamma^{(2)}(x,\tau)=\begin{bmatrix}
\mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4})\\
\mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-3})
\end{bmatrix}_{4\times4}.$$

Let $\Omega=\Omega^+$ be a bounded domain with a simply connected boundary $S=\partial\Omega^+$ and $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega}$. Introduce the generalized single layer potentials

$$V_S^{(j)}(g^{(j)})(x)=\int\limits_S \Gamma^{(j)}(x-y,\tau)\,g^{(j)}(y)\,dS_y,\quad j=1,2,\quad x\in\mathbb{R}^3\setminus S, \tag{5.2}$$

$g^{(1)} = (g_1^{(1)}, \dots, g_6^{(1)})^\top$ and $g^{(2)} = (g_1^{(2)}, \dots, g_4^{(2)})^\top$ are the density vector functions defined on S.

Theorem 5.1. *Let $S \in C^{m,\alpha'}$, $0 < \alpha < \alpha' \leqslant 1$, and let $m \geqslant 1$ and $k \leqslant m-1$ be nonnegative integers. Then the operators*

$$V_S^{(1)} \;:\; [C^{k,\alpha}(S)]^6 \to [C^{k+1,\alpha}(\overline{\Omega^\pm})]^6,$$
$$V_S^{(2)} \;:\; [C^{k,\alpha}(S)]^4 \to [C^{k+1,\alpha}(\overline{\Omega^\pm})]^4,$$

are continuous.

For any $g^{(1)} \in [C^{0,\alpha}(S)]^6$ and $g^{(2)} \in [C^{0,\alpha}(S)]^4$, and for any $x \in S$, the following jump relations

$$\{V_S^{(1)}(g^{(1)})(x)\}^\pm = \mathcal{H}_S^{(1)}\, g^{(1)}(x), \tag{5.3}$$
$$\{\mathcal{T}^{(1)}(\partial_x, n(x), \tau)\, V_S^{(1)}(g^{(1)})(x)\}^\pm = [\,\mp 2^{-1} I_6 + \mathcal{K}_S^{(1)}\,]\, g^{(1)}(x), \tag{5.4}$$
$$\{V_S^{(2)}(g^{(2)})(x)\}^\pm = \mathcal{H}_S^{(2)}\, g^{(2)}(x), \tag{5.5}$$
$$\{\mathcal{T}^{(2)}(\partial_x, n(x), \tau)\, V_S^{(2)}(g^{(2)})(x)\}^\pm = [\,\mp 2^{-1} I_4 + \mathcal{K}_S^{(2)}\,]\, g^{(2)}(x) \tag{5.6}$$

hold, where

$$\mathcal{H}_S^{(j)}\, g^{(j)}(x) := \int\limits_S \Gamma^{(j)}(x-y,\tau)\, g^{(j)}(y)\, dS_y\,, \quad x \in S, \quad j = 1,2, \tag{5.7}$$
$$\mathcal{K}_S^{(j)}\, g^{(j)}(x) := \int\limits_S [\,\mathcal{T}^{(j)}(\partial_x, n(x), \tau)\, \Gamma^{(j)}(x-y,\tau)\,]\, g^{(j)}(y)\, dS_y\,, \quad x \in S, \quad j = 1,2. \tag{5.8}$$

The following operators

$$\begin{aligned} \mathcal{H}_S^{(1)} &: [C^{k,\alpha}(S)]^6 \to [C^{k+1,\alpha}(S)]^6\,, \qquad & \mathcal{H}_S^{(2)} &: [C^{k,\alpha}(S)]^4 \to [C^{k+1,\alpha}(S)]^4\,, \\ \mathcal{K}_S^{(1)} &: [C^{k,\alpha}(S)]^6 \to [C^{k,\alpha}(S)]^6\,, & \mathcal{K}_S^{(2)} &: [C^{k,\alpha}(S)]^4 \to [C^{k,\alpha}(S)]^4 \end{aligned} \tag{5.9}$$

are continuous. Moreover, the operators (5.9) *are invertible.*

Theorem 5.2. *Let S be a Lipschitz surface. The operators $V_S^{(j)}$, $\mathcal{H}_S^{(j)}$, and $\mathcal{K}_S^{(j)}$, $j = 1,2$, defined by* (5.2), (5.7), *and* (5.8), *can be extended to the continuous mappings*

$$\begin{aligned} V_S^{(1)} &: \big[H_2^{-\frac12}(S)\big]^6 \to \big[H_2^1(\Omega^+)\big]^6, \qquad & V_S^{(1)} &: \big[H_2^{-\frac12}(S)\big]^6 \to \big[H_{2,\,\mathrm{loc}}^1(\Omega^-)\big]^6, \\ V_S^{(2)} &: \big[H_2^{-\frac12}(S)\big]^4 \to \big[H_2^1(\Omega^+)\big]^4, & V_S^{(2)} &: \big[H_2^{-\frac12}(S)\big]^4 \to \big[H_{2,\,\mathrm{loc}}^1(\Omega^-)\big]^4 \cap \mathbf{Z}_\tau(\Omega^-). \\ \mathcal{H}_S^{(1)} &: [H_2^{-\frac12}(S)]^6 \to [H_2^{\frac12}(S)]^6\,, & \mathcal{K}_S^{(1)} &: [H_2^{-\frac12}(S)]^6 \to [H_2^{-\frac12}(S)]^6\,, \\ \mathcal{H}_S^{(2)} &: [H_2^{-\frac12}(S)]^4 \to [H_2^{\frac12}(S)]^4\,, & \mathcal{K}_S^{(2)} &: [H_2^{-\frac12}(S)]^4 \to [H_2^{-\frac12}(S)]^4\,. \end{aligned}$$

Moreover, the operators

$$\mathcal{H}_S^{(1)} \;:\; [H_2^{-\frac12}(S)]^6 \to [H_2^{\frac12}(S)]^6, \qquad \mathcal{H}_S^{(2)} \;:\; [H_2^{-\frac12}(S)]^4 \to [H_2^{\frac12}(S)]^4\,,$$

are invertible.

Theorem 5.3. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. The operators $V_S^{(j)}$, $\mathcal{H}_S^{(j)}$, and $\mathcal{K}_S^{(j)}$, $j = 1, 2$, can be extended to the following continuous operators*

$$\begin{aligned}
V_S^{(1)} &: [B^s_{p,p}(S)]^6 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6 \quad \Big[\, [B^s_{p,p}(S)]^6 \to [H_{p,\,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-)]^6 \,\Big],\\
&: [B^s_{p,q}(S)]^6 \to [B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)]^6 \quad \Big[\, [B^s_{p,q}(S)]^6 \to [B_{p,q,\,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-)]^6 \,\Big],\\
V_S^{(2)} &: [B^s_{p,p}(S)]^4 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^4 \quad \Big[\, [B^s_{p,p}(S)]^4 \to [H_{p,\,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-)]^4 \cap \mathbf{Z}_\tau(\Omega^-) \,\Big],\\
&: [B^s_{p,q}(S)]^4 \to [B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)]^4 \quad \Big[\, [B^s_{p,q}(S)]^4 \to [B_{p,q,\,\mathrm{loc}}^{s+1+\frac{1}{p}}(\Omega^-)]^4 \cap \mathbf{Z}_\tau(\Omega^-) \,\Big],\\
\mathcal{H}_S^{(1)} &: [H^s_p(S)]^6 \to [H^{s+1}_p(S)]^6 \quad \Big[\, [B^s_{p,q}(S)]^6 \to [B^{s+1}_{p,q}(S)]^6 \,\Big],\\
\mathcal{K}_S^{(1)} &: [H^s_p(S)]^6 \to [H^{s}_p(S)]^6 \quad \Big[\, [B^s_{p,q}(S)]^6 \to [B^{s}_{p,q}(S)]^6 \,\Big],\\
\mathcal{H}_S^{(2)} &: [H^s_p(S)]^4 \to [H^{s+1}_p(S)]^4 \quad \Big[\, [B^s_{p,q}(S)]^4 \to [B^{s+1}_{p,q}(S)]^4 \,\Big],\\
\mathcal{K}_S^{(2)} &: [H^s_p(S)]^4 \to [H^{s}_p(S)]^4 \quad \Big[\, [B^s_{p,q}(S)]^4 \to [B^{s}_{p,q}(S)]^4 \,\Big].
\end{aligned}$$

For $s > -1$ the jump relations (5.3)–(5.6) *remain valid in appropriate function spaces.*
The operators

$$\begin{aligned}
\mathcal{H}_S^{(1)} &: [B^s_{p,q}(S)]^6 \to [B^{s+1}_{p,q}(S)]^6, \qquad s \in \mathbb{R},\ \ p > 1,\ \ q \ge 1,\\
\mathcal{H}_S^{(2)} &: [B^s_{p,q}(S)]^4 \to [B^{s+1}_{p,q}(S)]^4, \qquad s \in \mathbb{R},\ \ p > 1,\ \ q \ge 1,
\end{aligned}$$

are invertible.

Corollary 5.4. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Arbitrary solutions to the homogeneous equations*

$$A^{(1)}(\partial, \tau) U^{(1)} = 0 \ \ in \ \ \Omega, \quad U^{(1)} \in [W^1_p(\Omega)]^6, \ \ p > 1,$$

and

$$A^{(2)}(\partial, \tau) U^{(2)} = 0 \ \ in \ \ \Omega, \quad U^{(2)} \in [W^1_p(\Omega)]^4, \ \ p > 1,$$

are uniquely representable in the form

$$U^{(1)} = V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1} g^{(1)}\big) \ \ with \ \ g^{(1)} = \{U^{(1)}\}^+ \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)]^6\,, \tag{5.10}$$

$$U^{(2)} = V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1} g^{(2)}\big) \ \ with \ \ g^{(2)} = \{U^{(2)}\}^+ \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)]^4. \tag{5.11}$$

The representations (5.10) *and* (5.11) *hold true for Lipschitz domain Ω and $p = 2$.*

Further, let us introduce the Steklov-Poincaré type operators

$$\begin{aligned}
\mathcal{A}_S^{(1)\pm} &:= \big(\mp 2^{-1}\, I_6 + \mathcal{K}_S^{(1)} \big) \big[\mathcal{H}_S^{(1)}\big]^{-1},\\
\mathcal{A}_S^{(2)\pm} &:= \big(\mp 2^{-1}\, I_4 + \mathcal{K}_S^{(2)} \big) \big[\mathcal{H}_S^{(2)}\big]^{-1},
\end{aligned}$$

which are related to the single layer potentials by the relations

$$\mathcal{A}_S^{(1)\pm} g^{(1)} = \Big\{ \mathcal{T}^{(1)}(\partial_x, n(x), \tau)\, V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1} g^{(1)}\big) \Big\}^{\pm}\,, \tag{5.12}$$

$$\mathcal{A}_S^{(2)\pm} g^{(2)} = \Big\{ \mathcal{T}^{(2)}(\partial_x, n(x), \tau)\, V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1} g^{(2)}\big) \Big\}^{\pm}\,. \tag{5.13}$$

Theorem 5.5. *Let S be a Lipschitz surface and $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. Then for all $g^{(1)} \in [H_2^{\frac{1}{2}}(S)]^6$ and $g^{(2)} \in [H_2^{\frac{1}{2}}(S)]^4$, the coercivity inequalities*

$$\begin{aligned}
\mathrm{Re}\, \langle (\pm\, \mathcal{A}_S^{(1)\pm} + \mathcal{C}^{(1)}) g^{(1)}\,,\, g^{(1)} \rangle_S &\ge C_1\, \| g^{(1)} \|^2_{[H_2^{\frac{1}{2}}(S)]^6}\,,\\
\mathrm{Re}\, \langle (\pm\, \mathcal{A}_S^{(2)\pm} + \mathcal{C}^{(2)}) g^{(2)}\,,\, g^{(2)} \rangle_S &\ge C_2\, \| g^{(2)} \|^2_{[H_2^{\frac{1}{2}}(S)]^4}
\end{aligned}$$

hold, where

$$\mathcal{C}^{(1)} \ : \ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6 \ \ and \ \ \mathcal{C}^{(2)} \ : \ [H_2^{\frac{1}{2}}(S)]^4 \to [H_2^{-\frac{1}{2}}(S)]^4$$

are compact operators and C_j, $j=1,2$, are positive constants.
The operators

$$\mathcal{A}_S^{(1)-}\ :\ [H_2^{\frac{1}{2}}(S)]^6\to[H_2^{-\frac{1}{2}}(S)]^6\,,$$
$$\mathcal{A}_S^{(2)\pm}\ :\ [H_2^{\frac{1}{2}}(S)]^4\to[H_2^{-\frac{1}{2}}(S)]^4\,,$$

are invertible, while

$$\mathcal{A}_S^{(1)+}\ :\ [H_2^{\frac{1}{2}}(S)]^6\to[H_2^{-\frac{1}{2}}(S)]^6$$

is the Fredholm one of index zero with the null space spanned over the vectors

$$\Psi^{(1)}=(0,0,0,1,0,0)^\top,\quad \Psi^{(2)}=(0,0,0,0,1,0)^\top. \tag{5.14}$$

Theorem 5.6. *Let $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$, $S\in C^\infty$. The operators*

$$\mathcal{A}_S^{(1)-}\ :\ [B_{p,q}^{s+1}(S)]^6\to[B_{p,q}^{s}(S)]^6,\qquad s\in\mathbb{R},\ \ p>1,\ \ q\ge1,$$
$$\mathcal{A}_S^{(2)\pm}\ :\ [B_{p,q}^{s+1}(S)]^4\to[B_{p,q}^{s}(S)]^4,\qquad s\in\mathbb{R},\ \ p>1,\ \ q\ge1,$$

are invertible, while the operator

$$\mathcal{A}_S^{(1)+}\ :\ [B_{p,q}^{s+1}(S)]^6\to[B_{p,q}^{s}(S)]^6,\qquad s\in\mathbb{R},\ \ p>1,\ \ q\ge1,$$

is Fredholm of zero index with a two-dimensional null space spanned over the vectors (5.14).

Acknowledgement

This work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSF) (Grant number FR-18-126).

References

1. M. Aouadi, On the coupled theory of thermo-magnetoelectroelasticity. *Quart. J. Mech. Appl. Math.* **60** (2007), no. 4, 443–456.
2. M. Aouadi, Some theorems in the generalized theory of thermo-magnetoelectroelasticity under Green–Lindsay's model. *Acta Mech.* **200** (2008), 25–43.
3. M. Avellaneda, G. Harshé, Magnetoelectric effect in piezoelectric/magnetostrictive multilayer (2-2) composites. *J. Intelligent Mater. Systems and Structures* **5** (1994), no. 4, 501–513.
4. Y. Benveniste, Magnetoelectric effect in fibrous composites with piezoelectric and piezomagnetic phases. *Phys. Rev.* B **51** (1995), 424–427.
5. L. P. M. Bracke, R. G. Van Vliet, A broadband magneto-electric transducer using a composite material. *Int. J. of Electronics Theoretical and Experimental* **51** (1981), no. 3, 255–262.
6. W. F. Jr. Brown, *Magnetoelastic Interactions.* Springer, New York, 1966.
7. T. Buchukuri, O. Chkadua, D. Natroshvili, Mathematical problems of generalized thermo-electro-magneto-elasticity theory. *Mem. Differ. Equ. Math. Phys.* **68** (2016), 1–166.
8. T. V. Burchuladze, T. G. Gegelia, *Development of the Potential Method in Elasticity Theory.* (Russian) Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR **79** (1985), 226 pp.
9. W. G. Cady, *Piezoelectricity.* McGraw Hill, New York, 1946.
10. D. S. Chandrasekharaiah, Thermoelasticity with second sound: A review. *Appl. Mech. Rev.* **39** (1986), no. 3, 355–376.
11. D. S. Chandrasekharaiah, Hyperbolic thermoelasticity: a review of recent literature. *Appl. Mech. Rev.* **51** (1998), no. 12, 705–729.
12. A. C. Eringen, *Mechanics of Continua.* Robert E. Krieger Pub. Co., Hungtington, New York, 1980.
13. C. A. Eringen, G. A. Maugin, *Electrodynamics of Continua I: Foundations and Solid Media.* Springer, Berlin, 1990.
14. A. E. Green, K. A. Lindsay, Thermoelasticity. *J. Elast.* **2** (1972), no. 1, 1–7.
15. C. A. Grimes, S. C. Roy, S. Rani, Q. Cai, Theory, Instrumentation and applications of magnetoelastic resonance sensors: A review. *J. Sensors* **11** (2011), no. 3, 2809–2844.
16. G. Harshe, J. P. Dougherty, R. E. Newnham, Theoretical modeling of multilayer magnetoelectric composites. *Int. J. Electromagn. Mater.* **4** (1993), no. 2, 145–159.
17. G. Harshe, J. P. Dougherty, R. E. Newnham, Theoretical modeling of $3-0/0-3$ magneto-electric composites. *Int. J. Electromagn. Mater.* **4** (1993), 161–171.
18. L. L. Hench, Bioceramics. *J. Am. Ceram. Soc.* **81** (1998), no. 7, 1705–1728.
19. R. B. Hetnarski, J. Ignaczak, Generalized thermoelasticity. *J. Thermal Stresses* **22** (1999), no. 4-5, 451–461.
20. J. Ignaczak, M. Ostoja-Starzewski, *Thermoelasticity with Finite Speeds*, Oxford University Press, 2009.

21. L. Jentsch, D. Natroshvili, Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies. I. *Mem. Differential Equations Math. Phys.* **17** (1999), 7–126.
22. L. Jentsch, D. Natroshvili, Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies. II. *Mem. Differential Equations Math. Phys.* **18** (1999), 1–50.
23. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, T. V. Burchuladze, *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity.* Translated from the second Russian edition. Edited by V. D. Kupradze. North-Holland Series in Applied Mathematics and Mechanics, 25. North-Holland Publishing Co., Amsterdam-New York, 1979.
24. S. Lang, Guide to the literature of piezoelectricity and pyroelectricity. 20. *Ferroelectrics* **297** (2003), 107–253.
25. S. Lang, Guide to the literature of piezoelectricity and pyroelectricity. 22. *Ferroelectrics* **308** (2004), 193–304.
26. S. Lang, Guide to the literature of piezoelectricity and pyroelectricity. 23. *Ferroelectrics* **321** (2005), 91–204.
27. S. Lang, Guide to the literature of piezoelectricity and pyroelectricity. 24. *Ferroelectrics* **322** (2005), 115–210.
28. S. Lang, Guide to the literature of piezoelectricity and pyroelectricity. 25. *Ferroelectrics* **330** (2006), 103–182.
29. S. Lang, Guide to the literature of piezoelectricity and pyroelectricity. 26. *Ferroelectrics* **332** (2006), 227–321.
30. S. Lang, Guide to the literature of piezoelectricity and pyroelectricity. 27. *Ferroelectrics* **350** (2007), 130–216.
31. S. Lang, Guide to the literature of piezoelectricity and pyroelectricity. 28. *Ferroelectrics* **361** (2007), 124–239.
32. S. Lang, Guide to the literature of piezoelectricity and pyroelectricity. 29. *Ferroelectrics* **366** (2008), 122–237.
33. J. Y. Li, Magnetoelectroelastic multi-inclusion and inhomogeneity problems and their applications in composite materials. *Int. J. Eng. Sci.* **38** (2000), no. 18, 1993–2011.
34. J. Y. Li, Uniqueness and reciprocity theorems for linear thermo-electro-magneto-elasticity. *Quart. J. Mech. Appl. Math.* **56** (2003), no. 1, 35–43.
35. J. Y. Li, M. L. Dunn, Anisotropic coupled-field inclusion and inhomogeneity problems. *Phil. Mag. A* **77** (1998), no. 5, 1341–1350.
36. J. Y. Li, M. L. Dunn, Micromechanics of magnetoelectroelastic composite materials: average fields and effective behaviuor. *J. Intelligent Mater. Syst. Struct.* **9** (1998), no. 6, 404–416.
37. J. L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications. vol. I.* Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York–Heidelberg, 1972.
38. W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations.* Cambridge University Press, Cambridge, 2000.
39. F. C. Moon, *Magneto-Solid Mechanics*, Wiley, New York, 1984.
40. C. W. Nan, Magnetoelectric effect in composites of piezoelectric and piezomagnetic phases. *Phys. Rev.* B **50** (1994), no. 9, 6082–6088.
41. C. W. Nan, M. I. Bichurin, S. Dong, D. Viehland, G. Srinivasan, Multiferroic magnetoelectric composites: Historical perspective, status, and future directions. *Journal of Applied Physics* **103** (2008), Issue 3:031101.
42. D. Natroshvili, Mathematical problems of thermo-electro-magneto-elasticity. *Lect. Notes TICMI* **12** (2011), 1–127.
43. J. Nečas, *Direct Methods in the Theory of Elliptic Equations.* Springer-Verlag Berlin Heidelberg, 2012.
44. W. Nowacki, *Efecty Electromagnetyczne w Stalych Cialach Odksztalcalnych.* Warszawa, Panstwowe Widawnictwo Naukowe, 1983.
45. Y. E. Pak, Linear electro-elastic fracture mechanics of piezoelectric materials. *International Journal of Fracture* **54** (1992), no. 1, 79–100.
46. Y. H. Pao, *Electromagnetic Forces in Deformable Continua.* In S. Nemat-Nasser (Ed.), Mechanics Today (vol. 4, pp. 209–306). Oxford: Pergamon Press, 1978.
47. Q. H. Qin, *Fracture Mechanics of Piezoelastic Materials.* WIT Press, 2001.
48. C. C. Silva, D. Thomazini, A. G. Pinheiro, N. Aranha, S. D. Figueiró, J. C. Góes, A. S. B. Sombra, Collagen-hydroxyapatite films: Piezoelectric properties. *Materials Science and Engineering B* **86** (2001), no. 3, 210–218.
49. B. Straughan, *Heat Waves.* Springer, New York, London, 2011.
50. R. A. Toupin, The elastic dielectrics. *J. Rat. Mech. Analysis* **5** (1956), 849–915.
51. R. A. Toupin, A dynamical theory of elastic dielectrics. *Internat. J. Engrg. Sci.* **1** (1963), 101–126.
52. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland Mathematical Library, 18. North-Holland Publishing Co., Amsterdam-New York, 1978.
53. S. Uddin, M. Mohamad, M. Rahimi-Gorji, R. Roslan, I. M. Alarifi, *Fractional Electro-Magneto Transport of Blood Modeled with Magnetic Particles in Cylindrical Tube without Singular Kernel.* Microsystem Technologies: Micro and Nanosystems Information Storage and Processing Systems, 2019, pp. 1–10; DOI: 10.1007/s00542-019-04494-0.
54. A. M. J. G. Van Run, D. R. Terrell, J. H. Scholing, An in situ grown eutectic magnetoelectric composite material. *Acta Mechanika* **154** (2002), 1–9.
55. W. Voight, *Lehrbuch Der Kristall-Physik.* Leipzig, Berlin, B. G. Teubner, 1910.
56. L. Wei, S. Yapeng, F. Daining, Magnetoelastic coupling on soft ferromagnetic solids with an interface crack. *Int. J. Electron.* **51** (1981), 255–262.

(Received 06.06.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 131–136

ALMOST BICOMPLEX STRUCTURES

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Abstract. Bicomplex numbers exist in real $4n$-dimensions like quaternions. Also, quaternions are, as is known, associated to some tensorial structures defined in $4n$-dimensions, called almost quaternionic structures. In this paper we search the presence of such structures, which we call (almost) bicomplex structures, associated to bicomplex numbers. However, we can see that bicomplex numbers don't present a relation with the (almost) bicomplex structures because bicomplex numbers can be defined only in $4n$-dimensions even though $2n$-dimensions are sufficient to define the almost bicomplex structures. Two examples for 4- and 6-dimensions show clearly this result. Finally, the integrability conditions for these structures are investigated.

