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ON THE ASYMPTOTICS OF SOLUTIONS OF NONLINEAR CYCLIC SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to the memory of Temur Chanturia

Abstract. The asymptotics for a class of solutions for cyclic nonlinear systems of ordinary differential equations of more general type than Emden–Fowler system are established.

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1. Statement of the Problem and Auxiliary Designations

We consider the system of differential equations

$$y_i' = \alpha_i p_i(t)\varphi_{i+1}(y_{i+1}) \quad (i = \overline{1,n}),^* \tag{1.1}$$

where $\alpha_i \in \{-1, 1\}$ $(i = \overline{1,n})$, $p_i : [a, \omega[\to]0, +\infty[$ $(i = \overline{1,n})$ are continuous functions, $-\infty < a < \omega \le +\infty$,† $\varphi_i : \Delta(Y_i^0) \to]0; +\infty[$ $(i = \overline{1,n})$ are continuously differentiable functions satisfying the conditions

$$\lim_{\substack{y_i \to Y_i^0 \\ y_i \in \Delta(Y_i^0)}} \frac{y_i \varphi_i'(y_i)}{\varphi_i(y_i)} = \sigma_i \quad (i = \overline{1,n}), \quad \prod_{i=1}^{n} \sigma_i \ne 1, \tag{1.2}$$

where Y_i^0 $(i \in \{1, \dots, n\})$ is equal either to 0, or to $\pm\infty$, $\Delta(Y_i^0)$ $(i \in \{1, \dots, n\})$ is a one-sided neighborhood of Y_i^0.

It follows from the conditions (1.2) that φ_i $(i = \overline{1,n})$ are regularly varying functions of orders σ_i as $y_i \to Y_i^0$, hence (see [1]) these functions admit the representation

$$\varphi_i(y_i) = |y_i|^{\sigma_i} \theta_i(y_i) \quad (i = \overline{1,n}), \tag{1.3}$$

where θ_i $(i = \overline{1,n})$ are slowly varying functions as $y_i \to Y_i^0$. According to the definition and properties of slowly varying functions and also in view of (1.2),

$$\lim_{y_i \to Y_i^0} \frac{\theta_i(\lambda y_i)}{\theta_i(y_i)} = 1 \ \text{ for any } \ \lambda > 0, \quad \lim_{y_i \to Y_i^0} \frac{y_i \theta_i'(y_i)}{\theta_i(y_i)} = 0 \ \ (i = \overline{1,n}), \tag{1.4}$$

and the first limits are uniform with respect to λ on any segment $[c, d] \in]0, +\infty[$.

If $\theta_i(y_i) \equiv 1$ $(i = \overline{1,n})$, then the system (1.1) is called an Emden–Fowler system. In case $n = 2$, the asymptotic behavior of its nonoscillating solutions is thoroughly investigated in [2–6].

In the present paper (as distinct from [2–6]), the system (1.1) is considered in the case where the functions $\varphi_i(y_i)$ $(i = \overline{1,n})$ are close to the power functions in the neighborhoods of Y_i^0 in the sense of the definition of regularly varying functions.

In T. A. Chanturia's paper [7], for systems of differential equations that are close to (1.1) in a certain sense the criteria for the existence of A and B-properties are established.

A solution $(y_i)_{i=1}^n$ of the system (1.1) is called $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution, if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the following conditions:

$$\begin{gathered} y_i(t) \in \Delta(Y_i^0) \ \text{ while } \ t \in [t_0, \omega[, \quad \lim_{t \uparrow \omega} y_i(t) = Y_i^0, \\ \lim_{t \uparrow \omega} \frac{y_i(t) y_{i+1}'(t)}{y_i'(t) y_{i+1}(t)} = \Lambda_i \quad (i = \overline{1, n-1}). \end{gathered} \tag{1.5}$$

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* Here and in the sequel, all functions and parameters with the index $n+1$ will be equivalent to the corresponding values with index 1.

† While $\omega = +\infty$ we consider $a > 0$.

The aim of this work is to establish sufficient and necessary conditions for the existence of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions for the system (1.1), and also to provide the asymptotic representation (when $t\uparrow\omega$) for these solutions, when Λ_i $(i=\overline{1,n-1})$ are real numbers, including those equal to zero, and $\Lambda_{n-1}\sigma_n=1$.

Remark 1.1. The definition of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution does not give the direct connection between the first and the n-th components of the solution, which appear in the n-th equation of the system. To establish this connection, we define the following functions:

$$\lambda_i(t)=\frac{y_i(t)y'_{i+1}(t)}{y'_i(t)y_{i+1}(t)}\quad (i=\overline{1,n}). \tag{1.6}$$

We have

$$\lambda_n(t)=\frac{y_n(t)y'_1(t)}{y'_n(t)y_1(t)}=\frac{y_n(t)y'_{n-1}(t)}{y'_n(t)y_{n-1}(t)}\cdot\frac{y_{n-1}(t)y'_{n-2}(t)}{y'_{n-1}(t)y_{n-2}(t)}\cdots\frac{y_2(t)y'_1(t)}{y'_2(t)y_1(t)}=$$
$$=\frac{1}{\lambda_1(t)\dots\lambda_{n-1}(t)}\,. \tag{1.7}$$

It follows from (1.5) that $\lim\limits_{t\uparrow\omega}\lambda_i(t)=\Lambda_i$ $(i=\overline{1,n-1})$. Therefore, if there are zeroes among Λ_i $(i=\overline{1,n-1})$, taking into account (1.7), we obtain

$$\Lambda_n=\lim_{t\uparrow\omega}\lambda_n(t)=\pm\infty.$$

In particular, it is evident that the case in which among all Λ_i $(i=1,\dots,n-1)$ there is a single $\pm\infty$, while all others are real different from zero numbers, can be transformed into the case described in this work. This transformation is carried out by cyclic redesignation of variables, functions and constants. For instance, if $\Lambda_l=\pm\infty$ $(l\in\{1,\dots,n-1\})$, the indices are redesignated as follows:

$$l\to n,\ \ l+1\to 1,\dots,n\to n-l,\ \ 1\to n-l+1,\dots,l-1\to n-1.$$

It is obvious that $\Lambda_i=0$, if $i=n-l$.

Further, we introduce some auxiliary notation.

First, if

$$\mu_i=\begin{cases}1, & \text{as } Y_i^0=+\infty,\\ & \text{or } Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is right neighborhood of } 0,\\ -1, & \text{as } Y_i^0=-\infty,\\ & \text{or } Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is left neighboorhood of } 0,\end{cases}$$

it is obvious that μ_i $(i=\overline{1,n})$ determine the signs of the components of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution in some left neighborhood of ω.

Further, we denote the sets

$$\mathfrak{I}=\{i\in\{1,\dots,n-1\}:\ 1-\Lambda_i\sigma_{i+1}\neq 1\}\,,\quad \overline{\mathfrak{I}}=\{1,\dots,n-1\}\setminus\mathfrak{I}$$

and suppose that

$$r = \max \mathfrak{I} < n-1.$$

Taking into account the fact that $r < n-1$, we denote auxiliary functions I_i, Q_i $(i = 1, \dots, n)$ and none-zero constants β_i $(i = 1, \dots, n)$, supposing that

$$I_i(t) = \begin{cases} \int\limits_{A_i}^{t} p_i(\tau)\,d\tau & \text{for } i \in \mathfrak{I}, \\ \int\limits_{A_i}^{t} p_i(\tau) I_{i+1}(\tau)\,d\tau & \text{for } i \in \overline{\mathfrak{I}}, \\ \int\limits_{A_n}^{t} p_n(\tau) q_{r+1}(\tau)\,d\tau & \text{for } i = n, \end{cases} \quad \beta_i = \begin{cases} 1 - \Lambda_i \sigma_{i+1}, & \text{for } i \in \mathfrak{I}, \\ \beta_{i+1}\Lambda_i, & \text{for } i \in \overline{\mathfrak{I}}, \\ 1 - \prod\limits_{k=1}^{n} \sigma_k & \text{for } i = n, \end{cases}$$

$$Q_i(t) = \begin{cases} \alpha_i \beta_i I_i(t) & \text{for } i \in \mathfrak{I} \cup \{n\}, \\ \dfrac{\alpha_i \beta_i I_i(t)}{I_{i+1}(t)} & \text{for } i \in \overline{\mathfrak{I}}, \end{cases}$$

where limits of integration $A_i \in \{\omega, a\}$ $(i \in \{1, \dots, n-1\})$, $A_n \in \{\omega, b\}$ $(b \in [a, \omega[)$ are chosen in such a way that the corresponding integral I_i tends either to zero, or to ∞ as $t \uparrow \omega$,

$$q_{r+1}(t) = \theta_1 \left(\mu_1 |I_1(t)|^{\frac{1}{\beta_1}} \right) |Q_r(t)|^{\prod\limits_{k=1}^{r} \sigma_k} \times \\ \times \prod_{k=1}^{r-1} \left| Q_k(t) \theta_{k+1} \left(\mu_{k+1} |I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}} \right) \right|^{\prod\limits_{i=1}^{k} \sigma_i}.$$

In addition, we introduce the numbers

$$\begin{aligned} A_i^* &= \begin{cases} 1, & \text{if } A_i = a, \\ -1, & \text{if } A_i = \omega \end{cases} \quad (i = 1, \dots, n-1), \\ A_n^* &= \begin{cases} 1, & \text{if } A_n = b, \\ -1, & \text{if } A_n = \omega. \end{cases} \end{aligned} \tag{1.8}$$

These numbers enable us to define the signs of the functions I_i $(i = 1, \dots, n-1)$ on the interval $]a, \omega[$ and the sign of the function I_n on the interval $]b, \omega[$.

We will define that the function φ_k $(k \in \{1, \dots, n\})$ satisfies the condition **S**, if for any continuously differentiable function $l : \Delta(Y_k^0) \to]0, +\infty[$ with the property

$$\lim_{\substack{z \to Y_k^0 \\ z \in \Delta(Y_k^0)}} \frac{z\, l'(z)}{l(z)} = 0,$$

the function θ_k admits the asymptotic representation

$$\theta_k(zl(z)) = \theta(z)[1 + o(1)] \quad \text{as} \quad z \to Y_k^0 \quad (z \in \Delta(Y_k^0)). \tag{1.9}$$

For instance, the **S**-condition is, obviously, satisfied by the functions φ_k of the type

$$\varphi_k(y_k) = |y_k|^{\sigma_k} |\ln y_k|^{\gamma_1}, \quad \varphi_k(y_k) = |y_k|^{\sigma_k} |\ln y_k|^{\gamma_1} |\ln|\ln y_k||^{\gamma_2},$$

where $\gamma_1, \gamma_2 \neq 0$. The **S**-condition is also satisfied by the functions φ_k which include the functions θ_k that have the eventual limit as $y_k \to Y_k^0$. The **S**-condition is also satisfied by many other functions.

Remark 1.2. If φ_k $(k \in \{1, \dots, n\})$ satisfies the **S**-condition and $y_k : [t_0, \omega[\to \Delta(Y_k^0)$ is a continuously differentiable function with the property

$$\lim_{t\uparrow\omega} y_k(t) = Y_k^0, \quad \frac{y_k'(t)}{y_k(t)} = \frac{\xi'(t)}{\xi(t)}\,[r + o(1)] \quad \text{as} \quad t \uparrow \omega,$$

where r is a non-zero real constant, ξ is a continuously differentiable in some left neighborhood of ω real function with $\xi'(t) \neq 0$, then

$$\theta_k(y_k(t)) = \theta_k\left(\mu_k|\xi(t)|^r\right)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega,$$

since in this case

$$y_k(t) = z(t)l(z(t)), \quad \text{where} \quad z(t) = \mu_k|\xi(t)|^r,$$

and

$$\lim_{\substack{z\to Y_0\\ z\in\Delta_{Y_0}}} \frac{z\,l'(z)}{l(z)} = \lim_{t\uparrow\omega} \frac{z(t)\,l'(z(t))}{l(z(t))} =$$

$$= \lim_{t\uparrow\omega} \frac{z(t)\left(\frac{y_k(t)}{z(t)}\right)'}{\left(\frac{y_k(t)}{z(t)}\right) z'(t)} = \lim_{t\uparrow\omega}\left[\frac{\xi(t)y_k'(t)}{r\xi'(t)y_k(t)} - 1\right] = 0.$$

2. Main Results

Theorem 2.1. *Let $\Lambda_i \in \mathbb{R}$ $(i = \overline{1, n-1})$ include those equal to zero, $m = \max\{i \in \Im :\ \Lambda_i = 0\}$ and $r = \max \Im < n-1$. Let also the functions φ_k $(k = \overline{1,r})$ satisfy the* **S**-*condition. Then for the existence of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions of* (1.1) *it is necessary and, if the algebraic equation*

$$\Big(\prod_{j=1}^{n} \sigma_j - 1 - \lambda\Big) \prod_{j=m+1}^{n-1} (M_j + \lambda) =$$

$$= \Big(\prod_{j=1}^{n} \sigma_j\Big)\Big(\sum_{k=m}^{r} \prod_{j=m+1}^{k} (M_j + \lambda) \prod_{s=k+2}^{n-1} M_s\Big)\lambda,^{*} \tag{2.1}$$

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* Here and in what follows, we assume that $\prod\limits_{j=s}^{l} = 1$, $\sum\limits_{j=s}^{l} = 0$ if $l < s$.

where

$$M_j = \Big(\prod_{i=j}^{n-1} \Lambda_i\Big)^{-1} \quad (j = \overline{m+1, n-1}),$$

have no roots with a zero real part, it is also sufficient that

$$\lim_{t\uparrow\omega} \frac{I_i(t)I'_{i+1}(t)}{I'_i(t)I_{i+1}(t)} = \Lambda_i \frac{\beta_{i+1}}{\beta_i} \quad (i = \overline{1, n-1}) \tag{2.2}$$

and for each $i \in \{1, \dots, n\}$ the following conditions be satisfied:

$$A_i^* \beta_i > 0 \ \ \textit{if} \ \ Y_i^0 = \pm\infty, \quad A_i^* \beta_i < 0 \ \ \textit{if} \ \ Y_i^0 = 0, \tag{2.3}$$

$$\operatorname{sign}\,[\alpha_i A_i^* \beta_i] = \mu_i. \tag{2.4}$$

Moreover, the components of each solution of that type admit asymptotic representation when $t \uparrow \omega$,

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = Q_i(t)[1 + o(1)] \quad (i = \overline{1, n-1}), \tag{2.5}$$

$$\frac{y_n(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod\limits_{i=1}^{r} \sigma_i}} = Q_n(t)[1 + o(1)], \tag{2.6}$$

and there exists the whole k-parametric family of these solutions if there are k positive roots among the solutions of the following algebraic equation:

$$\gamma_i = \begin{cases} \beta_i A_i^* & \textit{if} \ \ i \in \mathfrak{I} \setminus \{m+1, \dots, n-1\}, \\ \beta_i A_i^* A_{i+1}^* & \textit{if} \ \ i \in \overline{\mathfrak{I}} \setminus \{m+1, \dots, n-1\}, \\ A_n^* \Big(\prod\limits_{j=1}^{n-1} \sigma_j - 1\Big) \operatorname{Re} \lambda_{i-m}^0 & \textit{if} \ \ i \in \{m+1, \dots, n\}, \end{cases} \tag{2.7}$$

where λ_j^0 $(j = \overline{1, n-m})$ are the roots of the algebraic equation (2.1) (*along with multiple*).

Remark 2.1. The algebraic equation (2.1) has, obviously, no roots with zero real part, if

$$\Big(\sum_{k=m+1}^{r+1} \prod_{j=k}^{n-1} |\Lambda_j|\Big) \prod_{k=1}^{n} |\sigma_k| < \Big|1 - \prod_{j=1}^{n} \sigma_j\Big|.$$

Proof of Theorem 2.1. *Necessity.* Let $y_i : [t_0, \omega[\to \Delta(Y_i^0)$ $(i = \overline{1, n})$ be an arbitrary $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution of (1.1). Then by virtue of (1.1), we obtain

$$\frac{y'_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i p_i(t) \quad (i = \overline{1, n}) \ \text{ as } \ t \in [t_0, \omega[\,. \tag{2.8}$$

When $i \in \Im$, integrating (2.8) over the interval from B_i to t, where $B_i = \omega$, if $A_i = \omega$, or $B_i = t_0$, if $A_i = a$, we get

$$\int\limits_{B_i}^{t} \frac{y_i'(\tau)}{\varphi_{i+1}(y_{i+1}(\tau))}\, d\tau = \alpha_i I_i(t)[1+o(1)] \text{ as } t \uparrow \omega. \tag{2.9}$$

In virtue of de L'Hospital's rule in the form of Stoltz, we get

$$\lim_{t\uparrow\omega} \frac{\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))}}{\int\limits_{B_i}^{t} \frac{y_i'(\tau)}{\varphi_{i+1}(y_{i+1}(\tau))}\, d\tau} = \lim_{t\uparrow\omega} \frac{\frac{y_i'(t)}{\varphi_{i+1}(y_{i+1}(t))} - \frac{y_i(t)\varphi_{i+1}'(y_{i+1}(t))y_{i+1}'(t)}{\varphi_{i+1}^2(y_{i+1}(t))}}{\frac{y_i'(t)}{\varphi_{i+1}(y_{i+1}(t))}} =$$

$$= 1 - \lim_{t\uparrow\omega} \frac{y_{i+1}(t)\varphi_{i+1}'(y_{i+1}(t))}{\varphi_{i+1}(y_{i+1}(t))} \lim_{t\uparrow\omega} \frac{y_i(t)y_{i+1}'(t)}{y_i'(t)y_{i+1}(t)} = 1 - \Lambda_i\sigma_{i+1} = \beta_i \neq 0.$$

Therefore, in view of (2.9), we have

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i\beta_i I_i(t)[1+o(1)] \text{ as } t \uparrow \omega. \tag{2.10}$$

Consequently, when $i \in \Im$, the asymptotic representation (2.5) is valid and, in virtue of (2.8) and (2.10),

$$\frac{y_i'(t)}{y_i(t)} = \frac{I_i'(t)}{\beta_i I_i(t)}[1+o(1)] \text{ as } t \uparrow \omega. \tag{2.11}$$

Further, taking into account that $r = \max \Im < n-1$, we consider the relations (2.8) consistently starting with the maximum $i \in \overline{\Im}$, that is lower than r, since $i \in \overline{\Im} \setminus \{r+1, \dots, n-1\}$. We consider these relations taking into account that the relations (2.11) are valid for bigger values of $i \le r$. Multiplying (2.8) by $I_{i+1}(t)$ and integrating over the interval from B_i to t, where B_i are chosen in the above way, we get

$$\int\limits_{B_i}^{t} \frac{y_i'(\tau)I_{i+1}(\tau)}{\varphi_{i+1}(y_{i+1}(\tau))}\, d\tau = \alpha_i I_i(t)[1+o(1)] \text{ as } t \uparrow \omega. \tag{2.12}$$

In virtue of de L'Hospital's rule in the form of Stoltz, using (2.11) and the definition of $P_\omega(\Lambda_1, \dots, \Lambda_{n-1})$- solution, we obtain

$$\lim_{t\uparrow\omega} \frac{\frac{y_i(t)I_{i+1}(t)}{\varphi_{i+1}(y_{i+1}(t))}}{\int\limits_{A_i}^{t} \frac{y_i'(\tau)I_{i+1}(\tau)}{\varphi_{i+1}(y_{i+1}(\tau))}\, d\tau} =$$

$$= \lim_{t\uparrow\omega} \frac{\frac{y_i'(t)I_{i+1}(t)}{\varphi_{i+1}(y_{i+1}(t))} + \frac{y_i(t)I_{i+1}'(t)}{\varphi_{i+1}(y_{i+1}(t))} - \frac{y_i(t)I_{i+1}(t)\varphi_{i+1}'(y_{i+1}(t))y_{i+1}'(t)}{\varphi_{i+1}^2(y_{i+1}(t))}}{\frac{y_i'(t)I_{i+1}(t)}{\varphi_{i+1}(y_{i+1}(t))}} =$$

$$= 1 + \lim_{t\uparrow\omega} \frac{y_i(t)I_{i+1}'(t)}{y_i'(t)I_{i+1}(t)} - \lim_{t\uparrow\omega} \frac{y_{i+1}(t)\varphi_{i+1}'(y_{i+1}(t))}{\varphi_{i+1}(y_{i+1}(t))} \lim_{t\uparrow\omega} \frac{y_i(t)y_{i+1}'(t)}{y_i'(t)y_{i+1}(t)} =$$

$$= 1 - \Lambda_i\sigma_{i+1} + \beta_{i+1} \lim_{t\uparrow\omega} \frac{y_i(t)y'_{i+1}(t)}{y'_i(t)y_{i+1}(t)} =$$
$$= \beta_{i+1} \lim_{t\uparrow\omega} \left[\frac{y_i(t)y'_{i+1}(t)}{y'_i(t)y_{i+1}(t)}\right] = \beta_{i+1}\Lambda_i = \beta_i.$$

Hence, with regard for (2.12), we get

$$\frac{y_i(t)}{\varphi_i(y_{i+1}(t))} = \frac{\alpha_i\beta_i I_i(t)}{I_{i+1}(t)}\,[1+o(1)] \text{ as } t\uparrow\omega. \tag{2.13}$$

Therefore, with regard for (2.8), the asymptotic formula (2.11) is valid. Consequently, the asymptotic representations (2.5) and (2.11) are admitted for all $i \in \overline{\mathfrak{I}} \setminus \{r+1, \dots, n-1\}$.

Taking into account that φ_i satisfy the **S**-condition for all $i \in \{1, \dots, r\}$ and asymptotic representations (2.11) are valid, in virtue of Remark 1.2, we get

$$\varphi_i(y_i(t)) = |y_i(t)|^{\sigma_i}\theta_i\left(\mu_i|I_i(t)|^{\frac{1}{\beta_i}}\right)[1+o(1)] \;\; (i=\overline{1,r}) \text{ as } t\uparrow\omega.$$

According to these representations and the asymptotic representations (2.5) for $i = \overline{1,r}$, we have

$$\varphi_1(y_1(t)) = |y_1(t)|^{\sigma_1}\theta_1\left(\mu_1|I_1(t)|^{\frac{1}{\beta_1}}\right)[1+o(1)] =$$
$$= \theta_1\left(\mu_1|I_1(t)|^{\frac{1}{\beta_1}}\right)\left||y_2(t)|^{\sigma_2}Q_1(t)\theta_2\left(\mu_2|I_2(t)|^{\frac{1}{\beta_2}}\right)\right|^{\sigma_1}[1+o(1)] =$$
$$= \theta_1\left(\mu_1|I_1(t)|^{\frac{1}{\beta_1}}\right)\left|Q_1(t)\theta_2\left(\mu_2|I_2(t)|^{\frac{1}{\beta_2}}\right)\right|^{\sigma_1} \times$$
$$\times \left||y_3(t)|^{\sigma_3}Q_2(t)\theta_3\left(\mu_3|I_3(t)|^{\frac{1}{\beta_3}}\right)\right|^{\sigma_1\sigma_2}[1+o(1)] = \dots =$$
$$= q_{r+1}(t)\,[\varphi_{r+1}(y_{r+1}(t))]^{\prod\limits_{i=1}^{r}\sigma_i}\,[1+o(1)] \text{ as } t\uparrow\omega.$$

From this and the last formula in (2.8), we conclude that

$$\frac{y'_n(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod\limits_{k=1}^{r}\sigma_k}} = \alpha_n p_n(t) q_{r+1}(t)[1+o(1)] \text{ as } t\uparrow\omega. \tag{2.14}$$

Integrating (2.14) over the interval from B_n to t, where $B_n = \omega$, if $A_n = \omega$, and $B_n = t_0$, if $A_n = b$, we obtain

$$\int\limits_{B_n}^{t} \frac{y'_n(\tau)}{[\varphi_{r+1}(y_{r+1}(\tau))]^{\prod\limits_{k=1}^{r}\sigma_k}}\,d\tau = \alpha_n I_n(t)[1+o(1)] \text{ as } t\uparrow\omega.$$

Using de L'Hospital's rule, with regard for (1.2), (1.5) and the conditions $1 - \Lambda_j\sigma_{j+1} = 0$ as $j = \overline{r+1, n-1}$, we get

$$\lim_{t\uparrow\omega}\frac{\dfrac{y_n(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod\limits_{k=1}^{r}\sigma_k}}}{\int\limits_{B_n}^{t}\dfrac{y_n'(\tau)}{[\varphi_{r+1}(y_{r+1}(\tau))]^{\prod\limits_{k=1}^{r}\sigma_k}}\,d\tau}=$$

$$=\lim_{t\uparrow\omega}\frac{\dfrac{y_n'(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod\limits_{k=1}^{r}\sigma_k}}\left[1-\Big(\prod\limits_{k=1}^{r}\sigma_k\Big)\dfrac{y_{r+1}(t)\varphi_{r+1}'(y_{r+1}(t))}{\varphi_{r+1}(y_{r+1}(t))}\,\dfrac{y_{r+1}'(t)y_n(t)}{y_{r+1}(t)y_n'(t)}\right]}{\dfrac{y_n'(t)}{[\varphi_n(y_n(t))]^{\prod\limits_{k=1}^{n-1}\sigma_k}}}=$$

$$=1-\Big(\prod_{j=1}^{r+1}\sigma_j\Big)\lim_{t\uparrow\omega}\frac{y_{r+1}'(t)y_{r+2}(t)}{y_{r+1}(t)y_{r+2}'(t)}\cdots\frac{y_{n-1}'(t)y_n(t)}{y_{n-1}(t)y_n'(t)}=$$

$$=1-\frac{\prod\limits_{j=1}^{r+1}\sigma_j}{\Lambda_{r+1}\dots\Lambda_{n-1}}=1-\prod_{j=1}^{n}\sigma_j=\beta_n.$$

The previous asymptotic representation yields

$$\frac{y_n(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod\limits_{k=1}^{r}\sigma_k}}=\alpha_n\beta_nI_n(t)[1+o(1)]\quad\text{as}\quad t\uparrow\omega.$$

Hence, the representation (2.6) is valid and, in virtue of (2.14), (2.11), it takes place when $i=n$.

Taking into account that the asymptotic representation (2.11) is valid for $i=n$, by the same reasoning (multiplying (2.8) by $I_{i+1}(t)$ and further integrating over the interval from B_i to t), we conclude that the asymptotic representations (2.5) and (2.11) are valid for all $i=\overline{r+1,n-1}$ starting with $i=\overline{r+1,n-1}$. The relations (2.11) are valid for $i=\overline{1,n}$ and the solution under consideration satisfies the last limiting condition from the definition of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution. Consequently, for all $i\in\{1,\dots,n-1\}$, the conditions (2.2) are valid. Moreover, from (2.11) it follows that

$$|y_i(t)|=|I_i(t)|^{\frac{1}{\beta_i}+o(1)}\quad(i=\overline{1,n})\ \text{ as }\ t\uparrow\omega.$$

On the basis of the above fact, from the condition $\lim\limits_{t\uparrow\omega}y_i(t)=Y_i^0$ in the definition of the $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution and from the definition of numbers A_i^*, there follow the sign conditions (2.3).

The validity of the sign conditions (2.4) follows immediately from (2.5), (2.6), if we consider the signs of the functions y_i and I_i $(i=\overline{1,n})$ over the interval $[t_0,\omega[$.

Sufficiency. Assume that the conditions (2.2)–(2.4) are satisfied and the algebraic equation (2.1) has no roots with zero real part. We will prove that the system (1.1) has at least one $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$- solution that admits the asymptotic representation (2.5), (2.6) as $t \uparrow \omega$. We will also study the question about the quantity of such solutions.

First, consider the system of the following relations:

$$\begin{cases} \dfrac{y_i}{\varphi_{i+1}(y_{i+1})} = Q_i(t)(1+v_i) & (i=\overline{1,n-1}), \\ \dfrac{y_n}{[\varphi_{r+1}(y_{r+1})]^{\prod\limits_{k=1}^{r}\sigma_k}} = Q_n(t)(1+v_n). \end{cases} \tag{2.15}$$

We will establish that this system on the sets $D = [t_0, \omega[\times \mathbb{R}^n_{\frac{1}{2}}$, where $t_0 \in [a, \omega[$ and $\mathbb{R}^n_{\frac{1}{2}} = \{\overline{x} \equiv (x_i, \dots, x_n) \in \mathbb{R}^n : \ |x_k| \le 1/2 \ (k = \overline{1,n})\}$, defines uniquely continuously differential functions $y_i = Y_i(t, \overline{v})$ $(i = \overline{1,n})$ of the type

$$Y_i(t,\overline{v}) = \mu_i |I_i(t)|^{\frac{1}{\beta_i}[1+z_i(t,\overline{v})]} \quad (i=\overline{1,n}), \tag{2.16}$$

where z_i $(i=\overline{1,n})$ are the following functions

$$|z_i(t,\overline{v})| \le \frac{1}{2} \ \text{ as } \ (t,\overline{v}) \in D$$

and

$$\lim_{t\uparrow\omega} z_i(t,\overline{v}) = 0 \ \text{ uniformly over } \ \overline{v} \in \mathbb{R}^n_{\frac{1}{2}}.$$

Setting in (2.15)

$$y_i = \mu_i |I_i(t)|^{\frac{1}{\beta_i}(1+z_i)} \quad (i=\overline{1,n}), \tag{2.17}$$

and taking into account (1.3), we obtain the following system of relations:

$$\begin{cases} \dfrac{|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}}{|I_{i+1}(t)|^{\frac{\sigma_{i+1}}{\beta_{i+1}}(1+z_{i+1})}} = \\ \quad = \mu_i Q_i(t)\theta_{i+1}\left(\mu_{i+1}|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}(1+z_{i+1})}\right)(1+v_i) \ (i=\overline{1,n-1}), \\ \dfrac{|I_n(t)|^{\frac{1}{\beta_n}(1+z_n)}}{|I_{r+1}(t)|^{\frac{\prod_{k=1}^{r+1}\sigma_k}{\beta_{r+1}}(1+z_{r+1})}} = \\ \quad = \mu_n Q_n(t)\left[\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_r)}\right)\right]^{\prod\limits_{k=1}^{r}\sigma_k}(1+v_n), \end{cases}$$

With regard to sign conditions (2.3), (2.4), the system is defined for all $|v_i| \le \frac{1}{2}$, $|z_i| \le \frac{1}{2}$ $(i=\overline{1,n})$ and t from some left neighborhood of ω.

