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CONTENTS

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ESTIMATION OF f-DIVERGENCE AND SHANNON ENTROPY BY LEVINSON TYPE INEQUALITIES VIA LIDSTONE INTERPOLATING POLYNOMIAL

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Abstract. Using Lidstone interpolating polynomial, some new generalizations of Levinson-type inequalities for $2p$-convex functions are obtained. In seek of applications to information theory, based on $\mathfrak{f}$-divergence, the estimates for new generalizations are also given. Moreover, inequalities for Shannon entropies are deduced.

1. Introduction and Preliminaries

The theory of convex functions has encountered a fast advancement. This can be attributed to a few causes: firstly, applications of convex functions are directly involved in the modern analysis, secondly, many important inequalities are applications of convex functions which are closely related to inequalities (see [24]).

Levinson generalized Ky Fan's inequality for 3-convex functions in [17] (see also [20, p.32, Theorem 1]) in the form of the following

Theorem 1.1. *Let $f : \mathbb{I} = (0, 2\lambda) \to \mathbb{R}$ be such that f is 3-convex. Also, let $0 < x_\rho < \lambda$ and $p_\rho > 0$. Then*

$$\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right) \leq \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(2\lambda - x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho (2\lambda - x_\rho)\right). \tag{1}$$

The difference of the right– and left-hand sides of (1) is the linear functional $J_1(f(\cdot))$, which can be written as follows:

$$J_1(f(\cdot)) = \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(2\lambda - x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho (2\lambda - x_\rho)\right) - \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) + f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right). \tag{2}$$

In [25], Popoviciu noticed that Levinson's inequality (1) is substantial on $(0, 2\lambda)$ for 3-convex functions, while in [9], (see additionally [20, p.32, Theorem 2]) Bullen gave distinctive confirmation of Popoviciu's result and furthermore the converse of (1).

Theorem 1.2. (a) *Let $f : \mathbb{I} = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a 3-convex function and $x_k, y_k \in [\zeta_1, \zeta_2]$ for $k = 1, 2, \ldots, \rho$ such that*

$$\max\{x_1 \ldots x_n\} \leq \min\{y_1 \ldots y_n\}, \ x_1 + y_1 = \cdots = x_n + y_n \tag{3}$$

and $p_\rho > 0$, then

$$\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right) \leq \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(y_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho y_\rho\right). \tag{4}$$

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(b) *If $p_\rho > 0$, inequality* (4) *is valid for all x_k, y_k satisfying condition* (3) *and the function f is continuous, then f is* 3*-convex.*

The difference of the right– and left-hand sides of (4) is the linear functional $J_2(f(\cdot))$, which can be written as follows:

$$J_2(f(\cdot)) = \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(y_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho y_\rho\right) - \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) + f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right). \tag{5}$$

Remark 1.1. It is essential to take note of the fact that under the suppositions of Theorem 1.1 and Theorem 1.2, if the function f is 3-convex, then $J_k(f(\cdot)) \geq 0$ for $k = 1, 2$, and $J_k(f(\cdot)) = 0$ for $f(x) = x$ or $f(x) = x^2$ or f is a constant function.

In the following result, Pečarić [21] (see also [20, p.32, Theorem 4]), proved inequality (4) by weakening condition (3).

Theorem 1.3. *Let $f : \mathbb{I} = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a* 3*-convex function, $p_\rho > 0$, and let $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ such that $x_\rho + y_\rho = 2\check{c}$, for $\rho = 1, \ldots, n$ $x_\rho + x_{n-\rho+1} \leq 2\check{c}$ and $\frac{p_\rho x_\rho + p_{n-\rho+1} x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leq \check{c}$. Then inequality* (4) *holds.*

In [19], Mercer replaced the symmetry by the equality of the variances of points and proved in the following result that inequality (4) still holds.

Theorem 1.4. *Let f be a* 3*-convex function on $[\zeta_1, \zeta_2]$, and let p_ρ be positive such that $\sum_{\rho=1}^{n} p_\rho = 1$. Also, let x_ρ, y_ϱ satisfy $\max\{x_1 \ldots x_n\} \leq \min\{y_1 \ldots y_n\}$ and*

$$\sum_{\rho=1}^{n} p_\rho \left(x_\rho - \sum_{\rho=1}^{n} p_\rho x_\rho\right)^2 = \sum_{\rho=1}^{n} p_\rho \left(y_\rho - \sum_{\rho=1}^{n} p_\rho y_\rho\right)^2, \tag{6}$$

then (4) *holds.*

In [22], Pečarić *et al.* gave probabilistic version of inequality (1) under condition (6). In [23] the operator version of probabilistic Levinsons inequality is discussed.
The following Lemma is given in [28].

Lemma 1.1. *If $f \in C^\infty[0,1]$, then*

$$f(t) = \sum_{l=0}^{p-1}\left[f^{(2l)}(0)\Theta_l(1-t) + f^{(2l)}(0)\Theta_l(t)\right] + \int_0^1 G_p(t,s) f^{(2p)}(t)dt,$$

where Θ_l is a polynomial of degree $2l+1$ defined by the relations

$$\Theta_0(t) = t,\ \Theta_p^{''}(t) = \Theta_{p-1}(t),\ \Theta_p(0) = \Theta_p(1) = 0,\ p \geq 1,$$

and

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & s \leq t; \\ (s-1)t, & t \leq s, \end{cases} \tag{7}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[0,1]$, and with the successive iterates of $G(t,s)$,

$$G_p(t,s) = \int_0^1 G_1(t,k) G_{p-1}(k,s)dk,\ p \geq 2. \tag{8}$$

The Lidstone polynomial can be expressed in terms of $G_p(t,s)$ as

$$\Theta_p(t) = \int_0^1 G_p(t,s)sds. \tag{9}$$

Lidstone series representation of $f \in C^{2p}[\zeta_1, \zeta_2]$ given in [7] as follows:

$$f(x) = \sum_{l=0}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_1)\Theta_l\left(\frac{\zeta_2-x}{\zeta_2-\zeta_1}\right) + \sum_{l=0}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2)\Theta_l\left(\frac{x-\zeta_1}{\zeta_2-\zeta_1}\right)$$
$$+ (\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\left(\frac{x-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\right) f^{(2p)}(t)dt \tag{10}$$

In [8], Gazić *et al.* considered the class of $2p$-convex functions and generalized Jensen's inequality and converses of Jensen's inequality by using Lidstone's interpolating polynomials. Some other, new and thought provoking results and their applications for various divergences, can be found in the literature (see, for example, [1–6]). All generalizations existing in literature are only for one type of data points. But in this paper and motivated by the above discussion, Levinson type inequalities are generalized via the Lidstone interpolating polynomial involving two types of data points for higher order convex functions. Moreover, a new functional is introduced based on f-divergence and then some estimates for new functional are obtained. Some inequalities for Shannon entropies are also deduced.

2. Main Results

Motivated by functional (5), we generalize the following results with the help of the Lidstone interpolating polynomial given by (10).

2.1. Generalization of Bullen type inequalities for $2p$-convex functions.

First, we define the following functional:
$\mathcal{F}$**:** Let $f : \mathbb{I}_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a function, $x_1, \ldots, x_n$ and $y_1, \ldots, y_m \in \mathbb{I}_1$ such that

$$\max\{x_1 \ldots x_n\} \le \min\{y_1 \ldots y_m\}. \tag{11}$$

Also, let $(p_1, \ldots, p_n) \in \mathbb{R}^n$ and $(q_1, \ldots, q_m) \in \mathbb{R}^m$ be such that $\sum_{\rho=1}^{n} p_\rho = 1$, $\sum_{\varrho=1}^{m} q_\varrho = 1$ and x_ρ, y_ϱ, $\sum_{\rho=1}^{n} p_\rho x_\rho$, $\sum_{\varrho=1}^{m} q_\varrho y_\varrho \in \mathbb{I}_1$. Then

$$\breve{J}(f(\cdot)) = \sum_{\varrho=1}^{m} q_\varrho f(y_\varrho) - f\left(\sum_{\varrho=1}^{m} q_\varrho y_\varrho\right) - \sum_{\rho=1}^{n} p_\rho f(x_\rho) + f\left(\sum_{\rho=1}^{n} p_\rho x_\rho\right). \tag{12}$$

Theorem 2.1. *Assume $\mathcal{F}$ with $f \in C^{2p}[\zeta_1, \zeta_2]$ $(p > 2)$ and let $\Theta_p(t)$ be the same as defined in Lemma 1.1. Then*

$$\breve{J}(f(\cdot)) = \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_1)\breve{J}\Big(\Theta_l(.)\Big) + \sum_{l=1}^{p-1}(\zeta_2-\zeta_2)^{2l} f^{(2l)}(\zeta_2)\breve{J}\Big(\ddot{\Theta}_l(\cdot)\Big)$$
$$+ (\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} \breve{J}\Big(G_p(t,\cdot)\Big) f^{(2p)}(t)dt, \tag{13}$$

where

$$\breve{J}\Big(\Theta_l(\cdot)\Big) = \sum_{\varrho=1}^{m} q_\varrho \Theta_l\left(\frac{\zeta_2 - y_\varrho}{\zeta_2 - \zeta_1}\right) - \Theta_l\left(\frac{\zeta_2 - \sum_{\varrho=1}^{m} q_\varrho y_\varrho}{\zeta_2 - \zeta_1}\right) - \sum_{\rho=1}^{n} p_\rho \Theta_l\left(\frac{\zeta_2 - x_\rho}{\zeta_2 - \zeta_1}\right) + \Theta_l\left(\frac{\zeta_2 - \sum_{\rho=1}^{n} p_\rho x_\rho}{\zeta_2 - \zeta_1}\right), \tag{14}$$

$$\breve{J}\Big(\ddot{\Theta}_l(\cdot)\Big) = \sum_{\varrho=1}^{m} q_\varrho \Theta_l\left(\frac{y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}\right) - \Theta_l\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}\right) - \sum_{\rho=1}^{n} p_\rho \Theta_l\left(\frac{x_\rho - \zeta_1}{\zeta_2 - \zeta_1}\right) + \Theta_l\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho - \zeta_1}{\zeta_2 - \zeta_1}\right) \tag{15}$$

and

$$\breve{J}\Big(G_p(t,\cdot)\Big) = \sum_{\varrho=1}^{m} q_\varrho G_p\left(\frac{y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) - G_p\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) - \sum_{\rho=1}^{n} p_\rho G_p\left(\frac{x_\rho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) + G_p\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right). \tag{16}$$

Proof. Using (10) in (12), we have

$$\begin{aligned}
\breve{J}(f(\cdot)) = &\sum_{\varrho=1}^{m} q_\varrho \Bigg[\sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) \Theta_l\left(\frac{\zeta_2 - y_\varrho}{\zeta_2 - \zeta_1}\right) + \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2) \\
&\times \Theta_l\left(\frac{y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}\right) + (\zeta_2 - \zeta_1)^{2p-1} \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) f^{(2p)}(t)dt\Bigg] \\
&- \Bigg[\sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) \Theta_l\left(\frac{\zeta_2 - \sum_{\varrho=1}^{m} q_\varrho y_\varrho}{\zeta_2 - \zeta_1}\right) \\
&+ \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2) \Theta_l\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}\right) \\
&+ (\zeta_2 - \zeta_1)^{2p-1} \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) f^{(2p)}(t)dt\Bigg] \\
&- \sum_{\rho=1}^{n} p_\rho \Bigg[\sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) \Theta_l\left(\frac{\zeta_2 - x_\rho}{\zeta_2 - \zeta_1}\right) + \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2) \\
&\times \Theta_l\left(\frac{x_\rho - \zeta_1}{\zeta_2 - \zeta_1}\right) + (\zeta_2 - \zeta_1)^{2p-1} \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{x_\rho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) f^{(2p)}(t)dt\Bigg] \\
&+ \Bigg[\sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) \Theta_l\left(\frac{\zeta_2 - \sum_{\rho=1}^{n} p_\rho x_\rho}{\zeta_2 - \zeta_1}\right) + \\
&+ \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2) \Theta_l\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho - \zeta_2}{\zeta_2 - \zeta_1}\right) \\
&+ (\zeta_2 - \zeta_1)^{2p-1} \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) f^{(2p)}(t)dt\Bigg]
\end{aligned}$$

$$+(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\Big(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)f^{(2p)}(t)dt\Big].$$

After some simple calculations, we have

$$\begin{aligned}
\breve{J}(f(\cdot)) =& \sum_{\varrho=1}^{m} q_\varrho\Big[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\Theta_l\Big(\frac{\zeta_2-y_\varrho}{\zeta_2-\zeta_1}\Big)\Big]-\Big[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\Theta_l\\
&\Big(\frac{\zeta_2-\sum_{\varrho=1}^{m}q_\varrho y_\varrho}{\zeta_2-\zeta_1}\Big)\Big]-\Big[\sum_{\rho=1}^{n}p_\rho\Big(\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\Theta_l\Big(\frac{\zeta_2-x_\rho}{\zeta_2-\zeta_1}\Big)\Big)\Big]\\
&+\Big[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\Theta_l\Big(\frac{\zeta_2-\sum_{\rho=1}^{n}p_\rho x_\rho}{\zeta_2-\zeta_1}\Big)\Big]\\
&+\Big[\sum_{\varrho=1}^{m} q_\varrho\Big[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\Theta_l\Big(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big]-\Big[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\Theta_l\\
&\Big(\frac{\sum_{\varrho=1}^{m}q_\varrho y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big]-\Big[\sum_{\rho=1}^{n}p_\rho\Big(\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\Theta_l\Big(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big)\Big]\\
&+\Big[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\Theta_l\Big(\frac{\sum_{\rho=1}^{n}p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big]\Big]\\
&+\sum_{\varrho=1}^{m} q_\varrho\Big[(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\Big(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)f^{(2p)}(t)dt\Big]\\
&-\Big[(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\Big(\frac{\sum_{\varrho=1}^{m}q_\varrho y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)f^{(2p)}(t)dt\Big]\\
&-\sum_{\rho=1}^{n} p_\rho\Big[(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\Big(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)f^{(2p)}(t)dt\Big]\\
&+\Big[(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\Big(\frac{\sum_{\rho=1}^{n}p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)f^{(2p)}(t)dt\Big].
\end{aligned}$$

$$\begin{aligned}
\breve{J}(f(\cdot)) =& \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\Big[\sum_{\varrho=1}^{m}q_\varrho\Theta_l\Big(\frac{\zeta_2-y_\varrho}{\zeta_2-\zeta_1}\Big)-\Theta_l\Big(\frac{\zeta_2-\sum_{\varrho=1}^{m}q_\varrho y_\varrho}{\zeta_2-\zeta_1}\Big)\\
&-\sum_{\rho=1}^{n}p_\rho\Theta_l\Big(\frac{\zeta_2-x_\rho}{\zeta_2-\zeta_1}\Big)+\Theta_l\Big(\frac{\zeta_2-\sum_{\rho=1}^{n}p_\rho x_\rho}{\zeta_2-\zeta_1}\Big)\Big]\\
&+\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\Big[\sum_{\varrho=1}^{m}q_\varrho\Theta_l\Big(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\Big)-\Theta_l\Big(\frac{\sum_{\varrho=1}^{m}q_\varrho y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\Big)\\
&-\sum_{\rho=1}^{n}p_\rho\Theta_l\Big(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1}\Big)+\Theta_l\Big(\frac{\sum_{\rho=1}^{n}p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1}\Big)\Big]\\
&+(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2}\Big[\sum_{\varrho=1}^{m}q_\varrho G_p\Big(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\Big)-
\end{aligned}$$

$$G_p\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)-\sum_{\rho=1}^{n} p_\rho G_p\left(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)$$
$$+G_p\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\Bigg] f^{(2p)}(t)dt.$$

Using definition of (14), (15) and (16), we get (13). □

As an application, we obtain a generalization of Bullen type inequality for $2p$-convex functions for $p>2$.

Theorem 2.2. *Assuming the conditions of Theorem* 2.1 *and*

$$\breve{J}\Big(G_p(t,\cdot)\Big)\geq 0. \tag{17}$$

If f is a $2p$-convex function, then

$$\breve{J}(f(\cdot))\geq\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2)\breve{J}\Big(\Theta_l(.)\Big)+\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2)\breve{J}\Big(\ddot{\Theta}_l(\cdot)\Big). \tag{18}$$

Proof. As the function f is $2p$-convex and $2p$-times differentiable, so

$$f^{(2p)}(x)\geq 0\ \forall\ x\in[\zeta_1,\zeta_2],$$

then using (17) in (13), we get (18). □

Remark 2.1.
(i) In Theorem 2.2, the reverse inequality in (17) leads to the reverse inequality in (18).
(ii) Inequality in (18) is also reversed if f is a $2p$-concave function.

If we put $m=n$, $p_\rho=q_\varrho$ and by using positive weights in (12), then $\breve{J}(\cdot)$ converts to the functional $J_2(\cdot)$ defined in (5), and also in this case, (13), (14), (15), (16), (17) and (18) become

$$J_2(f(\cdot))=\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_1) J_2\Big(\Theta_l(\cdot)\Big)+\sum_{l=1}^{p-1}(\zeta_2-\zeta_2)^{2l} f^{(2l)}(\zeta_2) J_2\Big(\ddot{\Theta}_l(\cdot)\Big)$$
$$+(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} J_2\Big(G_p(t,\cdot)\Big) f^{(2p)}(t)dt, \tag{19}$$

$$J_2\Big(\Theta_l(\cdot)\Big)=\sum_{\rho=1}^{n} p_\rho\Theta_l\left(\frac{\zeta_2-y_\varrho}{\zeta_2-\zeta_1}\right)-\Theta_l\left(\frac{\zeta_2-\sum_{\rho=1}^{n} p_\rho y_\varrho}{\zeta_2-\zeta_1}\right)$$
$$-\sum_{\rho=1}^{n} p_\rho\Theta_l\left(\frac{\zeta_2-x_\rho}{\zeta_2-\zeta_1}\right)+\Theta_l\left(\frac{\zeta_2-\sum_{\rho=1}^{n} p_\rho x_\rho}{\zeta_2-\zeta_1}\right), \tag{20}$$

$$J_2\Big(\ddot{\Theta}_l(\cdot)\Big)=\sum_{\rho=1}^{n} p_\rho\Theta_l\left(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\right)-\Theta_l\left(\frac{\sum_{\rho=1}^{n} p_\rho y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\right)$$
$$-\sum_{\rho=1}^{n} p_\rho\Theta_l\left(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1}\right)+\Theta_l\left(\frac{\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1}\right), \tag{21}$$

$$J_2\Big(G_p(t,\cdot)\Big)=\sum_{\rho=1}^{n} p_\rho G_p\left(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)-G_p\left(\frac{\sum_{\rho=1}^{n} p_\rho y_\varrho-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)$$
$$-\sum_{\rho=1}^{n} p_\rho G_p\left(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)+G_p\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\right), \tag{22}$$

$$J_2\Big(G_p(t,\cdot)\Big)\geq 0, \tag{23}$$

and
$$J_2(f(\cdot)) \geq \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2) J_2\Big(\Theta_l(\cdot)\Big) + \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2) J_2\Big(\ddot{\Theta}_l(\cdot)\Big). \tag{24}$$

Theorem 2.3. *Let $f : \mathbb{I}_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a $2p$ $(p > 2)$-convex function. Also, let $(p_1, \ldots, p_n)$ be positive real numbers such that $\sum_{\rho=1}^{n} p_\rho = 1$. Then for the functional $J_2(\cdot)$ defined in (5), we have the following:*

(i) (24) *holds for every $2p$-convex function if p is odd.*

(ii) *Let* (24) *hold. If the function*
$$F(x) = \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_1)\Theta_l\left(\frac{\zeta_2 - x}{\zeta_2-\zeta_1}\right) + \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2)\Theta_l\left(\frac{x-\zeta_1}{\zeta_2-\zeta_1}\right) \tag{25}$$
is 3-convex, then the right-hand side of (24) *is non-negative and we have the inequality*
$$J_2(f(\cdot)) \geq 0. \tag{26}$$

Proof.

(i) By (7), $G_1(t,s) \leq 0$, for $0 \leq t,\ s \leq 1$. By using (8), we have $G_p(t,s) \leq 0$ for odd p and $G_p(t,s) \geq 0$ for even p. Now, as G_1 is 3-convex and G_{p-1} is positive for odd p, therefore by using (8), G_p is 3-convex in the first variable if p is odd. Similarly, G_p is 3-concave in the first variable if p is even.

Hence if p is odd, then by Remark 1.1,
$$J_2\Big(G_p(t,\cdot)\Big) \geq 0,$$
therefore (24) holds.

(ii) $J_2(\cdot)$ is a linear functional, so we can write the right-hand side of (24) in the form $J_2(F(x))$, where F is defined in (25). Since F is assumed to be 3-convex, therefore using the given conditions and by Remark 1.1, the non-negativity of the right-hand side of (24) is immediate and we have (26) for n-tuples. □

In the next result we give generalization of Levinson's type inequality given in [21] (see also [20]).

Theorem 2.4. *Let $f \in C^{2p}[\zeta_1, \zeta_1]$ $(p > 2)$, $(p_1, \ldots, p_n)$ be positive real numbers such that $\sum\limits_{\rho=1}^{n} p_\rho = 1$. Also, let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n \in \mathbb{I}_1$ be such that $x_\rho + y_\rho = 2\check{c}$, $x_\rho + x_{n-\rho+1} \leq 2\check{c}$ and $\frac{p_\rho x_\rho + p_{n-\rho+1}x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leq \check{c}$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma* 1.1*, then* (19) *holds.*

Proof. The Proof is similar to that of Theorem 2.1 by assuming the conditions given in the statement. □

As an application, we give generalizations of Levinson's type inequalities for $2p$-convex functions $(p > 2)$.

Theorem 2.5. *Let $f \in C^{2p}[\zeta_1, \zeta_2]$ $(p > 2)$, $(p_1, \ldots, p_n)$ be positive real numbers such that $\sum\limits_{\rho=1}^{n} p_\rho = 1$. Also, let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n \in \mathbb{I}_1$ be such that $x_\rho + y_\rho = 2\check{c}$, $x_\rho + x_{n-\rho+1} \leq 2\check{c}$ and $\frac{p_\rho x_\rho + p_{n-\rho+1}x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leq \check{c}$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma* 1.1*. If* (23) *is valid, then* (24) *is also valid.*

Proof. Proof is similar to that of Theorem 2.2. □

Theorem 2.6. *Let $f \in C^{2p}[\zeta_1, \zeta_2]$ $(p > 2)$, $(p_1, \ldots, p_n)$ be positive real numbers such that $\sum\limits_{\rho=1}^{n} p_\rho = 1$. Also, let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n \in \mathbb{I}_1$ such that $x_\rho + y_\rho = 2\check{c}$ and $x_\rho + x_{n-\rho+1}$, $\frac{p_\rho x_\rho + p_{n-\rho+1}x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leq \check{c}$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma* 1.1*. Then:*

(i) *If p is odd, then for every $2p$-convex function $f : [\zeta_1, \zeta_2] \to \mathbb{R}$,* (24) *holds.*

(ii) *Let inequality* (24) *be satisfied. If the function* (25) *is* 3*-convex, the R.H.S of* (24) *is non-negative, we have inequality* (26).

Proof. Proof is similar to that of Theorem 2.5. □

In the next result, Levinson's type inequality is given (for positive weights) under Mercer's condition.

Corollary 2.1. *Let* $f : \mathbb{I}_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be a* $2p$*-convex function,* x_ρ*,* y_ϱ *satisfy* (6) *and the* $\max\{x_1 \ldots x_n\} \leq \min\{y_1 \ldots y_n\}$*. Also, let* $(p_1, \ldots, p_n) \in \mathbb{R}^n$ *such that* $\sum\limits_{\rho=1}^{n} p_\rho = 1$*. Then* (19) *is valid.*

Remark 2.2. Cebyšev, Grüss and Ostrowski-type new bounds related to the obtained generalizations can also be discussed. Moreover, we can also give the related mean value theorems by using non-negative functional (13) to construct new families of n-exponentially convex functions and Cauchy means related to these functionals such as given in Section 4 of [10].

3. Application to Information Theory

The idea of Shannon entropy is the central job of information speculation now and again implied as measure of uncertainty. The entropy of a random variable is described with respect to the probability distribution and can be shown to be a decent measure of randomness. Shannon entropy grants to assess the typical least number of bits expected to encode a progression of pictures subject to the letters all together size and the repeat of the symbols.

Divergences between probability distributions have been familiar with measure of the difference between them. An assortment of sorts of divergences exist, for example the $\mathfrak{f}$-divergences (especially, Kullback–Leibler divergences, Hellinger distance and total variation distance), Rényi divergences, Jensen–Shannon divergences, etc. (see [18, 27]). There are a lot of papers overseeing inequalities and entropies, see, e.g., [1, 14, 16, 26] and references therein. The Jensen inequality is an essential job in a bit of these inequalities. Regardless, Jensen's inequality manages one kind of data points and Levinson's inequality deals with two types of data points.

3.1. Csiszár divergence.

In [12, 13], Csiszár gave the following

Definition 3.1. *Let* f *be a convex function from* $\mathbb{R}^+$ *to* $\mathbb{R}^+$*. Let* $\tilde{\mathbf{r}}, \tilde{\mathbf{k}} \in \mathbb{R}^n_+$ *be such that* $\sum\limits_{\rho=1}^{n} r_\rho = 1$ *and* $\sum\limits_{\rho=1}^{n} k_\rho = 1$*. Then the* f*-divergence functional is defined by*

$$\mathbb{I}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\rho=1}^{n} k_\rho f\left(\frac{r_\rho}{k_\rho}\right).$$

By defining

$$f(0) := \lim_{x \to 0^+} f(x), \qquad 0f\left(\frac{0}{0}\right) := 0, \qquad 0f\left(\frac{a}{0}\right) := \lim_{x \to 0^+} xf\left(\frac{a}{x}\right), \quad a > 0,$$

he stated that non-negative probability distributions can also be used.

Using the definition of the f-divergence functional, Horv́ath *et al.* [15] gave the following functional.

Definition 3.2. *Let* $\mathbb{I}$ *be an interval contained in* $\mathbb{R}$ *and* $f : \mathbb{I} \to \mathbb{R}$ *be a function. Also, let* $\tilde{\mathbf{r}} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ *and* $\tilde{\mathbf{k}} = (k_1, \ldots, k_n) \in (0, \infty)^n$ *be such that*

$$\frac{r_\rho}{k_\rho} \in \mathbb{I}, \qquad \rho = 1, \ldots, n.$$

Then

$$\hat{\mathbb{I}}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\rho=1}^{n} k_\rho f\left(\frac{r_\rho}{k_\rho}\right). \tag{27}$$

We apply Theorem 2.2 for the $2p$-convex functions to $\hat{\mathbb{I}}_f(\tilde{\mathbf{r}},\tilde{\mathbf{k}})$.

Theorem 3.1. *Let* $\tilde{\mathbf{r}}=(r_1,\ldots,r_n)\in\mathbb{R}^n$*,* $\tilde{\mathbf{w}}=(w_1,\ldots,w_m)\in\mathbb{R}^m$*,* $\tilde{\mathbf{k}}=(k_1,\ldots,k_n)\in(0,\infty)^n$ *and* $\tilde{\mathbf{t}}=(t_1,\ldots,t_m)\in(0,\infty)^m$ *be such that*

$$\frac{r_\rho}{k_\rho}\in\mathbb{I},\qquad \rho=1,\ldots,n,$$

and

$$\frac{w_\varrho}{t_\varrho}\in\mathbb{I},\qquad \varrho=1,\ldots,m.$$

Also, let $f\in C^{2p}[\zeta_1,\zeta_2]$ *be such that* f *is* $2p$*-convex function (for odd* p*), then*

$$J_{cis}(f(\cdot))\geq\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)J\Big(\Theta_l(\cdot)\Big)+\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)J\Big(\ddot{\Theta}_l(\cdot)\Big),\tag{28}$$

where

$$\begin{aligned}J_{cis}(f(\cdot))=&\frac{1}{\sum_{\varrho=1}^m t_\varrho}\hat{\mathbb{I}}_f(\tilde{\mathbf{w}},\tilde{\mathbf{t}})-f\bigg(\sum_{\varrho=1}^m\frac{w_\varrho}{\sum_{\varrho=1}^m t_\varrho}\bigg)-\frac{1}{\sum_{\rho=1}^n k_\rho}\hat{\mathbb{I}}_f(\tilde{\mathbf{r}},\tilde{\mathbf{k}})\\&+f\bigg(\sum_{\rho=1}^n\frac{r_\rho}{\sum_{\rho=1}^n k_\rho}\bigg),\end{aligned}\tag{29}$$

$$\begin{aligned}J\Big(\Theta_l(\cdot)\Big)=&\sum_{\rho=1}^m\frac{t_\varrho}{\sum_{\varrho=1}^m t_\varrho}\Theta_l\bigg(\frac{\zeta_2-\frac{w_\varrho}{t_\varrho}}{\zeta_2-\zeta_1}\bigg)-\Theta_l\bigg(\frac{\zeta_2-\sum_{\rho=1}^m\frac{w_\varrho}{\sum_{\varrho=1}^m t_\varrho}}{\zeta_2-\zeta_1}\bigg)\\&-\sum_{\rho=1}^n\frac{k_\rho}{\sum_{\rho=1}^n k_\rho}\Theta_l\bigg(\frac{\zeta_2-\frac{r_\rho}{k_\rho}}{\zeta_2-\zeta_1}\bigg)+\Theta_l\bigg(\frac{\zeta_2-\sum_{\rho=1}^n\frac{r_\rho}{\sum_{\rho=1}^n k_\rho}}{\zeta_2-\zeta_1}\bigg),\end{aligned}\tag{30}$$

$$\begin{aligned}J\Big(\ddot{\Theta}_l(\cdot)\Big)=&\sum_{\rho=1}^m\frac{t_\varrho}{\sum_{\varrho=1}^m t_\varrho}\Theta_l\bigg(\frac{\frac{w_\varrho}{t_\varrho}-\zeta_1}{\zeta_2-\zeta_1}\bigg)-\Theta_l\bigg(\frac{\sum_{\rho=1}^m\frac{w_\varrho}{\sum_{\varrho=1}^m t_\varrho}-\zeta_1}{\zeta_2-\zeta_1}\bigg)\\&-\sum_{\rho=1}^n\frac{k_\rho}{\sum_{\rho=1}^n k_\rho}\Theta_l\bigg(\frac{\frac{r_\rho}{k_\rho}-\zeta_1}{\zeta_2-\zeta_1}\bigg)+\Theta_l\bigg(\frac{\sum_{\rho=1}^n\frac{r_\rho}{\sum_{\rho=1}^n k_\rho}-\zeta_1}{\zeta_2-\zeta_1}\bigg)\end{aligned}\tag{31}$$

and

$$\begin{aligned}J\Big(G_p(t,\cdot)\Big)=&\sum_{\rho=1}^m\frac{t_\varrho}{\sum_{\varrho=1}^m t_\varrho}G_p\bigg(\frac{\frac{w_\varrho}{t_\varrho}-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)-G_p\bigg(\frac{\sum_{\rho=1}^m\frac{w_\varrho}{\sum_{\varrho=1}^m t_\varrho}-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)\\&-\sum_{\rho=1}^n\frac{k_\rho}{\sum_{\rho=1}^n k_\rho}G_p\bigg(\frac{\frac{r_\rho}{k_\rho}-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg)+G_p\bigg(\frac{\sum_{\rho=1}^n\frac{r_\rho}{\sum_{\rho=1}^n k_\rho}-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\bigg).\end{aligned}\tag{32}$$

Proof. Since G_1 is 3-convex and G_{p-1} is positive for odd p, therefore by using (8), G_p is 3-convex in first variable if p is odd. Hence (17) holds. So using $p_\rho=\frac{k_\rho}{\sum_{\rho=1}^n k_\rho}$, $x_\rho=\frac{r_\rho}{k_\rho}$, $q_\varrho=\frac{t_\varrho}{\sum_{\varrho=1}^m t_\varrho}$, $y_\varrho=\frac{w_\varrho}{t_\varrho}$ in Theorem 2.2, (18) becomes (28), where $\hat{\mathbb{I}}_f(\tilde{\mathbf{r}},\tilde{\mathbf{k}})$ is defined in (27) and

$$\hat{\mathbb{I}}_f(\tilde{\mathbf{w}},\tilde{\mathbf{t}}):=\sum_{\varrho=1}^m t_\varrho f\bigg(\frac{w_\varrho}{t_\varrho}\bigg).\tag{33}$$

The theorem is proved. $\square$

3.2. **Shannon Entropy.**

Definition 3.3 (see [15]). *The Shannon entropy of the positive probability distribution* $\tilde{\mathbf{k}}=(k_1,\ldots,k_n)$ *is defined by*

$$\mathcal{S} := -\sum_{\rho=1}^{n} k_\rho \log(k_\rho). \tag{34}$$

Corollary 3.1. *Let* $\tilde{\mathbf{k}} = (k_1,\ldots,k_n)$ *and* $\tilde{\mathbf{t}} = (t_1,\ldots,t_m)$ *be the positive probability distributions. Also, let* $\tilde{\mathbf{r}} = (r_1,\ldots,r_n) \in (0,\infty)^n$ *and* $\tilde{\mathbf{w}} = (w_1,\ldots,w_m) \in (0,\infty)^m$.
If the base of log *is greater than* 1 *and p=odd* $(n = 3,5,\ldots)$, *then*

$$\begin{aligned} J_s(\cdot) \le & \sum_{l=1}^{p-1} (\zeta_2-\zeta_1)^{2l} \frac{(-1)^{2l-1}(2l-1)!}{(\zeta_1)^{2l}} J\Big(\Theta_l(\cdot)\Big) \\ & + \sum_{l=1}^{p-1} (\zeta_2-\zeta_2)^{2l} \frac{(-1)^{2l-1}(2l-1)!}{(\zeta_2)^{2l}} J\Big(\ddot{\Theta}_l(\cdot)\Big), \end{aligned} \tag{35}$$

where

$$\begin{aligned} J_s(\cdot) = & \sum_{\varrho=1}^{m} t_\varrho \log(w_\varrho) + \tilde{\mathcal{S}} - \log\left(\sum_{\varrho=1}^{m} w_\varrho\right) - \sum_{\rho=1}^{n} k_\rho \log(r_\rho) - \mathcal{S} \\ & + \log\left(\sum_{\rho=1}^{n} r_\rho\right) \end{aligned} \tag{36}$$

and $J\Big(\Theta_l(\cdot)\Big)$, $J\Big(\ddot{\Theta}_l(\cdot)\Big)$, $J\Big(G_p(t,\cdot)\Big)$ *are the same as defined in* (30), (31) *and* (32), *respectively.*

Proof. The function $f(x) = \log(x)$ is $2p$-concave for odd p $(p > 2)$ and the base of log is greater than 1. So, by using Remark 2.1(ii), (18) holds in reverse direction. Therefore using $f(x) = \log(x)$ and $p_\rho = \frac{k_\rho}{\sum_{\rho=1}^{n} k_\rho}$, $x_\rho = \frac{r_\rho}{k_\rho}$, $q_\varrho = \frac{t_\varrho}{\sum_{\varrho=1}^{m} t_\varrho}$, $y_\varrho = \frac{w_\varrho}{t_\varrho}$ in reversed inequality (18), we have (35), where $\mathcal{S}$ is defined in (34) and

$$\tilde{\mathcal{S}} = -\sum_{\varrho=1}^{m} t_\varrho \log(t_\varrho). \qquad \square$$

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EXTREME QUANTILE REGRESSION IN A PROPORTIONAL TAIL FRAMEWORK

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A paper devoted to the 75th birthday of Estate Khmaladze

Abstract. The model of heteroscedastic extremes initially introduced by Einmahl et al. (JRSSB, 2016) describes the evolution of a nonstationary sequence whose extremes evolve over time. We revisit this model and adapt it into a general extreme quantile regression framework. We provide estimates for the extreme value index and the integrated skedasis function and prove their joint asymptotic normality. Our results are quite similar to those developed for heteroscedastic extremes, but with a different proof approach emphasizing coupling arguments. We also propose a pointwise estimator of the skedasis function and a Weissman estimator of conditional extreme quantiles and prove the asymptotic normality of both estimators.

1. Introduction and Main Results

1.1. **Framework.** One of the main goals of the extreme value theory is to propose estimators of extreme quantiles: given an i.i.d. sample $Y_1, \dots, Y_n$ with distribution F, one wants to estimate the quantile of order $1-\alpha_n$ defined as $q(\alpha_n) := F^{\leftarrow}(1-\alpha_n)$, with $\alpha_n \to 0$ as $n \to \infty$ and

$$F^{\leftarrow}(u) := \inf\{x \in \mathbb{R} : F(x) \geq u\},\ u \in (0,1)$$

denotes the quantile function. The extreme regime corresponds to the case for $\alpha_n < 1/n$ in which case extrapolation beyond the sample maximum is needed. Considering an application in hydrology, these mathematical problems correspond to the following situation: given a record over $n = 50$ years of the level of a river, can we estimate the 100-year return level? The answer to this question is provided by the univariate extreme value theory and we refer to the monographs by Coles [6], Beirlant et al. [2] or de Haan and Ferreira [8] for a general background.

In many situations, auxiliary information is available and represented by a covariate X taking values in $\mathbb{R}^d$ and, given $x \in \mathbb{R}^d$, one wants to estimate $q(\alpha_n|x)$, the conditional $(1-\alpha_n)$-quantile of Y with respect to some given values of the covariate $X = x$. This is an extreme quantile regression problem. Recent advances in extreme quantile regression include the works by Chernozhukov [5], El Methni et al. [13] or Daouia et al. [7].

In this paper we develop the proportional tail framework for extreme quantile regression. It is an adaptation of the heteroscedastic extremes developed by Einmahl et al. [12], where the authors propose a model for the extremes of independent, but nonstationary observations whose distribution evolves over time, a model which can be viewed as a regression framework with time as covariate and deterministic design with uniformly distributed observation times $1/n, 2/n, \dots, 1$. In our setting, the covariate X takes values in $\mathbb{R}^d$ and is random with arbitrary distribution. The main assumption, directly adapted from Einmmahl et al. [12], is the so-called proportional tail assumption formulated in Equation (1) and stating that the conditional tail function of Y for the given $X = x$ is asymptotically proportional to the unconditional tail. The proportionality factor is given by the so-called skedasis function $\sigma(x)$ that accounts for the dependency of the extremes of Y with respect to the covariate X. Furthermore, as it is standard in the extreme value theory, the unconditional distribution of Y is assumed to be regularly varying. Together with the proportional tail assumption, this implies that

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all the conditional distributions are regularly varying with the same extreme value index. Hence the proportional tail framework appears suitable for modeling covariate dependent extremes, where the extreme value index is constant, but the scale parameter depends on the covariate X in a nonparametric way related to the skedasis function $\sigma(x)$. Note that this framework is also considered by Gardes [14] for the purpose of estimation of the extreme value index.

Our main results are presented in the following subsections. Section 1.2 considers the estimation of the extreme value index and integrated skedasis function in the proportional tail model, and our results of asymptotic normality are similar to those in Einmahl et al. [9], but with a different proof emphasizing coupling arguments. Section 1.3 considers both the pointwise estimation of the skedasis function and the conditional extreme quantile estimation with Weissman estimators and states their asymptotic normality. Section 2 develops some coupling arguments used in the proofs of the main theorems, proofs gathered in Section 3. Finally, an appendix states a technical lemma and its proof.

1.2. **The proportional tail model.** Let (X, Y) be a generic random couple taking values in $\mathbb{R}^d \times \mathbb{R}$. Define the conditional cumulative distribution function of Y given $X = x$ by

$$F_x(y) := \mathbb{P}(Y \leq y | X = x), \quad y \in \mathbb{R}, \ \ x \in \mathbb{R}^d.$$

The main assumption of *the proportional tail model* is

$$\lim_{y\to\infty} \frac{1 - F_x(y)}{1 - F^0(y)} = \sigma(x) \quad \text{uniformly in } x \in \mathbb{R}^d, \tag{1}$$

where F^0 is some baseline distribution function and σ is the so-called skedasis function following the terminology introduced in [12]. By integration, the unconditional distribution F of Y satisfies

$$\lim_{y\to\infty} \frac{1 - F(y)}{1 - F^0(y)} = \int_{\mathbb{R}^d} \sigma(x)\mathbf{P}_X(dx).$$

We can hence suppose without loss of generality that $F = F^0$ and that $\int \sigma d\mathbf{P}_X = 1$.
We also make the assumption that F is of $1/\gamma$-regular variation,

$$1 - F(y) = y^{-1/\gamma}\ell(y), \quad y \in \mathbb{R},$$

with ℓ, slowly varying at infinity. Together with the proportional tail condition (1) with $F = F^0$, this implies that F_x is also of $1/\gamma$-regular variation for each $x \in \mathbb{R}^d$. This is a strong consequence of the model assumptions. In this model, the extremes are driven by two parameters: the common extreme value index $\gamma > 0$ and the skedasis function $\sigma(\cdot)$. Following [12], we consider the usual ratio estimator (see, e.g., [16, p. 198]) for γ and propose a nonparametric estimator of the integrated (or cumulative) skedasis function

$$C(x) := \int_{\{u \leq x\}} \sigma(u)\mathbf{P}_X(du), \quad x \in \mathbb{R}^d,$$

where $u \leq x$ stands for the componentwise comparison of vectors. Note that - putting aside the case, where X is discrete - the function C is easier to estimate than σ, in the same way that a cumulative distribution function is easier to estimate than a density function. Estimation of C is useful to derive tests, while estimation of σ will be considered later on for the purpose of extreme quantile estimation.

Let $(X_i, Y_i)_{1\leq i\leq n}$ be i.i.d. copies of (X, Y). The estimators are built with observations (X_i, Y_i) for which Y_i exceeds a high threshold $\mathbf{y}_n$. Note that in this article, $(\mathbf{y}_n)_{n\in\mathbb{N}}$ may be deterministic or data driven. For the purpose of asymptotics, $\mathbf{y}_n$ depends on the sample size $n \geq 1$ in a way such that

$$\mathbf{y}_n \to \infty \quad \text{and} \quad N_n \to \infty \quad \text{in probability},$$

with $N_n := \sum_{i=1}^n \mathbb{1}_{\{Y_i > \mathbf{y}_n\}}$, the (possibly random) number of exceedances. The extreme value index $\gamma > 0$ is estimated by the ratio estimator

$$\hat{\gamma}_n := \frac{1}{N_n} \sum_{i=1}^n \log\left(\frac{Y_i}{\mathbf{y}_n}\right) \mathbb{1}_{\{Y_i > \mathbf{y}_n\}}.$$

The integrated skedasis function C can be estimated by the following empirical pseudo distribution function

$$\widehat{C}_n(x) := \frac{1}{N_n}\sum_{i=1}^{n} \mathbb{1}_{\{Y_i > \mathbf{y}_n,\ X_i \le x\}}, \quad x \in \mathbb{R}^d.$$

When Y is continuous and $\mathbf{y}_n := Y_{n-k_n:n}$ is the (k_n+1)-th highest order statistic, then $N_n = k$ and $\hat{\gamma}_n$ coincides with the usual Hill estimator.

Our first result addresses the joint asymptotic normality of $\hat{\gamma}_n$ and $\widehat{C}_n$, namely,

$$v_n \begin{pmatrix} \widehat{C}_n(\cdot) - C(\cdot) \\ \hat{\gamma}_n - \gamma \end{pmatrix} \xrightarrow{\mathscr{L}} \mathbb{W}, \tag{2}$$

where $\mathbb{W}$ is a Gaussian Borel probability measure on $L^\infty(\mathbb{R}^d) \times \mathbb{R}$, and $v_n \to \infty$ is a deterministic rate. To prove the asymptotic normality, the threshold $\mathbf{y}_n$ must scale suitably with respect to the rates of convergence in the proportional tail and domain of attraction conditions. More precisely, we assume the existence of a positive function A converging to zero and such that as $y \to \infty$,

$$\sup_{x \in \mathbb{R}^d} \left| \frac{\bar{F}_x(y)}{\sigma(x)\bar{F}(y)} - 1 \right| = \mathrm{O}\left(A\left(\frac{1}{\bar{F}(y)} \right) \right), \tag{3}$$

and

$$\sup_{z > \frac{1}{2}} \left| \frac{\bar{F}(zy)}{z^{-1/\gamma}\bar{F}(y)} - 1 \right| = \mathrm{O}\left(A\left(\frac{1}{\bar{F}(y)} \right) \right), \tag{4}$$

with $\bar{F}(y) := 1 - F(y)$ and $\bar{F}_x(y) := 1 - F_x(y)$. Our main result can then be stated as follows. When reading the present article, the reader probably notices that the domain $\{z > 1/2\}$ in (4) can be replaced by any domain $\{z > c\}$ for some $c \in]0,1[$.

Theorem 1.1. *Assume that assumptions* (3) *and* (4) *hold and* $\mathbf{y}_n/y_n \to 1$ *in probability for some deterministic sequence* y_n *such that* $p_n := \bar{F}(y_n)$ *satisfies*

$$p_n \to 0, \quad np_n \to \infty \quad \textit{and} \quad \sqrt{np_n}^{1+\varepsilon} A(1/p_n) \to 0 \textit{ for some } \varepsilon > 0.$$

Then the asymptotic normality (2) *holds with*

$$v_n := \sqrt{np_n} \quad \textit{and} \quad \mathbb{W} \stackrel{\mathscr{L}}{=} \begin{pmatrix} B \\ N \end{pmatrix},$$

with B *a* C*-Brownian bridge on* $\mathbb{R}^d$ *and* N *a centered Gaussian random variable with variance* γ^2 *and independent of* B.

Under the C-Brownian bridge we here mean a centered Gaussian process on $\mathbb{R}^d$ with the covariance function

$$\operatorname{cov}(B(x), B(x')) := \int_{\mathbb{R}^d} \mathbb{1}_{]-\infty,x]}\mathbb{1}_{]-\infty,x']} dC - C(x)C(x').$$

Remark. Theorem 1.1 extends Theorem 2.1 of Einmhal *et al.* [12] in two directions: first, it states that their estimators and theoretical results have natural counterparts in the framework of proportional tails. We also could go past their univariate dependency $i/n \to \sigma(i/n)$ to a multivariate dependency $x \to \sigma(x),\ x \in \mathbb{R}^d$. Second, it shows that general data-driven thresholds can be used. Those extensions come at the price of a slightly more stringent condition upon the bias control. Indeed, their condition $\sqrt{k_n}A(n/k_n) \to 0$ corresponds to our condition $\sqrt{np_n}^{1+\varepsilon}A(1/p_n) \to 0$ with $\varepsilon = 0$. We believe that this loss is small in regard to the gain on the practical side: the threshold $\mathbf{y}_n$ in $(\hat{\gamma}_n, \hat{C}_n)$ may be data-driven. Take, for example, $\mathbf{y}_n := Y_{n-k_n:n}$, which is equivalent in probability to $y_n := F^{\leftarrow}(1 - k_n/n)$ is $k_n \to \infty$. As a consequence, Theorem 1.1 holds for this choice of $\mathbf{y}_n$ if

$$k_n \to \infty, \ \frac{k_n}{n} \to 0, \text{ and } \sqrt{k_n}^{1+\varepsilon} A\left(\frac{n}{k_n}\right) \to 0.$$

An example where (3) and (4) hold: The reader might wonder if a model imposing (3) and (4) is not too restrictive for modeling. First, note that condition (4) has been well studied as the second order condition holding uniformly over intervals (see, e.g., [8, p. 383, Section B.3], [1, 11]). A generic example of the regression model, where (3) and (4) hold, is given as follows: take a c.d.f H fulfilling the second order heavy tail condition (4) on any domain $\{z > c\}$. Then assume that the laws of $Y \mid X = x$ obey a location scale model in the sense that

$$F_x(y) = H\left(\frac{y - \mu(x)}{\Delta(x)}\right),$$

for some functions $\mu(\cdot)$ and $\Delta(\cdot)$ that are uniformly bounded on $\mathbb{R}^d$. Then, since $1 - \Delta(x)\mu(x)/y \to 1$ uniformly in x as $y \to \infty$, condition (4) entails

$$\sup_{x\in\mathbb{R}^d}\left|\frac{\overline{F}_x(y)}{\Delta(x)^{1/\gamma}\overline{H}(y)} - 1\right| = O(A(1/\overline{H}(y))), \quad \text{as} \quad y \to \infty.$$

Integrating in x gives $\overline{H}(y) = \theta\overline{F}(y)$ as $y \to \infty$ for some $\theta > 0$, which yields (3) with the choice of $\sigma(\cdot) := \theta\Delta(\cdot)^{1/\gamma}$.

1.3. **Extreme quantile regression.** In this subsection, we restrict ourselves to the case where $\mathbf{y}_n$ is deterministic i.e. $\mathbf{y}_n = y_n$ according to the notations of Theorem 1.1. We now address the estimation of extreme conditional quantiles in the proportional tail model, namely

$$q(\alpha_n|x) := F_x^{\leftarrow}(1 - \alpha_n),$$

for some $x \in \mathbb{R}^d$ that will be fixed once for all in this section, and for a sequence $\alpha_n = O(1/n)$. To that aim, we shall borrow the heuristics behind the Weissman estimator [19], for which we here write a short reminder. It is known that $F \in D(G_\gamma)$ is equivalent to

$$\lim_{t\to\infty}\frac{U(tz)}{U(t)} = z^\gamma, \quad \text{for each} \quad z > 0,$$

with $U(t) = F^{\leftarrow}(1 - 1/t)$, $t > 1$. Recall that $p_n = \bar{F}(y_n)$. Since U is of γ-regular variation, the unconditional quantile $q(\alpha_n) := F^{\leftarrow}(1 - \alpha_n)$ is approximated by

$$q(\alpha_n) = U(1/p_n)\frac{U(1/\alpha_n)}{U(1/p_n)} \approx y_n\left(\frac{p_n}{\alpha_n}\right)^\gamma,$$

leading to the Weissman-type quantile estimator

$$\hat{q}(\alpha_n) := y_n\left(\frac{\hat{p}_n}{\alpha_n}\right)^{\hat{\gamma}_n},$$

where $\hat{p}_n := \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{Y_i > y_n\}}$ is the empirical counterpart of p_n.

Now going back to quantile regression in the proportional tail model, it is readily verified that assumption (1) implies

$$q(\alpha_n \mid x) \sim q\left(\frac{\alpha_n}{\sigma(x)}\right) \quad \text{as } n \to \infty.$$

This immediately leads to the plug-in estimator

$$\hat{q}(\alpha_n|x) := \hat{q}\left(\frac{\alpha_n}{\hat{\sigma}_n(x)}\right) = y_n\left(\frac{\hat{p}_n\hat{\sigma}_n(x)}{\alpha_n}\right)^{\hat{\gamma}_n},$$

where $\hat{\sigma}_n(x)$ denotes a consistent estimator of $\sigma(x)$.

In the following, we propose a kernel estimator of $\sigma(x)$ and prove its asymptotic normality before deriving the asymptotic normality of the extreme conditional quantile estimator $\hat{q}(\alpha_n|x)$. The proportional tail assumption (1) implies

$$\sigma(x) = \lim_{n\to\infty}\frac{\overline{F}_x(y_n)}{\overline{F}(y_n)}.$$

We propose the simplest kernel estimator with bandwidth $h_n > 0$,

$$\frac{\sum_{i=1}^n \mathbb{1}_{\{|x-X_i|<h_n\}}\mathbb{1}_{\{Y_i>y_n\}}}{\sum_{i=1}^n \mathbb{1}_{\{|x-X_i|<h_n\}}}$$

as an estimator of $\overline{F}_x(y_n)$, while the denominator is estimated by $\hat{p}_n$. Combining the two estimators yields

$$\hat{\sigma}_n(x) := n\frac{\sum_{i=1}^n \mathbb{1}_{\{|x-X_i|<h_n\}}\mathbb{1}_{\{Y_i>y_n\}}}{\sum_{i=1}^n \mathbb{1}_{\{|x-X_i|<h_n\}}\sum_{i=1}^n \mathbb{1}_{\{Y_i>y_n\}}}.$$

Our next result states the asymptotic normality of $\hat{\sigma}_n(x)$. The more general case of a random threshold is left for future works.

Theorem 1.2. *Take the notations of Theorem* 1.1, *and let $h_n \to 0$ be deterministic and positive. Assume that*

$$np_n h_n^d \to \infty, \quad \sqrt{np_n h_n^d}A\left(1/p_n\right) \to 0.$$

Assume that the law of X is continuous on a neighborhood of x. Also assume that σ is continuous and positive on a neighborhood of $x \in \mathbb{R}^d$, and that some version f of the density of X also shares those properties. Then, under assumption (3), *we have*

$$\sqrt{np_n h_n^d}\Big(\hat{\sigma}_n(x) - \sigma(x)\Big) \xrightarrow{\mathscr{L}} \mathcal{N}\left(0, \frac{\sigma(x)}{f(x)}\right).$$

The asymptotic normality of the extreme quantile estimate $\hat{q}(\alpha_n \mid x)$ is deduced from the asymptotic normality of $\hat{\gamma}_n$ and $\hat{\sigma}_n(x)$ stated respectively in Theorems 1.1 and 1.2. This is stated in our next theorem, which has to be seen as the counterpart of [8, p.138, Theorem 4.3.8] for conditional extreme quantiles. See also [16, p. 170, Theorem 9.8] for a similar result when $\log(p_n/\alpha_n) \to d \in \mathbb{R}$.

Theorem 1.3. *Under assumptions of Theorems* 1.1 *and* 1.2, *if $\sqrt{h_n^d}\log(p_n/\alpha_n) \to \infty$, we have*

$$\frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\log\left(\frac{\hat{q}(\alpha_n|x)}{q(\alpha_n|x)}\right) \xrightarrow{\mathscr{L}} \mathcal{N}\left(0, \gamma^2\right).$$

The condition $\sqrt{h_n^d}\log(p_n/\alpha_n)$ requires the bandwidth to be of larger order than $1/\log(p_n/\alpha_n)$, so the error in the estimation of $\sigma(x)$ is negligible. As a consequence of Theorem 1.3, the consistency

$$\frac{\hat{q}(\alpha_n|x)}{q(\alpha_n|x)} \xrightarrow{\mathbb{P}} 1.$$

That condition seems to state a limit for the extrapolation: α_n cannot be too small or one might lose consistency.

2. A Coupling Approach

We will first prove Theorem 1.1 when $\mathbf{y}_n$ is deterministic (i.e., $\mathbf{y}_n \equiv y_n$). In this case, N_n is binomial (n, p_n). Moreover, $N_n/np_n \to 1$ in probability, since $np_n \to \infty$.
A simple calculus shows that for each A Borel and $t \geq 1$, (1) entails

$$\mathbb{P}\left(\frac{Y}{\mathbf{y}} \geq t, X \in A \middle| Y \geq \mathbf{y}\right) \longrightarrow \int_t^\infty \int_A \mathbf{y}^{-1/\gamma}\sigma(x)\mathrm{d}\mathbf{y}\mathbf{P}_X(\mathrm{d}x), \text{ as } \mathbf{y} \to \infty, \tag{5}$$

defining a "limit model" for $(X, Y/\mathbf{y})$, the law

$$Q := \sigma(x)\mathbf{P}_X \otimes Pareto(1/\gamma)$$

with independent marginals. Fix $n \geq 1$. Using the heuristic of (5), we shall build an explicit coupling between $(X, Y/y_n)$ and the limit model Q. Define the conditional tail quantile function as $U_x(t) := F_x^{\leftarrow}(1 - 1/t)$ and recall that the total variation distance between two Borel probability measures on $\mathbb{R}^d$ is defined as

$$||P_1 - P_2|| := \sup_{B \text{ Borel}} |P_1(B) - P_2(B)|.$$

This distance is closely related to the notion of optimal coupling detailed in [15]. The following fundamental result is due to Dobrushin [10].

Lemma 2.1 (Dobrushin, 1970)**.** *For two probability measures P_1 and P_2 defined on the same measurable space, there exist two random variables (V_1, V_2) on a probability set $(\Omega, \mathcal{A}, \mathbb{P})$ such that*

$$V_1 \sim P_1, \quad V_2 \sim P_2 \quad \text{and} \quad ||P_1 - P_2|| = \mathbb{P}(V_1 \neq V_2).$$

This lemma will be a crucial tool of our coupling construction, which is described as follows.

Coupling construction: Fix $n \geq 1$. Let $(E_{i,n})_{1\leq i\leq n}$ be i.i.d. Bernoulli random variables with $\mathbb{P}(E_{i,n} = 1) = \bar{F}(y_n)$ and $(Z_i)_{1\leq i\leq n}$ i.i.d. with distribution Pareto(1) and independent of $(E_{i,n})_{1\leq i\leq n}$. For each $1 \leq i \leq n$, construct $(\tilde{X}_{i,n}, \tilde{Y}_{i,n}, X^*_{i,n}, Y^*_{i,n})$ as follows.

- ▶ If $E_{i,n} = 1$, then
  - ▷ Take $\tilde{X}_{i,n} \sim P_{X|Y>y_n}$, $X^*_{i,n} \sim \sigma(x)\mathbf{P}_X(dx)$ on the same probability space, satisfying $\mathbb{P}(\tilde{X}_{i,n} \neq X^*_{i,n}) = \|\mathbf{P}_{X|Y>y_n} - \sigma(x)\mathbf{P}_X(dx)\|$. Their existence is guaranteed by Lemma 2.1.
  - ▷ Set $\tilde{Y}_{i,n} := U_{\tilde{X}_{i,n}}(\frac{Z_i}{\bar{F}_{\tilde{X}_{i,n}}(y_n)})$, $Y^*_{i,n} := y_n Z_i^{\gamma}$.
- ▶ If $E_{i,n} = 0$, then
  - ▷ Randomly generate $(\tilde{X}_{i,n}, \tilde{Y}_{i,n}) \sim \mathbf{P}_{(X,Y)|Y\leq y_n}$.
  - ▷ Randomly generate $(X^*_{i,n}, Y^*_{i,n}/y_n) \sim \sigma(x)\mathbf{P}_X(\mathrm{d}x) \otimes Pareto(1/\gamma)$.

The following proposition states the properties of our coupling construction, which will play an essential role in our proof of Theorem 1.1.

Proposition 2.2. *For each $n \geq 1$, the coupling $(\tilde{X}_{i,n}, \tilde{Y}_{i,n}, X^*_{i,n}, Y^*_{i,n})_{1\leq i\leq n}$ has the following properties:*

1. *$(\tilde{X}_{i,n}, \tilde{Y}_{i,n})_{1\leq i\leq n}$ has the same law as $(X_i, Y_i)_{1\leq i\leq n}$.*
2. *$(X^*_{i,n}, Y^*_{i,n}/y_n) \rightsquigarrow Q$.*
3. *$(X^*_{i,n}, Y^*_{i,n})_{1\leq i\leq n}$ and $(E_{i,n})_{1\leq i\leq n}$ are independent. Moreover, $(Y^*_{i,n})_{1\leq i\leq n}$ are i.i.d. and independent of $(\tilde{X}_{i,n}, X^*_{i,n})$.*
4. *There exists $M > 0$ such that*

$$\max_{\substack{1\leq i\leq n,\\ E_{i,n}=1}} \left| \frac{Y^*_{i,n}}{\tilde{Y}_{i,n}} - 1 \right| \leq MA\left(1/p_n\right) \tag{6}$$

and

$$\mathbb{P}\left(\tilde{X}_{1,n} \neq X^*_{1,n} | E_{i,n} = 1\right) \leq MA\left(1/p_n\right), \tag{7}$$

where A is given by assumptions (3) and (4).

Proof. To prove Point 1, it is sufficient to see that

$$\mathscr{L}((\tilde{X}_{1,n}, \tilde{Y}_{1,n}) | E_{i,n} = 1) = \mathscr{L}((X, Y) | Y > y_n).$$

Since $U_x(z/(1 - F_x(y_n))) \leq y$ if and only if $1 - (1 - F_x(y_n))/z \leq F_x(y)$, then for $y \geq y_n$, we have

$$\begin{aligned} &\int_1^{\infty} \mathbb{1}_{\{U_x(z/(1-F_x(y_n)))\leq y\}} \frac{\mathrm{d}z}{z^2} \\ &= \int_1^{\infty} \mathbb{1}_{\{1-(1-F_x(y_n))/z\leq F_x(y)\}} \frac{\mathrm{d}z}{z^2} \\ &= \int_{F_x(y_n)}^{1} \mathbb{1}_{\{t\leq F_x(y)\}} \frac{\mathrm{d}t}{1 - F_x(y_n)} \end{aligned}$$

$$= \int_{F_x(y_n)}^{F_x(y)} \frac{\mathrm{d}t}{1-F_x(y_n)} = \frac{F_x(y)-F_x(y_n)}{1-F_x(y_n)},$$

with the second equality given by the change of variable $t = 1-(1-F_x(y_n))/z$. We can deduce from this computation that for a Borel set B and $y \geq y_n$,

$$\begin{aligned}
&\mathbb{P}\Big(\tilde{X}_{1,n} \in B, U_{\tilde{X}_{1,n}}\Big(\frac{Z}{1-F_{\tilde{X}_{1,n}}(y_n)}\Big) \leq y \Big| E_{1,n} = 1\Big) \\
&= \int_{x\in B} \int_1^\infty \mathbb{1}_{\{U_x(z/(1-F_x(y_n)))\leq y\}} \frac{\mathrm{d}z}{z^2} \mathrm{d}P_{X|Y>y_n}(x) \\
&= \int_{x\in B} \frac{F_x(y)-F_x(y_n)}{1-F_x(y_n)} \mathrm{d}P_{X|Y>y_n}(x) \\
&= \int_{x\in B} \mathbb{P}\big(Y \leq y | Y > y_n, X = x\big) \mathrm{d}P_{X|Y>y_n}(x) \\
&= \mathbb{P}\big(X \in B, Y \leq y | Y > y_n\big).
\end{aligned}$$

□

This proves Point 1. Points 2 and 3 are immediate.

Point 4 will be proved with the two following lemmas.

Lemma 2.3. *Under conditions* (3) *and* (4), *we have*

$$\sup_{z\geq 1/2} \sup_{x\in\mathbb{R}^p} \left| \frac{1}{z^\gamma y} U_x\left(\frac{z}{\bar{F}_x(y)}\right) - 1 \right| = \mathrm{O}\left(A\left(\frac{1}{\bar{F}(y)}\right)\right), \ \textit{as } y \to \infty.$$

Proof. According to assumptions (3) and (4), there exists a constant M such that

$$\begin{aligned}
\left|\frac{\bar{F}_x(y)}{\sigma(x)\bar{F}(y)} - 1\right| &\leq MA\left(\frac{1}{\bar{F}(y)}\right), \text{ uniformly in } x \in \mathbb{R}^d, \text{ and} \\
\left|\frac{\bar{F}(zy)}{z^{-1/\gamma}\bar{F}(y)} - 1\right| &\leq MA\left(\frac{1}{\bar{F}(y)}\right), \text{ uniformly in } z \geq 1/2.
\end{aligned} \tag{8}$$

From the definition of U_x, we have

$$\begin{aligned}
U_x\big(\tfrac{Z}{\bar{F}_x(y)}\big) &= F_x^{\leftarrow}\Big(1 - \tfrac{\bar{F}_x(y)}{z}\Big) \\
&= \inf\Big\{w \in \mathbb{R} : F_x(w) \geq 1 - \tfrac{\bar{F}_x(y)}{z}\Big\} \\
&= \inf\Big\{w \in \mathbb{R} : z\tfrac{\bar{F}_x(w)}{\bar{F}_x(y)} \leq 1\Big\}.
\end{aligned}$$

Hence for any $w^- < w^+$, one has

$$z\frac{\bar{F}_x(w^+)}{\bar{F}_x(y)} < 1 < z\frac{\bar{F}_x(w^-)}{\bar{F}_x(y)} \Rightarrow U_x\left(\frac{z}{\bar{F}_x(y)}\right) \in \left[w^-, w^+\right]. \tag{9}$$

Now write $\epsilon(y) := MA(1/\bar{F}(y))$ and choose $w^{\pm} := z^{\gamma}y\,(1 \pm 4\gamma\epsilon(y))$, so one can write

$$
\begin{aligned}
z\frac{\bar{F}_x(w^-)}{\bar{F}_x(y)} =& z\frac{\sigma(x)\bar{F}(\omega^-)(1-\epsilon(y))}{\sigma(x)\bar{F}(y)(1+\epsilon(y))} \\
\geq& z\frac{1-\epsilon(y)}{1+\epsilon(y)}\frac{1}{\bar{F}(y)}\bar{F}\left(z^{\gamma}y(1-4\gamma\epsilon(y))\right) \\
\geq& z\frac{1-\epsilon(y)}{1+\epsilon(y)}\frac{1}{\bar{F}(y)}\bar{F}(y)(1-\epsilon(y))\left(z^{\gamma}(1-4\gamma\epsilon(y))\right)^{-1/\gamma},\ \text{by (8)} \\
\geq& \frac{(1-\epsilon(y))^2}{1+\epsilon(y)}\left(1-4\gamma\epsilon(y)\right)^{-1/\gamma}.
\end{aligned}
$$

A similar computation gives

$$
z\frac{\bar{F}_x(w^+)}{\bar{F}_x(y)} \leq \frac{(1+\epsilon(y))^2}{1-\epsilon(y)}\left(1+4\gamma\epsilon(y)\right)^{-1/\gamma}.
$$

As a consequence, the condition before "$\Rightarrow$" in (9) holds if

$$
4\gamma \geq \frac{1}{\epsilon(y)}\max\left\{1-\left(\frac{(1-\epsilon(y))^2}{1+\epsilon(y)}\right)^{\gamma};\left(\frac{(1+\epsilon(y))^2}{1-\epsilon(y)}\right)^{\gamma}-1\right\}.
$$

But a Taylor expansion of the right hand side shows that it is $3\gamma + o(1)$ as $y \to \infty$. This concludes the proof of Lemma 2.3. □

Applying Lemma 2.3 with $z := Z_i$ and $y := y_n$, we have

$$
\max_{i:E_{i,n}=1}\left|\frac{Y^*_{i,n}}{\tilde{Y}_{i,n}} - 1\right| = \mathrm{O}\left(A\left(1/p_n\right)\right).
$$

Now, by the construction of $(\tilde{X}_{1,n}, X^*_{1,n})$, when $E_{1,n} = 1$, we see that (7) is a consequence of the following

Lemma 2.4. *Under conditions* (3) *and* (4), *we have*

$$
||P_{X|Y>y} - \sigma(x)\mathbf{P}_X(dx)|| = O\left(A\left(\frac{1}{\bar{F}(y)}\right)\right),\ \textit{as } y \to \infty.
$$

Proof. For $B \in \mathbb{R}^d$, we have

$$
\begin{aligned}
&|P(X\in B|Y>y) - \int_B \sigma(x)\mathbf{P}_X(dx)| \\
=& \left|\frac{\int_B \bar{F}_x(y)P_X(dx)}{\bar{F}(y)} - \int_B \sigma(x)\mathbf{P}_X(dx)\right| \\
\leq& \int_B \left|\frac{\bar{F}_x(y)}{\bar{F}(y)} - \sigma(x)\right|\mathbf{P}_X(dx) \\
=& O\left(A\left(\frac{1}{\bar{F}(y)}\right)\right),\ \text{by (3)}.
\end{aligned}
$$

□

This proves (7) and hence concludes the proof of Proposition 2.2.

3. Proofs

3.1. **Proof of Theorem 1.1. Change of notation:** Since for each n, the law of $(\tilde{X}_{i,n}, \tilde{Y}_{i,n})_{i=1,\dots,n}$ is $\mathbf{P}^{\otimes n}_{X,Y}$, we shall confound them with $(X_i, Y_i)_{i=1,\dots,n}$ to unburden notations.

3.1.1. *Proof when $\mathbf{y}_n = y_n$ is deterministic.* Fix $0 < \varepsilon < \frac{1}{2}$ and $0 < \beta < \varepsilon/(2\gamma)$. We consider the empirical process defined for every $x \in \mathbb{R}^d$ and $y \geq 1/2$ as

$$\mathbb{G}_n(x, y) := \sqrt{np_n}(\mathbb{F}_n(x, y) - \mathbb{F}(x, y)),$$

with

$$\mathbb{F}_n(x, y) := \frac{1}{N_n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq x\}} \mathbb{1}_{\{Y_i/y_n > y\}} E_{i,n},$$

and

$$\mathbb{F}(x, y) := C(x)V_\gamma(y) = Q\left(]-\infty, x]\times]y, +\infty[\right),$$

where $V_\gamma(y) := y^{-1/\gamma}$ for $y \geq 1$ and $V_\gamma(y) := 1$, otherwise.
Note that neither $\mathbb{F}$, nor any realisation of $\mathbb{F}_n$ is a cumulative distribution function in the strict sense, since they are decreasing in y. Their roles should, however, be seen as the same as for c.d.f. Now denote by $L^{\infty,\beta}(\mathbb{R}^d \times [1/2, \infty[)$ the (closed) subspace of $L^\infty(\mathbb{R}^d \times [1/2, \infty[)$ of all f satisfying

$$\begin{aligned}
&\|f\|_{\infty,\beta} := \sup_{x\in\mathbb{R}^d, y\geq 1/2} |y^\beta f(x, y)| < \infty,\\
&f(\infty, y) := \lim_{\min\{x_1,\ldots,x_d\}\to\infty} f(x, y) \text{ exists for each } y \geq 1,\\
&\{y \mapsto f(\infty, y)\} \text{ is Càdlàg (see e.g., [4], p. 121).}
\end{aligned}$$

Simple arguments show that $\mathbb{G}_n$ takes values in $L^{\infty,\beta}(\mathbb{R}^d \times [1/2, \infty[)$.
First note that $\widehat{C}_n - C$ and $\hat{\gamma}_n - \gamma$ are images of $\mathbb{G}_n$ by the following map φ.

$$\begin{aligned}
\varphi : L^{\infty,\beta}(\mathbb{R}^d \times [1/2, \infty[) &\to L^\infty(\mathbb{R}^d) \times \mathbb{R}\\
f &\mapsto \left(\{x \mapsto f(x, 1)\}, \int_1^\infty y^{-1} f(\infty, y)\mathrm{d}y\right),
\end{aligned}$$

and remark that φ is continuous, since $\beta > 0$. By the continuous mapping theorem, we hence see that Theorem 1.1 will be a consequence of

$$\mathbb{G}_n \xrightarrow{\mathscr{L}} \mathbf{W} \text{ in } L^{\infty,\beta}(\mathbb{R}^d \times [1/2, \infty[), \tag{10}$$

where $\mathbf{W}$ is the centered Gaussian process with a covariance function

$$\operatorname{cov}\big(\mathbf{W}(x_1, y_1), \mathbf{W}(x_2, y_2)\big) = C(x_1 \wedge x_2)V_\gamma(y_1) \wedge V_\gamma(y_2) - C(x_1)C(x_2)V_\gamma(y_1)V_\gamma(y_2),$$

and where $x_1 \wedge x_2$ is understood componentwise.

The proof is divided into two steps. In *step 1*, we prove (10) for the counterpart of $\mathbb{G}_n$, that is, we build on the Q sample $(X^*_{i,n}, Y^*_{i,n})_{1\leq i\leq n}$. Our proof relies on a standard argument from empirical processes. In *step 2*, we use the coupling properties of Proposition (2.2) to deduce (10) for the original sample $(X_i, Y_i)_{1\leq i\leq n}$.
Step 1: Define

$$\mathbb{F}^*_n(x, y) := \frac{1}{N_n}\sum_{i=1}^{n} \mathbb{1}_{\{X^*_i \leq x\}} \mathbb{1}_{\{Y^*_{i,n}/y_n > y\}} E_{i,n} \;\; x \in \mathbb{R}^d,\; y \geq 1/2.$$

The following proposition is a Donsker theorem in weighted topology for $\mathbb{G}^*_n := \sqrt{np_n}(\mathbb{F}^*_n - \mathbb{F})$.

Proposition 3.1. *If* (3) *and* (4) *hold, then*

$$\mathbb{G}^*_n \xrightarrow{\mathscr{L}} \mathbf{W},\ \textit{in } L^{\infty,\beta}(\mathbb{R}^d \times [1/4, \infty[).$$

Proof. Write $\delta_x(A) = 1$ if $x \in A$ and 0, otherwise.
Since $(X^*_{i,n}, Y^*_{i,n})_{1\leq i\leq n}$ is independent of $(E_{i,n})_{1\leq i\leq n}$, Lemma 4.1 entails the following equality in laws

$$\sum_{i=1}^{n} \delta_{\left(X^*_{i,n}, \frac{Y^*_{i,n}}{y_n}\right)} E_{i,n} \overset{\mathscr{L}}{=} \sum_{i=1}^{\nu(n)} \delta_{\left(X^*_{i,n}, \frac{Y^*_{i,n}}{y_n}\right)},$$

where $\nu(n) \sim \mathcal{B}(n, p_n)$ is independent of $(X^*_{i,n}, Y^*_{i,n})_{1\le i\le n}$.
Since $(X^*_{i,n}, Y^*_{i,n}/y_n) \rightsquigarrow Q$ and since $\nu(n) \xrightarrow{\mathbb{P}} \infty$, $\nu(n)/np_n \xrightarrow{\mathbb{P}} 1$ and $\nu(n)$ independent of $(X^*_{i,n}, Y^*_{i,n})_{1\le i\le n}$, we see that $\mathbb{G}_n \xrightarrow{\mathscr{L}} \mathbf{W}$ will be a consequence of

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{k} \mathbb{1}_{\{U_i \le .,V_i > .\}} - \mathbb{F}(.,.)\right) \xrightarrow{\mathscr{L}}_{k\to\infty} \mathbf{W} \text{ in } L^{\infty,\beta}\big(\mathbb{R}^d \times [1/4, \infty[\big),$$

where the (U_i, V_i) are i.i.d. with distribution Q. Now consider the following class of functions on $\mathbb{R}^d \times [1/4, \infty[$:

$$\mathcal{F}_\beta := \big\{ f_{x,y} : (u, v) \mapsto y^\beta \mathbb{1}_{(-\infty,x]}(u)\mathbb{1}_{]y,\infty[}(v), x \in \mathbb{R}^d, y \ge 1/4\big\}.$$

Using the isometry

$$\begin{aligned} L^{\infty,\beta}(\mathbb{R}^d \times [1/4, \infty[\,) &\to L^\infty(\mathcal{F}_\beta) \\ g &\mapsto \{\Psi : f_{x,y} \mapsto g(x, y)\}, \end{aligned}$$

it is enough to prove that the abstract empirical process indexed by $\mathcal{F}_\beta$ converges weakly to the Q-Brownian bridge indexed by $\mathcal{F}_\beta$. In other words, we need to verify that $\mathcal{F}_\beta$ is Q-Donsker. This property can be deduced from two remarks:

(1) $\mathcal{F}_\beta$ is a VC-subgraph class of functions (see, e.g., Van der Vaart and Wellner [18], p.141). To see this, note that

$$\mathcal{F}_\beta \subset \big\{ f_{x,s,z} : (u, v) \mapsto z\mathbb{1}_{(-\infty,x]}(u)\mathbb{1}_{]y,\infty[}(v), x \in \mathbb{R}^d, s \in [1/4, \infty[, z \in \mathbb{R}\big\}$$

which is a VC-subgraph class: the subgraph of each of its members is a hypercube of $\mathbb{R}^{d+2}$.