1. Introduction

Hypercomplex numbers [4,16] or division algebras [3] are of great importance in physics. Of course, quaternions play a pioneer role in this sense, i.e., the solutions of the $SU(2)$ Yang-Mills theory [1]. As is well known, the generators of the group $SU(2)$, that is, the Pauli matrices, present a quaternionic structure and therefore the $SU(2)$-valued gauge potentials (or connections) are indeed quaternions (or quaternion valued 1-forms). Other kind of these numbers is known as bicomplex numbers existing in the real $4n$-dimensions [14, 15], and the system of bicomplex numbers is the first non-trivial Clifford commutative [2] complex.

For similar to the quaternions and corresponding (almost) quaternionic structures there arises the question: are there some tensorial structures associated to the bicomplex numbers? In this paper we search an answer to this question. Our result is that bicomplex numbers aren't associated to any tensorial structures like quaternions. So, bicomplex numbers can be defined only in $4n$-dimensions even though $2n$-dimensions are sufficient to define the (almost) bicomplex structures defined in this paper. The reason of this result is via Proposition 4.1 given by Obata [13] and Theorem 4.2 by Hoffmann and Kunze [6]. The bicomplex structures are easily seen in 4– and 6-dimensions in this paper. Finally, the integrability conditions for these structures are investigated.

2. Bicomplex Numbers

Consider complex numbers field $\mathbb{C}$ with imaginary unit $i = \sqrt{-1}$. Let $j = \sqrt{-1}$ be another imaginary unit satisfying commutative product rule $ij = ji = k$. Given a set of $\mathbb{R}$-linear tensor products $\mathbb{B} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}^4$. Therefore, an element of this space is called a bicomplex number and is written as

$$q = z^1 + jz^2, \quad z^1, z^2 \in \mathbb{C}, \quad \text{or}$$
$$q = (x^1 + ix^2) + j(x^3 + ix^4), \quad x^1, x^2, x^3, x^4 \in \mathbb{R}.$$

Bicomplex numbers have the following multiplication rules:

$$ij = ji = k, \quad ik = ki = -j, \quad jk = kj = -i, \quad k^2 = +1$$

and the addition and subtraction operations are like in the real and complex numbers fields. Also, the zero and unit (or identity) elements of the bicomplex numbers are

$$0_{\mathbb{B}} = (0 + i0) + j(0 + i0) = 0,$$
$$1_{\mathbb{B}} = (1 + i0) + j(0 + i0) = 1.$$

2010 *Mathematics Subject Classification.* 17A35, 53C15.

Key words and phrases. Bicomplex numbers; Almost bicomplex structure; Integrability.

There is an important difference between $\mathbb{C}$ and $\mathbb{B}$: as the complex numbers form a field, the bicomplex numbers don't, since they contain the divisors of zero, i.e.,

$$(1+ij)(1-ij) = (1-ij)(1+ij) = 0.$$

Therefore bicomplex numbers space $\mathbb{B}$ is a commutative ring with unit and its algebraic properties can be seen in [14].

There are three conjugations in bicomplex numbers. Here, $(\bar{\bullet})$ denotes the complex conjugation in the complex numbers field $\mathbb{C}$. Then, $\forall z_1, z_2 \in \mathbb{C}$, we have the following i, j and ij conjugations, respectively:

$$\bar{q} = \bar{z}_1 + j\bar{z}_2, \quad q^* = z^1 - jz^2, \quad q^\dagger = \bar{z}_1 - j\bar{z}_2,$$

Moduli in the bicomplex numbers are defined for two bicomplex numbers $w = z^1 + z^2 j = x^1 + ix^2 + jx^3 + ijx^4$ in two ways as real $\mathbb{B}\times\mathbb{B} \to \mathbb{R}$ and complex $\mathbb{B}\times\mathbb{B} \to \mathbb{C}$. They are written, respectively, as follows:

$$\begin{aligned} \|w\|^2 =& (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2, \\ \|w\|_*^2 =& w^* w = (z^1)^2 + (z^2)^2. \end{aligned}$$

More details of the algebra of these numbers and the analysis of bicomplex holomorphic functions can be found in Refs. [2, 14, 15].

3. Some Tensorial Structure on Manifolds

Let M be a smooth manifold of real even n-dimensions. If we write a smooth tensorial field I of rank $(1,1)$ on this manifold satisfying the relation

$$I^2 = \epsilon \mathbb{I},$$

where $\mathbb{I}$ is the identity matrix and ϵ is $\{-1, +1\}$, then we say that I is

- an almost complex structure for $\epsilon = -1$, or
- an almost product structure for $\epsilon = +1$.

Definition 3.1. Given three smooth tensorial fields I_1, I_2, I_3 of rank $(1,1)$ on an even dimensional manifold M which satisfy the following rules:

$$\begin{aligned} &I_1^2 = \epsilon_1 \mathbb{I}, \quad I_2^2 = \epsilon_2 \mathbb{I}, \quad I_3^2 = \epsilon_3 \mathbb{I}, \\ &I_2 I_1 = \epsilon I_1 I_2 = -\epsilon_3 I_3, \\ &I_3 I_2 = \epsilon I_2 I_3 = -\epsilon_1 I_1, \\ &I_1 I_3 = \epsilon I_3 I_1 = -\epsilon_2 I_2, \end{aligned}$$

where

$$\epsilon = \epsilon_1 \epsilon_2 \epsilon_3.$$

Therefore we mention the following cases from Ref. [7]:

I. If $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$, then the triplet (I_1, I_2, I_3) is called an almost quaternionic structure,

II. If $\epsilon_1 = \epsilon_2 = -1$ and $\epsilon_3 = +1$, we will say that the triplet (I_1, I_2, I_3) is ***an almost bicomplex structure***,

III. If $\epsilon_1 = \epsilon_2 = \epsilon_3 = +1$, then the triplet (I_1, I_2) is called an almost product structure.

If we denote the local coordinates on the manifold M by $\{x^\mu\} = (x^i, y^i) \in \mathbb{R}^{2n}$, where $i, j = 1, \ldots, n$, then the acting of an almost complex structure on their local coordinate bases is written as

$$I(\frac{\partial}{\partial x^i}) = -\frac{\partial}{\partial y^i}, \qquad I(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i}.$$

Then, we can show the almost complex structure I on $\mathbb{R}^{2n}$ by a block matrix (or canonical) representation such that

$$I = \begin{pmatrix} \mathbf{0} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbf{0} \end{pmatrix},$$

where $\mathbb{I}_{n\times n}$ and $\mathbf{O}$ are the unit and zero $n \times n$ matrices, respectively.

4. Almost Bicomplex Structure

The Case II that we call bicomplex structure was handle by Hsu [7] and Liberman [11]. The mutual point of these authors is that this structure is considered as the distribution of a tangent bundle on a $4n$-dimensional manifold, since this structure is handled in the perspective of an almost quaternionic structure. If there exists a pair of two complex structures (I, J) commuting each other, $K = IJ = JI$, then we call it the almost bicomplex structure to the triple (I, J, K). Our claim mentioned above is whether this structure is associated to bicomplex numbers. Therefore, first we have to present Obata' s proposition.

Proposition 4.1 (Obata [13])**.** *Let $\tilde{J} \in GL(n, \mathbb{C})$ be a non-singular complex matrix such that*

$$\tilde{J} = A + \mathrm{i}B,$$

where $A, B \in GL(n, \mathbb{R})$. Then the correspondence

$$\tilde{J} \to J = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in GL(2n, \mathbb{R})$$

is an isomorphism. The matrix J is commutated by a matrix, independent of n odd or even such that

$$I = \begin{pmatrix} \mathbf{0} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbf{0} \end{pmatrix} \in GL(2n, \mathbb{R}),$$

so,

$$IJ = JI.$$

If A is unitary, B is orthogonal, and vice versa.

Since an almost complex structure is a diagonalizable matrix, Hoffman and Kunze' s theorem is valid.

Theorem 4.2 (Hoffman-Kunze [6])**.** *A set of commuting diagonalizable matrices are simultaneously diagonalizable.*

Therefore, we can say that the matrix J is also a diagonalizable matrix, and so we have the following

Corollary 4.3. *Let I and J be two almost complex structures commuting each other on a smooth manifold of real $2n$-dimensions, hence*

$$IJ = JI = K, \quad IK = KI = -J, \quad JK = KJ = -I, \quad K^2 = +\mathbb{I}_{2n\times 2n}. \tag{1}$$

Therefore, if the $n \times n$ matrices A and B satisfy the relations

$$AB + BA = \mathbf{0}_{n\times n},$$
$$A^2 - B^2 = -\mathbb{I}_{n\times n},$$

then I, J, K are constructed as independent of n odd or even as follows:

$$I = \begin{pmatrix} \mathbf{0} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbf{0} \end{pmatrix}, \quad J = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad K = \begin{pmatrix} -B & A \\ -A & -B \end{pmatrix}. \tag{2}$$

For this result we can give two examples in 4– and 6-dimensions to this result. In 4-dimensions, with respect to equation (2), we get

$$I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 & p \\ 0 & 0 & p & 0 \\ 0 & -p & 0 & 0 \\ -p & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -p & 0 & 0 \\ -p & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & -p & 0 \end{pmatrix}, \tag{3}$$

where $p^2 = 1$. On the other hand, in 6-dimensions, we have

$$I = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ -s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s \\ 0 & -q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & s \\ 0 & -q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & 0 & 0 \\ 0 & 0 & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & 0 \\ s & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

where $q^2 = s^2 = 1$.

5. Integrability of Almost Bicomplex Structures

In this paper, we use the following geometrical preliminaries . Let M be a smooth manifold of real even dimensions n with local coordinates $\{x^\mu\} \in \mathbb{R}^n$, $(\mu = 1, \ldots, n)$. Given a connection on a tangent bundle TM by the map $\nabla : \mathcal{C}^\infty(TM) \to \Lambda^1(TM)$ together with the covariant derivative

$$\nabla = \mathrm{d} + [\Gamma, \cdot],$$

where $\Gamma \in \Lambda^1(\mathrm{End}(TM))$ is the 1-form connection and d is an exterior derivative operator. The curvature of this connection is

$$R = \nabla\Gamma = \mathrm{d}\Gamma + \Gamma \wedge \Gamma \in \Lambda^2(\mathrm{End}(TM)).$$

Suppose that let M be an almost complex manifold with the almost complex structure I. Given two vector fields X, Y on this manifold. The torsion tensor of the almost complex structure I, called also Nijenhuis tensor, is defined as follows:

$$N_I(X,Y) = [I,I](X,Y) = 2\left\{[IX, IY] - [X,Y] - I[X, IY] - I[IX, Y]\right\},$$

where $[X,Y] = X(Y) - Y(X)$ is the Lie bracket.

Kobayashi and Nomizu [10] say that every almost complex manifold M admits an almost complex affine connection such that its torsion T is given by $N = 8T$, where N is the torsion of the almost complex structure I on M. Then, an almost complex structure is said to be integrable, $\mathrm{d}I = 0$, if its torsion vanishes ($N = 0$) and is parallel, $\nabla I = 0$, with respect to the connection ∇. Thus a complex structure on $\mathbb{R}^{n=2m}$ is equivalent to a torsion-free $GL(m, C)$-structure [8].

Definition 5.1. Let I_1 and I_2 be two tensor fields of $(1,1)$ type on an even dimensional manifold satisfying $I_1^2 = \epsilon_1\mathbb{I}$, $I_2^2 = \epsilon_2\mathbb{I}$ and $I_1 I_2 = \epsilon I_3$ for some constants $\epsilon, \epsilon_1, \epsilon_2$. They are covariant constant tensors with respect to the connection ∇ if

$$\nabla I_1 = 0, \quad \nabla I_2 = 0, \quad (also\ \nabla I_3 = 0).$$

Thus ∇ is called the (I_1, I_2)-connection (and, consequently, I_3-connection in view of $I_1 I_2 = \epsilon_3 I_3$).

Suppose for a short time that we have an almost quaternionic structure induced by three almost complex structures I_1, I_2, I_3 such that $I_1 I_2 = -I_2 I_1 = I_3$. The integrability conditions of the almost quaternionic structures are shortly given by six vanishing Lie brackets $[I_i, I_j] = 0$, $(i, j = 1, 2, 3)$ and vanishing curvature tensor of symmetric affine connection, that is, Levi-Civita, [12, 17]. Indeed, we can generalize this for a unique almost complex structure by the following

Theorem 5.2. *Let I be an almost complex structure which is a tensor field of $(1,1)$ type on a manifold. If this tensor field (or almost complex structure) is parallel with respect to a connection ∇ on this manifold, i.e., $\nabla I = 0$, then this connection is likewise flat.*

Proof. Let $\nabla = \mathrm{d} + [\Gamma, \dot{]}$ be the covariant derivative of the connection ∇. If $\nabla I = 0$, then $\mathrm{d}I + \Gamma I - I\Gamma = 0$. The exterior derivative of this expression reads as $RI = IR$, where $R = \mathrm{d}\Gamma + \Gamma \wedge \Gamma$ is the curvature of the connection. Also, we can write $R = I^{-1}RI = I^{-1}IR = \pm R$ from $I^2 = \pm\mathbb{I}$. Therefore, if $I^2 = -\mathbb{I}$, then the curvature of a connection, compatible by (or parallel to) almost complex structure I, is flat: $R = 0$. □

From all the above and Theorem 5.2, we can give for the integrability of almost complex structure the following

Corollary 5.3. *An almost complex manifold M admits a torsion free almost complex affine connection if and only if an almost complex structure has no torsion* [10]*. On an almost complex manifold there exists an affine connection whose almost complex structure is a covariant constant* [5, 13]*, and any connection which is compatible by this almost complex structure is flat.*

We have defined the almost bicomplex structure for two almost complex structures I and J in real $2n$-dimensions which commute each other as follows:

$$IJ = JI = K, \quad IK = KI = -J, \quad JK = KJ = -I, \quad K^2 = +\mathbb{I}_{2n\times 2n}.$$

One can easily see from Definition 5.1 that the almost bicomplex structure (I, J) is parallel with respect to an affine connection on the manifold. As a natural consequence, $K = IJ = JI$ is also parallel with respect to the same connection. When this connection is symmetric, that is, Levi Civita, this almost bicomplex structure is integrable.

On the other hand, in order to investigate another integrability condition of this structure we need the Lie brackets $[I, J]$, $[I, K]$, $[J, K]$ and $[K, K]$ as well as the Nijenhuis tensors $[I, I]$ and $[J, J]$ of the almost complex structures I and J. Therefore, we consider the following proposition due to Kobayashi and Nomizu.

Proposition 5.4 (Kobayashi [9])**.** *Let A and B be tensor fields of type $(1,1)$ and $X, Y \in \Gamma(M)$ vector fields on the manifold M. Set*

$$\begin{aligned}S(X,Y) = [P,Q](X,Y) =& [PX,QY] + [QX,PY] - P[X,QY] - P[QX,Y] \\ &- Q[X,PY] - Q[PX,Y] + (PQ+QP)[X,Y].\end{aligned}$$

Then the mapping $S : \Gamma(M)\times\Gamma(M) \to \Gamma(M)$ is a skew-symmetric tensor field of type $(1,2)$, $S(X,Y) = -S(Y,X)$.

Using Proposition 5.4 and following [17], we investigate the integrability properties of almost bicomplex structures. Similar theorems were obtained for almost quaternionic manifolds by Yano [17].

On the other hand, we write the following relation:

$$\begin{aligned}[P,QR](X,Y) =& [PX,QRY] + [QRX,PY] - P([X,QRY] + [QRX,Y]) \\ &- QR([X,PY] + [PX,Y]) - (PQR + QRP)[X,Y]. \end{aligned} \tag{5}$$

i) If we choose $P = Q = I$ and $R = J$, considering the commutation relation $IJ = K$ from the equation (5), we get

$$[I,K](X,Y) = I[I,J](X,Y) + \frac{1}{2}([I,I](JX,Y) + [I,I](X,JY]).$$

Similarly, for $P = Q = J$ and $R = I$, we have

$$[J,K](X,Y) = J[J,I](X,Y) + \frac{1}{2}([J,J](IX,Y) + [J,J](X,IY]).$$

Therefore, if $[I,I] = 0$, then

$$[I,K](X,Y) = [I,IJ](X,Y) = I[I,J](X,Y), \tag{6}$$

and if $[J,J] = 0$, then

$$[J,K](X,Y) = [J,JI](X,Y) = J[J,I](X,Y), \tag{7}$$

ii) If we choose $P = I$, $Q = J$ and $R = K$, we get

$$\begin{aligned}-[I,I](X,Y) - [J,J](X,Y) =& [I,J](KX,Y) + [I,J](X,KY) \\ &+ I[J,K](X,Y) + J[I,K](X,Y)\end{aligned}$$

If $[I,I] = 0$ and $[J,J] = 0$ simultaneously, because of equations (6) and (7), then we get

$$[I,J](KX,Y) + [I,J](X,KY) = 0,$$

or shortly,

$$[I,J] = [J,I] = [K,K] = IJ - JI = K - K = 0.$$

Then we can see that the almost complex structures I and J must simultaneously be integrable. Thus, we have following theorem:

Theorem 5.5. *If I and J are two almost complex structures on a smooth manifold of real $2n$-dimensions which commute each other, $IJ = JI = K$, then I, J and K must simultaneously be integrable as follows:*

$$[I, I] = [J, J] = 0, \quad [I, J] = [J, I] = [K, K] = 0, \quad [I, K] = [J, K] = 0.$$

6. Conclusion

When one compares quaternions and bicomplex numbers handled in this paper, although these numbers live in the real $4n$ dimensions, the associated almost complex structures to these numbers behave different concept. So, almost quaternionic structure has to be defined in $4n$-dimensions, but any even dimension is sufficient for the almost bicomplex structure because of Proposition 4.1 given by Obata [13] and Theorem 4.2 by Hoffmann and Kunze [6]. This means that the almost bicomplex structures in the concept of this paper don't relate to the bicomplex numbers. In the quaternions two anticommuting almost complex structures induce the third almost complex structure, however, two commuting almost complex structures cannot induce a third almost complex structure. As is shown from equation (1), if I and J are two almost complex structures having commutations relationship $IJ = JI = K$, then $K^2 = +\mathbb{I}$, that is K isn't an almost complex structure. Thus the triplet (I, J, K) cannot be associated to the bicomplex numbers. Although, in this case, we have used the term "almost bicomplex structure" for this triplet. We have shown clearly this situation on the almost complex structures obtained in 4– and 6-dimensions given in equations (3) and (4), respectively. Consequently, by Theorem (5.5) we have presented the integrability of the bicomplex structure.

References

1. M. F. Atiyah, *Geometry on Yang-Mills Fields.* Scuola Normale Superiore Pisa, Pisa, 1979.
2. P. Baird, J. C. Wood, Harmonic morphisms and bicomplex manifolds. *J. Geom. Phys.* **61** (2011), no. 1, 46–61.
3. J. M. Figueroa-O'Farrill, Gauge theory and the division algebras. *J. Geom. Phys.* **32** (1999), no. 2, 227–240.
4. G. Frobenius, Theorie der Hypercomplexen Größen. *Sitz. Kön. Preuss. Akad. Wiss.* (1903), 504-537; Gesammelte Abhandlungen, art. 70, 284–317.
5. A. Frölicher, Zur Differentialgeometrie der komplexen Strukturen. (German) *Math. Ann.* 129 (1955), 50–95.
6. K. Hoffman, R. Kunze, *Linear Algebra.* Second edition Prentice-Hall, Inc., Englewood Cliffs, N.J. 1971.
7. C.-u. Hsu, On some structures which are similar to the quaternion structure. *Tohoku Math. J. (2)* **12** (1960), 403–428.
8. D. Joyce, *Compact Manifolds with Special Holonomy.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.
9. S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry.* vol I. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.
10. S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry.* vol. II. Interscience Tracts in Pure and Applied Mathematics, No. 15 vol. II Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.
11. P. Libermann, Sur le probléme d'équivalence de certaines structures infinitésimales. (French) *Ann. Mat. Pura Appl. (4)* **36** (1954), 27–120.
12. M. Obata, Affine connections on manifolds with almost complex, quaternion or Hermitian structure. *Jap. J. Math.* **26** (1956), 43–77.
13. M. Obata, Affine transformations in an almost complex manifold with a natural affine connection. *J. Math. Soc. Japan* **8** (1956), 345–362.
14. G. B. Price, *An Introduction to Multicomplex Spaces and Functions.* With a foreword by Olga Taussky Todd. Monographs and Textbooks in Pure and Applied Mathematics, 140. Marcel Dekker, Inc., New York, 1991.
15. S. Rönn, Bicomplex algebra and function theory. arXiv preprint math/0101200 (2001).
16. E. Study, Über Systeme complexer Zahlen und ihre Anwendung in der Theorie der Transformationsgruppen. (German) *Monatsh. Math. Phys.* **1** (1890), no. 1, 283–354.
17. K. Yano, M. Ako, Integrability conditions for almost quaternion structures. *Hokkaido Math. J.* **1** (1972), 63–86.

(Received 21.09.2017)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 137–145

TOPSIS APPROACH TO MULTI-OBJECTIVE EMERGENCY SERVICE FACILITY LOCATION SELECTION PROBLEM UNDER Q-RUNG ORTHOPAIR FUZZY INFORMATION

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Abstract. A new model based on q-rung orthopair fuzzy sets (q-ROFS) has been presented to manage the uncertainty in real-world multi-criteria decision-making problems. q-ROFS has much stronger ability than Pythagorean fuzzy set (PFS) or intuitionistic fuzzy set (IFS) to model such uncertainty. A q-rung orthopair fuzzy TOPSIS approach for formation and representing experts knowledge on the parameters of emergency service facility location planning is developed. In this approach, we propose a score function based on the comparison method to identify the q-rung orthopair fuzzy positive ideal solution and the q-rung orthopair fuzzy negative ideal solution. Based on the constructed fuzzy TOPSIS aggregation, a new objective function is formulated. The constructed criterion maximizes service centers' selection index. This criterion together with the second criterion - minimization of a number of selected centers creates the multi-objective facility location set covering problem. The approach is illustrated by the simulation example of emergency service facility location planning for a city in Georgia. More exactly, the example looks into the problem of planning fire stations locations to serve emergency situations in specific demand points critical infrastructure objects.

1. Introduction

Multi-criteria decision making (MCDM) is to find an optimal alternative that has the highest degree of satisfaction from a set of feasible alternatives characterized with multiple criteria, and these kinds of MCDM problems arise in many real-world situations. Considering the inherent vagueness of human preferences as well as the objects being fuzzy and uncertain, Bellman and Zadeh [2] introduced the theory of fuzzy sets in the MCDM problems. Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) developed by Hwang and Yoon [7] (1981) is one of the most useful distance measure based on the classical approaches to multi-criteria/multi-attribute decision making (MCDM/MADM) problems. It is a practical and useful technique for ranking and selection of a number of externally determined alternatives through distance measures. The basic principle used in the TOPSIS is that the chosen alternative should have the shortest distance from positive-ideal solution (PIS) and farthest from the negative-ideal solution (NIS). There exists a large amount of literature involving TOPSIS theory and applications. In the TOPSIS, the performance ratings and the weights of the criteria are given as crisp values. In classical TOPSIS methods, crisp numerical values are used to express the performance rating and criteria weights. But human judgment, preference values and criteria weights are often ambiguous and cannot be represented by using crisp numerical value in real-life situation. To resolve the ambiguity frequently arising in information from human judgment and preference, the fuzzy set theory has been successfully used to handle imprecision and uncertainty in decision making problems. In this work, a novel decision-making TOPSIS approach is developed to deal effectively with the interactive MCDM problems with q-rung orthopair fuzzy information.

Intuitionistic fuzzy sets (IFS) were introduced by Atanassov [1], as a generalization of a Zadeh's fuzzy sets (FS). Since to each element of IFS, as Intuitionistic fuzzy number (IFN) (μ,ν), is assigned a membership degree (μ), a non-membership degree (ν) and a hesitancy degree $(1-\mu-\nu)$, IFS is more powerful in dealing with uncertainty and imprecision than FS. The IFS theory has been widely

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2010 *Mathematics Subject Classification.* 28E10, 47S40, 62C86, 90B50, 90C70.