Hence, taking the logarithm, we get

$$\begin{cases} \dfrac{1}{\beta_i}\,(1+z_i)\ln|I_i(t)| - \dfrac{\sigma_{i+1}}{\beta_{i+1}}\,(1+z_{i+1})\ln|I_{i+1}(t)| = \\ \qquad = \ln|Q_i(t)| + \ln\theta_{i+1}\Big(\mu_{i+1}|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}(1+z_{i+1})}\Big) + \ln|1+v_i| \\ \qquad\qquad (i=\overline{1,n-1}), \\ \dfrac{1}{\beta_n}\,(1+z_n)\ln|I_n(t)| - \dfrac{\prod\limits_{k=1}^{r+1}\sigma_k}{\beta_{r+1}}\,(1+z_{r+1})\ln|I_{r+1}(t)| = \\ \qquad = \ln|Q_n(t)| + \Big(\prod\limits_{k=1}^{r}\sigma_k\Big)\ln\theta_{r+1}\Big(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_{r+1})}\Big) + \\ \qquad + \ln|1+v_n|. \end{cases}$$

Therefore

$$\begin{cases} 1+z_i - \dfrac{\beta_i\sigma_{i+1}}{\beta_{i+1}}\,\dfrac{\ln|I_{i+1}(t)|}{\ln|I_i(t)|}\,(1+z_{i+1}) = \\ \qquad = \dfrac{\beta_i\ln|Q_i(t)|}{\ln|I_i(t)|} + \dfrac{\beta_i\ln\theta_{i+1}\Big(\mu_{i+1}|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}(1+z_{i+1})}\Big)}{\ln|I_i(t)|} + \\ \qquad + \dfrac{\beta_i\ln|1+v_i|}{\ln|I_i(t)|} \quad (i=\overline{1,n-1}), \\ 1+z_n - \dfrac{\beta_n\prod\limits_{k=1}^{r+1}\sigma_k}{\beta_{r+1}}\,\dfrac{\ln|I_{r+1}(t)|}{\ln|I_n(t)|}(1+z_{r+1}) = \\ \qquad = \dfrac{\beta_n\ln|Q_n(t)|}{\ln|I_n(t)|} + \\ \qquad + \dfrac{\beta_n\Big(\prod\limits_{k=1}^{r}\sigma_k\Big)\ln\theta_{r+1}\Big(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_{r+1})}\Big)}{\ln|I_n(t)|} + \\ \qquad + \dfrac{\beta_n\ln|1+v_n|}{\ln|I_n(t)|}. \end{cases}$$

Solving partly this system (as a system of nonhomogeneous linear equations with variables $1+z_i$ $(i=\overline{1,n})$), we obtain

$$z_i = a_i(t) + b_i(t,\overline{v}) + Z_i(t,\overline{z}) \quad (i=\overline{1,n}), \tag{2.18}$$

where the functions a_i, b_i, Z_i $(i=\overline{1,n})$ are defined by the following recurrent relations:

$$a_{r+1}(t) = -1 + \left[1 - \Big(\prod_{k=1}^{r+1}\sigma_k\Big)\frac{\beta_n\ln|I_{r+1}(t)|}{\beta_{r+1}\ln|I_n(t)|}\prod_{k=r}^{n-1}\frac{\sigma_{k+1}\beta_k\ln|I_{k+1}(t)|}{\beta_{k+1}\ln|I_k(t)|}\right]^{-1} \times$$

$$\times \sum_{k=r+1}^{n} \frac{\beta_k \ln|Q_k(t)|}{\ln|I_k(t)|} \prod_{j=r+1}^{k-1} \frac{\beta_j \sigma_{j+1} \ln|I_{j+1}(t)|}{\beta_{j+1} \ln|I_j(t)|},$$

$$b_{r+1}(t,\overline{v}) = \left[1 - \Big(\prod_{k=1}^{r+1} \sigma_k\Big) \frac{\beta_n \ln|I_{r+1}(t)|}{\beta_{r+1} \ln|I_n(t)|} \prod_{k=r}^{n-1} \frac{\sigma_{k+1} \beta_k \ln|I_{k+1}(t)|}{\beta_{k+1} \ln|I_k(t)|}\right]^{-1} \times$$

$$\times \sum_{k=r+1}^{n} \frac{\beta_k \ln|1+v_k|}{\ln|I_k(t)|} \prod_{j=r+1}^{k-1} \frac{\beta_j \sigma_{j+1} \ln|I_{j+1}(t)|}{\beta_{j+1} \ln|I_j(t)|},$$

$$Z_{r+1}(t,\overline{z}) = \left[1 - \Big(\prod_{k=1}^{r+1} \sigma_k\Big) \frac{\beta_n \ln|I_{r+1}(t)|}{\beta_{r+1} \ln|I_n(t)|} \prod_{k=r}^{n-1} \frac{\sigma_{k+1} \beta_k \ln|I_{k+1}(t)|}{\beta_{k+1} \ln|I_k(t)|}\right]^{-1} \times$$

$$\times \left[\sum_{k=r+1}^{n-1} \frac{\beta_k \ln\theta_{k+1}\left(\mu_{k+1} |I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}(1+z_{k+1})}\right)}{\ln|I_k(t)|} \prod_{j=r+1}^{k-1} \frac{\beta_j \sigma_{j+1} \ln|I_{j+1}(t)|}{\beta_{j+1} \ln|I_j(t)|} + \right.$$

$$+ \frac{\beta_n\Big(\prod\limits_{k=1}^{r} \sigma_k\Big) \ln\theta_{r+1}\left(\mu_{r+1} |I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_{r+1})}\right)}{\ln|I_n(t)|} \times$$

$$\left. \times \prod_{j=r+1}^{n-1} \frac{\beta_j \sigma_{j+1} \ln|I_{j+1}(t)|}{\beta_{j+1} \ln|I_j(t)|}\right],$$

$$a_n(t) = -1 + \frac{\beta_n \prod\limits_{k=1}^{r+1} \sigma_k}{\beta_{r+1}} \frac{\ln|I_{r+1}(t)|}{\ln|I_n(t)|} \left[1 + a_{r+1}(t)\right] + \frac{\beta_n \ln|Q_n(t)|}{\ln|I_n(t)|},$$

$$b_n(t,\overline{v}) = \frac{\beta_n \prod\limits_{k=1}^{r+1} \sigma_k}{\beta_{r+1}} \frac{\ln|I_{r+1}(t)|}{\ln|I_n(t)|} b_{r+1}(t,\overline{v}) + \frac{\beta_n \ln|1+v_n|}{\ln|I_n(t)|},$$

$$Z_n(t,\overline{z}) = \frac{\beta_n \prod\limits_{k=1}^{r+1} \sigma_k}{\beta_{r+1}} \frac{\ln|I_{r+1}(t)|}{\ln|I_n(t)|} Z_{r+1}(t,\overline{z}) +$$

$$+ \frac{\beta_n\Big(\prod\limits_{k=1}^{r} \sigma_k\Big) \ln\theta_{r+1}\left(\mu_{r+1} |I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_{r+1})}\right)}{\ln|I_n(t)|},$$

$$a_i(t) = -1 + \frac{\beta_i \sigma_{i+1}}{\beta_{i+1}} \frac{\ln|I_{i+1}(t)|}{\ln|I_i(t)|} \left[1 + a_{i+1}(t)\right] + \frac{\beta_i \ln|Q_i(t)|}{\ln|I_i(t)|} \quad \text{if } i \in \{1,\dots,n-1\} \setminus \{r+1\},$$

$$b_i(t,\overline{v}) = \frac{\beta_i \sigma_{i+1}}{\beta_{i+1}} \frac{\ln|I_{i+1}(t)|}{\ln|I_i(t)|} b_{i+1}(t,\overline{v}) + \frac{\beta_i \ln|1+v_i|}{\ln|I_i(t)|} \quad \text{if } i \in \{1,\dots,n-1\} \setminus \{r+1\},$$

$$Z_i(t,\overline{z}) = \frac{\beta_i\sigma_{i+1}}{\beta_{i+1}}\,\frac{\ln|I_{i+1}(t)|}{\ln|I_i(t)|}\,Z_{i+1}(t,\overline{z}) + \\ + \frac{\beta_i \ln\theta_{i+1}\left(\mu_{i+1}|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}(1+z_{i+1})}\right)}{\ln|I_i(t)|} \\ \text{as } \; i\in\{1,\dots,n-1\}\setminus\{r+1\}.$$

Here $\lim\limits_{t\uparrow\omega} I_i(t)$ $(i=\overline{1,n})$ is equal either to zero, or to $\pm\infty$. Moreover, by de L'Hospital's rule, (2.2), (1.4) and by the above-introduced notation β_i $(i=\overline{1,n})$, we get

$$\lim_{t\uparrow\omega}\frac{\beta_i\ln|I_{i+1}(t)|}{\beta_{i+1}\ln|I_i(t)|} = \lim_{t\uparrow\omega}\frac{\beta_i I_i(t) I'_{i+1}(t)}{\beta_{i+1} I'_i(t) I_{i+1}(t)} = \Lambda_i \;\; (i=\overline{1,n-1}),$$

$$\lim_{t\uparrow\omega}\frac{\beta_n\ln|I_{r+1}(t)|}{\beta_{r+1}\ln|I_n(t)|} = \lim_{t\uparrow\omega}\frac{\beta_n I'_{r+1}(t) I_n(t)}{\beta_{r+1} I_{r+1}(t) I'_n(t)} = (\Lambda_{r+1}\cdots\Lambda_{n-1})^{-1} = \prod_{k=r+2}^{n}\sigma_k,$$

$$\lim_{t\uparrow\omega}\frac{\beta_i\ln|Q_i(t)|}{\ln|I_i(t)|} = \begin{cases} \beta_i = 1-\Lambda_i\sigma_{i+1} & \text{if } \; i\in\Im, \\ \beta_n = 1-\prod\limits_{k=1}^{n}\sigma_k & \text{if } \; i=n, \end{cases}$$

$$\lim_{t\uparrow\omega}\frac{\beta_i\ln|Q_i(t)|}{\ln|I_i(t)|} = \beta_i\lim_{t\uparrow\omega}\left(1-\frac{I_i(t)I'_{i+1}(t)}{I'_i(t)I_{i+1}(t)}\right) = \beta_i\Big(1-\frac{\beta_{i+1}}{\beta_i}\Lambda_i\Big) = 0 \;\text{ if }\; i\in\overline{\Im},$$

$$\lim_{t\uparrow\omega}\frac{\ln\theta_i\left(\mu_i|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}\right)}{\ln|I_i(t)|} = \frac{1}{\beta_i}\,(1+z_i)\lim_{t\uparrow\omega}\frac{\ln\theta_i\left(\mu_i|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}\right)}{\ln\left|\mu_i|I_i(t)|\right|^{\frac{1}{\beta_i}(1+z_i)}} =$$

$$= \frac{1}{\beta_i}\,(1+z_i)\lim_{y\to Y_i^0}\frac{\ln\theta_i(y)}{\ln|y|} =$$

$$= \frac{1}{\beta_i}\,(1+z_i)\lim_{y\to Y_i^0}\frac{y\theta'_i(y)}{\theta_i(y)} = 0 \;\text{ uniformly over }\; |z_i|\le\frac{1}{2}\,.$$

From these limiting relations, starting with $i=r+1$, and further for $i=r,r-1,\dots,1$ and $i=r+2,\dots,n$, we obtain

$$\lim_{t\uparrow\omega} a_i(t) = 0 \;\; (i=\overline{1,n}), \tag{2.19}$$

$$\lim_{t\uparrow\omega} b_i(t,\overline{v}) = 0 \;\; (i=\overline{1,n}) \;\text{ uniformly over }\; \overline{v}\in\mathbb{R}^n_{\frac{1}{2}}\,, \tag{2.20}$$

$$\lim_{t\uparrow\omega} Z_i(t,\overline{z}) = 0 \;\; (i=\overline{1,n}) \;\text{ uniformly over }\; d\overline{z}\in\mathbb{R}^n_{\frac{1}{2}}\,. \tag{2.21}$$

Moreover, for each $i \in \{1, \dots, n\}$

$$\frac{1}{\ln|I_i(t)|}\frac{\partial\left[\ln\theta_i\left(\mu_i|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}\right)\right]}{\partial z_i} =$$

$$= \frac{1}{\beta_i}\,\frac{\mu_i|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}\theta_i'\left(\mu_i|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}\right)}{\theta_i\left(\mu_i|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}\right)}$$

and, therefore, this relation because of (1.4), tends to zero as $t \uparrow \omega$ uniformly over $|z_i| \le \frac{1}{2}$. Taking this fact into account, starting with $i = r+1$ (by the same method), we obtain

$$\lim_{t\uparrow\omega}\frac{\partial Z_i(t,\overline{z})}{\partial z_k} = 0 \ \ (i,k=\overline{1,n}) \ \text{ uniformly over } \ \overline{z} \in \mathbb{R}^n_{\frac{1}{2}}\,. \tag{2.22}$$

By conditions (2.19)–(2.22), there exists a number $t_0 \in [a,\omega[$ such that the following inequalities are valid:

$$\big|a_i(t) + b_i(t,\overline{v}) + Z_i(t,\overline{z})\big| \le \frac{1}{2n} \ \ (i=\overline{1,n}) \quad \text{as } \ (t,\overline{v},\overline{z}) \in [t_0,\omega[\,\times\mathbb{R}^n_{\frac{1}{2}}\times\mathbb{R}^n_{\frac{1}{2}} \tag{2.23}$$

and Lipschitz conditions are valid

$$\big|Z_i(t,\overline{z}^1) - Z_i(t,\overline{z}^2)\big| \le \frac{1}{n+1}\sum_{k=1}^{n}|z_k^1 - z_k^2| \ \ (i=\overline{1,n}) \tag{2.24}$$

as $(t,\overline{z}^1),(t,\overline{z}^2) \in [t_0,\omega[\,\times\mathbb{R}^n_{\frac{1}{2}}$.

Choosing the number t_0 by this method, let $\mathbf{B}$ denote the Banach space of vector-functions $z = (z_i)_{i=1}^n$; each its component, z_i $(i \in \{1,\dots,n\})$, is defined, continuous and bounded on the set $D = [t_0,\omega[\,\times\mathbb{R}^n_{\frac{1}{2}}$, with the norm

$$\|z\| = \sup\left\{\sum_{i=1}^{n}|z_i(t,\overline{v})| : \ (t,\overline{v}) \in D \ (i=\overline{1,n})\right\}.$$

Let us select from this space the subspace $\mathbf{B}_0$ of the functions from $\mathbf{B}$ with the property $\|z\| \le \frac{1}{2}$, and consider its elements, arbitrarily choosing the number $\nu \in (0,1)$ and the operator $\Phi = (\Phi_i)_{i=1}^n$, defined by the relations

$$\Phi_i(z)(t,\overline{v}) = z_i(t,\overline{v}) -$$

$$-\nu\left[z_i(t,\overline{v}) - a_i(t) - b_i(t,\overline{v}) - Z_i(t,z_1(t,\overline{v}),\dots z_n(t,\overline{v}))\right] \ \ (i=\overline{1,n}), \tag{2.25}$$

For each $z \in \mathbf{B}_0$, by the conditions (2.23) we get

$$|\Phi_i(z)(t,\overline{v})| \le (1-\nu)|z_i(t,\overline{v})| + \frac{\nu}{2n} \ \ (i=\overline{1,n}) \ \text{ as } \ (t,\overline{v}) \in D.$$

Therefore, if $(t,\overline{v}) \in D$,

$$\sum_{i=1}^{n} |\Phi_i(z)(t,\overline{v})| \le (1-\nu)\sum_{i=1}^{n} |z_i(t,\overline{v})| + \frac{1}{2}\nu \le$$
$$\le (1-\nu)\|z\| + \frac{1}{2}\,\nu \le (1-\nu)\,\frac{1}{2} + \nu\,\frac{1}{2} = \frac{1}{2}\,.$$

This yields that $\|\Phi(z)\| \le \frac{1}{2}$, i.e., $\Phi(\mathbf{B}_0) \subset \mathbf{B}_0$.

Suppose $z,\, \widetilde{z} \in \mathbf{B}_0$. Then, from (2.24), if $(t,\overline{v}) \in D$,

$$\left|\Phi_i(z)(t,\overline{v}) - \Phi_i(\widetilde{z})(t,\overline{v})\right| \le (1-\nu)|z_i(t,\overline{v}) - \widetilde{z}_i(t,\overline{v})| +$$
$$+\,\nu\left|Z_i(t, z_1(t,\overline{v}),\dots,z_n(t,\overline{v}) - Z_i(t,\widetilde{z}_1(t,\overline{v}),\dots,\widetilde{z}_n(t,\overline{v})\right| \le$$
$$\le (1-\nu)|z_i(t,\overline{v}_i) - \widetilde{z}_i(t,\overline{v}_i)| + \frac{\nu}{n+1}\sum_{k=1}^{n} |z_k(t,\overline{v}) - \widetilde{z}_k(t,\overline{v})| \ \ (i=\overline{1,n}).$$

Thus, if $(t,\overline{v}) \in D$ $(i=\overline{1,n})$,

$$\sum_{i=1}^{n} |\Phi_i(z)(t,\overline{v}_i) - \Phi_i(\widetilde{z})(t,\overline{v}_i)| \le$$
$$\le \left(1-\frac{\nu}{n+1}\right)\sum_{i=1}^{n} |z_i(t,\overline{v}) - \widetilde{z}_i(t,\overline{v})| \le \left(1-\frac{\nu}{n+1}\right)\|z-\widetilde{z}\|,$$

consequently,

$$\|\Phi(z) - \Phi(\widetilde{z})\| \le \left(1-\frac{\nu}{n+1}\right)\|z-\widetilde{z}\|.$$

Thus, the operator Φ maps the space $\mathbf{B}_0$ into itself and is a contraction operator on this space. Then, according to the contraction mapping principle, there exists a unique vector-function $z \in \mathbf{B}_0$ such that $z = \Phi(z)$. By (2.25), this vector-function with continuous components $z_i : D \to \mathbb{R}$ $(i=\overline{1,n})$ is the only solution of the system (2.18) that satisfies the conditions $\|z\| \le \frac{1}{2}$. From (2.18) together with the above condition, and from (2.19)–(2.21) it follows that the components $z_i(t,\overline{v})$ $(i=\overline{1,n})$ of this solution tend to zero when $t \uparrow \omega$ uniformly over $\overline{v} \in \mathbb{R}^n_{\frac{1}{2}}$. Continuous differentiability of these components on some set $[t_1,\omega[\times\mathbb{R}^n_{\frac{1}{2}}$, where $t_1 \in [t_0,\omega[$, follows immediately from the well-known local theorem about the existence of implicit functions defined by the system of relations. According to the transformation (2.17), the obtained vector-function $z = (z_i)_{i=1}^n$ corresponds to the continuously differentiable vector-function $(Y_i)_{i=1}^n : [t_1,\omega[\,\times\mathbb{R}^n_{\frac{1}{2}}$ with components of the type (2.16). This vector-function is a solution of the system (2.15). Moreover, according to (2.16) and the sign conditions (2.3), (2.4),

$$\lim_{t\uparrow\omega} Y_i(t,\overline{v}) = Y_i^0 \ \text{ uniformly over } \ \overline{v} \in \mathbb{R}^n_{\frac{1}{2}} \ \ (i=\overline{1,n}). \tag{2.26}$$

Moreover, from (2.15) it follows

$$\begin{cases} \dfrac{(Y_i(t,\overline{v}))'_t}{Y_i(t,\overline{v})} = \dfrac{O'_i(t)}{Q_i(t)} + \dfrac{Y_{i+1}(t,\overline{v})\varphi'_{i+1}(Y_{i+1}(t,\overline{v}))}{\varphi_{i+1}(Y_{i+1}(t,\overline{v}))} \, \dfrac{(Y_{i+1}(t,\overline{v}))'_t}{Y_{i+1}(t,\overline{v})} \\ \qquad\qquad\qquad\qquad\qquad\qquad (i=\overline{1,n-1}), \\ \dfrac{(Y_n(t,\overline{v}))'_t}{Y_n(t,\overline{v})} = \dfrac{O'_n(t)}{Q_n(t)} + \\ \qquad + \Big(\prod\limits_{k=1}^{r}\sigma_k\Big) \dfrac{Y_{r+1}(t,\overline{v})\varphi'_{r+1}(Y_{r+1}(t,\overline{v}))}{\varphi_{r+1}(Y_{r+1}(t,\overline{v}))} \, \dfrac{(Y_{r+1}(t,\overline{v}))'_t}{Y_{r+1}(t,\overline{v})}\,. \end{cases} \tag{2.27}$$

Here by virtue of (2.26) and (1.2),

$$\lim_{t\uparrow\omega} \frac{Y_i(t,\overline{v}_i)\varphi'_i\left(Y_i(t,\overline{v}_i)\right)}{\varphi_i\left(Y_i(t,\overline{v}_i)\right)} = \sigma_i \ \ (i=\overline{1,n}) \ \text{ uniformly over } \ \overline{v}\in\mathbb{R}^n_{\frac{1}{2}}, \tag{2.28}$$

and according to the form of the functions Q_i $(i=\overline{1,n})$,

$$\frac{Q'_i(t)}{Q_i(t)} = \begin{cases} \dfrac{I'_i(t)}{I_i(t)} & \text{as } \ i\in\mathfrak{I}\cup\{n\}, \\ \dfrac{I'_i(t)}{I_i(t)} - \dfrac{I'_{i+1}(t)}{I_{i+1}(t)} & \text{as } \ i\in\overline{\mathfrak{I}}. \end{cases} \tag{2.29}$$

First, from (2.27) we obtain

$$\frac{(Y_{r+1}(t,\overline{v}))'_t}{Y_{r+1}(t,\overline{v})} = \bigg[1 - \Big(\prod_{k=1}^{r}\sigma_k\Big)\prod_{k=r+1}^{n}\frac{Y_k(t,\overline{v})\varphi'_k\left(Y_k(t,\overline{v})\right)}{\varphi_k\left(Y_k(t,\overline{v})\right)}\bigg]^{-1} \times$$
$$\times\bigg(\sum_{k=r+1}^{n}\frac{Q'_k(t)}{Q_k(t)}\prod_{j=r+1}^{k-1}\frac{Y_{j+1}(t,\overline{v})\varphi'_{j+1}\left(Y_{j+1}(t,\overline{v})\right)}{\varphi_j\left(Y_j(t,\overline{v})\right)}\bigg).$$

Hence, according to (2.28), (2.29) and (2.2), we get

$$\lim_{t\uparrow\omega}\frac{I_{r+1}(t)\left(Y_{r+1}(t,\overline{v})\right)'_t}{I'_{r+1}(t)Y_{r+1}(t,\overline{v})} = \frac{1}{\beta_{r+1}} \quad \text{uniformly over} \quad \overline{v}\in\mathbb{R}^n_{\frac{1}{2}}.$$

Further, by virtue of this limiting condition, from (2.27), consistently, starting from $i=n$ to $i=r+2$, and then, starting from $i=r$ to $i=1$, we get, (using (2.28), (2.29), (2.2))

$$\lim_{t\uparrow\omega}\frac{I_i(t)\left(Y_i(t,\overline{v})\right)'_t}{I'_i(t)Y_i(t,\overline{v})} = \frac{1}{\beta_i} \ \text{ uniformly over } \ \overline{v}\in\mathbb{R}^n_{\frac{1}{2}}. \tag{2.30}$$

Applying now to the system of differential equations (1.1) the transformation

$$y_i(t) = Y_i(t,\overline{v}_i(t)) \ \ (i=\overline{1,n}) \tag{2.31}$$

and taking into consideration that the vector-function $(Y_i(t,\overline{v}(t)))_{i=1}^n$ with $t \in [t_1,\omega[$ and $\overline{v}(t) \in \mathbb{R}^n_{\frac{1}{2}}$ is a solution of the system

$$\begin{cases} \dfrac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = Q_i(t)[1+v_i(t)] \qquad\qquad (i=\overline{1,n-1}), \\ \dfrac{y_n(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod\limits_{k=1}^{r}\sigma_k}} = Q_n(t)[1+v_n(t)], \end{cases} \tag{2.32}$$

we obtain the system of differential equations of the type

$$\begin{cases} v_i' = \dfrac{I_i'(t)}{\beta_i I_i(t)} - \dfrac{Q_i'(t)}{Q_i(t)}\,(1+v_i)- \\ \quad -\dfrac{I_{i+1}'(t)}{\beta_{i+1} I_{i+1}(t)} \cdot \dfrac{1+v_i}{1+v_{i+1}}\, H_{i+1}(t,\overline{v}) \;\; (i=\overline{1,n-2}), \\ v_{n-1}' = \dfrac{I_{n-1}'(t)}{\beta_{n-1} I_{n-1}(t)} - \dfrac{Q_{n-1}'(t)}{Q_{n-1}(t)}\,(1+v_{n-1})- \\ \qquad -\dfrac{1+v_{n-1}}{1+v_n}\, H_n(t,\overline{v})\, \dfrac{H(t,\overline{v})}{Q_n(t)}\,, \\ v_n' = \dfrac{H(t,\overline{v})}{Q_n(t)} - \dfrac{Q_n'(t)}{Q_n(t)}\,(1+v_n)- \\ \quad -\Big(\prod\limits_{k=1}^{r}\sigma_k\Big)\dfrac{1+v_n}{1+v_{r+1}}\, H_{r+1}(t,\overline{v})\, \dfrac{I_{r+1}'(t)}{\beta_{r+1} I_{r+1}(t)}\,, \end{cases} \tag{2.33}$$

where

$$H_i(t,\overline{v}) = \frac{Y_i(t,\overline{v})\varphi_i'(Y_i(t,\overline{v}))}{\varphi_i(Y_i(t,\overline{v}))} \quad (i=\overline{1,n}),$$

$$H(t,\overline{v}_1) = \frac{\alpha_n p_n(t)\varphi_1(Y_1(t,\overline{v}))}{[\varphi_{r+1}(Y_{r+1}(t,\overline{v}))]^{\prod\limits_{k=1}^{r}\sigma_k}}\,.$$

Since the conditions (2.28), (2.30) are valid and the functions φ_i $(i = 1,\dots,r)$ satisfy the **S** - condition, by virtue of Remark (2.2), we obtain

$$H_i(t,\overline{v}) = \sigma_i + R_i(t,\overline{v}) \;\; (i=\overline{1,n}),$$

$$H(t,\overline{v}) = \alpha_n p_n(t) q_{r+1}(t) \prod_{k=1}^{r} |1+v_k|^{\prod_{j=1}^{k}\sigma_j}[1+R(t,\overline{v})],$$

where

$$\lim_{t\uparrow\omega} R_i(t,\overline{v}) = 0 \;\text{ uniformly over }\; \overline{v}\in\mathbb{R}^n_{\frac{1}{2}} \;\; (i=\overline{1,n}),$$

$$\lim_{t\uparrow\omega} R(t,\overline{v}) = 0 \;\text{ uniformly over }\; \overline{v}\in\mathbb{R}^n_{\frac{1}{2}}.$$

By virtue of these representations and the conditions (2.2), the system (2.33) can be rewritten in the form

$$\begin{cases} v_i' = h_i(t)\,[f_i(t,\overline{v}) - v_i + \Lambda_i\sigma_{i+1}v_{i+1} + V_i(\overline{v})] \quad (i=\overline{1,n-2}), \\ v_{n-1}' = h_{n-1}(t)\left[f_{n-1}(t,\overline{v}) - \sum\limits_{k=1}^{r} a_{0k}v_k - v_{n-1} + v_n + V_{n-1}(\overline{v})\right], \\ v_n' = h_n(t)\left[f_n(t,\overline{v}) + \sum\limits_{k=1}^{r} a_{0k}v_k + a_{0n}v_{r+1} - v_n + V_n(\overline{v})\right], \end{cases} \tag{2.34}$$

where

$$h_i(t) = \frac{I_i'(t)}{\beta_i I_i(t)} \quad (i=\overline{1,n}),$$

$$a_{0k} = \prod_{j=1}^{k} \sigma_j \quad (k=\overline{1,n}),$$

$$\lim_{t\uparrow\omega} f_i(t,\overline{v}) = 0 \ \text{ uniformly over } \ \overline{v} \in \mathbb{R}^n_{\frac{1}{2}} \quad (i=\overline{1,n}),$$

$$V_i(\overline{v}) = -\Lambda_i\sigma_{i+1}\left[\frac{1+v_i}{1+v_{i+1}} - 1 - v_i + v_{i+1}\right] \quad (i=\overline{1,n-2}),$$

$$V_{n-1}(\overline{v}) = -\left[\frac{1+v_{n-1}}{1+v_n}\prod_{k=1}^{r}|1+v_k|^{a_{0k}} - 1 - \sum_{k=1}^{r} a_{0k}v_k - v_{n-1} + v_n\right],$$

$$V_n(\overline{v}) = \prod_{k=1}^{r}|1+v_k|^{a_{0k}} - a_{0n}\frac{1+v_n}{1+v_{r+1}} - 1 + a_{0n} - \\ - \sum_{k=1}^{r} a_{0k}v_k - a_{on}v_{r+1} + a_{0n}v_n.$$

Here

$$\lim_{|v_1|+\cdots+|v_n|\to 0} \frac{\partial V_i(\overline{v})}{\partial v_k} = 0 \quad (i,k=\overline{1,n}),$$

and, taking into consideration that $\lim\limits_{t\uparrow\omega} I_i(t)$ $(i=\overline{1,n})$ is equal either to zero, or to $\pm\infty$, the following conditions are satisfied:

$$\int_{t_1}^{\omega} h_i(t)\,dt = \pm\infty \quad (i=\overline{1,n}).$$

Since $m = \max\{i \in \Im : \ \Lambda_i = 0\} < n-1$ and the conditions (2.2) are valid, when $i = \overline{m+1, n-1}$, we have

$$h_i(t) = h_n(t)\frac{h_i(t)}{h_n(t)} = h_n(t)\frac{\beta_n I_n(t) I_i'(t)}{\beta_i I_i(t) I_n'(t)} =$$
$$= h_n(t)\frac{\beta_{i+1} I_i'(t) I_{i+1}(t)}{\beta_i I_i(t) I_{i+1}'(t)}\,\frac{\beta_{i+2} I_{i+1}'(t) I_{i+2}(t)}{\beta_{i+1} I_{i+1}(t) I_{i+2}'(t)}\cdots\frac{\beta_n I_{n-1}'(t) I_n(t)}{\beta_{n-1} I_{n-1}(t) I_n'(t)} =$$
$$= \frac{h_n(t)[1+o(1)]}{\Lambda_i \Lambda_{i+1}\ldots\Lambda_{n-1}} \ \text{ as } \ t \uparrow \omega.$$

Therefore, the system (2.34) can be rewritten in the form

$$\begin{cases} v_i' = h_i(t)\Big[f_i(t,\overline{v}) - v_i + \Lambda_i \sigma_{i+1} v_{i+1} + V_i(\overline{v})\Big] \ \ (i = \overline{1, m-1}), \\[2mm] v_m' = h_m(t)\big[f_m(t,\overline{v}) - v_m\big], \\[2mm] v_i' = h_n(t)\Bigg[\widetilde{f}_i(t,\overline{v}) - \dfrac{v_i}{\Lambda_i\cdots\Lambda_{n-1}} + \\ \quad + \dfrac{\sigma_{i+1}}{\Lambda_{i+1}\cdots\Lambda_{n-1}}\, v_{i+1} + \dfrac{V_i(\overline{v})}{\Lambda_i\cdots\Lambda_{n-1}}\Bigg] \ \ (i = \overline{m+1, n-2}), \\[2mm] v_{n-1}' = h_n(t)\Bigg[\overline{f}_{n-1}(t,\overline{v}) - \sigma_n \sum\limits_{k=1}^{r} a_{0k} v_k - \sigma_n v_{n-1} + \\ \qquad\qquad + \sigma_n v_n + \sigma_n V_{n-1}(\overline{v})\Bigg], \\[2mm] v_n' = h_n(t)\left[f_n(t,\overline{v}) + \sum\limits_{k=1}^{r} a_{0k} v_k + a_{0n} v_{r+1} - v_n + V_n(\overline{v})\right], \end{cases} \tag{2.35}$$

where the functions $\overline{f}_i$ $(i = \overline{m+1, n-1})$ have the same properties as the functions f_i $(i = \overline{m+1, n-1})$ in the system (2.34).