(2) $\mathcal{F}_\beta$ has a square integrable envelope F. This is proved by noting that for fixed $(u, v) \in \mathbb{R}^d \times [1/4, \infty[$.

$$F^2(u, v) = \sup_{x\in\mathbb{R}^d,\ y\ge 1/4} y^{2\beta}\mathbb{1}_{[0,x]}(u)\mathbb{1}_{]y,\infty[}(v) = v^{2\beta}$$

as a consequence F^2 is Q-integrable, since $\beta < (2\gamma)^{-1}$.

This concludes the proof of Proposition 3.1. □

Step 2: We show here that the two empirical processes $\mathbb{G}_n$ and $\mathbb{G}^*_n$ must have the same weak limit by proving the next proposition.

Proposition 3.2. *Under Assumptions* (3) *and* (4)*, we have*

$$\sup_{x\in\mathbb{R}^d,\ y\ge 1/2} y^\beta \sqrt{np_n}|\mathbb{F}^*_n(x, y) - \mathbb{F}_n(x, y)| = o_{\mathbb{P}}(1).$$

Proof. Adding and subtracting

$$\mathbb{F}^\sharp_n(x, y) := \frac{1}{N_n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i\le x\}}\mathbb{1}_{\{Y^*_{i,n}/y_n>y\}}E_{i,n}$$

in $|\mathbb{F}_n(x, y) - \mathbb{F}^*_n(x, y)|$, the triangle inequality entails, almost surely,

$$\begin{aligned} &|\mathbb{F}_n(x, y) - \mathbb{F}^*_n(x, y)| \\ =&|\mathbb{F}_n(x, y) - \mathbb{F}^\sharp_n(x, y) + \mathbb{F}^\sharp_n(x, y) - \mathbb{F}^*_n(x, y)| \\ \le&\frac{1}{N_n}\sum_{i=1}^{n}|\mathbb{1}_{\{X_i\le x\}} - \mathbb{1}_{\{X^*_{i,n}\le x\}}|\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}E_{i,n} \\ &+ \frac{1}{N_n}\sum_{i=1}^{n}|\mathbb{1}_{\left\{\frac{Y_i}{y_n}>y\right\}} - \mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}|\mathbb{1}_{\{X_i\le x\}}E_{i,n} \\ \le&\frac{1}{N_n}\sum_{i=1}^{n}\mathbb{1}_{\{X_i\ne X^*_{i,n}\}}\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}E_{i,n} + \frac{1}{N_n}\sum_{i=1}^{n}|\mathbb{1}_{\left\{\frac{Y_i}{y_n}>y\right\}} - \mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}|E_{i,n}. \end{aligned}$$

Let us first focus on the first term. Notice that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d,\ y\geq 1/2} \frac{y^\beta\sqrt{np_n}}{N_n}\sum_{i=1}^n \mathbb{1}_{\{X_i\neq X^*_{i,n}\}}\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}E_{i,n} \\
= &\sup_{y\geq 1/2} \frac{y^\beta\sqrt{np_n}}{N_n}\sum_{i=1}^n \mathbb{1}_{\{X_i\neq X^*_{i,n}\}}\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}E_{i,n} \\
\leq &\sup_{y\geq 1/2} \frac{y^\beta\sqrt{np_n}}{N_n}\left(\max_{i=1,\dots,n}\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}E_{i,n}\right)\sum_{i=1}^n \mathbb{1}_{\{X_i\neq X^*_{i,n}\}}E_{i,n}.
\end{aligned}$$

Now notice that

$$\begin{aligned}
\sup_{y\geq 1/2}\max_{i=1,\dots,n} y^\beta\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}E_{i,n} &= \max_{i=1,\dots,n}\sup_{y\geq 1/2} y^\beta \mathbb{1}_{[1,Y^*_{i,n}/y_n]}(y)E_{i,n} \\
&= \max_{i=1,\dots,n}\left(\frac{Y^*_{i,n}}{y_n}\right)^\beta E_{i,n}.
\end{aligned}$$

By the independence between $E_{i,n}$ and $Y^*_{i,n}/y_n$, Lemma 4.1 in the Appendix gives

$$\max_{i=1,\dots,n}\left(\frac{Y^*_{i,n}}{y_n}\right)^\beta E_{i,n} \overset{\mathscr{L}}{=} \max_{i=1,\dots,\nu(n)}\left(\frac{Y^*_{i,n}}{y_n}\right)^\beta$$

where $Y^*_{i,n}/y_n$ in the right-hand side have a Pareto$(1/\gamma)$ distribution, whence

$$\max_{i=1,\dots,n}\left(\frac{Y^*_{i,n}}{y_n}\right)^\beta E_{i,n} = O_{\mathbb{P}}(\nu(n)^{\beta\gamma}) = O_{\mathbb{P}}((np_n)^{\beta\gamma}). \tag{11}$$

Moreover, writing $A_n := A(1/p_n)$, one has

$$\mathbb{E}\left(\sum_{i=1}^n \mathbb{1}_{\{X_i\neq X^*_{i,n}\}}E_{i,n}\right) = np_nA_n,$$

which entails

$$\frac{1}{np_nA_n}\sum_{i=1}^n \mathbb{1}_{\{X_i\neq X^*_{i,n}\}}E_{i,n} = O_{\mathbb{P}}(1). \tag{12}$$

As a consequence,

$$\begin{aligned}
&\frac{\sqrt{np_n}}{N_n}\max_{i=1,\dots,n}\left(\frac{Y^*_{i,n}}{y_n}\right)^\beta E_{i,n}\left(\sum_{i=1}^n \mathbb{1}_{\{X_i\neq X^*_{i,n}\}}E_{i,n}\right) \\
&= \frac{np_n}{N_n}\max_{i=1,\dots,n}\left(\frac{Y^*_{i,n}}{y_n}\right)^\beta E_{i,n}\left(\frac{1}{np_nA_n}\sum_{i=1}^n \mathbb{1}_{\{X_i\neq X^*_{i,n}\}}E_{i,n}\right)\sqrt{np_n}A_n \\
&= O_{\mathbb{P}}(1)O_{\mathbb{P}}((np_n)^{\beta\gamma})O_{\mathbb{P}}(1)\sqrt{np_n}A_n, \text{ by (11) and (12)} \\
&= o_{\mathbb{P}}(1), \text{ by the assumption of Theorem 1.1, and since } \beta\gamma < \frac{\varepsilon}{2}.
\end{aligned}$$

Let us now focus on the convergence

$$\sup_{x\in\mathbb{R}^d,\ y\geq 1/2} y^\beta\sqrt{np_n}\frac{1}{N_n}\sum_{i=1}^n\left|\mathbb{1}_{\left\{\frac{Y_i}{y_n}>y\right\}} - \mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}\right|E_{i,n} \xrightarrow{\mathbb{P}} 0.$$

We deduce from Proposition 2.2 that, almost surely, writing $\epsilon_n := MA_n$:

$$(1-\epsilon_n)\frac{Y_i}{y_n}E_{i,n} \leq \frac{Y^*_{i,n}}{y_n}E_{i,n} \leq (1+\epsilon_n)\frac{Y_i}{y_n}E_{i,n},$$

which entails, almost surely, for all $y \geq 1$:

$$E_{i,n}\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}\geq(1+\epsilon_n)y\right\}} \leq E_{i,n}\mathbb{1}_{\left\{\frac{Y_i}{y_n}\geq y\right\}} \leq E_{i,n}\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}\geq(1-\epsilon_n)y\right\}},$$

implying

$$\left|\mathbb{1}_{\left\{\frac{Y_i}{y_n}>y\right\}}-\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}\right|E_{i,n}\leq\left|\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>(1-\epsilon_n)y\right\}}-\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>(1+\epsilon_n)y\right\}}\right|E_{i,n}.$$

This entails

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d,\ y\geq 1/2} y^\beta\sqrt{np_n}\frac{1}{N_n}\sum_{i=1}^n\left|\mathbb{1}_{\left\{\frac{Y_i}{y_n}>y\right\}}-\mathbb{1}_{\left\{\frac{Y^*_{i,n}}{y_n}>y\right\}}\right|E_{i,n}\\ \leq&\sup_{x\in\mathbb{R}^d,\ y\geq 1/2} y^\beta\sqrt{np_n}\left|\mathbb{F}^*_n(\infty,(1-\epsilon_n)y)-\mathbb{F}^*_n(\infty,(1+\epsilon_n)y)\right|.\end{aligned}$$

Consequently, we have, adding and subtracting expectations:

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d,\ y\geq 1/2} y^\beta\sqrt{np_n}\frac{1}{N_n}\sum_{i=1}^n\left|\mathbb{1}_{\{\frac{Y_i}{y_n}>y\}}-\mathbb{1}_{\{\frac{Y^*_{i,n}}{y_n}>y\}}\right|E_{i,n}\\ \leq&\sup_{x\in\mathbb{R}^d,\ y\geq 1/2} y^\beta\left|\tilde{\mathbb{G}}^*_n((1-\epsilon_n)y)-\tilde{\mathbb{G}}^*_n((1+\epsilon_n)y)\right| \quad (13)\\ &+\sqrt{np_n}\sup_{y\geq 1/2} y^\beta(V_\gamma((1-\epsilon_n)y)-V_\gamma((1+\epsilon_n)y)), \quad (14)\end{aligned}$$

where we write $\tilde{\mathbb{G}}^*_n(y):=\mathbb{G}^*_n(\infty,y)$.
We first prove that (14) converges to 0. For $y\geq 1$, we can bound

$$\begin{aligned}&y^\beta(V_\gamma((1-\epsilon_n)y)-V_\gamma((1+\epsilon_n)y))\\ \leq&y^\beta|1-((1+\epsilon_n)y)^{-1/\gamma}|\mathbb{1}_{\{(1-\epsilon_n)y<1\}}\\ &+y^\beta|((1-\epsilon_n)y)^{-1/\gamma}-((1+\epsilon_n)y)^{-1/\gamma}|\mathbb{1}_{\{(1-\epsilon_n)y\geq 1\}}. \quad (15)\end{aligned}$$

In the first term of the right-hand side, since $(1-\epsilon_n)y<1$, we can write

$$\begin{aligned}&y^\beta|1-((1+A_n)y)^{-1/\gamma}|\mathbb{1}_{\{(1-\epsilon_n)y<1\}}\\ &\leq y^{\beta-1/\gamma}|y^{1/\gamma}-(1+A_n)^{-1/\gamma}|\mathbb{1}_{\{(1-\epsilon_n)y<1\}}\\ &\leq y^{\beta-1/\gamma}|(1-\epsilon_n)^{-1/\gamma}-(1+A_n)^{-1/\gamma}|\mathbb{1}_{\{(1-\epsilon_n)y<1\}}\\ &\leq 4\gamma^{-1}\epsilon_n,\ \text{since } \beta-1/\gamma<0.\end{aligned}$$

The second term of (15) is bounded by similar arguments, from where we have

$$\begin{aligned}&\sqrt{np_n}\sup_{x\in\mathbb{R}^d,\ y\geq 1/2} y^\beta|V_\gamma((1-\epsilon_n)y)-V_\gamma((1+\epsilon_n)y)|\\ &\leq 8\gamma^{-1}M\sqrt{np_n}A_n,\end{aligned}$$

which converges in probability to 0 by assumptions of Theorem 1.1.
We now prove that (13) converges to zero in probability. By Proposition 3.1, the continuous mapping theorem together with the Portmanteau theorem entail

$$\begin{aligned}\forall\varepsilon>0,\forall\rho>0,\ \overline{\lim}\ \mathbb{P}&\left(\sup_{y\geq 1/2,\delta<\rho} y^\beta|\tilde{\mathbb{G}}^*_n((1-\delta)y)-\tilde{\mathbb{G}}^*_n((1+\delta)y)|\geq\varepsilon\right)\\ \leq&\mathbb{P}\left(\sup_{y\geq 1/2,\delta<\rho} y^\beta|\tilde{\mathbf{W}}((1-\delta)y)-\tilde{\mathbf{W}}((1+\delta)y)|\geq\varepsilon\right),\end{aligned}$$

where $\tilde{\mathbf{W}}(y):=\mathbf{W}(\infty,y)$ is the centered Gaussian process with the covariance function

$$\operatorname{cov}(\tilde{\mathbf{W}}(y_1),\tilde{\mathbf{W}}(y_2)):=V_\gamma(y_1)\wedge V_\gamma(y_2)-V_\gamma(y_1)V_\gamma(y_2),\ (y_1,y_2)\in[1/4,\infty[^2.$$

With Proposition 3.1 together with the continuous mapping theorem, we see that the proof of Proposition 3.2 will be concluded if we establish the following lemma.

Lemma 3.3. *We have*

$$\sup_{y\geq 1/2,\delta<\rho} y^{\beta}|\tilde{\mathbf{W}}((1-\delta)y)-\tilde{\mathbf{W}}((1+\delta)y)| \xrightarrow[\rho\to 0]{\mathbb{P}} 0.$$

Proof. Let $\mathbb{B}_0$ be the standard Brownian bridge with $\mathbb{B}_0$ identically zero on $[1,\infty[$. $\tilde{\mathbf{W}}$ has the same law as $\{y\mapsto \mathbb{B}_0(y^{-1/\gamma})\}$ (see [17], p. 99), from where

$$\begin{aligned}\sup_{y\geq 1/2,\delta<\rho} & y^{\beta}|\tilde{\mathbf{W}}((1-\delta)y)-\tilde{\mathbf{W}}((1+\delta)y)| \\ \overset{\mathscr{L}}{=} & \sup_{y\geq 1/2,\delta<\rho} y^{\beta}\left|\mathbb{B}_0(((1-\delta)y)^{-1/\gamma})-\mathbb{B}_0(((1+\delta)y)^{-1/\gamma})\right| \\ \leq & \sup_{0\leq y\leq 2,\delta<\rho} y^{-\beta\gamma}\left|\mathbb{B}_0((1-\delta)^{-1/\gamma}y)-\mathbb{B}_0((1+\delta)^{-1/\gamma}y)\right|, \text{ almost surely.}\end{aligned}$$

Since $\beta\gamma<1/2$, the process $\mathbb{B}_0$ is a.s-$\beta\gamma$-Hölder continuous on $[0,+\infty[$. Consequently, for an a.s finite random variable H one has with probability one:

$$\begin{aligned}\sup_{0\leq y\leq 2,\delta<\rho} & y^{-\beta\gamma}|\mathbb{B}_0((1-\delta)^{-1/\gamma}y)-\mathbb{B}_0((1+\delta)^{-1/\gamma}y)| \\ \leq & \sup_{0\leq y\leq 2} y^{-\beta\gamma}|(1-\rho)^{-1/\gamma}-(1+\rho)^{-1/\gamma}|^{\beta\gamma}y^{\beta\gamma}H \\ = & |2(1-\rho)^{-1/\gamma}-2(1+\rho)^{-1/\gamma}|^{\beta\gamma}H \\ = & (4\tfrac{\rho}{\gamma})^{\beta\gamma}H. \end{aligned}$$

□

The preceding lemma concludes the proof of Proposition 3.2, which, combined with Proposition (3.1), proves (10). This concludes the proof of Theorem 1.1 when $\mathbf{y}_n\equiv y_n$. □

3.1.2. *Proof of Theorem* 1.1 *in the general case.* We now drop the assumption $\mathbf{y}_n\equiv y_n$ and relax it to $\frac{\mathbf{y}_n}{y_n}\xrightarrow{\mathbb{P}}1$ to achieve the proof of Theorem 1.1 in its full generality. We use the results of §3.1.1. Define

$$\overset{\vee}{\mathbb{F}}_n(x,y):=\frac{1}{\sum_{i=1}^{n}\mathbb{1}_{\{Y_i>y_n\}}}\sum_{i=1}^{n}\mathbb{1}_{\{X_i\leq x\}}\mathbb{1}_{\{Y_i/y_n>y\}}$$

and

$$\overset{\vee}{\mathbb{G}}_n(x,y):=\sqrt{np_n}\left(\overset{\vee}{\mathbb{F}}_n(x,y)-\mathbb{F}(x,y)\right).$$

Now write $u_n:=\frac{\mathbf{y}_n}{y_n}$. From §3.1.1, we know that

$$\left(\overset{\vee}{\mathbb{G}}_n,u_n\right)\xrightarrow{\mathscr{L}}(\mathbf{W},1) \text{ in } \mathbf{D}\times]0,+\infty[, \text{ where } \mathbf{D}:=L^{\infty,\beta}(\mathbb{R}^d\times[1/2,\infty[).$$

Moreover, as pointed out in Lemma 3.3, $\mathbf{W}$ almost surely belongs to

$$\mathbf{D}_0=\left\{\varphi\in L^{\infty,\beta}(\mathbb{R}^d\times[1/2,\infty[), \sup_{x\in\mathbb{R}^d,y,y'>1/2}\frac{|\varphi(x,y)-\varphi(x,y')|}{|y-y'|^{\beta\gamma}}<\infty\right\}.$$

Consider the followings maps $(g_n)_{n\in\mathbb{N}}$ and g from $\mathbf{D}$ to $L^{\infty,\beta}(\mathbb{R}^d\times[1,\infty[)$

$$g_n:(\varphi,u)\mapsto\sqrt{np_n}\left(\frac{\mathbb{F}(.,u.)+\frac{1}{\sqrt{np_n}}\varphi(.,u.)}{\mathbb{F}(\infty,u)+\frac{1}{\sqrt{np_n}}\varphi(\infty,u)}-\mathbb{F}(.,.)\right),$$

and

$$g:(\varphi,u)\mapsto u^{1/\gamma}\Big(\varphi(.,u.)-\varphi(\infty,u)\mathbb{F}(.,.)\Big).$$

Notice that $\mathbb{G}_n=g_n(\overset{\vee}{\mathbb{G}}_n,u_n)$ and $g(\mathbf{W},1)=\mathbf{W}$. The achievement of the proof of Theorem 1.1 hence boils down to making use of the extended continuous mapping theorem (see, e.g., Theorem 1.11.1, p. 67 in [18]) which is applicable to the sequence $(g_n,\mathbb{G}_n)$ provided that we establish the following

Lemma 3.4. *For any sequence φ_n of elements of* $\mathbf{D}$ *that converges to some* $\varphi \in \mathbf{D}_0$, *and for any sequence* $u_n \to 1$ *one has* $g_n(\varphi_n, u_n) \to g(\varphi, 1)$ *in* $L^{\infty,\beta}(\mathbb{R}^d \times [1, \infty[)$. *Here, the convergence in* $L^{\infty,\beta}(\mathbb{R}^d \times [1, \infty[)$ *is understood as with respect to* $\|\cdot\|$, *the restriction of* $\|\cdot\|_{\infty,\beta}$ *to* $\mathbb{R}^d \times [1, \infty[$.

Proof. For fixed $(x, y) \in \mathbb{R}^d \times [1/2, \infty[$ and $n \geq 1$, with the writing $t_n := (np_n)^{-1/2}$, we have

$$\begin{aligned}
&|g_n(\varphi_n, u_n)(x, y) - g(\varphi, 1)(x, y)| \\
&= \left| \frac{1}{t_n} \left(\frac{\mathbb{F}(x, u_n y) + t_n \varphi_n(x, u_n y)}{\mathbb{F}(\infty, u_n) + t_n \varphi_n(\infty, u_n)} - \mathbb{F}(x, y) \right) - \Big(\varphi(x, y) - \varphi(\infty, 1)\mathbb{F}(x, y) \Big) \right|.
\end{aligned}$$

Now, elementary algebra using $\mathbb{F}(x, yu_n)/\mathbb{F}(\infty, u_n) = \mathbb{F}(x, y)$ shows that

$$\begin{aligned}
&\frac{\mathbb{F}(x, u_n y) + t_n \varphi_n(x, u_n y)}{\mathbb{F}(\infty, u_n) + t_n \varphi_n(\infty, u_n)} - \mathbb{F}(x, y) \\
&= \mathbb{F}(x, y) \left(\frac{1 + t_n \frac{\varphi_n(x, u_n y)}{\mathbb{F}(x, u_n y)}}{1 + t_n \frac{\varphi_n(\infty, u_n)}{\mathbb{F}(\infty, u_n)}} - 1 \right) \\
&= \mathbb{F}(x, y) \left(\left(1 + t_n \frac{\varphi_n(x, u_n y)}{\mathbb{F}(x, u_n y)} \right) \Big(1 - t_n u_n^{1/\gamma} \varphi_n(\infty, u_n)(1 + \epsilon_n) \Big) (1 + \theta_n(x, y)) - 1 \right) \\
&= \mathbb{F}(x, y) \left(t_n \left(\frac{\varphi_n(x, u_n y)}{\mathbb{F}(x, u_n y)} - u_n^{1/\gamma} \varphi_n(\infty, u_n) \right) + R_n(x, y) \right),
\end{aligned}$$

with $\epsilon_n \to 0$ a sequence of real numbers, not depending on x and y, and with

$$R_n(x, y) := t_n u_n^{1/\gamma} \varphi_n(\infty, u_n)\epsilon_n + (t_n u_n^{1/\gamma})^2 \varphi_n(\infty, u_n) \frac{\varphi_n(x, u_n y)}{\mathbb{F}(x, y)} (1 + \epsilon_n).$$

This implies that

$$\|g_n(\varphi_n, u_n) - g(\varphi, 1)\| \leq B_{1,n} + B_{2,n} + B_{3,n} + B_{4,n},$$

where the four terms $B_{1,n}, \ldots, B_{4,n}$ are detailed below and will be proved to converge to zero as $n \to \infty$.

First term

$$\begin{aligned}
B_{1,n} :=& \|u_n^{1/\gamma} \varphi_n(., u_n.) - \varphi(., .)\| \\
\leq& \|u_n^{1/\gamma} \varphi_n(., u_n.) - \varphi_n(., u_n.) + \|\varphi_n(., u_n.) - \varphi(., .)\| \\
=& |u_n^{1/\gamma} - 1| \|\varphi_n(., u_n.)\| + \|\varphi_n(., u_n.) - \varphi(., .)\| \\
\leq& |u_n^{1/\gamma} - 1| \|\varphi_n(., u_n.)\| + \|\varphi_n(., u_n.) - \varphi(., u_n.)\| \\
&+ \|\varphi(., u_n.) - \varphi(., .)\| \\
\leq& |u_n^{1/\gamma} - 1| \|\varphi_n(., u_n.)\| + u_n^{-\beta} ||\varphi_n(x, y) - \varphi(x, y)||_{\infty,\beta} \\
&+ H_\varphi |u_n - u|^{\beta\gamma},
\end{aligned}$$

where $H_\varphi := \sup\{|y - y'|^{-\beta\gamma} |\varphi(x, y) - \varphi(x, y')|, x \in \mathbb{R}^d, y, y' \geq 1/2\}$ is finite since $\varphi \in \mathbf{D}_0$. The first two terms converge to 0, since $u_n \to 1$ and $\varphi_n \to \varphi$ in $\mathbf{D}$. The third term converges to zero, since H_φ is finite.

Second term

$$\begin{aligned}
B_{2,n} :=& \|(u_n^{1/\gamma} \varphi_n(\infty, u_n) - \varphi(\infty, 1))\mathbb{F}\| \\
&\leq \big(|u_n^{1/\gamma} \varphi_n(\infty, u_n) - \varphi_n(\infty, u_n)| + |\varphi_n(\infty, u_n) - \varphi(\infty, 1)| \big) \|\mathbb{F}\|.
\end{aligned}$$

But $\|\mathbb{F}\|$ is finite since $\beta\gamma < \varepsilon < 1/2$, from where $B_{2,n} \to 0$ by similar arguments as those used for $B_{1,n}$.

Third term

$$B_{3,n} := \|u_n^{1/\gamma}\varphi_n(\infty, u_n)\epsilon_n \mathbb{F}\| \leq |u_n^{1/\gamma}\varphi_n(\infty, u_n)| \times |\epsilon_n| \times \|\mathbb{F}\|.$$

Since $\|\mathbb{F}\|$ is finite, since $|u_n^{1/\gamma}\varphi_n(\infty, u_n)|$ is a converging sequence, and since $|\epsilon_n| \to 0$, we deduce that $B_{3,n} \to 0$.

Fourth term

$$\begin{aligned} B_{4,n} :=& \big(1+|\epsilon_n|\big)\Big\|(t_n u_n^{1/\gamma})^2 \varphi_n(\infty, u_n)\varphi_n(., u_n.)\Big\| \\ \leq& \big(1+|\epsilon_n|\big)\Big|(t_n u_n^{1/\gamma})^2 \varphi_n(\infty, u_n)\Big| \times \|\varphi_n(., u_n.)\|. \end{aligned}$$

Since $\varphi_n \to \varphi$ in $L^{\infty,\beta}(\mathbb{R}^d \times [1/2, \infty[)$, the same arguments as for $B_{3,n}$ entail the convergence to zero of $B_{4,n}$. □

3.2. **Proof of Theorem 1.2.** Let $x \in \mathbb{R}^d$, which will be kept fixed in all this section. To prove the asymptotic normality of $\hat{\sigma}_n(x)$, we first establish the asymptotic normality of the numerator and the denominator separately. Note that we don't need to study their joint asymptotic normality, because only the numerator will rule the asymptotic normality of $\hat{\sigma}_n(x)$, as its rate of convergence is the slowest.

Proposition 3.5. *Assume that $(p_n)_{n\geq 1}$ and $(h_n)_{n\geq 1}$ both converge to 0 and satisfy $np_nh_n^d \to 0$. We have*

$$\frac{1}{\sqrt{np_nh_n^d}} \sum_{i=1}^{n} \frac{\mathbb{1}_{\{|X_i - x| \leq h_n, Y_i > \mathbf{y}_n\}} - \mathbb{P}(|X_i - x| \leq h_n, Y_i > \mathbf{y}_n)}{\sqrt{\sigma(x)f(x)}} \xrightarrow{\mathscr{L}} \mathcal{N}(0,1), \tag{16}$$

$$\frac{1}{\sqrt{nh_n^d}} \sum_{i=1}^{n} \frac{\mathbb{1}_{\{|X_i - x| \leq h_n\}} - \mathbb{P}(|X_i - x| \leq h_n)}{\sqrt{f(x)}} \xrightarrow{\mathscr{L}} \mathcal{N}(0,1), \tag{17}$$

and

$$\frac{1}{\sqrt{np_n}} \sum_{i=1}^{n} (\mathbb{1}_{\{Y_i > \mathbf{y}_n\}} - p_n) \xrightarrow{\mathscr{L}} \mathcal{N}(0,1). \tag{18}$$

Proof. Note that (18) is the central limit theorem for binomial(n, p_n) sequences with $p_n \to 0$ and $np_n \to \infty$, while (17) is the well known pointwise asymptotic normality of the Parzen-Rosenblatt density estimator. The proof of (16) is a straghtforward use of the Lindeberg-Levy Theorem (see, e.g [3], Theorem 27.2 p. 359). First, we define

$$Z_{i,n} := \frac{\mathbb{1}_{\{|X_i - x| \leq h_n, Y_i > \mathbf{y}_n\}} - \mathbb{P}(|X_i - x| \leq h_n, Y_i > \mathbf{y}_n)}{\sqrt{np_nh_n^d}\sqrt{\sigma(x)f(x)}}$$

and remark that $\mathbb{E}(Z_{i,n}) = 0$. Moreover, we can write

$$\begin{aligned} \mathbb{E}\left(\mathbb{1}_{\{|X_i - x| \leq h_n, Y_i > \mathbf{y}_n\}}\right) =& \int_{B(x,h)} \mathbb{P}(Y_i > \mathbf{y}_n | X_i = z) P_X(dz) \\ \approx& \int_{B(x,h)} \sigma(z) p_n P_X(dz) && \text{(a)} \\ \approx& \sigma(x) f(x) p_n h_n^d, && \text{(b)} \end{aligned}$$

where (a) is a consequence of the uniformity in assumption (3), while equivalence (b) holds by our assumptions upon the regularity of both f and σ in Theorem 1.2. We conclude that $\sup\{|n\mathrm{Var}(Z_{i,n}) - 1|,\ i = 1, \ldots, n\} \to 0$. Note that we can invoke the Lindeberg-Levy Theorem if for all $\varepsilon > 0$, we have

$$\sum_{i=1}^{n} \int_{\{Z_{i,n} > \varepsilon\}} Z_{i,n}^2 P_X(dx) \to 0.$$

This convergence holds since the set $\{Z_{i,n} > \varepsilon\}$ can be rewritten as

$$\Big\{|\mathbb{1}_{\{|X_i-x|\le h_n, Y_i>\mathbf{y}_n\}} - \mathbb{P}(|X_i - x| \le h_n, Y_i > \mathbf{y}_n)| \ge \varepsilon\sqrt{\sigma(x)f(x)}\sqrt{np_nh_n^d}\Big\},$$

which is empty when n is large enough, since $np_nh_n^d \to \infty$. This proves (16). □

Now, writing

$$\hat{\sigma}_n(x) = \frac{n}{\sum_{i=1}^n \mathbb{1}_{\{Y_i>\mathbf{y}_n\}}} \times \frac{\sum_{i=1}^n \mathbb{1}_{\{|x-X_i|<h_n\}}\mathbb{1}_{\{Y_i>\mathbf{y}_n\}}}{\sum_{i=1}^n \mathbb{1}_{\{|x-X_i|<h_n\}}},$$

we have

$$\hat{\sigma}_n(x) = \frac{1}{1+\frac{1}{\sqrt{np_n}}\sum\limits_{i=1}^n Z^\sharp_{i,n}} \times \frac{\frac{\mathbb{P}(|X-x|\le h_n, Y>\mathbf{y}_n)}{p_nh_n^d} + \sqrt{\frac{f(x)\sigma(x)}{np_nh_n^d}}\sum\limits_{i=1}^n Z_{i,n}}{\frac{\mathbb{P}(|X-x|\le h_n)}{h_n^d} + \sqrt{\frac{f(x)}{nh_n^d}}\sum\limits_{i=1}^n \tilde{Z}_{i,n}},$$

where

$$\tilde{Z}_{i,n} := \frac{\mathbb{1}_{\{|X_i-x|\le h_n\}} - \mathbb{P}(|X_i - x| \le h_n)}{\sqrt{f(x)}\sqrt{nh_n^d}},$$

and

$$Z^\sharp_{i,n} := \frac{\mathbb{1}_{\{Y_i>\mathbf{y}_n\}} - p_n}{\sqrt{np_n}}.$$

Now, we write

$$\sigma_{h_n}(x) := \frac{\mathbb{P}(|X-x| \le h_n, Y > \mathbf{y}_n)}{p_nh_n^d f(x)}.$$

Since f is continuous and bounded away from zero on a neighbourhood of x, we have

$$\hat{\sigma}_n(x) = \frac{1}{1+\frac{1}{\sqrt{np_n}}\sum\limits_{i=1}^n Z^\sharp_{i,n}} \frac{\sigma_{h_n}(x)f(x)(1+\varepsilon_{n,1}) + \sqrt{\frac{f(x)\sigma(x)}{np_nh_n^d}}\sum\limits_{i=1}^n Z_{i,n}}{f(x)(1+\varepsilon_{n,2}) + \sqrt{\frac{f(x)}{nh_n^d}}\sum\limits_{i=1}^n \tilde{Z}_{i,n}},$$

with $|\varepsilon_{n,1}| \vee |\varepsilon_{n,2}| \to 0$. Now a Taylor expansion of the denominator gives

$$\begin{aligned}\hat{\sigma}_n(x) = &\frac{1}{1+\frac{1}{\sqrt{np_n}}\sum\limits_{i=1}^n Z^\sharp_{i,n}} \left(\sigma_{h_n}(x) + \sqrt{\frac{\sigma(x)}{np_nh_n^d f(x)}}\sum_{i=1}^n Z_{i,n}\right) \\ &\times \left(1 - \sqrt{\frac{1}{nh_n^d f(x)}}\sum_{i=1}^n \tilde{Z}_{i,n} + o_{\mathbb{P}}\left(\sqrt{\frac{1}{nh_n^d f(x)}}\right)\right).\end{aligned}$$

By similar arguments, remarking that $(nh_n^d)^{-1} = o\Big((np_nh_n^d)^{-1}\Big)$, by (16) and (17), we have

$$\hat{\sigma}_n(x) = \frac{1}{1+\frac{1}{\sqrt{np_n}}\sum\limits_{i=1}^n Z^\sharp_{i,n}} \left(\sigma_{h_n}(x) + \sqrt{\frac{\sigma(x)}{np_nh_n^d f(x)}}\sum_{i=1}^n Z_{i,n} + o_{\mathbb{P}}\left(\frac{1}{\sqrt{np_nh_n^d}}\right)\right).$$

Moreover, with one more Taylor expansion of the denominator, by (18), we have

$$\hat{\sigma}_n(x) = \sigma_{h_n}(x) + \sqrt{\frac{\sigma(x)}{np_nh_n^d f(x)}}\sum_{i=1}^n Z_{i,n} + o_{\mathbb{P}}\left(\frac{1}{\sqrt{np_nh_n^d}}\right),$$

which entails

$$\sqrt{np_nh_n^d}(\hat{\sigma}_n(x) - \sigma_{h_n}(x)) = \sqrt{\frac{\sigma(x)}{f(x)}}\sum_{i=1}^n Z_{i,n} + o_{\mathbb{P}}(1).$$

The asymptotic normality of $\sum_{i=1}^{n} Z_{i,n}$ gives

$$\sqrt{np_n h_n^d}(\hat{\sigma}_n(x) - \sigma_{h_n}(x)) \xrightarrow{\mathscr{L}} \mathcal{N}\left(0, \frac{\sigma(x)}{f(x)}\right).$$

The proof is achieved by noticing that assumption (3) entails

$$\begin{aligned}\sqrt{np_n h_n^d}|\sigma_{h_n}(x) - \sigma(x)| &= \sqrt{np_n h_n^d}\left|\frac{\mathbb{P}(|X-x| \le h_n, Y > \mathbf{y}_n)}{f(x)h_n^d \mathbb{P}(Y > \mathbf{y}_n)} - \sigma(x)\right| \\ &= \sqrt{np_n h_n^d}\left|\frac{\mathbb{P}(Y > \mathbf{y}_n | X \in B(x, h_n))}{\mathbb{P}(Y > \mathbf{y}_n)} - \sigma(x)\right| \\ &= O\left(\sqrt{np_n h_n^d} A\left(1/p_n\right)\right) \to 0.\end{aligned}$$

3.3. **Proof of Theorem 1.3.** For the sake of clarity, we first express conditions (3) and (4) in terms of the tail quantile function U: we have, uniformly in x,

$$\left|\frac{U_x(1/\alpha_n)}{U(\sigma(x)/\alpha_n)} - 1\right| = O(A_n) \text{ and } \left|\frac{U(1/\alpha_n)}{xU(x^{-1/\gamma}/\alpha_n)} - 1\right| = O(A_n),$$

where $A_n := A(1/p_n)$. Start the proof by splitting the quantity of interest into four parts,

$$\begin{aligned}\log\left(\frac{\hat{q}(\alpha_n|x)}{q(\alpha_n|x)}\right) &= \log\left(\frac{\mathbf{y}_n}{q(\alpha_n|x)}\left(\frac{\hat{p}_n\hat{\sigma}_n(x)}{\alpha_n}\right)^{\hat{\gamma}_n}\right) \\ &= \log\left(\frac{\mathbf{y}_n}{q(\alpha_n|x)}\left(\frac{p_n\hat{\sigma}_n(x)}{\alpha_n}\right)^{\hat{\gamma}_n}\left(\frac{\hat{p}_n}{p_n}\right)^{\hat{\gamma}_n}\right) \\ &= \log\left(\frac{\mathbf{y}_n}{q(\alpha_n|x)}\right) + \hat{\gamma}_n \log\left(\frac{p_n}{\alpha_n}\right) + \hat{\gamma}_n \log(\hat{\sigma}_n(x)) + \hat{\gamma}_n \log\left(\frac{\hat{p}_n}{p_n}\right) \\ &= \log\left(\frac{\mathbf{y}_n}{q(\alpha_n|x)}\left(\frac{p_n}{\alpha_n}\right)^{\gamma}\right) + (\hat{\gamma}_n - \gamma)\log\left(\frac{p_n}{\alpha_n}\right) \\ &\quad + \hat{\gamma}_n \log(\hat{\sigma}_n(x)) + \hat{\gamma}_n \log\left(\frac{\hat{p}_n}{p_n}\right).\end{aligned}$$

Moreover, we can see that

$$\begin{aligned}\log\left(\frac{\mathbf{y}_n}{q(\alpha_n|x)}\left(\frac{p_n}{\alpha_n}\right)^{\gamma}\right) &= \log\left(\frac{U(1/p_n)}{U_x(1/\alpha_n)}\left(\frac{p_n}{\alpha_n}\right)^{\gamma}\right) \\ &= \log\left(\frac{U(1/p_n)}{U(1/\alpha_n)}\left(\frac{p_n}{\alpha_n}\right)^{\gamma}\right) + \log\left(\frac{U(1/\alpha_n)}{U_x(1/\alpha_n)}\right)\end{aligned}$$

Further, we write

$$\frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\log\left(\frac{\hat{q}(\alpha_n|x)}{q(\alpha_n|x)}\right) = Q_{1,n} + Q_{2,n} + Q_{3,n} + Q_{4,n}, \text{ with}$$

$$\begin{aligned}Q_{1,n} &:= \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\log\left(\frac{U(1/p_n)}{U(1/\alpha_n)}\left(\frac{p_n}{\alpha_n}\right)^{\gamma}\right), \\ Q_{2,n} &:= \sqrt{np_n}(\hat{\gamma}_n - \gamma), \\ Q_{3,n} &:= \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\left(\hat{\gamma}_n \log(\hat{\sigma}_n(x)) + \log\left(\frac{U(1/\alpha_n)}{U_x(1/\alpha_n)}\right)\right), \\ Q_{4,n} &:= \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\hat{\gamma}_n \log\left(\frac{\hat{p}_n}{p_n}\right).\end{aligned}$$

First, condition (4) entails

$$\begin{aligned}
Q_{1,n} \sim & \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\left(\frac{U(1/p_n)}{U(1/\alpha_n)}\left(\frac{p_n}{\alpha_n}\right)^\gamma - 1\right) \\
\sim & \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\left(\frac{U((\alpha_n/p_n)^{\alpha\gamma}/\alpha_n)}{U(1/\alpha_n)}\left(\frac{p_n}{\alpha_n}\right)^\gamma - 1\right) \\
= & \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)} O(A_n).
\end{aligned}$$

Since $\alpha_n = o(p_n)$, we see that $\log(p_n/\alpha_n)^{-1} \to 0$ together with $\sqrt{np_n}A_n \to 0$ entails that $Q_{1,n} \to 0$.
Second, we know by Theorem 1.1 that $Q_{2,n} \xrightarrow{\mathscr{L}} \mathcal{N}(0,\gamma^2)$.
Now $Q_{3,n}$ is studied remarking that

$$\log\left(\frac{U(1/\alpha_n)}{U_x(1/\alpha_n)}\right) = \log\left(\frac{U(\sigma(x)/\alpha_n)}{U_x(1/\alpha_n)}\right) + \log\left(\frac{U(1/\alpha_n)}{\sigma(x)^{-\gamma}U(\sigma(x)/\alpha_n)}\right) - \gamma\log(\sigma(x)).$$

Together with (3) and (4), one has

$$\log\left(\frac{U(1/\alpha_n)}{U_x(1/\alpha_n)}\right) = O(A_n) - \gamma\log(\sigma(x)).$$

Consequently,

$$Q_{3,n} = \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)} O(A_n) + \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\Big(\hat\gamma_n \log(\hat\sigma_n(x)) - \gamma\log(\sigma(x))\Big).$$

Hence, the asymptotic behavior of $Q_{3,n}$ is ruled by that of $\hat\gamma_n\log(\hat\sigma_n(x)) - \gamma\log(\sigma(x))$, which we split into

$$(\hat\gamma_n - \gamma)\log(\hat\sigma_n(x)) + \gamma\log(\hat\sigma_n(x)) - \gamma\log(\sigma(x)).$$

Now, Theorem 1.1 entails

$$\frac{\log(\hat\sigma_n(x))}{\log(p_n/\alpha_n)}\sqrt{np_n}(\hat\gamma_n - \gamma) \xrightarrow{\mathbb{P}} 0.$$

Moreover, Theorem 1.2 together with the delta-method show that

$$\begin{aligned}
& \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}(\gamma\log(\hat\sigma_n(x)) - \gamma\log(\sigma(x))) \\
& \quad = \frac{\sqrt{np_nh_n^d}}{\sqrt{h_n^d}\log(p_n/\alpha_n)}(\gamma\log(\hat\sigma_n(x)) - \gamma\log(\sigma(x))) \xrightarrow{\mathbb{P}} 0.
\end{aligned}$$

Finally, using the notation introduced in the proof of Theorem 1.2, we have

$$\begin{aligned}
\frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\log\left(\frac{\hat p_n}{p_n}\right) = & \frac{\sqrt{np_n}}{\log(p_n/\alpha_n)}\log\left(1 + \frac{1}{\sqrt{np_n}}\sum_{i=1}^n Z^\sharp_{i,n}\right) \\
\sim & \frac{1}{\log(p_n/\alpha_n)}\sum_{i=1}^n Z^\sharp_{i,n} + o_{\mathbb{P}}\left(\frac{1}{\log(p_n/\alpha_n)}\right) \\
\xrightarrow{\mathbb{P}} & \, 0,
\end{aligned}$$

which proves that $Q_{4,n} \xrightarrow{\mathbb{P}} 0$, since $\hat\gamma_n \xrightarrow{\mathbb{P}} \gamma$.

4. Appendix

Lemma 4.1. *For fixed $n \geq 1$, let $(Y_i)_{1\leq i\leq n}$ be a sequence of i.i.d. random variables taking values in $(\mathfrak{X}, \mathcal{X})$. Let $E = (E_i)_{1\leq i\leq n}$ be an n-uple of independent Bernoulli random variables independent of Y_i. Write*

$$\nu(k) := \sum_{i=1}^k E_i, \ k \leq n.$$

Then we have

$$\sum_{i=1}^{n} \delta_{Y_i} E_i \stackrel{\mathscr{L}}{=} \sum_{i=1}^{\nu(n)} \delta_{Y_i}, \tag{19}$$

where the equality in law is understood as on the sigma algebra spanned by all Borel positive functions on $(\mathfrak{X}, \mathcal{X})$. Moreover, if the (Y_i) are almost surely positive, then

$$\max_{i=1,\ldots,n} Y_i E_i \stackrel{\mathscr{L}}{=} \max_{i=1,\ldots,\nu(n)} Y_i. \tag{20}$$

Proof. Note that (19) is exactly Khinchin's equality (see [16, p. 307, (14.6)]). We shall now prove (20). $e \in \{0,1\}^n$, and let g be a real measurable and positive function. Since the variables $(Y_i)_{1\le i\le n}$ are i.i.d. and independent of E, for any given permutation σ of $[\![1,n]\!]$,

$$wehave (Y_1,\ldots,Y_n) \stackrel{\mathscr{L}}{=} (Y_1,\ldots,Y_n)|_{E=e} \stackrel{\mathscr{L}}{=} (Y_{\sigma(1)},\ldots,Y_{\sigma(n)})|_{E=e}$$

by exchangeability. Now, define σ by

$$\sigma(k) := \begin{cases} \sum_{j=1}^{i} e_j \text{ if } e_i = 1 \\ n - \sum_{j=1}^{i} (1-e_j) \text{ if } e_i = 0 \end{cases} \qquad 1 \le i \le n.$$

Write $s(e) := \sum_{i=1}^{n} s(e_i)$ for the total number of ones in $(e_1,\ldots,e_n)$. By construction, the indices i for which $e_i = 1$ are mapped injectively to the set of first indices $[\![1, s(e)]\!]$, while those for which $e_i = 0$ are injectively mapped into $[\![s(e)+1, n]\!]$. Since e has fixed and nonrandom coordinates, we have

$$(Y_1 e_1,\ldots,Y_n e_n)|_{E=e} \stackrel{\mathscr{L}}{=} (Y_{\sigma(1)} e_1,\ldots,Y_{\sigma(n)} e_n)|_{E=e}.$$

Hence

$$\begin{aligned}
\max_{i=1,\ldots,n} Y_i e_i \;|_{E=e} &\stackrel{\mathscr{L}}{=} \max_{i=1,\ldots,n} Y_i e_i \\
&\stackrel{\mathscr{L}}{=} \max_{i=1,\ldots,n} Y_{\sigma(i)} e_{\sigma(i)} \\
&\stackrel{\mathscr{L}}{=} \max_{i=1,\ldots,s(e)} Y_{\sigma(i)} && \text{(a)} \\
&\stackrel{\mathscr{L}}{=} \max_{i=1,\ldots,s(e)} Y_i && \text{(b)} \\
&\stackrel{\mathscr{L}}{=} \max_{i=1,\ldots,s(e)} Y_i \;|_{E=e} \\
&\stackrel{\mathscr{L}}{=} \max_{i=1,\ldots,s(E)} Y_i \;|_{E=e},
\end{aligned}$$

where (a) holds because $e_{\sigma(i)} = 0$ for $i > s(e)$ by construction and the Y_i are a.s. positive, while (b) is obtained by noticing that $F_e(Y_{\sigma(1)},\ldots,Y_{\sigma(n)}) \stackrel{\mathscr{L}}{=} F_e(Y_1,\ldots,Y_n)$ with

$$F_e : (y_1,\ldots,y_n) \mapsto \max_{i=1,\ldots,s(e)} y_i.$$

Unconditioning upon E gives (20). □

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MODELLING POLLUTION OF RADIATION VIA TOPOLOGICAL MINIMAL STRUCTURES

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Abstract. The model of a generalized variable precision rough set is one of the variable precision rough sets used to solve some problems and measurements confront us that was difficult from the view point of science. The behavior of the radio contaminants in the environment is one of these measurements. Throughout this paper, we introduce and study a generalization variable precision rough set via a topological minimal structure. Some characteristics related to generalized upper and lower approximation with a variable precision by minimal structures will be discussed. A dispersion model which is the necessity to predict atmospheric path and danger from an atmospheric plume of hazardous materials will be applied with different types of examples.

1. Introduction and Preliminaries

Pawlak in [17] and [18] introduced rough sets as a formal tool to deal with uncertainty in the data analysis. It was based on the equivalence relation and crisp sets. Dudois et al. [4] introduced the notions of fuzzy roughness of information system in decision making. The connection between rough sets and topological spaces was investigated in [16] and [12]. Ziarko [27] extended rough sets through variable precision rough sets which are not only solve the problems with uncertain data, but also relax the strict definition of a rough set. He also studied the relative error limit of the partition blocks with the inclusion order $A \subseteq_\beta B$ if and only if $C(A,B) \leq \beta$, where β is called the majority inclusion relation, $0 \leq \beta \leq 0.5$. Both of the concept analyses [2] and rough sets are two significant mathematical creatures for the data analysis and knowledge processing. In 2000, Popa et al. [20] introduced the notion of minimal structure. Also, they introduced the notion of m_x-open sets and m_x- closed sets and characterized those sets using m_x-closure and m_x- interior operators, respectively. They defined in [15] separation axioms using the concept of minimal structure spaces and studied m_x^1 m_x^2-open in bimanual structure spaces. Recently, the neighborhood systems and rough sets on information system are used to represent structures such as self-similar fractals [7] and Human Heart [6] which are useful in physics and medicine, respectively. Also, the reduction of information system can be calculated by similarity as in [5].

Many researchers were interested in the atmospheric modeling as Santos et al. in [21] that is shown in Figure 1 for GPU-based implementation of a real-time model for atmospheric dispersion of radionuclides. Also, Cherradi et al. in [3] introduced the model of an atmospheric dispersion modeling microservice for HazMat transportation shown in Figure 2. These models were used by mathematicians to rephrase these models for each aspect of their study.

The aim of this paper is to use the model of Multilevel Meteorological Tower in National Center for Nuclear Safety and Radiation control, AEA, Egypt. We introduce the notion of generalized variable precision rough sets using a minimal structure. Some properties of generalized lower approximation (resp., generalized upper approximation) $\underline{R}_m^\beta$ (resp., $\overline{R}_m^\beta$) operator with e_x (resp., d_x) will be investigated. Atmospheric dispersion models which are important to predict path and danger from an atmospheric plume of hazardous materials with numerous kinds of examples will be applied.

Definition 1.1 ([19])**.** If U is a finite set of objects called universe, R a finite set of equivalence relations on U, called attributes, then the pair $K(U,R)$ is called an information system.

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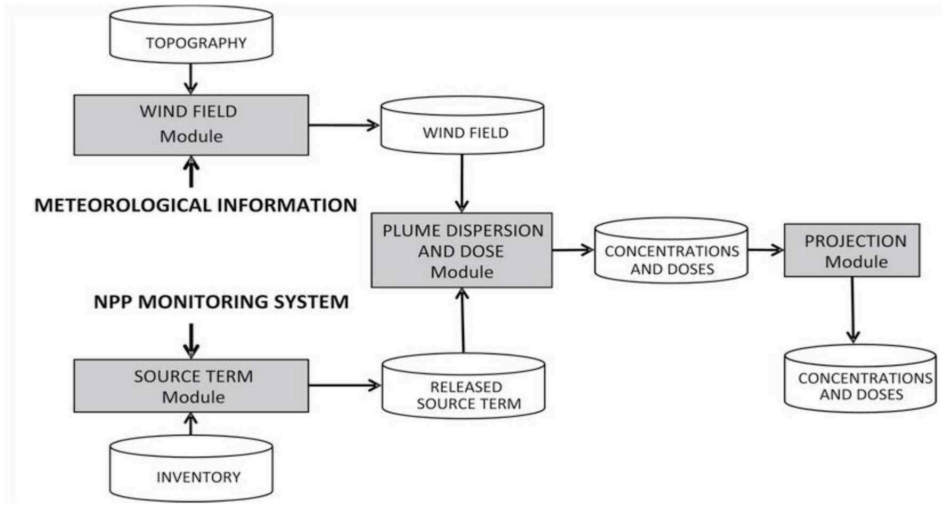

FIGURE 1. CNAAA site and th5e region covered by the ADR system

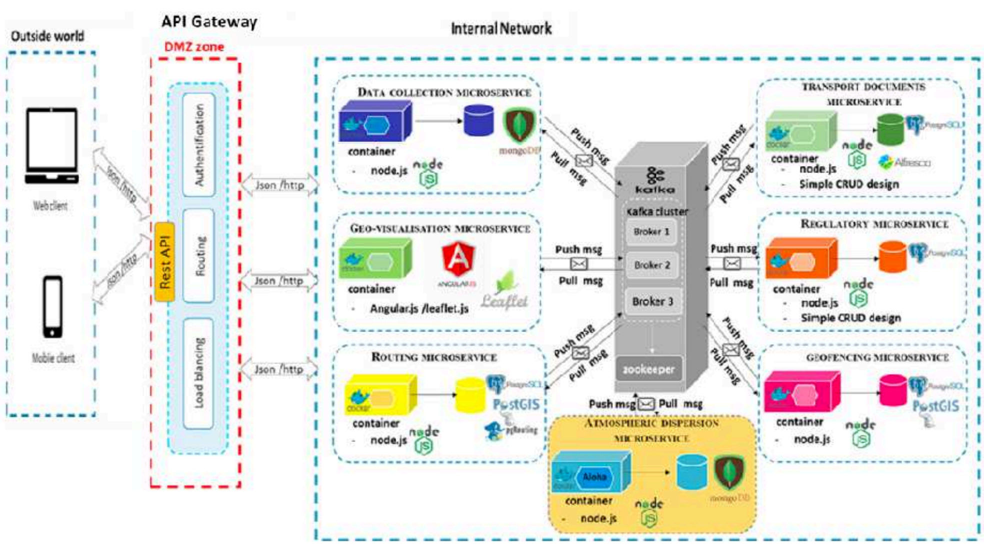

FIGURE 2. Overall system architecture

Definition 1.2 ([19])**.** Let X be a subset of U, the lower and upper approximation of X in U is defined by $\underline{R}(X) = \bigcup\{Y \in U/R : Y \subseteq X\}$, $\overline{R}(X) = \cup\{Y \in U/R : Y \cap X \neq \phi\}$.

Definition 1.3 ([27])**.** Let X and Y be a nonempty subset of a finite universe U. The measure of the relative degree of misclassification of the set X with respect to the set Y, denoted by $C(X, Y)$, is defined by

$$C(X,Y) = \begin{cases} \frac{1-\mathrm{Card}(X\cap Y)}{\mathrm{Card}(X)}, & \text{if} \quad \mathrm{Card}(X) > 0; \\ 0, & \text{if} \quad \mathrm{Card}(X) = 0, \end{cases}$$

where $\mathrm{Card}(X)$ denotes to cardinality of X.

Definition 1.4 ([27])**.** Let X and Y be nonempty subsets of a finite universe set U. $X \subseteq_\beta Y$ if $C(X, Y) \le \beta$.

Definition 1.5 ([27]). Let $X \subseteq Y$ be a subset of U. The β-lower $\underline{R}_\beta$ and β-upper $\overline{R}^\beta$ approximation of X under a relation R are defined by
$\underline{R}_\beta = \bigcup\{E \in U/R : C(E, X) \leq \beta\}$;
$\overline{R}^\beta = \bigcup\{E \in U/R : C(E, X) \leq 1 - \beta\}$.

Definition 1.6 ([14]). Let X be a nonempty set and $P(X)$ the power set of X. A subfamily of m_x is called a minimal structure (m-structure) on X if $\phi \in m_x$ and $X \in m_x$. By (X, m_x), we denote a nonempty set X with an m-structure m_x on X and call it an m-space. Each member of m_x is said to be m_x-open and the complement of an m_x-open set is said to be m_x-closed.

Definition 1.7 ([14]). Let X be a nonempty set and m_x be an m-structure on X. For a subset A of X, the m_x-closure (resp., m_x-interior) of A which is denoted by m_x-$Cl(A)$ (resp., m_x-Int(A)) is defined by m_x-$Cl(A) = \cap\{F : A \subseteq F, X - F \in m_x\}$ (resp., m_x-Int$(A) = \cap\{U : U \subseteq A, U \in m_x\}$).

Lemma 1.8 ([13]). *Let X be a nonempty set and m_x be an m-structure on X. For a subsets $A, B \subseteq X$, the following properties hold:*

(1) $m_x - Cl(X - A) = X - (m_x - \mathrm{Int}(A))$, $m_x - \mathrm{Int}(X - A) = X - (m_x - Cl(A))$.
(2) *If* $X - A \in m_x$ *then,* $m_x - Cl(A) = A$, *if* $A \in m_x$, *then* $m_x - \mathrm{Int}(A) = A$.
(3) $m_x - Cl(\phi) = \phi$, $m_x - Cl(X) = X$, $m_x - \mathrm{Int}(\phi) = \phi$, $m_x - \mathrm{Int}(X) = X$.
(4) *If* $A \subseteq B$, *then* $m_x - Cl(A) \subseteq m_x - Cl(B)$ *and* $m_x - \mathrm{Int}(A) \subseteq m_x - \mathrm{Int}(B)$.
(5) $m_x - Cl(m_x - Cl(A)) = m_x - Cl(A)$, $m_x - \mathrm{Int}(m_x - \mathrm{Int}(A)) = m_x - \mathrm{Int}(A)$.
(6) $A \subseteq m_x - Cl(A)$ *and* $m_x - \mathrm{Int}(A) \subseteq A$.

Lemma 1.9 ([13]). *For the subsets A on m_x an m-structure on X, $x \in m_x - Cl(A)$ if and only if $U \cap A \neq \phi$, for every m_x-open set U containing x.*

2. Generalized Granular Variable Precision Approximation Operators

Definition 2.1. Let $R \subset X \times X$ be a relation and $\beta \in [0, 1]$. Two maps $\underline{R}^\beta_m(A)$ and $R^\beta_m(A)$ are defined as follows, $A \subseteq X$:
$\underline{R}^\beta_m(A) = \cup\{E : E \in m_x, e_x(E, A) \geq \beta\}$,
$\overline{R}^\beta_m(A) = \cap\{E^c : E \in m_x, d_x(E^c, A) \leq \beta\}$,
where e_x, d_x are defined as: $e_x(A, B) = \frac{n(A^c \cup B)}{n(X)}$, $d_x(A, B) = \frac{n(A^c \cap B)}{n(X)}$ and $n(X)$ denotes a number of elements of X . Then $\underline{R}^\beta_m(A)$ and $\overline{R}^\beta_m(A)$ are said to be a generalized lower approximation (resp., upper approximation) operator. $(\underline{R}^\beta_m(A), R^\beta_m(A))$ is said to be a generalized variable precision rough set which is determined by e_x and d_x.

Remark 2.2.
(i) If $\beta = 1$, i.e., $e_x(E, A) = 1$, then $E^c \cup A = X$, i.e., $E \subseteq A$ such that $E \in m_x, A \subseteq X$. Hence $\underline{R}^1_m(A) = \bigcup\{E : E \in m_x, E \subset A\}$.
(ii) If $\beta = 0$, i.e., $d_x(E^c, A) = 0$, then $E \cap A = \phi$. So, $A \subseteq E^c$, and then $\overline{R}^0_m(A) = \bigcap\{E^c : E \in m_x, A \subseteq E^c\}$.

Theorem 2.3. *Let R be a relation on X and $\beta \in [0, 1]$. The following hold:*

(i) $d_x(E^c, A^c) = 1 - e_x(E, A)$ *for each* $A \subseteq X$.
(ii) $\underline{R}^\beta_m(A) = (\overline{R}^{1-\beta}_m(A^c))^c$ *for each* $A \subseteq X$.
(iii) $\underline{R}^1_m(\phi) = \phi$, $\overline{R}^0_m(X) = X$.
(iv) $\underline{R}^1_m(X) = \cup_{E \subseteq X} E$, $\overline{R}^0_m(\phi) = \cap_{E \subseteq X} E^c$.
(v) *If* $\forall\ x \in X$, $\exists\ E \subseteq X$ *s. t.* $x \in E$, *then* $\underline{R}^1_m(X) = X$, $\overline{R}^0_m(\phi) = \phi$.

Proof. (i) Follows from $d_x(E^c, A^c) = \frac{n(E \cap A^c)}{n(X)} = \frac{n(X) - n(E^c \cup A)}{n(X)} = 1 - e_x(E, A)$.
(ii) Deduced by $(\overline{R}^{1-\beta}_m(A^c))^c = (\bigcap\{E^c : E \in m_x, d_x(E^c, A^c) \leq 1 - \beta\})^c = \bigcup\{E : d_x(E^c, A^c) \leq 1 - \beta\} = \bigcup\{E : 1 - e_x(E, A) \leq 1 - \beta\} = \underline{R}^\beta_m(A)$.
(iii) Follows from Remark 2.2.

(iv) Since $\underline{R}^1_m(X) = \bigcup\{E : E \in m_x, E \subseteq X\} = \bigcup_{E\subseteq X} E$, $\underline{R}^0_m(\phi) = \bigcap\{E^c : E \in m_x, \phi \subset E^c\} = \bigcap_{E\subset X} E^c$.
(v) For each $x \in X$, there exists $E \subseteq X$ such that $x \in\in E$ and $X = \bigcup_{E\subset X} E$. Hence $\underline{R}^1_m(X) = X$, $\overline{R}^0_m(\phi) = \phi$. $\square$

Theorem 2.4. *Let R be a relation on X and $\beta \in [0,1]$, $E \in m_x$, $A \subset X$. The following hold:*
(i) $\underline{R}^1_m(A) \subseteq A$ *and* $A \subseteq \overline{R}^0_m(A)$.
(ii) *If* $\beta_1 \leq \beta_2$, *then* $\underline{R}^{\beta_2}_m(A) \subseteq \underline{R}^{\beta_1}_m(A)$.
(iii) *If* $A_1 \subseteq A_2$, *then* $\underline{R}^\beta_m(A_1) \subseteq \underline{R}^\beta_m(A_2)$.
(iv) *If* $\beta_1 \leq \beta_2$, *then* $\overline{R}^{\beta_2}_m(A) \subseteq \overline{R}^{\beta_1}_m(A)$.
(v) *If* $A_1 \subseteq A_2$, *then* $\overline{R}^\beta_m(A_1) \subseteq \overline{R}^\beta_m(A_2)$.
(vi) $\underline{R}^1_m(E) = E$, $\overline{R}^0_m(E^c) = E^c$, $E \in m_x$.
(vii) $\underline{R}^1_m(\underline{R}^\beta_m(A)) = \underline{R}^\beta_m(A)$ *and* $\overline{R}^0_m(\underline{R}^\beta_m(A)) = \underline{R}^\beta_m(A)$.

Proof. (i) By Remark 2.2, we have $\underline{R}^1_m(A) = \cup\{E : E \in m_x, E \subset A\} \subset A$. Moreover, $\overline{R}^0_m(A) = \cap\{E^c : E \in m_x, E \subset A^c\} \supset A$.
(ii), (iii), (iv) and (v) are obvious from Definition 2.1.
(vi) Since $e_x(E,E) = \frac{n(E^c\cup E)}{n(X)} = 1$, $\underline{R}^1_m(E) = \bigcup\{E : E \in m_x, E \subseteq E\} \supseteq E$. By (i), $\underline{R}^1_m(E) \subseteq E$, we have $\underline{R}^1_m(E) = E \forall E \in m_x$. Since $d_x(E^c,E^c) = \frac{n(E\cap E^c)}{n(X)} = 0$, $\overline{R}^0_m(E^c) = \bigcap\{E^c : E \in m_x, E^c \subseteq E^c\} \subseteq E^c$. By (i), $\overline{R}^0_m(E^c) \supseteq E^c$. Hence $\overline{R}^0_m(E^c) = E^c \forall E \in m_x$.
(vii) For $E \subseteq \underline{R}^\beta_m(A)$, $e_x(E,\underline{R}^\beta_m(A)) = \frac{n(E^c\cup\underline{R}^\beta_m(A))}{n(X)} = 1$, $\underline{R}^1_m(\underline{R}^\beta_m(A)) = \bigcup\{E : E \in m_x, E \subseteq \underline{R}^\beta_m(A)\} \supseteq \underline{R}^\beta_m(A) \supseteq \underline{R}^\beta_m(A)$. By (i), $\underline{R}^1_m(\underline{R}^\beta_m(A)) \subseteq \underline{R}^\beta_m(A)$. Hence $\underline{R}^1_m(\underline{R}^\beta_m(A)) = \underline{R}^\beta_m(A)$. For $E^c \supset \underline{R}^\beta_m(A)$, $d_x(E^c,\underline{R}^\beta_m(A)) = \frac{n(E\cap\underline{R}^\beta_m(A))}{n(X)} = 0$, $\overline{R}^0_m(\underline{R}^\beta_m(A)) = \bigcap\{E^c : E \in m_x, \underline{R}^\beta_m(A) \subseteq E^c\} \subseteq \underline{R}^\beta_m(A)$. By (i)$\overline{R}^0_m(\overline{R}^\beta_m(A)) \supseteq \overline{R}^0_m(A)$. Hence $\overline{R}^0_m(\overline{R}^\beta_m(A)) = \overline{R}^0_m(A)$. $\square$

Theorem 2.5. *Let R be a relation on X and $\beta \in [0,1]$. Define $T_X, F_X \subseteq P(X)$ as $T_X = \{A \subseteq X : A = \underline{R}^\beta_m(A)$, $F_X = \{A \subseteq X : A = \overline{R}^0_m(A)\}$*
(i) $\phi \in T_X, E \in T_X, \forall E \in m_x, \underline{R}^\beta_m(A) \in T_X$ *for each* $A \subset X$, $\beta \in [0,1]$.
(ii) *If* $A_i \in T_X$ *for each* $i \in I$, *then* $\cup_{i\in I} A_i \in T_X$.
(iii) $X \in F_X, E^c \in F_X \forall E \in m_x$ *and* $\underline{R}^\beta_m(A) \in F_X A \subset X$, $\beta \in [0,1]$.
(iv) *If* $A_i \in F_X$ *for each* $i \in I$, *then* $\cap_{i\in I} A_i \in F_X$.
(v) $A \in T_X$ *if and only if* $A^c \in F_X$.
(vi) *If* $\forall\ x \in X$, $\exists\ E \in m_x$ *s.t.* $x \in E$, *then* $X \in T_X$ *and* $\phi \in F_X$.

Proof. Follows directly by Theorems 2.3 and 2.4. $\square$

Definition 2.6. Let R be a relation on X and $\beta \in [0,1]$, $P, Q \subseteq R$ be families of open sets, say, attributes. Then Q is said to depend in degree $\gamma(P,Q,\beta)$ on attributes P. $\gamma(P,Q,\beta) = \frac{|Pos(P,Q,\beta)|}{|X|}$, $\gamma(P,Q,\beta) = \bigcup_{E\in Q} \underline{P}_m(E)$ is P_m-positive region of Q.

Example 2.7. *Define R such that xR_By if $\sum\limits_{i\in B} \frac{|i(x)-i(y)|}{|B|} < \lambda$, where $B \subseteq \{a,b,c,f\}$. Attributes in Table 2 are similar by $\mid c(x_i) - c(y_i) \mid$, where $i,j \in \{1,2,3,4,5,6\}$ for $B = \{c\}$.*

Consider $xR_{\{c\}}y$ if and only if $\mid c(x) - c(y) \mid < 5$, $X = \{x_1, x_2, x_3, x_4, x_5, x_6,\}$, $m_x = \{\phi, X, \{x_1, x_4, x_6\}, \{x_2, x_3, x_5\}, \{x_2, x_3, x_4, x_5, x_6\}, \{x_1, x_3, x_4, x_5, x_6\}\}$, $X - m_x = \{\phi, X, \{x_1, x_4, x_6\}, \{x_2, x_3, x_5\}, \{x_1\}\}$, $\underline{R}^\beta_m(A) = \bigcup\{E : E \in m_x,\ e_x(E,A) \geq \beta\} = \bigcup\{E : E \in m_x,\ \frac{n(E^c\cup A)}{n(X)} \geq \beta\}$, $\overline{R}^\beta_m(A) = \cap\{E^c : E \in m_x,\ d_x(E^c,A) \leq \beta\} = \bigcap\{E^c : E \in m_x,\ \frac{n(E\cap A)}{n(X)} \leq \beta\}$.

If $\beta = 0.8$, then $\underline{R}^{0.8}_m(A) = \{\phi\} \cup \{x_1, x_4, x_6\} = \{x_1, x_4, x_6\}$, $\overline{R}^{0.2}_m(A^c) = \overline{R}^{0.2}_m(\{x_2, x_4, x_5\} = X \cap \{x_2, x_3, x_5\} = \{x_2, x_3, x_5\}$, $\underline{R}^{0.8}_m(A) = (\overline{R}^{0.2}_m(A^c)^c = \{x_1, x_4, x_6,\}$,$\underline{R}^1_m(\underline{R}^{0.8}_m(A)) = \underline{R}^1_m(\{x_1, x_4, x_6,\} = \{x_1, x_4, x_6\} = \overline{R}^{0.2}_m(A) = \overline{R}^{0.2}_m\{x_1, x_3, x_6\} = \{x_1, x_4, x_6\}$, $\underline{R}^{0.8}_m(A^c) = \underline{R}^{0.8}_m\{x_2, x_4, x_5\} =$

$\{x_2, x_3, x_5\}$, $\underline{R}_m^{0.8}(A^c) = (\overline{R}_m^{0.2}(A))^c = \{x_2, x_3, x_5\}$,$\underline{R}_m^{1}(\underline{R}_m^{0.8}(A^c)) = \underline{R}_m^{1}(\{x_2, x_3, x_5,\} = \{x_2, x_3, x_5\} = \underline{R}_m^{0.8}(A^c)$,$\overline{R}_m^{0}(\overline{R}_m^{0.2}(A)) = \overline{R}_m^{0}(\{x_1, x_4, x_6\} = \{x_1, x_4, x_6\} = \overline{R}_m^{0.2}(A)$.

TABLE 1. IS with original data

	a	b	c	f
x_1	81	77	84	83
x_2	100	81	93	85
x_3	71	78	89	60
x_4	93	82	88	60
x_5	97	87	91	85
x_6	93	68	87	90

TABLE 2. IS with similarity

c	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	9	5	4	7	3
x_2	9	0	4	5	2	6
x_3	5	4	0	1	2	2
x_4	4	5	1	0	3	1
x_5	7	2	2	3	0	4
x_6	3	6	2	1	4	0

3. APPLICATION ON POLLUTION OF RADIATION

As a matter of fact, each industry has its own waste. Most of it are hazardous to a man and his environment. Due to the increasing importance of nuclear power industries and the release of different effluents to air through different pathways such as water and soil, later, they will find their way to a man. So, it becomes important to study the behavior of radio contaminants in environment. Several pathways exist, along which radionuclide can be transported to a man; the atmosphere is a pathway for transport of radioactive release from a nuclear power plant to environment and thereby to a man. Atmospheric dispersion models are the necessity to predict path and danger from an atmospheric plume of hazardous materials.

Atmospheric dispersion models are the computer simulation programs, which combine information about the source of a release and observations of wind and weather considerations with theories of atmospheric behaviour to predict the spread and travel of contaminants. The most widely used model is the Gaussian Plume Model [16]. Most of the countries participating in the NATO plume used this model in Germany. The ground level air concentrations are given from the equation

$$C(x,y) = \frac{2Q}{(2\pi\sigma_y\sigma_x + C_w A)u} \exp(\frac{-\lambda x}{u}) * (\exp\int_0^x \frac{dx}{\exp(\frac{H^2}{2\sigma^2})})^{-(\frac{2}{\pi})^{.5}(vd/u)} * \exp[-\frac{1}{2}(\frac{H}{\sigma_x^2})^2 - \frac{1}{2}(\frac{y}{\sigma_y})^2]$$

according to the Gaussian Plume model such that $C(Bq.m^{-3})$, $Q(Bq.s^{-1})$, $u(ms^{-1})$, $\sigma_y(m)$, $\sigma_z(m)$, $x(m)$, $y(m)$, $z(m)$, $H(m)$, $\exp(\frac{-\lambda x}{u})$, A and C_w denotes CAP, continuous point source strength, WS at height H, lateral dispersion parameter, VDP, HD in direction of downwind, LD from plume center, height above ground, EH of plume above ground, RD for the specified nuclide, CSA of building normal to wind and shape factor that represents the fraction of A over which the plume is dispersed, respectively. $C_w = 0.5$ is commonly used and V_d denotes the deposition velocity. The dispersion parameters σ_y and σ_z can be estimated from $\sigma_y = cx^d$, where a, b, c, and d are shown in Table 3 [10].

Table 3. Parameters Values

Stability	a	b	c	d
$A-B$	1.46	0.71	0.01	1.54
C	1.52	0.69	0.04	1.17
D	1.36	0.67	0.09	0.95
$E-F$	0.75	0.70	0.40	0.67

Table 4. An Experimental Data

Date(2009)	$Exps$	u	$Stability$	H	$Working$	Obs	Est
14-Mar	x_1	4	A	49	48	0.025	0.114
14-Mar	x_2	4	A	48	49	0.037	0.129
02-May	x_3	4	B	45	1.5	0.091	0.122
03-May	x_4	3	C	46	22	0.197	0.219
03-May	x_5	3	A	45	23	0.272	0.307
03-May	x_6		D	45	24	0.188	0.473
04-May	x_7		E	47	48	0.447	0.484
04-May	x_8		C	46	48.7	0.123	0.456
14-May	x_9		A	47	48.25	0.032	0.109
10-Jan	x_{10}		D	28	5	0.42	0.933
22-Jan	x_{11}		B	28.3	7	0.442	0.756
15-Mar	x_{12}		A	30.8	2	0.67	1.144
15-Mar	x_{13}		A	30.6	4	0.67	1.056

3.1. **Experimental data.** In the following subsection we use some experimental data. Air samples were gathered from 92m to 184m around two industrial locations. The area under consideration is flat and prevailed by a sand soil with poor vegetation cover. Also, it was separated into 16 sectors (with $22.5C^\circ$) width for each sector), origin from the north direction. Aerosols were gathered at a height of 0.7 m above the ground of 10.3 cm diameter filter paper with a desired collection efficiency ($\frac{3.4}{100}$) using a high volume air sample with 220V/50Hz bias.

Air sample had air flow rate of approximately $\frac{0.7m^3}{min}$ ($\frac{25ft^3}{min}$). Sample collection time was 30 min with an air volume of $21.2m^3(750ft^3)$. Air volume was adjusted to standard conditions (25 C° and 1013 mb). Filter paper was directly evaluated by energy and efficiency calibrated HPGe detectors relative to 3"x3" NaI (TI) detector were 15.6 and $\frac{30}{100}$ measured at $1.332MeV$ with source to detector distance of 25 cm.

Meteorological data were allowed for Ins has meteorological tower for four months at a smooth flat site (Ins has area, Egypt) for the year (2006). Vertical temperature gradient $\frac{\triangle T}{\triangle Z}$ was decided by measuring temperature at $10-60$m levels from the multilevel meteorological tower of Ins has sitting and Environment Department, National Center for Nuclear Safety and Radiation control, AEA, Egypt. This tower is placed near to our search.

Table 3.1 contains the working data, 13 experiments (Exps), wind speed, stability, the effective height (H) for two locations, observed (Obs) and estimated (Est) concentrations $\frac{Bq}{m^3}$ of Iodine 131(I-131) with the working hours.

3.2. **Analysis of data using topological minimal structure.** In the following, we explain that the observed data, satisfying the properties of our generalization, are used to determine the degree of dependence of the estimated and observed data. Define a relation xRy if and only if $\leftrightarrow|x_i-x_j|\leq 0.1$, $i,j=1-13$. Table 3.2 shows the relation R_{Obs} between the observed data. Table 3.2 shows the relation R_{Est} between the observed data.