Key words and phrases. Emergency Service Facility Location planning; q-Rung orthopair fizzy sets; Fuzzy TOPSIS; Critical infrastructure.

studied and applied to a variety of areas. But an IFN (μ, ν) has a significant restriction - the sum of the degrees of membership and the non-membership is equal to or less than 1. In some cases, a decision maker (DM) may provide data for some attribute that the sum of two degrees is greater than $1(\mu + \nu > 1)$. Yager in [13, 14] presented the concept of the Pythagorean fuzzy set (PFS) as extension of an IFS, where the pair of a Pythagorean fuzzy number (PFN) (μ, ν) has a less significant restrict – a square sum of the degrees of membership and the non-membership is equal to or less than $1(\mu^2 + \nu^2 \leq 1)$. In general, for practical problems, the PFSs can decide significant ones that IFSs cannot do. Therefore, PFSs are more able to process uncertain information and solve complex decision making problems. PFNs have much less, but significant restriction. When the evaluation psychology of a DM is too complicated and contradictory for complex decision making, the attribute's corresponding information is still difficult to express with PFNs. Recently, again Yager decided this problem in [15,16]. He proposed a concept of a q-rung orthopair fuzzy set (q-ROFS), where $q \geq 1$ and the sum of the qth power of the degrees of membership and the non-membership cannot be greater than 1. For a q-rung orthopair fuzzy number (q-ROFN) we have $(\mu^q + \nu^q \leq 1)$. It is obvious that the q-ROFSs are more general than IFSs and PFSs. The IFSs and PFSs are the special cases of the q-ROFSs when $q = 1$ and $q = 2$, respectively. Therefore, q-ROFNs are more convenient and able to describe DM's evaluation information than IFNs and PFNs.

Defnition 1 ([15])**.** Let S be a fixed ordinary set. A q-rung orthopair fuzzy set A on S is defined as membership grades:

$$A = \{\langle s, \mu_A(s), \nu_A(s)\rangle / (s \in S)\},$$

where the functions $\mu_A(s)$ indicate support for membership of $s \in A$ and $\nu_A(s)$ indicates support against membership of $s \in A$, where

$$q \geq 1, \;\; 0 \leq \mu_A(s) \leq 1, \;\; 0 \leq \nu_A(s) \leq 1, \;\; 0 \leq (\mu_A(s))^q + (\nu_A(s))^q \leq 1.$$

$\mathrm{Hes}_q(s) = (1 - (\mu_A(s))^q + (\nu_A(s))^q)^{1/q}$ is called a hesitancy associated with a q-rung orthopair membership grades and $\mathrm{Str}_q(s) = ((\mu_A(s))^q + (\nu_A(s))^q)^{1/q}$ is called a strength of commitment viewed at rung q.

In [15], Yager showed that Attanassov's intuitionistic fuzzy sets [1] are $q = 1$-rung orthopair and Yager's Pythagorean fuzzy sets [14] are $q = 2$ rung orthopair fuzzy sets. For convenience, the authors for every $s \in S$ called $\alpha = \langle s, \mu_\alpha(s), \nu_\alpha(s)\rangle$ a q-rung orthopair fuzzy number (q-ROFN) denoted by $\alpha = (\mu_\alpha, \nu_\alpha)$.

Let us denote by L the lattice of non-empty intervals $L = \{[a; b])/(a, b) \in [0, 1]^2, \; a \leq b\}$. The partial order relation $\leq_L$ is defined as $[a; b] \leq_L [c; d] \Leftrightarrow a \leq c$ and $b \leq d$. The top and bottom elements are $1_L = [1; 1]$ and $0_L = [0; 0]$, respectively. For the lattice of all q-ROFNs the corresponding partial order relation $\leq_{L_{q\text{-ROFNs}}}$ is defined as

$$(\mu_1, \nu_1) \leq_{L_{q\text{-ROFNs}}} (\mu_2, \nu_2) \Leftrightarrow \mu_1 \leq \mu_2 \;\text{ and }\; \nu_1 \geq \nu_2.$$

The top and bottom elements are $1_{L_{q\text{-ROFNs}}} = (1; 0)$ and $0_{L_{q\text{-ROFNs}}} = (0; 1)$, respectively.

Defnition 2 ([15])**.** Suppose $\alpha = (\mu_\alpha, \nu_\alpha)$ is a q-ROFN. a) A score function Sc of α is defined as

$$Sc(\alpha) = \mu_\alpha^q - \nu_\alpha^q; \tag{1}$$

b) An accuracy function Ac of α is defined as follows:

$$Ac(\alpha) = \mu_\alpha^q + \nu_\alpha^q. \tag{2}$$

Based on these definitions, a comparison method of q-ROFNs (total order relation $\leq_t$ on the lattice $L_{q\text{-ROFNs}}$) is defined.

Defnition 3 ([15]). Suppose $\alpha = (\mu_\alpha, \nu_\alpha)$ and $\beta = (\mu_\beta, \nu_\beta)$ are any two q-ROFNs and $Sc(\alpha)$, $Sc(\beta)$ are the score functions and $Ac(\alpha)$, $Ac(\beta)$ are the accuracy functions of α and β, respectively, then:

a) if $Sc(\alpha) > Sc(\beta)$, then $\beta <_t \alpha$;

b) if $Sc(\alpha) = Sc(\beta)$, then:
if $Ac(\alpha) > Ac(\beta)$, then $\beta <_t \alpha$;
if $Ac(\alpha) = Ac(\beta)$, then $\beta =_t \alpha$.

On the lattice $L_{q\text{-ROFNs}}$, the following basic operations can be defined.

Defnition 4 ([15]). Suppose for $\alpha = (\mu_\alpha, \nu_\alpha)$, $\alpha_1, \alpha_2 \in L_{q\text{-ROFNs}}$ we have:

1. $\alpha^c = (\nu_\alpha, \mu_\alpha)$;
2. $\alpha_1 \oplus_q \alpha_2 = \big((\mu_{\alpha_1}^q + \mu_{\alpha_2}^q - \mu_{\alpha_1}^q \cdot \mu_{\alpha_2}^q)^{1/q}, \nu_{\alpha_1}\dot{\nu}_{\alpha_2}\big)$;
3. $\alpha_1 \otimes_q \alpha_2 = \big(\mu_{\alpha_1} \cdot \mu_{\alpha_2}, (\nu_{\alpha_1}^q + \nu_{\alpha_2}^q - \nu_{\alpha_1}^q \cdot \nu_{\alpha_2}^q)^{1/q}\big)$;
4. $\mathrm{Min}(\alpha_1, \alpha_2) = \big(\min(\mu_{\alpha_1}, \mu_{\alpha_2}), \max(\nu_{\alpha_1}, \nu_{\alpha_2})\big)$;
5. $\mathrm{Max}(\alpha_1, \alpha_2) = \big(\max(\mu_{\alpha_1}, \mu_{\alpha_2}), \min(\nu_{\alpha_1}, \nu_{\alpha_2})\big)$;
6. $\lambda \cdot \alpha = ((1 - (1 - \mu_\alpha^q)^\lambda)^{1/q}, \nu_\alpha^\lambda), \quad \lambda > 0$;
7. $\alpha^\lambda = (\mu_\alpha^\lambda, (1 - (1 - \nu_\alpha^q)^\lambda)^{1/q}), \quad \lambda > 0$.

We define the distance between the q-rung orthopiar fuzzy numbers $\alpha_1, \alpha_2 \in L_{q\text{-ROFNs}}$:

$$d_q(\alpha_1, \alpha_2) = 1/2 \cdot (|(\mu_{\alpha_1})^q - (\mu_{\alpha_2})^q + |(\nu_{\alpha_1})^q - (\nu_{\alpha_2})^q|). \tag{3}$$

It is not difficult to prove that this measure satisfies all properties of a distance function.

2. Description of TOPSIS Approach to Facility Location Selection Problem with Q-Rung Orthopair Fuzzy Information

Location planning for candidate centers is vital in minimizing traffic congestion arising from facility movement in extreme environment. In recent years, transport activity has grown tremendously and this has undoubtedly affected the travel and living conditions in difficult and extreme urban areas. Considering the growth in the number of freight movements and their negative impacts on residents and the environment, municipal administrations are implementing sustainable freight regulations like restricted delivery timing, dedicated delivery zones, congestion charging etc. With the implementation of these regulations, the logistics operators are facing new challenges in location planning for service centers. For example, if service centers are located close to customer locations, then they increase traffic congestion in the urban areas. If they are located far from customer locations, then the service costs for the operators result to be very high. Under these circumstances, it is clear that the location planning for service centers in extreme environment is a complex decision that involves consideration of multiple attributes like maximum customer coverage, minimum service costs, least impacts on geographical points' residents and the environment, and conformance to freight regulations of these points.

Timely servicing from emergency service centers to the affected geographical areas (demand points as customers, for example, critical infrastructure objects) is a key task of the emergency management system. Scientific research in this area focuses on distribution networks decision-making problems, which are known as a Facility Location Problem (FLP) [4]. FLP's models have to support the generation of optimal locations of service centers in complex and uncertain situations. There are several publications about application of fuzzy methods in the FLP. However, all of them have a common approach. They represent parameters as fuzzy values (triangular fuzzy numbers [5] and others) and develop methods for facility location problems called in this case Fuzzy Facility Location Problem (FFLP). Fuzzy TOPSIS approaches for facility location selection problem for different fuzzy environments are developed in [3,8,10,12,17,18]. In this work we consider a new model of FFLP based on the q-rung orthopair fuzzy TOPSIS approach for the optimal selection of facility location centers.

This section first introduces the MCDM problem under q-rung orthopair fuzzy environment. Then, an effective decision-making approach is proposed to deal with such MCDM problems. At length, an algorithm of the proposed method is also presented

First, we are focusing on a multi-attribute decision making approach for location planning for service centers under uncertain and extreme environment. We develop a fuzzy multi-attribute decision making approach for the service center location selection problem for which a fuzzy TOPSIS approach is used.

The formation of expert's input data for construction of attributes is an important task of the centers' selection problem. To decide on the location of service centers, it is assumed that a set of *candidate sites* (CSs) already exists. This set is denoted by $CS = \{cs_1, cs_2, \dots, cs_m\}$, where we can locate service centers, and $S = \{s_1, s_2, \dots, s_n\}$ is the set of all attributes (transformed in benefit attributes) which define CS's selection. For example: "access by public and special transport modes to the candidate site", "security of the candidate site from accidents, theft and vandalism", "connectivity of the location with other modes of transport (highways, railways, seaports, airports, etc.)", "costs in vehicle resources, required products and etc. for the location of a candidate site", "impact of the candidate site location on the environment, such as important objects of Critical Infrastructure, air pollution and others", "proximity of the candidate site location from the central locations", "proximity of the candidate site location from customers", "availability of raw material and labor resources in the candidate site", "ability to conform to sustainable freight regulations imposed by managers for e.g. restricted delivery hours, special delivery zones", "ability to increase size to accommodate growing customers" and others. Let $W = \{w_1, w_2, \dots, w_n\}$ be the weights of attributes. For each expert e_k from invited group of experts (service dispatchers and so on) $E = \{e_1, e_2, \dots, e_t\}$, let α_{ij}^kbe the fuzzy rating of his/her evaluation in q-ROFNs for each candidate site cs_i $(i = 1, \dots, m)$, with respect to each attribute s_j $(j = 1, \dots, n)$. For the expert e_k we construct binary fuzzy relation $A_k = \{\alpha_{ij}^k,\ i = 1, \dots, m;\ j = 1, \dots, n\}$ decision making matrix, elements of which are represented in q-ROFNs. If some attribute s_j is cost type, then we transform experts' evaluations and α_{ij}^k is changed by $(\alpha_{ij}^k)^c$. Experts' data must be aggregated in etalon decision making matrix $-A = \{\alpha_{ij},$ $i = 1, \dots, m;\ j = 1, \dots, n\}$. Our task is to build fuzzy TOPSIS approach, which for each candidate site cs_i $(i = 1, \dots, m)$ aggregates presented objective and subjective data into scalar values − site's selection index. This aggregation can be formally represented as a TOPSIS "relative closeness of the alternative" defined on α_{ij}, $j = 1, \dots, n$:

$$\begin{aligned}\delta_i &= \text{relative closeness of the alternative } (cs_i)\\ &\equiv \text{TOPSIS aggregation } (\alpha_{i1}, \dots, \alpha_{in}),\ \ i = 1, \dots, m.\end{aligned} \tag{4}$$

The proposed framework of location planning for candidate sites comprises the following steps:

Step 1: Selection of location attributes. Involves the selection of location attributes for evaluating potential locations for candidate sites. These attributes are obtained from discussion with experts and members of the city transportation group. We use five attributes (n=5) defined above by short names: s_1 ="Accessibility", s_2 ="Security", s_3 ="Connectivity to multimodal transport", s_4 ="Costs", s_5 ="Proximity to customers". The fourth attribute is cost type and the others are benefit types. As mentioned above, cost type evaluation data must be transformed in the benefit forms.

Step 2: Selection of candidate location sites. Involves selection of potential locations for implementing service centers. The decision makers use their knowledge, prior experience in transportation or other aspects of the geographical area of extreme events and the presence of sustainable freight regulations to identify candidate locations for implementing service centers. For example, if certain areas are restricted for delivery by municipal administration, then these areas are barred from being considered as potential locations for implementing urban service centers. Ideally, the potential locations are those that cater for the interest of all city stakeholders, which are city residents, logistics operators, municipal administrations, etc.

Step 3: Assignment of ratings to the attributes with respect to the candidate sites. Let $A_k = \{\alpha_{ij}^k \in$ q-ROFNs, $i = 1, \dots, m;\ j = 1, \dots, n\}$ be the performance ratings of each expert e_k $(k = 1, 2, \dots, t)$ for each candidate site cs_i $(i = 1, 2, \dots, m)$ with respect to attributes s_j $(j = 1, 2, \dots, n)$.

Step 4: Computation of the q-ROF decision matrix for the attributes and the candidate sites. Let the ratings of all experts be described by positive numbers ω_k, $\omega_k > 0$, $k = 1, \dots, t$. If ratings of the attributes evaluated by the k-th expert are α_{ij}^k, then the aggregated fuzzy ratings (α_{ij}) of candidate sites with respect to each attribute are given by q-ROF weighted sum

$$\alpha_{ij} = \sum_{k=1}^{t} {}^{\oplus_q} \alpha_{ij}^k \left(\omega_k \Big/ \sum_{l=1}^{t} \omega_l \right). \tag{5}$$

The fuzzy decision matrix $\{\alpha_{ij}\}$ for the candidate sites CS and the attributes S is constructed as follows:

$$\begin{array}{c} \\ cs_1 \\ cs_2 \\ \dots \\ cs_m \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & \dots & s_n \end{array} \\ \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix} \end{array}$$

Construct the q-rung fuzzy decision matrix $\{\alpha_{ij}\}$ and calculate Sc and Ac functions values (Definition 2) of elements α_{ij}.

Step 5: Identification of q-rung orthopair fuzzy PIS and NIS. TOPSIS approach starts with the definition of the q-rung orthopair fuzzy PIS and the q-rung orthopair fuzzy NIS. Using formulas (1), (2) the PIS is defined as a q-rung orthopair fuzzy set on attributes S: $cs^+ = \{s_j,\ \alpha_j^+ \equiv \text{Max}_i[(\alpha_{ij})] \mid j = 1, 2, \dots, n\}$ and the NIS is defined as a q-rung orthopair fuzzy set on attributes S: $cs^- = \{s_j,\ \alpha_j^- \equiv \text{Min}_i[(\alpha_{ij})] \mid j = 1, 2, \dots, n\}$. In the real MCDM models PIS and NIS are usually not be feasible alternatives. They are extreme alternatives.

Step 6. Calculate the distances between the alternative candidate location site and the q-rung orthopair fuzzy PIS, as well as q-rung orthopair fuzzy NIS, respectively. Then, we proceed to calculate the distances between each alternative and q-rung orthopair fuzzy PIS and NIS. Using equation (3), we define distances between the alternative cs_i and the q-rung orthopair fuzzy PIS and NIS, as a weighted sums of distances between extreme and evaluated q-ROFNs:

$$D(cs_i, sc^+) = \sum_{j=1}^{n} w_j d_q(\alpha_{ij}, \alpha_j^+) = 1/2 \cdot \sum_{j=1}^{n} w_j (|(\mu_{\alpha_{ij}})^q - (\mu_{\alpha_j^+})^q| + |(\nu_{\alpha_{ij}})^q - (\nu_{\alpha_j^+})^q|),$$

$$D(cs_i, sc^-) = \sum_{j=1}^{n} w_j d_q(\alpha_{ij}, \alpha_j^-) = 1/2 \cdot \sum_{j=1}^{n} w_j (|(\mu_{\alpha_{ij}})^q - (\mu_{\alpha_j^-})^q| + |(\nu_{\alpha_{ij}})^q - (\nu_{\alpha_j^-})^q|),$$

Step 7. Calculate the revised closeness or TOPSIS aggregation as a site's selection index for every alternative. In general, the bigger $D(cs_i, sc^-)$ and the smaller $D(cs_i, sc^+)$ the better is the alternative cs_i. In the classical TOPSIS method, the authors usually need to calculate the relative closeness (RC) of the alternative cs_i. We define candidate site's selection index as RC with respect to the q-rung orthopair PIS sc^+ as follows:

$$\delta_i \equiv RC(cs_i) = \frac{D(cs_i, cs^+)}{D(cs_i, cs^+) + D(cs_i, cs^-)}, \quad i = 1, \dots, m. \tag{6}$$

3. Multi-Objective Optimization Model of Facility Location Set Covering Problem

The location set covering problem (LSCP) proposed by C. Toregas and C. Revell in 1972, seeks for a solution for locating the least number of facilities to cover all demand points within the service distance. In some of our works we are focusing on the multi-objective fuzzy set covering problems [9, 11] for extreme conditions. In this work, we construct new fuzzy LSCP model for emergency service facility location planning.

As we discussed in the previous section, the constructed Fuzzy TOPSIS technology forms center's selection rational index. The center's index reflects expert evaluations with respect to the center,

TABLE 1. Fuzzy travel times $\widetilde{t}_{ij}$ from fire stations to critical infrastructure objects (in minutes)

	a_1	a_2	a_3	a_4	a_5	a_6
cs_1	$(3,5,7)$	$(2,4,6)$	$(4,6,7)$	$(4,7,9)$	$(1,3,5)$	$(1,3,4)$
cs_2	$(6,10,14)$	$(4,9,14)$	$(2,4,6)$	$(5,7,10)$	$(1,4,8)$	$(1,4,5)$
cs_3	$(4,8,12)$	$(4,7,11)$	$(4,6,9)$	$(2,4,7)$	$(4,7,10)$	$(4,6,8)$
cs_4	$(4,7,10)$	$(7,11,15)$	$(6,9,13)$	$(4,6,8)$	$(2,4,6)$	$(1,3,5)$
cs_5	$(1,3,5)$	$(2,4,6)$	$(1,3,6)$	$(2,4,7)$	$(4,6,8)$	$(5,9,12)$

considering all actual attributes. If $x = \{x_1, x_2, \ldots, x_m\}$ is the Boolean decision vector that defines some selection from candidate centers $CS = \{cs_1, cs_2, \ldots, cs_m\}$ for facility location, we can build centers' selection index as a linear sum of $\delta_j x_j$ values: as a result, new objective function – *centers' selection index* $\sum_{j=1}^{m} \delta_j x_j$ is constructed. Maximizing it, we will be able to select a group of centers with the best total ranking from admissible covering selections. Classical facility location set covering problem tries to *minimize the number of centers*, where service facilities can be located – $\sum_{j=1}^{m} x_j$. The problem aims to locate service facilities in minimal travel time from candidate centers. Let customers covered by service centers in distribution networks be denoted by $A = \{a_1, \ldots, a_k\}$. The problem aims to locate service facilities in minimal travel time from candidate sites. Let experts evaluated movement fuzzy times (evaluated in triangular fuzzy numbers (TFNs) [5]) between customer and candidate sites be $\widetilde{t}_{ij}$, $a_i \in A$; $cs_j \in CS$. In extreme environment for emergency planning a radius of service center is defined based not on distance but on maximum allowed time T for movement, since the rapid help and servicing is crucial for customers in such situations. Respectively, a set of candidate sites N_i, covering customer $a_i \in A$, is defined as $N_i = \{cs_j, cs_j \in CS/E(\widetilde{t}_{ij}) \leq T\}$, $i = 1, \ldots, m$, where

$$E(\widetilde{t}_{ij}) = \widetilde{t}_{ij}^2 + (\widetilde{t}_{ij}^3 - 2\widetilde{t}_{ij}^2 + \widetilde{t}_{ij}^1)/4,$$

is an expected value of a TFN $\widetilde{t}_{ij} \equiv (\widetilde{t}_{ij}^1, \widetilde{t}_{ij}^2, \widetilde{t}_{ij}^3)$. Then we can state bi-objective facility location set covering problem:

$$\min z_1 = \sum_{j=1}^{m} x_j, \quad \max z_2 = \sum_{j=1}^{m} \delta_j x_j, \tag{7}$$

$$\sum_{s_j \in N_i} x_j \geq 1 \ \ (i = 1, 2, \ldots, k); \quad x_j \in \{0, 1\}, \quad j = 1, 2, \ldots, m.$$

Based on the epsilon-constraint approach, an algorithm of finding all Pareto solutions [6] is constructed (omitted here).

4. Numerical Simulation of Emergency Service Facility Location Model

We illustrate the effectiveness of the constructed optimization model by the numerical example. Let us consider an emergency management administration of a city in Georgia that wishes to locate some fire stations with respect to timely servicing of critical infrastructure objects. Assume that there are 6 demand points as customers (critical infrastructure objects) and 5 candidate facility centers (fire stations) in the urban area. Let there be 4 experts from Emergency Management Agency (EMA) of Georgia for the evaluation of travel times and the ranking of candidate facility centers. The travel times between demand points and candidate centers are evaluated in triangular fuzzy numbers (see Table 1). According to the standards of EMA (Georgia), the principle of location fire stations is that the fire station can reach the area edge within 5 minutes after receiving the dispatched instruction. Therefore, we set covering radius $T = 5$ minutes.

Table 2. Appraisal matrix A_1 by expert-1

	s_1	s_2	s_3	s_4	s_5
cs_1	$(0.7, 0.5)$	$(0.8, 0.3)$	$(0.7, 0.4)$	$(0.7, 0.4)$	$(0.8, 0.4)$
cs_2	$(0.6, 0.5)$	$(0.7, 0.4)$	$(0.4, 0.6)$	$(0.8, 0.4)$	$(0.7, 0.4)$
cs_3	$(0.7, 0.5)$	$(0.9, 0.5)$	$(0.9, 0.7)$	$(0.7, 0.4)$	$(0.8, 0.5)$
cs_4	$(0.6, 0.5)$	$(0.8, 0.4)$	$(0.8, 0.5)$	$(0.9, 0.5)$	$(0.8, 0.5)$
cs_5	$(0.8, 0.6)$	$(0.7, 0.4)$	$(0.9, 0.5)$	$(0.7, 0.4)$	$(0.8, 0.6)$

Table 3. Appraisal matrix A_2 by expert-2

	s_1	s_2	s_3	s_4	s_5
cs_1	$(0.7, 0.5)$	$(0.8, 0.4)$	$(0.6, 0.3)$	$(0.6, 0.3)$	$(0.7, 0.4)$
cs_2	$(0.6, 0.5)$	$(0.7, 0.3)$	$(0.7, 0.4)$	$(0.9, 0.4)$	$(0.8, 0.4)$
cs_3	$(0.8, 0.5)$	$(0.9, 0.5)$	$(0.6, 0.4)$	$(0.8, 0.4)$	$(0.6, 0.2)$
cs_4	$(0.6, 0.4)$	$(0.8, 0.3)$	$(0.9, 0.6)$	$(0.7, 0.3)$	$(0.6, 0.2)$
cs_5	$(0.9, 0.7)$	$(0.7, 0.4)$	$(0.9, 0.4)$	$(0.7, 0.3)$	$(0.9, 0.6)$

Table 4. Appraisal matrix A_3 by expert-3

	s_1	s_2	s_3	s_4	s_5
cs_1	$(0.7, 0.4)$	$(0.8, 0.3)$	$(0.7, 0.5)$	$(0.7, 0.4)$	$(0.9, 0.5)$
cs_2	$(0.6, 0.5)$	$(0.7, 0.4)$	$(0.5, 0.3)$	$(0.7, 0.2)$	$(0.6, 0.3)$
cs_3	$(0.6, 0.2)$	$(0.9, 0.6)$	$(0.7, 0.5)$	$(0.7, 0.3)$	$(0.6, 0.3)$
cs_4	$(0.8, 0.4)$	$(0.9, 0.4)$	$(0.8, 0.5)$	$(0.8, 0.5)$	$(0.8, 0.3)$
cs_5	$(0.9, 0.7)$	$(0.6, 0.3)$	$(0.9, 0.5)$	$(0.9, 0.6)$	$(0.7, 0.4)$

Covering sets of candidate sites N_i are defined (omitted here). Let experts generated the attributes weights as values of overall importance be based on the consensus:

$$w_1 = 0.25; \quad w_2 = 0.15; \quad w_3 = 0.25; \quad w_4 = 0.20; \quad w_5 = 0.15.$$

Each expert e_k $(k = 1, 2, 3)$ presented the ratings r_{ij}^k for each candidate center s_i $(i = 1, \dots, 5)$ with respect to each attribute s_j $(j = 1, \dots, 5)$.