The important peculiarity of the system is that the coefficient at v_{m+1} is equal to zero.

Suppose that B_{m+1} is a constant matrix of order $(n-m)\times(n-m)$. This matrix consists of the coefficients at $v_{m+1}, \ldots, v_n$ in the last standing in brakets $n-m$ equations of the system (2.35). Its characteristic equation is $\det[B_{m+1} - \lambda E_{n-m}] = 0$, where E_{n-m} is the unit matrix of order $(n-m)\times(n-m)$ and is represented by (2.1). Taking into consideration the conditions of the theorem, it is evident that this equation has no roots with zero real part. Therefore, using the proof of Theorem 2.1 in [8], we conclude that there exists a nonsingular constant matrix D_{m+1} of order $(n-m)\times(n-m)$ and there exists a nonsingular continuously differentiable and bounded (together with its inverse matrix) on the interval $[t_0, \omega[$ matrix

$L_{m+1}(t)$ such that

$$L_{m+1}^{-1}(t)D_{m+1}^{-1}B_{m+1}D_{m+1}L_{m+1}(t)-\frac{1}{h_n(t)}\,L^{-1}(t)L'(t)=C_{m+1},$$

where C_{m+1} is the upper triangular matrix of the form

$$C_{m+1}=\begin{pmatrix} \operatorname{Re}\lambda_1^0 & c_{m+1m+2} & \dots & c_{m+1n-1} & c_{m+1n} \\ 0 & \operatorname{Re}\lambda_2^0 & \dots & c_{m+2n-1} & c_{m+2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \operatorname{Re}\lambda_{n-m-1}^0 & c_{n-1n} \\ 0 & 0 & \dots & 0 & \operatorname{Re}\lambda_{n-m}^0 \end{pmatrix},$$

where λ_i^0 $(i=\overline{1,n-m})$ are all roots (with multipliciting) of the algebraic equation (2.1), all c_{ik} $(k=\overline{i+1,n})$ as $i\in\{m+1,\dots,n\}$ are equal to zero, except for a single one that equals 1.

In virtue of this fact, by means of the transformation

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}=\begin{pmatrix} E_m & O_1 \\ O_2 & D_{m+1}L_{m+1}(t) \end{pmatrix}\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \tag{2.36}$$

where O_1, O_2 are zero-matrices of orders $m\times(n-m)$ and $(n-m)\times m$ (respectively), E_m is the unit matrix of order $m\times m$, the system of differential equations (2.35) takes the form

$$\begin{cases} w_i'=h_i(t)\,[f_{1i}(t,\overline{w})-w_i+\Lambda_i\sigma_{i+1}w_{i+1}+f_{2i}(\overline{w})] \quad (i=\overline{1,m-1}), \\ w_m'=h_m(t)[f_{1m}(t,\overline{w})-w_m], \\ w_i'=h_n(t)\Big[f_{1i}(t,\overline{w})+\displaystyle\sum_{k=1}^{m}c_{ik}(t)w_k+\big(\operatorname{Re}\lambda_{i-m}^0\big)\,w_i+ \\ \qquad\qquad +\displaystyle\sum_{k=i+1}^{n}c_{ik}w_k+f_{2i}(t,\overline{w})\Big] \quad (i=\overline{m+1,n-1}), \\ w_n'=h_n(t)\Big[f_{1n}(t,\overline{w})+\displaystyle\sum_{k=1}^{m}c_{nk}(t)w_k+\big(\operatorname{Re}\lambda_{n-m}^0\big)\,w_n+f_{2n}(t,\overline{w})\Big], \end{cases} \tag{2.37}$$

where the functions c_{ik} $(i=\overline{m+1,n},\ k\in\{1,\dots,m\})$ are continuous and bounded on the interval $[t_1,\omega[$, the functions $f_{1i}:[t_1,\omega[\,\times\mathbb{R}_\delta^n\to\mathbb{R}$ $(i=\overline{1,n})$, $f_{2i}:\mathbb{R}_\delta^n\to\mathbb{R}$ $(i=\overline{1,m-1})$, the functions $f_{2i}:[t_1,\omega[\,\times\mathbb{R}_\delta^n\to\mathbb{R}$ $(i=\overline{m+1,n})$ are continuous, where $\mathbb{R}_\delta^k=\{(x_1,\dots,x_k)\in\mathbb{R}^k:\ |x_j|\le\delta\}$,

$\delta > 0$ is sufficiently small and satisfy the conditions

$$\lim_{t\uparrow\omega} f_{1i}(t,\overline{w}) = 0 \ \ (i=\overline{1,m}) \ \text{ uniformly over } \ \overline{w}\in\mathbb{R}^n_\delta,$$

$$\lim_{|w_1|+\cdots+|w_{i+1}|\to 0} \frac{f_{2i}(\overline{w})}{|w_1‘+\cdots+|w_n|} = 0 \ \ (i=\overline{1,m-1}),$$

$$\lim_{|w_1|+\cdots+|w_n|\to 0} \frac{f_{2i}(t,\overline{w})}{|w_1|+\cdots+|w_n|} = 0 \ \ (i=\overline{m+1,n})$$

$$\text{uniformly over } \ t\in[t_1,\omega[\,.$$

Since the functions c_{ik} $(i=\overline{m+1,n},\ k\in\{1,\dots,m\})$ are bounded on the interval $[t_1,\omega[$, there exists a number $\varepsilon > 0$ such that the constants B_i^0 $(i=\overline{m+1,n})$, defined (starting with $i=n$) by the recurrent relations

$$B_n^0 = \frac{\varepsilon}{|\mathrm{Re}\lambda^0_{n-m}|}\sum_{k=1}^m c^0_{nk},$$

$$B_i^0 = \frac{1}{|\mathrm{Re}\,\lambda^0_{i-m}|}\left(\varepsilon\sum_{k=1}^m c^0_{ik} + \sum_{i+1}^n |c_{ik}|B_k^0\right) \ \ (i=\overline{m+1,n-1}),$$

where

$$c^0_{ik} = \limsup_{t\uparrow\omega} |c_{ik}(t)| \ \ (i=\overline{m+1,n}, \ \ k\in\{1,\dots,m\}),$$

satisfy the inequalities $B_i^0 < 1$ $(i=\overline{m+1,n})$.

With this choice of the constant $\varepsilon > 0$, the system (2.37) by means of the transformation

$$w_i = \varepsilon z_i \ \ (i=\overline{1,m}), \quad w_i = z_i \ \ (i=\overline{m+1,n}) \tag{2.38}$$

is reduced to a system of differential equations that satisfies all the conditions of Theorem 1.2 in [7]. According to this theorem, this system has at least one solution $(z_i)_{i=1}^n : [t_2,\omega[\,\mathbb{R}^n$ $(t_2\in[t_1,\omega[)$, which tends to zero when $t\uparrow\omega$. Moreover, there exists the whole k-parametric family of solutions, if there are k positive numbers among the numbers (2.7). In virtue of the transformations (2.38), (2.36) and (2.31), each of these solutions corresponds to the solution of the system (1.1), satisfying (as $t\uparrow\omega$) the asymptotic representations (2.5), (2.6). Furthermore, taking into consideration the form of functions (2.31) and conditions (2.2)–(2.4), it is easy to see that all these solutions are the $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions of the system (1.1). Thus the theorem is proved. □

Consider now the conditions that give an opportunity to rewrite the asymptotic representations (2.5), (2.6) in an explicit form.

Theorem 2.2. *Let* $\Lambda_i\in\mathbb{R}$ $(i=\overline{1,n-1})$ *include those equal zero,* $m=\max\{i\in\mathfrak{I}:\ \Lambda_i=0\}$ *and* $r=\max\mathfrak{I}<n-1$. *Moreover, let all the functions* φ_k $(k=\overline{1,n})$ *satisfy the* **S**-*condition. Then each* $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-*solution (in case it exists) of the system* (1.1) *admits for* $t\uparrow\omega$ *asymptotic*

representations

$$y_{r+1}(t) = \mu_{r+1} \prod_{k=r+1}^{n-1} \left| q_k(t)\theta_{k+1}\left(\mu_{k+1}|I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}}\right)\right|^{\frac{\prod\limits_{j=r+2}^{k}\sigma_j}{1-\prod\limits_{j=1}^{n}\sigma_j}} \times$$

$$\times \left| Q_n(t)\left[\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}}\right)\right]^{\prod\limits_{j=1}^{r}\sigma_j} \right|^{\frac{\prod\limits_{j=r+2}^{n}\sigma_j}{1-\prod\limits_{j=1}^{n}\sigma_j}} [1+o(1)],$$

$$y_i(t) = \mu_i \prod_{k=i}^{r} \left| Q_k(t)\theta_{k+1}\left(\mu_{k+1}|I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}}\right)\right|^{\prod\limits_{j=i+1}^{k}\sigma_j} \times$$

$$\times |y_{r+1}(t)|^{\prod\limits_{j=i+1}^{r+1}\sigma_j}[1+o(1)] \quad (i=\overline{1,r}), \tag{2.39}$$

$$y_i(t) = \mu_i \prod_{k=i}^{n-1} \left| Q_k(t)\theta_{k+1}\left(\mu_{k+1}|I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}}\right)\right|^{\prod\limits_{j=i+1}^{k}\sigma_j} \times$$

$$\times \left| Q_n(t)\left[\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}}\right)\right]^{\prod\limits_{j=1}^{r}\sigma_j} \right|^{\prod\limits_{j=i+1}^{n}\sigma_j} \times$$

$$\times |y_{r+1}(t)|^{\prod\limits_{j=1}^{r+1}\sigma_j \prod\limits_{j=i+1}^{n}\sigma_j}[1+o(1)] \quad (i=\overline{r+2,n}).$$

Proof. In Theorem 2.1, it is proved, that for the existence of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions in (1.1), it is necessary that the conditions (2.2)–(2.4) valid, and each solution of that type admit for $t\uparrow\omega$ the asymptotic representations (2.5), (2.6). Moreover, the asymptotic representation (2.11) for these solutions was obtained. Since all functions φ_i $(i=\overline{1,n})$ satisfy the **S**-condition, in virtue of (2.11) and Remark 1.2, we get

$$\theta_i(y_i(t)) = \theta_i\left(\mu_i\,|I_i(t)|^{\frac{1}{\beta_i}}\right)[1+o(1)] \quad (i=\overline{1,n}) \ \text{ as } \ t\uparrow\omega.$$

That is why the asymptotic representations (2.5), (2.6) can be rewritten in the form

$$\frac{y_i(t)}{|y_{i+1}(t)|^{\sigma_{i+1}}} =$$

$$= Q_i(t)\theta_{i+1}\left(\mu_{i+1}\,|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}}\right)[1+o(1)] \quad (i=\overline{1,n-1}) \ \text{ as } \ t\uparrow\omega,$$

$$\frac{y_n(t)}{|y_{r+1}(t)|^{\prod\limits_{j=1}^{r+1}\sigma_j}} =$$

$$= Q_n(t)\left[\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}}\right)\right]^{\prod\limits_{j=1}^{r}\sigma_j}[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

Hence, consistently, starting with $i = n$, we obtain the asymptotic representations (2.39). The theorem is proved. □

3. Conclusions

In this paper, for cyclic system (1.1) with regularly varying non-linearities, the class of the so-called $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions is introduced and the question of the existence of such solutions in special case (when $\Lambda_i \in \mathbb{R}$ $(i = \overline{1, n-1})$ include zeroes) is discovered. Peculiarity of this case demands both the validity of the additional **S** -condition for all nonlinearities of the system, except one, and the assumption that $\Lambda_{n-1} \in \mathfrak{I}$. As a result, the necessary and sufficient conditions for the existence of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions for (1.1) are obtained. Implicit asymptotic formulas for components of these solutions (when $t \uparrow \omega$ $(\omega \le +\infty)$) are established. Explicit asymptotic formulas for components of these solutions are established, provided all nonlinearities satisfy the **S**-condition.

The results may be used, for instance, to establish the asymptotics of solutions for sufficiently nonlinear differential equations of the type

$$y'' = p(t)\varphi_1(y)\varphi_2(y') \ \text{ and } \ y^{(n)} = p(t)\varphi(y),$$

where $p : [a, \omega[\to \mathbb{R} \setminus \{0\}$ is a continuous function and $\varphi, \varphi_1 : \Delta(Y_0^0) \to]0, +\infty[$, $\varphi_2 : \Delta(Y_1^0) \to]0, +\infty[$, $\Delta(Y_i^0)$ is a one-sided neighborhood, Y_i^0 are continuously differentiable and regularly varying functions of certain orders (when $y \to Y_0^0$ and $y' \to Y_1^0$).

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EMPHATIC CONVERGENCE AND SEQUENTIAL SOLUTIONS OF GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

Dedicated to the memory of Temur Chanturia

Abstract. This contribution deals with systems of generalized linear differential equations of the form

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \;\; k \in \mathbb{N},$$

where $-\infty < a < b < \infty$, X is a Banach space, $L(X)$ is the Banach space of linear bounded operators on X, $\widetilde{x}_k \in X$, $A_k : [a,b] \to L(X)$ have bounded variations on $[a,b]$, $f_k : [a,b] \to X$ are regulated on $[a,b]$ and the integrals are understood in the Kurzweil–Stieltjes sense.

Our aim is to present new results on continuous dependence of solutions to generalized linear differential equations on the parameter k. We continue our research from [18], where we were assuming that A_k tends uniformly to A and f_k tends uniformly to f on $[a,b]$. Here we are interested in the cases when these assumptions are violated.

Furthermore, we introduce a notion of a sequential solution to generalized linear differential equations as the limit of solutions of a properly chosen sequence of ODE's obtained by piecewise linear approximations of functions A and f. Theorems on the existence and uniqueness of sequential solutions are proved and a comparison of solutions and sequential solutions is given, as well.

The convergence effects occurring in our contribution are, in some sense, very close to those described by Kurzweil and called by him emphatic convergence.

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Key words and phrases. Generalized linear differential equation, sequential solution, emphatic convergence.

რეზიუმე. განხილულია განზოგადებულ წრფივ დიფერენციალურ განტოლებათა სისტემა

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \;\; k \in \mathbb{N},$$

სადაც $-\infty < a < b < \infty$, X არის ბანახის სივრცე, $L(X)$ არის X-ის შეუღლებული სივრცე, $\widetilde{x}_k \in X$, $A_k : [a,b] \to L(X)$ -ს აქვს შემოსაზღვრული ვარიაცია, $f_k : [a,b] \to X$ რეგულირებადია, ხოლო ინტეგრალი გაიგება კურცვეილ–სტილტიესის აზრით.

მიღებულია ახალი შედეგები ამონახსნების ნატურალური k პარამეტრისაგან უწყვეტად დამოკიდებულების შესახებ, როცა $k \to \infty$. შემოღებულია განზოგადებული წრფივი დიფერენციალური განტოლების სეკვენციალური ამონახსნის ცნება და დამტკიცებულია თეორემები ასეთი ამონახსნის არსებობისა და ერთადერთობის შესახებ.

ნაშრომში დადგენილი ამონახსნთა მიმდევრობის კრებადობა გარკვეული აზრით ახლოს არის ი. კურცვეილის მიერ აღწერილ კრებადობასთან, რომელსაც იგი ემფატიკურს უწოდებს.

1. Introduction

Generalized differential equations were introduced in 1957 by J. Kurzweil in [14]. Since then they were studied by many authors. (See e.g. the monographs by Schwabik, Tvrdý and Vejvoda [29], [25], [32] or the papers by Ashordia [2], [3] or Fraňková [7] and the references therein). Closely related and fundamental is also the contribution by Hildebrandt [10]. Furthermore, during the recent decades, the interest in their special cases like equations with impulses or discrete systems increased considerably (cf. e.g. the monographs [21], [33], [4], [24] or [1]).

Concerning integral equations in a general Banach space, it is worth to highlight the monograph by Hönig [11] having as a background the interior (Dushnik) integral. On the other hand, dealing with the Kurzweil–Stieltjes integral, the contributions by Schwabik in [27] and [28] are essential for this paper. It is well-known that the theory of generalized differential equations in Banach spaces enables the investigation of continuous and discrete systems, including the equations on time scales and the functional differential equations with impulses, from the common standpoint. This fact can be observed in several papers related to special kinds of equations, such as e.g. those by Imaz and Vorel [12], Oliva and Vorel [19], Federson and Schwabik [6].

In this paper we consider linear generalized differential equations of the form

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \quad k \in \mathbb{N}, \tag{1.1}$$

and

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + f(t) - f(a), \quad t \in [a,b]. \tag{1.2}$$

In particular, we are interested in finding conditions ensuring the convergence of the solutions x_k of (1.1) to the solution x of (1.2). We continue our research from [9] and [18], where we supposed a.o. that A_k tends uniformly to A and f_k tends uniformly to f on $[a,b]$. Here we will deal, similarly to [31] and [8], with the situation when this assumption is not satisfied.

In the paper we use the following notation:

$\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers and $\mathbb{R}$ stands for the space of real numbers. If $-\infty < a < b < \infty$, then $[a,b]$ and (a,b) denote the corresponding closed and open intervals, respectively. Furthermore, $[a,b)$ and $(a,b]$ are the corresponding half-open intervals.

X is a Banach space equipped with the norm $\|\cdot\|_X$ and $L(X)$ is the Banach space of linear bounded operators on X equipped with the usual operator norm. For an arbitrary function $f : [a,b] \to X$, we set

$$\|f\|_\infty = \sup\big\{\|f(t)\|_X;\ t \in [a,b]\big\}.$$

If $f_k : [a,b] \to X$ for $k \in \mathbb{N}$ and $f : [a,b] \to X$ are such that

$$\lim_{k\to\infty} \|f_k - f\|_\infty = 0,$$

we say that f_k tends to f *uniformly* on $[a,b]$ and write $f_k \rightrightarrows f$ on $[a,b]$. If $J \subset \mathbb{R}$ and $f_k \rightrightarrows f$ on $[a,b]$ for each $[a,b]\subset J$, we say that f_k tends to f *locally uniformly* on J and write $f_k \rightrightarrows f$ locally on J.

If for each $t \in [a,b)$ and $s \in (a,b]$, the function $f : [a,b] \to X$ possesses the limits

$$f(t+) := \lim_{\tau\to t+} f(\tau), \quad f(s-) := \lim_{\tau\to s-} f(\tau),$$

we say that f is *regulated* on $[a,b]$. The set of all functions with values in X which are regulated on $[a,b]$ is denoted by $G([a,b],X)$. Furthermore,

$$\Delta^+ f(t) = f(t+) - f(t) \quad \text{for } t \in [a,b), \quad \Delta^+ f(b) = 0,$$
$$\Delta^- f(s) = f(s) - f(s-) \quad \text{for } s \in (a,b], \quad \Delta^- f(a) = 0$$

and

$$\Delta f(t) = f(t+) - f(t-) \quad \text{for } t \in (a,b).$$

Clearly, each function, regulated on $[a,b]$, is bounded on $[a,b]$.

The set $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \subset [a,b]$, where $m \in \mathbb{N}$, is called a *division* of the interval $[a,b]$, if $a = \alpha_0 < \alpha_1 < \dots < \alpha_m = b$. The set of all divisions of the interval $[a,b]$ is denoted by $\mathcal{D}[a,b]$. For a function $f : [a,b] \to X$ and a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \in \mathcal{D}[a,b]$, we put

$$\nu(D) := m, \quad |D| = \max\big\{\alpha_i - \alpha_{i-1};\ i = 1,2,\dots,m\big\},$$
$$v(f,D) := \sum_{j=1}^{m} \|f(\alpha_j) - f(\alpha_{j-1})\|_X$$

and

$$\operatorname{var}_a^b f := \sup\big\{v(f,D);\ D \in \mathcal{D}[a,b]\big\}$$

is the *variation* of f over $[a,b]$. We say that f has a bounded variation on $[a,b]$ if $\operatorname{var}_a^b f < \infty$. The set of X-valued functions of bounded variation on $[a,b]$ is denoted by $BV([a,b],X)$ and $\|f\|_{BV} = \|f(a)\|_X + \operatorname{var}_a^b f$. Finally, $C([a,b],X)$ is the set of functions $f : [a,b] \to X$ which are continuous on $[a,b]$. Obviously,

$$BV([a,b],X) \subset G([a,b],X) \text{ and } C([a,b],X) \subset G([a,b],X).$$

The integral which occurs in this paper is the abstract Kurzweil–Stieltjes integral (in short the KS-integral) as defined by Schwabik in [26]. (For its further properties see also our previous paper [17]). For the reader's convenience, let us recall the definition of the KS-integral.

Let $-\infty < a < b < \infty$, $m \in \mathbb{N}$,

$$D = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \in \mathcal{D}[a,b] \text{ and } \xi = (\xi_1, \xi_2, \dots, \xi_m) \in [a,b]^m.$$

Then the couple $P = (D, \xi)$ is called a *partition* of $[a,b]$ if

$$\alpha_{j-1} \le \xi_j \le \alpha_j \quad \text{for} \quad j = 1,2,\dots,m.$$

The set of all partitions of the interval $[a,b]$ is denoted by $\mathcal{P}[a,b]$. An arbitrary function $\delta : [a,b] \to (0,\infty)$ is called a *gauge* on $[a,b]$. Given a gauge δ on $[a,b]$, the partition

$$P = (D,\xi) = \big(\{\alpha_0,\alpha_1,\dots,\alpha_m\},(\xi_1,\xi_2,\dots,\xi_m)\big) \in \mathcal{P}[a,b]$$

is said to be δ-fine, if

$$[\alpha_{j-1},\alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for} \quad j = 1,2,\dots,m.$$

The set of all δ-fine partitions of $[a,b]$ is denoted by $\mathcal{A}(\delta;[a,b])$.

For the functions $f : [a,b] \to X$, $G : [a,b] \to L(X)$ and a partition $P \in \mathcal{P}[a,b]$,

$$P = (\{\alpha_0,\alpha_1,\dots,\alpha_m\},(\xi_1,\xi_2,\dots,\xi_m)),$$

we define

$$\Sigma(\Delta G f; P) = \sum_{j=1}^{m} \big[G(\alpha_j) - G(\alpha_{j-1})\big] f(\xi_j).$$

We say that $q \in X$ is the KS-integral of f with respect to G from a to b if

$$\begin{cases} \text{for each } \ \varepsilon > 0 \ \text{ there is a gauge } \delta \text{ on } [a,b] \text{ such that} \\ \qquad \big\|q - \Sigma(\Delta G f; P)\big\|_X < \varepsilon \ \text{ for all } \ P \in \mathcal{A}(\delta;[a,b]). \end{cases}$$

In such a case we write

$$q = \int_a^b \mathrm{d}[G(t)] f(t) \quad \text{or, more briefly,} \quad q = \int_a^b \mathrm{d}[G] f.$$

Analogously we define the integral $\int_a^b F \,\mathrm{d}[g]$ for $F : [a,b] \to L(X)$ and $g : [a,b] \to X$.

The following assertion summarizes the properties of the KS-integral needed later. (For the proofs, see [26] and [17].)

Theorem 1.1. *Let $f \in G([a,b],X), G \in G([a,b],L(X))$ and let at least one of the functions f, G have a bounded variation on $[a,b]$. Then there exists the integral $\int_a^b \mathrm{d}[G] f$. Furthermore,*

$$\Big\| \int_a^b \mathrm{d}[G] f \Big\|_X \le 2\|G\|_\infty \big(\|f(a)\|_X + \operatorname{var}_a^b f\big) \ \textit{ if } \ f \in BV([a,b],X), \tag{1.3}$$

$$\Big\| \int_a^b \mathrm{d}[G] f \Big\|_X \le (\operatorname{var}_a^b G) \|f\|_\infty \ \textit{ if } \ G \in BV([a,b],L(X)), \tag{1.4}$$

$$\left.\begin{aligned}&\int_a^t \mathrm{d}[G]f_k \rightrightarrows \int_a^t \mathrm{d}[G]f \quad on \quad [a,b]\\ &\qquad if\, G\in BV([a,b],L(X)),\ f_k\in G([a,b],X)\ for\ k\in\mathbb{N}\ and\ f_k\rightrightarrows f,\end{aligned}\right\} \tag{1.5}$$

$$\left.\begin{aligned}&\int_a^t \mathrm{d}[G_k]f \rightrightarrows \int_a^t \mathrm{d}[G]f \quad on \quad [a,b]\\ &\qquad if\, f\in BV([a,b],X),\ G_k\in G([a,b],L(X))\ for\ k\in\mathbb{N}\ and\ g_k\rightrightarrows g,\end{aligned}\right\} \tag{1.6}$$

$$\left.\begin{aligned}&\int_a^t \mathrm{d}[G_k]f_k \rightrightarrows \int_a^t \mathrm{d}[G]f\\ &\qquad if\ G_k\in BV([a,b],L(X)),\ \ f_k\in G([a,b],X)\ \ for\ \ k\in\mathbb{N},\\ &\qquad \sup\{\mathrm{var}_a^b\, G_k;\ k\in\mathbb{N}\}<\infty\ \ and\ \ f_k\rightrightarrows f,\ \ G_k\rightrightarrows G\ \ on\ \ [a,b].\end{aligned}\right\} \tag{1.7}$$

Remark 1.2. An assertion analogous to that of Theorem 1.1 holds also for the integrals

$$\int_a^b F\,\mathrm{d}[g],\ \int_a^b F_k\,\mathrm{d}[g],\ \int_a^b F\,\mathrm{d}[g_k],\ \int_a^b F_k\,\mathrm{d}[g_k],\ k\in\mathbb{N},$$

where $F,\ F_k : [a,b]\to L(X)$ and $g,\ f_k : [a,b]\to X$.

2. Generalized Differential Equations

Let $A\in BV([a,b],L(X))$, $f\in G([a,b],X)$ and $\widetilde{x}\in X$. Consider the *generalized linear differential equation* (1.2). We say that a function $x : [a,b]\to X$ is a *solution* of (1.2) on the interval $[a,b]$ if the integral $\int_a^b \mathrm{d}[A]x$ has a sense and equality (1.2) is satisfied for all $t\in[a,b]$.

Obviously, the generalized differential equation (1.2) is equivalent to the equation

$$x(t)=\widetilde{x}+\int_a^t \mathrm{d}[B]x+g(t)-g(a)$$

whenever $B-A$ and $g-f$ are constant on $[a,b]$. Therefore, without loss of generality we may assume that

$$A(a)=A_k(a)=0 \ \text{ and } \ f(a)=f_k(a)=0 \ \text{ for } \ k\in\mathbb{N}.$$

For our purposes the following property is crucial:

$$\big[I-\Delta^- A(t)\big]^{-1}\in L(X) \ \text{ for each } \ t\in(a,b]. \tag{2.1}$$

Its importance is well illustrated by the following assertion which summarizes some of the basic properties of generalized linear differential equations in abstract spaces. (For the proof see [18, Lemma 3.2].)