TABLE 5. Observation Data by R_{Obs}

Obs	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}
x_1	0	0.012	0.066	0.172	0.247	0.163	0.422	0.098	0.007	0.39	0.395	0.64	0.645
x_2	0.012	0	0.054	0.16	0.235	0.151	0.41	0.086	0.005	0.38	0.383	0.63	0.633
x_3	0.066	0.054	0	0.106	0.181	0.097	0.356	0.032	0.059	0.32	0.329	0.57	0.579
x_4	0.172	0.16	0.106	0	0.075	0.009	0.25	0.074	0.165	0.22	0.223	0.47	0.473
x_5	0.247	0.235	0.181	0.075	0	0.084	0.175	0.149	0.24	0.14	0.148	0.39	0.398
x_6	0.163	0.151	0.097	0.009	0.084	0	0.259	0.065	0.156	0.23	0.232	0.48	0.482
x_7	0.422	0.41	0.356	0.25	0.175	0.259	0	0.324	0.415	0.02	0.027	0.22	0.223
x_8	0.098	0.086	0.032	0.074	0.149	0.065	0.324	0	0.091	0.29	0.297	0.54	0.547
x_9	0.007	0.005	0.059	0.156	0.24	0.156	0.415	0.091	0	0.38	0.388	0.63	0.638
x_{10}	0.395	0.383	0.329	0.223	0.148	0.232	0.027	0.297	0.388	0	0	0.25	0.25
x_{11}	0.395	0.383	0.329	0.223	0.148	0.232	0.027	0.297	0.388	0	0	0.25	0.25
x_{12}	0.645	0.633	0.579	0.473	0.398	0.482	0.223	0.547	0.638	0.25	0.25	0	0
x_{13}	0.645	0.633	0.579	0.473	0.398	0.482	0.223	0.547	0.638	0.25	0.25	0	0

TABLE 6. Observation Data by R_{Est}

Est	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}
x_1	0	0.015	0.008	0.105	0.193	0.359	0.37	0.342	0.005	0.719	0.642	1.03	0.942
x_2	0.015	0	0.007	0.09	0.178	0.344	0.455	0.327	0.02	0.704	0.627	1.015	0.927
x_3	0.008	0.007	0	0.097	0.185	0.351	0.362	0.334	0.013	0.711	0.634	1.022	0.934
x_4	0.105	0.09	0.097	0	0.088	0.254	0.265	0.237	0.11	0.614	0.537	0.925	0.837
x_5	0.193	0.178	0.185	0.088	0	0.166	0.177	0.149	0.198	0.526	0.449	0.837	0.749
x_6	0.359	0.344	0.351	0.254	0.166	0	0.11	0.017	0.364	0.36	0.283	0.671	0.583
x_7	0.37	0.355	0.362	0.256	0.177	0.011	0	0.028	0.375	0.349	0.272	0.66	0.572
x_8	0.342	0.327	0.334	0.237	0.149	0.017	0.028	0	0.347	0.377	0.3	0.688	0.6
x_9	0.005	0.02	0.013	0.11	0.198	0.364	0.375	0.347	0	0.742	0.647	1.035	0.947
x_{10}	0.719	0.704	0.711	0.614	0.526	0.36	0.349	0.377	0.742	0	0.077	0.311	0.223
x_{11}	0.642	0.627	0.634	0.537	0.449	0.283	0.272	0.3	0.647	0.077	0	0.388	0.03
x_{12}	1.03	1.015	1.022	0.925	0.837	0.671	0.66	0.688	1.035	0.311	0.388	0	0.088
x_{13}	0.645	0.633	0.579	0.473	0.398	0.482	0.223	0.547	0.638	0.25	0.25	0	0

If $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}$, $m_{X(Obs)} = \{\phi, X, \{x_1, x_2, x_3, x_8, x_9\}, \{x_1, x_2, x_3, x_6, x_8, x_9\}, \{x_4, x_5, x_6, x_8\}, \{x_4, x_5, x_6\}, \{x_3, x_4, x_5, x_6, x_8\}, \{x_1, x_{10}, x_{11}\}, \{x_1, x_2, x_3, x_4, x_6, x_8, x_9\}, \{x_{12}, x_{13}\}\}$, and then $X(Obs) - m_{X(Obs)} = \{\phi, X, \{x_4, x_5, x_6, x_7, x_{10}, x_{11}, x_{12}, x_{13}\}, \{x_4, x_5, x_7, x_{10}, x_{11}, x_{12}, x_{13}\}, \{x_1, x_2, x_3, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}, \{x_1, x_2, x_3, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}, \{x_1, x_{10}, x_{11}\}, \{x_1, x_2, x_3, x_4, x_6, x_8, x_9\}, \{x_{12}, x_{13}\}, \{x_1, x_2, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}, \{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{12}, x_{13}\}, \{x_5, x_7, x_{10}, x_{11}, x_{12}, x_{13}\}, \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}\}$.
Let $A \subset X, \beta \in [0,1]$, $A = \{x_1, x_2, x_5, x_6\}$. If $\beta = 0.8$, then $\underline{R}_m^{0.8}(A) = \{x_4, x_5, x_6, x_8, x_{12}, x_{13}\}$. If $\beta = 0.2$, then $\overline{R}_m^{0.2}(A^c) = \{x_1, x_2, x_3, x_7, x_9, x_{10}, x_{11}\}$. $\underline{R}_m^{0.8}(A) = (\overline{R}_m^{0.2}(A^c))^c = \{x_4, x_5, x_6, x_8, x_{12}, x_{13}\}$. $\underline{R}_m^{1}(\underline{R}_m^{0.8}(A)) = \underline{R}_m^{1}(\{x_4, x_5, x_6, x_8, x_{12}, x_{13}\}) = \{x_4, x_5, x_6, x_8, x_{12}, x_{13}\} = \underline{R}_m^{0.8}(A)$. $\overline{R}_m^{0}(\overline{R}_m^{0.2}(A^c)) = \overline{R}_m^{0}(\{x_1, x_2, x_3, x_7, x_9, x_{10}, x_{11}\} = \{x_1, x_2, x_3, x_7, x_9, x_{10}, x_{11}\} = \overline{R}_m^{0.2}(A^c)$. $\overline{R}_m^{0.2}(A) = \overline{R}_m^{0.2}(\{x_1, x_2, x_5, x_6\}) = \phi$. $\underline{R}_m^{0.8}(A^c) = \underline{R}_m^{0.8}(\{x_3, x_4, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}) = X$. $\underline{R}_m^{0.8}(A^c) = (\overline{R}_m^{0.2}(A))^c = X$. $\underline{R}_m^{1}(\underline{R}_m^{0.8}(A^c)) = X = \underline{R}_m^{0.8}(A^c)$. $\overline{R}_m^{0}(\overline{R}_m^{0.2}(A)) = \phi = \overline{R}_m^{0.2}(A)$.

Now we use the approach to determine the degree of dependence of the estimated and observed data $m_{X(Est)} = \{\phi, X, \{x_1, x_2, x_3, x_9\}, \{x_1, x_2, x_3, x_4, x_9\}, \{x_4, x_5\}, \{x_6, x_7, x_8\}, \{x_{10}, x_{11}\}, \{x_{12}, x_{13}\}\}$, $X(Est) - m_{X(Est)} = \{\phi, X, \{x_4, x_5, x_6, x_7, x_{10}, x_{11}, x_{12}, x_{13}\}, \{x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}\}, \{x_1,$

$x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}$, $\{x_1, x_2, x_3, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}$, $\{x_1, x_2, x_3, x_4, x_5, x_9, x_{10}, x_{11}$, $\{x_{12}, x_{13}\}$, $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{12}, x_{13}\}$, $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}\}$. Therefore, $\gamma(P, Q, 0.3) = \frac{|POS(P,Q,0.3)|}{|X|} = 1$.

4. Conclusion and Discussion

The field of mathematical science which goes under the name of minimal structure is concerned with all questions, related directly or indirectly to topology. Therefore, the theory of rough sets is one of the subjects, most important in topology. Also, we give an atmospheric dispersion modeling which is essential to predict the path and danger from an atmospheric plume of hazardous materials. The approach used here can be applied in any IS with quantitative or qualitative data. Moreover, the concepts proposed in this paper can be extended to fuzzy topological structures [1] and thus one can get a more affirmative solution in decision making problems [9, 22–24] in real life solutions.

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A SIMPLE DERIVATION OF THE KEY EQUATION IN JANASHIA–LAGVILAVA METHOD

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Abstract. We provide a simple derivation of the key system of equations for the corresponding boundary value problem in the Janashia–Lagvilava matrix spectral factorization method.

1. Introduction

Let

$$S(t)=\begin{pmatrix} s_{11}(t) & s_{12}(t) & \cdots & s_{1r}(t)\\ s_{21}(t) & s_{22}(t) & \cdots & s_{2r}(t)\\ \vdots & \vdots & \vdots & \vdots\\ s_{r1}(t) & s_{r2}(t) & \cdots & s_{rr}(t)\end{pmatrix}, \tag{1}$$

$|t|=1$, be a positive definite (a.e.) matrix function with integrable entries, $s_{ij}\in L^1(\mathbb{T})$, defined on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$.

Wiener's matrix spectral factorization theorem [9] asserts that if

$$\int_{\mathbb{T}} \log\det S(t)\,dt>-\infty, \tag{2}$$

then S admits the factorization

$$S(t)=S_+(t)S_+^*(t), \tag{3}$$

where S_+ can be analytically extended inside the unit disk $\mathbb{D}$, and $S_+^*(t)$ is the Hermitian conjugate to $S_+(t)$. Furthermore, the entries of S_+ are the square integrable functions and, actually, belong to the Hardy space $H^2=H^2(\mathbb{D})$ (as usual, the functions from the Hardy space and their boundary values are identified). Representation (3) is unique (up to a constant unitary factor) under the additional requirement that the analytic function S_+ is *outer* (for the definition, see §2). Condition (2) is necessary and sufficient for the spectral factorization (3) to exist.

An approximate computation of the factor S_+ for the given matrix function (1) is an important challenging problem due to its practical applications. Therefore, different authors have developed dozens of methods for such factorization as the Levinson–Durbin algorithm, Bauer method (by Toeplitz matrix decomposition), Wilsons algorithm (based on Newton–Raphson iterations), symmetric factor extraction, solutions via algebraic Riccati equation, etc. (see [7,8]).

The Janashia–Lagvilava algorithm [4,5] is a relatively new method of a matrix spectral factorization which proved to be effective [3].

In this algorithm, the computational complexity of the problem is reduced to the minimum by intelligent manipulations. The algorithm starts with the LU triangular factorization

$$S(t)=M(t)M^*(t),$$

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with
$$M(t)=\begin{pmatrix} f_1^+(t) & 0 & \cdots & 0 & 0\\ \xi_{21}(t) & f_2^+(t) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \xi_{r-1,1}(t) & \xi_{r-1,2}(t) & \cdots & f_{r-1}^+(t) & 0\\ \xi_{r1}(t) & \xi_{r2}(t) & \cdots & \xi_{r,r-1}(t) & f_r^+(t)\end{pmatrix},$$
where f_j^+, $j=1,2,\dots,r$, are outer analytic functions in H^2 (denoted as $f_j^+\in H^2_O$) and $\xi_{ij}\in L^2(\mathbb{T})$, $2\le i\le r$, $1\le j<j$. Then the algorithm performs step-by-step spectral factorization of principal leading submatrices of S (see [5]).

A key component of this scheme is the constructive proof of the following

Theorem 1. *Let*
$$F(t)=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \cdots & \zeta_{m-1}(t) & f^+(t)\end{pmatrix} \tag{4}$$
be an $m\times m$ matrix, where $f^+\in H^2_O$ and $\zeta_j\in L^2(\mathbb{T})$, $j=1,2,\dots,m-1$. Then, there exists an $m\times m$ unitary matrix function U of the special structure
$$U(t)=\begin{pmatrix} u_{11}^+(t) & u_{12}^+(t) & \cdots & u_{1,m-1}^+(t) & u_{1m}^+(t)\\ u_{21}^+(t) & u_{22}^+(t) & \cdots & u_{2,m-1}^+(t) & u_{2m}^+(t)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ u_{m-1,1}^+(t) & u_{m-1,2}^+(t) & \cdots & u_{m-1,m-1}^+(t) & u_{m-1,m}^+(t)\\ \overline{u_{m1}^+(t)} & \overline{u_{m2}^+(t)} & \cdots & \overline{u_{m,m-1}^+(t)} & \overline{u_{mm}^+(t)}\end{pmatrix},\quad u_{ij}^+\in H^\infty, \tag{5}$$
with
$$\det U(t)=1 \ \text{ for a.a. } t\in\mathbb{T} \tag{6}$$
such that the entries of the product FU are analytic functions in H^2, i.e.,
$$FU\in H^2(\mathbb{D})^{m\times m}. \tag{7}$$

The existence of such a unitary matrix function U follows from the general existence theorem of the matrix spectral factorization and is demonstrated in [1]. The most important finding of Janashia and Lagvilava was, however, the observation that the columns of U can be constructed separately, independently of each other. In particular, the following theorem holds.

Theorem 2. *Let F and U be as in Theorem* 1*. Then, the columns of U* (*more specifically, taking $x_i^+=u_{ij}^+$, $i=1,2,\dots,m$, for each $j=1,2,\dots,m$*), *are the solutions of the following multi-dimensional boundary value problem*
$$\begin{cases} \zeta_1(t)x_m^+(t)-f^+(t)\overline{x_1^+(t)}=\varphi_1^+(t),\\ \zeta_2(t)x_m^+(t)-f^+(t)\overline{x_2^+(t)}=\varphi_2^+(t),\\ \vdots\\ \zeta_{m-1}(t)x_m^+(t)-f^+(t)\overline{x_{m-1}^+(t)}=\varphi_{m-1}^+(t),\\ \zeta_1(t)x_1^+(t)+\zeta_2(t)x_2^+(t)+\ldots+\zeta_{m-1}(t)x_{m-1}^+(t)+f^+(t)\overline{x_m^+(t)}=\varphi_m^+(t),\end{cases} \tag{8}$$
where ζ_i and f^+ are the entries of F, and $x_i^+\in H^\infty$ and $\varphi_i^+\in H^2$ are the unknowns.

Actually, the Janashia–Lagvilava algorithm approximates the solution of the above system for the given matrix function F. This task is not anymore as difficult as the discovery of system (8) itself.

A long sequence of transformations which derives system (8) from condition (6) is presented in [1]. In the present paper, we deduce the same system much easier by using a more transparent way.

The paper is organized as follows. In the next section we introduce the necessary notation and formulate the well-known theorems used afterwards. Although the proof of Theorem 1 based on the Wiener's existence theorem of the matrix spectral factorization is outlined in [1], for the readers convenience, we present the detailed proof of this theorem in Section 3. This makes the paper more self-contained. The proof of Theorem 2 is given in Section 4.

2. Notation and Preliminary Observations

Let $L^p(\mathbb{T})$, $p>0$, be the Lebesgue space of p integrable functions on the unit circle $\mathbb{T}:=\{z\in\mathbb{C}: |z|=1\}$ and

$$H^p=H^p(\mathbb{D}):=\left\{f\in\mathcal{A}(\mathbb{D}): \sup_{r<1}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta<\infty\right\}$$

be the Hardy space of analytic functions on the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$.

For $f\in H^p$ and $t=e^{i\theta}\in\mathbb{T}$, we assume that

$$f(t)=f(z)|_{z=t}:=\lim\nolimits_{r\to1}f(re^{i\theta})$$

(which is defined a.e. on $\mathbb{T}$); the class of the boundary value functions of all functions from H^p is denoted by L^p_+. It is well known that $L^p_+\subset L^p$ and, for $p\geq 1$,

$$L_p^+=\{f\in L^p(\mathbb{T}): c_k\{f\}=0 \text{ for } k<0\},$$

where $c_k\{f\}$ stands for the k-th Fourier coefficient of f. Furthermore, there is a one-to-one correspondence

$$L^p_+\longleftrightarrow H^p,\quad p>0, \tag{9}$$

which allows these two classes to be naturally identified. In particular, one can speak about the values of $f\in L^p_+$ inside the unit disk. The relation (9) can be strengthened by claiming that the function $f\in L^p_+$ cannot be equal to zero on a subset of $\mathbb{T}$ of positive measure and, furthermore, for each $f\in L^p_+$, we have

$$\int\limits_{\mathbb{T}}\log|f(t)|\,dt>-\infty.$$

That is why condition (2) is necessary for the existence of factorization (3) and Wiener proved its sufficiency, as well.

We use Smirnov's theorem (see, e.g., [6]) which claims that if a function $f\in H^p$ and its boundary values function belongs to L^q $(q>p)$, then $f\in H^q$. This theorem can be briefly formulated as

$$f\in H^p\cap L^q_+\Longrightarrow f\in H^q. \tag{10}$$

A nonzero function f is called *outer* if it can be reconstructed from the absolute values of its boundary values, namely,

$$f(z)=c\cdot\exp\left(\frac{1}{2\pi}\int_{\mathbb{T}}\frac{t+z}{t-z}\log\big|f(t)\big|\,dt\right),\qquad |c|=1. \tag{11}$$

The class of outer functions in H^p is denoted by H^p_O. Formula (11) implies that if $f,g\in H^p_O$ and $|f(t)|=|g(t)|$ for a.a. $t\in\mathbb{T}$, then $f=cg$ for some constant c with absolute value 1. The product of two outer functions is again outer and Hölder's inequality guarantees that if $f\in H^p_O$ and $g\in H^q_O$, then $fg\in H^{pq/(p+q)}_O$.

For any set $\mathcal{S}$, we denote by $\mathcal{S}^{m\times n}$ the set of $m\times n$ matrices with entries from $\mathcal{S}$.

A matrix function $G\in H^2(\mathbb{D})^{m\times m}$ is called *outer*, and we write $G\in H^2(\mathbb{D})^{m\times m}_O$, if the determinant of G is outer, i.e., $\det G\in H^{2/m}_O$ (cf. [2]).

For any matrix $M\in\mathbb{C}^{m\times m}$, we use the standard notation M^T, $M^*:=\overline{M}^T$, $\mathrm{Cof}(M)$, and $\mathrm{Adj}(M):=\mathrm{Cof}(M)^T$ for the transpose, the Hermitian conjugate, the cofactor matrix and the adjugate. The same notation is used for the matrix functions, as well.

A matrix function $U\in L^\infty(\mathbb{T})^{m\times m}$ is called *unitary* if

$$U(t)U^*(t)=I_m\quad\text{a.e.},$$

where I_m stands for the $m\times m$ unit matrix.

3. Proof of Theorem 1

Since $F\in L^2(\mathbb{T})^{m\times m}$ and $\det F=f^+\in H^2_O$, we have $FF^*\in L^1(\mathbb{T})^{m\times m}$ and

$$\int\limits_{\mathbb{T}}\log\det F(t)F^*(t)\,dt=2\int\limits_{\mathbb{T}}\log|f^+(t)|\,dt>-\infty.$$

Therefore, by virtue of the matrix spectral factorization theorem,

$$F(t)F^*(t)=G_+(t)G_+^*(t),$$

where $G_+\in H^2(\mathbb{D})^{m\times m}_O$. Since $\det G_+\in H^{2/m}_O$ and $|\det G_+(t)|=|\det F(t)|$ for a. a. $t\in\mathbb{T}$, we have $\det G_+(z)=c\,(\det F)(z)=cf^+(z)$ for $z\in\mathbb{D}$, with $|c|=1$ and it can be assumed that $c=1$, i.e.,

$$\det G_+=f^+. \tag{12}$$

Let

$$U(t)=F^{-1}(t)G_+(t). \tag{13}$$

We have

$$UU^*=F^{-1}G_+G_+^*(F^{-1})^*=F^{-1}FF^*(F^*)^{-1}=I_m\ \text{ a.e. on } \mathbb{T},$$

which implies that U is a unitary matrix function, and therefore,

$$U\in L^\infty(\mathbb{T})^{m\times m}. \tag{14}$$

We also know that (6) holds because of equations (13) and (12).

Note that

$$F^{-1}=\begin{pmatrix}1&0&0&\cdots&0&0\\0&1&0&\cdots&0&0\\0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\cdots&1&0\\-\zeta_1/f^+&-\zeta_2/f^+&-\zeta_3/f^+&\cdots&-\zeta_{m-1}/f^+&1/f^+\end{pmatrix}. \tag{15}$$

Therefore, it follows from (13) that the entries in the first $m-1$ rows of U and G_+ coincide. Since we know that these entries belong to H^2 and also (14) holds, it follows from Smirnov's theorem that

$$u_{ij}\in H^\infty,\quad 1\le i\le m-1,\ \ 1\le j\le m.$$

For the entries of the last row of U, we have

$$\overline{u_{mj}}=\operatorname{cof}(u_{mj})\in H^\infty,$$

since $U^*=U^{-1}=\operatorname{Adj}(U)=\operatorname{Cof}(U)^T$. Hence, the structure of U has the form (5), and Theorem 1 is proved.

4. Proof of Theorem 2

Assume

$$F(t)U(t)=\Phi_+, \tag{16}$$

where F is the matrix function (4), U is the unitary matrix function (5) satisfying (6) and

$$\Phi_+\in H^2(\mathbb{D})^{m\times m}_O$$

(the determinant of Φ^+ is outer because $f^+\in H^2_O$ and (6) holds). Then the last equation in (8) follows immediately from (16). It also follows from (16) that

$$U^*(t)F^{-1}(t)=\Phi_+^{-1}(t)=\frac{1}{f^+}\operatorname{Adj}\Phi_+,$$

i.e.,

$$\begin{pmatrix} \overline{u_{11}^+} & \overline{u_{21}^+} & \cdots & \overline{u_{m-1,\,1}^+} & u_{m1}^+ \\ \overline{u_{12}^+} & \overline{u_{22}^+} & \cdots & \overline{u_{m-1,\,2}^+} & u_{m2}^+ \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{u_{1m}^+} & \overline{u_{2m}^+} & \cdots & \overline{u_{m-1,\,m}^+} & u_{mm}^+ \end{pmatrix} \begin{pmatrix} f^+ & 0 & \cdots & 0 & 0 \\ 0 & f^+ & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & f^+ & 0 \\ -\zeta_1 & -\zeta_2 & \cdots & -\zeta_{m-1} & 1 \end{pmatrix} = \operatorname{Adj} \Phi_+.$$

Then, we conclude that for each $j = 1, 2, \dots, m$,

$$\begin{cases} f^+ \overline{u_{1j}^+} - \zeta_1 \, u_{mj}^+ = \phi_{j1}^+ \\ f^+ \overline{u_{2j}^+} - \zeta_2 \, u_{mj}^+ = \phi_{j2}^+ \\ \vdots \\ f^+ \overline{u_{m-1,\,j}^+} - \zeta_{m-1} \, u_{mj}^+ = \phi_{j,m-1}^+, \end{cases} \tag{17}$$

where we know that each ϕ_{jk}^+ belongs to $H^{2/(m-1)}$ as they are the entries of $\operatorname{Adj} \Phi_+$. However, equations (17) suggest that $\phi_{jk}^+ \in L^2(\mathbb{T})$ and applying Smirnov's theorem, we can conclude that $\phi_{jk}^+ \in H^2$.

Thus Theorem 2 is proved.

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INVERTIBILITY OF FOURIER CONVOLUTION OPERATORS WITH PIECEWISE CONTINUOUS SYMBOLS ON BANACH FUNCTION SPACES

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Dedicated to Professor Roland Duduchava on the occasion of his 75th birthday

Abstract. We extend results on the invertibility of Fourier convolution operators with piecewise continuous symbols on the Lebesgue space $L^p(\mathbb{R})$, $p \in (1, \infty)$, obtained by Roland Duduchava in the late 1970s, to the setting of a separable Banach function space $X(\mathbb{R})$ such that the Hardy–Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. We specify our results in the case of rearrangement-invariant spaces with suitable Muckenhoupt weights.

1. Introduction

Let PC be the C^*-algebra of all bounded piecewise continuous functions on the one-point compactification of the real line $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. By definition, $a \in PC$ if and only if $a \in L^\infty(\mathbb{R})$ and the finite one-sided limits

$$a(x_0 - 0) := \lim_{x \to x_0 - 0} a(x), \quad a(x_0 + 0) := \lim_{x \to x_0 + 0} a(x)$$

exist for each $x_0 \in \dot{\mathbb{R}}$. The set of all discontinuities (i.e., jumps) of a function $a \in PC$ is at most countable (see, e.g., [5, Chap. II. Section 3, Theorem 3]).

We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz class of all infinitely differentiable and rapidly decaying functions (see, e.g., [16, Section 2.2.1]). Let $\mathcal{F}$ denote the Fourier transform defined on $\mathcal{S}(\mathbb{R})$ by

$$(\mathcal{F}f)(x) := \int_{\mathbb{R}} f(t) e^{itx}\, dt, \quad x \in \mathbb{R},$$

and let $\mathcal{F}^{-1}$ be the inverse of $\mathcal{F}$ defined on $\mathcal{S}(\mathbb{R})$ by

$$(\mathcal{F}^{-1}g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} g(x) e^{-itx}\, dx, \quad t \in \mathbb{R}.$$

It is well known that these operators extend uniquely to the space $L^2(\mathbb{R})$. As usual, we will use the symbols $\mathcal{F}$ and $\mathcal{F}^{-1}$ for the direct and inverse Fourier transform on $L^2(\mathbb{R})$. It is well known (see, e.g., [16, Theorem 2.5.10]) that the Fourier convolution operator

$$W^0(a) := \mathcal{F}^{-1} a \mathcal{F} \tag{1.1}$$

is bounded on the space $L^2(\mathbb{R})$ for every $a \in L^\infty(\mathbb{R})$. The function a is called the symbol of the operator $W^0(a)$.

Let $X(\mathbb{R})$ be a Banach function space and $X'(\mathbb{R})$ be its associate space. Their technical definitions are postponed to Section 2.1. The class of Banach function spaces is very large. It includes Lebesgue, Orlicz, Lorentz spaces, variable Lebesgue spaces and their weighted analogues (see, e.g., [1, 7]). Let $\mathcal{B}(X(\mathbb{R}))$ denote the Banach algebra of all bounded linear operators acting on $X(\mathbb{R})$.

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Recall that the (non-centered) Hardy–Littlewood maximal function $\mathcal{M}f$ of a function $f \in L^1_{\rm loc}(\mathbb{R})$ is defined by

$$(\mathcal{M}f)(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ of finite length containing x. The Hardy–Littlewood maximal operator $\mathcal{M}$ defined by the rule $f \mapsto \mathcal{M}f$ is a sublinear operator.

If $X(\mathbb{R})$ is separable, then $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$ (see, e.g., [12, Lemma 2.2]). A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $X(\mathbb{R})$ if the convolution operator $W^0(a)$ defined by (1.1) maps $L^2(\mathbb{R}) \cap X(\mathbb{R})$ into $X(\mathbb{R})$ and extends to a bounded linear operator on $X(\mathbb{R})$. The function a is called the symbol of the Fourier convolution operator $W^0(a)$. The set $\mathcal{M}_{X(\mathbb{R})}$ of all Fourier multipliers on $X(\mathbb{R})$ is a unital normed algebra under pointwise operations and the norm

$$\|a\|_{\mathcal{M}_{X(\mathbb{R})}} := \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))}.$$

If, in addition, the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$, or on its associate space $X'(\mathbb{R})$, then for all $a \in \mathcal{M}_{X(\mathbb{R})}$,

$$\|a\|_{L^\infty(\mathbb{R})} \le \|a\|_{\mathcal{M}_{X(\mathbb{R})}}. \tag{1.2}$$

The constant 1 on the right-hand side of (1.2) is best possible (see [19, Corollary 4.2] and [20, Theorem 2.3]). Once (1.2) is available, one can show that $\mathcal{M}_{X(\mathbb{R})}$ is a Banach algebra (see [20, Corollary 2.4]).

Suppose that $a : \mathbb{R} \to \mathbb{C}$ is a function of the finite total variation $V(a)$ given by

$$V(a) := \sup \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})|,$$

where the supremum is taken over all partitions of $\mathbb{R}$ of the form

$$-\infty < x_0 < x_1 < \dots < x_n < +\infty$$

with $n \in \mathbb{N}$. The set $V(\mathbb{R})$ of all functions of finite total variation on $\mathbb{R}$ with the norm

$$\|a\|_{V(\mathbb{R})} := \|a\|_{L^\infty(\mathbb{R})} + V(a)$$

is a unital Banach algebra. By [13, Theorem 3.27], $V(\mathbb{R}) \subset PC$.

Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. It follows from [17, Theorem 4.3] that if $a \in V(\mathbb{R})$, then the convolution operator $W^0(a)$ is bounded on the space $X(\mathbb{R})$, and

$$\|W^0(a)\|_{\mathcal{B}(X(\mathbb{R}))} \le c_X \|a\|_{V(\mathbb{R})} \tag{1.3}$$

where c_X is a positive constant depending only on $X(\mathbb{R})$.

For the Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, inequality (1.3) is usually called Stechkin's inequality. Its proofs can be found, e.g., in [3, Theorem 17.1], [9, Theorem 2.11], [10, Theorem 6.2.5].

For a subset S of a Banach space E, let $\operatorname{clos}_E(S)$ denote the closure of S with respect to the norm of E. Let $P\mathbb{C}^0$ denote the set of all piecewise constant functions with finitely many jumps. It is clear that $P\mathbb{C}^0 \subset V(\mathbb{R}) \subset PC$. It follows from [9, Lemma 2.10] that $PC = \operatorname{clos}_{L^\infty(\mathbb{R})}\left(P\mathbb{C}^0\right)$. Hence

$$PC = \operatorname{clos}_{L^\infty(\mathbb{R})}\left(P\mathbb{C}^0\right) = \operatorname{clos}_{L^\infty(\mathbb{R})}\left(V(\mathbb{R})\right). \tag{1.4}$$

For a separable Banach function space $X(\mathbb{R})$ such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$, consider the following Banach algebras of Fourier multipliers:

$$PC^0_{X(\mathbb{R})} := \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R})}}\left(P\mathbb{C}^0\right), \quad PC_{X(\mathbb{R})} := \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R})}}\left(V(\mathbb{R})\right).$$

It follows from (1.2) and (1.4) that

$$PC^0_{X(\mathbb{R})} \subset PC_{X(\mathbb{R})} \subset PC.$$

Therefore, it is natural to refer to $PC^0_{X(\mathbb{R})}$ and $PC_{X(\mathbb{R})}$ as algebras of piecewise continuous Fourier multipliers. For $1 < p < \infty$, the algebras $PC^0_{L^p(\mathbb{R})}$ and $PC_{L^p(\mathbb{R})}$ were introduced by Duduchava (see [9, Chap. 1, Section 2]).

The aim of this paper is to study the invertibility of convolution operators $W^0(a)$ with piecewise continuous symbols $a \in PC^0_{X(\mathbb{R})}$ on the Banach function spaces. Our main result is the following

Theorem 1.1. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Suppose that $a \in PC^0_{X(\mathbb{R})}$. For the operator $W^0(a)$ to be invertible on the space $X(\mathbb{R})$, it is necessary and sufficient that*

$$\operatorname{ess\,inf}_{t\in\mathbb{R}} |a(t)| > 0.$$

For the Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, the above result was obtained by Roland Duduchava in [9, Theorem 2.18].

Question 1.2. Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Is it true that $PC^0_{X(\mathbb{R})} = PC_{X(\mathbb{R})}$?

Note that for the Lebesgue spaces $L^p(\mathbb{R})$, the positive answer follows from [9, Remark 2.12]:

$$PC^0_{L^p(\mathbb{R})} = PC_{L^p(\mathbb{R})}, \quad 1 < p < \infty. \tag{1.5}$$

Let $\mathcal{P}(\mathbb{R})$ denote the set of all measurable a.e. finite functions $p(\cdot) : \mathbb{R} \to [1, \infty]$ such that

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}} p(x) =: p_+ < \infty.$$

By $L^{p(\cdot)}(\mathbb{R})$ we denote the set of all complex-valued measurable functions f on $\mathbb{R}$ such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a separable and reflexive Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R})} := \inf \left\{ \lambda > 0 : I_{p(\cdot)}(f/\lambda) \le 1 \right\},$$

and its associate space is isomorphic to the space $L^{p'(\cdot)}(\mathbb{R})$, where

$$1/p(x) + 1/p'(x) = 1 \quad \text{for a.e.} \quad x \in \mathbb{R}$$

(see, e.g., [7, Chap. 2] or [8, Chap. 3]). It is easy to see that if p is constant, then $L^{p(\cdot)}(\mathbb{R})$ is nothing but the standard Lebesgue space $L^p(\mathbb{R})$. The space $L^{p(\cdot)}(\mathbb{R})$ is referred to as a variable Lebesgue space. By [8, Theorem 5.7.2], the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R})$ if and only if it is bounded on $L^{p'(\cdot)}(\mathbb{R})$. As it is shown in [18, Theorem 4.2], in this case

$$PC^0_{L^{p(\cdot)}(\mathbb{R})} = PC_{L^{p(\cdot)}(\mathbb{R})}.$$

The proof of this equality is based on an analogue of the Riesz-Thorin interpolation theorem for variable Lebesgue spaces. In Section 3, we show that the answer to Question 1.2 is positive also for rearrangement-invariant Banach function spaces with suitable Muckenhoupt weights. Our proof is based on the Boyd interpolation theorem [6].

For general Banach function spaces, interpolation tools are not available. Hence one cannot prevent that the answer to Question 1.2 might be negative. In this situation it would be interesting to answer the following.

Question 1.3. Does Theorem 1.1 remain true for the algebra $PC_{X(\mathbb{R})}$ in the place of $PC^0_{X(\mathbb{R})}$?

The paper is organized as follows. Section 2 contains definitions and properties of a Banach function space and its associate space (see, e.g., [23] and [1, Chap. 1]), of a rearrangement-invariant Banach function space (see, e.g., [1, Chap. 3]) and its Boyd indices [6], and of a weighted rearrangement-invariant Banach function space with a suitable Muckenhoupt weight (see, e.g., [2, Chap. 2]). In

Section 3, we first prove that the answer to Question 1.2 is positive for the Lebesgue spaces $L^p(\mathbb{R}, w)$, $1 < p < \infty$, with Muckenhoupt weights $w \in A_p(\mathbb{R})$ using the stability of Muckenhoupt weights and the Stein-Weiss interpolation theorem. Further, we extend this result to the case of a weighted Banach function space $X(\mathbb{R}, w)$ built upon a separable rearrangement-invariant space $X(\mathbb{R})$ with the Boyd indices $\alpha_X, \beta_X \in (0,1)$ and a suitable Muckenhoupt weight $w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. In Section 4, we recall the definition of M-equivalence of elements of a Banach algebra and formulate the Gohberg-Krupnik local principle [14, 15]. We apply it two times. First, we show that the algebra $PC^0_{X(\mathbb{R})}$ is inverse closed in the algebra $L^\infty(\mathbb{R})$. Finally, we prove Theorem 1.1 employing the local principle.

We would like to dedicate this work to Roland Duduchava, whose ideas penetrate the entire paper. This work was started as the Undergraduate Research Opportunity Project of the third author at the NOVA University of Lisbon in January-February of 2020 under the supervision of the second author.

2. Preliminaries

2.1. **Banach function spaces.** Let $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{S} \in \{\mathbb{R}_+, \mathbb{R}\}$. The set of all Lebesgue measurable complex-valued functions on $\mathbb{S}$ is denoted by $\mathfrak{M}(\mathbb{S})$. Let $\mathfrak{M}^+(\mathbb{S})$ be the subset of functions in $\mathfrak{M}(\mathbb{S})$ whose values lie in $[0, \infty]$. The Lebesgue measure of a measurable set $E \subset \mathbb{S}$ is denoted by $|E|$ and its characteristic function is denoted by χ_E. Following [23, p. 3] and [1, Chap. 1, Definition 1.1], a mapping $\rho : \mathfrak{M}^+(\mathbb{S}) \to [0, \infty]$ is called a Banach function norm if, for all functions f, g, f_n $(n \in \mathbb{N})$ in $\mathfrak{M}^+(\mathbb{S})$, for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{S}$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\quad \rho(af) = a\rho(f), \quad \rho(f + g) \leq \rho(f) + \rho(g)$,
(A2) $0 \leq g \leq f$ a.e. $\Rightarrow$ $\rho(g) \leq \rho(f)$ (the lattice property),
(A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow$ $\rho(f_n) \uparrow \rho(f)$ (the Fatou property),
(A4) E is bounded $\Rightarrow \rho(\chi_E) < \infty$,
(A5) E is bounded $\Rightarrow \int_E f(x)\, dx \leq C_E \rho(f)$

with $C_E \in (0, \infty)$ which may depend on E and ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set $X(\mathbb{S})$ of all functions $f \in \mathfrak{M}(\mathbb{S})$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\mathbb{S})$, the norm of f is defined by

$$\|f\|_{X(\mathbb{S})} := \rho(|f|).$$

Under the natural linear space operations and under this norm, the set $X(\mathbb{S})$ becomes a Banach space (see [23, Chap. 1, §1, Theorem 1] or [1, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathfrak{M}^+(\mathbb{S})$ by

$$\rho'(g) := \sup \left\{ \int_{\mathbb{S}} f(x) g(x)\, dx \ : \ f \in \mathfrak{M}^+(\mathbb{S}),\ \rho(f) \leq 1 \right\}, \quad g \in \mathfrak{M}^+(\mathbb{S}).$$

It is a Banach function norm itself [23, Chap. 1, §1] or [1, Chap. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{S})$ determined by the Banach function norm ρ' is called the associate space (Köthe dual) of $X(\mathbb{S})$. The associate space $X'(\mathbb{S})$ is naturally identified with a subspace of the (Banach) dual space $[X(\mathbb{S})]^*$.

Remark 2.1. We note that our definition of a Banach function space is slightly different from that found in [1, Chap. 1, Definition 1.1]. In particular, in Axioms (A4) and (A5) we assume that the set E is a bounded set, whereas it is sometimes assumed that E merely satisfies $|E| < \infty$. We do this so that the weighted Lebesgue spaces with Muckenhoupt weights satisfy Axioms (A4) and (A5). Moreover, it is well known that all main elements of the general theory of Banach function spaces work with (A4) and (A5) stated for bounded sets [23] (see also the discussion at the beginning of Chapter 1 on page 2 of [1]). Unfortunately, we overlooked that the definition of a Banach function space in our previous works [11, 12, 17, 19, 21] had to be changed by replacing in Axioms (A4) and (A5) the requirement of $|E| < \infty$ by the requirement that E is a bounded set to include weighted Lebesgue spaces in our considerations. However, the results proved in the above papers remain true.

2.2. **Rearrangement-invariant Banach function spaces.** Suppose that $\mathbb{S} \in \{\mathbb{R}, \mathbb{R}_+\}$. Let $\mathfrak{M}_0(\mathbb{S})$ and $\mathfrak{M}_0^+(\mathbb{S})$ be the classes of a.e. finite functions in $\mathfrak{M}(\mathbb{S})$ and $\mathfrak{M}^+(\mathbb{S})$, respectively. The distribution function μ_f of $f \in \mathfrak{M}_0(\mathbb{S})$ is given by

$$\mu_f(\lambda) := \big|\{x \in \mathbb{S} : |f(x)| > \lambda\}\big|, \quad \lambda \geq 0.$$

Two functions $f, g \in \mathfrak{M}_0(\mathbb{S})$ are said to be equimeasurable if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$. The non-increasing rearrangement of $f \in \mathfrak{M}_0(\mathbb{S})$ is the function defined by

$$f^*(t) := \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \quad t \geq 0.$$

We here use the standard convention that $\inf \emptyset = +\infty$.

A Banach function norm $\rho : \mathfrak{M}^+(\mathbb{S}) \to [0, \infty]$ is called rearrangement-invariant if for every pair of equimeasurable functions $f, g \in \mathfrak{M}_0^+(\mathbb{S})$ the equality $\rho(f) = \rho(g)$ holds. In that case, the Banach function space $X(\mathbb{S})$ generated by ρ is said to be a rearrangement-invariant Banach function space (or simply, rearrangement-invariant space). The Lebesgue, Orlicz, and Lorentz spaces are classical examples of rearrangement-invariant Banach function spaces (see, e.g., [1] and references therein). By [1, Chap. 2, Proposition 4.2], if a Banach function space $X(\mathbb{S})$ is rearrangement-invariant, then its associate space $X'(\mathbb{S})$ is rearrangement-invariant, too.

2.3. **Boyd indices.** Suppose $X(\mathbb{R})$ is a rearrangement-invariant Banach function space generated by a rearrangement-invariant Banach function norm ρ. In this case, the Luxemburg representation theorem [1, Chap. 2, Theorem 4.10] provides a unique rearrangement-invariant Banach function norm $\overline{\rho}$ over the half-line $\mathbb{R}_+$ equipped with the Lebesgue measure, defined by

$$\overline{\rho}(h) := \sup\left\{ \int_{\mathbb{R}_+} g^*(t) h^*(t)\, dt : \ \rho'(g) \leq 1 \right\},$$

and such that $\rho(f) = \overline{\rho}(f^*)$ for all $f \in \mathfrak{M}_0^+(\mathbb{R})$. The rearrangement-invariant Banach function space generated by $\overline{\rho}$ is denoted by $\overline{X}(\mathbb{R}_+)$.

For each $t > 0$, let E_t denote the dilation operator defined on $\mathfrak{M}(\mathbb{R}_+)$ by

$$(E_t f)(s) = f(st), \quad 0 < s < \infty.$$

With $X(\mathbb{R})$ and $\overline{X}(\mathbb{R}_+)$ as above, let $h_X(t)$ denote the operator norm of $E_{1/t}$ as an operator on $\overline{X}(\mathbb{R}_+)$. By [1, Chap. 3, Proposition 5.11], for each $t > 0$, the operator E_t is bounded on $\overline{X}(\mathbb{R}_+)$ and the function h_X is increasing and submultiplicative on $(0, \infty)$. The Boyd indices of $X(\mathbb{R})$ are the numbers α_X and β_X defined by

$$\alpha_X := \sup_{t \in (0,1)} \frac{\log h_X(t)}{\log t}, \quad \beta_X := \inf_{t \in (1,\infty)} \frac{\log h_X(t)}{\log t}.$$

By [1, Chap. 3, Proposition 5.13], $0 \leq \alpha_X \leq \beta_X \leq 1$. The Boyd indices are said to be nontrivial if $\alpha_X, \beta_X \in (0, 1)$. The Boyd indices of the Lebesgue space $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, are both equal to $1/p$. Note that the Boyd indices of a rearrangement-invariant space may be different [1, Chap. 3, Exercises 6, 13].

The next theorem follows from the Boyd interpolation theorem [6, Theorem 1] for quasi-linear operators of weak types (p, p) and (q, q). Its proof can also be found in [1, Chap. 3, Theorem 5.16] and [22, Theorem 2.b.11].

Theorem 2.2. *Let $1 \leq q < p \leq \infty$ and $X(\mathbb{R})$ be a rearrangement-invariant Banach function space with the Boyd indices α_X, β_X satisfying*

$$1/p < \alpha_X \leq \beta_X < 1/q.$$

Then there exists a constant $C_{p,q} \in (0, \infty)$ with the following property. If a linear operator $T : \mathfrak{M}(\mathbb{R}) \to \mathfrak{M}(\mathbb{R})$ is bounded on the Lebesgue spaces $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, then it is also bounded on the rearrangement-invariant Banach function space $X(\mathbb{R})$ and

$$\|T\|_{\mathcal{B}(X(\mathbb{R}))} \leq C_{p,q} \max\big\{ \|T\|_{\mathcal{B}(L^p(\mathbb{R}))}, \|T\|_{\mathcal{B}(L^q(\mathbb{R}))} \big\}. \tag{2.1}$$

Notice that estimate (2.1) is not stated explicitly in [1, 6, 22]. However, it can be extracted from the proof of the Boyd interpolation theorem.

2.4. **Lebesgue spaces with Muckenhoupt weights.** A measurable function $w : \mathbb{R} \to [0,\infty]$ is called a weight if the set $w^{-1}(\{0,\infty\})$ has measure zero. For $1 < p < \infty$, the Muckenhoupt class $A_p(\mathbb{R})$ is defined as the class of all weights $w : \mathbb{R} \to [0,\infty]$ such that $w \in L^p_{\mathrm{loc}}(\mathbb{R})$, $w^{-1} \in L^{p'}_{\mathrm{loc}}(\mathbb{R})$ and

$$\sup_I \left(\frac{1}{|I|}\int_I w^p(x)\,dx\right)^{1/p}\left(\frac{1}{|I|}\int_I w^{-p'}(x)\,dx\right)^{1/p'} < \infty,$$

where $1/p + 1/p' = 1$ and the supremum is taken over all intervals $I \subset \mathbb{R}$ of finite length $|I|$. Since $w \in L^p_{\mathrm{loc}}(\mathbb{R})$ and $w^{-1} \in L^{p'}_{\mathrm{loc}}(\mathbb{R})$, the weighted Lebesgue space

$$L^p(\mathbb{R}, w) := \{f \in \mathfrak{M}(\mathbb{R}) : fw \in L^p(\mathbb{R})\}$$

is a separable Banach function space (see, e.g., [21, Lemma 2.4]) with the norm

$$\|f\|_{L^p(\mathbb{R},w)} := \left(\int_{\mathbb{R}} |f(x)|^p w^p(x)\,dx\right)^{1/p}.$$

2.5. **Rearrangement-invariant Banach function spaces with suitable Muckenhupt weights.** Let $X(\mathbb{R})$ be a Banach function space generated by a Banach function norm ρ. We say that $f \in X_{\mathrm{loc}}(\mathbb{R})$ if $f\chi_E \in X(\mathbb{R})$ for every bounded measurable set $E \subset \mathbb{R}$.

Lemma 2.3 ([21, Lemma 2.4])**.** *Let $X(\mathbb{R})$ be a Banach function space generated by a Banach function norm ρ, let $X'(\mathbb{R})$ be its associate space, and let $w : \mathbb{R} \to [0,\infty]$ be a weight. Suppose that $w \in X_{\mathrm{loc}}(\mathbb{R})$ and $1/w \in X'_{\mathrm{loc}}(\mathbb{R})$. Then*

$$\rho_w(f) := \rho(fw), \quad f \in \mathfrak{M}^+(\mathbb{R}),$$

is a Banach function norm and

$$X(\mathbb{R}, w) := \{f \in \mathfrak{M}(\mathbb{R}) : fw \in X(\mathbb{R})\}$$

is a Banach function space generated by the Banach function norm ρ_w. The space $X'(\mathbb{R}, w^{-1})$ is the associate space of $X(\mathbb{R}, w)$.

Lemma 2.4 ([11, Lemma 2.3])**.** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space and $X'(\mathbb{R})$ be its associate space. Suppose that the Boyd indices of $X(\mathbb{R})$ satisfy $0 < \alpha_X, \beta_X < 1$ and let*

$$w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R}). \tag{2.2}$$

Then

(a) *$w \in X_{\mathrm{loc}}(\mathbb{R})$ and $1/w \in X'_{\mathrm{loc}}(\mathbb{R})$;*
(b) *the Banach function space $X(\mathbb{R}, w)$ is separable;*
(c) *the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the Banach function space $X(\mathbb{R}, w)$ and on its associate space $X'(\mathbb{R}, w^{-1})$.*

We say that a weight w is suitable for a rearrangement-invariant Banach function space $X(\mathbb{R})$ with the Boyd indices α_X, β_X satisfying $\alpha_X, \beta_X \in (0, 1)$ if (2.2) is fulfilled.

3. $PC_{X(\mathbb{R},w)}$ as the Closure of the Set of Piecewise Constant Functions

3.1. **The case of Lebesgue spaces with Muckenhoupt weights.** Let us start with an important lemma due to Duduchava.

Lemma 3.1 ([9, Lemma 2.10])**.** *For every function $a \in V(\mathbb{R})$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $P\mathbb{C}^0$ such that*

$$\lim_{n\to\infty} \|a_n - a\|_{L^\infty(\mathbb{R})} = 0, \quad \sup_{n\in\mathbb{N}} V(a_n) \le V(a).$$

We now extend equality (1.5) to the case of Lebesgue spaces with Muckenhoupt weights.

Theorem 3.2. *Let $1 < p < \infty$ and $w \in A_p(\mathbb{R})$. Then*

$$PC^0_{L^p(\mathbb{R},w)} = PC_{L^p(\mathbb{R},w)}. \tag{3.1}$$

Proof. The proof is analogous to that of [18, Theorem 4.2] (see also [11, Lemma 3.1]). First of all, we observe that if $w \in A_p(\mathbb{R})$, then the Stechkin-type inequality (1.3) is fulfilled in $L^p(\mathbb{R},w)$ (see [3, Theorem 17.1] and also Lemma 2.4).

Since $P\mathbb{C}^0 \subset V(\mathbb{R})$, we, obviously, have

$$PC^0_{L^p(\mathbb{R},w)} \subset PC_{L^p(\mathbb{R},w)}. \tag{3.2}$$

If $w \in A_p(\mathbb{R})$, then $w^{1+\delta_2} \in A_{p(1+\delta_1)}(\mathbb{R})$ whenever $|\delta_1|$ and $|\delta_2|$ are sufficiently small (see, e.g., [2, Theorem 2.31]). If $p \geq 2$, then one can find sufficiently small $\delta_1, \delta_2 > 0$ and a small number $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p(1+\delta_1)}, \quad w = 1^{1-\theta} w^{(1+\delta_2)\theta}, \quad w^{1+\delta_2} \in A_{p(1+\delta_1)}(\mathbb{R}). \tag{3.3}$$

If $1 < p < 2$, then one can find a sufficiently small number $\delta_2 > 0$, a number $\delta_1 < 0$ with sufficiently small $|\delta_1|$, and a number $\theta \in (0,1)$ such that all conditions in (3.3) are fulfilled. Let us use the following abbreviations:

$$\mathcal{M}_p := \mathcal{M}_{L^p(\mathbb{R},w)}, \qquad \mathcal{M}_{p_\theta} := \mathcal{M}_{L^{p(1+\delta_1)}(\mathbb{R},w^{1+\delta_2})},$$

$$\mathcal{B}_p := \mathcal{B}(L^p(\mathbb{R},w)), \quad \mathcal{B}_{p_\theta} := \mathcal{B}(L^{p(1+\delta_1)}(\mathbb{R},w^{1+\delta_2})).$$

Let $a \in PC_{L^p(\mathbb{R},w)}$ and $\varepsilon > 0$. Then there exists $b \in V(\mathbb{R})$ such that

$$\|a-b\|_{\mathcal{M}_p} < \varepsilon/2. \tag{3.4}$$

By Lemma 3.1, there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ in $P\mathbb{C}^0$ such that

$$\lim_{n\to\infty} \|b_n - b\|_{L^\infty(\mathbb{R})} = 0, \quad \sup_{n\in\mathbb{N}} V(b_n) \leq V(b). \tag{3.5}$$

Then there exists $N \in \mathbb{N}$ such that

$$\sup_{n\geq N} \|b_n\|_{V(\mathbb{R})} \leq 2\|b\|_{V(\mathbb{R})}. \tag{3.6}$$

It follows from the Stechkin-type inequality (1.3) and inequality (3.6) that for all $n \geq N$,

$$\|b_n - b\|_{\mathcal{M}_{p_\theta}} \leq \|b_n\|_{\mathcal{M}_{p_\theta}} + \|b\|_{\mathcal{M}_{p_\theta}} \leq 3c_\theta \|b\|_{V(\mathbb{R})}, \tag{3.7}$$

where $c_\theta := c_{L^{p(1+\delta_1)}(\mathbb{R},w^{1+\delta_2})}$.

Taking into account (3.3), we obtain from the Stein-Weiss interpolation theorem (see, e.g., [1, Chap. 3, Theorem 3.6]) that for all $n \in \mathbb{N}$,

$$\begin{aligned} \|b_n - b\|_{\mathcal{M}_p} &= \|W^0(b_n - b)\|_{\mathcal{B}_p} \\ &\leq \|W^0(b_n - b)\|^{1-\theta}_{\mathcal{B}(L^2(\mathbb{R}))} \|W^0(b_n - b)\|^{\theta}_{\mathcal{B}_{p_\theta}} \\ &= \|b_n - b\|^{1-\theta}_{L^\infty(\mathbb{R})} \|b_n - b\|^{\theta}_{\mathcal{M}_{p_\theta}}. \end{aligned} \tag{3.8}$$

Combining (3.5), (3.7) and (3.8), we see that there exists $n_0 \in \mathbb{N}$ such that

$$\|b_{n_0} - b\|_{\mathcal{M}_p} < \varepsilon/2. \tag{3.9}$$

Inequalities (3.4) and (3.9) imply that for every $\varepsilon > 0$, there exists $c = b_{n_0} \in P\mathbb{C}^0$ such that $\|a - c\|_{\mathcal{M}_p} < \varepsilon$, whence $a \in \operatorname{clos}_{\mathcal{M}_p}\left(P\mathbb{C}^0\right) = PC^0_{L^p(\mathbb{R},w)}$. Then

$$PC_{L^p(\mathbb{R},w)} \subset PC^0_{L^p(\mathbb{R},w)}. \tag{3.10}$$

Gathering embeddings (3.2) and (3.10), we arrive at equality (3.1). ∎

3.2. **The case of separable rearrangement-invariant spaces with suitable Muckenhoupt weights.** We are now in a position to prove the main result of this section and to answer Question 1.2 for separable rearrangement-invariant spaces Banach function spaces with suitable Muckenhoupt weights.

Theorem 3.3. *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X,\ \beta_X < 1$. Suppose that a weight w belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Then*

$$PC^0_{X(\mathbb{R},w)} = PC_{X(\mathbb{R},w)}. \tag{3.11}$$

Proof. Since $P\mathbb{C}^0 \subset V(\mathbb{R})$, we, obviously, have

$$PC^0_{X(\mathbb{R},w)} \subset PC_{X(\mathbb{R},w)}. \tag{3.12}$$

To prove the reverse inclusion, let $a \in PC_{X(\mathbb{R},w)}$ and $\varepsilon > 0$. Then there exists $b \in V(\mathbb{R})$ such that

$$\|a - b\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon/2. \tag{3.13}$$

Since $\alpha_X, \beta_X \in (0,1)$ and $w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$, it follows from [2, Theorem 2.31] that there exist p and q such that

$$1 < q < 1/\beta_X \le 1/\alpha_X < p < \infty, \quad w \in A_p(\mathbb{R}) \cap A_q(\mathbb{R}). \tag{3.14}$$

Let $C_{p,q}$ be the constant from estimate (2.1). As in the proof of inequality (3.9) (see the proof of the previous theorem), it can be shown that there exists $b_{n_0} \in P\mathbb{C}^0$ for some $n_0 \in \mathbb{N}$ such that

$$\|b - b_{n_0}\|_{\mathcal{M}_{L^p(\mathbb{R},w)}} < \frac{\varepsilon}{2C_{p,q}}, \quad \|b - b_{n_0}\|_{\mathcal{M}_{L^q(\mathbb{R},w)}} < \frac{\varepsilon}{2C_{p,q}}. \tag{3.15}$$

It follows from (3.14), (3.15) and Theorem 2.2 that

$$\begin{aligned}
\|b - b_{n_0}\|_{\mathcal{M}_{X(\mathbb{R},w)}} &= \|W^0(b - b_{n_0})\|_{\mathcal{B}(X(\mathbb{R},w))} \\
&= \|wW^0(b - b_{n_0})w^{-1}I\|_{\mathcal{B}(X(\mathbb{R}))} \\
&\le C_{p,q} \max\left\{\|wW^0(b - b_{n_0})w^{-1}I\|_{\mathcal{B}(L^p(\mathbb{R}))}, \|wW^0(b - b_{n_0})w^{-1}I\|_{\mathcal{B}(L^q(\mathbb{R}))}\right\} \\
&= C_{p,q} \max\left\{\|W^0(b - b_{n_0})\|_{\mathcal{B}(L^p(\mathbb{R},w))}, \|W^0(b - b_{n_0})\|_{\mathcal{B}(L^q(\mathbb{R},w))}\right\} \\
&= C_{p,q} \max\left\{\|b - b_{n_0}\|_{\mathcal{M}_{L^p(\mathbb{R},w)}}, \|b - b_{n_0}\|_{\mathcal{M}_{L^q(\mathbb{R},w)}}\right\} < \varepsilon/2.
\end{aligned} \tag{3.16}$$

Inequalities (3.13) and (3.16) imply that for every $\varepsilon > 0$, there exists $c = b_{n_0} \in P\mathbb{C}^0$ such that $\|a - c\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon$, whence $a \in \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}\left(P\mathbb{C}^0\right) = PC^0_{X(\mathbb{R},w)}$. Then

$$PC_{X(\mathbb{R},w)} \subset PC^0_{X(\mathbb{R},w)}. \tag{3.17}$$

The desired equality (3.11) follows now from the embeddings (3.12) and (3.17). ☐

Combining Theorem 3.3 with Theorem 1.1, we arrive at the following

Corollary 3.4. *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X,\ \beta_X < 1$. Suppose that a weight w belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Suppose that $a \in PC_{X(\mathbb{R},w)}$. For the operator $W^0(a)$ to be invertible on the Banach function space $X(\mathbb{R}, w)$, it is necessary and sufficient that*

$$\operatorname{ess\,inf}_{t\in\mathbb{R}} |a(t)| > 0.$$

4. The Gohberg-Krupnik Local Principle in Action

4.1. **M-equivalence.** Let $\mathcal{A}$ be a unital Banach algebra. A subset $M \subset \mathcal{A}$ is called a localizing class if $0 \notin M$ and for any $f_1, f_2 \in M$ there exists a third element $f \in M$ such that $f_j f = f f_j = f$ for $j = 1, 2$.

Two elements $a, b \in \mathcal{A}$ are said to be M-equivalent from the left (resp., from the right) if

$$\inf_{f\in M} \|(a - b)f\| = 0 \quad \Big(\text{resp.} \quad \inf_{f\in M} \|f(a - b)\| = 0\Big).$$

If a and b are both M-equivalent from the left and from the right, then they are said to be M-equivalent. In this case we write $a \overset{M}{\sim} b$.

Because of the completeness, let us give a simple proof of the continuity of M-equivalence, which was mentioned implicitly in [9, p. 21].

Proposition 4.1. *Let $\mathcal{A}$ be a unital Banach algebra, M a localizing class of $\mathcal{A}$ and $\{x_n\}_{n\in\mathbb{N}}$, and $\{y_n\}_{n\in\mathbb{N}}$ be the sequences of elements of $\mathcal{A}$, convergent to x and y, respectively. Suppose that*

$$\sup_{a\in M} \|a\| < \infty.$$

If $x_n \overset{M}{\sim} y_n$ for all $n \in \mathbb{N}$, then $x \overset{M}{\sim} y$.

Proof. Fix $\varepsilon > 0$. Let $a \in M$ and $L := \sup_{a\in M} \|a\|$. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|(x-y)a\| &= \|(x-x_n)a + (y_n - y)a + (x_n - y_n)a\| \\ &\le \|(x-x_n)a\| + \|(y_n-y)a\| + \|(x_n-y_n)a\| \\ &\le \|x-x_n\|\,\|a\| + \|y_n-y\|\,\|a\| + \|(x_n-y_n)a\| \\ &\le L(\|x-x_n\| + \|y-y_n\|) + \|(x_n-y_n)a\|. \end{aligned} \tag{4.1}$$

Since $x_n \to x$ and $y_n \to y$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\|x - x_n\| < \frac{\varepsilon}{4L} \quad \text{and} \quad \|y - y_n\| < \frac{\varepsilon}{4L}. \tag{4.2}$$

On the other hand, $x_n \overset{M}{\sim} y_n$ for every $n \in \mathbb{N}$. In particular, $x_{N+1} \overset{M}{\sim} y_{N+1}$. Therefore, by the definition of M-equivalence from the left, there is an $a' \in M$ such that

$$\|(x_{N+1} - y_{N+1})a'\| < \frac{\varepsilon}{2}. \tag{4.3}$$

Combining inequalities (4.1)–(4.3), we get

$$\forall\varepsilon > 0 \quad \exists a' \in M \,:\, \|(x-y)a'\| < \varepsilon,$$

i.e., x and y are M-equivalent from the left. Similarly, we prove that x and y are M-equivalent from the right. Thus x and y are M-equivalent. □

4.2. **The local principle.** Let $\mathcal{A}$ be a unital Banach algebra and M be a localizing class in $\mathcal{A}$. An element $a \in \mathcal{A}$ is called M-invertible from the left (resp., from the right) if there are the elements $b \in \mathcal{A}$ and $f \in M$ such that $baf = f$ (resp., $fab = f$). Finally, $a \in \mathcal{A}$ is said to be M-invertible if it is M-invertible from the left and from the right.

Let T be an index set. A system $\{M_\tau\}_{\tau\in T}$ of localizing classes is said to be covering if from each choice $\{f_\tau\}_{\tau\in T}$ with $f_\tau \in M_\tau$ there can be selected a finite number of elements $f_{\tau_1}, \dots, f_{\tau_m}$ whose sum is invertible in $\mathcal{A}$.

Let $M := \cup_{\tau\in T} M_\tau$ and let $\operatorname{Com} M$ stand for the commutant of M, that is, the set of all $a \in \mathcal{A}$ which commute with every element in M.

The following theorem was obtained by Gohberg and Krupnik [15]. Its proof can be found in several books (see, e.g., [4, Theorem 1.32], [14, Section 5.1], [24, Theorem 2.4.5]).

Theorem 4.2 (Gohberg, Krupnik)**.** *Let $\mathcal{A}$ be a unital Banach algebra, let T be an index set, let $\{M_\tau\}_{\tau\in T}$ be a covering system of localizing classes, and let $a \in \operatorname{Com} M$. Suppose that, for each $\tau \in T$, the element a is M_τ-equivalent from the left (resp., from the right) to $a_\tau \in \mathcal{A}$. Then the element a is left-invertible (resp., right-invertible) in $\mathcal{A}$ if and only if a_τ is M_τ-invertible from the left (resp., from the right) for all $\tau \in T$.*

4.3. **The algebra $PC^0_{X(\mathbb{R})}$ is inverse closed in the algebra $L^\infty(\mathbb{R})$.** The first step in the proof of Theorem 1.1 consists in establishing the inverse closedness of the Banach algebra of piecewise continuous Fourier multipliers $PC^0_{X(\mathbb{R})}$ in the C^*-algebra $L^\infty(\mathbb{R})$. Although the proof of the following lemma is similar to that of [9, Lemma 2.17], because of the completeness of presentation, we give it here.

Lemma 4.3. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If $a \in PC^0_{X(\mathbb{R})}$ and*

$$\operatorname{ess\,inf}_{t\in\mathbb{R}} |a(t)| > 0, \tag{4.4}$$

then $a^{-1} \in PC^0_{X(\mathbb{R})}$.

Proof. For each $x \in \dot{\mathbb{R}}$, consider the sets of characteristic functions of intervals of $\dot{\mathbb{R}}$ given by

$$\mathrm{M}_x^- := \left\{\chi_{[c,x]} : c \in \mathbb{R}, c < x\right\}, \quad \mathrm{M}_x^+ := \left\{\chi_{[x,d]} : d \in \mathbb{R}, x < d\right\}, \quad x \in \mathbb{R}, \tag{4.5}$$

and

$$\mathrm{M}_\infty^+ := \left\{\chi_{\{\infty\}\cup(-\infty,d]} : d \in \mathbb{R}\right\}, \quad \mathrm{M}_\infty^- := \left\{\chi_{[c,+\infty)\cup\{\infty\}} : c \in \mathbb{R}\right\}, \quad x = \infty. \tag{4.6}$$

We claim that $\left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x\in\dot{\mathbb{R}}}$ constitutes a covering system of localizing classes of $PC^0_{X(\mathbb{R})}$ (here the index set T coincides with the union of two copies of $\dot{\mathbb{R}}$, one of which corresponds to the left neighborhoods and the other corresponds to the right neighborhoods of $x \in \dot{\mathbb{R}}$). First note that every element of $\mathrm{M}_x^\pm$ is a characteristic function that, obviously, belongs to $P\mathbb{C}^0$. Therefore $\mathrm{M}_x^\pm \subset PC^0_{X(\mathbb{R})}$ for all $x \in \dot{\mathbb{R}}$. Moreover, by the definition of $\mathrm{M}_x^\pm$, we have $0 \notin \mathrm{M}_x^\pm$ for all $x \in \dot{\mathbb{R}}$.

Now fix $x \in \dot{\mathbb{R}}$ and $\chi_1, \chi_2 \in \mathrm{M}_x^\pm$. By the definition, χ_1 and χ_2 are characteristic functions of intervals I_1 and I_2, respectively. Let χ_3 be the characteristic function of $I_3 := I_1 \cap I_2$. We find that $\chi_3 \in \mathrm{M}_x^\pm$ and

$$\chi_1\chi_3 = \chi_2\chi_3 = \chi_3 = \chi_3\chi_2 = \chi_3\chi_1.$$

Therefore, $\left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x\in\dot{\mathbb{R}}}$ is a family of localizing classes of $PC^0_{X(\mathbb{R})}$.

Consider now an arbitrary choice of elements

$$\left\{\chi_x^-, \chi_x^+\right\}_{x\in\dot{\mathbb{R}}} \subseteq \left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x\in\dot{\mathbb{R}}}.$$

In view of the compactness of $\dot{\mathbb{R}}$, there exist a finite number of points $x_1, x_2, \dots, x_n$ in $\dot{\mathbb{R}}$ such that the functions $\chi_{x_1}^-, \chi_{x_2}^-, \dots, \chi_{x_n}^-, \chi_{x_1}^+, \chi_{x_2}^+, \dots, \chi_{x_n}^+$ satisfy the following property:

$$g(t) := \sum_{j=1}^n \left(\chi_{x_j}^-(t) + \chi_{x_j}^+(t)\right) \geq 1 \quad \text{for all} \quad t \in \dot{\mathbb{R}}. \tag{4.7}$$

Since g is a linear combination of characteristic functions of intervals of $\dot{\mathbb{R}}$, we see that $g : \dot{\mathbb{R}} \to \mathbb{N}$ and $g \in P\mathbb{C}^0$. Moreover, since $g \geq 1$, we have $g^{-1} = 1/g \in P\mathbb{C}^0$. Hence, by the definition of $PC^0_{X(\mathbb{R})}$, we conclude that $g, g^{-1} \in PC^0_{X(\mathbb{R})}$. Therefore, $\left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x\in\dot{\mathbb{R}}}$ is a covering system of localizing classes of $PC^0_{X(\mathbb{R})}$.

We have

$$a \overset{\mathrm{M}_x^\pm}{\sim} a(x \pm 0) \quad \text{for} \quad x \in \dot{\mathbb{R}}, \quad a \in P\mathbb{C}^0, \tag{4.8}$$

since for each $x \in \dot{\mathbb{R}}$, there exist the functions $\chi_x^\pm \in \mathrm{M}_x^\pm$ such that

$$[a(t) - a(x \pm 0)]\, \chi_x^\pm(t) = 0 \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

It is clear that if $x \in \dot{\mathbb{R}}$, then for each $\chi_x^\pm \in \mathrm{M}_x^\pm$, we have $\|\chi_x^\pm\|_{V(\mathbb{R})} = 3$. Therefore, by the Stechkin-type inequality (1.3),

$$\sup\left\{\|\chi_x^\pm\|_{\mathcal{M}_{X(\mathbb{R})}} : \chi_x^\pm \in \mathrm{M}_x^\pm\right\} \leq 3c_X < \infty. \tag{4.9}$$

If $a \in PC^0_{X(\mathbb{R})}$, then there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $P\mathbb{C}^0$ such that

$$\|a - a_n\|_{\mathcal{M}_{X(\mathbb{R})}} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.10}$$

Then, in view of inequality (1.2), we conclude that

$$\|a - a_n\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.11}$$

Since $a, a_n \in PC$, for each $x \in \dot{\mathbb{R}}$ there exist finite one-sided limits $a(x \pm 0)$ and $a_n(x \pm 0)$ and the sets of discontinuities of a and a_n are at most countable. Hence (4.11) implies that for $x \in \dot{\mathbb{R}}$ one has $a_n(x \pm 0) \to a(x \pm 0)$, as $n \to \infty$. Thus

$$\|a_n(x \pm 0) - a(x \pm 0)\|_{\mathcal{M}_{X(\mathbb{R})}} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.12}$$

Combining (4.8)–(4.10) and (4.12) with Proposition 4.1, we conclude that

$$a \overset{\mathrm{M}_x^{\pm}}{\sim} a(x \pm 0) \quad \text{for} \quad x \in \dot{\mathbb{R}}, \quad a \in PC^0_{X(\mathbb{R})}. \tag{4.13}$$

On the other hand, by the hypothesis (4.4), we get $a(x \pm 0) \neq 0$ for $x \in \dot{\mathbb{R}}$. Therefore the constant functions $a(x \pm 0)$ are invertible in the Banach algebra $PC^0_{X(\mathbb{R})}$ and $a(x \pm 0)^{-1} \in PC^0_{X(\mathbb{R})}$ for $x \in \dot{\mathbb{R}}$. Hence $a(x \pm 0)$ is $\mathrm{M}_x^{\pm}$-invertible for every $x \in \dot{\mathbb{R}}$. Finally, taking into account (4.13) and applying the Gohberg-Krupnik Local Principle (Theorem 4.2), we get that a is invertible in the algebra $PC^0_{X(\mathbb{R})}$, i.e., $a^{-1} \in PC^0_{X(\mathbb{R})}$. □

4.4. **Proof of Theorem 1.1.** The proof presented below follows that of [9, Theorem 2.18].

If (4.4) is fulfilled, then by Lemma 4.3, we have $a^{-1} \in PC^0_{X(\mathbb{R})}$. From the general properties of the Fourier convolution operators on $X(\mathbb{R})$, we get

$$W^0(a)W^0(a^{-1}) = W^0(a^{-1})W^0(a) = I.$$

Therefore, the operator $W^0(a)$ is invertible on $X(\mathbb{R})$ and $(W^0(a))^{-1} = W^0(a^{-1})$.

Suppose now that the operator $W^0(a)$ is invertible on the space $X(\mathbb{R})$. For each $x \in \dot{\mathbb{R}}$, let $\mathrm{M}_x^{\pm}$ be defined by (4.5)–(4.6) and

$$\mathrm{M}_x^{0,\pm} := \left\{ W^0(g) \in \mathcal{B}(X(\mathbb{R})) \ : \ g \in \mathrm{M}_x^{\pm} \right\}.$$

We claim that $\left\{\mathrm{M}_x^{0,-}, \mathrm{M}_x^{0,+}\right\}_{x \in \dot{\mathbb{R}}}$ constitutes a covering system of localizing classes in the Banach algebra of bounded linear operators $\mathcal{B}(X(\mathbb{R}))$. Knowing that $\mathrm{M}_x^{\pm}$ is a localizing class in $PC^0_{X(\mathbb{R})}$, we have $0 \notin \mathrm{M}_x^{\pm}$. Therefore, $0 \notin \mathrm{M}_x^{0,\pm}$ for all $x \in \dot{\mathbb{R}}$.

Consider now $W^0(g_1), W^0(g_2) \in \mathrm{M}_x^{0,\pm}$. Then $g_1, g_2 \in \mathrm{M}_x^{\pm}$. Since $\mathrm{M}_x^{\pm}$ is a localizing class of $PC^0_{X(\mathbb{R})}$, there exists $g_3 \in \mathrm{M}_x^{\pm}$ such that

$$g_1 g_3 = g_2 g_3 = g_3 = g_3 g_2 = g_3 g_1.$$

Therefore, $W^0(g_3) \in \mathrm{M}_x^{0,\pm}$ and

$$W^0(g_1)W^0(g_3) = W^0(g_2)W^0(g_3) = W^0(g_3) = W^0(g_3)W^0(g_2) = W^0(g_3)W^0(g_1).$$

Hence $\left\{\mathrm{M}_x^{0,-}, \mathrm{M}_x^{0,+}\right\}_{x \in \dot{\mathbb{R}}}$ is a family of localizing classes in the Banach algebra of bounded linear operators $\mathcal{B}(X(\mathbb{R}))$.

Consider an arbitrary choice of elements

$$\left\{W^0(g_x^-), W^0(g_x^+)\right\}_{x \in \dot{\mathbb{R}}} \subseteq \left\{\mathrm{M}_x^{0,-}, \mathrm{M}_x^{0,+}\right\}_{x \in \dot{\mathbb{R}}}.$$

Bearing in mind that $\left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x \in \dot{\mathbb{R}}}$ is a covering system of localizing classes of the Banach algebra $PC^0_{X(\mathbb{R})}$ (see the proof of Lemma 4.3), there exist the points $x_1, x_2, \dots, x_n \in \dot{\mathbb{R}}$ such that $g_{x_i}^- \in \mathrm{M}_{x_i}^-$ and $g_{x_i}^+ \in \mathrm{M}_{x_i}^+$ for $i \in \{1, 2, \dots, n\}$, and the function

$$g := \sum_{i=1}^{n} \left(g_{x_i}^- + g_{x_i}^+\right)$$

is invertible in the algebra $PC^0_{X(\mathbb{R})}$. It follows that the operator $W^0(g)$ is invertible in the algebra $\mathcal{B}(X(\mathbb{R}))$ and its inverse is equal to $W^0(g^{-1}) \in \mathcal{B}(X(\mathbb{R}))$. Thus, $\left\{\mathrm{M}_x^{0,-}, \mathrm{M}_x^{0,+}\right\}_{x \in \dot{\mathbb{R}}}$ forms a covering system of localizing classes in the Banach algebra $\mathcal{B}(X(\mathbb{R}))$.

It follows from (4.13) that for all $x \in \dot{\mathbb{R}}$,

$$W^0(a) \overset{\mathrm{M}_x^{0,\pm}}{\sim} W^0(a(x \pm 0)) = a(x \pm 0)I.$$

If there exists some $x^* \in \dot{\mathbb{R}}$ such that $a(x^*-0) = 0$ or $a(x^*+0) = 0$, then $W^0(a) \overset{\mathrm{M}^{0,-}_{x^*}}{\sim} 0$ or $W^0(a) \overset{\mathrm{M}^{0,+}_{x^*}}{\sim} 0$. Since $W^0(a)$ is invertible, applying Gohberg-Krupnik's local principle (Theorem 4.2), we conclude that 0 is $\mathrm{M}^{0,-}_{x^*}$-invertible or $\mathrm{M}^{0,+}_{x^*}$-invertible. Therefore, $0 \in \mathrm{M}^{0,-}_{x^*} \cup \mathrm{M}^{0,+}_{x^*}$ which is a contradiction, since $\mathrm{M}^{0,-}_{x^*}$ and $\mathrm{M}^{0,+}_{x^*}$ are localizing classes of $\mathcal{B}(X(\mathbb{R}))$. Thus, $a(x \pm 0) \neq 0$ for all $x \in \dot{\mathbb{R}}$. Consequently, (4.4) is fulfilled. □

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ON OSCILLATIONS OF REAL-VALUED FUNCTIONS

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Abstract. We consider the question whether a given real-valued non-negative upper semi-continuous function on a topological space E is the oscillation function of a Borel real-valued function defined on the same space E.

Let E be a topological space, let $\mathbf{R}$ denote the real line and let $f : E \to \mathbf{R}$ be a function. Suppose that f is locally bounded at each point x of E, i.e., there exists a neighborhood $U(x)$ of x such that the restriction $f|U(x)$ is bounded. Then there exist two values

$$f^*(x) = \limsup_{y \to x} f(y), \quad f_*(x) = \liminf_{y \to x} f(y),$$

and the difference

$$O_f(x) = \limsup_{y \to x} f(y) - \liminf_{y \to x} f(y)$$

is called the oscillation of f at x. As is known, the real-valued function

$$f^*(x) = \limsup_{y \to x} f(y) \ (x \in E)$$

is upper semi-continuous on E and the real-valued function

$$f_*(x) = \liminf_{y \to x} f(y) \ (x \in E)$$

is lower semi-continuous on E (see, e.g., [1], [3], [5]). Consequently, the produced function

$$O_f(x) = f^*(x) - f_*(x) \ (x \in E)$$

is non-negative and upper semi-continuous on E.

Let us mention some facts concerning the behavior of the oscillations of real-valued functions.