Let experts have equal ratings $\{\omega_j = 1/3\}$. Using formula (5), experts' evaluations are aggregated in decision making matrix $\{\alpha_{ij}\}$ (Table 5).

Using the algorithm from Section 2 of new fuzzy TOPSIS, we calculated values of candidate centers' selection indices: $\delta_1 = 0.472$, $\delta_2 = 0.803$, $\delta_3 = 0.441$, $\delta_4 = 0.455$, $\delta_5 = 0.377$. After these calculations

Table 5. Accumulated q-rung orthopair fuzzy decision matrix $\{\alpha_{ij}\}$

	s_1	s_2	s_3	s_4	s_5
cs_1	$(0.70, 0.46)$	$(0.80, 0.33)$	$(0.67, 0.39)$	$(0.67, 0.36)$	$(0.83, 0.43)$
cs_2	$(0.60, 0.50)$	$(0.70, 0.36)$	$(0.58, 0.42)$	$(0.83, 0.32)$	$(0.72, 0.36)$
cs_3	$(0.72, 0.37)$	$(0.90, 0.53)$	$(0.79, 0.52)$	$(0.74, 0.36)$	$(0.70, 0.31)$
cs_4	$(0.70, 0.43)$	$(0.84, 0.36)$	$(0.84, 0.53)$	$(0.83, 0.42)$	$(0.76, 0.31)$
cs_5	$(0.88, 0.66)$	$(0.67, 0.36)$	$(0.90, 0.46)$	$(0.80, 0.42)$	$(0.83, 0.52)$

the following Combinatorial Programming Problem (7) has been constructed:

$$\begin{cases} f_1 = x_1 + x_2 + x_3 + x_4 + x_5 \Rightarrow \min, \\ f_2 = 0.472x_1 + 0.803x_2 + 0.441x_3 + 0.455x_4 + 0.377x_5 \Rightarrow \max, \\ x_1 + x_5 \geq 1, \\ x_2 + x_5 \geq 1, \\ x_3 + x_5 \geq 1, \\ x_1 + x_2 + x_4 \geq 1, \\ x_i \in 0, 1, \quad i = 1, 2, 3, 4, 5. \end{cases} \tag{8}$$

Based on the developed software for problem (8), the Pareto solutions [6] are founded. They are:

a) $cs_1, cs_5, \quad f_1 = 2; \quad f_2 = 1.18,$

b) $cs_1, cs_2, cs_3, \quad f_1 = 3; \quad f_2 = 1.716,$

c) $cs_1, cs_2, cs_3, cs_4, \quad f_1 = 4; \quad f_2 = 2.171,$

d) $cs_1, cs_2, cs_3, cs_4, cs_5, \quad f_1 = 5, \quad f_2 = 2.548.$

It is clear that increasing of fire stations number in Pareto solutions results in a more better level of the second objective function – *fire stations' selection index*. But the decision on the choice of the fire stations as service centers depends on the decision making person's preferences with respect to risks of administrative actions.

5. Conclusions

The paper presented new approach for the facility location problem for selection of the locations of service centers in extreme and uncertain situations. The approach utilizes experts knowledge represented by q-rung orthopair fuzzy numbers and considers the suitability of central location (i.e., affordability, security, etc.) using constructed new fuzzy TOPSIS approach. On the other hand, the model also considers the necessity to reach all critical infrastructure points and time that is required to reach them, presented by triangular fuzzy numbers. As a result, the bi-objective set covering problem is obtained. The constructed approach is illustrated by a numerical example for locating fire stations servicing critical infrastructure points in a city in Georgia. For the constructed problem, the Pareto solutions are obtained. For the large-dimension cases of the problem, the epsilon-constraint approach for the Pareto front obtaining is constructed.

Acknowledgments

This work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG) [FR-18-466].

References

1. K. T. Atanassov, Intuitionistic fuzzy sets. *Fuzzy Sets and Systems* **20** (1986), no. 1, 87–96.
2. R. E. Bellman, L. A. Zadeh, Decision-making in a fuzzy environment. *Management Sci.* **17** (1970/71), B141–B164.

3. T. C. Chu, Facility location selection using fuzzy TOPSIS under group decisions. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems* **10** (2002), no. 6, 687–701.
4. M. S. Daskin, *Network and Discrete Location. Models, Algorithms, and Applications*. Second edition. John Wiley & Sons, Inc., Hoboken, NJ, 2013.
5. D. Dubois, H. Prade, *Possibility Theory. An Approach to Computerized Processing of Uncertainty*. Plenum Press, New York, 1988.
6. M. Ehrgott, *Multicriteria Optimization*. Second edition. Springer-Verlag, Berlin, 2005.
7. C. L. Hwang, K. Yoon, *Multiple Attribute Decision Making. Methods and Applications*. A state-of-the-art survey. Lecture Notes in Economics and Mathematical Systems, 186. Springer-Verlag, Berlin-New York, 1981.
8. G. R. Jahanshahloo, F. Hosseinzadeh Lotfi, M. Izadikhah, Extension of the TOPSIS method for decision-making problems with fuzzy data. *Applied Mathematics and Computation* **181** (2006), no. 2, 1544–1551.
9. G. Sirbiladze, B. Ghvaberidze, B. Matsaberidze, Bicriteria fuzzy vehicle routing problem for extreme environment. *Bull. Georgian Natl. Acad. Sci. (N.S.)* **8** (2014), no. 2, 41–48.
10. G. Sirbiladze, B. Ghvaberidze, B. Matsaberidze, A. Sikharulidze, Multi-objective emergency service facility location problem based on fuzzy TOPSIS. *Bull. Georgian Natl. Acad. Sci. (N.S.)* **11** (2017), no. 1, 23–30.
11. G. Sirbiladze, A. Sikharulidze, B. Ghvaberidze, B. Matsaberidze, Fuzzy-probabilistic aggregations in the discrete covering problem. *Int. J. Gen. Syst.* **40** (2011), no. 2, 169–196.
12. Y. J. Wang, H. S. Lee, Generalizing TOPSIS for fuzzy multiple-criteria group decision-making. *Comput. Math. Appl.* **53** (2007), no. 11, 1762–1772.
13. R. R. Yager, Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems* **22** (2013), no. 4, 958–965.
14. R. R. Yager, Pythagorean fuzzy subsets. *Joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS)*, pp. 57-61. IEEE, 2013.
15. R. R. Yager, Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems* **25** (2016), no. 5, 1222–1230.
16. R. R. Yager, N. Alajlan, Y. Bazi, Aspects of generalized orthopair fuzzy sets. *International Journal of Intelligent Systems* **33** (2018), no. 11, 2154–2174.
17. D. Yong, Plant location selection based on fuzzy TOPSIS. *The International Journal of Advanced Manufacturing Technology* **28** (2006), no. 7-8, 839–844.
18. X. Zhang, Z. Xu, Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *International Journal of Intelligent Systems* **29** (2014), no. 12, 1061–1078.

(Received 01.08.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 147–157

BI–LAPLACE–BELTRAMI EQUATION ON A HYPERSURFACE

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Abstract. We investigate the boundary value problems for the bi–Laplace–Beltrami equation on a smooth bounded surface $\mathscr{C}$ with a smooth boundary in non-classical setting in the Bessel potential space $\mathbb{H}^s_p(\mathscr{C})$ for $s > \frac{1}{p}$, $1 < p < \infty$. To the initial BVP we apply a quasi-localization and obtain a model BVP for the bi-Laplacian. The model BVP on the half-plane is investigated by the potential method and is reduced to an equivalent system in Sobolev–Slobodečkii space. Boundary integral equations are investigated in both Bessel potential and Sobolev–Slobodečkii spaces. The property of the obtained system in the non-classical setting is derived, as well.

Introduction

Let $\mathscr{S} \subset \mathbb{R}^3$ be some smooth closed orientable surface, bordering a compact inner Ω^+ and an outer $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$ domain. By $\mathscr{C}$ we denote a subsurface of $\mathscr{S}$, which has two faces $\mathscr{C}^-$ and $\mathscr{C}^+$ and inherits the orientation from $\mathscr{S}$: $\mathscr{C}^+$ borders the inner domain Ω^+ and $\mathscr{C}^-$ borders the outer domain Ω^-. $\mathscr{C}$ has the smooth boundary $\Gamma := \partial\mathscr{C}$.

Let $\boldsymbol{\nu}(\omega) = \big(\nu_1(\omega), \nu_2(\omega), \nu_3(\omega)\big)^\top$, $\omega \in \overline{\mathscr{C}}$ be the unit normal vector field on the surface $\mathscr{C}$ and $\partial_{\boldsymbol{\nu}} = \sum\limits_{j=1}^{3} \nu_j \partial_j$ be the normal derivative. Let us consider the bi–Laplace–Beltrami operator in $\mathscr{C}$ written in terms of the Günter's tangent derivatives (see [7, 9, 10] for more details)

$$\Delta^2_{\mathscr{C}} := \sum_{j,k=1}^{3} \mathscr{D}^2_j \mathscr{D}^2_k, \quad \mathscr{D}_j := \partial_j - \nu_j \partial_{\boldsymbol{\nu}}, \quad j = 1, 2, 3. \tag{0.1}$$

Let $\boldsymbol{\nu}_\Gamma(t) = (\nu_{\Gamma,1}(t), \nu_{\Gamma,2}(t), \nu_{\Gamma,3}(t))^\top$, $t \in \Gamma$, be the unit normal vector field on the boundary Γ, which is tangential to the surface $\mathscr{C}$ and directed outside of the surface. Let, finally, $\partial_{\boldsymbol{\nu}_\Gamma} := \sum\limits_{j=1}^{3} \nu_{\Gamma,j} \mathscr{D}_j$ denote the corresponding normal derivative on the boundary Γ.

We study the following boundary value problem for the bi–Laplace–Beltrami equation

$$\begin{cases} \Delta^2_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(s) = g(s), & \text{on} \quad \Gamma, \\ (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(s) = h(s), & \text{on} \quad \Gamma, \end{cases} \tag{0.2}$$

where u^+ and $(\partial_{\boldsymbol{\nu}_\Gamma} u)^+$ denote the traces on the boundary.

We need the Bessel potential $\mathbb{H}^s_p(\mathscr{S})$, $\mathbb{H}^s_p(\mathscr{C})$, $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ and Sobolev–Slobodečkii $\mathbb{W}^s_p(\mathscr{S})$, $\mathbb{W}^s_p(\mathscr{C})$, $\widetilde{\mathbb{W}}^s_p(\mathscr{C})$ spaces, where $\mathscr{S}$ is a closed smooth surface (without boundary), which contains $\mathscr{C}$ as a subsurface, $1 < p < \infty$, $s \in \mathbb{R}$. Let us commence with the definition of the Bessel potential space on the Euclidean space $\mathbb{H}^s_p(\mathbb{R}^n)$, defined as a subset of the space of Schwartz distributions $\mathbb{S}'(\mathbb{R}^n)$ endowed with the norm (see [14])

$$\big\|u\big|\mathbb{H}^s_p(\mathbb{R}^n)\big\| := \big\|\langle D\rangle^s u\big|L_p(\mathbb{R}^n)\big\|,$$

2010 *Mathematics Subject Classification.* 35J40, 35M12, 35J58, 45E05.
Key words and phrases. Bi–Laplace–Beltrami equation; Günter's tangential derivatives; Boundary value problems; Boundary condition; Bessel potential spaces.

where $\langle D\rangle^s := \mathscr{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathscr{F}$ is the Bessel potential and $\mathscr{F}$, $\mathscr{F}^{-1}$ are the Fourier transformations. For the definition of the Sobolev–Slobodečkii space $\mathbb{W}^s_p(\mathbb{R}^n) = \mathbb{B}^s_{p,p}(\mathbb{R}^n)$ (see [14]).

The spaces $\mathbb{H}^s_p(\mathscr{S})$ and $\mathbb{W}^s_p(\mathscr{S})$ are defined, in general, by a partition of the unity $\{\psi_j\}_{j=1}^{\ell}$ subordinated to some covering $\{Y_j\}_{j=1}^{\ell}$ of $\mathscr{S}$ and local coordinate diffeomorphisms (see [12, 14] for details)

$$\varkappa_j : X_j \to Y_j, \quad X_j \subset \mathbb{R}^2, \quad j = 1, \dots, \ell.$$

The space $\mathbb{W}^s_p(\mathscr{S})$ coincides with the trace space of $\mathbb{H}^{s+\frac{1}{p}}_p(\mathbb{R}^3)$ on $\mathscr{S}$ and it is known that $\mathbb{W}^s(\mathscr{S}) = \mathbb{H}^s(\mathscr{S})$ for $s \geq 0$, $1 < p < \infty$ (see [14]).

We use, as common, the notation $\mathbb{H}^s(\mathscr{S})$ and $\mathbb{W}^s(\mathscr{S})$ for the spaces $\mathbb{H}^s_2(\mathscr{S})$ and $\mathbb{W}^s_2(\mathscr{S})$ (the case $p = 2$).

The space $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$ is defined as the subspace of $\mathbb{H}^s_p(\mathscr{S})$ of those functions $\varphi \in \mathbb{H}^s_p(\mathscr{S})$, which are supported in the closed sub-surface $\operatorname{supp}\varphi \subset \overline{\mathscr{C}}$, whereas $\mathbb{H}^s_p(\mathscr{C})$ denotes the quotient space $\mathbb{H}^s_p(\mathscr{C}) := \mathbb{H}^s_p(\mathscr{S}) \big/ \widetilde{\mathbb{H}}^s_p(\mathscr{C}^c)$, and $\mathscr{C}^c := \mathscr{S} \setminus \overline{\mathscr{C}}$ is the complemented sub-surface. For $s > 1/p - 1$ the space $\mathbb{H}^s_p(\mathscr{C})$ can be identified with the space of those distributions φ on $\mathscr{C}$ which admit extensions $\ell\varphi \in \mathbb{H}^s_p(\mathscr{S})$, while $\mathbb{H}^s_p(\mathscr{C})$ is identified with the space $r_{\mathscr{C}}\mathbb{H}^s_p(\mathscr{S})$, where $r_{\mathscr{C}}$ is the restriction to the sub-surface $\mathscr{C}$ of $\mathscr{S}$.

For $s < 0$, the space is defined by duality, e.g., $\mathbb{H}^s_p(\mathscr{C}) = \big(\widetilde{\mathbb{H}}^{-s}_q(\mathscr{C})\big)'$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. The spaces $\widetilde{\mathbb{W}}^s_p(\mathscr{C})$ and $\mathbb{W}^s_p(\mathscr{C})$ are defined similarly.

The Bessel potential $\mathbb{H}^s_p(\Gamma)$, $\mathbb{H}^s_p(\Gamma_0)$, $\widetilde{\mathbb{H}}^s_p(\Gamma_0)$ and Sobolev–Slobodečkii $\mathbb{W}^s_p(\Gamma)$, $\mathbb{W}^s_p(\Gamma_0)$, $\widetilde{\mathbb{W}}^s_p(\Gamma_0)$ spaces on a closed contour Γ and an open arc Γ_0 are defined also similarly.

It is worth noting that for an integer $m = 1, 2, \dots$ the Bessel potential $\mathbb{H}^m_p(\mathscr{S})$ and Sobolev $\mathbb{W}^m_p(\mathscr{S})$ spaces coincide and the equivalent norm in both spaces is defined with the help of the Günter's derivatives (see [6, 7, 9] and cf. (0.1) for the Günter's derivatives $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3$):

$$\big\| u \,\big|\, \mathbb{W}^m_p(\mathscr{S}) \big\| := \Big[\sum_{|\alpha| \leqslant m} \big\| \mathscr{D}^\alpha u \,\big|\, L_p(\mathscr{S}) \big\|^p \Big]^{\frac{1}{p}}, \quad \text{where} \quad \mathscr{D}^\alpha := \mathscr{D}_1^{\alpha_1}\mathscr{D}_2^{\alpha_2}\mathscr{D}_3^{\alpha_3}.$$

Let us also consider $\widetilde{\mathbb{H}}^{-2}_0(\mathscr{C})$, a subspace of $\widetilde{\mathbb{H}}^{-2}(\mathscr{C})$, orthogonal to

$$\widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}) := \Big\{ f \in \widetilde{\mathbb{H}}^{-2}(\Omega) \;\big|\; (f, \varphi)_{\mathbb{L}^2(\Omega)} = 0, \quad \varphi \in \mathbb{C}^\infty_0(\Omega) \Big\}.$$

$\widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C})$ consists of those distributions from $\widetilde{\mathbb{H}}^{-2}(\mathscr{C})$ which are supported on Γ and $\widetilde{\mathbb{H}}^{-2}(\mathscr{C})$ decomposes into the following direct sum of the subspaces:

$$\widetilde{\mathbb{H}}^{-2}(\mathscr{C}) = \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}) \oplus \widetilde{\mathbb{H}}^{-2}_0(\mathscr{C}).$$

The space $\widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C})$ is nontrivial (see [12, §5.1]) and if the right-hand side f is chosen from the orthogonal subspace, the space $\widetilde{\mathbb{H}}^{-2}_0(\mathscr{C})$ guarantees the unique solvability of BVPs (cf. [12] and the next Theorem 0.1).

The Lax–Milgram Lemma applied to the BVP (0.2) gives the following result. Similar proofs see in [15].

Theorem 0.1. *The BVP* (0.2) *has a unique solution in the classical weak setting:*

$$u \in \mathbb{H}^2(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}^{-2}_\Gamma(\mathscr{C}), \quad g \in \mathbb{H}^{3/2}(\Gamma), \quad h \in \mathbb{H}^{1/2}(\Gamma). \tag{0.3}$$

From Theorem 0.1 we cannot even conclude that a solution is continuous. If we succeed in proving that a solution u belongs to the space $\mathbb{H}^2_p(\mathscr{C})$ for some $2 < p < \infty$, we can enjoy even a Hölder continuity of u. It is very important to know maximal smoothness of a solution as, for example, in designing approximation methods. To this end, we investigate the solvability properties of the BVP (0.2) in the following non-classical setting:

$$\begin{gathered} u \in \mathbb{H}^s_p(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}^{s-4}_p(\mathscr{C}) \cap \widetilde{\mathbb{H}}^{-2}_0(\mathscr{C}), \quad g \in \mathbb{H}^{s-1/p}_p(\Gamma), \\ h \in \mathbb{H}^{s-1-1/p}_p(\Gamma), \quad 1 < p < \infty, \quad s > \frac{1}{p} \end{gathered} \tag{0.4}$$

and find necessary and sufficient conditions of solvability.

To formulate the main theorem of the present work we need the following definition.

Definition 0.2. The BVP (0.2), (0.4) is Fredholm one if the homogeneous problem $f = g = h = 0$ has a finite number of linearly independent solutions and only a finite number of orthogonality conditions on the data f, g, h ensure the solvability of the BVP.

Theorem 0.3. *Let conditions* (0.4) *hold:*

a) *Then a solution to the BVP* (0.2) *is represented by the formula*

$$\begin{aligned} u(x) =& \boldsymbol{N}_{\mathscr{C}} f(x) + \boldsymbol{W}_{(0,\Gamma)} g(x) - \boldsymbol{W}_{(-1,\Gamma)} h(x) + \boldsymbol{W}_{(-2,\Gamma)} \varphi(x) \\ & - \boldsymbol{W}_{(-3,\Gamma)} \psi(x), \quad u \in \mathbb{H}^2_{\#}(\mathscr{C}), \quad x \in \mathscr{C}. \end{aligned} \tag{0.5}$$

Here $\boldsymbol{N}_{\mathscr{C}}$, $\boldsymbol{W}_{(j,\Gamma)}$, $\quad j = \overline{-3,1}$ *are the Newton's and layer potentials, defined below* (*see* (1.5)) *and* φ, ψ *in* (0.5) *are solutions to the following system of boundary pseudodifferential equations*

$$\begin{cases} \boldsymbol{V}^0_{(-2,\Gamma)} \varphi - \boldsymbol{V}^0_{(-3,\Gamma)} \psi = G & on \quad \Gamma, \\ \boldsymbol{V}^1_{(-1,\Gamma)} \varphi - \boldsymbol{V}^1_{(-2,\Gamma)} \psi = H & on \quad \Gamma, \end{cases} \tag{0.6}$$

$$\varphi \in \widetilde{\mathbb{H}}^r_p(\Gamma), \quad \psi \in \widetilde{\mathbb{H}}^{r-1}_p(\Gamma), \quad G \in \mathbb{H}^r_p(\Gamma), \quad H \in \mathbb{H}^{r-1}_p(\Gamma), \tag{0.7}$$

where $r = s - 1/p$, G *and* H *are the functions given in terms of* f, g, *and* h *in* (1.11) *in* §*1 below.*

b) *Vice versa: if* u *is a solution to the BVP* (0.2) *in the setting* (0.4), *then* $\varphi := u^+$, $\psi := (\partial_{\boldsymbol{\nu}} u)^+$ *are solutions to the system* (0.6).

c) *The system of equations* (0.6) *has a unique pair of solutions* $\varphi \in \widetilde{\mathbb{W}}^{3/2}(\Gamma)$ *and* $\psi \in \widetilde{\mathbb{W}}^{1/2}(\Gamma)$ *in the classical setting for* $p = 2$, $s = 2$.

The proof of Theorem 0.3 is exposed in §1.

The system of boundary pseudodifferential equations (0.6) we will consider also in the Sobolev–Slobodečkii space setting

$$\varphi \in \widetilde{\mathbb{W}}^r_p(\Gamma), \quad \psi \in \widetilde{\mathbb{W}}^{r-1}_p(\Gamma), \quad G \in \mathbb{W}^r_p(\Gamma), \quad H \in \mathbb{W}^{r-1}_p(\Gamma). \tag{0.8}$$

To formulate the theorem, consider the following model system of singular integral equations (SIEs) in two settings:

$$\begin{cases} iS_{\mathbb{R}} \psi_0(t) = G_0(t), \\ iS_{\mathbb{R}} \varphi_0(t) = H_0(t), & t \in \mathbb{R} \end{cases} \tag{0.9}$$

in the Sobolev–Slobodečkii

$$\varphi_0, \psi_0 \in \widetilde{\mathbb{W}}^{r-1}_p(\mathbb{R}), \quad G_0, H_0 \in \mathbb{W}^{r-1}_p(\mathbb{R}) \tag{0.10a}$$

and the Bessel potential space

$$\varphi_0, \psi_0 \in \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}), \quad G_0, H_0 \in \mathbb{H}^{r-1}_p(\mathbb{R}) \tag{0.10b}$$

settings. Here

$$S_{\mathbb{R}} v(t) := \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{v(\tau) d\tau}{\tau - t}, \quad v \in \mathbb{L}_p(\mathbb{R}) \tag{0.11}$$

is understood in the sense of Cauchy's principal value.