Theorem 2.1. *Let* $A \in BV([a,b], L(X))$ *satisfy* (2.1). *Then for each* $\widetilde{x} \in X$ *and each* $f \in G([a,b], X)$ *the equation* (1.2) *has a unique solution* x *on* $[a,b]$ *and* $x \in G([a,b], X)$. *Moreover,* $x - f \in BV([a,b], X)$

$$0 < c_A := \sup\Big\{ \big\|[I - \Delta^- A(t)]^{-1}\big\|_{L(X)}\,;\ t \in (a,b]\Big\} < \infty, \tag{2.2}$$

$$\|x(t)\|_X \leq c_A\,(\|\widetilde{x}\|_X + \|f(a)\|_X + \|f\|_\infty)\exp(c_A \operatorname{var}_a^t A) \ \ \textit{for}\ \ t \in [a,b] \tag{2.3}$$

and

$$\operatorname{var}_a^b(x - f) \leq c_A(\operatorname{var}_a^b A)\big(\|\widetilde{x}\|_X + 2\|f\|_\infty\big)\exp(c_A \operatorname{var}_a^b A). \tag{2.4}$$

The following result was proved in [18, Theorem 3.4].

Theorem 2.2. *Let* $A, A_k \in BV([a,b], L(X))$ $f, f_k \in G([a,b], X)$, $\widetilde{x}, \widetilde{x}_k \in X$ *for* $k \in \mathbb{N}$. *Assume* (2.1),

$$\alpha^* := \sup\{\operatorname{var}_a^b A_k; k \in \mathbb{N}\} < \infty, \tag{2.5}$$

$$A_k \rightrightarrows A \quad \textit{on}\ \ [a,b], \tag{2.6}$$

$$f_k \rightrightarrows f \quad \textit{on}\ \ [a,b] \tag{2.7}$$

and

$$\lim_{k\to\infty} \widetilde{x}_k = \widetilde{x}. \tag{2.8}$$

Then equation (1.2) *has a unique solution* x *on* $[a,b]$. *Furthermore, for each* $k \in \mathbb{N}$ *sufficiently large, there exists a unique solution* x_k *on* $[a,b]$ *for the equation* (1.1) *and*

$$x_k \rightrightarrows x \quad \textit{on}\ \ [a,b]. \tag{2.9}$$

Remark 2.3. If (2.5) is not true, but (2.6) is replaced by a stronger notion of convergence in the sense of Opial ([20, Theorem 1]) (cf. [13, Theorem 1.4.1] for extension to functional differential equations), the conclusion of Theorem 2.2 remains true (see [18, Theorem 4.2]). If (2.6) or (2.7) does not hold, the situation becomes rather more difficult (see [7], [8] and [31]). The next section deals with such a case.

3. Emphatic Convergence

The proofs of the next two lemmas follow the ideas of the proof of [8, Theorem 2.2].

Lemma 3.1. *Let* $A, A_k \in BV([a,b], L(X))$, $f, f_k \in G([a,b], X)$, $\widetilde{x}, \widetilde{x}_k \in X$ *for* $k \in \mathbb{N}$. *Assume* (2.1), (2.8),

$$\left.\begin{aligned} &[I - \Delta^- A_k(t)]^{-1} \in L(X) \\ &\qquad \textit{for all}\ \ t \in (a,b]\ \ \textit{and}\ \ k \in \mathbb{N}\ \ \textit{sufficiently large,} \end{aligned}\right\} \tag{3.1}$$

$$A_k \rightrightarrows A \ \ \textit{and}\ \ f_k \rightrightarrows f \ \ \textit{locally on}\ \ (a,b]. \tag{3.2}$$

Then there exists a unique solution x *of* (1.2) *on* $[a,b]$ *and, for each* $k \in \mathbb{N}$, *sufficiently large, there exists a unique solution* x_k *on* $[a,b]$ *to the equation* (1.1).

Moreover, let (2.5) *and*

$$\left.\begin{array}{l}\forall\,\varepsilon>0\ \ \exists\,\delta>0\ \ \textit{such that}\ \ \forall\,t\in(a,a+\delta)\ \ \exists\,k_0=k_0(t)\in\mathbb{N}\\ \qquad \textit{such that}\ \ \|x_k(t)-\widetilde{x}_k-\Delta^+A(a)\widetilde{x}-\Delta^+f(a)\|_X<\varepsilon\\ \qquad \textit{for all}\ \ \ k\ge k_0\end{array}\right\}\tag{3.3}$$

hold. Then

$$\lim_{k\to\infty} x_k(t)=x(t)\tag{3.4}$$

is true for $t\in[a,b]$, *while* $x_k\rightrightarrows x$ *locally on* $(a,b]$.

Proof. By (3.1), the solutions x_k of (1.1) exist on $[a,b]$ for all k sufficiently large. Let $\varepsilon>0$ be given and let $\delta>0$ and $k_1\in\mathbb{N}$ be such that

$$\|x(t)-x(a+)\|_X<\varepsilon\ \text{ for }\ t\in(a,a+\delta)\ \text{ and }\ \|\widetilde{x}_k-\widetilde{x}\|_X<\varepsilon\ \text{ for }\ k\ge k_1.$$

We may choose δ in such way that (3.3) holds. In view of this, for $t\in(a,a+\delta)$, let $k_0\in\mathbb{N}, k_0\ge k_1$, be such that

$$\|x_k(t)-\widetilde{x}_k-\Delta^+A(a)\widetilde{x}-\Delta^+f(a)\|_X<\varepsilon\ \text{ for }\ k\ge k_0.$$

Then, taking into account the relations

$$x(a+)=x(a)+\Delta^+A(a)x(a)+\Delta^+f(a)\ \text{ and }\ x(a)=\widetilde{x},$$

we get

$$\begin{aligned}&\|x_k(t)-x(t)\|_X=\\ &\quad=\|(x_k(t)-\widetilde{x}_k)+(\widetilde{x}_k-\widetilde{x})+(\widetilde{x}-x(a+))+(x(a+)-x(t))\|_X\le\\ &\quad\le\|x_k(t)-\widetilde{x}_k-x(a+)+\widetilde{x}\|_X+\|\widetilde{x}-\widetilde{x}_k\|_X+\|x(t)-x(a+)\|_X=\\ &\qquad\qquad=\|x_k(t)-\widetilde{x}_k-\Delta^+A(a)\widetilde{x}-\Delta^-f(a)\|_X+\\ &\qquad\qquad\qquad+\|\widetilde{x}-\widetilde{x}_k\|_X+\|x(t)-x(a+)\|_X<3\varepsilon.\end{aligned}$$

This means that (3.4) holds for $t\in[a,a+\delta)$.

Now, let an arbitrary $c\in(a,a+\delta)$ be given. We can use Theorem 2.2 to show that the solutions x_k to

$$x_k(t)=x_k(c)+\int_c^t \mathrm{d}[A_k]x_k+f_k(t)-f(t)$$

exist on $[c,b]$ and $x_k\rightrightarrows x$ on $[c,b]$. The assertion of the lemma follows easily. $\square$

Lemma 3.2. *Let* $A,A_k\in BV([a,b],L(X))$, $f,f_k\in G([a,b],X)$, $\widetilde{x},\widetilde{x}_k\in X$ *for* $k\in\mathbb{N}$. *Assume* (2.1), (2.8), (3.1) *and*

$$A_k\rightrightarrows A\ \ \textit{and}\ \ f_k\rightrightarrows f\ \ \textit{locally on}\ \ [a,b).\tag{3.5}$$

Then there exists a unique solution x of (1.2) *on* $[a,b]$ *and, for each* $k\in\mathbb{N}$ *sufficiently large, there exists a unique solution* x_k *on* $[a,b]$ *to the equation* (1.1).

Moreover, let (2.5) *and*

$$\left.\begin{aligned} &\forall\,\varepsilon > 0,\ \delta > 0\ \exists\,\tau \in (b-\delta, b),\ \ k_0 \in \mathbb{N}\ \textit{ such that}\\ &\Big| x_k(b) - x_k(\tau) - \Delta^- A(b)\,[I - \Delta^- A(b)]^{-1}\,x(b-) -\\ &\qquad - [I - \Delta^- A(b)]^{-1}\,\Delta^- f(b)\Big| < \varepsilon\ \textit{ for all }\ k \geq k_0 \end{aligned}\right\} \tag{3.6}$$

hold. Then (3.4) *is true, while* $x_k \rightrightarrows x$ *locally on* $[a, b)$.

Proof. Due to (2.1) and (3.1), there exists a unique solution x of (1.2) on $[a, b]$, there exists $k_1 \in \mathbb{N}$ such that (1.1) has a unique solution x_k on $[a, b]$ for each $k \geq k_1$. Furthermore, by Theorem 2.2, $x_k \rightrightarrows x$ locally on $[a, b)$. It remains to show that

$$\lim_{k\to\infty} x_k(b) = x(b) \tag{3.7}$$

is true, as well. Let $\varepsilon > 0$, $\delta \in (0, b-a)$ be given and let $\tau \in (b-\delta, b)$ and $k_0 \geq k_1$ be such that (3.6) is true. We have

$$\begin{aligned} &\|x_k(b) - x(b)\|_X =\\ &= \|(x_k(b) - x_k(\tau)) + (x_k(\tau) - x(\tau)) + (x(\tau) - x(b-)) + (x(b-) - x(b))\|_X \leq\\ &\leq \|x_k(b) - x_k(\tau) - x(b) + x(b-)\|_X + \|x(\tau) - x(b-)\|_X + \|x_k(\tau) - x(\tau)\|_X, \end{aligned}$$

wherefrom, having in mind that $x(b) = x(b-) + \Delta^- A(b)x(b) + \Delta^- f(b)$, i.e.,

$$x(b) = [I - \Delta^- A(b)]^{-1} x(b-) + [I - \Delta^- A(b)]^{-1} \Delta^- f(b)$$

and

$$\begin{aligned} x(b) - x(b-) = {} & \Delta^- A(b)[I - \Delta^- A(b)]^{-1} x(b-) +\\ & + \big[I + \Delta^- A(b)[I - \Delta^- A(b)]^{-1}\big]\Delta^- f(b), \end{aligned}$$

we deduce that

$$\begin{aligned} \|x_k(b) - x(b)\|_X \leq {} & \|x_k(b) - x(\tau) - \Delta^- A(b)[I - \Delta^- A(b)]^{-1} x(b-) -\\ & - \big[I + \Delta^- A(b)[I - \Delta^- A(b)]^{-1}\big]\Delta^- f(b)\|_X +\\ & + \|x(\tau) - x(b-)\|_X + \|x_k(\tau) - x(\tau)\|_X. \end{aligned}$$

We can choose δ and k_0 in such a way that $\|x(t) - x(b-)\|_X < \varepsilon$ for each $t \in (b-\delta, b)$ and $\|x_k(\tau) - x(\tau)\|_X < \varepsilon$ for $k \geq k_0$, as well. Furthermore, notice that if $B \in L(X)$ is such that $[I - B]^{-1} \in L(X)$, then $[I - B]^{-1} = I + B[I - B]^{-1}$. Thus, using (3.6), we get

$$\begin{aligned} &\|x_k(b) - x(b)\|_X \leq \|x_k(b) - x(\tau) - \Delta^- A(b)[I - \Delta^- A(b)]^{-1} x(b-) -\\ &- [I - \Delta^- A(b)]^{-1}\Delta^- f(b)\|_X + \|x(\tau) - x(b-)\|_X + \|x_k(\tau) - x(\tau)\|_X < 3\varepsilon. \end{aligned}$$

It follows that (3.7) is true and this completes the proof. □

The assertion below may be deduced from Lemmas 3.1 and 3.2

Theorem 3.3. *Let $A, A_k \in BV([a,b], L(X))$, $f, f_k \in G([a,b], X)$, $\widetilde{x}, \widetilde{x}_k \in X$ for $k \in \mathbb{N}$. Assume* (2.1), (2.8) *and* (3.1). *Furthermore, let there exist a division $D = \{s_0, s_2, \dots, s_m\}$ of the interval $[a,b]$ such that*

$$A_k \rightrightarrows A, f_k \rightrightarrows f \ \textit{ locally on each } \ (s_{i-1}, s_i), \quad i = 1, 2, \dots, m. \tag{3.8}$$

Then there exists a unique solution x of (1.2) *on $[a,b]$ and, for each $k \in \mathbb{N}$ sufficiently large, there exists a unique solution x_k on $[a,b]$ to the equation* (1.1).

Moreover, assume (2.5) *and let*

$$\left.\begin{aligned} &\forall\, \varepsilon > 0 \ \ \exists\, \delta_i \in (0, s_i - s_{i-1}) \ \textit{ such that } \forall\, t \in (s_{i-1}, s_{i-1} + \delta_i) \\ &\exists\, k_i = k_i(t) \in \mathbb{N} \ \textit{ such that} \\ &\qquad \|x_k(t) - x_k(s_{i-1}) - \Delta^+ A(s_{i-1}) x(s_{i-1}) - \Delta^+ f(s_{i-1})\|_X < \varepsilon \\ &\textit{for all } \ k \geq k_i \end{aligned}\right\} \tag{3.9}$$

and

$$\left.\begin{aligned} &\forall\, \varepsilon > 0, \delta \in (0, s_i - s_{i-1}) \exists\, \tau_i \in (s_i - \delta, s_i), \ell_i \in \mathbb{N} \ \textit{ such that} \\ &\quad \|x_k(s_i) - x_k(\tau_i) - \Delta^- A(s_i)\, [I - \Delta^- A(s_i)]^{-1}\, x(s_i-) - \\ &\qquad - [I - \Delta^- A(s_i)]^{-1}\, \Delta^- f(s_i)\|_X < \varepsilon \ \textit{ for all } \ k \geq \ell_i \end{aligned}\right\} \tag{3.10}$$

hold for each $i = 1, 2, \dots, m$.

Then (3.4) *is true for all $t \in [a,b]$, while $x_k \rightrightarrows x$ locally on each (s_{i-1}, s_i), $i = 1, 2, \dots, m$.*

Proof. Obviously, there is a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ of $[a,b]$ such that for each subinterval $[\alpha_{j-1}, \alpha_j]$, $j = 1, 2, \dots, r$, either the assumptions of Lemma 3.1 or the assumptions of Lemma 3.2 are satisfied with α_{j-1} in place of a and α_k in place of b. Hence the proof follows by Lemmas 3.1 and 3.2. □

4. Sequential Solutions

The aim of this section is to disclose the relationship between the solutions of generalized linear differential equation and limits of solutions of approximating sequences of linear ordinary differential equations generated by piecewise linear approximations of the coefficients A, f.

Let us introduce the following notation.

Notation 4.1. For $A \in BV([a,b], L(X))$, $f \in G([a,b], X)$ and

$$D = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \in \mathcal{D}[a,b],$$

we define

$$A_D(t) = \begin{cases} A(t) \quad \text{if } \ t \in D, \\ A(\alpha_{i-1}) + \dfrac{A(\alpha_i) - A(\alpha_{i-1})}{\alpha_i - \alpha_{i-1}}\,(t - \alpha_{i-1}) \\ \qquad\qquad \text{if } \ t \in (\alpha_{i-1}, \alpha_i) \ \text{ for some } \ i \in \{1, 2, \dots, m\}, \end{cases} \tag{4.1}$$

and

$$f_D(t) = \begin{cases} f(t) \quad \text{if } \ t \in D, \\ f(\alpha_{i-1}) + \dfrac{f(\alpha_i) - f(\alpha_{i-1})}{\alpha_i - \alpha_{i-1}}\,(t - \alpha_{i-1}) \\ \qquad\qquad \text{if } \ t \in (\alpha_{i-1}, \alpha_i) \ \text{ for some } \ i \in \{1, 2, \dots, m\}. \end{cases} \tag{4.2}$$

The following lemma presents some direct properties for the functions defined in (4.1) and (4.2).

Lemma 4.2. *Assume that* $A \in BV([a,b], L(X))$, $f \in G([a,b], X)$. *Furthermore, let* $D \in \mathcal{D}[a,b]$, $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\}$, *and let* A_D *and* f_D *be defined by* (4.1) *and* (4.2), *respectively. Then* A_D *and* f_D *are strongly absolutely continuous on* $[a,b]$ *and*

$$\operatorname{var}_a^b A_D \le \operatorname{var}_a^b A \ \ \textit{and} \ \ \|f_D\|_\infty \le \|f\|_\infty.$$

Proof. It is clear that A_D and f_D are strongly absolutely continuous on (α_{i-1}, α_i), for each $i = 1, \dots, m$. Since both functions are continuous on $[a,b]$, the absolute continuity holds on the closed intervals $[\alpha_{i-1}, \alpha_i], i = 1, \dots, m$ (cf. [30, Theorem 7.1.10]).

Let $\varepsilon > 0$ be given. For each $i = 1, \dots, m$, there exists $\eta_i > 0$ such that

$$\sum_{j=1}^{p} \|A_D(b_j) - A_D(a_j)\|_{L(X)} < \frac{\varepsilon}{m}, \quad \text{whenever} \quad \sum_{j=1}^{p} (b_j - a_j) < \eta_i,$$

where $[a_j, b_j]$, $j = 1, \dots, p$, are non-overlapping subintervals of $[\alpha_{i-1}, \alpha_i]$.

Let $\eta < \min\{\eta_i;\ i = 1, \dots, m\}$. Consider $\mathcal{F} = \{[c_j, d_j];\ j = 1, \dots, p\}$, a collection of non-overlapping subintervals of $[a,b]$, such that

$$\sum_{j=1}^{p} (d_j - c_j) < \eta.$$

Without loss of generality, we may assume that for each $j = 1, \dots, p$, $[c_j, d_j] \subset [\alpha_{k_j - 1}, \alpha_{k_j}]$, for some $k_j \in \{1, \dots, m\}$. Thus

$$\mathcal{F} = \bigcup_{i=1}^{m} \mathcal{F}_i, \ \ \text{with} \ \ \mathcal{F}_i = \Big\{[c,d] \in \mathcal{F}; [c,d] \cap [\alpha_{i-1}, \alpha_i] \ne \varnothing\Big\},$$

and $\sum_{[c,d]\in\mathcal{F}_i} (d-c) < \eta_i$, $i = 1,\dots,m$. In view of this, we get

$$\sum_{j=1}^{p} \|A_D(d_j) - A_D(c_j)\|_{L(X)} \le \\ \le \sum_{i=1}^{m} \sum_{[c,d]\in\mathcal{F}_i} \|A_D(d) - A_D(c)\|_{L(X)} < \sum_{i=1}^{m} \frac{\varepsilon}{m} = \varepsilon,$$

which shows that A_D is strongly absolutely continuous on $[a,b]$. Similarly we prove for f_D.

Furthermore, for each $\ell = 1,2,\dots,m$ and each $t \in [\alpha_{\ell-1},\alpha_\ell]$ we have

$$\operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell} A_D = \|A(\alpha_\ell) - A(\alpha_{\ell-1})\|_{L(X)} \le \operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell} A$$

and

$$\|f_D(t)\|_X = \Big\| f(\alpha_{\ell-1}) + \frac{f(\alpha_\ell) - f(\alpha_{\ell-1})}{\alpha_\ell - \alpha_{\ell-1}}(t-\alpha_{\ell-1}) \Big\|_X = \\ = \Big\| f(\alpha_{\ell-1}) \frac{\alpha_\ell - t}{\alpha_\ell - \alpha_{\ell-1}} + f(\alpha_\ell) \frac{t - \alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}} \Big\|_X \le \|f\|_\infty.$$

Therefore,

$$\operatorname{var}_a^b A_D = \sum_{\ell=1}^{m} \operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell} A_D \le \\ \le \sum_{\ell=1}^{m} \operatorname{var}_{\alpha_{\ell-1}}^{\alpha_\ell} A = \operatorname{var}_a^b A \quad \text{and} \quad \|f_D\|_\infty \le \|f\|_\infty. \qquad \square$$

Remark 4.3. Notice that the functions A_D, f_D, defined in (4.1) and (4.2), respectively, are differentiable on $(\alpha_{i-1},\alpha_i), i = 1,\dots,m$, and their derivatives are given by

$$A'_D(t) = \frac{A(\alpha_i) - A(\alpha_{i-1})}{\alpha_i - \alpha_{i-1}} \quad \text{if } t \in (\alpha_{i-1},\alpha_i) \text{ for some } i \in \{1,2,\dots,m\},$$

$$f'_D(t) = \frac{f(\alpha_i) - f(\alpha_{i-1})}{\alpha_i - \alpha_{i-1}} \quad \text{if } t \in (\alpha_{i-1},\alpha_i) \text{ for some } i \in \{1,2,\dots,m\}.$$

By Lemma 4.2, recalling that A_D and f_D are strongly absolutely continuous on $[a,b]$, the Bochner integral (cf. [30, Definition 7.4.16]) exists and hence also the strong McShane and the strong Kurzweil-Henstock integrals (cf. [30, Theorem 5.1.4] and [30, Proposition 3.6.3]). Moreover,

$$A_D(t) = \int_a^t A'_D(s)\,\mathrm{d}s, \quad f_D(t) = \int_a^t f'_D(s)\,\mathrm{d}s \quad \text{for } t \in [a,b],$$

(cf. [30, Theorem 7.3.10]). Consequently,

$$\int_a^t \mathrm{d}[A_D(s)]x(s) = \int_a^t A'_D(s)x(s)\mathrm{d}s$$

holds for each $x \in G([a,b],X)$ and $t \in [a,b]$. Hence, the generalized differential equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A_D(s)]x(s) + f_D(t) - f_D(a)$$

is equivalent to the initial value problem for the ordinary differential equation (in the Banach space X)

$$x'(t) = A'_D(t)x + f'_D(t), \quad x(a) = \widetilde{x}.$$

Theorem 4.4. *Let $A \in BV([a,b],L(X)) \cap C([a,b],L(X))$, $f \in C([a,b],X)$ and $\widetilde{x} \in X$. Furthermore, let $\{D_k\}$ be a sequence of divisions of the interval $[a,b]$ such that*

$$D_{k+1} \supset D_k \ \text{ for } \ k \in \mathbb{N} \ \text{ and } \ \lim_{k\to\infty} |D_k| = 0. \tag{4.3}$$

Finally, let the sequences $\{A_k\}$ and $\{f_k\}$ be given by

$$A_k = A_{D_k} \ \text{ and } \ f_k = f_{D_k} \ \text{ for } \ k \in \mathbb{N}, \tag{4.4}$$

where A_{D_k} and f_{D_k} are defined as in (4.1) *and* (4.2).

Then equation (1.2) *has a unique solution x on $[a,b]$. Furthermore, for each $k \in \mathbb{N}$, equation* (1.1) *has a solution x_k on $[a,b]$ and* (2.9) *holds.*

Proof. Step 1. Since A is uniformly continuous on $[a,b]$, we have

$$\left.\begin{array}{c} \text{for each } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that } \|A(t)-A(s)\|_{L(X)} < \dfrac{\varepsilon}{2} \\ \text{holds for all } \ t,s \in [a,b] \ \text{ such that } \ |t-s| < \delta. \end{array}\right\} \tag{4.5}$$

By (4.3), we can choose $k_0 \in \mathbb{N}$ such that $|D_k| < \delta$, for every $k \geq k_0$.

Given $t \in [a,b]$ and $k \geq k_0$, let $\alpha_{\ell-1}, \alpha_\ell \in \mathcal{D}_k$ be such that $t \in [\alpha_{\ell-1}, \alpha_\ell)$. Notice that $|\alpha_\ell - \alpha_{\ell-1}| < \delta$. So, according to (4.1), (4.4) and (4.5), we get

$$\begin{aligned} \|A_k(t) - A(t)\|_{L(X)} \leq \|A(\alpha_\ell) - A(\alpha_{\ell-1})\|_{L(X)} \Big[\frac{t-\alpha_{\ell-1}}{\alpha_\ell - \alpha_{\ell-1}}\Big] + \\ + \|A(\alpha_{\ell-1}) - A(t)\|_{L(X)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As k_0 was chosen independently of t, we can conclude that (2.6) is true.

Step 2. Analogously we can show that (2.7) is true, as well.

Step 3. By Lemma 4.2, (2.5) holds. Moreover, as A and A_k, $k \in \mathbb{N}$, are continuous, the equations (1.2) and (1.1) have unique solutions by Theorem 2.1 and we can complete the proof by using Theorem 2.2. $\square$

Notation 4.5. For the given $f \in G([a,b],X)$ and $k \in \mathbb{N}$, we denote

$$\mathcal{U}_k^+(f) = \Big\{t \in [a,b] : \ \|\Delta^+ f(t)\|_X \ge \frac{1}{k}\Big\},$$
$$\mathcal{U}_k^-(f) = \Big\{t \in [a,b] : \ \|\Delta^- f(t)\|_X \ge \frac{1}{k}\Big\},$$
$$\mathcal{U}_k(f) = \mathcal{U}_k^+(f) \cup \mathcal{U}_k^-(f) \quad \text{and} \quad \mathcal{U}(f) = \bigcup_{k=1}^{\infty} \mathcal{U}_k(f).$$

(Thus $\mathcal{U}(f)$ is a set of points of discontinuity of the function f in $[a,b]$.) Analogous symbols are used also for the operator valued function.

Definition 4.6. Let $A \in BV([a,b],L(X))$, $f \in G([a,b],X)$ and let $\{P_k\}$ be a sequence of divisions of $[a,b]$ such that

$$|P_k| = (1/2)^k \ \text{ for } \ k \in \mathbb{N}. \tag{4.6}$$

We say that the sequence $\{A_k, f_k\}$ is a *piecewise linear approximation* ($\mathcal{PL}$-approximation) of (A,f) if there exists a sequence $\{D_k\} \subset \mathcal{D}[a,b]$ of divisions of the interval $[a,b]$ such that

$$D_k \supset P_k \cup \mathcal{U}_k(A) \cup \mathcal{U}_k(f) \ \text{ for } \ k \in \mathbb{N} \tag{4.7}$$

and A_k, f_k are for $k \in \mathbb{N}$ defined by (4.1), (4.2) and (4.4).

Remark 4.7. Consider the case where $\dim X < \infty$ and let $\{A_k, f_k\}$ be a $\mathcal{PL}$-approximation of (A,f). Then by Lemma 4.2,

$$\operatorname{var}_a^b A_k \le \operatorname{var}_a^b A \quad \text{and} \quad \|f_k\|_\infty \le \|f\|_\infty.$$

Furthermore, as A_k are continuous, due to (2.2), we have $c_{A_k} = 1$ for all $k \in \mathbb{N}$. Hence, (2.4) yields

$$\operatorname{var}_a^b(x_k - f_k) \le \operatorname{var}_a^b A\,(\|\widetilde{x}\|_X + 2\|f\|_\infty)\exp(\operatorname{var}_a^b A) < \infty \quad \text{for all } \ k \in \mathbb{N}$$

and, by Helly's theorem, there is a subsequence $\{k_\ell\}$ of $\mathbb{N}$ and $w \in G([a,b],X)$ such that

$$\lim_{\ell\to\infty}(x_{k_\ell}(t) - f_{k_\ell}(t)) = w(t) - f(t) \quad \text{for } \ t \in [a,b].$$

In particular, $\lim\limits_{\ell\to\infty} x_{k_\ell}(t) = w(t)$ for all $t \in [a,b]$ such that $\lim\limits_{\ell\to\infty} f_{k_\ell}(t) = f(t)$.

In this context, it is worth mentioning that if the set $\mathcal{U}(f)$ has at most a finite number of elements, then

$$\lim_{k\to\infty} f_k(t) = f(t) \quad \text{for all } \ t \in [a,b].$$

Definition 4.8. Let $A \in BV([a,b],L(X))$, $f \in G([a,b],X)$ and $\widetilde{x} \in X$. We say that $x^* : [a,b] \to X$ is a *sequential solution* to equation (1.2) on the interval $[a,b]$ if there is a $\mathcal{PL}$-approximation $\{A_k, f_k\}$ of (A,f) such that

$$\lim_{k\to\infty} x_k(t) = x^*(t) \quad \text{for } \ t \in [a,b] \tag{4.8}$$

holds for solutions $x_k, k \in \mathbb{N}$, of the corresponding approximating initial value problems

$$x_k' = A_k'(t)x_k + f_k'(t), \quad x_k(a) = \widetilde{x}, \;\; k \in \mathbb{N}. \tag{4.9}$$

Remark 4.9. Notice that using the language of Definitions 4.6 and 4.8, we can translate Theorem 4.4 into the following form:
Let $A \in BV([a,b], L(X)) \cap C([a,b], L(X))$, $f \in C([a,b], X)$ *and* $\widetilde{x} \in X$. *Then equation* (1.2) *has a unique sequential solution* x^* *on* $[a,b]$ *and* x^* *coincides on* $[a,b]$ *with the solution of* (1.2).

In the rest of this paper we consider the case where the set $\mathcal{U}(A) \cup \mathcal{U}(f)$ of discontinuities of A, f is non-empty. We will start with the simplest case $\mathcal{U}(A) \cup \mathcal{U}(f) = \{b\}$.

The following natural assertion will be useful for our purposes and, in our opinion, it is not available in literature.

Lemma 4.10. *Let* $A \in BV([a,b], L(X))$. *Then*

$$\left.\begin{aligned} &\lim_{s \to t-} \frac{1}{t-s}\Big(\int_s^t \exp\Big([A(t)-A(s)]\,\frac{t-r}{t-s}\Big)\,\mathrm{d}r\Big) = \\ &\qquad = \int_0^1 \exp\big(\Delta^- A(t)(1-\sigma)\big)\,\mathrm{d}\sigma \quad \text{if } t \in (a,b] \end{aligned}\right\} \tag{4.10}$$

and

$$\left.\begin{aligned} &\lim_{s \to t+} \frac{1}{s-t}\Big(\int_t^s \exp\Big([A(s)-A(t)]\frac{s-r}{s-t}\Big)\,\mathrm{d}r\Big) = \\ &\qquad = \int_0^1 \exp\big(\Delta^+ A(t)(1-\sigma)\big)\,\mathrm{d}\sigma \quad \text{if } t \in [a,b). \end{aligned}\right\} \tag{4.11}$$

where the integrals are the Bochner ones.