(a) f is continuous at a point $x \in E$ if and only if $O_f(x) = 0$.

In particular, if x is an isolated point of E, then $O_f(x) = 0$ for an arbitrary $f : E \to \mathbf{R}$.

(b) $O_{tf}(x) = |t| O_f(x)$ for any real number t and for each point $x \in E$;

(c) $O_{f_1+f_2} \leq O_{f_1} + O_{f_2}$.

Actually, (c) implies the finite sub-additivity of the operator $O : f \to O_f$.

(d) If a series $\sum\{f_n : n \in \mathbf{N}\}$ of real-valued locally bounded functions on E converges uniformly to f, then $O_f \leq \sum\{O_{f_n} : n \in \mathbf{N}\}$.

(e) If a sequence $\{f_n : n \in \mathbf{N}\}$ of real-valued locally bounded functions on E converges uniformly to f, then the corresponding sequence of oscillations $\{O_{f_n} : n \in \mathbf{N}\}$ converges uniformly to the oscillation O_f.

Notice that (e) is a generalization of the well-known theorem of mathematical analysis, according to which the limit of a uniformly convergent sequence of real-valued continuous functions is also continuous.

In connection with the above facts, there arises the following natural question:

For a given real-valued non-negative upper semi-continuous function g on E, is it true that there exists a locally bounded function $f : E \to \mathbf{R}$ such that $O_f = g$?

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In case the answer to this question is positive, as far as g has a good descriptive structure (namely, g is upper semi-continuous), it is natural to try to find an f satisfying $O_f = g$ and also having good descriptive properties (e.g., a real-valued Borel measurable function f on E for which $O_f = g$).

Exemple 1. Let $E = \mathbf{R}$ and $g : \mathbf{R} \to \{1\}$. The widely known Dirichlet function $\chi : \mathbf{R} \to \{0,1\}$ satisfies the equality $O_\chi = g$. Recall that f takes value 1 at all rational points of $\mathbf{R}$ and takes value 0 at all irrational points of $\mathbf{R}$. Obviously, χ is a Borel function. Denoting by $\mathbf{c}$ the cardinality of the continuum, there are $2^{\mathbf{c}}$ many functions $f : \mathbf{R} \to \mathbf{R}$ such that $O_f = g$. Clearly, most of such f are not Borel functions.

The main goal of the present communication is to consider the above-formulated question and to give its solution for some classes of topological spaces E.

First of all, let us remark that the trivial necessary condition for the existence of f is the equality $g(x) = 0$ for all isolated points x in E.

Suppose that this condition is satisfied and denote by E' the closure of the set of all isolated points in E. Further, put $U = E \setminus E'$ and observe that the open set U does not contain isolated points. Denote by $g|U$ the restriction of g to U.

Lemma 1. *Assume that there exists a function $\phi : U \to \mathbf{R}$ such that $O_\phi = g|U$ and the relation $0 \le \phi \le g|U$ holds true.*

Then there exists a function $f : E \to [0, +\infty[$ such that $O_f = g$. Moreover, if ϕ is Borel, then f can be chosen to be Borel, too.

Proof. We define the required f as follows:

$f(x) = g(x)$ if x belongs to the set E';

$f(x) = \phi(x)$ if x belongs to the set U.

Let us verify that $O_f(x) = g(x)$ for each point $x \in E$.

If $x \in E'$, then it is easy to see that $f_*(x) = 0$ and $f^*(x) \ge g(x)$. At the same time, keeping in mind the relation $0 \le \phi \le g|U$ and the upper semi-continuity of g, we infer that $f^*(x) \le g(x)$, which implies

$$f^*(x) = g(x), \ O_f(x) = f^*(x) - f_*(x) = g(x) - 0 = g(x).$$

If $x \in U$, then using the equality $O_\phi = g|U$ and taking into account that U is an open set, we conclude that $O_f(x) = g(x)$, which completes the proof. □

In many cases, the above lemma enables one to reduce the formulated problem to those topological spaces E which do not contain isolated points.

Lemma 2. *Let E be a topological space, let $g : E \to \mathbf{R}$ be a non-negative upper semi-continuous function, and let $\{U_i : i \in I\}$ be a disjoint family of nonempty open subsets of E such that the union $\cup\{U_i : i \in I\}$ is everywhere dense in E. Suppose also that for each index $i \in I$, there exists a function $\phi_i : U_i \to \mathbf{R}$ satisfying these two conditions:*

(1) *$0 \le \phi_i \le g|U_i$ and the set $\{x \in U_i : \phi_i(x) = 0\}$ is everywhere dense in U_i;*

(2) *$g|U_i = O_{\phi_i}$.*

Let a function $f : E \to \mathbf{R}$ be defined by the formula

$f(x) = \phi_i(x)$ if $x \in U_i$, and $f(x) = g(x)$ if $x \in E \setminus \cup\{U_i : i \in I\}$.

Then the equality $g = O_f$ holds true.

The proof of Lemma 2 is similar to that of Lemma 1.

Using the above lemmas, one can deduce the following statement.

Theorem 1. *Let E be a locally compact metric space and let $g : E \to \mathbf{R}$ be a non-negative upper semi-continuous function such that $g(x) = 0$ for any isolated point x of E.*

Then there exists a Borel function $f : E \to \mathbf{R}$ for which $g = O_f$.

Theorem 2. *Let E be a topological space satisfying the following condition:*

There exists an infinite base $\mathcal{B}$ of E such that the cardinality of any nonempty set $U \in \mathcal{B}$ is strictly greater than card$(\mathcal{B})$.

Then for every non-negative upper semi-continuous function $g : E \to \mathbf{R}$, there exists a function $f : E \to \mathbf{R}$ such that $O_f = g$.

The proof of Theorem 2 essentially uses one auxiliary notion and Lemma 3 presented below.

Let $\mathbf{b}$ be an infinite cardinal and let E be a topological space.

A point $x \in E$ is called a $\mathbf{b}$-point in E if there exists a neighborhood $U(x)$ of x whose cardinality does not exceed $\mathbf{b}$.

Lemma 3. *If E is a topological space with a base whose cardinality does not exceed $\mathbf{b}$, then the cardinality of the set of all $\mathbf{b}$-points in E does not exceed $\mathbf{b}$.*

Lemma 3 enables one to make appropriate changes in the graph of a given real-valued non-negative upper semi-continuous function $g : E \to \mathbf{R}$ in order to obtain a function $f : E \to \mathbf{R}$ such that $O_f = g$.

In general, those changes produce a function f with bad descriptive properties. However, if E fulfils certain additional assumptions, then the required f can be chosen to be Borel.

Theorem 3. *Let E be a metric space satisfying the condition of Theorem 2.*

Then for every non-negative upper semi-continuous function $g : E \to \mathbf{R}$, there exists a Borel function $f : E \to \mathbf{R}$ such that $O_f = g$.

The proof of Theorem 3 is based on the following fact which is valid for any metric space E satisfying the condition of Theorem 2:

If $X \subset E$ has cardinality, strictly less than $\mathrm{card}(E)$, then there exists an everywhere dense set $Y \subset E$ of type F_σ in E such that

$$X \cap Y = \emptyset, \quad \mathrm{card}(Y) < \mathrm{card}(E).$$

For certain topological groups, we have the next statement.

Theorem 4. *Let E be a non-discrete locally compact σ-compact topological group and let $g : E \to \mathbf{R}$ be a non-negative upper semi-continuous function.*

Then there exists a function $f : E \to \mathbf{R}$ such that $O_f = g$.

The proof of Theorem 4 is based on the following important equality

$$\mathrm{card}(E) = 2^{w(E)},$$

where $w(E)$ denotes the topological weight of E (see, e.g., [2]). This equality implies that the assumption of Theorem 2 is automatically satisfied.

Recall that a topological space E is resolvable (in the sense of E. Hewitt) if there exists a partition $\{A, B\}$ of E such that both sets A and B are everywhere dense in E (see [4]). Otherwise, E is called an irresolvable space. Resolvable spaces have a number of interesting properties. For instance, the following assertions are valid.

(1) Any open subspace of a resolvable space is resolvable.

(2) The topological product of a family $\{E_i : i \in I\}$ of nonempty topological spaces is resolvable whenever at least one E_i is resolvable.

(3) The topological sum of a family $\{E_i : i \in I\}$ of nonempty topological spaces is resolvable if and only if all E_i $(i \in I)$ are resolvable.

(4) If E possesses a pseudo-base all members of which are resolvable, then E itself is resolvable.

(5) Any nonempty locally compact space without isolated points is resolvable.

(6) Any metric space without isolated points is resolvable.

(7) If E is resolvable, then for each F_σ-subset X of E there exists a function $f : E \to \mathbf{R}$ such that X coincides with the set of all points of discontinuity of f.

In this context, it makes sense to notice that the topological product of a family of irresolvable spaces can be resolvable, a closed subspace of a resolvable space can be irresolvable, and a continuous image of a resolvable space can be irresolvable.

Exemple 2. Let E be a topological space and let $g : E \to \mathbf{R}$ be a real-valued non-negative upper semi-continuous function. Suppose that the graph of g is a resolvable subspace of the product space $E \times \mathbf{R}$. Then there exists a function $f : E \to \mathbf{R}$ such that $g = O_f$. In particular, if E is a resolvable space, then for any real-valued non-negative constant function $g : E \to \mathbf{R}$, there exists a function $f : E \to \mathbf{R}$ such that $O_f = g$.

Theorem 5. *Let E be a metric space and let $g : E \to \mathbf{R}$ be a real-valued non-negative upper semi-continuous function.*

If the graph of g considered as a subspace of $E \times \mathbf{R}$ does not contain isolated points, then there exists a Borel function $f : E \to \mathbf{R}$ such that $O_f = g$.

Exemple 3. Let E be an infinite set, let $\mathcal{J}$ be a σ-ideal of subsets of E, and let $\mathcal{F} = \{X \subset E : E \setminus X \in \mathcal{J}\}$ be the dual filter of $\mathcal{J}$. Suppose that the following two conditions are fulfilled:

($*$) $\mathrm{card}(X) = \mathrm{card}(E)$ for each set $X \in \mathcal{F}$;

($**$) there exists a base $\mathcal{B}$ of $\mathcal{J}$ with $\mathrm{card}(\mathcal{B}) \leq \mathrm{card}(E)$.

Denote $\mathcal{T} = \{\emptyset\} \cup \mathcal{F}$. Then $\mathcal{T}$ is a topology on E such that:

(i) the space $(E, \mathcal{T})$ is resolvable;

(ii) for a function $g : E \to \{1\}$, there exists a function $f : E \to \mathbf{R}$ satisfying $O_f = g$;

(iii) for the same function $g : E \to \{1\}$, there exists no Borel function $h : E \to \mathbf{R}$ satisfying $O_h = g$.

Exemple 4. Take an infinite set E equipped with a nontrivial ω_1-complete ultrafilter Φ of subsets of E (this condition is equivalent to the existence of a two-valued measurable cardinal number). Equip E with the topology

$$\mathcal{T} = \{\emptyset\} \cup \Phi.$$

The obtained topological space $(E, \mathcal{T})$ has the following property:

For any function $f : E \to \mathbf{R}$, there exists a set $X \in \Phi$ such that the restriction $f|X$ is constant.

Therefore, for every function $f : E \to \mathbf{R}$, there are points x in E at which f is continuous and, consequently, $O_f(x) = 0$.

The latter implies that if g is a real-valued strictly positive constant function on E, then there is no $f : E \to \mathbf{R}$ such that $O_f = g$.

So, Example 4 shows us that certain restrictions on a general topological space E are necessary if one wants to have a positive solution to the question discussed in this note.

Let E be a topological space, (M, ρ) be a bounded metric space and let $f : E \to M$ be a function. For each point $x \in E$, one can define

$$O_f(x) = \inf\{\mathrm{diam}(f(U(x))) : U(x) \in \mathcal{F}(x)\},$$

where $\mathcal{F}(x)$ is the filter of all neighborhoods of x and $\mathrm{diam}(f(U(x)))$ denoting the diameter of the set $f(U(x))$. The obtained function $O_f : E \to \mathbf{R}$ called also the oscillation of f, is non-negative and upper semi-continuous.

Notice that the assertions (a) and (e) remain true for this more general concept of O_f.

The question analogous to the considered above can be formulated in terms of the pair (E, M).

Namely, one can ask about a characterization of all those pairs (E, M) for which any non-negative upper semi-continuous function $g : E \to \mathbf{R}$ admits a (Borel) function $f : E \to M$ such that $O_f = g$.

This question seems to be of interest from the viewpoint of mathematical analysis and general topology.

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MATHEMATICAL MODELING OF STOCHASTIC SYSTEMS USING THE GENERALIZED NORMAL SOLUTION METHOD

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Abstract. Operation of complex engineering systems gives rise to various physical processes, including thermal, electrical, hydrodynamic, mechanical, electromagnetic, etc. The parameters of the elements of an engineering system and the processes going on in the same are stochastic, which results both from a stochastic nature of the parameters of the elements and from a random nature of the parameters of the environment and external influencing factors. Mathematical modeling of stochastic engineering systems developed in this paper relies on a universal structural conceptual model of an engineering system represented as a directed graph which reflects the structure of the engineering system and the modeled physical processes. State variables in a structural conceptual model of a system are the potentials in the graph nodes and flows at the graph edges, which edges may contain elements modeling the processes of energy dissipation, potential energy accumulation and kinetic energy storage, and also independent sources such as potential and physical quantity flow with the a priori known value. Stochastic processes in a graph of an engineering system model for each elementary event ω from the space of elementary events Ω are described through the mathematical model $H(\omega)X(\omega) = Y(\omega)$, $\omega \in \Omega$ with a stochastic matrix $H(\omega) = AG(\omega)A^T$, where A is an incidence matrix and $G(\omega)$ is a stochastic diagonal matrix of such parameters of the elements of a graph as conductance. The present paper offers a method based on the generalized normal solution concept, known also as pseudosolution, pseudoinverse matrix and generalized inverse matrix method allowing one to determine an equation for statistical measures (expectations, covariances, dispersions, standard deviations) of the stochastic solution $X(\omega)$ of the mathematical model of a stochastic engineering system under the a priori known statistical measures of the matrix of system elements $G(\omega)$ and the stochastic right-hand side vector $Y(\omega)$. Utilization of the method in modeling of stochastic thermal processes and statistical measures for complex electronic systems has shown that the method is adequate and efficient.

Various physical processes going on in complex engineering systems are stochastic in the majority of practically important cases resulting both from a random nature of the internal parameters of the elements and structure of the system and from the external influencing factors. To allow mathematical modeling of physical processes, the design of an engineering system should, first of all, be represented as a structural conceptual model (SC model) constructed as a graph, which replaces the real design of the engineering system by a simplified, but still sufficiently adequate model reflecting both the structure of the engineering system and the physical processes going on in the same [3, 6, 14]. State variables in the SC model graph are the quantities such as potentials in the graph nodes and flows at the graph edges. The graph nodes are connected to each other by edges including such elements as R, where energy is dissipated; C accumulates potential energy; L stores kinetic energy, and also the elements that determine independent sources of state variables potential or physical quantity flow with the a priori known values [3, 6, 13, 14]. The SC models of engineering systems are sufficiently universal and allow to get efficient and highly adequate modeling of complex engineering systems and various physical processes (thermal, hydrodynamic, mechanical, electrical, etc.) going on in the same.

A mathematical model of a stationary stochastic physical process in a SC model of an engineering system is represented by a stochastic matrix equation [1, 6]

$$H(\omega)X(\omega) = Y(\omega), \quad \omega \in \Omega, \tag{1}$$

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where $H(\omega)$ is a stochastic $n \times n$-square matrix reflecting the structure of the SC model graph and component relations between state variables and graph elements; $X(\omega)$ is an n-column vector of stochastic state variables; $Y(\omega)$ is an n-column vector of independent stochastic sources of state variables; ω are elementary events from the space of elementary events Ω in the probability space $\{\Omega, U, P\}$, U is the σ-algebra, P is probability in U. It should be noted that random elements of the stochastic matrix $H(\omega)$ and vector $X(\omega)$ are stochastically interdependent and statically independent of the elements of the stochastic vector $Y(\omega)$.

The stochastic vectors $X(\omega)$, $Y(\omega)$ and the matrix $H(\omega)$ in equation (1) are the interval stochastic [7–9] quantities $\xi(\omega)$, whose values are evenly distributed within the interval of values $[\xi_{down}, \xi_{up}]$ with a density $p_\xi = \Delta_\xi^{-1}$, $\xi(\omega) \in [\xi_{down}, \xi_{up}]$ and $p_\xi = 0$, $\xi(\omega) \notin [\xi_{down}, \xi_{up}]$, where $\Delta_\xi = \xi_{up} - \xi_{down}$ is the length of the interval $[\xi_{down}, \xi_{up}]$; ξ_{up} and ξ_{down} are the upper and lower interval limits.

A random process is fully characterized by the sequence of all its distribution laws of various order over time [1, 11]. At the same time, it is impossible to determine the laws for the stochastic vector $X(\omega)$, which is a solution to the matrix equation (1), as the task is extremely difficult. However, modeling of stochastic processes going on in engineering systems does not require any knowledge of distribution laws, as the most informative and most important ones in the engineering practice are statistical measures of the vector of stochastic state variables $X(\omega)$, in particular:

–n-column vector of expectations $\bar{X} = E\{X(\omega)\}$ with elements $\bar{x}_i = E\{x_i(\omega)\}$, $i = 1, 2, \ldots, n$, where $E\{\cdot\}$ is the expectation operator;

– covariance $n \times n$-matrix $K_{XX} = E\{\mathring{X}(\omega)\mathring{X}^T(\omega)\}$ with elements ij, equal to $k_{ij} = E\{\mathring{x}_i(\omega)\mathring{x}_j(\omega)\}$, $i, j = 1, 2, \ldots, n$, where $\mathring{x}_i(\omega) = x_i(\omega) - \bar{x}_i$ is a centered stochastic quantity with a zero expectation, $(\cdot)^T$ is the operation of transposition;

– n-column vector of dispersions D_X, equal to diagonal elements $d_{x,i}$ of the correlation matrix K_{XX}, i.e., $d_{x,i} = k_{ii} = E\{(\mathring{x}_i(\omega))^2\}$, $i = 1, 2, \ldots, n$;

– n-column vector of standard deviations σ_X with elements $\sigma_{x,i} = \sqrt{d_{x,i}}$, $i = 1, 2, \ldots, n$.

The determined vectors of statistical measures $\bar{X} = \{\bar{x}_i\}_1^n$ and $\sigma_X = \{\sigma_{x,i}\}_1^n$ of the stochastic vector $X(\omega) = \{x_i\}_1^n$ allow to determine the vectors of the lower $X_{down} = \{x_{i,down}\}_1^n$ and upper $X_{up} = \{x_{i,up}\}_1^n$ interval limits $[x_{i,down}, x_{i,up}]$, $i = 1, 2, \ldots, n$, which will contain real values of the interval stochastic quantities $x_i(\omega) \in [x_{i,down}, x_{i,up}]$.

In the simplest case, where the matrix H of the set of equations [6] is deterministic, while external perturbations being a part of the right-hand side vector $Y(\omega)$ are stochastic only, the statistical measures $\bar{X}$ and K_{XX} of the stochastic vector $X(\omega)$ are determined by using the equations $\bar{X} = H^{-1}\bar{Y}$ and $K_{XX} = H^{-1}K_{YY}H^{-1}$, where H^{-1} is the deterministic inverse of the matrix H, $\bar{Y} = E\{Y(\omega)\}$ is the vector of expectations of the stochastic vector $Y(\omega)$, $K_{YY} = E\{\mathring{Y}(\omega)\mathring{Y}^T(\omega)\}$ is the covariance matrix of the stochastic vector $Y(\omega)$.

If the matrix $H(\omega)$ in equation (1) is stochastic, it is impossible to determine statistical measures of the vector $X(\omega)$ by a direct impact of the expectation operator on the both sides of equation [6] in view of the statistical relationship between the stochastic elements of the matrix $H(\omega)$ and vector $X(\omega)$; so, $E\{H(\omega)X(\omega)\} \neq E\{H(\omega)\} \cdot E\{X(\omega)\}$. In this case, to determine statistical measures of the stochastic vector $X(\omega)$, papers [1, 6] represent the stochastic matrix of the system $H(\omega)$ as $H(\omega) = \bar{H}(I + \bar{H}^{-1}\mathring{H}(\omega))$, and the stochastic inverse of the matrix $H^{-1}(\omega)$ for each $\omega \in \Omega$ is expanded along an infinite almost surely uniformly convergent series [1]

$$H^{-1}(\omega) = (I + \bar{H}^{-1}\mathring{H}(\omega))^{-1} \cdot \bar{H}^{-1} = \sum_{k=1}^{\infty}(-1)^k(\bar{H}^{-1}\mathring{H}(\omega))^k \cdot \bar{H}^{-1}, \tag{2}$$

provided the condition $||\bar{H}^{-1}\mathring{H}(\omega)|| < 1$ is satisfied for all realizations of $\omega \in \Omega$. Here, $||\cdot||$ is the matrix norm [5]; $\mathring{H}(\omega) = H(\omega) - \bar{H}$ is the centered stochastic $n \times n$-matrix with a zero expectation; $\bar{H} = E\{H(\omega)\}$ is the expectation of the stochastic matrix $H(\omega)$; $\bar{H}^{-1}$ is the inverse of the deterministic matrix $\bar{H}$, which can be easily determined.

Then, the statistical measures $\bar{X}$ and K_{XX} of the stochastic vector $X(\omega) = H^{-1}(\omega) \cdot Y(\omega)$, being a solution to equation (1), will be determined by using the following equations:

$$\bar{X} = E\{H^{-1}(\omega)\} \cdot \bar{Y} \quad \text{and} \quad K_{XX} = E\{H^{-1}(\omega)Y(\omega)Y^T(\omega)(H^T(\omega))^{-1}\},$$

where $H^{-1}(\omega)$ is the stochastic inverse matrix to be determined by equation (2).

Practical calculations are limited to the terms of the infinite series (2) containing a matrix $\mathring{H}(\omega)$ of degree max. 2. The above method, known also as a stochastic inverse matrix method, allows to obtain the results with errors, sufficient to be used in practice and not exceeding $5-7\%$ [1,6]. At the same time, the range of applicability of the method is subject to the condition $||\bar{H}^{-1}\mathring{H}(\omega)|| < 1$, which imposes significant limitations on the allowable values of the parameters of the engineering system.

It should be also noted that the use of the perturbation and hierarchy methods [1] described in literature to determine the statistical measures $\bar{X}$ and K_{XX} of the stochastic processes in engineering systems have not found practical use. The reason is that the first of the above methods is good only for extremely small perturbations, which do not occur in real practice, while the second one is heuristic and does not have mathematical justification. The assumptions concerning special types of random processes, such as Wiener or white noise, presented in a great number of papers, allow to obtain final solutions for statistical measures in many cases, but are unrealistic and cannot exist in practice of engineering systems operation. The use of the statistical test method [12] may not be recommended for designing engineering systems, as far as it requires a huge input of machine time and memory caused by the necessity to solve multiple (up to several tens of thousands) simultaneous equations (1) to achieve an acceptable accuracy [11].

This paper offers a method to determine the statistical measures of the stochastic vector $X(\omega)$, which is a solution to the matrix equation (1) describing physical processes in SC models of engineering systems. The method is based on the generalized normal solution concept and allows to get final closed-form equations for statistical measures of stochastic processes in engineering (electronic) systems of any complexity, which are adequately simulated by SC models, being free from the above defects. The developed method is used in modeling of stochastic thermal processes in real electronic systems and has proven to be adequate and efficient.

The method developed in this paper is based on the generalized normal solution concept, which is also known in relation to the matrix equation as pseudosolution, pseudoinverse matrix method, Moore–Penrose generalized inverse matrix method [4,5,10]. The essence of the method consists in the following.

If the matrix $A = \{a_{ij}\}_1^n$ in the matrix equation $Ax = y$ is square and nonsingular, then an inverse matrix A^{-1} and a unique solution to the equation $x = A^{-1}y$ exist. If the matrix A is square, but singular, or the matrix $A = \{a_{ij}\}_{(n\times m)}$ is a rectangular $n\times m$-matrix (n and m are the number of lines and columns), then the matrix A is known to have no inverse. At the same time, a unique so-called pseudoinverse matrix A^+ can be constructed for such matrix, which pseudoinverse matrix allows one to obtain the best approximate solution $x^0 = A^+y,\ x^0 = (x_1^0, x_2^0, \ldots, x_m^0)^T$ to the equation $Ax = y$ in terms of the minimum value of the residual norm (Euclidian l_2-norm) square achieved for $x = x^0$, in particular [4],

$$\min_x ||y - Ax||^2 = \min_x \sum_{i=1}^n \left| y_i - \sum_{j=1}^m a_{ij}x_j \right|^2. \tag{3}$$

In this case, the vector of the best approximate solution x^0 has the lowest length, i.e., $||x^0||^2 = (x^0)^T \cdot x^0 = \min$, where $x^T \cdot x$ is the scalar product of the vector x. It should be noted that if the matrix A is square and nonsingular, then the inverse A^{-1} is the same as the pseudoinverse matrix A^+. Further, it may be shown [4,5] that a rectangular $n\times m$-matrix of the A range $r = \min\{n,m\}$ can always be represented as the so-called skeleton decomposition $A = BC$, i.e., as a product of two rectangular matrices, in particular, the $n\times r$-matrix B and the $r\times m$-matrix C. In this case, the pseudoinverse matrix A^+ is determined by using the equation $A^+ = C^+B^+$, where $C^+ = C^T(CC^T)^{-1}$ and $B^+ = (B^TB)^{-1}B^T$ [4]. Despite the fact that the skeleton decomposition $A = BC$ provides no unambiguous determination of the multiplier matrices B and C, the equation $A^+ = C^+B^+$ determines the unique pseudoinverse matrix with any skeleton decompositions [4].

Let us apply the generalized normal solution method to the stochastic matrix equation (1) describing the physical processes in a SC model of an engineering system. Toward this end, let us represent the $n \times n$-matrix of the $H(\omega)$ graph of the SC model for each $\omega \in \Omega$ as a product of three matrices $H(\omega) = AG(\omega)A^T$, in particular, as a deterministic incidence $n \times m$-matrix of the A graph of the SC model (n, m are the number of nodes and edges, respectively), which contains elements 0 and 1 only, and a stochastic diagonal $m \times m$-matrix of elements $G(\omega)$ such as random conductance in the graph edges. The decomposition $H(\omega) = AG(\omega)A^T$ can be always performed for a random graph in a single way only [2, 6]. Then the stochastic equation (1) can be written as follows:

$$AG(\omega)A^T X(\omega) = Y(\omega), \quad \omega \in \Omega, \tag{4}$$

where $G(\omega) = \mathrm{diag}(g_1(\omega), g_2(\omega), \dots, g_m(\omega))$ is the stochastic diagonal $m \times m$-matrix with stochastic elements $g_i(\omega)$, $i = 1, 2, \dots, m$ at m edges of the graph of a SC model of an engineering system, which elements are expressed through physical stochastic parameters of the engineering system and the process going on in the same [2, 6].

Let us apply the generalized normal solution method to the stochastic equation (4). To do this, let us represent equation (4) as $AZ(\omega) = Y(\omega)$, $\omega \in \Omega$ with a stochastic column vector $Z(\omega) = G(\omega)A^T X(\omega)$ and multiply both right-hand sides by the transposed incidence matrix A^T. We obtain the equation $A^T AZ(\omega) = A^T Y(\omega)$ with a singular square $n \times n$-matrix $A^T A = B$, for which there exists no inverse matrix. At the same time, the product $A^T A$ is, in fact, a skeleton decomposition of the matrix $B = A^T A$, thus we can build a pseudoinverse deterministic matrix B^+ [4]

$$B^+ = A^T (AA^T)^{-1} (AA^T)^{-1} A, \tag{5}$$

and use the pseudoinverse matrix method to get the best approximate solution to the equation $A^T AZ(\omega) = BZ(\omega) = A^T Y(\omega)$, in particular,

$$Z^0(\omega) = B^+ A^T Y(\omega), \quad \omega \in \Omega, \tag{6}$$

which is understood as the minimum residual norm square $\min_Z ||Y(\omega) - AZ(\omega)||^2$ (3) for each realization of $\omega \in \Omega$ and has the lowest length $||Z^0||^2$.

If we write equation [8] as $G(\omega)A^T X^0(\omega) = B^+ A^T Y(\omega)$ considering that $Z^0(\omega) = G(\omega)A^T X^0(\omega)$ and successively multiply both its right-hand sides by the stochastic inverse matrix $G^{-1}(\omega)$ and then by the deterministic incidence matrix A, we get

$$AA^T X^0(\omega) = AG^{-1}(\omega) B^+ A^T Y(\omega), \quad \omega \in \Omega. \tag{7}$$

Note that the stochastic inverse $m \times m$-matrix $G^{-1}(\omega)$ is diagonal and easily determinable for each $\omega \in \Omega$; in particular, $G^{-1}(\omega) = \mathrm{diag}(g_1^{-1}(\omega), g_2^{-1}(\omega), \dots, g_m^{-1}(\omega))$, $i = 1, 2, \dots, m$.

Considering that the matrix AA^T is square, symmetrical and, hence, has an inverse matrix $(AA^T)^{-1}$, we get the final stochastic solution $X^0(\omega)$ to the stochastic equation (7) understood in terms of the generalized normal solution (3) as:

$$X^0(\omega) = (AA^T)^{-1} AG^{-1}(\omega) A^T (AA^T)^{-1} Y(\omega), \quad \omega \in \Omega, \tag{8}$$

which can after the introduction of the deterministic matrix

$$C = (AA^T)^{-1} A, \quad C^T = A^T ((AA^T)^{-1})^T \tag{9}$$

be written more compactly as follows:

$$X^0(\omega) = CG^{-1}(\omega) C^T Y(\omega), \quad \omega \in \Omega. \tag{10}$$

The statistical measures of the stochastic vector $X^0(\omega)$, in particular, the expectation vector $\bar{X}^0$ and the covariance matrix $K_{X^0 X^0}$, are determined from the stochastic solution (10) considering the stochastic independence of the elements of the stochastic vector $Y(\omega)$ and stochastic matrix $G(\omega)$, and appear to be presented as follows:

– the expectation vector $\bar{X}^0 = E\{X^0(\omega)\}$

$$\bar{X}^0 = C\bar{G}^{-1} C^T \bar{Y},$$

where $\bar{G}^{-1} = E\{G^{-1}(\omega)\} = \mathrm{diag}(E\{g_1^{-1}(\omega)\}, E\{g_2^{-1}(\omega)\}, \dots, E\{g_m^{-1}(\omega)\})$ is the diagonal matrix of expectations with elements $E\{g_i^{-1}(\omega)\}$, $i = 1, 2, \dots, m$; $\bar{Y} = E\{Y(\omega)\}$ is the vector of expectations of the stochastic vector $Y(\omega)$. As the elements $g_i(\omega)$ are interval stochastic ones, i.e., evenly distributed within the intervals $[g_{down,i}, g_{up,i}]$ with the length $\Delta_{g_i} = g_{up,i} - g_{down,i}$, $E\{g_i^{-1}(\omega)\} = \frac{1}{\Delta_{g_i}} \ln \frac{g_{up,i}}{g_{down,i}}$;

– the covariance matrix $K_{X^0X^0} = E\{(\overset{\circ}{X}{}^0)(\omega)\overset{\circ}{X}{}^{0T}(\omega)\}$

$$K_{X^0X^0} = CE\{G^{-1}(\omega)C^T M_{YY} CG^{-1}(\omega)\}C^T - \bar{X}^0\bar{X}^{0T}$$

where $M_{YY} = E\{Y(\omega)Y^T(\omega)\}$ is the matrix of moments about the origin of the stochastic vector $Y(\omega)$. Note that the diagonal structure of the matrix G allows easy calculation of the equation $E\{G^{-1}(\omega)C^T M_{YY} CG^{-1}(\omega)\}$ in its final form.

Let us estimate the relative error of the stochastic generalized normal solution $X^0(\omega)$ of (8), (10) relatively to the accurate stochastic solution $X(\omega)$ of equation (4). We determine the relative error δ as an expectation of the stochastic error $\delta(\omega)$, equal to the difference ratio between the norms $||X^0(\omega)||$ and $||X(\omega)||$ of the compared stochastic solutions and the norm $||X(\omega)||$ of the accurate solution to equation (4), i.e.,

$$|\bar{\delta}| = |E\{\delta(\omega)\}| = \left|E\left\{\frac{||X(\omega)|| - ||X^0(\omega)||}{||X(\omega)||}\right\}\right|, \tag{11}$$

where $||\Theta(\omega)||$ is the stochastic $l_2(\omega)$-norm determined for each realization of $\omega \in \Omega$, for the stochastic vector $\Theta(\omega) = (\Theta_1(\omega), \Theta_2(\omega), \ldots, \Theta_n(\omega))^T$ or the stochastic diagonal matrix $\Theta(\omega) = \{_{ij}(\omega)\}_{i,j=1}^{n,m}$ according to the equations

$$l_2^{vector}(\omega) = \left(\sum_{i=1}^{n} \Theta_i^2(\omega)\right)^{1/2}, \quad l_2^{matrix}(\omega) = \left(\sum_{i,j=1}^{nm} \Theta_{ij}^2(\omega)\right)^{1/2}.$$

It can be shown that the estimate of the stochastic error $\delta(\omega)$ (11) satisfies the inequality

$$\left|1 - \frac{||I||_G^2 \cdot ||I||_{AA^T}^2}{||G^{-1}(\omega)|| \cdot ||G(\omega)||}\right| \leq |\delta(\omega)|, \quad \omega \in \Omega,$$

where $||I||_G^2$ and $||I||_{AA^T}^2$ are Euclidean l_2-norms of single matrices I, one of which has the shape of the matrix G, and the other has the shape of the matrix AA^T.

Considering that the product of the norms $||G^{-1}(\omega)||\cdot||G(\omega)||$ is equal to the stochastic conditioning number $\mu_G(\omega)$ of the stochastic matrix $G(\omega)$, and the l_2-norms of single matrices I are equal to $||I||_G^2 = m$ and $||I||_{AA^T}^2 = n$ (n, m are the number of nodes and edges of the graph of the SC model of the system), we get the following estimate of the stochastic error $\delta(\omega)$ between the generalized normal and the accurate solutions $|1 - m \cdot n/\mu_G(\omega)| \leq |\delta(\omega)|$, $\omega \in \Omega$. If we expand the equation $1/\mu_G(\omega)$ along the Taylor series retaining the first-order terms only and applying the expectation operator to the resulting equation, we find that the expectation of the relative error $\bar{\delta}$ satisfies the inequality $|1 - m \cdot n/\mu_{\bar{G}}| \leq |\bar{\delta}|$, where $\mu_{\bar{G}} = ||\bar{G}^{-1}|| \cdot ||\bar{G}||$ is the conditioning number of the matrix of expectations $\bar{G}$ with all elements being equal to their expectations $\bar{G} = \mathrm{diag}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_m)$ and the l_2-norms $||\bar{G}||$ and $||\bar{G}^{-1}||$ being equal to $||\bar{G}|| = (\sum_{i=1}^{m} \bar{g}_i^2)^{1/2}$ and $||\bar{G}^{-1}|| = (\sum_{i=1}^{m} \bar{g}_i^{-2})^{1/2}$, respectively. The equation for the expectation of the stochastic estimate of the error shows that $\bar{\delta}$ depends on the conditioning number $\mu_{\bar{G}}$ of the matrix $\bar{G}$ and the number of edges (m) and nodes (n) in the graph of the SC model of the engineering system.

The developed method is used in modeling of thermal processes in complex electronic systems and has proven to be efficient.

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ON THE LERAY–HIRSCH THEOREM

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Abstract. In [7], E. Spanier directly proved that for the total pair $(E, \dot{E})$ of a fiber-bundle pair with base B and fiber pair $(F, \dot{F})$ such that $H_*(F, \dot{F}, R)$ is free and finitely generated over R and θ is a cohomology extension of the fiber, the homomorphism

$$\Phi_* : H_*(E, \dot{E}, G) \longrightarrow H_*(B, G) \otimes H_*(F, \dot{F}, R),$$

where H_* is the singular homology, is an isomorphism for all R modules G ([7, Theorem 5.7.9]), where R is a commutative ring with a unit.

About the homomorphism

$$\Phi^* : H^*(B, G) \otimes H^*(F, \dot{F}, R) \longrightarrow H^*(E, \dot{E}, R),$$

where H^* is the singular cohomology, he said that a similar argument does not appear possible, because it is not true that $H^*(B, R)$ is isomorphic to the inverse limit $\varprojlim \{H^*(U, R)\}_{U \in \mathcal{U}}$.

In [8], R. Switzer, using the spectral sequence of Serre, proved that the homomorphism Φ^* is an isomorphism ([8, Theorem 15.47]).

In [1], the Leray–Hirsch theorem (Theorem 4D.1) is proved, not using the spectral sequence, however, the base B is an infinite-dimensional CW complex.

In this paper, we give another proof of the fact that the homomorphism Φ^* is an isomorphism not using the spectral sequence of Serre.

Below, we give the brief summaries of some results used in the paper.

Let Ab be the category of abelian groups and homomorphisms.

Lemma 1 ([7, Lemma 5.5.6])**.** *If B is a finitely generated free abelian group, then for arbitrary abelian groups A and G, μ is an isomorphism*

$$\mu : \operatorname{Hom}(A, G) \otimes \operatorname{Hom}(B, \mathbb{Z}) \approx \operatorname{Hom}(A \otimes B, G).$$

Lemma 2 ([7, Corollary 5.5.4])**.** *If (X, A) is a topological pair such that $H_*(X, A)$ is finitely generated, then the free subgroups of $H^*(X, A)$ and $H_*(X, A)$ are isomorphic and the torsion subgroups of $H^*(X, A)$ and $H_{*-1}(X, A)$ are isomorphic, where H_* (H^*) is the integral singular homology* (*cohomology*) *theory.*

Lemma 3 ([6, Lemma 5.2])**.** *Given a short exact sequence of abelian groups*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

and an abelian group B, if A'' or B is torsion free (*where being torsion free is equivalent to being free*), *there is a short exact sequence*

$$0 \longrightarrow A' \otimes B \longrightarrow A \otimes B \longrightarrow A'' \otimes B \longrightarrow 0.$$

Lemma 4 ([3, V.1])**.** *If A and B are free abelian groups, then $A \otimes B$ is a free abelian group.*

Lemma 5 ([8, 10.36])**.** *Let $\{X^\alpha,\ \alpha \in \Lambda\}$ be a directed set $(\alpha \le \beta \Rightarrow X^\alpha \subset X^\beta)$ of subspaces of topological space X such that for any compact $C \subset X$ there exists $\alpha \in \Lambda$ with $C \subset X^\alpha$. The inclusions $i_\alpha : X^\alpha \to X$, $\alpha \in \Lambda$, induce an isomorphism*

$$\{i_{\alpha,*}\} : \varinjlim H_*(X^\alpha, G) \xrightarrow{\sim} H_*(X, G).$$

Theorem 1 ([4, Theorem 11.32])**.** *Let*

$$0 \longrightarrow \underline{X}' \longrightarrow \underline{X} \longrightarrow \underline{X}'' \longrightarrow 0$$

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be a short exact sequence of inverse systems. Then there exists an exact sequence

$$0 \longrightarrow \varprojlim \underline{X}' \longrightarrow \varprojlim \underline{X} \longrightarrow \varprojlim \underline{X}'' \longrightarrow \varprojlim{}^{(1)} \underline{X}' \longrightarrow \cdots \longrightarrow \varprojlim{}^{(n)} \underline{X}' \longrightarrow \varprojlim{}^{(n)} \underline{X} \longrightarrow \varprojlim{}^{(n)} \underline{X}'' \longrightarrow \cdots,$$

where $\varprojlim{}^{(i)}$, $i \geq 1$, *is a derived functor.*

Lemma 6. *If B is a free and finitely generated abelian group and $\{A_\alpha\}$ is an inverse system of abelian groups A_α, then there is an isomorphism*

$$\varprojlim{}^{(i)} \{A_\alpha\} \otimes B \approx \varprojlim{}^{(i)} \{A_\alpha \otimes B\}, \quad i \geq 0.$$

Proof. Let $A = \varprojlim \{A_\alpha\}$ be an inverse limit of abelian groups A_α. Since B is a free and finitely generated abelian group, there is an isomorphism

$$B \approx \mathbb{Z}^n.$$

Hence, for all α, we have an isomorphism

$$A_\alpha \otimes B \approx A_\alpha \otimes \mathbb{Z}^n \approx (A_\alpha \otimes \mathbb{Z})^n \approx (A_\alpha)^n.$$

a) By Lemma 11.24 [4], the functor $\varprojlim$ preserves finite products. Therefore there is

$$\varprojlim \{A_\alpha \otimes B\} \approx \varprojlim (A_\alpha)^n = \big(\varprojlim A_\alpha\big)^n = A^n$$
$$\approx (A \otimes \mathbb{Z})^n \approx A \otimes \mathbb{Z}^n \approx A \otimes B = \varprojlim \{A_\alpha\} \otimes B.$$

b) By Corollary 12.15 [4], for $i \geq 1$, we have

$$\varprojlim{}^{(i)} \{A_\alpha \otimes B\} \approx \varprojlim{}^{(i)} \{A_\alpha \otimes \mathbb{Z}^n\} \approx \varprojlim{}^{(i)} \{(A_\alpha \otimes \mathbb{Z})^n\} \approx \varprojlim{}^{(i)} \{A_\alpha\}^n$$
$$\approx \big(\varprojlim{}^{(i)} \{A_\alpha\}\big)^n \approx \big(\varprojlim{}^{(i)} \{A_\alpha\} \otimes \mathbb{Z}\big)^n \approx \varprojlim{}^{(i)} \{A_\alpha\} \otimes \mathbb{Z}^n \approx \varprojlim{}^{(i)} \{A_\alpha\} \otimes B. \qquad \square$$

Lemma 7 ([2, Proposition 1.2])**.** *For any direct system $\{A_\alpha\}$ of abelian groups A_α, there are an exact sequence*

$$\text{a)} \quad 0 \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(A_\alpha, G) \longrightarrow \operatorname{Ext}(\varinjlim A_\alpha, G) \longrightarrow \varprojlim \operatorname{Ext}(A_\alpha, G) \longrightarrow \varprojlim{}^{(2)} \operatorname{Hom}(A_\alpha, G) \longrightarrow 0$$

and an isomorphism

$$\text{b)} \quad \varprojlim{}^{(i)} \operatorname{Ext}(A_\alpha, G) \approx \varprojlim{}^{(i+2)} \operatorname{Hom}(A_\alpha, G), \quad i \geq 1.$$

Lemma 8 ([7, Theorem 5.1.9])**.** *The tensor-product functor commutes with direct limits, i.e., there is an isomorphism*

$$\varinjlim \{A_\alpha\} \otimes B \approx \varinjlim \{A_\alpha \otimes B\}.$$

Lemma 9 ([5, Exercise 3, §A.3])**.** *If $\{A_\alpha\}$ is a direct system of abelian groups A_α, then there is an isomorphism*

$$\operatorname{Hom}\big(\varinjlim \{A_\alpha\}, B\big) \approx \varprojlim \operatorname{Hom}(A_\alpha, B).$$

A fiber-bundle pair with the base space B consists of a total pair $(E, \dot{E})$, a fiber pair $(F, \dot{F})$ and a projection $p : E \to B$ such that there exist an open covering $\{V\}$ of B and, for each $V \in \{V\}$, a homeomorphism $\varphi_V : V \times (F, \dot{F}) \to (p^{-1}(V), p^{-1}(V) \cap \dot{E})$ such that the composite

$$V \times F \xrightarrow{\varphi_V} p^{-1}(V) \xrightarrow{p} V$$

is the projection to the first factor. If $A \subset B$, we suppose $E_A = p^{-1}(A)$ and $\dot{E}_A = p^{-1}(A) \cap \dot{E}$, and if $b \in B$, then $(E_b, \dot{E}_b)$ is the fiber pair over b.

Given a fiber-bundle pair with a total pair $(E, \dot{E})$ and a fiber pair $(F, \dot{F})$, a cohomology extension of the fiber is a homomorphism $\theta : H^*(F, \dot{F}) \to H^*(E, \dot{E})$ of graded abelian groups (of degree 0) such that for each $b \in B$ the composite

$$H^*(F, \dot{F}) \xrightarrow{\theta} H^*(E, \dot{E}) \longrightarrow H^*(E_b, \dot{E}_b)$$

is an isomorphism, where H^* is the integral singular cohomology.

Let $\overline{p} : B \times (F, \dot{F}) \to (F, \dot{F})$ be the projection to the second factor. Then

$$\theta = \overline{p}^{\,*} : H^*(F, \dot{F}) \longrightarrow H^*(B \times (F, \dot{F}))$$

is a cohomology extension of the fiber of the product-bundle pair.

Theorem of Leray–Hirsch. *Let $(E, \dot{E})$ be the total pair of a fiber-bundle pair with the base B and fiber pair $(F, \dot{F})$. Assume that $H_*(F, \dot{F})$ is free and finitely generated over $\mathbb{Z}$ and that θ is a cohomology extension of the fiber. Then the homomorphism*

$$\Phi^* : H^*(B, C) \otimes H^*(F, \dot{F}) \longrightarrow H^*(E, \dot{E}, G)$$

is an isomorphism for all abelian groups G, where $\Phi^*(u \otimes v) = p^*(u) \smile \theta(v)$, $\smile$ is the cup-product homomorphism.

Proof. By Lemma 5.7.1 [7], it suffices to prove the result for the map Φ^* in the case $G = \mathbb{Z}$.

For any subset $A \subset B$, let θ_A be the composite

$$H^*(F, \dot{F}) \xrightarrow{\theta} H^*(E, \dot{E}) \longrightarrow H^*(E_A, \dot{E}_A).$$

Then θ_A is a cohomology extension of the fiber in the induced bundle over A. It follows from Lemma 5.7.8 [7] that if the induced bundle over A is homeomorphic to the product-bundle pair $A \times (F, \dot{F})$, then

$$\Phi_A^* : H^*(A) \otimes H^*(F, \dot{F}) \xrightarrow{\sim} H^*(E_A, \dot{E}_A).$$

Hence Φ_A^* is a cohomology extension of the fiber in the induced bundle over A.

Using the exact Mayer–Vietoris sequences, property 5.6.20 [7] and also the fact that $H^*(F, \dot{F})$ is a free and finitely generated abelian group, we find that Φ_U^* is an isomorphism for any U which is a finite union of sufficiently small open sets. Let $\mathcal{U} = \{U\}$ be the collection of these sets. Since any compact subset of B lies in some element of $\mathcal{U}$, by Lemma 5, there is an isomorphism

$$H_*(B) \approx \varinjlim_{U \in \mathcal{U}} H_*(U).$$

Also, any compact subset of E lies in some element of $E_{\mathcal{U}} = \{E_U\}$, where $E_U = p^{-1}(U)$, $U \in \mathcal{U}$. Therefore, by Lemma 5, there is an isomorphism

$$H_*(E, \dot{E}) \approx \varinjlim H_*(E_U, \dot{E}_U). \tag{1}$$

Since $C_*(E_U, \dot{E}_U)$ is a subcomplex of the free chain complex $C_*(E_U)$, for the pair $(E_U, \dot{E}_U)$ there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow H^*(E_U, \dot{E}_U) \longrightarrow \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow 0. \tag{2}$$

The collection $\mathcal{U} = \{U\}$ generates the collection $E_{\mathcal{U}} = \{(E_U, \dot{E}_U)\}$ directed by inclusions. Hence the exact sequence (2) induces an exact sequence of inverse systems

$$0 \longrightarrow \{\operatorname{Ext}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z})\} \longrightarrow \{H^*(E_U, \dot{E}_U)\} \longrightarrow \{\operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z})\} \longrightarrow 0.$$

By Theorem 1, there is an exact sequence

$$0 \longrightarrow \varprojlim \operatorname{Ext}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow \varprojlim H^*(E_U, \dot{E}_U) \longrightarrow \varprojlim \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow$$
$$\longrightarrow \varprojlim{}^{(1)} \operatorname{Ext}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow \cdots .$$

Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_{*-1}(E, \dot{E}), \mathbb{Z}) \arrow[r] \arrow[d, "\varphi'"] & H^*(E, \dot{E}) \arrow[r] \arrow[d, "\varphi"] & \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & \varprojlim \operatorname{Ext}(H_{*-1}(E, \dot{E}), \mathbb{Z}) \arrow[r] & \varprojlim H^*(E, \dot{E}) \arrow[r] & \varprojlim \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) \arrow[r] & {}
\end{tikzcd}$$
$$\longrightarrow \varprojlim{}^{(1)} \operatorname{Ext}(H_{*-1}(E, \dot{E}), \mathbb{Z}) \longrightarrow \cdots . \tag{3}$$

Since there is the isomorphism (1), by Lemma 9, using the connection between the functors $\operatorname{Hom}(-, \mathbb{Z})$ and $\varinjlim$, we have an isomorphism

$$\operatorname{Hom}(H_*(E, \dot{E}), \mathbb{Z}) \approx \operatorname{Hom}(\varinjlim H_*(U, \dot{U}), \mathbb{Z}) \approx \varprojlim \operatorname{Hom}(H_*(U, \dot{U}), \mathbb{Z}).$$

Hence in diagram (3), the homomorphism φ'' is an isomorphism, and also, the isomorphisms

$$\operatorname{Ker}\varphi' \approx \operatorname{Ker}\varphi, \quad \operatorname{Coker}\varphi' \approx \operatorname{Coker}\varphi.$$

By Lemma 7 a), there are isomorphisms

$$\operatorname{Ker}\varphi \approx \operatorname{Ker}\varphi' \approx \varprojlim{}^{(1)} \operatorname{Hom}(H_{*-1}(E_U,\dot E_U),\mathbb{Z}), \tag{4}$$

$$\operatorname{Coker}\varphi \approx \operatorname{Coker}\varphi' \approx \varprojlim{}^{(2)} \operatorname{Hom}(H_{*-1}(E_U,\dot E_U),\mathbb{Z}). \tag{5}$$

Using isomorphisms (4) and (5), we have an exact sequence

$$\begin{aligned}0 \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(H_{*-1}(E_U,\dot E_U),\mathbb{Z}) \longrightarrow H^*(E,\dot E) \longrightarrow \varprojlim H^*(E_U,\dot E_U) \longrightarrow \\ \longrightarrow \varprojlim{}^{(2)} \operatorname{Hom}(H_{*-1}(E_U,\dot E_U),\mathbb{Z}) \longrightarrow 0.\end{aligned} \tag{6}$$

Using Lemma 2 [6], for each $U\in\mathcal{U}$, there is the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(B_{*-1},\mathbb{Z}) \arrow[r] \arrow[d] & Z_U^* \arrow[r] \arrow[d] & \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Ext}(H_{*-1}(E_U,\dot E_U),\mathbb{Z}) \arrow[r] & H_U^* \arrow[r] & \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z}) \arrow[r] & 0,
\end{tikzcd}$$

where $B_{*-1}=B_{*-1}(E_U,\dot E_U)$, $Z_U^*=Z^*(E_U,\dot E_U)$, $H_U^*=H^*(E_U,\dot E_U)$, which induces, by Theorem 1, a long commutative diagram with exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \varprojlim{}^{(i)} \operatorname{Hom}(B_{*-1},\mathbb{Z}) \arrow[r] \arrow[d] & \varprojlim{}^{(i)} Z_U^* \arrow[r] \arrow[d] & \varprojlim{}^{(i)} \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \varprojlim{}^{(i)} \operatorname{Ext}(H_{*-1},\mathbb{Z}) \arrow[r] & \varprojlim{}^{(i)} H_U^* \arrow[r] & \varprojlim{}^{(i)} \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z}) \arrow[r] & \cdots,
\end{tikzcd} \tag{7}$$

where $H_{*-1}=H_{*-1}(E_U,\dot E_U)$.

By Lemma 7 b), for $i\geq 1$, there is an isomorphism

$$\varprojlim{}^{(i)} \operatorname{Ext}(B_{*-1},\mathbb{Z}) \approx \varprojlim{}^{(i+2)} \operatorname{Hom}(B_{*-1},\mathbb{Z}). \tag{8}$$

Since B_{*-1} is a free abelian group, there is the equality [3, Theorem 3.5]

$$\operatorname{Ext}(B_{*-1},\mathbb{Z})=0. \tag{9}$$

Using isomorphism (8) and equality (9), for $k\geq 3$, we have the equality

$$\varprojlim{}^{(k)} \operatorname{Hom}(B_{*-1},\mathbb{Z})=0. \tag{10}$$

By Lemma 7 a) and equality (9), there is the equality

$$\varprojlim{}^{(2)} \operatorname{Hom}(B_{*-1},\mathbb{Z})=0. \tag{11}$$

From the commutative diagram (7) and equalities (10), (11), for $i\geq 2$, we have an isomorphism

$$\varprojlim{}^{(i)} Z_U^* \approx \varprojlim{}^{(i)} \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z})$$

and a split exact sequence

$$0 \longrightarrow \varprojlim{}^{(i)} \operatorname{Ext}(H_{*-1}(E_U,\dot E_U),\mathbb{Z}) \longrightarrow \varprojlim{}^{(i)} H^*(E_U,\dot E_U) \longrightarrow \varprojlim{}^{(i)} \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z}) \longrightarrow 0. \tag{12}$$

Using the exact sequence (6), the split exact sequence (12) for $i\geq 2$, the isomorphism φ'' from the commutative diagram (3), we have an exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)} \operatorname{Ext}(H_{*-1}(E_U,\dot E_U),\mathbb{Z}) \longrightarrow \varprojlim{}^{(1)} H^*(E_U,\dot E_U) \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z}) \longrightarrow 0,$$

and using the Yoneda method, we also have a finite exact sequence

$$
\begin{aligned}
0 \longrightarrow \varprojlim{}^{(2*-3)} H^1_U \longrightarrow \cdots \longrightarrow \varprojlim{}^{(1)} H^{*-1}_U \longrightarrow H^*(E,\dot E) \longrightarrow \varprojlim H^*_U \longrightarrow \\
\longrightarrow \varprojlim{}^{(2)} H^{*-1}_U \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2*-2)} H^1_U \longrightarrow 0,
\end{aligned} \tag{13}
$$

where $H^*_U = H^*(E_U, \dot E_U)$.

For the base B, there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{*-1}(B),\mathbb{Z}) \longrightarrow H^*(B) \longrightarrow \operatorname{Hom}(H_*(B),\mathbb{Z}) \longrightarrow 0.$$

Since $H_*(F,\dot F)$ is free and finitely generated over $\mathbb{Z}$, there is an isomorphism $\operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \approx H^*(F,\dot F)$, and, by Lemma 3, there is a short exact sequence

$$
\begin{aligned}
0 \longrightarrow \operatorname{Ext}(H_{*-1}(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow H^*(B)\otimes H^*(F,\dot F) \xrightarrow{\xi} \\
\xrightarrow{\xi} \operatorname{Hom}(H_*(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow 0.
\end{aligned}
$$

Denote $\operatorname{Ker}\xi = \operatorname{Ext}(H_{*-1}(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})$. For each $U \in \mathcal{U} = \{U\}$, there is an exact sequence

$$0 \longrightarrow \operatorname{Ker}\xi_U \longrightarrow H^*(U)\otimes H^*(F,\dot F) \longrightarrow \operatorname{Hom}(H_*(U),\mathbb{Z}) \otimes H^*(F,\dot F) \longrightarrow 0,$$

where $\operatorname{Ker}\xi_U = \operatorname{Ext}(H_{*-1}(U),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})$.

By Lemma 1, there is an isomorphism

$$\operatorname{Hom}(H_*(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \approx \operatorname{Hom}(H_*(B)\otimes H_*(F,\dot F),\mathbb{Z}). \tag{14}$$

The family $\mathcal{U} = \{U\}$ induces an exact sequence of inverse systems

$$0 \longrightarrow \{\operatorname{Ker}\xi_U\} \longrightarrow \{H^*(U)\otimes H^*(F,\dot F)\} \longrightarrow \{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\} \longrightarrow 0,$$

which, by Theorem 1, generate an exact sequence

$$
\begin{aligned}
0 \longrightarrow \varprojlim \{\operatorname{Ker}\xi_U\} \longrightarrow \varprojlim \{H^*(U)\otimes H^*(F,\dot F)\} \longrightarrow \varprojlim \{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\} \longrightarrow \\
\longrightarrow \varprojlim{}^{(1)} \{\operatorname{Ker}\xi_U\} \longrightarrow \cdots .
\end{aligned}
$$

Using isomorphism (14) and Lemma 8, we have an isomorphism

$$
\begin{aligned}
\operatorname{Hom}(H_*(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \approx \operatorname{Hom}(H_*(B)\otimes H_*(F,\dot F),\mathbb{Z}) \\
\approx \operatorname{Hom}(\varinjlim \{H_*(U)\} \otimes H_*(F,\dot F),\mathbb{Z}) \approx \varprojlim \operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z}).
\end{aligned} \tag{15}
$$

Hence, using isomorphism (15) and Lemma 9, there is a commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_{*-1}(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \arrow[r] \arrow[d, "\psi'"] & H^*(B)\otimes H^*(F) \arrow[r] \arrow[d, "\psi"] & {} \\
0 \arrow[r] & \varprojlim \{\operatorname{Ext}(H_*(U),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\} \arrow[r] & \varprojlim \{H^*(U)\otimes H^*(F,\dot F)\} \arrow[r] & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & \operatorname{Hom}(H_*(B)\otimes \operatorname{Hom}(F,\dot F),\mathbb{Z}) \arrow[r] \arrow[d, "\psi''", "\approx"'] & 0 \\
{} \arrow[r] & \varprojlim \{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\} \arrow[r] & 0.
\end{tikzcd} \tag{16}
$$

Since ψ'' is an isomorphism, by the commutative diagram (16), there are isomorphisms

$$\operatorname{Ker}\psi' \approx \operatorname{Ker}\psi, \tag{17}$$

$$\operatorname{Coker}\psi' \approx \operatorname{Coker}\psi. \tag{18}$$

Using Lemmas 6 a) and 9, we have an exact sequence

$$0 \longrightarrow \big(\varprojlim{}^{(1)}\{\mathrm{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow \mathrm{Ext}(H_{*-1}(B),\mathbb{Z})\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow$$
$$\longrightarrow \big(\varprojlim\{\mathrm{Ext}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow$$
$$\longrightarrow \big(\varprojlim{}^{(2)}\{\mathrm{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow 0.$$

Since

$$\mathrm{Ker}\,\psi' = \big(\varprojlim{}^{(1)}\{\mathrm{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z})$$

and

$$\mathrm{Coker}\,\psi' = \big(\varprojlim{}^{(2)}\{\mathrm{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}),$$

using isomorphisms (17) and (18), there is an exact sequence

$$0 \longrightarrow \big(\varprojlim{}^{(1)}\{\mathrm{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow H^*(B)\otimes H^*(F,\dot F) \longrightarrow$$
$$\longrightarrow \varprojlim\{H^*(U)\otimes H_*(F,\dot F)\} \longrightarrow \big(\varprojlim{}^{(2)}\{\mathrm{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow 0. \tag{19}$$

By Lemma 1 [6], for each $U\in\mathcal{U}=\{U\}$, there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}(B_{*-1}(U),\mathbb{Z}) \longrightarrow Z^*(U) \longrightarrow \mathrm{Hom}(H_*(U),\mathbb{Z}) \longrightarrow 0.$$

Since $\mathrm{Hom}(H_*(F,\dot F),\mathbb{Z})$ is free and finitely generated, by Lemma 3, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(B_{*-1}(U),\mathbb{Z})\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow Z^*(U)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}(H_*(U),\mathbb{Z})\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow 0. \tag{20}$$

Using Lemma 1, for the exact sequence (20), there is an exact sequence of inverse systems

$$0 \longrightarrow \Big\{\mathrm{Hom}(B_{*-1}(U))\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow \Big\{Z^*(U)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow$$
$$\longrightarrow \Big\{\mathrm{Hom}(H_*(U))\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow 0,$$

which, by Theorem 1, generates an exact sequence

$$0 \longrightarrow \varprojlim \Big\{\mathrm{Hom}(B_{*-1}(U))\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow \varprojlim \Big\{Z^*(U)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow$$
$$\longrightarrow \varprojlim \Big\{\mathrm{Hom}(H_*(U))\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow \cdots. \tag{21}$$

Since $B_{*-1}(U)$ and $H_*(F,\dot F)$ are free abelian groups, by Lemma 4, $B_{*-1}(U)\otimes H_C(F,\dot F)$ is a free abelian group. Using Lemma 7, we have

a) an epimorphism

$$\varprojlim \mathrm{Ext}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) \longrightarrow \varprojlim{}^{(2)} \mathrm{Hom}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) \longrightarrow 0;$$

b) an isomorphism

$$\varprojlim{}^{(i)} \mathrm{Ext}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) \approx \varprojlim{}^{(i+2)} \mathrm{Hom}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) \quad \text{for } \ i\geq 1;$$

Therefore, there is the equality

$$\varprojlim{}^{(i)} \mathrm{Hom}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) = 0, \quad i\geq 2. \tag{22}$$

From the exact sequence (21) and equality (22) it follows that for $i\geq 2$, there is an isomorphism

$$\varprojlim{}^{(i)}\{Z^*(U)\otimes \mathrm{Hom}(H_*(F,\dot F),\mathbb{Z})\} \approx \varprojlim{}^{(i)}\{\mathrm{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\}.$$

Consider the commutative diagram

$$\begin{tikzcd}
\varprojlim^{(i)}\{Z^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\} \arrow[r, phantom, "\approx"] \arrow[d] & \varprojlim^{(i)}\{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\} \arrow[d, equal] \\
\varprojlim^{(i)}\{H^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\} \arrow[r] & \varprojlim^{(i)}\{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\}.
\end{tikzcd}$$

For $i\ge 2$, there is a split exact sequence

$$\begin{gathered}
0\longrightarrow \varprojlim^{(i)}\{\operatorname{Ext}(H_{*-1}(U),\mathbb{Z})\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\}\longrightarrow \varprojlim^{(i)}\{H^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\}\longrightarrow\\
\longrightarrow \varprojlim^{(i)}\{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\}\longrightarrow 0.
\end{gathered}\tag{23}$$

Using the exact sequence (19), the split exact sequence (23), Lemma 6 b), Lemma 7 b) and the Yoneda method, we have a finite exact sequence

$$\begin{gathered}
0\longrightarrow \varprojlim^{(2*-3)}\{H^1(U)\otimes H^*(F,\dot F)\}\longrightarrow\cdots\longrightarrow \varprojlim^{(1)}\{H^{*-1}(U)\otimes H^*(F,\dot F)\}\longrightarrow\\
\longrightarrow H^*(B)\otimes H^*(F,\dot F)\longrightarrow \varprojlim\{H^*(U)\otimes H_*(F,\dot F)\}\longrightarrow \varprojlim^{(2)}\{H^{*-1}(U)\otimes H^*(F,\dot F)\}\longrightarrow\\
\longrightarrow\cdots\longrightarrow \varprojlim^{(2*-2)}\{H^1(U)\otimes H^*(F,\dot F)\}\longrightarrow 0.
\end{gathered}\tag{24}$$

Exact sequences (13), (24) and the homomorphisms Φ^* and $\{\Phi_U^*\}$ induce a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \varprojlim^{(3)}\left\{H^{*-2}(U)\otimes H^*(F,\dot F)\right\} \arrow[r] \arrow[d, "\varprojlim^{(3)}\Phi_U^*"'] & \varprojlim^{(1)}\left\{H^{*-1}(U)\otimes H^*(F,\dot F)\right\} \arrow[r] \arrow[d, "\varprojlim^{(1)}\Phi_U^*"'] & H^*(B)\otimes H^*(F,\dot F) \arrow[r] \arrow[d, "\Phi^*"'] & {} \\
\cdots \arrow[r] & \varprojlim^{(3)}H^{*-2}(E_U,\dot E_U) \arrow[r] & \varprojlim^{(1)}H^{*-1}(E_U,\dot E_U) \arrow[r] & H^*(E,\dot E) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \varprojlim\left\{H^*(U)\otimes H^*(F,\dot F)\right\} \arrow[r] \arrow[d, "\varprojlim\Phi_U^*"'] & \varprojlim^{(2)}\left\{H^{*-1}(U)\otimes H^*(F,\dot F)\right\} \arrow[r] \arrow[d, "\varprojlim^{(2)}\Phi_U^*"'] & \cdots \\
{} \arrow[r] & \varprojlim H^*(E,\dot E) \arrow[r] & \varprojlim^{(2)}H^{*-1}(E,\dot E) \arrow[r] & \cdots.
\end{tikzcd}\tag{25}$$

By Theorem 5.7.10 [7], for each $U\in\mathcal{U}=\{U\}$, there is an isomorphism

$$\Phi_U^*: H^*(U)\otimes H^*(F,\dot F)\xrightarrow{\sim} H^*(E_U,\dot E_U).$$

Hence the homomorphism $\{\Phi_U^*\}$ of inverse systems

$$\{\Psi_U^*\}:\{H^*(U)\otimes H^*(F,\dot F)\}\longrightarrow\{H^*(E_U,\dot E_U)\}$$

is an isomorphism and for $i\ge 0$ induces an isomorphism

$$\varprojlim^{(i)}\{H^*(U)\otimes H^*(F,\dot F)\}\xrightarrow{\sim}\varprojlim^{(i)}\{H^*(E_U,\dot E_U)\}.\tag{26}$$

By five Lemma [7, Lemma 4.5.11] and isomorphisms (26), from the commutative diagram (25) it follows that Φ^* is an isomorphism. $\square$

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A COMPUTATIONAL METHOD FOR SOLVING THE SYSTEM OF HAMILTON–JACOBI–BELLMAN PDES IN NONZERO-SUM FIXED-FINAL-TIME DIFFERENTIAL GAMES

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Abstract. In this study, the shifted Chebyshev-Gauss collocation method (SC-GCM) is used for finding the Nash equilibrium solution of nonzero-sum differential games with fixed-final-time. The search for the Nash equilibrium solutions in a feedback form usually leads to a nonlinear system of Hamilton–Jacobi–Bellman (HJB) PDEs. In the proposed approach, by applying the SC-GCM and pursuing the idea of value functions approximation, the system of HJB PDEs is reduced to a system of algebraic equations. By this method, a Nash equilibrium solution can be approximated as a function of the time and the current state by Chebyshev polynomials. The main advantage of this method is that the boundary conditions of the system of HJB PDEs can be included explicitly in the chosen approximations of value functions, which states that the boundary conditions are satisfied automatically. In view of the convergence of the method, several examples are given to demonstrate the accuracy and efficiency of the proposed method.

1. Introduction

Dynamic game is a practically significant discipline in many different fields such as engineering, ecology, management and economics. Differential game studies the situation that involves several Decision-Makers (or Players) with different objectives, where each Player looks for minimization (or maximization) of his own individual criterion. Nonzero-sum games were introduced in the works of Starr and Ho [44, 45]. For a detail treatment of differential games, we refer the reader to Nash [38], Basar and Olsder [4], Engwerda [13], Friesz [19], Yeung and Petrosyan [49] and Bressan [10].

Research in differential games is focused in the first place on extending control theory to incorporate strategic behavior [49]. Bellman's dynamic programming for solving optimal control problems leads to the Hamilton–Jacobi–Bellman (HJB) equation, which is challenging due to its inherently nonlinear nature. HJB equations have been solved by using different techniques. For example, variational iteration method was applied for nonlinear quadratic optimal control problems in [33]. Saberi and Effati [41] proposed a computational method to generate suboptimal solutions for a class of nonlinear optimal control problems.

The feedback Nash equilibrium strategies in non-zero sum games, where the strategies of players are allowed to depend on time and also on the current state, can be found by solving a highly nonlinear system of Hamilton–Jacobi–Bellman (HJB) PDEs, which are derived from the principle of dynamic programming (see, for example, [4, 10, 13, 19, 33, 41, 49]).

Due to the difficulty in solving nonlinear HJB PDEs, the existence and continuity of the feedback Nash equilibria are mainly considered in linear-quadratic dynamic games. Starr and Ho in [45] derived the sufficient conditions of the existence of a linear feedback equilibrium for a finite-horizon planning, which can be obtained via solving a system of Riccati equations. For more details on nonzero-sum linear-quadratic games see [1, 14–16, 32].

Compared to the linear-quadratic case, not many works are devoted to the nonlinear differential games. Jiménez-Lizárraga et al. [36] studied the state-dependent Riccati equations for a certain

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class of nonlinear polynomial games to obtain open-loop quasi-equilibrium. Kossiorisa et al. [34] provided a solution in a particular case of a nonlinear game representing a pollution and resource management problem. Nikooeinejad et al. employed the pseudospectral method to compute the open-loop Nash and saddle point equilibria for nonlinear nonzero-sum differential games and min-max optimal control problems (M-MOCPs) with uncertainty, respectively [39, 40]. An iterative adaptive dynamic programming method for solving a class of nonlinear zero-sum differential games is used to obtain saddle point of the zero-sum differential games (see [51]). The synchronous PI method in [47] was generalized to solve a multi-player nonzero-sum game for nonlinear continuous-time dynamic systems.

To the extent of our knowledge, the focus of the above paper is on the theoretical analysis rather than the numerical algorithms.

Although, setting up the system of HJB PDEs to obtain feedback Nash equilibrium solutions is not difficult, but in general the difficulty in solving the system of HJB equations remains the biggest problem to the practical application of nonlinear systems.

The methods from numerical analysis, such as Galerkin's method, can be used to convert the HJB equations from a continuous operator to a discrete problem. The existing references in this area to solve the Hamilton-Jacobi-Isaacs (HJI) equations for zero-sum differential games include Georges [20], Beard [5–7], Alamir [2], and Ferreira [17]. Disadvantage of Galerkin's method is that the evaluation of coefficients depends on the computation of definite integrals.

Our goal of this paper is to introduce a simple computational method that is able to address nonlinear system dynamics. The pseudospectral or collocation methods are the one of best tools for solving ordinary or partial differential equations with a high accuracy [11, 21–31, 37, 50]. A simple way to approximate the value functions of each player is by defining as a linear combinations of polynomial basis functions, and equalizing the residual functions to zero at collocation points to search for the associated coefficients. In this approach, Runge's phenomenon shows that the selection of nodes and the choice of basis function play an important role in the quality of the approximation. The shifted Jacobi polynomials are a well-known class of polynomials exhibiting exponential or sometimes super-exponential convergence, of which particular examples are the first and second kinds of Chebyshev and Legendre polynomials [8, 12, 43]. It is shown that by selecting a limited number of shifted Chebyshev collocation points, the excellent numerical results are obtained. The solution to the system of HJB PDEs (or the value functions for each Player) must be satisfied in the boundary conditions, therefore, the boundary conditions play a much more crucial role in the chosen form for the value functions approximation. In the present paper we intend to extend a simple and efficient numerical method based on value functions approximation and shifted Chebyshev-Gauss collocation method for finding Nash equilibrium solutions of nonzero-sum differential games.

The remainder of this paper is organized as follows. In Section 2, we introduce the nonzero-sum dynamic games and the formulation of the system of HJB PDEs. Some preliminary details about the SC-GCM are given in Section 3. In Section 4, the presented technique is used to approximate the value functions and the Nash equilibrium solutions of nonzero-sum dynamic games. Some numerical examples are given in Section 5 to show the efficiency of the proposed method. Finally, a brief conclusion is drawn in Section 6.

2. Problem Statement

Consider an $n-$person nonzero-sum differential game, where the players' dynamics is governed by the following nonlinear differential equation [4, 49]:

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), u_1(t), u_2(t), \dots, u_n(t)), \quad t \in [t_0, T], \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where $f(t, x(t), u_1(t), u_2(t), \dots, u_n(t)) = f_0(x(t)) + \sum_{j=1}^{n} g_j(x(t))u_j(t)$. We assume that $f_0(0) = 0$, $f_0(x)$ and $g_j(x)$ are Lipschitz continuous on a compact set $\Omega \in \mathbb{R}^m$ containing the origin, and the system is stabilizable on Ω. Define the finite horizon cost functions associated with Player i as:

$$J_k(u_1, u_2, \ldots, u_n) = \int_{t_0}^{T} L_k(t, x(t), u_1(t), \ldots, u_n(t))dt + \psi_k(x(T)),$$

where $L_k(t, x(t), u_1(t), \ldots, u_n(t)) = x^T Q_k x + \sum_{j=1}^{n} u_j^T R_{kj} u_j$, $x(t) \in \mathbb{R}^m$ is the state vector of the game, $u_k(t) \in U_k \subset \mathbb{R}^{m_k}$ is the control function implemented by the k-th Player and $Q_k \in \mathbb{R}^{m\times m}$, $R_{kj} \in \mathbb{R}^{m_j\times m_j}$ are symmetric positive definite matrices. Also, the functions $f_0(x)$, $g_k(x)$ and $\psi_k(x)$ for $k = 1, 2, \ldots, n$ are the differentiable functions.

It is desirable to find the optimal control vector $\{u_1^*, u_2^*, \ldots, u_n^*\}$ such that for $k = 1, 2, \ldots, n$, controls u_k^* are continuous, u_k^* stabilize (1) on Ω, $\forall x_0 \in \Omega$, $J_k(u_1^*, u_2^*, \ldots, u_n^*)$ are finite, and the cost functions (2) are minimized.

The control vector $\{u_1^*, u_2^*, \ldots, u_n^*\}$ corresponds to the Nash equilibrium solution of the game.

To find Nash equilibrium solutions, we need to consider a family of problems having a unique Nash equilibrium solution. Here we describe an important class of problems where this assumption is satisfied.