Theorem 0.4. *Let* $1 < p < \infty$, $r = s - \dfrac{1}{p} > -1$. *The system of boundary pseudodifferential equations* (0.6) *is Fredholm one in the Sobolev–Slobodečkii* (0.7) *and Bessel potential* (0.8) *space settings if the system of boundary integral equations* (0.9) *is locally invertible at* 0 *in the settings* (0.10a) *and* (0.10b), *respectively.*

Remark 0.5. Theorem 0.4 is proved at the end of §1. For the proof we apply a quasi-localization of the BVP (0.2) with the corresponding model BVP on the half-space (see Lemma 1.5). The constraint $r > -1$ is then natural since we deal with the boundary value problem.

In a forthcoming paper the problem will be treated by a direct application of the local quasi-equivalence to the equation (0.6).

A quasi-localization means "freezing coefficients" and "rectifying" underling contours and surfaces. For details of a quasi-localization we refer the reader to papers [13] and [1], where the quasi-localization is well described for singular integral operators and for BVPs, respectively. We also refer to [8, §3], where a short introduction to quasi-localization is exposed.

In the present case under consideration we get 2 different model problems by localizing the mixed BVP (0.2) to:

1 inner points of $\mathscr{C}$;
2 inner points on the boundary Γ.

The model BVPs obtained by a quasi-localization, are well investigated in the first case and such model problems have unique solutions without additional constraints. In the second case we get a mixed BVP on the half-plane for the bi-Laplace equation (cf. (1.13) below). System (0.9) is related to this model problem (1.13) just as the BVP (0.2) is related to system (0.6).

1. Potential Operators and Boundary Integral Equations

Let $\mathscr{S}$ be a closed, sufficiently smooth orientable surface in $\mathbb{R}^n$. We use the notation $\mathbb{X}^s_p(\mathscr{S})$ for either the Bessel potential $\mathbb{H}^s_p(\mathscr{S})$ or the Sobolev–Slobodečkii $\mathbb{W}^s_p(\mathscr{S})$ spaces for $\mathscr{S}$ closed or open and a similar notation $\widetilde{\mathbb{X}}^s_p(\mathscr{S})$ for $\mathscr{S}$ open.

Consider the space

$$\mathbb{X}^s_{p,\#}(\mathscr{S}) := \big\{\varphi \in \mathbb{X}^s_p(\mathscr{S}) \,:\, (\varphi, 1) = 0\big\}, \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the duality pairing between the adjoint spaces. It is obvious that $\mathbb{X}^s_{p,\#}(\mathscr{S})$ does not contain nonzero constants: if $c_0 = \mathrm{const} \in \mathbb{X}^s_{p,\#}(\mathscr{S})$ then

$$0 = (c_0, 1) = c_0(1, 1) = c_0 \mathrm{mes}\, \mathscr{S}$$

and $c_0 = 0$. Moreover, $\mathbb{X}^s_p(\mathscr{S})$ decomposes into the direct sum

$$\mathbb{X}^s_p(\mathscr{S}) = \mathbb{X}^s_{p,\#}(\mathscr{S}) + \{\mathrm{const}\} \tag{1.2}$$

and the dual (adjoint) space is

$$\big(\mathbb{X}^s_{p,\#}(\mathscr{S})\big)^* = \mathbb{X}^{-s}_{p',\#}(\mathscr{S}), \quad p' := \frac{p}{p-1}. \tag{1.3}$$

The following is a part of Theorem 10 proved in [10].

Theorem 1.1. *Let $\mathscr{S}$ be ℓ-smooth, $\ell = 1, 2, \dots,$ $1 < p < \infty$, and $|s| \leqslant \ell$. Let $\mathbb{X}^s_{p,\#}(\mathscr{S})$ be the same as in* (1.1)–(1.3). *The bi–Laplace–Beltrami operator $\Delta^2_{\mathscr{S}} := \Delta_{\mathscr{S}}\Delta_{\mathscr{S}}$ is invertible between the spaces with detached constants*

$$\Delta^2_{\mathscr{S}} \,:\, \mathbb{X}^{s+1}_{p,\#}(\mathscr{S}) \to \mathbb{X}^{s-1}_{p,\#}(\mathscr{S}), \tag{1.4}$$

i.e., has the fundamental solution $\mathscr{K}_{\mathscr{S}}$ in the setting (1.4).

Let $\mathscr{C} \subset \mathscr{S}$ be a subsurface with a smooth boundary $\Gamma := \partial\mathscr{C}$. With the fundamental solution $\mathscr{K}_{\mathscr{S}}$ of the bi–Laplace–Beltrami operator at hand we can consider on the surface $\mathscr{C}$ the standard layer potentials:

$$\begin{aligned}
\boldsymbol{N}_{\mathscr{C}} v(x) &:= \int\limits_{\mathscr{C}} \mathscr{K}_{\mathscr{S}}(x,y) v(y)\, d\sigma \\
\boldsymbol{W}_{(0,\Gamma)} v(x) &:= \int\limits_{\Gamma} (\partial_{\boldsymbol{\nu}_\Gamma} \Delta \mathscr{K}_{\mathscr{S}})(x,\tau) v(\tau) d\tau, \\
\boldsymbol{W}_{(-1,\Gamma)} v(x) &:= \int\limits_{\Gamma} (\Delta \mathscr{K}_{\mathscr{S}})(x,\tau) v(\tau) d\tau, \quad x \in \mathscr{C}, \\
\boldsymbol{W}_{(-2,\Gamma)} v(x) &:= \int\limits_{\Gamma} (\partial_{\boldsymbol{\nu}_\Gamma(\tau)} \mathscr{K}_{\mathscr{S}})(x,\tau) v(\tau) d\tau, \quad x \in \mathscr{C}, \\
\boldsymbol{W}_{(-3,\Gamma)} v(x) &:= \int\limits_{\Gamma} \mathscr{K}_{\mathscr{S}}(x,\tau) v(\tau) d\tau, \quad x \in \mathscr{C}.
\end{aligned} \tag{1.5}$$

The potential operators, defined above, have standard boundedness properties

$$\begin{aligned}
\boldsymbol{N}_{\mathscr{C}} &: \mathbb{H}^s_{p,\#}(\mathscr{C}) \longrightarrow \mathbb{H}^{s+4}_{p,\#}(\mathscr{C}), \\
\boldsymbol{W}_{(0,\Gamma)} &: \mathbb{H}^s_{p,\#}(\Gamma) \longrightarrow \mathbb{H}^{s+3+\frac{1}{p}}_{p,\#}(\mathscr{C}), \\
\boldsymbol{W}_{(-1,\Gamma)} &: \mathbb{H}^s_{p,\#}(\Gamma) \longrightarrow \mathbb{H}^{s+2+\frac{1}{p}}_{p,\#}(\mathscr{C}), \\
\boldsymbol{W}_{(-2,\Gamma)} &: \mathbb{H}^s_{p,\#}(\Gamma) \longrightarrow \mathbb{H}^{s+1+\frac{1}{p}}_{p,\#}(\mathscr{C}), \\
\boldsymbol{W}_{(-3,\Gamma)} &: \mathbb{H}^s_{p,\#}(\Gamma) \longrightarrow \mathbb{H}^{s+\frac{1}{p}}_{p,\#}(\mathscr{C})
\end{aligned}$$

and any solution to the mixed BVP (0.2) in the space $\mathbb{H}^2_{\#}(\mathscr{C}) := \mathbb{H}^2_{2,\#}(\mathscr{C})$ is represented as follows:

$$\begin{aligned}
u(x) =& \boldsymbol{N}_{\mathscr{C}} f(x) + \boldsymbol{W}_{(0,\Gamma)} u^+(x) - \boldsymbol{W}_{(-1,\Gamma)} (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(x) + \boldsymbol{W}_{(-2,\Gamma)} (\Delta u)^+(x) \\
&- \boldsymbol{W}_{(-3,\Gamma)} (\partial_{\boldsymbol{\nu}_\Gamma} \Delta u)^+(x), \quad u \in \mathbb{H}^2_{\#}(\mathscr{C}), \quad x \in \mathscr{C}.
\end{aligned} \tag{1.6}$$

Since $\mathbb{X}^s_p = \mathbb{X}^s_{p,\#} + \{\text{const}\}$, we can extend layer potentials to the entire space as follows:

$$\begin{aligned}
&\text{for} \quad \varphi = \varphi_0 + c, \quad \varphi_0 \in \mathbb{X}^s_{p,\#}, \quad c = \text{const}, \\
&\text{we set} \quad \boldsymbol{W}_{(j,\Gamma)} \varphi = \boldsymbol{W}_{(j,\Gamma)} \varphi_0 + c, \quad \boldsymbol{N}_{\mathscr{C}} f = \boldsymbol{N}_{\mathscr{C}} f_0 + c, \quad j = \overline{-3,0}
\end{aligned} \tag{1.7}$$

i.e., by setting $\boldsymbol{W}_{(j,\Gamma)} c = \boldsymbol{N}_{\mathscr{C}} c = c$.

Lemma 1.2. *The representation formula* (1.6) *remains valid for a solution in the space* $\mathbb{H}^2(\mathscr{C})$*, provided the potentials are extended as in* (1.7).

Proof. Indeed, since $u = u_0 + c$, $u_0 \in \mathbb{H}^s_{p,\#}(\mathscr{C})$, $u \in \mathbb{H}^s_p(\mathscr{C})$, we apply the extension formulae (1.7), the representation formula (1.6) for a solution in the space $\mathbb{H}^2_{\#}(\mathscr{C})$ and get the representation formula (1.6) for a solution in the space $\mathbb{H}^2(\mathscr{C})$:

$$\begin{aligned}
u(x) &= u_0(x) + c = \boldsymbol{N}_{\mathscr{C}} f_0(x) + \boldsymbol{W}_{(0,\Gamma)} u_0^+(x) - \boldsymbol{W}_{(-1,\Gamma)} (\partial_{\boldsymbol{\nu}_\Gamma} u_0)^+(x) \\
&\quad + \boldsymbol{W}_{(-2,\Gamma)} (\Delta u_0)^+(x) - \boldsymbol{W}_{(-3,\Gamma)} (\partial_{\boldsymbol{\nu}_\Gamma} \Delta u_0)^+(x) + c \\
&= \boldsymbol{N}_{\mathscr{C}} (f(x) - c) + \boldsymbol{W}_{(0,\Gamma)} (u - c)^+(x) - \boldsymbol{W}_{(-1,\Gamma)} (\partial_{\boldsymbol{\nu}_\Gamma} (u - c))^+(x) \\
&\quad + \boldsymbol{W}_{(-2,\Gamma)} (\Delta (u - c))^+(x) - \boldsymbol{W}_{(-3,\Gamma)} (\partial_{\boldsymbol{\nu}_\Gamma} \Delta (u - c))^+(x) + c \\
&= \boldsymbol{N}_{\mathscr{C}} f(x) + \boldsymbol{W}_{(0,\Gamma)} u^+(x) - \boldsymbol{W}_{(-1,\Gamma)} (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(x) \\
&\quad + \boldsymbol{W}_{(-2,\Gamma)} (\Delta u)^+(x) - \boldsymbol{W}_{(-3,\Gamma)} (\partial_{\boldsymbol{\nu}_\Gamma} \Delta u)^+(x), \quad u \in \mathbb{H}^1(\mathscr{C}), \quad x \in \mathscr{C}.
\end{aligned} \tag{1.8}$$

The lemma is proof. □

Proof of Theorem 0.3. Let us recall the Plemelji formulae

$$
\begin{aligned}
&(\boldsymbol{W}_{(0,\Gamma)}v)^{\pm}(t)=\pm\frac{1}{2}v(t)+\boldsymbol{W}_{(0,\Gamma)}v(t), \quad (\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{V}^0_{(0,\Gamma)}\psi)^{\pm}(t)=\boldsymbol{V}^1_{(+1,\Gamma)}v(t),\\
&(\boldsymbol{W}_{(-1,\Gamma)}v)^{\pm}(t)=\boldsymbol{V}^0_{(-1,\Gamma)}v(t), \quad (\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{(-1,\Gamma)})^{\pm}(t)=\mp\frac{1}{2}v(t)+\boldsymbol{V}^1_{(0,\Gamma)}v(t),\\
&(\boldsymbol{W}_{(-2,\Gamma)}v)^{\pm}(t)=\boldsymbol{V}^0_{(-2,\Gamma)}v(t), \quad (\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{(-2,\Gamma)})^{\pm}(t)=\boldsymbol{V}^1_{(-1,\Gamma)}v(t),\\
&(\boldsymbol{W}_{(-3,\Gamma)}v)^{\pm}(t)=\boldsymbol{V}^0_{(-3,\Gamma)}v(t), \quad (\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{(-3,\Gamma)})^{\pm}(t)=\boldsymbol{V}^1_{(-2,\Gamma)}v(t),
\end{aligned}
\tag{1.9}
$$

where $t\in\partial\Omega_\alpha$ and

$$
\begin{aligned}
\boldsymbol{V}^0_{(-3,\Gamma)}v(t)&:=\int\limits_\Gamma \mathscr{K}_{\mathscr{S}}(t,\tau)v(\tau)d\tau, \qquad t\in\Gamma,\\
\boldsymbol{V}^0_{(-2,\Gamma)}v(t)&:=\int\limits_\Gamma (\partial_{\boldsymbol{\nu}_{\Gamma(\tau)}}\mathscr{K}_{\mathscr{S}})(t,\tau)v(\tau)d\tau,\\
\boldsymbol{V}^1_{(-2,\Gamma)}v(t)&:=\int\limits_\Gamma (\partial_{\boldsymbol{\nu}_{\Gamma(t)}}\mathscr{K}_{\mathscr{S}})(t,\tau)v(\tau)d\tau,\\
\boldsymbol{V}^1_{(-1,\Gamma)}v(t)&:=\int\limits_\Gamma (\partial_{\boldsymbol{\nu}_{\Gamma(t)}}\partial_{\boldsymbol{\nu}_{\Gamma(\tau)}}\mathscr{K}_{\mathscr{S}})(t,\tau)v(\tau)d\tau,
\end{aligned}
\tag{1.10}
$$

are pseudodifferential operators on Γ, have orders -3, -2, -2 and -1, respectively, and represent the direct values of the corresponding potentials $\boldsymbol{W}_{-3,\Gamma}$, $\boldsymbol{W}_{-2,\Gamma}$, $\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{-3,\Gamma}$ and $\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{W}_{-2,\Gamma}$.

By applying the Plemelji formulae (1.9) to (1.6), we get

$$
\begin{cases}
u^+(t)=g(t)=(\boldsymbol{N}_{\mathscr{C}}f)^+ +\dfrac{1}{2}g(t)+\boldsymbol{V}^0_{(0,\Gamma)}g(t)-\boldsymbol{V}^0_{(-1,\Gamma)}h(t)\\
\quad +\boldsymbol{V}^0_{(-2,\Gamma)}\varphi(t)-\boldsymbol{V}^0_{(-3,\Gamma)}\psi(t),\\
(\partial_{\boldsymbol{\nu}_\Gamma}u)^+(t)=h(t)=(\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{N}_{\mathscr{C}}f)^+ +\boldsymbol{V}^1_{(+1,\Gamma)}g(t)+\dfrac{1}{2}h(t)-\boldsymbol{V}^1_{(0,\Gamma)}h(t)\\
\quad +\boldsymbol{V}^1_{(-1,\Gamma)}\varphi(t)-\boldsymbol{V}^1_{(-2,\Gamma)}\psi(t), \quad t\in\Gamma.
\end{cases}
$$

We obtain system (0.6), where

$$
\begin{aligned}
G&:=\left[\frac{1}{2}g-(\boldsymbol{N}_{\mathscr{C}}f)^+-\boldsymbol{V}^0_{(0,\Gamma)}g+\boldsymbol{V}^0_{(-1,\Gamma)}h\right]\in\mathbb{H}^{s-1/p}_p(\Gamma),\\
H&:=\left[\left(\frac{1}{2}h-\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{N}_{\mathscr{C}}f\right))^+-\boldsymbol{V}^1_{(0,\Gamma)}g+\boldsymbol{V}^1_{(-1,\Gamma)}h\right]\in\mathbb{H}^{s-1-1/p}_p(\Gamma).
\end{aligned}
\tag{1.11}
$$

Thus, we have proved the inverse assertion of Theorem 0.3: if u is a solution to the BVP (0.2), the functions φ and ψ are solutions to system (0.6).

The direct assertion is even easier to prove:

- the function in (1.8) represented by the potentials, satisfies the equation (0.2);
- if φ and ψ are solutions to system (0.6), using Plemelji formulae (1.9), it can easily be verified that u in (1.8) satisfies the boundary conditions in (0.2).

The existence and uniqueness of a solution to the BVP (0.2) in the classical setting (0.3) is stated in Theorem 0.1, while for system (0.6) it follows from the equivalence with the BVP (0.2). □

The remainder of the paper is devoted to the proof of solvability properties of the system (0.6) in the non-classical setting (0.4).

On the 2-dimensional Euclidean space we consider the following equation:

$$
\Delta^2u=f^0 \quad \text{on} \quad \mathbb{R}^2, \quad u\in\mathbb{H}^s_p(\mathbb{R}^2), \quad f^0\in\mathbb{H}^{s-4}_p(\mathbb{R}^2),
\tag{1.12}
$$

also the model

$$\begin{cases}\Delta^2 u(x) = f_1(x), & x \in \mathbb{R}^2_+,\\ u^+(t) = g_1(t), & t \in \mathbb{R},\\ -(\partial_2 u)^+(t) = h_1(t), & t \in \mathbb{R}.\end{cases} \tag{1.13}$$

boundary value problems for the Laplace equation on the upper half plane $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}^+$, where $\partial_{\boldsymbol{\nu}_\Gamma} = -\partial_2$ is the normal derivative on the boundary of $\mathbb{R}^2_+$.

The BVP (1.13) will be treated in the non-classical setting:

$$f_1 \in \widetilde{\mathbb{H}}^{s-4}_p(\mathbb{R}^2_+) \cap \widetilde{\mathbb{H}}^{-2}_0(\mathbb{R}^2_+), \quad g_1 \in \mathbb{H}^{s-1/p}_p(\mathbb{R}), \quad h_1 \in \mathbb{H}^{s-1-1/p}_p(\mathbb{R}), \\ 1 < p < \infty, \quad s > \frac{1}{p}. \tag{1.14}$$

Proposition 1.3. *The bi-Laplace equation* (1.12) *has a unique solution, as well.*

Proof. The assertion is a well-known classical result available in many textbooks on partial differential equations (see e.g. [12]). ☐

As a particular case of Theorem 0.1 (can easily be proved with the Lax–Milgram Lemma), we have the following

Proposition 1.4. *The BVP* (1.13) *has a unique solution u in the classical weak setting*

$$u \in \mathbb{H}^2(\mathbb{R}^2_+), \quad f_1 \in \widetilde{\mathbb{H}}^{-2}_0(\mathbb{R}^2_+), \quad g_1 \in \mathbb{H}^{3/2}(\mathbb{R}), \quad h_1 \in \mathbb{H}^{1/2}(\mathbb{R}),$$

Lemma 1.5. *The BVP* (0.2) *is Fredholm one in the non-classical setting* (0.4) *if the model mixed BVP* (1.13) *is locally Fredholm (i.e., is locally invertible) at* 0 *in the non-classical setting* (1.14).

Proof. We apply quasi-localization of the boundary value problem (0.2) in the more general non-classical setting (0.4), which includes the classical setting (0.3) as a particular case (see [1, 3] for details of quasi-localization of boundary value problems and also [2, 11, 13] for general results on localization and quasi-localization).

Prior proceeding with the quasi-localization let us explain shortly why the quasi-localization can be performed in the Bessel potential (Besov) spaces.

Localizing classes consist of multiplication operators by smooth functions and since localization is performed in quotient spaces modulo compact operators it suffices to note that smooth functions commute with the Bessel potential in quotient algebra (see [3]) and, therefore, their norms coincide with the norm in $\mathbb{L}_p$-space, i.e., with the supremum-norm. This makes localization (“freezing coefficients”) easy.

Concerning the “rectification”: since the difference of “pull-back” of the original operator and its local representative is locally compact in $\mathbb{L}_p$ and is bounded in the Bessel potential spaces $\mathbb{H}^s_p$, it is locally compact in all $\mathbb{H}^r_p$-spaces for $r < s$ (Krasnoselskij theorem).

By quasi-localization at the point $\omega \in \overline{\mathscr{C}}$ we first localize to the tangential plane $\mathbb{R}^2(\omega)$ (tangential half- plane $\mathbb{R}^2_+(\omega)$) to $\mathscr{C}$ at $\omega \in \mathscr{C}$ (at $\omega \in \Gamma = \partial\mathscr{C}$, respectively). The differential operators remain the same

$$\Delta^2_{\mathbb{R}^2} := \sum_{j,k=1}^{3} \mathscr{D}^2_j \mathscr{D}^2_k, \quad \mathscr{D}_j = \partial_j - \nu_j \partial_{\boldsymbol{\nu}}, \\ \partial_{\boldsymbol{\nu}} = \sum_{j=1}^{3} \nu_j \partial_j, \quad \partial_{\boldsymbol{\nu}_\Gamma} = \sum_{j=1}^{3} \nu_{\Gamma,j} \mathscr{D}_j, \tag{1.15}$$

but the normal vector $\boldsymbol{\nu}(\omega)$ to the tangent plane $\mathbb{R}^2$ and the normal vector $\boldsymbol{\nu}_\Gamma(\omega)$ to the boundary of the tangent plane $\mathbb{R}(\omega) = \partial\mathbb{R}^2_+(\omega)$ are now constant. Next, we rotate the tangent planes $\mathbb{R}^2(\omega)$ and $\mathbb{R}^2_+(\omega)$ to match them to the planes $\mathbb{R}^2$ and $\mathbb{R}^2_+$. The normal vector fields will transform into $\boldsymbol{\nu} = (0, 0, 1)$ and $\boldsymbol{\nu}_\Gamma = (0, -1, 0)$. The rotation is an isomorphism of the spaces $\mathbb{W}^r_p(\mathbb{R}^2(\omega)) \to \mathbb{W}^r_p(\mathbb{R}^2)$,

$\mathbb{W}^r_p(\mathbb{R}^2_+(\omega)) \to \mathbb{W}^r_p(\mathbb{R}^2_+)$, $\widetilde{\mathbb{W}}^r_p(\mathbb{R}^2_+(\omega)) \to \widetilde{\mathbb{W}}^r_p(\mathbb{R}^2_+)$ etc., and transforms the operators in (1.15) into the operators

$$\Delta^2_{\mathbb{R}^2(\omega)} \ \to\ \Delta^2 := \sum_{j,k=1}^{2} \partial_j^2\partial_k^2, \quad \mathscr{D}_j \ \to\ \partial_j, \quad j,k=1,2, \quad \mathscr{D}_3 \ \to\ 0,$$
$$\partial_{\boldsymbol{\nu}(\omega)} \ \to\ \partial_3, \quad \partial_{\boldsymbol{\nu}_\Gamma(\omega)} \ \to\ -\partial_2,$$

and we get (1.12), (1.13) as a local representatives of BVP (0.2).

For the BVP (0.2) in the non-classical setting (0.4) we get the following local quasi-equivalent equations and BVPs at different points of the surface $\omega \in \overline{\mathscr{C}}$:

i. the equation (1.12) at 0 if $\omega \in \mathscr{C}$ is an inner points of the surface;

iv. the mixed BVP (1.13) in the non-classical setting (1.14) at 0 if $\omega \in \Gamma$.