Proof. (i) Let $t \in (a,b]$ and $\varepsilon \in (0,1)$ be given. Then there is a $\delta > 0$ such that

$$\|A(t-) - A(s)\|_{L(X)} < \varepsilon \quad \text{whenever } \; t-\delta < s < t.$$

Taking now into account that

$$\big\|\exp(C\tau) - \exp(D\tau)\big\|_{L(X)} \le \|C-D\|_{L(X)} \exp\big((\|C\|_{L(X)} + \|D\|_{L(X)})\tau\big)$$

holds for all $C, D \in L(X)$, $\tau \in \mathbb{R}$, (cf. [22, Corollary 3.1.3]), we get

$$\Big\|\frac{1}{t-s}\int_s^t \Big[\exp\Big([A(t)-A(s)]\,\frac{t-r}{t-s}\Big) - \exp\Big(\Delta^- A(t)\,\frac{t-r}{t-s}\Big)\Big]\,\mathrm{d}r\Big\|_X \le$$

$$\le \frac{1}{t-s}\,\|A(t-) - A(s)\|_{L(X)} \int_s^t \exp\big(\varepsilon + 2\|\Delta^- A(t)\|_{L(X)}\big)\,\mathrm{d}r =$$

$$= \|A(t-) - A(s)\|_{L(X)} \exp\big(\varepsilon + 2\|\Delta^- A(t)\|_{L(X)}\big) \le$$

$$\le \varepsilon \exp\big(1 + 2\|\Delta^- A(t)\|_{L(X)}\big) \quad \text{for } t-\delta < s < t.$$

Therefore,

$$\lim_{s\to t-} \frac{1}{t-s}\left(\int_s^t \exp\Big([A(t)-A(s)]\,\frac{t-r}{t-s}\Big)\,\mathrm{d}r\right) =$$

$$= \lim_{s\to t-} \frac{1}{t-s}\left(\int_s^t \exp\Big(\Delta^- A(t)\,\frac{t-r}{t-s}\Big)\,\mathrm{d}r\right) \quad \text{for } t \in (a,b].$$

It is now easy to see that the substitution $\sigma = 1 - \frac{t-r}{t-s}$ into the second integral yields (4.10).
(ii) The relation (4.11) can be justified similarly. □

Lemma 4.11. *Let $A \in BV([a,b],L(X))$ and $f \in G([a,b],X)$ be continuous on $[a,b)$. Let $\widetilde{x} \in X$ and let x be a solution of* (1.2) *on $[a,b)$.*

Then equation (1.2) *has a unique sequential solution x^* on $[a,b]$.*

Moreover, x^* is continuous on $[a,b)$, $x^* = x$ on $[a,b)$ and $x^*(b) = v(1)$, where v is a solution on $[0,1]$ of the initial value problem

$$v' = [\Delta^- A(b)]v + [\Delta^- f(b)], \quad v(0) = x(b-). \tag{4.12}$$

Proof. Let $\{A_k, f_k\}$ be an arbitrary $\mathcal{PL}$-approximation of (A,f) and let $\{D_k\}$ be the corresponding sequence of divisions of $[a,b]$ fulfilling (4.6) and (4.7). Notice that under our assumptions, $D_k = P_k$ for $k \in \mathbb{N}$. For $k \in \mathbb{N}$, we put

$$\tau_k = \max\{t \in P_k;\ t < b\}.$$

By (4.3), we have $b - \frac{b-a}{2^k} \le \tau_k < b$ for $k \in \mathbb{N}$, and hence

$$\lim_{k\to\infty} \tau_k = b. \tag{4.13}$$

Now, for $k \in \mathbb{N}$ and $t \in [a,b]$, let us define

$$\widetilde{A}_k(t) = \begin{cases} A_k(t) & \text{if } t \in [a,\tau_k], \\[2ex] A(\tau_k) + \dfrac{A(b-) - A(\tau_k)}{b-\tau_k}\,(t-\tau_k) & \text{if } t \in (\tau_k, b], \end{cases}$$

$$\widetilde{f}_k(t) = \begin{cases} f_k(t) & \text{if } t \in [a,\tau_k], \\[2ex] f(\tau_k) + \dfrac{f(b-) - f(\tau_k)}{b-\tau_k}\,(t-\tau_k) & \text{if } t \in (\tau_k, b]. \end{cases}$$

Furthermore, let

$$\widetilde{A}(t) = \begin{cases} A(t) & \text{if } t \in [a,b), \\ A(b-) & \text{if } t = b, \end{cases} \qquad \widetilde{f}(t) = \begin{cases} f(t) & \text{if } t \in [a,b), \\ f(b-) & \text{if } t = b. \end{cases} \tag{4.14}$$

It is easy to see that for $k \in \mathbb{N}$, $\widetilde{A}_k$ $\widetilde{f}_k$ are strongly absolutely continuous and differentiable a.e. on $[a,b]$, $\widetilde{A} \in BV([a,b],L(X)) \cap C([a,b],L(X))$ and $\widetilde{f} \in C([a,b],X)$.

Step 1. Consider the problems

$$y_k' = \widetilde{A}_k'(t)y_k + \widetilde{f}_k'(t), \quad y_k(a) = \widetilde{x}, \quad k \in \mathbb{N}, \tag{4.15}$$

and

$$y(t) = \widetilde{x} + \int_a^t \mathrm{d}[\widetilde{A}]y + \widetilde{f}(t) - \widetilde{f}(a). \tag{4.16}$$

Taking into account Theorem 4.4 and Remark 4.9, we find that the equation (4.16) possesses a unique solution y on $[a,b]$ and

$$\lim_{k\to\infty} \|y_k - y\|_\infty = 0. \tag{4.17}$$

where for each $k \in \mathbb{N}$, y_k is the solution on $[a,b]$ of (4.15).

Note that y is continuous on $[a,b]$ and $y = x$ on $[a,b)$. Let $\{x_k\}$ be a sequence of solutions of the problems (4.9) on $[a,b]$. We can see that $x_k = y_k$ on $[a,\tau_k]$ for each $k \in \mathbb{N}$, and, due to (4.13), we have

$$\lim_{k\to\infty} x_k(t) = \lim_{k\to\infty} y_k(t) = y(t) = x(t) \quad \text{for } t \in [a,b). \tag{4.18}$$

Step 2. Next, we prove that

$$\lim_{k\to\infty} x_k(\tau_k) = y(b). \tag{4.19}$$

Indeed, let $\varepsilon > 0$ be given and let $\delta > 0$ be such that

$$\|y(t) - y(b)\|_X < \frac{\varepsilon}{2} \quad \text{for } t \in [b-\delta,b].$$

Further, by (4.17), there is a $k_0 \in \mathbb{N}$ such that

$$\tau_k \in [b-\delta,b) \quad \text{and} \quad \|y_k - y\|_\infty < \frac{\varepsilon}{2} \quad \text{whenever } k \ge k_0.$$

Consequently,

$$\begin{aligned}\|x_k(\tau_k) - y(b)\|_X &\le \|x_k(\tau_k) - y(\tau_k)\|_X + \|y(\tau_k) - y(b)\|_X = \\ &= \|y_k(\tau_k) - y(\tau_k)\|_X + \|y(\tau_k) - y(b)\|_X < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

holds for $k \ge k_0$. This completes the proof of (4.19).

Step 3. On the intervals $[\tau_k,b]$, the equations from (4.9) reduce to the equations with constant coefficients

$$x_k' = B_k x_k + e_k, \tag{4.20}$$

where

$$B_k = \frac{A(b) - A(\tau_k)}{b - \tau_k} \quad \text{and} \quad e_k = \frac{f(b) - f(\tau_k)}{b - \tau_k}.$$

Their solutions x_k are on $[\tau_k, b]$ given by

$$x_k(t) = \exp\left(B_k(t-\tau_k)\right) x_k(\tau_k) + \left(\int_{\tau_k}^{t} \exp\left(B_k(t-r)\right) \mathrm{d}r\right) e_k,$$

(cf. [5, Chapter II]). In particular,

$$x_k(b) = \exp\left(A(b) - A(\tau_k)\right) x_k(\tau_k) + \\ + \frac{1}{b-\tau_k}\left(\int_{\tau_k}^{b} \exp\left([A(b) - A(\tau_k)]\,\frac{b-r}{b-\tau_k}\right) \mathrm{d}r\right)[f_k(b) - f_k(\tau_k)].$$

By Lemma 4.10, we have

$$\lim_{k\to\infty} \frac{1}{b-\tau_k}\left(\int_{\tau_k}^{b} \exp\left([A(b) - A(\tau_k)]\,\frac{b-r}{b-\tau_k}\right) \mathrm{d}r\right)[f(b) - f(\tau_k)] =$$

$$= \lim_{k\to\infty} \frac{1}{b-\tau_k}\left(\int_{\tau_k}^{b} \exp\left(\Delta^- A(b)\,\frac{b-r}{b-\tau_k}\right) \mathrm{d}r\right)[f(b) - f(\tau_k)] =$$

$$= \left(\int_{0}^{1} \exp\left(\Delta^- A(b)(1-s)\right) \mathrm{d}s\right)\Delta^- f(b).$$

To summarize,

$$\lim_{k\to\infty} x_k(b) = \exp\left(\Delta^- A(b)\right) y(b) + \left(\int_{0}^{1} \exp\left(\Delta^- A(b)(1-s)\right) \mathrm{d}s\right)\Delta^- f(b),$$

i.e.,

$$\lim_{k\to\infty} x_k(b) = v(1), \tag{4.21}$$

where v is a solution of (4.12) on $[0,1]$.

Step 4. Define

$$x^*(t) = \begin{cases} y(t) & \text{if } t \in [a,b), \\ v(1) & \text{if } t = b. \end{cases}$$

Then $x^*(t) = \lim_{k\to\infty} x_k(t)$ for $t \in [a,b]$ due to (4.19) and (4.21). Therefore, x^* is a sequential solution of (1.2). Since it does not depend on the choice of the approximating sequence $\{A_k, f_k\}$, we can see that x^* is also the unique sequential solution of (1.2). This completes the proof. □

The following assertion concerns a situation, symmetric to that treated by Lemma 4.11. Similarly to the proof of Lemma 4.11, we will deal with the modified equation

$$y(t) = \widetilde{y} + \int_a^t \mathrm{d}[\widetilde{A}]y + \widetilde{f}(t) - \widetilde{f}(a), \tag{4.22}$$

where $\widetilde{y} \in X$ and

$$\widetilde{A}(t) = \begin{cases} A(a+) & \text{if } t = a, \\ A(t) & \text{if } t \in (a,b] \end{cases} \quad \text{and} \quad \widetilde{f}(t) = \begin{cases} f(a+) & \text{if } t = a, \\ f(t) & \text{if } t \in (a,b]. \end{cases} \tag{4.23}$$

Lemma 4.12. *Let $A \in BV([a,b], L(X))$ and $f \in G([a,b], X)$ be continuous on $(a,b]$. Then for each $\widetilde{x} \in X$, equation (1.2) has a unique sequential solution x^* on $[a,b]$ which is continuous on $(a,b]$.*

Furthermore, let w be a solution of the initial value problem

$$w' = [\Delta^+ A(a)]w + [\Delta^+ f(a)], \quad w(0) = \widetilde{x} \tag{4.24}$$

and let y be a solution on $[a,b]$ of equation (4.22), where $\widetilde{y} = w(1)$. Then x^* coincides with y on $(a,b]$.

Proof. Let $\{A_k, f_k\}$ be an arbitrary $\mathcal{PL}$-approximation of (A,f) and let $\{D_k\}$ be the corresponding sequence of divisions of $[a,b]$ fulfilling (4.1) and (4.2). Just as in the previous proof, $D_k = P_k$ for $k \in \mathbb{N}$.

For $k \in \mathbb{N}$, we put

$$\tau_k = \min\{t \in P_k : \ t > a\}.$$

By (4.3), we have $a + \frac{b-a}{2^k} \geq \tau_k > a$ for $k \in \mathbb{N}$, and hence

$$\lim_{k \to \infty} \tau_k = a.$$

Let $\{x_k\}$ be a sequence of solutions of the approximating initial value problems (4.9) on $[a,b]$.

Step 1. On the intervals $[a, \tau_k]$, the equations from (4.9) reduce to the equations (4.20) with the coefficients

$$B_k = \frac{A(\tau_k) - A(a)}{\tau_k - a}, \quad e_k = \frac{f(\tau_k) - f(a)}{\tau_k - a}.$$

Their solutions x_k are on $[a, \tau_k]$ given by

$$x_k(t) = \exp(B_k(t-a))\widetilde{x} + \left(\int_a^t \exp\left(B_k(t-r)\right) \mathrm{d}r \right) e_k,$$

(cf. [5, Chapter II]). In particular,

$$x_k(\tau_k) = \exp\left(A(\tau_k) - A(a)\right)\widetilde{x} +$$
$$+ \frac{1}{\tau_k - a} \left(\int_a^{\tau_k} \exp\left([A(\tau_k) - A(a)] \frac{\tau_k - r}{\tau_k - a} \right) \mathrm{d}r \right) [f(\tau_k) - f(\tau_k)].$$

By Lemma 4.10, we have

$$\lim_{k\to\infty}\frac{1}{\tau_k-a}\biggl(\int_a^{\tau_k}\exp\Bigl([A(\tau_k)-A(a)]\,\frac{\tau_k-r}{\tau_k-a}\Bigr)\mathrm{d}r\biggr)[f(\tau_k)-f(a)]=$$

$$=\biggl(\int_0^1\exp(\Delta^+A(a)(1-s))\,\mathrm{d}s\biggr)\Delta^+f(a).$$

Thus, $\lim\limits_{k\to\infty} x_k(\tau_k)=w(1)$, where w is the solution of (4.24) on $[0,1]$.

Step 2. Consider equation (4.22) with $\widetilde{y}=w(1)$. By Theorem 2.1, it has a unique solution y on $[a,b]$, y is continuous on $[a,b]$ and, by an argument analogous to that used in *Step* 1 of the proof of Lemma 4.11, we can show that the relation

$$\lim_{k\to\infty}x_k(t)=y(t)\quad\text{for }\ t\in(a,b]$$

is true.

Step 3. Analogously to *Step* 4 of the proof of Lemma 4.11, we can complete the proof by showing that the function

$$x^*(t)=\begin{cases}\widetilde{x} & \text{if }\ t=a,\\ y(t) & \text{if }\ t\in(a,b],\end{cases}$$

is the unique sequential solution of (1.2). □

Remark 4.13. Notice that if $a<c<b$ and the functions x_1^* and x_2^* are, respectively, the sequential solutions to

$$x(t)=\widetilde{x}_1+\int_a^t\mathrm{d}[A]x+f(t)-f(a),\quad t\in[a,c],$$

and

$$x(t)=\widetilde{x}_2+\int_c^t\mathrm{d}[A]x+f(t)-f(c),\quad t\in[c,b],$$

where $\widetilde{x}_2=x_1^*(c)$, then the function

$$x^*(t)=\begin{cases}x_1^*(t) & \text{if }\ t\in[a,c],\\ x_2^*(t) & \text{if }\ t\in(c,b]\end{cases}$$

is a sequential solution to (1.2).

Theorem 4.14. *Assume that* $A\in BV([a,b],L(X))$, $f\in G([a,b],X)$ *and*

$$\mathcal{U}(A)\cup\mathcal{U}(f)=\{s_1,s_2,\dots,s_m\}\subset[a,b].$$

Then for each $\widetilde{x}\in X$, *there is exactly one sequential solution* x^* *of equation* (1.2) *on* $[a,b]$.

Moreover,

$$x^*(t) = w_\ell(1) + \int_{s_\ell}^{t} \mathrm{d}[\widetilde{A}_\ell]x^* + \widetilde{f}_\ell(t) - \widetilde{f}_\ell(s_\ell) \quad \textit{for} \;\; t \in [s_\ell, s_{\ell+1}), \quad \ell \in \mathbb{N} \cap [0, m],$$

$$x^*(t) = v_\ell(1) \;\; \textit{for} \;\; t = s_\ell, \quad \ell \in \mathbb{N} \cap [1, m+1],$$

where $s_0 = a, s_{m+1} = b, w_0(1) = \widetilde{x}$ *and, for* $\ell \in \mathbb{N} \cap [0, m]$,

$$\widetilde{A}_\ell(t) = \begin{cases} A(s_\ell+) & \textit{if} \;\; t = s_\ell, \\ A(t) & \textit{if} \;\; t \in (s_\ell, s_{\ell+1}], \end{cases} \qquad \widetilde{f}_\ell(t) = \begin{cases} f(s_\ell+) & \textit{if} \;\; t = s_\ell, \\ f(t) & \textit{if} \;\; t \in (s_\ell, s_{\ell+1}] \end{cases}$$

and v_ℓ *and* w_ℓ *denote, respectively, the solutions on* $[0, 1]$ *of the initial value problems*

$$v_\ell' = [\Delta^- A(s_\ell)]v_\ell + [\Delta^- f(s_\ell)], \quad v_\ell(0) = x^*(s_\ell-)$$

and

$$w_\ell' = [\Delta^+ A(s_\ell)]w_\ell + [\Delta^+ f(s_\ell)], \quad w_\ell(0) = x^*(s_\ell).$$

Proof. Having in mind Remark 4.13, we deduce the assertion of Theorem 4.14 by a successive use of Lemmas 4.11 and 4.12. Towards this end, it suffices to choose a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ of $[a, b]$ such that for each subinterval $[\alpha_{k-1}, \alpha_k], k = 1, 2, \dots, r$, either the assumptions of Lemma 4.11 or those of Lemma 4.12 are satisfied with α_{k-1} in place of a and α_k in place of b. □

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CONDITIONAL WELL-POSEDNESS OF NONLOCAL PROBLEMS FOR FOURTH ORDER LINEAR HYPERBOLIC EQUATIONS WITH SINGULARITIES

Dedicated to the blessed memory of Professor T. Chanturia

Abstract. Unimprovable in a sense sufficient conditions of well-posedness of nonlocal problems are established for fourth order linear hyperbolic equations with singular coefficient.

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რეზიუმე. მეოთხე რიგის წრფივი ჰიპერბოლური განტოლებებისათვის სინგულარული კოეფიციენტებით დადგენილია არალოკალურ ამოცანათა პირობითად კორექტულობის არაგაუმჯობესებადი საკმარისი პირობები.

1. Formulation of the Main Results

1.1. **Statement of the problem.** In the rectangle $\Omega = [0,a]\times[0,b]$ consider the linear hyperbolic equation

$$u^{(2,2)} = \sum_{i=1}^{2}\sum_{k=1}^{2} h_{ik}(x,y)u^{(i-1,k-1)} + h(x,y) \tag{1.1}$$

with the nonlocal boundary conditions

$$\begin{aligned}&\int_0^a u(s,y)\,d\alpha_i(s) = 0 \ \text{ for } \ 0\le y\le b \ \ (i=1,2),\\ &\int_0^b u(x,t)\,d\beta_k(t) = 0 \ \text{ for } \ 0\le x\le a \ \ (k=1,2).\end{aligned} \tag{1.2}$$

Here

$$u^{(i,k)}(x,y) = \frac{\partial^{i+k}u(x,y)}{\partial x^i \partial y^k} \ \ (i,\,k=0,1,2),$$

$h_{ik}:\Omega\to\mathbb{R}$ $(i,\,k=1,2)$ are measurable functions, $h\in L(\Omega)$, and $\alpha_i : [0,a]\to\mathbb{R}$ and $\beta_i:[0,b]\to\mathbb{R}$ $(i=1,2)$ are functions of bounded variation.

We will use the following notation.

$L(\Omega)$ is the Banach space of Lebesgue integrable functions $v:\Omega\to\mathbb{R}$ with the norm

$$\|v\|_L = \int_0^a\int_0^b |v(x,y)|\,dx\,dy.$$

$C^{1,1}(\Omega)$ is the space of functions $u:\Omega\to\mathbb{R}$, continuous together with $u^{(i-1,k-1)}$ $(i,\,k=1,2)$, with the norm

$$\|u\|_{C^{1,1}} = \max\bigg\{\sum_{i=1}^{2}\sum_{k=1}^{2}\big|u^{(i-1,k-1)}(x,y)\big| : \ (x,y)\in\Omega\bigg\}.$$

$\widetilde{C}^{1,1}(\Omega)$ is the space of functions $u\in C^{1,1}(\Omega)$ for which $u^{(1,1)}$ is absolutely continuous (see, e.g., [1,4]).

The function $u\in\widetilde{C}^{1,1}(\Omega)$ is said to be a *solution of equation* (1.1) if it satisfies that equation almost everywhere on Ω.

A solution of equation (1.1) satisfying boundary conditions (1.2) is called a *solution of problem* (1.1), (1.2).

Along with the equation (1.1) consider the corresponding homogeneous and perturbed equations

$$u^{(2,2)} = \sum_{i=1}^{2}\sum_{k=1}^{2} h_{ik}(x,y)u^{(i-1,k-1)}, \tag{1.1$_0$}$$

$$u^{(2,2)} = \sum_{i=1}^{2}\sum_{k=1}^{2} h_{ik}(x,y)u^{(i-1,k-1)} + \widetilde{h}(x,y), \tag{1.1$'$}$$

with the nonhomogeneous boundary conditions

$$\begin{aligned}
&\int\limits_0^a u(s,y)\,d\alpha_i(s) = \int\limits_0^a c(s,y)\,d\alpha_i(s) \ \text{ for } \ 0 \le y \le b \ \ (i=1,2),\\
&\int\limits_0^b u(x,t)\,d\beta_k(t) = \int\limits_0^b c(x,t)\,d\beta_k(t) \ \text{ for } \ 0 \le x \le a \ \ (k=1,2).
\end{aligned} \tag{1.2$'$}$$

Following [2], introduce the definitions.

Definition 1.1. Problem $(1.1), (1.2)$ is said to be *well-posed* if for arbitrary $\widetilde{h} \in L(\Omega)$ and $c \in \widetilde{C}^{1,1}(\Omega)$ problem $(1.1'), (1.2')$ is uniquely solvable, and there exists a positive constant r independent of $\widetilde{h}$ and c such that

$$\|\widetilde{u} - u\|_{C^{1,1}} \le r\big(\|c\|_{C^{1,1}} + \|\widetilde{h} - h\|_L\big),$$

where u and $\widetilde{u}$, respectively, are solutions of problems $(1.1), (1.2)$ and $(1.1')$, $(1.2')$.

Definition 1.2. Problem $(1.1), (1.2)$ is said to be *conditionally well-posed* if for an arbitrary $\widetilde{h} \in L(\Omega)$ problem $(1.1'), (1.2)$ is uniquely solvable, and the exists a positive constant r independent of $\widetilde{h}$ such that

$$\|\widetilde{u} - u\|_{C^{1,1}} \le r\|\widetilde{h} - h\|_L,$$

where u and $\widetilde{u}$, respectively, are solutions of problems $(1.1), (1.2)$ and $(1.1')$, (1.2).

In the case where the coefficients of equation (1.1) are continuous functions sufficient conditions of well-posedness of problems of type $(1.1), (1.2)$ are established in [3–7]. We are interested in the singular case, where some of the coefficients h_{ik} $(i,\, k = 1,2)$ are nonintegrable on Ω. Until recently, for singular equations only the Dirichlet problem has been studied [8].

General theorems on conditional well-posedness of nonlocal problems for higher order linear hyperbolic equations with singular coefficients are proved in [2]. In the present paper effective and unimprovable in a sense conditions, guaranteeing conditional well-posedness of the singular problem $(1.1), (1.2)$, are established on the basis of those results.

The following boundary conditions are the particular cases of (1.2):

$$\begin{aligned}
&u(0,y) = 0, \ \ u(a,y) = 0 \ \text{ for } \ 0 \le y \le b,\\
&u(x,0) = 0, \ \ u(x,b) = 0 \ \text{ for } \ 0 \le x \le a
\end{aligned} \tag{1.2$_1$}$$

and

$$\begin{aligned}
&u(0,y) = 0, \ \ \int\limits_0^a u(s,y)\,d\alpha(s) = 0 \ \text{ for } \ 0 \le y \le b,\\
&u(x,0) = 0, \ \ \int\limits_0^b u(x,t)\,d\beta(t) = 0 \ \text{ for } \ 0 \le x \le a,
\end{aligned} \tag{1.2$_2$}$$

where $\alpha : [0, a] \to \mathbb{R}$ and $\beta : [0, b] \to \mathbb{R}$ are functions of bounded variation.

The theorems proved below imply new sufficient conditions of conditional well-posedness of problems $(1.1), (1.2_1)$ and $(1.1), (1.2_2)$.

1.2. **Theorems on the Conditional Well-Posedness of Problem (1.1), (1.2).** Let

$$\begin{aligned} \Delta_1(x) &= \alpha_2(a) \int\limits_x^a \alpha_1(s)\,ds - \alpha_1(a) \int\limits_x^a \alpha_2(s)\,ds, \\ \Delta_2(y) &= \beta_2(b) \int\limits_y^b \beta_1(t)\,dt - \beta_1(b) \int\limits_y^b \beta_2(t)\,dt. \end{aligned} \tag{1.3}$$

We study problem $(1.1), (1.2)$ in the case, where

$$\alpha_i(0) = 0, \;\; \beta_i(0) = 0, \;\; \Delta_i(0) \neq 0 \;\;\; (i = 1, 2). \tag{1.4}$$

Introduce the functions

$$\chi(t, s) = \begin{cases} 1 & \text{for } \; s \leq t, \\ 0 & \text{for } \; s > t, \end{cases} \tag{1.5}$$

$$\begin{aligned} g_1(x, s) = \frac{1}{\Delta_1(0)} \Bigg[& \int\limits_0^a \alpha_1(\tau)\,d\tau \int\limits_s^a \alpha_2(\tau)\,d\tau - \int\limits_s^a \alpha_1(\tau)\,d\tau \int\limits_0^a \alpha_2(\tau)\,d\tau + \\ & + (s - a)\Delta_1(0) + (a - x)\Delta_1(s) \Bigg] + \chi(x, s)(x - s) \;\; \text{for } \; 0 \leq x, \, s \leq a, \end{aligned} \tag{1.6}$$

$$\begin{aligned} g_2(y, t) = \frac{1}{\Delta_2(0)} \Bigg[& \int\limits_0^b \beta_1(\tau)\,d\tau \int\limits_t^b \beta_2(\tau)\,d\tau - \int\limits_t^b \beta_1(\tau)\,d\tau \int\limits_0^b \beta_2(\tau)\,d\tau + \\ & + (t - b)\Delta_2(0) + (b - y)\Delta_1(t) \Bigg] + \chi(y, t)(y - t) \;\; \text{for } \; 0 \leq y, \, t \leq b, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \varphi_{11}(x) &= \max \big\{ |g_1(x, s)| : \; 0 \leq s \leq a \big\}, \\ \varphi_{12}(x) &= \sup \Big\{ \big| g_1^{(1,0)}(x, s) \big| : \; 0 \leq s \leq a, \; s \neq x \Big\}, \end{aligned} \tag{1.8}$$

$$\begin{aligned} \varphi_{21}(y) &= \max \big\{ |g_2(y, t)| : \; 0 \leq t \leq b \big\}, \\ \varphi_{22}(y) &= \sup \Big\{ \big| g_2^{(1,0)}(y, t) \big| : \; 0 \leq t \leq b, \; t \neq y \Big\}. \end{aligned} \tag{1.9}$$

Theorem 1.1. *If along with* (1.4) *the inequalities*

$$\int\limits_0^b \int\limits_0^a \varphi_{1i}(x)\varphi_{2k}(y) |h_{ik}(x, y)|\,dx\,dy < +\infty \;\; (i, \, k = 1, 2) \tag{1.10}$$

hold, then problem $(1.1), (1.2)$ *is conditionally well-posed if and only if the corresponding homogeneous problem* $(1.1_0), (1.2)$ *has only the trivial solution.*

Theorem 1.2. *If along with* (1.4) *the condition*

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b\int_0^a \varphi_i(x)\psi_k(y)|h_{ik}(x,y)|\,dx\,dy<1 \tag{1.11}$$

holds, then problem (1.1), (1.2) *is conditionally well-posed. Moreover, if*

$$h_{ik}\in L(\Omega)\ \ (i,k=1,2), \tag{1.12}$$

then problem (1.1), (1.2) *is well-posed.*

Theorem 1.3. *If conditions* (1.4) *and* (1.11) *hold, and*

$$\int_0^b\int_0^a |h_{11}(x,y)|\,dx\,dy=+\infty, \tag{1.13}$$

then problem (1.1), (1.2) *is conditionally well-posed but not well-posed.*

1.3. **Corollaries for problem (1.1), (1.2_1).**

Corollary 1.1. *If*

$$\int_0^b\int_0^a\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k}|h_{ik}(x,y)|\,dx\,dy<+\infty\ \ (i,\,k=1,2) \tag{1.14}$$

hold, then problem (1.1), (1.2_1) *is conditionally well-posed if and only if the corresponding homogeneous problem* (1.1_0), (1.2_1) *has only the trivial solution.*

Corollary 1.2. *Let either*

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b\int_0^a\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k}|h_{ik}(x,y)|\,dx\,dy<1, \tag{1.15}$$

or

$$\operatorname{ess\,sup}\Big\{\sum_{i=1}^{2}\sum_{k=1}^{2}\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k}|h_{ik}(x,y)|:(x,y)\in\Omega\Big\}<$$
$$<\frac{4}{ab}\,. \tag{1.16}$$

Then problem (1.1), (1.2_1) *is conditionally well-posed. Moreover, if along with* (1.15) (*along with* (1.16)) *condition* (1.12) *holds, then problem* (1.1), (1.2_1) *is well-posed.*

Corollary 1.3. *Let along with condition* (1.13) *either of conditions* (1.15) *and* (1.16) *hold. Then problem* (1.1), (1.2_1) *is conditionally well-posed but not well-posed.*

1.4. **Corollaries for problem (1.1), (1.2$_2$).** We study problem (1.1), (1.2$_2$) in the case, where

$$\begin{aligned}&\alpha(0)=0,\ \ \alpha(x)\le\alpha(a)\ \text{ a.e. on } [0,a],\ \ \int\limits_0^a\alpha(x)\,dx<a\alpha(a),\\&\beta(0)=0,\ \ \beta(y)\le\beta(b)\ \text{ a.e. on } [0,b],\ \ \int\limits_0^b\beta(y)\,dy<b\beta(b).\end{aligned}\tag{1.17}$$

Corollary 1.4. *If along with* (1.17) *the condition*

$$\int\limits_0^b\int\limits_0^a x^{2-i}y^{2-k}|h_{ik}(x,y)|\,dx\,dy<+\infty\ \ (i,\,k=1,2)\tag{1.18}$$

holds, then problem (1.1), (1.2$_2$) *is conditionally well-posed if and only if the corresponding homogeneous problem* (1.1$_0$), (1.2$_2$) *has only the trivial solution.*

Corollary 1.5. *If along with* (1.17) *the condition*

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int\limits_0^b\int\limits_0^a x^{2-i}y^{2-k}|h_{ik}(x,y)|\,dx\,dy<1\tag{1.19}$$

holds, then problem (1.1), (1.2$_2$) *is conditionally well-posed. Moreover, if along with* (1.17) *and* (1.19) *condition* (1.12) *holds, then problem* (1.1), (1.2$_2$) *is well-posed.*

Corollary 1.6. *If along with* (1.17) *and* (1.19) *condition* (1.13) *holds, then problem* (1.1), (1.2$_2$) *is conditionally well-posed but not well-posed.*

1.5. **Examples.** The examples below demonstrate that in Theorem 1.2 (in Corollary 1.2) condition (1.11) (condition (1.15), as well as condition (1.16)) is unimprovable in a sense.