Lemma 2.1 ([10]). *Assume that the dynamics and the running costs take the decoupled form*

$$f(t, x, u_1, \ldots, u_n) = f_0(x) + \sum_{k=1}^{n} g_k(x) u_k, \tag{2}$$

$$L_k(t, x, u_1, \ldots, u_n) = \sum_{j=1}^{n} L_{kj}(t, x, u_j), \quad k = 1, \ldots, n.$$

Also, assume that

(i) *The domains $U_k (k = 1, \ldots, n)$ are closed and the convex subsets of R^{m_k} are, possibly, unbounded.*

(ii) *The functions $g_k(x)$ depend continuously on t, x.*

(iii) *The functions $u_k \mapsto L_{kk}(t, x, u_k)$ are strictly convex.*

(iv) *For each $k = 1, \ldots, n$, either U_k is compact, or L_{kk} has superlinear growth*

$$\lim_{|\omega|\to\infty} \frac{L_{kk}(t, x, \omega)}{u_k} = +\infty, \quad k = 1, \ldots, n.$$

Then for every $(t, x) \in [0, T] \times R^m$ and any vector $p_k \in R^m (k = 1, \ldots, n)$, there exists a unique set $(u_1^*(t), \ldots, u_n^*(t)) \in U_1 \times \cdots \times U_n$ such that

$$u_k^* = \arg\min_{\omega\in U_k} \{L_{kk}(t, x, \omega) + p_k . g_k(x)\omega\}.$$

We consider here the case, where both players can observe the current state of the system. The value functions $V_k(t, x)$, $k = 1, 2, \ldots, n$ associated with the admissible control policies $u_k \in U_k$ are defined as follows:

$$V_k(t, x) = \min_{u_k\in U_k} \left\{ \int_{t}^{T} L_k(t, x, u_1, \ldots, u_n)dt + \psi_k(x(T)) \right\}. \tag{3}$$

Assume that the value functions (3) are continuously differentiable. By Bellman's optimality and the dynamic programming principle, the optimal cost functions defined in (3) are satisfied the following system of Hamilton–Jacobi–Bellman (HJB) PDEs:

$$\begin{aligned} 0 =& \frac{\partial}{\partial t} V_k(t, x) \\ &+ \min_{u_k\in U_k} \left\{ L_k(t, x, u_1, \ldots, u_n) + \Big(\frac{\partial}{\partial x} V_k(t, x)\Big)^T \Big(f_0(x) + \sum_{j=1}^{n} g_j(x) u_j \Big) \right\}, \\ k =& 1, 2, \ldots, n, \end{aligned} \tag{4}$$

with the boundary conditions $V_k(T,x)=\psi_k(x)$, $k=1,2,\ldots,n$. Define the Hamiltonian functions as

$$H_k\left(t,x,u_1,\ldots,u_n,\frac{\partial}{\partial x}V_k\right)=\left(\frac{\partial}{\partial x}V_k\right)^T\left(f_0(x)+\sum_{j=1}^{n}g_j(x)u_j\right)$$
$$+L_k(t,x,u_1,\ldots,u_n),\quad k=1,2,\ldots,n.$$

Then the associated state feedback control policies can be obtained by

$$\frac{\partial H_k}{\partial u_k}=0\Rightarrow u_k^*(t,x)=-\frac{1}{2}R_{kk}^{-1}g_k^T(x)\frac{\partial}{\partial x}V_k(t,x),\quad k=1,2,\ldots,n. \tag{5}$$

Substitution of (5) into (4) yields the following n-coupled HJB equations:

$$\begin{aligned}0=&\frac{\partial}{\partial t}V_k(t,x)+x^TQ_kx+\left(\frac{\partial}{\partial x}V_k(t,x)\right)^T f_0(x)\\&-\frac{1}{2}\left(\frac{\partial}{\partial x}V_k(t,x)\right)^T\sum_{j=1}^{n}g_j(x)R_{jj}^{-1}g_j^T(x)\frac{\partial}{\partial x}V_j(t,x)\\&+\frac{1}{4}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x}V_j(t,x)\right)^T g_j(x)R_{jj}^{-1}R_{jj}R_{jj}^{-1}g_j^T(x)\frac{\partial}{\partial x}V_j(t,x),\\&k=1,2,\ldots,n,\end{aligned} \tag{6}$$

with the boundary conditions

$$V_k(T,x)=\psi_k(x),\quad k=1,2,\ldots,n. \tag{7}$$

The system of Hamilton–Jacobi–Bellman PDE equations (6) with boundary conditions (7) cannot generally be solved due to its nonlinear nature. We intend to solve system (6) and (7) by the shifted Chebyshev-Gauss collocation method (SC-GCM).

3. Some Properties of Chebyshev Polynomials

In this section, we introduce some basic properties of the Chebyshev polynomials that we use in the CSCM as the function approximation structures.

The Chebyshev polynomials $T_n(z), n=0,1,2,\ldots$ are the eigenfunctions of the singular Sturm-Liouville problem

$$(1-z^2)T_n''(z)-zT_n'(z)+n^2T_n(z)=0.$$

They are orthogonal with respect to the L_w^2 inner product on the interval $[-1,1]$ with the weight function $w(z)=\frac{1}{\sqrt{1-z^2}}$. The Chebyshev polynomials satisfy the recurrence formula as follows:

$$T_{n+1}(z)=2zT_n(z)-T_{n-1}(z),\quad n=1,2,\ldots,$$

where $T_0(z)=1$ and $T_1(z)=z$. For practical use of the Chebyshev polynomials on the interval $[a,b]$, it is necessary to shift the defining domain by the following variable substitution:

$$z=\frac{2}{b-a}t-\frac{b+a}{b-a}.$$

Let the shifted Chebyshev polynomials $T_n(\frac{2}{b-a}t-\frac{b+a}{b-a})$ be denoted by $T_n^*(t)$. Then these polynomials can be obtained by using the following recurrence formula:

$$T_{n+1}^*(t)=\left(4\left(\frac{t}{b-a}\right)-2\left(\frac{b+a}{b-a}\right)\right)T_n^*(t)-T_{n-1}^*(t),\quad n=1,2,\ldots,$$

where $T_0^*(t)=1$ and $T_1^*(t)=\frac{2}{b-a}t-\frac{b+a}{b-a}$.

Now, let the shifted Chebyshev polynomials $T_n(\frac{2}{b-a}t-\frac{b+a}{b-a})$ and $T_n(\frac{2}{d-c}x-\frac{d+c}{d-c})$ be denoted by $T_n^*(t)$ and $T_n^*(x)$, respectively.

Similarly, an arbitrary function of two variables $f(t,x) \in L^2_w([a,b]\times[c,d])$, can be approximated by the shifted Chebyshev polynomials as:

$$f(t,x) \simeq \tilde{f}(t,x) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} f_{ij} T_i^*(t) T_j^*(x),$$

with

$$f_{ij} = \frac{4}{\pi^2 c_i c_j} \int_{-1}^{1}\int_{-1}^{1} \frac{f(\frac{b-a}{2}t + \frac{b+a}{2}, \frac{d-c}{2}x + \frac{d+c}{2}) T_i(t) T_j(x)}{\sqrt{1-t^2}\sqrt{1-x^2}} dt dx,$$
$$i = 0,1,\dots,N_1, \quad j = 0,1,\dots,N_2.$$

The fundamental results of the proposed method are based on the remarkable Weierstrass Theorem and approximability of orthogonal polynomials [9, 43].

Theorem 3.1 ([42])**.** *If the function $f(t,x)$ has the second order continuous derivatives, then*

$$|f_{i,0}| \le \frac{2\gamma_{2,0}}{(i-1)^2}, \quad |f_{i,1}| \le \frac{8\gamma_{2,0}}{\pi(i-1)^2}, \quad i > 1,$$
$$|f_{0,j}| \le \frac{2\gamma_{0,2}}{(j-1)^2}, \quad |f_{1,j}| \le \frac{8\gamma_{0,2}}{\pi(j-1)^2}, \quad j > 1,$$

where $f(t,x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij} T_i(t) T_j(x)$, $\tilde{f}(t,x) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} f_{ij} T_i(t) T_j(x)$, $\gamma_{2,0} \ge \max\{|\frac{\partial^2 f}{\partial t^2}(t,x)| : t,x \in [-1,1]\}$, *and* $\gamma_{0,2} \ge \max\{|\frac{\partial^2 f}{\partial x^2}(t,x)| : t,x \in [-1,1]\}$.

Theorem 3.2 ([42])**.** *If the function $f(t,x)$ has the second order continuous partial derivatives, then* $\lim_{N_1,N_2\to\infty} \tilde{f}(t,x) = f(t,x)$ *uniformly in* $[-1,1]$ *and*

$$|f(t,x) - \tilde{f}(t,x)| \le \sqrt{6}\left(\frac{20\gamma_{0,2}^2}{(N_2-1)^2} + \frac{20\gamma_{2,0}^2}{(N_1-1)^2} + \frac{\pi^2\gamma_{1,1}^2}{6N_2} + \frac{\pi^2\gamma_{1,1}^2}{6N_1}\right)^{\frac{1}{2}}.$$

For obtaining the first partial derivatives $\frac{\partial}{\partial t}\tilde{f}(t,x)$ and $\frac{\partial}{\partial x}\tilde{f}(t,x)$, we can rewrite $\tilde{f}(t,x)$ as:

$$\tilde{f}(t,x) = \sum_{i=0}^{N_1} A_i(x) T_i^*(t), \quad \text{with} \quad A_i(x) = \sum_{j=0}^{N_2} f_{ij} T_j^*(x),$$

or

$$\tilde{f}(t,x) = \sum_{j=0}^{N_2} B_j(t) T_j^*(x), \quad \text{with} \quad B_j(t) = \sum_{i=0}^{N_1} f_{ij} T_i^*(t).$$

Then the first partial derivatives of $\tilde{f}(t,x)$ can be obtained as:

$$\frac{\partial}{\partial t}\tilde{f}(t,x) = \frac{2}{b-a}\sum_{i=0}^{N_1} A_i^{(1)}(x) T_i^*(t), \tag{8}$$

$$\frac{\partial}{\partial x}\tilde{f}(t,x) = \frac{2}{d-c}\sum_{j=0}^{N_2} B_j^{(1)}(t) T_i^*(x), \tag{9}$$

where the coefficients $A_i^{(1)}(x)$, $i = 0,1,\dots,N_1$ and $B_j^{(1)}(t)$, $j = 0,1,\dots,N_2$ are:

$$A_i^{(1)}(x) = \frac{2}{c_i} \sum_{\substack{p=i+1 \\ (p+i)\ odd}}^{N_1} p A_p(x), \ i = 0,\dots,N_1-1, \ A_{N_1}^{(1)}(x) = 0, \tag{10}$$

$$B_j^{(1)}(t) = \frac{2}{c_j} \sum_{\substack{q=j+1 \\ (p+j)\ odd}}^{N_2} q B_q(t), \ j = 0,\dots,N_2-1, \ B_{N_2}^{(1)}(t) = 0. \tag{11}$$

4. Nonlinear Fixed-Final-Time n-Coupled HJB Solution by the Collocation Method

The n-coupled HJB equations (6) and (7) are difficult to solve for the cost functions $V_k(t,x)$. In this section, the direct collocation method is used to solve approximately the value functions in (6) over Ω by approximating the cost functions $V_k(t,x)$ and their partial derivatives as Chebyshev polynomials. We assume that $V_k(t,x)$, $k=1,2,\dots,n$ are smooth. Therefore, one can use approximate cost functions $V_k(t,x)$ for $t\in[0,T]$ and a compact set $\Omega\subset\mathbb{R}^m$ as follows:

$$\begin{aligned} V_k(t,x) \simeq &\widetilde{V}_k(t,x)\\ =&(t-t_{N_1})\left(\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} v_{ij}^k T_i^*(t)T_j^*(x)\right)+\psi_k(x), \quad k=1,2,\dots,n. \end{aligned} \tag{12}$$

Before describing the method, it should be pointed out that this method is introduced for $x\in\mathbb{R}$, however, it can be extended easily to $x\in\mathbb{R}^m$. Our aim is to approximate the solution of system (6) and (7) for the time horizon $[t_0,T]$ and the state domain $\Omega=[x_{\min},x_{\max}]$. So, we define:

$$\begin{aligned} t_r =&\frac{T-t_0}{2}\left(\cos\left(\frac{(N_1-r)\pi}{N_1}\right)\right)+\frac{T+t_0}{2}, \quad r=0,1,\dots,N_1, \qquad (13)\\ x_s =&\frac{x_{\max}-x_{\min}}{2}\left(\cos\left(\frac{(N_2-s)\pi}{N_2}\right)\right)+\frac{x_{\max}+x_{\min}}{2}, \quad s=0,1,\dots,N_2, \end{aligned}$$

which are named as shifted Chebyshev–Gauss–Lobatto nodes. In fact, these points are zeros of the $(t-t_0)(T-t)\dot{T}^*_{N_1}(t)$ and $(x-x_{\min})(x_{\max}-x)\dot{T}^*_{N_2}(x)$, respectively.

By the grid points defined in (13), and substituting t_{N_1} into (12), we have:

$$\widetilde{V}_k(t_{N_1},x)=\widetilde{V}_k(T,x)=\psi_k(x), \quad k=1,2,\dots,n,$$

which guarantee the boundary conditions for the cost functions $V_k(t,x)$ and for $k=1,2,\dots,n$ are satisfied automatically.

In addition, from equations (8) and (9), we can get the partial derivatives $\frac{\partial}{\partial t}\widetilde{V}_k(t,x)$ and $\frac{\partial}{\partial x}\widetilde{V}_k(t,x)$ as follows:

$$\begin{aligned} \frac{\partial}{\partial t}\widetilde{V}_k(t,x) =&\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} v_{ij}^k T_i^*(t)T_j^*(x)+\frac{2(t-t_{N_1})}{T-t_0}\sum_{i=0}^{N_1} A_i^{(1)}(x)T_i^*(t),\\ \frac{\partial}{\partial x}\widetilde{V}_k(t,x) =&\frac{2(t-t_{N_1})}{x_{\max}-x_{\min}}\sum_{j=0}^{N_2} B_j^{(1)}(t)T_j^*(x)+\frac{\partial\psi_k(x)}{\partial x},\\ k =&1,2,\dots,n, \end{aligned}$$

where the coefficients $A_i^{(1)}(x)$, $i=0,1,\dots,N_1$ and $B_j^{(1)}(t)$, $j=0,1,\dots,N_2$ can be obtained from equations (10) and (11).

Approximating $V_k(t,x)$, $\frac{\partial}{\partial t}V_k(t,x)$ and $\frac{\partial}{\partial x}V_k(t,x)$ in the n-coupled HJB equations (6) by $\widetilde{V}_k(t,x)$, $\frac{\partial}{\partial t}\widetilde{V}_k(t,x)$ and $\frac{\partial}{\partial x}\widetilde{V}_k(t,x)$, respectively, we have

$$\begin{aligned} \mathrm{Res}_{V_k}(t,x) =&\frac{\partial}{\partial t}\widetilde{V}_k(t,x)+x^TQ_ix+\left(\frac{\partial}{\partial x}\widetilde{V}_k(t,x)\right)^T f_0(x)\\ &-\frac{1}{2}\left(\frac{\partial}{\partial x}\widetilde{V}_k(t,x)\right)^T\sum_{j=1}^{n} g_j(x)R_{jj}^{-1}g_j^T(x)\frac{\partial}{\partial x}\widetilde{V}_j(t,x)\\ &+\frac{1}{4}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x}\widetilde{V}_j(t,x)\right)^T g_j(x)R_{jj}^{-1}R_{jj}R_{jj}^{-1}g_j^T(x)\frac{\partial}{\partial x}\widetilde{V}_j(t,x),\\ &k=1,2,\dots,n, \end{aligned}$$

where $\mathrm{Res}_{V_k}(t,x)$, $k=1,2,\dots,n$ are the residual equations error. To find the coefficients v_{ij}^ks, the method of weighted residuals is used.

Consider the expression

$$< \mathrm{Res}_{V_k}(t,x), W_{r,s} >= \Big[\int_{t_0}^{T}\int_{\Omega} \mathrm{Res}_{V_k}(t,x).W_{r,s}\, d\Omega dt\Big], \tag{14}$$

where $W_{r,s}$, $r = 0, 1, \dots, N_1$, $s = 0, 1, \dots, N_2$ are the suitable functions.

The coefficients v_{ij}^ks will be selected to minimize residual equations error in a collocation sense over a set of points sampled from a compact set $[t_0, T] \times \Omega$.

To this end, the coefficients v_{ij}^ks are determined by projecting the residual errors onto the Dirac delta function and setting the results to zero $\forall x \in \Omega$ and $t \in [0, T]$.

Setting $P^{rs} = (t_r, x_s)$, we define:

$$W_{r,s} = \delta(P^{rs}), \quad r = 0, 1, \dots, N_1, \quad s = 0, 1, \dots, N_2, \tag{15}$$

where $\delta(P^{rs})$ is the Dirac delta function. By substituting (15) into (14), the coefficients v_{ij}^ks are obtained from equalizing $\mathrm{Res}_{V_k}(t_r, x_s)$ to zero at the collocation points as follows:

$$\begin{aligned} &< \mathrm{Res}_{V_k}(t,x), \delta(P^{rs}) >= \mathrm{Res}_{V_k}(t_r, x_s) = 0, \\ &r = 0, 1, \dots, N_1, \quad s = 0, 1, \dots, N_2, \quad k = 1, 2, \dots, n. \end{aligned} \tag{16}$$

Equations (16) generate a set of $n(N_1 + 1)(N_2 + 1)$ nonlinear algebraic equations that can be solved by the Newton method for the unknown coefficients v_{ij}^ks. Consequently, the cost functions $\widetilde{V}_k(t, x)$, $k = 1, 2, \dots, n$ can be calculated.

From (5), the corresponding Nash equilibrium solutions as a function of the time and the state are approximated as:

$$\widetilde{u}_k(t,x) = -\frac{1}{2} R_{kk}^{-1} g_k^T(x) \frac{\partial}{\partial x} \widetilde{V}_k(t,x), \quad k = 1, 2, \dots, n.$$

5. Illustrative Example

To demonstrate the application of the shifted Chebyshev-Gauss collocation method (SC-GCM) and its performance for finding feedback Nash equilibrium solution of nonzero-sum dynamic games, several examples are examined in this section. Example 5.1 is a linear-quadratic dynamic game that can be solved analytically. This allows one to verify the validity of the method by comparing with the results of exact solution. The analytic solution for Examples 5.2 and 5.3 is unachievable. It should be noted that for Example 5.2, the results obtained by the proposed method coincide with those obtained by the variables separation method.

Example 5.1. Consider the linear-quadratic nonzero-sum differential game defined by the system [4]

$$\dot{x}(t) = \sqrt{2}u_1(t) - u_2(t), \quad x(0) = 1, \quad 0 \le t \le T = 2,$$

and the performance criteria of Players 1 and 2 as follows:

$$\begin{aligned} \min_{u_1} J_1 &= \int_0^T (u_1^2(t) - u_2^2(t))dt + \frac{1}{2}x(T)^2, \\ \min_{u_2} J_2 &= \int_0^T (u_2^2(t) - u_1^2(t))dt - \frac{1}{2}x(T)^2. \end{aligned}$$

The exact solution for the feedback Nash equilibrium of this problem is

$$\begin{aligned} V_1(t,x) &= -V_2(t,x) = \frac{x^2}{2(3-t)}, \\ u_1^*(t,x) &= \frac{-\sqrt{2}x}{3-t}, \\ u_2^*(t,x) &= \frac{-x}{3-t}. \end{aligned}$$

TABLE 1. The numerical optimal value of cost functionals $J_i, i = 1, 2$ obtained by using the SC-GCM as compared with the exact solutions for Example 5.1

$(N_1,\ N_2)$	J_1	J_2	$\mid J_i - J_i^* \mid, i = 1, 2$
(2, 2)	0.1708622317	-0.1708622317	0.0041955650
(4, 2)	0.1666009734	-0.1666009734	0.0000656933
(6, 2)	0.1666658930	-0.1666658930	7.73×10^{-7}
(8, 2)	0.1666666471	-0.1666666471	1.96×10^{-8}
(10, 2)	0.1666666660	-0.1666666660	7.00×10^{-10}

As is discussed in Section 2, the HJB equations system for this problem has the following form:

$$\begin{aligned} V_{1,t}(t,x) + \min_{u_1}\left\{\frac{1}{2}(u_1(t)^2 - u_2(t)^2) + V_{1,x}(t,x)(\sqrt{2}u_1(t) - u_2(t))\right\} = 0, \\ V_{2,t}(t,x) + \min_{u_2}\left\{\frac{1}{2}(u_2(t)^2 - u_1(t)^2) + V_{2,x}(t,x)(\sqrt{2}u_1(t) - u_2(t))\right\} = 0, \end{aligned} \tag{17}$$

with the boundary conditions

$$V_1(2, x(2)) = -V_2(2, x(2)) = \frac{1}{2}x^2(2).$$

The corresponding Hamiltonian functions are given in the form

$$\begin{aligned} H_1(t, x, u_1, u_2, V_{1,x}) &= \frac{1}{2}(u_1(t)^2 - u_2(t)^2) + V_{1,x}(t,x)(\sqrt{2}u_1(t) - u_2(t)), \\ H_2(t, x, u_1, u_2, V_{2,x}) &= \frac{1}{2}(u_2(t)^2 - u_1(t)^2) + V_{2,x}(t,x)(\sqrt{2}u_1(t) - u_2(t)). \end{aligned}$$

Differentiating $H_1(t, x, u_1, u_2, V_{1,x})$ and $H_2(t, x, u_1, u_2, V_{2,x})$ with respect to u_1 and u_2, respectively, and by finding the functions u_1 and u_2, where these derivatives tend to zero, we have

$$\begin{aligned} u_1^*(t,x) &= -\sqrt{2}V_{1,x}(t,x), \\ u_2^*(t,x) &= V_{2,x}(t,x). \end{aligned}$$

Now, by substituting u_1^* and u_2^* into HJB equations system (17), we have the following partial differential equations:

$$\begin{cases} V_{1,t}(t,x) - V_{1,x}(t,x)^2 - V_{2,x}(t,x)^2 - V_{1,x}(t,x)V_{2,x}(t,x) = 0, \\ V_{2,t}(t,x) - V_{1,x}(t,x)^2 - V_{2,x}(t,x)^2 - V_{1,x}(t,x)V_{2,x}(t,x) = 0, \\ V_{1,t}(t,x) = -V_{2,t}(t,x) = \frac{1}{2}x^2(2). \end{cases} \tag{18}$$

We intend to solve the PDEs system (18) using the SC-GCM (as discussed in section 4). The numerical approximation of optimal value functions, the control solutions and state trajectory are plotted using SC-GCM for $N_1 = 10$ and $N_2 = 2$ on the computational domain $[0, 2] \times [-2, 2]$ in Figure 1. The graphs of the absolute error are also show in the same Figure 1. The exact optimal value cost functionals are $J_1^* = -J_2^* = 0.1666666667$.

Comparison of the optimal cost functionals $J_i,\ i = 1, 2$ for the SC-GCM with the exact solutions are shown in Table 1.

Example 5.2. In this example, we consider the application of differential games in competitive advertising in Sorger. There are two firms in a market and the profit of firm1 and that of 2 are respectively [49]:

$$\max_{u_1} J_1(u_1, u_2) = \int_0^T e^{-r_1 t}[q_1 x(t) - \frac{c_1}{2}u_1^2(t)]dt + e^{-r_1 T}S_1 x(T), \tag{19}$$

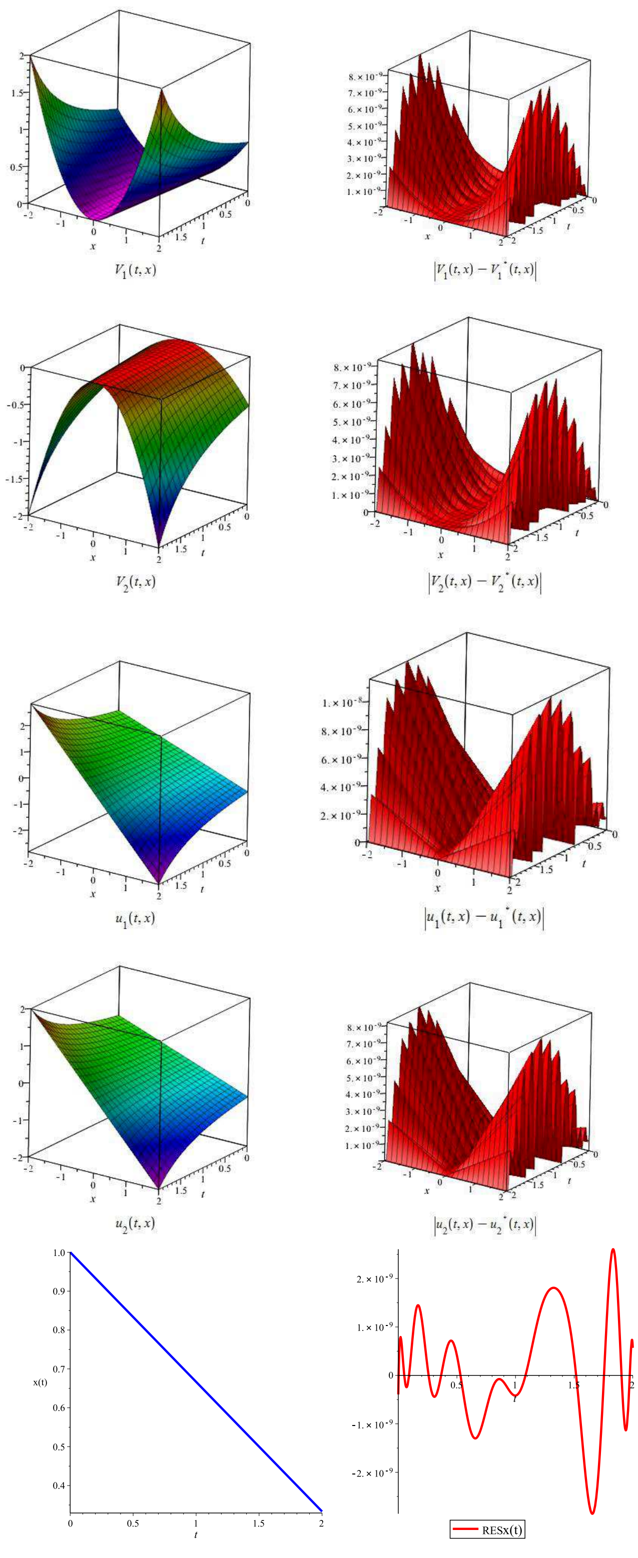

FIGURE 1. The numerical approximation of optimal value functions $V_i(t,x), i = 1,2,$ control solutions $u_i(t,x), i = 1,2,$ state trajectory $x(t)$ and absolute error functions, using the SC-GCM for $N_1 = 10$ and $N_2 = 2$ on the domain $[0,2] \times [-2,2]$ for Example 5.1.

and

$$\max_{u_2} J_2(u_1,u_2) = \int_0^T e^{-r_2 t}[q_2(1-x(t)) - \frac{c_2}{2}u_2^2(t)]dt + e^{-r_2 T}S_2(1-x(T)),$$

where r_i, q_i, c_i and S_i for $i=1,2$, are the positive constants. The dynamics of firms market share is governed by

$$\dot{x}(t) = u_1(t)\sqrt{1-x(t)} - u_2(t)\sqrt{x(t)}, \quad x(0)=1, \quad 0 \le x \le 1, \tag{20}$$

where $x(t)$ is the market share of firm1 at time t, $[1-x(t)]$ is that of firm2, $u_i(t)$ is advertising rate for firm $i=1,2$. A feedback solution which allows the firm to choose its advertising rates contingent upon the state of the game is a realistic approach to this problem. Invoking the dynamic programming principle, a feedback Nash equilibrium solution to the game (19)–(20) has to satisfy the following conditions:

$$\begin{aligned}
V_{1,t}(t,x) + &\max_{u_1}\{e^{-r_1 t}[q_1 x(t) - \frac{c_1}{2}u_1^2] \\
&+ V_{1,x}(t,x)(u_1\sqrt{1-x(t)} - u_2^*(t,x)\sqrt{x(t)})\} = 0, \\
V_{2,t}(t,x) + &\max_{u_2}\{e^{-r_2 t}[q_2(1-x(t)) - \frac{c_2}{2}u_2^2(t)] \\
&+ V_{2,x}(t,x)(u_1^*(t,x)\sqrt{1-x(t)} - u_2\sqrt{x(t)})\} = 0, \\
&V_1(T,x) = e^{-r_1 T}S_1 x(T), \\
&V_2(T,x) = e^{-r_2 T}S_2(1-x(T)).
\end{aligned} \tag{21}$$

Performing the indicated maximization in (21) yields

$$u_1^*(t,x) = \frac{V_{1,x}(t,x)}{c_1}\sqrt{1-x(t)}\exp(rt), \tag{22}$$

$$u_2^*(t,x) = \frac{-V_{2,x}(t,x)}{c_2}\sqrt{x(t)}\exp(rt). \tag{23}$$

If $q_i = S_i = 1, i=1,2$ and $T = r_1 = r_2 = 2$, upon substituting $u_1^*(t,x)$ and $u_2^*(t,x)$ from (22) and (23) into (21), we have the following system of PDEs:

$$\begin{aligned}
V_{1,t}(t,x) + &x\exp(-2t) \\
&+ \frac{1}{2}(1-x)\exp(2t)V_{1,x}(t,x)^2 + x\exp(2t)V_{1,x}(t,x)V_{2,x}(t,x) = 0, \\
V_{2,t}(t,x) + &(1-x)\exp(-2t) \\
&+ \frac{1}{2}x\exp(2t)V_{2,x}(t,x)^2 + (1-x)\exp(2t)V_{1,x}(t,x)V_{2,x}(t,x) = 0, \\
V_1(2,x) = &e^{-4}x \\
V_2(2,x) = &e^{-4}(1-x).
\end{aligned} \tag{24}$$

The numerical approximation of optimal value functions, residual errors and numerical approximation of control solutions for Player1 and Player2 are plotted by using the SC-GCM for $N_1 = 10$ and $N_2 = 2$ on the computational domain $[0,2]\times[0,1]$ in Figure 2. To solve the partial differential equations system (24) by separation variables method, we try a solution of the form

$$\begin{aligned}
V_1(t,x) &= \exp(-2t)[A_1(t)x + B_1(t)], \\
V_2(t,x) &= \exp(-2t)[A_2(t)x + B_2(t)],
\end{aligned}$$

where $A_1(t), B_1(t), A_2(t)$ and $B_2(t)$ satisfy

$$\dot{A}_1(t) = \frac{1}{2}A_1(t)^2 - A_1(t)A_2(t) - 2A_1(t) + 1, \ A_1(2) = 1, \tag{25}$$

$$\dot{B}_1(t) = -\frac{1}{2}A_1(t)^2 + B_1(t), \ B_1(2) = 0, \tag{26}$$

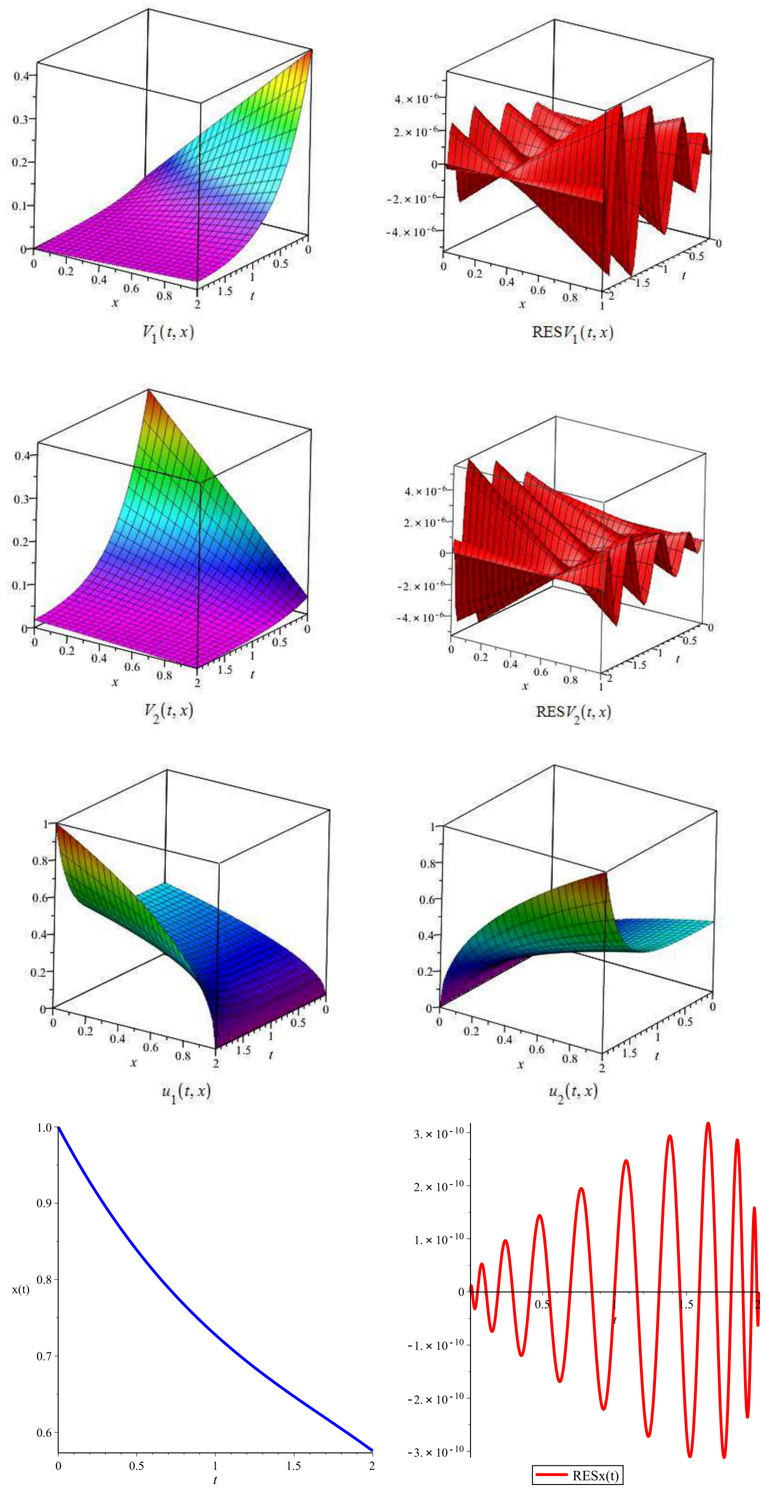

FIGURE 2. The numerical approximation of optimal value functions $V_i(t,x), i = 1,2,$ the residual errors $RES_{V_i}(t,x), i = 1,2,$ control solutions $u_i(t,x), i = 1,2,$ state trajectory $x(t)$ and $RES_x(t)$, using the SC-GCM for $N_1 = 10$ and $N_2 = 2$ on the domain $[0,2] \times [0,1]$ for Example 5.2.

Table 2. Optimal value of cost functionals $J_i, i = 1, 2$ obtained using the SC-GCM as compared with that of obtained by the SVM for Example 5.2

(N_1, N_2)	$J_{1SC-GCM}$	$J_{2SC-GCM}$
(2, 1)	0.4288819515	0.0406674857
(4, 1)	0.4285865391	0.0402552676
(6, 1)	0.4285683563	0.0402927736
(8, 1)	0.4285702876	0.0402936397
(10, 2)	0.4285703954	0.0402937017

$J_{1SVM} = 0.4285704042,\ J_{2SVM} = 0.0402937073$

$$\dot{A}_2(t) = -\frac{1}{2}A_2(t)^2 + A_1(t)A_2(t) + 2A_2(t) + 1,\ A_2(2) = -1, \tag{27}$$

$$\dot{B}_2(t) = -A_1(t)A_2(t) + 2B_2(t) - 1,\ B_2(2) = 1. \tag{28}$$

If ordinary differential equations system (25)–(28) has a solution, then the optimal control strategies as a function of the time and the current state are given in the form

$$u_1^*(t, x) = A_1(t)\sqrt{1-x}\exp(2t),$$
$$u_2^*(t, x) = -A_2(t)\sqrt{x}\exp(2t).$$

The SC-GCM method is also applied to solve the ordinary differential equations system (25)–(28) for $N = 10$ on the domain $[0, 2]$. Comparison of the optimal cost functionals $J_i, i = 1, 2$ obtained by the SC-GCM and the separation variables method (SVM) is shown in Table 2.

Example 5.3. The following example corresponds to a nonlinear electrical circuit managed by two electric companies, which employ different costs for the consumed electric energy. The purpose of the game problem is to minimize the energy cost for each company [36].

Consider the following nonlinear polynomial game:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\ x_1(0) = 1,\\ \dot{x}_2(t) &= x_1^2(t) + u_1(t) + u_2(t),\ x_2(0) = 1,\end{aligned} \tag{29}$$

with the finite-time quadratic cost functions

$$\begin{aligned}\min_{u_1} J_1(u_1, u_2) =& \frac{1}{2}(0.1x_1^2(T) + x_2^2(T))\\ &+\frac{1}{2}\int_0^T (0.1x_1^2(t) + x_2^2(t) + u_1^2(t) + u_2^2(t))dt,\end{aligned} \tag{30}$$

$$\begin{aligned}\min_{u_2} J_2(u_1, u_2) =& \frac{1}{2}(x_1^2(T) + 0.1x_2^2(T))\\ &+\frac{1}{2}\int_0^T (x_1^2(t) + 0.1x_2^2(t) + u_1^2(t) + u_2^2(t))dt.\end{aligned} \tag{31}$$

Invoking dynamic programming principle, a feedback Nash equilibrium solution to the game (29)–(31) has to satisfy the following conditions:

$$\begin{aligned}V_{1,t}(t, x_1, x_2)+&\min_{u_1}\Big\{\frac{1}{2}(0.1x_1^2 + x_2^2 + u_1^2 + (u_2^*(t, x_1, x_2)^2) + V_{1,x_1}(t, x_1, x_2)(x_2)\\ &+V_{1,x_2}(t, x_1, x_2)\left(x_1^2 + u_1 + u_2^*(t, x_1, x_2)\right)\Big\} = 0,\\ V_{2,t}(t, x_1, x_2)+&\min_{u_2}\Big\{\frac{1}{2}\left(x_1^2 + 0.1x_2^2 + (u_1^*(t, x_1, x_2)^2 + u_2^2\right) + V_{2,x_1}(t, x_1, x_2)\,(x_2)\\ &+V_{2,x_2}(t, x_1, x_2)\left(x_1^2 + u_1^*(t, x_1, x_2 + u_2\right)\Big\} = 0,\end{aligned}$$

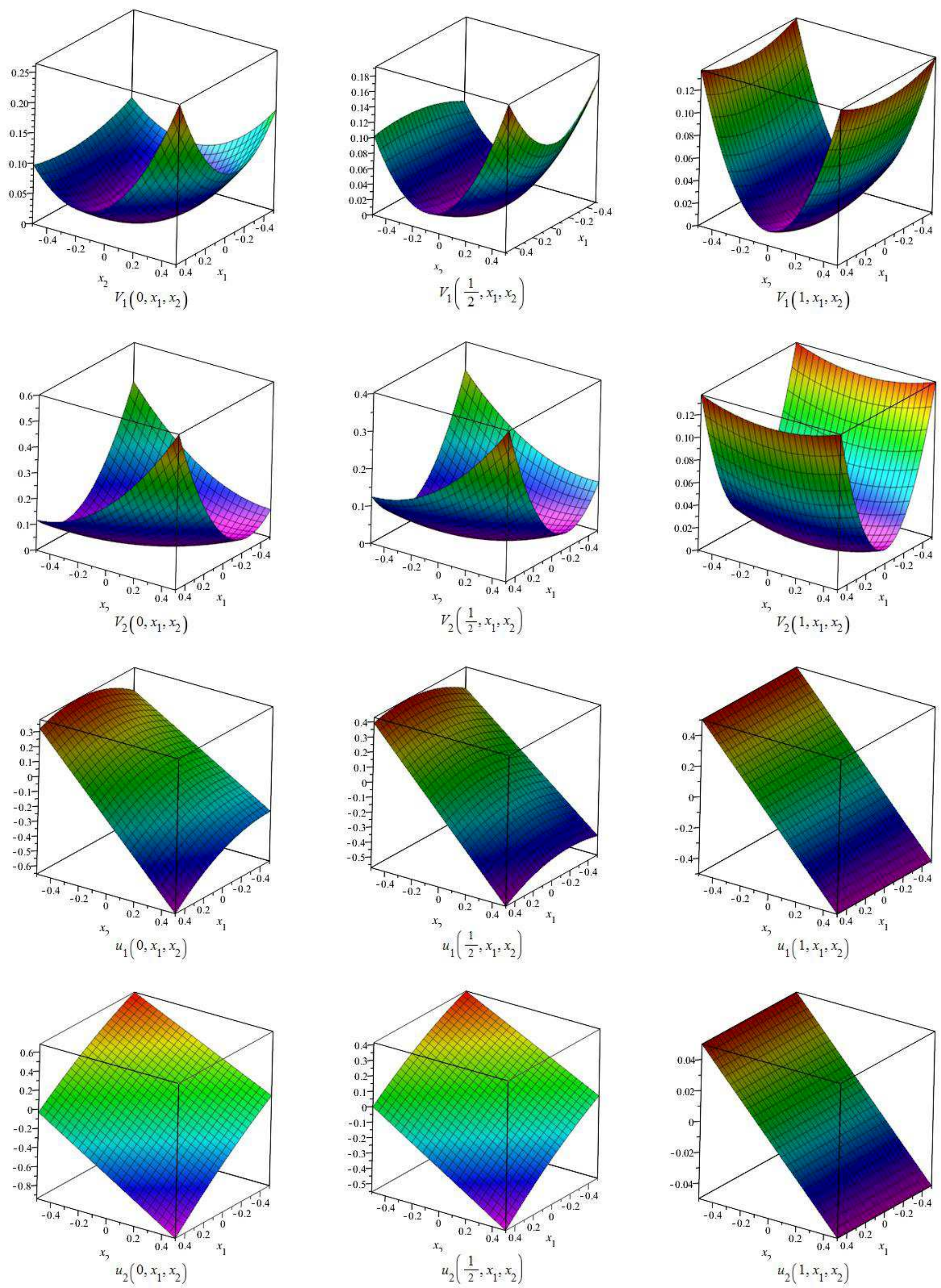

FIGURE 3. The numerical approximation of optimal value functions $V_i(t, x_1, x_2), i = 1, 2$, control solutions $u_i(t, x_1, x_2)$, $i = 1, 2$ using the SC-GCM for $t = 0, \frac{1}{2}, 1$, and $N_1 = 6, N_2 = 4, N_3 = 4$ on the domain $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ for Example 5.3.

$$
\begin{aligned}
V_1(T, x_1, x_2) =& \frac{1}{2}\left(0.1x_1^2 + x_2^2\right), \\
V_2(T, x_1, x_2) =& \frac{1}{2}\left(x_1^2 + 0.1x_2^2\right).
\end{aligned} \tag{32}
$$

Performing the indicated minimization in (32) yields:

$$
u_1^*(t, x_1, x_2) = -V_{1,x_2}(t, x_1, x_2),
$$

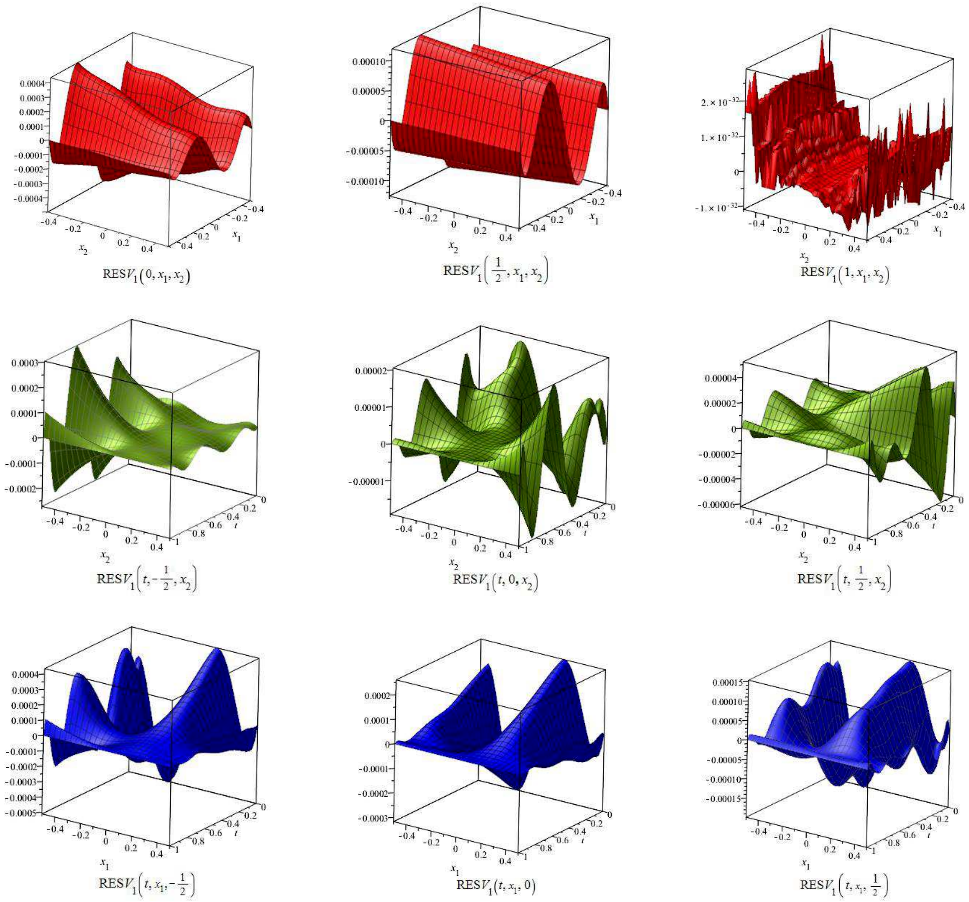

FIGURE 4. The residual errors $RES_{V_1}(t, x_1, x_2)$ for $t = 0, \frac{1}{2}, 1,\ x_1 = -\frac{1}{2}, 0, \frac{1}{2},\ x_2 = -\frac{1}{2}, 0, \frac{1}{2}$ using the SC-GCM at $N_1 = 6, N_2 = 4, N_3 = 4$ on the domain $[0, 1]\times[-\frac{1}{2}, \frac{1}{2}]\times[-\frac{1}{2}, \frac{1}{2}]$ for Example 5.3.

$$u_2^*(t, x_1, x_2) = -V_{2,x_2}(t, x_1, x_2). \tag{33}$$

Upon substituting $u_1^*(t, x_1, x_2)$ and $u_2^*(t, x_1, x_2)$ into (32), we have the following system of PDEs:

$$\begin{aligned}
V_{1,t}(t, x_1, x_2) &+ \frac{1}{20}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}\left(V_{2,x_2}(t, x_1, x_2)^2 - V_{1,x_2}(t, x_1, x_2)^2\right)\\
&+ V_{1,x_1}(t, x_1, x_2)x_2 + V_{1,x_2}(t, x_1, x_2)x_1^2\\
&- V_{2,x_2}(t, x_1, x_2)V_{1,x_2}(t, x_1, x_2) = 0,\\
V_{2,t}(t, x_1, x_2) &+ \frac{1}{2}x_1^2 + \frac{1}{20}x_2^2 + \frac{1}{2}\left(V_{1,x_2}(t, x_1, x_2)^2 - V_{2,x_2}(t, x_1, x_2)^2\right)\\
&+ V_{2,x_1}(t, x_1, x_2)x_2 + V_{2,x_2}(t, x_1, x_2)x_1^2\\
&- V_{2,x_2}(t, x_1, x_2)V_{1,x_2}(t, x_1, x_2) = 0,\\
V_1(T, x_1, x_2) &= \frac{1}{2}\left(0.1x_1^2 + x_2^2\right),\\
V_2(T, x_1, x_2) &= \frac{1}{2}\left(x_1^2 + 0.1x_2^2\right).
\end{aligned} \tag{34}$$

Substituting the relevant partial derivatives of $V_1(t, x_1, x_2)$ and $V_2(t, x_1, x_2)$ from (34) into (33), we get the feedback Nash equilibrium strategies $u_1^*(t, x_1, x_2) = \phi_1^*(t, x_1, x_2)$ and $u_2^*(t, x_1, x_2) = \phi_2^*(t, x_1, x_2)$.

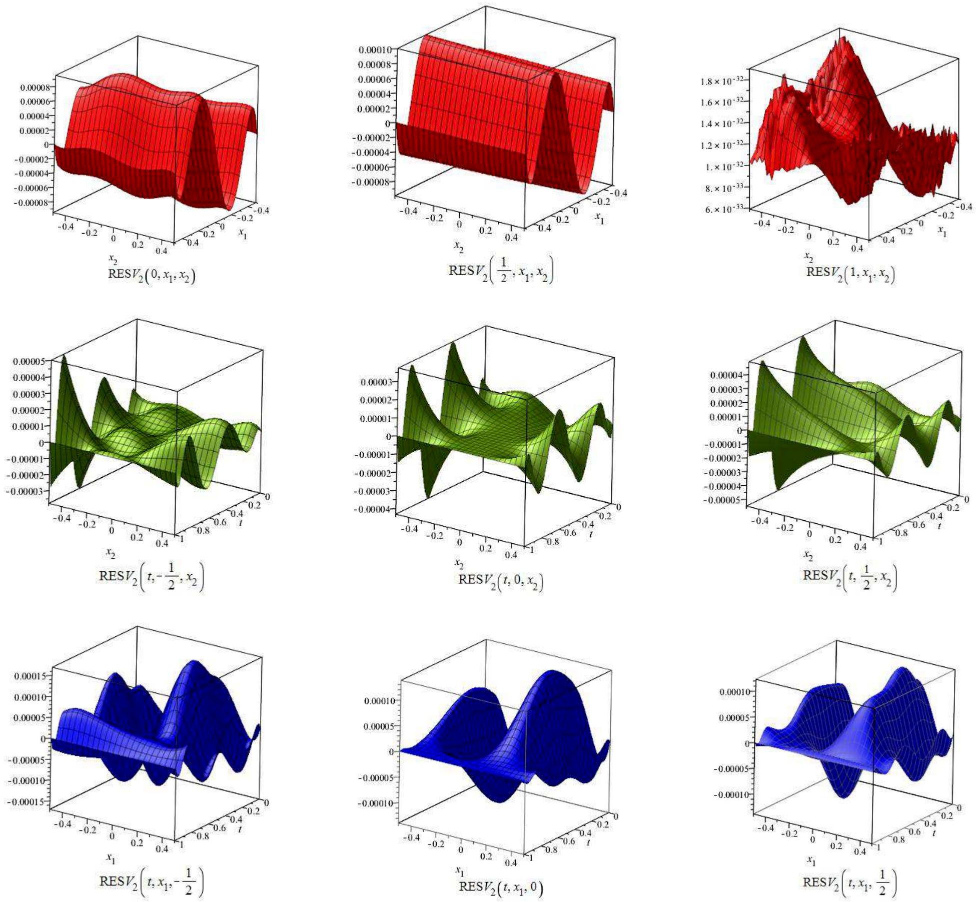

FIGURE 5. The residual errors $RES_{V_2}(t, x_1, x_2)$ for $t = 0, \frac{1}{2}, 1,\ x_1 = -\frac{1}{2}, 0, \frac{1}{2},\ x_2 = -\frac{1}{2}, 0, \frac{1}{2}$ using the SC-GCM at $N_1 = 6, N_2 = 4, N_3 = 4$ on the domain $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ for Example 5.3.

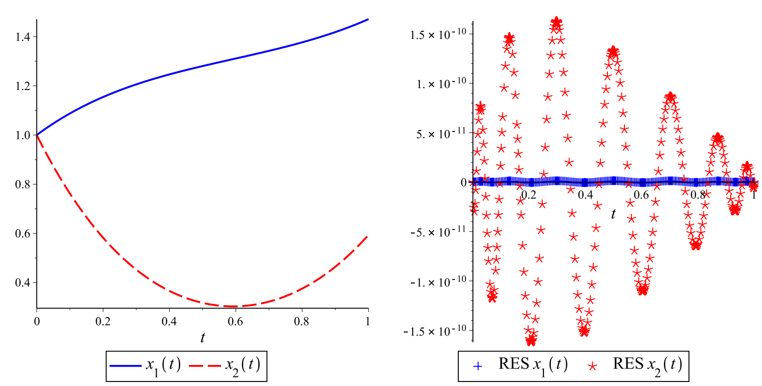

FIGURE 6. The numerical approximation of optimal state trajectories $x_i^*(t), i = 1, 2$ and the residual errors $RES_{x_i}(t), i = 1, 2,$ using the SC-GCM by $N = 10$ on the domain $[0, 1]$ for Example 5.3.

TABLE 3. Optimal value of cost functionals $J_i,\ i = 1, 2$ is obtained by using the SC-GCM, for Example 5.3.

(N_1, N_2, N_3)	J_1	J_2
(4, 2, 2)	1.32637334	2.47275222
(4, 3, 3)	1.379273134	2.884931842
(6, 4, 4)	1.694880607	3.001789476
(8, 4, 4)	1.694872219	3.001803382

After substituting $\phi_1^*(t, x_1(t), x_2(t))$ and $\phi_2^*(t, x_1(t), x_2(t))$ into the system of differential equations (29) and solving, we obtain the optimal state trajectories $x_1^*(t)$ and $x_2^*(t)$.

The SC-GCM method is applied to obtain the numerical approximation of optimal value functions $V_i(t, x_1, x_2)$ and the Nash equilibrium strategies $u_i(t, x_1, x_2)$ for $i = 1, 2$, for $t = 0, \frac{1}{2}, 1$, using a $6\times4\times4$ grid discreditization scheme on the computational domain $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, the obtained results are shown in Figure 3. The residual errors $RES_{V_1}(t, x_1, x_2)$ for $t = 0, \frac{1}{2}, 1,\ x_1 = \frac{-1}{2}, 0, \frac{1}{2}$, and $x_2 = -\frac{1}{2}, 0, \frac{1}{2}$ using SC-GCM for $N_1 = 6,\ N_2 = 4,\ N_3 = 4$ on the domain $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ are plotted in Figures 4 and 5.

The SC-GCM method is also applied to obtain the numerical optimal state trajectories $x_1^*(t)$ and $x_2^*(t)$, using $M = 10$ on the computational domain $[0, 1]$ (see Figure 6).

In Table 3, the computational results of the performance index of Player1 and Player2 for different values of N_1, N_2 and N_3 are reported. It should be noted that small values for $N_i, i = 1, 2, 3$ are needed to obtain a satisfactory convergence.

6. Conclusion

In this paper, we have proposed the SC-GCM to solve the HJB equations system of nonlinear nonzero-sum differential games for finding the feedback Nash equilibrium solution of these games. The main advantage of this method is that the boundary conditions of the system of HJB PDEs can be included implicitly in the chosen approximations of value functions. The majority of numerical methods are grid based suffer from the so-called "curse-of-dimensionality". However, the SC-GCM is also a grid based method, but with the Chebyshev–Gauss–Lobatto nodes the results show that selecting a limited number of collocation points, excellent numerical results are obtained.

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BASIS AND DIMENSION OF EXPONENTIAL VECTOR SPACE

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Abstract. Exponential vector space [shortly *evs*] is an algebraic order extension of a vector space in the sense that every evs contains a vector space and, conversely, every vector space can be embedded into such a structure. This evs structure consists of a semigroup structure, a scalar multiplication and a partial order. In this paper, we have developed the concepts of a basis and dimension of an evs by introducing the ideas of an orderly independent set and generating set with the help of partial order and algebraic operations. We have found that like a vector space, an evs does not contain basis always. We have established a necessary and sufficient condition for an evs to have a basis. It was shown that the equality of dimension is an evs property, but the converse is not true. We have studied the dimension of a subevs and found that every evs contains a subevs with all possible lower dimensions. Lastly, we have computed the basis and dimension of some evs which help us to explore the theory of basis by creating counter-examples in different aspects.

1. Introduction

Exponential vector space is an algebraic ordered extension of a vector space. The word 'extension' is used because of the fact that every exponential vector space contains a vector space and, conversely, every vector space can be embedded into such a structure. This structure comprises a semigroup structure, a scalar multiplication and a compatible partial order. We now start with the definition of evs.

Definition 1.1 ([7])**.** Let $(X, \leq)$ be a partially ordered set, '$+$' be a binary operation on X [called *addition*] and '$\cdot$': $K \times X \longrightarrow X$ be another composition [called *scalar multiplication*, K being a field]. If the operations and the partial order satisfy the axioms below, then $(X, +, \cdot, \leq)$ is called an *exponential vector space* (in short *evs*) over K [This structure was initiated under the name *quasi-vector space* or *qvs* by S. Ganguly et al. in [1]].

$A_1 : (X, +)$ is a commutative semigroup with identity θ

$.A_2 : x \leq y \ (x, y \in X) \Rightarrow x + z \leq y + z$ and $\alpha \cdot x \leq \alpha \cdot y, \ \forall z \in X, \forall \alpha \in K$

$.A_3 :$ (i) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$;

(ii) $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$;

(iii) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$;

(iv) $1 \cdot x = x$, where '1' is the multiplicative identity in K,

$\forall\, x, y \in X, \ \forall\, \alpha, \beta \in K$;

$A_4 : \alpha \cdot x = \theta$ iff $\alpha = 0$ or $x = \theta$;

$A_5 : x + (-1) \cdot x = \theta$ iff $x \in X_0 := \{z \in X : \ y \not\leq z, \, \forall\, y \in X \smallsetminus \{z\}\}$;

$A_6 :$ For each $x \in X$, $\exists\, p \in X_0$ such that $p \leq x$.

In the above definition, axiom A_3 (iii) indicates a rapid growth of the elements of X due to the fact that $x + x \geq 2x$ and axiom A_6 gives some positive sense of each element. These two facts express the exponential behaviour of the elements of an evs.

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In axiom A_5, we can notice that X_0 is precisely the set of all minimal elements of the evs X with respect to the partial order on X and forms the maximal vector space (within X) over the same field as that of X ([1]). We call this vector space X_0 as the '*primitive space*' or '*zero space*' of X and the elements of X_0 as '*primitive elements*'.

Also, given any vector space V over some field K, an evs X can be constructed (as is shown below) such that V is isomorphic to X_0. In this sense, an "exponential vector space" can be considered as an algebraic ordered extension of a vector space.

Example 1.2 ([7])**.** Let $X := \{(r,a) \in \mathbb{R} \times V : r \geq 0, a \in V\}$, where V is a vector space over some field K. Define operations and partial order on X as follows: for $(r,a),(s,b) \in X$ and $\alpha \in K$,
(i) $(r,a)+(s,b) := (r+s, a+b)$;
(ii) $\alpha(r,a) := (r, \alpha a)$, if $\alpha \neq 0$ and $0(r,a) := (0,\theta)$, θ being the identity in V;
(iii) $(r,a) \leq (s,b)$, iff $r \leq s$ and $a = b$.
Then X becomes an exponential vector space over K with the primitive space $\{0\} \times V$ which is evidently isomorphic to V.

Initially, the idea of this structure was given by S. Ganguly et al. under the name "quasi-vector space" in [1] and the following example of the hyperspace was the main motivation behind this new structure.

Example 1.3 ([1])**.** Let $\mathscr{C}(\mathcal{X})$ be the topological hyperspace consisting of all non-empty compact subsets of a Hausdörff topological vector space $\mathcal{X}$ over the field $\mathbb{K}$ of real or complex numbers. Then $\mathscr{C}(\mathcal{X})$ becomes an evs with respect to the operations and partial order defined as follows. For $A, B \in \mathscr{C}(\mathcal{X})$ and $\alpha \in \mathbb{K}$,
(i) $A + B := \{a + b : a \in A, b \in B\}$;
(ii) $\alpha A := \{\alpha a : a \in A\}$;
(iii) the usual set-inclusion as the partial order.

We now topologise an exponential vector space. For this we need the following concept.

Definition 1.4 ([5])**.** Let '$\leq$' be a preorder in a topological space Z; the preorder is said to be *closed* if its graph $G_{\leq}(Z) := \{(x,y) \in Z \times Z : x \leq y\}$ is closed in $Z \times Z$ (endowed with the product topology).

Theorem 1.5 ([5])**.** *A partial order '$\leq$' in a topological space Z will be a closed order iff for any $x, y \in Z$ with $x \nleq y$, $\exists$ open neighbourhoods U, V of x, y respectively in Z such that $(\uparrow U) \cap (\downarrow V) = \emptyset$, where $\uparrow U := \{z \in Z : z \geq u$ for some $u \in U\}$ and $\downarrow V := \{z \in Z : z \leq v$ for some $v \in V\}$.*

Definition 1.6 ([7])**.** An exponential vector space X over the field $\mathbb{K}$ of real or complex numbers is said to be a *topological exponential vector space* if there exists a topology on X with respect to which the addition and the scalar multiplication are continuous and the partial order '$\leq$' is closed (here, $\mathbb{K}$ is equipped with the usual topology).

Remark 1.7. If X is a topological exponential vector space, then its primitive space X_0 becomes a topological vector space, since the restriction of a continuous function is continuous. Moreover, the closedness of the partial order '$\leq$' in a topological exponential vector space X readily implies (in view of Theorem 1.5) that X is Hausdörff and hence X_0 becomes a Hausdörff topological vector space.

Example 1.8 ([2])**.** Let $X := [0,\infty) \times V$, where V is a vector space over the field $\mathbb{K}$ of real or complex numbers. Define operations and partial order on X as follows: for $(r,a),(s,b) \in X$ and $\alpha \in \mathbb{K}$,
(i) $(r,a)+(s,b) := (r+s, a+b)$;
(ii) $\alpha(r,a) := (|\alpha| r, \alpha a)$;
(iii) $(r,a) \leq (s,b)$ iff $r \leq s$ and $a = b$.
Then $[0,\infty) \times V$ becomes an exponential vector space with the primitive space $\{0\} \times V$ which is clearly isomorphic to V.

In this example, if we consider V as a Hausdörff topological vector space, then $[0,\infty) \times V$ becomes a topological exponential vector space with respect to the product topology, where $[0,\infty)$ is equipped with the subspace topology inherited from the real line $\mathbb{R}$.

If instead of V we take the trivial vector space $\{\theta\}$ in this example, then the resulting topological evs is $[0,\infty) \times \{\theta\}$ which can be clearly identified with the half-ray $[0,\infty)$ of the real line.

In this paper, we have developed the concept of the basis and dimension of an evs. We know that the basis of a vector space is a minimal part of it which generates the entire space. But in an evs it is impossible to express every element as a linear combination of some particular elements due to the exponential behaviour of its elements. In this paper, with the help of partial order we have developed the ideas of generating sets, orderly independent sets which allow us to define the basis. It has been shown that the basis of an evs is identified by a minimal generating set, whereas a maximal orderly independent set fails to form a basis [shown by a counter example], though every basis is a maximal orderly independent set. The main difference between a vector space and an evs in this respect is that an evs may not have a basis always (like a vector space). But for a topological evs, we have shown that if it has a basis, then it contains uncountably many bases. We have found out a property of every element of a basis which helped us to give a necessary and sufficient condition for an evs to have a basis. After that we have introduced the concept of dimension of an evs and shown that equality of dimension is an evs property, though two non-order-isomorphic evs may have the same dimension.

Lastly, we have studied the dimension of subevs and shown that every evs contains subevs(s) with all possible lower dimensions. In the last section of this paper, computations of the basis and dimension of some evs are given.

2. Prerequisites

In this section, we have discussed some definitions, results and examples of an exponential vector space which are very important in developing the main context. We now start with the definition of a subevs.

Definition 2.1 ([4])**.** A subset Y of an exponential vector space X is said to be a *sub- exponential vector space* (*subevs* in short) if Y itself is an exponential vector space with respect to the compositions of X being restricted to Y.

Note 2.2 ([4])**.** A subset Y of an exponential vector space X over a field K is a sub exponential vector space, iff Y satisfies the following:
(i) $\alpha x + y \in Y,\ \forall\, \alpha \in K, \forall\, x, y \in Y$;
(ii) $Y_0 \subseteq X_0 \bigcap Y$, where $Y_0 := \big\{z \in Y : y \nleq z, \forall\, y \in Y \smallsetminus \{z\}\big\}$;
(iii) for any $y \in Y$, $\exists\, p \in Y_0$ such that $p \leq y$.

If Y is a subevs of X, then actually $Y_0 = X_0 \cap Y$, since for any $Y \subseteq X$, we have $X_0 \cap Y \subseteq Y_0$. $[0, \infty) \times \{\theta\}$ is, clearly, a subevs of the evs $[0, \infty) \times V$.

We have used the following result to form a non-topological exponential vector space.

Result 2.3 ([3])**.** *In a topological evs X, if $a = a + x$ for some $a, x \in X$, then $x = \theta$.*

To talk about an evs property of this space we have to know the idea of an order-morphism.

Definition 2.4 ([2])**.** A mapping $f : X \longrightarrow Y$ (X, Y being two exponential vector spaces over the field K) is called an *order-morphism* if
(i) $f(x + y) = f(x) + f(y),\ \forall\, x, y \in X$;
(ii) $f(\alpha x) = \alpha f(x),\ \forall\, \alpha \in K, \forall\, x \in X$;
(iii) $x \leq y\ (x, y \in X) \Rightarrow f(x) \leq f(y)$;
(iv) $p \leq q\ \big(p, q \in f(X)\big) \Rightarrow f^{-1}(p) \subseteq \downarrow f^{-1}(q)$ and $f^{-1}(q) \subseteq \uparrow f^{-1}(p)$.

A bijective (injective, surjective) order-morphism is called an *order-isomorphism* (*order-monomorphism*, *order-epimorphism*, respectively).

If X, Y are two topological evs over $\mathbb{K}$, then an order-isomorphism $f : X \longrightarrow Y$ is said to be a *topological order-isomorphism* if f is a homeomorphism.

Definition 2.5. A property of an evs is called an *evs property* if it remains invariant under an order-isomorphism.

The concept of order-isomorphism is competent enough to extract the structural beauty of an evs by judging the invariance of its various properties. Since the composition of two order-isomorphisms, the inverse of an order-isomorphism and the identity map are again order-isomorphisms, the concept thereby produces a partition on the collection of all evs over some common field; this helps one to distinguish two evs belonging to two different classes under this partition.

Definition 2.6 ([4])**.** In an evs X, the *primitive* of $x \in X$ is defined as the set
$$P_x := \{p \in X_\circ : p \leq x\}.$$
Axiom A_6 in Definition 1.1 ensures that the primitive of each element of an evs is nonempty.

Definition 2.7 ([4])**.** An evs X is said to be a *single primitive evs* if P_x is a singleton set for each $x \in X$. Also, in a single primitive evs X, $P_{x+y} = P_x + P_y$ and $P_{\alpha x} = \alpha P_x$, $\forall\, x, y \in X$ and for all scalar α.

Single primitivity is an evs property [4].

Definition 2.8 ([4])**.** An evs X is said to be a *comparable evs* if $\forall x, y \in X$, $P_x = P_y \Rightarrow x$ and y are comparable with respect to the partial order of X.

This is also an evs property [4].

We now give some examples of an exponential vector space to build up some counter-examples of the main section.

Example 2.9 ([3])**. (Arbitrary product of exponential vector spaces)** Let $\{X_i : i \in \Lambda\}$ be an arbitrary family of exponential vector spaces over a common field K and $X := \prod_{i \in \Lambda} X_i$ be the Cartesian product. Then X becomes an exponential vector space over K with respect to the following operations and partial order:

For $x = (x_i)_i, y = (y_i)_i \in X$ and $\alpha \in K$ we define (i) $x + y := (x_i + y_i)_i$, (ii) $\alpha x := (\alpha x_i)_i$, (iii) $x \ll y$ if $x_i \leq y_i$, $\forall\, i \in \Lambda$.
Here the notation $x = (x_i)_i \in X$ means that the point $x \in X$ is the map $x : i \mapsto x_i$ $(i \in \Lambda)$, where $x_i \in X_i$, $\forall\, i \in \Lambda$. The additive identity of X is given by $\theta = (\theta_i)_i$, θ_i being the additive identity in X_i. Also, the primitive space of X is given by $X_0 = \prod_{i \in \Lambda} [X_i]_0$.

This product space X becomes a topological exponential vector space over the field $\mathbb{K}$ whenever each factor space X_i is a topological evs over $\mathbb{K}$ and X is endowed with the product topology, which is the weakest topology on X so that each projection map $p_i : X \longrightarrow X_i$ given by $p_i : x \longmapsto x_i$ is continuous.

Thus for any cardinal number β, $[0, \infty)^\beta$ becomes a topological evs.

Example 2.10. Let X be an evs over the field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) and V be a vector space over the same field $\mathbb{K}$. We now give operations on $X \times V$ like $[0, \infty) \times V$, i.e.,
for $(x_1, e_1), (x_2, e_2), (x, e) \in X \times V$ and $\alpha \in \mathbb{K}$:
(i) $(x_1, e_1) + (x_2, e_2) := (x_1 + x_2, e_1 + e_2)$.
(ii) $\alpha(x, e) := (\alpha x, \alpha e)$.
The partial order '$\leq$' is defined as: $(x_1, e_1) \leq (x_2, e_2)$, iff $x_1 \leq x_2$ and $e_1 = e_2$. Then $X \times V$ becomes an evs over the field $\mathbb{K}$. Justification of this is straightforward.

Example 2.11. Let $\mathscr{C}_\theta(\mathcal{X})$ be the collection of all compact subsets of a Hausdörff topological vector space $\mathcal{X}$ containing θ (the identity in $\mathcal{X}$). So, $\mathscr{C}_\theta(\mathcal{X}) \subseteq \mathscr{C}(\mathcal{X})$. If we take any two members $A, B \in \mathscr{C}_\theta(\mathcal{X})$ and any $\alpha \in \mathbb{K}$, then $\alpha A + B \in \mathscr{C}_\theta(\mathcal{X})$. Again, $[\mathscr{C}_\theta(\mathcal{X})]_0 = \big\{\{\theta\}\big\} = [\mathscr{C}(\mathcal{X})]_0 \cap \mathscr{C}_\theta(\mathcal{X})$. For any $A \in \mathscr{C}_\theta(\mathcal{X})$, $\{\theta\} \subseteq A$ which shows that $\mathscr{C}_\theta(\mathcal{X})$ is a subevs of $\mathscr{C}(\mathcal{X})$ [by note 2.2].

Example 2.12 ([4])**.** Let $\mathcal{X}$ be a vector space over the field $\mathbb{K}$ of real or complex numbers. Let $\mathscr{L}(\mathcal{X})$ be the set of all linear subspaces of $\mathcal{X}$. We now define $+, \cdot, \leq$ on $\mathscr{L}(\mathcal{X})$ as follows: For $\mathcal{X}_1, \mathcal{X}_2 \in \mathscr{L}(\mathcal{X})$ and $\alpha \in \mathbb{K}$ we define
(i) $\mathcal{X}_1 + \mathcal{X}_2 := \operatorname{span}(\mathcal{X}_1 \cup \mathcal{X}_2)$, (ii) $\alpha \cdot \mathcal{X}_1 := \mathcal{X}_1$, if $\alpha \neq 0$ and $\alpha \cdot \mathcal{X}_1 := \{\theta\}$, if $\alpha = 0$ (θ being the additive identity of $\mathcal{X}$), (iii) $\mathcal{X}_1 \leq \mathcal{X}_2$ iff $\mathcal{X}_1 \subseteq \mathcal{X}_2$.

Then $\big(\mathscr{L}(\mathcal{X}), +, \cdot, \leq\big)$ is an exponential vector space over $\mathbb{K}$.

Since every element of $\mathscr{L}(\mathcal{X})$ is an idempotent $\big[\because \mathcal{X}_1 + \mathcal{X}_1 = \mathcal{X}_1$, for all $\mathcal{X}_1 \in \mathscr{L}(\mathcal{X})\big]$, we can say that there is no topology with respect to which $\mathscr{L}(\mathcal{X})$ can be a topological evs $\big[$since a topological evs cannot contain any idempotent element, as follows from the Result 2.3$\big]$. $\square$

Example 2.13 ([6])**.** Let us consider $\mathscr{D}^2([0,\infty)) := [0,\infty) \times [0,\infty)$. We define $+, \cdot, \leq$ on $\mathscr{D}^2([0,\infty))$ as follows:
For $(x_1, y_1), (x_2, y_2) \in \mathscr{D}^2([0,\infty))$ and $\alpha \in \mathbb{C}$, we define
(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$;
(ii) $\alpha \cdot (x_1, y_1) = (|\alpha| x_1, |\alpha| y_1)$;
(iii) $(x_1, y_1) \leq (x_2, y_2) \Longleftrightarrow$ either $x_1 < x_2$ or if $x_1 = x_2$, then $y_1 \leq y_2$ [*dictionary order*].
Then $\left(\mathscr{D}^2([0,\infty)), +, \cdot, \leq\ \right)$ becomes a non-topological exponential vector space over the complex field $\mathbb{C}$.

Note 2.14 ([6])**.** For a ***well-ordered*** set I and an evs X, if we consider $\mathscr{D}(X : I) := X^I$, then also, like the above example, it forms a non-topological evs with dictionary order. If $I = \{1, 2, \ldots, n\}$, we usually denote the evs $\mathscr{D}(X : I)$ as $\mathscr{D}^n(X)$. We can also generalise this by taking different evs i.e., $\mathscr{D}(X_\alpha : \alpha \in I) := \prod\limits_{\alpha \in I} X_\alpha$, which also becomes a non-topological evs with dictionary order.

3. Basis and Dimension: General Discussion

In this section, we have introduced the concepts of a basis and dimension of an exponential vector space. These concepts are different from those already given in a vector space. Like a vector space, it is not true that every evs contains a basis, rather it behaves like a module in this respect. We have found a necessary as well as sufficient condition for an evs to have a basis. Finally, we have computed a basis and dimension of some particular evs.

Definition 3.1. Let X be an evs over the field K and $x \in X \smallsetminus X_0$. Define
$$L(x) := \{z \in X : z \geq \alpha x + p, \alpha \in K^*, p \in X_0\}, \quad \text{where } K^* \equiv K \smallsetminus \{0\}.$$
We name these sets $L(x)$ for different x's in $X \smallsetminus X_0$ as *testing sets* of X.

We discuss below some properties of $L(x)$. First of all note that $L(x) = \uparrow (K^* x + X_0)$.

Proposition 3.2. (i) $\forall\, x \in X \smallsetminus X_0$*,* $x \in L(x)$ *and* $\uparrow L(x) = L(x)$*.*
(ii) $x \leq y$ $(x, y \in X \smallsetminus X_0) \Rightarrow L(x) \supseteq L(y)$*.*
(iii) *If* $x = \alpha y + p$ *for some* $\alpha \in K^*$*,* $p \in X_0$ *and* $y \in X \smallsetminus X_0$*, then* $L(x) = L(y)$*.*
(iv) $L(x) \cap X_0 = \emptyset$*.*
(v) *If* $a \in L(b)$*, then* $L(a) \subseteq L(b)$*.*
(vi) *For any* $x, y \in X \smallsetminus X_0$*,* $L(x) \cap L(y) \neq \emptyset$*.*

Proof. **(i)** Immediate from definition.

(ii) Let $z \in L(y) \Rightarrow \exists\ \alpha \in K^*$ and $p \in X_0$ such that $\alpha y + p \leq z$. Now, $x \leq y \Rightarrow \alpha x + p \leq \alpha y + p \leq z$ $\Rightarrow z \in L(x)$.

(iii) As $y \in X \smallsetminus X_0$, so $x \in X \smallsetminus X_0$. Therefore we can talk about $L(x)$. Let $z \in L(y) \Rightarrow \exists\ \alpha_z \in K^*$ and $p_z \in X_0$ such that $\alpha_z y + p_z \leq z \Rightarrow \alpha_z \alpha^{-1}(x - p) + p_z \leq z \Rightarrow \alpha_z \alpha^{-1} x + (p_z - \alpha_z \alpha^{-1} p) \leq z \Rightarrow$ $z \in L(x)$. Therefore $L(y) \subseteq L(x)$. Again, $x = \alpha y + p \Rightarrow y = \alpha^{-1}(x - p)$. So, by the above argument, we also have $L(x) \subseteq L(y)$. Thus $L(x) = L(y)$.

