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A NOTE ON THE SUMMATION THEOREM FOR ${}_4F_3[-m,\alpha,\lambda+1,\mu+2;\beta,\lambda,\mu;1]$

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Abstract. This article aims to obtain a summation theorem for ${}_4F_3[-m,\alpha,\lambda+1,\mu+2;\beta,\lambda,\mu;1]$. Further, a general series identity is derived. Applications of the results in terms of interesting Kummer's type transformation formulas are given. Some numerical examples are also discussed.

1. Introduction and Preliminaries

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$
{}_pF_q\begin{bmatrix}(\alpha_p); & \\ & z\\ (\beta_q); & \end{bmatrix} := {}_pF_q\begin{bmatrix}\alpha_1,\alpha_2,\ldots,\alpha_p; & \\ & z\\ \beta_1,\beta_2,\ldots,\beta_q; & \end{bmatrix} = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!} \tag{1.1}
$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here, p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$
\beta_j \neq 0,-1,-2,\ldots;\quad j=1,2,\ldots,q.
$$

In contracted notation, the sequence of p numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ is denoted by (α_p) with a similar interpretation for others throughout this paper.

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z|<\infty$, if $p\le q$,
(ii) converges for $|z|<1$, if $p=q+1$,
(iii) diverges for all z, $z\neq 0$, if $p>q+1$.

Chu–Vandermonde theorem [7, p. 69, Q.No. 4]:

$$
{}_2F_1\begin{bmatrix}-M,\ A & ; & \\ & & 1\\ B & ; & \end{bmatrix} = \frac{(B-A)_M}{(B)_M};\quad M=0,1,2,\ldots, \tag{1.2}
$$

such that the ratio of Pochhammer symbols in r.h.s. is well defined and $A,\,B\in\mathbb{C}\setminus\mathbb{Z}_0^-$.

Just as the Gaussian ${}_2F_1$ function was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the four Appell functions were unified and generalized by Kampé de Fériet [1,4] who defined a general hypergeometric function of two variables.

We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation [11, p. 423, Eq. (26)]:

$$
F^{p:\,q;\,k}_{\ell:\,m;\,n}\begin{bmatrix}(a_p)\ :\ (b_q)\ ;\ (c_k)\ ; & \\ & x,\ y\\ (\alpha_\ell)\ :\ (\beta_m)\ ;\ (\gamma_n)\ ; & \end{bmatrix} = \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\,\frac{x^r}{r!}\,\frac{y^s}{s!},
$$

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where for the convergence,

(i) $p+q<\ell+m+1, \quad p+k<\ell+n+1, \quad |x|<\infty, \quad |y|<\infty$, or

(ii) $p+q=\ell+m+1, \quad p+k=\ell+n+1$ and

$$\begin{cases} |x|^{1/(p-\ell)}+|y|^{1/(p-\ell)}<1, & \text{if } p>\ell \\ \max\{|x|,|y|\}<1, & \text{if } p\leq\ell. \end{cases}$$

Lemma 1.1 ([10, p.100])**.**

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Omega(m,n)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\Omega(m-n,n),$$

provided that the multiple series involved are absolutely convergent.

It is well known that whenever a generalized hypergeometric function reduces to quotients of the products of the gamma functions, the results are very important from the point of view of applications in numerous areas of physical, mathematical and statistical sciences including (for example) in series systems of symbolic computer algebra manipulation [9].

An important development has been made by various authors in generalizations of the summation and transformation theorems (see [5, 6, 8, 12]). Very recently, several remarkable transformation theorems for the q-series have been proved by W. Chu in [2]. Further, by making use of divided differences, new proofs have been presented in [3] for Dougall's summation theorem for well-poised ${}_7F_6$-series and Whipple's transformation between well-poised ${}_7F_6$-series and balanced ${}_4F_3$-series.

In this work, our main motive is to find the summation theorem for ${}_4F_3[-m,\alpha,\lambda+1,\mu+2;\beta,\lambda,\mu;1]$ and to obtain some applications.

2. Summation Theorem

Theorem 2.1. *If ρ, δ, σ are the non-vanishing zeros of the cubic polynomial Cm^3+Dm^2+Em+G and α, β, λ, μ, $-\rho$, $-\delta$, $-\sigma\in\mathbb{C}\setminus\mathbb{Z}_0^-$; $m\in\mathbb{N}_0$, then the following summation theorem holds true:*

$${}_4F_3\left[\begin{matrix}-m,\ \alpha,\ \lambda+1,\ \mu+2 & ; \\ & 1 \\ \beta,\ \lambda,\ \mu & ;\end{matrix}\right]=\frac{(-\rho+1)_m\ (-\delta+1)_m\ (-\sigma+1)_m\ (\beta-\alpha-3)_m}{(-\rho)_m\ (-\delta)_m\ (-\sigma)_m\ (\beta)_m}, \tag{2.1}$$

where the coefficients C, D, E and G are the polynomials in α, β, λ, μ given as follows:

$$C=\alpha^2-\alpha^3-\alpha\lambda+\alpha^2\lambda-\alpha\mu+2\alpha^2\mu+\lambda\mu-2\alpha\lambda\mu-\alpha\mu^2+\lambda\mu^2, \tag{2.2}$$

$$\begin{aligned}D=&-7\alpha^2-\alpha^3+4\alpha^2\beta+6\alpha\lambda-\alpha^2\lambda-\alpha^3\lambda-3\alpha\beta\lambda+\alpha^2\beta\lambda+7\alpha\mu-4\alpha^2\mu-2\alpha^3\mu-4\alpha\beta\mu\\&+2\alpha^2\beta\mu-6\lambda\mu+7\alpha\lambda\mu+4\alpha^2\lambda\mu+3\beta\lambda\mu-4\alpha\beta\lambda\mu+5\alpha\mu^2+2\alpha^2\mu^2-2\alpha\beta\mu^2-6\lambda\mu^2\\&-3\alpha\lambda\mu^2+3\beta\lambda\mu^2,\end{aligned} \tag{2.3}$$

$$\begin{aligned}E=&-4\alpha+6\alpha\beta-2\alpha\beta^2-9\alpha\lambda-6\alpha^2\lambda-\alpha^3\lambda+9\alpha\beta\lambda+3\alpha^2\beta\lambda-2\alpha\beta^2\lambda-12\alpha\mu\\&-7\alpha^2\mu-\alpha^3\mu+13\alpha\beta\mu+4\alpha^2\beta\mu-3\alpha\beta^2\mu+11\lambda\mu-7\alpha^2\lambda\mu-2\alpha^3\lambda\mu-12\beta\lambda\mu\\&+4\alpha\beta\lambda\mu+4\alpha^2\beta\lambda\mu+3\beta^2\lambda\mu-2\alpha\beta^2\lambda\mu-6\alpha\mu^2-5\alpha^2\mu^2-\alpha^3\mu^2+5\alpha\beta\mu^2\\&+2\alpha^2\beta\mu^2-\alpha\beta^2\mu^2+11\lambda\mu^2+12\alpha\lambda\mu^2+3\alpha^2\lambda\mu^2-12\beta\lambda\mu^2-6\alpha\beta\lambda\mu^2+3\beta^2\lambda\mu^2,\end{aligned} \tag{2.4}$$

$$\begin{aligned}G=&-6\lambda\mu-11\alpha\lambda\mu-6\alpha^2\lambda\mu-\alpha^3\lambda\mu+11\beta\lambda\mu+12\alpha\beta\lambda\mu+3\alpha^2\beta\lambda\mu-6\beta^2\lambda\mu-3\alpha\beta^2\lambda\mu\\&+\beta^3\lambda\mu-6\lambda\mu^2-11\alpha\lambda\mu^2-6\alpha^2\lambda\mu^2-\alpha^3\lambda\mu^2+11\beta\lambda\mu^2+12\alpha\beta\lambda\mu^2+3\alpha^2\beta\lambda\mu^2\\&-6\beta^2\lambda\mu^2-3\alpha\beta^2\lambda\mu^2+\beta^3\lambda\mu^2\\=&-C\rho\delta\sigma\\=&\lambda\ \mu\ (\mu+1)\ (\beta-\alpha-1)\ (\beta-\alpha-2)\ (\beta-\alpha-3).\end{aligned} \tag{2.5}$$

Proof. Suppose the l.h.s. of equation (2.1) is denoted by Δ, then we have

$$
\begin{aligned}
\Delta =&\sum_{r=0}^{m} \frac{(-m)_r\ (\alpha)_r\ (\lambda+1)_r\ (\mu+2)_r}{(\beta)_r\ (\lambda)_r\ (\mu)_r\ r!}\\
=&\sum_{r=0}^{m} \frac{(-m)_r\ (\alpha)_r}{(\beta)_r\ r!}\left[1+\frac{r(2\lambda+\mu+2)}{\lambda\mu}+\frac{r(r-1)(\lambda+2\mu+4)}{\lambda\mu(\mu+1)}+\frac{r(r-1)(r-2)}{\lambda\mu(\mu+1)}\right]\\
=&{}_2F_1\left[\begin{matrix}-m,\ \alpha & ; & \\ & & 1\\ \beta & ; & \end{matrix}\right]+\frac{(2\lambda+\mu+2)}{\lambda\mu}\sum_{r=0}^{m-1}\frac{(-m)_{r+1}\ (\alpha)_{r+1}}{(\beta)_{r+1}\ r!}\\
&+\frac{(\lambda+2\mu+4)}{\lambda\mu(\mu+1)}\sum_{r=0}^{m-2}\frac{(-m)_{r+2}\ (\alpha)_{r+2}}{(\beta)_{r+2}\ r!}+\frac{1}{\lambda\mu(\mu+1)}\sum_{r=0}^{m-3}\frac{(-m)_{r+3}\ (\alpha)_{r+3}}{(\beta)_{r+3}\ r!}\\
=&{}_2F_1\left[\begin{matrix}-m,\ \alpha & ; & \\ & & 1\\ \beta & ; & \end{matrix}\right]+\frac{(2\lambda+\mu+2)}{\lambda\mu}\frac{(-m)_1\ (\alpha)_1}{(\beta)_1}{}_2F_1\left[\begin{matrix}-(m-1),\ \alpha+1 & ; & \\ & & 1\\ \beta+1 & ; & \end{matrix}\right]\\
&+\frac{(\lambda+2\mu+4)}{\lambda\mu(\mu+1)}\frac{(-m)_2\ (\alpha)_2}{(\beta)_2}{}_2F_1\left[\begin{matrix}-(m-2),\ \alpha+2 & ; & \\ & & 1\\ \beta+2 & ; & \end{matrix}\right]\\
&+\frac{1}{\lambda\mu(\mu+1)}\frac{(-m)_3\ (\alpha)_3}{(\beta)_3}{}_2F_1\left[\begin{matrix}-(m-3),\ \alpha+3 & ; & \\ & & 1\\ \beta+3 & ; & \end{matrix}\right].
\end{aligned}
\tag{2.6}
$$

Using the Chu–Vandermonde theorem 1.2 in r.h.s. of equation (2.6), we obtain

$$
\begin{aligned}
\Delta =&\frac{(\beta-\alpha)_m}{(\beta)_m}+\frac{(2\lambda+\mu+2)}{\lambda\mu}\frac{(-m)_1\ (\alpha)_1}{(\beta)_1}\frac{(\beta-\alpha)_{m-1}}{(\beta+1)_{m-1}}\\
&+\frac{(\lambda+2\mu+4)}{\lambda\mu(\mu+1)}\frac{(-m)_2\ (\alpha)_2}{(\beta)_2}\frac{(\beta-\alpha)_{m-2}}{(\beta+2)_{m-2}}+\frac{1}{\lambda\mu(\mu+1)}\frac{(-m)_3\ (\alpha)_3}{(\beta)_3}\frac{(\beta-\alpha)_{m-3}}{(\beta+3)_{m-3}}\\
=&\frac{(\beta-\alpha)_m}{(\beta)_m}+\frac{(2\lambda+\mu+2)(-m)_1\ (\alpha)_1}{\lambda\mu}\frac{(\beta-\alpha)_{m-1}}{(\beta)_m}\\
&+\frac{(\lambda+2\mu+4)(-m)_2\ (\alpha)_2}{\lambda\mu(\mu+1)}\frac{(\beta-\alpha)_{m-2}}{(\beta)_m}+\frac{(-m)_3\ (\alpha)_3}{\lambda\ \mu\ (\mu+1)}\frac{(\beta-\alpha)_{m-3}}{(\beta)_m}\\
=&\frac{(\beta-\alpha)_m}{(\beta)_m}\left[1-\frac{(2\lambda+\mu+2)\ m\ \alpha}{\lambda\ \mu\ (\beta-\alpha+m-1)}+\frac{(\lambda+2\mu+4)(-m)_2\ (\alpha)_2}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-2)_2}\right.\\
&\left.+\frac{(-m)_3\ (\alpha)_3}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-3)_3}\right]\\
=&\frac{(\beta-\alpha)_m}{(\beta)_m}\left[\frac{\Psi(\alpha,\beta,\lambda,\mu,m)}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right],
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
\Psi(\alpha,\beta,\lambda,\mu,m) =&\lambda\ \mu\ (\mu+1)(\beta-\alpha+m-1)(\beta-\alpha+m-2)(\beta-\alpha+m-3)\\
&-m\alpha(2\lambda+\mu+2)(\mu+1)(\beta-\alpha+m-2)(\beta-\alpha+m-3)\\
&+(-m)(-m+1)(\lambda+2\mu+4)(\alpha)(\alpha+1)(\beta-\alpha+m-3)\\
&+(-m)(-m+1)(-m+2)(\alpha)(\alpha+1)(\alpha+2).
\end{aligned}
$$

Equation (2.7) can be written as

$$
\Delta = \frac{(\beta-\alpha)_m}{(\beta)_m}\left[\frac{Cm^3+Dm^2+Em+G}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right],
\tag{2.8}
$$

Since $\rho,\ \delta,\ \sigma$ are the roots of the cubic equation $Cm^3+Dm^2+Em+G=0$, therefore equation (2.8) can be written as follows:

$$\Delta=\frac{(\beta-\alpha)_m}{(\beta)_m}\left[\frac{C(m-\rho)(m-\delta)(m-\sigma)}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right].$$

On simplification, we get the assertion (2.1). □

3. General Double Series Identity

The application of the summation Theorem 2.1 is given by proving the following general double series identity:

Theorem 3.1. *Let* $\{\Phi(\ell)\}_{\ell=1}^{\infty}$ *be a bounded sequence of arbitrary complex numbers such that* $\Phi(0)\neq 0$ *and* $\alpha,\ \beta,\ \lambda,\ \mu,\ -\rho,\ -\delta,\ -\sigma\in\mathbb{C}\setminus\mathbb{Z}_0^-$. *Then*

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{\Phi(m+n)\ (\alpha)_n\ (\lambda+1)_n\ (\mu+2)_n\ (-1)^n}{(\beta)_n\ (\lambda)_n\ (\mu)_n}\frac{z^{m+n}}{m!\ n!}\\ &=\sum_{m=0}^{\infty}\frac{\Phi(m)\ (-\rho+1)_m\ (-\delta+1)_m\ (-\sigma+1)_m\ (\beta-\alpha-3)_m}{(-\rho)_m\ (-\delta)_m\ (-\sigma)_m\ (\beta)_m}\frac{z^m}{m!},\end{aligned}\tag{3.1}$$

where $\rho,\ \delta,\ \sigma$ *are the roots of the cubic equation* $Cm^3+Dm^2+Em+G=0$ *and* C,D,E,G *are given by equations* (2.2)–(2.5) *with each absolutely convergent multiple series involved.*

Proof. Suppose l.h.s. of equation (3.1) is denoted by Ξ. Then in view of Lemma 1.1, we have

$$\begin{aligned}\Xi&=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{\Phi(m)\ (\alpha)_n\ (\lambda+1)_n\ (\mu+2)_n\ (-1)^n}{(\beta)_n\ (\lambda)_n\ (\mu)_n}\frac{z^m}{(m-n)!\ n!}\\ &=\sum_{m=0}^{\infty}\Phi(m)\frac{z^m}{m!}\sum_{n=0}^{m}\frac{(-m)_n\ (\alpha)_n\ (\lambda+1)_n\ (\mu+2)_n}{(\beta)_n\ (\lambda)_n\ (\mu)_n\ n!}\\ &=\sum_{m=0}^{\infty}\Phi(m)\frac{z^m}{m!}\ {}_4F_3\left[\begin{matrix}-m,\ \alpha,\ \lambda+1,\ \mu+2\ ;\\ \\ \beta,\ \lambda,\ \mu\ ;\end{matrix}\ 1\right].\end{aligned}$$

Using Theorem 2.1 in r.h.s. of the above equation, relation (3.1) follows. □

4. Applications

If $\rho,\ \delta,\ \sigma$ are the roots of the cubic equation $Cm^3+Dm^2+Em+G=0$ and $\alpha,\ \beta,\ \lambda,\ \mu,\ -\rho,\ -\delta,\ -\sigma\in\mathbb{C}\setminus\mathbb{Z}_0^-$; $m\in\mathbb{N}_0$, then we have the following applications:

I. Taking $\Phi(p)=\frac{\prod_{i=1}^{A}(a_i)_p}{\prod_{i=1}^{B}(b_i)_p}$ in equation (3.1), we get the following reduction formula:

$$\begin{aligned}&F^{A:0;3}_{B:0;3}\left[\begin{matrix}(a_A)\ :\ -\ ;\ \alpha,\ \lambda+1,\ \mu+2\ ;\\ \\ (b_B)\ :\ -\ ;\ \beta,\ \lambda,\ \mu\ ;\end{matrix}\ z,\ -z\right]\\ &={}_{A+4}F_{B+4}\left[\begin{matrix}a_1,\dots,a_A,\ -\rho+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3\ ;\\ \\ b_1,\dots,b_B,\ -\rho,\ -\delta,\ -\sigma,\ \beta\ ;\end{matrix}\ z\right],\end{aligned}$$

subject to the convergence conditions:

$$\begin{cases}|z|<\frac{1}{2}, & \text{if } A=B+1\\ |z|<\infty, & \text{if } A\leq B.\end{cases}$$

II. Taking $A = 1$, $a_1 = d$, $B = 0$ in equation (3.1) and putting $z = \frac{-y}{1-y}$, we get the following Pfaff-Kummer-type transformation formula:

$$
{}_4F_3\left[\begin{matrix} d,\ \alpha,\ \lambda+1,\ \mu+2 & ; \\ & & y \\ \beta,\ \lambda,\ \mu & ; \end{matrix}\right]
$$

$$
=(1-y)^{-d}\ {}_5F_4\left[\begin{matrix} d,\ -\rho+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3 & ; \\ & & \frac{-y}{1-y} \\ -\rho,\ -\delta,\ -\sigma,\ \beta & ; \end{matrix}\right], \tag{4.1}
$$

where $|y| < 1$, $Re(y) < \frac{1}{2}$ and $\alpha,\ \beta,\ \lambda,\ \mu,\ -\rho,\ -\delta,\ -\sigma \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

III. Taking $A = B = 0$ in equation (3.1) and putting $z = -y$, we get the following Kummer's type first transformation formula:

$$
{}_3F_3\left[\begin{matrix} \alpha,\ \lambda+1,\ \mu+2 & ; \\ & & y \\ \beta,\ \lambda,\ \mu & ; \end{matrix}\right]
$$

$$
=\exp(y)\ {}_4F_4\left[\begin{matrix} -\rho+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3 & ; \\ & & -y \\ -\rho,\ -\delta,\ -\sigma,\ \beta & ; \end{matrix}\right], \tag{4.2}
$$

where $|y| < \infty$ and $\alpha,\ \beta,\ \lambda,\ \mu,\ -\rho,\ -\delta,\ -\sigma \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

In this section, we consider some numerical examples.

5. Numerical Examples

Taking $\alpha = 2$, $\beta = \frac{3}{2}$, $\lambda = \frac{5}{4}$, $\mu = \frac{7}{3}$ in equations (2.2)–(2.5), the numerical values of C, D, E and G are obtained as follows:

$$
C = -\frac{1}{3}, \quad D = \frac{14}{3}, \quad E = -\frac{1579}{24}, \quad G = -\frac{6125}{48}.
$$

The cubic polynomial equation $Cm^3 + Dm^2 + Em + G = 0$ becomes

$$
16m^3 - 224m^2 + 3158m + 6125 = 0. \tag{5.1}
$$

The roots ρ, δ and σ of equation (5.1) are obtained approximately as:

$$
\rho = -1.70749, \quad \delta = 7.85375 + i12.7481, \quad \sigma = 7.85375 - i12.7481.
$$

Now, substituting the values of α, β, λ, μ, ρ, δ and σ in equations (4.1) and (4.2), we get the following Kummer-type transformation formulas:

$$
{}_4F_3\left[\begin{matrix} d,\ 2,\ \frac{9}{4},\ \frac{13}{3} & ; \\ & & y \\ \frac{3}{2},\ \frac{5}{4},\ \frac{7}{3} & ; \end{matrix}\right]
$$

$$
=(1-y)^{-d}\ {}_5F_4\left[\begin{matrix} d,\ 2.70749,\ -6.85375-i12.7481,\ -6.85375+i12.7481,\ -\frac{7}{2} & ; \\ & & \frac{-y}{1-y} \\ 1.70749,\ -7.85375-i12.7481,\ -7.85375+i12.7481,\ \frac{3}{2} & ; \end{matrix}\right],
$$

where $|y| < 1$, $Re(y) < \frac{1}{2}$ and

$$
{}_3F_3\left[\begin{matrix} 2,\ \frac{9}{4},\ \frac{13}{3} & ; \\ & & y \\ \frac{3}{2},\ \frac{5}{4},\ \frac{7}{3} & ; \end{matrix}\right]
$$

$$
=\exp(y)\ {}_4F_4\left[\begin{matrix} 2.70749,\ -6.85375-i12.7481,\ -6.85375+i12.7481,\ -\frac{7}{2} & ; \\ & & -y \\ 1.70749,\ -7.85375-i12.7481,\ -7.85375+i12.7481,\ \frac{3}{2} & ; \end{matrix}\right],
$$

where $|y| < \infty$.

Several other examples can be obtained in a similar manner by considering different values of α, β, λ and μ. The extensions of the summation theorems to hypergeometric functions containing arbitrary number of pairs of numerator and denominator parameters will be taken as future aspect.

Appendix

The roots ρ, δ, σ of the cubic equation $Cm^3 + Dm^2 + Em + G = 0$ are calculated by using *Wolfram Mathematica 9.0* Software. The values of ρ, δ and σ are given as follows:

$$\begin{aligned}
\rho = &-\frac{D}{3C}\\
&-\frac{2^{1/3}\left(-D^2+3CE\right)}{3C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}\\
&+\frac{\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{3\times 2^{1/3}C}\\
\delta = &-\frac{D}{3C}\\
&+\frac{\left(1+i\sqrt{3}\right)\left(-D^2+3CE\right)}{3\times 2^{2/3}C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}\\
&-\frac{\left(1-i\sqrt{3}\right)\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{6\times 2^{1/3}C}\\
\sigma = &-\frac{D}{3C}\\
&+\frac{\left(1-i\sqrt{3}\right)\left(-D^2+3CE\right)}{3\times 2^{2/3}C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}\\
&-\frac{\left(1+i\sqrt{3}\right)\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{6\times 2^{1/3}C}.
\end{aligned}$$

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References

1. P. Appell, J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques: Polynômes d' Hermite*. Gauthier-Villars, Paris, 1926.
2. W. Chu, Infinite series formulae related to Gauss and Bailey ${}_2F_1(\frac{1}{2})$-sums. *Acta Math. Sci. Ser. B (Engl. Ed.)* **40** (2020), no. 2, 293–315.

3. W. Chu, Divided differences and terminating well-poised hypergeometric series. *Integral Transforms Spec. Funct.* **31** (2020), no. 3, 1–14.
4. J. Kampé, de Fériet, Les fonctions hypergéométriques d'ordre supérieur à deux variables. *C. R. Acad. Sci. Paris*, **173** (1921), 401–404.
5. A. R. Miller, R. B. Paris, Certain transformations and summations for generalized hypergeometric series with integral parameter differences. *Integral Transforms Spec. Funct.* **22** (2011), no. 1, 67–77.
6. A. R. Miller, R. B. Paris, Transformation formulas for the generalized hypergeometric function with integral parameter difference, *Rocky Mountain J. Math.*, **43** (2013), no. 1, 291–327.
7. E. D. Rainville, *Special Functions.* Reprint of 1960 first edition. Chelsea Publishig Co., Bronx, New York, 1971.
8. M. A. Rakha, A. K. Rathie, Extensions of Euler type II transformation and Saalschütz's theorem. *Bull. Korean Math. Soc.* **48** (2011), no. 1, 151–156.
9. H. M. Srivastava, B. R. K. Kashyap, *Special Functions in Queuing Theory and Related Stochastic Processes.* Academic Press, New York-London, 1982.
10. H. M. Srivastava, H. L. Manocha, *A Treatise on Generating Functions.* Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1984.
11. H. M. Srivastava, R. Panda, An integral representation for the product of two Jacobi polynomials, *J. London Math. Soc. (Series II)*, **12** (1976), no. 4, 419–425.
12. H. M. Srivastava, Y. Vyas, K. Fatawat, Extensions of the classical theorems for very well-poised hypergeometric functions. *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat. RACSAM* **113** (2019), no. 2, 367–397.

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MIXED BOUNDARY-TRANSMISSION PROBLEMS OF THE GENERALIZED THERMO-ELECTRO-MAGNETO-ELASTICITY THEORY FOR PIECEWISE HOMOGENEOUS COMPOSED STRUCTURES

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Abstract. The paper is devoted to the investigation of mixed boundary-transmission problems for composed elastic structures consisting of two contacting anisotropic bodies occupying two three-dimensional adjacent regions with a common contacting interface, being a proper part of their boundaries. It is assumed that the contacting elastic bodies are subject to different mathematical models. In particular, we consider Green-Lindsay's model of generalized thermo-electro-magneto-elasticity in one elastic component, while in the other one, we considered Gree Lindsay's model of generalized thermo-elasticity. The interaction of the thermo-mechanical and electro-magnetic fields in the composed piecewise elastic structure is described by the fully coupled systems of partial differential equations of pseudo-oscillations, obtained from the corresponding dynamical models by the Laplace transform. These systems are equipped with the appropriate mixed boundary-transmission conditions which cover the conditions arising in the case of interfacial cracks. Using the potential method and the theory of pseudodifferential equations on manifolds with a boundary, the uniqueness and existence theorems in suitable function spaces are proved, the regularity of solutions is analyzed and singularities of the corresponding thermo-mechanical and electro-magnetic fields near the interfacial crack edges are characterized. The explicit expressions for the stress singularity exponents are derived and it is shown that they depend essentially on the material parameters. A special class of composed elastic structures is considered, where the so-called oscillating stress singularities do not occur.

1. Introduction

In the present paper, we consider a boundary-transmission problem for a composed elastic structure consisting of two contacting bodies occupying two three-dimensional adjacent regions $\overline{\Omega^{(1)}}$ and $\overline{\Omega^{(2)}}$ with a common contacting interface, being a proper part of the boundaries $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ (see Figure 1). We analyze the case in which contacting elastic bodies are subject to different mathematical models. In particular, we consider *Green-Lindsay's model of generalized thermo-electro-magneto-elasticity* in $\Omega^{(1)}$ and *Green-Lindsay's model of generalized thermo-elasticity* in $\Omega^{(2)}$. Theoretical study of such problems attracts great attention due to the widespread application of modern sensing and actuating devices based on the ability to transform mechanical, electric, magnetic and thermal energies from one form to another. Therefore, the mathematical models having regard to the coupling effects between thermo-mechanical and electro-magnetic fields in elastic composites became very popular over the last decades (see, e.g., [1, 28, 29, 34], and references therein).

A remarkable feature of the generalized Green-Lindsay's model is a finite speed of heat propagation in contrast to an infinite speed of heat transfer occurring in the classical heat equation theory (see, e.g., [32]).

We investigate a general mixed boundary-transmission problem for the above described two-component elastic structure with the appropriate boundary and transmission conditions which cover the conditions arising in the case of interfacial cracks. In each region we consider the corresponding

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system of partial differential equations of pseudo-oscillations containing a complex parameter τ. These systems are obtained from the corresponding dynamical models by the Laplace transform.

Using the potential method and the theory of pseudodifferential equations on manifolds with a boundary, we study the mixed boundary-transmission problems and prove the uniqueness and existence of solutions in appropriate function spaces. Further, we analyze regularity of solutions and characterize singularities of the corresponding thermo-mechanical and electro-magnetic fields near the exceptional curves (crack edges, lines where the different type boundary conditions collide, and interface edges). In the upcoming papers, we plan to use the obtained results in the study of asymptotic properties of solutions of the corresponding dynamical problems.

Remark that in [8], we have investigated the mixed type boundary value problems of the theory of generalized thermo-electro-magneto-elasticity for homogeneous anisotropic materials with interior cracks. The interfacial crack problems for multilayered piecewise homogeneous anisotropic nested elastic structures, when all interacting components are subject to generalized thermo-electro-magneto-elasticity model with distinct material parameters in distinct elastic components, are considered in the reference [26]. The present investigation can be considered as a continuation of papers [5, 8–10, 24] and [26], but it turned out to be more difficult as far as it refers to the interaction between different dimensional physical fields (for the six-dimensional field in $\Omega^{(1)}$ and four-dimensional field in $\Omega^{(2)}$ see the problem setting in Subsection 2.4).

The paper is organized as follows. In Section 2, we describe the geometrical structure of the elastic composite body consisting of two interacting components, write down the governing pseudo-oscillation equations of Green-Lindsay's model of generalized thermo-electro-magneto-elasticity (GTEME model) and generalized thermo-elasticity (GTE model), formulate the mixed boundary-transmission problem and prove the uniqueness theorem in appropriate function spaces. In Section 3, we reduce equivalently the boundary-transmission problem to the system of boundary pseudodifferential equations, investigate the mapping properties of the corresponding pseudodifferential operator and prove the invertibility of the pseudodifferential operator in appropriate Bessel potential and Besov spaces. Further, we prove the theorem on the existence of solutions to the original mixed boundary-transmission problem, study asymptotic properties of solutions and their derivatives near the exceptional curves and evaluate explicitly the corresponding stress singularity exponents. It should be mentioned that in our analysis, we essentially use some approaches and results presented in [7] and [8]. In Section 4, we consider a particular case when an elastic solid medium occupying the region $\Omega^{(1)}$ belongs to the 422 (Tetragonal) or 622 (Hexagonal) classes of crystals or to the class of transversally isotropic materials, while the solid medium occupying the domain $\Omega^{(2)}$ is an isotropic material. These types of media includeF some key polymers and bio-materials (see [31]). For this particular problem, we analyze the asymptotic properties of solutions near the interfacial crack edges and derive explicit expressions for stress singularity exponents, playing an essential role in fracture mechanics. The stress singularity exponents essentially depend on the elastic, piezoelectric, piezomagnetic, dielectric and permeability constants. We prove that unlike the classical elasticity theory, in the case under consideration we have no oscillating stress singularities for physical fields near the interfacial crack edges. However, it should be mentioned that in comparison with the classical elasticity case, the stress singularity exponents increase and are greater than $\frac{1}{2}$, in general.

In Appendix, for the reader's convenience, we collected some auxiliary results used in the main text of the paper.

2. Formulation of the Mixed Boundary-Transmission Problem

2.1. **Geometrical configuration of the composite.** Let $\Omega^{(1)}$ and $\Omega^{(2)}$ be the bounded disjoint domains of the three-dimensional Euclidean space $\mathbb{R}^3$ with boundaries $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$, respectively. Moreover, let $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ have a nonempty, simply connected intersection $\overline{\Gamma} := \partial\Omega^{(1)} \cap \partial\Omega^{(2)}$ of positive measure. From now on, Γ will be referred to as an *interface*. Throughout the paper, $n = n^{(1)}$ and $\nu = n^{(2)}$ stand for the outward unit normal vectors to $\partial\Omega^{(1)}$ and to $\partial\Omega^{(2)}$, respectively. Clearly, $n(x) = -\nu(x)$ for $x \in \Gamma$.

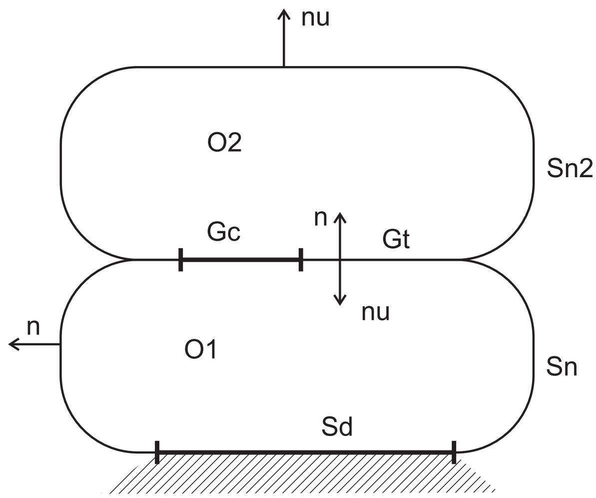

FIGURE 1. Composed body.

Further, let $\overline{\Gamma} = \overline{\Gamma_T} \cup \overline{\Gamma_C}$, where Γ_C is an open, simply connected proper part of Γ. Moreover, $\Gamma_T \cap \Gamma_C = \varnothing$ and $\partial\Gamma \cap \overline{\Gamma_C} = \varnothing$.

We set $S_N^{(2)} := \partial\Omega^{(2)} \setminus \overline{\Gamma}$ and $S^{(1)} := \partial\Omega^{(1)} \setminus \overline{\Gamma}$. Further, we denote by $S_D^{(1)}$ some open, nonempty, proper sub-manifold of $S^{(1)}$ and put $S_N^{(1)} := S^{(1)} \setminus \overline{S_D^{(1)}}$. Thus, we have the following dissections of the boundary surfaces (see Figure 1):

$$\partial\Omega^{(1)} = \overline{\Gamma_T} \cup \overline{\Gamma_C} \cup \overline{S_N^{(1)}} \cup \overline{S_D^{(1)}}, \quad \partial\Omega^{(2)} = \overline{\Gamma_T} \cup \overline{\Gamma_C} \cup \overline{S_N^{(2)}}.$$

Throughout the paper, for simplicity, we assume that $\partial\Omega^{(2)}$, $\partial\Omega^{(1)}$, $\partial S_N^{(2)}$, $\partial\Gamma_T$, $\partial\Gamma_C$, $\partial S_D^{(1)}$, $\partial S_N^{(1)}$ are C^∞-smooth and $\partial\Omega^{(2)} \cap \overline{S_D^{(1)}} = \varnothing$.

Let $\Omega^{(1)}$ be occupied by an anisotropic homogeneous elastic medium revealing thermo-electro-magnetic properties described by Green-Lindsay's model of generalized thermo-electro-magneto-elasticity and $\Omega^{(2)}$ be filled by an anisotropic homogeneous elastic medium (e.g. metallic solid) with properties described by Green-Lindsay's generalized thermo-elasticity model. These two bodies interact along the interface Γ with the interfacial crack Γ_C. Moreover, it is assumed that the composed body is fixed along the sub-surface $S_D^{(1)}$ (the Dirichlet part of the boundary $\partial\Omega^{(1)}$), while on the sub-manifolds $S_N^{(2)}$ and $S_N^{(1)}$ we have the Neumann type boundary conditions.

In the domain $\Omega^{(1)}$ we have a six-dimensional physical field described by the displacement vector $u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top$, the electric potential $u_4^{(1)} = \varphi^{(1)}$, the magnetic potential $u_5^{(1)} = \psi^{(1)}$, and the temperature distribution function $u_6^{(1)} = \vartheta^{(1)}$, while in the domain $\Omega^{(2)}$ we have a four-dimensional thermoelastic field represented by the displacement vector $u^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)})^\top$ and temperature distribution function $u_4^{(2)} = \vartheta^{(2)}$. The superscript $(\cdot)^\top$ denotes transposition operation.

Throughout the paper, the summation over the repeated indices is meant from 1 to 3, unless otherwise stated.

2.2. **GTE Model.** In the domain $\Omega^{(2)}$ of the composed body, the system of pseudo-oscillation equations obtained from the dynamical equations of the generalized Green-Lindsay's linear model of thermoelasticity in matrix form reads as (see [7,11])

$$A^{(2)}(\partial_x, \tau)\, U^{(2)}(x, \tau) = \Phi^{(2)}(x, \tau)\,,$$

where $U^{(2)}=(u_1^{(2)},u_2^{(2)},u_3^{(2)},u_4^{(2)})^\top:=(u^{(2)},\vartheta^{(2)})^\top$ is a sought for complex-valued vector function, $\Phi^{(2)}=(\Phi_1^{(2)},\dots,\Phi_4^{(2)})^\top$ is a given vector function, and

$$A^{(2)}(\partial_x,\tau)=\Big[A_{pq}^{(2)}(\partial_x,\tau)\Big]_{4\times4}$$
$$:=\begin{bmatrix} [c_{rjkl}^{(2)}\partial_j\partial_l-\varrho^{(2)}\tau^2\delta_{rk}]_{3\times3} & [-(1+\nu_0^{(2)}\tau)\lambda_{rj}^{(2)}\partial_j]_{3\times1} \\ [-\tau\lambda_{kl}^{(2)}\partial_l]_{1\times3} & \eta_{jl}^{(2)}\partial_j\partial_l-\tau d_0^{(2)}-\tau^2h_0^{(2)}\end{bmatrix}_{4\times4}. \tag{2.1}$$

Here, $\tau=\sigma+i\omega$ is a complex parameter, $u^{(2)}=(u_1^{(2)},u_2^{(2)},u_3^{(2)})^\top$ is the displacement vector, $u_4^{(2)}:=\vartheta^{(2)}=T^{(2)}-T_0$ is the relative temperature (temperature increment), $\varrho^{(2)}$ is the mass density, $c_{ijkl}^{(2)}$ are the elastic constants, $\varkappa_{kj}^{(2)}$ are the thermal conductivity constants, $\lambda_{rj}^{(2)}$ are the coefficients, coupling thermal, electric and magnetic fields, $\nu_0^{(2)}$ and $h_0^{(2)}$ are two relaxation times, $d_0^{(2)}$ is the constitutive coefficient; $T_0>0$ is the initial temperature, i.e., the temperature in the natural state in the absence of deformation and electromagnetic fields. We employ the notation $\partial=\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$.

For an isotropic medium we have (see [22]):

$$c_{ijlk}^{(2)}=\lambda^{(2)}\delta_{ij}\,\delta_{lk}+\mu^{(2)}(\delta_{il}\,\delta_{jk}+\delta_{ik}\,\delta_{jl}),\quad \lambda_{ij}^{(2)}=\lambda^{(2)}\delta_{ij},\ \ \eta_{ij}^{(2)}=\eta^{(2)}\delta_{ij}, \tag{2.2}$$

where $\lambda^{(2)}$ and $\mu^{(2)}$ are the Lamé constants and δ_{ij} is Kronecker's delta.

The stress operator in the generalised thermo-elasticity theory has the form

$$\mathcal{T}^{(2)}(\partial_x,\nu,\tau)=\big[\,\mathcal{T}_{pq}^{(2)}(\partial_x,\nu,\tau)\,\big]_{4\times4}$$
$$:=\begin{bmatrix} [c_{rjkl}^{(2)}\nu_j\partial_l]_{3\times3} & [-(1+\nu_0^{(2)}\tau)\lambda_{rj}^{(2)}\nu_j]_{3\times1} \\ [0]_{1\times3} & \eta_{jl}^{(2)}\nu_j\partial_l\end{bmatrix}_{4\times4}.$$

Note that for a four-dimensional vector $U^{(2)}=(u_1^{(2)},u_2^{(2)},u_3^{(2)},u_4^{(2)})^\top$ we have

$$\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}=(\sigma_{1j}^{(2)}\nu_j,\sigma_{2j}^{(2)}\nu_j,\sigma_{3j}^{(2)}\nu_j,-T_0^{-1}q_j^{(2)}\nu_j)^\top,$$

where $\sigma_{kj}^{(2)}$, $k,j=1,2,3$, are components of the stress tensor, $\sigma^{(2)}=(\sigma_{1j}^{(2)}\nu_j,\sigma_{2j}^{(2)}\nu_j,\sigma_{3j}^{(2)}\nu_j)^\top$ is the mechanical stress vector and $q=q_j^{(2)}\nu_j$ is the heat flow across the surface element with normal ν (for details see [7]).

The constants involved in the above equations satisfy the following symmetry conditions:

$$c_{ijkl}^{(2)}=c_{jikl}^{(2)}=c_{klij}^{(2)},\ \ \lambda_{ij}^{(2)}=\lambda_{ji}^{(2)},\ \ \eta_{ij}^{(2)}=\eta_{ji}^{(2)},\ \ i,j,k,l=1,2,3. \tag{2.3}$$

Moreover, from physical considerations related to the positive definiteness of the potential energy, it follows that there exist positive constants c_0 and c_1 such that

$$c_{ijkl}^{(2)}\,\xi_{ij}\,\xi_{kl}\geqslant c_0\,\xi_{ij}\,\xi_{ij},\ \ \eta_{ij}^{(2)}\xi_i\xi_j\geqslant c_1\,\xi_i\,\xi_i\quad\text{for all }\ \xi_{ij}=\xi_{ji}\in\mathbb{R},\ \ \xi_j\in\mathbb{R},\ i,j=1,2,3. \tag{2.4}$$

In particular, the first inequality above implies that the density of potential energy

$$E^{(2)}\big(u^{(2)},u^{(2)}\big)=c_{ijlk}^{(2)}\,s_{ij}^{(2)}\,s_{lk}^{(2)},$$

corresponding to the real-valued displacement vector $u^{(2)}$, is positive definite with respect to the symmetric components of the strain tensor $s_{lk}^{(2)}=s_{kl}^{(2)}=2^{-1}(\partial_k\,u_j^{(2)}+\partial_j\,u_k^{(2)})$.

By $A^{(2,0)}(-i\xi)$ with $\xi\in\mathbb{R}^3$ we denote the principal homogeneous symbol matrix of the operator $A^{(2)}(\partial_x,\tau)$,

$$A^{(2,0)}(-i\xi)=A^{(2,0)}(i\xi)=-A^{(2,0)}(\xi)=-\begin{bmatrix} [c_{rjkl}^{(2)}\xi_j\xi_l]_{3\times3} & [0]_{3\times1} \\ [0]_{1\times3} & \eta_{jl}^{(2)}\xi_j\xi_l\end{bmatrix}_{4\times4}.$$

The symmetry conditions (2.3) and inequalities (2.4) imply that the matrix $A^{(2,0)}(\xi)$ is positive definite, i.e., there is a positive constant C depending only on the material parameters such that

$$\big(A^{(2,0)}(\xi)\zeta\cdot\zeta\big)=\big(-A^{(2,0)}(-i\xi)\zeta\cdot\zeta\big)=\Big(\sum_{k,j=1}^{4}A^{(2,0)}_{kj}(\xi)\zeta_j\overline{\zeta}_k\Big)\geq C|\xi|^2|\zeta|^2$$
$$\text{for all } \ \xi\in\mathbb{R}^3 \ \text{ and for all } \ \zeta\in\mathbb{C}^4.$$

Here and in what follows, the central dot denotes the scalar product in the space of complex-valued vectors $\mathbb{C}^m$ and the over bar denotes complex conjugation.

2.3. **GTEME Model.** In $\Omega^{(1)}$, the thermo-mechanical and electro-magnetic fields are governed by the following pseudo-oscillation system of equations of Green-Lindsay's thermo-electro-magneto-elasticity theory (see [7]):

$$A^{(1)}(\partial_x,\tau)U^{(1)}(x,\tau)=\Phi^{(1)}(x,\tau),$$

where

$$\begin{aligned}A^{(1)}(\partial_x,\tau)&=\Big[A^{(1)}_{pq}(\partial_x,\tau)\Big]_{6\times 6}\\ &:=\begin{bmatrix}[c^{(1)}_{rjkl}\partial_j\partial_l-\varrho^{(1)}\tau^2\delta_{rk}]_{3\times3} & [e^{(1)}_{lrj}\partial_j\partial_l]_{3\times1} & [q^{(1)}_{lrj}\partial_j\partial_l]_{3\times1} & [-(1+\nu^{(1)}_0\tau)\lambda^{(1)}_{rj}\partial_j]_{3\times1}\\ [-e^{(1)}_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa^{(1)}_{jl}\partial_j\partial_l & a^{(1)}_{jl}\partial_j\partial_l & -(1+\nu^{(1)}_0\tau)p^{(1)}_j\partial_j\\ [-q^{(1)}_{jkl}\partial_j\partial_l]_{1\times3} & a^{(1)}_{jl}\partial_j\partial_l & \mu^{(1)}_{jl}\partial_j\partial_l & -(1+\nu^{(1)}_0\tau)m^{(1)}_j\partial_j\\ [-\tau\lambda^{(1)}_{kl}\partial_l]_{1\times3} & \tau p^{(1)}_l\partial_l & \tau m^{(1)}_l\partial_l & \eta^{(1)}_{jl}\partial_j\partial_l-\tau^2h^{(1)}_0-\tau d^{(1)}_0\end{bmatrix}_{6\times6}\end{aligned}\tag{2.5}$$

is the differential operator associated with the pseudo-oscillation equations of the thermo-electro-magneto-elasticity theory, obtained by the Laplace transform from the corresponding dynamical equations, $U^{(1)}=(u^{(1)}_1,u^{(1)}_2,u^{(1)}_3,u^{(1)}_4,u^{(1)}_5,u^{(1)}_6)^\top:=(u^{(1)},\varphi^{(1)},\psi^{(1)},\vartheta^{(1)})^\top$ is the sought for complex-valued vector function, $u^{(1)}=(u^{(1)}_1,u^{(1)}_2,u^{(1)}_3)^\top$ denotes the displacement vector, $\varphi^{(1)}$ and $\psi^{(1)}$ stand for the electric and magnetic potentials and $\vartheta^{(1)}=T^{(1)}-T_0$ is the relative temperature (temperature increment), and $\Phi^{(1)}=\big(\Phi^{(1)}_1,\dots,\Phi^{(1)}_6\big)^\top$ is a given vector function. Here we also employ the following notation: $\varrho^{(1)}$ is the mass density, $c^{(1)}_{rjkl}$ are the elastic constants, $e^{(1)}_{jkl}$ are the piezoelectric constants, $q^{(1)}_{jkl}$ are the piezomagnetic constants, $\varkappa^{(1)}_{jk}$ are the dielectric (permittivity) constants, $\mu^{(1)}_{jk}$ are the magnetic permeability constants, $a^{(1)}_{jk}$ are the electromagnetic coupling coefficients, $p^{(1)}_j$, $m^{(1)}_j$, and $\lambda^{(1)}_{rj}$ are the coefficients, coupling thermal field with displacement, electric and magnetic fields, $\eta^{(1)}_{jk}$ are the heat conductivity coefficients, T_0 is the initial reference temperature, that is, the temperature in the natural state in the absence of deformation and electromagnetic fields, $\nu^{(1)}_0$ and $h^{(1)}_0$ are two relaxation times, $a^{(1)}_0$ and $d^{(1)}_0$ are some constitutive coefficients.

Throughout the paper, we assume that the time relaxation parameters $\nu^{(1)}_0$ and $\nu^{(2)}_0$ involved in operators (2.5) and (2.1) are the same and we set

$$\nu^{(1)}_0=\nu^{(2)}_0=\nu_0.$$

The constants involved in the above equations satisfy the following symmetry conditions:

$$\begin{gathered}c^{(1)}_{rjkl}=c^{(1)}_{jrkl}=c^{(1)}_{klrj},\ \ e^{(1)}_{klj}=e^{(1)}_{kjl},\ \ q^{(1)}_{klj}=q^{(1)}_{kjl},\\ \varkappa^{(1)}_{kj}=\varkappa^{(1)}_{jk},\ \ \lambda^{(1)}_{kj}=\lambda^{(1)}_{jk},\ \ \mu^{(1)}_{kj}=\mu^{(1)}_{jk},\ \ a^{(1)}_{kj}=a^{(1)}_{jk},\ \ \eta^{(1)}_{kj}=\eta^{(1)}_{jk},\ \ r,j,k,l=1,2,3.\end{gathered}\tag{2.6}$$

From physical considerations it follows that (see, e.g., [3,27,32]):

$$\begin{gathered}c^{(1)}_{rjkl}\xi_{rj}\xi_{kl}\geq\delta_0\xi_{kl}\,\xi_{kl},\ \ \varkappa^{(1)}_{kj}\xi_k\xi_j\geq\delta^{(1)}_1|\xi|^2,\ \ \mu^{(1)}_{kj}\xi_k\xi_j\geq\delta_2|\xi|^2,\ \ \eta^{(1)}_{kj}\xi_k\xi_j\geq\delta_3|\xi|^2,\\ \text{for all } \ \xi_{kj}=\xi_{jk}\in\mathbb{R} \ \text{ and for all } \ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3,\end{gathered}\tag{2.7}$$

$$\nu_0>0,\ \ h^{(1)}_0>0,\ \ d^{(1)}_0\nu_0-h^{(1)}_0>0,\tag{2.8}$$

where δ_0, δ_1, δ_2, and δ_3 are the positive constants depending on material parameters.

Due to the symmetry conditions (2.6), with the help of (2.7), we easily derive

$$c^{(1)}_{rjkl}\zeta_{rj}\overline{\zeta_{kl}} \geq \delta_0\zeta_{kl}\overline{\zeta_{kl}}, \;\; \varkappa^{(1)}_{kj}\zeta_k\overline{\zeta_j} \geq \delta_1|\zeta|^2, \;\; \mu^{(1)}_{kj}\zeta_k\overline{\zeta_j} \geq \delta_2|\zeta|^2, \;\; \eta^{(1)}_{kj}\zeta_k\overline{\zeta_j} \geq \delta_3|\,\zeta|^2,$$
$$\text{for all } \zeta_{kj}=\zeta_{jk}\in\mathbb{C} \text{ and for all } \zeta=(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3. \tag{2.9}$$

More careful analysis related to the positive definiteness of the potential energy and the thermodynamical laws insure that the following 8×8 matrix

$$M=[M_{kj}]_{8\times 8} := \begin{bmatrix} [\varkappa^{(1)}_{jl}]_{3\times 3} & [a^{(1)}_{jl}]_{3\times 3} & [p^{(1)}_j]_{3\times 1} & [\nu_0 p^{(1)}_j]_{3\times 1} \\ [a^{(1)}_{jl}]_{3\times 3} & [\mu^{(1)}_{jl}]_{3\times 3} & [m^{(1)}_j]_{3\times 1} & [\nu_0 m_j]_{3\times 1} \\ [p^{(1)}_j]_{1\times 3} & [m^{(1)}_j]_{1\times 3} & d^{(1)}_0 & h^{(1)}_0 \\ [\nu_0 p^{(1)}_j]_{1\times 3} & [\nu_0 m^{(1)}_j]_{1\times 3} & h^{(1)}_0 & \nu_0 h^{(1)}_0 \end{bmatrix}_{8\times 8} \tag{2.10}$$

is positive definite (see [7]). Note that the positive definiteness of M remains valid if the parameters $p^{(1)}_j$ and $m^{(1)}_j$ in (2.10) are replaced by the opposite ones, $-p^{(1)}_j$ and $-m^{(1)}_j$. Moreover, it follows that the matrices

$$\Lambda^{(1)} := \begin{bmatrix} [\varkappa^{(1)}_{kj}]_{3\times 3} & [a^{(1)}_{kj}]_{3\times 3} \\ [a^{(1)}_{kj}]_{3\times 3} & [\mu^{(1)}_{kj}]_{3\times 3} \end{bmatrix}_{6\times 6}, \quad \Lambda^{(2)} := \begin{bmatrix} d^{(1)}_0 & h^{(1)}_0 \\ h^{(1)}_0 & \nu_0 h_0 \end{bmatrix}_{2\times 2} \tag{2.11}$$

are positive definite as well, i.e.,

$$\varkappa^{(1)}_{kj}\zeta'_k\overline{\zeta'_j} + a^{(1)}_{kj}(\zeta'_k\overline{\zeta''_j} + \overline{\zeta'_k}\zeta''_j) + \mu^{(1)}_{kj}\zeta''_k\overline{\zeta''_j} \geq \kappa^{(1)}_1\big(|\zeta'|^2+|\zeta''|^2\big) \;\; \forall\,\zeta',\zeta''\in\mathbb{C}^3, \tag{2.12}$$
$$d^{(1)}_0|z_1|^2 + h_0\big(z_1\overline{z_2}+\overline{z_1}z_2\big) + \nu_0 h^{(1)}_0|z_2|^2 \geq \kappa^{(1)}_2\big(|z_1|^2+|z_2|^2\big) \;\; \forall\, z_1,z_2\in\mathbb{C}, \tag{2.13}$$

with some positive constants $\kappa^{(1)}_1$ and $\kappa^{(1)}_2$ depending on the material parameters involved in (2.11) (for details see [7]).

The stress operator $\mathcal{T}^{(1)}(\partial_x,n,\tau)$ in the generalized thermo-electro-magneto-elasticity theory reads as

$$\mathcal{T}^{(1)}(\partial_x,n,\tau) = \big[\,\mathcal{T}^{(1)}_{pq}(\partial_x,n,\tau)\,\big]_{6\times 6}$$
$$:= \begin{bmatrix} [c^{(1)}_{rjkl}n_j\partial_l]_{3\times 3} & [e^{(1)}_{lrj}n_j\partial_l]_{3\times 1} & [q^{(1)}_{lrj}n_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda^{(1)}_{rj}n_j]_{3\times 1} \\ [-e^{(1)}_{jkl}n_j\partial_l]_{1\times 3} & \varkappa^{(1)}_{jl}n_j\partial_l & a^{(1)}_{jl}n_j\partial_l & -(1+\nu_0\tau)p^{(1)}_j n_j \\ [-q^{(1)}_{jkl}n_j\partial_l]_{1\times 3} & a^{(1)}_{jl}n_j\partial_l & \mu^{(1)}_{jl}n_j\partial_l & -(1+\nu_0\tau)m^{(1)}_j n_j \\ [0]_{1\times 3} & 0 & 0 & \eta^{(1)}_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}.$$

Note that for a vector $U^{(1)} := (u^{(1)},\varphi^{(1)},\psi^{(1)},\vartheta^{(1)})^\top$, the components of the corresponding generalized stress vector $\mathcal{T}^{(1)}U^{(1)}$ have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the forth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector, respectively, with opposite sign, and finally, the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector (for details see [7, Ch.2]).

Denote by $A^{(1,0)}(-i\xi)$ with $\xi\in\mathbb{R}^3$ the principal homogeneous symbol matrix of the differential operator $A^{(1)}(\partial_x,\tau)$. We have

$$A^{(1,0)}(-i\xi) = -A^{(1,0)}(\xi) = \begin{bmatrix} [-c^{(1)}_{rjkl}\xi_j\xi_l]_{3\times 3} & [-e^{(1)}_{lrj}\xi_j\xi_l]_{3\times 1} & [-q^{(1)}_{lrj}\xi_j\xi_l]_{3\times 1} & [0]_{3\times 1} \\ [e^{(1)}_{jkl}\xi_j\xi_l]_{1\times 3} & -\varkappa^{(1)}_{jl}\xi_j\xi_l & -a^{(1)}_{jl}\xi_j\xi_l & 0 \\ [q^{(1)}_{jkl}\xi_j\xi_l]_{1\times 3} & -a^{(1)}_{jl}\xi_j\xi_l & -\mu^{(1)}_{jl}\xi_j\xi_l & 0 \\ [0]_{1\times 3} & 0 & 0 & -\eta^{(1)}_{jl}\xi_j\xi_l \end{bmatrix}_{6\times 6}.$$

From the symmetry conditions (2.6), inequalities (2.7) and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.11) it follows that there is a positive constant C depending only on the material parameters such that

$$\operatorname{Re}\big(-A^{(1,0)}(-i\xi)\zeta\cdot\zeta\big)=\operatorname{Re}\Big(\sum_{k,j=1}^{6}A^{(1,0)}_{kj}(\xi)\zeta_j\overline{\zeta}_k\Big)\geq C|\xi|^2|\zeta|^2$$
$$\text{for all }\ \xi\in\mathbb{R}^3\ \text{ and for all }\ \zeta\in\mathbb{C}^6.$$

Therefore, $-A^{(1)}(\partial_x,\tau)$ is a non-selfadjoint strongly elliptic differential operator.

2.4. **Formulation of the Mixed Boundary-Transmission problem.** By W^r_p, H^s_p and $B^s_{p,q}$ with $r\geqslant 0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\leqslant q\leqslant\infty$, we denote the Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively, (see, e.g., [33]). Recall that $H^r_2=W^r_2=B^r_{2,2}$, $H^s_2=B^s_{2,2}$, $W^t_p=B^t_{p,p}$, and $H^k_p=W^k_p$, for any $r\geqslant 0$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a proper sub-manifold $\mathcal{M}\subset\mathcal{M}_0$, we denote by $\widetilde{H}^s_p(\mathcal{M})$ and $\widetilde{B}^s_{p,q}(\mathcal{M})$ the subspaces of $H^s_p(\mathcal{M}_0)$ and $B^s_{p,q}(\mathcal{M}_0)$, respectively,

$$\widetilde{H}^s_p(\mathcal{M})=\big\{g:\ g\in H^s_p(\mathcal{M}_0),\ \operatorname{supp} g\subset\overline{\mathcal{M}}\big\},$$
$$\widetilde{B}^s_{p,q}(\mathcal{M})=\big\{g:\ g\in B^s_{p,q}(\mathcal{M}_0),\ \operatorname{supp} g\subset\overline{\mathcal{M}}\big\},$$

while $H^s_p(\mathcal{M})$ and $B^s_{p,q}(\mathcal{M})$ stand for the spaces of restrictions on $\mathcal{M}$ of functions from $H^s_p(\mathcal{M}_0)$ and $B^s_{p,q}(\mathcal{M}_0)$, respectively,

$$H^s_p(\mathcal{M})=\big\{r_{\mathcal{M}}f:\ f\in H^s_p(\mathcal{M}_0)\big\},\qquad B^s_{p,q}(\mathcal{M})=\big\{r_{\mathcal{M}}f:\ f\in B^s_{p,q}(\mathcal{M}_0)\big\},$$

where $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$.

Now we formulate the mixed boundary-transmission problem: *Find vector functions*

$$U^{(1)}=(u^{(1)},\varphi^{(1)},\psi^{(1)},\vartheta^{(1)})^\top=(u^{(1)}_1,\dots,u^{(1)}_6)^\top\ :\ \Omega^{(1)}\to\mathbb{C}^6,$$
$$U^{(2)}=(u^{(2)},\vartheta^{(2)})^\top=(u^{(2)}_1,\dots,u^{(2)}_4)^\top\ :\ \Omega^{(2)}\to\mathbb{C}^4,$$

belonging, respectively, to the spaces $[W^1_p(\Omega^{(2)})]^4$ *and* $[W^1_p(\Omega^{(1)})]^6$ *with* $1<p<\infty$ *and satisfying*
(i) *the systems of partial differential equations*:

$$A^{(1)}(\partial_x,\tau)\,U^{(1)}=0\ \text{ in }\ \Omega^{(1)},\tag{2.14}$$
$$A^{(2)}(\partial_x,\tau)\,U^{(2)}=0\ \text{ in }\ \Omega^{(2)},\tag{2.15}$$

(ii) *the boundary conditions*:

$$\big\{\mathcal{T}^{(1)}(\partial_x,n,\tau)\,U^{(1)}\big\}^+=Q^{(1)}\ \text{ on }\ S^{(1)}_N,\tag{2.16}$$
$$\big\{\mathcal{T}^{(2)}(\partial_x,\nu,\tau)\,U^{(2)}\big\}^+=Q^{(2)}\ \text{ on }\ S^{(2)}_N,\tag{2.17}$$
$$\big\{U^{(1)}\big\}^+=f^{(1)}\ \text{ on }\ S^{(1)}_D,\tag{2.18}$$
$$\{u^{(1)}_4\}^+=f_4\ \text{ on }\ \Gamma_T,\tag{2.19}$$
$$\{u^{(1)}_5\}^+=f_5\ \text{ on }\ \Gamma_T,\tag{2.20}$$

(iii) *the transmission conditions on* Γ_T *:*

$$\Big\{u^{(1)}_j\Big\}^+-\big\{u^{(2)}_j\big\}^+=f_j\quad\text{on }\ \Gamma_T,\qquad j=1,2,3,\tag{2.21}$$
$$\Big\{u^{(1)}_6\Big\}^+-\big\{u^{(2)}_4\big\}^+=f_6\quad\text{on }\ \Gamma_T,\tag{2.22}$$
$$\big\{\big[\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big]_j\big\}^++\big\{[\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}]_j\big\}^+=F_j,\quad\text{on }\ \Gamma_T,\qquad j=1,2,3,\tag{2.23}$$
$$\big\{\big[\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big]_6\big\}^++\big\{[\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}]_4\big\}^+=F_4,\quad\text{on }\ \Gamma_T,\tag{2.24}$$

(iv) *the interfacial crack conditions on* Γ_C:

$$\big\{\mathcal{T}^{(1)}(\partial_x, n, \tau)\, U^{(1)}\big\}^+ = \widetilde{Q}^{(1)} \quad \text{on } \ \Gamma_C, \tag{2.25}$$

$$\big\{\mathcal{T}^{(2)}(\partial_x, \nu, \tau)\, U^{(2)}\big\}^+ = \widetilde{Q}^{(2)} \quad \text{on } \ \Gamma_C, \tag{2.26}$$

where $n = -\nu$ *on* Γ,

$$\begin{aligned}
Q^{(1)} &= (Q_1^{(1)}, Q_2^{(1)}, Q_3^{(1)}, Q_4^{(1)}, Q_5^{(1)}, Q_6^{(1)})^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(S_N^{(1)})\,\big]^6, \\
\widetilde{Q}^{(1)} &= (\widetilde{Q}_1^{(1)}, \widetilde{Q}_2^{(1)}, \widetilde{Q}_3^{(1)}, \widetilde{Q}_4^{(1)}, \widetilde{Q}_5^{(1)}, \widetilde{Q}_6^{(1)})^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(\Gamma_C)\,\big]^6, \\
Q^{(2)} &= \big(Q_1^{(2)}, Q_2^{(2)}, Q_3^{(2)}, Q_4^{(2)}\big)^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(S_N^{(2)})\,\big]^4, \\
\widetilde{Q}^{(2)} &= \big(\widetilde{Q}_1^{(2)}, \widetilde{Q}_2^{(2)}, \widetilde{Q}_3^{(2)}, \widetilde{Q}_4^{(2)}\big)^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(\Gamma_C)\,\big]^4, \\
f^{(1)} &= (f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, f_4^{(1)}, f_5^{(1)}, f_6^{(1)})^\top \in \big[\, B_{p,p}^{1-\frac{1}{p}}(S_D^{(1)})\,\big]^6, \\
f &= \big(f_1, f_2, f_3, f_4, f_5, f_6\big)^\top \in \big[\, B_{p,p}^{1-\frac{1}{p}}(\Gamma_T)\,\big]^6, \\
F &= \big(F_1, F_2, F_3, F_4\big)^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(\Gamma_T)\,\big]^4.
\end{aligned} \tag{2.27}$$

Note that, in addition, the functions F_j, $Q_j^{(1)}$, $\widetilde{Q}_j^{(1)}$, $\widetilde{Q}_j^{(2)}$ and $Q_j^{(2)}$ have to satisfy some evident compatibility conditions (see Subsection 3.1, inclusion (3.22), (3.23)).
We have the following uniqueness theorem for $p = 2$.

Theorem 2.1. *Let* $\Omega^{(1)}$ *and* $\Omega^{(2)}$ *be the Lipschitz domains and either* $\tau = \sigma + i\omega$ *with* $\sigma > 0$ *or* $\tau = 0$. *Then the mixed boundary transmission problem* (2.14)–(2.26) *has at most one solution pair* $(U^{(1)}, U^{(2)})$ *in the space* $[W_2^1(\Omega^{(1)})]^6 \times [W_2^1(\Omega^{(2)})]^4$, *provided* $\operatorname{mes} S_D^{(1)} > 0$.

Proof. Proof of the theorem is quite similar to that of Theorem 1.1 in reference [6]. □

Later we will prove the uniqueness theorem for $p \neq 2$.

To prove the existence of solutions to the above formulated mixed boundary-transmission problem, we use the potential method and the theory of pseudodifferential equations. To this end, we introduce the following single layer potentials:

$$V_\tau^{(1)}(h^{(1)})(x) = \int\limits_{\partial\Omega^{(1)}} \Gamma^{(1)}(x-y, \tau)\, h^{(1)}(y)\, d_y S,$$

$$V_\tau^{(2)}(h^{(2)})(x) = \int\limits_{\partial\Omega^{(2)}} \Gamma^{(2)}(x-y, \tau)\, h^{(2)}(y)\, d_y S,$$

where $\Gamma^{(1)}(x, \tau)$ and $\Gamma^{(2)}(x, \tau)$ are the fundamental matrices of the differential operators $A^{(1)}(\partial_x, \tau)$ and $A^{(2)}(\partial_x, \tau)$, respectively, $h^{(1)} = (h_1^{(1)}, \dots, h_6^{(1)})^\top$ and $h^{(2)} = (h_1^{(2)}, \dots, h_4^{(2)})^\top$ are the density vector functions. The explicit expressions of the fundamental matrices $\Gamma^{(1)}(x, \tau)$ and $\Gamma^{(2)}(x, \tau)$ and their properties can be found in references [7] and [8].

We introduce also the following boundary integral operators generated by the single layer potentials

$$\mathcal{H}_\tau^{(1)}(h^{(1)})(z) = \int\limits_{\partial\Omega^{(1)}} \Gamma^{(1)}(z-y, \tau)\, h^{(1)}(y)\, d_y S, \quad z \in \partial\Omega^{(1)}, \tag{2.28}$$

$$\mathcal{K}_\tau^{(1)}(h^{(1)})(z) = \int\limits_{\partial\Omega^{(1)}} \mathcal{T}^{(1)}(\partial_z, n(z), \tau)\Gamma^{(1)}(z-y, \tau)\, h^{(1)}(y)\, d_y S, \quad z \in \partial\Omega^{(1)}, \tag{2.29}$$

$$\mathcal{H}_\tau^{(2)}(h^{(2)})(z) = \int\limits_{\partial\Omega^{(2)}} \Gamma^{(2)}(z-y, \tau)\, h^{(2)}(y)\, d_y S, \quad z \in \partial\Omega^{(2)}, \tag{2.30}$$

$$\mathcal{K}_\tau^{(2)}(h^{(2)})(z) = \int\limits_{\partial\Omega^{(2)}} \mathcal{T}^{(2)}(\partial_z, n(z), \tau)\Gamma^{(2)}(z-y,\tau)\, h^{(2)}(y)\, d_yS, \quad z \in \partial\Omega^{(2)}. \tag{2.31}$$

Note that $\mathcal{H}_\tau^{(1)}$ and $\mathcal{H}_\tau^{(2)}$ are pseudodifferential operators of order -1, while $\mathcal{K}_\tau^{(1)}$ and $\mathcal{K}_\tau^{(2)}$ are pseudodifferential operators of order 0, i.e., singular integral operators (for details see Appendix).

Now, we formulate several auxiliary lemmas proved in reference [8].

Lemma 2.2. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution vector* $U^{(2)} \in [\,W_p^1(\Omega^{(2)})\,]^4$ *to the homogeneous equation* $A^{(2)}(\partial,\tau)\,U^{(2)} = 0$ *in* $\Omega^{(2)}$, *can be uniquely represented by the single layer potential*

$$U^{(2)} \;=\; V_\tau^{(2)}\Big(\big[\,P_\tau^{(2)}\big]^{-1}\chi^{(2)}\Big) \;\; \text{in } \; \Omega^{(2)},$$

where

$$P_\tau^{(2)} := -2^{-1}\, I_4 + \mathcal{K}_\tau^{(2)}, \quad \chi^{(2)} = \big\{\, \mathcal{T}^{(2)} U^{(2)}\,\big\}^+ \in \big[\, B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(2)})\,\big]^4, \tag{2.32}$$

and $\mathcal{K}_\tau^{(2)}$ *is defined by* (2.31).

For the mapping properties and invertibility of the operator $P_\tau^{(2)}$ in appropriate function spaces see Theorem 5.4.

Lemma 2.3. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and*

$$P_\tau^{(1)} := -2^{-1}\, I_6 + \mathcal{K}_\tau^{(1)} + \beta\, \mathcal{H}_\tau^{(1)}, \tag{2.33}$$

where $\mathcal{K}_\tau^{(1)}$ *and* $\mathcal{H}_\tau^{(1)}$ *are defined by* (2.29) *and* (2.28), *respectively, and* β *is a smooth real-valued scalar function on* $S^{(1)}$, *not vanishing identically and satisfying the conditions*

$$\beta \geqslant 0, \qquad \operatorname{supp}\beta \subset S_D^{(1)}. \tag{2.34}$$

Then the operators

$$\begin{aligned} P_\tau^{(1)} \;&:\; \big[\, H_p^s(\partial\Omega^{(1)})\,\big]^6 \to \big[\, H_p^s(\partial\Omega^{(1)})\,\big]^6, \\ P_\tau^{(1)} \;&:\; \big[\, B_{p,q}^s(\partial\Omega^{(1)})\,\big]^6 \to \big[\, B_{p,q}^s(\partial\Omega^{(1)})\,\big]^6 \end{aligned}$$

are invertible for all $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, *and* $s \in \mathbb{R}$.