Example 1.1. Let ε be an arbitrary positive number and $\gamma>1$ be sufficiently large number such that

$$\Big(\frac{\gamma+1}{\gamma-1}\Big)^2<1+\varepsilon.\tag{1.20}$$

Set

$$h_0(t)=\begin{cases}(\gamma+1)t^{\gamma-2}-t^{2\gamma-2} & \text{for } 0\le t\le 1,\\(\gamma+1)(2-t)^{\gamma-2}-(2-t)^{2\gamma-2} & \text{for } 1<t\le 2,\end{cases}\tag{1.21}$$

$$w_0(t)=\begin{cases}t\exp\Big(-\dfrac{t^\gamma}{\gamma}\Big) & \text{for } 0\le t\le 1,\\(2-t)\exp\Big(-\dfrac{(2-t)^\gamma}{\gamma}\Big) & \text{for } 1<t\le 2,\end{cases}$$

and consider the differential equation (1.1), where $h \in L(\Omega)$ and

$$h_{11}(x,y) = \frac{16}{a^2b^2} h_0\Big(\frac{2x}{a}\Big) h_0\Big(\frac{2y}{b}\Big), \quad h_{ik}(x,y) = 0 \quad \text{for } (x,y) \in \Omega, \ \ i+k \neq 2. \tag{1.22}$$

Then problem $(1.1), (1.2_1)$ is *not* conditionally well-posed since the corresponding homogeneous problem $(1.1_0), (1.2_1)$ has the nontrivial solution

$$u(x,y) = w_0\Big(\frac{2x}{a}\Big) w_0\Big(\frac{2y}{b}\Big).$$

On the other hand, according to (1.21) and (1.22) we have

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int\limits_0^b\int\limits_0^a \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} |h_{ik}(x,y)|\,dx\,dy =$$
$$= \frac{16}{a^2b^2}\int\limits_0^a x\Big(1-\frac{x}{a}\Big) h_0\Big(\frac{2x}{a}\Big)\,dx \int\limits_0^b y\Big(1-\frac{y}{b}\Big) h_0\Big(\frac{2y}{b}\Big)\,dy \leq$$
$$\leq \frac{1}{ab}\int\limits_0^a h_0\Big(\frac{2x}{a}\Big)\,dx \int\limits_0^b h_0\Big(\frac{2y}{b}\Big)\,dy = \frac{1}{4}\int\limits_0^2 h_0(t)\,dt =$$
$$= \Big(\int\limits_0^1 h_0(t)\,dt\Big)^2 < \Big(\frac{\gamma+1}{\gamma-1}\Big)^2.$$

Hence, by (1.20) it follows that

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int\limits_0^b\int\limits_0^a \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} |h_{ik}(x,y)|\,dx\,dy < 1+\varepsilon. \tag{1.23}$$

Consequently, in Corollary 1.2 condition (1.15) cannot be replaced by the condition

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\int\limits_0^b\int\limits_0^a \varphi_{1i}(x)\varphi_{2k}(y)|h_{ik}(x,y)|\,dx\,dy < 1+\varepsilon$$

no matter how small $\varepsilon > 0$ might be.

Example 1.2. Let

$$h_{11}(x,y) = \frac{4}{x(a-x)y(b-y)}, \quad h_{ik}(x,y) = 0 \ \text{ for } \ (x,y) \in \Omega, \ \ i+k > 2.$$

Then

$$\operatorname{ess\,sup}\left\{\sum_{i=1}^{2}\sum_{k=1}^{2}\left[x\Big(1-\frac{x}{a}\Big)\right]^{2-i}\left[y\Big(1-\frac{y}{b}\Big)\right]^{2-k}|h_{ik}(x,y)| : (x,y)\in\Omega\right\} = \\ = \frac{4}{ab}\,. \tag{1.24}$$

On the other hand, problem $(1.1), (1.2_1)$ is *not* conditionally well-posed, since its corresponding homogeneous problem $(1.1_0), 1.2_1()$ has the nontrivial solution

$$u(x,y) = x(x-a)y(y-b).$$

Consequently, in Corollary 1.2 inequality (1.16) cannot be replaced by equality (1.24).

2. Auxiliary Statements

By $L([0,T])$ we denote the space of Lebesgue integrable functions $v : [0,T] \to \mathbb{R}$ endowed with the norm

$$\|v\|_L = \int_0^T |v(t)|\,dt,$$

and by $\widetilde{C}^1([0,T])$ we denote the space of continuously differentiable functions $u : [0,T] \to \mathbb{R}$ for which u' is absolutely continuous.

Also, we will need to consider the second order ordinary differential equation

$$u'' = q(t) \tag{2.1}$$

with the nonlocal boundary conditions

$$\int_0^T u(t)\,d\,\gamma_i(t) = 0 \ \ (i = 1,2), \tag{2.2}$$

where $q \in L([0,T])$, and $\gamma_i : [0,T] \to \mathbb{R}$ $(i = 1,2)$ are functions of bounded variation such that

$$\gamma_i(0) = 0 \quad (i = 1,2). \tag{2.3}$$

A solution of problem $(2.1), (2.2)$ will be sought in the space $\widetilde{C}^1([0,T])$.

2.1. **Lemmas on estimates of solutions to problems of type (2.1), (2.2).** Let

$$\Delta(t) = \gamma_2(T)\int_t^T \gamma_1(s)\,ds - \gamma_1(T)\int_t^T \gamma_2(s)\,ds \ \text{ for } \ 0 \le t \le T. \tag{2.4}$$

If $\Delta(0) \neq 0$, then set

$$g(t,s) = \frac{1}{\Delta(0)}\Big[\int_0^T \gamma_1(\tau)\,d\tau \int_s^T \gamma_2(\tau)\,d\tau - \int_s^T \gamma_1(\tau)\,d\tau \int_0^T \gamma_2(\tau)\,d\tau\Big]+$$
$$+\frac{1}{\Delta(0)}\big[(s-T)\Delta(0)+(T-t)\Delta(s)\big]+\chi(t,s)(t-s) \ \text{ for } \ 0\le t,s\le T, \quad (2.5)$$

where χ is the function given by equality (1.5).

Lemma 2.1. *Problem* (2.1) *is uniquely solvable if and only if*

$$\Delta(0) \neq 0. \quad (2.6)$$

Moreover, is condition (2.6) *holds, then the function g given equality* (2.5) *is the Green's function of the boundary value problem*

$$u'' = 0; \quad \int_0^T u(t)\,d\gamma_i(t) = 0 \quad (i=1,2), \quad (2.7)$$

and a solution u of problem (2.1), (2.2) *admits the estimates*

$$|u^{(i-1)}(t)| \le \varphi_i(t)\|h\|_L \ \ for \ \ 0 \le t \le T (i=1,2), \quad (2.8)$$

where

$$\begin{aligned} \varphi_1(t) &= \max\big\{|g(t,s)| : \ 0 \le s \le T\big\}, \\ \varphi_2(t) &= \sup\Big\{|g^{(1,0)}(t,s)| : \ 0 \le s \le T, \ \ s \neq t\Big\}. \end{aligned} \quad (2.9)$$

Proof. An arbitrary solution of equation (2.1) admits the representation

$$u(t) = c_1 + c_2 t + \int_0^t (t-s)q(s)\,ds \ \text{ for } \ 0 \le t \le T. \quad (2.10)$$

In view of (2.3) the function u is a solution of problem (2.1), (2.2) if and only if (c_1, c_2) is a solution of the system of linear algebraic equation

$$\gamma_i(T)c_1 + \Big(\int_0^T \tau\,d\gamma_i(\tau)\Big)c_2 = \int_0^T \Big(\int_0^s (\tau - s)q(\tau)\,d\tau\Big)d\gamma_i(s) \ \ (i=1,2). \quad (2.11)$$

However,

$$\int_0^T \tau\,d\gamma_i(\tau) = T\gamma_i(T) - \int_0^T \gamma_i(\tau)\,d\tau \ \ (i=1,2),$$
$$\int_0^T \Big(\int_0^s (\tau - s)q(\tau)\,d\tau\Big)d\gamma_i(s) =$$

$$= \gamma_i(T)\int_0^T (s-T)q(s)\,ds + \int_0^T \Big(\int_0^s q(\tau)\,d\tau\Big)\gamma_i(s)\,ds =$$

$$= \gamma_i(T)\int_0^T (s-T)q(s)\,ds + \int_0^T \Big(\int_s^T \gamma_i(\tau)\,d\tau\Big)q(s)\,ds =$$

$$= \int_0^T \Big(\int_s^T \gamma_i(\tau)\,d\tau - \gamma_i(T)(T-s)\Big)q(s)\,ds \quad (i=1,2).$$

Therefore system (2.11) is equivalent to system

$$\gamma_i(T)c_1 + \Big(T\gamma_i(T) - \int_0^T \gamma_i(\tau)\,d\tau\Big)c_2 =$$

$$= \int_0^T \Big(\int_s^T \gamma_i(\tau)\,d\tau - \gamma_i(T)(T-s)\Big)q(s)\,ds \quad (i=1,2).$$

In view of notation (2.4) the latter system is uniquely solvable if and only if inequality (2.6) holds. Besides, if this inequality holds, then

$$c_1 = \frac{1}{\Delta(0)}\int_0^T \Big[\int_0^T \gamma_1(\tau)\,d\tau \int_s^T \gamma_2(\tau)\,d\tau - \int_s^T \gamma_1(\tau)\,d\tau \int_0^T \gamma_2(\tau)\,d\tau\Big]q(s)\,ds +$$

$$+\frac{1}{\Delta(0)}\int_0^T [T\Delta(s) + (s-T)\Delta(0)]q(s)\,ds, \quad c_2 = -\int_0^T \frac{\Delta(s)}{\Delta(0)}q(s)\,ds.$$

Substituting c_1 and c_2 in (2.10) and taking into account (2.5), we get

$$u(t) = \int_0^T g(t,s)q(s)\,ds \ \text{ for } \ 0 \le t \le T.$$

Consequently g is the Green's function of problem (2.7). On the other hand, the obtained representation of a solution of problem (2.1), (2.2) implies estimates (2.8), where φ_i $(i=1,2)$ are the functions given by equalities (2.9). □

Lemma 2.2. *If inequality* (2.6) *holds, then the functions* φ_1 *and* φ_2*, given by equalities* (2.9)*, are continuous on* $[0,T]$*. Moreover,* φ_1 *has at most two zeros, and* φ_2 *is positive in* $[0,T]$*.*

Proof. According to equalities (2.4) and (2.5) the function $g : [0,T] \times [0,T] \to \mathbb{R}$ is continuous, that guarantees continuity of function φ_1. On the other hand

$$g^{(1,0)}(t,s) = \begin{cases} 1 - \dfrac{\Delta(s)}{\Delta(0)} & \text{for } \ 0 \le s < t \le T, \\ -\dfrac{\Delta(s)}{\Delta(0)} & \text{for } \ 0 \le t < s \le T. \end{cases} \tag{2.12}$$

Therefore

$$\varphi_2(t) = \frac{1}{2}(\varphi_{21}(t) + \varphi_{22}(t) + |\varphi_{22}(t) - \varphi_{21}(t)|) \ \text{ for } \ 0 \le t \le T,$$

where

$$\varphi_{21}(t) = \max\left\{\Big|1 - \frac{\Delta(s)}{\Delta(0)}\Big| : \ 0 \le s \le t\right\}, \ \ \varphi_{22}(t) = \max\left\{\Big|\frac{\Delta(s)}{\Delta(0)}\Big| : \ t \le s \le T\right\}.$$

Consequently, in view of continuity if the function Δ, the functions φ_{21}, φ_{22} and φ_2 are continuous. Besides,

$$\varphi_2(t) \ge \frac{1}{2}(\varphi_{21}(t) + \varphi_{22}(t)) \ge \frac{1}{2}\left(\Big|1 - \frac{\Delta(t)}{\Delta(0)}\Big| + \Big|\frac{\Delta(t)}{\Delta(0)}\Big|\right) \ge \frac{1}{2} \ \text{ for } \ 0 \le t \le T.$$

To complete the proof it remains to show that the function φ_1 has at most two zeros in $[0,T]$. Assume the contrary that φ_1 has at least three zeros t_1, t_2 and t_3, where $0 \le t_1 < t_2 < t_3 \le T$. Let $s_0 \in (t_1,t_2)$ be arbitrarily fixed and set

$$v(t) = g(t,s_0) \ \text{ for } \ 0 \le t \le T.$$

Then, in view of the equalities $\varphi_1(t_i) = 0$ $(i = 1,2,3)$, we have $v(t_i) = 0$ $(i = 1,2,3)$. Hence, in view of equality (2.12), it follows that $v'(t) = 1 - \frac{\Delta(s_0)}{\Delta(0)} = 0$ for $t_2 \le t \le t_3$. Consequently,

$$v'(t) = \begin{cases} -1 & \text{for } \ t_1 \le t < s_0, \\ 0 & \text{for } \ s_0 < t \le t_2. \end{cases}$$

But this is impossible since $v(t_1) = v(t_2) = 0$. The obtained contradiction proves the lemma. $\square$

If

$$\gamma_1(t) = \begin{cases} 0 & \text{for } \ t = 0 \\ 1 & \text{for } \ 0 < t \le T \end{cases}, \quad \gamma_2(t) = \gamma(t) \ \text{ for } \ 0 \le t \le T, \tag{2.13}$$

where $\gamma : [0,T] \to \mathbb{R}$ is a function of bounded variation, then boundary condition (2.2) receives the form

$$u(0) = 0, \quad \int\limits_0^T u(s)\, d\gamma(s) = 0. \tag{2.14}$$

Lemma 2.3. *If*

$$\gamma(0)=0,\ \ \gamma(t)\le\gamma(T)\ \text{ a.e. on } [0,T],\quad \int\limits_0^T \gamma(s)\,ds < T\gamma(T), \tag{2.15}$$

then problem (2.1), (2.14) *is uniquely solvable and the Green's function of the problem*

$$u''=0;\quad u(0)=0,\quad \int\limits_0^T u(s)\,d\gamma(s)=0$$

admits the estimates

$$\begin{aligned}&\max\big\{|g(t,s)|:\ 0\le s\le T\big\}\le t,\\ &\qquad\sup\Big\{|g^{(1,0)}(t,s)|:\ 0\le s\le T,\ s\ne t\Big\}\le 1\ \textit{ for }\ 0\le t\le T.\end{aligned} \tag{2.16}$$

Proof. According to conditions (2.13) and (2.15) from inequalities (2.4) and (2.5) we find

$$\Delta(0)=T\gamma(T)-\int\limits_0^T\gamma(s)\,ds>0, \tag{2.17}$$

$$0\le\Delta(t)=(T-t)\gamma(T)-\int\limits_t^T\gamma(s)\,ds\le\Delta(0)\ \text{ for }\ 0\le t\le T, \tag{2.18}$$

$$g(t,s)=-\frac{\Delta(s)}{\Delta(0)}t+\chi(t,s)(t-s)\ \text{ for }\ 0\le t,s,\le T. \tag{2.19}$$

By Lemma 2.1, inequality (2.17) guarantees unique solvability of problem (2.1), (2.14). On the other hand, by virtue of inequalities (2.17) and (2.18), estimates (2.16) follow from representation (2.19). □

In conclusion of this subsection consider equation (2.1) with the Dirichlet boundary conditions

$$u(0)=0,\quad u(T)=0. \tag{2.20}$$

Lemma 2.4. *Problem* (2.1), (2.20) *is uniquely solvable and the Green's function of the problem*

$$u''=0;\quad u(0)=0,\ \ u(T)=0$$

admits the estimates

$$\begin{aligned}&\max\big\{|g(t,s)|:\ 0\le s\le T\big\}\le t\Big(1-\frac{t}{T}\Big),\\ &\sup\Big\{\big|g^{(1,0)}(t,s)\big|:\ 0\le s\le T,\ s\ne t\Big\}\le 1\ \textit{ for }\ 0\le t\le T,\end{aligned} \tag{2.21}$$

$$\int_0^T \left|g^{(i-1,0)}(t,s)\right| ds \le \frac{T}{2}\Big[t\Big(1-\frac{t}{T}\Big)\Big]^{2-i} \ \ \textit{for} \ \ 0 \le t \le T \ \ (i=1,2). \tag{2.22}$$

Proof. Boundary condition (2.20) follow from conditions (2.2) in the case where

$$\gamma_1(t) = \begin{cases} 0 & \text{for } \ t=0 \\ 1 & \text{for } \ 0<t\le T \end{cases}, \quad \gamma_2(t) = \begin{cases} 0 & \text{for } \ 0\le t<T \\ 1 & \text{for } \ t=T \end{cases}. \tag{2.23}$$

Therefore equalities (2.4) and (2.5) imply

$$\Delta(t) = T-t \ \text{ for } \ 0\le t\le T, \ \ \Delta(0)=T>0$$

and

$$g(t,s) = \begin{cases} s\Big(\dfrac{t}{T}-1\Big) & \text{for } \ 0\le s\le t\le T \\ t\Big(\dfrac{s}{T}-1\Big) & \text{for } \ 0\le t<s\le T \end{cases}. \tag{2.24}$$

By Lemma 2.1 problem (2.1), (2.2) is uniquely solvable. On the other hand, estimates (2.21) and (2.22) immediately follow from representation (2.24).

2.2. **Lemma on estimates of functions satisfying conditions** (1.2_1)**.**

Lemma 2.5. *Let* $u \in \widetilde{C}^{1,1}(\Omega)$ *be a function satisfying boundary conditions* (1.2_1)*. Then*

$$|u^{(i-1,k-1)}(x,y)| \le$$
$$\le \|u^{(2,2)}\|_L \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \ \ \textit{for} \ \ (x,y)\in\Omega \ (i,k=1,2). \tag{2.25}$$

Moreover, if

$$\rho = \operatorname{ess\,sup}\{|u^{(2,2)}(x,y)| : \ (x,y)\in\Omega\} < +\infty, \tag{2.26}$$

then

$$|u^{(i-1,k-1)}(x,y)| \le$$
$$\le \frac{ab}{4}\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k}\rho \ \ \textit{for} \ \ (x,y)\in\Omega \ \ (i,k=1,2). \tag{2.27}$$

Proof. By Lemma 2.6 from [2], the function u satisfies inequality (2.25) and admits the representation

$$u(x,y) = \int_0^b\int_0^a g_2(y,t)g_1(x,s)u^{(2,2)}(s,t)\,ds\,dt \ \ \text{for} \ \ (x,y)\in\Omega, \tag{2.28}$$

where $g_1 : [0,a]\times[0,a]\to\mathbb{R}$ and $g_2 : [0,b]\times[0,b]\to\mathbb{R}$, respectively, are the Green's functions of the boundary value problems

$$v''=0; \quad v(0)=0, \quad v(a)=0 \tag{2.29}$$

and

$$w'' = 0; \quad w(0) = 0, \quad w(b) = 0. \tag{2.30}$$

On the other hand, according to Lemma 2.4, the functions g_1 and g_2 admit the estimates

$$\big|g_1^{(i-1,0)}(x,s)\big| \le \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i} \ \text{ for } \ 0\le x,s\le a, \ \ x\ne s \ \ (i=1,2), \tag{2.31}$$

$$\big|g_2^{(0,k-1)}(y,t)\big| \le \Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \ \text{ for } \ 0\le y,t\le b, \ \ y\ne t \ \ (k=1,2), \tag{2.32}$$

and

$$\int\limits_0^a \big|g_1^{(i-1,0)}(x,s)\big|\,ds \le \frac{a}{2}\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i} \ \text{ for } \ 0\le x\le a \ \ (i=1,2), \tag{2.33}$$

$$\int\limits_0^b \big|g_2^{(0,k-1)}(y,t)\big|\,dt \le \frac{b}{2}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \ \text{ for } \ 0\le y\le b \ \ (k=1,2). \tag{2.34}$$

In view of estimates (2.31) and (2.32), estimates (2.25) follow from (2.28).

Now assume that the function u satisfies condition (2.26). Then representation (2.28) yields

$$\big|u^{(i-1,k-1)}(x,y)\big| \le \bigg(\int\limits_0^a |g_1^{(i-1,0)}(x,s)|\,ds\bigg)\bigg(\int\limits_0^b |g_2^{(0,k-1)}(y,t)|\,dt\bigg)\rho$$
$$\text{for } \ (x,y)\in\Omega \ \ (i,k=1,2),$$

whence, by inequalities (2.33) and (2.34), estimates (2.27) follow. □

2.3. **Lemmas on conditional well-posedness of problem (1.1), (1.2).** Let there exist continuous functions $\psi_{1i} : [0,a] \to [0,\infty)$, $\psi_{2i} : [0,b] \to [0,+\infty)$ $(i=1,2)$ such that

$$\psi_{1i}(x) > 0 \ \text{ a.e. on } \ [0,a], \quad \psi_{2i}(y) > 0 \ \text{ a.e. on } \ [0,b], \tag{2.35}$$

and arbitrary functions $v \in \widetilde{C}^1([0,a])$ and $w \in \widetilde{C}^1([0,b])$, satisfying the boundary conditions

$$\int\limits_0^a v(x)\,d\alpha_i(x) = 0, \quad \int\limits_0^b w(y)\,d\beta_i(y) = 0 \ \ (i=1,2) \tag{2.36}$$

admit the estimates

$$\begin{aligned} |v^{(i-1)}(x)| &\le \psi_{1i}(x)\|v''\|_L \ \text{ for } \ 0\le x\le a \ \ (i=1,2),\\ |w^{(i-1)}(y)| &\le \psi_{2i}(y)\|w''\|_L \ \text{ for } \ 0\le y\le b \ \ (i=1,2). \end{aligned} \tag{2.37}$$

Then Theorems 1.4, 1.5 and 1.10 from [2] imply the following lemmas.

Lemma 2.6. *If*

$$\int_0^b\int_0^a \psi_{1i}(x)\psi_{2k}(y)|h_{ik}(x,y)|\,dx\,dy < +\infty \quad (i,k=1,2), \tag{2.38}$$

then problem (1.1), (1.2) *is conditionally well-posed if and only if the homogeneous problem* (1.1_0), (1.2) *has only the trivial solution.*

Lemma 2.7. *If*

$$\sum_{i=1}^2\sum_{k=1}^2\int_0^b\int_0^a \psi_{1i}(x)\psi_{2k}(y)|h_{ik}(x,y)|\,dx\,dy < 1, \tag{2.39}$$

then problem (1.1), (1.2) *is conditionally well-posed. Moreover, if along with* (2.39) *condition* (1.12) *holds, then problem* (1.1), (1.2) *is well-posed.*

Lemma 2.8. *If conditions* (1.13) *and* (2.39) *hold, then problem* (1.1), (1.2) *is conditionally well-posed but not well-posed.*

3. Proofs of the Main Results

Proof of Theorem 1.1. Set

$$\psi_{1i}(x)=\varphi_{1i}(x) \ \text{ for } \ 0\le x\le a, \ \ \psi_{2i}(y)=\varphi_{2i}(y) \ \text{ for } \ 0\le y\le b \ \ (i=1,2).$$

Then by conditions (1.4), (1.10) and Lemma 2.2, the functions ψ_{1i} and ψ_{2i} $(i=1,2)$ are continuous and satisfy conditions (2.35) and (2.38). On the other hand, according to Lemma 2.1, functions $v\in\widetilde{C}^1([0,a])$ and $w\in\widetilde{C}^1([0,b])$ satisfying boundary conditions (2.36) admit estimates (2.37). Therefore Theorem 1.1 immediately follows from Lemma 2.6. ☐

Theorem 1.2 follows from Lemmas 2.1, 2.2 and 2.7, while Theorem 1.3 follows from Lemmas 2.1, 2.2 and 2.8.

Proof of Corollary 1.1. Boundary conditions (1.2_1) follow from the conditions (1.2), where

$$\alpha_1(x)=\begin{cases}0 & \text{for } \ x=0\\ 1 & \text{for } \ 0<x\le a\end{cases}, \quad \alpha_2(x)=\begin{cases}0 & \text{for } \ 0\le x<a\\ 1 & \text{for } \ x=a\end{cases},$$

$$\beta_1(y)=\begin{cases}0 & \text{for } \ y=0\\ 1 & \text{for } \ 0<y\le b\end{cases}, \quad \beta_2(y)=\begin{cases}0 & \text{for } \ 0\le y<b\\ 1 & \text{for } \ y=b\end{cases}.$$

In this case, by Lemmas 2.1 and 2.4, the functions g_1 and g_2, given by equalities (1.6) and (1.7), are Green's functions of problems (2.29) and (2.30), respectively, and the functions φ_{ik} $(i,k=1,2)$, given by equalities (1.8) and (1.9), admit the estimates

$$\varphi_{1i}(x)\le\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i} \ \text{ for } \ 0\le x\le a \ \ (i=1,2),$$

$$\varphi_{2i}(y) \le \Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \ \text{ for } \ 0 \le y \le b \ \ (k=1,2).$$

According to those estimates, inequalities (1.10) follow from inequalities (1.14). Now applying Theorem 1.1, the validity of Corollary 1.1 becomes evident. □

Proof of Corollary 1.2. In view of Corollary 1.1, in order to prove Corollary 1.2 it is sufficient to show that problem $(1.1_0), (1.2_1)$ has only the trivial solution provided that inequality (1.15) (inequality (1.16)) holds.

Let u be an arbitrary solution of problem $(1.1_0), (1.2)$. Then, in view of Lemma 2.5, estimates (2.25) are valid. Therefore from (1.1_0) we deduce

$$\|u^{(2,2)}\|_L \le \bigg(\sum_{i=1}^{2}\sum_{k=1}^{2}\int_0^b\int_0^a \Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \times \\ \times |h_{ik}(x,y)|\,dx\,dy\bigg)\|u^{(2,2)}\|_L. \quad (3.1)$$

If inequality (1.15) holds, then (3.1) and (2.25) imply that $\|u^{(2,2)}\|_L = 0$ and $u(x,y) \equiv 0$.

To complete the proof it remains to consider the case, where inequality (1.16) holds. In that case according to estimates (2.25) we have

$$\rho = \operatorname{ess\,sup}\{|u^{(2,2)}(x,y)| :\ (x,y) \in \Omega\} \le l\|u^{(2,2)}\|_L < +\infty, \quad (3.2)$$

where

$$l = \operatorname{ess\,sup}\bigg\{\sum_{i=1}^{2}\sum_{k=1}^{2}\Big[x\Big(1-\frac{x}{a}\Big)\Big]^{2-i}\Big[y\Big(1-\frac{y}{b}\Big)\Big]^{2-k} \times \\ \times |h_{ik}(x,y)| :\ (x,y) \in \Omega\bigg\} < \frac{4}{ab}. \quad (3.3)$$

But, by Lemma 2.5, condition (3.2) guarantees the validity of estimates (2.27). Taking in account those estimates from (1.1_0) we obtain

$$\rho \le \frac{ab}{4}\,l\,\rho. \quad (3.4)$$

In view of inequality (3.3), (3.4) and (2.27) imply that $\rho = 0$ and $u(x,y) \equiv 0$. □

Corollary 1.3 follows from Theorem 1.3 and Lemmas 2.1 and 2.4.

Corollaries 1.4 and 1.5 can be proved in the same manner as Corollaries 1.1 and 1.2. The only difference between the proofs is that instead of Lemma 2.4 one should use Lemma 2.3.

Corollary 1.6 follows from Theorem 1.3 and Lemmas 2.1 and 2.3.

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ON OSCILLATION OF SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to the memory of Professor T. Chanturia

Abstract. A new oscillation criterion is proved for second-order linear ordinary differential equations with locally integrable coefficients. It is also shown that a certain generalization of the Hartman–Wintner theorem can be derived from the result obtained.

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1. Introduction

In the present paper we consider the second-order linear differential equation

$$u'' = -p(t)u + g(t)u', \tag{1.1}$$

where $p, g\colon \mathbb{R}_+ \to \mathbb{R}$ are locally integrable functions such that

$$\int\limits_0^{+\infty} \exp\bigg(\int\limits_0^s g(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = +\infty. \tag{1.2}$$

As usual, in the Carathéodory case, a function $u\colon \mathbb{R}_+ \to \mathbb{R}$ is said to be *a solution to equation* (1.1) if it is absolutely continuous together with the first derivative on every compact interval contained in $\mathbb{R}_+$ and satisfies

$$u''(t) = -p(t)u(t) + g(t)u'(t) \quad \text{for a. e. } t \geq 0.$$

Equation (1.1) is said to be *oscillatory* if every solution of this equation has a sequence of zeros tending to infinity.