(iv) Let $y \in L(x) \Rightarrow \exists\ \alpha \in K^*$ and $p \in X_0$ such that $\alpha x + p \leq y$. If $y \in X_0$, then $\alpha x + p \in X_0 \Rightarrow$ $x \in X_0$. This contradiction proves that $L(x) \cap X_0 = \emptyset$.

(v) $a \in L(b) \Rightarrow a \in X \smallsetminus X_0$ and $\exists\, \alpha \in K^*, p \in X_0$ such that $\alpha b + p \leq a \Rightarrow L(a) \subseteq L(\alpha b + p) = L(b)$ [by (ii) and (iii) above].

(vi) For any $p \in X_0$ with $p \leq y$, $x + p \leq x + y \Rightarrow x + y \in L(x)$. Similarly, we can say that $x + y \in L(y)$. So, $x + y \in L(x) \cap L(y) \Rightarrow L(x) \cap L(y) \neq \emptyset$. □

Definition 3.3. A subset B of $X \smallsetminus X_0$ is said to be a *generator* of $X \smallsetminus X_0$ if
$$X \smallsetminus X_0 = \bigcup_{x \in B} L(x).$$

Note 3.4. The set $X \smallsetminus X_0$ always generates $X \smallsetminus X_0$. So, a generator always exists for $X \smallsetminus X_0$. It is clear that any superset of a generator of $X \smallsetminus X_0$ is also a generator of $X \smallsetminus X_0$.

Definition 3.5. Two elements $x, y \in X \setminus X_0$ are said to be *orderly dependent* if either $x \in L(y)$ or $y \in L(x)$.

Definition 3.6. Two elements $x, y \in X \setminus X_0$ are said to be *orderly independent* if they are not orderly dependent, i.e., neither $x \in L(y)$, nor $y \in L(x)$.

A subset B of $X \setminus X_0$ is said to be *orderly independent* if any two members $x, y \in B$ are orderly independent.

Remark 3.7. Let Y be a subevs of an evs X. Then any two orderly dependent elements of $Y \setminus Y_0$ are also orderly dependent in $X \setminus X_0$ because of the fact that $Y_0 \subseteq X_0$. In other words, any two elements of $Y \setminus Y_0$ which are orderly independent in $X \setminus X_0$ are also orderly independent in $Y \setminus Y_0$. But the converse is not true in general, i.e., two orderly independent elements in $Y \setminus Y_0$ may not be orderly independent in $X \setminus X_0$ [in contrast to the case of linear independence in vector space]. For example, $\{0, 2, 5\}$ and $\{0, -2, 3\}$ are orderly independent in $\mathscr{C}_\theta(\mathbb{R})$ [see Example 2.11], since $\nexists$ any $\alpha \in \mathbb{R}^*$ such that $\alpha\{0, 2, 5\} \subseteq \{0, -2, 3\}$ or $\alpha\{0, -2, 3\} \subseteq \{0, 2, 5\}$. $\big[$Here, $[\mathscr{C}_\theta(\mathbb{R})]_0 = \big\{\{0\}\big\}\ \big]$. But these two elements are not orderly independent in $\mathscr{C}(\mathbb{R})$, as we can write $\{0, -2, 3\} = \{0, 2, 5\} + \{-2\}$, where $\{-2\} \in [\mathscr{C}(\mathbb{R})]_0$.

In the above context, it should thus be noted that while discussing the orderly independence of two elements of a subevs Y of an evs X, there are two types of orderly independence — one with respect to Y, and the other with respect to X; while considering orderly independence with respect to Y, the testing sets should be of the form

$$L_Y(y) := \{z \in Y : z \geq \alpha y + p, \alpha \in K^*, p \in Y_0\} \text{ for any } y \in Y \setminus Y_0,$$

and when considering orderly independence with respect to X, the testing sets should be of the form

$$L_X(y) := \{z \in X : z \geq \alpha y + p, \alpha \in K^*, p \in X_0\} \text{ for any } y \in Y \setminus Y_0.$$

Since $Y_0 \subseteq X_0$, we have $L_Y(y) \subseteq L_X(y)$, for any $y \in Y \setminus Y_0$. Thus it follows that an orderly independent set in $Y \setminus Y_0$ need not be orderly independent in $X \setminus X_0$. However, a set $B\,(\subseteq Y \setminus Y_0)$ which is orderly independent in $X \setminus X_0$ must be such in $Y \setminus Y_0$.

Definition 3.8. A subset B of $X \setminus X_0$ is said to be a *basis* of $X \setminus X_0$ if B is orderly independent and generates $X \setminus X_0$.

Note 3.9. For each $x \in X$, either $x \in X_0$, or $x \in X \setminus X_0$. If $x \in X_0$, then it can be expressed as a finite linear combination of some basic vectors of some basis of X_0 [as a vector space]. If $x \in X \setminus X_0$, then there exists a member of some basis [if exists] of $X \setminus X_0$ which generates x. So, we can say that to represent an evs X it is necessary to consider a basis of $X \setminus X_0$ together with a basis of X_0 [in the sense of a vector space]. Thus a basis of an evs X should be composed of two components, one for X_0 and the other for $X \setminus X_0$. To express this fact in an easiest way we shall represent a basis of an evs X as $[B : B_0]$, where B is a basis of $X \setminus X_0$ and B_0 is a basis of X_0 [as a vector space]. If for an evs X, $X_0 = \{\theta\}$ then we shall consider $B_0 = \{\theta\}$, since in that case X_0 has no basis.

Theorem 3.10. *For a topological evs X, $X \setminus X_0$ either has no basis or has uncountably many bases.*

Proof. Let B be a basis of $X \setminus X_0$. Then $G_\alpha := \{\alpha x : x \in B\}$ and $H_p := \{x + p : x \in B\}$ are also bases of $X \setminus X_0$, for any $\alpha \in \mathbb{K}^*$ and any $p \in X_0$. This holds because of the result $L(x) = L(\alpha x + p)$ [Proposition 3.2]. If $G_\alpha = G_\beta$ for any $\alpha, \beta \in \mathbb{K}^*$, then $\alpha x = \beta x$, $\forall\, x \in B$ [$\because \alpha x \neq \beta y$ for any $x, y \in B$ as B is orderly independent]. If we choose $\alpha, \beta \in \mathbb{K}^*$ such that $|\alpha| < |\beta|$, then using the continuity of the scalar multiplication of the topological evs X, we must have $x = \theta$ [$\because \alpha x = \beta x \Rightarrow (\alpha\beta^{-1})^n x = x$, $\forall\, n \in \mathbb{N}$ which implies by taking limit $n \to \infty$ that $x = \theta$, as $|\alpha\beta^{-1}| < 1$] is a contradiction. Thus, it follows that for any $\alpha, \beta \in \mathbb{K}^*$ with $|\alpha| < |\beta|$ we must have $G_\alpha \neq G_\beta$. This immediately justifies that there are uncountably many bases of $X \setminus X_0$.

If X_0 contains more than one element, then for $p, q \in X_0$ we may consider H_p, H_q. If $H_p = H_q$, then B, being orderly independent, we must have $x + p = x + q$, $\forall\, x \in B$. Then by Result 2.3, it follows that $p = q$. Since X is a topological evs, X_0 is a Hausdörff topological vector space. So, if $X_0 \neq \{\theta\}$, then it should be uncountable and hence ensure the existence of uncountably many bases of $X \setminus X_0$. $\square$

For a non-topological evs it may happen that $G_\alpha = B$ for every $\alpha \in K^*$ [this will be discussed in the next section]. However, an evs (topological or not) need not have a basis. We show in the next section that the evs $\mathscr{D}\big([0,\infty) : \mathbb{N}\big)$ discussed in Note 2.14 cannot have a basis. The following result shows that having a basis is an evs property.

Result 3.11. *Let $\phi : X \longrightarrow Y$ be an order-isomorphism. Then*
(1) *for any generator B of $X \smallsetminus X_0$, $\phi(B)$ is a generator of $Y \smallsetminus Y_0$;*
(2) *for any orderly independent subset B of $X \smallsetminus X_0$, $\phi(B)$ is also an orderly independent subset of $Y \smallsetminus Y_0$.*

Thus, for a basis B of $X \smallsetminus X_0$, $\phi(B)$ becomes a basis of $Y \smallsetminus Y_0$.

Proof. (1) $B \subseteq X \smallsetminus X_0 \Rightarrow \phi(B) \subseteq Y \smallsetminus Y_0$ [As $\phi(X_0) = Y_0$]. Let $y \in Y \smallsetminus Y_0 \Rightarrow \phi^{-1}(y) \in X \smallsetminus X_0$ $\Rightarrow \exists\, b \in B$ and $\alpha \in K^*$, $p \in X_0$ such that $\phi^{-1}(y) \geq \alpha b + p \Rightarrow y \geq \alpha\phi(b) + \phi(p) \Rightarrow y \in L(\phi(b)) \subseteq \bigcup\limits_{b\in B} L(\phi(b))$. Therefore $Y \smallsetminus Y_0 \subseteq \bigcup\limits_{b\in B} L(\phi(b))$. Again, by Proposition 3.2, $L(\phi(b)) \cap Y_0 = \emptyset$, $\forall\, b \in B$. So $Y \smallsetminus Y_0 = \bigcup\limits_{b\in B} L(\phi(b))$. Thus $\phi(B)$ is a generator of $Y \smallsetminus Y_0$.

(2) We first show that for any two orderly dependent members y_1, y_2 of $Y \smallsetminus Y_0$, $\phi^{-1}(y_1), \phi^{-1}(y_2)$ are orderly dependent in $X \smallsetminus X_0$. As y_1, y_2 are orderly dependent, so without loss of generality, we can take $y_1 \in L(y_2) \Rightarrow \exists\, \alpha \in K^*$ and $p \in Y_0$ such that $\alpha y_2 + p \leq y_1$. Then ϕ^{-1}, also being an order-isomorphism, we have $\phi^{-1}(\alpha y_2 + p) \leq \phi^{-1}(y_1) \Rightarrow \alpha\phi^{-1}(y_2) + \phi^{-1}(p) \leq \phi^{-1}(y_1) \Rightarrow \phi^{-1}(y_1) \in L(\phi^{-1}(y_2))$ [as $\phi^{-1}(p) \in X_0$]. This justifies our assertion. Then contra-positively, the result follows. $\square$

The next theorem characterises a basis (if exists) of $X \smallsetminus X_0$, for any evs X.

Theorem 3.12. *A subset of $X \smallsetminus X_0$ is a basis of $X \smallsetminus X_0$, iff it is a minimal generating subset of $X \smallsetminus X_0$.* $\big[$Here, a minimal generating subset B of $X \smallsetminus X_0$ means that there does not exist any proper subset of B which can generate $X \smallsetminus X_0$.$\big]$

Proof. Suppose B is a basis of $X \smallsetminus X_0$. Then B generates $X \smallsetminus X_0$. Now B, being an orderly independent subset of $X \smallsetminus X_0$, if we take an element $x \in B$ then $\forall\, y \in B \smallsetminus \{x\}$, x and y are orderly independent. Therefore $x \notin L(y)$, $\forall\, y \in B \smallsetminus \{x\}$. This shows that $B \smallsetminus \{x\}$ cannot generate $X \smallsetminus X_0$ and this holds for any $x \in B$. Therefore B is a minimal generator of $X \smallsetminus X_0$.

Conversely, suppose B is a minimal generator of $X \smallsetminus X_0$. For any two members $x, y \in B$ if $x \in L(y)$ then by Proposition 3.2, $L(x) \subseteq L(y) \Longrightarrow B \smallsetminus \{x\}$ also generates $X \smallsetminus X_0$, which contradicts that B is a minimal generator of $X \smallsetminus X_0$. Again, if $y \in L(x)$, we get a similar contradiction. So, neither $x \in L(y)$, nor $y \in L(x) \Longrightarrow x, y$ are orderly independent. Arbitrariness of x, y shows that B is an orderly independent subset of $X \smallsetminus X_0$. Consequently, B is a basis of $X \smallsetminus X_0$. $\square$

Result 3.13. *Every basis of $X \smallsetminus X_0$ is a maximal orderly independent subset of $X \smallsetminus X_0$.* $\big[$Here, a maximal orderly independent subset B of $X \smallsetminus X_0$ means that there does not exist any orderly independent subset of $X \smallsetminus X_0$ containing B.$\big]$

Proof. Let B be a basis of $X \smallsetminus X_0$. Then for any $x \in X \smallsetminus (B \cup X_0)$, $\exists\, b \in B$ such that $x \in L(b)$. This shows that $B \cup \{x\}$ is not orderly independent. Thus B is maximal orderly independent in $X \smallsetminus X_0$. $\square$

The converse of the above result is not true in general, i.e., a maximal orderly independent subset of $X \smallsetminus X_0$ may not be a basis of $X \smallsetminus X_0$. For example, in the evs $\mathscr{C}_\theta(\mathcal{X})$ [discussed in 2.11], let us consider the collection
$$\mathscr{G} := \big\{A \in \mathscr{C}_\theta(\mathcal{X}) : A \text{ consists of three distinct elements of } \mathcal{X}\big\}.$$
Then $\mathscr{G} \subset \mathscr{C}_\theta(\mathcal{X}) \smallsetminus \big\{\{\theta\}\big\}$. Now we define a relation '$\sim$' on $\mathscr{G}$ by "$A \sim B$, iff $A = \alpha B$ for some $\alpha \in \mathbb{K}^*$". Then this relation becomes an equivalence relation on $\mathscr{G}$. Let us consider the subcollection $\mathscr{H}$ of $\mathscr{G}$ taking exactly one member from each equivalence class produced by the equivalence relation '$\sim$'. Then $\mathscr{H}$ becomes an orderly independent subset of $\mathscr{C}_\theta(\mathcal{X}) \smallsetminus \big\{\{\theta\}\big\}$, because any two elements $A, B \in \mathscr{G}$ are orderly dependent, iff $A = \alpha B$ for some $\alpha \in \mathbb{K}^*$ and hence belong to the same equivalence class. For any member $C \in \mathscr{C}_\theta(\mathcal{X}) \setminus (\mathscr{H} \cup [\mathscr{C}_\theta(\mathcal{X})]_0)$ if $\mathrm{card}(C) \geq 3$, then there exists a member $A_C \in \mathscr{H}$ and $\alpha \in \mathbb{K}^*$ such that $\alpha A_C \subseteq C$. If $\mathrm{card}(C) = 2$, then also there exists $\beta \in \mathbb{K}^*$ and $A_C \in \mathscr{H}$ such

that $C \subseteq \beta A_C$. This shows that $\mathscr{H} \cup \{C\}$ is orderly dependent. So, we can say that $\mathscr{H}$ forms a maximal orderly independent set in $\mathscr{C}_\theta(\mathcal{X}) \smallsetminus \big\{\{\theta\}\big\}$. But it does not generate $\mathscr{C}_\theta(\mathcal{X}) \smallsetminus \big\{\{\theta\}\big\}$. In fact, for any $D \in \mathscr{C}_\theta(\mathcal{X})$ with $\operatorname{card}(D) = 2$ there does not exist any $A \in \mathscr{H}$ such that $D \in L(A)$, since each member of $L(A)$ contains three or more elements of $\mathcal{X}$. Hence $\mathscr{H}$ cannot be a basis of $\mathscr{C}_\theta(\mathcal{X}) \smallsetminus [\mathscr{C}_\theta(\mathcal{X})]_0$, although it is maximal orderly independent $\big[$note here that $[\mathscr{C}_\theta(\mathcal{X})]_0 = \big\{\{\theta\}\big\}\big]$.

Remark 3.14. If A is an orderly independent set in $X \smallsetminus X_0$, then for any $a_1, a_2 \in A$ with $a_1 \neq a_2$ neither $a_1 \in L(a_2)$, nor $a_2 \in L(a_1)$. In other words, if $a_1 \in L(a_2)$ for some $a_1, a_2 \in A$, then $a_1 = a_2$. Moreover, any two elements of an orderly independent set A must be incomparable with respect to the partial order '$\leq$' of the evs X; in fact, $x \leq y \Rightarrow y \in L(x)$.

Lemma 3.15. *Let A and B be two bases of $X \smallsetminus X_0$. Then for any $a \in A$, there exists one and only one $b_a \in B$ such that $L(a) = L(b_a)$.*

Proof. As B is a basis of $X \smallsetminus X_0$, so for the member $a \in A$, there must exist some $b \in B$ such that $a \in L(b)$. Let us suppose, $\exists\ b_1, b_2 \in B$ such that $a \in L(b_1) \cap L(b_2) \Rightarrow L(a) \subseteq L(b_1) \cap L(b_2)$ [by Proposition 3.2] — $(*)$. Again, $a \in L(b_1) \Rightarrow \exists\ \alpha \in K^*$ and $p \in X_0$ such that $\alpha b_1 + p \leq a$ — $(**)$. Now, since A is a basis, so for $b_1, b_2 \in B$, $\exists\ a_1, a_2 \in A$ such that $b_1 \in L(a_1)$ and $b_2 \in L(a_2) \Rightarrow L(b_1) \subseteq L(a_1)$ and $L(b_2) \subseteq L(a_2)$ [by Proposition 3.2]. By $(*)$, $L(a) \subseteq L(a_1)$ and $L(a) \subseteq L(a_2) \Rightarrow a \in L(a_1)$ and $a \in L(a_2)$. As a, a_1, a_2 are the members of the basis A, so we can say that $a_2 = a = a_1$ [by the above Remark 3.14]. Therefore $b_1, b_2 \in L(a) \Rightarrow \exists\ \alpha_1, \alpha_2 \in K^*$ and $p_1, p_2 \in X_0$ such that $\alpha_1 a + p_1 \leq b_1$ and $\alpha_2 a + p_2 \leq b_2$ — $(***)$. From $(**)$ and $(***)$, we get $b_2 \geq \alpha_2 a + p_2 \geq \alpha_2(\alpha b_1 + p) + p_2 = \alpha_2 \alpha b_1 + (\alpha_2 p + p_2)$. $\therefore b_2 \in L(b_1)$ [as $\alpha_2 \alpha \in K^*$ and $\alpha_2 p + p_2 \in X_0$]. Since b_1, b_2 are the members of the basis B so $b_2 = b_1$ [by the above Remark 3.14]. Thus there exists one and only one member (say) $b_a \in B$ such that $a \in L(b_a)$ and also, $b_a \in L(a) \Rightarrow L(a) \subseteq L(b_a)$ and $L(b_a) \subseteq L(a) \Rightarrow L(a) = L(b_a)$. ☐

Theorem 3.16. *If A and B are two bases of $X \smallsetminus X_0$, then card(A) = card(B).*

Proof. From the proof of the above Lemma 3.15, we can say that for each $a \in A$, $\exists$ unique $b \in B$ such that $L(b) = L(a)$. This property creates a one-to-one correspondence between A and B. Hence the theorem is complete. ☐

This theorem motivates us to introduce the concept of dimension of an evs.

Definition 3.17. For an evs X we define *dimension* of $X \smallsetminus X_0$ as

$$\dim(X \smallsetminus X_0) := \operatorname{card}(B), \text{ where } B \text{ is a basis of } X \smallsetminus X_0.$$

Then we represent *dimension* of the evs X as $\dim X := [\dim(X \smallsetminus X_0) : \dim X_0]$. If $X_0 = \{\theta\}$, dimension of X_0 will be taken as 0, since then X_0 has no basis [as a vector space].

Note 3.18. Theorem 3.16 makes the above definition well-defined. From Result 3.11 we can say that if X and Y are order-isomorphic evs, then $\dim X = \dim Y$. Here, by "$\dim X = \dim Y$" we mean that $\dim(X \smallsetminus X_0) = \dim(Y \smallsetminus Y_0)$, as well as $\dim X_0 = \dim Y_0$. However, the converse of this is not true in general, i.e., there are the evs X, Y such that $\dim X = \dim Y$, but X, Y are not order-isomorphic. This will be clear in the next section, in computing the dimension of some particular evs.

Result 3.19. *Let X be an evs and B be a basis of $X \smallsetminus X_0$. Then $\downarrow x \smallsetminus X_0 \subseteq L(x)$, for each $x \in B$.*

Proof. Let $x \in B$ and $y \in \downarrow x \smallsetminus X_0$. Since B is a basis of $X \smallsetminus X_0$, $\exists\ x_1 \in B$ such that $y \in L(x_1) \Rightarrow \exists\ \alpha_1 \in K^*$ and $p_1 \in X_0$ such that $\alpha_1 x_1 + p_1 \leq y \Rightarrow \alpha_1 x_1 + p_1 \leq x$ [$\because y \leq x$] $\Rightarrow x \in L(x_1)$. Since B is orderly independent and both $x, x_1 \in B$, we can say by Remark 3.14 that $x = x_1$. Therefore $y \in L(x)$. Thus $\downarrow x \smallsetminus X_0 \subseteq L(x)$, for each $x \in B$. ☐

This result reveals an important property of each member of a basis of $X \smallsetminus X_0$ which helps us to set up a precise domain of basic elements of $X \smallsetminus X_0$. The collection of all x in $X \smallsetminus X_0$ satisfying the property stated in the above Result 3.19 makes our task of finding a basis of $X \smallsetminus X_0$ easier. To make this assertion precise, let us consider the following.

For an evs X, let

$$Q(X) := \big\{x \in X \smallsetminus X_0 : (\downarrow x \smallsetminus X_0) \subseteq L(x)\big\}.$$

From Result 3.19, we can say that $B \subseteq Q(X)$, for any basis B of $X \smallsetminus X_0$. It is thus enough to find any basis of $X \smallsetminus X_0$ within $Q(X)$. We call this set $Q(X)$ as a *feasible set* of X. At this point it is important to note that $Q(X)$ may be empty; in fact, if for an evs X, $Q(X) = \emptyset$, then such evs X cannot have any basis (as we have claimed earlier). We shall encounter such evs later. If for an evs X, $Q(X) \neq \emptyset$, then also X may not have a basis. In fact, we shall prove a theorem shortly in terms of $Q(X)$ which identifies an evs when it has a basis or not. We now prove a lemma that will be useful in the sequel.

Lemma 3.20. *For an evs X, if $x \in Q(X)$ then for each $y \in \downarrow x \smallsetminus X_0$, $L(x) = L(y)$.*

Proof. $y \in \downarrow x \smallsetminus X_0 \Rightarrow y \leq x$. So, by Proposition 3.2, we have $L(x) \subseteq L(y)$. Again, by the construction of $Q(X)$, $x \in Q(X) \Rightarrow \downarrow x \smallsetminus X_0 \subseteq L(x) \Rightarrow y \in L(x) \Rightarrow L(y) \subseteq L(x)$ [by Proposition 3.2]. Thus $L(x) = L(y)$. □

The following theorem may be compared with the so-called '*Replacement theorem*' in the context of the basis of a vector space.

Theorem 3.21. *For an evs X, let B be a basis of $X \smallsetminus X_0$ and $x \in B$. Then for any $y \in \downarrow x \smallsetminus X_0$, $\big(B \smallsetminus \{x\}\big) \cup \{y\}$ is also a basis of $X \smallsetminus X_0$.*

Proof. Let $A = \big(B \smallsetminus \{x\}\big) \cup \{y\}$. As $y \in \downarrow x \smallsetminus X_0$ and $x \in B \subseteq Q(X)$ so, by Lemma 3.20, $L(x) = L(y)$. Therefore $X \smallsetminus X_0 = \bigcup\limits_{z \in B} L(z) = \bigcup\limits_{z \in A} L(z) \Rightarrow A$ generates $X \smallsetminus X_0$. To show that A is orderly independent it is sufficient to show that for any $z \in B \smallsetminus \{x\}$, z, y are orderly independent. If not, then for some $z_1 \in B \smallsetminus \{x\}$ either $y \in L(z_1)$ or $z_1 \in L(y)$. Now, if $y \in L(z_1)$, then by Proposition 3.2, $L(y) \subseteq L(z_1) \Rightarrow x \in L(x) = L(y) \subseteq L(z_1)$ which contradicts that x, z_1 are two members of the basis B. Again, if $z_1 \in L(y)$, then $z_1 \in L(y) = L(x)$ which again contradicts that x, z_1 are orderly independent. Thus it follows that A is orderly independent. □

The above theorem makes it convenient to construct a new basis from the old one. The following theorem is the key to ensure the existence of a basis of an evs.

Theorem 3.22. *An evs X has a basis, iff $Q(X)$ is a generator of $X \smallsetminus X_0$.*

Proof. Suppose that X has a basis $[B : B_0]$, where B and B_0 are bases of $X \smallsetminus X_0$ and X_0 respectively. Then by Result 3.19, $B \subseteq Q(X)$. As B is a generator of $X \smallsetminus X_0$, so $Q(X)$ is also a generator of $X \smallsetminus X_0$.

Conversely, suppose $Q(X)$ is a generator of $X \smallsetminus X_0 \Rightarrow Q(X) \neq \emptyset$. We now give a relation $\sim$ in $Q(X)$ as follows: For $x, y \in Q(X)$, we say $x \sim y \Leftrightarrow L(x) = L(y)$. Then, obviously, this becomes an equivalence relation on $Q(X)$. Let us consider a collection taking exactly one representative from each equivalence class and denote this collection as B. Then $B \subseteq Q(X) \subseteq X \smallsetminus X_0$. Also, $x, y \in B$ with $x \neq y \Leftrightarrow x, y \in Q(X)$ and $L(x) \neq L(y)$. We claim that B is a basis of $X \smallsetminus X_0$. Let $z \in X \smallsetminus X_0 \Rightarrow \exists\ x_z \in Q(X)$ [$\because Q(X)$ is a generator] such that $z \in L(x_z) \Rightarrow \exists$ an element $x'_z \in B$ such that $L(x_z) = L(x'_z)$ and hence $z \in L(x'_z) \Rightarrow B$ generates $X \smallsetminus X_0$. Now, we have to show that B is orderly independent. Suppose not, then $\exists$ two distinct elements $x_1, x_2 \in B$ such that they are orderly dependent. So, without loss of generality, we can think that $x_1 \in L(x_2)$. Now, $x_1 \in L(x_2) \Rightarrow \exists\ \alpha \in \mathbb{K}^*$ and $p \in X_0$ such that $\alpha x_2 + p \leq x_1$. Since $x_1 \in Q(X)$ [as $B \subseteq Q(X)$] and $\alpha x_2 + p \in \downarrow x_1 \smallsetminus X_0$, using Lemma 3.20 we can say that $L(\alpha x_2 + p) = L(x_1)$ and then by Proposition 3.2, we have $L(x_2) = L(\alpha x_2 + p) = L(x_1)$ which contradicts that x_1, x_2 are two distinct elements of B. Thus B becomes a basis of $X \smallsetminus X_0$. Let us take any basis B_0 of X_0. Then $[B : B_0]$ becomes a basis of X. □

From the proof of the above theorem it is clear that the hypothesis of $Q(X)$, being a generator of $X \smallsetminus X_0$, is used only to justify that B, constructed by using the equivalence relation $\sim$ within $Q(X)$, is a generator of $X \smallsetminus X_0$; to ensure the orderly independence of B, the structure of $Q(X)$ is enough. Thus we can conclude that for any evs X, $Q(X)$ (if nonempty) always contains an orderly independent set like B (as constructed in the proof of the above Theorem 3.22). This orderly independent set is also a maximal orderly independent set in $Q(X)$. In fact, if D is another orderly independent set in

$Q(X)$ such that $B \subset D$, then for any $x \in D$, $\exists\, z \in B$ such that $L(x) = L(z) \Rightarrow x = z$ [$\because x, z \in D$ and D is orderly independent] and hence $x \in B$. Thus $B = D$. Summarising all these facts, we get the following

Theorem 3.23. *For an evs X, if $Q(X) \neq \emptyset$, then it contains a maximal orderly independent set.*

The next theorem is useful in finding a basis of $X \smallsetminus X_0$, for any evs X.

Theorem 3.24. *For an evs X, every maximal orderly independent set of $Q(X)$ is a basis of $X \smallsetminus X_0$, provided $Q(X)$ generates $X \smallsetminus X_0$.*

Proof. Let B be a maximal orderly independent set in $Q(X)$. Since $Q(X)$ generates $X \smallsetminus X_0$, for any $x \in X \smallsetminus X_0$, $\exists\, z \in Q(X)$ such that $x \in L(z)$. If $z \in B$, we are done. If $z \notin B$, then B, being a maximal orderly independent set in $Q(X)$, $B \cup \{z\}$ is orderly dependent. So, $\exists\, b \in B$ such that either $z \in L(b)$ or $b \in L(z)$. If $z \in L(b)$ then by Proposition 3.2, $x \in L(z) \subseteq L(b)$. If $b \in L(z)$, $\exists\, \alpha \in \mathbb{K}^*$ and $p \in X_0$ such that $b \geq \alpha z + p$, then $b \in B \subseteq Q(X) \Rightarrow L(z) = L(b)$ [by Lemma 3.20] $\Rightarrow x \in L(b)$. Thus B generates $X \smallsetminus X_0$. Consequently, B is a basis of $X \smallsetminus X_0$. $\square$

The above Theorem 3.24 shows the converse of Result 3.13 to some extent; as we have explained, just after Result 3.13, through the example of $\mathscr{C}_\theta(\mathcal{X})$ that every maximal orderly independent subset of $X \smallsetminus X_0$ need not be a basis of $X \smallsetminus X_0$, the above Theorem 3.24 shows that every maximal orderly independent subset of $Q(X)$ [but *not only* of $X \smallsetminus X_0$] becomes a basis of $X \smallsetminus X_0$, provided of course, $Q(X)$ generates $X \smallsetminus X_0$ [note that the necessity of $Q(X)$ being a generator of $X \smallsetminus X_0$ is the principal key for an evs X to have a basis]. From Remark 3.14, we may recall one more point that while finding a basis of $X \smallsetminus X_0$, we have to gather only suitable incomparable elements from $Q(X)$. In this context it should also be noted that any two elements of $Q(X)$ need not be orderly independent. In fact, for any $x \in Q(X)$, if $y \in \downarrow x \smallsetminus X_0$, then $y \in Q(X)$, as well [see Result 3.26(ii)]. Clearly, these x, y are orderly dependent, since $L(x) = L(y)$.

Result 3.25. *If X and Y are order-isomorphic, then $Q(X)$ and $Q(Y)$ are in a one-to-one correspondence.*

Proof. Let $\phi : X \longrightarrow Y$ be an order-isomorphism. We now show that $\phi(Q(X)) = Q(Y)$. Let $x_0 \in Q(X) \Rightarrow \downarrow x_0 \smallsetminus X_0 \subseteq L(x_0)$. Also let $y \in \downarrow \phi(x_0) \smallsetminus Y_0 \Rightarrow y \leq \phi(x_0)$ and $y \notin Y_0 \Rightarrow \phi^{-1}(y) \leq x_0$ and $\phi^{-1}(y) \notin X_0 \Rightarrow \phi^{-1}(y) \in \downarrow x_0 \smallsetminus X_0 \subseteq L(x_0) \Rightarrow \exists\, \alpha \in \mathbb{K}^*$ and $p \in X_0$ such that $\alpha x_0 + p \leq \phi^{-1}(y)$ $\Rightarrow \alpha\phi(x_0) + \phi(p) \leq y \Rightarrow y \in L(\phi(x_0))$ [$\because \phi(p) \in Y_0$]. Therefore $\downarrow \phi(x_0) \smallsetminus Y_0 \subseteq L(\phi(x_0)) \Rightarrow$ $\phi(Q(X)) \subseteq Q(Y)$. Similarly, we can say that $\phi^{-1}(Q(Y)) \subseteq Q(X)$ [$\because \phi^{-1}$ is an order-isomorphism from Y onto X] $\Rightarrow Q(Y) \subseteq \phi(Q(X))$.
$\therefore \phi(Q(X)) = Q(Y)$. Thus $Q(X)$ and $Q(Y)$ are in a one-to-one correspondence. $\square$

Result 3.26. (i) *If $x \in Q(X)$, then for any $\alpha \in \mathbb{K}^*$ and $p \in X_0$, $\alpha x + p \in Q(X)$, i.e., $Q(X)$ is closed under dilation and translation by primitive elements.*
(ii) *If $x \in Q(X)$, then $\downarrow x \smallsetminus X_0 \subseteq Q(X)$, i.e., $\downarrow Q(X) \smallsetminus X_0 \subseteq Q(X)$.*

Proof. (i) $x \in Q(X) \Rightarrow \downarrow x \smallsetminus X_0 \subseteq L(x)$. We now show that $\downarrow (\alpha x + p) \smallsetminus X_0 \subseteq L(\alpha x + p)$. Let $y \in \downarrow (\alpha x + p) \smallsetminus X_0 \Rightarrow y \leq \alpha x + p$ and $y \notin X_0 \Rightarrow \alpha^{-1}(y - p) \leq x$ and $y \notin X_0 \Rightarrow \alpha^{-1}(y - p) \in \downarrow x \smallsetminus X_0$ $\Rightarrow \alpha^{-1}(y - p) \in L(x) \Rightarrow \exists\, \beta \in \mathbb{K}^*$ and $q \in X_0$ such that $\beta x + q \leq \alpha^{-1}(y - p) \Rightarrow \alpha(\beta x + q) + p \leq y \Rightarrow$ $\alpha\beta x + \alpha q + p \leq y \Rightarrow y \in L(x)$ [$\because \alpha q + p \in X_0$] $\Rightarrow y \in L(\alpha x + p)$ [$\because L(\alpha x + p) = L(x)$, by Proposition 3.2]. Therefore $\alpha x + p \in Q(X)$.

(ii) Let $y \in \downarrow x \smallsetminus X_0$. Then by Lemma 3.20, $L(x) = L(y)$. Now, for each $z \in \downarrow y \smallsetminus X_0$, we have $z \leq y \leq x$ with $z \notin X_0 \Rightarrow L(z) = L(x)$ [by Lemma 3.20] $\Rightarrow z \in L(x) = L(y)$. Thus $\downarrow y \smallsetminus X_0 \subseteq L(y)$. Consequently, $y \in Q(X)$ and hence $\downarrow x \smallsetminus X_0 \subseteq Q(X)$. $\square$

As we have explained in Remark 3.7 regarding orderly independence in a subevs of an evs, the theory of basis of a subevs does not behave nicely like the theory of basis of a subspace of a vector space. However, we have the following theorems and examples which reveal some technical aspects of the dimension theory of evs.

Theorem 3.27. *Every evs contains a subevs of dimension* $[1 : 0]$.

Proof. Let X be an evs over $\mathbb{K}$ and $B(x) := \left\{ \sum_{i=1}^{n} \alpha_i x : \alpha_i \in \mathbb{K}, n \in \mathbb{N} \right\}$, where $x \in \uparrow \theta \smallsetminus \{\theta\}$. Then for any $\alpha, \beta \in \mathbb{K}$ and any $\sum_{i=1}^{n} \alpha_i x, \sum_{j=1}^{m} \beta_j x \in B(x)$, we have $\alpha \sum_{i=1}^{n} \alpha_i x + \beta \sum_{j=1}^{m} \beta_j x = \sum_{i=1}^{n} \alpha\alpha_i x + \sum_{j=1}^{m} \beta\beta_j x \in B(x)$. Also, $[B(x)]_0 = \{\theta\} = B(x) \cap X_0$ and for any $y \in B(x)$, $\theta \leq y$ [$\because \theta \leq x$]. So, $B(x)$ forms a subevs of X for any $x \in \uparrow \theta \smallsetminus \{\theta\}$. In this case, $\{x\}$ forms a basis of $B(x) \smallsetminus [B(x)]_0$. In fact, for any $\sum_{i=1}^{n} \alpha_i x \in B(x)$, $\alpha_j x + \theta \leq \sum_{i=1}^{n} \alpha_i x$ for some $j \in \{1, 2, \ldots, n\}$ for which $\alpha_j \neq 0$. Again, any singleton set consisting of non-zero elements is always orderly independent. So, we can say that $\dim B(x) = [1 : 0]$. ☐

The following example shows that corresponding to any cardinal α, there exists an evs of dimension $[\alpha : 0]$.

Example 3.28. For any cardinal number α, let us consider the evs $[0, \infty)^\alpha$, discussed in Example 2.9. Let us take a set I such that $\text{card}(I) = \alpha$. We now show that $B := \{e_i : i \in I\}$ is a basis of $[0, \infty)^\alpha$, where $e_i = (\delta_j^i)_{j \in I}$ and $\delta_j^i = \begin{cases} 1, & \text{when } i = j, \\ 0, & \text{when } i \neq j. \end{cases}$

For any $x \in [0, \infty)^\alpha \smallsetminus \big[[0, \infty)^\alpha\big]_0$ $\Big[$here $\big[[0, \infty)^\alpha\big]_0 = \{\theta\}$ and $\theta = (z_j)_{j \in I}$, where $z_j = 0, \forall\, j \in I\Big]$ with representation $x = (x_j)_{j \in I}$, $\exists\, p \in I$ such that $x_p \neq 0 \Rightarrow x_p e_p \leq x$ [$\because x_j \geq 0, \forall\, j \in I$] $\Rightarrow x_p e_p + \theta \leq x$ $\Rightarrow x \in L(e_p) \Rightarrow B$ generates $[0, \infty)^\alpha \smallsetminus \big[[0, \infty)^\alpha\big]_0$. Now clearly, any two members of B are orderly independent in $[0, \infty)^\alpha \smallsetminus \big[[0, \infty)^\alpha\big]_0$. This shows that B is a basis of $[0, \infty)^\alpha \smallsetminus \big[[0, \infty)^\alpha\big]_0$. Therefore $\dim[0, \infty)^\alpha = [\alpha : 0]$, since $\text{card}(B) = \text{card}(I) = \alpha$ and $\dim \big[[0, \infty)^\alpha\big]_0 = \dim\{\theta\} = 0$.

Thus any two cardinal numbers α, β with $\alpha \neq \beta$, $[0, \infty)^\alpha$ and $[0, \infty)^\beta$ cannot be order-isomorphic, since they are of different dimension. We now show that for any two cardinal numbers α, β, there exists an evs of dimension $[\alpha : \beta]$. Toward this end, we need first the following

Theorem 3.29. *For an evs X and a vector space V, both being over the common field $\mathbb{K}$, the evs $Y := X \times V$ has a basis, iff the evs X has a basis.* [The evs $X \times V$ is discussed in Example 2.10]. *Also,* $\dim(X \times V) = [\dim(X \smallsetminus X_0) : \dim X_0 + \dim V]$.

Proof. Let X has a basis. We first show that $A := \{(b, \theta_V) : b \in B\}$ is a basis of $Y \smallsetminus Y_0$, where B is a basis of $X \smallsetminus X_0$ and θ_V is the identity of V. As B is orderly independent in $X \smallsetminus X_0$, we can say that any two members of A are orderly independent $\Rightarrow A$ is an orderly independent set in $Y \smallsetminus Y_0$. Let $(x, v) \in Y \smallsetminus Y_0 \Rightarrow x \in X \smallsetminus X_0$ [$\because Y_0 = X_0 \times V$]. Since B generates $X \smallsetminus X_0$, for this x, $\exists\, b \in B$ such that $x \in L(b) \Rightarrow \alpha b + p \leq x$ for some $\alpha \in \mathbb{K}^*$ and $p \in X_0 \Rightarrow \alpha(b, \theta_V) + (p, v) = (\alpha b + p, v) \leq (x, v)$ and $(p, v) \in [X \times V]_0 \Rightarrow (x, v) \in L\big((b, \theta_V)\big) \Rightarrow A$ is a generator of $Y \smallsetminus Y_0$. So A becomes a basis of $Y \smallsetminus Y_0$. Consequently, Y has a basis. Now $\dim(Y \smallsetminus Y_0) = \text{card}(A) = \text{card}(B) = \dim(X \smallsetminus X_0)$ and $\dim[X \times V]_0 = \dim(X_0 \times V) = \dim X_0 + \dim V$. Therefore $\dim(X \times V) = [\dim(X \smallsetminus X_0) : \dim X_0 + \dim V]$.

Conversely, suppose $Y := X \times V$ has a basis. Let B be a basis of $Y \smallsetminus Y_0$. Now consider $B' := \{x : (x, v_x) \in B$ for some $v_x \in V\}$. Then $x \in B' \Rightarrow x \notin X_0$. Therefore $B' \subseteq X \smallsetminus X_0$. We now show that B' forms a basis of $X \smallsetminus X_0$. For any $z \in X \smallsetminus X_0$, $(z, \theta_V) \in Y \smallsetminus Y_0$. As B is a basis of $Y \smallsetminus Y_0$, $\exists$ $(x, v_x) \in B$ such that $\alpha(x, v_x) + (p, v) \leq (z, \theta_V)$ for some $\alpha \in \mathbb{K}^*$ and $(p, v) \in [X \times V]_0 = X_0 \times V \Rightarrow$ $(\alpha x + p, \alpha v_x + v) \leq (z, \theta_V) \Rightarrow \alpha x + p \leq z \Rightarrow z \in L(x)$. So, B' generates $X \smallsetminus X_0$. If two members of B', say x', z', are orderly dependent, then without loss of generality, we can take $x' \in L(z') \Rightarrow \exists\, \alpha \in \mathbb{K}^*$ and $p \in X_0$ such that $\alpha z' + p \leq x' \Rightarrow \alpha(z', v_{z'}) + (p, v_{x'} - \alpha v_{z'}) \leq (x', v_{x'}) \Rightarrow (x', v_{x'})$ and $(z', v_{z'})$ are orderly dependent in $Y \smallsetminus Y_0$. Therefore we can say that B' is orderly independent in $X \smallsetminus X_0$ as B is orderly independent in $Y \smallsetminus Y_0$. So B' becomes a basis of $X \smallsetminus X_0$. Consequently, X has a basis. ☐

Example 3.30. For any two cardinal numbers α, β there exists an evs X such that $\dim X = [\alpha : \beta]$. For example, if we consider the evs $X := Y \times E$, where Y is an evs whose dimension is $[\alpha : 0]$ (existence of such evs has been established in Example 3.28) and E is a vector space with dimension β, then by the above Theorem 3.29, $\dim X = [\alpha : \beta]$.

Theorem 3.31. *Let X be an evs whose dimension is $[\alpha : \beta]$. Also, let γ and δ be two cardinal numbers such that $\gamma \leq \alpha$ and $\delta \leq \beta$. Then $\exists$ a subevs Y of X such that $\dim Y = [\gamma : \delta]$.*

Proof. Let B be a basis of $X \smallsetminus X_0$. Then $\text{card}(B) = \alpha$. Since $\gamma \leq \alpha$, there exists $C \subseteq B$ such that $\text{card}(C) = \gamma$. For each $c \in C$, we choose one element $p_c \in P_c$ and fix it.
Case 1: If $\delta < \gamma$, then $\exists\ E \subsetneqq C$ such that $\text{card}(E) = \delta$. Consider the set

$$D := E \cup \{c - p_c : c \in C \smallsetminus E\}.$$

Since C is orderly independent, it follows that $\text{card}(D) = \text{card}(C) = \gamma$. As $L(c - p_c) = L(c)$, it follows that D is an orderly independent set in $X \smallsetminus X_0$. Also, consider for any $d \in D$, $q_d = p_d$ if $d \in E$, otherwise $q_d = \theta$. Then there exists a subspace W of the vector space X_0 such that $q_d \in W$, $\forall\, d \in D$ and $\dim W = \delta$.
Case 2: If $\gamma \leq \delta$, then consider $D = C$ and $q_d = p_d$, $\forall\, d \in D$. Then also there exists a subspace W of X_0 such that $q_d \in W$, $\forall\, d \in D$ and $\dim W = \delta$.
Thus for both cases we get
(i) an orderly independent set D in $X \smallsetminus X_0$ whose cardinality is γ.
(ii) a subspace W of X_0 such that $q_d \in W$, where $q_d < d$,[1] $\forall\, d \in D$ and $\dim W = \delta$.
Now we consider the set

$$G(D) := \left\{ \sum_{i=1}^{n} \alpha_i d_i + p : \alpha_i \in \mathbb{K}, d_i \in D, p \in W, n \in \mathbb{N} \right\}.$$

Step 1: In this step we prove that $G(D)$ becomes a subevs of X with $D \subseteq G(D)$ and $[G(D)]_0 = W$. For any $d \in D$, $d = 1.d + \theta \in G(D) \Rightarrow D \subseteq G(D)$. Also, for any $p \in W$, $0.d + p \in G(D) \Rightarrow W \subseteq G(D)$. For any two elements $x = \sum_{i=1}^{m} \alpha_i d_i + p$, $y = \sum_{j=1}^{n} \beta_j d_j + q$ in $G(D)$ and any two scalars α, β, $\alpha x + \beta y = \sum_{i=1}^{m} \alpha\alpha_i d_i + \sum_{j=1}^{n} \beta\beta_j d_j + (\alpha p + \beta q) \in G(D)$ [as W is a subspace]. Let $y \in [G(D)]_0$. Then y is a minimal element of $G(D)$. As $y \in G(D)$, y can be written as $y = \sum_{i=1}^{n} \alpha_i d_i + p$. Our claim is that all $\alpha_i = 0$. If not, there exists $j \in \{1, 2, \ldots, n\}$ such that $\alpha_j \neq 0$. Then there exists $q_{d_j} \in W$ such that $q_{d_j} < d_j \Rightarrow \sum_{i=1}^{n} \alpha_i q_{d_i} + p < y$ which contradicts that $y \in [G(D)]_0$, as $\sum_{i=1}^{n} \alpha_i q_{d_i} + p \in W \subseteq G(D)$. So, all $\alpha_i = 0$. Therefore $y = p \in W \Rightarrow [G(D)]_0 \subseteq W \subseteq G(D) \cap X_0$, and hence $[G(D)]_0 = G(D) \cap X_0 = W$ [by Note 2.2]. Also, for any $x = \sum_{i=1}^{n} \alpha_i d_i + p \in G(D)$, $\sum_{i=1}^{n} \alpha_i q_{d_i} + p \in W = [G(D)]_0$ such that $x \geq \sum_{i=1}^{n} \alpha_i q_{d_i} + p$. Thus it follows that $G(D)$ is a subevs of X.
Step 2: In this step we show that D is a basis of $G(D) \smallsetminus [G(D)]_0$. Since D is an orderly independent subset of $X \smallsetminus X_0$ and $G(D)$ is a subevs of X containing D, by Remark 3.7, we can say that D is orderly independent in $G(D) \smallsetminus [G(D)]_0$. Now, let $y \in G(D) \smallsetminus [G(D)]_0$. Then y can be written as $y = \sum_{i=1}^{n} \alpha_i d_i + p$, where not all $\alpha_i = 0$. Let $\alpha_j \neq 0$. Then $\alpha_j d_j + \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i q_{d_i} + p \right) \leq y$. As $\left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i q_{d_i} + p \right) \in W = [G(D)]_0$, so $y \in L(d_j)$ in $G(D) \smallsetminus [G(D)]_0$. Thus D becomes a basis of

[1] Here the notation '$q_d < d$' is used to mean that $q_d \leq d$ but $q_d \neq d$.

$G(D) \smallsetminus [G(D)]_0$.
Therefore $\dim G(D) = [\mathrm{card}(D) : \dim W] = [\gamma : \delta]$. □

4. Computation of Basis and Dimension of Some Exponential Vector Space

In this section, we discuss the existence of a basis of some particular evs and thereby compute their dimensions. We show that there are evs which do not have basis.

Theorem 4.1. *Let X be a single-primitive comparable topological evs. Then X has a basis and* $\dim X = [1 : \dim X_0]$.

Proof. Since X is single-primitive, for each $z \in X$ let us write $P_z = \{p_z\}$. Let $x \in\uparrow \theta$ with $x \neq \theta$. Then $P_x = \{p_x\} = \{\theta\}$. Now for $y \in X \smallsetminus X_0$, $y - p_y \in\uparrow \theta$. Then X, being comparable evs, x and $y - p_y$ are comparable as $P_x = P_{y-p_y} = \{\theta\}$. If $x \leq y - p_y$, then $x + p_y \leq y \Rightarrow y \in L(x)$. If $x > y - p_y$, our claim is that there exists $\alpha \in \mathbb{K}^*$ such that $\alpha x \leq y - p_y$ with $|\alpha| < 1$. For, otherwise we can choose a sequence $\{\alpha_n\}$ in $\mathbb{K}^*$ such that $y - p_y < \alpha_n x \ \forall n \in \mathbb{N}$ and $\alpha_n \to 0$ as $n \to \infty$. Since X is a topological evs, we then have $y - p_y \leq \theta$ [taking the limit $n \to \infty$], which is a contradiction. So, there must exist one $\alpha \in \mathbb{K}^*$ such that $\alpha x \leq y - p_y \Rightarrow y \in L(x)$. Thus $L(x) = X \smallsetminus X_0$. Clearly, $\{x\}$ is orderly independent. Therefore $\{x\}$ is a basis of $X \smallsetminus X_0$. Consequently, X has a basis and $\dim X = [1 : \dim X_0]$. □

As the evs $[0, \infty) \times V$ is a single primitive comparable evs by the above theorem, we can say that $\dim([0, \infty) \times V) = [1 : \dim V]$, for any Hausdörff topological vector space V. So, in particular, if $V = \{\theta\}$, then the resulting evs is order-isomorphic to $[0, \infty)$ and hence $\dim[0, \infty) = [1 : 0]$. We have shown in the previous section that $\dim[0, \infty)^\alpha = [\alpha : 0]$, for any cardinal α. This can also be justified from the following more general example.

Example 4.2. Let $\{X_i : i \in I\}$ be an arbitrary collection of exponential vector spaces, over the common field $\mathbb{K}$, each having a basis. Let B_i be a basis of $X_i \smallsetminus [X_i]_0$, $\forall i \in I$. Consider the product evs $X := \prod_{i \in I} X_i$ [see Example 2.9]. Then $X_0 = \prod_{i \in I} [X_i]_0$. For any $j \in I$, consider the set $D_j := \prod_{i \in I} C_i$, where $C_i := \begin{cases} \{\theta_{X_i}\}, & \text{when } i \neq j, \\ B_j, & \text{when } i = j. \end{cases}$ Here, θ_{X_i} is the identity in X_i. Then $D_j \subseteq X \smallsetminus X_0$, $\forall j \in I$. Let $D := \bigcup_{j \in I} D_j$. Then $D \subseteq X \smallsetminus X_0$. Now, two different members in different D_i are orderly independent. As each B_i is a basis of X_i, so two different members of one D_i are orderly independent. Thus any two different members of D are orderly independent and hence D is orderly independent in $X \smallsetminus X_0$. We now show that D is a basis of $X \smallsetminus X_0$. For any $x = (x_i)_{i \in I} \in X \smallsetminus X_0$, $\exists$ some $k \in I$ such that $x_k \in X_k \smallsetminus [X_k]_0 \Rightarrow \exists\ b_k \in B_k$, $\alpha_k \in \mathbb{K}^*$ and $p_k \in [X_k]_0$ such that $\alpha_k b_k + p_k \leq x_k$. Now for $i \neq k$, $\exists$ $p_i \in [X_i]_0$ such that $p_i \leq x_i$. Let $b = (b_i)_{i \in I}$, where $b_i = \theta_{X_i}$ for $i \neq k$ and $p = (p_i)_{i \in I} \in X_0$. Then $\alpha_k b + p = (\alpha_k b_i + p_i)_{i \in I} \leq (x_i)_{i \in I} = x$ and $b \in D_k \subset D \Rightarrow x \in L(b)$. This shows that D generates $X \smallsetminus X_0$ and hence is a basis of $X \smallsetminus X_0$. Consequently, X has a basis and $\dim X = [\mathrm{card}(D) : \dim X_0]$.

If I is finite, then $\mathrm{card}(D) = \sum_{i \in I} \mathrm{card}(D_i) = \sum_{i \in I} \mathrm{card}(B_i) = \sum_{i \in I} \dim\big(X_i \smallsetminus [X_i]_0\big)$ and $\dim X_0 = \sum_{i \in I} \dim[X_i]_0$. For any four cardinal number $\alpha, \beta, \gamma, \delta$, if we use the notation $[\alpha + \gamma : \beta + \delta] = [\alpha : \beta] + [\gamma : \delta]$, then we can write

$$\dim \prod_{i \in I} X_i = \left[\sum_{i \in I} \dim\left(X_i \smallsetminus [X_i]_0\right) : \sum_{i \in I} \dim[X_i]_0\right] = \sum_{i \in I} \big[\dim\left(X_i \smallsetminus [X_i]_0\right) : \dim[X_i]_0\big].$$

If I is infinite, then also we get the similar expression as above, provided the sums (over I) are properly defined.

If all X_i's are the same, say $X_i = Y$, $\forall i \in I$ and $\mathrm{card}(I) = \alpha$, then we have

$$\dim(Y^\alpha) = [\alpha \cdot \dim(Y \smallsetminus Y_0) : \alpha \cdot \dim Y_0].$$

Thus it follows that for any cardinal α, $\dim[0, \infty)^\alpha = [\alpha : 0]$, since $\dim[0, \infty) = [1 : 0]$.

Theorem 4.3. *For every Hausdörff topological vector space $\mathcal{X}$, $\mathscr{C}(\mathcal{X})$ [discussed in 1.3] has a basis.*

Proof. Let us consider the relation '$\sim$' on $\mathcal{X} \smallsetminus \{\theta\}$, defined as

$$x \sim y \Leftrightarrow \exists\ \alpha \in \mathbb{K}^* \text{ such that } x = \alpha y.$$

Then '$\sim$' becomes an equivalence relation on $\mathcal{X} \smallsetminus \{\theta\}$. Let us construct a set $\mathcal{X}'$ taking exactly one representative from each equivalence class, relative to '$\sim$', and consider the set

$$\mathscr{N} := \big\{\{\theta, x\} : x \in \mathcal{X}'\big\}.$$

We now show that $\mathscr{N}$ becomes a basis of $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$. If $A \in \mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$, then there must exist two elements x, y of $\mathcal{X}$ with $\{x, y\} \subseteq A$ and $x \neq y$. Then $\{\theta, x-y\} + \{y\} = \{x, y\} \subseteq A$. Now, $x - y \in \mathcal{X} \smallsetminus \{\theta\} \Rightarrow \exists\ z \in \mathcal{X}'$ and $\alpha \in \mathbb{K}^*$ such that $x - y = \alpha z$. So we can write $\alpha\{\theta, z\} + \{y\} \subseteq A$ $\Rightarrow A \in L(\{\theta, z\})$. Therefore $\mathscr{N}$ generates $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$. We now show that $\mathscr{N}$ is an orderly independent set in $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(X)]_0$. For any two elements $\{\theta, x\}$ and $\{\theta, y\}$ in $\mathscr{N}$, if $\{\theta, x\} \in L(\{\theta, y\})$, then $\exists\ \alpha \in K^*$ such that $\alpha\{\theta, y\} + \{z\} \subseteq \{\theta, x\}$ for some $z \in \mathcal{X} \Rightarrow \{z, \alpha y + z\} = \{\theta, x\}$ [$\because z \neq \alpha y + z$] $\Rightarrow$ either $z = \theta$ or $z = x$. If $z = \theta$, then $\alpha y = x$ which means that x, y belong to the same equivalence class, relative to '$\sim$', and hence $\{\theta, x\}, \{\theta, y\}$ cannot be two distinct elements of $\mathscr{N}$, which is not the case. If $z = x$, then $\alpha y + x = \theta \Rightarrow x = -\alpha y$ which again leads to the same contradiction. This proves that any two elements of $\mathscr{N}$ are orderly independent. Therefore $\mathscr{N}$ is orderly independent in $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(X)]_0$ and hence becomes a basis of $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$. Consequently, $\mathscr{C}(\mathcal{X})$ has a basis and $\dim \mathscr{C}(\mathcal{X}) = [\operatorname{card}(\mathscr{N}) : \dim \mathcal{X}]$. □

Remark 4.4. We have shown in the above Theorem 4.3 that $\mathscr{N}$ forms a basis of $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$. We now show that this basis depends on a basis (as vector space) of $\mathcal{X}$.

(i) If $\mathcal{X}$ is a Hausdörff topological vector space of dimension 1, then any non-zero element of $\mathcal{X}$ is a scalar multiple of a single basic vector of $\mathcal{X}$ and hence $\mathscr{N}$ contains exactly one element. So, $\dim \mathscr{C}(\mathcal{X}) = [1 : 1]$. For that reason dimension of $\mathscr{C}(\mathbb{R})$ over $\mathbb{R}$ is $[1 : 1]$ and dimension of $\mathscr{C}(\mathbb{C})$ over $\mathbb{C}$ is $[1 : 1]$.

(ii) Let $\mathcal{X}$ be a Hausdörff topological vector space of dimension 2 and $B = \{a, b\}$ be a basis of $\mathcal{X}$. We first show that $\mathcal{X}' = \{a + \beta b : \beta \in \mathbb{K}\} \cup \{b\}$, where $\mathcal{X}'$ is as defined in the proof of Theorem 4.3. Any two distinct elements $a + \beta_1 b, a + \beta_2 b \in \mathcal{X} \smallsetminus \{\theta\}$ must lie in two different equivalence classes, relative to '$\sim$', since for any $\alpha \in \mathbb{K}^*$, if $\alpha(a + \beta_1 b) = a + \beta_2 b$, we have $\alpha = 1$ and hence $\beta_1 = \beta_2$ [as $\{a, b\}$ is a linearly independent subset of $\mathcal{X}$] — this contradicts that $a + \beta_1 b \neq a + \beta_2 b$. Also, the linear independence of a, b implies that $a + \beta b$ and b must lie in two different equivalence classes, relative to '$\sim$', for any $\beta \in \mathbb{K}$. Now, for any non-zero element $x \in \mathcal{X}$, $\exists\, \alpha, \beta \in \mathbb{K}$ (not both zero) such that $x = \alpha a + \beta b$ [since $\{a, b\}$ is a basis of $\mathcal{X}$]. If $\alpha \neq 0$, then $x = \alpha(a + \beta\alpha^{-1} b) \Rightarrow x$ lies in the class (relative to '$\sim$'), whose representative is $(a + \beta\alpha^{-1} b)$. If $\alpha = 0$, then x lies in the equivalence class (relative to '$\sim$'), whose representative is b. Therefore $\mathcal{X}' = \{a + \beta b : \beta \in \mathbb{K}\} \cup \{b\}$. Now, the map $\alpha \longmapsto a + \alpha b$ creates a bijection between $\mathbb{K}$ and $\mathcal{X}' \smallsetminus \{b\}$. So, we can say that the cardinality of $\mathcal{X}'$ and hence that of $\mathscr{N}$ is c, the cardinality of the set of real numbers $\mathbb{R}$. Therefore $\dim \mathscr{C}(\mathcal{X}) = [c : 2]$. For that reason dimension of $\mathscr{C}(\mathbb{C})$ over $\mathbb{R}$ is $[c : 2]$.

(iii) In a similar manner as above, we can show that for a *well-ordered* basis B of a Hausdörff topological vector space $\mathcal{X}$,

$$\mathcal{X}' = (e_1 + < B_1 >) \cup (e_2 + < B_2 >) \cup \cdots \cup (e_n + < B_n >) \cup \cdots,$$

where $B = \{e_1, e_2, \dots, e_n, \dots\}$, $B_1 = B \smallsetminus \{e_1\}$, $B_n = B_{n-1} \smallsetminus \{e_n\}$, $\forall\, n \geq 2$ and $< B_i >$ denotes the linear span of B_i in $\mathcal{X}$, $\forall\, i$.

Theorem 4.5. *For every vector space $\mathcal{X}$, the evs $\mathscr{L}(\mathcal{X})$ has a basis.* [The evs $\mathscr{L}(\mathcal{X})$ is discussed in Example 2.12]

Proof. Let $\mathscr{T}$ be the collection of all one-dimensional subspaces of $\mathcal{X}$. We now show that $\mathscr{T}$ forms a basis of $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$. For any non-trivial subspace $\mathcal{Y}$ of $\mathcal{X}$, there exists a non-zero element $x \in \mathcal{Y}$ such that $< x > \subseteq \mathcal{Y}$ $\big[$here, $< x >$ denotes the linear span of x in $\mathcal{X}\big]$. So, $\mathcal{Y} \in L(< x >)$. Also, $< x > \in \mathscr{T}$. Thus $\mathscr{T}$ generates $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$. For any two distinct elements $< x >, < y > \in \mathscr{T}$, if $\alpha < x > \subseteq < y >$ for some $\alpha \in \mathbb{K}^*$, then $< x > = \alpha < x > \subseteq < y > \Rightarrow < x > = < y >$ which contradicts that $< x >$ and $< y >$ are distinct. So, we can say that $\mathscr{T}$ is an orderly independent subset of $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$. Therefore $\mathscr{T}$ forms a basis of $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$. Consequently, $\mathscr{L}(\mathcal{X})$ has a basis and $\dim \mathscr{L}(\mathcal{X}) = [\operatorname{card}(\mathscr{T}) : 0]$, since $[\mathscr{L}(\mathcal{X})]_0 = \big\{\{\theta\}\big\}$. $\square$

From the above theorem, we can immediately get the following result. Also, $\mathscr{T}$ is the *only* basis of $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$.

Result 4.6. $\dim \mathscr{L}(\mathcal{X}) = [1 : 0]$*, when* $\dim \mathcal{X} = 1$ *and* $\dim \mathscr{L}(\mathcal{X}) = [c : 0]$*, when* $\dim \mathcal{X} = 2$*, c being the cardinality of the set of all reals* $\mathbb{R}$*.*

Note 4.7. From the previous result we can say that $\dim \mathscr{L}(\mathbb{R}) = [1 : 0]$ which is the same with the $\dim[0, \infty)$. But $\mathscr{L}(\mathbb{R})$ and $[0, \infty)$ are not order-isomorphic as the first one is non-topological evs, whereas the second one is a topological evs and, being topological, is an evs property. This example shows that a converse part of the statement that equality of dimension is an evs property which we have discussed in 3.18, is not true.

Theorem 4.8. *For any* $n \in \mathbb{N}$*,* $\mathscr{D}^n[0, \infty)$ *has a basis and* $\dim \mathscr{D}^n[0, \infty) = [1 : 0]$.

Proof. We first show that $(0, 0, \dots, 0, 1)$ generates $\mathscr{D}^n[0, \infty) \smallsetminus [\mathscr{D}^n[0, \infty)]_0$. Let $x = (x_1, \dots, x_n) \in \mathscr{D}^n[0, \infty) \smallsetminus [\mathscr{D}^n[0, \infty)]_0$. Since $[\mathscr{D}^n[0, \infty)]_0 = \big\{(0, \dots, 0)\big\}$, there exists $i \in \{1, 2, \dots, n\}$ such that $x_i \neq 0$ and $x_j = 0$, for all $j < i$. If $i < n$, then, obviously, $(0, 0, \dots, 0, 1) \leq x$. If $i = n$, then $\frac{x_i}{2}(0, 0, \dots, 0, 1) \leq x$. In any case, $x \in L\big((0, 0, \dots, 1)\big)$. Since $\big\{(0, \dots, 0, 1)\big\}$ is orderly independent, it follows that $\big\{(0, \dots, 0, 1)\big\}$ is a basis of $\mathscr{D}^n[0, \infty) \smallsetminus [\mathscr{D}^n[0, \infty)]_0$ and hence $\dim \mathscr{D}^n[0, \infty) = [1 : 0]$. $\square$

The following example shows that there exists an evs which has no basis.

Theorem 4.9. $X := \mathscr{D}\big([0, \infty) : \mathbb{N}\big)$ *has no basis.*

Proof. Let $x = (x_i)_{i \in \mathbb{N}} \in X \smallsetminus X_0$. Since here $X_0 = \big\{(0, 0, \dots)\big\}$, there must exist a least positive integer p such that $x_p \neq 0$. If we consider $y = (y_i)_{i \in \mathbb{N}}$, where $y_i = x_i$, $\forall\, i \neq p, p+1$ and $y_p = 0$, $y_{p+1} = 1$, then $y \leq x$ and $y \notin X_0$; but there does not exist any $\alpha \in \mathbb{K}^*$ such that $\alpha x \leq y$, which means that $y \notin L(x)$. This shows that $x \notin Q(X)$ and this holds for any non-zero element x of X. Therefore $Q(X) = \emptyset$. So, $\mathscr{D}\big([0, \infty) : \mathbb{N}\big)$ has no basis. $\square$

Looking at the proof of the above theorems, we can get the following generalised theorem.

Theorem 4.10. *For a well-ordered set* I*,* $\mathscr{D}(X : I)$ *has a basis, iff* I *has a maximum element.*

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RESTRICTED TESTING FOR THE HARDY–LITTLEWOOD MAXIMAL FUNCTION ON ORLICZ SPACES

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Abstract. In this short note, we formulate and prove a restricted testing condition for the Hardy–Littlewood maximal function acting between the weighted Orlicz spaces.

1. Introduction

Our interest in this note is for the two weight inequalities for the Hardy–Littlewood maximal function acting between the weighted Orlicz spaces of $\mathbb{R}^d$.

Recall that a weight ω on $\mathbb{R}^d$ is any positive locally integrable function. The Hardy–Littlewood maximal function is defined by

$$\mathcal{M}f(x) := \sup_{x\in Q} \frac{1_Q(x)}{|Q|} \int\limits_Q |f(y)|dy \tag{1}$$

with Q a cube whose sides are parallel to the coordinate axes, and $|Q|$ is the Lebesgue measure of Q.

In 1982, E. T. Sawyer (see [8]) obtained the following two weight characterizations for the Hardy–Littlewood maximal function.

Theorem 1.1. *For $1 < p < \infty$ and for any pair of weights (ω, σ), we have the inequality*

$$\|\mathcal{M}(\sigma f)\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\sigma)} \tag{2}$$

if and only if

$$\sup_{Q,\,\sigma(Q)>0} \sigma(Q)^{-1/p}\|1_Q\mathcal{M}(\sigma 1_Q)\|_{L^p(\omega)} < \infty. \tag{3}$$

Sawyer's result summarizes as follows: for (2) to hold for any $f \in L^p(\sigma)$, it suffices for it to hold on characteristic functions of cubes.

Pretty recently, it has been observed that the supremum in (3) doesn't need to be taken on all cubes provided the so-called A_p condition for the pair of weights (ω, σ) holds ([2, 3]). To be more precise, this type of new characterizations was first considered in [5, 7] for various operators. In particular, in [7], the authors introduced the restricted testing to doubling cubes in the two weight inequalities for the maximal operator $\mathcal{M}$. In [3], W. Chen and M. T. Lacey obtained similar conditions providing also a short proof. More recently, in [2], the authors exploited these new ideas to obtain corresponding results for the multilinear maximal operator.

Recall that the pair of weights (ω, σ) is said to satisfy the A_p condition if

$$[\omega, \sigma]_p := \sup_Q \langle\omega\rangle_Q^{1/p}\langle\sigma\rangle_Q^{1/p'} < \infty$$

where $\langle\omega\rangle_Q = |Q|^{-1}\int_Q \omega dx$, and $\frac{1}{p} + \frac{1}{p'} = 1$.

We recall the following definition.

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Definition 1.2. Let $1 < \rho, p, D < \infty$. We say the pair of weights (ω, σ) satisfies a (ρ, p, D) parent doubling testing condition if there is a positive finite constant $\mathcal{P} = \mathcal{P}_{\rho,D} = \mathcal{P}(\omega, \sigma, d, p, \rho, D)$, so that we have

$$\sigma(Q)^{-1/p}\|1_Q\mathcal{M}(\sigma 1_Q)\|_{L^p(\omega)} \leq \mathcal{P}$$

for every cube Q for which there is another cube $R \supset Q$, with $\ell R \geq \rho \ell Q$, and $\sigma(R) \leq D\sigma(Q)$.

Above and all over the test, $\ell Q = |Q|^{1/d}$ is the side length of the cube Q.

The result obtained by W. Chen and M. T. Lacey in [3] is the following

Theorem 1.3. *Let $1 < p, \rho < \infty$. Then there exists a constant $D = D_{d,p,\rho}$ such that for any pair of weights (ω, σ), we have*

$$\|\mathcal{M}(\sigma\cdot)\|_{L^p(\sigma)\to L^p(\omega)} \approx [\omega, \sigma]_p + \mathcal{P}_{\rho,D}.$$

Our aim here is to formulate and prove an analogue of Theorem 1.3 when the Lebesgue spaces are replaced by the Orlicz spaces.

By a growth function we will mean a continuous and nondecreasing function Φ from $[0, \infty)$ onto itself. We note that this implies, in particular, that $\Phi(0) = 0$.

The growth function Φ is said to satisfy the Δ_2-condition if there exists a constant $K > 1$ such that for any $t \geq 0$,

$$\Phi(2t) \leq K\Phi(t). \tag{4}$$

Given a convex growth function Φ satisfying the Δ_2-condition and a weight ω, the weighted Orlicz space $L^\Phi_\omega(\mathbb{R}^d)$ is defined to be the space of all functions f on $\mathbb{R}^d$ such that

$$\|f\|_{\Phi,\omega} := \int_{\mathbb{R}^d} \Phi(|f(t)|)\omega(t)dt < \infty.$$

Let us note that when $\Phi(t) = t^p$, $1 \leq p < \infty$, the above space is just the usual weighted Lebesgue $L^p_\omega(\mathbb{R}^d)$.

Recall also that the complementary function Ψ of the convex growth function Φ is the function defined from $\mathbb{R}_+$ onto itself by

$$\Psi(s) = \sup_{t\in\mathbb{R}_+}\{ts - \Phi(t)\}. \tag{5}$$

A growth function Φ is said to satisfy the ∇_2-condition whenever both Φ and its complementary function satisfy the Δ_2-conditon.

Given a convex growth function Φ, we define $\phi(t) = \frac{\Phi(t)}{t}$ and observe that ϕ is nondecreasing. We then say a pair of weights (ω, σ) satisfies the A_Φ condition whenever

$$[\omega, \sigma]_\Phi := \sup_Q \langle\omega\rangle_Q \phi(\langle\sigma\rangle_Q) < \infty.$$

Let us now introduce the following

Definition 1.4. Let Φ be a convex growth function and $1 < \rho, D < \infty$. We say the pair of weights (ω, σ) satisfies a (ρ, Φ, D) parent doubling testing condition if there is a positive finite constant $\mathcal{P} = \mathcal{P}_{\rho,D} = \mathcal{P}(\omega, \sigma, d, \Phi, \rho, D)$ so that we have

$$\int_Q \Phi\left(\mathcal{M}(\sigma 1_Q)\right)\omega dx \leq \mathcal{P}\sigma(Q) \tag{6}$$

for every cube Q for which there is another cube $R \supset Q$ with $\ell R \geq \rho \ell Q$, and $\sigma(R) \leq D\sigma(Q)$.

Let us denote by $\mathscr{C}$ the set of all convex growth functions. We then define $\mathscr{C}'$ as the set of all $\Phi \in \mathscr{C} \cap \mathcal{C}^1$ such that $\Phi'(t) \approx \frac{\Phi(t)}{t}$.

We say a growth function Φ satisfies the Δ'-condition if there exists a constant $C_1 > 0$ such that for any $0 < s, t < \infty$,

$$\Phi(st) \leq C_1\Phi(s)\Phi(t). \tag{7}$$

Obviously, power functions satisfy (7). As a nontrivial example of a growth function satisfying (7), we have the function $t \mapsto t^q \log^\alpha(C + t)$, where $q \geq 1$, $\alpha > 0$ and the constant $C > 0$ is large enough.

It is not difficult to prove the following extension of Theorem 1.1 (see also [9]).

Theorem 1.5. *Let $\Phi \in \mathcal{C}'$ be a growth function satisfying both the Δ'-condition and the ∇_2 condition. Then*

$$\sup_{0\neq f\in L^\Phi(\sigma)} \frac{\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}(\sigma f)(x)\right)\omega(x)dx}{\int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx} < \infty \tag{8}$$

holds if and only if

$$[\omega,\sigma]_{S_\Phi} := \sup_{Q,\,\sigma(Q)>0} \sigma(Q)^{-1} \int_Q \Phi\left(\mathcal{M}(\sigma 1_Q)\right)\omega dx < \infty. \tag{9}$$

Moreover,

$$\sup_{0\neq f\in L^\Phi(\sigma)} \frac{\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}(\sigma f)(x)\right)\omega(x)dx}{\int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx} \approx [\omega,\sigma]_{S_\Phi}. \tag{10}$$

It is obvious that (8) implies (9). The converse can be proved as in the power functions case, using Theorem 2.4 in the next section.

Our result here is about restricting the global testing condition (9). It is given as follows.

Theorem 1.6. *Let $\Phi \in \mathcal{C}'$ be a growth function satisfying both the Δ'-condition and the ∇_2 condition, and let $1 < \rho < \infty$. Then there exists a constant $D = D_{d,\Phi,\rho}$ such that for any pair of weights (ω,σ), we have*

$$\sup_{0\neq f\in L^\Phi(\sigma)} \frac{\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}(\sigma f)(x)\right)\omega(x)dx}{\int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx} \approx [\omega,\sigma]_\Phi + \mathcal{P}_{\rho,D}. \tag{11}$$

2. Preliminaries

2.1. **Indices of a Growth function.** We recall that for Φ a $\mathcal{C}^1$ growth function, the lower index of Φ is defined by

$$a = a_\Phi := \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

Following [4, Lemma 2.6], we find that if a convex growth function Φ satisfies the ∇_2-condition, then $1 < a_\Phi < \infty$.

It is easy to see that if Φ is a $\mathcal{C}^1$ growth function, then the function $\frac{\Phi(t)}{t^{a_\Phi}}$ is increasing.

2.2. **Dyadic grids and sparse families.** The standard dyadic grid $\mathcal{D}$ in $\mathbb{R}^d$ is the collection of all cubes of the form

$$2^{-k}\left([0,1)^d + m\right), \quad k\in\mathbb{Z}, m\in\mathbb{Z}^d.$$

Definition 2.1. A (general) dyadic grid $\mathcal{D}^\beta$ in $\mathbb{R}^d$ is any collection of cubes such that:

(i) the sidelength ℓQ of any cube $Q \in \mathcal{D}^\beta$ is 2^k for some $k \in \mathbb{Z}$;
(ii) for $Q, Q' \in \mathcal{D}^\beta$, $Q\cap Q' \in \{Q, Q', \emptyset\}$;
(iii) for each $k\in\mathbb{Z}$, the family $\mathcal{D}^\beta_k := \{Q\in\mathcal{D}^\beta : \ell Q = 2^k\}$ forms a partition of $\mathbb{R}^d$.

We say a collection of dyadic cubes $\mathcal{S}^\beta = \{Q_{j,k}\}_{j,k\in\mathbb{Z}} \subset \mathcal{D}^\beta$ is a sparse family if

(i) for each fixed k, the family $\{Q_{j,k}\}_{j\in\mathbb{Z}}$ is pairwise disjoint;
(ii) if $A_k = \cup_{j\in\mathbb{Z}} Q_{j,k}$, then $A_{k+1}\subset A_k$;
(iii) $|A_{k+1}\cap Q_{j.k}| \leq \frac{1}{2}|Q_{j,k}|$.

In particular, given a sparse family $\mathcal{S}^\beta = \{Q_{j,k}\}_{j,k\in\mathbb{Z}} \subset \mathcal{D}^\beta$, if we define for $Q_{j,k}\in\mathcal{S}^\beta$, the set $E_{Q_{j,k}} := Q_{j,k}\setminus A_{k+1}$, then we find that the family $\{E_Q\}_{Q\in\mathcal{S}^\beta}$ is pairwise disjoint.