As a consequence, we have the following

Lemma 2.4. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution* $U^{(1)} \in [\,W_p^1(\Omega^{(1)})\,]^6$ *to the homogeneous equation* $A^{(1)}(\partial_x,\tau)U^{(1)} = 0$ *in* $\Omega^{(1)}$ *can be uniquely represented by the single layer potential*

$$U^{(1)} = V_\tau^{(1)}\big(\,\big[P_\tau^{(1)}\big]^{-1}\chi\big) \;\; \text{in } \; \Omega^{(1)},$$

where

$$\chi = \{\mathcal{T}^{(1)}U^{(1)}\}^+ + \beta\,\{U^{(1)}\}^+ \in [\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(1)})\,]^6.$$

3. The Existence and Regularity Results

3.1. **Reduction to boundary equations.** Let us return to problem (2.14)–(2.26) and derive the equivalent boundary integral formulation. Keeping in mind (2.27), let

$$\begin{gathered} G^{(1)} := \begin{cases} Q^{(1)} & \text{on } S_N^{(1)}, \\ \widetilde{Q}^{(1)} & \text{on } \Gamma_C, \end{cases} \quad G^{(2)} := \begin{cases} Q^{(2)} & \text{on } S_N^{(2)}, \\ \widetilde{Q}^{(2)} & \text{on } \Gamma_C, \end{cases} \\ G^{(1)} \in \big[\, B_{p,p}^{-1/p}(S_N^{(1)} \cup \Gamma_C)\,\big]^6, \quad G^{(2)} \in \big[\, B_{p,p}^{-1/p}(S_N^{(2)} \cup \Gamma_C)\,\big]^4, \end{gathered} \tag{3.1}$$

and

$$G_0^{(1)} = (G_{01}^{(1)}, \dots, G_{06}^{(1)})^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(1)})\,\big]^6, \quad G_0^{(2)} = (G_{01}^{(2)}, \dots, G_{04}^{(2)})^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(2)})\big]^4 \tag{3.2}$$

be some fixed extensions of the vector functions $G^{(1)}$ and $G^{(2)}$, respectively, onto $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ preserving the space. It is evident that arbitrary extensions of the same vector functions can then be represented as

$$G^{(1)*} = G_0^{(1)} + \psi + h^{(1)}, \quad G^{(2)*} = G_0^{(2)} + h^{(2)},$$

where

$$\begin{aligned} \psi &:= (\psi_1, \dots, \psi_6)^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D^{(1)})\,\big]^6, \\ h^{(1)} &:= (h_1^{(1)}, \dots, h_6^{(1)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)\,\big]^6, \\ h^{(2)} &:= (h_1^{(2)}, \dots, h_4^{(2)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)\,\big]^4 \end{aligned} \tag{3.3}$$

are arbitrary vector functions.

We look for a solution pair $(U^{(1)}, U^{(2)})$ of the mixed boundary-transmission problem (2.14)–(2.26) in the form of single layer potentials

$$U^{(1)} = (u_1^{(1)}, \dots, u_6^{(1)})^\top = V_\tau^{(1)}\big([\,P_\tau^{(1)}]^{-1}\,\big[\,G_0^{(1)} + \psi + h^{(1)}\big]\,\big) \ \text{ in } \ \Omega^{(1)}, \tag{3.4}$$

$$U^{(2)} = (u_1^{(2)}, \dots, u_4^{(2)})^\top = V_\tau^{(2)}\big([\,P_\tau^{(2)}\,]^{-1}\,\big[\,G_0^{(2)} + h^{(2)}\big]\,\big) \ \text{ in } \ \Omega^{(2)}, \tag{3.5}$$

where $P_\tau^{(1)}$ and $P_\tau^{(2)}$ are given by (2.33) and (2.32), and $h^{(1)}$, $h^{(2)}$ and ψ are the unknown vector functions satisfying inclusions (3.3).

Keeping in mind (2.34), we see that the homogeneous differential equations (2.14), (2.15), the boundary conditions (2.16), (2.17) and the crack conditions (2.25), (2.26) are satisfied automatically.

The remaining boundary and transmission conditions (2.21)–(2.24) lead to the system of pseudodifferential equations for the unknown vector functions ψ, $h^{(1)}$ and $h^{(2)}$,

$$r_{S_D^{(1)}}\,\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,\big(G_0^{(1)} + \psi + h^{(1)}\big)\,\big] = f^{(1)} \ \text{ on } \ S_D^{(1)}, \tag{3.6}$$

$$r_{\Gamma_T}\,\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\,G_0^{(1)} + \psi + h^{(1)})\,\big]_j = f_j \ \text{ on } \ \Gamma_T, \ \ j = 4, 5, \tag{3.7}$$

$$r_{\Gamma_T}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(G_0^{(1)} + \psi + h^{(1)})\big]_j - r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(G_0^{(2)} + h^{(2)})\big]_j = f_j \ \text{ on } \ \Gamma_T, \quad j = 1, 2, 3, \tag{3.8}$$

$$r_{\Gamma_T}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(G_0^{(1)} + \psi + h^{(1)})\big]_6 - r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(G_0^{(2)} + h^{(2)})\big]_4 = f_6 \ \text{ on } \ \Gamma_T, \tag{3.9}$$

$$r_{\Gamma_T}[G_0^{(1)} + \psi + h^{(1)}]_j + r_{\Gamma_T}[G_0^{(2)} + h^{(2)}]_j = F_j \ \text{ on } \ \Gamma_T, \ \ j = 1, 2, 3, \tag{3.10}$$

$$r_{\Gamma_T}[G_0^{(1)} + \psi + h^{(1)}]_6 + r_{\Gamma_T}[G_0^{(2)} + h^{(2)}]_4 = F_4 \ \text{ on } \ \Gamma_T. \tag{3.11}$$

After some rearrangement we get the system of pseudodifferential equations

$$r_{S_D^{(1)}}\,\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,\big(\psi + h^{(1)}\big)\,\big] = \widetilde{f}^{(1)} \ \text{ on } \ S_D^{(1)}, \tag{3.12}$$

$$r_{\Gamma_T}\,\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\,\psi + h^{(1)})\,\big]_j = \widetilde{f}_j \ \text{ on } \ \Gamma_T, \ \ j = 4, 5, \tag{3.13}$$

$$r_{\Gamma_T}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\psi + h^{(1)})\big]_j - r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(\,h^{(2)})\big]_j = \widetilde{f}_j \ \text{ on } \ \Gamma_T, \ \ j = 1, 2, 3, \tag{3.14}$$

$$r_{\Gamma_T}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\psi + h^{(1)})\big]_6 - r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(\,h^{(2)})\big]_4 = \widetilde{f}_6 \ \text{ on } \ \Gamma_T, \tag{3.15}$$

$$r_{\Gamma_T} h_j^{(1)} + r_{\Gamma_T} h_j^{(2)} = \widetilde{F}_j \ \text{ on } \ \Gamma_T, \ \ j = 1, 2, 3, \tag{3.16}$$

$$r_{\Gamma_T} h_6^{(1)} + r_{\Gamma_T} h_4^{(2)} = \widetilde{F}_4 \ \text{ on } \ \Gamma_T, \tag{3.17}$$

where

$$\widetilde{f}_k^{(1)} := f_k^{(1)} - r_{S_D^{(1)}}\,\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\,\big]_k \in B_{p,p}^{1-\frac{1}{p}}(S_D^{(1)}), \ \ k = \overline{1,6}, \tag{3.18}$$

$$\widetilde{f}_j := f_j - r_{\Gamma_T}\,\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\big]_j \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T), \ \ j = 4, 5, \tag{3.19}$$

$$\widetilde{f}_j := f_j + r_{\Gamma_T}\,\big[\,\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}\,G_0^{(2)}\big]_j - r_{\Gamma_T}\,\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\big]_j \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T), \ \ j = 1, 2, 3, \tag{3.20}$$

$$\widetilde{f}_6 := f_6 + r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}\,G_0^{(2)}\big]_4 - r_{\Gamma_T}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\big]_6 \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T), \tag{3.21}$$

$$\widetilde{F}_j := F_j - r_{\Gamma_T} G_{0j}^{(1)} - r_{\Gamma_T} G_{0j}^{(2)} \in r_{\Gamma_T}\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T), \quad j=1,2,3, \tag{3.22}$$

$$\widetilde{F}_4 := F_4 - r_{\Gamma_T} G_{06}^{(1)} - r_{\Gamma_T} G_{04}^{(2)} \in r_{\Gamma_T}\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T). \tag{3.23}$$

Inclusions (3.22), (3.23) are the *compatibility conditions* for the mixed boundary-transmission problem under consideration. Therefore, in what follows, we assume that $\widetilde{F}_j$ are extended from Γ_T onto the manifold $\partial\Omega^{(2)} \cup \partial\Omega^{(1)} \setminus \Gamma_T$ by zero, i.e., $\widetilde{F}_j \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)$, $j = \overline{1,4}$.

Introduce the Steklov–Poincaré type 6×6 matrix pseudodifferential operators

$$\mathcal{A}_\tau^{(1)} := \mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}, \quad \mathcal{A}_\tau^{(2)} := \mathcal{H}_\tau^{(2)}\,\big(P_\tau^{(2)}\big)^{-1}.$$

Let

$$\mathcal{B}_\tau^{(2)} := \begin{bmatrix} (A_\tau^{(2)})_{11} & (A_\tau^{(2)})_{12} & (A_\tau^{(2)})_{13} & 0 & 0 & (A_\tau^{(2)})_{14} \\ (A_\tau^{(2)})_{21} & (A_\tau^{(2)})_{22} & (A_\tau^{(2)})_{23} & 0 & 0 & (A_\tau^{(2)})_{24} \\ (A_\tau^{(2)})_{31} & (A_\tau^{(2)})_{32} & (A_\tau^{(2)})_{33} & 0 & 0 & (A_\tau^{(2)})_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ (A_\tau^{(2)})_{41} & (A_\tau^{(2)})_{42} & (A_\tau^{(2)})_{43} & 0 & 0 & (A_\tau^{(2)})_{44} \end{bmatrix}_{6\times 6}.$$

Taking into account equations (3.16) and (3.17), we can rewrite equations (3.13), (3.14), (3.15) in a matrix form and, finally, the whole system (3.12)-(3.17) can be rewritten as follows:

$$r_{S_D^{(1)}}\,\mathcal{A}_\tau^{(1)}\,(\psi + h^{(1)}) = \widetilde{f}^{(1)} \ \text{ on } \ S_D^{(1)}, \tag{3.24}$$

$$r_{\Gamma_T}\,\mathcal{A}_\tau^{(1)}\,(\psi + h^{(1)}) + r_{\Gamma_T}\,\mathcal{B}_\tau^{(2)} h^{(1)} = \widetilde{g} \ \text{ on } \ \Gamma_T, \tag{3.25}$$

$$r_{\Gamma_T}\,h_j^{(1)} + r_{\Gamma_T}\,h_j^{(2)} = \widetilde{F}_j \ \text{ on } \ \Gamma_T, \quad j=\overline{1,3}, \tag{3.26}$$

$$r_{\Gamma_T} h_6^{(1)} + r_{\Gamma_T} h_4^{(2)} = \widetilde{F}_4 \ \text{ on } \ \Gamma_T, \tag{3.27}$$

where

$$\widetilde{f}^{(1)} := (\widetilde{f}_1^{(1)}, \ldots, \widetilde{f}_6^{(1)})^\top \in \big[B_{p,p}^{1-\frac{1}{p}}(S_D^{(1)})\big]^6, \tag{3.28}$$

$$\widetilde{g} := (\widetilde{g}_1, \ldots, \widetilde{g}_6)^\top \in \big[B_{p,p}^{1-\frac{1}{p}}(\Gamma_T)\big]^6, \tag{3.29}$$

$$\widetilde{g}_j := \widetilde{f}_j + r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}[P_\tau^{(2)}]^{-1}\,\widetilde{F}\big]_j, \quad j=\overline{1,3}, \tag{3.30}$$

$$\widetilde{g}_4 = \widetilde{f}_4, \ \ \widetilde{g}_5 = \widetilde{f}_5, \ \ \widetilde{g}_6 = \widetilde{f}_6 + r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}[P_\tau^{(2)}]^{-1}\,\widetilde{F}\big]_4,$$

$$\widetilde{F} := (\widetilde{F}_1, \ldots, \widetilde{F}_4)^\top \in \big[\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)\big]^4. \tag{3.31}$$

It is easy to see that the simultaneous equations (3.12)–(3.17) and (3.24)–(3.27), where the right-hand sides are related by equalities (3.18)–(3.23) and (3.28)–(3.31), are equivalent in the following sense: if the triplet $(\psi, h^{(1)}, h^{(2)}) \in [\widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D^{(1)})]^6 \times [\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)]^6 \times [\widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)]^4$ solves the system (3.24)–(3.27), then $(\psi, h^{(1)}, h^{(2)})$ solves the system (3.12)–(3.17), and vice versa.

3.2. **The Existence theorems and regularity of solutions.** Here we show that the system of pseudodifferential equations (3.24)–(3.27) is uniquely solvable in appropriate function spaces. To this end, let us introduce the notation

$$\mathcal{N}_\tau := \begin{bmatrix} r_{S_D^{(1)}}\,\mathcal{A}_\tau^{(1)} & r_{S_D^{(1)}}\,\mathcal{A}_\tau^{(1)} & r_{S_D^{(1)}}\,[\,0\,]_{6\times 4} \\ r_{\Gamma_T}\,\mathcal{A}_\tau^{(1)} & r_{\Gamma_T}[\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}] & r_{\Gamma_T}\,[\,0\,]_{6\times 4} \\ r_{\Gamma_T}\,[\,0\,]_{4\times 6} & r_{\Gamma_T}\,I_{4\times 6} & r_{\Gamma_T}\,I_4 \end{bmatrix}_{16\times 16},$$

$$I_{4\times 6} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 6}.$$

Further, let

$$\begin{aligned}
\Phi :&= (\psi,\ h^{(1)},\ h^{(2)})^\top, \quad Y := (\widetilde{f},\ \widetilde{g},\ \widetilde{F})^\top, \\
\mathbf{X}_p^s &:= \big[\, \widetilde{B}^s_{p,p}(S_D^{(1)})\,\big]^6 \times \big[\, \widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^6 \times \big[\, \widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^4, \\
\mathbf{Y}_p^s &:= \big[\, B^{s+1}_{p,p}(S_D^{(1)})\,\big]^6 \times \big[\, B^{s+1}_{p,p}(\Gamma_T)\,\big]^6 \times \big[\, \widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^4, \\
\mathbf{X}_{p,q}^s &:= \big[\, \widetilde{B}^s_{p,q}(S_D^{(1)})\,\big]^6 \times \big[\, \widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^6 \times \big[\, \widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^4, \\
\mathbf{Y}_{p,q}^s &:= \big[\, B^{s+1}_{p,q}(S_D^{(1)})\,\big]^6 \times \big[\, B^{s+1}_{p,q}(\Gamma_T)\,\big]^6 \times \big[\, \widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^4.
\end{aligned}$$

Note that

$$\mathbf{X}_2^s = \mathbf{X}_{2,2}^s, \quad \mathbf{Y}_2^s = \mathbf{Y}_{2,2}^s, \quad \forall s \in \mathbb{R}.$$

System (3.24)–(3.27) can be rewritten as follows:

$$\mathcal{N}_\tau\, \Phi = Y, \tag{3.32}$$

where $\Phi \in \mathbf{X}_p^s$ is the sought for vector function and $Y \in \mathbf{Y}_p^s$ is a given vector function.

Due to Theorems 5.3 and 5.4, the operator $\mathcal{N}_\tau$ has the following mapping properties:

$$\begin{aligned}
\mathcal{N}_\tau\ &:\ \mathbf{X}_p^s \to \mathbf{Y}_p^s, \\
\mathcal{N}_\tau\ &:\ \mathbf{X}_{p,q}^s \to \mathbf{Y}_{p,q}^s,
\end{aligned} \tag{3.33}$$

for all $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$.

As it will become clear later, the operator (3.33) is not invertible for all $s \in \mathbb{R}$. The interval $a < s < b$ of invertibility depends on p and on some parameters γ' and γ'' (see (3.40)–(3.43)), which are determined by the eigenvalues of special matrices constructed by means of the principal homogeneous symbol matrices of the operators $\mathcal{A}_\tau^{(1)}$ and $\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}$. Note that the numbers γ' and γ'' define also Hölder's smoothness exponents for the solutions to the original mixed boundary-transmission problem in the neighbourhood of the exceptional curves $\partial S_D^{(1)}$, $\partial\Gamma_C$ and $\partial\Gamma$. We start with the following

Theorem 3.1. *Let the conditions*

$$1 < p < \infty, \quad 1 \leqslant q \leqslant \infty, \quad \frac{1}{p} - 1 + \gamma'' < s + \frac{1}{2} < \frac{1}{p} + \gamma' \tag{3.34}$$

be satisfied with γ' and γ'' given by (3.43)*. Then the operators in* (3.33) *are invertible.*

Proof. We prove the theorem in several steps. First, we show that the operators (3.33) are Fredholm ones with a zero index and afterwards we establish that the corresponding null-spaces are trivial.

Step 1. Let us note that the operators

$$\begin{aligned}
r_{_{S_D^{(1)}}}\, \mathcal{A}_\tau^{(1)}\ &:\ \big[\, \widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^6 \to \big[\, B^{s+1}_{p,q}(S_D^{(1)})\,\big]^6, \\
r_{_{\Gamma_T}}\, \mathcal{A}_\tau^{(1)}\ &:\ \big[\, \widetilde{B}^s_{p,q}(S_D^{(1)})\,\big]^6 \to \big[\, B^{s+1}_{p,q}(\Gamma_T)\,\big]^6
\end{aligned} \tag{3.35}$$

are compact since $S_D^{(1)}$ and Γ_T are disjoint, $\overline{S_D^{(1)}} \cap \overline{\Gamma_T} = \varnothing$. Further, we establish that the operators

$$\begin{aligned}
r_{_{S_D^{(1)}}}\, \mathcal{A}_\tau^{(1)}\ &:\ \big[\, \widetilde{H}_2^{-\frac{1}{2}}(S_D^{(1)})\,\big]^6 \to \big[\, [H_2^{\frac{1}{2}}(S_D^{(1)})\,\big]^6, \\
r_{_{\Gamma_T}}\big[\, \mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}\,\big]\ &:\ \big[\, \widetilde{H}_2^{-\frac{1}{2}}(\Gamma_T)\,\big]^6 \to \big[\, H_2^{\frac{1}{2}}(\Gamma_T)\,\big]^6
\end{aligned} \tag{3.36}$$

are strongly elliptic Fredholm pseudodifferential operators of order -1 with a index zero. We note that the principal homogeneous symbol matrices of these operators are strongly elliptic.

Using Green's formula and Korn's inequality, for an arbitrary solution vector $U^{(1)}\in[H_2^1(\Omega^{(1)})]^6=[W_2^1(\Omega^{(1)})]^6$ to the homogeneous equation

$$A^{(1)}(\partial_x,\,\tau)U^{(1)}=0\quad\text{in}\quad\Omega^{(1)},$$

by the standard arguments we derive (see, e.g., [7,8])

$$\operatorname{Re}\big\langle[U^{(1)}]^+,\,[\mathcal{T}^{(1)}U^{(1)}]^+\big\rangle_{\partial\Omega^{(1)}}\geqslant c_1\,\|\,U^{(1)}\,\|^2_{[H_2^1(\Omega^{(1)})]^6}-c_2\,\|\,U^{(1)}\,\|^2_{[H_2^0(\Omega^{(1)})]^6}\,,\tag{3.37}$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega^{(1)}}$ denotes the duality pairing between the spaces $\big[H^{\frac12}(\partial\Omega^{(1)})\big]^6$ and $\big[H^{-\frac12}(\partial\Omega^{(1)})\big]^6$.

Substitute here $U^{(1)}=V_\tau^{(1)}([P_\tau^{(1)}]^{-1}\zeta)$ with $\zeta\in[H_2^{-\frac12}(\partial\Omega^{(1)})]^6$. Due to the equality

$$\zeta=P_\tau^{(1)}[\mathcal{H}_\tau^{(1)}]^{-1}\{U^{(1)}\}^+$$

and boundedness of the operators involved, we have

$$\|\zeta\|^2_{[H_2^{-\frac12}(\partial\Omega^{(1)})]^6}\leqslant c^*\|\{U^{(1)}\}^+\|^2_{[H_2^{\frac12}(\partial\Omega^{(1)})]^6}$$

with some positive constant c^*. By the properties of single layer potentials, we have

$$\big\{U^{(1)}\big\}^+=\mathcal{H}_\tau^{(1)}\big[P_\tau^{(1)}\big]^{-1}\zeta,\quad\big\{\mathcal{T}^{(1)}U^{(1)}\big\}^+=\big(-\frac12 I_6+\mathcal{K}_\tau^{(1)}\big)\big[P_\tau^{(1)}\big]^{-1}\zeta.$$

By the trace theorem, from (3.37), we deduce

$$\begin{aligned}\operatorname{Re}\langle\mathcal{H}_\tau^{(1)}[P_\tau^{(1)}]^{-1}\zeta,\,\big(-2^{-1}I_6+\mathcal{K}_\tau^{(1)}+\beta\,\mathcal{H}_\tau^{(1)}\big)\big[P_\tau^{(1)}\big]^{-1}\zeta\rangle_{\partial\Omega^{(1)}}\geqslant c_1'\,\|\,\zeta\,\|^2_{[H_2^{-\frac12}(\partial\Omega^{(1)})]^6}\\+\big\|\,\beta\,\mathcal{H}^{(1)}[P_\tau^{(1)}]^{-1}\zeta\big\|^2_{[H_2^{\frac12}(\partial\Omega^{(1)})]^6}-c_2\,\big\|\,V_\tau^{(1)}([P_\tau^{(1)}]^{-1}\zeta)\big\|^2_{[H_2^0(\Omega^{(1)})]^6}.\end{aligned}$$

Thus we have

$$\begin{aligned}\operatorname{Re}\langle\mathcal{H}_\tau^{(1)}[P_\tau^{(1)}]^{-1}\zeta,\,\zeta\rangle_{\partial\Omega^{(1)}}\geqslant c_1'\,\|\,\zeta\,\|^2_{[H_2^{-\frac12}(\partial\Omega^{(1)})]^6}\\+\big\|\,\beta\,\mathcal{H}^{(1)}[P_\tau^{(1)}]^{-1}\zeta\big\|^2_{[H_2^{\frac12}(\partial\Omega^{(1)})]^6}-c_2\,\big\|\,V_\tau^{(1)}([P_\tau^{(1)}]^{-1}\zeta)\big\|^2_{[H_2^0(\Omega^{(1)})]^6}.\end{aligned}$$

In particular, in view of Theorem 5.1, for arbitrary $\zeta\in[\widetilde{H}_2^{-\frac12}(S_D^{(1)})]^6$, we have

$$\|\,U^{(1)}\,\|^2_{[H_2^0(\Omega^{(1)})]^6}\leqslant c^{**}\,\|\,\zeta\,\|^2_{[\widetilde{H}_2^{-\frac32}(S_D^{(1)})]^6}\,,$$

and, consequently,

$$\operatorname{Re}\big\langle r_{S_D^{(1)}}\mathcal{H}_\tau^{(1)}[P_\tau^{(1)}]^{-1}\zeta,\,\zeta\big\rangle_{\partial\Omega^{(1)}}\geqslant c_1'\,\|\,\zeta\,\|^2_{[\widetilde{H}_2^{-\frac12}(S_D^{(1)})]^6}-c_2''\,\|\,\zeta\,\|^2_{[\widetilde{H}_2^{-\frac32}(S_D^{(1)})]^6}\,.\tag{3.38}$$

From (3.38), it follows that

$$r_{S_D^{(1)}}\mathcal{A}_\tau^{(1)}=r_{S_D^{(1)}}\mathcal{H}_\tau^{(1)}[P_\tau^{(1)})]^{-1}\;:\;\big[\,\widetilde{H}_2^{-\frac12}(S_D^{(1)})\,\big]^6\to\big[\,H_2^{\frac12}(S_D^{(1)})\,\big]^6$$

is a strongly elliptic pseudodifferential Fredholm operator with index zero (see [21,23]).

Then the same is true for the operator (3.36), since the principal homogeneous symbol matrix of the operator $\mathcal{B}_\tau^{(2)}$ is nonnegative (see [25]). Therefore, the operator (3.33) is Fredholm with index zero for $s=-1/2$, $p=2$ and $q=2$ due to the compactness of operators (3.35).

Step 2. With the help of the uniqueness Theorem 2.1, via representation formulas (3.4) and (3.5) with $G_0^{(1)}=0$ and $G_0^{(2)}=0$, we can easily show that the operator (3.33) is injective for $s=-1/2$, $p=2$ and $q=2$. Since its index is zero, we conclude that it is surjective. Thus the operator (3.33) is invertible for $s=-1/2$, $p=2$ and $q=2$.

Step 3. To complete the proof for the general case we proceed as follows. The following block-wise lower triangular operator

$$\mathcal{N}_\tau^{(0)} := \begin{bmatrix} r_{S_D^{(1)}} \mathcal{A}_\tau^{(1)} & r_{S_D^{(1)}} [\,0\,]_{6\times 6} & r_{S_D^{(1)}} [\,0\,]_{6\times 4} \\ r_{\Gamma_T} [\,0\,]_{6\times 6} & r_{\Gamma_T} [\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}] & r_{\Gamma_T} [\,0\,]_{6\times 4} \\ r_{\Gamma_T} [\,0\,]_{4\times 6} & r_{\Gamma_T} I_{4\times 6} & r_{\Gamma_T} I_4 \end{bmatrix}_{16\times 16}$$

is a compact perturbation of the operator $\mathcal{N}_\tau$. Let us analyze the properties of the diagonal entries

$$r_{S_D^{(1)}} \mathcal{A}_\tau^{(1)} \ : \ [\,\widetilde{B}^s_{p,q}(S_D^{(1)})\,]^6 \to \big[\,B^{s+1}_{p,q}(S_D^{(1)})\,\big]^6,$$

$$r_{\Gamma_T} \big[\,\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}\,\big] \ : \ \big[\,\widetilde{B}^s_{p,q}(\Gamma_T)\big]^6 \to \big[\,B^{s+1}_{p,q}(\Gamma_T)\,\big]^6.$$

Let

$$\mathfrak{S}_1(x,\xi_1,\xi_2) := \mathfrak{S}(\mathcal{A}_\tau^{(1)}; x,\xi_1,\xi_2)$$

be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau^{(1)}$ and let $\lambda_j^{(1)}(x)$ $(j=\overline{1,6})$ be the eigenvalues of the matrix

$$\mathcal{D}_1(x) := \big[\mathfrak{S}_1(x,0,+1)\big]^{-1}\, \mathfrak{S}_1(x,0,-1), \quad x\in\partial S_D^{(1)}.$$

Similarly, let

$$\mathfrak{S}_2(x,\xi_1,\xi_2) = \mathfrak{S}(\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}; x,\xi_1,\xi_2)$$

be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}$ and let $\lambda_j^{(2)}(x)$ $(j=\overline{1,6})$ be the eigenvalues of the corresponding matrix

$$\mathcal{D}_2(x) := \big[\mathfrak{S}_2(x,0,+1)\big]^{-1}\, \mathfrak{S}_2(x,0,-1), \quad x\in\partial\Gamma_T. \tag{3.39}$$

Note that the curve $\partial\Gamma_T$ is the union of the curves, where the interface intersects the exterior boundary $\partial\Gamma$, and the crack edge $\partial\Gamma_C$, $\partial\Gamma_T = \partial\Gamma\cup\partial\Gamma_C$.

Further, we set

$$\gamma_1' := \inf_{x\in\partial S_D^{(1)},\,1\leqslant j\leqslant 6} \frac{1}{2\pi}\,\arg\lambda_j^{(1)}(x), \quad \gamma_1'' := \sup_{x\in\partial S_D^{(1)},\,1\leqslant j\leqslant 6} \frac{1}{2\pi}\,\arg\lambda_j^{(1)}(x), \tag{3.40}$$

$$\gamma_2' := \inf_{x\in\partial \Gamma_T,\,1\leqslant j\leqslant 6} \frac{1}{2\pi}\,\arg\lambda_j^{(2)}(x), \quad \gamma_2'' := \sup_{x\in\partial \Gamma_T,\,1\leqslant j\leqslant 6} \frac{1}{2\pi}\,\arg\lambda_j^{(2)}(x). \tag{3.41}$$

It can be shown that one of the eigenvalues is equal to 1, say $\lambda_6^{(1)}=1$ (for details see [6, Subsection 4.4], [7, Subsection 5.7]) and [8, Theorem 4.7]. Therefore we have

$$\gamma_1'\leqslant 0, \quad \gamma_1''\geqslant 0. \tag{3.42}$$

Note that γ_j' and γ_j'' $(j=1,2)$ depend on the material parameters, in general, and belong to the interval $(-\frac{1}{2}\,,\,\frac{1}{2})$. We put

$$\gamma' := \min\{\gamma_1',\,\gamma_2'\}, \quad \gamma'' := \max\{\gamma_1'',\,\gamma_2''\}. \tag{3.43}$$

In view of (3.42), we have

$$-\frac{1}{2} < \gamma' \leqslant 0 \leqslant \gamma'' < \frac{1}{2}\,. \tag{3.44}$$

From Theorem 5.5, we conclude that if the parameters $r_1,r_2\in\mathbb{R}$, $1<p<\infty$, $1\leqslant q\leqslant\infty$, satisfy the conditions

$$\frac{1}{p}-1+\gamma_1'' < r_1+\frac{1}{2} < \frac{1}{p}+\gamma_1', \quad \frac{1}{p}-1+\gamma_2'' < r_2+\frac{1}{2} < \frac{1}{p}+\gamma_2',$$

then the operators

$$r_{S_D^{(1)}} \mathcal{A}_\tau^{(1)} \ : \ \big[\,\widetilde{H}^{r_1}_p(S_D^{(1)})\,\big]^6 \to \big[\,H^{r_1+1}_p(S_D^{(1)})\,\big]^6,$$

$$r_{S_D^{(1)}} \mathcal{A}_\tau^{(1)} \ : \ \big[\,\widetilde{B}^{r_1}_{p,q}(S_D^{(1)})\big]^6 \to \big[[B^{r_1+1}_{p,q}(S_D^{(1)})\,\big]^6,$$

$$r_{\Gamma_T}\big[\,\mathcal{A}_\tau^{(1)}+\mathcal{B}_\tau^{(2)}\,\big]\ :\ \big[\,\widetilde{H}_p^{r_2}(\Gamma_T)\,\big]^6\to\big[\,H_p^{r_2+1}(\Gamma_T)\,\big]^6,$$

$$r_{\Gamma_T}\big[\,\mathcal{A}_\tau^{(1)}+\mathcal{B}_\tau^{(2)}\,\big]\ :\ \big[\,\widetilde{B}_{p,q}^{r_2}(\Gamma_T)\,\big]^6\to\big[\,B_{p,q}^{r_2+1}(\Gamma_T)\,\big]^6$$

are the Fredholm operators with index zero.

Therefore, if conditions (3.34) are satisfied, then the above operators are Fredholm ones with a zero index. Consequently, operators (3.33) are Fredholm with zero index and are invertible due to the results obtained in Step 2 (see [2]) . □

Now we formulate the basic existence and uniqueness results for the mixed boundary-transmission problem under consideration.

Theorem 3.2. *Let inclusions* (2.27) *and compatibility conditions* (3.22), (3.23) *hold and let*

$$\frac{4}{3-2\gamma''}<p<\frac{4}{1-2\gamma'} \tag{3.45}$$

with γ' *and* γ'' *be defined in* (3.43). *Then the mixed boundary-transmission problem* (2.14)–(2.26) *has a unique solution*

$$(U^{(1)},U^{(2)})\in\big[\,W_p^1(\Omega^{(1)})\,\big]^6\times[\,W_p^1(\Omega^{(2)})\,]^4,$$

which can be represented by the formulas

$$U^{(1)}=V_\tau^{(1)}\big([P_\tau^{(1)}]^{-1}\big[\,G_0^{(1)}+\psi+h^{(1)}\big]\big)\ \ in\ \ \Omega^{(1)}, \tag{3.46}$$

$$U^{(2)}=V_\tau^{(2)}\big(\big[\,P_\tau^{(2)}\,\big]^{-1}\big[\,G_0^{(2)}+h^{(2)}\big]\big)\ \ in\ \ \Omega^{(2)}, \tag{3.47}$$

where the densities ψ, $h^{(1)}$ *and* $h^{(2)}$ *are to be determined from system* (3.6)–(3.11) (*or from system* (3.24)–(3.27)), *while* $G_0^{(1)}$ *and* $G_0^{(2)}$ *are some fixed extensions of the vector functions* $G^{(1)}$ *and* $G^{(2)}$, *respectively, onto* $\partial\Omega^{(1)}$ *and* $\partial\Omega^{(2)}$, *preserving the space* (*see* (3.1) *and* (3.2)).

Moreover, the vector functions $G_0^{(1)}+\psi+h^{(1)}$ *and* $G_0^{(2)}+h^{(2)}$ *are defined uniquely by the above systems and are independent of the extension operators.*

Proof. From Theorems 5.1, 5.2 and 3.1 with p satisfying (3.45) and $s=-1/p$ it follows immediately that the pair $(U^{(1)},U^{(2)})\in[W_p^1(\Omega^{(1)})]^6\times[W_p^1(\Omega^{(2)})]^4$ given by (3.46), (3.47) represents a solution to the mixed boundary-transmission problem (2.14)–(2.26). Next, we show the uniqueness of solutions.

Due to inequalities (3.44), we have

$$p=2\in\Big(\frac{4}{3-2\gamma''}\,,\,\frac{4}{1-2\gamma'}\Big).$$

Therefore the unique solvability for $p=2$ is a consequence of Theorem 2.1.

To show the uniqueness result for all other values of p from the interval (3.45), we proceed as follows. Let a pair

$$(U^{(1)},U^{(2)})\in\big[\,W_p^1(\Omega^{(1)})\,\big]^6\times[\,W_p^1(\Omega^{(2)})\,]^4$$

with p satisfying (3.45), be a solution to the homogeneous mixed boundary-transmission problem. Then it is evident that

$$\big\{\,U^{(1)}\,\big\}^+\in\big[\,B_{p,p}^{1-\frac1p}(\partial\Omega^{(1)})\,\big]^6,\quad \big\{\,U^{(2)}\,\big\}^+\in\big[\,B_{p,p}^{1-\frac1p}(\partial\Omega^{(2)})\,\big]^4,$$

$$\big\{\,\mathcal{T}^{(1)}U^{(1)}\,\big\}^+\in\big[\,B_{p,p}^{-\frac1p}(\partial\Omega^{(1)})\,\big]^6,\quad \big\{\,\mathcal{T}^{(2)}U^{(2)}\,\big\}^+\in\big[\,B_{p,p}^{-\frac1p}(\partial\Omega^{(2)})\,\big]^4.$$

By Lemmas 2.2 and 2.3, the vectors $U^{(2)}$ and $U^{(1)}$ in $\Omega^{(2)}$ and $\Omega^{(1)}$, respectively, are representable in the form

$$U^{(2)}=V_\tau^{(2)}\big(\big[\,P_\tau^{(2)}\big]^{-1}h^{(2)}\big)\ \text{ in }\ \Omega^{(2)},\quad h^{(2)}=\big\{\mathcal{T}^{(2)}U^{(2)}\,\big\}^+,$$

$$U^{(1)}=V_\tau^{(1)}\big([P_\tau^{(1)}]^{-1}\chi\big)\ \text{ in }\ \Omega^{(1)},\quad \chi=\big\{\mathcal{T}^{(1)}U^{(1)}\big\}^++\beta\,\big\{U^{(1)}\big\}^+.$$

Moreover, due to the homogeneous boundary and transmission conditions, we have

$$h^{(2)}\in\big[\,\widetilde{B}_{p,p}^{-\frac1p}(\Gamma_T)\,\big]^4,\ \ \chi=h^{(1)}+\psi\in[B_{p,p}^{-\frac1p}(S^{(1)})\,\big]^6,\ \ h^{(1)}\in\big[\,\widetilde{B}_{p,p}^{-\frac1p}(\Gamma_T)\,\big]^6,\ \ \psi\in\big[\,\widetilde{B}_{p,p}^{-\frac1p}(S_D^{(1)})\,\big]^6.$$

By the same arguments as above we arrive at the homogeneous system

$$\mathcal{N}_\tau\,\Phi = 0 \ \text{ with } \ \Phi := (\psi,\ h^{(1)},\ h^{(2)})^\top \in \mathbf{X}_p^{-\frac{1}{p}}\,.$$

Due to Theorem 3.1, $\Phi = 0$ and we conclude that $U^{(2)} = 0$ in $\Omega^{(2)}$ and $U^{(1)} = 0$ in $\Omega^{(1)}$.

The last assertion of the theorem is trivial and is an easy consequence of the fact that if the single layer potentials (3.46)and (3.47) vanish identically in $\Omega^{(2)}$ and $\Omega^{(1)}$, then the corresponding densities vanish, as well. □

The following regularity result is true.

Theorem 3.3. *Let the inclusions* (2.27) *and compatibility conditions* (3.22), (3.23) *hold and let* $1<r<\infty$, $1\leqslant q\leqslant\infty$,

$$\frac{4}{3-2\gamma''} < p < \frac{4}{1-2\gamma'}\,, \quad \frac{1}{r}-\frac{1}{2}+\gamma'' < s < \frac{1}{r}+\frac{1}{2}+\gamma', \tag{3.48}$$

with γ' *and* γ'' *defined in* (3.43).

Further, let $U^{(1)}\in[W_p^1(\Omega^{(1)})]^6$ *and* $U^{(2)}\in[W_p^1(\Omega^{(2)})]^4$ *be a unique solution pair to the mixed boundary-transmission problem* (2.14)–(2.26). *Then the following items hold:*

(i) *if*

$$Q_k^{(1)}\in B_{r,r}^{s-1}(S_N^{(1)}),\quad Q_j^{(2)}\in B_{r,r}^{s-1}(S_N^{(2)}),\quad f_k^{(1)}\in B_{r,r}^{s}(S_D^{(1)}),\quad f_k\in B_{r,r}^{s}(\Gamma_T),\quad F_j\in B_{r,r}^{s-1}(\Gamma_T),$$
$$\widetilde{Q}_j^{(2)}\in B_{r,r}^{s-1}(\Gamma_C),\quad \widetilde{Q}_k^{(1)}\in B_{r,r}^{s-1}(\Gamma_C),\ \ k=\overline{1,6},\ \ j=\overline{1,4},$$

and the compatibility conditions

$$\widetilde{F}_j := F_j - r_{\Gamma_T}\,G_{0j}^{(1)} - r_{\Gamma_T}G_{0j}^{(2)} \in r_{\Gamma_T}\widetilde{B}_{r,r}^{s-1}(\Gamma_T),\ \ j=\overline{1,3},$$
$$\widetilde{F}_4 := F_4 - r_{\Gamma_T}\,G_{06}^{(1)} - r_{\Gamma_T}G_{04}^{(2)} \in r_{\Gamma_T}\widetilde{B}_{r,r}^{s-1}(\Gamma_T),$$

are satisfied, then

$$U^{(1)}\in[\,H_r^{s+\frac{1}{r}}(\Omega^{(1)})\,]^6,\ \ U^{(2)}\in[\,H_r^{s+\frac{1}{r}}(\Omega^{(2)})\,]^4;$$

(ii) *if*

$$Q_k^{(1)}\in B_{r,q}^{s-1}(S_N^{(1)}),\quad Q_j^{(2)}\in B_{r,q}^{s-1}(S_N^{(2)}),\quad f_k^{(1)}\in B_{r,q}^{s}(S_D^{(1)}),\quad f_k\in B_{r,q}^{s}(\Gamma_T),\quad F_j\in B_{r,q}^{s-1}(\Gamma_T),$$
$$\widetilde{Q}_j^{(2)}\in B_{r,q}^{s-1}(\Gamma_C),\quad \widetilde{Q}_k^{(1)}\in B_{r,q}^{s-1}(\Gamma_C),\ \ k=\overline{1,6},\ \ j=\overline{1,4},$$

and the compatibility conditions

$$\widetilde{F}_j := F_j - r_{\Gamma_T}\,G_{0j}^{(1)} - r_{\Gamma_T}G_{0j}^{(2)} \in r_{\Gamma_T}\widetilde{B}_{r,q}^{s-1}(\Gamma_T),\ \ j=\overline{1,3},$$
$$\widetilde{F}_4 := F_4 - r_{\Gamma_T}\,G_{06}^{(1)} - r_{\Gamma_T}G_{04}^{(2)} \in r_{\Gamma_T}\widetilde{B}_{r,q}^{s-1}(\Gamma_T),$$

are satisfied, then

$$U^{(1)}\in\big[\,B_{r,q}^{s+\frac{1}{r}}(\Omega^{(1)})\big]^6,\quad U^{(2)}\in\big[\,B_{r,q}^{s+\frac{1}{r}}(\Omega^{(2)})\big]^4;$$

(iii) *if* $\alpha>0$ *is not integer and*

$$Q_k^{(1)}\in B_{\infty,\infty}^{\alpha-1}(S_N^{(1)}),\quad Q_j^{(2)}\in B_{\infty,\infty}^{\alpha-1}(S_N^{(2)}),\quad f_k^{(1)}\in C^{\alpha}(\overline{S_D^{(1)}}),\quad f_k\in C^{\alpha}(\overline{\Gamma_T}),$$
$$F_j\in B_{\infty,\infty}^{\alpha-1}(\Gamma_T),\quad \widetilde{Q}_j^{(2)}\in B_{\infty,\infty}^{\alpha-1}(\Gamma_C),\quad \widetilde{Q}_k^{(1)}\in B_{\infty,\infty}^{\alpha-1}(\Gamma_C),\ \ k=\overline{1,6},\ \ j=\overline{1,4},$$

and the compatibility conditions

$$\widetilde{F}_j := F_j - r_{\Gamma_T}\,G_{0j}^{(1)} - r_{\Gamma_T}G_{0j}^{(2)} \in r_{\Gamma_T}\widetilde{B}_{\infty,\infty}^{\alpha-1}(\Gamma_T),\ \ j=\overline{1,3},$$
$$\widetilde{F}_4 := F_4 - r_{\Gamma_T}\,G_{06}^{(1)} - r_{\Gamma_T}G_{04}^{(2)} \in r_{\Gamma_T}\widetilde{B}_{\infty,\infty}^{\alpha-1}(\Gamma_T),$$

are satisfied, then

$$U^{(1)}\in\bigcap_{\alpha'<\kappa}\big[\,C^{\alpha'}(\overline{\Omega^{(1)}}\,)\big]^6,\quad U^{(2)}\in\bigcap_{\alpha'<\kappa}\big[\,C^{\alpha'}(\overline{\Omega^{(2)}}\,)\big]^4,$$

where $\kappa = \min\{\alpha,\ \gamma' + \frac{1}{2}\} > 0$.

Proof. It is word for word repeats the proof of Theorem 5.22 in [7]. $\square$

Regularity results for $u_6^{(1)} = \vartheta^{(1)}$ and $u_4^{(2)} = \vartheta^{(2)}$ are refined in Proposition 3.4 (see also Theorem 4.1).

Proposition 3.4. *Let the conditions of Theorem* 3.3 *(i) and* (3.48) *hold, then*

$$u_6^{(1)} \in C^{\frac{1}{2}-\varepsilon}(\overline{\Omega^{(1)}}), \quad u_4^{(2)} \in C^{\frac{1}{2}-\varepsilon}(\overline{\Omega^{(2)}}), \tag{3.49}$$

where ε *is an arbitrarily small positive number.*

Proof. Due to Theorem 3.3.(i), we deduce

$$U^{(1)} \in \big[\, H_r^{s+\frac{1}{r}}(\Omega^{(1)})\,\big]^6, \quad U^{(2)} \in \big[\, H_r^{s+\frac{1}{r}}(\Omega^{(2)})\,\big]^4,$$

where s and r satisfy (3.48). Note that $u_6^{(1)} = \vartheta^{(1)}$ and $u_4^{(2)} = \vartheta^{(2)}$ solve the following mixed boundary-transmission problem:

$$\begin{cases} \eta_{il}^{(1)}\partial_i\partial_l u_6^{(1)} - \tau^2 h_0^{(1)} u_6^{(1)} = Q^{(1)*} \ \text{ in } \ \Omega^{(1)}, \\ \eta_{il}^{(2)}\partial_i\partial_l u_4^{(2)} - \tau^2 h_0^{(2)} u_4^{(1)} = Q^{(2)*} \ \text{ in } \ \Omega^{(2)}, \\ r_{\Gamma_T}\{u_6^{(1)}\}^+ - r_{\Gamma_T}\{u_4^{(2)}\}^+ = f_6 \ \text{ on } \ \Gamma_T, \\ r_{\Gamma_T}\big\{[\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}]_6\big\}^+ + r_{\Gamma_T}\big\{[\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}]_4\big\}^+ = F_4 \ \text{ on } \ \Gamma_T, \\ r_{S_N^{(1)}\cup\Gamma_C}\big\{[\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}]_6\big\}^+ = G_6^{(1)} \ \text{ on } \ S_N^{(1)}\cup\Gamma_C, \\ r_{S_N^{(2)}\cup\Gamma_C}\big\{[\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}]_4\big\}^+ = G_4^{(2)} \ \text{ on } \ S_N^{(2)}\cup\Gamma_C, \\ r_{S_D^{(1)}}\{u_6^{(1)}\}^+ = f_6^{(1)} \ \text{ on } \ S_D^{(1)}, \end{cases} \tag{3.50}$$

where

$$\big[\,\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\,\big]_6 = \eta_{il}^{(1)}\, n_i\,\partial_l\vartheta^{(1)}, \quad \big[\,\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}\,\big]_4 = \eta_{il}^{(2)}\,\nu_i\,\partial_l\vartheta^{(2)},$$
$$Q^{(1)*} = \tau\lambda_{kl}^{(1)}\partial_l u_k^{(1)} - \tau p_l^{(1)}\partial_l\varphi^{(1)} - \tau m_l^{(1)}\partial_l\psi^{(1)} + \tau d_0^{(1)}\vartheta^{(1)} \in H_r^{s+\frac{1}{r}-1}(\Omega^{(1)}),$$
$$Q^{(2)*} = \tau\lambda_{kl}^{(1)}\partial_l u_k^{(2)} + \tau d_0^{(2)}\vartheta^{(2)} \in H_r^{s+\frac{1}{r}-1}(\Omega^{(2)}),$$
$$f_6 \in B_{r,r}^{s'}(\Gamma_T), \quad F_4 \in B_{r,r}^{s'-1}(\Gamma_T), \quad f_6^{(1)} \in B_{r,r}^{s'}(S_D^{(1)}), \quad G_6^{(1)} \in B_{r,r}^{s'-1}(S_N^{(1)}\cup\Gamma_C),$$
$$G_4^{(2)} \in B_{r,r}^{s'-1}(S_N^{(2)}\cup\Gamma_C), \quad s < s' < \frac{1}{r} + \frac{1}{2}, \ \ 1 < r < \infty.$$

Since the symbols of the differential operators $-\eta_{il}^{(1)}\partial_i\partial_j$ and $-\eta_{il}^{(2)}\partial_i\partial_j$ are positive, the above problem can be reduced to the strongly elliptic system of pseudodifferential equations. Moreover, the corresponding pseudodifferential operator is positive definite. Therefore (see [25])

$$u_6^{(1)} \in H_r^{s'+\frac{1}{r}}(\Omega^{(1)}), \ \ u_4^{(2)} \in H_r^{s'+\frac{1}{r}}(\Omega^{(2)}), \ \ s < s' < \frac{1}{r} + \frac{1}{2}\,, \ \ 1 < r < \infty.$$

Due to the embedding theorem (see [33]), for sufficiently small $\delta > 0$, sufficiently large r and $s' > 1/2 + 1/r - \delta$ we have

$$H_r^{s'+\frac{1}{r}}(\Omega^{(1)}) \subset C^{\frac{1}{2}-\frac{1}{r}-\delta}(\overline{\Omega^{(1)}}), \quad H_r^{s'+\frac{1}{r}}(\Omega^{(2)}) \subset C^{\frac{1}{2}-\frac{1}{r}-\delta}(\overline{\Omega^{(2)}}).$$

Therefore (3.49) holds with $\varepsilon = 1/r + \delta$. $\square$

3.3. **Asymptotic behaviour of solutions near the exceptional curves.** Here, we study the asymptotic properties of solutions to the mixed boundary-transmission problem near the interfacial crack edge $\partial\Gamma_C$ and at the curve $\partial\Gamma$, where the interface intersects the exterior boundary. Let us set $\ell := \partial\Gamma_C \cup \partial\Gamma = \partial\Gamma_T$.

Note that the regularity and the asymptotic behaviour of solutions near the collision curve $\partial S_D^{(1)}$ were studied in details in [8].

For the sake of simplicity of description of the method, we assume that the boundary data and the geometrical characteristics of the problem are infinitely smooth. In particular,

$$Q_k^{(1)} \in C^\infty(\overline{S}_N^{(1)}), \;\; Q_j^{(2)} \in C^\infty(\overline{S_N^{(2)}}), \;\; f_k^{(1)} \in C^\infty(\overline{S}_D^{(1)}),$$
$$f_k \in C^\infty(\overline{\Gamma_T}), \;\; F_j \in C^\infty(\overline{\Gamma_T}), \;\; \widetilde{Q}_j^{(1)} \in C^\infty(\overline{\Gamma_C}),$$
$$\widetilde{F}_i := F_i - r_{\Gamma_T} G_{0i}^{(1)} - r_{\Gamma_T} G_{0i}^{(2)} \in C_0^\infty(\overline{\Gamma_T}), \quad \widetilde{F}_4 := F_4 - r_{\Gamma_T} G_{06}^{(1)} - r_{\Gamma_T} G_{04}^{(2)} \in C_0^\infty(\overline{\Gamma_T}),$$
$$\widetilde{Q}_j^{(2)} \in C^\infty(\overline{\Gamma_C}), \;\; i = \overline{1,3}, \;\; j = \overline{1,4}, \;\; k = \overline{1,6},$$

where $C_0^\infty(\overline{\Gamma_T})$ denotes a space of infinitely differentiable functions vanishing on $\partial\Gamma_T$ along with all tangential derivatives.

We have already shown that the mixed boundary-transmission problem is uniquely solvable and the pair of solution vectors $(U^{(1)}, U^{(2)})$ are represented by (3.46), (3.47) with the densities defined by the system of pseudodifferential equations (3.6)–(3.11) i.e., (3.24)–(3.27).

Let $\Phi := (\psi, h^{(1)}, h^{(2)})^\top \in \mathbf{X}_p^s$ be a solution of the system (3.24)–(3.27) which is written in matrix form (3.32)

$$\mathcal{N}_\tau \Phi = Y,$$

where

$$Y \in \big[\, C^\infty(\overline{S}_D) \,\big]^6 \times \big[\, C^\infty(\overline{\Gamma_T}) \,\big]^6 \times \big[\, C_0^\infty(\overline{\Gamma_T}) \,\big]^4.$$

To establish asymptotic properties of the solution vectors $U^{(1)}$ and $U^{(2)}$ near the exceptional curve $\ell = \partial\Gamma_T$, we rewrite the representations (3.46), (3.47) in the form

$$U^{(1)} = V_\tau^{(1)}([P_\tau^{(1)}]^{-1}\psi) + V_\tau^{(1)}([P_\tau^{(1)}]^{-1}h^{(1)}) + R^{(1)} \;\text{ in }\; \Omega^{(1)},$$
$$U^{(2)} = V_\tau^{(2)}([P_\tau^{(2)}]^{-1}\widetilde{h}^{(2)}) + R^{(2)} \;\text{ in }\; \Omega^{(2)},$$

where

$$\psi \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D^{(1)}) \,\big]^6, \;\; h^{(1)} = (h_1^{(1)}, \dots, h_6^{(1)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T) \,\big]^6,$$
$$\widetilde{h}^{(2)} = -(h_1^{(1)}, h_2^{(1)}, h_3^{(1)}, h_6^{(1)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T) \,\big]^4, \quad R^{(1)} := V_\tau^{(1)}([P_\tau^{(1)}]^{-1}G_0^{(1)}) \in \big[\, C^\infty(\overline{\Omega^{(1)}}) \,\big]^6,$$
$$R^{(2)} := V_\tau^{(2)}([P_\tau^{(2)}]^{-1}G_0^{(2)}) + V_\tau^{(2)}([P_\tau^{(2)}]^{-1}\widetilde{F}) \in \big[\, C^\infty(\overline{\Omega^{(2)}}) \,\big]^4, \;\; \widetilde{F} = \big(\widetilde{F}_1, \dots, \widetilde{F}_4\big)^\top.$$

The vectors $h^{(1)} = (h_1^{(1)}, \dots, h_6^{(1)})^\top$ and $\psi = (\psi_1, \dots, \psi_6)^\top$ solve the following strongly elliptic system of pseudodifferential equations (see (3.24)-(3.27)):

$$r_{S_D^{(1)}} \mathcal{A}_\tau^{(1)} \psi = \Phi^{(1)} \qquad \text{on} \qquad S_D^{(1)},$$
$$r_{\Gamma_T} (\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}) h^{(1)} = \Phi^{(2)} \qquad \text{on} \qquad \Gamma_T,$$

where

$$\Phi_k^{(1)} = f_k^{(1)} - r_{S_D^{(1)}} [\mathcal{A}_\tau^{(1)} G_0^{(1)}]_k - r_{S_D^{(1)}} [\mathcal{A}_\tau^{(1)} h^{(1)}]_k, \;\; k = \overline{1,6},$$
$$\Phi^{(1)} = (\Phi_1^{(1)}, \dots, \Phi_6^{(1)})^\top \in \big[\, C^\infty(\overline{S}_D^{(1)})\big]^6,$$
$$\Phi_j^{(2)} = f_j + r_{\Gamma_T} \big[\, \mathcal{H}_\tau^{(2)} (P_\tau^{(2)})^{-1} G_0^{(2)} \,\big]_j - r_{\Gamma_T} [\mathcal{A}_\tau^{(1)} G_0^{(1)}]_j$$
$$\qquad + r_{\Gamma_T} \big[\, \mathcal{H}_\tau^{(2)} (P_\tau^{(2)})^{-1} \widetilde{F} \,\big]_j - r_{\Gamma_T} [\mathcal{A}_\tau^{(1)} \psi]_j, \;\; j = 1, 2, 3,$$
$$\Phi_j^{(2)} = f_j - r_{\Gamma_T} [\mathcal{A}_\tau^{(1)} G_0^{(1)}]_j - r_{\Gamma_T} [\mathcal{A}_\tau^{(1)} \psi]_j \,, \;\; j = 4, 5,$$

$$\Phi_6^{(2)} = f_6 + r_{\Gamma_T}\left[\mathcal{H}_\tau^{(2)}(P_\tau^{(2)})^{-1}\widetilde{F}\right]_4 - r_{\Gamma_T}\,[\mathcal{A}_\tau^{(1)}\psi]_6,$$
$$\Phi^{(2)} = (\Phi_1^{(2)},\dots,\Phi_6^{(2)})^\top \in \left[C^\infty(\overline{\Gamma_T})\right]^6.$$

Applying a partition of unity, natural local coordinate systems and standard rectifying technique based on canonical diffeomorphisms, we can assume that $\ell = \partial\Gamma_T$ is rectified. Then we identify a one-sided neighbourhood on Γ_T of an arbitrary point $\widetilde{x} \in \ell = \partial\Gamma_T$ as a part of the half-plane $x_2 > 0$. Thus we assume that $(x_1, 0) = \widetilde{x} \in \ell = \partial\Gamma_T$ and $(x_1, x_{2,+}) \in \Gamma_T$ for $0 < x_{2,+} < \varepsilon$ with some positive ε.

Denote by m_j the algebraic multiplicities of $\lambda_j^{(2)}(x_1)$, where $\lambda_j^{(2)}$, $j = \overline{1,6}$, are the eigenvalues of the matrix $\mathcal{D}_2(x_1)$ (see (3.39)). Let $\mu_1(x_1),\dots,\mu_l(x_1)$, $1 \leqslant l \leqslant 6$, be the distinct eigenvalues. Evidently, m_j and l depend on x_1, in general, and $m_1 + \dots + m_l = 6$.

It is well known that the matrix $\mathcal{D}_2(x_1)$ in (3.39) admits the following decomposition (see, e.g., [19]):

$$\mathcal{D}_2(x_1) = \mathcal{D}(x_1)\,\mathcal{J}_{\mathcal{D}_2}(x_1)\left[\mathcal{D}(x_1)\right]^{-1}, \quad (x_1,0) \in \ell = \partial\Gamma_T, \tag{3.51}$$

where $\mathcal{D}$ is the 6×6 nondegenerate matrix with infinitely differentiable entries and $\mathcal{J}_{\mathcal{D}_2}$ is block diagonal

$$\mathcal{J}_{\mathcal{D}_2}(x_1) := \operatorname{diag}\Big\{\mu_1(x_1)B^{(m_1)}(1),\dots,\mu_l(x_1)B^{(m_l)}(1)\Big\}.$$

Here, $B^{(r)}(t)$, $r \in \{m_1,\dots,m_l\}$ are upper triangular matrices,

$$B^{(r)}(t) = \big\|b_{jk}^{(r)}(t)\big\|_{r\times r}, \quad b_{jk}^{(r)}(t) = \begin{cases} \dfrac{t^{k-j}}{(k-j)!}, & j<k,\\ 1, & j=k,\\ 0, & j>k.\end{cases}$$

Denote

$$B_0(t) := \operatorname{diag}\big\{B^{(m_1)}(t),\,\dots,\,B^{(m_l)}(t)\big\}. \tag{3.52}$$

Applying the results from reference [15], we derive the following asymptotic expansion:

$$\begin{aligned} h^{(1)}(x_1,x_{2,+}) &= \mathcal{D}(x_1)\,x_{2,+}^{-\frac12+\Delta(x_1)}\,B_0\Big(-\frac{1}{2\pi i}\log x_{2,+}\Big)\Big(\mathcal{D}(x_1)\Big)^{-1} b_0(x_1)\\ &\quad + \sum_{k=1}^{M}\mathcal{D}(x_1)\,x_{2,+}^{-\frac12+\Delta(x_1)+k}\,B_k\big(x_1,\log x_{2,+}\big) + h_{M+1}^{(1)}(x_1,x_{2,+}),\end{aligned} \tag{3.53}$$

where $b_0 \in \left[C^\infty(\ell)\right]^6$, $h_{M+1}^{(1)} \in [\,C^\infty(\ell_\varepsilon^+)\,]^6$, $\ell_\varepsilon^+ = \ell\times[0,\varepsilon]$,

$$B_k(x_1,t) = B_0\Big(-\frac{t}{2\pi i}\Big)\sum_{j=1}^{k(2m_0-1)} t^j\,d_{kj}(x_1);$$

$m_0 = \max\{m_1,\dots,m_l\}$, the coefficients $d_{kj} \in [C^\infty(\ell)]^6$, $\Delta := (\Delta_1^{(2)},\dots,\Delta_6^{(2)})^\top$,

$$\Delta_j^{(2)}(x_1) = \frac{1}{2\pi i}\log\lambda_j^{(2)}(x_1) = \frac{1}{2\pi}\,\arg\lambda_j^{(2)}(x_1) + \frac{1}{2\pi i}\,\log|\lambda_j^{(2)}(x_1)|,$$
$$-\pi < \arg\lambda_j^{(2)}(x_1) < \pi, \quad (x_1,0)\in\ell, \quad j = \overline{1,6},$$

and

$$x_{2,+}^{-\frac12+\Delta(x_1)+k} := \operatorname{diag}\Big\{x_{2,+}^{-\frac12+\Delta_1^{(2)}(x_1)+k},\,\dots,\,x_{2,+}^{-\frac12+\Delta_6^{(2)}(x_1)+k}\Big\}.$$

Now, having in hand the above asymptotic expansion for the density vector function $h^{(1)}$, we can apply the results of [14] and write the spatial asymptotic expansions of the solution vectors $U^{(1)}$ and $U^{(2)}$:

$$U^{(1)}(x) = \sum_{\mu=\pm1}\sum_{s=1}^{l_0^{(1)}}\Big\{\sum_{j=0}^{n_s^{(1)}-1} x_3^j\Big[d_{sj}^{(1)}(x_1,\mu)(z_{s,\mu}^{(1)})^{\frac12+\Delta(x_1)-j}\,B_0(\zeta^{(1)})\Big]\,c_j(x_1)$$

$$+\sum_{\substack{k,l=0\\ k+l+j+p\geq 1}}^{M+2}\sum_{j+p=0}^{M+2-l} x_2^l\, x_3^j d^{(1)}_{sljp}(x_1,\mu)\,(z^{(1)}_{s,\mu})^{\frac12+\Delta(x_1)+p+k}\,B^{(1)}_{skjp}(x_1,\log z^{(1)}_{s,\mu})\Big\}+U^{(1)}_{M+1}(x)\,, \tag{3.54}$$

$$x_3>0,\ \ \zeta^{(1)}:=-\frac{1}{2\pi i}\,\log z^{(1)}_{s,\mu},$$

$$U^{(2)}(x)=\sum_{\mu=\pm1}\sum_{s=1}^{l_0^{(2)}}\Big\{\sum_{j=0}^{n_s^{(2)}-1} x_3^j\Big[d^{(2)}_{sj}(x_1,\mu)(z^{(2)}_{s,\mu})^{\frac12+\Delta(x_1)-j}\,B_0(\zeta^{(2)})\Big]\,c_j(x_1)$$

$$+\sum_{\substack{k,l=0\\ k+l+j+p\geq 1}}^{M+2}\sum_{j+p=0}^{M+2-l} x_2^l\, x_3^j d^{(2)}_{sljp}(x_1,\mu)\,(z^{(2)}_{s,\mu})^{\frac12+\Delta(x_1)+p+k}\,B^{(2)}_{skjp}(x_1,\log z^{(2)}_{s,\mu})\Big\}+U^{(2)}_{M+1}(x)\,, \tag{3.55}$$

$$x_3>0,\ \ \zeta^{(2)}:=-\frac{1}{2\pi i}\,\log z^{(2)}_{s,\mu}\,.$$

The coefficients $d^{(1)}_{sj}(\cdot\,,\mu)$, $d^{(2)}_{sj}(\cdot\,,\mu)$, $d^{(1)}_{sljp}(\cdot\,,\mu)$ and $d^{(2)}_{sljp}(\cdot\,,\mu)$ are the matrices with entries from the space $C^\infty(\ell)$, $B^{(1)}_{skjp}(x_1,t)$ and $B^{(2)}_{skjp}(x_1,t)$ are polynomials in t with vector coefficients which depend on the variable x_1 and have the order $\nu_{kjp}=k(2m_0-1)+m_0-1+p+j$ with $m_0=\max\{m_1,\dots,m_l\}$,

$$\begin{gathered}
c_j\in[C^\infty(\ell)]^6,\ \ U^{(1)}_{M+1}\in\big[\,C^{M+1}(\overline{\Omega^{(1)}})\,\big]^6,\ \ U^{(2)}_{M+1}\in\big[\,C^{M+1}(\overline{\Omega^{(2)}})\,\big]^4,\\
(z^{(1)}_{s,\mu})^{\kappa+\Delta(x_1)}:=\operatorname{diag}\Big\{(z^{(1)}_{s,\mu})^{\kappa+\Delta^{(2)}_1(x_1)},\,\dots,\,(z^{(1)}_{s,\mu})^{\kappa+\Delta^{(2)}_6(x_1)}\Big\},\\
(z^{(2)}_{s,\mu})^{\kappa+\Delta(x_1)}:=\operatorname{diag}\Big\{(z^{(2)}_{s,\mu})^{\kappa+\Delta^{(2)}_1(x_1)},\,\dots,\,(z^{(2)}_{s,\mu})^{\kappa+\Delta^{(2)}_6(x_1)}\Big\},\\
\kappa\in\mathbb{R},\ \ \mu=\pm1,\ \ (x_1,\,0)\in\ell,\\
z^{(1)}_{s,+1}=-x_2-x_3\tau^{(1)}_{s,+1},\ \ z^{(1)}_{s,-1}=x_2-x_3\tau^{(1)}_{s,-1},\\
z^{(2)}_{s,+1}=-x_2-x_3\tau^{(2)}_{s,+1},\ \ z^{(2)}_{s,-1}=x_2-x_3\tau^{(2)}_{s,-1},\\
-\pi<\arg z_{s,\pm1}<\pi,\quad -\pi<\arg z^{(2)}_{s,\pm1}<\pi,\\
\{\tau^{(1)}_{s,\pm1}\}_{s=1}^{l_0^{(1)}}\in C^\infty(\ell),\quad \{\tau^{(2)}_{s,\pm1}\}_{s=1}^{l_0^{(1)}}\in C^\infty(\ell).
\end{gathered}\tag{3.56}$$

Here, $\{\tau^{(1)}_{s,\pm1}\}_{s=1}^{l_0^{(1)}}$ (respectively, $\{\tau^{(2)}_{s,\pm1}\}_{s=1}^{l_0^{(2)}}$) are the different roots of multiplicity $n_s^{(1)}$, $s=1,\dots,l_0^{(1)}$, (respectively, $n_s^{(2)}$, $s=1,\dots,l_0^{(2)}$) of the polynomial in ζ, $\det A^{(1,0)}([J^\top_{\varkappa^{(1)}}(x_1,0,0)]^{-1}\eta_\pm)$ (respectively, $\det A^{(2,0)}([J^\top_{\varkappa_2}(x_1,0,0)]^{-1}\eta_\pm)$) with $\eta_\pm=(0,\pm1,\zeta)^\top$, satisfying the condition $\operatorname{Re}\tau^{(1)}_{s,\pm1}<0$ (respectively, $\operatorname{Re}\tau^{(2)}_{s,\pm1}<0$). The matrix $J_{\varkappa_1}$ (respectively, $J_{\varkappa_2}$) stands for the Jacobian matrix corresponding to the canonical diffeomorphism $\varkappa_1$ (respectively, $\varkappa_2$) related to the local coordinate system. Under this diffeomorphism, the curve ℓ is locally rectified and we assume that $(x_1,0,0)\in\ell$, $x_2=\operatorname{dist}(x_{_T},\ell)$, $x_3=\operatorname{dist}(x,\Gamma_T)$, where $x_{_T}$ is the projection of the reference point $x\in\Omega^{(1)}$ (respectively, $x\in\Omega^{(2)}$) on the plane corresponding to the image of Γ_T under the diffeomorphism $\varkappa_1$ (respectively, $\varkappa_2$).

Note that the coefficients $d^{(1)}_{sj}(\,\cdot\,,\mu)$ and $d^{(2)}_{sj}(\,\cdot\,,\mu)$ can be calculated explicitly, whereas the coefficients c_j can be expressed by means of the first coefficient b_0 in the asymptotic expansion of (3.53) (see [14]),

$$\begin{aligned}
&d^{(1)}_{sj}(x_1,+1)=\frac{1}{2\pi}\,G_{\varkappa_1}(x_1,0)\,P^{+(1)}_{sj}(x_1)\,\mathcal{D}(x_1),\\
&d^{(1)}_{sj}(x_1,-1)=\frac{1}{2\pi}\,G_{\varkappa_1}(x_1,0)\,P^{-(1)}_{sj}(x_1)\,\mathcal{D}(x_1)\,e^{i\pi(\frac12-\Delta(x_1))},\qquad s=\overline{1,l_0^{(1)}},\ \ j=\overline{0,n_s^{(1)}-1},\\
&d^{(2)}_{sj}(x_1,\,+1)=\frac{1}{2\pi}\,G_{\varkappa_2}(x_1,0)\,P^{+(2)}_{sj}(x_1)\,\widetilde{\mathcal{D}}(x_1),\\
&d^{(2)}_{sj}(x_1,\,-1)=\frac{1}{2\pi}\,G_{\varkappa_2}(x_1,0)\,P^{-(2)}_{sj}(x_1)\,\widetilde{\mathcal{D}}(x_1)\,e^{i\pi(\frac12-\Delta(x_1))},\qquad s=\overline{1,l_0^{(2)}},\ \ j=\overline{0,n_s^{(2)}-1},
\end{aligned}$$

where $\widetilde{\mathcal{D}} = \|\mathcal{D}_{kj}\|_{4\times 6}$, $k = 1, 2, 3, 6$, $j = \overline{1, 6}$, is composed of the entries of matrix $\mathcal{D}$ (see (3.51)),

$$P^{\pm(1)}_{sj}(x_1) := V^{(1),s}_{-1,\,j}(x_1, 0, 0, \pm 1)\Big[\mathfrak{S}\big(-\frac{1}{2}\,I_6 + \mathcal{K}^{(1)}_\tau; x_1, 0, 0, \pm 1\big)\Big]^{-1},$$

$$P^{\pm(2)}_{sj}(x_1) := V^{(2),s}_{-1,\,j}(x_1, 0, 0, \pm 1)\Big[\mathfrak{S}\big(-\frac{1}{2}\,I_4 + \mathcal{K}^{(2)}_\tau; x_1, 0, 0, \pm 1\big)\Big]^{-1},$$

$$V^{(1),s}_{-1,\,j}(x_1, 0, 0, \pm 1) := -\frac{i^{j+1}}{j!(n^{(1)}_s - 1 - j)!}\,\frac{d^{n^{(1)}_s - 1 - j}}{d\zeta^{n^{(1)}_s - 1 - j}}\,(\zeta - \tau^{(1)}_{s,\pm 1})^{n^{(1)}_s} \times \Big(A^{(1,0)}\big((J^\top_{\varkappa_1}(x_1, 0))^{-1}\big)\cdot(0, \pm 1, \zeta)^\top\Big)^{-1}\Big|_{\zeta = \tau^{(1)}_{s,\pm 1}},$$

$$V^{(2),s}_{-1,\,j}(x_1, 0, 0, \pm 1) := -\frac{i^{j+1}}{j!(n^{(2)}_s - 1 - j)!}\,\frac{d^{n^{(2)}_s - 1 - j}}{d\zeta^{n^{(2)}_s - 1 - j}}\,(\zeta - \tau^{(2)}_{s,\pm 1})^{n^{(2)}_s} \times \Big(A^{(2,\,0)}\big((J^\top_{\varkappa_2}(x_1, 0))^{-1}\big)\cdot(0, \pm 1, \zeta)^\top\Big)^{-1}\Big|_{\zeta = \tau^{(2)}_{s,\pm 1}},$$

$G_{\varkappa_1}(x_1, 0)$ and $G_{\varkappa_2}(x_1, 0)$ are smooth scalar functions explicitly written in terms of diffeomorphisms $\varkappa_1$ and $\varkappa_2$, respectively, and

$$c_j(x_1) = a_j(x_1)\,B^-_0\Big(-\frac{1}{2} + \Delta(x_1)\Big)\,\mathcal{D}^{-1}(x_1)b_0(x_1),$$
$$j = 0, \dots, n^{(1)}_s - 1, \quad (j = 0, \dots, n^{(2)}_s - 1),$$

where

$$B^-_0\Big(-\frac{1}{2} + \Delta(x_1)\Big) = \mathrm{diag}\Big\{B^{m_1}_-\Big(-\frac{1}{2} + \Delta^{(2)}_1(x_1)\Big), \dots, B^{m_l}_-\Big(-\frac{1}{2} + \Delta^{(2)}_l(x_1)\Big)\Big\},$$

$$B^{m_q}_-(t) = \big\|\widetilde{b}^{m_q}_{kp}(t)\big\|_{m_q\times m_q}, \quad q = 1, \dots, l,$$

$$\widetilde{b}^{m_q}_{kp}(t) = \begin{cases} \Big(\dfrac{1}{2\pi i}\Big)^{p-k}\,\dfrac{(-1)^{p-k}}{(p-k)!}\,\dfrac{d^{p-k}}{dt^{p-k}}\,\Gamma(t+1)\,e^{\frac{i\pi(t+1)}{2}}, & \text{for } k \leqslant p,\\ 0, & \text{for } k > p,\end{cases}$$

and $\Gamma(t+1)$ is the Euler integral,

$$a_j(x_1) = \mathrm{diag}\Big\{a^{m_1}(\alpha^{(j)}_1), \dots, a^{m_l}(\alpha^{(j)}_l)\Big\},$$

$$\alpha^{(j)}_q(x_1) = -\frac{3}{2} - \Delta^{(2)}_q(x_1) + j, \quad q = \overline{1, l}, \quad j = \overline{0, n^{(1)}_s - 1} \;\; (j = \overline{0, n^{(2)}_s - 1}),$$

$$a^{m_q}(\alpha^{(j)}_q) = \big\|a^{m_q}_{kp}(\alpha^{(j)}_q)\big\|_{m_q\times m_q},$$

$$a^{m_q}_{kp}(\alpha^{(j)}_q) = \begin{cases} -i\displaystyle\sum_{l=k}^{p}\frac{(-1)^{p-k}(2\pi i)^{l-p}\widetilde{b}^{m_q}_{kl}(\mu_q)}{(\alpha^{(0)}_q + 1)^{p-l+1}}, \quad j = 0, \quad k \leqslant p,\\ (-1)^{p-k}\widetilde{b}^{m_q}_{kp}(\alpha^{(j)}_q), \quad j = \overline{1, n^{(1)}_s - 1} \;\; (j = \overline{1, n^{(2)}_s - 1}), \quad k \leqslant p,\\ 0, \quad k > p,\end{cases}$$

$$\mu_q = -\frac{1}{2} - \Delta^{(2)}_q(x_1), \quad -1 < \mathrm{Re}\,\mu_q < 0.$$

Analogous investigation for the basic mixed and interior crack problems for homogeneous piezoelectric bodies has been carried out in reference [8], where the asymptotic properties of solutions have been established near the interior crack's edges and the curves, where the different boundary conditions collide. In [8], it is shown that the stress singularity exponents at the interior crack edges do not depend on the material parameters and are equal to -0.5, while they depend essentially on the material parameters at the collision curves, where different boundary conditions collide.