The main conclusion of the present theorem on Fredholm properties of BVPs (0.2) and (1.13) follows from Proposition 1.3 and the general theorem on quasi-localizaion (see [1–3,11,13]): *The BVP* (0.2), (0.4) *is Fredholm one if all local representatives* (1.12) *and* (1.13) *in non-classical settings are locally Fredholm (i.e., are locally invertible).* □

Now we concentrate on the model mixed BVP (1.13). To this end, let us recall that the function

$$\mathscr{K}^2_\Delta(x) := \frac{1}{8\pi}|x-y|^2 \ln|x-y|$$

is the fundamental solution to the bi-Laplace's equation in two variables

$$\begin{aligned} &\Delta^2 \mathscr{K}^2_\Delta(x) = \delta(x), \qquad x \in \mathbb{R}^2, \\ &\Delta^2 = (\partial_1^2 + \partial_2^2)^2 = (\partial_{\boldsymbol{\nu}}^2 + \partial_\ell^2)^2. \end{aligned} \tag{1.16}$$

Standard Newton and layer potential operators (cf. (1.5)) acquire the following forms:

$$\boldsymbol{N}_{\mathbb{R}^2_+} v(x) := \frac{1}{8\pi} \int\limits_{\mathbb{R}^2_+} |x-y|^2 \ln|x-y| v(y)\,dy,$$
$$\boldsymbol{W}_{(0,\mathbb{R})} v(x) := -\frac{1}{8\pi} \int\limits_{\mathbb{R}} \partial_{y_2}(\partial^2_{y_1} + \partial^2_{y_2}) \big|(x_1,x_2) - (\tau,y_2)\big|^2 \ln\big|(x_1,x_2) - (\tau,y_2)\big|\Big|_{y_2=0} v(\tau)\,d\tau,$$
$$\boldsymbol{W}_{(-1,\mathbb{R})} v(x) \ = \frac{1}{8\pi} \int\limits_{\mathbb{R}} (\partial^2_{y_1} + \partial^2_{y_2}) \big|(x_1,x_2) - (\tau,y_2)\big|^2 \ln\big|(x_1,x_2) - (\tau,y_2)\big|\Big|_{y_2=0} v(\tau)\,d\tau,$$
$$\boldsymbol{W}_{(-2,\mathbb{R})} v(x) := -\frac{1}{8\pi} \int\limits_{\mathbb{R}} \partial_{y_2} \big|(x_1,x_2) - (\tau,y_2)\big|^2 \ln\big|(x_1,x_2) - (\tau,y_2)\big|\Big|_{y_2=0} v(\tau)\,d\tau,$$
$$\boldsymbol{W}_{(-3,\mathbb{R})} v(x) := \frac{1}{8\pi} \int\limits_{\mathbb{R}} \big|x - (\tau,0)\big|^2 \ln\big|x-(\tau,0)\big| v(\tau)\,d\tau.$$

The pseudodifferential operators on $\boldsymbol{V}^0_{-3,\mathbb{R}}$, $\boldsymbol{V}^0_{-2,\mathbb{R}}$, $\boldsymbol{V}^1_{-2,\mathbb{R}}$ and $\boldsymbol{V}^1_{-1,\mathbb{R}}$ associated with the layer potentials (see (1.10)), acquire the form

$$\boldsymbol{V}^0_{(-3,\mathbb{R})} v(x) := \frac{1}{8\pi} \int\limits_{\mathbb{R}} \big((x_1-\tau)^2 + x_2^2\big) \ln\big((x_1-\tau)^2 + x_2^2\big)^{1/2} v(\tau)d\tau, \quad t \in \mathbb{R},$$
$$\begin{aligned} \boldsymbol{V}^0_{(-2,\mathbb{R})} v(x) &:= -\frac{1}{8\pi} \int\limits_{\mathbb{R}} \partial_{y_2} \big((x_1-\tau)^2 + (x_2-y_2)^2\big) \ln\big((x_1-\tau)^2 + (x_2-y_2)^2\big)^{1/2}\Big|_{y_2=0} v(\tau)d\tau \\ &:= \frac{(x_2-y_2)}{8\pi} \int\limits_{\mathbb{R}} \big(2\ln\big((x_1-\tau)^2 + (x_2-y_2)^2\big)^{1/2} + 1\big)\Big|_{y_2=0} v(\tau)d\tau \end{aligned}$$

$$:= \frac{x_2}{8\pi}\int\limits_{\mathbb{R}} \big(2\ln\big((x_1-\tau)^2+x_2^2\big)^{1/2}+1\big)v(\tau)d\tau,$$

$$\boldsymbol{V}^1_{(-2,\mathbb{R})}v(x) := -\frac{1}{8\pi}\int\limits_{\mathbb{R}} \partial_{x_2}\big((x_1-\tau)^2+(x_2-y_2)^2\big)\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}\Big|_{y_2=0}v(\tau)d\tau$$

$$:= \frac{(y_2-x_2)}{8\pi}\int\limits_{\mathbb{R}} \big(2\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}+1\big)\Big|_{y_2=0}v(\tau)d\tau$$

$$:= -\frac{x_2}{8\pi}\int\limits_{\mathbb{R}} \big(2\ln\big((x_1-\tau)^2+x_2^2\big)^{1/2}+1\big)v(\tau)d\tau,$$

$$\boldsymbol{V}^1_{(-1,\mathbb{R})}v(x) := -\frac{1}{8\pi}\int\limits_{\mathbb{R}} \partial^2_{x_2}\big((x_1-\tau)^2+(x_2-y_2)^2\big)\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}\Big|_{y_2=0}v(\tau)d\tau$$

$$:= -\frac{1}{4\pi}\int\limits_{\mathbb{R}} \Big(\ln\big((x_1-\tau)^2+(x_2-y_2)^2\big)^{1/2}+\frac{(y_2-x_2)^2}{(x_1-\tau)^2+(x_2-y_2)^2}+\frac{1}{2}\Big)\Big|_{y_2=0}v(\tau)d\tau$$

$$:= -\frac{1}{4\pi}\int\limits_{\mathbb{R}} \Big(\ln\big((x_1-\tau)^2+x_2^2\big)^{1/2}+\frac{x_2^2}{(x_1-\tau)^2+x_2^2}+\frac{1}{2}\Big)v(\tau)d\tau,$$

and when $x_2\to 0$, we get

$$\begin{aligned}
&\boldsymbol{V}_{(-3,\mathbb{R})}v(t) := \lim_{x_2\to 0}\boldsymbol{V}^0_{(-3,\mathbb{R})}v(x) = \frac{1}{8\pi}\int\limits_{\mathbb{R}}(t-\tau)^2\ln|t-\tau|v(\tau)d\tau,\\
&\boldsymbol{V}_{(-2,\mathbb{R})}v(t) := \lim_{x_2\to 0}\boldsymbol{V}^0_{(-2,\mathbb{R})}v(x) = 0, \quad \boldsymbol{V}^*_{(-2,\mathbb{R})}v(t) := \lim_{x_2\to 0}\boldsymbol{V}^1_{(-2,\mathbb{R})}v(x) = 0, \qquad (1.17)\\
&\boldsymbol{V}_{(-1,\mathbb{R})}v(t) := \lim_{x_2\to 0}\boldsymbol{V}^1_{(-1,\mathbb{R})}v(x) = -\frac{1}{4\pi}\int\limits_{\mathbb{R}}\Big(\ln|t-\tau|+\frac{1}{2}\Big)v(\tau)d\tau.
\end{aligned}$$

Now we prove the following

Lemma 1.6. *Let $1<p<\infty$, $s>\frac{1}{p}$. Let $g_1\in\mathbb{H}^{s-1/p}_p(\mathbb{R})$ and $h_1\in\mathbb{H}^{s-1-1/p}_p(\mathbb{R})$ (non-classical formulation (1.14)). A solution to the BVP (1.13) is represented by the formula*

$$\begin{aligned}
u(x) = \boldsymbol{N}_{\mathbb{R}^2_+}f(x)+\boldsymbol{W}_{(0,\mathbb{R})}g_1(x)-\boldsymbol{W}_{(-1,\mathbb{R})}h_1(x)+\boldsymbol{W}_{(-2,\mathbb{R})}\varphi_0(x)&\\
-\boldsymbol{W}_{(-3,\mathbb{R})}\psi_0(x), \quad x\in\mathbb{R}^2& \qquad (1.18)
\end{aligned}$$

and φ^0 and ψ^0 are the solutions to the system of pseudodifferential equations

$$\begin{cases}\boldsymbol{V}^0_{(-2,\mathbb{R})}\varphi_0-\boldsymbol{V}^0_{(-3,\mathbb{R})}\psi_0 = G_0 & \text{on} \quad \mathbb{R},\\ \boldsymbol{V}^1_{(-1,\mathbb{R})}\varphi_0-\boldsymbol{V}^1_{(-2,\mathbb{R})}\psi_0 = H_0 & \text{on} \quad \mathbb{R},\end{cases} \qquad (1.19)$$

$$\begin{aligned}
&\varphi_0\in\widetilde{\mathbb{H}}^{s-1/p}_p(\mathbb{R}), \quad \psi_0\in\widetilde{\mathbb{H}}^{s-1-1/p}_p(\mathbb{R}),\\
&G_0\in\mathbb{H}^{s-1/p}_p(\mathbb{R}), \quad H_0\in\mathbb{H}^{s-1-1/p}_p(\mathbb{R}), \qquad (1.20)
\end{aligned}$$

where

$$\begin{aligned}
G_0 &:= \Big[\frac{1}{2}g_1-(\boldsymbol{N}_{\mathbb{R}^2_+}f)^+-\boldsymbol{V}^0_{(0,\mathbb{R})}g_1+\boldsymbol{V}^0_{(-1,\mathbb{R})}h_1\Big]\in\mathbb{H}^{s-1/p}_p(\mathbb{R}),\\
H_0 &:= \Big[\Big(\frac{1}{2}h_1-\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{N}_{\mathbb{R}^2_+}f\Big)^+-\boldsymbol{V}^1_{(0,\mathbb{R})}g_1+\boldsymbol{V}^1_{(-1,\mathbb{R})}h_1\Big]\in\mathbb{H}^{s-1-1/p}_p(\Gamma).
\end{aligned}$$

The system of boundary pseudodifferential equations (1.19) has a unique pair of solutions φ_0 and ψ_0 in the classical setting $p=2$, $s=1$.

Proof. By repeating word by word the proof of Theorem 0.3, we prove the equivalence via the representation formulae (1.18) of the BVP (1.13) in the non-classical setting (1.14) and of the system (1.19).

The existence and uniqueness of a solution to the BVP (1.13) in the classical setting (1.14) is stated in Proposition 1.4, while for system (1.19) it follows from the proved equivalence with the BVP (1.13). □

Lemma 1.7. *Let* $1 < p < \infty$, $s > \dfrac{1}{p}$. *The system of boundary pseudodifferential equations* (1.19) *is locally invertible at* 0 *if and only if the system* (0.9) *is locally invertible at* 0 *in the non-classical setting* (0.10a) *and the space parameters are related as follows:* $r = s - \dfrac{1}{p} > 0$.

Proof. Due to the equalities (1.17) $\boldsymbol{V}^0_{(-2,\mathbb{R})}\varphi_0 = 0$, $\boldsymbol{V}^1_{(-2,\mathbb{R})}\psi_0 = 0$ and the equation in (1.19) acquires the form

$$\begin{cases} -\dfrac{1}{8\pi}\displaystyle\int_{\mathbb{R}}(t-\tau)^2\ln|t-\tau|\psi_0(\tau)d\tau = G(t), \quad t\in\mathbb{R},\\ -\dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}}\big(\ln|t-\tau|+\dfrac{1}{2}\big)\varphi_0(\tau)d\tau = H(t), \quad t\in\mathbb{R}. \end{cases}$$

Multiply both equations by -4, apply to the first equation the differentiation ∂_t^3 and to the second equation ∂_t. We get

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}}\dfrac{\psi_0(\tau)d\tau}{\tau-t} = G(t),\\ \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}}\dfrac{\varphi_0(\tau)d\tau}{\tau-t} = H(t), \quad t\in\mathbb{R}. \end{cases}$$

The obtained equation coincides with system (0.9).

Invertibility of the singular integral operator follows from the following equality (see [4,5,11])

$$\mathscr{F}S_{\mathbb{R}}\varphi(\xi) = -\operatorname{sign}\xi\varphi(\xi),$$

since $S_{\mathbb{R}}\varphi(\xi) = \mathscr{F}^{-1}(-\operatorname{sign}\xi)\mathscr{F}$, we get

$$\begin{aligned} S^2_{\mathbb{R}}\varphi(\xi) &= \mathscr{F}^{-1}(-\operatorname{sign}\xi)\mathscr{F}\mathscr{F}^{-1}(-\operatorname{sign}\xi)\mathscr{F}\varphi(\xi)\\ &= \mathscr{F}^{-1}(-\operatorname{sign}\xi)^2\mathscr{F}\varphi(\xi) = \mathscr{F}^{-1}\mathscr{F}\varphi(\xi) = \varphi(\xi). \end{aligned}$$

Here

$$\mathscr{F}u(\xi) := \int_{\mathbb{R}^n} e^{i\xi x}u(x)dx, \quad \xi\in\mathbb{R}^n,$$

is the Fourier transform and

$$\mathscr{F}^{-1}v(\xi) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\xi x}v(\xi)d\xi, \quad x\in\mathbb{R}^n,$$

is its inverse transform.

To prove the local equivalence at 0 of systems (1.19) and (0.9) we note that the differentiation

$$\partial_t := \frac{d}{dt} : \mathbb{H}^r_p(\mathbb{R}) \to \mathbb{H}^{r-1}_p(\mathbb{R}), \quad \partial_t : \widetilde{\mathbb{H}}^r_p(\mathbb{R}) \to \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R})$$

is invertible at any finite point $x\in\mathbb{R}$ and the inverse operator is

$$\Big(\frac{d}{dt}\Big)^{-1}\varphi(t) = \int_{-\infty}^{t}\varphi(\tau)d\tau.$$

□

Proof of Theorem 0.4. By Theorem 0.3, system (0.6) is Fredholm one in the Bessel potential space setting (0.7) if the BVP (0.2) is Fredholm in the non-classical setting (0.4). On the other hand, by Lemma 1.5 the BVP (0.2) is Fredholm in the non-classical setting (0.4) if the BVP (1.13) is locally invertible at 0 in the non-classical setting (1.14). And, finally, by Lemma 1.6 and Lemma 1.7, the BVP (1.13) is locally invertible in the non-classical setting (1.14) if the system of boundary integral equations (0.9) is locally invertible at 0 in the Bessel potential space setting (0.10b). □

Acknowledgement

The author was supported by the grant of the Shota Rustaveli Georgian National Science Foundation PhD–F–17–197.

References

1. L. P. Castro, R. Duduchava, F. O. Speck, Localization and minimal normalization of some basic mixed boundary value problems. *Factorization, singular operators and related problems (Funchal, 2002)*, 73–100, *Kluwer Acad. Publ. Dordrecht*, 2003.
2. V. D. Didenko, B. Silbermann, *Approximation of Additive Convolution-Like Operators. Real C^*-Algebra Approach.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2008.
3. R. Duduchava, On multidimensional singular integral operators I–II. *J. Operator Theory* **11** (1984), 41–76, 199–214.
4. R. Duduchava, On general singular integral operators of the plane theory of elasticity. *Rend. Sem. Mat. Univ. Politec. Torino* **42** (1984), no. 3, 15–41.
5. R. Duduchava, General singular integral equations and fundamental problems basic theorems of the plane theory of elasticity. (Russian) *Trudi Tbiliss. Mat. Inst. Razmadze Acad. Nauk Gruzin. SSR* **82** (1986), 45–89.
6. R. Duduchava, The Green formula and layer potentials. *Integral Equations Operator Theory* **41** (2001), no. 2, 127–178.
7. R. Duduchava, Partial differential equations on hypersurfaces. *Mem. Differential Equations Math. Phys.* **48** (2009), 19–74.
8. R. Duduchava, Mellin convolution operators in Bessel potential spaces with admissible meromorphic kernels. *Mem. Differ. Equ. Math. Phys.* **60** (2013), 135–177.
9. R. Duduchava, D. Mitrea, M. Marius, Differential operators and boundary value problems on hypersurfaces. *Math. Nachr.* **279** (2006), 996–1023.
10. R. Duduchava, M. Tsaava, T. Tsutsunava, Mixed boundary value problem on hypersurfaces. *Int. J. Differ. Equ.* 2014, Art. ID 245350, 8 pp.
11. I. Gohberg, N. Krupnik, One-Dimensional Linear Singular Integral Equations, I-II, Oper. Theory Adv. Appl. **53-54**, Birkhäuser, Basel, 1979.
12. G. C. Hsiao, W. L. Wendland, *Boundary Integral Equations, Applied Mathematical Sciences.* vol. **164**, Springer, Berlin Heidelberg 2008, ISBN 978-3-540-15284-2.
13. I. B. Simonenko, A new general method of investigating linear operator equations of singular integral equation type. I. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **29**, 1965, 567–586.
14. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators.* Second edition, Johann Ambrosius Barth, Heidelberg, 1995.
15. M. Tsaava, The boundary value problems for the bi-Laplace-Beltrami equation. *Mem. Differ. Equ. Math. Phys.* **77** (2019), 93–103.

(Received 04.09.2019)

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Transactions of A. Razmadze Mathematical Institute
Volume 173, 2019, issue 3, 161–178

SHORT COMMUNICATIONS

Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 161–162

APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS IN THE FRAMEWORK OF WEIGHTED FULLY MEASURABLE GRAND LORENTZ SPACES

VAKHTANG KOKILASHVILI

Abstract. In this note we present the fundamental Bernstein and Nikol'skii type inequalities in weighted fully measurable grand Lorentz spaces. These inequalities we apply to obtain the direct and inverse approximation theorems in approximable subspaces of aforementioned function spaces.

Let $1 < p < \infty$. By Φ_p we denote the set of positive measurable functions φ defined on $(0, p-1]$ which are nondecreasing, bounded with a condition $\lim\limits_{x\to 0+} \varphi(x) = 0$.

Let $\varphi \in \Phi_p$. The fully measureble weighted grand Lebesgue space $L_w^{p),s,\varphi}(\mathbb{T})$ is defined as a set of all measurable 2π-periodic functions $f : \mathbb{T} \longrightarrow \mathbb{R}^1$, for which the norm

$$\|f\|_{L_w^{p),s,\varphi}(\mathbb{T})} = \sup_{0<\varepsilon<p-1} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(s \int\limits_0^\infty \left(w\Big(x \in \mathbb{T} : \, |f(x)| > \lambda \Big) \right)^{s/p-\varepsilon} \lambda^{s-1} d\lambda \right)^{1/s}$$

is finite.

The space $L_w^{p),s,\varphi}(\mathbb{T})$ is a non-reflexive, non-separable Banach function space.

The subspace of $L_w^{p),s,\varphi}(\mathbb{T})$ in which the smooth functions are dense, is characterized by the equality

$$\lim_{\varepsilon\to 0} (\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(s \int\limits_0^\infty \left(w\Big(x \in \mathbb{T} : \, |f(x)| > \lambda \Big) \right)^{s/p-\varepsilon} \lambda^{s-1} d\lambda \right)^{1/s} = 0.$$

Denote this subspace by $\widetilde{L}_w^{p),s,\varphi}(\mathbb{T})$. First, we treat the fundamental inequalities for trigonometric polynomials in $L_w^{p),s,\varphi}(\mathbb{T})$.

In the sequel we assume that the weights w belong to the well-known Muckenhoupt A_p class.

Let us give the Bernstein type inequality for the Weyl's fractional derivative of trigonometric polynomials.

Theorem 1. *Let $1 < p, s < \infty$, $\varphi \in \Phi_p$ and $w \in A_p$. For an arbitrary trigonometric polynomial T_n and a number $\alpha > 0$ the following inequality*

$$\|T_n^{(\alpha)}\|_{L_w^{p),s,\varphi}} \le c n^\alpha \|T_n\|_{L_w^{p),s,\varphi}}$$

holds, where the constant c is independent of n and T_n.

The next theorem deals with the Nikol'skii type inequality

Theorem 2. *Let $1 < p < q < \infty$, $1 < s < pq/q-p$ and $r = s/\big(1 - \big(\frac{1}{p} - \frac{1}{q}\big)s\big)$. Assume that $\varphi(x) = x^\theta$, $\theta > 0$ and $w \in A_{1+\frac{q}{p'}}$, $p' = \frac{p}{p-1}$.*

Then the inequality

$$\|T_n n^{\frac{1}{p}-\frac{1}{q}}\|_{L_w^{q),r,\theta q/p}} \le c n^{\frac{1}{p}-\frac{1}{q}} \|T_n\|_{L_w^{p),s,\theta}}$$

holds.

2010 *Mathematics Subject Classification.* 42A17, 41A10, 42B35, 46E30.

Key words and phrases. Trigonometric approximation; Bernstein and Nikol'skii inequalities; Weighted fully measurable Grand Lebesgue spaces; Direct and inverse theorems.

In the space $\widetilde{L}_w^{p)s}(\mathbb{T})$, we introduce the structural and constructive characteristics of functions: the moduli of smoothness

$$\Omega(f,\delta)_{L_w^{p)s,\varphi}} = \sup_{0<h\le\delta}\left\|\frac{1}{2h}\int\limits_{x-h}^{x+h} f(t)dt - f(x)\right\|_{L_w^{p)s,\varphi}}$$

and the best approximation by trigonometric polynomial

$$E_n(f)_{L_w^{p)s,\varphi}} = \inf \|f-T_k\|_{L_w^{p)s,\varphi}},$$

where the infimum is taken over all trigonometric polynomials T_k of degree $k\le n$.

The following analogy of Jackson's theorem is valid.

Theorem 3. *Let $1<p<\infty$, $\varphi\in\Phi_p$, $\alpha\ge 0$, and $w\in A_p(\mathbb{T})$.*

Then for some positive constant c and all f, $f^{(\alpha)}\in \widetilde{L}_w^{p)s,\varphi}$ the following inequality

$$E_n(f)_{L_w^{p)s,\varphi}} \le \frac{c}{(n+1)^\alpha}\Omega\left(f^{(\alpha)},\frac{1}{n}\right)_{L_w^{p)s,\varphi}}$$

holds.

In the next statement we announce the inverse inequality.

Theorem 4. *Let $1<p<\infty$, $\varphi\in\Phi_p$ and $w\in A_p(\mathbb{T})$.*

Then the following inequality:

$$\Omega\left(f,\frac{1}{n}\right)_{L_w^{p)s,\varphi}} \le \frac{c}{n^2}\sum_{k=0}^{n-1}(k+1)E_k(f)_{L_w^{p)s,\varphi}},$$

for $f\in \widetilde{L}w^{p)s,\varphi}$ holds, where the constant c is independent of f and n.

Applying the Nikol'skii type inequality, we prove the following statement.

Theorem 5. *Let the conditions of Theorem 2 be satisfied. Assume that for $f\in\widetilde{L}_w^{p)s,\varphi}$*

$$\sum_{k=1}^{\infty} k^{1/p-1/q-1}E_k(f)_{L_w^{p)s,\varphi}} < \infty.$$

Then $f\in\widetilde{L}_w^{q)r,\theta q/p}$ and

$$E_n(f\cdot w^{1/p-1/q})_{L_w^{q)r,\theta q/p}} \le c\left\{n^{1/p-1/q}E_n(f)_{L_w^{p)s,\theta}} + \sum_{k=n+1}^{\infty}k^{1/p-1/q-1}E_k(f)_{L_w^{p)s,\varphi}}\right\}.$$

The proof of thr above-mentioned theorems are based essentially on the results obtained in [1].

For the similar results in weighted grand Lebesgue spaces we refer the readers to paper [2].

The detiled proofs will be published in Georgian Mathematikal journal.

Acknowledgement

This work was supported by Shota Rustaveli National Foundation grant (contract number: FR-2499).

References

1. V. M. Kokilashvili, A. N. Meskhi, Extrapolation in weighted classical and grand Lorentz spaces. Application to the boundedness of integral operators. arXiv:1910.01362v1 [math.FA] 3 Oct, 2019.
2. V. M. Kokilashvili, A. N. Meskhi, Weighted extrapolation in Iwaniec-Sbordone spaces. Applications to integral operators and approximation theory. (Russian) *Proc. Steklov Inst. Math.* **293** (2016), no. 1, 161–185.

(Received 09.07.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 163–166

TRIGONOMETRIC APPROXIMATION BY ANGLE IN CLASSICAL WEIGHTED LORENTZ AND GRAND LORENTZ SPACES

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Abstract. In this paper, we present our results on an angular trigonometric approximation of functions of two variables in weighted Lorentz spaces and in that subspace of weighted grand Lorentz spaces in which C_0^∞ with a compact support is dense.