We refer to [6] for the following

Lemma 2.2. *There are 2^d dyadic grids $\mathcal{D}^\beta$ such that for any cube $Q\in\mathbb{R}^d$, there exists a cube $R\in\mathcal{D}^\beta$ for some β such that $Q\subset R$ and $\ell R\leq 6\ell Q$.*

2.3. **Extended Carleson embedding lemma.** Recall that for σ a weight, the weighted (dyadic) Hardy–Littlewood maximal function is defined by

$$\mathcal{M}_\sigma^{\mathcal{D}^\beta} f(x) := \sup_{Q\in\mathcal{D}^\beta} \frac{1_Q(x)}{\sigma(Q)} \int_Q |f(s)|\sigma(s)ds.$$

We have the following easy fact.

Theorem 2.3. *Let Φ be a convex growth function in $\mathcal{C}'$ satisfying the ∇_2-condition, and let σ be a weight in $\mathbb{R}^d$. Then there exists a constant $C = C_\Phi > 0$ such that for any $f \in L^\Phi_\sigma(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}_\sigma^{\mathcal{D}^\beta} f(x)\right)\sigma(x)dx \le C \int_{\mathbb{R}^d} \Phi\left(|f(x)|\right)\sigma(x)dx. \tag{12}$$

The following Carleson embeddding result can be proved as in the power functions case (see [1,6]).

Theorem 2.4. *Let Φ be a growth function in $\mathcal{C}'$ satisfying the ∇_2-condition. Let σ be a weight on $\mathbb{R}^d$ and let $\{\lambda_Q\}_{Q\in\mathcal{D}^\beta}$ be a sequence of positive numbers indexed over the set of dyadic cubes $\mathcal{D}^\beta$ in $\mathbb{R}^d$. Then the following assertions are equivalent.*

(a) $\{\lambda_Q\}_{Q\in\mathcal{D}^\beta}$ *is a σ-Carleson sequence, i.e., there is a constant $A > 0$ such that*

$$\sum_{Q\subseteq R, Q\in\mathcal{D}^\beta} \lambda_Q \le A\sigma(R).$$

(b) *There exists a constant $C > 0$ such that for any function f,*

$$\sum_{Q\in\mathcal{D}^\beta} \lambda_Q \Phi\left(\frac{1}{\sigma(Q)}\int_Q |f(x)|\sigma(x)dx\right) \le CA\int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx. \tag{13}$$

3. Proof of Theorem 1.6

First, the fact that the condition (8) implies the A_Φ condition for the pair (ω,σ) is obvious. From Theorem 1.5, we have that (8) implies (9). Hence the heart of the matter is to prove the existence of a sufficiently large (doubling) constant D such that for any pair of weights (ω,σ) and for any $f \in L^\Phi(\sigma)$,

$$\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}(\sigma f)(x)\right)\omega(x)dx \lesssim \left([\omega,\sigma]_\Phi + \mathcal{P}_{\rho,D}\right)\int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx. \tag{14}$$

As in the power function case, we restrict our proof to $1 < \rho \le 2$ and use dyadic grids introduced above as for $r = 3, 4, \dots$, and choices of $r-1 < \rho \le r$, the proof proceeds by replacing dyadic grids by r–ary grids.

Indeed, following Lemma 2.2, it is enough to prove (14) with $\mathcal{M}^{\mathcal{D}^\beta}$ in place of $\mathcal{M}$. We only consider the standard grid $\mathcal{D}$ as the proof doesn't depend on the choice of the dyadic grid.

We set $D = 2^{d\frac{a+1}{a-1}}$, where a is the lower index of Φ. We note that if $\mathcal{S} = \{Q_{j,k}\}_{j,k\in\mathbb{Z}} \subset \mathcal{D}$ is the set of all maximal dyadic cubes $Q_{j,k} \in \mathcal{D}$ (with respect to the inclusion) such that

$$\frac{1}{|Q_{j,k}|}\int_{Q_{j,k}} |f(y)|\sigma(y)dy > 2^k,$$

then $\mathcal{S}$ is a sparse family. Moreover,

$$A_k = \bigcup_{j\in\mathbb{Z}} Q_{j,k} = \{x \in \mathbb{R}^d : \mathcal{M}^{\mathcal{D}} f(x) > 2^k\}.$$

As Φ satisfies the Δ'-condition, we obtain

$$\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}^{\mathcal{D}}(\sigma f)(x)\right)\omega(x)dx \lesssim \sum_{Q\in\mathcal{D}} \lambda_Q \Phi\left(\frac{1}{\sigma(Q)}\int_Q |f(x)|\sigma(x)dx\right),$$

where

$$\lambda_Q = \begin{cases} \Phi\left(\frac{\sigma(Q)}{|Q|}\right)\omega(E_Q) & \text{if} \quad Q \in \mathcal{S} \\ 0 & \text{if} \quad Q \in \mathcal{D} \setminus \mathcal{S}. \end{cases}$$

By Theorem 2.4, (14) follows provided the sequence $\{\lambda_Q\}_{Q\in\mathcal{D}}$ satisfies

$$\sum_{Q\subset R,\, Q\in\mathcal{D}} \lambda_Q = \sum_{Q\in\mathcal{S}_R} \lambda_Q \lesssim ([\omega,\sigma]_\Phi + \mathcal{P}_{\rho,D})\,\sigma(R), \quad \forall R \in \mathcal{D}, \tag{15}$$

where $\mathcal{S}_R := \{Q \in \mathcal{D} : Q \subset R\}$.

We partition $\mathcal{S}_R$ into the following four subcollections.

- (The Testing Collection). Let $\mathcal{T}$ be the subcollection of cubes in $\mathcal{S}_R$ such that the testing inequality (6) is satisfied.
- (The Top Cubes). Let $\mathcal{U} := \{Q \in \mathcal{S}_R \setminus \mathcal{T} : 2^k \ell Q \geq \ell R\}$, where k is chosen large enough so that $2^{dk}k^{-a} > 1$. One can observe that this collection has at most $2^{1+d(k+1)}$ cubes.
- (The Small A_Φ Cubes). Let $\mathcal{A}$ be the set of cubes in $Q \in \mathcal{S}_R \setminus (\mathcal{T} \cup \mathcal{U})$ such that

$$\langle\omega\rangle_Q \phi(\langle\sigma\rangle_Q) \leq \frac{[\omega,\sigma]_\Phi}{(\log_2 \ell R/\ell Q)^a}. \tag{16}$$

- (The Remaining Cubes). Let $\mathcal{L} := \mathcal{S}_R \setminus (\mathcal{T} \cup \mathcal{U} \cup \mathcal{A})$.

We now show that the estimate (15) holds when the sum is restricted to each of the above subcollections.

Starting with the Testing Collection, we easily obtain

$$\begin{aligned} \sum_{Q\in\mathcal{T}} \omega(E_Q)\Phi\left(\frac{\sigma(Q)}{|Q|}\right) &\leq \sum_{Q\in\mathcal{T}} \int_{E_Q} \Phi\left(\mathcal{M}^{\mathcal{D}}(\sigma 1_Q)\right)\omega(x)dx \\ &\leq \int_R \Phi\left(\mathcal{M}^{\mathcal{D}}(\sigma 1_Q)\right)\omega(x)dx \\ &\leq \mathcal{P}_{\rho,D}\sigma(R). \end{aligned}$$

Recalling that the Top Collection $\mathcal{U}$ has at most $2^{1+d(k+1)}$ cubes, that $\phi(t) = \frac{\Phi(t)}{t}$, and using our definition of A_Φ, we obtain

$$\begin{aligned} \sum_{Q\in\mathcal{U}} \omega(E_Q)\Phi\left(\frac{\sigma(Q)}{|Q|}\right) &\leq \sum_{Q\in\mathcal{U}} \sigma(Q)\frac{\omega(Q)}{|Q|}\phi\left(\frac{\sigma(Q)}{|Q|}\right) \\ &\lesssim_k [\omega,\sigma]_\Phi\sigma(R). \end{aligned}$$

Here, the notation $\lesssim_k$ means that the implied constant depends on the integer k.

The Small A_Φ Cubes are handled by using the condition (16) defining them as follows:

$$\begin{aligned} \sum_{Q\in\mathcal{A}} \omega(E_Q)\Phi\left(\frac{\sigma(Q)}{|Q|}\right) &\leq \sum_{Q\in\mathcal{A}} \sigma(Q)\frac{\omega(Q)}{|Q|}\phi\left(\frac{\sigma(Q)}{|Q|}\right) \\ &\lesssim [\omega,\sigma]_\Phi \sum_{Q\in\mathcal{A}} \frac{\sigma(Q)}{(\log_2 \ell R/\ell Q)^a} \\ &= [\omega,\sigma]_\Phi \sum_{s>k}\ \sum_{Q\in\mathcal{A},\, \ell R = 2^s \ell Q} \frac{\sigma(Q)}{(\log_2 \ell R/\ell Q)^a} \\ &= [\omega,\sigma]_\Phi \sum_{s>k} \frac{1}{s^a} \sum_{Q\in\mathcal{A},\, \ell R = 2^s \ell Q} \sigma(Q) \\ &\lesssim [\omega,\sigma]_\Phi\sigma(R). \end{aligned}$$

It now remains to deal with the last subcollection. We will prove that $\mathcal{L}$ is also empty in our case. Indeed, suppose that $\mathcal{L} \neq \emptyset$. Then there is a cube $Q \in \mathcal{S}_R$ such that $2^k\ell Q < \ell R$ and (16) fails and no ancestor of Q contained in R has a doubling parent in the sense of Definition 1.2.

Denote by $Q^{(1)}$ the D-parent of Q and let $Q^{(j+1)} = \left(Q^{(j)}\right)^{(1)}$. Let k_0 be the integer such that $R = Q^{(k_0)}$. Observe that for any $1 \le j < k_0$, $\sigma(Q^{(j+1)}) > D\sigma(Q^{(j)})$. Hence $\sigma(R) \ge D^{k_0}\sigma(Q)$.

We recall that the function $\frac{\Phi(t)}{t^a} = \frac{\phi(t)}{t^{a-1}}$ is increasing. From this and the above observations, we obtain

$$\begin{aligned}
[\omega,\sigma]_\Phi &\ge \langle\omega\rangle_R \phi\left(\frac{\langle\sigma\rangle_R}{|R|}\right)\\
&\ge \frac{\omega(Q)}{|Q^{(k_0)}|}\phi\left(\frac{D^{k_0}\sigma(Q)}{|Q^{(k_0)}|}\right)\\
&= \frac{\omega(Q)}{2^{dk_0}|Q|}\phi\left(\left(\frac{D}{2^d}\right)^{k_0}\frac{\sigma(Q)}{|Q|}\right)\\
&\ge \frac{\omega(Q)}{2^{dk_0}|Q|}\left(\frac{D}{2^d}\right)^{k_0(a-1)}\phi\left(\frac{\sigma(Q)}{|Q|}\right)\\
&\ge 2^{-dk_0}\left(\frac{D}{2^d}\right)^{k_0(a-1)}\frac{[\omega,\sigma]_\Phi}{(\log_2 \ell R/\ell Q)^a}\\
&= [\omega,\sigma]_\Phi 2^{-dk_0}\left(\frac{D}{2^d}\right)^{k_0(a-1)} k_0^{-a}\\
&= 2^{dk_0}[\omega,\sigma]_\Phi k_0^{-a}.
\end{aligned}$$

The last line follows from our choice of D. We easily deduce that $k_0 < k$, which implies that the cube Q belongs to $\mathcal{U}$. This is a contraction. Thus $\mathcal{L} = \emptyset$ and the proof is complete.

Remark 3.1. Following the equivalence (10), one could have chosen to prove directly that under the conditions in Theorem 1.6,

$$[\omega,\sigma]_{S_\Phi} \approx [\omega,\sigma]_\Phi + \mathcal{P}_{\rho,D}.$$

This can be done combining the ideas in this text with those in [3]. Our choice of the method in this text is motivated only by the fact that as the proof of Theorem 1.5 is left to the reader, we wanted the reader to have an idea of how the extended Carleson embedding result can be used.

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STEADY VIBRATIONS PROBLEMS IN THE THEORY OF THERMOVISCOELASTIC POROUS MIXTURES

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Abstract. In this paper, the linear theory of thermoviscoelastic binary porous mixtures is considered and the basic boundary value problems (BVPs) of steady vibrations are investigated. Namely, the fundamental solution of the system of equations of steady vibrations is constructed explicitly and its basic properties are established. Green's identities are obtained and the uniqueness theorems for classical solutions of the internal and external basic BVPs of steady vibrations are proved. The surface and volume potentials are constructed and their basic properties are given. The determinants of symbolic matrices of the singular integral operators are calculated explicitly and the BVPs are reduced to the always solvable singular integral equations for which Fredholm's theorems are valid. Finally, the existence theorems for classical solutions of the internal and external BVPs of steady vibrations are proved by means of the potential method and the theory of singular integral equations.

1. Introduction

The prediction of the mechanical properties of viscoelastic materials has been one of hot topics of continuum mechanics for more than 100 years. The construction of mathematical models of viscoelastic continua arise by an extensive use of viscous materials in many branches of engineering, technology and biomechanics (see Lakes [19], Brinson and Brinson [5] and references therein).

In the past two decades there has been much effort to develop mathematical models of thermoviscoelastic mixtures. Indeed, Ieşan [12] has presented the theory of thermoelasticity of binary porous mixtures in Lagrangian description, and the classical Kelvin–Voigt viscoelastic model is generalized by using a mixture theory. The existence and exponential decay of a solution in the linear variant of this theory is studied by Quintanilla [23]. The theory of thermoviscoelastic composites modelled as interacting Cosserat continua is introduced by Ieşan [14]. A mathematical model of porous thermoviscoelastic binary mixtures is presented by Ieşan and Quintanilla [16], where the individual components are modelled as Kelvin–Voigt viscoelastic materials. In [15], a nonlinear theory of heat conducting mixtures is introduced. A mixture theory for microstretch thermoviscoelastic solids is developed by Chiriţă and Galeş [6]. The theory of microstretch thermoviscoelastic composite materials is constructed by Passarella et al. [21]. A continuum theory for a thermoviscoelastic composite with the help of an entropy production inequality proposed by Green and Laws is presented by Ieşan and Scalia [17]. Recently, a nonlinear theory is derived for a thermoviscoelastic diffusion composite which is modeled as a binary mixture consisting of two Kelvin–Voigt viscoelastic materials by Aouadi et al. [2].

The basic problems of these theories are intensively investigated by scientists of several research groups in the series of papers [1,3,7–11,13,22]. Moreover, in [25,26], the basic properties of plane waves are established, the uniqueness and existence theorems are proved in the theories of viscoelasticity and thermoviscoelasticity for binary mixtures without pores. Recently, the potential method is developed in the theory of viscoelastic binary porous mixtures by Svanadze [27].

For an extensive review of the works and basic results in the theory of mixtures see the books of Bowen [4] and Rajagopal and Tao [24].

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In this paper, the linear theory of thermoviscoelastic binary porous mixtures (see Ieşan [12]) is considered and the basic BVPs of steady vibrations are investigated. Indeed, the fundamental solution of the system of equations of steady vibrations in the considered theory is constructed explicitly and its basic properties are established. Green's identities are obtained and the uniqueness theorems for classical solutions of the internal and external basic BVPs of steady vibrations are proved. The surface and volume potentials are constructed and their basic properties are given. The determinants of symbolic matrices are calculated explicitly. The BVPs are reduced to the always solvable singular integral equations for which Fredholm's theorems are valid. Finally, the existence theorems for classical solutions of the internal and external BVPs of steady vibrations are proved by means of the potential method and the theory of singular integral equations.

2. Basic Equations

We consider a thermoelastic binary porous mixture of constituents $s^{(1)}$ and $s^{(2)}$ that occupies the region Ω of the Euclidean three-dimensional space $\mathbb{R}^3$, where $s^{(1)}$ and $s^{(2)}$ are a Kelvin–Voigt material and an isotropic elastic solid, respectively. Let $\mathbf{x}=(x_1,x_2,x_3)$ be a point of $\mathbb{R}^3$ and let t denote the time variable. We assume that subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range (1,2,3) and the dot denotes differentiation with respect to t.

Let $\hat{\mathbf{u}}(\mathbf{x},t)$ and $\hat{\mathbf{w}}(\mathbf{x},t)$ be the partial displacements of constituents $s^{(1)}$ and $s^{(2)}$, respectively; $\hat{\mathbf{u}}=(\hat{u}_1,\hat{u}_2,\hat{u}_3)$, $\hat{\mathbf{w}}=(\hat{w}_1,\hat{w}_2,\hat{w}_3)$. We denote by $\hat{\varphi}(\mathbf{x},t)$ and $\hat{\psi}(\mathbf{x},t)$ the changes of volume fraction fields from the reference configuration for the constituents $s^{(1)}$ and $s^{(2)}$, respectively. Let $\hat{\theta}(\mathbf{x})$ be the temperature measured from some constant absolute temperature T_0 ($T_0>0$).

The governing system of field equations of motion in the linear theory of thermoviscoelastic binary porous mixtures consists of the following equations (see Ieşan [12]):

1. *The constitutive equations*

$$
\begin{gathered}
t_{jl}=(\lambda+\nu)e_{rr}\delta_{jl}+2(\mu+\zeta)e_{jl}+(\alpha+\nu)g_{rr}\delta_{jl}+(2\kappa+\zeta)g_{jl}+(2\gamma+\zeta)g_{lj}\\
+(m^{(1)}+l^{(1)})\hat{\varphi}\,\delta_{jl}+(m^{(2)}+l^{(2)})\hat{\psi}\,\delta_{jl}-(\beta^{(1)}+\beta^{(2)})\,\hat{\theta}\,\delta_{jl}+\lambda^*\dot{e}_{rr}\delta_{jl}+2\mu^*\dot{e}_{jl},\\
s_{jl}=\nu e_{rr}\delta_{jl}+2\zeta e_{lj}+\alpha g_{rr}\delta_{jl}+2\kappa g_{lj}+2\gamma g_{jl}+(l^{(1)}\hat{\varphi}+l^{(2)}\hat{\psi})\,\delta_{jl}-\beta^{(2)}\,\hat{\theta}\,\delta_{jl},\\
h_l^{(1)}=\alpha^{(1)}\hat{\varphi}_{,l}+\alpha^{(3)}\hat{\psi}_{,l}+b\,d_l,\qquad h_l^{(2)}=\alpha^{(3)}\hat{\varphi}_{,l}+\alpha^{(2)}\hat{\psi}_{,l}+c_0d_l,\\
g^{(1)}=-m^{(1)}e_{rr}-l^{(1)}g_{rr}-\zeta^{(1)}\hat{\varphi}-\zeta^{(3)}\hat{\psi}+b^{(1)}\,\hat{\theta},\\
g^{(2)}=-m^{(2)}e_{rr}-l^{(2)}g_{rr}-\zeta^{(3)}\hat{\varphi}-\zeta^{(2)}\hat{\psi}+b^{(2)}\,\hat{\theta},\\
p_l=\xi d_l+\xi^*\dot{d}_l+b\hat{\varphi}_{,l}+c_0\hat{\varphi}_{,l}+b^*\hat{\theta}_{,l},\qquad \rho\eta=\beta^{(1)}e_{rr}+\beta^{(2)}g_{rr}+b^{(1)}\,\hat{\varphi}+b^{(2)}\,\hat{\psi}+a\hat{\theta},\\
q_l=k\theta_{,l}+f^*\dot{d}_l,\qquad l,j=1,2,3,
\end{gathered}
\tag{1}
$$

where t_{jl} and s_{jl} are the components of the partial stresses of the constituents $s^{(1)}$ and $s^{(2)}$, respectively; λ, μ, α, γ, ζ, ν, κ, ξ, $\beta^{(1)}$, $\beta^{(2)}$, a, b, c_0, k, $b^{(1)}$, $b^{(2)}$, $m^{(1)}$, $m^{(2)}$, $l^{(1)}$, $l^{(2)}$, $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$, $\zeta^{(1)}$, $\zeta^{(2)}$, $\zeta^{(3)}$, λ^*, μ^*, ξ^*, b^*, f^* are the constitutive coefficients and $a\neq 0$, δ_{jl} is the Kronecker delta and

$$
e_{lj}=\frac{1}{2}(\hat{u}_{l,j}+\hat{u}_{j,l}),\quad g_{lj}=\hat{u}_{j,l}+\hat{w}_{l,j},\quad d_l=\hat{u}_l-\hat{w}_l,\quad l,j=1,2,3. \tag{2}
$$

2. *The equations of motion*

$$
\begin{gathered}
t_{jl,j}-p_l=\rho_1\left(\ddot{\hat{u}}_l-\hat{F}_l^{(1)}\right),\qquad s_{jl,j}+p_l=\rho_2\left(\ddot{\hat{w}}_l-\hat{F}_l^{(2)}\right),\qquad l=1,2,3,\\
h_{j,j}^{(1)}+g^{(1)}=\rho_1\left(\kappa_1\ddot{\hat{\varphi}}-\hat{L}^{(1)}\right),\qquad h_{j,j}^{(2)}+g^{(2)}=\rho_2\left(\kappa_2\ddot{\hat{\psi}}-\hat{L}^{(2)}\right),
\end{gathered}
\tag{3}
$$

where $\hat{L}^{(r)}$, κ_r, ρ_r and $\hat{\mathbf{F}}^{(r)}=(\hat{F}_1^{(r)},\,\hat{F}_2^{(r)},\,\hat{F}_3^{(r)})$ are the extrinsic equilibrated body force, the coefficient of the equilibrated inertia, the mass density and the partial body force of the constituent $s^{(r)}$, respectively; $\rho_r>0$, $\kappa_r>0$ and $r=1,2$.

3. *The heat transfer equation*

$$\rho T_0 \dot{\eta} = q_{l,l} + \rho \hat{s}, \tag{4}$$

where $\rho = \rho_1 + \rho_2$ and $\hat{s}$ is the heat source.

Substituting equations (1) and (2) into (3) and (4), we obtain the following system of equations of motion in the linear theory of thermoviscoelastic binary porous mixtures expressed in terms of the partial displacement vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{w}}$, the changes of volume fractions $\hat{\varphi}(\mathbf{x},t)$, $\hat{\psi}(\mathbf{x},t)$ and the change of temperature $\hat{\theta}$:

$$\begin{gathered}
\hat{\alpha}_1 \Delta \hat{\mathbf{u}} + \hat{\alpha}_2 \nabla \mathrm{div}\, \hat{\mathbf{u}} + \beta_1 \Delta \hat{\mathbf{w}} + \beta_2 \nabla \mathrm{div}\, \hat{\mathbf{w}} - \hat{\xi}(\hat{\mathbf{u}} - \hat{\mathbf{w}}) + \sigma_1 \nabla \hat{\varphi} + \sigma_2 \nabla \hat{\psi} - m_1 \nabla \hat{\theta} = \rho_1 \left(\ddot{\hat{\mathbf{u}}} - \hat{\mathbf{F}}^{(\mathbf{1})} \right), \\
\beta_1 \Delta \hat{\mathbf{u}} + \beta_2 \nabla \mathrm{div}\, \hat{\mathbf{u}} + \gamma_1 \Delta \hat{\mathbf{w}} + \gamma_2 \nabla \mathrm{div}\, \hat{\mathbf{w}} + \hat{\xi}(\hat{\mathbf{u}} - \hat{\mathbf{w}}) + \tau_1 \nabla \hat{\varphi} + \tau_2 \nabla \hat{\psi} - m_2 \nabla \hat{\theta} = \rho_2 \left(\ddot{\hat{\mathbf{w}}} - \hat{\mathbf{F}}^{(\mathbf{2})} \right), \\
\alpha^{(1)} \Delta \hat{\varphi} + \alpha^{(3)} \Delta \hat{\psi} - \sigma_1 \mathrm{div}\, \hat{\mathbf{u}} - \tau_1 \mathrm{div}\, \hat{\mathbf{w}} - \zeta^{(1)} \hat{\varphi} - \zeta^{(3)} \hat{\psi} + b^{(1)} \hat{\theta} = \rho_1 \left(\kappa_1 \ddot{\hat{\varphi}} - \hat{L}^{(1)} \right), \\
\alpha^{(3)} \Delta \hat{\varphi} + \alpha^{(2)} \Delta \hat{\psi} - \sigma_2 \mathrm{div}\, \hat{\mathbf{u}} - \tau_2 \mathrm{div}\, \hat{\mathbf{w}} - \zeta^{(3)} \hat{\varphi} - \zeta^{(2)} \hat{\psi} + b^{(2)} \hat{\theta} = \rho_2 \left(\kappa_2 \ddot{\hat{\varphi}} - \hat{L}^{(2)} \right), \\
k \Delta \hat{\theta} - a T_0 \dot{\hat{\theta}} - a_1 \mathrm{div}\, \dot{\hat{\mathbf{u}}} - a_2 \mathrm{div}\, \dot{\hat{\mathbf{w}}} - b^{(1)} T_0 \hat{\varphi} - b^{(2)} T_0 \hat{\psi} = -\rho \hat{s},
\end{gathered} \tag{5}$$

where Δ is the Laplacian operator,

$$\hat{\alpha}_1 = \alpha_1 + \mu^* \frac{\partial}{\partial t}, \quad \hat{\alpha}_2 = \alpha_2 + (\lambda^* + \mu^*) \frac{\partial}{\partial t}, \quad \hat{\xi} = \xi + \xi^* \frac{\partial}{\partial t}$$

and

$$\begin{gathered}
\alpha_1 = \mu + 2\kappa + 2\zeta, \quad \alpha_2 = \lambda + \mu + \alpha + 2\nu + 2\gamma + 2\zeta, \quad \beta_1 = 2\gamma + \zeta, \\
\beta_2 = \alpha + \nu + 2\kappa + \zeta, \quad \gamma_1 = 2\kappa, \quad \gamma_2 = \alpha + 2\gamma, \quad m_1 = \beta^{(1)} + \beta^{(2)} + b^*, \\
m_2 = \beta^{(2)} - b^*, \qquad \sigma_1 = m^{(1)} + l^{(1)} - b, \quad \sigma_2 = m^{(2)} + l^{(2)} - c_0, \\
\tau_1 = l^{(1)} + b, \quad \tau_2 = l^{(2)} + c_0, \quad a_1 = T_0(\beta^{(1)} + \beta^{(2)}) - f^*, \quad a_2 = T_0 \beta^{(2)} + f^*.
\end{gathered} \tag{6}$$

If the functions $\hat{\mathbf{u}}$, $\hat{\mathbf{w}}$, $\hat{\varphi}$, $\hat{\psi}$, $\hat{\theta}$, $\hat{\mathbf{F}}^{(1)}$, $\hat{\mathbf{F}}^{(2)}$, $\hat{L}^{(1)}$, $\hat{L}^{(2)}$ and $\hat{s}$ are postulated to have a harmonic time variation, that is,

$$\{\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\varphi}, \hat{\psi}, \hat{\theta}, \hat{\mathbf{F}}^{(1)}, \hat{\mathbf{F}}^{(2)}, \hat{L}^{(1)}, \hat{L}^{(2)}, \hat{s}\}(\mathbf{x}, t) = \mathrm{Re}\left[\{\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta, \mathbf{F}^{(1)}, \mathbf{F}^{(2)}, L^{(1)}, L^{(2)}, s\}(\mathbf{x})\, e^{-i\omega t} \right],$$

then from the system of equations of motion (5), we obtain the following system of equations of steady vibrations in the theory under consideration:

$$\begin{gathered}
(\alpha_1' \Delta + \eta_1')\mathbf{u} + \alpha_2' \nabla \mathrm{div}\, \mathbf{u} + (\beta_1 \Delta + \xi')\mathbf{w} + \beta_2 \nabla \mathrm{div}\, \mathbf{w} + \sigma_1 \nabla \varphi + \sigma_2 \nabla \psi - m_1 \nabla \theta = -\rho_1 \mathbf{F}^{(1)}, \\
(\beta_1 \Delta + \xi')\mathbf{u} + \beta_2 \nabla \mathrm{div}\, \mathbf{u} + (\gamma_1 \Delta + \eta_2')\mathbf{w} + \gamma_2 \nabla \mathrm{div}\, \mathbf{w} + \tau_1 \nabla \varphi + \tau_2 \nabla \psi - m_2 \nabla \theta = -\rho_2 \mathbf{F}^{(2)}, \\
(\alpha^{(1)} \Delta + \eta_1)\varphi + (\alpha^{(3)} \Delta - \zeta^{(3)})\psi - \sigma_1 \mathrm{div}\, \mathbf{u} - \tau_1 \mathrm{div}\, \mathbf{w} + b^{(1)} \theta = -\rho_1 L^{(1)}, \\
(\alpha^{(3)} \Delta - \zeta^{(3)})\varphi + (\alpha^{(2)} \Delta + \eta_2)\psi - \sigma_2 \mathrm{div}\, \mathbf{u} - \tau_2 \mathrm{div}\, \mathbf{w} + b^{(2)} \theta = -\rho_2 L^{(2)}, \\
(k \Delta + a')\, \theta + i\omega a_1 \mathrm{div}\, \mathbf{u} + i\omega a_2 \mathrm{div}\, \mathbf{w} + i\omega b^{(1)} T_0 \varphi + i\omega b^{(2)} T_0 \psi = -\rho s,
\end{gathered} \tag{7}$$

where

$$\begin{gathered}
\alpha_1' = \alpha_1 - i\omega\mu^*, \quad \alpha_2' = \alpha_2 - i\omega(\lambda^* + \mu^*), \quad \xi' = \xi - i\omega\, \xi^*, \\
\eta_1' = \rho_1 \omega^2 - \xi', \quad \eta_2' = \rho_2 \omega^2 - \xi', \quad \eta_1 = \rho_1 \kappa_1 \omega^2 - \zeta^{(1)}, \\
\eta_2 = \rho_2 \kappa_2 \omega^2 - \zeta^{(2)}, \quad a' = i\omega a T_0
\end{gathered} \tag{8}$$

and ω is the oscillation frequency ($\omega > 0$).

We introduce the matrix differential operator $\mathbf{A}(\mathbf{D_x}) = (A_{rq}(\mathbf{D_x}))_{9\times 9}$, where

$$A_{lj}(\mathbf{D_x}) = (\alpha_1'\Delta + \eta_1')\delta_{lj} + \alpha_2' \frac{\partial^2}{\partial x_l \partial x_j},$$

$$A_{l;j+3}(\mathbf{D_x}) = A_{l+3;j}(\mathbf{D_x}) = (\beta_1\Delta + \xi')\delta_{lj} + \beta_2 \frac{\partial^2}{\partial x_l \partial x_j}, \quad A_{l;r+6}(\mathbf{D_x}) = -A_{r+6;l}(\mathbf{D_x}) = \sigma_r \frac{\partial}{\partial x_l},$$

$$A_{l9}(\mathbf{D_x}) = -m_1 \frac{\partial}{\partial x_l}, \quad A_{l+3;j+3}(\mathbf{D_x}) = (\gamma_1\Delta + \eta_2')\delta_{lj} + \gamma_2 \frac{\partial^2}{\partial x_l \partial x_j},$$

$$A_{l+3;r+6}(\mathbf{D_x}) = -A_{r+6;l+3}(\mathbf{D_x}) = \tau_r \frac{\partial}{\partial x_l}, \quad A_{l+3;9}(\mathbf{D_x}) = -m_2 \frac{\partial}{\partial x_l},$$

$$A_{77}(\mathbf{D_x}) = \alpha^{(1)}\Delta + \eta_1, \quad A_{78}(\mathbf{D_x}) = A_{87}(\mathbf{D_x}) = \alpha^{(3)}\Delta + \zeta^{(3)},$$

$$A_{88}(\mathbf{D_x}) = \alpha^{(1)}\Delta + \eta_1, \quad A_{9l}(\mathbf{D_x}) = i\omega a_1 \frac{\partial}{\partial x_l}, \quad A_{9;l+3}(\mathbf{D_x}) = i\omega a_2 \frac{\partial}{\partial x_l},$$

$$A_{9;r+6}(\mathbf{D_x}) = i\omega b^{(r)} T_0, \quad A_{99}(\mathbf{D_x}) = k\Delta + a', \quad l,j = 1,2,3, \quad r = 1,2.$$

Obviously, system (7) can be written as follows:

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{9}$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta)$, $\mathbf{F} = (-\rho_1 \mathbf{F}^{(1)}, -\rho_2 \mathbf{F}^{(2)}, -\rho_1 L^{(1)}, -\rho_2 L^{(2)}, -\rho s)$ and $\mathbf{x} \in \Omega$.

3. Fundamental Solution

In this section, the fundamental solution of system (7) is constructed explicitly and its basic properties are established.

Definition 1. The fundamental solution of system (7) is the matrix $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{lj}(\mathbf{x}))_{9\times 9}$ satisfying the following equation in the class of generalized functions:

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J},$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{9\times 9}$ is the unit matrix and $\mathbf{x} \in \mathbb{R}^3$.

We denote by

$$\begin{gathered} \alpha_0' = \alpha_1' + \alpha_2', \quad \beta_0 = \beta_1 + \beta_2, \quad \gamma_0 = \gamma_1 + \gamma_2, \\ k_0 = \alpha_0'\gamma_0 - \beta_0^2, \quad k_1 = \alpha_1'\gamma_1 - \beta_1^2, \quad \alpha_0 = \alpha^{(1)}\alpha^{(2)} - \left(\alpha^{(3)}\right)^2. \end{gathered} \tag{10}$$

In this section, we assume that

$$\alpha_0 k k_0 k_1 \neq 0. \tag{11}$$

We introduce the following notation:

i)

$$\mathbf{B}(\Delta) = (B_{lj}(\Delta))_{5\times 5}$$

$$= \begin{pmatrix} \alpha_0'\Delta + \eta_1' & \beta_0\Delta + \xi' & -\sigma_1\Delta & -\sigma_2\Delta & i\omega a_1\Delta \\ \beta_0\Delta + \xi' & \gamma_0\Delta + \eta_2' & -\tau_1\Delta & -\tau_2\Delta & i\omega a_2\Delta \\ \sigma_1 & \tau_1 & \alpha^{(1)}\Delta + \eta_1 & \alpha^{(3)}\Delta - \zeta^{(3)} & i\omega b^{(1)}T_0 \\ \sigma_2 & \tau_2 & \alpha^{(3)}\Delta - \zeta^{(3)} & \alpha^{(2)}\Delta + \eta_2 & i\omega b^{(2)}T_0 \\ -m_1 & -m_2 & b^{(1)} & b^{(2)} & k\Delta + a' \end{pmatrix}_{5\times 5}.$$

ii)

$$\Lambda_1(\Delta) = \frac{1}{\alpha_0 k k_0} \det \mathbf{B}(\Delta) = \prod_{j=1}^{5} (\Delta + \lambda_j^2),$$

where $\lambda_j^2 (j = 1, 2, \ldots, 5)$ are the roots of the equation $\Lambda_1(-\tilde{\lambda}) = 0$ (with respect to $\tilde{\lambda}$).

iii)

$$\Lambda_2(\Delta)=\frac{1}{k_1}\det\begin{pmatrix}\alpha_1'\Delta+\eta_1' & \beta_1\Delta+\xi'\\ \beta_1\Delta+\xi' & \gamma_1\Delta+\eta_2'\end{pmatrix}_{2\times 2}=(\Delta+\lambda_6^2)(\Delta+\lambda_7^2),$$

where λ_6^2 and λ_7^2 are the roots of the equation $\Lambda_2(-\tilde{\lambda})=0$ (with respect to $\tilde{\lambda}$). We assume that $\operatorname{Im}\lambda_l>0$ and $\lambda_l\neq\lambda_j$ $(l,j=1,2,\dots,7)$.

iv)

$$n_{l1}(\Delta)=\frac{1}{\alpha_0 k k_0 k_1}\sum_{j=1}^{5}C_j B_{lj}^*(\Delta),\quad n_{l2}(\Delta)=\frac{1}{\alpha_0 k k_0 k_1}\sum_{j=1}^{5}C_{j+5}B_{lj}^*(\Delta),$$
$$n_{lr}(\Delta)=\frac{1}{\alpha_0 k k_0}B_{lr}^*(\Delta),\quad l=1,2,\dots,5,\quad r=3,4,5,$$

where B_{lj}^* is the cofactor of element B_{lj} of the matrix $\mathbf{B}$ and

$$C_1=\beta_2(\beta_1\Delta+\xi')-\alpha_2'(\gamma_1\Delta+\eta_2'),\quad C_2=\gamma_2(\beta_1\Delta+\xi')-\beta_2(\gamma_1\Delta+\eta_2'),$$
$$C_3=\sigma_1(\gamma_1\Delta+\eta_2')-\tau_1(\beta_1\Delta+\xi'),\quad C_4=\sigma_2(\gamma_1\Delta+\eta_2')-\tau_2(\beta_1\Delta+\xi'),$$
$$C_5=i\omega[a_2(\beta_1\Delta+\xi')-a_1(\gamma_1\Delta+\eta_2')],\quad C_6=\alpha_2'(\beta_1\Delta+\xi')-\beta_2(\alpha_1'\Delta+\eta_1'),$$
$$C_7=\beta_2(\beta_1\Delta+\xi')-\gamma_2(\alpha_1'\Delta+\eta_1'),\quad C_8=\tau_1(\alpha_1'\Delta+\eta_1')-\sigma_1(\beta_1\Delta+\xi'),$$
$$C_9=\tau_2(\alpha_1'\Delta+\eta_1')-\sigma_2(\beta_1\Delta+\xi'),\quad C_{10}=i\omega[a_1(\beta_1\Delta+\xi')-a_2(\alpha_1'\Delta+\eta_1')].$$

v)

$$\mathbf{\Lambda}(\Delta)=\left(\Lambda_{lj}\left(\Delta\right)\right)_{9\times 9},\quad \Lambda_{11}(\Delta)=\Lambda_{22}(\Delta)=\cdots=\Lambda_{66}(\Delta)=\Lambda_2(\Delta),$$
$$\Lambda_{77}(\Delta)=\Lambda_{88}(\Delta)=\Lambda_{99}(\Delta)=\Lambda_1(\Delta),\quad \Lambda_{lj}(\Delta)=0,$$
$$l\neq j,\quad l,j=1,2,\dots,9.$$

vi)

$$\begin{aligned}
&\mathbf{L}\left(\mathbf{D_x}\right)=\left(L_{lj}\left(\mathbf{D_x}\right)\right)_{9\times 9},\\
&L_{lj}\left(\mathbf{D_x}\right)=\frac{1}{k}(\gamma_1\Delta+\eta_2')\Lambda_1(\Delta)\,\delta_{lj}+n_{11}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
&L_{l;j+3}\left(\mathbf{D_x}\right)=-\frac{1}{k}(\beta_1\Delta+\xi')\Lambda_1(\Delta)\,\delta_{lj}+n_{12}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
&L_{l+3;j}\left(\mathbf{D_x}\right)=-\frac{1}{k}(\beta_1\Delta+\xi')\Lambda_1(\Delta)\,\delta_{lj}+n_{21}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
&L_{l+3;j+3}\left(\mathbf{D_x}\right)=\frac{1}{k}(\alpha_1'\Delta+\eta_1')\Lambda_1(\Delta)\,\delta_{lj}+n_{22}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
&L_{lr}\left(\mathbf{D_x}\right)=n_{1;r-4}(\Delta)\frac{\partial}{\partial x_l},\qquad L_{l+3;r}\left(\mathbf{D_x}\right)=n_{2;r-4}(\Delta)\frac{\partial}{\partial x_l},\\
&L_{rl}\left(\mathbf{D_x}\right)=n_{r-4;1}(\Delta)\frac{\partial}{\partial x_l},\qquad L_{r;l+3}\left(\mathbf{D_x}\right)=n_{r-4;2}(\Delta)\frac{\partial}{\partial x_l},\\
&L_{rm}\left(\mathbf{D_x}\right)=n_{r-4;m-4}(\Delta),\qquad l,j=1,2,3,\quad r,m=7,8,9.
\end{aligned}\tag{12}$$

vii)

$$
\begin{gathered}
\mathbf{Y}(\mathbf{x}) = (Y_{lj}(\mathbf{x}))_{9\times 9}\,, \\
Y_{11}(\mathbf{x}) = Y_{22}(\mathbf{x}) = \cdots = Y_{66}(\mathbf{x}) = \sum_{j=1}^{7} \eta_{2j}\gamma^{(j)}(\mathbf{x}), \\
Y_{77}(\mathbf{x}) = Y_{88}(\mathbf{x}) = Y_{88}(\mathbf{x}) = \sum_{j=1}^{5} \eta_{1j}\gamma^{(j)}(\mathbf{x}), \\
Y_{lj}(\mathbf{x}) = 0, \quad l \neq j, \quad l, j = 1, 2, \ldots, 9,
\end{gathered}
\tag{13}
$$

where

$$\gamma^{(j)}(\mathbf{x}) = -\frac{e^{i\lambda_j|\mathbf{x}|}}{4\pi\,|\mathbf{x}|}$$

and

$$
\begin{gathered}
\eta_{1m} = \prod_{l=1,\, l\neq m}^{5} (\lambda_l^2 - \lambda_m^2)^{-1}, \quad \eta_{2j} = \prod_{l=1,\, l\neq j}^{7} (\lambda_l^2 - \lambda_j^2)^{-1}, \\
m = 1, 2, \ldots, 5, \quad j = 1, 2 \ldots, 7.
\end{gathered}
$$

It is not difficult to prove

Lemma 1. *If the condition* (11) *is satisfied, then:*
a) *the following identity*

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{L}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta)$$

is valid;
b) *the matrix* $\mathbf{Y}(\mathbf{x})$ *is the fundamental solution of the operator* $\mathbf{\Lambda}(\Delta)$*, i.e.,*

$$\mathbf{\Lambda}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.$$

Lemma 1 leads to the following

Theorem 1. *If the condition* (11) *is satisfied, then the matrix* $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{lj}(\mathbf{x}))_{9\times 9}$ *defined by*

$$\mathbf{\Gamma}\left(\mathbf{x}\right) = \mathbf{L}\left(\mathbf{D_x}\right)\mathbf{Y}\left(\mathbf{x}\right) \tag{14}$$

is the fundamental solution of system (7) (*the fundamental matrix of the operator* $\mathbf{A}(\mathbf{D_x})$)*, where the matrices* $\mathbf{L}\left(\mathbf{D_x}\right)$ *and* $\mathbf{Y}\left(\mathbf{x}\right)$ *are given by* (12) *and* (13)*, respectively.*

We now formulate the basic properties of the matrix $\mathbf{\Gamma}\left(\mathbf{x}\right)$. Theorem 1 has the following consequences.

Theorem 2. *Each column of the matrix* $\mathbf{\Gamma}(\mathbf{x})$ *is a solution of the homogeneous equation*

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \mathbf{0}$$

at every point $\mathbf{x} \in \mathbb{R}^3$*, except the origin.*

Theorem 3. *The relations*

$$
\begin{gathered}
\Gamma_{lj}\left(\mathbf{x}\right) = O\left(|\mathbf{x}|^{-1}\right), \quad \Gamma_{rm}\left(\mathbf{x}\right) = O\left(|\mathbf{x}|^{-1}\right), \quad \Gamma_{99}\left(\mathbf{x}\right) = O\left(|\mathbf{x}|^{-1}\right), \\
\Gamma_{le}\left(\mathbf{x}\right) = O\left(1\right), \quad \Gamma_{el}\left(\mathbf{x}\right) = O\left(1\right), \quad \Gamma_{r9}\left(\mathbf{x}\right) = O\left(1\right), \\
\Gamma_{9r}\left(\mathbf{x}\right) = O\left(1\right), \quad l, j = 1, 2, \ldots, 6, \quad r, m = 7, 8, \quad e = 7, 8, 9
\end{gathered}
$$

hold in the neighborhood of the origin.

We introduce the notation:

i)

$$\mathbf{A}^{(0)}(\mathbf{D_x}) = \left(A^{(0)}_{lj}(\mathbf{D_x})\right)_{9\times 9}, \quad A^{(0)}_{lj}(\mathbf{D_x}) = \alpha'_1 \Delta\delta_{lj} + \alpha'_2 \frac{\partial^2}{\partial x_l \partial x_j},$$
$$A^{(0)}_{l;j+3}(\mathbf{D_x}) = A^{(0)}_{l+3;j}(\mathbf{D_x}) = \beta_1 \Delta\delta_{lj} + \beta_2 \frac{\partial^2}{\partial x_l \partial x_j},$$
$$A^{(0)}_{l+3;j+3}(\mathbf{D_x}) = \gamma_1 \Delta\delta_{lj} + \gamma_2 \frac{\partial^2}{\partial x_l \partial x_j}, \quad A^{(0)}_{77}(\mathbf{D_x}) = \alpha^{(1)}\Delta,$$
$$A^{(0)}_{78}(\mathbf{D_x}) = A^{(0)}_{87}(\mathbf{D_x}) = \alpha^{(3)}\Delta, \quad A^{(0)}_{88}(\mathbf{D_x}) = \alpha^{(2)}\Delta, \quad A^{(0)}_{99}(\mathbf{D_x}) = k\Delta,$$
$$A^{(0)}_{mr}(\mathbf{D_x}) = A^{(0)}_{rm}(\mathbf{D_x}) = A^{(0)}_{e9}(\mathbf{D_x}) = A^{(0)}_{9e}(\mathbf{D_x}) = 0.$$

ii)

$$\mathbf{\Gamma}^{(0)}(\mathbf{x}) = \left(\Gamma^{(0)}_{lj}(\mathbf{x})\right)_{9\times 9},$$
$$\Gamma^{(0)}_{lj}(\mathbf{x}) = -\frac{1}{8\pi}\left(\frac{\gamma_0}{k_0} + \frac{\gamma_1}{k_1}\right)\frac{\delta_{lj}}{|\mathbf{x}|} + \frac{1}{8\pi}\left(\frac{\gamma_0}{k_0} - \frac{\gamma_1}{k_1}\right)\frac{x_l x_j}{|\mathbf{x}|^3},$$
$$\Gamma^{(0)}_{l;j+3}(\mathbf{x}) = \Gamma^{(0)}_{l+3;j}(\mathbf{x}) = \frac{1}{8\pi}\left(\frac{\beta_0}{k_0} + \frac{\beta_1}{k_1}\right)\frac{\delta_{lj}}{|\mathbf{x}|} - \frac{1}{8\pi}\left(\frac{\beta_0}{k_0} - \frac{\beta_1}{k_1}\right)\frac{x_l x_j}{|\mathbf{x}|^3},$$
$$\Gamma^{(0)}_{l+3;j+3}(\mathbf{x}) = -\frac{1}{8\pi}\left(\frac{\alpha'_0}{k_0} + \frac{\alpha'_1}{k_1}\right)\frac{\delta_{lj}}{|\mathbf{x}|} + \frac{1}{8\pi}\left(\frac{\alpha'_0}{k_0} - \frac{\alpha'_1}{k_1}\right)\frac{x_l x_j}{|\mathbf{x}|^3},$$
$$\Gamma^{(0)}_{77}(\mathbf{x}) = -\frac{\alpha^{(2)}}{4\pi\alpha_0}\frac{1}{|\mathbf{x}|}, \quad \Gamma^{(0)}_{78}(\mathbf{x}) = \Gamma^{(0)}_{87}(\mathbf{x}) = \frac{\alpha^{(3)}}{4\pi\alpha_0}\frac{1}{|\mathbf{x}|}, \quad \Gamma^{(0)}_{88}(\mathbf{x}) = -\frac{\alpha^{(1)}}{4\pi\alpha_0}\frac{1}{|\mathbf{x}|},$$
$$\Gamma^{(0)}_{99}(\mathbf{x}) = -\frac{1}{4\pi k}\frac{1}{|\mathbf{x}|}, \quad \Gamma^{(0)}_{mr}(\mathbf{D_x}) = \Gamma^{(0)}_{rm}(\mathbf{D_x}) = \Gamma^{(0)}_{e9}(\mathbf{D_x}) = \Gamma^{(0)}_{9e}(\mathbf{D_x}) = 0,$$

where $l, j = 1, 2, 3$, $m = 1, 2, \ldots, 6$, $e = 7, 8$ and $r = 7, 8, 9$.

Theorem 1 leads directly to the following basic properties of the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x})$.

Theorem 4. *The fundamental solution of the equation*

$$\mathbf{A}^{(0)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$$

is the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x})$, *and the following relations:*

$$\Gamma^{(0)}_{lj}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \quad \Gamma^{(0)}_{mr}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \quad \Gamma^{(0)}_{99}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right),$$
$$l, j = 1, 2, \ldots, 6, \quad m, r = 7, 8$$

hold in the neighborhood of the origin.

Theorem 5. *The relations*

$$\Gamma_{lj}(\mathbf{x}) - \Gamma^{(0)}_{lj}(\mathbf{x}) = \text{const} + O(|\mathbf{x}|), \quad l, j = 1, 2, \ldots, 9 \tag{15}$$

hold in the neighborhood of the origin.

Thus, on the basis of Theorem 5 the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$ in the neighborhood of the origin.

4. Basic Boundary Value Problems

Let S be the smooth closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $S \in C^{2,\nu'}$, $0 < \nu' \leq 1$; $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, $\overline{\Omega^-} = \Omega^- \cup S$. We denote by $\mathbf{n}(\mathbf{z})$ the external (with respect to the Ω^+) unit vector, normal to S at $\mathbf{z}$.

Definition 2. A vector function $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta) = (U_1, U_2, \dots, U_9)$ is called *regular* in Ω^- (or Ω^+) if:

1)
$$U_j \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-}) \quad (\text{or } U_j \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})),$$

2)
$$U_j(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad U_{j,l}(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \quad \text{for} \quad |\mathbf{x}| \gg 1, \tag{16}$$

where $j = 1, 2, \dots, 9$ and $l = 1, 2, 3$.

In the sequel, we use the matrix differential operators

1)
$$\begin{gathered}
\mathbf{R}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = (R_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}))_{9\times 9},\\
R_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \alpha_1' \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \alpha_2' n_l \frac{\partial}{\partial x_j} + \epsilon_1 \mathcal{M}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}),\\
R_{l;j+3}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = R_{l+3;j}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \beta_1 \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \beta_2 n_l \frac{\partial}{\partial x_j} + \epsilon_2 \mathcal{M}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}),\\
R_{l+3;j+3}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \gamma_1 \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \gamma_2 n_l \frac{\partial}{\partial x_j} + \epsilon_3 \mathcal{M}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}),\\
R_{l7}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \left(m^{(1)} + l^{(1)}\right) n_l, \quad R_{l8}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \left(m^{(2)} + l^{(2)}\right) n_l,\\
R_{l9}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -(\beta^{(1)} + \beta^{(2)})\, n_l, \quad R_{l+3;7}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = l^{(1)} n_l, \quad R_{l+3;8}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = l^{(2)} n_l,\\
R_{l+3;9}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -\beta^{(2)}\, n_l, \quad R_{7l}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -R_{7;l+3}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = b\, n_l,\\
R_{8l}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -R_{8;l+3}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = c_0\, n_l, \quad R_{77}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \alpha^{(1)} \frac{\partial}{\partial \mathbf{n}},\\
R_{78}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = R_{87}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \alpha^{(3)} \frac{\partial}{\partial \mathbf{n}}, \quad R_{88}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \alpha^{(2)} \frac{\partial}{\partial \mathbf{n}},\\
R_{9l}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -R_{9;l+3}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -i\omega f^*\, n_l, \quad R_{99}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = k \frac{\partial}{\partial \mathbf{n}},\\
R_{m9}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -R_{9m}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = 0, \quad l, j = 1, 2, 3, \quad m = 7, 8,
\end{gathered} \tag{17}$$

where $\mathbf{n} = (n_1, n_2, n_3)$, $\frac{\partial}{\partial \mathbf{n}}$ is the derivative along the vector $\mathbf{n}$ and

$$\mathcal{M}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = n_j \frac{\partial}{\partial x_l} - n_l \frac{\partial}{\partial x_j}, \quad \epsilon_1 = \mu - i\omega\mu^* + 2\gamma + 2\zeta, \quad \epsilon_2 = 2\kappa + \zeta, \quad \epsilon_3 = 2\gamma.$$

The basic internal and external BVPs of steady vibrations in the linear theory of thermoviscoelastic binary porous mixtures are formulated as follows.

Find a regular (classical) solution to (9) for $\mathbf{x} \in \Omega^\pm$ satisfying the boundary condition

$$\lim_{\Omega^\pm \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^\pm = \mathbf{f}(\mathbf{z})$$

in *Problem* $(I)^\pm_{\mathbf{F},\mathbf{f}}$, and

$$\lim_{\Omega^\pm \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D}_\mathbf{x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^\pm = \mathbf{f}(\mathbf{z})$$

in *Problem* $(II)^\pm_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are the prescribed nine-component vector functions and $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-.

5. Green's Identities

In this section, Green's identities in the linear theory of thermoviscoelasticity for binary porous mixtures are established.

Let $u'_l, w'_l, \varphi', \psi', \theta'$ $(l = 1,2,3)$ be complex functions, $\mathbf{u}' = (u'_1, u'_2, u'_3)$, $\mathbf{w}' = (w'_1, w'_2, w'_3)$, $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \varphi', \psi', \theta')$. We introduce the notation

$$\begin{gathered}
W^{(0)}(\mathbf{u},\mathbf{u}') = \frac{1}{4}\sum_{l,j=1;\, l\neq j}^{3} (u_{j,l}+u_{l,j})\left(\overline{u'_{j,l}}+\overline{u'_{l,j}}\right) + \frac{1}{6}\sum_{l,j=1}^{3}\left(\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right)\left(\frac{\overline{\partial u'_l}}{\partial x_l}-\frac{\overline{\partial u'_j}}{\partial x_j}\right),\\
W^{(1)}(\mathbf{u},\mathbf{u}') = \frac{1}{3}(\alpha'_1+3\alpha'_2-2\epsilon_1)\operatorname{div}\mathbf{u}\operatorname{div}\overline{\mathbf{u}'} + \frac{1}{2}(\alpha'_1-\epsilon_1)\operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{u}'\\
+(\alpha'_1+\epsilon_1)W^{(0)}(\mathbf{u},\mathbf{u}') - \eta'_1\mathbf{u}\cdot\mathbf{u}',\\
W^{(2)}(\mathbf{u},\mathbf{w}') = \frac{1}{3}(\beta_1+3\beta_2-2\epsilon_2)\operatorname{div}\mathbf{u}\operatorname{div}\overline{\mathbf{w}'} + \frac{1}{2}(\beta_1-\epsilon_2)\operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{w}'\\
+(\beta_1+\epsilon_2)W^{(0)}(\mathbf{u},\mathbf{w}') - \xi'\mathbf{u}\cdot\mathbf{w}',\\
W^{(3)}(\mathbf{w},\mathbf{w}') = \frac{1}{3}(\gamma_1+3\gamma_2-2\epsilon_3)\operatorname{div}\mathbf{w}\operatorname{div}\overline{\mathbf{w}'} + \frac{1}{2}(\gamma_1-\epsilon_3)\operatorname{curl}\mathbf{w}\cdot\operatorname{curl}\mathbf{w}'\\
+(\gamma_1+\epsilon_3)W^{(0)}(\mathbf{w},\mathbf{w}') - \eta'_2\mathbf{w}\cdot\mathbf{w}'.
\end{gathered}\tag{18}$$

Using Green's first identity of the classical theory of elasticity (see e.g., Kupradze et al. [18]), it is a simple matter to verify that

$$\begin{gathered}
\int\limits_{\Omega^+}\left[A_{lj}(\mathbf{D_x})\,u_j\overline{u'_l}+W^{(1)}(\mathbf{u},\mathbf{u}')\right]d\mathbf{x} = \int\limits_S R_{lj}(\mathbf{D_z},\mathbf{n})u_j(\mathbf{z})\overline{u'_l(\mathbf{z})}\,d_{\mathbf{z}}S,\\
\int\limits_{\Omega^+}\left[A_{l;j+3}(\mathbf{D_x})\,w_j\overline{u'_l}+W^{(2)}(\mathbf{w},\mathbf{u}')\right]d\mathbf{x} = \int\limits_S R_{l;j+3}(\mathbf{D_z},\mathbf{n})w_j(\mathbf{z})\overline{u'_l(\mathbf{z})}\,d_{\mathbf{z}}S,\\
\int\limits_{\Omega^+}\left[A_{l+3;j}(\mathbf{D_x})\,u_j\overline{w'_l}+W^{(2)}(\mathbf{u},\mathbf{w}')\right]d\mathbf{x} = \int\limits_S R_{l+3;j}(\mathbf{D_z},\mathbf{n})u_j(\mathbf{z})\overline{w'_l(\mathbf{z})}\,d_{\mathbf{z}}S,\\
\int\limits_{\Omega^+}\left[A_{l+3;j+3}(\mathbf{D_x})\,w_j\overline{w'_l}+W^{(3)}(\mathbf{w},\mathbf{w}')\right]d\mathbf{x} = \int\limits_S R_{l+3;j+3}(\mathbf{D_z},\mathbf{n})w_j(\mathbf{z})\overline{w'_l(\mathbf{z})}\,d_{\mathbf{z}}S.
\end{gathered}\tag{19}$$

On the basis of (18) and identity

$$\int\limits_{\Omega^+}\left[\nabla\varphi(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x})+\varphi(\mathbf{x})\operatorname{div}\overline{\mathbf{u}'(\mathbf{x})}\right]d\mathbf{x} = \int\limits_S \varphi(\mathbf{z})\mathbf{n}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S,\tag{20}$$

from (19), it follows that

$$\begin{gathered}
\int\limits_{\Omega^+}\left[(A_{lj}u_j+A_{l;j+3}w_j+A_{l7}\varphi+A_{l8}\psi+A_{l9}\theta)\,\overline{u'_l}+W_1(\mathbf{U},\mathbf{u}')\right]d\mathbf{x}\\
=\int\limits_S\left[R_{lj}u_j+R_{l;j+3}w_j+R_{l7}\varphi+R_{l8}\psi+R_{l9}\theta\right]\overline{u'_l}d_{\mathbf{z}}S,\\
\int\limits_{\Omega^+}\left[(A_{l+3;j}u_j+A_{l+3;j+3}w_j+A_{l+3;7}\varphi+A_{l+3;8}\psi+A_{l+3;9}\theta)\,\overline{w'_l}+W_2(\mathbf{U},\mathbf{w}')\right]d\mathbf{x}\\
=\int\limits_S(R_{l+3;j}u_j+R_{l+3;j+3}w_j+R_{l+3;7}\varphi+R_{l+3;8}\psi+R_{l+3;9}\theta)\,\overline{w'_l}d_{\mathbf{z}}S,\quad l=1,2,3,
\end{gathered}\tag{21}$$

where

$$\begin{aligned} W_1(\mathbf{U},\mathbf{u}') &= W^{(1)}(\mathbf{u},\mathbf{u}') + W^{(2)}(\mathbf{w},\mathbf{u}') \\ &+ \left[\left(m^{(1)}+l^{(1)}\right)\varphi + \left(m^{(2)}+l^{(2)}\right)\psi - (\beta^{(1)}+\beta^{(2)})\theta\right]\operatorname{div}\overline{\mathbf{u}'} \\ &+\nabla(b\varphi + c_0\psi + b^*\theta)\cdot\mathbf{u}', \\ W_2(\mathbf{U},\mathbf{w}') &= W^{(2)}(\mathbf{u},\mathbf{w}') + W^{(3)}(\mathbf{w},\mathbf{w}') + \left(l^{(1)}\varphi + l^{(2)}\psi - \beta^{(2)}\theta\right)\operatorname{div}\overline{\mathbf{w}'} \\ &-\nabla(b\varphi + c_0\psi + b^*\theta)\cdot\mathbf{w}' \end{aligned} \tag{22}$$

and $W^{(1)}(\mathbf{u},\mathbf{u}'), W^{(2)}(\mathbf{w},\mathbf{u}')$ and $W^{(3)}(\mathbf{w},\mathbf{w}')$ are defined by (18).

Now, taking into account the identities (20) and

$$\int\limits_{\Omega^+}\left[\Delta\varphi(\mathbf{x})\,\overline{\psi'(\mathbf{x})} + \nabla\varphi(\mathbf{x})\cdot\nabla\psi'(\mathbf{x})\right]d\mathbf{x} = \int\limits_S \frac{\partial\varphi(\mathbf{z})}{\partial\mathbf{n}(\mathbf{z})}\overline{\psi'(\mathbf{z})}\,d_{\mathbf{z}}, S$$

we deduce that

$$\begin{aligned} &\int\limits_{\Omega^+}\left[(A_{7j}u_j + A_{7;j+3}w_j + A_{77}\varphi + A_{78}\psi + A_{79}\theta)\,\overline{\varphi'} + W_3(\mathbf{U},\varphi')\right]d\mathbf{x} \\ &\qquad = \int\limits_S (R_{7j}u_j + R_{7;j+3}w_j + R_{77}\varphi + R_{78}\psi)\,\overline{\varphi'}d_{\mathbf{z}}S, \\ &\int\limits_{\Omega^+}\left[(A_{8j}u_j + A_{8;j+3}w_j + A_{87}\varphi + A_{88}\psi + A_{89}\theta)\,\overline{\psi'} + W_4(\mathbf{U},\psi')\right]d\mathbf{x} \\ &\qquad = \int\limits_S (R_{8j}u_j + R_{8;j+3}w_j + R_{87}\varphi + R_{88}\psi)\,\overline{\psi'}d_{\mathbf{z}}S, \\ &\int\limits_{\Omega^+}\left[(A_{9j}u_j + A_{9;j+3}w_j + A_{97}\varphi + A_{98}\psi + A_{99}\theta)\,\overline{\theta'} + W_5(\mathbf{U},\theta')\right]d\mathbf{x} \\ &\qquad = \int\limits_S (R_{9j}u_j + R_{9;j+3}w_j + R_{99}\theta)\,\overline{\theta'}d_{\mathbf{z}}S, \end{aligned} \tag{23}$$

where

$$\begin{aligned} W_3(\mathbf{U},\varphi') &= \left[\nabla(\alpha^{(1)}\varphi + \alpha^{(3)}\psi) + b(\mathbf{u}-\mathbf{w})\right]\cdot\nabla\varphi' \\ &+ \left[(m^{(1)}+l^{(1)})\operatorname{div}\mathbf{u} + l^{(1)}\operatorname{div}\mathbf{w} - \eta_1\varphi + \zeta^{(3)}\psi - b^{(1)}\theta\right]\overline{\varphi'}, \\ W_4(\mathbf{U},\psi') &= \left[\nabla(\alpha^{(3)}\varphi + \alpha^{(2)}\psi) + c_0(\mathbf{u}-\mathbf{w})\right]\cdot\nabla\psi' \\ &+ \left[(m^{(2)}+l^{(2)})\operatorname{div}\mathbf{u} + l^{(2)}\operatorname{div}\mathbf{w} + \zeta^{(3)}\varphi - \eta_2\psi - b^{(2)}\theta\right]\overline{\psi'}, \\ W_5(\mathbf{U},\theta') &= k\nabla\theta\cdot\nabla\theta' - a'\theta\overline{\theta'} \\ &- i\omega T_0\left[(\beta^{(1)}+\beta^{(2)})\operatorname{div}\mathbf{u} + \beta^{(2)}\operatorname{div}\mathbf{w} + b^{(1)}\varphi + b^{(2)}\psi\right]\overline{\theta'} - i\omega f^*(\mathbf{u}-\mathbf{w})\cdot\nabla\theta'. \end{aligned} \tag{24}$$

Finally, we combine the relations (21) and (23) to deduce the identity

$$\int\limits_{\Omega^+}\left[\mathbf{A}(\mathbf{D_x})\,\mathbf{U}\cdot\mathbf{U}' + W(\mathbf{U},\mathbf{U}')\right]d\mathbf{x} = \int\limits_S \mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S, \tag{25}$$

where

$$W(\mathbf{U},\mathbf{U}') = W_1(\mathbf{U},\mathbf{u}') + W_2(\mathbf{U},\mathbf{w}') + W_3(\mathbf{U},\varphi') + W_4(\mathbf{U},\psi') + W_5(\mathbf{U},\theta'). \tag{26}$$

Hence, the following theorem is proved.

Theorem 6. *If* $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta)$ *and* $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \varphi', \psi', \theta')$ *are regular vectors in* Ω^+*, then the identity* (25) *is valid, where* $\mathbf{R}(\mathbf{D_z}, \mathbf{n})$ *and* $W(\mathbf{U}, \mathbf{U}')$ *are defined by* (17) *and* (26)*, respectively.*

Quite similarly as in Theorem 6, on the basis of (16), we obtain the following

Theorem 7. *If* $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta)$ *and* $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \varphi', \psi', \theta')$ *are regular vectors in* Ω^-*, then*

$$\int_{\Omega^-} [\mathbf{A}(\mathbf{D_x})\, \mathbf{U} \cdot \mathbf{U}' + W(\mathbf{U}, \mathbf{U}')]\, d\mathbf{x} = -\int_S \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \cdot \mathbf{U}'(\mathbf{z})\, d_{\mathbf{z}}S. \tag{27}$$

Formulas (25) and (27) are Green's first identity in the linear theory of thermoviscoelasticity of binary porous mixtures for domains Ω^+ and Ω^-, respectively.

We introduce the matrix differential operator $\tilde{\mathbf{A}}(\mathbf{D_x})$, where $\tilde{\mathbf{A}}(\mathbf{D_x}) = \mathbf{A}^\top(-\mathbf{D_x})$ and $\mathbf{A}^\top$ is the transpose of the matrix $\mathbf{A}$. Obviously, the fundamental matrix of the operator $\tilde{\mathbf{A}}(\mathbf{D_x})$ is $\tilde{\mathbf{\Gamma}}(\mathbf{x})$, where

$$\tilde{\mathbf{\Gamma}}(\mathbf{x}) = \mathbf{\Gamma}^\top(-\mathbf{x}). \tag{28}$$

Let $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta)$ and the vector $\tilde{\mathbf{U}}_j$ be the j-th column of the matrix $\tilde{\mathbf{U}} = (\tilde{U}_{lj})_{9\times 9}$. By a direct calculation we obtain the following results.

Theorem 8. *If* U *and* $\tilde{U}_j$ $(j = 1, 2, \ldots, 9)$ *are regular vectors in* Ω^+*, then*

$$\begin{aligned}
&\int_{\Omega^+} \Big\{[\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y})\Big\}\, d\mathbf{y} \\
&= \int_S \Big\{[\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z})\Big\}\, d_{\mathbf{z}}S,
\end{aligned} \tag{29}$$

where the operator $\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})$ *is defined by*

$$\begin{gathered}
\tilde{\mathbf{R}}(\mathbf{D_x}, \mathbf{n}) = (\tilde{R}_{lj}(\mathbf{D_x}, \mathbf{n}))_{9\times 9}, \quad \tilde{R}_{lj}(\mathbf{D_x}, \mathbf{n}) = R_{lj}(\mathbf{D_x}, \mathbf{n}), \\
\tilde{R}_{r9}(\mathbf{D_x}, \mathbf{n}) = -i\omega T_0 \left(\beta^{(1)} + \beta^{(1)}\right) n_r, \quad \tilde{R}_{r+3;9}(\mathbf{D_x}, \mathbf{n}) = -i\omega T_0 \beta^{(2)} n_r, \\
\tilde{R}_{9r}(\mathbf{D_x}, \mathbf{n}) = -\tilde{R}_{9;r+3}(\mathbf{D_x}, \mathbf{n}) = b^* n_r, \quad \tilde{R}_{99}(\mathbf{D_x}, \mathbf{n}) = k \frac{\partial}{\partial \mathbf{n}}, \\
l, j = 1, 2, \ldots, 8, \quad r = 1, 2, 3.
\end{gathered} \tag{30}$$

Theorem 9. *If* U *and* $\tilde{U}_j$ $(j = 1, 2, \ldots, 9)$ *are regular vectors in* Ω^-*, then*

$$\begin{aligned}
&\int_{\Omega^-} \Big\{[\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y})\Big\}\, d\mathbf{y} \\
&= -\int_S \Big\{[\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z})\Big\}\, d_{\mathbf{z}}S.
\end{aligned} \tag{31}$$

Formulas (30) and (31) are Green's second identities in the linear theory of thermoviscoelasticity of binary porous mixtures for the domains Ω^+ and Ω^-, respectively.

With the help of the relations (28), (29) and (31) we can derive the following useful consequences.

Theorem 10. *If* U *is a regular vector in* Ω^+*, then*

$$\begin{aligned}
\mathbf{U}(\mathbf{x}) = &\int_S \Big\{[\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\mathbf{\Gamma}^\top(\mathbf{x} - \mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x} - \mathbf{z})\, \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z})\Big\}\, d_{\mathbf{z}}S \\
&+ \int_{\Omega^+} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\, \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) d\mathbf{y} \quad \textit{for} \quad \mathbf{x} \in \Omega^+.
\end{aligned} \tag{32}$$

Theorem 11. *If U is a regular vector in Ω^-, then*

$$\mathbf{U}(\mathbf{x}) = -\int\limits_S \left\{[\tilde{\mathbf{R}}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{z})]^\top\mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x}-\mathbf{z})\,\mathbf{R}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{U}(\mathbf{z})\right\} d_\mathbf{z}S$$
$$+\int\limits_{\Omega^-} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\,\boldsymbol{A}(\mathbf{D}_\mathbf{y})\mathbf{U}(\mathbf{y})d\mathbf{y} \quad \textit{for} \quad \mathbf{x}\in\Omega^-. \tag{33}$$

Formulas (32) and (33) are integral representations of the regular vector (Green's third identity) in the linear theory of thermoviscoelasticity of binary porous mixtures for the domains Ω^+ and Ω^-, respectively.

6. Uniqueness Theorems

In this section, on the basis of Green's first identity we prove the uniqueness of regular (classical) solutions of BVPs $(K)^+_{\mathbf{F},\mathbf{f}}$ and $(K)^-_{\mathbf{F},\mathbf{f}}$, where $K = I, II$. The scalar product of two vectors $\mathbf{U} = (U_1, U_2, \ldots, U_l)$ and $\mathbf{V} = (V_1, V_2, \ldots, V_l)$ is denoted by $\mathbf{U}\cdot\mathbf{V} = \sum\limits_{j=1}^{l} U_j\overline{V_j}$, where $\overline{V_j}$ is the complex conjugate of V_j.

We have the following

Theorem 12. *If the conditions*

$$\begin{gathered}\mu^* > 0, \qquad 3\lambda^* + 2\mu^* > 0, \qquad \xi^* > 0,\\ 4k\xi^* T_0 > (b^* T_0 + f^*)^2, \qquad \sigma_1\tau_2 - \sigma_2\tau_1 \neq 0\end{gathered} \tag{34}$$

are satisfied, then the internal BVP $(I)^+_{\mathbf{F},\mathbf{f}}$ admits at most one regular solution.

Proof. Suppose that there are two regular solutions of the problem $(I)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ corresponds to the zero data ($\mathbf{F} = \mathbf{f} = \mathbf{0}$), i.e., $\mathbf{U}$ is a regular solution of the homogeneous equation

$$\mathbf{A}(\mathbf{D}_\mathbf{x})\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{35}$$

for $\mathbf{x}\in\Omega^+$ and satisfies the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0}. \tag{36}$$

Then, employing the conditions (35) and (36), we can derive from (21) and (23)

$$\begin{gathered}\int\limits_{\Omega^+} W_1(\mathbf{U},\mathbf{u})d\mathbf{x} = 0, \quad \int\limits_{\Omega^+} W_2(\mathbf{U},\mathbf{w})d\mathbf{x} = 0, \quad \int\limits_{\Omega^+} W_3(\mathbf{U},\varphi)d\mathbf{x} = 0,\\ \int\limits_{\Omega^+} W_4(\mathbf{U},\psi)d\mathbf{x} = 0, \quad \int\limits_{\Omega^+} W_5(\mathbf{U},\theta)d\mathbf{x} = 0,\end{gathered} \tag{37}$$

where $W_1, W_2, \ldots, W_5$ are defined by (22) and (24).