In [7], the following oscillation criterion is proved for the equation

$$u'' = -p(t)u. \tag{1.3}$$

Theorem 1.1 ([7])**.** *Let the condition*

$$\limsup_{t\to+\infty} \frac{1}{t^\alpha} \int\limits_0^t (t-s)^\alpha p(s)\,\mathrm{d}s = +\infty \tag{1.4}$$

hold for some $\alpha > 1$*. Then equation* (1.3) *is oscillatory.*

It is also mentioned therein that the well-known Wintner criterion (see [10])

$$\lim_{t\to+\infty} \frac{1}{t} \int\limits_0^t \bigg(\int\limits_0^s p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = +\infty \tag{1.5}$$

follows from this result, because equality (1.5) guarantees the validity of relation (1.4) with $\alpha = 2$. Theorem 1.1 has been then generalized for the second-order equations, e. g., in [8, 9] (see also references therein). For higher-order equations, the integral oscillation criteria have been proved in [2–4].

The aim of the present paper is to establish a new oscillation criterion, which is applicable to the case where the "Kamenev-type" upper limit (1.4) is finite. The main result (namely, Theorem 2.1) and some further remarks are given in Section 2, and the proofs are given in Section 3. Moreover, a certain generalization of the Hartman–Wintner theorem (namely, Corollary 2.1) is derived in Section 2.

2. Main Results

Let

$$\sigma(g)(t) := \exp\left(\int_0^t g(s)\,\mathrm{d}s\right), \quad f(t) := \int_0^t \sigma(g)(s)\,\mathrm{d}s \quad \text{for } t \geq 0. \tag{2.1}$$

For any $\alpha > 1$, $\beta > 0$, and $\lambda < 1$, we put

$$k(t;\alpha,\beta,\lambda) := \frac{1}{f^{\alpha\beta}(t)} \int_0^t \left(f^\beta(t) - f^\beta(s)\right)^\alpha \frac{f^\lambda(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s \quad \text{for } t > 0 \tag{2.2}$$

and

$$c(t;\lambda) := \frac{1-\lambda}{f^{1-\lambda}(t)} \int_0^t \frac{\sigma(g)(s)}{f^\lambda(s)} \left(\int_0^s \frac{f^\lambda(\xi)p(\xi)}{\sigma(g)(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}s \quad \text{for } t > 0. \tag{2.3}$$

We are now in a position to formulate our main result.

Theorem 2.1. *Let $\alpha > 1$, $\beta > 0$, $\lambda < 1$, condition* (1.2) *hold, and either*

$$\limsup_{t\to+\infty} k(t;\alpha,\beta,\lambda) = +\infty \tag{2.4}$$

or

$$-\infty < \limsup_{t\to+\infty} k(t;\alpha,\beta,\lambda) < +\infty, \tag{2.5}$$

the function $c(\cdot\,;\lambda)$ does not possess a finite limit as $t \to +\infty$. (2.6)

Then equation (1.1) *is oscillatory.*

Observe that condition (2.4) with $\beta = 1$, $\lambda = 0$, and $g \equiv 0$ reduces to the Kamenev condition (1.4). Therefore, Theorem 2.1 can be regarded as an extension of Theorem 1.1 to the case where condition (1.4) is violated.

It is well-known that oscillatory properties of equation (1.1) can be also described in terms of lower and upper limits of the function c. We mention, in particular, the following Hartman–Wintner theorem (see A. Wintner [10] and P. Hartman [5,6] for $\lambda = 0$ and $g \equiv 0$).

Theorem 2.2 (Hartman–Wintner)**.** *Let $\lambda < 1$, condition* (1.2) *hold, and either*

$$\lim_{t\to+\infty} c(t;\lambda) = +\infty,$$

or

$$-\infty < \liminf_{t\to+\infty} c(t;\lambda) < \limsup_{t\to+\infty} c(t;\lambda)$$

be satisfied. Then equation (1.1) *is oscillatory.*

It is clear that for the given $\lambda < 1$, the following two cases remain uncovered in the previous theorem:

$$\text{there exists a finite limit } \lim_{t\to+\infty} c(t;\lambda) \tag{2.7}$$

and

$$\liminf_{t\to+\infty} c(t;\lambda) = -\infty. \tag{2.8}$$

The case, where (2.7) holds, is already studied in literature (see, e. g., [1] and references therein), but the authors know that there is still a broad field for further investigation if (2.8) is satisfied. Corollary 2.1 below gives a new oscillation criterion which is applicable also to the case where (2.8) holds.

For any $\lambda < 1$, we put

$$h(t;\lambda) := \frac{2(1-\lambda)}{f^{2(1-\lambda)}(t)} \int_0^t \sigma(g)(s) f^{1-2\lambda}(s) c(s;\lambda)\, \mathrm{d}s \quad \text{for } t > 0.$$

Theorem 2.1 yields

Corollary 2.1. *Let* $\lambda < 1$*, condition* (1.2) *hold, and either*

$$\limsup_{t\to+\infty} h(t;\lambda) = +\infty$$

or

$$-\infty < \limsup_{t\to+\infty} h(t;\lambda) < +\infty,$$

the function $c(\cdot\,;\lambda)$ *does not possess a finite limit as* $t \to +\infty$*.*

Then equation (1.1) *is oscillatory.*

This statement can be regarded as a generalization of Theorem 2.2. Indeed, it is not difficult to verify that if there exists a (finite or infinite) limit $\lim_{t\to+\infty} c(t;\lambda)$, then there exists also a limit $\lim_{t\to+\infty} h(t;\lambda)$ and both limits coincide. Moreover, if

$$\liminf_{t\to+\infty} c(t;\lambda) > -\infty \tag{2.9}$$

then

$$\liminf_{t\to+\infty} h(t;\lambda) > -\infty.$$

Therefore, if the assumptions of Theorem 2.2 are satisfied then the assumptions of Corollary 2.1 hold, as well. Note also that the assumptions of Theorem 2.2 require necessarily the validity of inequality (2.9). The following example shows that in some cases can be applied Corollary 2.1, while condition (2.9) is violated (i. e., (2.8) holds).

Example 2.1. Let $g \equiv 0$ and $p(t) = (2 - t^2)\cos(t) - 4t\sin(t)$ for $t \geq 0$. Then

$$c(t;0) = t\cos(t), \quad h(t;0) = 2\sin(t) + \frac{4}{t}\cos(t) - \frac{4}{t^2}\sin(t) \quad \text{for } t \geq 0,$$

and thus

$$\liminf_{t\to+\infty} c(t;0) = -\infty, \quad \limsup_{t\to+\infty} c(t;0) = +\infty,$$
$$\liminf_{t\to+\infty} h(t;0) = -2, \quad \limsup_{t\to+\infty} h(t;0) = 2.$$

Consequently, Theorem 2.2 with $\lambda = 0$ cannot be applied in this case. However, Corollary 2.1 yields that equation (1.1) is oscillatory.

3. Proofs

In order to prove Theorem 2.1, we need the following two lemmas.

Lemma 3.1. *Let* $\alpha > 1$, $\beta > 0$, $\lambda < 1$, *condition* (1.2) *hold, and* u *be a solution to equation* (1.1) *satisfying the relation*

$$u(t) \neq 0 \quad \text{for } t \geq t_0 \tag{3.1}$$

with $t_0 > 0$. *Then*

$$\limsup_{t\to+\infty} k(t;\alpha,\beta,\lambda) < +\infty. \tag{3.2}$$

If, in addition, the inequality

$$\limsup_{t\to+\infty} k(t;\alpha,\beta,\lambda) > -\infty \tag{3.3}$$

is satisfied, then

$$\int_{t_0}^{+\infty} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)} \left[f(s)\varrho(s) - \frac{\lambda}{2} \right]^2 \mathrm{d}s < +\infty, \tag{3.4}$$

where

$$\varrho(t) := \frac{u'(t)}{u(t)\sigma(g)(t)} \quad \text{for } t \geq t_0. \tag{3.5}$$

Proof. In view of (1.1), relation (3.5) yields that

$$\varrho'(t) = -\frac{p(t)}{\sigma(g)(t)} - \sigma(g)(t)\varrho^2(t) \quad \text{for a.e. } t \geq t_0, \tag{3.6}$$

whence we get

$$\begin{aligned}
&\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha f^\lambda(s)\varrho'(s)\,\mathrm{d}s = \\
&\qquad = -\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha \frac{f^\lambda(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s - \\
&\qquad - \int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha f^\lambda(s)\sigma(g)(s)\varrho^2(s)\,\mathrm{d}s \quad \text{for } t \geq t_0.
\end{aligned}$$

Integration by parts on the left-hand side of the latter equality results in

$$\begin{aligned}
&-\left(f^\beta(t) - f^\beta(t_0)\right)^\alpha f^\lambda(t_0)\varrho(t_0) + \\
&\qquad + \alpha\beta \int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^{\alpha-1} f^{\beta-1}(s) f^\lambda(s)\sigma(g)(s)\varrho(s)\,\mathrm{d}s -
\end{aligned}$$

$$-\lambda \int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha \frac{\sigma(g)(s)}{f^{1-\lambda}(s)}\, \varrho(s)\,\mathrm{d}s =$$

$$= -\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha \frac{f^\lambda(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s -$$

$$-\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha f^\lambda(s)\sigma(g)(s)\varrho^2(s)\,\mathrm{d}s \quad \text{for } t \geq t_0. \quad (3.7)$$

We now point out that

$$-\frac{1}{2}\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha f^\lambda(s)\sigma(g)(s)\varrho^2(s)\,\mathrm{d}s +$$

$$+\lambda \int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha \frac{\sigma(g)(s)}{f^{1-\lambda}(s)}\, \varrho(s)\,\mathrm{d}s =$$

$$= -\frac{1}{2}\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha \frac{\sigma(g)(s)}{f^{2-\lambda}(s)} \left[f(s)\varrho(s) - \lambda\right]^2 \mathrm{d}s +$$

$$+\frac{\lambda^2}{2}\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\,\mathrm{d}s \quad \text{for } t \geq t_0$$

and

$$-\frac{1}{2}\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha f^\lambda(s)\sigma(g)(s)\varrho^2(s)\,\mathrm{d}s -$$

$$-\alpha\beta \int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^{\alpha-1} f^{\beta-1}(s) f^\lambda(s)\sigma(g)(s)\varrho(s)\,\mathrm{d}s =$$

$$= -\frac{1}{2}\int_{t_0}^{t} f^\lambda(s)\left(f^\beta(t) - f^\beta(s)\right)^{\alpha-2} \times$$

$$\times\, \sigma(g)(s)\left[\left(f^\beta(t) - f^\beta(s)\right)\varrho(t) + \alpha\beta f^{\beta-1}(s)\right]^2 \mathrm{d}s +$$

$$+\frac{\alpha^2\beta^2}{2}\int_{t_0}^{t} f^\lambda(s)\left(f^\beta(t) - f^\beta(s)\right)^{\alpha-2} f^{2(\beta-1)}(s)\sigma(g)(s)\,\mathrm{d}s \quad \text{for } t \geq t_0.$$

Therefore relation (3.7) yields

$$
\begin{aligned}
k(t;\alpha,\beta,\lambda) \leq & -\frac{1}{2}\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2 \mathrm{d}s+ \\
& +\frac{\lambda^2}{2}\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\,\mathrm{d}s+ \\
& +\frac{\alpha^2\beta^2}{2f^{\alpha\beta}(t)}\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha-2}f^{2(\beta-1)+\lambda}(s)\sigma(g)(s)\,\mathrm{d}s+ \\
& +\int\limits_{0}^{t_0}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s+ \\
& +\left(1-\left[\frac{f(t_0)}{f(t)}\right]^{\beta}\right)^{\alpha}f^{\lambda}(t_0)\varrho(t_0) \quad \text{for } t\geq t_0. \qquad (3.8)
\end{aligned}
$$

Since assumption (1.2) and notation (2.1) guarantee that

$$
\lim_{t\to+\infty} f(t)=+\infty, \qquad (3.9)
$$

it is easy to get

$$
\lim_{t\to+\infty}\int\limits_{0}^{t_0}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s=\int\limits_{0}^{t_0}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s \qquad (3.10)
$$

and

$$
\lim_{t\to+\infty}\left(1-\left[\frac{f(t_0)}{f(t)}\right]^{\beta}\right)^{\alpha}f^{\lambda}(t_0)\varrho(t_0)=f^{\lambda}(t_0)\varrho(t_0). \qquad (3.11)
$$

On the other hand, we have

$$
\begin{aligned}
&\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\,\mathrm{d}s \leq \\
&\qquad\leq \int\limits_{t_0}^{t}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\,\mathrm{d}s \leq \frac{1}{(1-\lambda)f^{1-\lambda}(t_0)} \quad \text{for } t\geq t_0 \qquad (3.12)
\end{aligned}
$$

and

$$
\frac{1}{f^{\alpha\beta}(t)}\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha-2}f^{2(\beta-1)+\lambda}(s)\sigma(g)(s)\,\mathrm{d}s \leq
$$

$$\leq \frac{1}{f^{\beta(\alpha-1)}(t)f^{1-\lambda}(t_0)} \int\limits_{t_0}^{t} f^{\beta-1}(s)\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha-2}\sigma(g)(s)\,\mathrm{d}s =$$

$$= \frac{1}{\beta(\alpha-1)f^{1-\lambda}(t_0)}\left(1-\left[\frac{f(t_0)}{f(t)}\right]^{\beta}\right)^{\alpha-1} \leq$$

$$\leq \frac{1}{\beta(\alpha-1)f^{1-\lambda}(t_0)} \quad \text{for } t \geq t_0. \tag{3.13}$$

Consequently, in view of (3.10)–(3.13), relation (3.8) implies that

$$\limsup_{t\to+\infty} k(t;\alpha,\beta,\lambda) \leq \frac{1}{2}\left(\frac{\lambda^2}{1-\lambda}+\frac{\alpha^2\beta}{\alpha-1}\right)\frac{1}{f^{1-\lambda}(t_0)}+ \\ + \int\limits_{0}^{t_0} \frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s + f^{\lambda}(t_0)\varrho(t_0),$$

and thus inequality (3.2) is satisfied.

Assume now that, in addition, relation (3.3) holds. We will show that inequality (3.4) is satisfied. It is obvious that either

$$\int\limits_{t_0}^{+\infty} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s = +\infty, \tag{3.14}$$

or

$$\int\limits_{t_0}^{+\infty} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s < +\infty. \tag{3.15}$$

Suppose that (3.14) holds. For any $\tau \geq a$ we have

$$\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s \geq \\ \geq \int\limits_{t_0}^{\tau}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s \quad \text{for } t \geq \tau$$

and thus

$$\liminf_{t\to+\infty}\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s \geq \\ \geq \int\limits_{a}^{\tau}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s \quad \text{for } \tau \geq t_0.$$

The last relation, by virtue of equality (3.14), guarantees that

$$\lim_{t\to+\infty}\int_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s=+\infty.$$

Therefore inequality (3.8), together with (3.10)–(3.13), yields

$$\limsup_{t\to+\infty} k(t;\lambda)=-\infty,$$

which contradicts assumption (3.3). The obtained contradiction proves that inequality (3.15) holds. Since the function $\sqrt{\frac{\sigma(g)(\cdot)}{f^{2-\lambda}(\cdot)}}$ is quadratically integrable on $[t_0,+\infty[$, relation (3.4) is fulfilled, as well. □

The next lemma belongs to P. Hartman in the case where $\lambda=0$ and $g\equiv 0$ (see, e. g., [5,6]).

Lemma 3.2. *Let $\lambda<1$, condition* (1.2) *hold, and u be a solution to equation* (1.1) *satisfying relation* (3.1) *with $t_0>0$. Moreover, let condition* (3.4) *be fulfilled, where the function ϱ is defined by formula* (3.5). *Then there exists a finite limit*

$$\lim_{t\to+\infty} c(t;\lambda). \tag{3.16}$$

Proof. In view of (1.1), from relation (3.5) we easily obtain equality (3.6). Multiplying both sides of (3.6) by the expression $f^{\lambda}(t)$ and integrating it by parts from t_0 to t, we arrive at

$$f^{\lambda}(t)\varrho(t)-f^{\lambda}(t_0)\varrho(t_0)-\lambda\int_{t_0}^{t}\frac{\sigma(g)(s)}{f^{1-\lambda}(s)}\,\varrho(s)\,\mathrm{d}s=$$
$$=-\int_{t_0}^{t}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s-\int_{t_0}^{t}f^{\lambda}(s)\sigma(g)(s)\varrho^2(s)\,\mathrm{d}s\quad\text{for } t\geq t_0,$$

whence we get

$$\frac{1}{f^{1-\lambda}(t)}\left[f(t)\varrho(t)-\frac{\lambda}{2}\right]=\varrho_1-\frac{\lambda(2-\lambda)}{4(1-\lambda)}\,\frac{1}{f^{1-\lambda}(t)}-\int_{0}^{t}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s+$$
$$+\int_{t}^{+\infty}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\frac{\lambda}{2}\right]^2\mathrm{d}s\quad\text{for } t\geq t_0, \tag{3.17}$$

where

$$\varrho_1 := f^{\lambda}(t_0)\varrho(t_0) + \frac{\lambda^2}{4(1-\lambda)f^{1-\lambda}(t_0)} + \\ + \int_0^{t_0} \frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s - \int_{t_0}^{+\infty} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]^2 \mathrm{d}s.$$

We now multiply both sides of equality (3.17) by the expression $f^{-\lambda}(t)\sigma(g)(t)$, integrate them by parts from t_0 to t and thus we get

$$\int_{t_0}^{t} \frac{\sigma(g)(s)}{f(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]\mathrm{d}s = -\int_0^t \frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_0^s \frac{f^{\lambda}(\xi)p(\xi)}{\sigma(g)(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}s + \\ + \int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_s^{+\infty} \frac{\sigma(g)(\xi)}{f^{2-\lambda}(\xi)}\left[f(\xi)\varrho(\xi) - \frac{\lambda}{2}\right]^2 \mathrm{d}\xi\right)\mathrm{d}s + \\ + \frac{\varrho_1}{1-\lambda}\,f^{1-\lambda}(t) - \frac{\lambda(2-\lambda)}{4(1-\lambda)}\ln\frac{f(t)}{f(t_0)} + \varrho_3 \quad \text{for } t \geq t_0, \tag{3.18}$$

where

$$\varrho_3 := \int_0^{t_0} \frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_0^s \frac{f^{\lambda}(\xi)p(\xi)}{\sigma(g)(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}s - \frac{\varrho_1}{1-\lambda}\,f^{1-\lambda}(t_0).$$

Since assumption (1.2) and notation (1.3) guarantee that relation (3.9) holds, by using the l'Hospital rule, it is easy to get

$$\lim_{t\to+\infty} \frac{1}{f^{1-\lambda}(t)}\ln\frac{f(t)}{f(t_0)}\,\mathrm{d}s = 0 \tag{3.19}$$

and

$$\lim_{t\to+\infty} \frac{1}{f^{1-\lambda}(t)}\int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_s^{+\infty} \frac{\sigma(g)(\xi)}{f^{2-\lambda}(\xi)}\left[f(\xi)\varrho(\xi) - \frac{\lambda}{2}\right]^2 \mathrm{d}\xi\right)\mathrm{d}s = 0. \tag{3.20}$$

On the other hand, by using the Hölder inequality, we obtain

$$\left(\int_{t_0}^{t} \frac{\sigma(g)(s)}{f(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]\mathrm{d}s\right)^2 \leq \\ \leq \int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{\lambda}(s)}\,\mathrm{d}s \int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]^2 \mathrm{d}s \leq \\ \leq \frac{f^{1-\lambda}(t)}{1-\lambda}\int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]^2 \mathrm{d}s \quad \text{for } t \geq t_0$$

and thus, by virtue of relations (3.4) and (3.9), we have

$$\lim_{t\to+\infty}\frac{1}{f^{1-\lambda}(t)}\int_{t_0}^{t}\frac{\sigma(g)(s)}{f(s)}\left[f(s)\varrho(s)-\frac{\lambda}{2}\right]\mathrm{d}s=0. \tag{3.21}$$

Consequently, in view of relations (3.9) and (3.19)–(3.21), it follows from equality (3.18) that

$$\lim_{t\to+\infty}c(t;\lambda)=\varrho_1.$$

□

Proof of Theorem 2.1. Assume, to the contrary, that there exists a solution u to equation (1.1) fulfilling relation (3.1) with $t_0>0$.

Then, according to Lemma 3.1, assumption (2.4) of Theorem 2.1 cannot be satisfied and thus assumptions (2.5) and (2.6) hold. By using Lemma 3.1, we obtain the validity of inequality (3.4) in which the function ϱ is defined by formula (3.5). However, Lemma 3.2 then guarantees that there exists a finite limit (3.16) which contradicts assumption (2.6). □

Proof of Corollary 2.1. By direct calculation we can check that

$$k(t;2,1-\lambda,\lambda)=\frac{1}{f^{2(1-\lambda)}(t)}\int_0^t\left(f^{1-\lambda}(t)-f^{1-\lambda}(s)\right)^2\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s=$$

$$=\frac{2(1-\lambda)}{f^{2(1-\lambda)}(t)}\int_0^t\left(f^{1-\lambda}(t)-f^{1-\lambda}(s)\right)\frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_0^s\frac{f^{\lambda}(\xi)p(\xi)}{\sigma(g)(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}s=$$

$$=\frac{2(1-\lambda)^2}{f^{2(1-\lambda)}(t)}\int_0^t\frac{\sigma(g)(s)}{f^{\lambda}(s)}\left[\int_0^s\frac{\sigma(g)(\xi)}{f^{\lambda}(\xi)}\left(\int_0^{\xi}\frac{f^{\lambda}(\eta)p(\eta)}{\sigma(g)(\eta)}\,\mathrm{d}\eta\right)\mathrm{d}\xi\right]\mathrm{d}s=h(t;\lambda)$$

for $t\geq 0$ and thus the validity of the corollary follows immediately from Theorem 2.1 with $\alpha=2$ and $\beta=1-\lambda$. □

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ASYMPTOTIC PROPERTIES OF HOMOCLINIC SOLUTIONS OF SOME SINGULAR NONLINEAR DIFFERENTIAL EQUATION

Dedicated to the memory of Professor Temuri Chanturia

Abstract. We investigate an asymptotic behaviour of homoclinic solutions of the singular differential equation $(p(t)u')' = p(t)f(u)$. Here f is Lipschitz continuous on $\mathbb{R}$ and has at least two zeros 0 and $L > 0$. The function p is continuous on $[0, \infty)$, has a positive continuous derivative on $(0, \infty)$ and $p(0) = 0$.

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რეზიუმე. გამოკვლეულია $(p(t)u')' = p(t)f(u)$ სინგულარული დიფერენციალური განტოლების ჰომოკლინიკური ამონახსნების ასიმპტოტური თვისებები. აქ $f : \mathbb{R} \to \mathbb{R}$ ლიპშიცურად უწყვეტი, ხოლო $p : [0, \infty) \to [0, \infty)$ უწყვეტი და $(0, \infty)$ შუალედში წარმოებადი ფუნქციაა. ამასთან $f(0) = f(L) = 0$, სადაც $L > 0$, $p(0) = 0$ და $p'(t) > 0$, როცა $t > 0$.

1. Introduction

We investigate the differential equation

$$(p(t)u')' = p(t)f(u), \quad t \in (0,\infty), \tag{1}$$

and throughout the paper it will be assumed that f satisfies

$$f \in Lip_{\text{loc}}(\mathbb{R}), \quad \exists L \in (0,\infty): \ f(L) = 0, \tag{2}$$

$$\exists L_0 \in [-\infty, 0): \quad xf(x) < 0, \ x \in (L_0, 0) \cup (0, L), \tag{3}$$

$$\exists \bar{B} \in (L_0, 0): F(\bar{B}) = F(L), \ \text{ where } F(x) = -\int_0^x f(z)\,\mathrm{d}z, \ \ x \in \mathbb{R}, \tag{4}$$

and p fulfils

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p(0) = 0, \tag{5}$$

$$p'(t) > 0, \ t \in (0,\infty), \quad \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{6}$$

Due to $p(0) = 0$, equation (1) has a singularity at $t = 0$.

Definition 1. A function $u \in C^1[0,\infty) \cap C^2(0,\infty)$ which satisfies equation (1) for all $t \in (0,\infty)$ is called a *solution* of equation (1).

Consider a solution u of equation (1). Since $u \in C^1[0,\infty)$, we have $u(0), u'(0) \in \mathbb{R}$, and the assumption $p(0) = 0$ yields $p(0)u'(0) = 0$. We can find that $M > 0$ and $\delta > 0$ such that $|f(u(t))| \le M$ for $t \in (0,\delta)$. Integrating equation (1) and using the fact that p is increasing, we get

$$|u'(t)| = \left| \frac{1}{p(t)} \int_0^t p(s) f(u(s))\,\mathrm{d}s \right| \le \frac{M}{p(t)} \int_0^t p(s)\,\mathrm{d}s \le Mt, \quad t \in (0,\delta).$$

Consequently, the condition $u'(0) = 0$ is necessary for each solution u of equation (1). Therefore the set of all solutions of equation (1) forms a one-parameter system of functions u satisfying $u(0) = A$, $A \in \mathbb{R}$.

Definition 2. Let u be a solution of equation (1) and let L be of (2) and (3). Denote $u_{\sup} = \sup\{u(t) : t \in [0,\infty)\}$. If $u_{\sup} = L$ ($u_{\sup} < L$ or $u_{\sup} > L$), then u is called a *homoclinic* solution (a *damped* solution or an *escape* solution).

The existence and properties of these three types of solutions have been investigated in [19]–[23]. In particular, we have proved that if $u(0) \in (0, L)$, than u is a damped solution ([22], Theorem 2.3). Clearly, for $u(0) = 0$ and $u(0) = L$, equation (1) has a unique solution $u \equiv 0$ and $u \equiv L$, respectively.

In this paper we focus our attention on homoclinic solutions. According to the above considerations, such solutions have to satisfy the initial conditions

$$u(0) = B, \quad u'(0) = 0, \quad B < 0. \tag{7}$$

Note that if we extend the function $p(t)$ in equation (1) from the half–line onto $\mathbb{R}$ (as an even function), then a homoclinic solution of (1) has the same limit L as $t \to -\infty$ and $t \to \infty$. This is a motivation for Definition 2.

We have proved in [21], Lemma 3.5, that a solution u of equation (1) is homoclinic if and only if u is strictly increasing and $\lim_{t\to\infty} u(t) = L$. If such homoclinic solution exists, then many important physical properties of corresponding models (see below) can be obtained. In particular, equation (1) is a generalization of the equation

$$u'' + \frac{k-1}{t} u' = f(u), \quad t \in (0, \infty), \tag{8}$$

and we can find in [16] that equation (8) with $k > 1$ and special forms of f arise in many areas, for example, in the study of phase transitions of Van der Waals fluids [3], [10], [24], in the population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [8], [9], in the homogeneous nucleation theory [1], in relativistic cosmology for description of particles which can be treated as domains in the universe [18], in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [7]. Numerical simulations of solutions of (8), where f is a polynomial with three zeros, have been presented in [6], [14], [17]. Close problems on the existence of positive solutions are investigated in [2], [4], [5].

The main result of the present paper is contained in Section 3, Theorem 12, where we deduce an asymptotic formula for homoclinic solutions of equation (1). Note that many important results dealing with asymptotic properties of various types of differential equations can be found in the monograph by I. Kiguradze and T. Chanturia [12].

2. The Existence of Homoclinic Solutions

Here we cite theorems on the existence of homoclinic solutions. Remind that assumptions (2)–(6) are common for all these theorems. For a given $B < 0$, we denote the solution of problem (1), (7) by u_B.

Theorem 3. *Assume that problem* (1), (7) *has an escape solution and let* $\bar{B}$ *be of* (4). *Then there exists* $B^* < \bar{B}$ *such that* u_{B^*} *is a homoclinic solution of problem* (1), (7) *with* $B = B^*$.

Proof. Theorem 2.3 in [22] shows that for any $B \in [\bar{B}, 0)$ there exists a unique solution u_B of problem (1), (7) and u_B is damped. Thus, if we denote by $\mathcal{M}_d$ a set of all $B < 0$ such that u_B is a damped solution of problem (1), (7), then we obtain $\mathcal{M}_d \neq \emptyset$. Moreover, $\mathcal{M}_d$ is open in $(-\infty, 0)$, due to Theorem 14 in [19]. Further, denote by $\mathcal{M}_e$ a set of all $B < 0$ such that u_B is an escape solution of problem (1), (7). By our assumption, we have $\mathcal{M}_e \neq \emptyset$ and, by Theorem 20 in [19], the set $\mathcal{M}_e$ is open in $(-\infty, 0)$, as well. Therefore, the set $\mathcal{M}_h = (-\infty, 0) \setminus (\mathcal{M}_d \cup \mathcal{M}_e)$ is non-empty. Let us choose $B^* \in \mathcal{M}_h$. Then $B^* < \bar{B}$, and by virtue of Definition 2, the supremum of the solution u_{B^*} on $(0,\infty)$ cannot be less than L and cannot be greater than L. Consequently, this supremum is equal to L, and u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$. $\square$

Theorem 4. *Assume that L_0 of (3) satisfies*

$$L_0 \in (-\infty, 0), \quad f(L_0) = 0. \tag{9}$$

Then there exists $B^* \in (L_0, \bar{B})$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$.

Proof. Define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for} \quad x \le L, \\ 0 & \text{for} \quad x > L, \end{cases}$$

and consider the auxiliary equation

$$(p(t)u')' = p(t)\tilde{f}(u), \quad t \in (0,\infty). \tag{10}$$

By Theorem 10 and Lemma 9 in [20], there exists $B \in (L_0, \bar{B})$ such that u_B is an escape solution of problem (10), (7). If we modify the proof of Theorem 3 working on $(L_0, 0)$ instead of on $(-\infty, 0)$, we get a homoclinic solution u_{B^*} of problem (10), (7) having its starting value B^* in $(L_0, \bar{B})$. Since u_{B^*} is increasing on $(0,\infty)$ (see e.g., Lemma 3.5 in [21]), we have

$$B^* \le u_{B^*}(t) < L, \quad t \in [0,\infty), \tag{11}$$

and u_{B^*} is likewise a solution of equation (1). $\square$

Theorem 4 assumes that f has the negative finite zero L_0. The following two theorems concern the case that $L_0 = -\infty$ and f is positive on $(-\infty, 0)$. Then a behavior of f near $-\infty$ plays an important role. Equations with f having sublinear or linear behavior near $-\infty$ are discussed in the following theorem.