As it is evident from the above exposed results, the stress singularity exponents at the interfacial crack edges and at the curves, where the interface intersects the exterior boundary, depend essentially on the material parameters, in general. More precise results for particular classes of solids are presented in the next section, where the stress singularity exponents are calculated explicitly.

4. Analysis of Singularities of Solutions

Here, we assume that Γ_T and ℓ are rectified with the help of the diffeomorphisms mentioned in the previous section and for $x' \in \ell = \partial\Gamma_T$ by $\Pi_{x'}$ we denote the plane passing trough the point x' and orthogonal to ℓ. We introduce the polar coordinates (r,α), $r \geqslant 0$, $-\pi \leqslant \alpha \leqslant \pi$, in the plane $\Pi_{x'}$ with the pole at the point x'. Denote by $\Gamma_T^{\pm}$ the two different faces of the surface Γ_T. It is evident that $(r,\pm\pi) \in \Gamma_T^{\pm}$.

The intersection of the plane $\Pi_{x'}$ and $\Omega^{(1)}$ is identified with the half-plane $r \geqslant 0$ and $-\pi \leqslant \alpha \leqslant 0$, while the intersection of the plane $\Pi_{x'}$ and $\Omega^{(2)}$ is identified with the half-plane $r \geqslant 0$ and $0 \leqslant \alpha \leqslant \pi$.

The roots given by (3.56) are represented as follows:

$$\begin{gathered}
z^{(1)}_{s,+1} = -r\big[\cos\alpha + \tau^{(1)}_{s,+1}(x')\sin\alpha\big], \quad z^{(1)}_{s,-1} = r\big[\cos\alpha - \tau^{(1)}_{s,-1}(x')\sin\alpha\big], \\
s = 1,\dots,l_0^{(1)}, \ \ x' \in \ell, \\
z^{(2)}_{s,+1} = -r\big[\cos\alpha + \tau^{(2)}_{s,+1}(x')\sin\alpha\big], \quad z^{(2)}_{s,-1} = r\big[\cos\alpha - \tau^{(2)}_{s,-1}(x')\sin\alpha\big], \\
s = 1,\dots,l_0^{(2)}, \ \ x' \in \ell.
\end{gathered}$$

From the asymptotic expansions (3.54) and (3.55) we get

$$U^{(1)}(x) = \sum_{\mu=\pm1}\sum_{s=1}^{l_0^{(1)}}\sum_{j=0}^{n_s^{(1)}-1} c^{(1)}_{sj\mu}(x',\,\alpha)\, r^{\gamma+i\delta} B_0(\zeta)\, \widetilde{c}^{(1)}_{sj\mu}(x',\,\alpha) + \cdots, \tag{4.1}$$

$$U^{(2)}(x) = \sum_{\mu=\pm1}\sum_{s=1}^{l_0^{(2)}}\sum_{j=0}^{n_s^{(2)}-1} c^{(2)}_{sj\mu}(x',\,\alpha)\, r^{\gamma+i\delta} B_0(\zeta)\, \widetilde{c}^{(2)}_{sj\mu}(x',\,\alpha) + \cdots, \tag{4.2}$$

where

$$\begin{gathered}
r^{\gamma+i\delta} = \mathrm{diag}\big\{r^{\gamma_1+i\delta_1},\,\dots,\,r^{\gamma_6+i\delta_6}\big\}, \ \ \zeta = -\frac{1}{2\pi i}\,\log r, \\
\gamma_j = \frac12 + \frac{1}{2\pi}\,\arg\lambda_j(x'), \ \ \delta_j = -\frac{1}{2\pi}\,\log|\lambda_j(x')|, \ \ x' \in \ell, \ \ j = \overline{1,6},
\end{gathered} \tag{4.3}$$

and $\lambda_j = \lambda_j^{(2)}$, $j = \overline{1,6}$, are eigenvalues of the matrix

$$\mathcal{D}_2(x') = \big[\mathfrak{S}_2(x',0,+1)\big]^{-1}\,\mathfrak{S}_2(x',0,-1), \ \ x' \in \ell. \tag{4.4}$$

Note that the subsequent terms in expansion (4.1) and (4.2) have higher regularity, i.e., the real parts of the corresponding exponents are greater than γ_j.

The coefficients $c^{(1)}_{sj\mu}$, $\widetilde{c}^{(1)}_{sj\mu}$, $c^{(2)}_{sj\mu}$ and $\widetilde{c}^{(2)}_{sj\mu}$ in asymptotic expansions (4.1) and (4.2) read as

$$\begin{gathered}
c^{(1)}_{sj\mu}(x',\,\alpha) = \sin^j\alpha\; d^{(1)}_{sj}(x',\,\mu)\big[\psi^{(1)}_{s,\mu}(x',\,\alpha)\big]^{\gamma+i\delta-j}, \quad \widetilde{c}^{(1)}_{sj\mu}(x',\,\alpha) = B_0\Big(-\frac{1}{2\pi i}\log\psi^{(1)}_{s,\mu}(x',\,\alpha)\Big)\, c_j(x'), \\
j = \overline{0, n_s^{(1)}-1}, \ \ \mu = \pm1, \ \ s = \overline{1, l_0^{(1)}}, \\
c^{(2)}_{sj\mu}(x',\,\alpha) = \sin^j\alpha\; d^{(2)}_{sj}(x',\,\mu)\big[\psi^{(2)}_{s,\mu}(x',\,\alpha)\big]^{\gamma+i\delta-j}, \quad \widetilde{c}^{(2)}_{sj\mu}(x',\,\alpha) = B_0\Big(-\frac{1}{2\pi i}\log\psi^{(2)}_{s,\mu}(x',\,\alpha)\Big)\, c_j(x'), \\
j = \overline{0, n_s^{(2)}-1}, \ \ \mu = \pm1, \ \ s = \overline{1, l_0^{(2)}},
\end{gathered}$$

where

$$\begin{gathered}
\psi^{(1)}_{s,\mu}(x',\,\alpha) = -\mu\cos\alpha - \tau^{(1)}_{s,\mu}(x')\sin\alpha, \ \ s = \overline{1, l_0^{(1)}}, \\
\psi^{(2)}_{s,\mu}(x',\,\alpha) = -\mu\cos\alpha - \tau^{(2)}_{s,\mu}(x')\sin\alpha, \ \ s = \overline{1, l_0^{(2)}},
\end{gathered}$$

$$c^{(1)}_{sj\mu}(x',\,\alpha) = \big\|c^{(1,kp)}_{sj\mu}(x',\,\alpha)\big\|_{6\times 6}, \quad c^{(2)}_{sj\mu}(x',\,\alpha) = \big\|c^{(2,kp)}_{sj\mu}(x',\,\alpha)\big\|_{4\times 6}.$$

In what follows, for special classes of elastic materials we will analyze the exponents $\gamma_j + i\delta_j$, which determine the behaviour of $U^{(1)}$ and $U^{(2)}$ near the line ℓ.

As it was mention above, $\lambda_6 = 1$ (for details see [7, Section 5.7]). Therefore, $\gamma_6 = 1/2$ and $\delta_6 = 0$ in accordance with (4.3). This implies that one could not expect better smoothness for solutions than $C^{1/2}$, in general.

More detailed analysis leads to the following refined asymptotic behaviour for the temperature functions (cf. [8]).

Theorem 4.1. *Near the exceptional curve ℓ the functions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ possess the following asymptotic behaviour:*

$$\vartheta^{(1)} = b^{(1)}_0 r^{\frac12} + \mathcal{R}^{(1)}, \tag{4.5}$$

$$\vartheta^{(2)} = b^{(2)}_0 r^{\frac12} + \mathcal{R}^{(2)}, \tag{4.6}$$

where $b^{(i)}_0 \in C^{1+\gamma'-\varepsilon}$, $\mathcal{R}^{(i)} \in C^{\frac32+\gamma'-\varepsilon}$, $i = 1, 2$, in the corresponding one-sided neighbourhoods of ℓ and $1+\gamma'-\varepsilon > \frac12$ for sufficiently small $\varepsilon > 0$.

Proof. Indeed, $u^{(1)}_6 = \vartheta^{(1)}$ and $u^{(2)}_4 = \vartheta^{(2)}$ are the solutions of the transmission problem (3.50) with C^∞ data. Since the matrices $[\eta^{(1)}_{ij}]_{3\times 3}$ and $[\eta^{(2)}_{ij}]_{3\times 3}$ are positive definite, this transmission problem can be reduced to a system of pseudodifferential equations, where the principal part is described by the scalar positive-definite invertible pseudodifferential operators

$$\begin{gathered}
\mathcal{H}^{(1)}_{scalar}\big(-2^{-1}I + \mathcal{K}^{(1)}_{scalar}\big)^{-1} + \mathcal{H}^{(2)}_{scalar}\big(-2^{-1}I + \mathcal{K}^{(2)}_{scalar}\big)^{-1} \;:\; \widetilde{H}^{s-1}_p(\Gamma_T) \to H^s_p(\Gamma_T)\\
\mathcal{H}^{(1)}_{scalar}\big(-2^{-1}I + \mathcal{K}^{(1)}_{scalar}\big)^{-1} + \mathcal{H}^{(2)}_{scalar}\big(-2^{-1}I + \mathcal{K}^{(2)}_{scalar}\big)^{-1} \;:\; \widetilde{B}^{s-1}_{p,p}(\Gamma_T) \to B^s_{p,p}(\Gamma_T),\\
\frac1p - \frac12 < s < \frac1p + \frac12\,,\ \ 1 < p < \infty,
\end{gathered}$$

where $\mathcal{K}^{(i)}_{scalar}$, $i = 1, 2$, are compact. These pseudodifferential operators have principal homogeneous symbol $-2\mathfrak{S}\big(\mathcal{H}^{(1)}_{scalar} + \mathcal{H}^{(2)}_{scalar}; x, \xi\big)$, which is positive and even in ξ. Hence we can establish refined explicit asymptotic relations of type (4.5), (4.6) for the temperature functions $u^{(1)}_6 = \vartheta^{(1)}$ and $u^{(2)}_4 = \vartheta^{(2)}$ in the corresponding one-sided neighbourhoods of ℓ (see [14, 15, 17, 18]). □

From (4.5) and (4.6), it follows that

(i) The leading exponents for $u^{(1)}_6 = \vartheta^{(1)}$ and $u^{(2)}_4 = \vartheta^{(2)}$ in the neighborhood of line ℓ are equal to $\frac12$;
(ii) Logarithmic factors are absent in the first terms of the asymptotic expansions of $\vartheta^{(1)}$ and $\vartheta^{(2)}$;
(iii) The temperature functions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ do not oscillate in the neighbourhood of the collision curve ℓ and for the heat flux vector we have no oscillating singularities;
(iv) The temperature functions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ belong to $C^{\frac12}(\overline{\Omega^{(1)}})$ and $C^{\frac12}(\overline{\Omega^{(2)}})$, respectively, (cf. [8], Theorem 6.4).

Non-zero parameters δ_j in (4.3) lead to the so-called oscillating singularities for the first order derivatives of $U^{(1)}$ and $U^{(2)}$, in general. In turn, this yields oscillating stress singularities, which sometimes lead to mechanical contradictions, for example, to an overlapping of materials. So, from the practical point of view, it is important to single out the classes of solids for which the oscillating singularities do not occur.

Let us consider the above investigated mixed boundary-transmission problem for particular elastic components. We assume that the medium occupying the domain $\Omega^{(1)}$ belongs to the **422** (Tetragonal) or **622** (Hexagonal) class of crystals. The corresponding system of differential equations reads as (see, e.g., [16])

$$
\begin{aligned}
&(c_{11}\,\partial_1^2+c_{66}\,\partial_2^2+c_{44}\,\partial_3^2)u_1^{(1)}+(\,c_{12}+c_{66}\,)\,\partial_1\partial_2 u_2^{(1)}+(\,c_{13}+c_{44}\,)\,\partial_1\partial_3 u_3^{(1)}\\
&\qquad\qquad -e_{14}\,\partial_2\partial_3\varphi^{(1)}-q_{15}\,\partial_2\partial_3\psi^{(1)}-\widetilde{\gamma}_1\,\partial_1\vartheta^{(1)}-\varrho^{(1)}\,\tau^2\,u_1^{(1)}=F_1,\\
&(\,c_{12}+c_{66}\,)\,\partial_2\partial_1 u_1^{(1)}+(\,c_{66}\,\partial_1^2+c_{11}\,\partial_2^2+c_{44}\,\partial_3^2\,)\,u_2^{(1)}+(\,c_{13}+c_{44}\,)\,\partial_2\partial_3 u_3^{(1)}\\
&\qquad\qquad +e_{14}\partial_1\partial_3\varphi^{(1)}+q_{15}\partial_1\partial_3\psi^{(1)}-\widetilde{\gamma}_1\partial_2\vartheta^{(1)}-\varrho^{(1)}\,\tau^2\,u_2^{(1)}=F_2,\\
&(\,c_{13}+c_{44}\,)\,\partial_3\partial_1 u_1^{(1)}+(\,c_{13}+c_{44}\,)\,\partial_3\partial_2 u_2^{(1)}+(\,c_{44}\,\partial_1^2+c_{44}\,\partial_2^2+c_{33}\,\partial_3^2\,)\,u_3^{(1)}\\
&\qquad\qquad -\widetilde{\gamma}_3\,\partial_3\vartheta^{(1)}-\varrho^{(1)}\,\tau^2\,u_3^{(1)}=F_3,\\
&e_{14}\partial_2\partial_3 u_1^{(1)}-e_{14}\partial_1\partial_3 u_2^{(1)}+(\,\varkappa_{11}\,\partial_1^2+\varkappa_{11}\,\partial_2^2+\varkappa_{33}\,\partial_3^2\,)\,\varphi^{(1)}-(1+\nu_0\tau)p_3\,\partial_3\vartheta^{(1)}=F_4,\\
&q_{15}\partial_2\partial_3 u_1^{(1)}-q_{15}\partial_1\partial_3 u_2^{(1)}+(\,\mu_{11}\,\partial_1^2+\mu_{11}\,\partial_2^2+\mu_{33}\,\partial_3^2\,)\,\psi^{(1)}-(1+\nu_0\tau)m_3\,\partial_3\vartheta^{(1)}=F_5,\\
&-\tau\,T_0\,(\,\widetilde{\gamma}_1\,\partial_1 u_1^{(1)}+\widetilde{\gamma}_1\,\partial_2 u_2^{(1)}+\widetilde{\gamma}_3\,\partial_3 u_3^{(1)}\,)+\tau\,T_0\,p_3\,\partial_3\varphi^{(1)}+\tau\,T_0\,m_3\,\partial_3\psi^{(1)}\\
&\qquad\qquad +(\,\eta_{11}\,\partial_1^2+\eta_{11}\,\partial_2^2+\eta_{33}\,\partial_3^2\,)\,\vartheta^{(1)}-(\tau\,d_0+\tau^2 h^{(1)})\,\vartheta^{(1)}=F_6,
\end{aligned}
$$

where $c_{11},\ c_{12},\ c_{13},\ c_{33},\ c_{44},\ c_{66}$ are the elastic constants, e_{14} is the piezoelectric constant, q_{15} is the piezomagnetic constant, $\varkappa_{11}$ and $\varkappa_{33}$ are the dielectric constants, μ_{11} and μ_{33} are the magnetic permeability constants, $\widetilde{\gamma}_1=(1+\nu_0\tau)\lambda_{11}=(1+\nu_0\tau)\lambda_{21}$ and $\widetilde{\gamma}_3=(1+\nu_0\tau)\lambda_{31}$ are the thermal strain constants, η_{11} and η_{33} are the thermal conductivity constants, p_3 is the pyroelectric constant and m_3 is the pyromagnetic constant. In the case of Hexagonal crystals (**622** class), we have $c_{66}=(c_{11}-c_{12})/2$.

Note that some important polymers and bio-materials are modelled by the above partial differential equations, for example, the *collagen-hydroxyapatite* is one example of such a material. This material is widely used in biology and medicine (see [31]). Another important example is TeO_2 [16].

In this model, the generalized stress operator is defined as

$$
T(\partial_x,n,\tau)=\big\|\,T_{jk}(\partial_x,n,\tau)\,\big\|_{6\times 6}
$$

with

$$
\begin{aligned}
&T_{11}(\partial_x,n,\tau)=c_{11}n_1\partial_1+c_{66}n_2\partial_2+c_{44}n_3\partial_3, && T_{12}(\partial_x,n,\tau)=c_{12}n_1\partial_2+c_{66}n_2\partial_1,\\
&T_{13}(\partial_x,n,\tau)=c_{13}n_1\partial_3+c_{44}n_3\partial_1, && T_{14}(\partial_x,n,\tau)=-e_{14}n_3\partial_2,\\
&T_{15}(\partial_x,n,\tau)=-q_{15}n_3\partial_2, && T_{16}(\partial_x,n,\tau)=-\widetilde{\gamma}_1\,n_1,\\
&T_{21}(\partial_x,n,\tau)=c_{66}n_1\partial_2+c_{12}n_2\partial_1, && T_{22}(\partial_x,n,\tau)=c_{66}n_1\partial_1+c_{11}n_2\partial_2+c_{44}n_3\partial_3,\\
&T_{23}(\partial_x,n,\tau)=c_{13}n_2\partial_3+c_{44}n_3\partial_2, && T_{24}(\partial_x,n,\tau)=e_{14}n_3\partial_1,\\
&T_{25}(\partial_x,n,\tau)=q_{15}n_3\partial_1, && T_{26}(\partial_x,n,\tau)=-\widetilde{\gamma}_1\,n_2,\\
&T_{31}(\partial_x,n,\tau)=c_{44}n_1\partial_3+c_{13}n_3\partial_1, && T_{32}(\partial_x,n,\tau)=c_{44}n_2\,\partial_3+c_{13}n_3\partial_2,\\
&T_{33}(\partial_x,n)=c_{44}n_1\partial_1+c_{44}n_2\partial_2+c_{33}n_3\partial_3, && T_{34}(\partial_x,n,\tau)=0,\\
&T_{35}(\partial_x,n,\tau)=0, && T_{36}(\partial_x,n,\tau)=-\widetilde{\gamma}_3\,n_3,\\
&T_{41}(\partial_x,n,\tau)=e_{14}n_2\partial_3, && T_{42}(\partial_x,n,\tau)=-e_{14}n_1\partial_3,\\
&T_{43}(\partial_x,n,\tau)=0, && T_{44}(\partial_x,n,\tau)=\varkappa_{11}(n_1\partial_1+n_2\partial_2)+\varkappa_{33}n_3\partial_3,\\
&T_{45}(\partial_x,n,\tau)=0, && T_{46}(\partial_x,n,\tau)=-p_3n_3,\\
&T_{51}(\partial_x,n,\tau)=q_{15}n_2\partial_3, && T_{52}(\partial_x,n,\tau)=-q_{15}n_1\partial_3,
\end{aligned}
$$

$$
\begin{aligned}
&T_{53}(\partial_x,n,\tau)=0, && T_{54}(\partial_x,n,\tau)=0,\\
&T_{55}(\partial_x,n,\tau)=\mu_{11}(n_1\partial_1+n_2\partial_2)+\mu_{33}n_3\partial_3, && T_{56}(\partial_x,n,\tau)=-m_3n_3,\\
&T_{6j}(\partial_x,n,\tau)=0,\ \text{for}\ j=\overline{1,5}, && T_{66}(\partial_x,n,\tau)=\eta_{11}(n_1\partial_1+n_2\partial_2)+\eta_{33}n_3\partial_3.
\end{aligned}
$$

The material constants satisfy the following system of inequalities

$$\begin{gathered} c_{11} > |c_{12}|, \quad c_{44} > 0, \quad c_{66} > 0, \quad c_{33}(c_{11}+c_{12}) > 2c_{13}^2, \\ \varkappa_{11} > 0, \quad \varkappa_{33} > 0, \quad \eta_{11} > 0, \quad \eta_{33} > 0, \quad \mu_{11} > 0, \quad \mu_{33} > 0, \end{gathered} \tag{4.7}$$

which are equivalent to the positive definiteness of the internal energy form (see (2.7), (2.8)).

From (2.9), (2.12), (2.13), and (4.7) it follows also that

$$\varkappa_{33} > p_3^2 T_0 d_0^{-1}, \quad \mu_{33} > m_3^2 T_0 d_0^{-1}, \quad c_{11}c_{33} > c_{13}^2. \tag{4.8}$$

Under these conditions the mixed boundary-transmission problem in question is uniquely solvable.

Furthermore, we assume that $e_{14} \neq 0$, $e_{15} \neq 0$, $\dfrac{\mu_{11}}{\varkappa_{11}} = \dfrac{\mu_{33}}{\varkappa_{33}} = \alpha$, the surface Γ_C is parallel to the plane of isotropy (i.e., to the plane $x_3 = 0$) in some neighbourhood of $\partial\Gamma_C$, and the domain $\Omega^{(2)}$ is occupied by an isotropic material modeled by the generalized thermoelasticity equations (see (2.1), (2.2))

$$\begin{gathered} \mu\Delta u^{(2)} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u^{(2)} - (1+\nu_0\tau)\lambda^{(2)}\,\mathrm{grad}\,\vartheta^{(2)} - \varrho^{(2)}\tau^2 u^{(2)} = 0, \\ \eta^{(2)}\Delta\vartheta^{(2)} - (\tau d_0^{(2)} + \tau^2 h_0^{(2)})\vartheta^{(2)} - \tau\lambda^{(2)} u^{(2)} = 0, \\ \mu > 0, \quad 3\lambda + 2\mu > 0, \quad \eta^{(2)} > 0, \quad h_0^{(2)} > 0, \quad d_0^{(2)} - \nu_0 h_0^{(2)} > 0. \end{gathered}$$

In the case of this particular mixed boundary-transmission problem we find the exponents involved in the asymptotic expansions of solutions explicitly in terms of the material constants. To this end, we find the eigenvalues of the matrix (4.4) explicitly and calculate the exponents $\gamma + i\delta$ involved in the asymptotic expansions (4.1) and (4.2).

Taking into account the relations

$$\mathfrak{S}\Big(-2^{-1}I_6 \pm \mathcal{K}_\tau^{(1)}; x', 0, 1\Big) = \mathfrak{S}\Big(-2^{-1}I_6 + \mathcal{K}_\tau^{(1)}; x', 0, \pm 1\Big), \quad \mathfrak{S}(\mathcal{H}_\tau^{(1)}; x', 0, -1) = \mathfrak{S}(\mathcal{H}_\tau^{(1)}; x', 0, 1),$$

for these symbol matrices we introduce the short notation

$$\sigma\Big(-2^{-1}I_6 \pm \mathcal{K}_\tau^{(1)}\Big) := \mathfrak{S}\Big(-2^{-1}I_6 \pm \mathcal{K}_\tau^{(1)}; x', 0, 1\Big) = \mathfrak{S}\Big(-2^{-1}I_6 + \mathcal{K}_\tau^{(1)}; x', 0, \pm 1\Big)$$

and

$$\sigma(\mathcal{H}_\tau^{(1)}) := \mathfrak{S}(\mathcal{H}_\tau^{(1)}; x', 0, \pm 1).$$

These symbols can be calculated explicitly (see [8], Appendix B):

$$\sigma\Big(-\frac{1}{2}I_6 \pm \mathcal{K}_\tau^{(1)}\Big) = \begin{bmatrix} -\frac12 & 0 & 0 & \pm A_{14} & \pm A_{15} & 0 \\ 0 & -\frac12 & \pm A_{23} & 0 & 0 & 0 \\ 0 & \pm A_{32} & -\frac12 & 0 & 0 & 0 \\ \pm A_{41} & 0 & 0 & -\frac12 & 0 & 0 \\ \pm A_{51} & 0 & 0 & 0 & -\frac12 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac12 \end{bmatrix},$$

where

$$A_{14} = -i\,\frac{e_{14}\,c_{66}\,(b_2-b_1)}{2\,b_1\,b_2\sqrt{B}} - i\,\frac{e_{14}q_{15}^2}{\alpha\varkappa_{11}\widetilde{e}_{14}^{\,2}}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1 b_2 + \varkappa_{11})}{\sqrt{B}}\right],$$

$$A_{15} = -i\,\frac{q_{15}\,c_{66}\,(b_2-b_1)}{2\alpha\,b_1\,b_2\sqrt{B}} - i\,\frac{q_{15}e_{14}^2}{\alpha\varkappa_{11}\widetilde{e}_{14}^{\,2}}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1 b_2 + \varkappa_{11})}{\sqrt{B}}\right],$$

$$A_{41} = -i\,\frac{e_{14}\,\varkappa_{33}\,(b_2-b_1)}{2\sqrt{B}}, \quad A_{51} = -i\,\frac{q_{15}\,\varkappa_{33}\,(b_2-b_1)}{2\sqrt{B}},$$

$$b_1 = \sqrt{\frac{A-\sqrt{B}}{2\,c_{44}\,\varkappa_{33}}}, \qquad b_2 = \sqrt{\frac{A+\sqrt{B}}{2\,c_{44}\,\varkappa_{33}}}, \quad \widetilde{e}_{14} = \Big(e_{14}^2 + \alpha^{-1}q_{15}^2\Big)^{1/2}, \quad \alpha = \frac{\mu_{11}}{\varkappa_{11}} = \frac{\mu_{33}}{\varkappa_{33}} > 0,$$

$$A = \widetilde{e}_{14}^2 + c_{44}\,\varkappa_{11} + c_{66}\,\varkappa_{33} > 0, \qquad B = A^2 - 4\,c_{44}\,c_{66}\,\varkappa_{11}\,\varkappa_{33} > 0, \quad A > \sqrt{B}.$$

Note that $b_1 b_2 = \sqrt{\dfrac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}}$.

It can be proved that $A_{14}A_{41} < 0$, $A_{15}A_{51} < 0$ (see [8], Appendix B).

Let us calculate the entries A_{23} and A_{32}. Introduce the notation

$$C := c_{11}\,c_{33} - c_{13}^2 - 2\,c_{13}\,c_{44}, \quad D := C^2 - 4\,c_{44}^2\,c_{33}c_{11}. \tag{4.9}$$

Consider two cases.

Case 1. Let $D > 0$. Then

$$A_{23} = i\,\frac{c_{44}\,(d_2 - d_1)\,(c_{11} - c_{13}\,d_1 d_2)}{2\,d_1\,d_2\sqrt{D}}, \quad A_{32} = -i\,\frac{c_{44}\,(d_2 - d_1)\,(c_{33}\,d_1\,d_2 - c_{13})}{2\,d_1\,d_2\sqrt{D}}, \tag{4.10}$$

where

$$d_1 = \sqrt{\frac{C - \sqrt{D}}{2\,c_{44}\,c_{33}}}, \qquad d_2 = \sqrt{\frac{C + \sqrt{D}}{2\,c_{44}\,c_{33}}}.$$

Inequalities (4.7) imply $C > \sqrt{D}$ and

$$d_1 d_2 = \frac{\sqrt{c_{11}}}{\sqrt{c_{33}}}, \quad (d_2 - d_1)^2 = \frac{C - 2c_{44}\sqrt{c_{33}}\sqrt{c_{11}}}{c_{44}c_{33}} > 0. \tag{4.11}$$

Then, from (4.10), we obtain $A_{23}\,A_{32} > 0$.

Case 2. Let $D < 0$. In this case,

$$A_{23} = i\,\frac{a\,c_{44}(\sqrt{c_{11}\,c_{33}} - c_{13})}{\sqrt{-D}}, \quad A_{32} = -i\,\frac{a\,c_{44}(\sqrt{c_{11}\,c_{33}} - c_{13})}{\sqrt{-D}}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}}, \tag{4.12}$$

where

$$a = \frac{1}{2}\sqrt{\frac{-C + 2c_{44}\sqrt{c_{11}c_{33}}}{c_{44}c_{33}}} > 0 \tag{4.13}$$

and we get again

$$A_{23}A_{32} = \frac{c_{44}^2\,a^2(\sqrt{c_{11}\,c_{33}} - c_{13})^2}{-D}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}} > 0.$$

The symbol matrix $\sigma(\mathcal{H}_\tau^{(1)})$ has the following block-wise structure:

$$\sigma(\mathcal{H}_\tau^{(1)}) = \begin{bmatrix} \mathbf{C}_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{C}_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{C}_{44} & \mathbf{C}_{45} & 0 \\ 0 & 0 & 0 & \mathbf{C}_{45} & \mathbf{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{C}_{66} \end{bmatrix}_{6\times 6},$$

where

$$\mathbf{C}_{11} = -\frac{b_2 - b_1}{2\sqrt{B}}\left(\varkappa_{33} + \frac{\varkappa_{11}}{b_1 b_2}\right),$$

$$\mathbf{C}_{22} = \begin{cases} -\dfrac{d_2 - d_1}{2\sqrt{D}}\left(c_{33} + c_{44}\sqrt{\dfrac{c_{33}}{c_{11}}}\right) & \text{if } D > 0, \\[2ex] -\dfrac{a}{\sqrt{D}}\left(c_{33} + c_{44}\sqrt{\dfrac{c_{33}}{c_{11}}}\right), & \text{if } D < 0, \end{cases}$$

$$\mathbf{C}_{33} = \begin{cases} -\dfrac{d_2 - d_1}{2\sqrt{D}}\,(c_{44} + \sqrt{c_{11}c_{33}}\,), & \text{if } D > 0, \\[2ex] -\dfrac{a}{\sqrt{D}}\,(c_{44} + \sqrt{c_{11}c_{33}}\,), & \text{if } D < 0, \end{cases}.$$

$$\mathbf{C}_{44} = -\left\{ \frac{b_2 - b_1}{2\sqrt{B}} \left(c_{44} + \frac{c_{66}}{b_1 b_2} \right) + \frac{q_{15}^2}{2\alpha \varkappa_{11} \widetilde{e}_{14}^2} \left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2 - b_1)(\varkappa_{33} b_1 b_2 + \varkappa_{11})}{\sqrt{B}} \right] \right\},$$

$$\mathbf{C}_{55} = -\left\{ \frac{b_2 - b_1}{2\sqrt{B}} \left(c_{44} + \frac{c_{66}}{b_1 b_2} \right) + \frac{e_{14}^2}{2\alpha \varkappa_{11} \widetilde{e}_{14}^2} \left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2 - b_1)(\varkappa_{33} b_1 b_2 + \varkappa_{11})}{\sqrt{B}} \right] \right\},$$

$$\mathbf{C}_{45} = \mathbf{C}_{54} = \frac{e_{14} q_{15}}{2\alpha \varkappa_{11} \widetilde{e}_{14}^2} \left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2 - b_1)(\varkappa_{33} b_1 b_2 + \varkappa_{11})}{\sqrt{B}} \right], \quad \mathbf{C}_{66} = -\frac{1}{2\sqrt{\eta_{11}\eta_{33}}}.$$

Remark that $C_{jj} < 0,\ j = \overline{1,6}$ (see [8], Appendix B).

The symbol matrix $\sigma^{\pm}(\mathcal{B}_\tau^{(2)}) := \mathfrak{S}(\mathcal{B}_\tau^{(2)}; x', 0, \pm 1)$ reads as

$$\sigma^{\pm}(\mathcal{B}_\tau^{(2)}) = \begin{bmatrix} \frac{1}{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & a & \pm ib & 0 & 0 & 0 \\ 0 & \mp ib & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 6}, \quad a := \frac{2(\lambda + 2\mu)\mu}{\lambda + \mu}, \quad b := \frac{\mu^2}{\lambda + 3\mu}.$$

Then the symbol matrix of the Poincaré–Steklov type operator has the form

$$\sigma^{\pm}(\mathcal{A}_\tau^{(1)}) := \sigma(\mathcal{H}_\tau^{(1)}) \sigma\Big(-\frac{1}{2} I_6 \pm \mathcal{K}_\tau^{(1)} \Big)^{-1} = \big[\mathcal{A}_{jk}^{\pm} \big]_{6\times 6},$$

where

$$\begin{aligned}
&\mathcal{A}_{11}^{\pm} = \mathcal{A}_{11} = \frac{2\mathbf{C}_{11}}{Q_1}, \quad \mathcal{A}_{12}^{\pm} = \mathcal{A}_{13}^{\pm} = \mathcal{A}_{16}^{\pm} = 0, \quad \mathcal{A}_{14}^{\pm} = \mp \frac{4A_{14}\mathbf{C}_{11}}{Q_1}, \quad \mathcal{A}_{15}^{\pm} = \pm \frac{4A_{15}\mathbf{C}_{11}}{Q_1}, \\
&\mathcal{A}_{21}^{\pm} = 0, \quad \mathcal{A}_{22}^{\pm} = \mathcal{A}_{22} = \frac{2\mathbf{C}_{22}}{Q_2}, \quad \mathcal{A}_{23}^{\pm} = \mp \frac{4A_{23}\mathbf{C}_{22}}{Q_2}, \quad \mathcal{A}_{24}^{\pm} = \mathcal{A}_{25}^{\pm} = \mathcal{A}_{26}^{\pm} = 0, \\
&\mathcal{A}_{31}^{\pm} = 0, \quad \mathcal{A}_{32}^{\pm} = \mp \frac{4A_{32}\mathbf{C}_{33}}{Q_2}, \quad \mathcal{A}_{33}^{\pm} = \mathcal{A}_{33} = \frac{2\mathbf{C}_{33}}{Q_2}, \quad \mathcal{A}_{34}^{\pm} = \mathcal{A}_{35}^{\pm} = \mathcal{A}_{36}^{\pm} = 0, \\
&\mathcal{A}_{41}^{\pm} = \mp \left(\frac{4A_{41}\mathbf{C}_{44}}{Q_1} + \frac{4A_{51}\mathbf{C}_{45}}{Q_1} \right), \quad \mathcal{A}_{42}^{\pm} = \mathcal{A}_{43}^{\pm} = \mathcal{A}_{46}^{\pm} = 0, \\
&\mathcal{A}_{44}^{\pm} = \mathcal{A}_{44} = \frac{(2 - 8A_{15}A_{51})\mathbf{C}_{44}}{Q_1} + \frac{8A_{14}A_{51}\mathbf{C}_{45}}{Q_1}, \\
&\mathcal{A}_{45}^{\pm} = \mathcal{A}_{45} = -\frac{8A_{15}A_{41}\mathbf{C}_{44}}{Q_1} - \frac{(2 - 8A_{14}A_{41})\mathbf{C}_{45}}{Q_1}, \\
&\mathcal{A}_{51}^{\pm} = \pm \left(\frac{4A_{41}\mathbf{C}_{45}}{Q_1} + \frac{4A_{51}\mathbf{C}_{55}}{Q_1} \right), \quad \mathcal{A}_{52}^{\pm} = \mathcal{A}_{53}^{\pm} = \mathcal{A}_{56}^{\pm} = 0, \\
&\mathcal{A}_{54}^{\pm} = \mathcal{A}_{54} = -\frac{(2 - 8A_{15}A_{51})\mathbf{C}_{45}}{Q_1} - \frac{8A_{14}A_{51}\mathbf{C}_{55}}{Q_1}, \\
&\mathcal{A}_{55}^{\pm} = \mathcal{A}_{55} = \frac{8A_{15}A_{41}\mathbf{C}_{45}}{Q_1} + \frac{(2 - 8A_{14}A_{41})\mathbf{C}_{55}}{Q_1}, \\
&\mathcal{A}_{61}^{\pm} = \mathcal{A}_{62}^{\pm} = \mathcal{A}_{63}^{\pm} = \mathcal{A}_{64}^{\pm} = \mathcal{A}_{65}^{\pm} = 0, \quad \mathcal{A}_{66}^{\pm} = \mathcal{A}_{66} = -2\mathbf{C}_{66}.
\end{aligned}$$

Introduce the notation

$$Q_1 := -1 + 4A_{14}A_{41} + 4A_{15}A_{51} < 0, \quad Q_2 := -1 + 4A_{23}A_{32}.$$

Lemma 4.2. *The following inequality* $Q_2 = -1 + 4A_{23}A_{32} < 0$ *holds.*

Proof. Consider two cases.

Case 1: $D > 0$. Then inequality $4A_{23}A_{32} < 1$ can be equivalently reduced to the inequality

$$c_{44}^2 (d_2 - d_1)^2 (c_{11} - c_{13} d_1 d_2)(c_{33} d_1 d_2 - c_{13}) < d_1^2 d_2^2 D.$$

By replacing here $d_1 d_2$ by its expression from (4.11), we get

$$c_{44}^2 (d_2 - d_1)^2 \big(\sqrt{c_{11}c_{33}} - c_{13}\big)^2 < \frac{\sqrt{c_{11}}}{\sqrt{c_{33}}} D.$$

Now, replace $(d_2 - d_1)^2$ and D by their expressions from (4.11) and (4.9), respectively, to obtain

$$c_{44}^2 \frac{(C - 2c_{44}\sqrt{c_{11}c_{33}})}{c_{44}c_{33}} (\sqrt{c_{11}c_{33}} - c_{13})^2 < \frac{\sqrt{c_{11}}}{\sqrt{c_{33}}} (C^2 - 4c_{44}^2 c_{33} c_{11}).$$

From the above inequality we deduce

$$c_{44}(C - 2c_{44}\sqrt{c_{11}c_{33}})(\sqrt{c_{11}c_{33}} - c_{13})^2 < \sqrt{c_{11}c_{33}}(C + 2c_{44}\sqrt{c_{11}c_{33}})(C - 2c_{44}\sqrt{c_{11}c_{33}}).$$

Substituting here the expression of C from (4.9) to obtain

$$c_{44}(\sqrt{c_{11}c_{33}} - c_{13})^2 < \sqrt{c_{11}c_{33}}(c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44} + 2c_{44}\sqrt{c_{11}c_{33}}),$$

i.e.,

$$c_{44}(\sqrt{c_{11}c_{33}} - c_{13})^2 < \sqrt{c_{11}c_{33}}\big[(\sqrt{c_{11}c_{33}} + c_{13})(\sqrt{c_{11}c_{33}} - c_{13}) + 2c_{44}(\sqrt{c_{11}c_{33}} - c_{13})\big],$$

we arrive at the inequality

$$c_{44}(\sqrt{c_{11}c_{33}} - c_{13}) < \sqrt{c_{11}c_{33}}(\sqrt{c_{11}c_{33}} + c_{13} + 2c_{44}). \tag{4.14}$$

But (4.14) holds, since

$$c_{44}(\sqrt{c_{11}c_{33}} - c_{13}) < 2c_{44}\sqrt{c_{11}c_{33}} < \sqrt{c_{11}c_{33}}(\sqrt{c_{11}c_{33}} + c_{13} + 2c_{44}),$$

due to the inequality $\sqrt{c_{11}c_{33}} > |c_{13}|$ (see (4.8)).

So, we finally obtain

$$Q_2 = -1 + 4A_{23}A_{32} < 0.$$

Case 2: $D < 0$. In this case, due to (4.12), we have

$$4A_{23}A_{32} = \frac{4a^2 c_{44}^2 (\sqrt{c_{11}c_{33}} - c_{13})^2}{-D} \frac{\sqrt{c_{33}}}{\sqrt{c_{11}}} < 1.$$

Therefore

$$4a^2 c_{44}^2 (\sqrt{c_{11}c_{33}} - c_{13})^2 \sqrt{c_{33}} < -D\sqrt{c_{11}}.$$

Inserting here a and D from (4.13) and (4.9), respectively, we rewrite the above inequality as

$$\left(\frac{-C + 2c_{44}\sqrt{c_{11}c_{33}}}{\sqrt{c_{33}}}\right) c_{44}(\sqrt{c_{11}c_{33}} - c_{13})^2 < \big(-C^2 + 4c_{44}^2 c_{33}c_{11}\big)\sqrt{c_{11}}.$$

Replacing here C with it's expression from (4.9), we get

$$c_{44}(\sqrt{c_{11}c_{33}} - c_{13})^2 < \big(2c_{44}\sqrt{c_{11}c_{33}} + c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}\big)\sqrt{c_{11}c_{33}},$$

implying

$$c_{44}(\sqrt{c_{11}c_{33}} - c_{13})^2 < \Big[2c_{44}(\sqrt{c_{11}c_{33}} - c_{13}) + (\sqrt{c_{11}c_{33}} + c_{13})(\sqrt{c_{11}c_{33}} - c_{13})\Big]\sqrt{c_{11}c_{33}}.$$

Dividing the inequality by $\sqrt{c_{11}c_{33}} - c_{13}$, we obtain

$$c_{44}(\sqrt{c_{11}c_{33}} - c_{13}) < \sqrt{c_{11}c_{33}}\,(2c_{44} + \sqrt{c_{11}c_{33}} + c_{13}). \tag{4.15}$$

Thus, the inequality $Q_2 < 0$ is equivalently reduced to the relation (4.15), which coincides with (4.14) and which is true as is shown above. This completes the proof. $\square$

Introduce the notation

$$\sigma_2^{\pm} = \sigma_2^{\pm}(\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}) := \mathfrak{S}\big(\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}; x', 0, \pm 1\big),\ x' \in \ell.$$

The characteristic polynomial of the matrix $(\sigma_2^+)^{-1}\sigma_2^-$ can be represented as follows:

$$\det\big(\sigma_2^- - \lambda\sigma_2^+\big) = \det\Big[\sigma_2^-\big(\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}\big) - \lambda\sigma_2^+\big(\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}\big)\Big]$$

$$= \det\left\{\Big[\sigma(\mathcal{H}_\tau^{(1)})\sigma\big(-\frac{1}{2}I_6-\mathcal{K}_\tau^{(2)}\big)^{-1}+\sigma^-(\mathcal{B}_\tau^{(2)})\Big]-\lambda\Big[\sigma(\mathcal{H}_\tau^{(1)})\sigma\big(-\frac{1}{2}I_6+\mathcal{K}_\tau^{(1)}\big)^{-1}+\sigma^+(\mathcal{B}_\tau^{(2)})\Big]\right\}$$

$$=\det\begin{bmatrix} (1-\lambda)\widetilde{\mathcal{A}}_{11} & 0 & 0 & -(1+\lambda)\widetilde{\mathcal{A}}_{14}^+ & -(1+\lambda)\widetilde{\mathcal{A}}_{15}^+ & 0\\ 0 & (1-\lambda)\widetilde{\mathcal{A}}_{22} & -(1+\lambda)\widetilde{\mathcal{A}}_{23}^+ & 0 & 0 & 0\\ 0 & -(1+\lambda)\widetilde{\mathcal{A}}_{32}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{33} & 0 & 0 & 0\\ -(1+\lambda)\widetilde{\mathcal{A}}_{41}^+ & 0 & 0 & (1-\lambda)\widetilde{\mathcal{A}}_{44} & (1-\lambda)\widetilde{\mathcal{A}}_{45}^+ & 0\\ -(1+\lambda)\widetilde{\mathcal{A}}_{51}^+ & 0 & 0 & (1-\lambda)\widetilde{\mathcal{A}}_{54}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & (1-\lambda)\widetilde{\mathcal{A}}_{66} \end{bmatrix}_{6\times 6}, \quad (4.16)$$

where

$$\begin{array}{llll} \widetilde{\mathcal{A}}_{11}=\mathcal{A}_{11}+\frac{1}{\mu}, & \widetilde{\mathcal{A}}_{14}^+=\mathcal{A}_{14}^+, & \widetilde{\mathcal{A}}_{15}^+=\mathcal{A}_{15}^+ & \widetilde{\mathcal{A}}_{22}=\mathcal{A}_{22}+a,\\ \widetilde{\mathcal{A}}_{23}^+=\mathcal{A}_{23}^++ib, & \widetilde{\mathcal{A}}_{32}^+=\mathcal{A}_{32}^+-ib, & \widetilde{\mathcal{A}}_{33}=\mathcal{A}_{33}+a, & \widetilde{\mathcal{A}}_{41}^+=\mathcal{A}_{41}^+,\\ \widetilde{\mathcal{A}}_{44}=\mathcal{A}_{44}, & \widetilde{\mathcal{A}}_{45}^+=\mathcal{A}_{45}, & \widetilde{\mathcal{A}}_{51}^+=\mathcal{A}_{51}^+, & \widetilde{\mathcal{A}}_{54}^+=\mathcal{A}_{54},\\ & \widetilde{\mathcal{A}}_{55}=\mathcal{A}_{55}, & \widetilde{\mathcal{A}}_{66}=\mathcal{A}_{66}+1. & \end{array}$$

From (4.16), one can easily deduce

$$\det\left(\sigma_2^- - \lambda\sigma_2^+\right) = \det\begin{bmatrix} (1-\lambda)\widetilde{\mathcal{A}}_{22} & -(1+\lambda)\widetilde{\mathcal{A}}_{23}^+,\\ -(1+\lambda)\widetilde{\mathcal{A}}_{32}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{33}\end{bmatrix}$$

$$\times\det\begin{bmatrix} (1-\lambda)\widetilde{\mathcal{A}}_{11} & -(1+\lambda)\mathcal{A}_{14}^+ & -(1+\lambda)\mathcal{A}_{15}^+\\ -(1+\lambda)\mathcal{A}_{41}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{44} & (1-\lambda)\mathcal{A}_{45}\\ -(1+\lambda)\mathcal{A}_{51}^+ & (1-\lambda)\mathcal{A}_{54} & (1-\lambda)\widetilde{\mathcal{A}}_{55}\end{bmatrix}(1-\lambda)\widetilde{\mathcal{A}}_{66}=0.$$

Therefore, one of the eigenvalues, say λ_6, is equal to 1 and other eigenvalues are defined by the following equations:

$$\det\begin{bmatrix} (1-\lambda)\widetilde{\mathcal{A}}_{22} & -(1+\lambda)\widetilde{\mathcal{A}}_{23}^+\\ -(1+\lambda)\widetilde{\mathcal{A}}_{32}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{33}\end{bmatrix} = (1-\lambda)^2\widetilde{\mathcal{A}}_{22}\widetilde{\mathcal{A}}_{33}-(1+\lambda)^2\widetilde{\mathcal{A}}_{23}^+\widetilde{\mathcal{A}}_{32}^+=0, \quad (4.17)$$

$$\det\begin{bmatrix} (1-\lambda)\widetilde{\mathcal{A}}_{11} & -(1+\lambda)\mathcal{A}_{14}^+ & -(1+\lambda)\mathcal{A}_{15}^+\\ -(1+\lambda)\mathcal{A}_{41}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{44} & (1-\lambda)\mathcal{A}_{45}\\ -(1+\lambda)\mathcal{A}_{51}^+ & (1-\lambda)\mathcal{A}_{54} & (1-\lambda)\widetilde{\mathcal{A}}_{55}\end{bmatrix}=0. \quad (4.18)$$

Equation (4.17) can be rewritten as

$$\left(\frac{1-\lambda}{1+\lambda}\right)^2 = \frac{\widetilde{\mathcal{A}}_{23}^+\widetilde{\mathcal{A}}_{32}^+}{\widetilde{\mathcal{A}}_{22}\widetilde{\mathcal{A}}_{33}}. \quad (4.19)$$

Lemma 4.3. *The expression* $q := \dfrac{\widetilde{\mathcal{A}}_{23}^+\widetilde{\mathcal{A}}_{32}^+}{\widetilde{\mathcal{A}}_{22}\widetilde{\mathcal{A}}_{33}}$ *is positive.*

Proof. We have

$$\widetilde{\mathcal{A}}_{23}^+=\mathcal{A}_{23}^++ib, \quad \widetilde{\mathcal{A}}_{32}^+=\mathcal{A}_{32}^+-ib, \quad \widetilde{\mathcal{A}}_{22}=\mathcal{A}_{22}+a, \quad \widetilde{\mathcal{A}}_{33}=\mathcal{A}_{33}+a,$$

where

$$\mathcal{A}_{23}^+=-\frac{4A_{23}\mathbf{C}_{22}}{Q_2}, \quad \mathcal{A}_{32}^+=-\frac{4A_{32}\mathbf{C}_{33}}{Q_2}, \quad \mathcal{A}_{22}=\frac{2\mathbf{C}_{22}}{Q_2}, \quad \mathcal{A}_{33}=\frac{2\mathbf{C}_{33}}{Q_2}.$$

Since

$$Q_2 = -1 + 4A_{23}A_{32} < 0, \quad \mathbf{C}_{22} < 0, \quad \mathbf{C}_{33} < 0, \quad a > 0,$$

we have

$$\widetilde{\mathcal{A}}_{22} > 0, \quad \widetilde{\mathcal{A}}_{33} > 0.$$

Further, we show that $\widetilde{\mathcal{A}}^+_{23}\widetilde{\mathcal{A}}^+_{32} > 0$. Using the relations

$$\mathbf{C}_{22} = \mathbf{C}_{33}\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}}, \quad A_{23} = -A_{32}\frac{\sqrt{c_{11}}}{\sqrt{c_{33}}},$$

we deduce $A_{23}\mathbf{C}_{22} = -\mathbf{C}_{33}A_{32}$ and, consequently, $\mathcal{A}^+_{32} = -\mathcal{A}^+_{23}$. Since $\mathcal{A}^+_{23}$ is pure imaginary, we get

$$\widetilde{\mathcal{A}}^+_{23}\widetilde{\mathcal{A}}^+_{32} = -\big(\mathcal{A}^+_{23} + ib\big)^2 > 0,$$

which implies $q > 0$. □

Now, consider equation (4.18),

$$\det\begin{bmatrix} (1-\lambda)\widetilde{\mathcal{A}}_{11} & -(1+\lambda)\mathcal{A}^+_{14} & -(1+\lambda)\mathcal{A}^+_{15} \\ -(1+\lambda)\mathcal{A}^+_{41} & (1-\lambda)\widetilde{\mathcal{A}}_{44} & (1-\lambda)\mathcal{A}_{45} \\ -(1+\lambda)\mathcal{A}^+_{51} & (1-\lambda)\mathcal{A}_{54} & (1-\lambda)\widetilde{\mathcal{A}}_{55} \end{bmatrix} = (1-\lambda)^3\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55}$$

$$- (1-\lambda)^3\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45} - (1+\lambda)^2(1-\lambda)\widetilde{\mathcal{A}}^+_{14}\widetilde{\mathcal{A}}^+_{41}\widetilde{\mathcal{A}}_{55} + (1+\lambda)^2(1-\lambda)\widetilde{\mathcal{A}}^+_{14}\widetilde{\mathcal{A}}^+_{51}\widetilde{\mathcal{A}}_{45}$$
$$+ (1+\lambda)^2(1-\lambda)\widetilde{\mathcal{A}}^+_{15}\widetilde{\mathcal{A}}^+_{41}\widetilde{\mathcal{A}}_{54} - (1+\lambda)^2(1-\lambda)\widetilde{\mathcal{A}}^+_{15}\widetilde{\mathcal{A}}^+_{51}\widetilde{\mathcal{A}}_{44} = 0,$$

which can be rewritten as

$$(1-\lambda)\big[(1-\lambda)^2A + (1+\lambda)^2B\big] = 0.$$

Consequently, we get $\lambda_5 = 1$ and two other eigenvalues are defined by the equation

$$\Big(\frac{1-\lambda}{1+\lambda}\Big)^2 = -\frac{B}{A} =: -p, \tag{4.20}$$

where

$$A = \widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55} - \widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45},$$
$$B = -\widetilde{\mathcal{A}}^+_{14}\widetilde{\mathcal{A}}^+_{41}\widetilde{\mathcal{A}}_{55} + \widetilde{\mathcal{A}}^+_{14}\widetilde{\mathcal{A}}^+_{51}\widetilde{\mathcal{A}}_{45} + \widetilde{\mathcal{A}}^+_{15}\widetilde{\mathcal{A}}^+_{41}\widetilde{\mathcal{A}}_{54} - \widetilde{\mathcal{A}}^+_{15}\widetilde{\mathcal{A}}^+_{51}\widetilde{\mathcal{A}}_{44}.$$

Lemma 4.4. *The inequality* $p = \frac{B}{A} > 0$ *holds.*

Proof. We have

$$\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55} - \widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45} = \bigg[\frac{(2-8A_{15}A_{51})\mathbf{C}_{44}}{Q_1} + \frac{8A_{14}A_{51}\mathbf{C}_{45}}{Q_1}\bigg]\bigg[\frac{8A_{15}A_{41}\mathbf{C}_{45}}{Q_1} + \frac{(2-8A_{14}A_{41})\mathbf{C}_{55}}{Q_1}\bigg]$$
$$- \bigg[\frac{(2-8A_{15}A_{51})\mathbf{C}_{45}}{Q_1} + \frac{8A_{14}A_{51}\mathbf{C}_{55}}{Q_1}\bigg]\bigg[\frac{8A_{15}A_{41}\mathbf{C}_{44}}{Q_1} + \frac{(2-8A_{14}A_{41})\mathbf{C}_{45}}{Q_1}\bigg]$$
$$= \frac{(2-8A_{15}A_{51})\mathbf{C}_{44}}{Q_1}\cdot\frac{(2-8A_{14}A_{41})\mathbf{C}_{55}}{Q_1} + \frac{64A_{14}A_{51}A_{15}A_{41}\mathbf{C}^2_{45}}{Q_1^2}$$
$$- \frac{(2-8A_{15}A_{51})\mathbf{C}_{45}}{Q_1}\cdot\frac{(2-8A_{14}A_{41})\mathbf{C}_{45}}{Q_1} - \frac{64A_{14}A_{51}A_{15}A_{41}\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}$$
$$= M(\mathbf{C}_{44}\mathbf{C}_{55} - \mathbf{C}^2_{45}) + N(\mathbf{C}^2_{45} - \mathbf{C}_{44}\mathbf{C}_{55}) = (\mathbf{C}_{44}\mathbf{C}_{55} - \mathbf{C}^2_{45})(M-N),$$

where

$$M := \frac{(2-8A_{15}A_{51})(2-8A_{14}A_{41})}{Q_1^2}, \quad N := \frac{64A_{14}A_{51}A_{15}A_{41}}{Q_1^2}.$$

Note that $M - N > 0$, since $A_{14}A_{41} < 0$ and $A_{15}A_{51} < 0$. Indeed, we have

$$M - N = \frac{(2-8A_{15}A_{51})(2-8A_{14}A_{41})}{Q_1^2} - \frac{64A_{14}A_{51}A_{15}A_{41}}{Q_1^2} = \frac{4}{Q_1^2}\big[1 - 4A_{14}A_{41} - 4A_{15}A_{51}\big] > 0.$$

Now we show that $\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2>0$. Rewrite $\mathbf{C}_{44}$, $\mathbf{C}_{55}$ and $\mathbf{C}_{45}$ in the form

$$\mathbf{C}_{44}=-(m+q_{15}^2n),\quad \mathbf{C}_{55}=-(m+e_{14}^2n),\quad \mathbf{C}_{45}=e_{14}q_{15}n,$$

where

$$m=\frac{(b_2-b_1)}{2\sqrt{B}}\left(c_{44}+\frac{c_{66}}{b_1b_2}\right)>0,\quad n=\frac{1}{2\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}}-\frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1b_2+\varkappa_{11})}{\sqrt{B}}\right]>0$$

(see [8], Appendix B) and

$$\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2=m^2+(e_{14}^2+q_{15}^2)mn>0.$$

Consequently,

$$\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55}-\widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45}>0$$

and, since $\widetilde{\mathcal{A}}_{11}>0$, we have

$$A=\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55}-\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45}>0.$$

Now, we show that

$$B=-\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45}+\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{54}-\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{44}>0.$$

First, we prove the inequality $-\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45}>0$. Indeed,

$$\begin{aligned}
&-\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45}=\widetilde{\mathcal{A}}_{14}^+(-\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45})\\
&=-\frac{4A_{14}\mathbf{C}_{11}}{Q_1}\Big[\Big(\frac{4A_{41}\mathbf{C}_{44}}{Q_1}+\frac{4A_{51}\mathbf{C}_{45}}{Q_1}\Big)\Big(\frac{8A_{15}A_{41}\mathbf{C}_{45}}{Q_1}+\frac{(2-8A_{14}A_{41})\mathbf{C}_{55}}{Q_1}\Big)\\
&-\Big(\frac{4A_{41}\mathbf{C}_{45}}{Q_1}+\frac{4A_{51}\mathbf{C}_{55}}{Q_1}\Big)\Big(\frac{8A_{15}A_{41}\mathbf{C}_{44}}{Q_1}+\frac{(2-8A_{14}A_{41})\mathbf{C}_{45}}{Q_1}\Big)\Big]\\
&=-\frac{4A_{14}\mathbf{C}_{11}}{Q_1}\Big[\frac{4A_{41}(2-8A_{14}A_{41})\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}+\frac{32A_{51}A_{15}A_{41}\mathbf{C}_{45}^2}{Q_1^2}\\
&-\frac{4A_{41}(2-8A_{14}A_{41})\mathbf{C}_{45}^2}{Q_1^2}-\frac{32A_{51}A_{15}A_{41}\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}\Big]\\
&=-\frac{32A_{14}A_{41}\mathbf{C}_{11}}{Q_1}\Big[\frac{(1-4A_{14}A_{41})}{Q_1^2}(\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2)+\frac{4A_{51}A_{15}}{Q_1^2}(\mathbf{C}_{45}^2-\mathbf{C}_{44}\mathbf{C}_{55})\Big]\\
&=-\frac{32A_{14}A_{41}\mathbf{C}_{11}}{Q_1}\Big[\frac{1-4A_{14}A_{41}-4A_{51}A_{15}}{Q_1^2}\Big](\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2)=\frac{32A_{14}A_{41}C_{11}}{Q_1^2}(C_{44}C_{55}-C_{45}^2).
\end{aligned}$$

Therefore, taking into account the inequalities $A_{14}A_{41}<0$, $\mathbf{C}_{11}<0$, $\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2>0$, we conclude that

$$-\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45}>0.$$

Further, we prove that

$$\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{54}-\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{44}>0.$$

Conducting algebraic transformations as in the previous case, we get

$$\begin{aligned}
&\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{54}-\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{44}=\widetilde{\mathcal{A}}_{15}^+(-\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{44}+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{54})\\
&=\frac{4A_{15}\mathbf{C}_{11}}{Q_1}\Big[-\Big(\frac{4A_{41}\mathbf{C}_{45}}{Q_1}+\frac{4A_{51}\mathbf{C}_{55}}{Q_1}\Big)\Big(\frac{(2-8A_{15}A_{51})\mathbf{C}_{44}}{Q_1}+\frac{8A_{14}A_{51}\mathbf{C}_{45}}{Q_1}\Big)\\
&+\Big(\frac{4A_{41}\mathbf{C}_{44}}{Q_1}+\frac{4A_{51}\mathbf{C}_{45}}{Q_1}\Big)\Big(\frac{(2-8A_{15}A_{51})\mathbf{C}_{45}}{Q_1}+\frac{8A_{14}A_{51}\mathbf{C}_{55}}{Q_1}\Big)\Big]\\
&=\frac{4A_{15}\mathbf{C}_{11}}{Q_1}\Big[-\frac{32A_{41}A_{14}A_{51}\mathbf{C}_{45}^2}{Q_1^2}-\frac{4A_{51}(2-8A_{15}A_{51})\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}\\
&+\frac{32A_{41}A_{14}A_{51}\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}+\frac{4A_{51}(2-8A_{15}A_{51})\mathbf{C}_{45}^2}{Q_1^2}\Big]\\
&=\frac{4A_{15}A_{51}\mathbf{C}_{11}}{Q_1}\Big[-\frac{32A_{14}A_{41}\mathbf{C}_{45}^2}{Q_1^2}-\frac{4(2-8A_{15}A_{51})\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{32A_{41}A_{14}\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}+\frac{4(2-8A_{15}A_{51})\mathbf{C}_{45}^2}{Q_1^2}\Big]\\
&=\frac{4A_{15}A_{51}\mathbf{C}_{11}}{Q_1}\Big[\frac{32A_{14}A_{41}}{Q_1^2}(-\mathbf{C}_{45}^2+\mathbf{C}_{44}\mathbf{C}_{55})-\frac{4(2-8A_{15}A_{51})}{Q_1^2}(-\mathbf{C}_{45}^2+\mathbf{C}_{44}\mathbf{C}_{55})\Big]\\
&=\frac{32A_{15}A_{51}\mathbf{C}_{11}}{Q_1}\Big[\frac{4A_{14}A_{41}}{Q_1^2}-\frac{(1-4A_{15}A_{51})}{Q_1^2}\Big](-\mathbf{C}_{45}^2+\mathbf{C}_{44}\mathbf{C}_{55})=\frac{32A_{15}A_{51}\mathbf{C}_{11}}{Q_1^2}(-\mathbf{C}_{45}^2+\mathbf{C}_{44}\mathbf{C}_{55}).
\end{aligned}
$$

Taking into account the inequalities $A_{51}A_{15}<0$, $\mathbf{C}_{11}<0$ and $\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2>0$, we obtain

$$\widetilde{\mathcal{A}}_{15}^{+}\widetilde{\mathcal{A}}_{41}^{+}\widetilde{\mathcal{A}}_{54}-\widetilde{\mathcal{A}}_{15}^{+}\widetilde{\mathcal{A}}_{51}^{+}\widetilde{\mathcal{A}}_{44}>0.$$

Thus, $B>0$ and, consequently, $p=\dfrac{A}{B}>0$. □

Due to (4.19) and (4.20), we have the following expressions for the eigenvalues of the matrix $(\sigma_2^+)^{-1}\sigma_2^-$ (i.e., the roots of polynomial (4.16) with respect to λ),

$$\lambda_1=\frac{1-i\sqrt{p}}{1+i\sqrt{p}},\quad \lambda_2=\lambda_1^{-1}=\overline{\lambda_1},\quad \lambda_3=\frac{1-\sqrt{q}}{1+\sqrt{q}},\quad \lambda_4=\lambda_3^{-1},\quad \lambda_5=\lambda_6=1.$$

Note that $|\lambda_1|=|\lambda_2|=1$. Moreover, since λ_3 and λ_4 are real, they are positive (see Appendix, Subsection 5.2).

Applying the above results, we can explicitly write the exponents of the first terms of the asymptotic expansions of the solutions (see (4.3)):

$$
\begin{aligned}
\gamma_1&=\frac12+\frac{1}{2\pi}\arg\lambda_1=\frac12+\frac{1}{2\pi}\arg\frac{1-i\sqrt{p}}{1+i\sqrt{p}}\\
&=\frac12+\frac{1}{2\pi}\Big(\arg(1-i\sqrt{p})-\arg(1+i\sqrt{p})\Big)=\frac12-\frac1\pi\arctan\sqrt{p},\\
\gamma_1&=\frac12-\frac1\pi\arctan\sqrt{p}, && \delta_1=0,\\
\gamma_2&=\frac12+\frac1\pi\arctan\sqrt{p}, && \delta_2=0,\\
\gamma_3&=\gamma_4=\frac12, && \delta_3=-\delta_4=\widetilde{\delta}=-\frac{1}{2\pi}\log\frac{1-\sqrt{q}}{1+\sqrt{q}},\\
\gamma_5&=\gamma_6=\frac12, && \delta_5=\delta_6=0.
\end{aligned}
$$

It is evident that $0<\gamma_1<\frac12$ and $\frac12<\gamma_2<1$.

Note that in this case $B_0(t)$ has the following form (see (3.52)):

$$B_0(t)=\begin{bmatrix} I_4 & [0]_{4\times 2}\\ [0]_{2\times 4} & B^{(2)}(t)\end{bmatrix},\qquad \text{where}\quad B^{(2)}(t)=\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}.$$

Now, we can draw the following conclusions:

(1) In view of Theorem 4.1, the solutions of the problem possess the following asymptotic behaviour near the edge curve $\ell=\partial\Gamma_T$:

$$
\begin{aligned}
(u^{(1)},\varphi^{(1)},\psi^{(1)})^\top &= c_0^{(1)}r^{\gamma_1}+c_1^{(1)}r^{\frac12}\ln r+c_2^{(1)}r^{\frac12+i\widetilde{\delta}}+c_3^{(1)}r^{\frac12-i\widetilde{\delta}}+c_4^{(1)}r^{\frac12}+c_5^{(1)}r^{\gamma_2}+\cdots,\\
\vartheta^{(1)} &= b_0^{(1)}r^{\frac12}+b_1^{(1)}r^{\gamma_2}+\cdots,\\
u^{(2)} &= c_0^{(2)}r^{\gamma_1}+c_1^{(2)}r^{\frac12}\ln r+c_2^{(2)}r^{\frac12+i\widetilde{\delta}}+c_3^{(2)}r^{\frac12-i\widetilde{\delta}}+c_4^{(2)}r^{\frac12}+c_5^{(2)}r^{\gamma_2}+\cdots,\\
\vartheta^{(2)} &= b_0^{(2)}r^{\frac12}+b_1^{(2)}r^{\gamma_2}+\cdots,
\end{aligned}
$$

where coefficients $c_j^{(1)}$, $j=0,...5$, are the 5-dimensional vectors, $c_j^{(2)}$, $j=0,...5$, are the 3-dimensional vectors and $b_j^{(k)}$, $j=0,1$, $k=1,2$, are scalars.

As we can see, the exponent γ_1 characterizing the behaviour of $u^{(1)}$, $\varphi^{(1)}$, $\psi^{(1)}$ and $u^{(2)}$ near the line ℓ depends on the elastic, piezoelectric, piezomagnetic, dielectric and permeability constants, and does not depend on the thermal constants. Moreover, γ_1 takes values from the interval $(0, \frac{1}{2})$.

For the general anisotropic case, these exponents also depend on the geometry of the line ℓ, in general.

(2) In general, we have the following smoothness of mechanical and electromagnetic fields:
$$(u^{(1)},\varphi^{(1)},\psi^{(1)})\in\big[C^{\gamma_1}(\overline{\Omega}_1)\big]^5,\quad u^{(2)}\in\big[C^{\gamma_1}(\overline{\Omega}_2)\big]^3,\ \ 0<\gamma_1<\frac{1}{2}.$$

(3) Since $\gamma_1<\frac{1}{2}$, we have no oscillating stress singularities for physical fields in the neighbourhood of the curve ℓ.

Note that in the classical elasticity theory (for both isotropic and anisotropic solids) for mixed boundary value and mixed transmission problems the dominant exponents are $\frac{1}{2}$, $\frac{1}{2}\pm i\,\widetilde{\delta}$ with $\widetilde{\delta}\neq 0$ and, consequently, there occur oscillating stress singularities at the line ℓ (for details see [12, 13]).

5. Appendix

5.1. **Properties of Potentials and Boundary Operators.** Here we collect some theorems describing the mapping properties of potentials and the corresponding boundary integral (pseudodifferential) operators. The proof of these theorems can be found in references [7, 8, 20].