1. Introduction

The study of angular trigonometric approximation of 2π-periodic multivariate functions in classical Lebesgue spaces L^p $(1 < p < \infty)$ was initiated by K. Potapov (see, e.g., [7–9] and the review article [10]). Recently, these results were extended to the L^p $(1 < p < \infty)$ spaces with Muckenhoupt weights [1,2]. We aim to generalize the results of [1,2] to the classical weighted Lorentz spaces and to the certain subspace of weighted grand Lebesgue spaces.

In the sequel, by $\mathbb{T}^2$ we denote the torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, where $\mathbb{T}$ is a circle $\{e^{i\varphi},\ \varphi \in [0, 2\pi)\}$. The function $w : \mathbb{T}^2 \longrightarrow \mathbb{R}^1$ is called a weight if w is a measurable on $\mathbb{T}^2$, positive almost everywhere and integrable. For a Borel measure $E \subset \mathbb{T}^2$, we define the absolute continuous measure

$$wE = \int\limits_E w(x,y)dxdy.$$

A weight function w is said to be of Muckenhoupt type class $\mathcal{A}_p(\mathbb{T}^2)$ if

$$\sup \bigg(\frac{1}{|J|} \int\limits_J w(x,y)dxdy\bigg)\bigg(\frac{1}{|J|} \int\limits_J w^{1-p'}(x,y)dxdy\bigg)^{p-1} < \infty, \quad p' = \frac{p}{p-1},$$

where the supremum is taken over all two-dimensional intervals with sides parallel to the coordinate axis.

In the sequel, we consider the set of measurable functions $f(x,y) : \mathbb{T}^2 \longrightarrow \mathbb{R}^1$ such that they are 2π-periodic with respect to each variable x and y.

2. Approximation in Weighted Lorentz Spaces

Definition 2.1. Let $1 < p, s < \infty$, w be the weight function defined on $\mathbb{T}^2$. We say that a measurable function f belongs to the weighted Lorentz space $L_w^{ps}(\mathbb{T}^2)$ (L_w^{ps} shortly) if the norm

$$\|f\|_{L_w^{ps}} = \bigg(s \int\limits_0^\infty \Big(w\Big((x,y) \in \mathbb{T}^2 :\ |f(x,y)| > \lambda\Big)\Big)^{s/p} \lambda^{s-1} d\lambda\bigg)^{1/s}$$

is finite.

The space L_w^{ps} is the Banach function space.

Let us introduce the notion a of modulus of smoothness

$$\Omega(f, \delta_1, \delta_2) = \sup \bigg\| \frac{1}{hk} \int\limits_{x-h}^{x+h} \int\limits_{y-k}^{y+k} f(t,s)dtds - f(x,y) \bigg\|_{L_w^{ps}}.$$

2010 *Mathematics Subject Classification.* Primary 42A10; Secondary 42A05, 42B08, 42B35.

Key words and phrases. Angular approximation; Weighted Lorentz spaces; Grand Lorentz spaces, weights.

By $P_{m,0}$ (respectively, by $P_{0,n}$) is denoted the set of all trigonometric polynomials of degree m (at most n) with respect to the variable x (variable y). Also, $P_{m,n}$ is defined as the set of all trigonometric polynomials of degree at most m with respect to the variable x and of degree at most n with respect to the variable y.

The best partial trigonometric approximation orders are defined as

$$E_{m,0}(f)_{L^{ps}_w} = \inf\{\|f-T\|_{L^{ps}_w} : T \in P_{m,0}\}.$$

Analogously,

$$E_{0,m}(f)_{L^{ps}_w} = \inf\{\|f-G\|_{L^{ps}_w} : G \in P_{0,n}\}.$$

Then the best angular approximation order is defined by the equality

$$E_{m,n}(f)_{L^{ps}_w} = \inf\{\|f-T-G\|_X : \mathbb{T} \in P_{m,0},\ G \in P_{0,n}\}.$$

The following assertions are true.

Theorem 2.1. *Let $1<p,s<\infty$, $w \in \mathcal{A}_p(\mathbb{T}^2)$. For $f \in L^{ps}_w$, the following inequality*

$$E_{m,n}(f)_{L^{ps}_w} \le c_1 \Omega\left(f, \frac{1}{m}, \frac{1}{n}\right)_{L^{ps}_w}$$

holds with a constant c_1, independent of f, m and n.

Theorem 2.2. *Let $1<p,s<\infty$, $w \in \mathcal{A}_p(\mathbb{T}^2)$, $f \in L^{ps}_w$. Then*

$$\Omega\left(f, \frac{1}{m}, \frac{1}{n}\right)_{L^{ps}_w} \le \frac{c}{m^2n^2} \sum_{i=0}^{m} \sum_{j=0}^{n} (i+1)(j+1) E_{ij}(f)_{L^{ps}_w}.$$

In what follows, we discuss some tools, contributing to the proving of aforementioned assertions.

Let $\sigma^{\alpha,\beta}_{mn}(f,x,y)$ ($\alpha>0$, $\beta>0$) be the Cesáro means of double Fourier trigonometric series of $f \in L^{ps}_w$.

Theorem 2.3. *Let $1<p,s<\infty$, $w \in \mathcal{A}_p(\mathbb{T}^2)$. Then*

$$\|\sigma^{\alpha,\beta}_{mn}(f)\|_{L^{ps}_w} \le c\|f\|_{L^{ps}_w}$$

and

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \|\sigma^{\alpha,\beta}_{mn}(f)-f\|_{L^{ps}_w} = 0.$$

For the partial sums of double Fourier trigonometric series, we have

$$\|S_{mn}(f)\|_{L^{ps}_w} \le c\|f\|_{L^{ps}_w}$$

with a constant c, independent of $m,n \in \mathbb{N}$ and $f \in L^{ps}_w$.

Further, for $f \in L^{ps}_w$,

$$\lim_{n\to\infty} \|S_{n,n}-f\|_{L^{ps}_w} = 0.$$

In the sequel, under the derivatives we assume those in Weyl's sense.

Theorem 2.4 (Bernstein type inequalities)**.** *Let $1<p,s<\infty$, $w \in \mathcal{A}_p(\mathbb{T}^2)$. Assume that $\alpha,\beta>0$.*

Let $T_1 \in P_{m,0}$, $T_2 \in P_{0,n}$ and $T_3 \in P_{mn}$. Then for α,β order Weyl's derivatives, we have

$$\left\|\frac{\partial^\alpha}{\partial x^\alpha} T_1\right\|_{L^{ps}_w} \le c_1 m^\alpha \|T_1\|_{L^{ps}_w}$$

$$\left\|\frac{\partial^\beta}{\partial y^\beta} T_2\right\|_{L^{ps}_w} \le c_2 n^\beta \|T_2\|_{L^{ps}_w}$$

and

$$\left\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} T_3\right\|_{L^{ps}_w} \le c_3 m^\alpha n^\beta \|T_3\|_{L^{ps}_w},$$

where the constants c_1, c_2 and c_3 are independent of m, n and of polynomial.

For one– and two-weighted Bernstein inequalities in Lebesgue spaces we refer to [5], Chapter 6.

Definition 2.2. Let $f \in L^{ps}_w$, $w \in \mathcal{A}_p(\mathbb{T}^2)$. The mixed K-functional is defined as

$$K(f,\delta,\varepsilon,p,s,w,2)_{L^{ps}_w}:$$

$$=\inf_{h_1 h_2, h}\left\{\|f-h_1-h_2-h\|_{L^{ps}_w}+\delta^2\left\|\frac{\partial^2 h_1}{\partial x^2}\right\|_{L^{ps}_w}+\varepsilon^2\left\|\frac{\partial^2 h_2}{\partial y^2}\right\|_{L^{ps}_w}+\delta^2\varepsilon^2\left\|\frac{\partial^4 h}{\partial x^2\partial y^2}\right\|_{L^{ps}_w}\right\},$$

where the infimum is taken from all h_1, h_2, h such that $h_1 \in W^{2,0}_{L^{ps}_w}$, $h_2 \in W^{0,2}_{L^{ps}_w}$, $h \in W^4_{L^{ps}_w}$.

Here we use the following notation:

$$W^{2,0}_{L^{ps}_w}=\left\{h_1:\frac{\partial^2 h_1}{\partial x^2}\in L^{ps}_w\right\},$$

$$W^{0,2}_{L^{ps}_w}=\left\{h_2:\frac{\partial^2 h_2}{\partial y^2}\in L^{ps}_w\right\}$$

and

$$W^4_{L^{ps}_w}=\left\{h:\frac{\partial^4 h}{\partial x^2\partial y^2}\in L^{ps}_w\right\}.$$

The following statement is true.

Theorem 2.5. *Let $f \in L^{ps}_w$, $1<p,s<\infty$, $w\in\mathcal{A}^p(\mathbb{T})^2$. Then the equivalence*

$$\Omega(f,\delta_1,\delta_2)_{L^{ps}_w}\approx K(f,\delta_1,\delta_2,p,s,w,2)_{L^{ps}_w}$$

holds with the equivalence constants, independent of f, δ_1 and δ_2.

It should be noted that the mixed K-functionals were explored in [6] and [11]. This notion turn out to be very useful in the approximation and interpolation theory.

3. Approximation in a Subspace of Weighted Grand Lorentz Spaces

Let $1<p<\infty$. By Φ_p we denote the set of positive measurable functions φ defined on $(0,p-1]$ which are nondecreasing, bounded with a condition $\lim\limits_{x\to 0+}\varphi(x)=0$.

Let $\varphi\in\Phi_p$. The fully measurable weighted grand Lebesgue space $L^{p),s,\varphi}_w(\mathbb{T}^2)$ is defined as a set of all measurable functions $f:\mathbb{T}^2\longrightarrow\mathbb{R}^1$, for which the norm

$$\|f\|_{L^{p),s,\varphi}_w(\mathbb{T}^2)}=\sup_{0<\varepsilon<p-1}(\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}}\|f\|_{L^{p-\varepsilon}_w(\mathbb{T}^2)},\quad 1<p<\infty,\quad \theta>0$$

is finite.

The space $L^{p),s,\varphi}_w(\mathbb{T}^2)$ is non-reflexive, non-separable Banach function space.

The subspace of $L^{p),s,\varphi}_w(\mathbb{T}^2)$, in which the smooth functions are dense, is characterized by the equality

$$\lim_{\varepsilon\to 0}(\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}}\|f\|_{L^{p-\varepsilon}_w(\mathbb{T}^2)}=0.$$

Denote this subspace by $\widetilde{L}^{p),s,\varphi}_w(\mathbb{T}^2)$. We treat the angular trigonometric approximation of a function of two variables in this subspace.

Analogously to the previous section, for $f\in\widetilde{L}^{p),s,\varphi}_w(\mathbb{T}^2)$ we introduce the structural and constructive characteristics $\Omega(f,\delta_1,\delta_2)_{\widetilde{L}^{p),s,\varphi}_w(\mathbb{T}^2)}$ and $E_{m,0}(f)_{\widetilde{L}^{p),s,\varphi}_w(\mathbb{T}^2)}$, $E_{(0,n)}(f)_{\widetilde{L}^{p),s,\varphi}_w(\mathbb{T}^2)}$ and $E_{m,n}(f)_{\widetilde{L}^{p),s,\varphi}_w(\mathbb{T}^2)}$.

We claim that for $\widetilde{L}^{p),s,\varphi}_w(\mathbb{T}^2)$, the statements similar to Theorem 2.1 and Theorem 2.2 are valid.

Remark 3.1. The results, analogous to the above-mentioned, are true both for the function of several variables and for the modulus of smoothness of fractional order.

The proofs of the presented results based essentially on the boundedness of integral operators in the classical weighted Lorentz spaces and grand Lorentz spaces have been obtained recently in [6].

The detiled proofs will be published in Georgian Mathematikal journal.

Acknowledgement

This work was supported by Shota Rustaveli National Foundation grant (contract number: Di–18–118).

References

1. R. Akgün, Mixed modulus of continuity of the Lebesgue spaces with Muckenhoupt weights and their properties. *Turkish J. Math.* **40** (2016), no. 6, 1169–1192.
2. R. Akgün, Mixed modulus of smoothness with Muckenhoupt weights and approximation by angle. *Complex Var. Elliptic Equ.* **64** (2019), no. 2, 330–351.
3. A. Benedek, R. Panzone, The space L^P with mixed norm. *Duke Math. J.* **28**(1961), 301–324.
4. C. Cottin, Mixed K-functionals: a measure of smoothness for blending-type approximation. *Math. Z.* **204** (1990), no. 1, 69–83.
5. V. Kokilashvili, A. Meskhi, L.-E. Persson, *Weighted Norm Inequalities for Integral Transforms with Product Kernels.* Hauppauge, Ney-York, Nova Science Publishers, Inc., 2009.
6. V. Kokilashvili, A. Meskhi, Extrapolation in weighted classical and grand Lorentz spaces. Application to the boundedness of integral operators. arXiv:1910.01362v1 [math.FA] 3 Oct, 2019.
7. M. K. Potapov, Approximation by "angle". (Russian) *Proceedings of the Conference on the Constructive Theory of Functions (Approximation Theory) (Budapest, 1969)*, 371–399. Akadémiai Kiadó, Budapest, 1972.
8. M. K. Potapov, Approximation "by angle", and imbedding theorems. (Russian) *Math. Balkanica* **2** (1972), 183–198.
9. M. K. Potapov, Imbedding of classes of functions with a dominating mixed modulus of smoothness. (Russian) Studies in the theory of differentiable functions of several variables and its applications, V. *Trudy Mat. Inst. Steklov.* **131** (1974), 199–210, 247.
10. M. K. Potapov, B. V. Simonov, S. Y. Tikhonov, Mixed moduli of smoothness in L_p, $1 < p < \infty$: a survey. *Surv. Approx. Theory* **8** (2013), 1–57.
11. K. V. Runovski, *Several questions of approximation theory.* Ph.D Thesis, Moscow State University MGU, Moscow 1989.

(Received 11.07.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 167–171

BICRITICAL POINTS IN PROBLEM ON THE STABILITY OF HEAT-CONDUCTING FLOWS BETWEEN HORIZONTAL POROUS CYLINDERS

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Abstract. The stability of heat-conducting flow between horizontal porous rotating cylinders with a constant azimuthal pressure gradient is studied. It is assumed that the flow is subjected to the action of a radial flow through the cylinder walls and a radial temperature gradient. The aim of this paper is to find the intersection points of neutral curves that correspond to flow instability and appearance of complex regimes.

1. Formulation of the Problem

We consider the steady heat-conducting flow between horizontal porous rotating cylinders with a constant azimuthal pressure gradient maintained by a pumping of a fluid around the annulus at cylinders. It is assumed that the cylinders heated up to different temperatures and the flow is subjected to the action of a radial converging and diverging fluid through the permeable cylinder walls and radial temperature gradient. External mass forces absent, the fluid inflow through the wall of one cylinder is equal to the fluid outflow through the other one.

We denote the radii, angular velocities and temperature of the inner and outer cylinders by R_1, Ω_1, T_1, and R_2, Ω_2, T_2, respectively. Assume that on the surface of the cylinders the following boundary conditions

$$\begin{aligned} &v_r' = U_0, \quad v_\varphi' = \Omega_1 R_1, \quad v_z' = 0, \quad T' = T_1 \ \ (r = R_1),\\ &v_r' = \frac{U_0}{R}\,, \quad v_\varphi' = \Omega_2 R_2, \quad v_z' = 0, \quad T' = T_2 \ \ (r = R_2) \end{aligned} \tag{1.1}$$

are fulfilled, where $R = \frac{R_2}{R_1}$, $V'(v_r', v_\theta', v_z')$ is the velocity vector, U_0 is the radial velocity through the wall of the inner cylinder.

Under the above assumption, using the Navier–Stokes system, heat transfer, continuity equations and an equation of state [5] in terms of cylindrical coordinates r, θ, z with z-axis coinciding with that of cylinders we obtain the following exact solution for the velocity V_0, temperature T_0, pressure Π_0:

$$V_0 = \{u_0(r), v_0(r), 0\}, \quad T_0 = c_1 + c_2 r^{\varkappa P_r},$$

$$u_0(r) = \frac{R_1 U_0}{r}\,, \quad v_0(r) = \begin{cases} \dfrac{K}{\varkappa}\left(a r^{\varkappa+1} + \dfrac{b}{r} - r\right) + A r^{\varkappa+1} + \dfrac{B}{r}, & \varkappa \neq -2,\\[2mm] \dfrac{K}{2}\left(\dfrac{a_1 \ln r + b_1}{r}\right) + \dfrac{A_1 \ln r + B_1}{r}, & \varkappa = -2, \end{cases} \tag{1.2}$$

$$\frac{\partial \Pi_0}{\partial r} = \frac{\rho(u_0^2 + v_0^2)}{r}\,,$$

where

$$K = \frac{1}{2\rho\nu}\left(\frac{\partial \Pi_0}{\partial \theta}\right)_0 = \text{const}\,, \quad a = \frac{R^2 - 1}{(R^{\varkappa+2} - 1) R_1^{\varkappa}}\,, \quad a_1 = \frac{R_1^2 (R^2 - 1)}{\ln R}\,,$$

$$b = \frac{R_2^2 (R^{\varkappa} - 1)}{R^{\varkappa+2} - 1}, \quad b_1 = -\frac{R_1^2 \ln R_2 - R_2^2 \ln R_1}{\ln R}\,,$$

2010 *Mathematics Subject Classification.* 76E30.

Key words and phrases. Heat-condacting flow; Porous cylinders, Instability; Liquid pumping.

$$A=\frac{\Omega_1(\Omega R^2-1)}{(R^{\varkappa+2}-1)R_1^{\varkappa}}\,,\quad A_1=\frac{\Omega_1R_1^2(\Omega R^2-1)}{\ln R}\,,$$
$$B=\frac{\Omega_1R_2^2(R^{\varkappa}-\Omega)}{R^{\varkappa+2}-1},\quad B_1=-\frac{\Omega_1R_1^2(\ln R_2-\Omega R^2\ln R_1)}{\ln R}\,,$$
$$c_1=\frac{T_1R^{\mathrm{Pr}\,\varkappa}-T_2}{R^{\varkappa\,\mathrm{Pr}}-1}\,,\quad c_2=\frac{T_2-T_1}{R_1^{\varkappa\,\mathrm{Pr}}(R^{\varkappa\,\mathrm{Pr}}-1)}\,,$$

$\varkappa=\frac{U_0R_1}{\nu}$ is the radial Reynolds number, $Pr=\frac{\nu}{\chi}$ is the Prandtl number, ρ is the fluid density, ν and χ are, respectively, the coefficients of kinematic viscosity and thermal diffusion. The radial flow is inward for $\varkappa<0$ (converging flow) and outward for $\varkappa>0$ (diverging flow).

The flow with the velocity vector V_0, temperature T_0 and pressure Π_0 is called the main stationary flow. This flow is a superposition of the heat-conducting flow in the transverse direction(maintained by a pumping fluid round the cylinders) and a distribution of angular velocities (maintained by the rotation of the two cylinders). Our aim is to find the intersection points of neutral curves which correspond to flow instability and appearance of complex regimes.

2. Neutral Curves

Let the perturbed state be taken as

$$V'=V_0+V(v_r,v_\theta,v_z),\quad T'=T_0+\tau,\quad \Pi'=\Pi_0+\Pi. \tag{2.1}$$

Taking into account that the main stationary flow consists a rotating shear flow, we denote rotation shear S by $\frac{V_m}{d}$, where V_m is an average velocity in the azimuthal direction, $d=R_2-R_1$ is a gap width between cylinders. Introducing dimensionless variables for time, velocity, temperature and pressure by S, R_2, SR_2, T_2-T_1, $\nu\rho' S$ in the system of Navier-Stokes equations, for the vector-functions $F=\{v_r,v_\theta,v_z,\tau\}$ and $F_1=\{u_r,u_\theta,u_z,T_1\}$, we obtain the following nonlinear problem of finding perturbations V, τ and Π:

$$\begin{gathered}\frac{\partial F}{\partial t}+NF-\frac{1}{\mathrm{Ta}}\,MF+\frac{1}{\mathrm{Ta}}\,\nabla_1\Pi=-\mathcal{L}(F,F_1),\\ (\nabla_1,rF)=0,\quad F\big|_{r=1,R}=0,\end{gathered} \tag{2.2}$$

where

$$MF=\Big\{\Delta_1v_r-\frac{1-\varkappa}{r^2}\,v_r-\frac{2}{r^2}\,\frac{\partial v_\theta}{\partial\theta},\Delta_1v_\theta-\frac{1+\varkappa}{r^2}v_\theta+\frac{2}{r^2}\,\frac{\partial v_r}{\partial\theta},\Delta_1v_z,\frac{1}{\mathrm{Pr}}\,\Delta_1\tau\Big\},$$
$$NF=\omega_1\frac{\partial F}{\partial\theta}+\Big\{\,\mathrm{Ra}\,\omega_2\tau-2\,\mathrm{Ta}\,\omega_1v_\theta,-g_1v_r,0,\frac{g_2}{\mathrm{Pr}}v_r,\Big\},$$
$$\mathcal{L}(F,F_1)=\Big\{(F,\nabla_1)u_r-\frac{v_\theta u_\theta}{r},\ (F,\nabla_1)u_\theta+\frac{v_ru_\theta}{r},(F,\nabla_1)u_z,(F,\nabla_1)T_1\Big\},$$
$$\Delta_1=\frac{\partial^2}{\partial r^2}+\frac{1-\varkappa}{r}\,\frac{\partial}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\,,\quad \nabla_1=\Big\{\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial\theta},\frac{\partial}{\partial z},0\Big\},$$
$$\mathrm{Ta}=\frac{\Omega_1R_2^2}{2}\ \text{ is Taylor number},$$
$$\mathrm{Ra}=\frac{\mu}{\lambda}\,,\quad \mu=\frac{\beta(T_2-T_1)}{2},\ \ \beta\ \text{ is coefficient of thermal expansion},$$
$$\lambda=\frac{V_m}{\Omega_1R_2}\ \text{ is ratio of the average velocities of pumping liquid and rotation},$$
$$V_m=K\,\frac{R_1R^2}{R-1}\,D(R),\quad D(R)=\frac{R^{\varkappa}-1}{R^{\varkappa+2}-1}\,\ln R-\frac{\varkappa(R^2-1)}{2R^2(\varkappa+2)}\,,$$
$$\omega_1=\frac{v_0(r)}{r}=\lambda g(r)+g_0(r),\quad \omega_2=\omega_1^2r,$$
$$g(r)=\frac{d}{R_2}\,\frac{D1(R)r^{\varkappa+2}+D2(R)-r^2}{rD(R)}\,,\quad g_0(r)=D3(R)r^{\varkappa+1}+\frac{D4(R)}{r}\,,$$

$$g_1(r) = \frac{dv_0}{dr} + \frac{v_0}{r} = \frac{d}{R_2}\,\frac{D1(R)(\varkappa+2)r^{\varkappa}-2}{D(R)} + (\varkappa+2)r^{\varkappa}D3(R),$$
$$D1(R) = \frac{(R^2-1)R^{\varkappa}}{R^{\varkappa+2}-1}\,, \quad D2(R) = 1 - D1(R),$$
$$D3(R) = \frac{(\Omega R^2-1)R^{\varkappa}}{R^{\varkappa+2}-1}\,, \quad D4(R) = \frac{R^{\varkappa}-\Omega}{R^{\varkappa+2}-1}\,,$$
$$g_2(r) = \frac{\varkappa\,\mathrm{Pr}^2\,R^{\varkappa\,\mathrm{Pr}}}{R^{\varkappa\,\mathrm{Pr}}-1}\,r^{\varkappa\,\mathrm{Pr}-1}.$$

Problem (2.2) is written in terms of the Boussinesq approximation, which is based on the assumption that the thermal expansion coefficient is small [1]. In the sequel it will always be assumed that the velocity, temperature and pressure components are periodic with respect to z and θ with the known periods $2\pi/\alpha$ and $2\pi/m$, respectively.

The theoretical and experimental studies have shown that after the loss of stability of main flow between rotating cylinders there occured secondary modes either axisymmetric or nonaxisymmetric disturbances as vortices and oscillatory modes in the form of traveling waves.