In view of the relations (6) and (8), we can write

$$\begin{gathered}\alpha_1' + 3\alpha_2' - 2\epsilon_1 = \alpha_1 + 3\alpha_2 - 2(\mu + 2\zeta + 2\gamma) - i\omega(3\lambda^* + 2\mu^*),\\ \alpha_1' + \epsilon_1 = \alpha_1 + \mu + 2\zeta + 2\gamma - 2i\omega\mu^*,\\ \eta_1' \left|\mathbf{u}\right|^2 + 2\xi'\mathrm{Re}\left[\mathbf{u}\cdot\mathbf{w})\right] + \eta_2' \left|\mathbf{w}\right|^2 = (\rho_1\omega^2 - \xi)\left|\mathbf{u}\right|^2 + 2\xi\mathrm{Re}\left[\mathbf{u}\cdot\mathbf{w})\right]\\ +(\rho_2\omega^2 - \xi)\left|\mathbf{w}\right|^2 + i\omega\xi^*\left|\mathbf{u}-\mathbf{w}\right|^2.\end{gathered} \tag{38}$$

Obviously, on the basis of (22), (24) and (38), it follows that

$$\begin{gathered}\mathrm{Im}\left[W_1(\mathbf{U},\mathbf{u}) + W_2(\mathbf{U},\mathbf{w}) + W_3(\mathbf{U},\varphi) + W_4(\mathbf{U},\psi)\right]\\ = -\frac{\omega}{3}(3\lambda^* + 2\mu^*)\left|\mathrm{div}\,\mathbf{u}\right|^2 - 2\omega W^{(0)}(\mathbf{u},\mathbf{u}) - \omega\xi^*|\mathbf{u}-\mathbf{w}|^2\\ -\mathrm{Im}\left\{\left[(\beta^{(1)} + \beta^{(2)})\mathrm{div}\,\overline{\mathbf{u}} + \beta^{(2)}\mathrm{div}\,\overline{\mathbf{w}} + b^{(1)}\overline{\varphi} + b^{(2)}\overline{\psi}\right]\theta - b^*\nabla\theta\cdot(\mathbf{u}-\mathbf{w})\right\},\end{gathered} \tag{39}$$

$$\frac{1}{\omega T_0}\mathrm{Re}W_5(\mathbf{U},\theta) = \frac{k}{\omega T_0}|\nabla\theta|^2 + \mathrm{Im}\left[(\beta^{(1)}+\beta^{(2)})\mathrm{div}\,\mathbf{u} + \beta^{(2)}\mathrm{div}\,\mathbf{w} + b^{(1)}\varphi + b^{(2)}\psi\right]\overline{\theta}$$
$$+\frac{1}{T_0}f^*\mathrm{Im}\left[(\mathbf{u}-\mathbf{w})\cdot\nabla\theta\right].$$

Clearly, from (39), we get

$$\begin{aligned}
&\frac{1}{\omega T_0}\mathrm{Re}W_5(\mathbf{U},\theta) - \mathrm{Im}\left[W_1(\mathbf{U},\mathbf{u}) + W_2(\mathbf{U},\mathbf{w}) + W_3(\mathbf{U},\varphi) + W_4(\mathbf{U},\psi)\right]\\
&\quad= \frac{\omega}{3}(3\lambda^*+2\mu^*)\,|\mathrm{div}\,\mathbf{u}|^2 + 2\omega W^{(0)}(\mathbf{u},\mathbf{u}) + \omega\xi^*|\mathbf{u}-\mathbf{w}|^2\\
&\quad\quad+\frac{k}{\omega T_0}|\nabla\theta|^2 - \left(b^* + \frac{f^*}{T_0}\right)\mathrm{Im}\left[(\mathbf{u}-\mathbf{w})\cdot\nabla\theta\right].
\end{aligned}\tag{40}$$

In view of (37) and (40), we have

$$\frac{\omega}{3}(3\lambda^*+2\mu^*)\,|\mathrm{div}\,\mathbf{u}|^2 + 2\omega W^{(0)}(\mathbf{u},\mathbf{u}) + \omega\xi^*|\mathbf{u}-\mathbf{w}|^2$$
$$+\frac{k}{\omega T_0}|\nabla\theta|^2 - \left(b^* + \frac{f^*}{T_0}\right)\mathrm{Im}\left[(\mathbf{u}-\mathbf{w})\cdot\nabla\theta\right] = 0.$$

By virtue of (34), the last equation leads to the following relations:

$$\begin{gathered}
\mathrm{div}\,\mathbf{u}(\mathbf{x}) = 0, \quad W^{(0)}(\mathbf{u},\mathbf{u}) = 0, \quad \mathbf{u}(\mathbf{x}) = \mathbf{w}(\mathbf{x}),\\
\nabla\theta(\mathbf{x}) = 0, \quad \mathbf{x}\in\Omega^+.
\end{gathered}\tag{41}$$

Then, employing (18), (22) and (41), we can derive from (37)

$$\begin{aligned}
W_1(\mathbf{u},\mathbf{u}) + W_2(\mathbf{u},\mathbf{u}) &= W^{(1)}(\mathbf{u},\mathbf{u}) + 2W^{(2)}(\mathbf{u},\mathbf{u}) + W^{(3)}(\mathbf{u},\mathbf{u})\\
= \frac{1}{2}\left[\alpha_1' - \epsilon_1 + 2(\beta_1-\epsilon_2) + \gamma_1 - \epsilon_3\right]&|\mathrm{curl}\,\mathbf{u}|^2 - (\eta_1' + 2\xi' + \eta_2')|\mathbf{u}|^2 = -\rho\omega^2|\mathbf{u}|^2 = 0
\end{aligned}\tag{42}$$

and consequently, from (42), we have

$$\mathbf{u}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x}\in\Omega^+. \tag{43}$$

Now, taking into account (41) and (43), from (35), we deduce the system

$$\sigma_1\nabla\varphi + \sigma_2\nabla\psi = 0, \quad \tau_1\nabla\varphi + \tau_2\nabla\psi = 0. \tag{44}$$

By virtue of the last relation of (34), from (44), we obtain $\nabla\varphi = \nabla\psi = 0$. Combining this relation with (41) and (43), we may further conclude that

$$\mathbf{u}(\mathbf{x}) = \mathbf{w}(\mathbf{x}) \equiv \mathbf{0}, \quad \varphi(\mathbf{x}) = c_1, \quad \psi(\mathbf{x}) = c_2, \quad \theta(\mathbf{x}) = c_3 \quad \text{for} \quad \mathbf{x}\in\Omega^+, \tag{45}$$

where c_1, c_2 and c_3 are arbitrary complex numbers. Finally, in view of the homogeneous boundary condition (36), from (45), we get $c_1 = c_2 = c_3 = 0$. Thus, $\mathbf{U}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x}\in\Omega^+$, and we have the desired result. □

Theorem 13. *If the conditions* (34) *and*

$$\det\begin{pmatrix} \eta_1 & -\zeta^{(3)} & b^{(1)}\\ -\zeta^{(3)} & \eta_2 & b^{(2)}\\ b^{(1)} & b^{(2)} & a \end{pmatrix}_{3\times 3} \neq 0 \tag{46}$$

are satisfied, then the internal BVP $(II)^+_{\mathbf{F},\mathbf{f}}$ *admits at most one regular solution.*

Proof. Suppose that there are two regular solutions of problem $(II)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ corresponds to zero data ($\mathbf{F} = \mathbf{f} = \mathbf{0}$), i.e., $\mathbf{U}$ is a regular solution of problem $(II)^+_{\mathbf{0},\mathbf{0}}$. Consequently, $\mathbf{U}$ is a regular solution of the system of homogeneous equations (35) satisfying the homogeneous boundary condition

$$\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for} \quad \mathbf{z}\in S.$$

In a similar manner as in Theorem 12 we obtain the relations (45). We now combine (45) with (35) to deduce the system

$$\begin{gathered}\eta_1 c_1 - \zeta^{(3)} c_2 + b^{(1)} c_3 = 0,\\ -\zeta^{(3)} c_1 + \eta_2 c_2 + b^{(2)} c_3 = 0,\\ b^{(1)} c_1 + b^{(2)} c_2 + a c_3 = 0.\end{gathered} \tag{47}$$

By virtue of (46), from (47), we obtain $c_1 = c_2 = c_3 = 0$ and, therefore, we get the relation $\mathbf{U}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x} \in \Omega^+$. Hence, the uniqueness of a regular solution to problem $(II)^+_{\mathbf{F},\mathbf{f}}$ follows. □

Quite similarly, on the basis of the condition (16) and the identity (27), we obtain the following

Theorem 14. *If condition* (34) *is satisfied, then the external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *admits at most one regular solution, where* $K = I, II$.

7. Surface and Volume Potentials

We introduce the following notation:

i) $\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g}) = \int\limits_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the single-layer potential,

ii) $\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) = \int\limits_S [\tilde{\mathbf{R}}(\mathbf{D}_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{y})]^\top \mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the double-layer potential, and

iii) $\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^\pm) = \int\limits_{\Omega^\pm} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\boldsymbol{\phi}(\mathbf{y})d\mathbf{y}$ is the volume potential,

where the matrices $\mathbf{\Gamma}(\mathbf{x})$ and $\tilde{\mathbf{R}}(\mathbf{D}_{\mathbf{x}})$ are given by (14) and (30), respectively; $\mathbf{g}$ and $\boldsymbol{\phi}$ are the nine-component vector functions.

Obviously, on the basis of Green's third identities (32) and (33), the regular vector $\mathbf{U}$ in Ω^+ is represented by the sum of the single-layer, double-layer and volume potentials as follows:

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x},\mathbf{U}) - \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{R}\mathbf{U}) + \mathbf{Q}^{(3)}(\mathbf{x},\mathbf{A}\mathbf{U},\Omega^+) \quad \text{for} \quad \mathbf{x} \in \Omega^+.$$

Similarly, the regular vector $\mathbf{U}$ in Ω^- is represented by the sum

$$\mathbf{U}(\mathbf{x}) = -\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{U}) + \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{R}\mathbf{U}) + \mathbf{Q}^{(3)}(\mathbf{x},\mathbf{A}\mathbf{U},\Omega^-) \quad \text{for} \quad \mathbf{x} \in \Omega^-.$$

On the basis of (14) and (15), we have the following results.

Theorem 15. *If* $S \in C^{m+1,\nu'}$, $\mathbf{g} \in C^{m,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, *and* m *is a non-negative integer, then:*

a)
$$\mathbf{Q}^{(1)}(\cdot,\mathbf{g}) \in C^{0,\nu''}(\mathbb{R}^3) \cap C^{m+1,\nu''}(\overline{\Omega^\pm}) \cap C^\infty(\Omega^\pm),$$

b)
$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g}) = \mathbf{0},$$

c)
$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})\}^\pm = \mp\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}), \tag{48}$$

d)
$$\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})$$

is a singular integral, where $\mathbf{z} \in S$, $\mathbf{x} \in \Omega^\pm$ *and*

$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})\}^\pm \equiv \lim_{\Omega^\pm \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D}_{\mathbf{x}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g}).$$

Theorem 16. *If* $S \in C^{m+1,\nu'}$, $\mathbf{g} \in C^{m,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, *then:*

a)
$$\mathbf{Q}^{(2)}(\cdot,\mathbf{g}) \in C^{m,\nu''}(\overline{\Omega^\pm}) \cap C^\infty(\Omega^\pm),$$

b)
$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) = \mathbf{0},$$

c)
$$\{\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{\pm} = \pm\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}) \tag{49}$$
for the non-negative integer m,

d) $\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})$ *is a singular integral, where* $\mathbf{z}\in S$,

e)
$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{+} = \{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{-}$$
for the natural number m, where $\mathbf{z}\in S$, $\mathbf{x}\in\Omega^{\pm}$ and
$$\{\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{\pm} \equiv \lim_{\Omega^{\pm}\ni\mathbf{x}\to\ \mathbf{z}\in S} \mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}).$$

Theorem 17. *If $S\in C^{1,\nu'}$, $\boldsymbol{\phi}\in C^{0,\nu''}(\Omega^{+})$, $0<\nu''<\nu'\le 1$, then:*

a)
$$\mathbf{Q}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^{+}) \in C^{1,\nu''}(\mathbb{R}^3)\cap C^{2}(\Omega^{+})\cap C^{2,\nu''}\left(\overline{\Omega_0^{+}}\right),$$

b)
$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^{+}) = \boldsymbol{\phi}(\mathbf{x}),$$
where $\mathbf{x}\in\Omega^{+}$, Ω_0^{+} is a domain in $\mathbb{R}^3$ and $\overline{\Omega_0^{+}}\subset\Omega^{+}$.

Theorem 18. *If $S\in C^{1,\nu'}$, $\mathrm{supp}\boldsymbol{\phi}=\Omega\subset\Omega^{-}$, $\boldsymbol{\phi}\in C^{0,\nu''}(\Omega^{-})$, $0<\nu''<\nu'\le 1$, then:*

a)
$$\mathbf{Q}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^{-}) \in C^{1,\nu''}(\mathbb{R}^3)\cap C^{2}(\Omega^{-})\cap C^{2,\nu''}(\overline{\Omega_0^{-}}),$$

b)
$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^{-}) = \boldsymbol{\phi}(\mathbf{x}),$$
where $\mathbf{x}\in\Omega^{-}$, Ω is a bounded domain in $\mathbb{R}^3$ and $\overline{\Omega_0^{-}}\subset\Omega^{-}$.

We here introduce the following notation:
$$\begin{gathered}
\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) \equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}), \qquad \mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) \equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(3)}\mathbf{g}(\mathbf{z}) \equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}), \quad \mathcal{K}^{(4)}\mathbf{g}(\mathbf{z}) \equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}_{\varsigma}\mathbf{g}(\mathbf{z}) \equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \varsigma\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}) \quad \text{for} \quad \mathbf{z}\in S,
\end{gathered} \tag{50}$$
where ς is a complex parameter. On the basis of Theorems 15 and 16, $\mathcal{K}^{(j)}$ $(j=1,2,3,4)$ and $\mathcal{K}_{\varsigma}$ are singular integral operators.

We introduce the notation
$$\begin{gathered}
\mu_1' = \mu_1 - i\omega\mu^{*}, \quad \mu_1 = \mu+\kappa+\gamma+2\zeta, \quad \mu_2 = \kappa+\gamma,\\
\mu_3 = \kappa+\gamma+\zeta, \quad \mu_0' = \mu_1'+\mu_2+3\mu_3, \quad e_1' = \alpha_2'+\kappa-\gamma,\\
e_1 = \alpha_2+\kappa-\gamma, \quad e_2 = \gamma_2+\kappa-\gamma, \quad e_3 = \beta_2-\kappa+\gamma,\\
e_0' = e_1'+e_2+e_3, \quad b_1 = \alpha_1\gamma_1-\beta_1^2, \quad b_2' = \mu_1'\mu_2-\mu_3^2,\\
b_2 = \mu_1\mu_2-\mu_3^2, \quad b_3' = e_1'e_2-e_3^2, \quad b_3 = e_1e_2-e_3^2.
\end{gathered} \tag{51}$$

Let $\boldsymbol{\sigma}^{(j)} = (\sigma_{lm}^{(j)})_{9\times 9}$ be the symbol of the singular integral operator $\mathcal{K}^{(j)}$ $(j=1,2,3,4)$. Taking into account (50) and (51), by a long calculation for $\det\boldsymbol{\sigma}^{(j)}$, we find that
$$\det\boldsymbol{\sigma}^{(j)} = -\frac{1}{512}\frac{k_2k_3}{k_0k_1}, \quad j=1,2,3,4, \tag{52}$$
where k_0 and k_1 are defined by (10) and
$$k_2 = (\alpha_0'+\alpha_1)(\gamma_0+\gamma_1)-(\beta_0+\beta_1)^2, \quad k_3 = k_1b_3'+(\kappa-\gamma)(\mu_0'b_3'+e_0'b_2'). \tag{53}$$
On the basis of (6), (8) and (51), from (10) and (53), we have
$$\mathrm{Im}k_0 = -\omega(\lambda^{*}+2\mu^{*})\gamma_0, \quad \mathrm{Im}k_1 = -\omega\mu^{*}\gamma_1, \quad \mathrm{Im}k_2 = -\omega(\lambda^{*}+3\mu^{*})(\gamma_0+\gamma_1),$$

$$\mathrm{Im}k_3 = -\omega(\lambda^* + \mu^*)b_1 e_2 - \omega\mu^* b_3 \gamma_1 - \omega(\kappa - \gamma)\left[\lambda^*(\mu_0 e_2 + b_2) + \mu^*(\mu_2 e_0 + b_3)\right].$$

Clearly, if

$$\mathrm{Im}k_l \neq 0, \quad l = 0,1,2,3, \tag{54}$$

then from (52), it follows that

$$\det \boldsymbol{\sigma}^{(j)} \neq 0 \tag{55}$$

which proves that the singular integral operator $\mathcal{K}^{(j)}$ is of the normal type, where $j = 1,2,3,4$. Hence we have the following

Theorem 19. *If condition* (54) *is satisfied, then the singular integral operator* $\mathcal{K}^{(j)}$ *is of the normal type, where* $j = 1,2,3,4$.

Let $\boldsymbol{\sigma}_\varsigma$ and $\mathrm{ind}\,\mathcal{K}_\varsigma$ be the symbol and the index of the operator $\mathcal{K}_\varsigma$, respectively. It can be easily shown that $\det \boldsymbol{\sigma}_\varsigma$ vanishes only at four points ς_j $(j = 1,2,3,4)$ of the complex plane. By virtue of (55) and $\det \boldsymbol{\sigma}_1 = \det \boldsymbol{\sigma}^{(1)}$, we get $\varsigma_j \neq 1$ for $j = 1,2,3,4$, and we obtain

$$\mathrm{ind}\,\mathcal{K}^{(1)} = \mathrm{ind}\,\mathcal{K}_1 = 0.$$

In a quite similar manner, we have the relation $\mathrm{ind}\,\mathcal{K}^{(2)} = 0$. We can easily verify that the operators $\mathcal{K}^{(3)}$ and $\mathcal{K}^{(4)}$ are the adjoint operators for $\mathcal{K}^{(2)}$ and $\mathcal{K}^{(1)}$, respectively. Consequently, we have

$$\mathrm{ind}\,\mathcal{K}^{(3)} = -\mathrm{ind}\,\mathcal{K}^{(2)} = 0, \quad \mathrm{ind}\,\mathcal{K}^{(4)} = -\mathrm{ind}\,\mathcal{K}^{(1)} = 0.$$

Hence, the singular integral operator $\mathcal{K}^{(j)}$ $(j = 1,2,3,4)$ is of the normal type with an index equal to zero, i.e., Fredholm's theorems are valid for $\mathcal{K}^{(j)}$. Thus, we have proved the following

Theorem 20. *If condition* (54) *is satisfied, then Fredholm's theorems are valid for the singular integral operator* $\mathcal{K}^{(j)}$, *where* $j = 1,2,3,4$.

Remark 1. The definitions of a normal type singular integral operator, the symbol and the index of the operator are given in [18,20]. In addition, in these books, one can find the method for calculating the symbol of singular integral operator.

8. Existence Theorems

In what follows, we assume that the constitutive coefficients satisfy the conditions (34), (46) and (54). Obviously, by Theorems 17 and 18, the volume potential $\mathbf{Q}^{(3)}(\mathbf{x},\mathbf{F},\Omega^\pm)$ is a partial regular solution of the nonhomogeneous equation (9), where $\mathbf{F} \in C^{0,\nu'}(\Omega^\pm)$, $0 < \nu' \leq 1$ and $\mathrm{supp}\,\mathbf{F}$ is a finite domain in Ω^-. Therefore, in this section, we prove the existence theorems for classical solutions of the BVPs $(K)^+_{\mathbf{0},\mathbf{f}}$ and $(K)^-_{\mathbf{0},\mathbf{f}}$, where $K = I, II$.

Problem $(I)^+_{\mathbf{0},\mathbf{f}}$**.** We are looking for a regular solution to this problem in the form of a double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) \quad \text{for} \quad \mathbf{x} \in \Omega^+, \tag{56}$$

where $\mathbf{g}$ is the required nine-component vector function. By Theorem 16, the vector function $\mathbf{U}$ is a solution of the homogeneous equation (35) for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition $\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$ and using (49), from (56), for determining the unknown vector $\mathbf{g}$, we obtain a singular integral equation

$$\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{57}$$

By Theorem 20, Fredholm's theorems are valid for the operator $\mathcal{K}^{(1)}$. We prove that (57) is always solvable for an arbitrary vector $\mathbf{f}$. Let us consider the adjoint homogeneous equation

$$\mathcal{K}^{(4)}\mathbf{h}_0(\mathbf{z}) = \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S, \tag{58}$$

where $\mathbf{h}_0$ is the required nine-component vector function.

Now we prove that (58) has only the trivial solution. Indeed, let $\mathbf{h}_0$ be a solution of the homogeneous equation (58). On the basis of Theorem 15 and equation (58), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{h}_0)$ is a regular solution of problem $(II)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem 14, problem $(II)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{59}$$

On the other hand, by Theorem 15 and equation (59), we get $\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0}$ for $\mathbf{z} \in S$, i.e., the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of problem $(I)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem 12, problem $(I)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, i.e.,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^+. \tag{60}$$

By virtue of (59), (60) and identity (48), we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ \equiv \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S.$$

Thus, the homogeneous equation (58) has only the trivial solution and, therefore, (57) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved

Theorem 21. *If $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{1,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, then a regular solution of problem $(I)^+_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by the double-layer potential* (56)*, where $\mathbf{g}$ is a solution of the singular integral equation* (57) *which is always solvable for an arbitrary vector $\mathbf{f}$.*

Problem $(II)^-_{\mathbf{0},\mathbf{f}}$**.** We are looking for a regular solution to this problem in the form of a single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}) \quad \text{for} \quad \mathbf{x} \in \Omega^-, \tag{61}$$

where $\mathbf{h}$ is the required nine-component vector function. Obviously, by Theorem 15, the vector function $\mathbf{U}$ is a solution of (35) for $\mathbf{x} \in \Omega^-$. Keeping in mind the boundary condition $\{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$ and using (48), from (61), for determining the unknown vector $\mathbf{h}$, we obtain a singular integral equation

$$\mathcal{K}^{(4)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{62}$$

It has been proved above that the corresponding homogeneous equation (58) has only the trivial solution. Hence, it follows that (62) is always solvable.

We have thereby proved

Theorem 22. *If $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{0,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, then a regular solution of problem $(II)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by a single-layer potential* (61)*, where $\mathbf{h}$ is a solution of the singular integral equation* (62) *which is always solvable for an arbitrary vector $\mathbf{f}$.*

Problem $(II)^+_{\mathbf{0},\mathbf{f}}$**.** We are looking for a regular solution to this problem in the form of a single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}) \quad \text{for} \quad \mathbf{x} \in \Omega^+, \tag{63}$$

where $\mathbf{h}$ is the required nine-component vector function. Obviously, by Theorem 15, the vector function $\mathbf{U}$ is a solution of (35) for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition $\{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$ and using (48), from (63), for determining the unknown vector $\mathbf{h}$, we obtain a singular integral equation

$$\mathcal{K}^{(2)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{64}$$

We now prove that (64) is always solvable for an arbitrary vector $\mathbf{f}$. Let $\mathbf{h}_0$ be a solution of the homogeneous equation

$$\mathcal{K}^{(2)}\mathbf{h}(\mathbf{z}) = \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S. \tag{65}$$

On the basis of Theorem 15 and equation (65), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of problem $(II)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem 13, problem $(II)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^+. \tag{66}$$

On the other hand, by Theorem 15 and equation (66), we get $\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0}$ for $\mathbf{z} \in S$, i.e., the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of problem $(I)^-_{\mathbf{0},\mathbf{0}}$. Now, using Theorem 14, problem $(I)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{67}$$

By virtue of (66), (67) and identity (48), we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ \equiv \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S.$$

Thus, the homogeneous equation (65) has only the trivial solution and, therefore, (64) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved

Theorem 23. *If $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{0,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, then a regular solution of problem $(II)^+_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by a single-layer potential* (63)*, where* $\mathbf{h}$ *is a solution of the singular integral equation* (64) *which is always solvable for an arbitrary vector* $\mathbf{f}$*.*

Problem $(I)^-_{\mathbf{0},\mathbf{f}}$. Finally, we are looking for a regular solution to this problem in the form of a double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) \quad \text{for} \quad \mathbf{x} \in \Omega^-, \tag{68}$$

where $\mathbf{g}$ is the required nine-component vector function. Obviously, by Theorem 16, the vector function $\mathbf{U}$ is a solution of (35) for $\mathbf{x} \in \Omega^-$. Keeping in mind the boundary condition $\{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$ and using (49), from (68), for determining the unknown vector $\mathbf{g}$, we obtain a singular integral equation

$$\mathcal{K}^{(3)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{69}$$

It has been proved above that the adjoint homogeneous equation (65) has only the trivial solution. Hence, it follows that (69) is always solvable.

Thus, we have thereby proved

Theorem 24. *If $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{1,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, then a regular solution of problem $(I)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by a double-layer potential* (68)*, where* $\mathbf{g}$ *is a solution of the singular integral equation* (69) *which is always solvable for an arbitrary vector* $\mathbf{f}$*.*

9. Concluding Remarks

In this paper, the linear theory of thermoviscoelasticity for binary porous mixtures is considered and the following results are obtained.

a) The fundamental solution of the system of equations of steady vibrations is constructed explicitly and its basic properties are established.

b) Green's identities are obtained.

c) The uniqueness theorems for classical solutions of the internal and external basic BVPs of steady vibrations are proved.

d) The surface and volume potentials are constructed and their basic properties are given.

e) The determinants of symbolic matrices are calculated explicitly.

f) The BVPs are reduced to the always solvable singular integral equations for which Fredholm's theorems are valid.

g) Finally, the existence theorem for classical solutions of the internal and external BVPs of steady vibrations are proved by means of the potential method and the theory of singular integral equations.

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ON THE F STRUCTURES OF THE SPACE $T(Lm(Vn))$

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Abstract. There are constructed lifts of tensor fields a^i_j, a^i_α, $a^i_{\overline{j}}$, $a^i_{\overline{\alpha}}$, $a^{\overline{i}}_j$, $a^{\overline{i}}_\alpha$, $a^{\overline{i}}_{\overline{j}}$, $a^{\overline{i}}_{\overline{\alpha}}$, a^β_j, a^β_α, $a^\beta_{\overline{j}}$, $a^\beta_{\overline{\alpha}}$, $a^{\overline{\beta}}_j$, $a^{\overline{\beta}}_\alpha$, $a^{\overline{\beta}}_{\overline{j}}$, $a^{\overline{\beta}}_{\overline{\alpha}}$. There are defined F structures on the space $a^{\overline{\beta}}_j$ and there is proved, that real-valued F structures exist only for $\lambda = -1$.

Consider a tangent bundle $T(Lm(Vn))$ with the local coordinates x^i, y^α, y^i, z^α where are the coordinates of the basis $T(Lm(Vn))$, and y^i, z^α are those of the layer T_z, $z \in Lm(Vn)$, in other words, the vector fields X,

$$X = y^i \frac{\partial}{\partial x^i} + z^\alpha \frac{\partial}{\partial y^\alpha},$$

generate the bundle $T(Lm(Vn))$. It is now evident that the local coordinates $(x^i, y^\alpha, y^i, z^\alpha)$ of the point of the space $T(Lm(Vn))$ are transformed as follows:

$$\overline{x}^i = \overline{x}^i(x^k), \quad \overline{y}^\alpha = A^\alpha_\beta(x) y^\beta, \quad \overline{y}^i = x^i_k(y^k), \quad \overline{z}^\alpha = A^\alpha_\beta z^\beta + A^\alpha_{\beta k} y^\beta y^k.$$

A complete equipment of the space $T(Lm(Vn))$ can be defined by means of the vectors D_i, D_α, D_k [1–4]:

$$D_i = \frac{\partial}{\partial y^i} - Q^\alpha_i \frac{\partial}{\partial z^\alpha}, \quad D_\alpha = \frac{\partial}{\partial y^\alpha} - E^\beta_\alpha \frac{\partial}{\partial z^\beta}, \quad D_k = \frac{\partial}{\partial x^k} - C^\alpha_k \frac{\partial}{\partial z^\alpha} - Q^\alpha_k \frac{\partial}{\partial y^\alpha} - E^i_k \frac{\partial}{\partial y^i}.$$

Note that the tensor field T can be represented as

$$\begin{aligned} T = {} & T^i_j dx^j \otimes \frac{\partial}{\partial x^i} + T^i_\alpha dx^\alpha \otimes \frac{\partial}{\partial x^i} + T^i_{j} dy^j \otimes \frac{\partial}{\partial x^i} + T^i_{\overline{\alpha}} dz^\alpha \otimes \frac{\partial}{\partial x^i} \\ & + T^\alpha_i dx^i \otimes \frac{\partial}{\partial y^\alpha} + T^\alpha_\beta dy^\beta \otimes \frac{\partial}{\partial y^\alpha} + T^\alpha_{\overline{i}} dy^i \otimes \frac{\partial}{\partial y^\alpha} + T^\alpha_{\overline{\beta}} dz^\beta \otimes \frac{\partial}{\partial y^\alpha} \\ & + T^{\overline{i}}_j dx^j \otimes \frac{\partial}{\partial y^i} + T^{\overline{i}}_\alpha dy^\alpha \otimes \frac{\partial}{\partial y^i} + T^{\overline{i}}_j dy^j \otimes \frac{\partial}{\partial y^i} + T^{\overline{i}}_{\overline{\alpha}} dz^\alpha \otimes \frac{\partial}{\partial y^i} \\ & + T^{\overline{\alpha}}_i dx^i \otimes \frac{\partial}{\partial z^\alpha} + T^{\overline{\alpha}}_\beta dy^\beta \otimes \frac{\partial}{\partial z^\alpha} + T^{\overline{\alpha}}_{\overline{i}} dy^i \otimes \frac{\partial}{\partial z^\alpha} + T^{\overline{\alpha}}_{\overline{\beta}} dz^\beta \otimes \frac{\partial}{\partial z^\alpha}. \end{aligned}$$

The tensor T in the equipped basis can be decomposed as follows:

$$\begin{aligned} T = {} & a^i_j dx^j \otimes D_i + a^i_\alpha Dy^\alpha \otimes D_i + a^i_{\overline{j}} Dy^j \otimes D_i + a^i_{\overline{\alpha}} Dz^\alpha \otimes D_i \\ & + a^\alpha_\beta Dy^\beta \otimes D_\alpha + a^\beta_j dx^j \otimes D_\beta + a^\beta_{\overline{j}} Dy^j \otimes D_\beta + a^\beta_{\overline{\alpha}} Dz^\alpha \otimes D_\beta \\ & + a^{\overline{i}}_j dx^j \otimes D_i + a^{\overline{i}}_\beta Dy^\beta \otimes D_{\overline{i}} + a^{\overline{i}}_{\overline{j}} Dy^j \otimes D_{\overline{i}} + a^{\overline{i}}_{\overline{\alpha}} Dz^\alpha \otimes D_{\overline{i}} \\ & + a^{\overline{\alpha}}_i dx^i \otimes D_{\overline{\alpha}} + a^{\overline{\alpha}}_\beta Dy^\beta \otimes D_{\overline{\alpha}} + a^{\overline{\alpha}}_{\overline{i}} Dy^i \otimes D_{\overline{\alpha}} + a^{\overline{\alpha}}_{\overline{\beta}} Dz^\beta \otimes D_{\overline{\alpha}}, \end{aligned}$$

where

$$\begin{aligned} Dy^\alpha &= dy^\alpha + C^\alpha_k dx^k, \\ Dy^i &= dy^i + \Gamma^i_j dx^j, \\ Dz^\alpha &= dz^\alpha + L^\alpha_k dx^k + C^\alpha_k dy^k + G^\alpha_\beta dy^\beta. \end{aligned}$$

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From the above equalities, after removing the parentheses, we obtain

$$\begin{gathered}
T=(a^i_j+a^i_{\overline{p}}\Gamma^p_j+a^i_\beta\Gamma^\beta_j+a^i_{\overline{\gamma}}L^\gamma_j)dx^j\otimes\frac{\partial}{\partial x^i}\\
+(a^\alpha_j-a^i_j\Gamma^\alpha_i-a^i_{\overline{p}}\Gamma^\alpha_i\Gamma^p_j-a^i_{\overline{\gamma}}\Gamma^\alpha_i\Gamma^\gamma_j+a^\alpha_\beta\Gamma^\beta_j+a^\alpha_{\overline{p}}\Gamma^p_j+a^\alpha_{\overline{\beta}}L^\beta_j)dx^j\otimes\frac{\partial}{\partial y^\alpha}\\
+(a^{\overline{k}}_j-a^i_j\Gamma^k_i-a^i_\beta\Gamma^k_i\Gamma^\beta_j-a^i_{\overline{p}}\Gamma^k_i\Gamma^p_j-a^i_{\overline{\gamma}}\Gamma^k_iL^\gamma_j+a^{\overline{k}}_\alpha\Gamma^\alpha_j+a^{\overline{k}}_{\overline{p}}\Gamma^p_j+a^{\overline{k}}_{\overline{\alpha}}L^\alpha_j)dx^j\otimes\frac{\partial}{\partial y^k}\\
+(a^{\overline{\beta}}_j-a^i_jC^\beta_i-a^i_\alpha C^\beta_i\Gamma^\alpha_j-a^i_{\overline{p}}C^\beta_i\Gamma^p_j-a^i_{\overline{\gamma}}C^\beta_iL^\gamma_j-a^\alpha_jG^\beta_\alpha-a^\gamma_\alpha\Gamma^\alpha_jG^\beta_\gamma-a^\gamma_{\overline{p}}G^\beta_\gamma\Gamma^p_j-a^\gamma_{\overline{\alpha}}G^\beta_\gamma L^\alpha_j-a^{\overline{i}}_j\Gamma^\beta_i\\
-a^{\overline{i}}_\alpha\Gamma^\alpha_j\Gamma^\beta_i-a^{\overline{i}}_{\overline{p}}\Gamma^p_j\Gamma^\beta_i-a^{\overline{i}}_{\overline{\alpha}}L^\alpha_j\Gamma^\beta_i+a^{\overline{\beta}}_\alpha\Gamma^\alpha_j+a^{\overline{\beta}}_{\overline{p}}\Gamma^p_j+a^{\overline{\beta}}_{\overline{\alpha}}L^\alpha_j)dx^j\otimes\frac{\partial}{\partial z^\beta}\\
+(a^i_\beta+a^i_{\overline{\gamma}}G^\gamma_\beta)dy^\beta\otimes\frac{\partial}{\partial x^i}+(a^\alpha_\beta-a^i_\beta\Gamma^\alpha_i-a^i_{\overline{\gamma}}\Gamma^\alpha_iG^\gamma_\beta\\
+a^\alpha_{\overline{\gamma}}G^\gamma_\beta)dy^\beta\otimes\frac{\partial}{\partial y^\alpha}+(a^{\overline{i}}_\alpha-a^k_\alpha\Gamma^i_k-a^k_{\overline{\gamma}}\Gamma^i_kG^\gamma_\alpha+a^{\overline{i}}_{\overline{\gamma}}G^\gamma_\alpha)dy^\alpha\otimes\frac{\partial}{\partial y^i}+(a^{\overline{\beta}}_\alpha-a^i_\alpha C^\beta_i\\
-a^\gamma_\alpha G^\beta_\gamma-a^i_{\overline{\gamma}}C^\beta_iG^\gamma_\alpha-a^\delta_{\overline{\gamma}}G^\beta_\delta G^\gamma_\alpha-a^{\overline{i}}_\alpha\Gamma^\beta_i-a^{\overline{i}}_{\overline{\gamma}}G^\gamma_\alpha\Gamma^\beta_i+a^{\overline{\beta}}_{\overline{\gamma}}G^\gamma_\alpha)dy^\alpha\otimes\frac{\partial}{\partial z^\beta}+(a^{\overline{i}}_{\overline{p}}+a^i_{\overline{\gamma}}\Gamma^\gamma_p)dy^p\otimes\frac{\partial}{\partial x^i}\\
+(a^\alpha_{\overline{p}}-a^i_{\overline{p}}\Gamma^\alpha_i-a^i_{\overline{\gamma}}\Gamma^\alpha_i\Gamma^\gamma_p+a^\alpha_{\overline{\beta}}\Gamma^\beta_p)dy^p\otimes\frac{\partial}{\partial y^\alpha}+(a^{\overline{i}}_{\overline{p}}-a^k_{\overline{p}}\Gamma^i_k-a^k_{\overline{\gamma}}\Gamma^i_k\Gamma^\gamma_p\\
+a^{\overline{i}}_{\overline{\alpha}}\Gamma^\alpha_p)dy^p\otimes\frac{\partial}{\partial y^i}+(a^{\overline{\alpha}}_{\overline{p}}-a^i_{\overline{p}}C^\alpha_i-a^i_{\overline{\gamma}}C^\alpha_i\Gamma^\gamma_p-a^\beta_{\overline{p}}G^\alpha_\beta-a^\beta_{\overline{\gamma}}G^\alpha_\beta\Gamma^\gamma_p-a^{\overline{i}}_{\overline{p}}\Gamma^\alpha_i+a^{\overline{\alpha}}_{\overline{\gamma}}\Gamma^\gamma_p\\
-a^{\overline{i}}_{\overline{\gamma}}\Gamma^\gamma_p\Gamma^\gamma_i)dy^p\otimes\frac{\partial}{\partial z^\alpha}+(a^\beta_{\overline{\alpha}}-a^i_{\overline{\alpha}}\Gamma^\beta_i)dz^\alpha\otimes\frac{\partial}{\partial y^\beta}+(a^{\overline{i}}_{\overline{\alpha}}-a^k_{\overline{\alpha}}\Gamma^\alpha_k)dz^\alpha\otimes\frac{\partial}{\partial y^i}\\
+a^i_{\overline{\alpha}}dz^\alpha\otimes\frac{\partial}{\partial x^i}+(a^{\overline{\beta}}_{\overline{\alpha}}-a^i_{\overline{\alpha}}C^\beta_i-a^\gamma_{\overline{\alpha}}G^\beta_\gamma-a^{\overline{i}}_{\overline{\alpha}}\Gamma^\beta_i)dz^\alpha\otimes\frac{\partial}{\partial z^\beta}.
\end{gathered}$$

Since the values $a^i_j, a^i_\alpha, \ldots, a^i_{\overline{\beta}}\overline{\alpha}$ are the tensors with respect to the group $GL(n,R)$, $GL(m,R)$, $GL(m,n,R)$ we find that the completely definite choice of sixteen tensors a^i_j, a^i_α, $a^i_{\overline{j}}$, $a^i_{\overline{\alpha}}$, a^α_i, a^α_β, is associated to the completely definite tensor T^A_B with respect to the first differential group $GL(2n,2m,R)$ of the space $T(Lm(Vn))$ as follows:

$$\begin{gathered}
T^i_j=a^i_j+a^i_\beta\Gamma^\beta_j+a_{\overline{k}^i}\Gamma^k_j+a^i_{\overline{\beta}}L^\beta_j,\ \ T^i_{\overline{\alpha}}=a^i_{\overline{\alpha}},\ \ T^i_\alpha=a^i_\alpha+a^i_{\overline{\beta}}G^\beta_\alpha,\ \ T^i_{\overline{j}}=a^i_{\overline{\gamma}}\Gamma^\gamma_j,\\
T^\alpha_j=a^\alpha_j-a^i_j\Gamma^\alpha_i-a^i_\beta\Gamma^\alpha_i\Gamma^\beta_j-a^i_{\overline{p}}\Gamma^\alpha_i\Gamma^p_j-a^i_{\overline{\gamma}}\Gamma^\alpha_iL^\gamma_j+a^\alpha_\beta\Gamma^\beta_j+a^\alpha_{\overline{p}}\Gamma^p_j+a^\alpha_{\overline{\beta}}L^\beta_j,\\
T^\alpha_\beta=a^\alpha_\beta-a^i_\beta\Gamma^\alpha_i-a^i_{\overline{\gamma}}\Gamma^\alpha_iG^\gamma_\beta+a^\alpha_{\overline{\gamma}}G^\gamma_\beta,\ \ T^\alpha_{\overline{j}}=a^\alpha_{\overline{j}}-a^i_{\overline{j}}\Gamma^\alpha_i-a^i_{\overline{\gamma}}\Gamma^\alpha_i\Gamma^\gamma_j+a^\alpha_{\overline{\gamma}}\Gamma^\beta_j,\\
T^{\overline{i}}_j=a^{\overline{i}}_j-a^k_j\Gamma^i_k-a^k_\beta\Gamma^i_k\Gamma^\beta_j-a^k_{\overline{p}}\Gamma^i_k\Gamma^p_j-a^k_{\overline{\gamma}}\Gamma^i_kL^\gamma_j+a^{\overline{i}}_\alpha\Gamma^\alpha_j+a^{\overline{i}}_{\overline{p}}\Gamma^p_j+a^{\overline{i}}_{\overline{\alpha}}L^\alpha_j,\\
T^{\overline{\alpha}}_{\overline{\beta}}=a^{\overline{\alpha}}_{\overline{\beta}}-a^i_{\overline{\beta}}C^\alpha_i-a^\gamma_{\overline{\beta}}G^\alpha_\gamma-a^{\overline{i}}_{\overline{\beta}}\Gamma^\alpha_i,\ \ T^{\overline{i}}_\alpha=a^{\overline{i}}_\alpha-a^k_\alpha\Gamma^i_k-a^k_{\overline{\gamma}}\Gamma^i_kG^\gamma_\alpha+a^{\overline{i}}_{\overline{\gamma}}G^\gamma_\alpha,\\
T^{\overline{i}}_{\overline{j}}=a^{\overline{i}}_{\overline{j}}-a^k_{\overline{j}}\Gamma^i_k-a^k_{\overline{\gamma}}\Gamma^i_k\Gamma^\gamma_j+a^{\overline{i}}_{\overline{\alpha}}\Gamma^\alpha_j,\ \ T^{\overline{i}}_{\overline{\alpha}}=a^{\overline{i}}_{\overline{\alpha}}-a^k_{\overline{\alpha}}\Gamma^i_k,\\
T^{\overline{\alpha}}_j=a^{\overline{\alpha}}_j-a^i_jC^\alpha_i-a^i_\beta C^\alpha_i\Gamma^\beta_j-a^i_{\overline{p}}C^\alpha_i\Gamma^p_j-a^i_{\overline{\gamma}}C^\alpha_iL^\gamma_j-a^\beta_jG^\alpha_\beta-a^\gamma_{\overline{p}}G^\alpha_\gamma\Gamma^p_j-a^\gamma_\beta\Gamma^\beta_jG^\alpha_\gamma-a^\gamma_{\overline{\beta}}G^\alpha_\gamma L^\beta_j\\
-a^{\overline{i}}_j\Gamma^\alpha_i-a^{\overline{i}}_\beta\Gamma^\beta_j\Gamma^\alpha_i-a^{\overline{i}}_{\overline{p}}\Gamma^p_j\Gamma^\alpha_i-a^{\overline{i}}_{\overline{\beta}}\Gamma^\alpha_iL^\beta_j+a^{\overline{\alpha}}_\beta\Gamma^\beta_j+a^{\overline{\alpha}}_{\overline{p}}\Gamma^p_j+a^{\overline{\alpha}}_{\overline{\beta}}L^\beta_j,\\
T^{\overline{\alpha}}_\beta=a^{\overline{\alpha}}_\beta-a^i_\beta C^\alpha_i-a^i_{\overline{\gamma}}C^\alpha_iG^\gamma_\beta-a^\gamma_\beta G^\alpha_\gamma-a^\delta_{\overline{\gamma}}G^\alpha_\delta G^\gamma_\beta-a^{\overline{i}}_\beta\Gamma^\alpha_i+a^{\overline{\alpha}}_{\overline{\gamma}}G^\gamma_\beta-a^{\overline{i}}_{\overline{\gamma}}G^\gamma_\beta\Gamma^\alpha_i,\\
T^{\overline{\alpha}}_{\overline{j}}=a^{\overline{\alpha}}_{\overline{j}}-a^i_{\overline{j}}C^\alpha_i-a^i_{\overline{\gamma}}C^\alpha_i\Gamma^\gamma_j-a^\beta_{\overline{j}}G^\alpha_\beta-a^\beta_{\overline{\gamma}}G^\alpha_\beta\Gamma^\gamma_j-a^{\overline{i}}_{\overline{j}}\Gamma^\alpha_i-a^{\overline{i}}_{\overline{\gamma}}\Gamma^\gamma_j\Gamma^\alpha_i+a^{\overline{\alpha}}_{\overline{\gamma}}\Gamma^\gamma_j.
\end{gathered}\tag{1}$$

Definition 1. The $GL(2n,2m,R)$-tensor field T^A_B defined by equalities (1) is called the Γ-lifting of ordered sixteen $GL(n,R)$, $GL(m,R)$, $GL(n,m,R)$-tensor fields $a^i_j,\ldots,a^{\overline{\alpha}}_{\overline{\beta}}$ defined on $Lm(Vn)$.

Definition 2. The space $T(Lm(Vn))$ on which is defined the tensor field satisfying conditions

$$T_B A T^B_C T^C_D + \lambda T^A_D = 0 \quad (\lambda = \pm 1) \tag{2}$$

is called the space with a F structure.

Equations (2) can be rewritten in the form

$$\begin{gathered}
T^i_k T^k_p T^p_j + T^i_k T^k_{\bar p} T^{\bar p}_j + T^i_k T^k_\alpha T^\alpha_j + T^i_k T^k_{\bar\alpha} T^{\bar\alpha}_j + T^i_{\bar k} T^{\bar k}_p T^p_j + T^i_{\bar k} T^{\bar k}_\alpha T^\alpha_j + T^i_{\bar k} T^{\bar k}_{\bar\alpha} T^{\bar\alpha}_j \\
+T^i_\alpha T^\alpha_p T^p_j + T^i_\alpha T^\alpha_{\bar p} T^{\bar p}_j + T^i_\alpha T^\alpha_\beta T^\beta_j + T^i_\alpha T^\alpha_{\bar\beta} T^{\bar\beta}_j + T^i_{\bar\alpha} T^{\bar\alpha}_p T^p_j + T^i_{\bar\alpha} T^{\bar\alpha}_{\bar p} T^{\bar p}_j + T^i_{\bar\alpha} T^{\bar\alpha}_{\bar\beta} T^\beta_j + T^i_{\bar\alpha} T^{\bar\alpha}_{\bar\beta} T^{\bar\beta}_j + \lambda T^i_j = 0, \\
T^k_i T^k_p T^p_j + T^i_k T^k_{\bar p} T^{\bar p}_{\bar j} + T^i_k T^k_\alpha T^\alpha_j + T^i_k T^k_{\bar\alpha} T^{\bar\alpha}_j + T^i_{\bar k} T^{\bar k}_p T^p_{\bar j} + T^i_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_{\bar j} + T^i_{\bar k} T^{\bar k}_\alpha T^\alpha_{\bar j} + T^i_{\bar k} T^{\bar k}_{\bar\alpha} T^{\bar\alpha}_{\bar j} \\
+T^i_\alpha T^\alpha_p T^p_{\bar j} + T^i_\alpha T^\alpha_{\bar p} T^{\bar p}_{\bar j} + T^i_\alpha T^\alpha_\beta T^\beta_{\bar j} + T^i_\alpha T^\alpha_{\bar\beta} T^{\bar\beta}_{\bar j} + T^i_{\bar\alpha} T^{\bar\alpha}_p T^p_{\bar j} + T^i_{\bar\alpha} T^{\bar\alpha}_{\bar p} T^{\bar p}_{\bar j} + T^i_{\bar\alpha} T^{\bar\alpha}_{\bar\beta} T^\beta_j + T^i_{\bar\alpha} T^{\bar\alpha}_{\bar\beta} T^{\bar\beta}_j + \lambda T^i_{\bar j} = 0, \\
T^i_k T^k_p T^p_\alpha + T^i_k T^k_{\bar p} T^{\bar p}_\alpha + T^i_k T^k_\beta T^\beta_\alpha + T^i_k T^k_{\bar\beta} T^{\bar\beta}_\alpha + T^i_{\bar k} T^{\bar k}_p T^p_\alpha + T^i_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_\alpha + T^i_{\bar k} T^{\bar k}_\beta T^\beta_\alpha + T^i_{\bar k} T^{\bar k}_{\bar\beta} T^{\bar\beta}_\alpha \\
+T^i_\beta T^\beta_p T^p_\alpha + T^i_\beta T^\beta_{\bar p} T^{\bar p}_\alpha + T^i_\gamma T^\gamma_\beta T^\beta_\alpha + T^i_\gamma T^\gamma_{\bar\beta} T^{\bar\beta}_\alpha + T^i_{\bar\beta} T^{\bar\beta}_p T^p_\alpha + T^i_{\bar\beta} T^{\bar\beta}_{\bar p} T^{\bar p}_\alpha + T^i_{\bar\gamma} T^{\bar\gamma}_\beta T^\beta_\alpha + T^i_{\bar\gamma} T^{\bar\gamma}_{\bar\beta} T^{\bar\beta}_\alpha + \lambda T^i_\alpha = 0, \\
T^i_k T^k_p T^p_{\bar\alpha} + T^i_k T^k_{\bar p} T^{\bar p}_{\bar\alpha} + T^i_k T^k_\beta T^\beta_{\bar\alpha} + T^i_k T^k_{\bar\beta} T^{\bar\beta}_{\bar\alpha} + T^i_{\bar k} T^{\bar k}_p T^p_{\bar\alpha} + T^i_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_{\bar\alpha} + T^i_{\bar k} T^{\bar k}_\beta T^\beta_{\bar\alpha} + T^i_{\bar k} T^{\bar k}_{\bar\beta} T^{\bar\beta}_{\bar\alpha} \\
+T^i_\beta T^\beta_p T^p_{\bar\alpha} + T^i_\beta T^\beta_{\bar p} T^{\bar p}_{\bar\alpha} + T^i_\gamma T^\gamma_\beta T^\beta_{\bar\alpha} + T^i_\gamma T^\gamma_{\bar\beta} T^{\bar\beta}_{\bar\alpha} + T^i_{\bar\beta} T^{\bar\beta}_p T^p_{\bar\alpha} + T^i_{\bar\beta} T^{\bar\beta}_{\bar p} T^{\bar p}_{\bar\alpha} + T^i_{\bar\gamma} T^{\bar\gamma}_\beta T^\beta_{\bar\alpha} + T^i_{\bar\gamma} T^{\bar\gamma}_{\bar\beta} T^{\bar\beta}_{\bar\alpha} + \lambda T^i_{\bar\alpha} = 0, \\
T^{\bar i}_k T^k_p T^p_j + T^{\bar i}_k T^k_{\bar p} T^{\bar p}_j + T^{\bar i}_k T^k_\alpha T^\alpha_j + T^{\bar i}_k T^k_{\bar\alpha} T^{\bar\alpha}_j + T^{\bar i}_{\bar k} T^{\bar k}_p T^p_j + T^{\bar i}_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_j + T^{\bar i}_{\bar k} T^{\bar k}_\alpha T^\alpha_j + T^{\bar i}_{\bar k} T^{\bar k}_{\bar\alpha} T^{\bar\alpha}_j \\
+T^{\bar i}_\alpha T^\alpha_p T^p_j + T^{\bar i}_\alpha T^\alpha_{\bar p} T^{\bar p}_j + T^{\bar i}_\alpha T^\alpha_\beta T^\beta_j + T^{\bar i}_\alpha T^\alpha_{\bar\beta} T^{\bar\beta}_j + T^{\bar i}_{\bar\alpha} T^{\bar\alpha}_p T^p_j + T^{\bar i}_{\bar\alpha} T^{\bar\alpha}_{\bar p} T^{\bar p}_j + T^{\bar i}_{\bar\alpha} T^{\bar\alpha}_\beta T^\beta_j + T^{\bar i}_{\bar\alpha} T^{\bar\alpha}_{\bar\beta} T^{\bar\beta}_j + \lambda T^{\bar i}_j = 0, \\
T^{\bar i}_k T^k_p T^p_{\bar j} + T^{\bar i}_k T^k_{\bar p} T^{\bar p}_{\bar j} + T^{\bar i}_k T^k_\alpha T^\alpha_{\bar j} + T^{\bar i}_k T^k_{\bar\alpha} T^{\bar\alpha}_{\bar j} + T^{\bar i}_{\bar k} T^{\bar k}_p T^p_{\bar j} + T^{\bar i}_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_{\bar j} + T^{\bar i}_{\bar k} T^{\bar k}_\alpha T^\alpha_{\bar j} + T^{\bar i}_{\bar k} T^{\bar k}_{\bar\alpha} T^{\bar\alpha}_{\bar j} \\
+T^{\bar i}_\alpha T^\alpha_p T^p_{\bar j} + T^{\bar i}_\alpha T^\alpha_{\bar p} T^{\bar p}_{\bar j} + T^{\bar i}_\alpha T^\alpha_\beta T^\beta_{\bar j} + T^{\bar i}_\alpha T^\alpha_{\bar\beta} T^{\bar\beta}_{\bar j} + T^{\bar i}_{\bar\alpha} T^{\bar\alpha}_p T^p_{\bar j} + T^{\bar i}_{\bar\alpha} T^{\bar\alpha}_{\bar p} T^{\bar p}_{\bar j} + T^{\bar i}_{\bar\alpha} T^{\bar\alpha}_\beta T^\beta_{\bar j} + T^{\bar i}_{\bar\alpha} T^{\bar\alpha}_{\bar\beta} T_{\bar j \bar\beta} + \lambda T^{\bar i}_{\bar j} = 0, \\
T^{\bar i}_k T^k_p T^p_\alpha + T^{\bar i}_k T^k_{\bar p} T^{\bar p}_\alpha + T^{\bar i}_k T^k_\beta T^\beta_\alpha + T^{\bar i}_k T^k_{\bar\beta} T^{\bar\beta}_\alpha + T^{\bar i}_{\bar k} T^{\bar k}_p T^p_\alpha + T^{\bar i}_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_\alpha + T^{\bar i}_{\bar k} T^{\bar k}_\beta T^\beta_\alpha + T^{\bar i}_{\bar k} T^{\bar k}_{\bar\beta} T^{\bar\beta}_\alpha \\
+T^{\bar i}_\beta T^\beta_p T^p_\alpha + T^{\bar i}_\beta T^\beta_{\bar p} T^{\bar p}_\alpha + T^{\bar i}_\gamma T^\gamma_\beta T^\beta_\alpha + T^{\bar i}_\gamma T^\gamma_{\bar\beta} T^{\bar\beta}_\alpha + T^{\bar i}_{\bar\beta} T^{\bar\beta}_{\bar p} T^p_\alpha + T^{\bar i}_{\bar\gamma} T^{\bar\gamma}_\beta T^\beta_\alpha + T^{\bar i}_{\bar\gamma} T^{\bar\gamma}_{\bar\beta} T^{\bar\beta}_\alpha + \lambda T^{\bar i}_\alpha = 0, \\
T^{\bar i}_k T^k_p T^p_{\bar\alpha} + T^{\bar i}_k T^k_{\bar p} T^{\bar p}_{\bar\alpha} + T^{\bar i}_k T^k_\beta T^\beta_{\bar\alpha} + T^{\bar i}_k T^k_{\bar\beta} T^{\bar\beta}_{\bar\alpha} + T^{\bar i}_{\bar k} T^{\bar k}_p T^p_{\bar\alpha} + T^{\bar i}_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_{\bar\alpha} + T^{\bar i}_{\bar k} T^{\bar k}_\beta T^\beta_{\bar\alpha} + T^{\bar i}_{\bar k} T^{\bar k}_{\bar\beta} T^{\bar\beta}_{\bar\alpha} + T^{\bar i}_\beta T^\beta_p T^p_{\bar\alpha} \\
+T^{\bar i}_\beta T^\beta_{\bar p} T^{\bar p}_{\bar\alpha} + T^{\bar i}_\gamma T^\gamma_\beta T^\beta_{\bar\alpha} + T^{\bar i}_\gamma T^\gamma_{\bar\beta} T^{\bar\beta}_{\bar\alpha} + T^{\bar i}_{\bar\beta} T^{\bar\beta}_p T^p_{\bar\alpha} + T^{\bar i}_{\bar\beta} T^\beta_{\bar p} T^{\bar p}_{\bar\alpha} + T^{\bar i}_{\bar\gamma} T^{\bar\gamma}_\beta T^\beta_{\bar\alpha} + T^{\bar i}_{\bar\gamma} T^{\bar\gamma}_{\bar\beta} T^{\bar\beta}_{\bar\alpha} + \lambda T^{\bar i}_{\bar\alpha} = 0, \\
T^\alpha_k T^k_p T^p_j + T^\alpha_k T^k_{\bar p} T^{\bar p}_j + T^\alpha_k T^k_\beta T^\beta_j + T^\alpha_k T^k_{\bar\beta} T^{\bar\beta}_j + T^\alpha_{\bar k} T^{\bar k}_p T^p_j + T^\alpha_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_j + T^\alpha_{\bar k} T^{\bar k}_\beta T^\beta_j + T^\alpha_{\bar k} T^{\bar k}_{\bar\beta} T^{\bar\beta}_j \\
+T^\alpha_\beta T^\beta_p T^p_j + T^\alpha_\beta T^\beta_{\bar p} T^{\bar p}_j + T^\alpha_\gamma T^\gamma_\beta T^\beta_j + T^\alpha_\gamma T^\gamma_{\bar\beta} T^{\bar\beta}_j + T^\alpha_{\bar\gamma} T^{\bar\gamma}_p T^p_j + T^\alpha_{\bar\gamma} T^{\bar\gamma}_{\bar p} T^{\bar p}_j + T^\alpha_{\bar\gamma} T^{\bar\gamma}_\beta T^\beta_j + T^\alpha_{\bar\gamma} T^{\bar\gamma}_{\bar\beta} T^{\bar\beta}_j + \lambda T^\alpha_j = 0, \\
T^\alpha_k T^k_p T^p_{\bar j} + T^\alpha_k T^k_{\bar p} T^{\bar p}_{\bar j} + T^\alpha_k T^k_\beta T^\beta_{\bar j} + T^\alpha_k T^k_{\bar\beta} T^{\bar\beta}_{\bar j} + T^\alpha_{\bar k} T^{\bar k}_p T^p_{\bar j} + T^\alpha_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_{\bar j} + T^\alpha_{\bar k} T^{\bar k}_\beta T^\beta_{\bar j} + T^\alpha_{\bar k} T^{\bar k}_{\bar\beta} T^{\bar\beta}_{\bar j} \\
+T^\alpha_\beta T^\beta_p T^p_{\bar j} + T^\alpha_\beta T^\beta_{\bar p} T^{\bar p}_{\bar j} + T^\alpha_\gamma T^\gamma_\beta T^\beta_{\bar j} + T^\alpha_\gamma T^\gamma_{\bar\beta} T^{\bar\beta}_{\bar j} + T^\alpha_{\bar\gamma} T^{\bar\gamma}_p T^p_{\bar j} + T^\alpha_{\bar\gamma} T^{\bar\gamma}_{\bar p} T^{\bar p}_{\bar j} + T^\alpha_{\bar\gamma} T^{\bar\gamma}_\beta T^\beta_{\bar j} + T^\alpha_{\bar\gamma} T^{\bar\gamma}_{\bar\beta} T^{\bar\beta}_{\bar j} + \lambda T^\alpha_{\bar j} = 0, \\
T^\alpha_k T^k_p T^p_\gamma + T^\alpha_k T^k_{\bar p} T^{\bar p}_\gamma + T^\alpha_k T^k_\beta T^\beta_\gamma + T^\alpha_k T^k_{\bar\beta} T^{\bar\beta}_\gamma + T^\alpha_{\bar k} T^{\bar k}_p T^p_\gamma + T^\alpha_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_\gamma + T^\alpha_{\bar k} T^{\bar k}_\beta T^\beta_\gamma + T^\alpha_{\bar k} T^{\bar k}_{\bar\beta} T^{\bar\beta}_\gamma \\
+T^\alpha_\beta T^\beta_p T^p_\gamma + T^\alpha_\beta T^\beta_{\bar p} T^{\bar p}_\gamma + T^\alpha_\delta T^\delta_\beta T^\beta_\gamma + T^\alpha_\delta T^\delta_{\bar\beta} T^{\bar\beta}_\gamma + T^\alpha_{\bar\delta} T^{\bar\delta}_p T^p_\gamma + T^\alpha_{\bar\delta} T^{\bar\gamma}_{\bar p} T^{\bar p}_\gamma + T^\alpha_{\bar\delta} T^{\bar\delta}_\beta T^\beta_\gamma + T^\alpha_{\bar\delta} T^{\bar\delta}_{\bar\beta} T^{\bar\beta}_\gamma + \lambda T^\alpha_\gamma = 0, \\
T^\alpha_k T^k_p T^p_{\bar\gamma} + T^\alpha_k T^k_{\bar p} T^{\bar p}_{\bar\gamma} + T^\alpha_k T^k_\beta T^\beta_{\bar\gamma} + T^\alpha_k T^k_{\bar\beta} T^{\bar\beta}_{\bar\gamma} + T^\alpha_{\bar k} T^{\bar k}_p T^p_{\bar\gamma} + T^\alpha_{\bar k} T^{\bar k}_{\bar p} T^{\bar p}_{\bar\gamma} + T^\alpha_{\bar k} T^{\bar k}_\beta T^\beta_{\bar\gamma} + T^\alpha_{\bar k} T^{\bar k}_{\bar\beta} T^{\bar\beta}_{\bar\gamma}
\end{gathered} \tag{3}$$

$$+T^{\alpha}_{\beta}T^{\beta}_{p}T^{p}_{\overline{\gamma}}+T^{\alpha}_{\beta}T^{\beta}_{\overline{p}}T^{\overline{p}}_{\overline{\gamma}}+T^{\alpha}_{\delta}T^{\delta}_{\beta}T^{\beta}_{\overline{\gamma}}+T^{\alpha}_{\delta}T^{\delta}_{\overline{\beta}}T^{\overline{\beta}}_{\overline{\gamma}}+T^{\alpha}_{\overline{\delta}}T^{\overline{\delta}}_{p}T^{p}_{\overline{\gamma}}+T^{\alpha}_{\overline{\delta}}T^{\overline{\gamma}}_{\overline{p}}T^{\overline{p}}_{\overline{\gamma}}+T^{\alpha}_{\overline{\delta}}T^{\overline{\delta}}_{\beta}T^{\beta}_{\overline{\gamma}}+T^{\alpha}_{\overline{\delta}}T^{\overline{\delta}}_{\overline{\beta}}T^{\overline{\beta}}_{\overline{\gamma}}+\lambda T^{\alpha}_{\overline{\gamma}}=0,$$

$$T^{\overline{\alpha}}_{k}T^{k}_{p}T^{p}_{j}+T^{\overline{\alpha}}_{k}T^{k}_{\overline{p}}T^{\overline{p}}_{j}+T^{\overline{\alpha}}_{k}T^{k}_{\beta}T^{\beta}_{j}+T^{\overline{\alpha}}_{k}T^{k}_{\overline{\beta}}T^{\overline{\beta}}_{j}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{p}T^{p}_{j}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\overline{p}}T^{\overline{p}}_{j}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\beta}T^{\beta}_{j}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\overline{\beta}}T^{\overline{\beta}}_{j}$$

$$+T^{\overline{\alpha}}_{\beta}T^{\beta}_{p}T^{p}_{j}+T^{\overline{\alpha}}_{\beta}T^{\beta}_{\overline{p}}T^{\overline{p}}_{j}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\beta}T^{\beta}_{j}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\overline{\beta}}T^{\overline{\beta}}_{j}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{p}T^{p}_{j}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{p}}T^{\overline{p}}_{j}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\beta}T^{\beta}_{j}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{\beta}}T^{\beta}_{j}+\lambda T^{\overline{\alpha}}_{j}=0,$$

$$T^{\overline{\alpha}}_{k}T^{k}_{p}T^{p}_{\overline{j}}+T^{\overline{\alpha}}_{k}T^{k}_{\overline{p}}T^{\overline{p}}_{\overline{j}}+T^{\overline{\alpha}}_{k}T^{k}_{\beta}T^{\beta}_{\overline{j}}+T^{\overline{\alpha}}_{k}T^{k}_{\overline{\beta}}T^{\overline{\beta}}_{\overline{j}}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{p}T^{p}_{\overline{j}}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\overline{p}}T^{\overline{p}}_{\overline{j}}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\beta}T^{\beta}_{\overline{j}}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\overline{\beta}}T^{\overline{\beta}}_{\overline{j}}$$

$$+T^{\overline{\alpha}}_{\beta}T^{\beta}_{p}T^{p}_{\overline{j}}+T^{\overline{\alpha}}_{\beta}T^{\beta}_{\overline{p}}T^{\overline{p}}_{\overline{j}}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\beta}T^{\beta}_{\overline{j}}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\overline{\beta}}T^{\overline{\beta}}_{\overline{j}}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{p}T^{p}_{\overline{j}}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{p}}T^{\overline{p}}_{\overline{j}}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\beta}T^{\beta}_{\overline{j}}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{\beta}}T^{\beta}_{\overline{j}}+\lambda T^{\overline{\alpha}}_{\overline{j}}=0,$$

$$T^{\overline{\alpha}}_{k}T^{k}_{p}T^{p}_{\beta}+T^{\overline{\alpha}}_{k}T^{k}_{\overline{p}}T^{\overline{p}}_{\beta}+T^{\overline{\alpha}}_{k}T^{k}_{\gamma}T^{\gamma}_{\beta}+T^{\overline{\alpha}}_{k}T^{k}_{\overline{\gamma}}T^{\overline{\gamma}}_{\beta}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{p}T^{p}_{\beta}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\overline{p}}T^{\overline{p}}_{\beta}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\gamma}T^{\gamma}_{\beta}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\beta}$$

$$+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{p}T^{p}_{\beta}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\overline{p}}T^{\overline{p}}_{\beta}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\delta}T^{\delta}_{\beta}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\overline{\delta}}T^{\overline{\delta}}_{\beta}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{p}T^{p}_{\beta}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{p}}T^{\overline{p}}_{\beta}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\delta}T^{\delta}_{\beta}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{\delta}}T^{\overline{\delta}}_{\beta}+\lambda T^{\overline{\alpha}}_{\beta}=0,$$

$$T^{\overline{\alpha}}_{k}T^{k}_{p}T^{p}_{\overline{\beta}}+T^{\overline{\alpha}}_{k}T^{k}_{\overline{p}}T^{\overline{p}}_{\overline{\beta}}+T^{\overline{\alpha}}_{k}T^{k}_{\gamma}T^{\gamma}_{\overline{\beta}}+T^{\overline{\alpha}}_{k}T^{k}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{\beta}}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{p}T^{p}_{\overline{\beta}}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\overline{p}}T^{\overline{p}}_{\overline{\beta}}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\gamma}T^{\gamma}_{\overline{\beta}}+T^{\overline{\alpha}}_{\overline{k}}T^{\overline{k}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{\beta}}$$

$$+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{p}T^{p}_{\overline{\beta}}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\overline{p}}T^{\overline{p}}_{\overline{\beta}}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\delta}T^{\delta}_{\overline{\beta}}+T^{\overline{\alpha}}_{\gamma}T^{\gamma}_{\overline{\delta}}T^{\overline{\delta}}_{\overline{\beta}}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{p}T^{p}_{\overline{\beta}}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{p}}T^{\overline{p}}_{\overline{\beta}}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\delta}T^{\delta}_{\overline{\beta}}+T^{\overline{\alpha}}_{\overline{\gamma}}T^{\overline{\gamma}}_{\overline{\delta}}T^{\overline{\delta}}_{\overline{\beta}}+\lambda T^{\overline{\alpha}}_{\overline{\beta}}=0.$$

Consider the case in which

$$a^{i}_{\alpha}=a^{i}_{\overline{j}}=a^{i}_{\overline{\alpha}}=a^{\alpha}_{i}=a^{\alpha}_{\overline{\beta}}=a^{\overline{i}}_{j}=a^{\overline{i}}_{\alpha}=a^{\overline{\alpha}}_{i}=a^{\overline{\alpha}}_{\beta}=a^{\overline{\alpha}}_{\overline{i}}=0,$$
$$a^{i}_{j}=a\delta^{i}_{j},\quad a^{\alpha}_{\beta}=c\delta^{\alpha}_{\beta},\quad a^{\overline{i}}_{\overline{j}}=b\delta^{i}_{j},\quad a^{\overline{\alpha}}_{\overline{\beta}}=a\delta^{\alpha}_{\beta}.$$

In this case, the tensors a^{i}_{j}, a^{α}_{β} can be associated with the tensor T^{A}_{B} as follows:

$$T^{i}_{j}=a\delta^{i}_{j},\quad T^{i}_{\alpha}=0,\quad T^{i}_{\overline{j}}=0,\quad T^{\alpha}_{i}=(c-a)\Gamma^{\alpha}_{i},\quad T^{\alpha}_{\beta}=c\delta^{\alpha}_{\beta},\quad T^{\alpha}_{\overline{i}}=0,\quad T^{\alpha}_{\overline{\beta}}=0,$$
$$T^{\overline{i}}_{j}=(b-a)\Gamma^{i}_{j},\quad T^{\overline{i}}_{\alpha}=0,\quad T^{\overline{i}}_{\overline{j}}=0,\quad T^{\overline{\alpha}}_{\beta}=(d-c)\Gamma^{\alpha}_{k},$$
$$T^{\overline{\alpha}}_{k}=-aC^{\alpha}_{k}-c\Gamma^{\gamma}_{k}\Gamma^{\alpha}_{\gamma}-b\Gamma^{i}_{k}\Gamma^{\alpha}_{i}+dL^{\alpha}_{k},\quad T^{\overline{\alpha}}_{\overline{\beta}}=d\delta^{\alpha}_{\beta}.$$

Equalities (3) imply that

$$(b-a)(a^{2}+ab+b^{2}+\lambda)=0,\quad a(a^{2}+\lambda)=0,$$
$$(c-a)(a^{2}+ac+c^{2}+\lambda)=0,\quad c(c^{2}+\lambda)=0,$$
$$(d-c)(c^{2}+cd+d^{2}+\lambda)=0,\quad d(d^{2}+\lambda)=0,$$
$$(d-b)(b^{2}+bd+d^{2}+\lambda)=0,\quad b(b^{2}+\lambda)=0,$$
$$(a^{2}+ad+d^{2}+\lambda)(-aC^{\alpha}_{i}-c\Gamma^{\lambda}_{i}\Gamma^{\alpha}_{\beta}-b\Gamma^{k}_{i}\Gamma^{\alpha}_{k}+dL^{\alpha}_{i})$$
$$+(a+b+d)(b-a)(d-b)\Gamma^{\alpha}_{k}\Gamma^{k}_{i}+(a+c+d)(d-c)(c-a)\Gamma^{\alpha}_{\gamma}\Gamma^{\gamma}_{i}.$$

It follows that

$$a^{2}+\lambda=0,\quad c^{2}+\lambda=0\quad d^{2}+\lambda=0,\quad b^{2}+\lambda=0.$$

Similar results are obtained for other cases. Thus, we have proved the following theorem:

Theorem. *In the tangent $T(Lm(Vn))$ space a real-valued F structures exist only for $\lambda=-1$.*

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ON THE GENERALIZED NONMEASURABILITY OF SOME CLASSICAL POINT SETS

ALEXANDER KHARAZISHVILI

Abstract. The generalized nonmeasurability of certain classical point sets (such as Vitali sets, Bernstein sets, and Hamel bases) is considered in connection with **CH** and **MA**.

This short note is a continuation of our paper [7]. It was shown in [7] that the nonmeasurability in Ulam's sense (i.e., the non-real-valued measurability) of the cardinality continuum is equivalent to some generalized nonmeasurability of Vitali subsets and Bernstein subsets of the real line $\mathbf{R}$. Here it is demonstrated that, assuming the Continuum Hypothesis (**CH**), it becomes possible to essentially strengthen the result obtained in [7], concerning the generalized nonmeasurability of Vitali sets and Bernstein sets.

According to the classical theorem of Erdös and Kakutani [3], the Continuum Hypothesis is equivalent to the following assertion:

There exists a countable family $\{H_i : i \in I\}$ of Hamel bases of $\mathbf{R}$ such that

$$\cup\{H_i : i \in I\} = \mathbf{R} \setminus \{0\}.$$

Starting with this result and using the Banach-Kuratowski matrix [1] (or Ulam's $(\omega\times\omega_1)$-matrix [12] or a countable base of a Luzin subspace of $\mathbf{R}$), one can prove the following statement.

Theorem 1. *Under* **CH**, *there exists a countable family $\{H_j : j \in J\}$ of Hamel bases of $\mathbf{R}$ such that, for every nonzero σ-finite diffused measure μ on $\mathbf{R}$, at least one member of $\{H_j : j \in J\}$ is nonmeasurable with respect to μ.*

In fact, the existence of $\{H_j : j \in J\}$ with the above property implies **CH** (cf., [4], where an analogous result in terms of nonzero σ-finite translation invariant measures on $\mathbf{R}$ is formulated and proved).

It makes sense to examine analogues of Theorem 1 for some other classical point sets. First of all, we mean here the Vitali subsets and Bernstein subsets of $\mathbf{R}$.

Recall that a Vitali set in $\mathbf{R}$ is any selector of the quotient group $\mathbf{R}/\mathbf{Q}$, where $\mathbf{Q}$ denotes the rational subgroup of the additive group $(\mathbf{R}, +)$.

Recall also that a Bernstein set in $\mathbf{R}$ is any set $B \subset \mathbf{R}$ which has the property that, for every nonempty perfect set $P \subset \mathbf{R}$, the relations

$$P \cap B \neq \emptyset,\ P \cap (\mathbf{R} \setminus B) \neq \emptyset$$

hold true.

In many works, the Vitali sets and Bernstein sets are discussed from the measure-theoretical and topological viewpoints (see, e.g., [2,5,6,8–11,13]). Usually, these sets are treated as pathological ones.

In particular, it is well known within **ZFC** set theory that:

(a) if μ is a measure on $\mathbf{R}$ extending the standard Lebesgue measure and invariant under all rational translations of $\mathbf{R}$, then no Vitali set is measurable with respect to μ (cf., [13]);

(b) if μ is the completion of a nonzero σ-finite diffused Borel measure on $\mathbf{R}$, then no Bernstein set is measurable with respect to μ.

Notice that μ in (a) carries some algebraic structure and μ in (b) carries some topological structure. At the same time, suppose that ν is an arbitrary σ-finite measure on $\mathbf{R}$ without any additional structure, and let $\{V_k : k \in K\}$ (respectively, $\{B_k : k \in K\}$) be a finite family of Vitali sets

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(respectively, Bernstein sets). Then there exists a measure ν' on $\mathbf{R}$ extending ν and such that all sets V_k (respectively, all sets B_k) become ν'-measurable. Actually, the same fact remains valid for ν and for an arbitrary finite family $\{Z_k : k \in K\}$ of subsets of $\mathbf{R}$ (see, for example, [5]).

Remark 1. There exists a Vitali set which is measurable with respect to some translation quasi-invariant extension of the Lebesgue measure on $\mathbf{R}$ (see [5, 6]).

Remark 2. If μ is a nonzero σ-finite diffused measure on $\mathbf{R}$ containing in its domain some Bernstein set, then μ cannot be a Radon measure.

For the class $\mathcal{M}(\mathbf{R})$ of all nonzero σ-finite diffused measures on $\mathbf{R}$, we have the next two results (similar to Theorem 1) which show us the generalized nonmeasurability of Vitali sets and Bernstein sets with respect to $\mathcal{M}(\mathbf{R})$.

Theorem 2. *Under* **CH**, *there exists a countable family $\{V_j : j \in J\}$ of Vitali subsets of $\mathbf{R}$ such that, for every nonzero σ-finite diffused measure μ on $\mathbf{R}$, at least one member of $\{V_j : j \in J\}$ is nonmeasurable with respect to μ.*

Theorem 3. *Under* **CH**, *there exists a countable family $\{B_j : j \in J\}$ of Bernstein subsets of $\mathbf{R}$ such that, for every nonzero σ-finite diffused measure μ on $\mathbf{R}$, at least one member of $\{B_j : j \in J\}$ is nonmeasurable with respect to μ.*

Both proofs of Theorems 2 and 3 are based on the following auxiliary statement.

Lemma 1. *Let $\{X_i : i \in I\}$ be a partition of a ground set E such that*

$$(\forall i \in I)(2 \leq \operatorname{card}(X_i) \leq \omega),$$

where ω denotes the least infinite cardinal number.

Then the union of any subfamily of $\{X_i : i \in I\}$ belongs to the σ-algebra generated by a countable family of selectors of $\{X_i : i \in I\}$.

Also, the following auxiliary statement is used in the proof of Theorem 3.

Lemma 2. *There exists a partition $\{Y_t : t \in T\}$ of $\mathbf{R}$ such that:*

(1) $2 \leq \operatorname{card}(Y_t) \leq \omega$ *for each index $t \in T$;*

(2) *all selectors of $\{Y_t : t \in T\}$ are Bernstein subsets of $\mathbf{R}$.*

Remark 3. The assertions of Theorems 2 and 3 can also be established under Martin's Axiom (**MA**). As widely known, **MA** is much weaker than the Continuum Hypothesis, because the conjunction **MA** & $\neg$**CH** is consistent with **ZFC** set theory. The proofs of the modified versions of Theorems 2 and 3 are based on Lemmas 1 and 2 and on some properties of so-called generalized Luzin subsets of $\mathbf{R}$. As indicated after Theorem 1, **CH** is equivalent to the existence of a countable family $\{H_j : j \in J\}$ of Hamel bases of $\mathbf{R}$ such that, for every nonzero σ-finite diffused measure μ on $\mathbf{R}$, at least one member of $\{H_j : j \in J\}$ is nonmeasurable with respect to μ. We thus see that the case of Hamel bases of $\mathbf{R}$ essentially differs from the cases of Vitali and Bernstein sets in $\mathbf{R}$.

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