Theorem 5.1. *Let $1<p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$. Then the single layer potentials can be extended to the following continuous operators:*
$$V_\tau^{(2)}\ :\big[B^s_{p,q}(S)\big]^4\to\big[B^{s+1+\frac{1}{p}}_{p,q}(\Omega^{(2)})\big]^4,\qquad V_\tau^{(1)}\ :\big[B^s_{p,p}(S)\big]^6\to\big[H^{s+1+\frac{1}{p}}_{p}(\Omega^{(1)})\big]^6,$$
$$V_\tau^{(2)}\ :\big[H^s_{p}(S)\big]^4\to\big[H^{s+1+\frac{1}{p}}_{p}(\Omega^{(2)})\big]^4,\qquad V_\tau^{(1)}\ :\big[H^s_{p}(S)\big]^6\to\big[H^{s+1+\frac{1}{p}}_{p}(\Omega^{(1)})\big]^6.$$

Theorem 5.2. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $h^{(2)}\in\big[B^{-\frac{1}{p}}_{p,q}(\partial\Omega^{(2)})\big]^4$, $h^{(1)}\in\big[B^{-\frac{1}{p}}_{p,q}(\partial\Omega^{(1)})\big]^6$. Then*
$$\big\{V_\tau^{(2)}(h^{(2)})\big\}^+=\big\{V_\tau^{(2)}(h^{(2)})\big\}^-=\mathcal{H}_\tau^{(2)}(h^{(2)})\ \ on\ \ \partial\Omega^{(2)},$$
$$\big\{\mathcal{T}^{(2)}(\partial,\nu,\tau)V_\tau^{(2)}(h^{(2)})\big\}^\pm=\big[\mp 2^{-1}I_4+\mathcal{K}_\tau^{(2)}\big](h^{(2)})\ \ on\ \ \partial\Omega^{(2)},$$
$$\big\{V_\tau^{(1)}(h^{(1)})\big\}^+=\big\{V_\tau^{(1)}(h^{(1)})\big\}^-=\mathcal{H}_\tau^{(1)}(h^{(1)})\ \ on\ \ \partial\Omega^{(1)},$$
$$\big\{\mathcal{T}^{(1)}(\partial,n,\tau)V_\tau^{(1)}(h^{(1)})\big\}^\pm=\big[\mp 2^{-1}I_6+\mathcal{K}_\tau^{(1)}\big](h^{(1)})\ \ on\ \ \partial\Omega^{(1)},$$
where I_k stands for the $k\times k$ unit matrix.

The operators $\mathcal{H}_\tau^{(1)}$, $\mathcal{H}_\tau^{(2)}$, $\mathcal{K}_\tau^{(1)}$ and $\mathcal{K}_\tau^{(2)}$ possess the mapping and the Fredholm properties [7].

Theorem 5.3. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $s\in\mathbb{R}$. The operators*
$$\mathcal{H}_\tau^{(2)}\ :\ \big[H^s_p(\partial\Omega^{(2)})\big]^4\to\big[H^{s+1}_p(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{H}_\tau^{(1)}\ :\ \big[H^s_p(\partial\Omega^{(1)})\big]^6\to\big[H^{s+1}_p(\partial\Omega^{(1)})\big]^6,$$
$$\mathcal{H}_\tau^{(2)}\ :\ \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4\to\big[B^{s+1}_{p,q}(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{H}_\tau^{(1)}\ :\ \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6\to\big[B^{s+1}_{p,q}(\partial\Omega^{(1)})\big]^6,$$
$$\mathcal{K}_\tau^{(2)}\ :\ \big[H^s_p(\partial\Omega^{(2)})\big]^4\to\big[H^{s}_p(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{K}_\tau^{(1)}\ :\ \big[H^s_p(\partial\Omega^{(1)})\big]^6\to\big[H^{s}_p(\partial\Omega^{(1)})\big]^6,$$
$$\mathcal{K}_\tau^{(2)}\ :\ \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4\to\big[B^{s}_{p,q}(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{K}_\tau^{(1)}\ :\ \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6\to\big[B^{s}_{p,q}(\partial\Omega^{(1)})\big]^6,$$
are continuous.

Theorem 5.4. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $s\in\mathbb{R}$ and $\tau=\sigma+i\,\omega$. The operators*
$$\mathcal{H}_\tau^{(2)}:\big[H^s_p(\partial\Omega^{(2)})\big]^4\to\big[H^{s+1}_p(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{H}_\tau^{(1)}\ :\big[H^s_p(\partial\Omega^{(1)})\big]^6\to\big[H^{s+1}_p(\partial\Omega^{(1)})\big]^6,$$

$$\mathcal{H}_\tau^{(2)} : \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4 \to \big[B^{s+1}_{p,q}(\partial\Omega^{(2)})\big]^4, \qquad \mathcal{H}_\tau^{(1)} : \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6 \to \big[B^{s+1}_{p,q}(\partial\Omega^{(1)})\big]^6,$$

are invertible if $\sigma > 0$ *or* $\tau = 0$.

The operators

$$\begin{aligned}
\pm 2^{-1} I_4 + \mathcal{K}_\tau^{(2)} &: \big[H^s_p(\partial\Omega^{(2)})\big]^4 \to \big[H^s_p(\partial\Omega^{(2)})\big]^4,\\
\pm 2^{-1} I_4 + \mathcal{K}_\tau^{(2)} &: \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4 \to \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4,\\
2^{-1} I_6 + \mathcal{K}_\tau^{(1)} &: \big[H^s_p(\partial\Omega^{(1)})\big]^6 \to \big[H^s_p(\partial\Omega^{(1)})\big]^6,\\
2^{-1} I_6 + \mathcal{K}_\tau^{(1)} &: \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6 \to \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6,
\end{aligned}$$

are invertible if $\sigma > 0$.

The operators

$$\begin{aligned}
-2^{-1} I_6 + \mathcal{K}_\tau^{(1)} &: \big[H^s_p(\partial\Omega^{(1)})\big]^6 \to \big[H^s_p(\partial\Omega^{(1)})\big]^6,\\
-2^{-1} I_6 + \mathcal{K}_\tau^{(1)} &: \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6 \to \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6
\end{aligned}$$

are Fredholm ones with the index, equal to zero for any $\tau \in \mathbb{C}$.

5.2. **Fredholm properties of pseudodifferential operators on manifolds with boundary.** Let $\mathcal{M}$ be a compact, n-dimensional, smooth, nonselfintersecting manifold with the smooth boundary $\partial\mathcal{M} \neq \varnothing$ and let $\mathbf{A}(x,D)$ be a strongly elliptic $N\times N$ matrix pseudodifferential operator of order $\nu\in\mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathbf{A};x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathbf{A}(x,D)$ in some local coordinate system ($x\in\overline{\mathcal{M}}$, $\xi\in\mathbb{R}^n\setminus\{0\}$).

Let $\lambda_1(x),\dots,\lambda_N(x)$ be the eigenvalues of the matrix

$$\big[\,\mathfrak{S}(\mathbf{A};x,0,\dots,0,+1)\,\big]^{-1}\,\big[\,\mathfrak{S}(\mathbf{A};x,0,\dots,0,-1)\,\big],\ \ x\in\partial\mathcal{M},$$

and introduce the notation

$$\delta_j(x) = \operatorname{Re}\big[\,(2\pi\, i)^{-1}\ln\lambda_j(x)\,\big],\ \ j=1,\dots,N.$$

Here $\ln\zeta$ denotes the branch of the logarithmic function, analytic in the complex plane cut along $(-\infty,0]$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system and the strong inequality $-1/2 < \delta_j(x) < 1/2$ holds for all $x\in\overline{\mathcal{M}}$, $j=\overline{1,N}$, due to the strong ellipticity of $\mathbf{A}$. In a particular case, when $\mathfrak{S}(\mathbf{A};x,\xi)$ is a positive definite matrix for every $x\in\overline{\mathcal{M}}$ and $\xi\in\mathbb{R}^n\setminus\{0\}$, we have $\delta_1(x)=\dots=\delta_N(x)=0$, since the eigenvalues $\lambda_1(x),\dots,\lambda_N(x)$ are positive for all $x\in\overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators on manifolds with boundary are characterized by the following theorem (see [2, 4, 18, 30]).

Theorem 5.5. *Let* $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$, *and let* $\mathbf{A}(x,D)$ *be a pseudodifferential operator of order* $\nu\in\mathbb{R}$ *with the strongly elliptic symbol* $\mathfrak{S}(\mathbf{A};x,\xi)$, *that is, there is a positive constant* c_0 *such that*

$$\operatorname{Re}\mathfrak{S}(\mathbf{A};x,\xi)\,\eta\cdot\eta \geqslant c_0\,|\eta|^2$$

for $x\in\overline{\mathcal{M}}$, $\xi\in\mathbb{R}^n$ *with* $|\xi|=1$, *and* $\eta\in\mathbb{C}^N$.

Then the operators

$$\begin{aligned}
\mathbf{A} &: \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \to \big[H^{s-\nu}_p(\mathcal{M})\big]^N\\
\mathbf{A} &: \big[\widetilde{B}^s_{p,q}(\mathcal{M})\big]^N \to \big[B^{s-\nu}_{p,q}(\mathcal{M})\big]^N
\end{aligned}\tag{5.1}$$

are Fredholm and have the trivial index $\operatorname{Ind}\mathbf{A}=0$ *if*

$$\frac1p - 1 + \sup_{\substack{x\in\partial\mathcal{M},\\ 1\leqslant j\leqslant N}}\delta_j(x) < s - \frac{\nu}{2} < \frac1p + \inf_{\substack{x\in\partial\mathcal{M},\\ 1\leqslant j\leqslant N}}\delta_j(x). \tag{5.2}$$

Moreover, the null-spaces and indices of the operators (5.1) *coincide for all values of the parameter* $q\in[1,+\infty]$ *provided* p *and* s *satisfy inequality* (5.2).

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References

1. J. Aboudi, Micromechanical analysis of fully coupled electro-magneto-thermo-elastic multiphase composites. *Smart Mater. Struct.* **10** (2001), 867–877.
2. M. S. Agranovich, Elliptic operators on closed manifolds. Partial differential equations. VI. Elliptic operators on closed manifolds. *Encycl. Math. Sci.* **63** (1994), 1–130 (1994); translation from *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya* **63** (1990), 5–129.
3. M. Aouadi, Some theorems in the generalized theory of thermo-magnetoelectroelasticity under Green–Lindsay's model. *Acta Mech.* **200** (2008), no. 1-2, 25–43.
4. A. V. Brenner, E. M. Shargorodsky, Boundary value problems for elliptic pseudodifferential operators. Translated from the Russian by Brenner. *Encyclopaedia Math. Sci.* **79**, *Partial differential equations*, IX, Springer, Berlin, 1997, 145–215.
5. T. Buchukuri, O. Chkadua, Boundary problems of thermopiezoelectricity in domains with cuspidal edges. *Georgian Math. J.* **7** (2000), no. 3, 441–460.
6. T. Buchukuri, O. Chkadua, R. Duduchava, D. Natroshvili, Interface crack problems for metallic-piezoelectric composite structures. *Mem. Differ. Equ. Math. Phys.* **55** (2012), 1–150.
7. T. Buchukuri, O. Chkadua, D. Natroshvili, Mathematical Problems of Generalized Thermo-Electro-Magneto-Elasticity Theory. *Mem. Differ. Equ. Math. Phys.* **68**(2016), 1–166.
8. T. Buchukuri, O. Chkadua, D. Natroshvili, Mixed boundary value problems of pseudo-oscillations of generalized thermo-electro-magneto-elasticity theory for solids with interior cracks. *Trans. A. Razmadze Math. Inst.* **170** (2016), no. 3, 308–351.
9. T. Buchukuri, O. Chkadua, D. Natroshvili, A. M. Sändig, Interaction problems of metallic and piezoelectric materials with regard to thermal stresses. *Mem. Differential Equations Math. Phys.* **45** (2008), 7–74.
10. T. Buchukuri, O. Chkadua, D. Natroshvili, A. M. Sändig, Solvability and regularity results to boundary-transmission problems for metallic and piezoelectric elastic materials. *Math. Nachr.* **282** (2009), no. 8, 1079–1110.
11. T. V. Burchuladze, T. G. Gegelia, Development of the potential method in elasticity theory. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR* **79** (1985), 226 pp.
12. O. Chkadua, Solvability and asymptotics of solutions of crack-type boundary-contact problems of the couple-stress elasticity. *Georgian Math. J.* **10** (2003), no. 3, 427–465.
13. O. Chkadua, R. Duduchava, Asymptotics of solutions to some boundary value problems of elasticity for bodies with cuspidal edges. *Mem. Differential Equations Math. Phys.* **15** (1998), 29–58.
14. O. Chkadua, R. Duduchava, Asymptotics of functions represented by potentials. *Russ. J. Math. Phys.* **7** (2000), no. 1, 15–47.
15. O. Chkadua, R. Duduchava, Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotic. *Math. Nachr.* **222** (2001), 79–139.
16. E. Dieulesaint, D. Royer, *Ondes Elastiques dans les Solides*. vol. 2 in Enseignement de la physique, 1999.
17. R. Duduchava, W. Wendland, The Wiener-Hopf method for systems of pseudodifferential equations with an application to crack problems. *Integral Equations Operator Theory* **23** (1995), no. 3, 294–335.
18. G. Eskin, *Boundary Value Problems for Elliptic Pseudodifferential Equations.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, 52. American Mathematical Society, Providence, R.I., 1981.
19. F. Gantmakher, *Theory of Matrices.* (Russian) Nauka, Moscow, 1967.
20. L. Hörmander, *The Analysis of Linear Partial Differential Operators.* III. Pseudodifferential operators. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 274. Springer-Verlag, Berlin, 1985.
21. G. C. Hsiao, W. L. Wendland, *Boundary Integral Equations.* Applied Mathematical Sciences, 164. Springer-Verlag, Berlin, 2008.
22. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, T. V. Burchuladze, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity.* Translated from the second Russian edition. Edited by V. D. Kupradze. North-Holland Series in Applied Mathematics and Mechanics, 25. North-Holland Publishing Co., Amsterdam-New York, 1979.
23. W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations.* Cambridge University Press, Cambridge, 2000.
24. D. Natroshvili, T. Buchukuri, O. Chkadua, Mathematical modelling and analysis of interaction problems for piezoelastic composites. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)* **30** (2006), 159–190.
25. D. Natroshvili, O. Chkadua, E. Shargorodskiĭ, Mixed problems for homogeneous anisotropic elastic media. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **39** (1990), 133–181.
26. D. Natroshvili, M. Mrevlishvili, Mixed boundarytransmission problems for composite layered elastic structures. *Mathematical Methods in the Applied Sciences*, 2020, `https://doi.org/10.1002/mma.6734`.

27. W. Nowacki, *Efekty Elektromagnetyczne w Stałych Ciałach Odkształcalnych*. Państwowe Wydawnictwo Naukowe, 1983.
28. Y. E. Pak, Linear electro-elastic fracture mechanics of piezoelectric materials. *International Journal of Fracture* **54** (1992), no. 1, 79–100.
29. Q. H. Qin, *Fracture Mechanics of Piezoelectric Materials*. Wit Press, Southampton, UK, 2001.
30. E. Shargorodsky, An L_p-analogue of the Vishik-Eskin theory. *Mem. Differential Equations Math. Phys.* **2** (1994), 41–146.
31. C. C. Silva, D. Thomazini, A. G. Pinheiro, N. Aranha, S. D. Figueiró, J. C. Góes, A. S. B. Sombra, Collagen-hydroxyapatite films: piezoelectric properties. *Materials Science and Engineering* **B86** (2001), no. 3, 210–218.
32. B. Straughan, *Heat Waves*. Applied Mathematical Sciences, 177. Springer, New York, 2011.
33. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland Mathematical Library, 18. North-Holland Publishing Co., Amsterdam-New York, 1978.
34. A. M. Zenkour, I. A. Abbas, Electro-magneto-thermo-elastic response of infinite functionally graded cylinders without energy dissipation. *Journal of Magnetism and Magnetic Materials*, **395** (2015), 123–129.

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NEW RESULTS ON SEMI-I-CONVERGENCE

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Abstract. In this article, we use the notions of semi-open, semi-I-open sets and S-I-convergence to show and study other properties on semi-I-convergence. Besides, some basic properties of semi-I-Fréchet–Urysohn space are shown. Moreover, the notions related to semi-I-sequential and semi-I-sequentially open spaces are proved. Furthermore, we show some relations of semi-I-irresolute functions between preserving semi-I-convergence functions and semi-I-covering functions.

1. Introduction

The notion of ideal was introduced by Kuratowski in 1933 [5], an ideal I on a space X is a collection of elements of X which satisfies: (1) $\emptyset \in I$, (2) if $A, B \in I$, then $A \cup B \in I$, and (3) if $B \subset I$ and $A \subset B$, then $A \in I$. This notion has been grown in several concepts of general topology. In 2019, Zhou and Lin [8] used the notion of ideal on the set $\mathbb{N}$ to extend the notion of I-convergence, the results were useful for the developing of this paper. Recently, in 2020, Guevara et.al. [3] have shown some basic properties of S-I-convergent sequences and studied the notions related to the compactness and cluster points by using semi-open sets, furthermore, they have proved that S-I-convergence implies I-convergence for any ideal I on $\mathbb{N}$. On the other hand, in 1963, Levine [6] introduced the concept of semi-open sets in topological spaces, and then in 2005, Hatir and Noiri [4] presented the idea of semi-I-open sets and semi-I-continuous functions in the ideal topological spaces. In this article, we took in the whole the notions mentioned above, define other properties on semi-I-convergence and study the relation between semi-I-sequentially open and semi-I-sequential spaces. Moreover, we define and study some basic properties of preserving semi-I-convergence functions and semi-I-covering functions, furthermore, we prove some relations with semi-I-irresolute functions. Besides, the idea of semi-I-Fréchet–Urysohn spaces is defined.

Throughout this paper, the terms (X, τ) and (Y, σ) mean topological spaces on which no separation axioms are assumed unless otherwise mentioned. Additionally, we sometimes write X or Y instead of (X, τ) or (Y, σ), respectively.

2. Semi-I-convergence

We first introduce some definitions.

Definition 2.1. Let (X, τ) be a topological space, $A \subset X$ and $x \in X$. Then A is said to be semi-neighbourhood if and only if there exits a semi-open set B such that $x \in B \subset A$.

Definition 2.2. A sequence $(x_n : n \in \mathbb{N})$ in a topological space X is called semi-I-convergent to a point $x \in X$, provided for any semi-neighbourhood U of x, it has $A_V = \{n \in \mathbb{N} : x_n \notin V\} \in I$, which is denoted by s-I-$\lim\limits_{n\to\infty} x_n = x$ or $x_n \to^{sI} x$, and the point x is called the s-I-limit of the sequence $(x_n : n \in \mathbb{N})$.

Definition 2.3. Let (X, τ) be a topological space and $A \subset X$. Then A is called semi-I-sequentially open if and only if no sequence in $X - A$ has a semi-I-limit in A. That is, the sequence cannot be semi-I-convergent outside of a semi-I-sequentially closed set.

Definition 2.4. Let I be an ideal on $\mathbb{N}$ and X be a topological space, then:

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(1) A subset J of X is said to be semi-I-closed if for each sequence $(x_n : n \in \mathbb{N}) \subseteq J$ with $x_n \to^{sI} x \in X$, then $x \in J$.
(2) A subset V of X is said to be semi-I-open if $X - V$ is semi-I-closed.
(3) X is said to be a semi-I-sequential space if each semi-I-closed set in X is closed.

Definition 2.5. Let (X, τ) be a topological space. Then X is semi-I-sequential, when any set A is semi-open, if and only if it is semi-I-sequentially open.

Now, we show some results.

Lemma 2.1 ([8]). *Let I be an ideal on $\mathbb{N}$ and X be a topological space. If a sequence $(X_n : x \in \mathbb{N})$ I-converges to a point $x \in X$ and $(y_n : n \in \mathbb{N})$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in I$, then the sequence $(y_n : n \in \mathbb{N})$ I-converges to $x \in X$*

Lemma 2.2 ([8]). *Let $I \subseteq J$ be two ideals of $\mathbb{N}$. If $(x_n : n \in \mathbb{N})$ is a sequence in a topological space X such that $x_n \to^I x$, then $x_n \to^J x$.*

Lemma 2.3. *Let (X, τ) be a topological space. Then $B \subset X$ is semi-I-sequentially open if and only if every sequence with semi-I-limit in B has all, but finitely many, terms in B, where the index set of the part in B of the sequence does not belong to I.*

Proof. Suppose that B is not a semi-I-sequentially open, then there is a sequence with terms in $X - B$, but semi-I-limit in B. Conversely, suppose that $(x_n : n \in \mathbb{N})$ is a sequence with infinitely many terms in $X - B$ such that semi-I-converges to $y \in A$ and the index set of the part in B of the sequence does not belong to I. Then $(x_n : n \in \mathbb{N})$ has a subsequence in $X - B$ that has still to converge to $y \in B$ and so, B is not semi-I-sequentially open. □

Lemma 2.4. *Let I and J be two ideals of $\mathbb{N}$, where $I \subseteq J$ and X is a topological space. If $V \subseteq X$ is semi-J-open, then it is semi-I-open.*

Proof. Let $V \subseteq X$ be semi-J-open. Then $X - V$ is semi-J-open, if $(x_n : n \in \mathbb{N})$ is a sequence in $X - V$ with $x_n \to^{sI} x$, thus by Lemma 2.2, it has $x \in X - V$. Therefore, V is semi-I-open. □

Corollary 2.1. *Let I and J be two ideals of $\mathbb{N}$. If a topological space X is semi-I-sequential, then it is semi-J-sequential.*

Lemma 2.5. *Let I be an ideal on $\mathbb{N}$ and X be a topological space. If a sequence $(X_n : x \in \mathbb{N})$ is semi-I-convergent to a point $x \in X$ and $(y_n : n \in \mathbb{N})$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in I$, then the sequence $(y_n : n \in \mathbb{N})$ is semi-I-convergent to $x \in X$.*

Proof. The proof is followed by Lemma 2.1 and Definition 2.2. □

Lemma 2.6. *Let X be a topological space X, $A \subset X$ and I be an ideal on $\mathbb{N}$. Then the following statements are equivalent.*

(1) *A is semi-I-open.*
(2) *$\{n \in \mathbb{N} : x_n \in A\} \notin I$ for each sequence $(x_n : n \in \mathbb{N})$ in X with $x_n \to^{sI} x \in A$.*
(3) *$|\{n \in \mathbb{N} : x_n \in A\}| = \theta$ for each sequence $(x_n : n \in \mathbb{N})$ in X with $x_n \to^{sI} x \in A$.*

Proof. (1) $\Rightarrow$ (2) : Suppose that A is a semi-I-open set of X and let $(x_n : n \in \mathbb{N})$ be a sequence in X satisfying $x_n \to^{sI} x \in A$. Now, take $N_0 = \{n \in \mathbb{N} : x_n \in A\}$. If $N_0 \in I$, then $N_0 \neq \mathbb{N}$ and so, $A \neq X$. Now, take a point $a \in X - A$ and define the sequence $(y_n : n \in \mathbb{N})$ in X by $y_n = a, n \in N_0$, thus $y_n = x_n$, $n \notin N_0$. By Lemma 2.5, the sequence $(y_n : n \in \mathbb{N})$ semi-I-converges to x. We can see that $X - A$ is semi-I-closed and $(y_n)_{n \in \mathbb{N}} \subseteq X - A$, as a consequence, $x \in X - A$, but this is a contradiction. Therefore, $N_0 \notin I$.

The implication (2) $\Rightarrow$ (3) follows from the notion that the ideal I is admissible.

Now, let us show the following implication. (3) $\Rightarrow$ (1) : Let A be nonsemi-I-open in X. Then $X - A$ is not semi-I-closed and there is a sequence $(x_n : n \in \mathbb{N}) \subseteq X - A$ with $x_n \to^{sI} x \in A$ and this is a contradiction. □

Theorem 2.1. *Every semi-I-sequential space is hereditary with respect to semi-I-open (semi-I-closed) subspaces.*

Proof. Let X be a semi-I-sequential space. Suppose now that A is a semi-I-open set of X. Then A is semi-open in X. Now, we can see that A is semi-I-sequential. Let V be a semi-I-open set in A, thus V is semi-open in X. Indeed, by Definition 2.5, if we show that V is semi-I-open in X, this will be sufficient. Now, suppose that there is a point $y \in Y - V$ and take an arbitrary $x \in V$ and a sequence $(x_n : n \in \mathbb{N}) \subseteq X$ with $x_n \to^{sI} x$ in X. Since A is semi-open in X and $x \in A$, the set $\{n \in \mathbb{N} : x_n \notin A\} \in I$. We define the sequence $y_n : n \in \mathbb{N})$ in X by $y_n = x_n$, $x_n \in A$, $y_n = y$, $x_n \notin A$. By Lemma 2.5, the sequence $(y_n : n \in \mathbb{N})$ is semi-I-convergent to x. Since $|\{n \in \mathbb{N} : x_n \notin V\}| = |\{n \in \mathbb{N} : y_n \notin V\}|$ and by Lemma 2.6, V is semi-I-open in X.

Now, let A be a semi-I-closed set of X. Then A is semi-closed in X. For any semi-I-closed set J of A we have to show that J is semi-closed in X, but since X is a semi-I-sequential space, it suffices for J to be semi-I-closed in X. Hence, let $(x_n : n \in \mathbb{N})$ be an arbitrary sequence in J with $x_n \to^{sI} x \in X$. Thus we obtain that $x \in J$. Indeed, since A is semi-closed, therefore $x \in A$ and so, $x \in J$, since J is a semi-I-closed set of A. □

Theorem 2.2. *Semi-I-sequential spaces are preserved by the topological sums.*

Proof. Let $\{X_\delta\}_{\delta\in\Delta}$ be a family of semi-I-sequential spaces. Take $X = \bigoplus_{\delta\in\Delta} X_\delta$, being the topological sum of $\{X_\delta\}_{\delta\in\Delta}$. We now show that the topological sum is a semi-I-sequential space. Let J be a semi-I-closed set in X. For each $\delta \in \Delta$, since X_δ is semi-closed in X, $J \cap X_\delta$ is semi-I-closed in X. We can see that $J \cap X_\delta \subseteq X_\delta$ and $J \cap X_\delta$ is semi-I-closed in X_δ. By the assumption, we have that $J \cap X_\delta$ is semi-closed in X_δ. By the definition of topological sums, we get that J is semi-closed in X. Therefore, the topological sum X is a semi-I-sequential space. □

Remark 2.1. The union of a family of semi-I-open sets is a topological space which is semi-I-open. Therefore, the intersection of finitely many sequentially semi-I-open sets is sequentially semi-I-open

Definition 2.6. Let I be an ideal on $\mathbb{N}$ and A be a subset of a topological space X. A sequence $(x_n : n \in \mathbb{N})$ in X is I-eventually in A [8] if there is $B \in I$ such that for all $n \in \mathbb{N} - B$, $x_n \in A$.

Proposition 2.1. *Let I be a maximal ideal on $\mathbb{N}$ and X be a topological space. Then A is a subset of X, where A is semi-I-open if and only if each semi-I-convergent sequence in X, converging to a point of A is I-eventually in A.*

Proof. Let A be a semi-I-open and $x_n \to^{sI} x \in A$. Since I is maximal, by Lemma 2.6, $B = \{n \in \mathbb{N} : x_n \notin A\} \in I$. Therefore, for each $n \in \mathbb{N} - B$, $x_n \in A$, i.e., the sequence $(x_n : n \in \mathbb{N})$ is I-eventually in A. □

Theorem 2.3. *Let I be a maximal ideal of $\mathbb{N}$ and X be a topological space. If V, W are two semi-I-open sets of X, then $V \cap W$ is semi-I-open.*

Proof. It will be shown that every semi-I-convergent sequence converging to a point in $V \cap W$ is I-eventually in it. Let $(x_n : n \in \mathbb{N})$ be a sequence in X such that $x_n \to^{sI} x \in V \cap W$. There are $A, S \in I$ such that for each $n \in \mathbb{N} - A$, $x_n \in V$ and for each $n \in \mathbb{N} - S$, $x_n \in W$. Since $A \cup S \in I$ and for each $n \in \mathbb{N} - (A \cup S)$, $x_n \in V \cap W$, we have $V \cap W$ is a semi-I-open set. □

3. Further Properties

3.1. **Semi-I-irresolute functions.** In this part, we introduce semi-I-irresolute functions and show some relations between continuous and semi-I-continuous functions.

Definition 3.1 ([1]). Let $f : (X, \tau) \to (Y, \sigma)$ be the functions. f is called sequentially continuous, provided V is sequentially open in Y, then $f^{-1}(V)$ is sequentially open in X.

Definition 3.2. Let I be an ideal on $\mathbb{N}$, $(X, \tau), (Y, \sigma)$ be topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function, then:

(1) f is said to be preserving semi-I-convergence, provided for each sequences $(x_n : n \in \mathbb{N})$ in X with $x_n \to^{sI} x$, the sequence $(f(x_n) : n \in \mathbb{N})$ is semi-I-convergent to $f(x)$.
(2) [4] f is said to be semi-I-irresolute if for each semi-I-open V in Y, then $f^{-1}(V)$ is semi-I-open in X.

Lemma 3.1 ([4]). *Every semi-I-irresolute function is semi-I-continuous.*

Theorem 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f is continuous, then f preserves semi-I-convergence.*

Proof. Suppose that f is continuous and let $(x_n : n \in \mathbb{N})$ be a sequence in X such that $x_n \to^{sI} x \in X$. Now, let V be an arbitrary semi-neighbourhood of $f(x)$ in Y. Since f is continuous, $f^{-1}(V)$ is a semi-neighbourhood of x. Therefore, we have $\{n \in \mathbb{N} : x_n \notin f^{-1}(V)\} \in I$. We can see that $\{n \in \mathbb{N} : f(x_n) \notin V\} = \{n \in \mathbb{N} : x_n \notin f^{-1}(V)\}$. This implies that $\{n \in \mathbb{N} : f(x_n) \notin V\} \in I$. Hence, $f(x_n) \to^{sI} f(x)$. □

Theorem 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f preserves the semi-I-convergence, then f is semi-I-irresolute.*

Proof. Suppose that f preserves semi-I-convergence and J is an arbitrary semi-I-closed set in Y. Let $(x_n : n \in \mathbb{N})$ be a sequence in $f^{-1}(J)$ such that $x_n \to^{sI} x \in X$. By the assumption, we have $f(x_n) \to^{sI} f(x)$. Since $(f(x_n) : n \in \mathbb{N}) \subseteq J$ and J is semi-I-closed in Y, hence $f(x) \in J$, i.e., $x \in f^{-1}(J)$. Therefore, $f^{-1}(J)$ is semi-I-closed in X and then f is semi-I-irresolute. □

Proposition 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If f preserves the semi-I-convergence, then f is semi-I-continuous.*

Proof. The proof is followed by Lemma 3.1 and Theorem 3.2. □

Theorem 3.3. *Let I be a maximal ideal on $\mathbb{N}$. Then a function $f : (X, \tau) \to (Y, \sigma)$ is semi-I-irresolute if and only if it preserves semi-I-convergent sequences.*

Proof. Assume that f is semi-I-irresolute and a sequence $x_n \to^{sI} x$ in X. We have to show that $f(x_n) \to^{sI} f(x)$ in Y. Now, let V be a semi-neighbourhood of $f(x)$. Then $x \in f^{-1}(V)$ is semi-I-open in X, because V is semi-I-open in Y. Hence, there is $B \in I$ such that for all $n \in \mathbb{N} - B$, $x_n \in f^{-1}(V)$. This means that for all $n \in \mathbb{N} - B$, $f(x_n) \in V$. Therefore, $\{n \in \mathbb{N} : f(x_n) \notin V\} \in I$ and hence, $f(x_n) \to^{sI} f(x)$. □

Theorem 3.4. *Let X be a semi-I-sequential space and $f(X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent.*

(1) *f is continuous.*
(2) *f preserves semi-I-convergence.*
(3) *f is semi-I-irresolute.*

Proof. (1) ⇔ (2) was proved in Theorems 3.1 and 3.2.

(3) ⇒ (1) : Let f be semi-I-irresolute and J be an arbitrary semi-closed set in Y. Then J is semi-I-closed in Y. Since f is semi-I-irresolute, $f^{-1}(J)$ is semi-U-closed in X. By the assumption, we find that $f^{-1}(J)$ is semi-closed in X. Therefore, f is continuous. □

Proposition 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and X be a semi-I-sequential space. Then the following statements are equivalent.*

(1) *f is continuous.*
(2) *f is semi-I-continuous.*

Proof. The proof is followed by Proposition 3.1 and Theorem 3.4. □

Lemma 3.2. *Let X be a semi-I-sequential space, then the function $f : (X, \tau) \to (Y, \sigma)$ is continuous if and only if it is sequentially continuous.*

Proof. Let X be a semi-I-sequential space, then every semi-I-closed set is closed, by [1] who proved that f is continuous if and only if f is sequentially continuous, indeed we have completed the proof. □

Corollary 3.1. *Let X be a semi-I-sequential space and for a function $f:(X,\tau)\to(Y,\sigma)$ the following statements are equivalent.*

(1) *f is continuous.*
(2) *f preserves semi-I-convergence.*
(3) *f is semi-I-continuous.*
(4) *f is sequentially continuous.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) was proved in Theorem 3.4, by Lemma 3.2, we have (1) $\Leftrightarrow$ (4). □

Lemma 3.3. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function and X be a semi-I-sequential space. Then the following statements are equivalent.*

(1) *f is sequentially continuous.*
(2) *f is semi-I-continuous.*

Proof. The proof is followed by Proposition 3.2 and Corollary 3.1. □

3.2. **Semi-I-irresolute and semi-I-covering functions.** Continuity and sequentially continuity are the ones of the most important tools for studying sequential spaces [7]. In this part, we define the concept of semi-I-covering functions and show some of their properties.

Definition 3.3 ([1])**.** Let $f:(X,\tau)\to(Y,\sigma)$ be a topological space. Then f is said to be sequentially continuous, provided $f^{-1}(V)$ is sequentially open in X, then V is sequentially open in Y.

Definition 3.4 ([1])**.** Let $f:(X,\tau)\to(Y,\sigma)$ be a topological space. Then f is said to be sequence-covering if, whenever $(y_n:n\in\mathbb{N})$ is a sequence in Y covering to y in Y, there exits a sequence $(x_n:n\in\mathbb{N})$ of points $x_n\in f^{-1}(y_n)$ for all $n\in\mathbb{N}$ and $x\in f^{-1}(y)$ such that $x_n\to x$.

Definition 3.5. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then f is said to be semi-I-covering if, whenever $(y_n:n\in\mathbb{N})$ is a sequence in Y, semi-I-converging to y in Y, there exits a sequence $(x_n:n\in\mathbb{N})$ of points $x_n\in f^{-1}(y_n)$ for all $n\in\mathbb{N}$ and $x\in f^{-1}(y)$ such that $x_n\to^{sI}x$.

Theorem 3.5. *Every semi-I-covering function is semi-I-irresolute.*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a function and f be a semi-I-covering function. Assume now that V is a non-semi-I-closed in Y. Then there exits a sequence $(y_n:n\in\mathbb{N})\subseteq V$ such that $y_n\to^{sI}y\notin V$. Since f is semi-I-covering, there exits a sequence $(x_n:n\in\mathbb{N})$ of points $x_n\in f^{-1}(y_n)$ for all $n\in\mathbb{N}$ and $x\in f^{-1}(y)$ such that $x_n\to^{sI}x$. We can see now that $(x_n:n\in\mathbb{N})\subseteq f^{-1}(V)$ and so, $x\notin f^{-1}(V)$, therefore $f^{-1}(V)$ is non-semi-I-closed. As a conclusion, f is semi-I-irresolute. □

Theorem 3.6. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following statements hold.*

(1) *If X is a semi-I-sequential space and f is continuous, then Y is a semi-I-sequential space and semi-I-irresolute.*
(2) *If Y is a semi-Y-sequential space and f is semi-I-irresolute, then f is continuous.*

Proof. (1) Let X be a semi-I-sequential space and f be continuous. Suppose that V is semi-I-open in Y. Since f is a continuous function and X is a semi-I-sequential space, take an arbitrary sequence $(x_n:n\in\mathbb{N})\subseteq X$ such that $x_n\to^{sI}x\in f^{-1}(V)$ in X. Since f is a continuous function, by Theorem 3.1, $f(x_n)\to^{sI}f(x)\in V$. Now, since V is semi-I-open in Y and by Lemma 2.6, we have $|\{n\in\mathbb{N}:f(x_n)\in V\}|=\theta$, i.e., $|\{n\in\mathbb{N}:x_n\in f^{-1}(V)\}|=\theta$, therefore, $f^{-1}(V)$ is semi-I-open in X.

Assume now that $V\subseteq Y$ such that $f^{-1}(V)$ is semi-I-open in X. Then $f^{-1}(V)$ is an open set of X, since X is semi-I-sequential space. As is well know, f is continuous, then V is open in Y. Hence, f is continuous.

(2) Let Y be a semi-I-sequential space and f be semi-I-irresolute. If $f^{-1}(V)$ is an open set of X, then $f^{-1}(V)$ is a semi-I-open set of X. Since f is semi-I-irresolute, V is a semi-I-open set of Y. Now, we know that Y is a semi-I-sequential space and so, V is an open set of Y. Therefore, f is continuous. □

By Theorems 3.4 and 3.6, we have the following result.

Corollary 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function, then f is continuous if and only if f is semi-I-irresolute and Y is a semi-I-sequential space.*

3.3. **Semi-I-Fréchet–Urysohn spaces.** A topological space X is said to be Fréchet–Urysohn [2] if for each $A \subseteq X$ and each $x \in Cl(A)$, there is a sequence in A converging to the point x in X. Now, in this part, we introduce the notion of semi-I-Fréchet–Urysohn and show a short result.

Definition 3.6. Let (X, τ) be a topological space. Then X is said to be semi-I-Fréchet–Urysohn or, simply, S-I-FU, if for each $A \subseteq X$ and each $x \in sCl(A)$, there exits a sequence in A, semi-I-converging to the point $x \in X$.

Lemma 3.4. *For two ideals I and J on $\mathbb{N}$, where $I \subseteq J$, if X is a S-I-FU-space, then it is a semi-J-FU-space.*

Proof. Let A be a subset of X and $x \in sCl(A)$. Since X is a S-I-FU-space, then there exits a sequence $(x_n : n \in \mathbb{N})$ in A such that $x_n \to^{sI} x$. As a consequence, $x_n \to^{sI} x$ in X, and so, X is a semi-J-FU-space. □

Theorem 3.7. *Let (X, τ) be a topological space. If X is a S-I-FU-space, then X is a semi-I-sequential space.*

Proof. Let $\{A_\delta : \delta \in \Delta\}$ be a family of semi-I-closed subsets of X, where $\delta \in \Delta \in X$, since X is a S-I-FU-space, by Definition 3.6, $A_\delta \subseteq X$ and each $x \in sCl(A_\delta)$. Now, since A_δ is semi-I-closed, $sCl(A_\delta) = A_\delta \in Cl(A)$, but by Definition 3.6, there exits a sequence semi-I-converging to the point $x \in sCl(A) \in Cl(A) \in X$, therefore $\{A_\delta : \delta \in \Delta\}$ is a closed set of X. As a consequence, X is a semi-I-sequential space. □

References

1. J. Boone, F. Siwiec, Sequentially quotient mappings. *Czechoslovak Math. J.* **26(101)** (1976), no. 2, 174–182.
2. S. Franklin, Spaces in which sequences suffice. *Fund. Math.* **57** (1965), 107–115.
3. A. Guevara, J. Sanabria, E. Rosas, S-$\mathcal{I}$-convergence of sequences. *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 1, 75–81.
4. E. Hatir, T. Noiri, On semi-I-open sets and semi-I-continuous functions. *Acta Math. Hungar.* **107** (2005), no. 4, 345–353.
5. K. Kuratowski, *Topologie I.* Monografie Matematyczne, Tom. 3, PWN-Polish Scientific Publishers, Warszawa, 1933.
6. N. Levine, Semi-open sets and semi-continuity in topological spaces. *Amer. Math. Monthly* **70** (1963), 36–41.
7. S. Lin, Z. Yun, *Generalized Metric Spaces and Mappings.* Atlantis Studies in Mathematics, 6. Atlantis Press, 2016.
8. X. Zhou, L. Liu, S. Lin, On topological spaces defined by $\mathcal{I}$-convergence. *Bull. Iranian Math. Soc.* **46** (2020), no. 3, 675–692.

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HAUSDORFF MEASURE OF NONCOMPACTNESS OF CERTAIN MATRIX OPERATORS ON ABSOLUTE NÖRLUND SPACES

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Abstract. The absolute Nörlund spaces $\left|N_p^u\right|_k, k \geq 1$, have more recently been introduced and studied by Hazar and Sarıgöl [On absolute Nörlund spaces and matrix operators, Acta Math. Sin. (Engl. Ser.), 34 (5) (2018), 812-826]. In the present paper, we characterize the classes of infinite matrix and compact operators transforming from $\left|N_p^u\right|_k$ into X and obtain some identities or estimates for the Hausdorff measures of noncompactness, where X is one of the spaces ℓ_∞, c and c_0.

1. Background, Notation and Preliminaries

A linear subspace of the space w, the space of all (real-- or) complex-valued sequences, is called a *sequence space*. We write ℓ_∞, c, c_0 and ϕ for the spaces of all bounded, convergent, null sequences and the set of all finite sequences, respectively. By $e^{(n)}$ and ℓ_k $(\ell_1 = \ell)$, we denote the sequence whose only non-zero term is 1 in n-th place for each $n \in \mathbb{N}$ and the space of all k-absolutely convergent series, respectively.

Let X, Y be two sequence spaces, $A = (a_{nv})$ be an infinite matrix of complex numbers and A_n be the sequence in the n-th row of A, that is, $A_n = (a_{n\nu})_{\nu=0}^{\infty}$ for each $n \in \mathbb{N}$. Then, we write $A(x) = (A_n(x))$, the A-transform of x, if

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v$$

converges for $n \geq 0$. If $A(x) = (A_n(x)) \in Y$ for all $x = (x_v) \in X$, then A is called a matrix transformation from X into Y, denoted by $A : X \to Y$, and we also denote the class of such maps by (X, Y).

For a sequence space X, the matrix domain X_A and the β-dual of X are introduced by

$$X_A = \{x \in w : A(x) \in X\}, \tag{1.1}$$
$$X^\beta = \{\varepsilon = (\varepsilon_v) \in w : \Sigma \varepsilon_v x_v \text{ converges for all } x \in X\},$$

respectively.

If $A = (a_{nv})$ is an infinite triangle matrix, *i.e.*, $a_{nn} \neq 0$, and $a_{nv} = 0$ for $v > n$, there exists its unique inverse [30]. Throughout the paper, k^* denotes the conjugate of $k > 1$, i.e., $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k = 1$.

A sequence space X is called a BK- space if it is a Banach space with continuous coordinates $P_n : X \to \mathbb{C}$ defined by $P_n(x) = x_n$ for $n \geq 0$, where $\mathbb{C}$ denotes the complex field. Also, a BK- space $X \supset \phi$ is said to have AK if every $x = (x_\nu) \in X$ has a unique representation $x = \sum_{v=0}^{\infty} x_v e^{(\nu)}$ [2]. For example, ℓ_∞, c and c_0 are BK-spaces according to the norm $\|x\|_{\ell_\infty} = \sup_{\nu \in \mathbb{N}} |x_v|$ and ℓ_k is a BK-space according to the norm $\|x\|_{\ell_k} = \left(\sum\limits_{v=0}^{\infty} |x_v|^k\right)^{1/k}$, $1 \leq k < \infty$. Moreover, the spaces c_0 and ℓ_k have the property AK under their natural norms [13].

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If $X \supset \phi$ is a BK- space and $a = (a_\nu) \in w$, then we write

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{v=0}^{\infty} a_v x_v \right|, \tag{1.2}$$

provided the statement on the right is defined and finite, which is satisfied whenever $a \in X^\beta$, where S_X denotes the unit sphere in X, i.e., $S_X = \{x \in X : \|x\| = 1\}$ [14].

If S and H are subsets of a metric space (X, d) and $\varepsilon > 0$, then S is called an ε-net of H, if, for every $h \in H$, there exists $s \in S$ such that $d(h, s) < \varepsilon$; if S is finite, then the ε-net S of H is called a finite *ε-net* of H. By $\mathcal{M}_X$ we denote the collection of all bounded subsets of X. If $Q \in \mathcal{M}_X$, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called the Hausdorff measure of noncompactness [21].

If X and Y are normed spaces, the set $\mathcal{B}(X, Y)$ states the set of all bounded linear operators $L : X \to Y$ and it is also a normed space to the norm $\|L\| = \sup_{x \in S_X} \|L(x)\|_Y$, where S_X is a unit sphere in X, and we write $\mathcal{B}(X) = \mathcal{B}(X, X)$. Further, let X and Y be Banach spaces. Then a linear operator $L : X \to Y$ is said to be compact if its domain is all of X and the sequence $(L(x_n))$ has a convergent subsequence in Y for every bounded sequence $x = (x_n) \in X$. We write $\mathcal{C}(X, Y)$ for the class of such operators. Studies on the Hausdorff measure noncompactness and compact operators can be found in [11, 13, 17–21].

The following results are important tool to compute the Hausdorff measure of noncompactness.

Lemma 1.1 ([13]). *Let X and Y be Banach spaces, $L \in \mathcal{B}(X, Y)$. Then the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, is defined by*

$$\|L\|_\chi = \chi(L(S_X)),$$

and L is compact, if and only if $\|L\|_\chi = 0$.

Lemma 1.2 ([21]). *Let Q be a bounded subset of the normed space X, where $X = \ell_k$ for $1 \leq k < \infty$. If $P_n : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, \ldots)$ for all $x \in X$, then*

$$\chi(Q) = \lim_{r \to \infty} \sup_{x \in Q} \|(I - P_r)(x)\|,$$

where I is the identity operator on X.

Also, we need the following known results for our investigations.

Lemma 1.3 ([13]). *Let $1 < k < \infty$ and $k^* = k/(k-1)$. Then we have $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$, $\ell_1^\beta = \ell_\infty$ and $\ell_k^\beta = \ell_{k^*}$. Furthermore, let X denote any of the spaces $\ell_\infty, c, c_0, \ell_1$ and ℓ_k. Then, we have $\|a\|_X^* = \|a\|_{X^\beta}$ for all $a \in X^\beta$, where $\|.\|_{X^\beta}$ is the natural norm on the dual space X^β.*

Lemma 1.4 ([13]). *Let X and Y be BK-spaces. Then we have $(X, Y) \subset \mathcal{B}(X, Y)$, i.e., every matrix $A \in (X, Y)$ defines a linear operator $L_A \in \mathcal{B}(X, Y)$ by $L_A(x) = A(x)$ for all $x \in X$.*

Lemma 1.5 ([7]). *s Let $X \supset \phi$ be a BK-space and Y be any of the spaces ℓ_∞, c, c_0. If $A \in (X, Y)$, then $\|L_A\| = \|A\|_{(X, \ell_\infty)} = \sup_n \|A_n\|_X^* < \infty$.*

Lemma 1.6 ([25]). *Let $1 < k < \infty$. Then $A \in (\ell_k, \ell)$ if and only if*

$$\|A\|'_{(\ell_k, \ell)} = \left\{ \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} \right\}^{1/k^*} < \infty,$$

and there exists $1 \leq \xi \leq 4$ such that $\|A\|'_{(\ell_k, \ell)} = \xi \|A\|_{(\ell_k, \ell)}$.

Lemma 1.7 ([12]). *Let $1 \leq k < \infty$. Then $A \in (\ell, \ell_k)$ if and only if*

$$\|A\|_{(\ell, \ell_k)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^k \right\}^{1/k} < \infty.$$

2. Absolute Nörlund Spaces

Let Σa_v be an infinite series with the n-th partial sum s_n and (u_n) be a sequence of nonnegative terms. The series Σa_v is said to be summable $|A, u_n|_k$, $k \geq 1$, if

$$\sum_{n=0}^{\infty} u_n^{k-1} |\Delta A_n(s)|^k < \infty, \ A_{-1}(s) = 0,$$

where $\Delta A_n(s) = A_n(s) - A_{n-1}(s)$, for $n \geq 0$, $A_{-1}(s) = 0$, [22]. If we take A as a matrix of weighted mean $\left(\overline{N}, p_n\right)$ $(resp., \ u_n = P_n/p_n)$, then the summability $|A, u_n|_k$ reduces to the summability $\left|\overline{N}, p_n, u_n\right|_k$ (resp., $\left|\overline{N}, p_n\right|_k$, [5]), [29]. Further, if $u_n = n$ for $n \geq 1$ and A is the matrix of Nörlund mean (N, p_n), then it is the same as the summability $|N, p_n|_k$, $k \geq 1$, given by Borwein and Cass [6], which also includes the summability $|C, \alpha|_k$ of Flett [9]. By a Nörlund matrix $A = (a_{nv})$, we mean

$$a_{nv} = \begin{cases} p_{n-v}/P_n, & 0 \leq v \leq n, \\ 0, & v > n, \end{cases}$$

where (p_n) is a sequence of complex numbers with $P_n = p_0 + p_1 + \cdots + p_n \neq 0$, $p_0 \neq 0$, $P_{-n} = 0$ for $n \geq 1$.

More recently, the space $\left|N_p^u\right|_k$ has been introduced as the set of all series, summable by the absolute Nörlund method $|N, p_n, u_n|_k$ for $k \geq 1$, *i.e.*,

$$\left|N_p^u\right|_k = \left\{a = (a_v) \in w : \sum_{n=1}^{\infty} u_n^{k-1} \left| \sum_{v=1}^{n} \left(\frac{P_{n-\nu}}{P_n} - \frac{P_{n-1-\nu}}{P_{n-1}} \right) a_\nu \right|^k < \infty \right\}, \ \left(\left|N_p^u\right|_1 = |N_p| \right).$$

Certain matrix operators on this space have been studied by Hazar and Sarıgöl [10] together with their norms, which is also generalized some known results in [6, 15, 24, 26]. Also, one can see some related works on sequences and series spaces in [1, 3, 4, 8, 11, 16, 23, 27].

Note that if the matrices $T^{(p)} = (t_{nv}^{(p)})$ and $E^{(k)} = (e_{nv}^{(k)})$, $1 \leq k < \infty$, are defined by

$$t_{nv}^{(p)} = \begin{cases} \dfrac{P_{n-v}}{P_n}, & 0 \leq v \leq n, \\ 0, & v > n, \end{cases} \tag{2.1}$$

$$e_{nv}^{(k)} = \begin{cases} -u_n^{1/k^*}, & v = n-1, \\ u_n^{1/k^*}, & v = n, \\ 0, & v \neq n, n-1, \end{cases} \tag{2.2}$$

respectively, then we may restate $\left|N_p^u\right|_k = (\ell_k)_{E^{(k)}oT^{(p)}}$ in view of the identity (1.1), where $1/k^* = 0$ for $k = 1$ [10]. Further, there exists the inverse matrix $S^{(p)}$ of $T^{(p)}$, since $T^{(p)}$ is triangle matrix. To obtain the matrix $S^{(p)}$, take p_0 as a non-zero. Then there exists a sequence (C_n) such that

$$\sum_{v=0}^{n} P_{n-v} C_v = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1, \end{cases} \tag{2.3}$$

which gives that

$$y_n = \frac{1}{P_n} \sum_{v=0}^{n} P_{n-v} x_v \quad \text{if and only if} \quad x_n = \sum_{v=0}^{n} C_{n-v} P_v y_v,$$

where $P_n = p_0 + p_1 + \cdots + p_n \neq 0$ for $n \geq 1$, and so, $S^{(p)} = (s_{nv}^{(p)})$ is defined by

$$s_{nv}^{(p)} = \begin{cases} C_{n-v} P_v, & 0 \leq v \leq n, \\ 0, & v > n. \end{cases} \tag{2.4}$$

Throughout the paper, for any sequence $x = (x_v) \in \left|N_p^u\right|_k$, we associate the sequence $z = (z_v)$ by $z = E^{(k)} o T^{(p)}(x)$. If we say that $T^{(p)}(x) = y$, then

$$z_n = u_n^{1/k^*} \Delta y_n = u_n^{1/k^*} \sum_{v=1}^{n} \left(\frac{P_{n-\nu}}{P_n} - \frac{P_{n-1-\nu}}{P_{n-1}} \right) x_\nu \tag{2.5}$$

for $n \geq 1$, $y_{-1} = 0$. So, it is trivial that $x \in \left|N_p^u\right|_k$, if and only if $z = E^{(k)} oT^{(p)}(x) \in \ell_k$, and $x \in S_{\left|N_p^u\right|_k}$ if and only if $z \in S_{\ell_k}$. In other words, $E^{(k)} oT^{(p)} : \left|N_p^u\right|_k \to \ell_k$ is a bijective linear map preserving norm [10].

Further, we recall that $\left|N_p^u\right|_k$ is a BK-space (see [10]) with respect to the norm

$$\|x\|_{\left|N_p^u\right|_k} = \left\| E^{(k)} oT^{(p)}(x) \right\|_{\ell_k}. \tag{2.6}$$

We require the following notations and lemmas.

$$G_{nv} = \sum_{r=v}^{n} P_r C_{n-r} \ ; v, n \geq 0,$$

$$D_1 = \left\{ \varepsilon = (\varepsilon_v) \in w : \lim_m \sum_{v=r}^{m} \varepsilon_v G_{vr} \text{ exists} \right\},$$

$$D_2 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sup_{m,r} \left| \sum_{v=r}^{m} \varepsilon_v G_{vr} \right| < \infty \right\},$$

$$D_3 = \left\{ \varepsilon = (\varepsilon_v) \in w : \sup_m \sum_{r=0}^{m} \left| u_r^{-1/k^*} \sum_{v=r}^{m} \varepsilon_v G_{vr} \right|^{k^*} < \infty \right\}.$$

Lemma 2.1.
a) $A \in (\ell, c) \Leftrightarrow$ (i) $\lim_n a_{nv}$ *exists,* $v \geq 0$, (ii) $\sup_{n,v} |a_{nv}| < \infty$.
b) $A \in (\ell, \ell_\infty) \Leftrightarrow$ (ii) *holds.*
c) *If* $1 < k < \infty$, *then* $A \in (\ell_k, c) \Leftrightarrow$ (i) *holds,* (iii) $\sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty$.
d) *If* $1 < k < \infty$, *then* $A \in (\ell_k, \ell_\infty) \Leftrightarrow$ (iii) *holds.*
e) *If* $1 < k < \infty$, *then* $A \in (\ell_k, c_0) \Leftrightarrow$ (iii) *holds, (iv)* $\lim_n a_{nv} = 0$, $v \geq 0$.
f) $A \in (\ell, c_0) \Leftrightarrow$ (ii) *and* (iv) *holds* [28].

Lemma 2.2. *Let* $1 \leq k < \infty$. *If* $a = (a_\nu) \in \left|N_p^u\right|_k^\beta$, *then* $\tilde{a} = (\tilde{a}_\nu) \in \ell_{k^*}$ *for* $k > 1$, *and* $\tilde{a} \in \ell_\infty$ *for* $k = 1$. *Moreover,*

$$\sum_{v=1}^{\infty} a_v x_v = \sum_{v=1}^{\infty} \tilde{a}_\nu z_v \tag{2.7}$$

holds for every $x = (x_k) \in \left|N_p^u\right|_k$, *where* $z = E^{(k)} oT^{(p)}(x)$ *is the associated sequence defined by* (2.5) *and*

$$\tilde{a}_\nu = u_v^{-1/k^*} \sum_{r=v}^{\infty} a_r G_{rv}. \tag{2.8}$$

Also, the following result is immediate by Lemma 2.2.

Lemma 2.3. *Let* (u_n) *be a sequence of nonnegative numbers. Then* $\left|N_p^u\right|_k^\beta = D_1 \cap D_3$ *for* $1 < k < \infty$ *and* $|N_p|_1^\beta = D_1 \cap D_2$ *for* $k = 1$ [10].

Lemma 2.4. *Let* $\tilde{a} = (\tilde{a}_\nu)$ *be defined as in* (2.8). *Then* $\|a\|^*_{\left|N_p^u\right|_k} = \|\tilde{a}\|_{\ell_{k^*}}$ *for* $1 < k < \infty$ *and* $\|a\|^*_{|N_p|} = \|\tilde{a}\|_{\ell_\infty}$ *for* $k = 1$, *where* $a \in \left|N_p^u\right|_k^\beta$.

Proof. Let $1<k<\infty$ and $a\in \left|N_p^u\right|_k^\beta$. Then by Lemma 2.2, we get $\tilde{a}=(\tilde{a}_\nu)\in \ell_{k^*}$, and equality (2.7) holds, and also, by (2.6), $x\in S_{\left|N_p^u\right|_k}$, if and only if $z\in S_{\ell_k}$. So, it follows from (1.2) and (2.7) that

$$\|a\|^*_{\left|N_p^u\right|_k}=\sup_{x\in S_{\left|N_p^u\right|_k}}\left|\sum_{v=1}^\infty a_v x_v\right|=\sup_{z\in S_{\ell_k}}\left|\sum_{v=1}^\infty \tilde{a}_\nu z_v\right|=\|\tilde{a}\|^*_{\ell_k}$$

and, since $\tilde{a}=(\tilde{a}_\nu)\in \ell_{k^*}$, by Lemma 1.3,

$$\|a\|^*_{\left|N_p^u\right|_k}=\|\tilde{a}\|^*_{\ell_k}=\|\tilde{a}\|_{\ell_{k^*}}.$$

This concludes the proof. □

The proof for $k=1$ is similar to the above, so it is omitted.

Lemma 2.5. *Let V be a sequence space and (u_n) be a sequence of nonnegative numbers. If $A\in\left(\left|N_p^u\right|_k,V\right)$, then $F^{(k)}\in(\ell_k,V)$, where the matrix $F^{(k)}=\left(f_{nv}^{(k)}\right)$ is defined by*

$$f_{nv}^{(k)}=u_v^{-1/k^*}\sum_{r=v}^\infty a_{nr}G_{rv}. \tag{2.9}$$

Proof. The proof is seen at once by Lemma 2.2. □

Now, we give some lemmas on the operator norms.

Lemma 2.6. *Let (u_n) be a sequence of nonnegative numbers and define the matrix $F^{(k)}=\left(f_{nv}^{(k)}\right)$ by (2.9). If A is in any of the classes $\left(\left|N_p^u\right|_k,c_0\right)$, $\left(\left|N_p^u\right|_k,c\right)$ and $\left(\left|N_p^u\right|_k,\ell_\infty\right)$, then for $1<k<\infty$,*

$$\|L_A\|=\|A\|_{\left(\left|N_p^u\right|_k,\ell_\infty\right)}=\sup_n\left\|F_n^{(k)}\right\|_{\ell_{k^*}},$$

and for $k=1$,

$$\|L_A\|=\|A\|_{(|N_p|,\ell_\infty)}=\sup_n\left\|F_n^{(1)}\right\|_{\ell_\infty}.$$

Proof. It follows immediately by combining Lemmas 1.4, 1.5 and 2.4. □

Lemma 2.7. *Let (u_n) be a sequence of nonnegative numbers and the matrix $F^{(k)}=\left(f_{nv}^{(k)}\right)$ be given by (2.9).*

a) *If $A\in(|N_p|,\ell_k)$, then for $k\geq 1$,*

$$\|L_A\|=\|A\|_{(|N_p|,\ell_k)}=\left\|F^{(1)}\right\|_{(\ell,\ell_k)}.$$

b) *If $A\in\left(\left|N_p^u\right|_k,\ell\right)$, then for $1<k<\infty$, there exists $1\leq\xi\leq 4$ such that*

$$\|L_A\|=\|A\|_{\left(\left|N_p^u\right|_k,\ell\right)}=\left\|F^{(k)}\right\|_{(\ell_k,\ell)}=\frac{1}{\xi}\left\|F^{(k)}\right\|'_{(\ell_k,\ell)}.$$

Proof. It follows by combining Lemmas 1.4, 1.6, 1.7 and 2.5. □

3. Compact Operators on Absolute Nörlund Spaces

In this section, we characterize the classes $\left(\left|N_p^u\right|_k,X\right)$ and $\mathcal{C}\left(\left|N_p^u\right|_k,X\right)$, and also obtain some identities or estimates for the Hausdorff measures of noncompactness in these classes, where X is one of the spaces ℓ_∞, c and c_0.

Theorem 3.1. *Let (u_n) be a sequence of nonnegative numbers and let $F^{(1)} = \left(f_{nv}^{(1)}\right)$ be given by*

$$f_{nv}^{(1)} = \lim_m \sum_{j=\nu}^{m} a_{nj} G_{j\nu}, \; n, v \geq 0.$$

a) $A \in (|N_p|, \ell_\infty)$, *if and only if*

$$\lim_m \sum_{j=\nu}^{m} a_{nj} G_{j\nu} \text{ exists for each } n, v \geq 0, \tag{3.1}$$

$$\sup_{m,v} \left| \sum_{r=v}^{m} a_{jr} G_{rv} \right| < \infty \text{ for each } j, \tag{3.2}$$

$$\sup_{n,j} \left| f_{nj}^{(1)} \right| < \infty. \tag{3.3}$$

b) $A \in (|N_p|, c)$, *if and only if* (3.1), (3.2), (3.3) *hold and*

$$\lim_n f_{nj}^{(1)} \text{ exists for each } j.$$

c) $A \in (|N_p|, c_0)$, *if and only if* (3.1), (3.2), (3.3) *hold and*

$$\lim_n f_{nj}^{(1)} = 0 \text{ for each } j.$$

Proof. **a)** $A \in (|N_p|, \ell_\infty)$, if and only if $(a_{nv})_{v=0}^{\infty} \in |N_p|^\beta$ for each n, and $A(x) \in \ell_\infty$ for every $x \in |N_p|$. Also, by Lemma 2.3, it is seen that $(a_{nj})_{j=0}^{\infty} \in D_1 \cap D_2$, i.e., (3.1) and (3.2) hold for each n. To prove the necessity and sufficiency of (3.3), let $x \in |N_p|$. Consider the composite operator $E^{(1)} o T^{(p)} : |N_p| \to \ell$ defined by (2.1) and (2.2). Then it is easy to see that $E^{(1)} o T^{(p)}$ is a bijective linear operator, since $T^{(p)}$ and $E^{(1)}$ are bijective linear operators (see, [10]). Now, we write $z \in \ell$, where $T^{(p)}(x) = y$ and $z = \left(E^{(1)} o T^{(p)}\right)(x)$, i.e., $z_n = \Delta y_n$ for $n \geq 0$, $y_{-1} = 0$, and also $y_n = \sum_{j=0}^{n} z_j$. Then, it follows from (2.3) and (2.4) that

$$\sum_{v=0}^{m} a_{nv} x_v = \sum_{j=0}^{m} \left(P_j \sum_{v=j}^{m} a_{nv} C_{v-j} \right) y_j = \sum_{j=0}^{m} \tilde{f}_{mj}^{(1)} z_j,$$

where

$$\tilde{f}_{mj}^{(1)} = \begin{cases} \sum_{v=j}^{m} a_{nv} G_{vj}, & 0 \leq j \leq m, \\ 0, & j > m. \end{cases}$$

Moreover, if any matrix $R = (r_{nv}) \in (\ell, c)$, then the series $R_n(x) = \Sigma_v r_{nv} x_v$ converges uniformly in n, since, by Lemma 2.1, the remaining term tends to zero uniformly in n, that is,

$$\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \sup_{n,v} |r_{nv}| \sum_{v=m}^{\infty} |x_\nu| \to 0 \text{ as } m \to \infty,$$

and so we get

$$\lim_n R_n(x) = \sum_{v=0}^{\infty} \lim_n r_{nv} x_v. \tag{3.4}$$

Hence, it is easily seen from (3.1) and (3.2) that $\tilde{F}^{(1)} = \left(\tilde{f}_{mj}^{(1)}\right) \in (\ell, c)$, and so, by (3.4), we have

$$A_n(x) = \sum_{j=0}^{\infty} \left(\lim_m \tilde{f}_{mj}^{(1)} \right) z_j = \sum_{j=0}^{\infty} f_{nj}^{(1)} z_j = F_n^{(1)}(z),$$

where $f_{nj}^{(1)} = \lim_m \tilde{f}_{mj}^{(1)}$. This results in $A(x) \in \ell_\infty$ for every $x \in |N_p|$, if and only if $F^{(1)}(z) \in \ell_\infty$ for every $z \in \ell$, which implies that $A \in (|N_p|, \ell_\infty)$ if and only if (3.1) and (3.2) hold, and $F^{(1)} \in (\ell, \ell_\infty)$.

Also, it follows from Lemma 2.1 that $F^{(1)} \in (\ell, \ell_\infty)$, if and only if (3.3) is satisfied. This concludes the proof of the part of a).

The parts b) and c) can be proved similarly, so we omit the detail. □

Theorem 3.2. *Let $k > 1$, (u_n) be a sequence of nonnegative numbers. Define the matrix $F^{(k)} = \left(f_{nv}^{(k)}\right)$ by*

$$f_{nv}^{(k)} = u_v^{-1/k^*} \sum_{j=\nu}^{\infty} a_{nj} G_{j\nu}, \; n, v \geq 0.$$

Then

a) $A \in \left(\left|N_p^u\right|_k, \ell_\infty\right)$ *if and only if* (3.1) *holds, and*

$$\sup_m \sum_{v=0}^{m} \left| u_v^{-1/k^*} \sum_{r=v}^{m} a_{nr} G_{rv} \right|^{k^*} < \infty, \tag{3.5}$$

$$\sup_n \sum_{\nu=0}^{\infty} \left| f_{nv}^{(k)} \right|^{k^*} < \infty. \tag{3.6}$$

b) $A \in \left(\left|N_p^u\right|_k, c\right)$, *if and only if* (3.1), (3.5), (3.6) *hold, and*

$$\lim_n f_{nv}^{(k)} \quad \textit{exists for each } \nu.$$

c) $A \in \left(\left|N_p^u\right|_k, c_0\right)$ *if and only if* (3.1), (3.5), (3.6) *hold, and*

$$\lim_n f_{nv}^{(k)} = 0, \quad \textit{for each } \nu.$$

Proof. **a)** Let $A \in \left(\left|N_p^u\right|_k, \ell_\infty\right)$. Then, equivalently, $(a_{nj})_{j=0}^{\infty} \in \left(\left|N_p^u\right|_k\right)^\beta$ and $A(x) \in \ell_\infty$ for every $x \in \left|N_p^u\right|_k$. Also, by Lemma 2.3, it is seen that $(a_{nj})_{j=0}^{\infty} \in \left(\left|N_p^u\right|_k\right)^\beta$, if and only if $(a_{nj})_{j=0}^{\infty} \in D_1 \cap D_3$ for each n, which is the same as (3.1) and (3.5). To prove the necessity and sufficiency of (3.6), by considering (2.1) and (2.2), we define the operator $E^{(k)} o T^{(p)} : \left|N_p^u\right|_k \to \ell_k$ by

$$\left(E^{(k)} o T^{(p)}\right)_n (x) = u_n^{1/k^*} \Delta T_n^{(p)}(x).$$

It is easy to see that a composite function $E^{(k)} o T^{(p)}$ is a bijective linear operator, since $T^{(p)}$ and $E^{(k)}$ are bijective linear operators (see, [10]). Given $x \in \left|N_p^u\right|_k$. If we say that $T^{(p)}(x) = y$ and $z = \left(E^{(k)} o T^{(p)}\right)(x)$, *i.e.*,$z_n = u_n^{1/k^*} \Delta y_n$ for $n \geq 0$, $y_{-1} = 0$, then we have $z \in \ell_k$, and since the space $\left|N_p^u\right|_k$ is isomorphic to ℓ_k, it follows that $x \in \left|N_p^u\right|_k$, if and only if $z \in \ell_k$. Further, $y_n = \sum_{j=0}^{n} u_j^{-1/k^*} z_j$. So, considering (2.3), as in the proof of Theorem 3.1, we obtain

$$\sum_{j=0}^{m} a_{nj} x_j = \sum_{j=0}^{m} \tilde{f}_{mj}^{(k)} z_j,$$

where

$$\tilde{f}_{mj}^{(k)} = \begin{cases} u_j^{-1/k^*} \sum\limits_{r=j}^{m} a_{nr} G_{rj}, & 0 \leq j \leq m, \\ 0, & j > m. \end{cases}$$

Furthermore, if any matrix $R = (r_{nv}) \in (\ell_k, c)$, then the series $R_n(x) = \Sigma_v r_{nv} x_v$ converges uniformly in n, by Lemma 2.1. In fact, applying Hölder's inequality to the remaining term, we get

$$\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \left(\sum_{v=m}^{\infty} |r_{nv}|^{k^*} \right)^{1/k^*} \left(\sum_{v=m}^{\infty} |x_\nu|^k \right)^{1/k}$$

and the right-hand side of this inequality tends to zero as $m \to \infty$, since $x \in \ell_k$. This means that the remaining term tends to zero uniformly in n, and so, $R_n(x) = \Sigma_v r_{nv} x_v$ converges uniformly in n, which implies

$$\lim_n R_n(x) = \sum_{v=0}^{\infty} \lim_n r_{nv} x_v. \tag{3.7}$$

Thus, it is easily seen from (3.1) and (3.5) that $\tilde{F}^{(k)} = \left(\tilde{f}_{mj}^{(k)}\right) \in (\ell_k, c)$, and so, by (3.7),

$$A_n(x) = \sum_{v=0}^{\infty} \left(\lim_m \tilde{f}_{mv}^{(k)}\right) z_v = \sum_{v=0}^{\infty} f_{nv}^{(k)} z_v = F_n^{(k)}(z),$$

where $\lim_m \tilde{f}_{mv}^{(k)} = f_{nv}^{(k)}$. This gives that $A(x) \in \ell_\infty$ for every $x \in \left|N_p^u\right|_k$, if and only if $F^{(k)}(z) \in \ell_\infty$ for every $z \in \ell_k$, which implies that $F^{(k)} \in (\ell_k, \ell_\infty)$, and so, it follows by applying Lemma 2.1 to the matrix $F^{(k)}$ for $k > 1$ that $F^{(k)} \in (\ell_k, \ell_\infty)$, if and only if (3.6) holds. This concludes the proof of the part of a).