To study the transition to complex regimes of special attention are the points of intersection of neutral curves, corresponding to the two above-mentioned kinds of the secondary flows, since at these points with a high probability may appear various regimes, including the complex one [2, 4].

Let $(\mathrm{Ra}_0, \mathrm{Ta}_0)$ be the point lying on the plane of parameters $(\mathrm{Ra}, \mathrm{Ta})$ and corresponding to the intersection of the neutral curves corresponding to the monotonic ($m = 0$) axisymmetric and oscillatory nonaxisymmetric loss of stability of main flow (1.2). Under the definite values of parameters of the problem, the neutral curves may be nonintersecting that indicates that under the corresponding values of parameters of the problem we cannot expect the appearance of complex regimes.

To construct neutral curves, we assume that the perturbations V, temperature τ and pressure Π are infinitely small. The neutral curves, which corresponds to the bifurcation of vortex and azimuthal waves are found by solving the spectral problems:

$$(M - \mathrm{Ta}\,N)\Phi_0 = \nabla_1 p_0, \quad (\nabla_1, r\Phi_0) = 0, \quad \Phi_0\big|_{r=1,R} = 0, \tag{2.3}$$

and

$$(M - \mathrm{Ta}\,N - ic\,\mathrm{Ta})\Phi_1 = \nabla_1 p_1, \quad (\nabla_1, r\Phi_1) = 0, \quad \Phi_1\big|_{r=1,R} = 0, \tag{2.4}$$

where

$$\Phi_0 = \big\{u_0(r), v_0(r), iw_0(r), \tau_0(r)\big\}e^{i\alpha z}, \quad p_0 = q_0(r)e^{i\alpha z}, \tag{2.5}$$
$$\Phi_1 = \big\{u_1(r), v_1(r), w_1(r), \tau_1(r)\big\}e^{-i(m\theta+\alpha z)}, \quad p_1 = q_1(r)e^{-i(m\theta+\alpha z)}, \tag{2.6}$$

c – unknow frequency of neutral azimuthal waves.

Problems of eigenvalues (2.3) and (2.4) have been solved by the shooting method for fixed λ, $\varkappa$, α,R, m, Pr, Ω. Thus, for the fixed values of these parameters we established the dependence of the critical value of the number Ta, Ra and the neutral mode frequency c corresponding to the bifurcation of vortices and azimuthal waves origination on a number Ω. Further, using the Newton method, we minimize the difference between the obtained critical values of Ta_0. This allows us to calculate with sufficient exactness the values Ta_0,Ra_0 and c_0 corresponding to the point of intersection of neutral curves.

The calculations in this paper were performed for the case $R = 2$ (radius of the outer cylinders is two times greater than that of the inner ones), $m = 0, 1$, for various values of axial wave number α, $\mathrm{Pr} = 7$ (the working medium is water) and for small absolute values $\varkappa$ ($-2 < \varkappa < 2$). The results of calculations are presented in Tables 1 and 2.

3. Conclusions

As our calculations show, these intersections of neutral curves take place especially when the liquid pumping is in the direction of the rotation inner cylinder.

When the liquid pumping and the inner cylinder rotate in the same direction we can expect the occurrence of complex modes. In the case where the outer cylinder is rest (Table 1) we find that intersections of neutral curves take place when temperature of the inner cylinder is higher than that of the outer for different axial wave numbers. If the liquid pumping and both cylinders rotate in the same direction, we can expect the occurrence of complex modes when temperature of the outer cylinder is higher than that of the inner one. But for the opposite rotating cylinders when the inner cylinder rotates in the same direction as of the pumping, there arise complex regimes, if temperature of the inner cylinder exceeds that of the outer one for both diverging and converging flows (Table 2).

When the pumping flow is directed to the opposite direction of the rotating inner cylinder, the neutral curves do not intersect and thus it is difficult to expect the occurrence of complex regimes. In this case we find very high frequency of neutral azimuthal waves.

Table 1. The points of intersection of neutral curves $\lambda = 1$, $\Omega = 0$

$\varkappa$	$\alpha = 5$			$\alpha = 8$		
	Ra_0	Ta_0	c_0	Ra_0	τ_0	c_0
-1.9	-2.2118	60.9357	2.5582	-0.5292	48.736	2.6613
-1.5	-0.607	59.501	2.5798	-0.209	49.592	2.6649
-1.1	-0.1559	60.7709	2.5911	-0.0695	51.14	2.665
-0.5	-0.03525	65.1397	2.5924	-0.0184	53.989	2.6618
-0.2	-0.02168	67.8398	2.59219	-0.0117	55.599	2.66
0.5	-0.01567	75.1248	2.5938	-0.0083	59.809	2.657
1	-0.02122	81.1659	2.59819	-0.01068	69.227	2.6567
1.5	-0.03486	87.971	2.60578	-0.01674	67.025	2.6587
2	-0.060456	495.619	2.617139	-0.0283	71.2452	2.6635

Table 2. The points of intersection of neutral curves $\lambda = 1$, $\alpha = 4$

$\varkappa$	$\Omega = 0.1$			$\varkappa$	$\Omega = -0.2$		
	Ra_0	Ta_0	c_0		Ra_0	Re_0	c_0
0.2	0.626	76.221	4.247	2	-0.105	115.42	2.48
0.18	0.6784	71.69	4.20716	1.5	-0.0602	104.8	2.459
0.16	0.7165	68.69	4.17436	1	-0.0035	95.57	2.445
-0.2	0.9096	57.835	4.003	-0.1	-0.0307	79.244	2.426
-0.5	1.1257	55.508	3.906	-1.5	-2.129	74.297	2.355
-0.8	1.4729	54.938	3.818	-1.9	-7.345	78.84	2.317

References

1. G. Z. Gershuni, E. M. Zhukhovitski, *Convective Stability of Incompressible Fluid.* Keter, Jerusalem/Wiley, 1976.
2. V. V. Kolesov, A. G. Khoperskiĭ, *Nonisothermal Couette–Taylor Problem.* (Russian) Publisher Yuj. Fed. Yniv. Rostov, 2009.
3. V. V. Kolesov, L. D. Shapakidze, Instabilities and Transition in Flows Between Two Porous Concentric Cylinders with Radial Flow and Radial Temperature Gradient. *Journal Phys. Fluids* **23** (2011), no. 1, 014107–1–014107–13. http://dx.doi.org/10.1063/1.3534026

4. V. V. Kolesov, V. I.Yudovich, Calculation of Oscillatory Regimes in Couette flow in the Neighborhood of the Point of Intersection of Bifurcations Initiating Taylor Vortices and Azimuthal Waves. *Journal Fluid Dynamics* **33** (1998), no. 4, 532–542. https://doi.org/10.1007/BF02698218
5. L. D. Landau, E. M. Lifshits, *Fluid Mechanics*. vol. 6 (2nd ed.). Butterworth–Heinemann, 1987.

(Received 25.07.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 173–176

PERIODICALLY MIXED SERIES AND APPROXIMATIONS OF MULTIVARIATE FUNCTIONS

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Abstract. In the present paper the notion of a periodically mixed power function series is introduced. A theorem asserting the existence of a universal periodically mixed power function series such that any continuous multivariate function can be uniformly approximated by the corresponding subsequence of partial sums of this series is formulated.

Introduction

Fekkete was the first (see [4]) who proved that there exists a real power series $\sum\limits_{n=1}^{\infty} a_n t^n$ on $[-1,1]$, such that to every continuous function g on $[-1,1]$ with $g(0)=0$ there exists an increasing sequence $(m_k)_{k=1}^{\infty}$ of positive integers such that $\sum\limits_{n=1}^{m_k} a_n t^n \to g(t)$ uniformly as $k\to\infty$. Later, Mazurkiewicz [3] and Sierpinski [5], proved, that there exists a real power series (see, also, [1, pp. 74–75])

$$\sum_{n=1}^{\infty} a_n t^n, \quad t\in[0,1]$$

such that to every continuous function f on $[0,1]$ there exists an increasing sequence $(m_k)_{k=1}^{\infty}$ of positive integers such that

$$\sum_{n=1}^{m_k} a_n t^n \to f(t)-f(0)$$

uniformly as $k\to\infty$.

Mentioned phenomenon is called a universality in the sense of uniform approximation (see [2]). In the present paper the notion of d-periodically mixed function series, where d is a natural number such that $d\geq 2$, is introduced and some properties of such a series are established.

Note, that a d-periodically mixed function series is a single function series and every term of this series is a function of one variable (see Definition 1, below). Notions of d-periodically mixed power type series and d-periodically mixed power series are also introduced (see, below Definition 2 and Definition 3, respectively).

The existence of a d-periodically mixed power type series such that for every continuous on $[0,1]^d$ function there exists a sequence of partial sums of this series which uniformly approximates this function on $[0,1]^d$ is established (see, Theorem 1, below). It holds the analogous proposition for d-periodically mixed power series (see, Theorem 2, below). So, there exists a universal single function series such that every term of this series is a function of one variable and every continuous multivariate function can be uniformly approximated by subsequences of this series. The latter is a generalization of the above mentioned known results.

1. Notation, Definitions, Theorems

Let N be the set of all positive integer numbers, d be a natural number such that $d\geq 2$, R^d be the d-dimensional Euclidean space, $[0,1]^d$ be a d-dimensional unit cube, $x=(x_1,\dots,x_d)$ be a point of $[0,1]^d$, $\theta=(0,\dots,0)\in[0,1]^d$. As usual $C[0,1]$ stands for the set of all continuous on $[0,1]$ functions

2010 *Mathematics Subject Classification.* 42B05, 42B08.
Key words and phrases. Universal series; Power series; Approximation of multivariate functions; Mixed series.

and $C[0,1]^d$ stands for the set of all continuous on $[0,1]^d$ functions. $\Phi = \{\varphi_n(t)\}_{n=1}^{\infty}$ be a system of functions defined on $[0,1]$.

Consider a series with respect to Φ, i. e.,

$$\sum_{n=1}^{\infty} a_n \varphi_n(t), \quad t \in [0,1]. \tag{1}$$

Let $S_m(t)$ be the m-th partial sum of this series. i. e.,

$$S_m(t) = \sum_{n=1}^{m} a_n \varphi_n(t).$$

Let $d \geq 2$ be a fixed natural number and

$$d(n) = n - \left[\frac{n-1}{d}\right] d$$

for every positive integer n.

Note, that $(d(n))_{n=1}^{\infty}$ is the following periodic sequence of natural numbers:

$$1, 2, \dots, d, 1, 2, \dots, d, 1, 2, \dots, d, \dots.$$

Every positive integer number n may uniquely be presented in the following form:

$$n = j + (i-1)d$$

where i and j are positive integer numbers and $1 \leq j \leq d$.

For every j, $1 \leq j \leq d$ consider the following set

$$N_j = \{n \in N: \ n = j + (i-1)d, \quad \text{where} \quad i \in N\}$$

then

$$N = N_1 \bigcup \cdots \bigcup N_d, \quad \text{where} \quad N_i \bigcap N_j = \emptyset, \quad \text{if} \quad i \neq j.$$

Therefore for every $n \in N$ there exists j, such that $n \in N_j$ and

$$d(n) = n - \left[\frac{n-1}{d}\right] d = j.$$

So, for every positive integer n and $x = (x_1, \dots, x_d) \in [0,1]^d$ we have

$$x_{d(n)} = x_j \in [0,1].$$

Definition 1. We say that a single series

$$\sum_{n=1}^{\infty} a_n \varphi_n\left(x_{d(n)}\right), \quad \text{where} \quad x = (x_1, \dots, x_d) \in [0,1]^d \tag{2}$$

is a d-periodically mixed series with respect to variables.

Let $S_m^{(d)}(x)$ be the m-th partial sum of a d-periodical mixed function series at the point $x \in [0,1]^d$, i. e.,

$$S_m^{(d)}(x) = S_m^{(d)}(x_1, \dots, x_d) = \sum_{n=1}^{m} a_n \varphi_n\left(x_{d(n)}\right).$$

It is obvious that d-periodical mixed function series (2) is a generalization of the series (1) in the sense that series (1) and (2) coincides with each other at points $t \in [0,1]$ and $(t, \dots, t) \in [0,1]^d$ respectively. So, it holds the following equality for the m-th partial sums of (1) and (2):

$$S_m(t) = S_m^{(d)}(t, \dots, t).$$

In the present paper we consider a system of functions $\Phi = (\varphi_n(t))_{n=1}^{\infty}$ with $\varphi_n(t) = t^{p_n}$, where $t \in [0,1]$, $n = 1, 2, \dots$ and $(p_n)_{n=1}^{\infty}$ is a strictly increasing sequence of positive real numbers.

Definition 2. We say that a series

$$\sum_{n=1}^{\infty} a_n t^{p_n}, \quad t \in [0,1] \tag{3}$$

where $(p_n)_{n=1}^{\infty}$ is an increasing sequence of positive real numbers is a power type series and a series

$$\sum_{n=1}^{\infty} a_n x_{d(n)}^{p_n}, \quad x \in [0,1]^d \tag{4}$$

is a d-periodically mixed power type series.

We denote by $\sigma_m(t)$ and $\sigma_m^{(d)}(x)$ the m-th partial sums of series (3) and (4) respectively. i. e.

$$\sigma_m(t) = \sum_{n=1}^{m} a_n t^{p_n}, \quad t \in [0,1],$$

and

$$\sigma_m^{(d)}(x) = \sum_{n=1}^{m} a_n x_{d(n)}^{p_n}, \quad x \in [0,1]^d.$$

If $p_n = n$ for any positive integer n, then the series (3) is the power series

$$\sum_{n=1}^{\infty} a_n t^n, \quad t \in [0,1] \tag{5}$$

and the series (4) is the series

$$\sum_{n=1}^{\infty} a_n x_{d(n)}^n, \quad x \in [0,1]^d. \tag{6}$$

Definition 3. We say that the series (6) is a d-periodically mixed power series.

We denote by $\tau_m(t)$ and $\tau_m^{(d)}(x)$ the m-th partial sums of series (5) and (6) recpectively, that is

$$\tau_m(t) = \sum_{n=1}^{m} a_n t^n, \quad t \in [0,1]$$

and

$$\tau_m^{(d)}(x) = \sum_{n=1}^{m} a_n x_{d(n)}^n, \quad x \in [0,1]^d.$$

It is obvious that if $t \in [0,1]$ and $(t,\dots,t) \in [0,1]^d$ then for every positive integer m we have:

$$\sigma_m(t) = \sigma_m^{(d)}(t,\dots,t) \quad \text{and} \quad \tau_m(t) = \tau_m^{(d)}(t,\dots,t).$$

If $(f_k(x))_{k=1}^{\infty}$ is a sequence of functions defined on $[0,1]^d$, then it is meant that there exists a limit $\lim_{k\to\infty} f_k(x) = t \in [0,1]$ at the point $x \in [0,1]^d$ if the symbol $S_m\left(\lim_{k\to\infty} f_k(x)\right)$ is applied.

For d-periodically mixed power type series it holds the following:

Theorem 1. *Let d be a natural number, such that $d \geq 2$ and $(p_n)_{n=1}^{\infty}$ be an increasing sequence of positive real numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty$$

then there exist a sequence of real numbers $(a_n)_{n=1}^{\infty}$ and a strictly increasing sequence of positive integers $(M_{k,q})_{k=1}^{\infty}$, where $q = 1, 2, \dots, 2d+1$, such that for the d-periodically mixed power type series (4) with a_n coefficients we have:

$$\left\{\lim_{k\to\infty} \sigma_{M_{k,q}}^{(d)}(x), \quad \textit{where} \quad x \in [0,1]^d\right\} = [0,1], \quad q = 1, 2, \dots, 2d+1$$

and also for any function $F \in C[0,1]^d$, *there exists an increasing sequence of positive integers* $(M_k)_{k=1}^{\infty}$ *such that:*

$$F(x) - F(\theta) = \sum_{q=1}^{2d+1} \lim_{k\to\infty} \sigma_{M_k} \left(\lim_{k\to\infty} \sigma_{M_{k,q}}^{(d)}(x) \right)$$

uniformly on $[0,1]^d$ *and* $[0,1]$ *for indicated limits respectively.*

Note, that one direct consequence of Theorem 1 is the following theorem related to d-periodically mixed power series.

Theorem 2. *Let* d *be a natural number, such that* $d \geq 2$, *then there exist a sequence of real numbers* $(a_n)_{n=1}^{\infty}$ *and a strictly increasing sequence of positive integers* $(M_{k,q})_{k=1}^{\infty}$, *where* $q = 1, 2, \dots, 2d+1$, *such that for the* d*-periodically mixed power series* (6) *with* a_n *coefficients we have:*

$$\left\{ \lim_{k\to\infty} \tau_{M_{k,q}}^{(d)}(x), \quad \textit{where} \quad x \in [0,1]^d \right\} = [0,1], \quad q = 1, 2, \dots, 2d+1$$

and also for any function $F \in C[0,1]^d$, *there exists an increasing sequence of positive integers* $(M_k)_{k=1}^{\infty}$ *such that:*

$$F(x) - F(\theta) = \sum_{q=1}^{2d+1} \lim_{k\to\infty} \tau_{M_k} \left(\lim_{k\to\infty} \tau_{M_{k,q}}^{(d)}(x) \right)$$

uniformly on $[0,1]^d$ *and* $[0,1]$ *for indicated limits respectively.*

Acknowledgement

Presented work was supported by the grant DI-18-118 of Shota Rustaveli National Science Foundation of Georgia.

References

1. B. R. Gelbaum, J. M. H. Olmsted, *Counterexamples in Analysis.* The Mathesis Series Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam, 1964.
2. K. G. Grosse-Erdmann, Universal families and hypercyclic operators. *Bull. Amer. Math. Soc. (N.S.)* **36** (1999), no. 3, 345–381.
3. S. Mazurkiewicz, Sur l'approximation des fonctions continues d'une variable réelle par les sommes partielles d'une série de puissances, *C. R. Soc. Sci. Lett. Varsovie Cl. III*, **30**(1937), 25–30. Zbl 17:204.
4. J. Pál, Zwei kleine bemerkungen, *Tôhoku Math. J.*, **6**(1914/15), 42–43.
5. W. Sierpiński, Sur une série de puissonces universelle pour les fonctions continues, *Studia Math.*, **7**(1938), 45–48. Zbl 18:114.

(Received 29.10.2019)

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Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), issue 3, 177–178

FUBINI'S TYPE PHENOMENON FOR CONVERGENT IN PRINGSHEIM SENSE MULTIPLE FUNCTION SERIES

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Abstract. In the present paper ε-uniqueness multiple function systems are considered. A theorem representing a possibility of calculation of the limit of a convergent in the Pringsheim sense multiple function series with respect to an ε-uniqueness multiple function system via application of iterated limits is formulated.

Let $d \geq 2$ be a natural number, R^d be the d-dimensional Euclidean space, Z_0^d be the set of all points in R^d with integer nonnegative coordinates. By $x = (x_1, \ldots, x_d)$ we denote the points of the unit cube $[0,1]^d$ and by $m = (m_1, \ldots, m_d)$ and $n = (n_1, \ldots, n_d)$ those from the set Z_0^d. The symbol $m \to \infty$ means that $m_j \to \infty$ for every j, $1 \leq j \leq d$ independently of each other. μ is the linear Lebesgue measure. $E_1 \times E_2 \times \cdots \times E_d$ is the Cartesian product of the sets E_j, where $j = 1, 2, \ldots, d$ and $E_j \subset [0,1]$.

Let $\phi = \{\varphi_i(t)\}_{i=0}^{\infty}$ be a system of measurable and finite functions defined on $[0,1]$. So,

$$|\varphi_i(t)| < \infty, \quad t \in [0,1], \quad i = 0, 1, 2, \ldots.$$

Definition 1. A set $A \subset [0,1]$ is called an U set of the system $\phi = \{\varphi_i(t)\}_{i=0}^{\infty}$ if the convergence of a series $\sum\limits_{i=0}^{\infty} a_i \varphi_i(t)$ to zero on the set $[0,1] \backslash A$ implies that $a_i = 0$ for every $i \geq 0$.

Definition 2. The system $\phi = \{\varphi_i(t)\}_{i=0}^{\infty}$ is called an ε-uniqueness system if the number $\varepsilon \in (0,1]$ and any set $A \subset [0,1]$ with $\mu A < \varepsilon$ is an U set of $\phi = \{\varphi_i(t)\}_{i=0}^{\infty}$.

The expression $\Phi \in U(\varepsilon)$ means, that Φ is an ε-uniqueness system.

Note, that if $0 < \varepsilon < \varepsilon_1 \leq 1$ and $\Phi \in U(\varepsilon_1)$, then $\Phi \in U(\varepsilon)$.

Examples of an ε-uniqueness systems are a lacunary trigonometric system defined on $[0,1]$, with $\varepsilon = 1$ (see [3]) and Rademacher system, with $\varepsilon = \frac{1}{2}$ (see [1]).

Let $\Phi^{(j)} = \left\{\varphi_{n_j}^{(j)}(x_j)\right\}_{n_j=0}^{\infty}$ be a system of measurable and finite on $[0,1]$ functions for every j, where $1 \leq j \leq d$.

Let

$$\phi_n(x) = \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j), \quad x = (x_1, \ldots, x_d) \in [0,1]^d$$

for every $n \in Z_0^d$.

Consider the d-multiple series with respect to the system $\overline{\phi} = \{\phi_n(x)\}_{n \in Z_0^d}$,

$$\sum_{n=0}^{\infty} a_n \phi_n(x) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_1,\ldots,n_d} \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j). \tag{1}$$

By $S_m(x)$ we denote rectangular partial sums of the series (1), i. e.,

$$S_m(x) = \sum_{n_1=0}^{m_1} \cdots \sum_{n_d=0}^{m_d} a_{n_1,\ldots,n_d} \prod_{j=1}^{d} \varphi_{n_j}^{(j)}(x_j).$$

2010 *Mathematics Subject Classification.* 42B05, 42B08.
Key words and phrases. Multiple function series; Pringsheim convergence.

The convergence of the series (1) at the point x means that there exists a finite Pringsheim limit, i. e.,

$$-\infty < \lim_{m\to\infty} S_m(x) < \infty.$$

Let $\{j_1, j_2, \dots, j_d\}$ be a rearrangement of $\{1, 2, \dots, d\}$, then it holds the following Fubini-type

Theorem. *Let for any j, $1 \le j \le d$, the system $\Phi^{(j)}$ be an ε_j-uniqueness system and a set $E_j \subset [0,1]$ be such that $\mu E_j > 1 - \varepsilon_j$. If there exists*

$$\lim_{m\to\infty} S_m(x), \quad \textit{when} \quad x \in E_1 \times E_2 \times \cdots E_d,$$

then for any $\{j_1, j_2, \dots, j_d\}$ there exists iterated limit

$$\lim_{m_{j_1}\to\infty} \left(\lim_{m_{j_2}\to\infty} \left(\cdots \left(\lim_{m_{j_d}\to\infty} S_m(x) \right) \cdots \right) \right) \quad \textit{when} \quad x \in E_1 \times E_2 \times \cdots E_d$$

and

$$\lim_{m\to\infty} S_m(x) = \lim_{m_{j_1}\to\infty} \left(\lim_{m_{j_2}\to\infty} \left(\cdots \left(\lim_{m_{j_d}\to\infty} S_m(x) \right) \cdots \right) \right)$$

for any $x \in E_1 \times E_2 \times \cdots E_d$.

Remark. Note, that the theorem presented in [2] is a direct consequence of the above formulated theorem when $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_d = \varepsilon$.

Acknowledgement. Presented work was supported by the grant FR18-2499 of Shota Rustaveli National Science Foundation of Georgia.

References

1. S. B. Stečkin, P. L. Ul'janov, On sets of uniqueness. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **26** (1962), 211–222.
2. Sh. Tetunashvili, On the Fubini's sets of multiple function series. *Proc. A. Razmadze Math. Inst.* **143** (2007), 138–139.
3. A. Zygmund, On lacunary trigonometric series. *Trans. Amer. Math. Soc.* **34** (1932), no. 3, 435–446.

(Received 23.10.2019)

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