Since b) and c) can be proved similarly, so we omit the details.

The following lemma is required to characterize a subclass of compact operators $\mathcal{K}\left(\left|N_p^u\right|_k, X\right)$, where X is one of the spaces ℓ_∞, c_0 and c. □

Lemma 3.3 ([19]). *Let $X \supset \phi$ be a BK-space. Then we have:*

a) *If $A \in (X, \ell_\infty)$, then*

$$0 \le \|L_A\|_\chi \le \lim_{n\to\infty} \sup \|A_n\|_X^* .$$

b) *If $A \in (X, c_0)$, then*

$$\|L_A\|_\chi = \lim_{n\to\infty} \sup \|A_n\|_X^* .$$

c) *If X has AK or $X = \ell_\infty$ and $A \in (X, c)$, then*

$$\frac{1}{2} \lim_{n\to\infty} \sup \|A_n - \alpha\|_X^* \le \|L_A\|_\chi \le \lim_{n\to\infty} \sup \|A_n - \alpha\|_X^* ,$$

where $\alpha = (\alpha_v)$ is given by $\alpha_v = \lim_{n\to\infty} a_{n\nu}$, for all $\nu \in \mathbb{N}$.

By using Lemma 3.3, we establish the following result.

Theorem 3.4. *Let $k \ge 1$ and (u_n) be a sequence of nonnegative numbers. Also, define the matrix $F^{(k)} = \left(f_{nv}^{(k)}\right)$ by (2.9).*
Then we have:

a) *If $A \in \left(\left|N_p^u\right|_k, \ell_\infty\right)$, then*

$$0 \le \|L_A\|_\chi \le \lim_{n\to\infty} \sup \left\|F_n^{(k)}\right\|_{\ell_k}^* \tag{3.8}$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty} \left\|F_n^{(k)}\right\|_{\ell_k}^* = 0. \tag{3.9}$$

b) *If $A \in \left(\left|N_p^u\right|_k, c_0\right)$, then*

$$\|L_A\|_\chi = \lim_{n\to\infty} \sup \left\|F_n^{(k)}\right\|_{\ell_k}^* , \tag{3.10}$$

$$L_A \text{ is compact, if and only if } \lim_{n\to\infty} \left\|F_n^{(k)}\right\|_{\ell_k}^* = 0. \tag{3.11}$$

c) *If $A \in \left(\left|N_p^u\right|_k, c\right)$, then*

$$\frac{1}{2} \lim_{n\to\infty} \sup \left\|F_n^{(k)} - \tilde{\alpha}\right\|_{\ell_k}^* \le \|L_A\|_\chi \le \lim_{n\to\infty} \sup \left\|F_n^{(k)} - \tilde{\alpha}\right\|_{\ell_k}^* , \tag{3.12}$$

$$L_A \text{ is compact, if and only if } \lim_{n\to\infty} \left\|F_n^{(k)} - \tilde{\alpha}\right\|_{\ell_k}^* = 0, \tag{3.13}$$

where $\tilde{\alpha} = (\tilde{\alpha}_v)$ *is given by* $\tilde{\alpha}_v = \lim_{n\to\infty} f_{nv}^{(k)}$, *for all* $\nu \in \mathbb{N}$.

Proof. First, by Lemma 1.1, we point out that (3.9), (3.11) and (3.13) are obtained from (3.8), (3.10) and (3.12), respectively. Also, since $\left|N_p^u\right|_k, k \geq 1$ is a BK-space, using parts a) and b) of Lemma 3.3 with Lemma 2.4, we get (3.8) and (3.10), respectively.

Finally, we see that (3.12) holds. In fact, if $A \in \left(\left|N_p^u\right|_k, c\right)$, we write $F^{(k)} \in (\ell_k, c)$ by using Lemma 2.5, where $A(x) = F^{(k)}(z)$ for all $x \in \left|N_p^u\right|_k$ and $z \in \ell_k$. So, since ℓ_k has AK, from part c) of Lemma 3.3, we get

$$\frac{1}{2} \lim_{n\to\infty} \sup \left\| F_n^{(k)} - \tilde{\alpha} \right\|_{\ell_k}^* \leq \|L_{F^{(k)}}\|_\chi \leq \lim_{n\to\infty} \sup \left\| F_n^{(k)} - \tilde{\alpha} \right\|_{\ell_k}^*, \tag{3.14}$$

where $\tilde{\alpha} = (\tilde{\alpha}_v)$ is given by $\tilde{\alpha}_v = \lim_{n\to\infty} f_{nv}^{(k)}$ for all $\nu \in \mathbb{N}$.

On the other hand, $x \in S_{\left|N_p^u\right|_k}$, if and only if $z \in S_{\ell_k}$ by (2.6). So, it follows from Lemmas 1.1, 1.4 and 2.5 that

$$\|L_A\|_\chi = \chi\left(AS_{\left|N_p^u\right|_k}\right) = \chi\left(F^{(k)} S_{\ell_k}\right) = \|L_{F^{(k)}}\|_\chi. \tag{3.15}$$

Hence (3.12) is obtained by (3.14) and (3.15), which completes the proof. □

Theorem 3.5. *Let* $F^{(k)} = \left(f_{nv}^{(k)}\right)$ *be defined as in* (2.9) *and* (u_n) *be a sequence of nonnegative numbers. Then we have:*

a) *If* $A \in (|N_p|, \ell_k)$, *then for* $1 \leq k < \infty$,

$$\|L_A\|_\chi = \lim_{r\to\infty} \sup_v \left(\sum_{n=r+1}^{\infty} \left| f_{nv}^{(1)} \right|^k \right)^{1/k} \tag{3.16}$$

and

$$L_A \text{ is compact, if and only if } \lim_{r\to\infty} \sup_v \sum_{n=r+1}^{\infty} \left| f_{nv}^{(1)} \right|^k = 0. \tag{3.17}$$

b) *If* $A \in \left(\left|N_p^u\right|_k, \ell\right)$, *then for* $k > 1$, *there exists* $1 \leq \xi \leq 4$ *such that*

$$\|L_A\|_\chi = \frac{1}{\xi} \lim_{r\to\infty} \left(\sum_{v=1}^{\infty} \left(\sum_{n=r+1}^{\infty} \left| f_{nv}^{(k)} \right| \right)^{k^*} \right)^{1/k^*}$$

and

$$L_A \text{ is compact iff } \lim_{r\to\infty} \sum_{v=1}^{\infty} \left(\sum_{n=r+1}^{\infty} \left| f_{nv}^{(k)} \right| \right)^{k^*} = 0.$$

Proof. **a)** Let $S_X = \{x \in X : \|x\| = 1\}$. Now, from (2.6), we can write that $x \in S_{|N_p|}$, if and only if $z \in S_\ell$ for all $x \in |N_p|$ and $z \in \ell$. For brevity, we write $S_{|N_p|} = S$ and $S_\ell = \bar{S}$. By Lemmas 1.1, 1.2, 1.4 and 1.7, we have

$$\begin{aligned}\|L_A\|_\chi &= \chi(AS) = \chi\left(F^{(1)} \bar{S}\right) \\ &= \lim_{r\to\infty} \sup_{z\in\bar{S}} \left\| (I - P_r) F^{(1)}(z) \right\|_{\ell_k} = \lim_{r\to\infty} \sup_v \left(\sum_{n=r+1}^{\infty} \left| f_{nv}^{(1)} \right|^k \right)^{1/k}\end{aligned}$$

where $P_r : \ell_k \to \ell_k$ is defined by $P_r(z) = (z_0, z_1, \ldots, z_r, 0, \ldots)$, which completes our assertions.

Also, (3.17) is derived from (3.16) by using Lemma 1.1.

Since part b) is proved easily as in part a) using Lemma 1.6 instead of 1.7, so, we omit the details. □

References

1. B. Altay, F. Başar, Generalization of the sequence space $l(p)$ derived by weighted mean. *J. Math. Anal. Appl.* **330** (2007), no. 1, 174–185.
2. B. Altay, F. Başar, E. Malkowsky, Matrix transformations on some sequence spaces related to strong Cesàro summability and boundedness. *Appl. Math. Comput.* **211** (2009), no. 2, 255–264.
3. F. Başar, E. Malkowsky, B. Altay, Matrix transformations on the matrix domains of triangles in the spaces of strongly C_1-summable and bounded sequences. Publ. Math. Debrecen **73** (2008), no. 1-2, 193–213.
4. M. Başarır, E. E. Kara, On the m^{th} order difference sequence space of generalized weighted mean and compact operators. *Acta Math. Sci. Ser. B (Engl. Ed.)* **33** (2013), no. 3, 797–813.
5. H. Bor, B. Thorpe, On some absolute summability methods. *Analysis* **7** (1987), no. 2, 145–152.
6. D. Borwein, F. P. Cass, Strong Nörlund summability. *Math. Z.* **103** (1968), 94–111.
7. I. Djolović, On compact operators on some spaces related to matrix $B(r, s)$. *Filomat* **24** (2010), no. 2, 41–51.
8. M. Et, M. Işık, On $p\alpha$-dual spaces of generalized difference sequence spaces. *Appl. Math. Lett.* **25** (2012), no. 10, 1486–1489.
9. T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley. *Proc. London Math. Soc. (3)* **7** (1957), 113–141.
10. G. C. Hazar, M. A. Sarıgöl, On absolute Nörlund spaces and matrix operators. *Acta Math. Sin. (Engl. Ser.)* **34** (2018), no. 5, 812–826.
11. G. C. Hazar, M. A. Sarıgöl, Compact and matrix operators on the space $|C, -1|_k$. *J. Comput. Anal. Appl.* **25** (2018), no. 6, 1014–1024.
12. I. J. Maddox, *Elements of Functional Analysis.* Cambridge University Press, London-New York, 1970.
13. E. Malkowsky, V. Rakočević, An introduction into the theory of sequence spaces and measures of noncompactness. *Zb. Rad. (Beogr.)* **9(17)** (2000), 143–234.
14. E. Malkowsky, V. Rakočević, S. Živković, Matrix transformations between the sequence space bv^p and certain BK spaces. *Bull. Cl. Sci. Math. Nat. Sci. Math.* **27** (2002), 33–46.
15. F. Mears, Absolute regularity and the Nrlund mean. *Ann. of Math. (2)* **38** (1937), no. 3, 594–601.
16. R. N. Mohapatra, G. Das, Summability factors for lower-semi-matrix transformations. *Monatsh. Math.* **79** (1975), no. 4, 307–315.
17. M. Mursaleen, V. Karakaya, H. Polat, N. Simşek, Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means. *Comput. Math. Appl.* **62** (2011), no. 2, 814–820.
18. M. Mursaleen, A. K. Noman, Applications of the Hausdorff measure of noncompactness in some sequence spaces of weighted means. *Comput. Math. Appl.* **60** (2010), no. 5, 1245–1258.
19. M. Mursaleen, A. K. Noman, Compactness by the Hausdorff measure of noncompactness. *Nonlinear Anal.* **73** (2010), no. 8, 2541–2557.
20. M. Mursaleen, A. K. Noman, The Hausdorff measure of noncompactness of matrix operators on some BK spaces. *Oper. Matrices* **5** (2011), no. 3, 473–486.
21. V. Rakočević, Measures of noncompactness and some applications. *Filomat* **12** (1998), part 2, 87–120.
22. M. A. Sarıgöl, On the local properties of factored Fourier series. *Appl. Math. Comput.* **216** (2010), no. 11, 3386–3390.
23. M. A. Sarıgöl, Matrix transformations on fields of absolute weighted mean summability. *Studia Sci. Math. Hungar.* **48** (2011), no. 3, 331–341.
24. M. A. Sarıgöl, Matrix operators on sequence A_k. *Math. Comput. Modelling* **55** (2012), no. 5-6, 1763–1769.
25. M. A. Sarıgöl, Extension of Mazhar's theorem on summability factors. *Kuwait J. Sci.* **42** (2015), no. 3, 28–35.
26. M. A. Sarıgöl, Spaces of series summable by absolute Cesàro and matrix operators. *Comm. Math. Appl.* **7** (2016), no. 1, 11–22.
27. M. A. Sarıgöl, H. Bor, Characterization of absolute summability factors. *J. Math. Anal. Appl.* **195** (1995), no. 2, 537–545.
28. M. Stieglitz, H. Tietz, Matrixtransformationen von Folgenräumen. Eine Ergebnisübersicht. (German) *Math. Z.* **154** (1977), no. 1, 1–16.
29. W. T. Sulaiman, On some summability factors of infinite series. *Proc. Amer. Math. Soc.* **115** (1992), no. 2, 313–317.
30. A. Wilansky, *Summability through Functional Analysis.* North-Holland Mathematics Studies, 85. Notas de Matemática [Mathematical Notes], 91. North-Holland Publishing Co., Amsterdam, 1984.

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SOLITONS ON A SHALLOW FLUID OF VARIABLE DEPTH

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Abstract. The results of numerical study of evolution of the solitons of gravity and gravity-capillary waves on the surface of a shallow fluid, when the characteristic wavelength is essentially greater than the depth, $\lambda \gg H$, are presented for the cases when dispersive parameter is a function of time, and the spatial coordinates $\beta = \beta(t, x, y)$. This corresponds to the problems when the relief of the bottom is changed in time and space. We use both the one-dimensional approach (the equations of the KdV-class) and also two-dimensional description (the equations of the KP-class), in case of need.

1. Basic Equations and General Properties of Solutions

Let us consider the models of the Korteweg – de Vries (KdV) and Kadomtsev – Petviashvili (KP) equations in their application to hydrodynamics, namely, to describe the gravity waves on the surface of an ideal incompressible fluid of small (compared to wavelength) depth. In this case, the generalized density and "sound" velocity in the general set of the hydrodynamic equations [3] acquire the sense of fluid depth H, and velocity $c = \sqrt{gH}$, the term $gH^2/2$ plays the role of the pressure, this corresponds to the effective adiabatic index $\gamma = 2$ [5]. Then the Boussinesq equations take the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{v} + \nabla gH + \frac{gh^2}{3}\nabla\Delta H = 0\,, \tag{1}$$

$$\frac{\partial H}{\partial t} + \nabla\,(H\mathbf{v}) = 0\,, \tag{2}$$

($h = \mathrm{const}$ is the depth of the fluid). It is easy to add into these equations the terms associated with the capillary effects. Assuming that the curvature of the surface is not too large and the additional pressure to the fluid caused by the surface tension is defined by the Laplace formula

$$\delta p = \sigma\left(R_1^{-1} + R_2^{-1}\right)\,,$$

where σ is the surface tension coefficient, R_1 and R_2 are the main curvature radii, we can write $\delta p = -\sigma\Delta\eta$, where $\eta\,(x, y, t)$ is the surface function (the value of is sufficiently small). Replacing $\rho g h$ in (1) by $\rho g H + \delta\rho$ (ρ is the fluid density), we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{v} + g\nabla H + \left(\frac{gh^2}{3} - \frac{\sigma}{\rho}\right)\nabla\Delta H = 0\,. \tag{3}$$

Equations (2), (3) are the Boussinesq equations having regard to the capillary effects [5]. The factor change at the dispersive term in the dispersion relation in its standard form [5] leads to the change of the dispersion equation and, instead of $\omega = c_0 k\left(1 - \frac{1}{6}H^2k^2 + \cdots\right)$, we have

$$\omega = c_0 k\left[1 - \frac{1}{6}\left(H^2 - \frac{3\sigma}{\rho g}\right)k^2 + \cdots\right]\,, \tag{4}$$

where $c_0 = \sqrt{gH}$. In this case, the dispersive factor is defined by

$$\beta = \frac{c_0}{6}\left(H^2 - \frac{3\sigma}{\rho g}\right). \tag{5}$$

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Using furthermore the results of [3], we transform (1) and (2) to the form

$$\partial_t u + \alpha u \partial_x u - \beta \partial_x^3 u = -(c_0/2) \int\limits_{-\infty}^{x} \partial_y^2 u dx \,,$$

that is, we obtain the KP equation for the gravity-capillary waves on the shallow fluid. Note that for sufficiently large $\sigma > \frac{1}{3}\rho H^2$, the dispersive parameter changes its sign that involves the qualitative change of the evolution character and the form of the solutions [1]. Consider now in more detail the following interesting case. Often there are the cases when the factor β is unusually small. As it follows from (5), $\beta = 0$ for $H = (3\sigma/\rho g)^{1/2} \approx 0.48$cm (for pure water). However, $\beta = 0$ does not mean that there is no dispersion in medium. It simply means that in this case the next terms in the Taylor expansion in k of the full dispersion relation must be taking into account. In addition, the corresponding additional terms appear in the equation. This generalization leads to the KP equation which can be written as

$$\partial_t u + \alpha u \partial_x u - \beta \partial_x^3 u + \gamma \partial_x^5 u = -(c_0/2) \int\limits_{-\infty}^{x} \partial_y^2 u dx, \tag{6}$$

where the coefficients are

$$\alpha = \frac{3}{2}\frac{c_0}{H}, \gamma = \frac{c_0}{6}\left[H^2\left(\frac{2}{5}H^2 - \frac{\sigma}{\rho g}\right) - \frac{1}{12}\left(\frac{3\sigma}{\rho g} - H^2\right)^2\right].$$

Using the methods based on the implicit and explicit difference schemes [1, 3], numerical integration of (6) enables us to investigate the structure of the one-dimensional (1D) and two-dimensional (2D) solitons on a shallow fluid in the case of anomalously weak dispersion. We have found that the qualitative form of the solutions depends significantly on the value of parameter $\varepsilon = (\beta/V)(-V/\gamma)^{1/2} \ll 2$, where V is the soliton's velocity in the reference frame moving along the x-axis with the phase velocity c_0. In the 1D case, for $\varepsilon = 0$, the structure of propagating solitons does not differ qualitatively from that of solitons of the usual KdV equation (see [5]), and in the 2D case − from the structure of the algebraic KP-solitons [1, 3]. Such solitons on the surface of a fluid have negative polarity (the hollow solitons). When $\varepsilon > 0$, for example, in the case of the increasing fluid depth, starting from the depth $H = (3\sigma/\rho g)^{1/2}$, the structure of solitons radically changes: by remaining to decay from their maximum to zero in the transverse direction as before, now their sign varies along the direction of their propagation (in addition, the amplitude of the 2D solitons falls from the maximum to zero in the transverse direction, as before). As $\varepsilon \to 2$, the number of oscillations in the tails increases and now the solitons become similar to the 1D and 2D high-frequency trains, respectively, i.e., envelope solitons[1]. Note that a similar structure is typical also for solitons of internal gravity waves, considered in detail in [2–4]. Separately for the cases 1D and 2D, let us consider now some our results of numerical simulation of the soliton dynamics on the surface of a shallow fluid which is describes by the standard KdV and KP equations (equation (6) with $\gamma = 0$) when the factor β is a function of the space coordinates and time.

2. Structure and Evolution of 1D Solitons of Gravity and Gravity-Capillary Waves with a Varying Relief of the Bottom

First, let us consider the evolution of the 1D solitons in the framework of model (6) with $\gamma = 0$ and right-hand side being equal to zero (the KdV equation):

$$\partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u = 0 \tag{7}$$

on the surface of a fluid with varying in time and space dispersive parameter $\beta = \beta(t, x)$. Such situation can take place, for example, in the problems on propagation of the gravity and gravity-capillary waves on the surface of a shallow fluid [1], when $\beta = c_0 H^2/6$ and $\beta = (c_0/6)\left[H^2 - 3\sigma/\rho g\right]$, respectively (see above). In these cases, if $H = H(t, x)$, the dispersive parameter becomes also the

[1]For the structure of the 1D solitons of the generalized KdV equation, see also [1–3].

function of the x coordinate and time. In [2, 3], it has been shown that the solutions of the KDV equation for $\beta = \text{const}$, depending on the value of β, are divided into two classes: for $|\beta| < u_0(0, x)\, l/12$ (where l is the characteristic wavelength of the initial disturbance), they have soliton character, in the opposite case, the solutions are the wave packets with asymptotes being proportional to the derivative of the Airy function (see also [5]). In these cases, the KdV equation can be integrated analytically by the inverse scattering transform (IST) method. But even in the 1D case, if $\beta = \beta(t, x)$, this approach is impossible principally, it is necessary to resort to a numerical simulation in the conforming problems.

Let us formulate the problem of numerical simulation of the KdV equation with $\beta = \beta(t, x)$ and consider some results of our numerical experiments in studying the structure and evolution of the solitary waves on the surface of a shallow fluid. To solve the initial problem for the KdV equation (7) with a variable dispersion, we have used an implicit difference scheme [3] with $O\left(\tau^2, h^4\right)$ approximation. Initial conditions were chosen in the form of the solitary disturbance

$$u(0, x) = u_0 \exp\left(-x^2/l^2\right), \tag{8}$$

and in the form of a "smoothed step"

$$u(0, x) = \frac{c}{1 + \exp(x/l)}, \tag{9}$$

with different values of parameters u_0, l and c, defined by the convenience of numerical calculation for specific sizes of the numerical integration area. The zero conditions on the boundaries of the computation region were imposed, and simulation has been conducted for a few types of model types of function β (see Figures 1 and 2) when for $t < t_{\text{cr}}$, $\beta = \beta_0 = \text{const}$, and for $t \geq t_{\text{cr}}$,

$$1)\ \ \beta(x) = \begin{cases} \beta_0, & x \leq a; \\ \beta_0 + c, & x > a; \end{cases} \tag{10}$$

$$2)\ \ \beta(x, t) = \begin{cases} \beta_0, & x \leq a; \\ \beta_0 + nc, \quad n = (t - t_{\text{cr}})/\tau = 1, 2, \ldots; & x > a; \end{cases} \tag{11}$$

$$\begin{gathered} 3)\ \ \beta(t) = \beta_0\left(1 + k_0 \bar{\beta} \sin \omega t\right), \quad \bar{\beta} = (\beta_{\max} - \beta_{\min})/2, \\ 0 < k_0 < 1, \quad \pi/2\tau < \omega < 2\pi/\tau; \end{gathered} \tag{12}$$

where a and c are the constants. In terms of the problem of the wave propagation on the surface of a shallow water that accordingly means that on reaching t_{cr} we have: 1) sudden "breaking up of the bottom", 2) gradual "changing of height" of the bottom area, and 3) "bottom oscillation" with time.

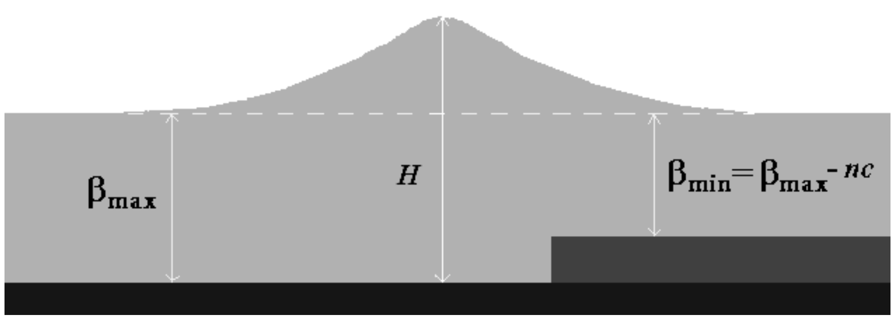

FIGURE 1. Dependence $\beta = \beta(t, x)$ of type of "step", models (10) and (11).

Consider briefly some results of numerical simulation for two types of initial conditions and different kinds of model function $\beta = \beta(t, x)$. In the first series of numerical experiments we investigate the evolution of the initial disturbance in the form of the solitary soliton-like pulse (8) for the models

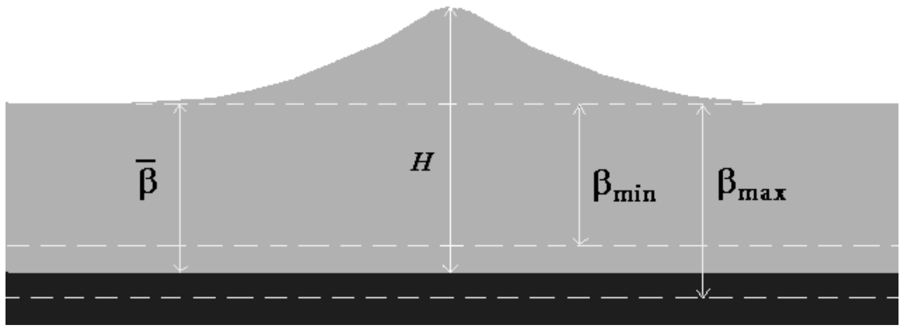

FIGURE 2. Dependence $\beta = \beta\,(t, x)$ of type of "bottom oscillations", models (12).

with spasmodic change of dispersion [models of "step" type bottom (10) and (11) with values of the parameter a corresponding (for $t = 0$) to the position of the "break" behind and ahead of soliton, and values $c < 0$ ("negative" step) and $c > 0$ ("positive" step). The obtained results show that in all cases the deformation of initial pulse occurs with time. If the step is located behind the soliton, in both cases $c < 0$ and $c > 0$, the waving tail which is not associated with the main maximum of the outgoing forward main pulse is formed, and its evolution is entirely determined by the value β in its location. In case $t = 0$, the "step" is located ahead front of the initial pulse, for $c > 0$, in the model (11) a steep front is formed quite rapidly, that leads to the overturning of the wave with time. For $c > 0$, we can observe the destruction of the soliton (Figure 3), which occurs due to the fact that in the region of localization of its front, the relative role of nonlinear effects falls due to the increase of the dispersive parameter here, and dispersive effects prevail.

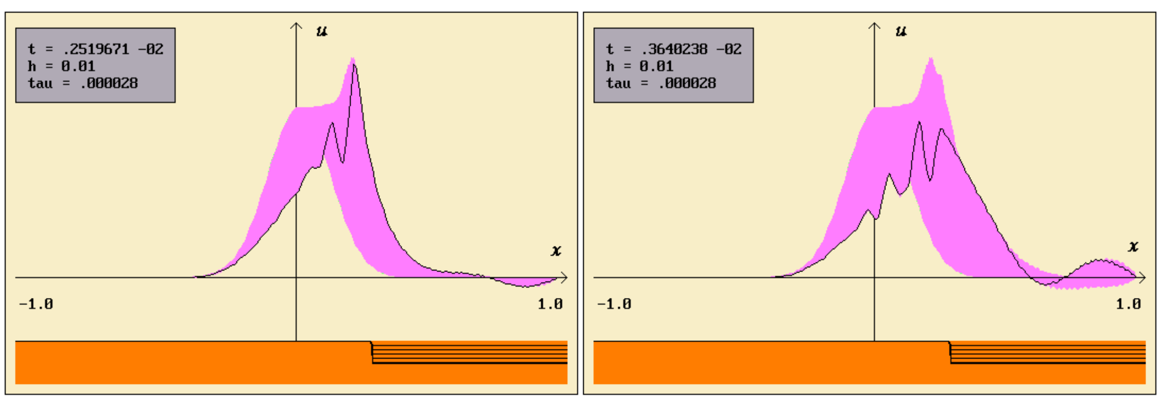

FIGURE 3. Evolution of the KdV soliton in model (11) with $c < 0$.

The second series of numerical experiments is devoted to the study of evolution of the initial disturbance of type (9) for the models of "bottom" (10), (11) for different values of parameters a and c.

Figure 4 shows the result of numerical simulation of evolution of the initial disturbance (9) for the model of "bottom" in the form of a positive step in the case if "break" is located directly under the region of the disturbance front of the fluid surface. It is seen that due to the fact that the development of perturbations occurs mainly in the region where the value of the dispersion parameter corresponds to the multi-soliton solution of the KdV equation [3, 5], the solitary disturbance propagates with the development of high-frequency oscillatory structure behind the shock front, and in the region of the soliton "tail", where dispersive effects dominate over the nonlinear ones, the high-frequency train of oscillations decays rather quickly to zero and it is limited in the region $x < 0$. Figure 5 shows the example of the results of simulation of the evolution of initial disturbance in the form of the "smoothed

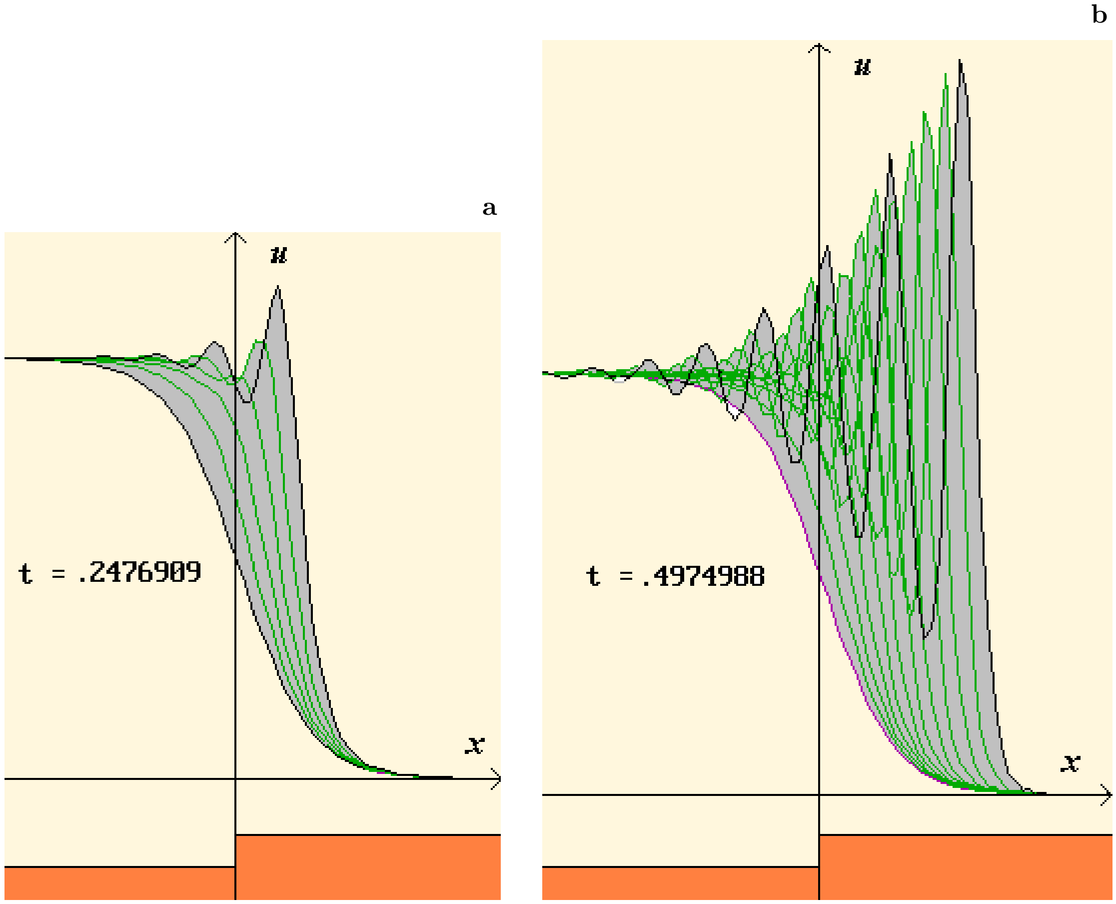

FIGURE 4. Evolution of "step" (9) in model (10) with $c > 0$: a – $t \approx 0.25$; b – $t \approx 0.5$.

step" (9) in case when the break of the "bottom" is negative and located in front of the localization region of the fluid surface disturbance. It can be seen that in this case, the front of the disturbance becomes more gentle with time, the oscillatory soliton structure in the front region is not formed, but the development of low-frequency oscillations behind the main maximum occurs. This result is easily explained within the framework of the similarity principle for the KdV equation [5]: the evolution of the "tail" of the initial disturbance occurs in the region of small values of the dispersive parameter, whereas in the front region, where the dispersion is relatively large, the formation of a shock wave does not occur.

As for the third law of change of β (harmonic oscillations of the parameter β with time on the whole x-axis), a series of numerical experiments for various $k_0 = \text{const}$ and a variable frequency ω [see law of change (12)] show that the stationary (locally) standing waves can be formed for some values of ω, in other cases, the formation of stationary periodic wave structures is possible, and in intermediate cases, a chaotic regime is usually realized.

3. Structure and Evolution of 2D Solitons of Gravity and Gravity-Capillary Waves with a Varying Relief of the Bottom

Let us now consider the problem of evolution of the 2D solitons in the framework of the standard KP equation

$$\partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u = \kappa \int_{-\infty}^{x} \partial_y u \, dx \tag{13}$$

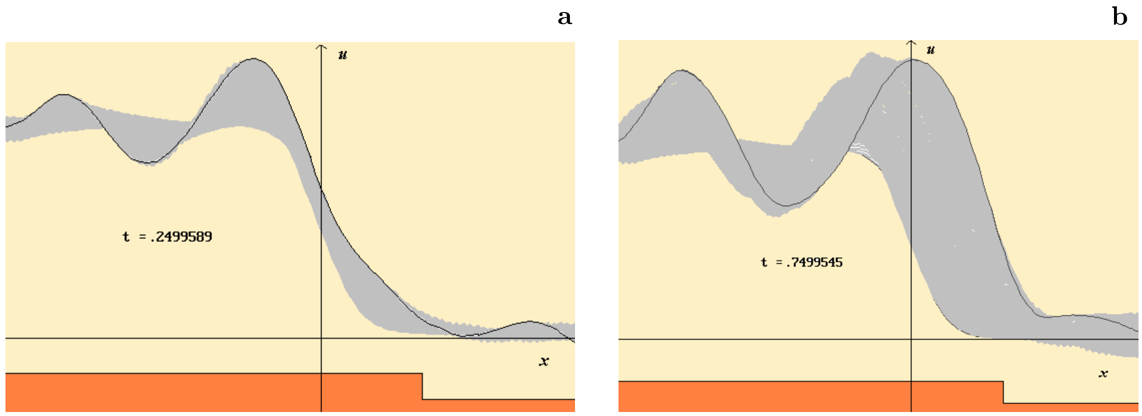

FIGURE 5. Evolution of "step" (9) in model (10) with $c < 0$: a – $t \approx 0.25$; b – $t \approx 0.75$.

with a varying in time and space dispersive parameter $\beta = \beta(t, x, y)$. This situation can take place in the problems dealing with the propagation of gravity and gravity-capillary waves on the surface of a shallow fluid [3] when the fluid depth is the function of the spatial coordinates and time $H = H(t, x, y)$.

Here, we have the same situation as for the 1D model of the KdV equation described above: if analytical solutions of the KP equation are known, in case $\beta = \beta(t, \mathbf{r})$, the dispersion term of equation becomes quasi-linear and the model is not exactly integrable (the IST method is not applicable) [3]. The problem of numerical simulation of the KP equation with $\beta = \beta(t, x, y)$ is formulated analogously to the problem for the KdV equation (see previous section). To solve the initial problem for the KP equation (13) with a variable dispersion (varying relief of the bottom), we use an implicit difference scheme [1] with $O\left(\tau^2, h^4\right)$ approximation. The initial conditions are chosen in the form of the exact 2D one-soliton solution of the KP equation [3], the complete absorption conditions on the boundaries of computation region [1, 3] are imposed, and simulation is conducted for the same types of model function as for the KdV equation [see formulae (10)–(12)]. Consider the basic results of the numerical experiments on the investigation of the structure and the evolution of 2D solitary waves on the fluid surface with a variable dispersion.

The first series of numerical experiments have been aimed at the study of soliton dynamics under spasmodic character of the dispersion change (the function $\beta = \beta(t, x, y)$ has the form of the "step"). First, we investigated evolution of the initial pulse when the spasmodic change of β for t_{cr} takes place behind the soliton ["negative" step when $c < 0$ in formulae (10), (11)]. In addition, we have studied the dependence of the spatial structure of a solution on the value of parameter a in models (10) and (11). The obtained results (see Example in Figure 6) show that in all the cases the evolution leads to the formation of waving tail which is not connected with the soliton going away and caused only by a local influence of sudden change of the "relief" $\beta = \beta(t, x)$. Consequently, the formation of oscillatory structure is connected not so much with decreasing of a role of the dispersion effects behind the soliton as with the spasmodic changing of β in space.

In the next series of numerical simulation, we considered the evolution of a 2D soliton when the sudden change of the dispersion parameter is located directly under or in front of an initial pulse ("negative" step). An example of the results of this series is given in Figure 7. Analysing the obtained results of the whole series, we can see that for such character of the relief of the function β the disturbance caused by sudden change of the dispersive parameter has also a local character, i.e., it doesn't propagate together with the going away soliton. But, unlike the cases considered in the first series of simulation, the asymptotes of a leaving soliton become oscillating (in any case, in the time limits of numerical experiment), besides, against a background of the long-wave oscillations of the waving tail we can also see the appearance of the wave fluctuations. The effects noted may be interpreted as a result of those that for the areas of the wave surface with different values of local wave number k_x the value of the dispersive effects is different. As a result, the intensity of the phase mixing of the Fourier-harmonics within the (x, y)-region varies with the coordinates and, therefore,

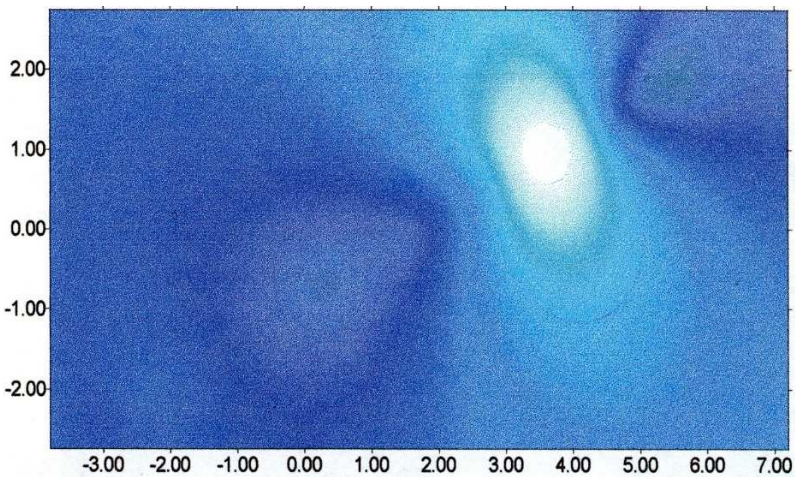

FIGURE 6. Solution of eq. (13) for the dispersion law (10) with $a = 5.0$, $c = -0.0038$ for $t = 0.6$.

it reacts differently to the nonlinear generation of the harmonics with various (in particular, large) wave-numbers k_x.

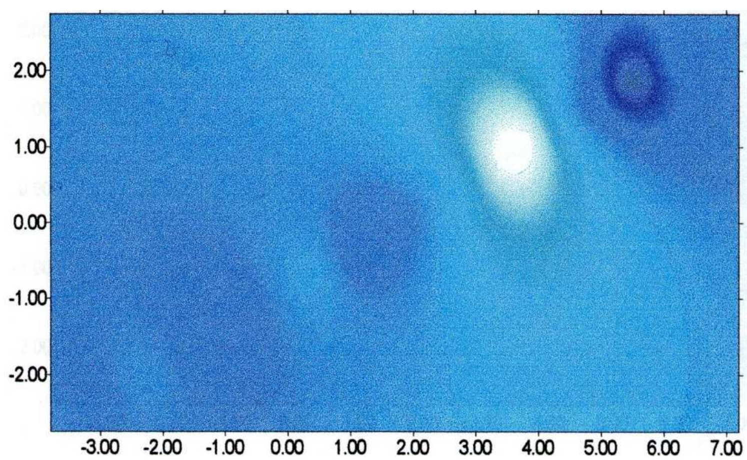

FIGURE 7. Solution of eq. (13) for the dispersion law (11) with $a = 4.0$, $c = -0.0038$ for $t = 0.6$.

In the third series of the experiments with dispersive parameter changing with the laws (10) and (11) we consider the cases of "positive" step [$c > 0$ in formulae (10) and (11)] being both in front of and behind the initial pulse for the wide diapason of values of parameter a. The examples of the most interesting results are shown in Figure 8. One can see that when "positive" step is far in front of maximum of the function $u\,(0, x, y)$, the soliton evolution at the initial stage does not practically differ qualitatively from that for $\beta = \text{const}$ (Figure 8a), but in the future, the evolution character is defined by the presence of the step, namely, the processes, caused by the same causes noted for the results of the second series of numerical simulation, begin to be developed (Figure 8b). As we can see in the figure, the appreciable change of the soliton structure which can lead to the wave falling is observed owing to an intensive generation of the harmonics with big k_x in the soliton front region, even for rather small height of the step (i.e., even if the value of parameter a in formulae (10), (11) is still rather small). Thus, as it follows from the results of this series, the disturbance of the propagating 2D soliton caused by sudden change in time and space of the dispersive parameter with $c > 0$ is also of local character.

As to the second law of the β change (model (12) – harmonic oscillation of the parameter β with time on the whole (x, y)-plane), the series of numerical simulations for different $k_0 =$ const and variable frequency ω [see law (12)] show that for some values of ω, the stationary (locally) standing waves can be formed, in other cases, the formation of the stationary periodical wave structures is possible, and in the intermediate cases, a chaotic regime is usually realized.

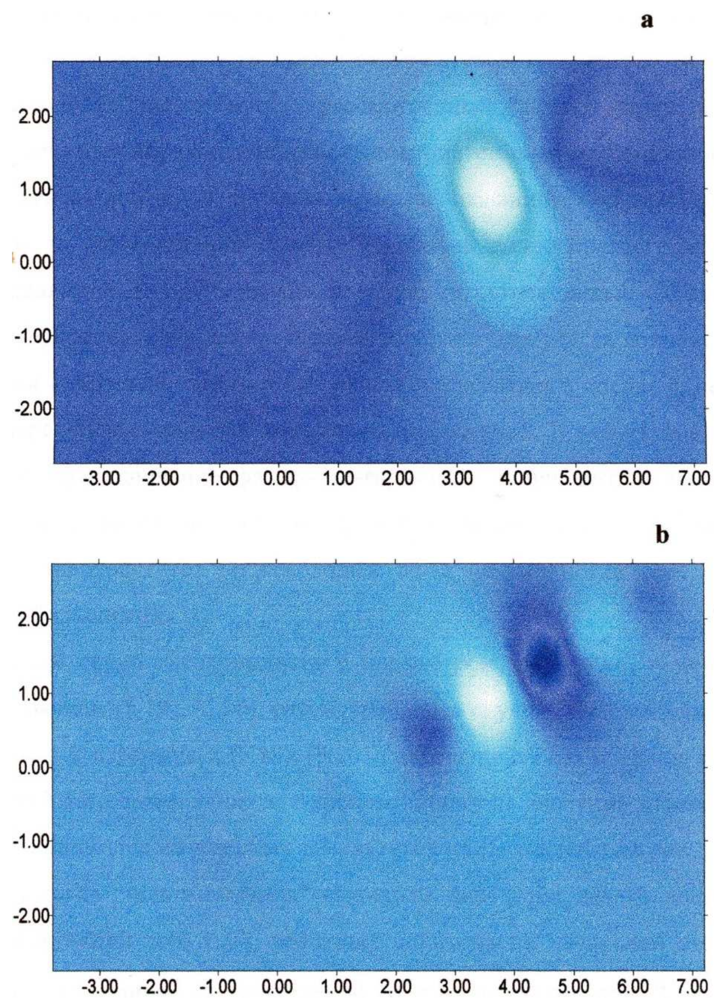

FIGURE 8. Evolution of soliton of eq. (13) for the dispersion law (11) with $a = 5.0$, $c = 0.0038$: (a) $t = 0.6$, (b) $t = 0.8$.

In the experiments, carried out for different values of the parameter k_0 and $\omega =$ const, we have found that the stable (in any case, in the limits of the numerical computation time) solutions can be derived only for $k_0 \leq \beta_0$ in formula (12), and the solutions are unstable in the other cases. An example of evolution of the 2D soliton, when its structure along the $x-$ – and $y-$axes acquires the wave character and the amplitude of its maximum decreases with time, is given in Figure 9.

Summing up the above, one can note that the numerical simulation of evolution of the 2D solitons describing by the model of the KP equation with $\beta = \beta\,(t, x, y)$ enable us to find different types of stable and unstable solutions including those of the mixed "soliton – non-soliton" type for various character of dispersion changes in time and space.

The obtained results open the new perspectives in the investigation of a number of applied problems of dynamics of the non-one-dimensional nonlinear waves in the specific physical media, including upper atmosphere (ionosphere), magneto-sphere and in a plasma [1–3, 5].

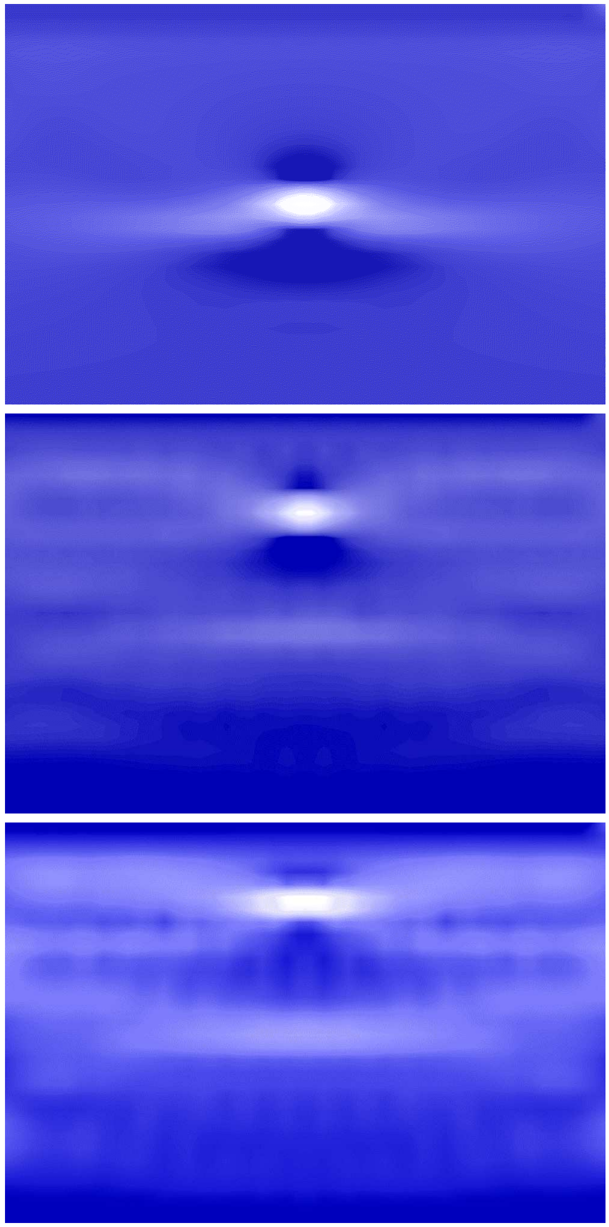

FIGURE 9. Evolution of 2D soliton of eq. (13) for $t = 0.4,\ 1.2,\ 2.0$.

4. Conclusion

In the paper, the results of numerical study of evolution of the solitons of gravity and gravity-capillary waves on the surface of a shallow fluid when the characteristic wavelength is essentially greater then depth, $\lambda \gg H$, were presented for the cases, when dispersive parameter is a function of time and spatial coordinates, $\beta = \beta\,(t, x, y)$. This corresponds to the problems when the relief of the bottom is changed in time and space. We have considered three cases of variable dispersion when the sudden "breaking up of the bottom", the gradual "changing of height" of the bottom area, and the "bottom oscillation" with time take place. To solve the problem, we have used both the 1D approach (the equations of the KdV-class) and also the 2D description (the equations of the KP-class). For all cases, numerical solutions of the problem in 1D and 2D geometry were presented. It was noted that the realized approach can be useful also in other applications of the nonlinear wave theory such as dynamics of 1D and multidimensional solitary waves in other specific physical media, including upper atmosphere (ionosphere), magnetosphere and in a plasma.

Acknowledgement

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References

1. V. Yu. Belashov, The KP equation and its generalizations. (Russian) In: *Theory and Applications*, NEISRI FEB RAS, Magadan, 1997.
2. V. Yu. Belashov, E. S. Belashova, *Solitons. Theory, Simulation, Applications.* (Russian) Kazan, RIZ "School", 2016.
3. V. Yu. Belashov, S. V. Vladimirov, *Solitary Waves in Dispersive Complex Media.* Theory, simulation, applications. Springer Series in Solid-State Sciences, 149. Springer-Verlag, Berlin, 2005.
4. V. Yu. Belashov, E. S. Belashova, O. A. Kharshiladze, Nonlinear wave structures of the soliton and vortex types in complex continuous media: theory, simulation, applications. *Lect. Notes TICMI* **18** (2017), 86 pp.
5. V. I. Karpman, *Non-Linear Waves in Dispersive Media.* vol. 71. Pergamon Press, Oxford-New York-Toronto, Ont., 1975.

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SCHUR-GEOMETRIC AND SCHUR-HARMONIC CONVEXITY OF WEIGHTED INTEGRAL MEAN

SANJA KOVAČ

Abstract. Recently, there have been many new results on Schur convexity of integral means. In this paper we investigate the necessary and sufficient conditions for the existence of Schur-geometric and Schur-harmonic properties in weighted integral means, weighted midpoint and weighted trapezoid quadrature formulas.

1. Introduction

Let us recall the definitions of convex, $n-$convex and Schur-convex functions.

Definition 1. A function f is *convex* on an interval I if for any two points $x, y \in I$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1.1}$$

If inequality (1.1) is reversed, then f is said to be *concave*.

Let $A \subset \mathbb{R}^n$. We introduce the following notion: for $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in A$, we write $\mathbf{x} \prec \mathbf{y}$, if

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \quad \text{and} \quad \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{for} \quad k = 1, \ldots, n-1,$$

where $x_{[i]}$ denotes the $i-$th-largest component in $\mathbf{x}$.

Definition 2. Function $F : A \subset \mathbb{R}^n \to \mathbb{R}$ is said to be **Schur-convex** on A if for every $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in A$ such that $\mathbf{x} \prec \mathbf{y}$, we have

$$F(x_1, \ldots, x_n) \leq F(y_1, \ldots, y_n).$$

Function F is said to be **Schur-concave** on A if $-F$ is Schur-convex.

Remark 1. Every convex and symmetric function is Schur-convex.

Numerous researchers have recently investigated Schur-geometric and Schur-harmonic convexities [2, 8, 9].

First, let us define for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\ln \mathbf{x} := (\ln x_1, \ldots, \ln x_n)$ and $\frac{1}{\mathbf{x}} := \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$.

Let us give the following definitions:

Definition 3. Function $F : A \subset \mathbb{R}^n_+ \to \mathbb{R}$ is said to be **Schur-geometrically convex** on A if for every $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in A$ such that $\ln \mathbf{x} \prec \ln \mathbf{y}$, we have

$$F(x_1, \ldots, x_n) \leq F(y_1, \ldots, y_n).$$

Function F is said to be **Schur-geometrically concave** on A if $-F$ is Schur-convex.

Definition 4. Function $F : A \subset \mathbb{R}^n \to \mathbb{R}$ is said to be **Schur-harmonically convex** on A if for every $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in A$ such that $\frac{1}{\mathbf{x}} \prec \frac{1}{\mathbf{y}}$, we have

$$F(x_1, \ldots, x_n) \leq F(y_1, \ldots, y_n).$$

Function F is said to be **Schur-harmonically concave** on A if $-F$ is Schur-convex.

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Schur-convexity has been investigated by numerous researchers. The following result was proved in [4] for the arithmetic integral mean.

Theorem 1. *Let f be a continuous function on an interval I with a non-empty interior. Then*

$$F(x,y)=\begin{cases} \frac{1}{y-x}\int_x^y f(t)dt & x,y\in I, x\neq y\\ f(x) & x=y\in I\end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

The next result for Schur-convexity of the weighted arithmetic integral mean was proved several years ago [7].

Theorem 2. *Let f be a continuous function on $I\subset\mathbb{R}$ and let w be a positive continuous weight on I. Then the function*

$$F_w(x,y)=\begin{cases} \frac{1}{\int_x^y w(t)dt}\int_x^y w(t)f(t)dt & x,y\in I, x\neq y\\ f(x) & x=y\in I\end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if the inequality

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\leq\frac{w(x)f(x)+w(y)f(y)}{w(x)+w(y)}$$

holds (reverses) for all $x,y\in I$.

The Schur-convex property of the functions

$$M(x,y)=\begin{cases} \frac{1}{y-x}\int_x^y f(t)dt-f\left(\frac{x+y}{2}\right) & x,y\in I, x\neq y\\ 0 & x=y\in I\end{cases}$$

$$T(x,y)=\begin{cases} \frac{f(x)+f(y)}{2}-\frac{1}{y-x}\int_x^y f(t)dt & x,y\in I, x\neq y\\ 0 & x=y\in\end{cases}$$

has been recently investigated (see [1,3]).

The objective of this paper is to give the necessary and sufficient condition for the function $F_w(x,y)$, function $M_w: I^2\to\mathbb{R}$ defined by

$$M_w(x,y)=\begin{cases} \frac{1}{\int_x^y w(t)dt}\int_x^y w(t)f(t)dt-f\left(\frac{x+y}{2}\right) & x,y\in I, x\neq y\\ 0 & x=y\in I\end{cases}$$

and function $T_w: I^2\to\mathbb{R}$ defined by

$$T_w(x,y)=\begin{cases} \frac{f(x)+f(y)}{2}-\frac{1}{\int_x^y w(t)dt}\int_x^y w(t)f(t)dt & x,y\in I, x\neq y\\ 0 & x=y\in I\end{cases}$$

to be Schur-geometrically convex (Schur-geometrically concave) and Schur-harmonically convex (Schur-harmonically concave) on I^2. The necessary and sufficient condition for the functions $M_w(x,y)$ and $T_w(x,y)$ to be Schur-convex on I^2 is given in [5].

Let us recall the weighted one-point quadrature formula [6]. If $f:[x,y]\to\mathbb{R}$ is such that $f^{(n)}$ is a piecewiese continuous function, then we have

$$\int_x^y w(t)f(t)dt=\sum_{j=1}^n A_j(z)f^{(j-1)}(z)+(-1)^n\int_x^y W_{n,w}(t,z)f^{(n)}(t)dt, \tag{1.2}$$

where for $j=1,\dots,n$

$$A_j(z)=\frac{(-1)^{j-1}}{(j-1)!}\int_x^y (z-s)^{j-1}w(s)ds$$

and

$$W_{n,w}(t,z)=\begin{cases} w_{1n}(t)=\frac{1}{(n-1)!}\int_x^t (t-s)^{n-1}w(s)ds & t\in[x,z],\\ w_{2n}(t)=\frac{1}{(n-1)!}\int_y^t (t-s)^{n-1}w(s)ds & t\in(z,y].\end{cases}$$

In order to prove our results, we shall use the following characterization of the Schur-geometric convexity and Schur-harmonic convexity [9]:

Lemma 1. *Let $f : I^2 \subset \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on I^2 and differentiable in the interior of I^2. Then f is Schur-geometrically convex (Schur-geometrically concave) on I^2 if and only if it is symmetric and*

$$(\log b-\log a)\left(b\frac{\partial f}{\partial b}-a\frac{\partial f}{\partial a}\right)\geq 0(\leq 0) \tag{1.3}$$

for all $a, b \in I$.

Lemma 2. *Let $f : I^2 \subset \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on I^2 and differentiable in the interior of I^2. Then f is Schur-harmonically convex (Schur-harmonically concave) on I^2 if and only if it is symmetric and*

$$(b-a)\left(b^2\frac{\partial f}{\partial b}-a^2\frac{\partial f}{\partial a}\right)\geq 0(\leq 0) \tag{1.4}$$

for all $a, b \in I$.

2. Main result

Theorem 3. *The function $F_w(x,y)$ is Schur-geometrically convex (concave) on $I^2 \subset \mathbb{R}_+^2$ if and only if the inequality*

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\leq\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)} \tag{2.1}$$

holds (reverses) for every $x, y \in I$.

Proof. Obviously, $F_w(x,y)$ is continuous on I^2, differentiable in the interior of I^2 and symmetric. Let $x, y \in I$, and without loss of generality, we can assume that $x \leq y$. After direct computation we get

$$\begin{aligned}
&(\log y-\log x)\left(y\frac{\partial f}{\partial y}-x\frac{\partial f}{\partial x}\right)\\
&=(\log y-\log x)\cdot\left(y\cdot\frac{w(y)f(y)\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot w(y)}{\left(\int_x^y w(t)dt\right)^2}\right.\\
&\left.-\ x\cdot\frac{-w(x)f(x)\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot(-w(x))}{\left(\int_x^y w(t)dt\right)^2}\right)\\
&=\frac{\log y-\log x}{\int_x^y w(t)dt}\cdot\left(yw(y)f(y)+xw(x)f(x)-\frac{(xw(x)+yw(y))\cdot\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right)\\
&=\frac{(\log y-\log x)\,(xw(x)+yw(y))}{\int_x^y w(t)dt}\left(\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)}-\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right),
\end{aligned} \tag{2.2}$$

so, the sign of the expression (2.2) depends on the sign of the term in the brackets. According to Lemma 1, the function F_w is Schur-geometrically convex (concave) if and only if (2.1) holds (reverses), so we have proved the theorem. □

Theorem 4. *The function $F_w(x,y)$ is Schur-harmonically convex (concave) on $I^2 \subset \mathbb{R}_+^2$ if and only if the inequality*

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\leq\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)} \tag{2.3}$$

holds (reverses) for every $x, y \in I$.

Proof. The function $F_w(x,y)$ is continuous on I^2, differentiable in the interior of I^2 and symmetric. Let $x,y\in I$, and without loss of generality, we can assume that $x\le y$. We compute

$$
\begin{aligned}
&(y-x)\left(y^2\frac{\partial F_w}{\partial y}-x^2\frac{\partial F_w}{\partial x}\right)\\
&=(y-x)\cdot\left(y^2\cdot\frac{w(y)f(y)\cdot\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot w(y)}{\left(\int_x^y w(t)dt\right)^2}\right.\\
&\left.-\,x^2\cdot\frac{-w(x)f(x)\cdot\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot(-w(x))}{\left(\int_x^y w(t)dt\right)^2}\right)\\
&=\frac{y-x}{\int_x^y w(t)dt}\cdot\left(x^2w(x)f(x)+y^2w(y)f(y)-\frac{\left(x^2w(x)+y^2w(y)\right)\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right)\\
&=\frac{(y-x)\left(x^2w(x)+y^2w(y)\right)}{\int_x^y w(t)dt}\left(\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)}-\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right). \qquad (2.4)
\end{aligned}
$$

The term $\frac{(y-x)\left(x^2w(x)+y^2w(y)\right)}{\int_x^y w(t)dt}$ is always positive, so the sign of the expression (2.4) depends only on the sign of the term in brackets. According to Lemma 2, function F_w is Schur-harmonically convex (concave) if and only if (2.3) holds (reverses), so we have proved the theorem. □

Remark 2. If $w(t)=\frac{1}{y-x}$ (the case of a uniform weight function), we get the following classification of Schur-geometrically and Schur-harmonically convexity (concavity):

$F(x,y)$ is Schur-geometrically convex (concave) $\Leftrightarrow \frac{\int_x^y f(t)dt}{y-x}\le\frac{xf(x)+yf(y)}{x+y}$, holds (reverses) for every $x,y\in I$.

$F(x,y)$ is Schur-harmonically convex (concave) $\Leftrightarrow \frac{\int_x^y f(t)dt}{y-x}\le\frac{x^2f(x)+y^2f(y)}{x^2+y^2}$, holds (reverses) for every $x,y\in I$.

Theorem 5. *The function $M_w(x,y)$ is Schur-geometrically convex (concave) if $f:I\to\mathbb{R}$ is decreasing (increasing) and the inequality*

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\le\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)} \qquad (2.5)$$

holds (reverses) for all $x,y\in I$.

Proof. It is easy to check that $M_w(x,y)$ is symmetric, contionuous on I^2 and differentiable on the interior of I^2. According to Lemma 1, we have to check that $M_w(x,y)$ satisfies condition (1.3). Let $x,y\in I$, and without loss of generallity we can assume that $x\le y$. Then we have

$$
\begin{aligned}
&(\log y-\log x)\left(y\frac{\partial M_w}{\partial y}-x\frac{\partial M_w}{\partial x}\right)\\
&=(\log y-\log x)\left(y\cdot\frac{w(y)f(y)\cdot\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot w(y)}{\left(\int_x^y w(t)dt\right)^2}-\frac{y}{2}f'\left(\frac{x+y}{2}\right)\right.\\
&\left.-\,x\cdot\frac{-w(x)f(x)\cdot\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot(-w(x))}{\left(\int_x^y w(t)dt\right)^2}+\frac{x}{2}f'\left(\frac{x+y}{2}\right)\right)\\
&=\frac{\log y-\log x}{\int_x^y w(t)dt}\left(xw(x)f(x)+yw(y)f(y)-(xw(x)+yw(y))\cdot\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right.\\
&\left.-\,\frac{y-x}{2}\int\limits_x^y w(t)dt\cdot f'\left(\frac{x+y}{2}\right)\right)=\frac{(\log y-\log x)(xw(x)+yw(y))}{\int_x^y w(t)dt}
\end{aligned}
$$

$$\times\left(\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)}-\frac{y-x}{2}\cdot\frac{\int_x^y w(t)dt}{xw(x)+yw(y)}\cdot f'(\frac{x+y}{2})-\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right)$$

(If the function f is decreasing (increasing),

then the middle term in the upper identity is ≥ 0 (≤ 0) so)

$$\geq(\leq)\frac{(\log y-\log x)(xw(x)+yw(y))}{\int_x^y w(t)dt}\left(\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)}-\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right).$$

Since (2.5) holds (reverses), the condition in Lemma 1 is satisfied and the proof is completed. □

Theorem 6. *The function $M_w(x,y)$ is Schur-harmonically convex (concave) if f is decreasing (increasing) and the inequality*

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\leq\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)} \tag{2.6}$$

holds (reverses) for all $x,y\in I$.

Proof. Since $M_w(x,y)$ is symmetric, continuous on I^2 and differentiable on the interior of I^2, according to Lemma 2 we have to check that $M_w(x,y)$ satisfies condition (1.4). Let $x,y\in I$, and without loss of generality, we can assume that $x\leq y$. Then we have

$$(y-x)\left(y^2\frac{\partial M_w}{\partial y}-x^2\frac{\partial M_w}{\partial x}\right)$$

$$=\frac{y-x}{\int_x^y w(t)dt}\cdot\left(x^2w(x)f(x)+y^2w(y)f(y)-\left(x^2w(x)+y^2w(y)\right)\cdot\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right.$$

$$\left.-\frac{y^2-x^2}{2}\int_x^y w(t)dt\cdot f'\left(\frac{x+y}{2}\right)\right)$$

$$=\frac{(y-x)(x^2w(x)+y^2w(y))}{\int_x^y w(t)dt}\cdot\left(\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)}-\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right.$$

$$\left.-\frac{y^2-x^2}{2}f'(\frac{x+y}{2})\frac{\int_x^y w(t)dt}{x^2w(x)+y^2w(y)}\right)$$

(If the function f is decreasing (increasing),

then the last term in the upper identity is ≥ 0 (≤ 0) so)

$$\geq(\leq)\frac{(y-x)(x^2w(x)+y^2w(y))}{\int_x^y w(t)dt}\left(\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)}-\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right).$$

Since (2.6) holds (reverses), the condition in Lemma 2 is satisfied and the proof is completed. □

Remark 3. For the case of the uniform weight function we have:

$M(x,y)$ is Schur-geometrically convex (concave) if f is decreasing (increasing) and $\frac{\int_x^y f(t)dt}{y-x}\leq\frac{xf(x)+yf(y)}{x+y}$, holds (reverses) for every $x,y\in I$.

$M(x,y)$ is Schur-harmonically convex (concave) if f is decreasing (increasing) and $\frac{\int_x^y f(t)dt}{y-x}\leq\frac{x^2f(x)+y^2f(y)}{x^2+y^2}$, holds (reverses) for every $x,y\in I$.

Theorem 7. *The function $T_w(x,y)$ is Schur-geometrically convex (concave) if $f:I\to\mathbb{R}$ is convex (concave), twice differentiable and*

$$\frac{\int_x^y tw(t)dt}{\int_x^y w(t)dt}=\frac{xw(x)+yw(y)}{w(x)+w(y)} \tag{2.7}$$

and

$$2\frac{w(x)w(y)(y-x)}{w(x)+w(y)} \leq \int\limits_x^y w(t)dt \tag{2.8}$$

holds (reverses) for all $x, y \in I$.

Proof. The function $T_w(x,y)$ is symmetric, continuous on I^2 and differentiable on the interior of I^2, so according to Lemma 1, we have to check if the condition (1.3) holds. Let us assume $x, y \in I$, $x < y$. We have

$$\begin{aligned}
&(\log y - \log x)\left(y\frac{\partial T_w}{\partial y} - x\frac{\partial T_w}{\partial x}\right) = (\log y - \log x)\cdot\left(\frac{yf'(y)}{2} - \frac{yw(y)f(y)}{\int_x^y w(t)dt}\right.\\
&+ \left.\frac{yw(y)\int_x^y w(t)f(t)dt}{\left(\int\limits_x^y w(t)dt\right)^2} - \frac{xf'(x)}{2} - \frac{xw(x)f(x)}{\int\limits_x^y w(t)dt} + \frac{xw(x)\int\limits_x^y w(t)f(t)dt}{\left(\int\limits_x^y w(t)dt\right)^2}\right)\\
&= \frac{(\log y - \log x)(xw(x)+yw(y))}{\int_x^y w(t)dt}\cdot\left(\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} - \frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)}\right.\\
&+ \left.\frac{\int_x^y w(t)dt}{xw(x)+yw(y)}\cdot\frac{yf'(y)-xf'(x)}{2}\right).
\end{aligned} \tag{2.9}$$

From (2.7), we have

$$\begin{aligned}
&(w(x)+w(y))\int\limits_x^y tw(t)dt = (xw(x)+yw(y))\int\limits_x^y w(t)dt\\
&\Rightarrow w(y)\int\limits_x^y (y-t)w(t)dt = w(x)\int\limits_x^y (t-x)w(t)dt.
\end{aligned} \tag{2.10}$$

Further, from (2.10), we have

$$\begin{aligned}
&w(y)\int\limits_x^y (y-t)w(t)dt = w(x)\int\limits_x^y (y-x-y+t)w(t)dt\\
&\Rightarrow w(y)\int\limits_x^y (y-t)w(t)dt = (y-x)w(x)\int\limits_x^y w(t)dt - w(x)\int\limits_x^y (y-t)w(t)dt\\
&\Rightarrow (w(x)+w(y))\cdot\int\limits_x^y (y-t)w(t)dt = (y-x)w(x)\int\limits_x^y w(t)dt\\
&\Rightarrow \frac{w(y)\int\limits_x^y (y-t)w(t)dt}{\int\limits_x^y w(t)dt} = \frac{w(x)w(y)(y-x)}{w(x)+w(y)}.
\end{aligned} \tag{2.11}$$

Applying (2.11) and according to the inequality (2.7), we have

$$\frac{\int_x^y w(t)dt}{2} - \frac{w(y)\int_x^y (y-t)w(t)dt}{\int_x^y w(t)dt} \geq 0.$$

If f is convex, we have $f''(t) \geq 0$, so function f' is increasing, and we have

$$0 < x < y \Rightarrow f'(x) \leq f'(y) \Rightarrow xf'(x) \leq xf'(y) \leq yf'(y). \tag{2.12}$$

Applying (2.10), (2.11) and (2.12), we have

$$\begin{aligned}&\frac{yw(y)f'(y)\int_x^y(y-t)w(t)dt-xw(x)f'(x)\int_x^y(t-x)w(t)dt}{(xw(x)+yw(y))\cdot\int_x^y w(t)dt}\\&=\frac{w(y)\int\limits_x^y(y-t)w(t)dt}{(xw(x)+yw(y))\int\limits_x^y w(t)dt}\cdot(yf'(y)-xf'(x))\\&\le\frac{\int_x^y w(t)dt}{xw(x)+yw(y)}\cdot\frac{yf'(y)-xf'(x)}{2}.\end{aligned}\tag{2.13}$$

On the other hand, if we apply (1.2) for $n=2$ and $z=x$ and multiply by $\frac{xw(x)}{xw(x)+yw(y)}$, and also for $z=y$, multiply by $\frac{yw(y)}{xw(x)+yw(y)}$, and then add those two identities, we obtain

$$\begin{aligned}&\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}-\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)}\\&+\frac{yw(y)f'(y)\int_x^y(y-t)w(t)dt-xw(x)f'(x)\int_x^y(t-x)w(t)dt}{(xw(x)+yw(y))\cdot\int_x^y w(t)dt}\\&=\frac{\int_x^y\left[xw(x)\cdot\int_t^y(s-t)w(s)ds+yw(y)\cdot\int_x^t(t-s)w(s)ds\right]f''(t)dt}{(xw(x)+yw(y))\cdot\int_x^y w(t)dt}.\end{aligned}\tag{2.14}$$

Now, we apply (2.13) in (2.9) and use (2.14) to get

$$\begin{aligned}&(\log y-\log x)\left(y\frac{\partial T_w}{\partial y}-x\frac{\partial T_w}{\partial x}\right)\ge\frac{(\log y-\log x)(xw(x)+yw(y))}{\int_x^y w(t)dt}\\&\times\frac{\int_x^y\left[xw(x)\cdot\int_t^y(s-t)w(s)ds+yw(y)\cdot\int_x^t(t-s)w(s)ds\right]f''(t)dt}{(xw(x)+yw(y))\cdot\int_x^y w(t)dt}\\&=\frac{(\log y-\log x)\cdot\int_x^y\left[xw(x)\cdot\int_t^y(s-t)w(s)ds+yw(y)\cdot\int_x^t(t-s)w(s)ds\right]f''(t)dt}{\left(\int_x^y w(t)dt\right)^2}.\end{aligned}$$

Since f is convex and the integrals in the brackets are non negative, we have proved that $(\log y-\log x)\left(y\frac{\partial T_w}{\partial y}-x\frac{\partial T_w}{\partial x}\right)\ge 0$, for all $x,y\in I$, $x<y$, so, the function T_w is Schur-geometrically convex.

The proof for the Schur-geometrically concave case is similar. □

Theorem 8. *The function $T_w(x,y)$ is Schur-harmonically convex (concave) if $f:I\to\mathbb{R}$ is convex (concave), twice differentiable and*

$$\frac{\int_x^y tw(t)dt}{\int_x^y w(t)dt}=\frac{xw(x)+yw(y)}{w(x)+w(y)}\tag{2.15}$$

and

$$2\frac{w(x)w(y)(y-x)}{w(x)+w(y)}\le\int\limits_x^y w(t)dt\tag{2.16}$$

holds (reverses) for all $x,y\in I$.

Proof. Since the function $T_w(x,y)$ is symmetric, continuous on I^2 and differentiable on the interior of I^2, according to Lemma 2, we have to check if the condition (1.4) holds. Let us assume $x,y\in I$,

$x < y$. We have

$$(y-x)\left(y^2\frac{\partial T_w}{\partial y}-x^2\frac{\partial T_w}{\partial x}\right)=(y-x)\cdot\left(\frac{y^2f'(y)}{2}-\frac{y^2w(y)f(y)}{\int_x^y w(t)dt}\right.$$
$$+\frac{y^2w(y)\int_x^y w(t)f(t)dt}{\left(\int\limits_x^y w(t)dt\right)^2}-\frac{x^2f'(x)}{2}-\frac{x^2w(x)f(x)}{\int\limits_x^y w(t)dt}+\left.\frac{x^2w(x)\int\limits_x^y w(t)f(t)dt}{\left(\int\limits_x^y w(t)dt\right)^2}\right)$$
$$=\frac{(y-x)(x^2w(x)+y^2w(y))}{\int_x^y w(t)dt}\cdot\left(\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}-\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)}\right.$$
$$\left.+\frac{\int_x^y w(t)dt}{x^2w(x)+y^2w(y)}\cdot\frac{y^2f'(y)-x^2f'(x)}{2}\right). \tag{2.17}$$

Again, as in the proof of Theorem 7, we conclude that (2.10), (2.11) and

$$\frac{\int_x^y w(t)dt}{2}-\frac{w(y)\int_x^y(y-t)w(t)dt}{\int_x^y w(t)dt}\geq 0$$

hold.

If f is convex, we have $f''(t)\geq 0$, so, the function f' is increasing, and we have

$$0<x<y\Rightarrow f'(x)\leq f'(y)\Rightarrow x^2f'(x)\leq x^2f'(y)\leq y^2f'(y). \tag{2.18}$$

Applying (2.10), (2.11) and (2.18), we have

$$\frac{y^2w(y)f'(y)\int_x^y(y-t)w(t)dt-x^2w(x)f'(x)\int_x^y(t-x)w(t)dt}{(x^2w(x)+y^2w(y))\cdot\int_x^y w(t)dt}$$
$$=\frac{w(y)\int\limits_x^y(y-t)w(t)dt}{(x^2w(x)+y^2w(y))\int\limits_x^y w(t)dt}\cdot\left(y^2f'(y)-x^2f'(x)\right)$$
$$\leq\frac{\int_x^y w(t)dt}{x^2w(x)+y^2w(y)}\cdot\frac{y^2f'(y)-x^2f'(x)}{2}. \tag{2.19}$$

On the other hand, if we apply (1.2) for $n=2$ and $z=x$ and multiply by $\frac{x^2w(x)}{x^2w(x)+y^2w(y)}$, and also for $z=y$, multiply by $\frac{y^2w(y)}{x^2w(x)+y^2w(y)}$, and then add those two identities, we obtain

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}-\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)}$$
$$+\frac{y^2w(y)f'(y)\int_x^y(y-t)w(t)dt-x^2w(x)f'(x)\int_x^y(t-x)w(t)dt}{(x^2w(x)+y^2w(y))\cdot\int_x^y w(t)dt}$$
$$=\frac{\int_x^y\left[x^2w(x)\cdot\int_t^y(s-t)w(s)ds+y^2w(y)\cdot\int_x^t(t-s)w(s)ds\right]f''(t)dt}{(x^2w(x)+y^2w(y))\cdot\int_x^y w(t)dt}. \tag{2.20}$$

Now, we apply (2.19) in (2.17) and use (2.20) to get

$$(y-x)\left(y^2\frac{\partial T_w}{\partial y}-x^2\frac{\partial T_w}{\partial x}\right)\geq\frac{(y-x)(x^2w(x)+y^2w(y))}{\int_x^y w(t)dt}$$
$$\times\frac{\int_x^y\left[x^2w(x)\cdot\int_t^y(s-t)w(s)ds+y^2w(y)\cdot\int_x^t(t-s)w(s)ds\right]f''(t)dt}{(x^2w(x)+y^2w(y))\cdot\int_x^y w(t)dt}$$

$$= \frac{(y-x)\cdot\int_x^y \left[x^2 w(x)\cdot\int_t^y (s-t)w(s)ds + y^2 w(y)\cdot\int_x^t (t-s)w(s)ds\right] f''(t)dt}{\left(\int_x^y w(t)dt\right)^2}.$$

Since f is convex and the integrals in the brackets are non negative, we have proved that $(y-x)\left(y^2\frac{\partial T_w}{\partial y} - x^2\frac{\partial T_w}{\partial x}\right) \geq 0$, for all $x, y \in I, x < y$, so, the function T_w is Schur-harmonically convex.

The proof for the Schur-harmonically concave case is similar. □

Remark 4. For $w(t) = \frac{1}{y-x}$ it is easy to check that conditions (2.7), (2.8), (2.15) and (2.16) are valid, so, if f is convex, then T is Schur-geometrically and Schur-harmonically convex.

References

1. Y. Chu, G. Wang, X. Zhang, Schur convexity and Hadamard's inequality. *Math. Inequal. Appl.* **13** (2010), no. 4, 725–731.
2. Y. Chu, X. Zhang, G. Wang, The Schur geometrical convexity of the extended mean values. *J. Convex Anal.* **15** (2008), no. 4, 707–718.
3. V. Čuljak, I. Franjić, R. Ghulam, J. Pečarić, Schur-convexity of averages of convex functions. *J. Inequal. Appl.* **2011**, Art. ID 581918, 25 pp.
4. N. Elezović, J. Pečarić, A note on Schur-convex functions. *Rocky Mountain J. Math.* **30** (2000), no. 3, 853–856.
5. S. Kovač, Schur-convexity of the weighted quadrature formula. *Math. Inequal. Appl.* **22** (2019), no. 4, 1397–1403.
6. S. Kovač, J. Pečarić, Weighted version of general integral formula of Euler type. *Math. Inequal. Appl.* **13** (2010), no. 3, 579–599.
7. F. Qi, J. Sandor, S. S. Dragomir, A. Sofo, Notes on the Schur-convexity of the extended mean values. *Taiwanese J. Math.* **9** (2005), no. 3, 411–420.
8. H.-N. Shi, J. Zhang, Q.-H. Ma, Schur-convexity, Schur-geometric and Schur-harmonic convexity for a composite function of complete symmetric function. *SpringerPlus* **5** (2016), no. 1, 1–8.
9. J. Sun, Z.-L. Sun, B.-Y. Xi, F. Qi, Schur-geometric and Schur-harmonic convexity of an integral mean for convex functions. *Turkish J. Anal. Number Theory* **3** (2015), 87–89.

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RATIONAL PÁL TYPE $(0,1;0)$-INTERPOLATION AND QUADRATURE FORMULA WITH CHEBYSHEV–MARKOV FRACTIONS

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Abstract. We present a Pál-type $(0,1;0)$-interpolation on an inter-scaled set of nodes, when Hermite and Lagrange data are prescribed on the zeros of Chebyshev–Markov sine fraction $U_n(x)$ and its derivative $U_n'(x)$, respectively. A quadrature formula based on the obtained Pál-type interpolation has been constructed. Coefficients of this quadrature are obtained in the explicit form.

1. Introduction

The study of different type interpolation processes has been a subject of interest for several mathematicians. In almost all the cases the interpolatory polynomials are considered on the nodes which are the zeros of certain classical orthogonal polynomials. The main idea of the present paper is to construct a rational interpolation process and its corresponding quadrature formula.

Let $\mathcal{R}_{2n-1}(a_0,a_1,a_2,\dots,a_{2n-1})$ be a rational space defined as

$$\mathcal{R}_{2n-1}(a_0,a_1,\dots,a_{2n-1}) := \left\{\frac{p_{2n-1}(x)}{\prod_{k=0}^{2n-1}(1+a_kx)}\right\},$$

where $p_{2n-1}(x)$ is a polynomial of degree $\le 2n-1$ and $\{a_k\}_{k=0}^{2n-1}$ are real and belong to $[-1,1]$, or are paired by a complex conjugation.

Chebyshev and Markov introduced rational cosine and sine fractions [9] which generalize Chebyshev polynomials, possess many similar properties [8, 16, 18] and are called Chebyshev–Markov rational fractions. More details on the rational generalization of Chebyshev polynomials can be found in [1–6, 19]. Let $U_n(x)$ be the rational Chebyshev–Markov sine fraction,

$$U_n(x) = \frac{\sin\mu_{2n}(x)}{\sqrt{1-x^2},} \tag{1.1}$$

where

$$\mu_{2n}(x) = \frac{1}{2}\sum_{k=0}^{2n-1}\arccos\frac{x+a_k}{1+a_kx},\quad \mu_{2n}'(x) = -\frac{\lambda_{2n}(x)}{\sqrt{1-x^2}},$$

$$\lambda_{2n}(x) = \frac{1}{2}\sum_{k=0}^{2n-1}\frac{\sqrt{1-a_k^2}}{1+a_kx},\quad n\in N. \tag{1.2}$$

The rational fraction $U_n(x)$ can be expressed as

$$U_n(x) = \frac{P_{n-1}(x)}{\sqrt{\Pi_{k=0}^{2n-1}(1+a_kx)}},$$

where $P_{n-1}(x)$ is an algebraic polynomial of degree $n-1$ with a real coefficient, and $\{a_k\}_{k=0}^{2n-1}$ are as defined above. The fraction $U_n(x)$ has $n-1$ zeros on the interval $(-1,1)$ given by

$$-1 < x_{n-1} < x_{n-2} < \cdots < x_2 < x_1 < 1,$$

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with

$$\mu_{2n}(x_k) = k\pi,\ \ k = 1, 2, \ldots, n-1.$$

Also, the rational function $\lambda_{2n}(x)$ can be expressed as

$$\lambda_{2n}(x) = \frac{p_{2n-1}(x)}{\prod_{k=0}^{2n-1}(1 + a_k x)},$$

where $p_{2n-1}(x)$ is a polynomial of degree atmost $2n-1$. It has no zeros in the interval $[-1, 1]$. On differentiating (1.1), we get

$$U_n'(x) = \frac{-\cos\mu_{2n}(x)\lambda_{2n}(x)\sqrt{1-x^2} + x\sin\mu_{2n}(x)}{(1-x^2)^{3/2}} \tag{1.3}$$

and

$$U_n'(x_k) = -\frac{\lambda_{2n}(x_k)}{(1-x_k^2)}. \tag{1.4}$$

In 1962, Rusak [15] initiated the study of interpolation processes by means of rational functions on the interval $[-1, 1]$. The nodes were taken to be the zeros of Chebyshev–Markov rational fractions. In [13], rational interpolation functions of Hermite–Fejér-type were constructed. Min [10] was the first, who considered the rational quasi-Hermite-type interpolation. He constructed the interpolated function and proved its uniform convergence for the continuous functions on the segment with the restriction that the poles of the approximating rational functions should not have limit points on the interval $[-1, 1]$. Based on the ideas of [13] and using the method, somewhat different from that of [10], Rouba et al. [12], [14] revisited the rational interpolation functions of Hermite–Fejér-type. They also proved the uniform convergence of the interpolation process for the function $f \in \mathcal{C}[-1, 1]$ and obtained explicitly its corresponding Lobatto type quadrature formula. Recently, Shrawan Kumar et al. [7] studied the Radau type quadrature for an almost quasi-Hermite–Fejér-type interpolation in rational spaces.

In this paper, we have considered the existence and explicit representation of a Pál type $(0, 1; 0)$-interpolation on the rational space $\mathcal{R}_{3n-3}(a_0, a_1, \ldots, a_{2n-1})$, when the Hermite and Lagrange data are prescribed on the zeros of $U_n(x)$ $(\{x_k\}_{k=1}^{n-1})$ and its derivative $U_n'(x)$ $(\{t_k\}_{k=1}^{n-2})$, respectively. These zeros are inter-scaled such that

$$-1 = x_n < x_{n-1} < t_{n-2} < x_{n-2} < \cdots < x_2 < t_1 < x_1 < 1 = x_0.$$

A quadrature formula corresponding to the interpolation process has also been obtained.

2. Explicit Representation of Pál type $(0, 1; 0)$-interpolation

For any function $f \in C[-1, 1]$ the Pál type (0,1;0)-interpolation function $W_n(x, f)$ satisfying the conditions

$$\begin{cases} W_n(x_k, f) = f(x_k), & k = 0, 1, \ldots, n, \\ W_n'(x_k, f) = \alpha_k, & k = 1, 2, \ldots, n-1, \\ W_n(t_k, f) = f(t_k), & k = 1, 2, \ldots, n-2, \end{cases} \tag{2.1}$$

can be explicitly represented as

$$W_n(x, f) = \sum_{k=0}^{n} f(x_k)E_k(x) + \sum_{k=1}^{n-1} \alpha_k D_k(x) + \sum_{k=1}^{n-2} f(t_k)C_k(x), \tag{2.2}$$

where α_k, $k = 1, 2, \ldots, n-1$ are arbitrarily given real numbers, $\{E_k(x)\}_{k=0}^{n}$, $\{D_k(x)\}_{k=1}^{n-1}$ and $\{C_k(x)\}_{k=1}^{n-2}$ are fundamental functions of the Pál type (0,1;0) interpolation $W_n(x, f)$, satisfying the following conditions: for $k = 1, 2, \ldots, n-2$,

$$\begin{cases} C_k(x_j) = 0, & j = 0, 1, \ldots, n, \\ C_k'(x_j) = 0, & j = 1, 2, \ldots, n-1, \\ C_k(t_j) = \delta_{jk}, & j = 1, 2, \ldots, n-2, \end{cases} \tag{2.3}$$

for $k = 1, 2, \dots, n-1$,

$$\begin{cases} D_k(x_j) = 0, & j = 0, 1, \dots, n, \\ D'_k(x_j) = \delta_{jk}, & j = 1, 2, \dots, n-1, \\ D_k(t_j) = 0, & j = 1, 2, \dots, n-2 \end{cases} \tag{2.4}$$

and for $k = 0, 1, 2, \dots, n$,

$$\begin{cases} E_k(x_j) = \delta_{jk}, & j = 0, 1, \dots, n, \\ E'_k(x_j) = 0, & j = 1, 2, \dots, n-1, \\ E_k(t_j) = 0, & j = 1, 2, \dots, n-2. \end{cases} \tag{2.5}$$

In the following lemmas, we give the explicit representation of these fundamental functions of the Pál type $(0,1;0)$-interpolation $W_n(x, f)$.

Lemma 1. *The fundamental functions* $\{C_k(x)\}_{k=1}^{n-2}$ *satisfying conditions* (2.3) *can be explicitly represented for* $k = 1, 2, \dots, n-2$, *as*

$$C_k(x) = \frac{(\lambda_{2n}(t_k))^{3/2}(1-x^2)U_n^2(x)L_k(x)}{(1-t_k^2)U_n^2(t_k)(\lambda_{2n}(x))^{3/2}}, \tag{2.6}$$

where $U_n(x)$ *are given by* (1.1), $\lambda_{2n}(x)$ *are given by* (1.2) *and* $\{L_k(x)\}_{k=1}^{n-2}$ *are given by*

$$L_k(x) = \frac{U'_n(x)}{(x-t_k)U''_n(t_k)}.$$

Proof. We will show that $\{C_k(x)\}_{k=1}^{n-2}$ given by (2.6) satisfies conditions (2.3). Obviously, for $k = 1, 2, \dots, n-2$, $C_k(x_j) = 0$, $j = 0, 1, \dots, n$ and $C'_k(x_j) = 0$, $j = 1, 2, \dots, n-1$. Also, for $j \neq k$, $C_k(t_j) = 0$, $j = 1, \dots, n-2$ and for $j = k$,

$$\lim_{x \to t_k} C_k(x) = \frac{(\lambda_{2n}(t_k))^{3/2}}{(1-t_k^2)U_n^2(t_k)} \frac{(1-t_k^2)U_n^2(t_k)}{(\lambda_{2n}(t_k))^{3/2}} \lim_{x \to t_k} L_k(x) = 1$$

which completes the proof of the Lemma. □

Lemma 2. *The fundamental functions* $\{D_k(x)\}_{k=1}^{n-1}$ *satisfying conditions* (2.4) *can be explicitly represented for* $k = 1, 2, \dots, n-1$, *as*

$$D_k(x) = \frac{(\lambda_{2n}(x_k))^{3/2}}{(1-x_k^2)(U'_n(x_k))^2} \frac{(1-x^2)U_n(x)U'_n(x)\ell_k(x)}{(\lambda_{2n}(x))^{3/2}}, \tag{2.7}$$

where $U'_n(x)$ *are given by* (1.3), $\lambda_{2n}(x)$ *are given by* (1.2) *and* $\{\ell_k(x)\}_{k=1}^{n-1}$ *are given by*

$$\ell_k(x) = \frac{U_n(x)}{(x-x_k)U'_n(x_k)}. \tag{2.8}$$

Proof. Obviously, for $k = 1, 2, \dots, n-1$, $D_k(x_j) = 0$, $j = 0, 1, \dots, n$ and for $j \neq k$, $D'_k(x_j) = 0$, $j = 1, 2, \dots, n-1$, for $j = k$,

$$\lim_{x \to y_k} D'_k(x) = \left(\frac{(\lambda_{2n}(x_k))^{3/2}}{(1-x_k^2)(U'_n(x_k))^2}\right)\left(\frac{(1-x_k^2)U'_n(x_k)}{(\lambda_{2n}(x_k))^{3/2}}\right) \lim_{x \to x_k} \left(\frac{U_n(x)}{x-x_k}\right) = 1.$$

Also, $D_k(t_j) = 0$, $j = 0, 1, \dots, n$, which shows that $\{D_k(x)\}_{k=1}^{n-1}$, given by (2.7), satisfies all conditions (2.4) and hence completes the proof of the Lemma. □

Lemma 3. *The fundamental functions* $\{E_k(x)\}_{k=0}^{n}$ *satisfying conditions* (2.5) *can be explicitly represented as*

$$E_0(x) = \frac{(\lambda_{2n}(1))^{3/2}}{2U_n^2(1)U'_n(1)} \frac{(1+x)U_n^2(x)U'_n(x)}{(\lambda_{2n}(x))^{3/2}}, \tag{2.9}$$

for $k = 1, 2, \dots, n-1$,

$$E_k(x) = \frac{(\lambda_{2n}(x_k))^{3/2}}{(1-x_k^2)U'_n(x_k)} \frac{(1-x^2)U'_n(x)}{(\lambda_{2n}(x))^{3/2}} \Big(1 + b_k(x-x_k)\Big)\ell_k^2(x), \tag{2.10}$$

where

$$b_k = -\frac{x_k}{1-x_k^2} - \frac{U_n''(x_k)}{U_n'(x_k)} + \frac{\lambda_{2n}'(x_k)}{2\lambda_{2n}(x_k)} \tag{2.11}$$

and

$$E_n(x) = \frac{(\lambda_{2n}(-1))^{3/2}}{2U_n^2(-1)U_n'(-1)} \frac{(1-x)U_n^2(x)U_n'(x)}{(\lambda_{2n}(x))^{3/2}}. \tag{2.12}$$

Proof. Obviously, for $j \neq k$, we have $E_k(x_j) = 0$, $j = 0, 1, \ldots, n$ and for $j = k$, using de L'Hospital's rule and (1.4), we have

$$\begin{aligned}\lim_{x\to x_k} E_k(x) =& \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)}\left(\lim_{x\to x_k}\frac{\sin\mu_{2n}(x)}{(x-x_k)}\right)^2\\ =& \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)}\left(\lim_{x\to x_k}\frac{-\lambda_{2n}(x)\cos\mu_{2n}(x)}{\sqrt{1-x^2}}\right)^2 = 1.\end{aligned}$$

Also, for $k = 1, 2, \ldots, n-1$, we have $E_k(t_j) = 0$, $j = 1, 2, \ldots, n-2$.

On differentiating (2.10) with respect to x and using (1.4), we get

$$\begin{aligned}E_k'(x) =& \frac{(1-x_k^2)}{U_n'(x_k)(\lambda_{2n}(x_k))^{1/2}}\left[\frac{2U_n'(x)\{1+b_k(x-x_k)\}}{(\lambda_{2n}(x))^{3/2}}\left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right)'\right.\\ &+\left(\frac{b_kU_n'(x)+\{1+b_k(x-x_k)\}U_n''(x)}{(\lambda_{2n}(x))^{3/2}}\right.\\ &\left.\left.-\frac{3\lambda_{2n}'(x)U_n'(x)\{1+b_k(x-x_k)\}}{2(\lambda_{2n}(x))^{5/2}}\right)\left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right)\right]\left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right),\end{aligned}$$

then for $j \neq k$, we have $E_k'(x_j) = 0$, $j = 1, 2, \ldots, n-1$ and for $j = k$,

$$\begin{aligned}\lim_{x\to x_k} E_k'(x) =& \frac{(1-x_k^2)}{U_n'(x_k)(\lambda_{2n}(x_k))^{1/2}}\left[\frac{2U_n'(x_k)}{(\lambda_{2n}(x_k))^{3/2}}\left(\lim_{x\to x_k}\left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right)\left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right)'\right)\right.\\ &\left.+\left(\frac{b_kU_n'(x_k)+U_n''(x_k)}{(\lambda_{2n}(x_k))^{3/2}} - \frac{3\lambda_{2n}'(x_k)U_n'(x_k)}{2(\lambda_{2n}^2(x_k))^{5/2}}\right)\left(\lim_{x\to x_k}\frac{\sin\mu_{2n}(x)}{x-x_k}\right)^2\right].\end{aligned}$$

We know that

$$\lim_{x\to x_k}\frac{\sin\mu_{2n}(x)}{(x-x_k)} = \mu_{2n}'(x_k)\cos\mu_{2n}(x_k) = -\frac{\lambda_{2n}(x_k)}{\sqrt{1-x_k^2}}$$

and

$$\lim_{x\to x_k}\left(\frac{\sin\mu_{2n}(x)}{x-x_k}\right)' = \frac{1}{2}\cos\mu_{2n}(x_k)\mu_{2n}''(x_k),$$

where

$$\mu_{2n}''(x) = -\frac{x\lambda_{2n}(x)+(1-x^2)\lambda_{2n}'(x)}{(1-x^2)^{3/2}},$$

therefore

$$\lim_{x\to x_k} E_k'(x) = \left[\frac{x_k}{1-x_k^2} + \frac{U_n''(x_k)}{U_n'(x_k)} - \frac{\lambda_{2n}'(x_k)}{2\lambda_{2n}(x_k)} + b_k\right] = 0,$$

due to (2.11) which shows that $\{E_k(x)\}_{k=1}^{n-1}$ given by (2.10) satisfy all the conditions given by (2.5) for $k = 1, 2, \ldots, n-1$.

Similarly, we can show that E_0 and $E_n(x)$ given by (2.9) and (2.12), respectively, satisfy conditions (2.5) for $k = 0$ and (2.5), for $k = n$, respectively, which completes the proof of the Lemma. □

Remark 4. The Pál type (0, 1; 0)-interpolation $W_n(f, x)$, satisfying conditions (2.1) can be explicitly represented as (2.2) with the help of Lemmas 1–3. Taking all a_i's as zero, $W_n(f, x)$ reduces to the interpolated polynomials of degree $\leq 3n-3$.

Theorem 5. *The function $W_n(f, x)$ is a rational function, that is,*

$$W_n(f, x) \in \mathcal{R}_{3n-3}(a_0, a_1, a_2, \ldots, a_{2n-1}).$$

Proof. Since $U_n \in \mathcal{R}_{n-1}(a_0, a_1, \ldots, a_{2n-1})$, we can express it as

$$U_n(x) := \frac{S_{n-1}(x)}{S_n^*(x)},$$

where $S_n^*(x) := \sqrt{\prod_{k=0}^{2n-1}(1 + xa_k)}$, $S_{n-1}(x) := c_{n-1}(x - x_1)(x - x_2) \ldots (x - x_{n-1})$ and c_{n-1} depends on n and $\{a_k\}_{k=0}^{2n-1}$. So, we have

$$\ell_k(x) = \frac{S_n^*(x_k)}{S_n^*(x)} q_k(x), \quad k = 1, 2, \ldots, n - 1,$$

where

$$q_k(x) := \frac{S_{n-1}(x)}{S'_{n-1}(x_k)(x - x_k)}, \quad k = 1, 2, \ldots, n - 1.$$

Thus $\ell_k(x) \in \mathcal{R}_{n-2}(a_0, a_1, \ldots, a_{2n-1})$. Similarly, we can express

$$U'_n(x) := \frac{Q_{n-2}(x)}{S_n^*(x)},$$

where $Q_{n-1}(x) := d_{n-1}(x - t_1)(x - t_2) \ldots (x - t_{n-2})$ and d_{n-1} depends on n and $\{a_k\}_{k=0}^{2n-1}$. Then

$$L_k(x) = \frac{S_n^*(t_k)}{S_n^*(x)} q_k^*(x), \quad k = 1, 2, \ldots, n - 1,$$

where

$$q_k^*(x) := \frac{Q_{n-2}(x)}{Q'_{n-2}(t_k)(x - t_k)}, \quad k = 1, 2, \ldots, n - 2.$$

Thus $L_k(x) \in \mathcal{R}_{n-3}(a_0, a_1, \ldots, a_{2n-1})$. Hence, by (2.6), (2.7) and (2.10) the lemma follows. □

Remark 6. Notice that the poles of the rational function $W_n(f, x)$ can be found from the equality $\lambda_{2n}(x) = 0$. They depend on the parameters $a_k, k = 0, 1, \ldots, 2n - 1$. The relationship between the zeros of the function $\lambda_{2n}(x)$ and the parameters a_k is described in [17].

3. Quadrature Formula

Under the same assumption on the parameters $a_1, a_2, \ldots, a_{2n1}$, we consider the following Pál type (0, 1; 0)-interpolation.

For the given function f defined on $[-1, 1]$, we define the function

$$V_n(f, x) = \sum_{k=0}^{n} f(x_k)\Omega_k(x) + \sum_{k=1}^{n-1} \alpha_k \sigma_k(x) + \sum_{k=1}^{n-2} f(t_k)\gamma_k(x), \tag{3.1}$$

where, for $k = 1, 2, \ldots, n - 1$,

$$\Omega_k(x) = \frac{(1 - x^2)U'_n(x)}{(1 - x_k^2)U'_n(x_k)} \left[1 - 2\left(\frac{U''_n(x_k)}{U'_n(x_k)} + \frac{x_k}{(1 - x_k^2)}\right)(x - x_k)\right] \ell_k^2(x),$$

$$\Omega_0(x) = \frac{(1 + x)U_n^2(x)U'_n(x)}{2U_n^2(1)U'_n(1)}, \quad \Omega_n(x) = \frac{(1 - x)U_n^2(x)U'_n(x)}{2U_n^2(-1)U'_n(-1)},$$

for $k = 1, 2, \ldots, n - 1$,

$$\sigma_k(x) = \frac{(1 - x^2)U_n(x)U'_n(x)\ell_k(x)}{(1 - x_k^2)(U'_n(x_k))^2}$$

and for $k = 1, 2, \ldots, n - 2$,

$$\gamma_k(x) = \frac{(1 - x^2)U_n^2(x)L_k(x)}{(1 - t_k^2)U_n^2(t_k)}.$$

The function $V_n(x)$ given by (3.1), satisfies conditions (2.1) and hence is the Pál type $(0,1;0)$-interpolation, and

$$V_n(f,x) \in \mathcal{R}_{3n-3}(a_1, a_2, \ldots, a_{2n-1}).$$

The quadrature formula corresponding to the interpolatory function (3.1) is given by

$$\begin{aligned}
&\int_{-1}^{1} (1-x^2) f(x)dx \approx \sum_{k=0}^{n} f(x_k) \int_{-1}^{1} (1-x^2)\Omega_k(x)dx \\
&+ \sum_{k=1}^{n-1} f'(x_k) \int_{-1}^{1} (1-x^2)\sigma_k(x)dx + \sum_{k=1}^{n-2} f(y_k) \int_{-1}^{1} (1-x^2)\gamma_k(x)dx \\
&\approx \sum_{k=0}^{n} E_k f(x_k) + \sum_{k=1}^{n-1} D_k f'(x_k) + \sum_{k=1}^{n-2} C_k f(y_k),
\end{aligned} \tag{3.2}$$

where

$$E_k = \int_{-1}^{1} (1-x^2)\Omega_k(x)dx, \quad k = 0, 1, \ldots, n, \tag{3.3}$$

$$D_k = \int_{-1}^{1} (1-x^2)\sigma_k(x)dx, \quad k = 1, 2, \ldots, n-1, \tag{3.4}$$

$$C_k = \int_{-1}^{1} (1-x^2)\gamma_k(x)dx, \quad k = 1, 2, \ldots, n-2. \tag{3.5}$$

Theorem 7. *The quadrature formula* (3.2) *can be expressed as*

$$\int_{-1}^{1} (1-x^2) f(x)dx = \sum_{k=1}^{n-1} \left(\frac{2\pi(1-x_k^2)^{3/2}}{\lambda_{2n}(x_k)} \right) f(x_k). \tag{3.6}$$

Remark 8. The quadrature formula (3.6) can be evaluated by finding the value of the integrals (3.3), (3.4) and (3.5). These integrals have singularities lying in the interval $[-1,1]$. The integrals are evaluated by performing suitable transformations and using the Cauchy residue theorem at the poles which lie in the interval.

To prove Theorem 7, we shall need the following lemmas below.

Lemma 9. *For D_k, $k = 1, 2, \ldots, n-1$, given by* (3.4), *we have*

$$D_k = \frac{1}{(1-x_k^2)(U_n'(x_k))^3} \int_{-1}^{1} \frac{(1-x^2)^2 U_n^2(x) U_n'(x)}{(x-x_k)} dx = 0.$$

Proof. D_k for $k = 1, 2, \ldots, n-1$, given by (3.4), can be represented as

$$D_k = \frac{1}{(1-x_k^2)(U_n'(x_k))^3} I_k, \tag{3.7}$$

where

$$I_k = \int_{-1}^{1} \frac{(1-x^2)^2 U_n^2(x) U_n'(x)}{(x-x_k)} dx$$

$$=\int_{-1}^{1}\frac{\sin^2\mu_{2n}(x)}{(x-x_k)}\left(\frac{x\sin\mu_{2n}(x)-\cos\mu_{2n}(x)\lambda_{2n}(x)\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)dx=I_{k1}-I_{k2},$$

where

$$I_{k1}=\int_{-1}^{1}\frac{x\sin^3\mu_{2n}(x)}{(x-x_k)\sqrt{1-x^2}}dx \tag{3.8}$$

and

$$I_{k2}=\int_{-1}^{1}\frac{\sin^2\mu_{2n}(x)\cos\mu_{2n}(x)\lambda_{2n}(x)}{(x-x_k)}dx. \tag{3.9}$$

Consider the transformation

$$x=\frac{1-y^2}{1+y^2} \tag{3.10}$$

which gives

$$dx=-\frac{4y}{(1+y^2)^2}dy, \tag{3.11}$$

$$\sqrt{1-x^2}=\frac{2y}{(1+y^2)}, \tag{3.12}$$

$$(x-x_k)=\frac{-2(y^2-y_k^2)}{(1+y^2)(1+y_k^2)}. \tag{3.13}$$

We know that

$$\sin\mu_{2n}\left(\frac{1-y^2}{1+y^2}\right)=\sin\phi_{2n}(y), \tag{3.14}$$

where $\sin\phi_{2n}(y)$ is Bernstein's sine fraction

$$\sin\phi_{2n}(y)=\frac{1}{2i}\Big(\chi_n(y)-\chi_n^{-1}(y)\Big), \tag{3.15}$$

where

$$\chi_n(y)=\prod_{j=0}^{2n-1}\frac{y-z_j}{y-\bar{z_j}}$$

and z_k are the roots of the equations $y^2+(1+a_k)(1-a_k)^{-1}=0$, $\mathcal{I}z_k>0$, $k=0,1,\ldots,2n-1$. Taking into account the assumptions on the parameters a_k, $k=0,1,\ldots,2n-1$, we have the following: 1) $z_0=i$, 2) if a_k and a_l are paired by a complex conjugation, then the corresponding numbers z_k and z_l are symmetric with respect to the imaginary axis. Besides, the function $\sin\phi_{2n}(y)$ has zeros at $\pm y_k,\ y_k=\sqrt{(1-x_k)/(1+x_k)}$, $k=1,2,\ldots,n-1$. Thus, by using transformation (3.10)–(3.13) and (3.14) in (3.8), we get

$$\begin{aligned}I_{k1}&=-\frac{1+y_k^2}{2}\int_{-\infty}^{\infty}\left(\frac{1-y^2}{1+y^2}\right)\frac{\sin^3\phi_{2n}(y)}{(y^2-y_k^2)}dy\\&=-\frac{1+y_k^2}{2}\lim_{z\to y_k,\mathcal{I}z_k>0}J_{k1}(z),\end{aligned}$$

where

$$J_{k1}(z)=\int_{-\infty}^{\infty}\left(\frac{1-y^2}{1+y^2}\right)\frac{\sin^3\phi_{2n}(y)}{(y^2-z^2)}dy.$$

From (3.14), we have

$$\sin^3\phi_{2n}(y)=-\frac{1}{8i}\left(\chi_n^3(y)-3\chi_n(y)+3\chi_n^{-1}(y)-\chi_n^{-3}(y)\right). \tag{3.16}$$

Thus
$$J_{k1}(z) = \frac{1}{8i}\Big(J_{k11}(z) - 3J_{k12}(z) + 3J_{k13}(z) - J_{k14}(z)\Big), \tag{3.17}$$
where
$$J_{k11}(z) = \int_{-\infty}^{\infty}\left(\frac{1-y^2}{1+y^2}\right)\frac{\chi_n^3(y)}{(y^2-z^2)}dy,$$
$$J_{k12}(z) = \int_{-\infty}^{\infty}\left(\frac{1-y^2}{1+y^2}\right)\frac{\chi_n^{-3}(y)}{(y^2-z^2)}dy,$$
$$J_{k13}(z) = \int_{-\infty}^{\infty}\left(\frac{1-y^2}{1+y^2}\right)\frac{\chi_n(y)}{(y^2-z^2)}dy$$
and
$$J_{k14}(z) = \int_{-\infty}^{\infty}\left(\frac{1-y^2}{1+y^2}\right)\frac{\chi_n^{-1}(y)}{(y^2-z^2)}dy.$$
Since $z_0 = i$, thus the integrand of $J_{k11}(z)$ has only a singular point $y = z$ in the upper half plane. Thus by the residue theorem, we have
$$\begin{aligned} J_{k11}(z) =& 2\pi i \lim_{y\to z}\left(\frac{1-y^2}{1+y^2}\right)\frac{\chi_n^3(y)}{(y+z)} \\ =& \left(\frac{1-z^2}{1+z^2}\right)\frac{\chi_n^3(z)}{z}\pi i. \end{aligned} \tag{3.18}$$
Similarly,
$$J_{k12}(z) = \left(\frac{1-z^2}{1+z^2}\right)\frac{\chi_n^{-3}(z)}{z}\pi i, \tag{3.19}$$
$$J_{k13}(z) = \left(\frac{1-z^2}{1+z^2}\right)\frac{\chi_n(z)}{z}\pi i \tag{3.20}$$
and
$$J_{k14}(z) = \left(\frac{1-z^2}{1+z^2}\right)\frac{\chi_n^{-1}(z)}{z}\pi i \tag{3.21}$$
Using (3.18), (3.19), (3.20) and (3.21) in (3.17), we get
$$\begin{aligned} J_{k1}(z) =& \frac{1}{8i}\Bigg(\left(\left(\frac{1-z^2}{1+z^2}\right)\frac{\chi_n^3(z)}{z}\pi i - \left(\frac{1-z^2}{1+z^2}\right)\frac{\chi_n^{-3}(z)}{z}\pi i\right. \\ & - 3\left(\frac{1-z^2}{1+z^2}\right)\frac{\chi_n(z)}{z}\pi i + 3\left(\frac{1-z^2}{1+z^2}\right)\frac{\chi_n^{-1}(z)}{z}\pi i\Bigg). \end{aligned}$$
Taking the limit as $\lim\limits_{z\to y_k, \mathcal{I}z_k>0}$ and using $\chi_n(y_k) = 1$, it follows that
$$I_{k1} = 0. \tag{3.22}$$
Now we evaluate I_{k2}, given by (3.9). Using (3.11) and (3.13) in (3.9), we get
$$I_{k2} \quad = \quad -(1+y_k^2)\int_{-\infty}^{\infty}\frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\sin^2\mu_{2n}\left(\frac{1-y^2}{1+y^2}\right)\cos\mu_{2n}\left(\frac{1-y^2}{1+y^2}\right)}{(1+y^2)(y^2-y_k^2)}\,dy.$$
We know that
$$\cos\mu_{2n}\left(\frac{1-y^2}{1+y^2}\right) = \cos\phi_{2n}(y),$$

where $\cos\phi_{2n}(y)$ is Bernstein's cosine fraction

$$\cos\phi_{2n}(y) = \frac{1}{2}\left(\chi_n(y) + \chi_n^{-1}(y)\right). \tag{3.23}$$

Thus by (3.14) and (3.23), we have

$$I_{k2} = -(1+y_k^2) \lim_{z\to y_k, \Im z_k>0} \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\sin^2\phi_{2n}(y)\cos\phi_{2n}(y)}{(1+y^2)(y^2-z^2)}\, dy.$$

By virtue of (3.14) and (3.23), we have

$$\sin^2\phi_{2n}(y)\cos\phi_{2n}(y) = -\frac{1}{8}\left(\chi_n^3(y) + \chi_n^{-3}(y) - \chi_n(y) - \chi_n^{-1}(y)\right). \tag{3.24}$$

Thus

$$I_{k2} = \frac{(1+y_k^2)}{8} \lim_{z\to y_k, \Im z_k>0} \Big(J_{k21}(z) + J_{k22}(z) - J_{k23}(z) - J_{k24}(z)\Big), \tag{3.25}$$

where

$$J_{k21}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(1+y^2)(y^2-z^2)} dy,$$

$$J_{k22}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^{-3}(y)}{(1+y^2)(y^2-z^2)} dy,$$

$$J_{k23}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n(y)}{(1+y^2)(y^2-z^2)} dy$$

and

$$J_{k24}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(1+y^2)(y^2-z^2)} dy.$$

Since $z_0 = i$, thus the integrand of $J_{k21}(z)$ has only a singular point $y = z$ in the upper half-plane. Hence by the residue theorem, we have

$$\begin{aligned} J_{k21}(z) =& 2\pi i \lim_{y\to z} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(1+y^2)(y+z)} \\ =& \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n^3(z)}{(1+z^2)}\pi i. \end{aligned} \tag{3.26}$$

Similarly,

$$J_{k22}(z) = \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n^{-3}(z)}{(1+z^2)}\pi i, \tag{3.27}$$

$$J_{k23}(z) = \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n(z)}{(1+z^2)}\pi i \tag{3.28}$$

and

$$J_{k24}(z) = \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n^{-1}(z)}{(1+z^2)}\pi i. \tag{3.29}$$

Putting the values of $J_{k21}(z), J_{k22}(z), J_{k23}(z)$ and $J_{k24}(z)$ from (3.26), (3.27), (3.28) and (3.29), respectively, in (3.25), we get

$$I_{k2} = \frac{(1+y_k^2)}{8} \lim_{z\to y_k, \Im z_k>0} \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)}{(1+z^2)} \left(\chi_n^3(z) + \chi_n^{-3}(z) - \chi_n(z) - \chi_n^{-1}(z)\right)\pi i.$$

Since $\chi_n(y_k) = 1$, thus

$$I_{k2} = 0. \tag{3.30}$$

Using (3.22) and (3.30) in (3.7), the Lemma follows. □

Lemma 10. *For $E_k, k = 1, 2, \ldots, n-1$ given by* (3.3)*, we have*

$$E_k = \frac{2\pi(1-x_k^2)^{3/2}}{\lambda_{2n}(x_k)}.$$

Proof. E_k for $k = 1, 2, \ldots, n-1$ given by (3.3), due to (2.8) and Lemma 2 can be represented as

$$\begin{aligned} E_k =& \frac{1}{(1-x_k^2)U_n'(x_k)} \int_{-1}^{1} (1-x^2)^2 U_n'(x)\ell_k^2(x)dx \\ =& \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)U_n'(x_k)} \int_{-1}^{1} \frac{(1-x^2)U_n'(x)\sin^2\mu_{2n}(x)}{(x-x_k)^2}dx. \end{aligned}$$

Since

$$U_n'(x) = \frac{-\cos\mu_{2n}(x)\lambda_{2n}(x)\sqrt{1-x^2} + x\sin\mu_{2n}(x)}{(1-x^2)^{3/2}},$$

we have

$$E_k(x) = \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)U_n'(x_k)} I_k, \tag{3.31}$$

where

$$I_k = I_{k1} - I_{k2} \tag{3.32}$$

with

$$I_{k1} = \int_{-1}^{1} \frac{x\sin^3\mu_{2n}(x)}{\sqrt{1-x^2}(x-x_k)^2}dx \tag{3.33}$$

and

$$I_{k2} = \int_{-1}^{1} \frac{\lambda_{2n}(x)\sin^2\mu_{2n}(x)\cos\mu_{2n}(x)}{(x-x_k)^2}dx. \tag{3.34}$$

Using transformation (3.10) and due to (3.11), (3.13) and (3.14), (3.33) can be transformed to

$$\begin{aligned} I_{k1} =& \frac{(1+y_k)^2}{2} \int_{-\infty}^{\infty} \frac{(1-y^2)\sin^3\phi_{2n}(y)}{(y^2-y_k^2)^2}\, dy \\ =& \frac{(1+y_k^2)^2}{2} \lim_{z\to y_k, \Im z_k>0} \int_{-\infty}^{\infty} \frac{(1-y^2)\sin^3\phi_{2n}(y)}{(y^2-z^2)^2}\, dy. \end{aligned} \tag{3.35}$$

Due to (3.16), (3.35) can be represented as

$$I_{k1} = -\frac{(1+y_k)^2}{16i} \lim_{z\to y_k, \Im z_k>0} \left(I_{k11}(z) - 3I_{k12}(z) + 3I_{k13}(z) - I_{k14}(z)\right), \tag{3.36}$$

where

$$I_{k11}(z) = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^3(y)}{(y^2-z^2)^2} dy,$$

$$I_{k12}(z) = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n(y)}{(y^2-z^2)^2} dy,$$

$$I_{k13}(z) = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^{-1}(y)}{(y^2-z^2)^2} dy$$

and

$$I_{k14}(z) = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^{-3}(y)}{(y^2-z^2)^2} dy.$$

Since $z_0 = i$, the integrand of $I_{k11}(z)$ has only a singular point $y = z$ in the upper half-plane. Thus by the residue theorem, we have

$$I_{k11}(z) = 2\pi i \lim_{y \to z} \frac{d}{dy} \frac{(1-y^2)\chi_n^3(y)}{(y+z)^2}$$

which implies

$$I_{k11}(z) = 2\pi i \left(\frac{z\{3(1-z^2)\chi_n^2(z)\chi_n'(z) - 2z\chi_n^3(z)\} - (1-z^2)\chi_n^3(z)}{4z^3} \right).$$

On simple calculations and using $\chi_n(y_k) = 1$, we get

$$\lim_{z \to y_k, \Im z_k > 0} I_{k11}(z) = \frac{\pi i}{2y_k^3} \left(3y_k(1-y_k^2) \sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} - (1+y_k^2) \right). \tag{3.37}$$

Similarly,

$$\lim_{z \to y_k, \Im z_k > 0} I_{k14}(z) = \frac{\pi i}{2y_k^3} \left(3y_k(1-y_k^2) \sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} - (1+y_k^2) \right). \tag{3.38}$$

Since the integrand of $I_{k12}(z)$ has only a singular point $y = z$ in the upper half-plane. Thus again, using the residue theorem, we have

$$I_{k12}(z) = 2\pi i \lim_{y \to z} \frac{d}{dy} \frac{(1-y^2)\chi_n(y)}{(y+z)^2}$$

which gives

$$I_{k12}(z) = 2\pi i \left(\frac{z\{(1-z^2)\chi_n'(z) - 2z\chi_n(z)\} - (1-z^2)\chi_n(z)}{4z^3} \right).$$

On simple calculations and using $\chi_n(y_k) = 1$, we get

$$\lim_{z \to y_k, \Im z_k > 0} I_{k12}(z) = \frac{\pi i}{2y_k^3} \left(y_k(1-y_k^2) \sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} - (1+y_k^2) \right). \tag{3.39}$$

Similarly,

$$\lim_{z \to y_k, \Im z_k > 0} I_{k13}(z) = \frac{\pi i}{2y_k^3} \left(y_k(1-y_k^2) \sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} - (1+y_k^2) \right) \tag{3.40}$$

Using (3.37), (3.38), (3.39), (3.40) in (3.36), we get

$$I_{k1} = 0. \tag{3.41}$$

Now we evaluate I_{k2} given by (3.34). Using the transformation (3.10) and due to (3.11), (3.13) and (3.14), (3.34) can be written as

$$I_{k2} = (1+y_k^2)^2 \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\sin^2\mu_{2n}\left(\frac{1-y^2}{1+y^2}\right)\cos\mu_{2n}\left(\frac{1-y^2}{1+y^2}\right)}{(y^2-y_k^2)^2}\, dy.$$

Thus by (3.15) and (3.23), we have

$$I_{k2} = (1+y_k^2)^2 \lim_{z\to y_k, \Im z_k>0} \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\sin^2\phi_{2n}(y)\cos\phi_{2n}(y)}{(y^2-z^2)^2}\, dy.$$

Using (3.24) in the above equation, we get

$$I_{k2} = -\frac{(1+y_k^2)^2}{8} \lim_{z\to y_k, \Im z_k>0} \left(I_{k21}(z)+I_{k22}(z)-I_{k23}(z)-I_{k24}(z)\right), \tag{3.42}$$

where

$$I_{k21}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(y^2-z^2)^2} dy,$$

$$I_{k22}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^{-3}(y)}{(y^2-z^2)^2} dy,$$

$$I_{k23}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n(y)}{(y^2-z^2)^2} dy$$

and

$$I_{k24}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^{-1}(y)}{(y^2-z^2)^2} dy.$$

Since $z_0 = i$, the integrand of $I_{k21}(z)$ has only a singular point $y = z$ in the upper half-plane. Thus by the residue theorem, we have

$$\begin{aligned} I_{k21}(z) =& 2\pi i \lim_{y\to z} \frac{d}{dy} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(y+z)^2} \\ =& \frac{\pi i}{2z}\left(3\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n^2(z)\chi_n'(z) + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_z \chi_n^3(z)\right). \end{aligned}$$

On simple calculations and using $\chi_n(y_k) = 1$, we get

$$\begin{aligned} \lim_{z\to y_k, \Im z_k>0} I_{k21}(z) =& \frac{\pi i}{2y_k}\left(3\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(y_k-z_j)(y_k-\bar{z}_j)}\right. \\ & \left. + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_{y_k}\right). \end{aligned} \tag{3.43}$$

Similarly,

$$\begin{aligned} \lim_{z\to y_k, \Im z_k>0} I_{k22}(z) =& \frac{\pi i}{2y_k}\left(3\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(y_k-z_j)(y_k-\bar{z}_j)}\right. \\ & \left. + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_{y_k}\right). \end{aligned} \tag{3.44}$$

The integrand of $I_{k23}(z)$ has only a singular point $y = z$ in the upper half-plane. Thus by the residue theorem, we have

$$\begin{aligned}I_{k23}(z) =&2\pi i \lim_{y\to z} \frac{d}{dy} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n(y)}{(y+z)^2}\\ =&\frac{\pi i}{2z}\left(\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n'(z) + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_z \chi_n(z)\right)\end{aligned}$$

which on simple calculations and using $\chi_n(y_k) = 1$, gives

$$\begin{aligned}\lim_{z\to y_k, \Im z_k>0} I_{k23}(z) =&\frac{\pi i}{2y_k}\left(\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)}\right.\\ &\left. + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_{y_k}\right) \qquad (3.45)\end{aligned}$$

$$\begin{aligned}\lim_{z\to y_k, \Im z_k>0} I_{k24}(z) =&\frac{\pi i}{2y_k}\left(\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)}\right.\\ &\left. + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_{y_k}\right). \qquad (3.46)\end{aligned}$$

Using (3.43), (3.44), (3.45) and (3.46) in (3.42), we get

$$I_{k2} = -\frac{(1+y_k^2)^2}{4y_k}\pi i\left(\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)}\right).$$

We know that

$$\sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} = -\frac{4\lambda_{2n}(x_k)}{i(1+y_k^2)},$$

hence

$$I_{k2} = \frac{2\pi\lambda_{2n}^2(x_k)}{\sqrt{1-x_k^2}}. \qquad (3.47)$$

Putting the values of I_{k1} and I_{k2} from (3.41) and (3.47), respectively, in (3.32), we get

$$I_k = -\frac{2\pi\lambda_{2n}^2(x_k)}{\sqrt{1-x_k^2}}.$$

Substituting this value of I_k in (3.31), the Lemma follows. □

Lemma 11. *For E_0 defined by* (3.3) *for $k = 0$, we have*

$$E_0 = 0.$$

Proof. For $k = 0$, (3.3) can be represented as

$$E_0 = \frac{1}{2U_n^2(1)U_n'(1)} I_0, \qquad (3.48)$$

where

$$\begin{aligned}I_0 =& \int_{-1}^{1} (1+x)(1-x^2)U_n^2(x)U_n'(x)dx\\ =& \int_{-1}^{1} (1+x)\sin^2\mu_{2n}(x)\left(\frac{x\sin\mu_{2n}(x) - \cos\mu_{2n}(x)\lambda_{2n}(x)\sqrt{1-x^2}}{(1-x^2)^{3/2}}\right)dx,\end{aligned}$$

$$I_0 = I_{01} - I_{02}, \tag{3.49}$$

where

$$I_{01} = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \left(\frac{x \sin^3 \mu_{2n}(x)}{(1-x^2)} \right) dx \tag{3.50}$$

and

$$I_{02} = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \left(\frac{\sin^2 \mu_{2n}(x) \cos \mu_{2n}(x) \lambda_{2n}(x)}{\sqrt{1-x^2}} \right) dx. \tag{3.51}$$

First, we evaluate I_{01}. Using transformations (3.10) and (3.11), (3.12) and (3.14) in (3.50), we have

$$I_{01} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{y^2} \left(\frac{1-y^2}{1+y^2} \right) \sin^3 \phi_{2n}(y) dy,$$

and due to (3.15), I_1 can be represented as

$$I_{01} = -\frac{1}{16i} \int_{-\infty}^{\infty} \Big(I_{011} - I_{012} - 3I_{013} + 3I_{014} \Big), \tag{3.52}$$

where

$$I_{011} = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^3(y)}{y^2(1+y^2)} dy, \tag{3.53}$$

$$I_{012} = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^{-3}(y)}{y^2(1+y^2)} dy,$$

$$I_{013} = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n(y)}{y^2(1+y^2)} dy \tag{3.54}$$

and

$$I_{014} = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^{-1}(y)}{y^2(1+y^2)} dy.$$

Since $z_0 = i$, the integrand of I_{011}, given by (3.53), has only a singular point $y = 0$ in the upper half-plane. Thus by the residue theorem, we have

$$I_{011} = 2\pi i \lim_{y \to 0} \frac{d}{dy} \frac{1-y^2}{1+y^2} \chi_n^3(y) = 6\pi i \sum_{j=0}^{2n-1} \left(\frac{1}{\bar{z_j}} - \frac{1}{z_j} \right) = -24\pi \lambda_{2n}(1). \tag{3.55}$$

Similarly,

$$I_{012} = -24\pi \lambda_{2n}(1). \tag{3.56}$$

Again, using the residue theorem for I_{013}, given by (3.54), we get

$$I_{013} = 2\pi i \lim_{y \to 0} \frac{d}{dy} \frac{1-y^2}{1+y^2} \chi_n(y) = 2\pi i \sum_{j=0}^{2n-1} \left(\frac{1}{\bar{z_j}} - \frac{1}{z_j} \right) = -8\pi \lambda_{2n}(1). \tag{3.57}$$

and, similarly,

$$I_{014} = -8\pi \lambda_{2n}(1). \tag{3.58}$$

Thus using (3.55), (3.56), (3.57) and (3.58) in (3.52), we get

$$I_{01} = 0. \tag{3.59}$$

Now, for I_{02}, given by (3.51), due to (3.10)–(3.12) and (3.14), on a simple calculation, we have

$$I_{02} = \int_{-\infty}^{\infty} \frac{\sin^2\phi_{2n}(y)\cos\phi_{2n}(y)\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)}{y(1+y^2)}dy$$

which, due to (3.24), can be represented as

$$I_{02} = -\frac{1}{8}\left(I_{021} + I_{022} - I_{023} - I_{024}\right), \tag{3.60}$$

where

$$I_{021} = \int_{-\infty}^{\infty} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{y(1+y^2)}dy,$$

$$I_{022} = \int_{-\infty}^{\infty} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^{-3}(y)}{y(1+y^2)}dy,$$

$$I_{023} = \int_{-\infty}^{\infty} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n(y)}{y(1+y^2)}dy$$

and

$$I_{024} = \int_{-\infty}^{\infty} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^{-1}(y)}{y(1+y^2)}dy.$$

Now, since for I_{021}, the only singularity on the upper half plane is $y=0$, hence by residue theorem, we have

$$I_{021} = 2\pi i \lim_{y\to 0} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(1+y^2)} = 2\pi i\lambda_{2n}(1). \tag{3.61}$$

Similarly,

$$I_{022} = 2\pi i\lambda_{2n}(1). \tag{3.62}$$

Again, for I_{023}, by the residue theorem, we have

$$I_{023} = 2\pi i \lim_{y\to 0} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n(y)}{(1+y^2)} = 2\pi i\lambda_{2n}(1) \tag{3.63}$$

and

$$I_{024} = 2\pi i\lambda_{2n}(1). \tag{3.64}$$

Substituting the values of (3.61), (3.62), (3.63) and (3.64) in (3.60), we get

$$I_{02} = 0. \tag{3.65}$$

Putting the value of I_{01} and I_{02} from (3.59) and (3.65), respectively, in (3.49), we get $I_0 = 0$ which, due to (3.48), implies

$$E_0 = 0.$$

□

Lemma 12. *For E_n defined by* (3.3) *for $k = n$, we have*

$$E_n = 0.$$

Proof. For $k = n$, (3.3) can be represented as

$$E_n(x) = \frac{1}{2U_n^2(-1)U_n'(-1)} I_n,$$

where

$$\begin{aligned} I_n &= \int_{-1}^{1} (1-x)(1-x^2)U_n^2(x)U_n'(x)dx \\ &= \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left(\frac{x\sin^3\mu_{2n}(x)}{(1-x^2)} - \frac{\sin^2\mu_{2n}(x)\cos\mu_{2n}(x)\lambda_{2n}(x)}{\sqrt{1-x^2}} \right) dx = I_{n1} - I_{n2}. \end{aligned}$$

Following the same steps as in Lemma 11, we get $I_{n1} = I_{n2} = 0$ which implies that $I_n = 0$ and hence the Lemma follows. □

Lemma 13. *For C_k given by* (3.5)*, we have $C_k = 0$, $k = 1, 2, \ldots, n-2$.*

Proof. C_k given by (3.5), can be represented as

$$C_k = \frac{1}{(1-t_k^2)U_n^2(t_k)U_n''(t_k)} I_k,$$

where

$$I_k = \int_{-1}^{1} \frac{(1-x^2)^2 U_n^2(x)U_n'(x)}{(x-t_k)} dx,$$

which reduces to

$$I_k = \int_{-1}^{1} \frac{1}{(x-t_k)} \left(\frac{x\sin^3\mu_{2n}(x)}{\sqrt{1-x^2}} - \sin^2\mu_{2n}(x)\cos\mu_{2n}(x)\lambda_{2n}(x) \right) dx = I_{k1} - I_{k2},$$

where

$$I_{k1} = \int_{-1}^{1} \frac{x\sin^3\mu_{2n}(x)}{(x-t_k)\sqrt{1-x^2}} dx$$

and

$$I_{k2} = \int_{-1}^{1} \frac{\sin^2\mu_{2n}(x)\cos\mu_{2n}(x)\lambda_{2n}(x)}{(x-t_k)} dx.$$

Proceeding as in the above lemmas, it follows that $I_{k1} = I_{k2} = 0$ which implies that $I_k = 0$, from which the Lemma follows. □

From Lemma 9–13 and (3.2), Theorem 7 follows.

4. Conclusion

Here, a quadrature formula corresponding to the Pál type $(0, 1; 0)$-interpolation in rational spaces has been obtained. This study may further be extended to the case of lacunary interpolation in rational spaces.

References

1. P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities.* Graduate Texts in Mathematics, 161. Springer-Verlag, New York, 1995.
2. C. Cesarano, Integral representations and new generating functions of Chebyshev polynomials. *Hacet. J. Math. Stat.* **44** (2015), no. 3, 535–546.
3. C. Cesarano, Multi-dimensional Chebyshev polynomials: a non-conventional approach. *Commun. Appl. Ind. Math.* **10** (2019), no. 1, 1–19.
4. C. Cesarano, C. Fornaro, A note on two-variable Chebyshev polynomials. *Georgian Math. J.* **24** (2017), no. 3, 339–349.
5. C. Cesarano, S. Pinelas, P. E. Ricci, The third and fourth kind pseudo-Chebyshev polynomials of half-integer degree. *Symmetry* no. 2, **11** (2019), 1–11.
6. C. Cesarano, P. E. Ricci, Orthogonality Properties of the Pseudo-Chebyshev Functions (Variations on a Chebyshev's Theme). *Mathematics* **7** (2019), no. 2, 1–11.
7. S. Kumar, N. Mathur, V. N. Mishra, P. Mathur, Radau Quadrature for an Almost Quasi-Hermite-Fejer-Type Interpolation in Rational Spaces. *Int. Jour. of Anal. and Appl.* **19** (2021), no. 2, 180–192.
8. A. L. Lukashov, Inequalities for the derivatives of rational functions on several intervals. (Russian) *translated from Izv. Ross. Akad. Nauk Ser. Mat.* **68** (2004), no. 3, 115–138 *Izv. Math.* **68** (2004), no. 3, 543–565.
9. A. A. Markov, *Izbrannye Trudy.* Teoriya čisel. Teoriya veroyatnosteĭ. (Russian) [Selected works. Theory of numbers. Theory of probability.] Izdat. Akad. Nauk SSSR, Leningrad, 1951.
10. G. Min, Lobatto-type quadrature formula in rational spaces. *J. Comput. Appl. Math.* **94** (1998), no. 1, 1–12.
11. E. A. Rouba, *Interpoljacija i Rjady Furie v Ratsionalnoj Approksimatsii.* GrSU, Grodno, 2001.
12. Y. Rouba, K. Smatrytski, Y. Dirvuk, Rational quasi-Hermite-Fejér-type interpolation and Lobatto-type quadrature formula with Chebyshev-Markov nodes. *Jaen J. Approx.* **7** (2015), no. 2, 291–308.
13. E. A. Rovba, Interpolation rational operators of Fejér and de la Vallée-Poussin type. (Russian) *Mat. Zametki* **53** (1993), no. 2, 114–121 *translated from Math. Notes.* **53** (1993), no.2, 195–200.
14. Y. A. Rovba, K. A. Smatrytski, Rational interpolation in the zeros of Chebyshev-Markov sine-fractions. (Russian) *Dokl. Nats. Akad. Nauk Belarusi* **52** (2008), no. 5, 11–15, 124.
15. V. N. Rusak, Interpolation by rational functions with fixed poles. (Russian) *Dokl. Akad. Nauk BSSR* **6** (1962), 548–550.
16. V. N. Rusak, On approximations by rational fractions. (Russian) *Dokl. Akad. Nauk BSSR* **8** (1964), 432–435.
17. V. N. Rusak, *Rational Functions as Approximation Apparatus.* (Russian) Beloruss. Gos. Univ. Minsk, 1979.
18. A. H. Turecki, *Teorija Interpolirovanija v Zadachakh.* Izdat Vyssh. Skola, Minsk, 1968.
19. J. Van Deun, Electrostatics and ghost poles in near best fixed pole rational interpolation. *Electron. Trans. Numer. Anal.* **26** (2007), 439–452.

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FRACTAL STRUCTURES FROM THE BAND MATRICES FOR MATRIX ALGORITHMS

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Abstract. The aim of the present paper is to construct a set of high order strong matrices for a key-exchange matrix algorithm on an open channel and to create a high-speed one-way matrix function. Fractal structures are synthesized from band matrices. Square matrices are considered over a Galois field of GF (2). Each initial n-th order square matrix is primitive (the degree value is equal to $2^n - 1$) or its degree value is a Mersenne number $2^j - 1$, when $j < n$ (except $n = 18$).

1. Introduction

In cryptographic algorithms, the main task is the question of reliability of the algorithm. In matrix algorithms, the process of encryption and decryption is implemented by matrices, and for the algorithm to be reliable it is necessary to create powerful high-order matrix sets.

Each cryptographic system uses its own procedure, types of keys, methods of their distribution and coding algorithms. The essence of the asymmetric cryptography consists in a specific character of a one-way function. The one-way function is a $y = f(x)$ function; its value can be obtain by computer calculations in case x is known, but it is impossible to get the value of x argument by means of the function $f(x)$ and computer calculations at a real time. This fact is clearly illustrated by an example of the Diffie–Hellman [2] one-way function $a^x = y(\mathrm{mod}\, p)$.

2. The Matrix Function

To implement a one-way matrix function, we have the $n \times n$ matrix A. For simplicity of the statement, the matrices are considered over the GF (2) field. Matrix A presents a secret parameter selected randomly from a group of high powered $\hat{A}$; thus, $A \in \hat{A}$, $v \in V_n$, where V_n is a vector space over GF (2) (v is an open parameter). Then the one-way matrix function looks as

$$vA = u, \tag{2.1}$$

where both $v \in V_n$ and u are open parameters.

It should be mentioned that if for Diffie–Hellman's algorithm the one-way function

$$a^x = y \bmod p \tag{2.2}$$

is based on the problem of a discrete logarithm, then the problem for that function appears to be the recursion inside the matrix [4, 7]. In a cryptographic algorithm, the use is made of a one-way function for solving the authentication and verification tasks for a certain period of time. We also use this function for solving the problem of stability of our matrix one-way function for a certain period of time. Towards this end, using the exponential one-way function, the key exchange takes place through the open channel. The result of this key exchange is the secret parameter $k = v$. At the same time period, the key exchange or other operations are performed with our algorithm. In this case, parameters v and A in (2.1) are secret and only parameter u is open [3].

In authors' opinion, after reading the next section there should be no doubt both about the high-speed of the matrix one-way function and about that of the key exchange algorithm on an open channel.

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The function (2.1) fundamentally differs from the function (2.2) by the fact that for the function (2.1) is used the operation of multiplication, whereas the function (2.2) is exponential.

3. The Matrix Algorithm

The Matrix Algorithm about Key-Exchange on an Open Channel is Implemented in the Following Way:

• Mariami chooses (randomly) an $n \times n$ matrix $A_1 \in \hat{A}$ and sends to George the following vector:

$$u_1 = vA_1;$$

• George chooses (randomly) an $n \times n$ matrix $A_2 \in \hat{A}$ and sends to Mariami the vector

$$u_2 = vA_2,$$

where n is a size of vector u (open), A_1 and A_2 are (secret) matrix keys.

• Mariami computes

$$k_1 = u_2A_1;$$

• George computes

$$k_2 = u_1A_2,$$

where, k_1 and k_2 are secret keys, $k_1 = k_2 = k$, because $k = vA_1A_2 = vA_2A_1$.

The one-way matrix function and a new matrix algorithm for the corresponding open channel key exchange considered in this paper have been first obtained and studied by the first author.

As is shown above, for the implementation of the key-exchange algorithm we need high power multiple $n \times n$ matrices which are, at the same time, commutative. A number commutation in Diffie–Hellman's algorithm is implemented naturally, in accordance with the construction of the commutative multiplicity of $\hat{A}$ for each value of dimension n, while for our algorithm this task is difficult.

In the given work, we present an effective and constructive solution. The characteristics of effective and constructive methods for construction of matrices are included in the following statements:

• For each $n > 1$ dimension, the initial $n \times n$ matrix should generate either a maximum number of matrices $(2^n - 1)$, or this number should be the number of Mersenne, meaning $2^j - 1$, where $j < n$.

• The method of synthesis of any $n \times n$ matrix of any dimension, should be the same (where n is probably implementable maximal dimension of the initial matrices). Hence the technology of the construction for initial matrices should be implementable and similar to any given dimension of n.

For simplicity, the n-th order square matrices and other structures are considered in the Galois GF(2) field. Obviously, of great importance is generation of a high power matrix set for the functioning of a new key exchange function. The synthesis of such matrix sets and their structural study attract particular attention [5, 6].

The new algorithm is an original cryptographic approach, especially, when its quickness is taken into account. However, at the same time, this algorithm needs analyzing in regard to its cracking and generating a required set of high order matrices. The study, analysis and software implementation of such issues is also the main goal.

4. Software Implementation

The object of our study is a matrix, finding such a structure, whose existence makes the matrix able to generate a multiplication cyclic group of matrices with a maximum value or a value equal to the Mersenne prime degree.

In order to find out such structures it is necessary to verify the matrices of different orders regarding whether this scheme gives such a multiplication group of matrices that is generated by any matrix constructed by this structure and its value of degree is maximal, i.e., whether this matrix is primitive (a matrix is primitive in case it generates a group with a maximum value of the degree). For this purpose a method for natural increase of matrix order has been introduced, i.e., a method for natural increase of the n order.

Several types of nondegenerate initial matrixes are experimentally tested. As a result, a general structure is obtained, the matrices generating multiplication groups, sometimes with maximum degree value and sometimes with a degree value equal to Mersenne prime are alo abtained. Only in a single case, (except for $n = 18$) [8–10], the matrix degree value is not a Mersenne prime and it is a subject to an individual structural study. The paper also deals with new original fractal matrix structures, banded matrices, etc.

The original matrix algorithm described in the paper is in some degree a similar model to the Diffie–Hellman open channel key exchange algorithm. When the Diffie–Hellman algorithm stability depends on the highest values of p simple number (i.e., stability depending on a real scale of time), the one-way matrix function stability also depends on the high value of the A set.

The research is carried out for the matrices that are free from the internal recursion. A high order matrix set consisting of primitive matrices is constructed (see Figures 1, 2, 3).

Matrix of the first Fractal structure:

$$n = 3,\ A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \quad n = 4, \quad A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}; \quad n = 5, \quad A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Matrix of the second Fractal structure:

$$n = 3, \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad n = 4, \quad A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad n = 5, \quad A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

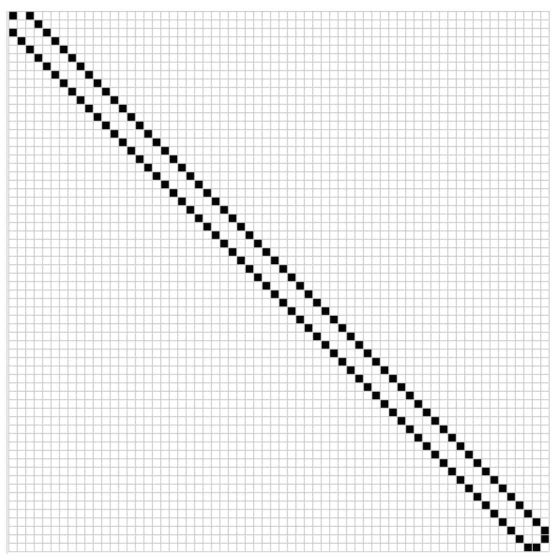

Figure 1. The fractal structure from the band matrix.

Matrix of the third Fractal structure:

$$n = 3, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}; \quad n = 4, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}; \quad n = 5, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

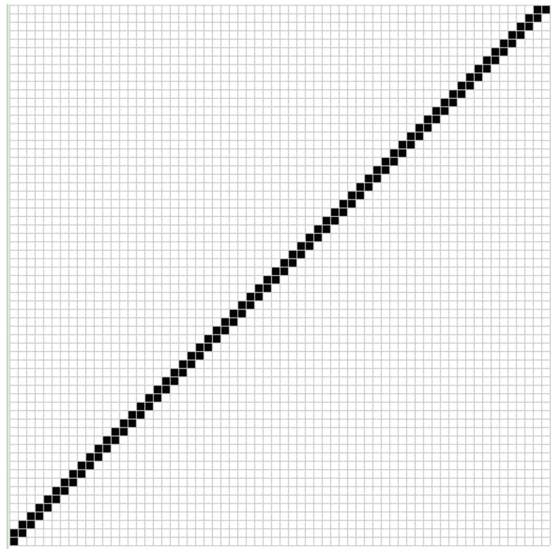

FIGURE 2. The fractal structure from the band matrix.

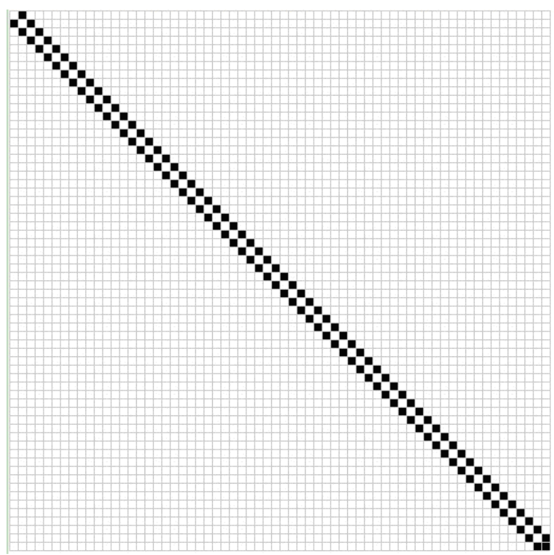

FIGURE 3. The fractal structure from the band matrix.

By using software, the orders of e were calculated for the initial normal $n \times n$ matrix structures and the results are shown in the table below (Table 1).

1. Each initial n order square matrix is primitive (the degree value is equal to $2^n - 1$) or its degree value is a Mersenne prime $2^j - 1$, when $j < n$ (except for $n = 18$).

2. The corresponding matrices of the pairs (3, 4), (7, 8), (15, 16), (31, 32), (63, 64), (127, 128), (255, 256) and (511, 512) of values $(n,\ n+1)$ are described by the following formulae:

$$A_{2^r-1}^{2^{r+1}-1} = E_{2^r-1}; \quad A_{2^r}^{2^{r+1}-1} = E_{2^r}, \quad \text{where } r \geq 2.$$

3. It is also noteworthy that nowadays in cryptographic algorithms the 2^{89} probable selection variants are very difficult even for the latest computers. We have calculated all matrices including the 1000×1000 size matrices. Each initial n order square matrix is primitive and its degree value is equal to $2^n - 1$ (Table 2).

TABLE 1. The results for calculated orders of e for the initial normal $n \times n$ matrices.

n	e	n	e	n	e	n	e	n	e	n	e	n	e	n	e
1	2^{1}-1	17	2^{12}-1	33	2^{33}-1	49	2^{15}-1	65	2^{65}-1	81	2^{81}-1	97	2^{12}-1	113	2^{113}-1
2	2^{2}-1	18	87381	34	2^{22}-1	50	2^{50}-1	66	2^{18}-1	82	2^{20}-1	98	2^{98}-1	114	2^{38}-1
3	2^{3}-1	19	2^{12}-1	35	2^{35}-1	51	2^{51}-1	67	2^{36}-1	83	2^{83}-1	99	2^{99}-1	115	2^{30}-1
4	2^{3}-1	20	2^{10}-1	36	2^{9}-1	52	2^{12}-1	68	2^{34}-1	84	2^{78}-1	100	2^{33}-1	116	2^{29}-1
5	2^{5}-1	21	2^{7}-1	37	2^{20}-1	53	2^{53}-1	69	2^{69}-1	85	2^{9}-1	101	2^{84}-1	117	2^{92}-1
6	2^{6}-1	22	2^{12}-1	38	2^{30}-1	54	2^{18}-1	70	2^{46}-1	86	2^{86}-1	102	2^{10}-1	118	2^{78}-1
7	2^{4}-1	23	2^{23}-1	39	2^{39}-1	55	2^{36}-1	71	2^{60}-1	87	2^{81}-1	103	2^{66}-1	119	2^{119}-1
8	2^{4}-1	24	2^{21}-1	40	2^{27}-1	56	2^{14}-1	72	2^{14}-1	88	2^{29}-1	104	2^{45}-1	120	2^{12}-1
9	2^{9}-1	25	2^{8}-1	41	2^{41}-1	57	2^{44}-1	73	2^{42}-1	89	2^{89}-1	105	2^{105}-1	121	2^{81}-1
10	2^{6}-1	26	2^{26}-1	42	2^{8}-1	58	2^{12}-1	74	2^{74}-1	90	2^{90}-1	106	2^{70}-1	122	2^{84}-1
11	2^{11}-1	27	2^{20}-1	43	2^{28}-1	59	2^{24}-1	75	2^{15}-1	91	2^{60}-1	107	2^{28}-1	123	2^{36}-1
12	2^{10}-1	28	2^{9}-1	44	2^{11}-1	60	2^{55}-1	76	2^{24}-1	92	2^{18}-1	108	2^{15}-1	124	2^{41}-1
13	2^{9}-1	29	2^{29}-1	45	2^{12}-1	61	2^{20}-1	77	2^{20}-1	93	2^{40}-1	109	2^{18}-1	125	2^{25}-1
14	2^{14}-1	30	2^{30}-1	46	2^{10}-1	62	2^{50}-1	78	2^{26}-1	94	2^{18}-1	110	2^{24}-1	126	2^{110}-1
15	2^{5}-1	31	2^{6}-1	47	2^{36}-1	63	2^{7}-1	79	2^{52}-1	95	2^{95}-1	111	2^{37}-1	127	2^{8}-1
16	2^{5}-1	32	2^{6}-1	48	2^{24}-1	64	2^{7}-1	80	2^{33}-1	96	2^{48}-1	112	2^{60}-1	128	2^{8}-1

TABLE 2. Higher order matrices.

119	278	438	639	809
131	281	441	641	810
134	293	443	645	818
135	299	453	650	831
146	303	470	651	833
155	306	473	653	834
158	309	483	659	846
173	323	491	683	866
174	326	495	686	870
179	329	509	690	873
183	330	515	713	879
186	338	519	719	891
189	350	530	723	893
191	354	531	725	911
194	359	543	726	923
209	371	545	741	930
210	377	554	743	933
221	378	558	746	935
230	386	561	749	938
231	393	575	755	939
233	398	585	761	950
239	410	593	765	953
243	411	606	771	965
245	413	611	774	974
251	414	614	779	975
254	419	615	783	986
261	426	618	785	989
270	429	629	791	993
273	431	638	803	998

4. It is noteworthy that these results completely coincide with the results of Ukrainian scientist, Professor Anatoly Beletsky. Although, as is well known, the initial matrices have completely different structures, [1] i.e., the structures that are derived from the generalized Gray Codes.

During the last decades overwhelming necessity has arisen for modern scientific-theoretical and technological studies in security (reliability) and high-speed performance and their practical use in asymmetric cryptography systems. The paper considers a new trend in asymmetric cryptography, namely, a single-sided matrix function and the issues of generation of matrix sets, necessary for its fulfilment, and also the problems of new fractal matrix structure synthesis. The above circumstances provide actuality of the issue and its immense theoretical and practical value.

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References

1. A. Belesky, D. Stetsenko, The order of the abelian cyclic group generated by the generalized transformations of Gray. *Electronics and control systems* **23** (2010), no. 1, 5–11.
2. W. Diffie, M. E. Hellman, New directions in cryptography. *IEEE Trans. Inform. Theory* **IT-22** (1976), no. 6, 644–654.
3. R. Megrelishvili, Analysis of the Matrix one-Way Function and Two Variants of Its Implementation. *International Journal of Multidisciplinary Research and Advances in Engineering (IJMRAE)* **5** (2013), no. 4, 99–105.
4. R. Megrelishvili, Tropical Cryptography and Analysis of Implementation of New Matrix One-Way Function. (Russia) Im: *Proceedings of the 2014 International Conference on Mathematical Models and Methods in Appled Sciences (MMAS '14). Saint Peterburg*, pp. 273–275, 2014.
5. R. Megrelishvili, Analysis of the Matrix One-Way Function and Two Variants of Its Implementation. *Computer Sciences and Telecommunication Reviewed Electronic Scientific Journal* **43** (2014), no. 3, 37–41.
6. R. Megrelishvili, Two New Versions of Numbers Fast Multiplication and Tropical Cryptography. *The Communications on Applied Electronics (CAE)* Foundation of Computer Science FCS, New York, USA, **7** (2017), no. 8, 12–15.
7. R. Megrelishvili, M. Chelidze, K. Chelidze, On the construction of secret and public-key cryptosystems. *Appl. Math. Inform. Mech.* **11** (2006), no. 2, 29–36, 91.
8. R. Megrelishvili, S. Shengelia, On the original one-way matrix function and the implementation of the key exchange protocol on open channel. *Appl. Math. Inform. Mech.* **17** (2012), no. 2, 20–25, 56–57.
9. R. Megrelishvili, S. Shengelia, Matrix function and its realization problems. In: *IV International Conference of the Georgian Mathematical Union, Batumi*, pp. 138–139, 2013.
10. R. Megrelishvili, S. Shengelia, Open Channel Key Exchange Algorithm and Fractal Structures Research. In: *Tbilisi International Conference on Computer Science and Applied Mathematics*, pp. 93–97, 2015.

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A COMPARISON BETWEEN BERNOULLI-COLLOCATION METHOD AND HERMITE–GALERKIN METHOD FOR SOLVING TWO-DIMENSIONAL MIXED VOLTERRA–FREDHOLM SINGULAR INTEGRAL EQUATIONS

DOAA SHOKRY MOHAMED AND DINA MOHAMED ABDESSAMI

Abstract. In this paper, a numerical solution of two-dimensional singular integral equations is proposed. For this, two operative methods are demonstrated, Bernoulli polynomials with collocation method and Hermite polynomials through Galerkin method which is a useful technique in two-dimensional integral equations. Various numerical examples are presented to illustrate the efficiency of these two methods. Maple 17 program will be used to solve the system numerically.

1. Introduction

In the last years, there was a significant importance of multidimensional singular integral equations (MSIE). Many problems in physical, biological and applied mathematics fields reduce to a singular integral equation. Such as hydrodynamics, population genetics, elasticity, and others. In 1928, F. G. Tricomi [20] was the first who proposed an important study concerning (MSIE). He considered double singular integrals. Recently, many researchers had studied the numerical solutions of singular integral equations in several formulas. For instance, [2, 7] involved solutions of the nonlinear singular integral equations, whereas in [6,15] with Hilbert kernel. J. Obaiyst et al. [11] deal with hypersingular integral equations. E. Hashim [5], V. A. Zisis and E. G. Ladopoulus [21] presented solutions for the singularity of linear integral equations. S. Banerjea et al. [3] worked on a weak singular kernel with a water wave problem as an application. There are different methods for solving two-dimensional integral equations (see [4, 8–10] and others). M. Rahman [14] discussed the solution of linear integral equations in one-dimension using the Hermite–Galerkin method. In our paper, we work on the solution of two-dimensional singular mixed Volterra–Fredholm integral equations using the Bernoulli-collocation method and Hermite–Galerkin method. One can observe that the Hermite–Galerkin method is a novel technique in the two-dimensional integral equations.

The aim of this paper is to convert the singular integral equation to a non-singular form by repealing the singularity and then converting it into a system of algebraic equations based on orthogonal polynomials.

The next sections are arranged as follows; some definitions and properties of Bernoulli and Hermite polynomials are introduced in Section 2. The description of the collocation and Galerkin methods with two-dimensional singular mixed Volterra–Fredholm integral equations are explained in Section 3. Section 4 includes some numerical examples that illustrate the above-mentioned methods. Finally, Section 5 gives the conclusions.

We list here some of the most important advantages of the proposed methods.

- The proposed methods are easy to implement, and it is a powerful mathematical tool to obtain the numerical solution of various kind of problems with little additional works.
- By using these methods, the problem under consideration is transformed into a system of algebraic equations which can be solved via a suitable numerical method.

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Key words and phrases. Mixed Volterra–Fredholm integral equations; Two-dimensional singular integral equations; Bernoulli polynomials; Collocation method; Hermite polynomials; Galerkin method.

2. Some Definitions and Properties

2.1. **Bernoulli Polynomials.** In many topics of mathematics, Bernoulli polynomials have a vital role, e.g., in the theory of numbers [13] and in complex differential equations [19].

The Bernoulli polynomials are expressed by the formula [19]

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k, \tag{2.1}$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ and $B_n(x)$ is the Bernoulli polynomial of n^{th} degree.

In a special case, if $x = 0$ in (2.1), then $B_n(0) = B_n$ are called Bernoulli numbers, and $B_0 = 1$.

The Bernoulli numbers can be calculated as follows:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (1-n)\, x^n, \quad n = 0, 1, 2, \dots .$$

The first few Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \quad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,$$
$$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 + \frac{1}{2}x^2 - \frac{1}{42}.$$

2.2. **Hermite Polynomials [12].** The differential equation $y'' - 2xy' + 2\lambda y = 0$ has polynomial solutions called Hermite polynomials which were introduced for the first time by Pierre–Simon Laplace in 1810. Charles Hermite defined the multidimensional polynomials. Hermite polynomials are a mutually orthogonal function with weight functions, which can be determined easily by using the Rodrigues formula

$$H_n(x) = (-1)^n\, e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}), \quad n = 0, 1, 2, \dots .$$

The first few Hermite polynomials are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12,$$
$$H_5(x) = 32x^5 - 160x^3 + 120x, \quad H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120.$$

Hermite polynomials have the generating function

$$w(x,t) = e^{2xt - x^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n, \quad |t| < \infty.$$

3. The Description of Methods

We are concerned with solving the two-dimensional singular mixed Volterra–Fredholm integral equations which have the form

$$u(x,t) = f(x,t) + \int_c^t \int_a^b (t-z)^{\alpha-1}\, \phi(x,y)\, u(y,z)\,dy\,dz, \tag{3.1}$$

where $0 < \alpha < 1$ and $(x,t) \in [a,b] \times [c,d]$, where $u(x,t)$ is an unknown function, $f(x,t)$ is a given function defined on $[a,b] \times [c,d]$ and $k(x,t,y,z) = (t-z)^{\alpha-1}\, \phi(x,y)$ is the singular kernel satisfying the discontinuity condition in the domain $([a,b] \times [c,d])^2$.

3.1. **Bernoulli-Collocation method [16,19].** This method is based on approximating the unknown function $u(x,t)$ in (3.1) on the form

$$u(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\ B_i(x)\ B_j(t), \tag{3.2}$$

where $B_i(x)$, $B_j(t)$ are Bernoulli polynomials and a_{ij} are unknown coefficients to be determined in order to obtain the approximate solution, in the following steps:

Curtailing the infinite series (3.2), we get

$$\tilde{u}(x,t) \simeq \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\ B_i(x)\ B_j(t), \tag{3.3}$$

Substituting from (3.3) into (3.1) we get

$$\sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\ [B_i(x)\ B_j(t) - \int_c^t\int_a^b (t-z)^{\alpha-1}\ \phi(x,y)\ B_i(y)\ B_j(z)dydz] = f(x,t). \tag{3.4}$$

Using the collocation points x_p, t_q of Bernoulli polynomials given by

$$x_p = a + \frac{b-a}{N}p\ , \quad t_q = c + \frac{d-c}{N}q, \tag{3.5}$$

for $p,q = 0,1,2,\ldots,N$ and $x_p \in [a,b]$, $t_q \in [c,d]$,

equation (3.4) would be written as

$$\sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\ [B_i(x_p)\ B_j(t_q) - \int_c^{t_q}\int_a^b (t_q-z)^{\alpha-1}\ \phi(x_p,y)\ B_i(y)\ B_j(z)dydz] = f(x_p,t_q), 0<\alpha<1. \tag{3.6}$$

Substituting collocation points (3.5) into (3.6), we get a system of algebraic equations which contains $(N+1)^2$ of a_{ij} unknown coefficients. Solving this system to obtain a_{ij} values, we get an approximate solution $\tilde{u}(x,t)$.

The accuracy of this method is given by the formula (see [18])

$$\|u(x,t) - \tilde{u}(x,t)\| \leq \gamma\ \lambda\ C\ N(2\pi)^{-N},$$

where

$$\lambda = \max_{0\leq x\leq b, c\leq t\leq d}\ |\ k(x,t,y,z)\ |,$$

λ is a positive constant, independent of N, and a bound for the partial derivative of $u(x,t), \gamma$ is a positive constant and C is the coefficient matrix.

3.2. **The Hermite–Galerkin method.** Assume that $u(x,t)$ is an approximate solution of (3.1). We use Hermite polynomials through the Galerkin method which has the form

$$\widetilde{u}(x,t) \simeq \sum_{i=0}^{N}\sum_{j=0}^{N} c_{i,j}\ H_i(x)\ H_j(t), \tag{3.7}$$

where $H_i(x)$, $H_j(t)$ are Hermite polynomials and $c_{i,j}$ are unknown Hermite coefficients to be determined in the following steps.

Substituting from (3.7) into (3.1), we get

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{i,j}\ [H_i(x)\ H_j(t) - \int_c^t\int_a^b (t-z)^{\alpha-1}\ \phi(x,y)\ H_i(y)\ H_j(z)dydz] = f(x,t), \tag{3.8}$$

where $0<\alpha<1$, $(x,t) \in [a,b]\times[c,d]$.

Multiplying both sides in equation (3.8) by $H_p(x)\ H_q(t)$, then integrate with respect to x and y from a to b and from c to d, respectively, such that p and $q = 0, 1, 2, \dots, N$. Hence, equation (3.8) becomes of the form

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{i,j} \int_c^d \int_a^b G_{ij}(x,t)\ H_p(x)\ H_q(t)\ dxdt = F_{pq}, \tag{3.9}$$

where

$$F_{pq} = \int_c^d \int_a^b f(x,t)\ H_p(x)\ H_q(t)\ dxdt,$$

$$G_{ij}(x,t) = H_i(x)\ H_j(t) - \int_c^t \int_a^b (t-z)^{\alpha-1}\ \phi(x,y)\ H_i(y)\ H_j(z) dydz.$$

Substituting $p, q = 0, 1, \dots, N$ into (3.9), we get a system of $(N+1)^2$ non-singular algebraic equations. By solving this system, we get Hermite coefficients $c_{i,j}$.

The accuracy of this method depends on reducing the error using low-degree interpolation polynomials without increasing time of calculation (see [17]). The error function is expressed by the formula

$$E(x,t) = |u(x,t) - \widetilde{u}(x,t|,$$

for $x_l \in [a,b]$ and $t_m \in [c,d]$, the error function can be written as follows:

$$E(x_l,t_m) = |u(x_l,t_m) - \widetilde{u}(x_l,t_m| \cong 0,$$

or $E(x_l,t_m) \leq 10^{-k_i}$, (k_i) is a positive integer,
if $\max(10^{-k_i}) = 10^{-k}$, k is a positive integer.

4. Numerical Examples

In this section some numerical examples of two-dimensional singular mixed Volterra–Fredholm integral equations are presented to illustrate the previous methods.

Example 1. Consider the singular VFIE [1]

$$u(x,t) = x^2t^2 - \frac{25}{156}t^{\frac{13}{5}} + \int_0^t \int_0^1 y^2(t-z)^{-0.4}\ u(y,z)dydz, \tag{4.1}$$

where $x, t \in [0,1]$ with the exact solution $u(x,t) = x^2t^2$.

In Table 1, we give the absolute error of equation (4.1) by the Bernoulli-collocation (BC) and Hermite–Galerkin (HG) methods for different values of x, t and $N = 2, 4, 6$ according to Section 3. Figures 1, 2, and 3 clarify the exact solution of (4.1), the absolute error for $N = 6$ by BC and HG methods, respectively. Moreover, these methods are compared to the Toeplitz matrix method [1] that given for $N = 40$.

Example 2. Consider the singular VFIE [1]

$$u(x,t) = x^2t^2 - \frac{125}{336}x^2t^{\frac{12}{5}} + \int_0^t \int_0^1 x^2\ y(t-z)^{-0.6}\ u(y,z)dydz, \tag{4.2}$$

where $x, t \in [0,1]$ with the exact solution $u(x,t) = x^2t^2$.

The absolute error of equation (4.2) for different values of x, t and $N = 2, 4, 6$ by BC and HG methods are obtained in Table 2. We plot Figures 4 and 5 to show the absolute error with $N = 6$ by our methods. Furthermore, these examples compared to Toeplitz matrix method [1] are solved for $N = 60$.

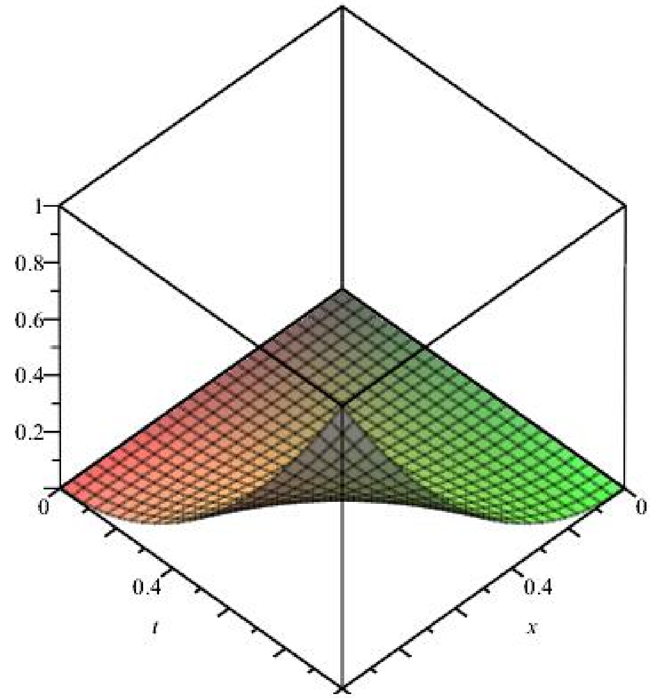

FIGURE 1. Exact solution of Examples 1 and 2.

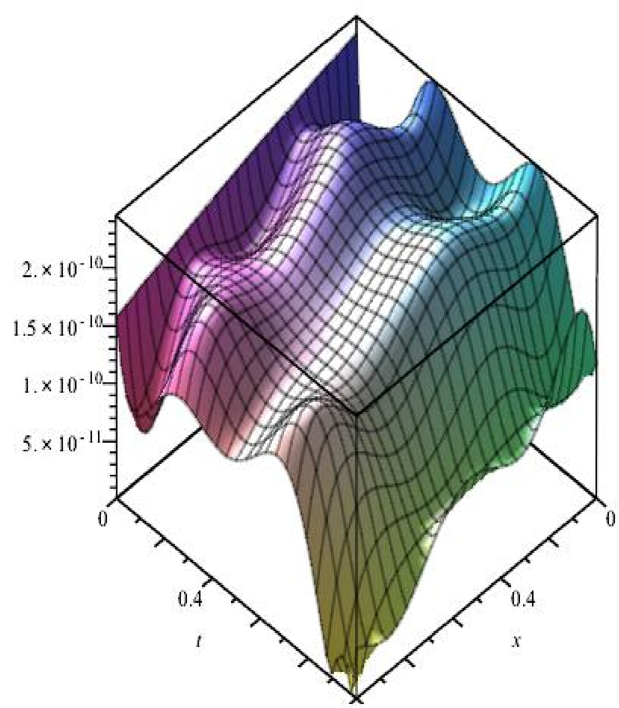

FIGURE 2. Absolute error of Example 1, $N = 6$ by BC method.

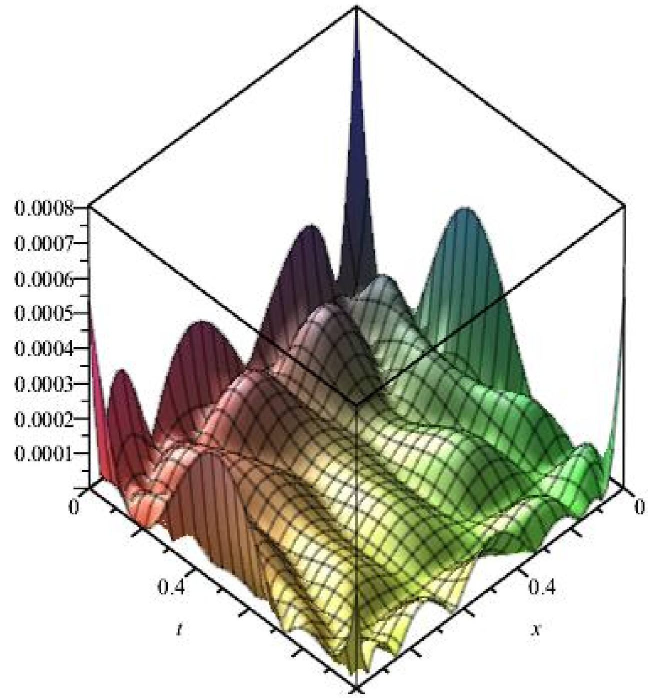

FIGURE 3. Absolute error of Example 1, $N = 6$ by HG method.

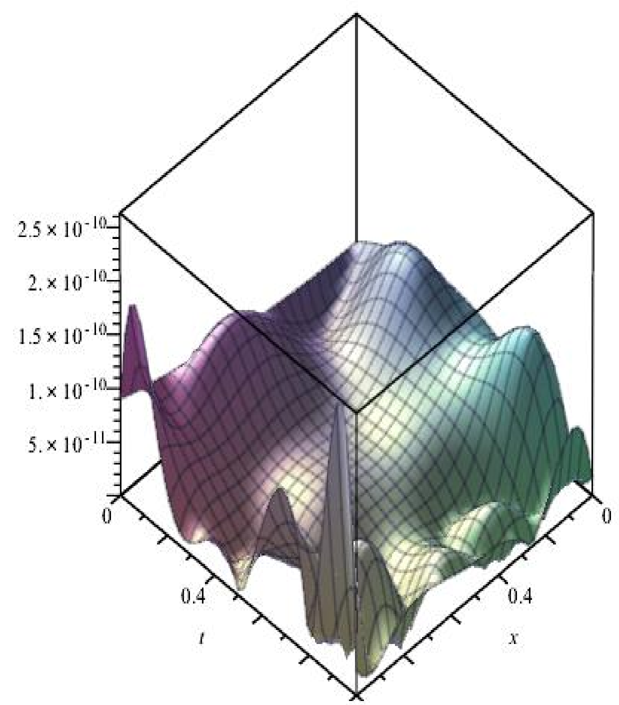

FIGURE 4. Absolute error of Example 2, $N = 6$ by BC method.

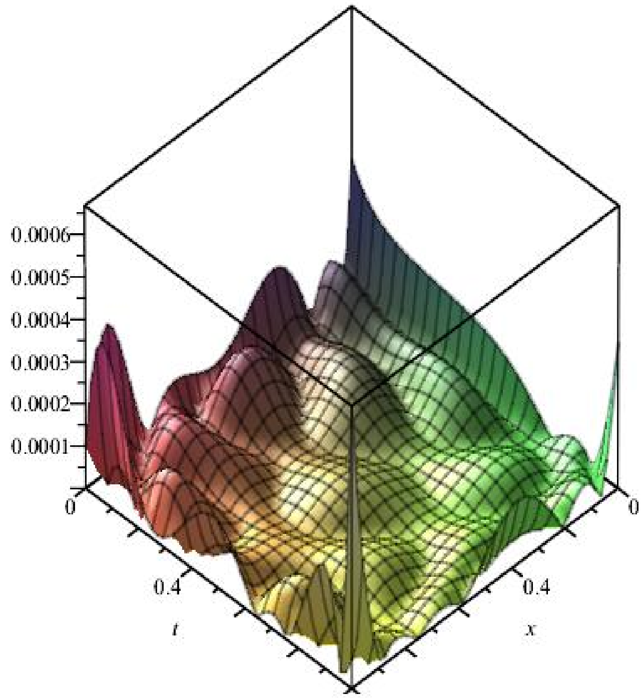

FIGURE 5. Absolute error of Example 2, $N = 6$ by HG method.

TABLE 1. Absolute Error of Example 1 by BC and HG methods for $N = 2, 4, 6$.

(x, y)	$n = 2$ BC method	$n = 2$ HG method	$n = 4$ BC method	$n = 4$ HG method	$n = 6$ BC method	$n = 6$ HG method
$(0, 0)$	1.2×10^{-10}	1.15×10^{-8}	1.0×10^{-10}	8.6399×10^{-6}	1×10^{-10}	3.1208×10^{-5}
$(0.1, 0.1)$	1.2×10^{-10}	3.6077×10^{-9}	6.970×10^{-11}	6.7453×10^{-7}	3.294×10^{-11}	2.9815×10^{-7}
$(0.2, 0.2)$	1.2×10^{-10}	4.545×10^{-10}	8.882×10^{-11}	1.8246×10^{-6}	1.255×10^{-10}	2.2109×10^{-6}
$(0.3, 0.3)$	1.2×10^{-10}	7.659×10^{-12}	1.388×10^{-10}	1.8611×10^{-7}	1.774×10^{-10}	1.1138×10^{-5}
$(0.4, 0.4)$	1.2×10^{-10}	7.046×10^{-10}	2.048×10^{-10}	5.9285×10^{-7}	2.531×10^{-10}	9.7149×10^{-6}
$(0.5, 0.5)$	1.2×10^{-10}	1.4533×10^{-9}	2.702×10^{-10}	1.6656×10^{-6}	3.322×10^{-10}	1.8397×10^{-6}
$(0.6, 0.6)$	1.2×10^{-10}	1.6322×10^{-9}	3.480×10^{-10}	8.6798×10^{-7}	4.016×10^{-10}	5.3769×10^{-6}
$(0.7, 0.7)$	1.2×10^{-10}	1.0899×10^{-9}	4.788×10^{-10}	6.5042×10^{-8}	5.385×10^{-10}	1.6465×10^{-5}
$(0.8, 0.8)$	1.2×10^{-10}	1.455×10^{-10}	6.838×10^{-10}	1.2693×10^{-6}	8.329×10^{-10}	1.0652×10^{-5}
$(0.9, 0.9)$	1.2×10^{-10}	4.114×10^{-10}	9.297×10^{-10}	6.7722×10^{-7}	1.0599×10^{-9}	1.2718×10^{-5}
$(1, 1)$	1.2×10^{-10}	6.791×10^{-10}	1.2191×10^{-9}	6.2889×10^{-6}	1.0661×10^{-9}	3.9371×10^{-5}

TABLE 2. Absolute Error of Example 2 by BC and HG methods for $N = 2, 4, 6$.

	$n=2$		$n=4$		$n=6$	
(x, y)	BC method	HG method	BC method	HG method	BC method	HG method
$(0, 0)$	8.10×10^{-11}	1.24×10^{-8}	1.100×10^{-10}	4.6954×10^{-6}	2×10^{-10}	1.0264×10^{-4}
$(0.1, 0.1)$	6.9×10^{-11}	2.9013×10^{-9}	6.996×10^{-11}	2.3239×10^{-8}	9.8772×10^{-11}	1.7232×10^{-5}
$(0.2, 0.2)$	5.6×10^{-11}	4.559×10^{-11}	1.181×10^{-10}	68951×10^{-7}	1.8759×10^{-10}	1.1683×10^{-5}
$(0.3, 0.3)$	4.1×10^{-11}	4.518×10^{-10}	2.444×10^{-10}	3.1453×10^{-7}	1.5683×10^{-10}	7.8709×10^{-6}
$(0.4, 0.4)$	2.4×10^{-11}	1.6951×10^{-9}	3.745×10^{-10}	1.5990×10^{-7}	1.3114×10^{-10}	9.7149×10^{-6}
$(0.5, 0.5)$	5×10^{-12}	2.3067×10^{-9}	5.165×10^{-10}	5.5695×10^{-8}	4.0122×10^{-12}	1.6494×10^{-5}
$(0.6, 0.6)$	1.6×10^{-11}	1.7741×10^{-9}	7.210×10^{-10}	2.9392×10^{-7}	4.6349×10^{-11}	9.1906×10^{-6}
$(0.7, 0.7)$	3.9×10^{-11}	5.408×10^{-10}	1.0063×10^{-9}	8.5861×10^{-8}	6.43557×10^{-11}	1.9329×10^{-6}
$(0.8, 0.8)$	6.4×10^{-11}	6.763×10^{-12}	1.3374×10^{-9}	1.2693×10^{-7}	$1.73078. \times 10^{-10}$	4.1436×10^{-6}
$(0.9, 0.9)$	9.1×10^{-11}	2.5278×10^{-9}	1.6814×10^{-9}	6.5965×10^{-8}	5.42449×10^{-10}	4.2443×10^{-6}
$(1, 1)$	1.2×10^{-10}	1.1416×10^{-8}	2.0952×10^{-9}	2.6867×10^{-6}	7.8985×10^{-10}	6.3423×10^{-5}

5. Conclusions and Discussions

In this paper, two methods are presented to solve the two-dimensional singular mixed Volterra–Fredholm integral equations by the results of the given two examples we compare between the two methods and establish the following deductions

1. The two methods are better and more effective than Toeplitz matrix method [1], numerically.
2. The Bernoulli-collocation method is more effective than the Hermite–Galerkin method in application.
3. The proposed computational methods could be further applied to the non-linear Volterra–Fredholm integral equations.

References

1. M. A. Abdou, M. A. Elsayed, Numerical solution of mixed linear fractional integro differential equation. *Inter. Review of Basic and Appl. Sci.* **3** (2015), 122–128.
2. H. Almasieh, J. Nazarimeleh, Numerical solution of a class of mixed two-dimensional nonlinear Volterra–Fredholm integral equations using multiquadric radial basis functions. *J. Comput. Appl. Math.* **260** (2014), 173–179.
3. S. Banerjea, B. Dutta, A. Chakrabarti, Solution of Singular Integral Equations Involving Logarithmically Singular Kernel with an Application in a Water Wave Problem. *ISRN Appl. Math.* **2011**, 16pp.
4. R. M. Hafez, E. H. Doha, A. H. Bhrawy, D. Baleanu, Numerical solutions of two-dimensional mixed Volterra-Freholm integral equations via Bernoulli collocation method. *Romanian J. Phy.* **62** (2017), 11p.
5. H. E. Hashim, T. M. Elzaki, Solving singular partial integro-differential equations using Taylor expansion. *IJISET* **2** (2015), no. 4, 2348–7968.
6. Du Jinyan, On the collocation methods for singular integral equation with Hilbert kernel. *Math. Comp.* **78** (2009), 891–928.
7. E. G. Ladopoulos, *Singular Integral Equations.* Linear and non-linear theory and its applications in science and engineering. Springer-Verlag, Berlin, 2000.
8. F. Mirzaee, S. Alipour, Solving two-dimensional non-linear quadratic integral equations of fractional order via operational matrix method. *Multidiscipline Modeling in Materials and Structures* **15** (2019), no. 6, 1136–1151.
9. F. Mirzaee, N. Samadyar, Convergence of 2D-orthonormal Bernstein collocation method for solving 2D-mixed Volterra–Fredholm integral equations. *Trans. A. Razmadze Math. Inst.* **172** (2018), no. 3, part B, 631–641.

10. F. Mirzaee, N. Samadyar, Numerical solution based on two-dimensional orthonormal Bernstein polynomials for solving some classes of two-dimensional nonlinear integral equations of fractional order. *Appl. Math. Comput.* **344/345** (2019), 191–203.
11. S. J. Obaiys, Z. Abbas, N. M. Long, A. F. Ahmad, A. Ahmedov, H. K. Raad. On the general solution of first-kind hypersingular integral equations. *Am. J. Eng. Applied Sci* **9** (2016), 195–201.
12. A. D. Poularikas, *The Handbook of Formulas and Tables for Signal Processing.* CRC Press LLC, 1999.
13. H. Rademacher, *Topics in Analytic Number Theory.* Edited by E. Grosswald, J. Lehner and M. Newman. Die Grundlehren der mathematischen Wissenschaften, Band 169. Springer-Verlag, New York-Heidelberg, 1973.
14. M. M. Rahman, Numerical solutions of Volterra integral equations using Galerkin method with Hermite polynomials. In: *Proceedings of the International Conference on Applied Mathematics and Computational Methods in Engineering*, 2013.
15. M. H. Saleh, S. M. Amer, On the approximate solution of singular integral equations with Hilbert kernel. *J. Appl. Sci.* **7** (2007), no. 20, 2957–2966.
16. M. H. Saleh, D. Sh. Mohamed, D. M. Abdessamiea, Numerical solution of two-dimensional Fredholm integral equation by Chebyshev and Legendre polynomials using collocation Method. *Universal Journal of Integral Equations* **6** (2018), 20–29.
17. Y. Salih, A. Müge, Hermite series soultions of linear Fredholm integral equations. *Mathematical and Computational Applications* **16** (2011), no. 2, 497–506.
18. F. Toutounian, E. Tohidi, A new Bernoulli matrix method for solving second order linear partial differential equations with the convergence analysis. *Appl. Math. Comput.* **223** (2013), 298–310.
19. F. Toutounian, E. Tohidi, S. Shateyi, A collocation method based on the Bernoulli operational matrix for solving high-order linear complex differential equations in a rectangular domain. *Abstr. Appl. Anal.* **2013**, Art. ID 823098, 12pp.
20. F. G. Tricomi, Equazioni integrali contenenti il valor principale di un integrale doppio. (Italian) *Math. Z.* **27** (1928), no. 1, 87–133.
21. V. A. Zisis, E. G. Ladopoulos, Two-dimensional singular integral equations: exact solutions. *J. Comput. Appl. Math.* **31** (1990), no. 2, 227–232.

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ON AN ABSTRACT FORMULATION OF A THEOREM OF SIERPIŃSKI

DEBASISH SEN[1] AND SANJIB BASU[2]

Abstract. In our earlier paper, we gave an abstract formulation of a theorem of Sierpiński in uncountable commutative groups. In this paper, we prove a result which generalizes the earlier formulation.

1. Introduction

Sierpiński [9] in one of his classical papers proved that there exist two Lebesgue measure zero sets in $\mathbb{R}$ whose algebraic sum is nonmeasurable. In establishing this result, he used the Hamel basis and Steinhaus famous theorem on a distance set. Several generalizations of Sierpiński's theorem are available in the literature. Kharazishvili [7] proved that for every σ-ideal $\mathcal{I}$ in $\mathbb{R}$ which is not closed with respect to the algebraic sum, and for every σ-algebra $\mathcal{S}(\supseteq \mathcal{I})$ for which the quotient algebra satisfies the countable chain condition, there exist X, $Y \in \mathcal{I}$ such that $X + Y \notin \mathcal{S}$. Now, instead of the real line $\mathbb{R}$, if we choose a commutative group G and any non-zero, σ-finite, complete, G-invariant (or, G-quasi-invariant) measure μ, then an analogue of Sierpiński's theorem can be established with respect to some extension of μ. In fact, it was shown by Kharazishvili [10] that for every uncountable commutative group G and for any σ-finite, left G-invariant (or, G-quasi-invariant) measure μ on G, there exists a left G-invariant (or, G-quasi-invariant) complete measure μ' extending μ and two sets $A, B \in \mathcal{I}(\mu')$ (the σ-ideal of μ'-measure zero sets) such that $A + B \notin \mathrm{dom}(\mu')$. In [1], the authors gave an abstract and generalized formulation of Sierpiński's theorem in uncountable commutative groups which do not involve any use of measure.

Most of the notations, definitions and results of this paper are taken from [1] (see also [2, 3]). Throughout the paper, we identify every infinite cardinal with the least ordinal representing it as $\mathrm{card}(E)$ for the cardinality of any set E, and use the symbols such as ξ, ρ, α, k etc. for any arbitrary infinite cardinal k and k^+ for the successor of k. Further, given an infinite group G and a set $A \subseteq G$, we denote by gA $(g \in G)$ the set $\{gx : x \in A\}$ and call a class $\mathcal{C}$ of subsets of G as G-invariant if $gA \in \mathcal{C}$ for every $g \in G$ and $A \in \mathcal{C}$.

Definition 1.1. A pair $(\Sigma, \mathcal{I})$ consisting of two non-empty classes of subsets of G is called a G-invariant, k-additive measurable structure on G if:
(i) Σ is an algebra and $\mathcal{I}$ $(\subseteq \Sigma)$ is a proper ideal in G;
(ii) both Σ and $\mathcal{I}$ are k-additive. This means that the both classes Σ and $\mathcal{I}$ are closed with respect to the union of at most k number of sets;
(iii) Σ and $\mathcal{I}$ are G-invariant.

A k-additive algebra Σ is diffused if $\{x\} \in \Sigma$ for every $x \in G$ and a k-additive measurable structure $(\Sigma, \mathcal{I})$ is called k^+-saturated if the cardinality of any arbitrary collection of mutually disjoint sets from $\Sigma \setminus \mathcal{I}$ is atmost k.

In the sixtieth of the past century, Riecan and Neubrunn developed the notion of small systems and used the same to give abstract formulations of several well-known theorems in classical measure and integration (see [12–14], etc.). Small systems have been used by several other authors in the subsequent periods [5, 6, 11, 15]. The following Definition introduces a modified and generalized version.

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Key words and phrases. G-invariant class; G-invariant k-small system; k-additive measurable structure; Admissible k-additive algebra; Saturated set.

Definition 1.2. For any infinite cardinal k, a transfinite k-sequence $\{\mathcal{N}_\alpha\}_{\alpha<k}$ of nonempty classes of sets in G is called a G-invariant, k-small system on G if:

(i) $\emptyset \in \mathcal{N}_\alpha$ for all $\alpha < k$;

(ii) each $\mathcal{N}_\alpha$ is a G-invariant class;

(iii) $E \in \mathcal{N}_\alpha$ and $F \subseteq E$ implies $F \in \mathcal{N}_\alpha$;

(iv) $E \in \mathcal{N}_\alpha$ and $F \in \bigcap_{\alpha<k} \mathcal{N}_\alpha$ implies $E \cup F \in \mathcal{N}_\alpha$;

(v) for any $\alpha < k$, there exists $\alpha^* > \alpha$ such that for any one-to-one correspondence $\beta \to \mathcal{N}_\beta$ with $\beta > \alpha^*$, $\bigcup_\beta E_\beta \in \mathcal{N}_\alpha$ whenever $E_\beta \in \mathcal{N}_\beta$;

(vi) for any $\alpha, \beta < k$, there exists $\gamma > \alpha, \beta$ such that $\mathcal{N}_\gamma \subseteq \mathcal{N}_\alpha$ and $\mathcal{N}_\gamma \subseteq \mathcal{N}_\beta$.

We further define

Definition 1.3. A G-invariant k-additive algebra $\mathcal{S}$ on G is admissible with respect to the k-small system $\{\mathcal{N}_\alpha\}_{\alpha<k}$ if for every $\alpha < k$:

(i) $\mathcal{S} \setminus \mathcal{N}_\alpha \neq \emptyset \neq \mathcal{S} \cap \mathcal{N}_\alpha$;

(ii) $\mathcal{N}_\alpha$ has an S-base, i.e., $E \in \mathcal{N}_\alpha$ is contained in some $F \in \mathcal{N}_\alpha \cap \mathcal{S}$;

(iii) $\mathcal{S} \setminus \mathcal{N}_\alpha$ satisfies the k-chain condition, i.e., the cardinality of any arbitrary collection of mutually disjoint sets from $\mathcal{S} \setminus \mathcal{N}_\alpha$ is at most k.

The above two Definitions have been used by the present authors in some of their recently done works (see, for example, [1–3]). We set $\mathcal{N}_\infty = \bigcap_{\alpha<k} \mathcal{N}_\alpha$. From conditions (ii), (iii) and (v) of Definition 1.2, it follows that $\mathcal{N}_\infty$ is a G-invariant, k-additive ideal in G and denote by $\widetilde{\mathcal{S}}$ the G-invariant k-additive algebra generated by $\mathcal{S}$ and $\mathcal{N}_\infty$. Every element of $\widetilde{\mathcal{S}}$ is of the form $(X \setminus Y) \cup Z$, where $X \in \mathcal{S}$ and $Y, Z \in \mathcal{N}_\infty$, and $(\widetilde{\mathcal{S}}, \mathcal{N}_\infty)$ turns out to be a G-invariant, k-additive measurable structure on G. Moreover, we have the following

Theorem 1.4. *If $\mathcal{S}$ is admissible with respect to $\{\mathcal{N}_\alpha\}_{\alpha<k}$, then the G-invariant, k-additive measurable structure $(\widetilde{\mathcal{S}}, \mathcal{N}_\infty)$ on G is k^+-saturated.*

A proof of the above theorem follows directly from condition (iv) of Definition 1.2 and from conditions (i), (ii) and (iii) of Definition 1.3 or, in short, from the admissibility of $\mathcal{S}$. Based on the above Definitions and Theorems, some combinatorial properties of sets [9, Ch. 7] and also on the important representation theorem for infinite commutative groups [9, Appendix 2], the present authors have proved in [1] the following

Theorem 1.5. *Let G be an uncountable commutative group with $\mathrm{card}(G) = k^+$. Let $\{\mathcal{N}_\alpha\}_{\alpha<k}$ be a G-invariant, k-small system on G and let $\mathcal{S}$ be a diffused, k-additive algebra on G which is also admissible with respect to $\{\mathcal{N}_\alpha\}_{\alpha<k}$. Then there exists a subset A of G such that $A \in \mathcal{N}_\infty$, but $A + A \notin \widetilde{\mathcal{S}}$.*

2. Result

Theorem 1.5 is an abstract formulation of Sierpiński's theorem given in terms of any diffused, G-invariant, k-additive measurable structure on a commutative group G to which we have referred to in the Introduction. In this section we prove a result which extends our previous formulation to the groups that are not necessarily commutative.

Definition 2.1 ([4])**.** Let $\mathcal{R}$ be an equivalence relation on a set X and $E \subseteq X$. The saturation of E in X with respect to the equivalence relation is the union of all equivalence classes of $\mathcal{R}$ whose intersection with E is nonvoid.

In other words, it is $\bigcup\{C : C \cap E \neq \varnothing \text{ and } C \in X/\mathcal{R}\}$.

It is easy to check that if H is a normal subgroup of any group G, then the saturation of any set E in G with respect to the equivalence relation generated by the quotient group G/H is the set HE. If E coincides with its saturation, then it is called saturated. Thus E is saturated if $HE = E$. A saturated set is also called H-invariant [8].

Theorem 2.2. *Let G be any uncountable group with* $\mathrm{card}(G) = k^+$. *Let $\{\mathcal{N}_\alpha\}_{\alpha<k}$ be a G-invariant, k-small system on G and $\mathcal{S}$ be a G-invariant, k-additive algebra on G which is admissible with respect to $\{\mathcal{N}_\alpha\}_{\alpha<k}$. We further assume that G has a normal subgroup $H \in \mathcal{S}$ such that G/H is commutative with* $\mathrm{card}(G/H) = k^+$ *and the saturation of any set E in G with respect to G/H also belongs to $\mathcal{S}$.*

Then there exists a subset A of G such that $A \in \mathcal{N}_\infty$ and $AA \notin \widetilde{\mathcal{S}}$.

Proof. We write $\Gamma = G/H$. By the hypothesis, Γ is commutative. Let $f : G \to \Gamma$ be the canonical homomorphism. We set $\mathcal{S}' = \{Y \subseteq \Gamma : f^{-1}(Y) \in \mathcal{S}\}$ and $\mathcal{N}'_\alpha = \{Y \subseteq \Gamma : f^{-1}(Y) \in \mathcal{N}_\alpha\}$ for any $\alpha < k$.

Since $\mathcal{S}$ is a G-invariant, k-additive algebra on G and f is a canonical homomorphism, so $\mathcal{S}'$ is a Γ-invariant, k-additive algebra on Γ. Also, since $H \in \mathcal{S}$, therefore $\mathcal{S}'$ is diffused.

Condition (i) of Definition 1.2 for $\{\mathcal{N}'_\alpha\}_{\alpha<k}$ is obvious. Let $h \in \Gamma$ and $F \in \mathcal{N}'_\alpha$. Then $h = f(x)$ for every $x \in gH$, where $g \in G$ and $f^{-1}(F) \in \mathcal{N}_\alpha$. Since $\mathcal{N}_\alpha$ is G-invariant, therefore $f^{-1}(hF) = xf^{-1}(F) \in \mathcal{N}_\alpha$. Hence $hF \in \mathcal{N}'_\alpha$ which proves condition (ii) of Definition 1.2 for $\{\mathcal{N}'_\alpha\}_{\alpha<k}$. Finally, from the Definition of $\mathcal{N}'_\alpha$ and some simple properties of inverse images of any function, it follows that conditions (iii)-(vi) of Definition 1.2 also hold for $\{\mathcal{N}'_\alpha\}_{\alpha<k}$. Thus $\{\mathcal{N}'_\alpha\}_{\alpha<k}$ is a Γ-invariant, k-small system on Γ.

We shall now show that $\mathcal{S}'$ is admissible with respect to $\{\mathcal{N}'_\alpha\}_{\alpha<k}$. Clearly, $\emptyset \in \mathcal{S}' \cap \mathcal{N}'_\alpha$ for $\alpha < k$. Since $\mathcal{S}$ is admissible with respect to $\{\mathcal{N}_\alpha\}_{\alpha<k}$, so by (i) of Definition 1.3, there exists for every $\alpha < k$, a set $A_\alpha \in \mathcal{S} \setminus \mathcal{N}_\alpha$. If A_α is saturated with respect to the equivalence relation generated by the quotient group G/H, then $A_\alpha = f^{-1}(B_\alpha)$ for some $B_\alpha \in \mathcal{S}' \setminus \mathcal{N}'_\alpha$. If A_α is not saturated, we replace it by HA_α which is saturated, and choose B_α such that $HA_\alpha = f^{-1}(B_\alpha)$. Consequently, $B_\alpha \in \mathcal{S}' \setminus \mathcal{N}'_\alpha$ and condition (i) of Definition 1.3 is satisfied.

Let $F \in \mathcal{N}'_\alpha$ and $E = f^{-1}(F)$. Then $E \in \mathcal{N}_\alpha$ by (ii) of Definition 1.3 there exists $A \in \mathcal{S} \cap \mathcal{N}_\alpha$ such that $E \subseteq A$. If A is saturated, then $A = f^{-1}(B)$ for some $B \in \mathcal{S}' \cap \mathcal{N}'_\alpha$ and $F \subseteq B$. If A is not saturated, we choose the saturation of $G \setminus A$, i.e., $H(G \setminus A)$ with respect to the equivalence relation generated by the quotient group G/H. But $H(G \setminus A) \in \mathcal{S}$ and so, $G \setminus H(G \setminus A) \in \mathcal{S}$. Moreover, $G \setminus H(G \setminus A)$ is a subset of A. Therefore $G \setminus H(G \setminus A) \in \mathcal{N}_\alpha \cap \mathcal{S}$. We choose $B (\subseteq \Gamma)$ such that $G \setminus H(G \setminus A) = f^{-1}(B)$. Then $F \subseteq B$ and $B \in \mathcal{S}' \cap \mathcal{N}'_\alpha$. This shows that $\mathcal{N}'_\alpha$ has an $\mathcal{S}'$-base for every $\alpha < k$ and condition (ii) of Definition 1.3 is proved. Lastly, any arbitrary collection of mutually disjoint sets from $\mathcal{S}' \setminus \mathcal{N}'_\alpha$ is at most k which follows directly from the fact that a similar result is true for the sets from $\mathcal{S} \setminus \mathcal{N}_\alpha$. This shows that $\mathcal{S}' \setminus \mathcal{N}'_\alpha$ satisfies the k-chain condition for every $\alpha < k$ which proves (iii) of Definition 1.3.

Thus we find that $\mathcal{S}'$ is a Γ-invariant, k-additive algebra on Γ which is diffused and admissible with respect to the Γ-invariant, k-small system $\{\mathcal{N}'_\alpha\}_{\alpha<k}$ on Γ.

Let $\mathcal{N}'_\infty = \bigcap\limits_{\alpha<k} \mathcal{N}'_\alpha$ and $\widetilde{\mathcal{S}'}$ be the Γ-invariant, k-additive algebra generated by $\mathcal{S}'$ and $\mathcal{N}'_\infty$. Thus $(\widetilde{\mathcal{S}'}, \mathcal{N}'_\infty)$ is a Γ-invariant, k-additive, measurable structure on the quotient group Γ which is k^+-saturated. Hence by Theorem 1.5, there exists $B \in \mathcal{N}'_\infty$ such that $BB \notin \widetilde{\mathcal{S}'}$. Let $A = f^{-1}(B)$. Then $AA = f^{-1}(B)f^{-1}(B) = f^{-1}(BB)$. So, AA is saturated. If possible, let $AA \in \widetilde{\mathcal{S}}$. Then $AA = E\Delta P$, where $E \in \mathcal{S}$, $P \in \mathcal{N}_\infty$ and E, P are both saturated. Hence $E = f^{-1}(F)$, $P = f^{-1}(Q)$, where $F \in \mathcal{S}'$, $Q \in \mathcal{N}'_\infty$ and therefore $AA = E\Delta P = f^{-1}(F)\Delta f^{-1}(Q) = f^{-1}(F\Delta Q) = f^{-1}(BB)$. But this implies that $BB \in \widetilde{\mathcal{S}'}$ – a contradiction. □

Remark. In general for Theorem 2.2, G need not be commutative. Let H' be a noncommutative group with $\mathrm{card}(H') = \omega$ (the first infinite cardinal) and A' be a commutative group with $\mathrm{card}(A') = \omega_1$ (the first uncountable cardinal). We set $G = H' \times A'$ as the external direct product of H' and A'. Then G is isomorphic with the internal direct product HA, where $H = \{(h, e_{A'}) : h \in H'\}$ and $A = \{(e_{H'}, a) : a \in A'\}$. Moreover, G is noncommutative having H as a normal subgroup and $G/H = A$ is commutative with $\mathrm{card}(G/H) = \omega_1$.

References

1. S. Basu, D. Sen, An abstract formulation of a theorem of Sierpiński on the nonmeasurable sum of two measure zero sets. *Georgian Math. J.* **26** (2019), no. 4, 617–620.
2. S. Basu, D. Sen, On a theorem by Pelc and Prikry on the nonexistence of invariant extensions of Borel measures. *Anal. Math.* (2021). https://arXiv:2002.00334v1.
3. S. Basu, D. Sen, A nonseparable invariant extensions of Lebesgue measure- A generalized and abstract approach. *Georgian Math. J.* (2021), https://doi.org/10.1515/gmj-2021-2096.
4. N. Bourbaki, *Elements of Mathematics. General Topology.* Part 1. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966.
5. J. Hejduk, E. Wajch, Compactness in the sense of the convergence with respect to a small system. *Math. Slovaca* **39** (1989), no. 3, 267–275.
6. R. A. Johnson, J. Niewiarowski, T. Światkowski, Small systems convergence and metrizability. *Proc. Amer. Math. Soc.* **103** (1988), no. 1, 105–112.
7. A. Kharazishvili, Some remarks on additive properties of invariant σ-ideals on the real line. *Real Anal. Exchange* **21** (1995/96), no. 2, 715–724.
8. A. Kharazishvili, *Transformation Groups and Invariant Measures.* Set-theoretical aspects. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.
9. A. Kharazishvili, *Nonmeasurable Sets and Functions.* North-Holland Mathematics Studies, 195. Elsevier Science B.V., Amsterdam, 2004.
10. A. Kharazishvili, A. Kirtadze, On algebraic sums of measure zero sets in uncountable commutative groups. *Proc. A. Razmadze Math. Inst.* **135** (2004), 97–103.
11. J. Niewiarowski, Convergence of sequences of real functions with respect to small systems. *Math. Slovaca* **38** (1988), no. 4, 333–340.
12. B. Riečan, Abstract formulation of some theorems of measure theory. *Mat.-Fyz. Časopis. Sloven. Akad. Vied.* **16** (1966), 268–273.
13. B. Riečan, Abstract formulation of some theorems of measure theory. II. *Mat. Časopis* **19** (1969), no. 2, 138–144.
14. B. Riečan, T. Neubrunn, *Integral, Measure, and Ordering.* Mathematics and its Applications, 411. Kluwer Academic Publishers, Dordrecht; Ister Science, Bratislava, 1997.
15. Z. Riečonová, On an abstract formulation of regularity. *Mat. Časopis Sloven. Akad. Vied* **21** (1971), 117–123.

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ON THE ABSOLUTE MATRIX SUMMABILITY FACTORS OF FOURIER SERIES

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Abstract. In this paper, a general theorem on the local property of the $\left|\bar{N}, p_n; \delta\right|_k$ summability of factored Fourier series, which generalizes some known results has been extended to absolute matrix summability factors of Fourier series.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (t_n) of the Riesz means or, simply, the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [4]).

The series $\sum a_n$ is said to be $\left|\bar{N}, p_n; \delta\right|_k$ summable, where $k \geq 1$ and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \mid t_n - t_{n-1} \mid^k < \infty.$$

In the special case, $p_n = 1$ for all n (resp., $\delta = 0$), the $\left|\bar{N}, p_n; \delta\right|_k$ summability is the same as the $|C, 1; \delta|_k$ $(resp., \left|\bar{N}, p_n\right|_k)$ summability (see [1]).

A sequence (λ_n) is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer n, where $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$ (see [9]).

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v \quad n = 0, 1, \ldots .$$

The series $\sum a_n$ is said to be $|A, p_n; \delta|_k$ summable, where $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

If we take $a_{nv} = \frac{p_v}{P_n}$, then the $|A, p_n; \delta|_k$ summability is the same as the $\left|\bar{N}, p_n; \delta\right|_k$ summability. If we take $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$, then the $|A, p_n; \delta|_k$ summability reduces to the $\left|\bar{N}, p_n\right|_k$ summability. Also, if we take $\delta = 0$, then the $|A, p_n; \delta|_k$ summability reduces to the $|A, p_n|_k$ summability (see [6]).

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Given a normal matrix $A=(a_{nv})$, we associate two lower semimatrices $\bar{A}=(\bar{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv}=\sum_{i=v}^{n} a_{ni}, \quad n,v=0,1,\dots \tag{1}$$

and

$$\hat{a}_{00}=\bar{a}_{00}=a_{00}, \quad \hat{a}_{nv}=\bar{a}_{nv}-\bar{a}_{n-1,v}, \quad n=1,2,\dots . \tag{2}$$

It should be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. So, we get

$$A_n(s)=\sum_{v=0}^{n} a_{nv}s_v=\sum_{v=0}^{n}\bar{a}_{nv}a_v$$

and

$$\bar{\Delta}A_n(s)=\sum_{v=0}^{n}\hat{a}_{nv}a_v.$$

2. The Known Results

Let f be a periodic function with period 2π, integrable (L) over $(-\pi,\pi)$. We may assume that the constant term of the Fourier series of f is zero, that is,

$$\int_{-\pi}^{\pi} f(t)dt=0,$$

$$f(t)\sim\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)\equiv\sum_{n=1}^{\infty}C_n(t).$$

In [3], Bor proved the following result dealing with the $\left|\bar{N},p_n;\delta\right|_k$ summability factors of Fourier series.

Theorem 2.1 ([3])**.** *Let $k\geq 1$ and $0\leq\delta<1/k$. If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent and*

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}P_v\Delta\lambda_v=O(1) \quad as \quad m\to\infty,$$

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}p_v\lambda_v=O(1) \quad as \quad m\to\infty,$$

$$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}=O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right),$$

then the $\left|\bar{N},p_n;\delta\right|_k$ summability of the series $\sum C_n(t)\lambda_nP_n$ at a point can be ensured by a local property.

In [7], Sulaiman has obtained a result from which a special case improved the result of [3] in the following form.

Theorem 2.2 ([7]). *Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let (φ_n) be a complex sequence. If $(|\lambda_n|)$ is non-increasing such that $\sum p_n|\lambda_n|$ is convergent and*

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{p_v}{P_v^k}|\lambda_v||\varphi_v|^k = O(1) \quad as \quad m \to \infty, \tag{3}$$

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta}|\Delta\lambda_v||\varphi_v| = O(1) \quad as \quad m \to \infty, \tag{4}$$

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{p_{v+1}^{k-1}}|\lambda_{v+1}||\Delta\varphi_v|^k = O(1) \quad as \quad m \to \infty, \tag{5}$$

$$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right),$$

then the $\left|\bar{N}, p_n; \delta\right|_k$ summability of the series $\sum C_n(t)\lambda_n P_n$ at a point can be ensured by a local property.

3. Main Result

The aim of this paper is to generalize Sulaiman's result in [7] for the $|A, p_n; \delta|_k$ summability method.

Theorem 3.1. *Let (φ_n) be a complex sequence. Let $k \geq 1$ and $0 \leq \delta < 1/k$. Suppose that $A = (a_{nv})$ is a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \ n = 0, 1, \ldots, \tag{6}$$

$$a_{n-1,v} \geq a_{nv}, \ \text{for } n \geq v+1, \tag{7}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right).$$

If $(|\lambda_n|)$ is non-increasing such that $\sum p_n|\lambda_n|$ is convergent and satisfy conditions (3)–(5) of Theorem 2.2 and the conditions

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{p_v}{P_v}\right) \quad as \quad m \to \infty, \tag{8}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\right) \quad as \quad m \to \infty \tag{9}$$

are satisfied, then the $|A, p_n; \delta|_k$ summability of the series

$$\sum_{n=1}^{\infty} C_n(t)\lambda_n\varphi_n,$$

at any point is a local property of f.

Lemma 3.1 ([8]). *From conditions* (1), (2) *and* (6), (7), *we have*

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})| \leq a_{nn},$$

$$|\hat{a}_{n,v+1}| \leq a_{nn}.$$

Lemma 3.2 ([7]). *If $(|\lambda_n|)$ is non-increasing such that $\sum p_n|\lambda_n| < \infty$, then $P_n|\lambda_n| = O(1)$, as $n \to \infty$.*

Lemma 3.3. *Let (φ_n) be a complex sequence. If (s_n) is bounded, and all the conditions of Theorem* 3.1 *are satisfied, then the series*

$$\sum_{n=1}^{\infty} a_n\lambda_n\varphi_n$$

is $|A,p_n;\delta|_k$ *summable, where* $k\geq 1$ *and* $0\leq\delta<1/k$*, and* $(|\lambda_n|)$ *is the same as in Theorem* 3.1.

Proof. Let (I_n) denotes the A-transform of the series $\sum_{n=1}^{\infty} a_n\lambda_n\varphi_n$, then

$$\bar{\Delta} I_n=\sum_{v=1}^{n}\hat{a}_{nv}a_v\lambda_v\varphi_v.$$

Applying Abel's transformation to this sum, we have

$$\begin{aligned}\bar{\Delta} I_n&=\sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv}\lambda_v\varphi_v)\sum_{r=1}^{v}a_r+a_{nn}\lambda_n\varphi_n\sum_{v=1}^{n}a_v\\&=\sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv}\lambda_v\varphi_v)s_v+a_{nn}\lambda_n\varphi_n s_n\\&=\sum_{v=1}^{n-1}\Delta(\hat{a}_{nv})\lambda_v\varphi_v s_v+\sum_{v=1}^{n-1}\Delta\lambda_v\varphi_v\hat{a}_{n,v+1}s_v+\sum_{v=1}^{n-1}\Delta\varphi_v\lambda_{v+1}\hat{a}_{n,v+1}s_v+a_{nn}\lambda_n\varphi_n s_n\\&=I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}.\end{aligned}$$

To complete the proof of Lemma 3.3, it suffices to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid I_{n,r}\mid^k<\infty,\quad\text{for}\quad r=1,2,3,4.$$

First, applying Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid I_{n,1}\mid^k&\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v||\varphi_v||s_v|\right\}^k\\&\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta(\hat{a}_{nv})|\,|s_v|^k|\lambda_v|^k|\varphi_v|^k\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}\\&=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|\varphi_v|^k\right\}\\&=O(1)\sum_{v=1}^{m}|\lambda_v|^k|\varphi_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\&=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{p_v}{P_v}|\lambda_v|^k|\varphi_v|^k\\&=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}(P_v|\lambda_v|)^{k-1}|\lambda_v||\varphi_v|^k\frac{p_v}{P_v^k}\\&=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\lambda_v||\varphi_v|^k\frac{p_v}{P_v^k}\\&=O(1)\quad\text{as}\quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Lemma 3.3 and by using condition (3) of Theorem 2.2, condition (8) of Theorem 3.1 and also taking into account Lemma 3.1 and Lemma 3.2. Now, using Hölder's inequality, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid I_{n,2}\mid^k\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v||\varphi_v||s_v|\right\}^k$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}|\hat{a}_{n,v+1}||\Delta\lambda_v||\varphi_v||s_v|^k\times\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{\delta}|\hat{a}_{n,v+1}||\Delta\lambda_v||\varphi_v|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}|\hat{a}_{n,v+1}||\Delta\lambda_v||\varphi_v|\times\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{\delta}|\Delta\lambda_v||\varphi_v|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left\{\sum_{v=1}^{n-1}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}|\hat{a}_{n,v+1}||\Delta\lambda_v||\varphi_v|\right\}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}|\Delta\lambda_v||\varphi_v|\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}\left(\frac{P_v}{p_v}\right)^{\delta k}|\Delta\lambda_v||\varphi_v|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta}|\Delta\lambda_v||\varphi_v|\\
&= O(1)\quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Lemma 3.3 and by taking condition (4) of Theorem 2.2 and also condition (9) of Theorem 3.1. Further, we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid I_{n,3}\mid^k &\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\varphi_v||\lambda_{v+1}||s_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta\varphi_v|^k|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\times\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|p_{v+1}\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\Delta\varphi_v|^k|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\times\left\{\sum_{v=1}^{n-1}|\lambda_{v+1}|p_{v+1}\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\sum_{v=1}^{n-1}|\Delta\varphi_v|^k|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\\
&= O(1)\sum_{v=1}^{m}|\Delta\varphi_v|^k\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\Delta\varphi_v|^k\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\\
&= O(1)\quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Lemma 3.3 and using condition (5) of Theorem 2.2, condition (9) of Theorem 3.1 and also taking Lemma 3.1 and Lemma 3.2. Finally, by virtue of the hypotheses of Lemma 3.3 and using condition (3) of Theorem 2.2 and taking Lemma 3.2, we have

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid I_{n,4}\mid^k &= \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|a_{nn}\lambda_n\varphi_n s_n|^k \le \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^k|\lambda_n|^k|\varphi_n|^k|s_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n|^k|\varphi_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{p_n}{P_n^k}|\lambda_n||\varphi_n|^k(P_n|\lambda_n|)^{k-1}
\end{aligned}
$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{p_n}{P_n^k}|\lambda_n||\varphi_n|^k = O(1) \quad \text{as} \quad m\to\infty,$$

which completes the proof of Lemma 3.3. □

Proof of Theorem 3.1. Since the convergence of Fourier series at a point is a local property of its generating function f, our theorem follows immediately from Lemma 3.3.

4. Conclusions

If we take $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we have a result of Theorem 2.2. Also, if we take $\delta = 0$ in Theorem 3.1, we have a new result dealing with the $|A, p_n|_k$ summability of Fourier series.

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References

1. H. Bor, On two summability methods. *Math. Proc. Cambridge Philos. Soc.* **97** (1985), no. 1, 147–149.
2. H. Bor, On local property of $|\overline{N}, p_n; \delta|_k$ summability of factored Fourier series. *J. Math. Anal. Appl.* **179** (1993), no. 2, 646–649.
3. H. Bor, A note on local property of factored Fourier series. *Nonlinear Anal.* **64** (2006), no. 3, 513–517.
4. G. H. Hardy, *Divergent Series.* Oxford, at the Clarendon Press, 1949.
5. H. S. Özarslan, H. N. Öğdük, Generalizations of two theorems on absolute summability methods. *Aust. J. Math. Anal. Appl.* **1** (2004), no. 2, Art. 13, 7 pp.
6. W. T. Sulaiman, Inclusion theorems for absolute matrix summability methods of an infinite series. IV. *Indian J. Pure Appl. Math.* **34** (2003), no. 11, 1547–1557.
7. W. T. Sulaiman, A study on local properties of Fourier series. *Bull. Allahabad Math. Soc.* **26** (2011), no. 1, 47–53.
8. W. T. Sulaiman, Some new factor theorem for absolute summability. *Demonstratio Math.* **46** (2013), no. 1, 149–156.
9. A. Zygmund, *Trigonometrical Series.* Instytut Matematyczny Polskiej Akademi Nauk, Warszawa-Lwów, 1935.

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SHORT COMMUNICATIONS

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ASYMPTOTIC ANALYSIS OF COUPLED OSCILLATORS EQUATIONS IN A NON-UNIFORM PLASMA

GRIGOL GOGOBERIDZE

Abstract. We study a set of coupled oscillators equations describing Alfvén's linear coupling and fast magnetosonic waves in a magnetized plasma. Using the methods of asymptotic analysis, we derive analytical expressions for the transformation coefficient, as well as Liouville–Green asymptotic solutions. The obtained results are compared with the mathematically similar Landau–Zener problem in quantum mechanics.

1. Introduction

The aim of the present paper is to study coupled evolution of linear plasma waves in a shear flow. This mechanism is expected to be responsible for generation of compressible perturbations in the solar wind [5].

In a plasma with the uniform background velocity shear $\mathbf{U}_0 = (Ay, 0, 0)$ equations that describe coupled evolution of the Alfvén waves (AW) and fast magnetosonic waves (FMW) are governed by the following coupled oscillators equations [4]:

$$\frac{\mathrm{d}^2 b_y}{\mathrm{d}\tau^2} + \left[1 + K_y^2(\tau)\right] b_y = -K_y(\tau) K_z b_z, \tag{1}$$

$$\frac{\mathrm{d}^2 b_z}{\mathrm{d}\tau^2} + \left[1 + K_z^2\right] b_z = -K_y(\tau) K_z b_y. \tag{2}$$

Here, b_y and b_z are the Fourier amplitudes of the corresponding magnetic field components, K_z is the dimensionless wave number $K_z = k_z/k_x$, k_z and k_x are the components of the wave number vector, $K_y(\tau) = K_y - S\tau$ is the dimensionless wave number, $S = A/k_x V_A$ is a dimensionless shear rate, V_A is the Alfvén speed and $\tau = V_A k_x t$ is a dimensionless time.

The solutions of the characteristic equation of the set of equations (1), (2) are

$$\Omega_F^2(\tau) = 1 + K_z^2 + K_y^2(\tau),\ \Omega_A^2 = 1. \tag{3}$$

They can be easily identified as the frequencies of FMW and AW, respectively.

In the next section we present detailed analysis of equations (1), (2). We study the phenomenon of a mutual transformation of wave modes and derive analytical expression for the transformation coefficient.

2. Asymptotic Analysis

It is well known from the theory of coupled oscillator systems that if inhomogeneity is weak enough (in the considered case the condition implies that the normalized shear rate should be small $S \ll \Omega_A = 1$) and the frequencies of the modes are not close to each other (in the case under consideration this condition of weak coupling implies [4] $\delta \equiv |K_z|/S^{1/3} \ll 1$), then the Liouville–Green approximation [2,6,7] is valid and the asymptotic solutions of equations (1), (2) are given by the following expressions:

$$\Psi_\pm = \frac{D_{F,A\pm}}{\sqrt{\Omega_{F,A}(\tau)}} e^{\pm i \int \Omega_{F,A}(\tau) \mathrm{d}\tau},$$

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where $D_{F,A\pm}$ are the Liouville–Green amplitudes of the corresponding oscillations determined by the initial conditions. It is well known [6] that the signs $\pm$ correspond to the waves propagating along and backward with respect to the x-axis, respectively.

If one considers equations (1), (2) in a complex τ-plane, then the Liouville–Green solution is valid everywhere, except some vicinities of turning points, where $\Omega_F = 0$, and the resonant points, where $\Omega_F = \Omega_A$. If the Liouville–Green approximation is valid, then there is no energy exchange between FMWs and AWs and the energy densities of the modes satisfy the standard relation $E_{F,A\pm} = \Omega_{F,A} D^2_{F,A\pm}$. Analysis of equation (3) shows that if $S \ll 1$, the turning points are not located close to the real τ-axis, i.e., physically speaking, in this case the wave reflection is absent [3]. When solving the equation $\Omega_F = \Omega_A$, one finds that there are two second order resonant points $K_y(\tau_{1,2}) = \pm i K_z$ (the resonant point τ_1 has the order n if $(\Omega_1 - \Omega_2) \sim (\tau_1 - \tau)^{n/2}$ in the neighborhood of τ_1).

As follows from equation (3), the frequencies are closest, i.e., an effective coupling is possible only in some vicinity at the time moment when $K_y(\tau) = 0$. This means that the Liouville–Green approximation is always valid far on the left– and right-hand sides of this point. This circumstance enables to study the wave coupling based on the asymptotic analysis that is usual in the scattering theory. Assume that at the initial moment of time $K_y(0) \ll 1$ and the initial amplitudes of the modes are $D^L_{F,A}$. Denote the amplitudes on the right of the resonant area by $D^R_{F,A}$. If so, the problem reduces to the derivation of the so-called transformation coefficient T_{FA} that connects the initial and final amplitudes $T_{FA} = (D^L_F)^2/(D^R_A)^2$. Physically, T_{FA} represents a part of energy of the initial FMW transformed into the AW energy.

If the condition for the effective coupling $\delta \equiv |K_z|/S^{1/3} < 1$ is not satisfied, the transformation coefficient is exponentially small, namely [4],

$$T_{FA} \approx \frac{\pi}{2} \exp\left(-\frac{\delta^3}{3}\right). \tag{4}$$

Analytical expression for the transformation coefficients can be derived also in the opposite limit $\delta \ll 1$. In this case, it can be readily shown that b_y and b_z coincide with the eigenfunctions of FMW and AW, accurate to the terms of order K_z^2. Consequently, the terms on the right-hand sides of equations (1), (2) represent the coupling terms of the same accuracy. Since $K_z \ll S^{1/3}$, the coupling is weak, and if initially there exists only FMW, one can neglect the feedback of AW to FMW. Then, using the well-known expressions for the solution of a linear inhomogeneous second-order differential equation, in the above-considered limit ($\delta \ll 1$), we obtain

$$T_{FA} \approx 2^{2/3} \delta \int\limits_0^\infty x \sin\left(\frac{x^3}{3} - \frac{\delta^2}{2^{2/3}} x\right) dx.$$

Note that

$$\int\limits_0^\infty x \sin\left(\frac{x^3}{3} - \gamma x\right) dx \equiv \pi \frac{\partial}{\partial \gamma} Ai(-\gamma),$$

and using the expansion of the Airy function $Ai(\gamma)$ into power series [1], we finally obtain

$$T_{FA} \approx \frac{2^{2/3}\pi}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} \delta \left(1 - \frac{\Gamma\left(\frac{1}{3}\right)}{2^{7/4} 3^{1/3} \Gamma\left(\frac{2}{3}\right)} \delta^4\right). \tag{5}$$

The results of numerical solution of the initial set of equations (1), (2) (solid line), as well as analytical expressions (4) (dash-dotted line) and (5) (dashed line) are presented in Figure 1. It shows that the transformation coefficient reaches its maximal value $(T^2_{FA})_{max} = 1/2$ at δ^{cr} that can be found numerically, or alternatively, by finding the maximum of the analytical expression presented by the equation (5):

$$\delta^{cr} = \left(\frac{2^{7/4} 3^{1/3} \Gamma\left(\frac{2}{3}\right)}{5\Gamma\left(\frac{1}{3}\right)}\right)^{1/4}. \tag{6}$$

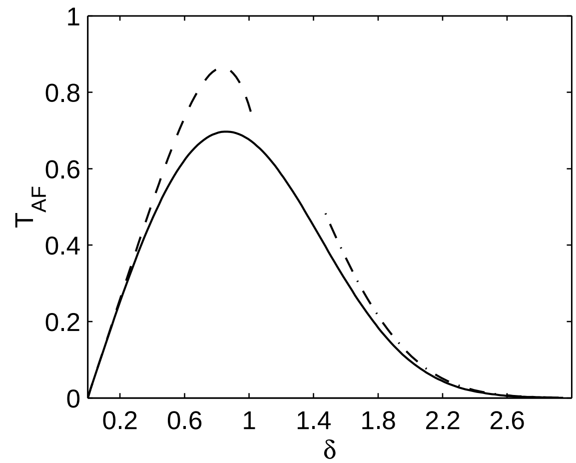

FIGURE 1. The transformation coefficient T_{FA} vs δ. Dash-dotted line and dashed line represent analytical expressions (4) and (5), respectively. Solid line is obtained by numerical solution of equations (1), (2).

Formula (6) is in a perfect accordance with the numerically calculated δ^{cr} (see Figure 1), despite the failure of equation (5) at $\delta \sim 1$. This fact can be explained as follows: the only reason why equation (5) fails is the neglect of the feedback mentioned above. The feedback changes the value of the transformation coefficient, but does not affect the value of δ^{cr}.

3. Discussion and Conclusions

It is well known (see [2, 7] and references therein) that if in the coupled oscillators system with eigenfrequencies $\Omega_{1,2}$, in the neighborhood of the real τ-axis there exist only a pair of complex conjugated first-order resonant points τ_1 and τ_2, the transformation coefficient can be derived from the exact asymptotic formula

$$T_{12} = \exp\left(- \left| \mathrm{Im} \int_{\tau_0}^{\tau_1} (\Omega_1 - \Omega_2) d\tau \right| \right). \tag{7}$$

We shall make two remarks about this equation. Firstly, it shows that in the case of the first-order resonant points only the eigenfrequencies are needed to derive the transformation coefficient. Secondly, equation (7) is valid in the case of strong wave interactions. For instance, if a complex conjugate resonant point of the first order tends to the real τ-axis, then T_{12} tends to unity, i.e., the energy of one wave mode is entirely transformed into another.

None of these properties remain valid in the case of the second order resonant points. Firstly, the transformation coefficient is small in the both limiting cases $\delta \gg 1$ and $\delta \ll 1$, i.e., when the resonant points are both close and far from the real τ-axis. Secondly, only the expressions of the eigenfrequencies are not sufficient for the derivation of the transformation coefficient, the problem needs deeper analysis. Thirdly, the maximum value of the transformation coefficient is $1/2$. This means that even in the optimal regime, only half of the energy of FMW can be transformed into AW, and vice versa. It has to be noted that the Landau–Zener theory [6] provides the same maximum value for the transition probability in the two-level quantum mechanical systems.

The last point we would like to discuss in the present paper is the comparison of our problem with the theory of quantum transitions in the two-level systems. First of all, note that equations (1), (2) correspond to the so-called quantum mechanical diabatic representation. On the other hand, the normal variables that were introduced in [3, 4], correspond to the adiabatic representation. As in the two-level quantum systems, both representations are useful for derivation of a transformation coefficient in different limits. One distinction that makes our problem different and generally more difficult is that in the area of the effective interaction the 'coupling terms' (terms on the right-hand side of equations (1), (2)) cannot be treated as constants. This circumstance does not allow to use another powerful asymptotic method, the so-called momentum representation [6, 7].

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References

1. M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions.* Dover Publications, Inc., New York, 1965.
2. M. V. Fedoriuk, *Asymptotic Methods for Linear Ordinary Differential Equations.* Nauka, Moskow, 1983.
3. G. Gogoberidze, Asymptotic analysis of an over-reflection equation in magnetized plasma. *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 1, 71–74.
4. G. Gogoberidze, G. Chagelishvili, R. Z. Sagdeev, J. Lominadze, Linear coupling and overreflection phenomena of magnetohydrodynamic waves in smooth shear flows. *Phys. Plasmas.* **10** (2004), no. 10, 4672–4685.
5. G. Gogoberidze, A. Rogava, S. Poedts, Quantifying shear-induced wave transformations in the solar wind. *Astrophys. J.* **664** (2007), no. 1, 549–555.
6. L. D. Landau, E. M. Lifschitz, *Quantum Mechanics, Non-Relativistic Theory.* Pergamon Press, Oxford, England, 1977.
7. F. W. J. Olver, *Asymptotics and Special Functions.* Academic Press, New York-London, 1974.

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ON NONMEASURABLE UNIFORM SUBSETS OF THE EUCLIDEAN PLANE

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Abstract. It is shown that the cardinality continuum is not measurable in the Ulam sense if and only if for every nonzero σ-finite diffused measure μ on $\mathbf{R}^2$ there is a μ-nonmeasurable uniform subset of $\mathbf{R}^2$. Several related results are also considered.

The main goal of this communication is to discuss briefly uniform subsets of the Euclidean plane $\mathbf{R}^2$ in the context of their nonmeasurability in some generalized sense.

Let l be a straight line in the plane $\mathbf{R}^2$ considered as a certain direction in $\mathbf{R}^2$.

A set $Z \subset \mathbf{R}^2$ is called uniform in direction l if any line of $\mathbf{R}^2$, parallel to l, meets Z at most at one point.

A set $Z \subset \mathbf{R}^2$ is called a graph in direction l if any line of $\mathbf{R}^2$, parallel to l, meets Z exactly at one point.

Accordingly, we say that a set $Z \subset \mathbf{R}^2$ is uniform in $\mathbf{R}^2$ (is a graph in $\mathbf{R}^2$) if there exists a line l in $\mathbf{R}^2$ such that Z is uniform (is a graph) in direction l.

There were established interesting properties of uniform subsets of the plane, which are closely related to the Continuum Hypothesis (**CH**) and to certain propositions in the plane geometry (see, e.g., [1–3, 8, 9]).

Some other properties of uniform sets in $\mathbf{R}^2$ are connected (more or less) with the notion of measurability. To illustrate the above-said, let us give several examples.

1. Every uniform set is G-negligible, where G denotes the group of all translations of $\mathbf{R}^2$ (see [5, 6]).
2. There exist uniform sets which are not G-absolutely negligible (see again [5, 6]).
3. For any straight line l in $\mathbf{R}^2$, there exists a G-invariant measure μ_l on $\mathbf{R}^2$ which extends the standard Lebesgue measure λ_2 on $\mathbf{R}^2$ and is such that all uniform sets in direction l belong to $\mathrm{dom}(\mu_l)$ (it is clear that if Z is uniform in direction l, then $\mu_l(Z) = 0$).
4. There exists a graph in direction l, which is a Hamel basis of $\mathbf{R}^2$. Since every Hamel basis of $\mathbf{R}^2$ is G-absolutely negligible (see [4]), one can conclude that there exist G-absolutely negligible graphs in $\mathbf{R}^2$.
5. No finite family of uniform subsets of $\mathbf{R}^2$ can be a covering of $\mathbf{R}^2$ (see [8]).

Observe that the last fact easily follows from Banach's classical result stating that there exists a finitely additive translation invariant measure on $\mathbf{R}^2$, which extends λ_2 and is defined for all bounded subsets of $\mathbf{R}^2$. Notice also that the analogous fact remains valid for uniform hyper-surfaces in the multi-dimensional Euclidean spaces.

In the sequel, we need a simple auxiliary proposition.

Let l be any fixed straight line in $\mathbf{R}^2$ and let $Z \subset \mathbf{R}^2$ be uniform in direction l. The following two assertions are valid:

(a) every subset of Z is uniform in the same direction l;

(b) $Z = Z_1 \cap Z_2$, where Z_1 and Z_2 are two graphs in the same direction l.

Recall that a measure μ defined on some σ-algebra of subsets of a ground set E is diffused (or continuous) if all singletons in E belong to $\mathrm{dom}(\mu)$ and μ vanishes on all of them.

Also, recall that a cardinal number $\mathbf{a}$ is measurable in Ulam's sense if there exists a probability diffused measure whose domain is the power set of $\mathbf{a}$.

Theorem 1. *Let $\{l_j : j \in J\}$ be a countably infinite family of pairwise non-parallel directions in $\mathbf{R}^2$. The following two assertions are equivalent:*

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Key words and phrases. Euclidean plane; Uniform set; Nonmeasurable set; Cardinality of the continuum.

(1) *the cardinality continuum* $\mathbf{c}$ *is not measurable in Ulam's sense;*

(2) *for any nonzero* σ*-finite diffused measure* μ *on* $\mathbf{R}^2$*, there exist a direction* l_j *and a graph in this direction, which is nonmeasurable with respect to* μ*.*

The proof of Theorem 1 is essentially based on the profound result of Davies [3].

Remark 1. Let $\{l_k : k \in K\}$ be a fixed finite family of pairwise non-parallel directions in $\mathbf{R}^2$ and suppose that for any nonzero σ-finite diffused measure μ on $\mathbf{R}^2$ there exist a direction l_k and a uniform set in this direction, which is nonmeasurable with respect to μ. Then, using the result from [1], it can be shown that $\mathbf{c} = \omega_n$ for some natural number n. So, in this case, $\mathbf{c}$ is substantially restricted in its size and automatically turns out to be nonmeasurable in Ulam's sense.

Theorem 2. *Assume Martin's Axiom* ($\mathbf{MA}$) *and let* $\{l_j : j \in J\}$ *be a countably infinite family of pairwise non-parallel directions in* $\mathbf{R}^2$*.*

Then there exists a countable family $\{Z_t : t \in T\}$ *of sets in the plane* $\mathbf{R}^2$ *such that:*

(1) *every set* Z_t *is a graph in some direction* $l_{j(t)}$*, where* $j(t) \in J$*;*

(2) *for any nonzero* σ*-finite diffused measure* μ *on* $\mathbf{R}^2$*, at least one set from the family* $\{Z_t : t \in T\}$ *is nonmeasurable with respect to* μ*.*

The proof of Theorem 2 is again based on the result of Davies [3] and on the fact that under $\mathbf{MA}$ there exists a countable family $\{B_i : i \in I\}$ of subsets of $\mathbf{R}^2$, which is absolutely nonmeasurable with respect to the family of all nonzero σ-finite diffused measures on $\mathbf{R}^2$. Actually, the role of $\{B_i : i \in I\}$ can be played by a countable topological base of some generalized Luzin subset of $\mathbf{R}^2$.

Remark 2. Under the assumption that $\mathbf{c}$ is not measurable in Ulam's sense, the problem of generalized nonmeasurability can be considered for other classes of point sets in $\mathbf{R}^2$, e.g., for the class of all Vitali subsets of $\mathbf{R}^2$, for the class of all Bernstein subsets of $\mathbf{R}^2$, or for the class of all Hamel bases of $\mathbf{R}^2$ (cf. [6, 7]).

References

1. F. Bagemihl, Decompositions of the plane into sets, and coverings of the plane with curves. *Czechoslovak Math. J.* **18(93)** (1968), 616–621.
2. R. O. Davies, The power of the continuum and some propositions of plane geometry. *Fund. Math.* **52** (1963), 277–281.
3. R. O. Davies, Covering the plane with denumerably many curves. *J. London Math. Soc.* **38** (1963), 433–438.
4. A. B. Kharazishvili, On a property of Hamel bases. (Russian) *Bull. Acad. Sci. GSSR* **95** (1979), no. 2, 277–280.
5. A. B. Kharazishvili, *Transformation Groups and Invariant Measures*. Set-theoretical aspects. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.
6. A. B. Kharazishvili, On Mazurkiewicz sets from the measure-theoretical point of view. *Bull. TICMI* **21** (2017), no. 1, 45–54.
7. A. B. Kharazishvili, On the generalized nonmeasurability of Vitali sets and Bernstein sets. *Georgian Math. J.* `https://doi.org/10.1515/gmj-2020-2083`.
8. S. Mazurkiewicz, Sur la décomposition du plan en courbes. *Fund. Math.* **21** (1933), 43–45.
9. W. Sierpiński, *Hypothèse du Continu.* Monografje Matematyczne, Tome 4, Warszawa-Lwow, Subwencji Funduszu Kultur, 1934.

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WEIGHTED EXTRAPOLATION IN MIXED NORM FUNCTION SPACES

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Abstract. Rubio de Francía's weighted extrapolation results for pairs of functions in mixed-norm Banach function spaces defined on the product of quasi-metric measure spaces are obtained. As a consequence, we formulate appropriate results for the mixed-norm Lebesgue, Lorentz, Orlicz and grand Lebesgue spaces. Here we treat only the weighted extrapolation in grand Lebesgue spaces with mixed norms.

1. Introduction and Preliminaries

In this note we formulate weighted extrapolation theorems for pairs of functions (f, g) in mixed norm spaces defined on the product of quasi-metric measure spaces with doubling measures (spaces of homogeneous type). Let (X, d, μ) and (Y, ρ, ν) be the spaces of homogeneous type. We showed that if the one-weight inequality holds in the classical weighted Lebesgue space for all weights from the "strong" Muckenhoupt class $A^{(S)}(X \times Y)$ defined with respect to products of balls $B_1 \times B_2$, $B_1 \subset X$, $B_2 \subset Y$, then appropriate inequality holds for the same pair of functions in mixed-norm Banach function spaces $(E_1(X), E_2(Y))$ provided that the Hardy–Littlewood maximal operators M_X and M_Y are bounded in the spaces $\big(E_1^{1/q_0}\big)'(X)$ and $\big(E_2^{1/q_0}\big)'(Y)$, respectively, for some $q_0 > 1$ (for a similar result in the Euclidean setting see [12]). We treat both cases: diagonal and off-diagonal ones.

Rubio de Francía's extrapolation theory is one of the important tools to study the boundedness of integral operators in the weighted function spaces.

By taking $(f, g) = (f, Tf)$, as a special case, one can obtain one-weighted inequalities for that multiple operator T of Harmonic Analysis for which the strong Muckenhoupt condition guarantees the one-weighted boundedness. To such operators belong, for example, strong maximal operators, Calderón–Zygmund singular integrals with product kernels and multiple fractional integral operators. Based on the extrapolation result for the mixed-norm Lebesgue spaces we can derive, for example, the one-weight mixed-norm inequality due to D. Kurtz [18] regarding the strong maximal operator in mixed-norm Lebesgue spaces under the $A_q(A_p)$ condition on weights, and formulate an appropriate weighted extrapolation result.

One of the novelties of this note is that together with the extrapolation results we determine weighted bounds in terms of the weighted Muckenhuopt characteristics. The derived extrapolation results are applied to the weighted extrapolation in the mixed-norm Lebesgue, Lorentz, Orlicz and grand Lebesgue spaces. It should be emphasized that the majority of the results are new even for the case of Euclidean spaces with the Lebesgue measure.

Let (X, d, μ) be a quasi-metric measure space with a quasi-metric d and measure μ. In what follows, we will assume that the balls $B(x, r) := \{y \in X;\, d(x, y) < r\}$ with center x and radius r are measurable with positive μ for all $x \in X$ and $r > 0$.

If μ satisfies the doubling condition $\mu(B(x, 2r)) \le C_d \mu(B(x, r))$, with a positive constant C_d, independent of x and r, then we say that (X, d, μ) is a space of homogeneous type (SHT, shortly).

We assume that (X, d, μ) and (Y, ρ, ν) are the spaces of homogeneous type without atoms.

For the definition, examples and some properties of an SHT see, e.g., [3]. We also assume that the class of continuous functions is dense in L^1 defined on an SHT.

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For a given quasi-metric measure space (X,d,μ) and q, satisfying $1<q<\infty$, we denote as usual by $L^q(\mu)=L^q(X,\mu)$ the Lebesgue space equipped with the standard norm.

Let (X,d,μ) be an SHT. The Hardy–Littlewood maximal function defined on X and given by the formula

$$M_X f(x)=\sup_{B\ni x}\frac{1}{\mu(B)}\int\limits_B |f(y)|d\mu(y) \tag{1}$$

is the Hardy–Littlewood maximal operator defined on an SHT (X,d,μ).

For the sharp bounds of the norm of the maximal operator M_X in terms of characteristics of weights we refer to [13] and references cited therein.

Let $1<r<\infty$. We say that a weight function w defined on $X\times Y$ belongs to the Muckenhoupt class $A_r^{(S)}$ if

$$[w]_{A_r^{(S)}}:=\sup_{B_1\times B_2}\bigg(\frac{1}{\mu(B_1)\nu(B_2)}\int\limits_{B_1\times B_2} w\,d\mu\times\nu\bigg)\bigg(\frac{1}{\mu(B_1)\nu(B_2)}\int\limits_{B_1\times B_2} w^{1-r'}\,d\mu\times\nu\bigg)^{r-1}<\infty,$$

where the supremum is taken over all products of the balls $B_1\times B_2\subset X\times Y$.

Let $1<p,q<\infty$. Suppose that ρ is a μ-a.e. positive function on $X\times Y$ such that ρ^q is locally integrable. We say that $\rho\in\mathcal{A}_{p,q}^{(S)}$ if

$$[\rho]_{\mathcal{A}_{p,q}^{(S)}}:=\sup_{B_1\times B_2}\bigg(\frac{1}{\mu(B_1)\nu(B_2)}\int\limits_{B_1\times B_1}\rho^q\;d\mu\times\nu\bigg)\bigg(\frac{1}{\mu(B_1)\nu(B_2)}\int\limits_{B}\rho^{-p'}\,d\mu\times\nu\bigg)^{q/p'}<\infty,$$

where the supremum is taken over all products of balls $B_1\times B_2\in X\times Y$.

If $p=q$, then we denote $\mathcal{A}_{p,q}^{(S)}$ by $\mathcal{A}_p^{(S)}$.

Let E be a Banach function space (BFS) on X (for the Definition and some essential properties of BFSs, see [1]). For a BFS E, we denote by E' its Köthe dual (or associated) space.

Now we define the mixed-norm space for BFSs E_1 and E_2 defined on quasi-metric measure spaces (X,d,μ) and (Y,ρ,ν) respectively. The mixed-norm space, denoted by $(E_1(X),E_2(Y))$ (or simply, (E_1,E_2)), is defined with respect to the norm defined for the $\mu\times\nu$- measurable function $f:X\times Y\to\mathbb{R}$:

$$\|f\|_{(E_1,E_2)}=\Big\|\|f\|_{E_1}\Big\|_{E_2}.$$

It can be checked that (E_1,E_2) is a BFS.

For a Banach space E and $0<p<\infty$, the p-convexification of E is defined as follows:

$$E^p=\{f:|f|^p\in E\}.$$

E^p may be equipped with the quasi-norm $\|f\|_{E^p}=\||f|^p\|_E^{1/p}$. It can be observed that if $1\le p<\infty$, then E^p is a Banach space, as well. For $1\le p<\infty$ and BFSs E_1 and E_2, we have

$$(E_1,E_2)^p=(E_1^p,E_2^p).$$

Before formulating the main results we recall that Rubio de Francía's extrapolation in the setting of a strong Muckenhoupt condition was treated in [6] (see also [12,15]).

We say that a BFS E belongs to $\mathbb{M}(X)$ if the maximal operator M_X defined with respect to the balls $B\subset X$ (see (1)) is bounded in E. The class $\mathbb{M}(Y)$ is defined similarly.

To formulate the main results, we need the following notation:

$$[M_X,M_Y]:=\|M_X\|_{(E_1^{1/q_0})'}\|M_Y\|_{(E_2^{1/q_0})'};\;\;\overline{[M_X,M_Y]}:=\|M_X\|_{(\overline{E}_1^{1/\widetilde{q}_0})'}\|M_Y\|_{(\overline{E}_2^{1/\widetilde{q}_0})'}.$$

2. Main Results

Now we formulate the main extrapolation results for the mixed-norm BFSs.

Theorem 2.1 (Diagonal Case)**.** *Let $\mathcal{F}$ be a family of pairs (f,g) of measurable functions f and g defined on $X\times Y$. Suppose that for some $1<p_0<\infty$ and for every $w\in A_{p_0}^{(S)}$ and $(f,g)\in\mathcal{F}$, the one-weight inequality*

$$\Bigg(\int\limits_{X\times Y} g^{p_0}(x,y)w(x,y)\,d\mu\times\nu\Bigg)^{\frac{1}{p_0}}\le CN\big([w]_{A_{p_0}^{(S)}}\big)\Bigg(\int\limits_{X\times Y} f^{p_0}(x,y)w(x,y)\,d\mu\times\nu\Bigg)^{\frac{1}{p_0}}, \tag{2}$$

with some non-decreasing function $s\to N(s)$, holds. Suppose that there exists $1<q_0<\infty$ such that E_1^{1/q_0} and E_2^{1/q_0} are again $BFSs$. If $(E_1^{1/q_0})'\in\mathbb{M}(X)$ and $(E_2^{1/q_0})'\in\mathbb{M}(Y)$, then for any $(f,g)\in\mathcal{F}$,

$$\|g\|_{(E_1,E_2)}\le 16^{1/q_0}C\widetilde{C}J\big([M_X,M_Y],p_0,q_0\big)\|f\|_{(E_1,E_2)},$$

where the positive constant C is defined in (2),

$$\begin{aligned}
&J\big([M_X,M_Y],p_0,q_0\big)\\
&:=\begin{cases} N\Big(2^{p_0-q_0}(\bar{c}q_0')^{2(p_0-q_0)}\big([M_X,M_Y]\big)^{(2((q_0)'-1)+1)(p_0-q_0)}\Big), & q_0<p_0\\ N\Big(2^{(q_0-p_0)/(q_0-1)}(\bar{c}(q_0)')^{2(q_0-p_0)/(q_0-1)}\big([M_X,M_Y]\big)^{(2q_0-p_0-1)(q_0-1)}\Big), & q_0>p_0\end{cases}
\end{aligned}$$

and $\widetilde{C}$ is defined by $\widetilde{C}=\max\Big\{2,2^{(q_0-p_0)/(q_0p_0-p_0)}\Big\}$.

Theorem 2.2 (Off-diagonal Case)**.** *Let $\mathcal{F}$ be a family of pairs (f,g) of measurable functions $f,g\in L^0(\mu\times\nu)$ defined on $X\times Y$. Suppose that for some $1<p_0\le q_0<\infty$ and for every $w\in A_{1+q_0/p_0'}^{(S)}$ and $(f,g)\in\mathcal{F}$, the one-weight inequality*

$$\Bigg(\int\limits_{X\times Y} g^{q_0}(x,y)w(x,y)\,d\mu\times\nu\Bigg)^{\frac{1}{q_0}}\le CN\big([w]_{A_{1+\frac{q_0}{p_0'}}^{(S)}}\big)\Bigg(\int\limits_{X\times Y} f^{p_0}(x,y)w^{\frac{p_0}{q_0}}(x,y)\,d\mu\times\nu\Bigg)^{\frac{1}{p_0}}, \tag{3}$$

with some positive constant C and non-decreasing function $s\to N(s)$, holds. Suppose that there exist $1<\widetilde{p}_0<\infty$, $1<\widetilde{q}_0<\infty$ such that

$$\frac{1}{\widetilde{p}_0}-\frac{1}{\widetilde{q}_0}=\frac{1}{p_0}-\frac{1}{q_0},$$

and $\overline{E}_1(X)^{1/\widetilde{q}_0}$, $E_1(X)^{1/\widetilde{p_0}}$, $\overline{E}_2(Y)^{1/\widetilde{q}_0}$, $E_2(Y)^{1/\widetilde{p_0}}$ are $BFSs$, and also the following condition

$$\big(\overline{E}_1(Y)^{1/\widetilde{q}_0}\big)'=\Big[\big(E_1(Y)^{1/\widetilde{p}_0}\big)'\Big]^{\widetilde{p}_0/\widetilde{q}_0};\ \ \big(\overline{E}_2(Y)^{1/\widetilde{q}_0}\big)'=\Big[\big(E_2(Y)^{1/\widetilde{p}_0}\big)'\Big]^{\widetilde{p}_0/\widetilde{q}_0}$$

is satisfied.

If $\Big(\overline{E}_1^{1/\widetilde{q}_0}\Big)'\in\mathbb{M}(X)$ and $\Big(\overline{E}_2^{1/\widetilde{q}_0}\Big)'\in\mathbb{M}(Y)$, then for any $(f,g)\in\mathcal{F}$,

$$\|g\|_{(\overline{E}_1,\overline{E}_2)}\le 16^{\widetilde{q}_0}\,C\overline{C}\overline{J}\big(\overline{[M_X,M_Y]},p_0,q_0,\widetilde{p}_0,\widetilde{q}_0\big)\|f\|_{(E_1,E_2)},$$

where the constant C is the same as in (3),

$$\begin{aligned}
&\overline{J}\big(\overline{[M_X,M_Y]},p_0,q_0,\widetilde{p}_0,\widetilde{q}_0\big)\\
&:=\begin{cases} N\Bigg[\Big(2\overline{c}^2\Big(1+\dfrac{\widetilde{q}_0'}{\widetilde{q}_0}\Big)^2\Big)^{\gamma(\widetilde{q}_0-q_0)}\big(\overline{[M_X,M_Y]}\big)^{1+2\frac{\gamma\widetilde{q}_0(q_0-q)}{\widetilde{q}_0'}}\Bigg], & \widetilde{q}_0<q_0,\\ N\Bigg[\Big(2\overline{c}^2\Big(1+\frac{\widetilde{q}_0}{\widetilde{p}_0'}\Big)^2\Big)^{\frac{\gamma(\widetilde{q}_0-q_0)}{\gamma\widetilde{q}_0-1}}\big(\overline{[M_X,M_Y]}\big)^{(2\gamma\widetilde{q}_0-\gamma q_0-1)/(\gamma\widetilde{q}_0-1)}\Bigg], & \widetilde{q}_0>q_0,\end{cases}
\end{aligned}$$

and $\overline{C}:=\max\Bigg\{2^{\gamma\widetilde{q}_0\left(\frac{1}{\widetilde{q}_0}-\frac{1}{q_0}\right)},2^{\gamma(\widetilde{p}_0)'\left(\frac{1}{p_0}-\frac{1}{\widetilde{p}_0}\right)}\Bigg\}$.

As corollaries, we have appropriate extrapolation results for the mixed-norm Lebesgue, Lorentz, Orlicz and grand Lebesgue spaces. Here, we give the statements about only grand Lebesgue spaces with mixed norms.

In 1992, T. Iwaniec and C. Sbordone [14], in their studies related to the integrability properties of the Jacobian in a bounded open set Ω, introduced a new type of function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. A generalized version of spaces $L^{p),\theta}(\Omega)$ can be found in the work of L. Greco, T. Iwaniec and C. Sbordone [11].

Harmonic analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), was intensively studied during the last years due to their various applications, we mention here, e.g., [2, 7–10], the monograph [17] and references therein.

To formulate and prove the main result of this section we need to introduce the following notation: let σ_i, $i = 1, 2$, be sufficiently small positive numbers and let $\psi_1(\cdot)$ and $\psi_2(\cdot)$ be n-tuple positive increasing functions on the intervals $(0, \sigma_i)$, $i = 1, 2$, such that $\lim\limits_{\lambda\to 0} \psi_i(\lambda) = 0$, $i = 1, 2$. In this case, we say that $\psi_i \in \Psi_{\sigma_i}$, $i = 1, 2$.

We say that for weight functions u and v on X and Y, respectively, a function $f : X \times Y$ belongs to $\big(L_u^{p_1),\psi_1(\cdot),\sigma_1}(X), L_v^{p_2),\psi_2(\cdot),\sigma_2}(Y)\big)$ $1 < p_1, p_2 < \infty$, if

$$\|f\|_{\left(L_u^{p_1),\psi_1(\cdot),\sigma_1}(X), L_v^{p_2),\psi_2(\cdot),\sigma_2}(Y)\right)}$$
$$= \sup_{0<\varepsilon_1<\sigma_1} \sup_{0<\varepsilon_2<\sigma_2} \left(\psi_2(\varepsilon_2) \int\limits_Y \left(\psi_1(\varepsilon_1) \int\limits_X |f(x,y))|^{p_1-\varepsilon_1} u(x) d\mu(x)\right)^{\frac{p_2-\varepsilon_2}{p_1-\varepsilon_1}} v(y) d\nu(y)\right)^{\frac{1}{p_2-\varepsilon_2}} < \infty.$$

If $\psi_i(\cdot) \equiv 1$, $i = 1, 2$, then the space $\big(L^{p_1),\psi_1(\cdot),\sigma_1}(X), L^{p_2),\psi_2(\cdot),\sigma_2}(Y)\big)$ is the mixed norm Lebesgue space. Further, if $\psi_i(\cdot) = \theta_i$, $i = 1, 2$, then we denote $\big(L^{p_1),\psi_1(\cdot),\sigma_1}(X), L^{p_2),\psi_2(\cdot),\sigma_2}(Y)\big)$ by $\big(L^{p_1),\theta_1,\sigma_1}(X),$ $L^{p_2),\theta_2,\sigma_2}(Y)\big)$.

Theorem 2.3. *Let $\mathcal{F}$ be a family of pairs (f, g) of non-negative functions $f, g \in L^0(\mu \times \nu)$ defined on $X \times Y$. Suppose that for some $1 \le p_0 < \infty$ and for every $w \in A_{p_0}^{(S)}(X \times Y)$ and $(f, g) \in \mathcal{F}$, the one-weight inequality*

$$\left(\int\limits_{X\times Y} g^{p_0}(x,y) w(x,y)\, d\mu \times \nu\right)^{\frac{1}{p_0}} \le CN\big([w]_{A_{p_0}^{(S)}}\big) \left(\int\limits_{X\times Y} f^{p_0}(x,y) w(x,y)\, d\mu \times \nu\right)^{\frac{1}{p_0}}, \tag{4}$$

with some non-decreasing function $s \to N(s)$, holds. Then there is a positive constant C such that for every $1 < p_1, p_2 < \infty$, $\psi_i \in \Psi_{\sigma_i}$, $i = 1, 2$, and $u \in A_{p_1}(X)$, $v \in A_{p_2}(Y)$,

$$\|g\|_{(L_u^{p_1),\psi_1(\cdot),\sigma_1}(X), L^{p_1),\psi_2(\cdot),\sigma_2}(Y))} \le C\|f\|_{(L_u^{p_1),\psi_1(\cdot),\sigma_1}(X), L^{p_1),\psi_2(\cdot),\sigma_2}(Y))},$$

where $\sigma_1, \sigma_2 > 0$ are the numbers such that $u \in A_{p_1-\sigma_1}(X)$, $u \in A_{p_2-\sigma_2}(X)$.

Theorem 2.4 (Off-diagonal Case)**.** *Let $\mathcal{F}$ be a family of pairs (f, g) of non-negative functions $f, g \in L^0(\mu\times\nu)$ defined on $X\times Y$. Suppose that for some $1 < p_0 \le q_0 < \infty$ and for every $w \in A_{1+q_0/p_0'}^{(S)}(X\times Y)$ and $(f, g) \in \mathcal{F}$, the one-weight inequality*

$$\left(\int\limits_{X\times Y} g^{q_0}(x,y) w(x,y)\, d\mu \times \nu\right)^{\frac{1}{q_0}} \le CN\big([w]_{A_{1+\frac{q_0}{p_0'}}^{(S)}}\big) \left(\int\limits_{X\times Y} f^{p_0}(x,y) w^{\frac{p_0}{q_0}}(x,y)\, d\mu \times \nu\right)^{\frac{1}{p_0}},$$

with some non-decreasing function $s \to N(s)$, holds.

Then for any $1 < p_1, q_1, p_2, q_2 < \infty$, satisfying the condition

$$\frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p_2} - \frac{1}{q_2} = \frac{1}{p_0} - \frac{1}{q_0},$$

any $\theta_1, \theta_2 > 0$, $u \in \mathcal{A}_{p_1,q_1}(X)$, $v \in \mathcal{A}_{p_2,q_2}(Y)$, and for all $(f, g) \in \mathcal{F}$

we have

$$\Big\| \|v(y)\|u(x)g(x,y)\|_{L_u^{q_1),\frac{\theta_1 q_1}{p_1},\sigma_1}(X)} \Big\|_{L_v^{q_2),\frac{\theta_2 q_2}{p_2},\sigma_2}(Y)} \le C \Big\| \|v(y)\|u(x)f(x,y)\|_{L_u^{p_1),\theta_1,\eta_1}} \Big\|_{L_v^{p_2),\theta_2,\eta_2}},$$

with a positive constant C independent of (f,g), constants σ_i and η_i, $i=1,2$ satisfying the condition

$$\frac{1}{p_1-\eta_1}-\frac{1}{q_1-\sigma_1}=\frac{1}{p_2-\eta_2}-\frac{1}{q_2-\sigma_2}=\frac{1}{p_0}-\frac{1}{q_0},$$

where σ_1, σ_2, η_1 and $\eta_2>0$ are positive numbers such that $u^{q_1}\in A_{1+(q_1-\sigma_1)/(p_1-\eta_2)'}(X)$, $v^{q_1}\in A_{1+(q_2-\sigma_2)/(p_2-\eta_2)'}(Y)$.

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References

1. C. Bennett, R. Sharpley, *Interpolation of Operators.* Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.
2. C. Capone, A. Fiorenza, On small Lebesgue spaces. *J. Funct. Spaces Appl.* **3** (2005), no. 1, 73–89.
3. R. R. Coifman, G. Weiss, *Analyse Harmonique Non-commutative sur Certains Espaces Homogénes.* (French) Lecture Notes in Mathematics, vol. 242. Springer-Verlag, Berlin-New York, 1971.
4. D. Cruz-Uribe, C. Pérez, Two weight extrapolation via the maximal operator. *J. Funct. Anal.* **174** (2000), no. 1, 1–17.
5. D. Cruz-Uribe, J. Martell, C. Perź, *Weights, Extrapolation and the Theory of Rubio de Francia.* Operator Theory: Advances and Applications, 215. Birkhäuser/Springer Basel AG, Basel, 2011.
6. J. Duoandikoetxea, Extrapolation of weights revisited: new proofs and sharp bounds. *J. Funct. Anal.* **260** (2011), no. 6, 1886–1901.
7. A. Fiorenza, Duality and reflexivity in grand Lebesgue spaces. *Collect. Math.* **51** (2000), no. 2, 131–148.
8. A. Fiorenza, G. E. Karadzhov, Grand and small Lebesgue spaces and their analogs. *Z. Anal. Anwendungen* **23** (2004), no. 4, 657–681.
9. A. Fiorenza, B. Gupta, P. Jain, The maximal theorem for weighted grand Lebesgue spaces. *Studia Math.* **188** (2008), no. 2, 123–133.
10. G. Di Fratta, A. Fiorenza, A direct approach to the duality of grand and small Lebesgue spaces. *Nonlinear Anal.* **70** (2009), no. 7, 2582–2592.
11. L. Greco, T. Iwaniec, C. Inverting the p-harmonic operator. *Manuscripta Math.* 92 (1997), no. 2, 249–258.
12. K.-P. Ho, Strong maximal operator on mixed-norm spaces. *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **62** (2016), no. 2, 275–291. `https://doi.10.1007/s11565-016-0245-z`.
13. T. P. Hytönen, C. Pérez, E. Rela, Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type. *J. Funct. Anal.* **263** (2012), no. 12, 3883–3899.
14. T. Iwaniec, C. Sbordone, On the integrability of the Jacobian under minimal hypotheses. *Arch. Rational Mech. Anal.* **119** (1992), no. 2, 129–143.
15. V. Kokilashvili, A. Meskhi, Extrapolation results in grand Lebesgue spaces defined on product sets. *Positivity* **22** (2018), no. 4, 1143–1163.
16. V. Kokilashvili, A. Meskhi, Extrapolation in weighted classical and grand Lorentz spaces. Application to the boundedness of integral operators. *Banach J. Math. Anal.* **14** (2020), no. 3, 1111–1142. `https://doi.10.1007/s43037-020-00054-1`.
17. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-standard Function Spaces.* vol. 2. Variable exponent Hölder, Morrey-Campanato and grand spaces. Operator Theory: Advances and Applications, 249. Birkhäuser/Springer, 2016.
18. D. Kurtz, Classical operators on mixed-normed spaces with product weights. *Rocky Mountain J. Math.* **37** (2007), no. 1, 269–283.

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ON POISSON TYPE INTEGRALS IN THE POLYBALL

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Abstract. We consider some natural extensions of Poisson integral in the unit ball to polyballs and extend some known classical results to the case of product domains (polyballs). In particular, we extend some known results in the unit ball on Poisson integrals related to BMO to the product domain case.

1. Introduction

Let $B_n = |z| < 1$ be the unit ball in $\mathbb{C}^n$, $S^n = \partial B$ be the unit sphere in $\mathbb{C}^n$.

Let $d(z,w) = |1 - \langle z, w\rangle|^{\frac{1}{2}}$, $z, w \in \bar{B}_n$, be the restriction of d on S_n; it is a non-isotropic metric (see [4,6]).

Let also $Q(\xi, r) = \left\{\eta \in S_n : |1 - \langle \xi, \eta\rangle|^{\frac{1}{2}} < r\right\}$, $r > 0$, $\xi \in S_n$. We call Q a d-ball putting r sometimes as a subscript for the extension to the ball, $S_n = \{|z| = 1\}$.

We denote various constants appearing in this paper by C, C_1, C_2, c. As usual, we define both a Poisson kernel $P(z,w)$, $z, w \in \bar{B}_n$, $P(z,w) = \frac{(1-|z|^2)^n}{|1-\langle \bar{z}, w\rangle|^{2n}}$, and a Poisson integral of a positive Borel measure μ $P(\mu)$ in a standard way (see, e.g., [6]). For some new and classical results on these objects we refer the reader to [4,6] and [2,5]. By $d\sigma$ we denote the Lebesgue measure on S_n. For the d function we refer to [4,6] (see also below).

In this note we discuss some problems in an open new research area of Poisson type integrals in product domains in $\mathbb{C}^n$. Some new objects in this note will be defined and some interesting new problems will be posed and solved. We provide first known results in a unit ball (see [6]).

It has been shown in [4,6] that for an f function of a certain class (these estimates were used in the study of analytic BMO)

$$\int\limits_{S_n} |f(\eta) - f(a)|^2 P(a,\eta) d\sigma(\eta) \geq \left(\frac{\tilde{c}}{\sigma(Q)} \int\limits_{Q} |f - f(a)|^2 d\sigma\right), \tag{1}$$

we can see that for all $\frac{3}{4} < |a| < 1$,

$$\int\limits_{S_n} |f(\xi) - f(a)|^2 P(a,\xi) d\sigma(\xi) \leq c(\sup)\Big(\frac{1}{\sigma(Q)}\Big) \int\limits_{Q} |f - f_Q|^2 d\sigma < \infty \tag{2}$$

(see [4,6]). Next, it has been shown (see [4,6]) that

$$\Big(\sup_{z\in B_n}\Big) \int\limits_{B_n} P(z,w) d\mu(w) \geq \left(\frac{\mu(Q_r(\xi))}{4^n r^{2n}}\right), \tag{3}$$

for a positive Borel measure μ in B.

And if $c = \Big(\sup\limits_{\xi, r}\Big) \frac{\mu(Q_r(\xi))}{r^{2n}}$, then we have

$$\Big(\sup_{|z|>\frac{3}{4}}\Big) \int\limits_{B_n} (P(z,w)) d\mu(w) \leq (c16^n) \sum_{k\geq 0}^{\infty} \Big(\frac{1}{2^{nk}}\Big), \tag{4}$$

(see [4,6]) for a positive μ Borel measure.

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The question is how to extend these and other similar results for a positive μ Borel measure to more general situation if, for example, we consider more general Poisson type kernels of the type $\widetilde{P}(\vec{z},\xi) = \frac{\prod^{n}(1-|z|^2)^{\beta_j}}{\prod\limits_{j=1}^{n}|1-z_j\xi|^{\alpha_j}}$, where

$$\sum_{i=1}^{m}\beta_j = n, \quad \sum_{i=1}^{m}\alpha_j = 2n,$$

with $z_j \in B$, $j = 1,\dots,n$, $\xi \in S$ (we assume sometimes $|z_j| = |z|$) and to a group of positive Borel (μ_j) measures, where $j = 1,\dots,n$.

In this paper, we have found some ways on how to extend (2)–(4) to this more general situation. These results may have various applications in the function theory. Complete proofs will be provided in a separate note. We simply modify already known proofs provided in one domain.

Indeed, a natural idea consists in finding some ways to modify old and known proofs to the product domain case. However, there exist some technical difficulties.

Following the proof for the case $m = 1$ (see [4,6]), for a positive Borel μ measure we find

$$\sup_{z_j\in B_n}\left(\int\limits_{B_n}\cdots\left(\int\limits_{B_n}\frac{\left[\prod\limits_{j=1}^{m}(1-|z_j|)^{\alpha_j}\right]^{p_1}d\mu(w_1)}{\prod\limits_{j=1}^{m}|1-\langle z_j,w_j\rangle|^{\beta_j p_1}}\right)^{\frac{p_2}{p_1}}\cdots\mu(w_m)\right)^{\frac{1}{p_m}} \geq c\frac{(\mu(Q_r(\xi)))^{\sum\limits_{i=1}^{m}\frac{1}{p_i}}}{r^{2n}}, \qquad (\widetilde{A})$$

where $Q_r(\xi) = \{z \in B_n : d(z,\xi) < r\}$, $\xi \in S_n$; $r > 0$, $\sum\limits_{j=1}^{m}\alpha_j = n$, $\sum\limits_{j=1}^{m}\beta_j = 2n$, $\beta_j > 0$; $\alpha_j > 0$, $j = 1,\dots,m$, $0 < p_i < \infty$, $i = 1,\dots,m$.

Next, we have the following known estimate (see [6])

$$\left(\sup_{|z|>\frac{3}{4}}\right)\int\limits_{B_n}P(z,w)d\mu(w) \leq 16^n c\left(\sum_{k=0}^{\infty}\frac{1}{2^{nk}}\right), \quad \text{where} \qquad (\tilde{\tilde{A}})$$

$$P(z,w) = \frac{(1-|z|^2)^n}{|1-\langle\overline{z},w\rangle|^{2n}},\ c = \left(\sup_{\xi,r}\right)\left(\frac{\mu(Q_r(\xi))}{r^{2n}}\right), \quad n \in \mathbb{N}, \quad z,w \in \mathbb{B}_n.$$

The natural question to give an extension of this $(\tilde{\tilde{A}})$ estimate to

$$\widetilde{M} = \left(\sup_{|z_j|>\frac{3}{4}}\right)\left(\int\limits_{B_n}\cdots\int\limits_{B_n}(\widetilde{P}(\vec{z},\vec{w}))d\mu_1(w_1)\dots d\mu_m(w_m)\right)^{\frac{1}{p_m}},$$

where

$$\widetilde{P}(\vec{z},\vec{w}) = \frac{\prod\limits_{j=1}^{m}(1-|z_j|)^{\alpha_j}}{\prod\limits_{j=1}^{m}(1-\langle z_j,w_j\rangle)^{\beta_j}},$$

and

$$\sum_{j=1}^{m}\alpha_j = n, \quad \sum_{j=1}^{m}\beta_j = 2n, \quad 0 < p_j < \infty, \quad z_j,\ w_j \in B_n, \quad j = 1,\dots,m,$$

following carefully one functional known proof (see [4,6]).

Note that here μ_j are positive Borel measures on B, $i = 1,\dots,m$.

We have found the following generalization:

$$\left(\sup_{\substack{|z_j|>\tau\\|z_j|=R}}\right)\int\limits_{B_n}\frac{\prod\limits_{j=1}^{m}(1-|z_j|)^{\alpha_j}d\mu(w)}{\prod\limits_{j=1}^{m}|(1-z_jw)^{\beta_j}|} \leq (16^n)\mathbb{C}\left(\sum_{k=0}^{\infty}\frac{1}{2^{nk}}\right) \leq C_n\cdot C; \qquad \text{(B)}$$

where $\sum_{j=1}^{m} \beta_j = 2n$, $\sum_{j=1}^{m} \alpha_j = n$, for some positive constant $\mathbb{C}_n$.

We mention another known estimate and provide below some similar type extensions to the polyballs.

Note (see [4,6]) that for $Q = Q(\frac{a}{|a|,\ \sqrt{1-|a|^2}})$, $a \in B$, $a \neq 0$,

$$I_a = \int_{S_n} |f(\xi) - f(a)|^2 \frac{(1-|a|^2)^n}{|1-\overline{\xi}a|^{2n}} d\sigma(\xi) \geq \frac{1}{4^n(1-|a|^2)^n}$$

and $\int_Q |f - f(a)|^2 d\sigma \geq \frac{\text{const}}{\sigma(Q_1)} \int_{Q_1} |f - f(a)|^2 d\sigma$, where $Q = Q\Big(\frac{a}{|a|}, \sqrt{1-|a|^2}\Big)$ as runs over $B_n/0$, whereas the above Q_1 runs over all d balls of radius less than 1 (see [4,6]). We provide some generalizations of such estimates.

Let now $f \in L^{p_1}(S_n \times \cdots \times S_n)$,

$$I_{\vec{a}} = \Bigg(\int_{S_n} \cdots \Bigg(\int_{S_n} |f(\xi_1,\ldots,\xi_n) - f(a_1,\ldots,a_m)|^{p_1} \times \frac{\prod_{i=1}^{m}(1-|a_i|^2)^{\alpha_i}}{\prod_{i=1}^{m}|1-\overline{\xi}_i a_i|^{\beta_i}} d\sigma(\xi_1)\Bigg)^{\frac{p_2}{p_1}} \cdots d\sigma(\xi_m)\Bigg)^{\frac{1}{p_m}},$$

$$0 < p_i < \infty, \quad i = 1,\ldots,m, \quad \sum_{i=1}^{m} \alpha_i = n, \quad \sum_{i=1}^{m} \beta_j = 2n, \quad \alpha_j, \beta_j > 0, \quad j = 1,\ldots,m.$$

Then

$$I_{\vec{a}} \geq \frac{4^{-\frac{n}{p_1}}}{\prod_{i=1}^{m}(1-|a_i|^2)^{\frac{(2\beta_i - \alpha_i)}{p_1}}} \Bigg(\int_Q \cdots \Bigg(\int_Q |f(\xi_1,\ldots,\xi_n) - f(a_1,\ldots,a_m)|^{p_1} d\sigma(\xi_1)\Bigg) \cdots d\sigma(\xi_m)\Bigg)^{\frac{1}{p_m}} \tag{C}$$

$$Q = Q\Big(\frac{a}{|a|}, \sqrt{1-|a|^2}\Big), \quad a \in B_n, \quad a \neq 0, \quad 2\beta_i - \alpha_i > 0, \quad i = 1,\ldots,m.$$

And for same parameters, let

$$\widetilde{I}_{\vec{a}} = \Bigg(\int_{S_n} \cdots \Bigg(\int_{S_n} \prod_{i=1}^{m} |f_i(\xi_i) - f_i(a_i)|^{p_1} \times \frac{\prod_{i=1}^{m}(1-|a_i|)^{\alpha_i}}{\prod_{i=1}^{m}|1-\overline{\xi}_i a_i|^{\beta_i}} d\sigma(\xi_1)\Bigg)^{\frac{p_2}{p_1}} \cdots d\sigma(\xi_m)\Bigg)^{\frac{1}{p_m}}.$$

Then we also have

$$\widetilde{I}_{\vec{a}} \geq \frac{4^{-\frac{n}{p_1}}}{\prod_{i=1}^{m}(1-|a_i|^2)^{\frac{(2\beta_i - \alpha_i)}{p_1}}} \prod_{i=1}^{m} \Bigg(\int_Q |f_i(\xi_i) - f_i(a_i)|^{p_i}\Bigg)^{\frac{1}{p_i}}, \tag{$\tilde{C}$}$$

where $f_i \in L^{p_1}(B)$, $i = 1,\ldots,m$; the proof is based on the estimate

$$(|1 - (\overline{a},\xi)|) = 1 - \Big(\frac{a}{|a|}, \xi\Big) + (1-|a|)\Big(\frac{a}{|a|}, \xi\Big)$$

and, hence, $|1 - a\overline{\xi}| \leq 2(1-|a|^2)$ (see [4,6]), where $\xi \in Q$ and is similar to one domain proof (see [4,6]).

The above estimates may have various applications. In this note we omit the details of proofs of the last estimates refereing to [4,6] for complete elegant proofs of simpler “one domain” cases.

We have also the following known estimate (see [4,6]):

$$\int_{S_n} |f(\xi) - f(a)|^P \frac{(1-|a|^2)}{|1-a\xi|^{2n}} d\sigma \leq c\, C_*^{t,p}, \tag{A}$$

$r_0 < |a| < 1$, for some constant C, where

$$C_*^{t,p} = (\sup)\Big(\frac{1}{\sigma(Q)^t}\Big)\int\limits_Q |f - f_Q|^P d\sigma < \infty, \quad f \in H^P(B), \quad (f_Q) = \frac{1}{\sigma(Q)}\Bigg(\int\limits_Q f d\sigma\Bigg).$$

We refer to [4,6] for $t = 1$, $p = 2$ case of (A).

We wish to extend (A) again using an extension of classical Poisson kernel $P(z, \xi)$ and carefully studying the classical known proof in [1,2]. We have found the following result, the main result of this note. Here we formulate this result.

Theorem 1. *Let $f \in L^P(S)$, $p \geq 1$. Then we have*

$$\Bigg(\sup_{\substack{z_0<|a_j|<1\\ j=1,\dots,m}}\Bigg)\int\limits_{S^n} |f(\xi) - f_{(Q_0)}|^P \times \prod_{j=1}^{m} \frac{(1-|a_j|)^{\alpha_j}}{|1-\xi a_j|^{\beta_j}} d\sigma(\xi) \leq \widetilde{C} C_*^{1,p}, \tag{1}$$

for some positive constant C, where α_j, $\beta_j > 0$, $j = 1, \dots, m$, $Q_0 = Q\left(\frac{a_1}{|\widetilde{a}|}, \sqrt{1-|\widetilde{a}|}\right)$, $a_1 \in B$ $|\widetilde{a}| > r_0$, $\sum\limits_{j=1}^{m} \beta_j = \sum\limits_{j=1}^{m} \alpha_j + n$.

Also,

$$\Big(\sup_j\Big)\int\limits_{S_n} |f(\xi) - f(a_j)|^P \cdot \widetilde{P}(\vec{a}, \xi) d\sigma(\xi) \leq C_\sigma \left(C_*^{1,p}\right), \tag{2}$$

for all $|a_j| \in [r_0, 1)$, $|a_j| = |\tilde{a}|$, $a_j = \tilde{a}|\varphi_j$, $\widetilde{P}(\vec{a}, \xi) = \tilde{P} = \left(\prod\limits_{j=1}^{m} \frac{(1-|a_j|)^{\alpha_j}}{|1-\xi a_j|^{\beta_j}}\right)$, where

$$\sum_{j\geq 1}^{m} \beta_j = \sum_{j\geq 1}^{m} \alpha_j + n, \;\; f \in H^P, \;\; 1 \leq p < \infty, \;\; j = 1, \dots, m.$$

Putting $f_Q^S = \frac{1}{\sigma(Q)^S}\Big(\int\limits_Q f d\sigma\Big)$, $S > 0$, we can similarly provide another version of our theorem with other restrictions on α_j, β_j, $j = 1, \dots, \bar{m}$, extending classical results.

It will be interesting to consider another object

$$f_Q^S = \frac{1}{\sigma(\tilde{Q})^S}\Bigg(\int\limits_{\tilde{Q}} f d\sigma\Bigg), \quad \tilde{Q} = Q \times Q, \quad S > 0,$$

on the product domains and to prove similar to our theorem result for those domains, as well.

References

1. I. Chyzhykov, O. Zolota, Growth of the Poisson-Stieltjes integral in a polydisc. *Zh. Mat. Fiz. Anal. Geom.* **7** (2011), no. 2, 141–157, 196, 198.
2. G. Kresin, V. Maz'ya, Sharp estimates for the gradient of the generalized Poisson integral for a half-space. *Georgian Math. J.* **25** (2018), no. 2, 283–290.
3. V. Maz'ya, G. Kresin, Generalized Poisson integral and sharp estimates for harmonic and biharmonic functions in the half-space. *Math. Model. Nat. Phenom.* **13** (2018), no. 4, 37pp.
4. W. Rudin, *Function Theory in the Unit Ball of C^n*. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008.
5. R. Shamoyan, A note on Poisson type integrals in pseudoconvex and convex domains of finite type and some related results. *Acta Univ. Apulensis Math. Inform.* **50** (2017), 19–27.
6. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball.* Graduate Texts in Mathematics, 226. Springer-Verlag, New York, 2005.

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TWO-DIMENSIONAL UNSTEADY PULSATION FLOW OF A VISCOUS INCOMPRESSIBLE FLUID BETWEEN THE POROUS WALLS

VARDEN TSUTSKIRIDZE

Abstract. Two-dimensional unsteady pulsation flow of a viscous incompressible fluid through a porous channel is considered. This motion gets excited from the periodical time change of a pressure drop and a percolation velocity.

1. Introduction

The problems of viscous conducting fluid flows in channels are classical problems of magnetic hydrodynamics. Beginning with the work of Hartmann, who considered the flow in a planar channel, up to present days, a considerable number of studies have been devoted to this issue. Recently, interest in such flows has increased due to applications to MHD generators. There we have to deal with the flow of a conducting fluid in a common rectangular cross-section channel with two non-conducting and two conducting walls, with a transverse magnetic field applied along the latter walls. A similar problem for perfectly conducting electrodes and ideally insulating sidewalls was solved in the articles [1,2,6,8,9,12]. However, its solution was either not obtained in a finite form, or it was impossible to obtain integral characteristics of the flow for large Hartmann numbers from a formal solution.

The approximate method presented below gives the possibility to find a solution in a practically convenient form, as well as take into account the final conductivity of the channel walls.

2. Basic Part

Let us consider the unsteady flow of a viscous fluid in a porous channel with a constant cross-section. If ox is directed in parallel to the walls, and the axis oy is perpendicular to them, then the equations of non-steady two-dimensional motion of a viscous incompressible fluid will be as in [3–5,7,10,11,13–15]:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right),\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.
\end{aligned}\tag{1}$$

The desired velocities $u(x,y,t)$ and $v(x,y,t)$ must satisfy the following limiting conditions:

$$\begin{aligned}
u(x,y,0)=&0, \quad v(x,y,0)=0,\\
u(x,-h,t)=&0, \quad v(x,-h,t)=v_{w_1}(t),\\
u(x,h,t)=&0, \quad v(x,h,t)=v_{w_2}(t).
\end{aligned}\tag{2}$$

Let us introduce the following dimensionless quantities:

$$u=u_0u_1,\quad v=v_0v_1,\quad x=lx_1=\frac{u_0}{v_0}hx_1,\quad y=hy_1,\quad t=\frac{A^2}{\nu}t_1,$$

$$\rho=\frac{\nu u_0^2}{v_0 h}P_1,\quad v_{w_1}=v_0v_{01},\quad v_{w_2}=v_0v_{02}.$$

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Then from system (1) we will have equations in a dimensionless form:

$$\begin{aligned}
&\left(\frac{v_0}{u_0}\right)^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial t}=R_0\Big(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\Big)+\frac{\partial p}{\partial x},\\
&\left(\frac{v_0}{u_0}\right)^2\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-\frac{\partial v}{\partial t}=R_0\Big(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\Big)+\Big(\frac{u_0}{v_0}\Big)\frac{\partial p}{\partial y},\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,
\end{aligned}\tag{3}$$

where u_0,v_0 - are the characteristic average velocity and rate of infiltration, accordingly;
l - is the length of the channel,
h - is half distance between the walls,
$R_0=\frac{v_0 h}{\nu}$ - is the number of Reynolds infiltration. In system (3) indices are down for the sake of simplicity.

We are looking for solutions of system (3) in the following form:

$$u(x,y,t)=(1-x)\frac{\partial f(y,t)}{\partial y},\quad v(x,y,t)=f(y,t).$$

Then it will be as

$$\begin{aligned}
\frac{\partial^3 f}{\partial y^3}-\frac{\partial^2 f}{\partial y\partial t}=&R_0\left[f\frac{\partial^2 f}{\partial y^2}-\Big(\frac{\partial f}{\partial y}\Big)^2\right]+\frac{1}{1-x}\frac{\partial p}{\partial x}\\
\frac{\partial^2 f}{\partial y^2}-\frac{\partial f}{\partial t}=&R_0 f\frac{\partial f}{\partial y}+\frac{u_0}{v_0}\frac{\partial p}{\partial y}.
\end{aligned}\tag{4}$$

Let the values $\frac{1}{1-x}\frac{\partial p}{\partial x}$, $v_{01}(t)$ and $v_{02}(t)$ vary according to the periodic law:

$$\begin{aligned}
\frac{1}{1-x}\frac{\partial p}{\partial x}&=a+\varepsilon e^{i\omega t}b,\\
v_{01}&=c\left(1+\varepsilon e^{i\omega t}\right),
\end{aligned}$$

where a , b are unknown constants determined from the boundary conditions;
c and d are the stated constants.

We will search for the function a$f(y,t)$ in the following form:

$$f(y,t)=\varphi(y)+\varepsilon e^{i\omega t}\phi(y).\tag{5}$$

Substituting (5) into system (4) and neglecting the terms containing and above from the first equation of system (4), we have

$$\varphi^{'''}=R_0\left(\varphi\varphi^{'}-\varphi^{'2}\right)+a,\tag{6}$$

$$\phi^{'''}-i\omega\phi^{'}=R_0\left(\phi\varphi^{''}+\varphi\phi^{''}-2\varphi^{'}\phi^{'}\right)+b,\tag{7}$$

and from (2), we have the following boundary conditions:

$$\begin{aligned}
\varphi(-1)=c,&\quad \varphi(1)=d\\
\varphi^{'}(-1)=0,&\quad \varphi^{'}(1)=0,\\
\phi(-1)=c,&\quad \phi(1)=d,\\
\phi^{'}(-1)=0,&\quad \phi^{'}(1)=0.
\end{aligned}$$

Assume that the Reynolds number of infiltration $R_0 = \frac{v_0 h}{\nu}$ is a small quantity. Here, we present the functions $\varphi(y)$ and $\phi(y)$, as well as the unknown constants a and b as the series on powers R_0:

$$\begin{gathered}\varphi(y) = \sum_{k=0}^{\infty} R_0^k, \quad \phi(y) = \sum_{k=0}^{\infty} R_0^k \phi_k(y), \\ a = \sum_{k=0}^{\infty} R_0^k a_k, \quad b = \sum_{k=0}^{\infty} R_0^k b_k.\end{gathered} \tag{8}$$

Substituting (8) into equations (6) and (7) and equating the coefficients at the same powers R_0, we get in the first two approximations:

$$\begin{aligned}&\varphi_0^{'''} = a_0, \\ &\phi_0^{'''} - i\omega\phi_0^{'} = b_0, \\ &\ldots\ldots\ldots\ldots\ldots \\ &\varphi_1^{'''} = \varphi_0\varphi_0^{''} - \varphi^{'2} + a_1, \\ &\phi_1^{'''} - i\omega\phi_1^{'} = b_1 + \varphi_0\phi_0^{''} + \varphi_0^{''}\phi_0 - 2\varphi_0^{'}\phi_0^{'}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned} \tag{9}$$

where the functions φ_0, ϕ_0, φ_1 and ϕ_1 functions must satisfy the following boundary conditions:

$$\begin{gathered}\varphi_0(-1) = c, \quad \varphi_0(1) = d, \quad \varphi_0^{'}(-1) = 0, \quad \varphi_0^{'}(1) = 0, \\ \phi_0(-1) = c, \quad \phi_0(1) = d, \quad \phi_0^{'}(-1) = 0, \quad \phi_0^{'}(1) = 0, \\ \varphi_k(-1) = 0, \quad \varphi_k(1) = 0, \quad \phi_k^{'}(-1) = 0, \quad \phi_0^{'}(1) = 0,\end{gathered}$$

where $k \geq 1, 2, \ldots$, and strokes show derivatives on y.

The solution of system (9) is not difficult. We find the functions φ_0, φ_1 and ϕ_0, as well as the values a_0, b_0, a_1. Finding the function φ_1 and value b_1 in the allowed approximation does not make sense, since the terms ϕ_1 and b_1 have coefficients as the product of two infinitely small values εR_0.

Thus, for φ_0, ϕ_0, φ_1, a_0, b_0 and a_1, we obtain the following expressions:

$$\begin{gathered}\varphi_0(y) = A\left(y^3 - 3y\right) + B, \\ \phi_0(y) = \frac{1}{D}\left[4A\left(sh\sqrt{i\omega}y - y\sqrt{i\omega}ch\sqrt{i\omega}\right) + B\right], \\ a_0 = 6A, \quad b_0 = \frac{1}{D}\left[4a(i\omega)^{3/2} ch\sqrt{i\omega}\right], \quad a_1 = \frac{324}{35}A^2 \\ \varphi_1 = \frac{AB}{4}\left(y^2 - 1\right)^2 - \frac{A^2}{70}\left(y^7 - 3y^3 + 2y\right),\end{gathered}$$

with the notation

$$A = \frac{c-d}{4}, \quad B = \frac{c+d}{2}, \quad D = 2\left(sh\sqrt{i\omega} - \sqrt{i\omega}ch\sqrt{i\omega}\right).$$

Finally, for the components of velocity and pressure drops along and across the main flow, in the proposed approximation, we will have

$$\begin{aligned}&u(x,y,t) = (1-x)\left[\varphi_0^{'}(y) + R_0\varphi_1^{'}(y) + \varepsilon e^{i\omega t}\phi_0^{'}(y)\right], \\ &v(x,y,t) = \varphi_0(y) + R_0\varphi_1(y) + \varepsilon e^{i\omega t}\left[\phi_0(y) + R_0\phi_1(y)\right], \\ &\frac{\partial p}{\partial x} = (1-x)[a_0 + R_0 a_1 + \varepsilon e^{i\omega t} b_0], \\ &\frac{\partial p}{\partial y} = \frac{v_o}{u_0}\left[\varphi_0^{''} + R_0\varphi_1^{''} + \varepsilon e^{i\omega t}\left(\phi_0^{''} - i\omega\phi_0\right) - R_0\varphi_0\varphi_0^{'}\right].\end{aligned}$$

3. Conclusion

Thus, the pulsating flow of a viscous incompressible fluid between porous walls was studied. Fluid flow is caused by pulsating pressure drop and pulsating movement of porous walls [3, 10, 11, 14].

References

1. K. V. Deshikachar, R. A. Ramachandra, Magnetohydrodynamic unsteady flow in a tube of variable cross section in an axial magnetic field. *The Physics of fluids* 30 (1987), no. 1, 278–279.
2. L. A. Jikidze, V. N. Tsutskiridze, Approximate Method for Solving an Unsteady Rotation Problem for a Porous Plate in the Conducting Fluid with Regard for the Heat Transfer in the Case of Electroconductivity. In: *Several Problems of Applied Mathematics and Mechanics*, pp. 157-164, Series: Science and Technology Mathematical Physics (ebook), New York, 2013.
3. L. A. Jikidze, V. N. Tsutskiridze, The unsteady flow of conductind liquid squeezed between two parallel infinite rotating porous disks taking into account the strong magnetic field and the heat transfer. *Transactions of GTU (Georgian Technical University)*, **510** (2018), no. 4, 126–135.
4. L. A. Landau, E. M. Lifshits, *Electrodynamics of Continua.* (Russian) GITTL, Moscow, 1982.
5. L. G. Loitsiansky, *Mechanic of a Fluid and Gas.* (Russian) Sciences, Moscow, 1987.
6. J. T. Ramos, N. S. Magnetohydrodynamic channel flow study. *Phys. Fluids* **29** (1986), no. 4, 992–997.
7. J. Sharikadze, V. Tsutskiridze, L. Jikidze, Pulsating flow of weakly conductive liquid with heat transfer. *Appl. Math. Inform. Mech.* **18** (2013), no. 1, 61–73.
8. J. V. Sharikadze, V. Tsutskiridze, L. Jikidze, The unsteady flow of incompressible fluid in a constant cross section pipes in an external uniform magnetic field. *International scientific journal of IFTOMM Problems of Mechanics.* **1** (2013), no. 50, 77–83.
9. V. Tsutskiridze, Hartmann problem for circular pipe. In: *The International scientific Conference on "Mechanics 2016". Proceedings of Mechanics 2016, Tbilisi*, 2016, pp. 89–92.
10. V. Tsutskiridze, Flow of viscous fluid in initial section of plane channel with porous walls. *International Scientific Journal of IFTOMM Problems of Mechanics.* **2** (2018), no. 71, 43–47.
11. V. Tsutskiridze, The stationary flow of laminar liquid in an circular pipe of infinite length. *Appl. Math. Inform. Mech.*, **23** (2018), no. 1, 47–51.
12. V. Tsutskiridze, L. Jikidze, The conducting liquid flow between porous walls with heat transfer. *Proc. A. Razmadze Math. Inst.* **167** (2015), 73–89.
13. V. Tsutskiridze, L. Jikidze, The nonstationary flow of a conducting fluid in a plane pipe in the presence of a transverse magnetic field. *Trans. A. Razmadze Math. Inst.* **170** (2016), no. 2, 280–286.
14. V. Tsutskiridze, L. Jikidze, The flow of weakly electroconductive liquid between porous walls with heat transfer. *AMIM*, Tbilisi **24** (2019), no. 1, 31–44.
15. A. B. Vatazin, G. A. Lubimov, C. A. Regirer, *Magnetohydrodynamic Flow in Channels.* (Russian) Nauka, Moscow, 1970.

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