Theorem 5. *Assume that $f(x) > 0$ for $x \in (-\infty, 0)$ and*

$$0 \le \limsup_{x \to -\infty} \frac{f(x)}{|x|} < \infty. \tag{12}$$

Then there exists $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$.

Proof. In the linear case, that is if we assume

$$0 < \limsup_{x \to -\infty} \frac{f(x)}{|x|} < \infty,$$

the assertion follows from Theorem 5.1 in [21]. Consider the sublinear case in which we work with the condition

$$\limsup_{x \to -\infty} \frac{f(x)}{|x|} = 0.$$

Assumption $f > 0$ on $(-\infty, 0)$ provides us with

$$\lim_{x \to -\infty} \frac{f(x)}{|x|} = 0,$$

and Theorem 19 in [19] guarantees the existence of $B < \bar{B}$ such that u_B is an escape solution of problem (10), (7). Theorem 3 and estimate (11) yield $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$. □

Theorem 6. *Assume that $f(x) > 0$ for $x \in (-\infty, 0)$ and there exists $k \geq 2$ such that*

$$\lim_{t \to 0+} \frac{p'(t)}{t^{k-2}} \in (0, \infty). \tag{13}$$

Further, let $r \in (1, \frac{k+2}{k-2})$ be such that f fulfils

$$\lim_{x \to -\infty} \frac{f(x)}{|x|^r} \in (0, \infty). \tag{14}$$

Then there exists $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$.

Proof. Theorem 2.10 in [23] guarantees the existence of $B < \bar{B}$ such that u_B is an escape solution of problem (10), (7) . Theorem 3 and estimate (11) yield $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$. □

Theorem 6 discusses a superlinear behavior of f near $-\infty$. Note that if $k = 2$, we can take arbitrary $r \in (0, \infty)$. The last existence theorem imposes an additional assumption on p only.

Theorem 7. *Assume that p satisfies*

$$\int_0^1 \frac{\mathrm{d}s}{p(s)} < \infty. \tag{15}$$

Then there exists $B^* < \bar{B}$ such that u_{B^*} is a homoclinic solution of problem (1), (7) with $B = B^*$.

Proof. Using Theorem 18 in [19] instead of Theorem 2.10 in [23], we argue just as in the proof of Theorem 6. □

In the next section, the use will be made of the generalized Matell's theorem which can be found as Theorem 6.5 in the monograph by I. Kiguradze [11]. For our purpose we provide its following special case.

Consider an interval $J \subset \mathbb{R}$. We write $AC(J)$ for the set of functions, absolutely continuous on J, and $AC_{loc}(J)$ for the set of functions belonging to $AC(I)$ for each compact interval $I \subset J$. Choose $T > 0$ and a function matrix $A(t) = (a_{i,j}(t))_{i,j\le 2}$ which is defined on (T,∞). Denote by $\lambda(t)$ and $\mu(t)$ the eigenvalues of $A(t)$, $t \in (T,\infty)$. Further, suppose that

$$\lambda = \lim_{t\to\infty} \lambda(t) \quad \text{and} \quad \mu = \lim_{t\to\infty} \mu(t)$$

are different eigenvalues of the matrix $A = \lim_{t\to\infty} A(t)$ and let $\mathbf{l}$ and $\mathbf{m}$ be eigenvectors of A corresponding to λ and μ, respectively.

Theorem 8 ([11])**.** *Assume that*

$$a_{i,j} \in AC_{loc}(T,\infty), \qquad \left|\int_T^\infty a'_{i,j}(t)\,\mathrm{d}t\right| < \infty, \quad i,j = 1,2, \tag{16}$$

and there exists $c_0 > 0$ *such that*

$$\int_s^t \operatorname{Re}(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau \le c_0, \quad T \le s < t, \tag{17}$$

or

$$\begin{gathered} \int_T^\infty \operatorname{Re}(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau = \infty, \\ \int_s^t \operatorname{Re}(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau \ge -c_0, \quad T \le s < t. \end{gathered} \tag{18}$$

Then the differential system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) \tag{19}$$

has a fundamental system of solutions $\mathbf{x}(t)$, $\mathbf{y}(t)$ *such that*

$$\lim_{t\to\infty} \mathbf{x}(t)\mathrm{e}^{-\int_T^t \lambda(\tau)\,\mathrm{d}\tau} = \mathbf{l}, \quad \lim_{t\to\infty} \mathbf{y}(t)\mathrm{e}^{-\int_T^t \mu(\tau)\,\mathrm{d}\tau} = \mathbf{m}. \tag{20}$$

3. Asymptotic Behavior of Homoclinic Solutions

In this section we assume that $B < \bar{B}$ is such that the corresponding solution u of the initial problem (1), (7) is homoclinic. Hence u fulfils

$$u(0) = B, \quad u'(0) = 0, \quad u'(t) > 0,\ t \in (0,\infty), \quad \lim_{t\to\infty} u(t) = L. \tag{21}$$

Moreover, due to (1),

$$u''(t) + \frac{p'(t)}{p(t)}u'(t) = f(u(t)), \quad t > 0, \tag{22}$$

and, by multiplication and integration over $[0,t]$,

$$\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)}u'^2(s)\,\mathrm{d}s = F(u(0)) - F(u(t)), \quad t > 0. \tag{23}$$

Therefore

$$0 \le \lim_{t\to\infty} \int\limits_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s \le F(B) - F(L) < \infty,$$

and hence there exists

$$\lim_{t\to\infty} \int\limits_0^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s.$$

Consequently, according to (23), $\lim\limits_{t\to\infty} u'^2(t)$ exists, as well. Since u is bounded on $[0,\infty)$, we get

$$\lim_{t\to\infty} u'^2(t) = \lim_{t\to\infty} u'(t) = 0. \tag{24}$$

In order to derive an asymptotic formula for u we have to characterize a behavior of p in ∞ and that of f near L more precisely. In particular, we put

$$h(x) := \frac{f(x)}{x-L}, \quad x < L,$$

and work with the following assumptions:

$$\exists c, \eta > 0: \quad h \in C^1[L-\eta, L], \quad \lim_{x\to L-} h(x) = h(L) = c, \tag{25}$$

$$p' \in AC_{loc}(0,\infty), \quad \exists n \ge 2: \quad \lim_{t\to\infty} \frac{p'(t)}{t^{n-2}} \in (0,\infty). \tag{26}$$

For the sake of simplicity we transform L to the origin by the substitution

$$z(t) = L - u(t), \quad t \in [0,\infty), \tag{27}$$

and put

$$g(y) = -f(L-y), \quad y > 0. \tag{28}$$

Then the function z given by (27) is a solution of the equation

$$(p(t)z')' = p(t)g(z), \quad t \in (0,\infty), \tag{29}$$

satisfies

$$z(0) = L + |B|, \quad z'(0) = 0, \quad z'(t) < 0,\ t \in (0,\infty), \tag{30}$$

$$\lim_{t\to\infty} z(t) = 0, \quad \lim_{t\to\infty} z'(t) = 0. \tag{31}$$

Lemma 9. *Assume the above condition* (25) *holds and let z be given by* (27)*. Then there exists $T > 0$ such that*

$$|z'(t)| > \sqrt{\frac{c}{2}}\, z(t), \quad t \ge T. \tag{32}$$

Proof. According to (29), the function z fulfils the following equation:

$$z''(t) = -\frac{p'(t)}{p(t)} z'(t) + g(z(t)), \quad t \in (0, \infty). \tag{33}$$

Define the Lyapunov function V by

$$V(t) = \frac{z'^2(t)}{2} + G(z(t)), \tag{34}$$

where

$$G(x) = -\int_0^x g(s)\,\mathrm{d}s.$$

Owing to (3), (4) and $B < \bar{B}$, the function G fulfils

$$G(L + |B|) = -\int_0^{L+|B|} g(s)\,\mathrm{d}s = \int_B^L f(s)\,\mathrm{d}s = F(B) - F(L) > 0.$$

Thus $V(0) = G(L + |B|) > 0$. Further, using (33), we have

$$V'(t) = z'(t)z''(t) - g(z(t))z'(t) = -\frac{p'(t)}{p(t)} z'^2(t) < 0, \quad t > 0.$$

Hence V is decreasing on $(0, \infty)$ and, by (31), (34), we get $\lim\limits_{t\to\infty} V(t) = 0$. Consequently, $V(t) > 0$ for $t \in [0, \infty)$ which implies that

$$\frac{z'^2(t)}{2} > -G(z(t)), \quad t > 0. \tag{35}$$

Let $y = L - x$. Then, using (25) and (28), we deduce

$$-\lim_{y\to 0+} \frac{G(y)}{y^2} = \lim_{y\to 0+} \frac{g(y)}{2y} = \frac{1}{2} \lim_{x\to L-} \frac{f(x)}{x - L} = \frac{c}{2}.$$

Hence by virtue of (31), there exists $T > 0$ such that

$$-\frac{G(z(t))}{z^2(t)} > \frac{c}{4}, \quad t \geq T.$$

This, together with (35), results in

$$\frac{z'^2(t)}{2} > \frac{c}{4} z^2(t), \quad t \geq T.$$

Consequently, we get (32). $\square$

Lemma 10. *Assume that the condition* (25) *holds and let z and g be given by* (27) *and* (28)*, respectively. Then*

$$\int_1^\infty \left| \frac{g(z(\tau))}{z(\tau)} - c \right| \mathrm{d}\tau < \infty. \tag{36}$$

Proof. Let us put

$$\tilde{h}(y) = \frac{g(y)}{y}, \quad y > 0. \tag{37}$$

By (25) and (28), we have

$$h(L-y) = \tilde{h}(y),\ y > 0, \quad \tilde{h} \in C^1[0,\eta], \quad \lim_{y\to 0+} \tilde{h}(y) = \tilde{h}(0) = c, \tag{38}$$

and there exists $M_0 \in (0,\infty)$ such that

$$\left| \frac{\mathrm{d}\tilde{h}(y)}{\mathrm{d}y} \right| \le M_0, \quad y \in [0,\eta].$$

The Mean Value Theorem guarantees the existence of $\theta \in (0,1)$ such that

$$\tilde{h}(y) = c + y\frac{\mathrm{d}\tilde{h}(\theta y)}{\mathrm{d}y}, \quad y \in (0,\eta].$$

By (31), there exists $T \ge 1$ such that $0 < z(t) \le \eta$ for $t \ge T$ and hence, according to (37),

$$\left| \frac{g(z(t))}{z(t)} - c \right| \le M_0 z(t), \quad t \ge T. \tag{39}$$

Using (2), (28) and $z > 0$ on $[1,T]$, we can find $M_1 \in (0,\infty)$ such that

$$\int_1^T \left| \frac{g(z(\tau))}{z(\tau)} - c \right| \mathrm{d}\tau \le M_1,$$

and, without loss of generality, we may assume that T is chosen in such a way that (32) is valid, as well. Therefore, using (32) and (39), we get

$$\begin{aligned}
\int_1^t \left| \frac{g(z(\tau))}{z(\tau)} - c \right| \mathrm{d}\tau \le M_1 + M_0 \int_T^t z(\tau)\,\mathrm{d}\tau < \\
< M_1 + \sqrt{\frac{2}{c}} M_0 \int_T^t |z'(\tau)|\,\mathrm{d}\tau = M_1 - \sqrt{\frac{2}{c}} M_0 \int_T^t z'(\tau)\,\mathrm{d}\tau = \\
= M_1 + \sqrt{2c} M_0 (z(T) - z(t)), \quad t \ge T.
\end{aligned}$$

Letting $t \to \infty$ and using (31), we obtain (36). □

Lemma 11. *Assume that the condition* (26) *holds. Then*

$$\int_1^\infty \left(\frac{p'(\tau)}{p(\tau)} \right)^2 \mathrm{d}\tau < \infty. \tag{40}$$

Proof. The condition (26) implies that there exists $c_0 \in (0,\infty)$ such that

$$\lim_{t\to\infty} \frac{p'(t)}{t^{n-2}} = c_0, \quad \lim_{t\to\infty} \frac{p(t)}{t^{n-1}} = \frac{c_0}{n-1}.$$

Therefore

$$\lim_{t\to\infty} t^2 \left(\frac{p'(t)}{p(t)}\right)^2 = (n-1)^2.$$

Hence we can find $T \geq 1$ such that

$$\left(\frac{p'(t)}{p(t)}\right)^2 < \frac{n^2}{t^2}, \quad t \geq T, \tag{41}$$

and due to (5) and (6), we can find $M_3 \in (0,\infty)$ such that

$$\int_1^T \left(\frac{p'(\tau)}{p(\tau)}\right)^2 \mathrm{d}\tau \leq M_3.$$

Consequently,

$$\int_1^t \left(\frac{p'(\tau)}{p(\tau)}\right)^2 \mathrm{d}\tau < M_3 + n^2 \int_T^t \frac{\mathrm{d}\tau}{\tau^2} = n^2 \left(\frac{1}{T} - \frac{1}{t}\right), \quad t \geq T.$$

Letting $t \to \infty$, we get (40). $\square$

The main result on the asymptotic behavior of homoclinic solutions is contained in the following theorem.

Theorem 12. *Assume that* (25) *and* (26) *hold. Let $B < \bar{B}$ be such that the corresponding solution u of the initial problem* (1), (7) *is homoclinic. Then u fulfils the equation*

$$\lim_{t\to\infty} (L - u(t))\mathrm{e}^{\sqrt{c}t} \sqrt{p(t)} \in (0,\infty). \tag{42}$$

Remark 13. A similar asymptotic formula for positive solutions of equation (8), where $k > 1$ and $f(x) = x - |x|^r \operatorname{sign} x$, $r > 1$, has been derived in [13], Theorem 6.1.

Proof. Step 1. *Construction of an auxiliary linear differential system.* Consider the function z given by (27). According to (29), z satisfies

$$z'' + \frac{p'(t)}{p(t)} z' = \frac{g(z(t))}{z(t)} z(t), \quad t \in (0,\infty). \tag{43}$$

Having z at hand, we introduce the linear differential equation

$$v'' + \frac{p'(t)}{p(t)} v' = \frac{g(z(t))}{z(t)} v, \tag{44}$$

and the corresponding linear differential system

$$x_1' = x_2, \quad x_2' = \frac{g(z(t))}{z(t)} x_1 - \frac{p'(t)}{p(t)} x_2. \tag{45}$$

Denote

$$A(t) = (a_{i,j}(t))_{i,j\le 2} = \begin{pmatrix} 0 & 1 \\ \frac{g(z(t))}{z(t)} & -\frac{p'(t)}{p(t)} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}.$$

By (6), (31), (37) and (38),

$$A = \lim_{t\to\infty} A(t).$$

Eigenvalues of A are the numbers $\lambda = \sqrt{c}$ and $\mu = -\sqrt{c}$, eigenvectors of A are $\mathbf{l} = (1, \sqrt{c})$ and $\mathbf{m} = (1, -\sqrt{c})$, respectively. Denote

$$D(t) = \left(\frac{p'(t)}{2p(t)}\right)^2 + \frac{g(z(t))}{z(t)}, \quad t \in (0,\infty). \tag{46}$$

Then eigenvalues of $A(t)$ have the form

$$\lambda(t) = -\frac{p'(t)}{2p(t)} + \sqrt{D(t)}, \ \ \mu(t) = -\frac{p'(t)}{2p(t)} - \sqrt{D(t)}, \ \ t \in (0,\infty). \tag{47}$$

We can see that

$$\lim_{t\to\infty} \lambda(t) = \lambda, \quad \lim_{t\to\infty} \mu(t) = \mu.$$

Step 2. *Verification of the Assumptions of Theorem* 8. Due to (31) and (38), we can find $T \ge 1$ such that

$$0 < z(t) \le \eta, \quad D(t) > 0, \quad t \in (T,\infty). \tag{48}$$

Therefore, by (37) and (38),

$$a_{21}(t) = \frac{g(z(t))}{z(t)} \in AC_{loc}(T,\infty),$$

and hence

$$\left|\int_T^\infty \left(\frac{g(z(t))}{z(t)}\right)' \mathrm{d}t\right| = \left|\lim_{t\to\infty} \frac{g(z(t))}{z(t)} - \frac{g(z(T))}{z(T)}\right| = \left|c - \frac{g(z(T))}{z(T)}\right| < \infty.$$

Further, by (26), $a_{22}(t) = -\frac{p'(t)}{p(t)} \in AC_{loc}(T,\infty)$. Hence due to (6),

$$\left|\int_T^\infty \left(\frac{p'(t)}{p(t)}\right)' \mathrm{d}t\right| = \left|\lim_{t\to\infty} \frac{p'(t)}{p(t)} - \frac{p'(T)}{p(T)}\right| = \frac{p'(T)}{p(T)} < \infty.$$

Since $a_{11}(t) \equiv 0$ and $a_{12}(t) \equiv 1$, it is not difficult to see that (16) is satisfied. Using (47), we get $\operatorname{Re}(\lambda(t) - \mu(t)) = 2\sqrt{D(t)} > 0$ for $t \in (T,\infty)$. Since $\lim_{t\to\infty} \sqrt{D(t)} = \sqrt{c} > 0$, we have

$$\int_T^\infty \operatorname{Re}(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau = \infty, \quad \int_s^t \operatorname{Re}(\lambda(\tau) - \mu(\tau))\,\mathrm{d}\tau > 0, \quad T \le s < t.$$

Consequently, (18) is valid.

Step 3. *Application of Theorem* 8. By Theorem 8, there exists a fundamental system $\mathbf{x}(t) = (x_1(t), x_2(t))$, $\mathbf{y}(t) = (y_1(t), y_2(t))$ of solutions of (45) such that (20) is valid. Hence

$$\lim_{t\to\infty} x_1(t)\mathrm{e}^{-\int_T^t \lambda(\tau)\,\mathrm{d}\tau} = 1, \quad \lim_{t\to\infty} y_1(t)\mathrm{e}^{-\int_T^t \mu(\tau)\,\mathrm{d}\tau} = 1. \tag{49}$$

Using (47), for $t \ge T$ we get

$$\begin{aligned}\exp\left(-\int_T^t \lambda(\tau)\,\mathrm{d}\tau\right) &= \exp\left(\int_T^t \left(\frac{p'(\tau)}{2p(\tau)} - \sqrt{D(\tau)}\right)\mathrm{d}\tau\right) = \\ &= \exp\left(\frac{1}{2}\ln\frac{p(t)}{p(T)}\right)\exp\left(-\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau\right) = \\ &= \sqrt{\frac{p(t)}{p(T)}}\exp\left(-\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau\right),\end{aligned}$$

and

$$\begin{aligned}\exp\left(-\int_T^t \mu(\tau)\,\mathrm{d}\tau\right) &= \exp\left(\int_T^t \left(\frac{p'(\tau)}{2p(\tau)} + \sqrt{D(\tau)}\right)\mathrm{d}\tau\right) = \\ &= \exp\left(\frac{1}{2}\ln\frac{p(t)}{p(T)}\right)\exp\left(\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau\right) = \\ &= \sqrt{\frac{p(t)}{p(T)}}\exp\left(\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau\right).\end{aligned}$$

Further,

$$\int_T^t \sqrt{D(\tau)}\,\mathrm{d}\tau = E_0(t) + \sqrt{c}(t-T),$$

where

$$E_0(t) = \int_T^t \frac{D(\tau)-c}{\sqrt{D(\tau)}+\sqrt{c}}\,\mathrm{d}\tau, \quad t \ge T. \tag{50}$$

Hence

$$\exp\left(-\int_T^t \lambda(\tau)\,\mathrm{d}\tau\right) = \sqrt{\frac{p(t)}{p(T)}}\mathrm{e}^{-E_0(t)}\mathrm{e}^{-\sqrt{c}(t-T)}, \quad t \ge T, \tag{51}$$

$$\exp\left(-\int_T^t \mu(\tau)\,\mathrm{d}\tau\right) = \sqrt{\frac{p(t)}{p(T)}}\mathrm{e}^{E_0(t)}\mathrm{e}^{\sqrt{c}(t-T)}, \quad t \ge T. \tag{52}$$

Using (36), (40) and (46), we can find $K_0 \in (0,\infty)$ such that for $t \geq T$,

$$\int_T^t \left| \frac{D(\tau) - c}{\sqrt{D(\tau)} + \sqrt{c}} \right| \mathrm{d}\tau \leq$$
$$\leq \frac{1}{\sqrt{c}} \left(\int_T^t \left(\frac{p'(\tau)}{2p(\tau)} \right)^2 \mathrm{d}\tau + \int_T^t \left| \frac{g(z(\tau))}{z(\tau)} - c \right| \mathrm{d}\tau \right) \leq K_0.$$

Consequently, due to (50),

$$\lim_{t\to\infty} E_0(t) = E_0 \in \mathbb{R}.$$

Therefore (49), (51) and (52) imply

$$1 = \lim_{t\to\infty} x_1(t) \sqrt{\frac{p(t)}{p(T)}} \mathrm{e}^{-E_0} \mathrm{e}^{-\sqrt{c}(t-T)},$$
$$1 = \lim_{t\to\infty} y_1(t) \sqrt{\frac{p(t)}{p(T)}} \mathrm{e}^{E_0} \mathrm{e}^{\sqrt{c}(t-T)}.$$

Since by (26),

$$\lim_{t\to\infty} \sqrt{p(t)} \mathrm{e}^{-\sqrt{c}t} = \lim_{t\to\infty} \sqrt{\frac{p(t)}{t^{n-1}}} t^{(n-1)/2} \mathrm{e}^{-\sqrt{c}t} = 0,$$
$$\lim_{t\to\infty} \sqrt{p(t)} \mathrm{e}^{\sqrt{c}t} = \infty,$$

we obtain

$$\lim_{t\to\infty} x_1(t) = \infty, \quad \lim_{t\to\infty} y_1(t) = 0. \tag{53}$$

Step 4. *Asymptotic Formula.* According to (43), z is likewise a solution of (44). Therefore there are $c_1, c_2 \in \mathbb{R}$ such that $z(t) = c_1 x_1(t) + c_2 y_1(t)$, $t \in (0,\infty)$. Having in mind (30), (31), (49) and (53), we get $c_1 = 0$, $c_2 y_1(t) > 0$ on $(0,\infty)$, and $c_2 \in (0,\infty)$. Consequently, $z(t) = c_2 y_1(t)$ and

$$1 = \lim_{t\to\infty} \frac{1}{c_2} z(t) \sqrt{\frac{p(t)}{p(T)}} \mathrm{e}^{E_0} \mathrm{e}^{\sqrt{c}(t-T)},$$

which together with (27) yields (42). □

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NONLOCAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

Dedicated to the memory of Professor Temuri Chanturia

Abstract. We present the existence principle which can be used for a large class of nonlocal fractional boundary value problems of the form $({}^c\!D^\alpha x)(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t))$, $\Lambda(x) = 0$, $\Phi(x) = 0$, where ${}^c\!D$ is the Caputo fractional derivative. Here, $\alpha \in (1, 2)$, $\mu \in (0, 1)$, f is a L^q-Carathéodory function, $q > \frac{1}{\alpha-1}$, and $\Lambda, \Phi : C^1[0, T] \to \mathbb{R}$ are continuous and bounded ones. The proofs are based on the Leray–Schauder degree theory. Applications of our existence principle are given.

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რეზიუმე. შემოთავაზებულია ამოხსნადობის პრინციპი წილადურ სასაზღვრო ამოცანათა ფართო კლასისათვის, რომელთაც აქვთ შემდეგი სახე $({}^c\!D^\alpha x)(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t))$, $\Lambda(x) = 0$, $\Phi(x) = 0$, სადაც ${}^c\!D$ არის კაპუტოს წილადური წარმოებული, $\alpha \in (1, 2)$, $\mu \in (0, 1)$, f არის L^q-კარათეოდორის ფუნქცია, $q > \frac{1}{\alpha-1}$, ხოლო $\Lambda, \Phi : C^1[0, T] \to \mathbb{R}$ უწყვეტი, შემოსაზღვრული ფუნქციონალებია. მოყვანილია, აგრეთვე, აღნიშნული პრინციპის გამოყენებები. დამტკიცებები ეფუძნება ლერე–შაუდერის თეორიას ასახვათა ხარისხების შესახებ.

1. Introduction

Let $T > 0$ and $\mathbb{R}_+ = [0, \infty)$. As usual, L^q $(q \geq 1)$ is the set of functions whose qth powers of modulus are integrable on $[0, T]$ equipped with the norm $\|x\|_q = \left(\int_0^T |x(t)|^q \,\mathrm{d}t\right)^{\frac{1}{q}}$. $C[0, T]$ is equipped with the norm $\|x\| = \max\{|x(t)| : t \in [0, T]\}$.
Let $\mathcal{A}$ be a set of functionals $\Lambda : C^1[0, T] \to \mathbb{R}$, which are

(a) continuous,

(b) bounded, that is, $\Lambda(\Omega)$ is bounded for any bounded $\Omega \subset C^1[0, T]$.

We say that $\Lambda, \Phi \in \mathcal{A}$ satisfy *the compatibility condition* if for each $\nu \in [0, 1]$ there exists a solution of the problem

$$x'' = 0, \quad \Lambda(x) - \nu\Lambda(-x) = 0, \quad \Phi(x) - \nu\Phi(-x) = 0.$$

This is true if and only if the system

$$\begin{aligned} &\Phi(a + bt) - \nu\Phi(-a - bt) = 0, \\ &\Psi(a + bt) - \nu\Psi(-a - bt) = 0 \end{aligned} \tag{1.1}$$

has a solution $(a, b) \in \mathbb{R}^2$ for each $\nu \in [0, 1]$.

We say that the functionals $\Phi, \Psi \in \mathcal{A}$ satisfy *the admissible compatibility condition* if $\Phi and \Psi$ satisfy the compatibility condition and there exists a positive constant $L = L(\Phi, \Psi)$ such that $|a| \leq L$ and $|b| \leq L$ for each $\nu \in [0, 1]$ and each solution $(a, b) \in \mathbb{R}^2$ of system (1.1).

Remark 1.1. If the functionals $\Phi, \Psi : C^1[0, T] \to \mathbb{R}$ are linear and continuous, then $\Phi, \Psi \in \mathcal{A}$ and satisfy the compatibility condition. Indeed, system (1.1) is of the form

$$\begin{aligned} a\Phi(1) + b\Phi(t) = 0, \\ a\Psi(1) + b\Psi(t) = 0 \end{aligned}$$

for each $\nu \in [0, 1]$, and we see that it is always solvable in $\mathbb{R}^2$. The set of all its solutions (a, b) is bounded (that is, Φ, Ψ satisfy the admissible compatibility condition) if and only if $\Phi(1)\Psi(t) - \Phi(t)\Psi(1) \neq 0$.

We investigate the fractional boundary value problem

$$({}^c\!D^\alpha x)(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)), \tag{1.2}$$

$$\Phi(x) = 0, \quad \Psi(x) = 0, \tag{1.3}$$

where $\alpha \in (1, 2)$, $\mu \in (0, 1)$, f is an L^q-Carathéodory function on $[0, T] \times \mathbb{R}^3$, $q > \frac{1}{\alpha - 1}$, and where $\Phi, \Psi \in \mathcal{A}$ satisfy the admissible compatibility condition.

We say that a function $x \in C^1[0, T]$ is *a solution of problem* (1.2), (1.3) if ${}^c\!D^\alpha x \in L^q[0, T]$, x satisfies the boundary conditions (1.3), and (1.2) holds for a.e. $t \in [0, T]$.

Note that if x is a solution of problem (1.2), (1.3), then ${}^c\!D^\mu x \in C[0, T]$ (see Lemma 2.5).

The Caputo fractional derivative ${}^{c}\!D^{\gamma}v$ *of order* $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $v : [0,T] \to \mathbb{R}$ is defined by the formula [10, 15, 18]

$$({}^{c}\!D^{\gamma}v)(t) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\gamma-1} \left(v(s) - \sum_{k=0}^{n-1} \frac{v^{(k)}(0)}{k!} s^k \right) \mathrm{d}s,$$

where $n = [\gamma] + 1$ and $[\gamma]$ denotes the integral part of γ, and Γ is the Euler gamma function.

We recall that a function $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is an *L^q-Carathéodory function on* $[0,T] \times \mathbb{R}^3$ if

(i) for each $(x,y,z) \in \mathbb{R}^3$, the function $f(\cdot,x,y,z) : [0,T] \to \mathbb{R}$ is measurable,

(ii) for a.e. $t \in [0,T]$, the function $f(t,\cdot,\cdot,\cdot) : \mathbb{R}^3 \to \mathbb{R}$ is continuous,

(iii) for each compact set $\mathcal{U} \subset \mathbb{R}^3$, there exists $w_{\mathcal{U}} \in L^q[0,T]$ such that $|f(t,x,y,z)| \le w_{\mathcal{U}}(t)$ for a.e. $t \in [0,T]$ and all $(x,y,z) \in \mathcal{U}$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. We can find numerous applications in porous media, electromagnetic, fluid mechanics, viskoelasticity, edge detection, and so on. (For examples and details, see [7, 8, 10, 13, 14, 15, 18, 23, 27] and references therein). There has been a significant development in the study of fractional differential equations in recent years. The authors discuss regular (see, e.g., [4, 6, 11, 12, 17]) and singular (see, e.g., [2, 5, 19, 26, 28]) fractional boundary value problems. These problems are usually investigated with the two-point boundary conditions, multipoint boundary conditions and also with nonlocal boundary conditions (see, e.g., [3, 6]). Paper [3] deals with the integral boundary conditions

$$ax(0) + bx'(0) = \int_0^1 q_1(x(s))\,\mathrm{d}s, \quad ax(1) + bx'(1) = \int_0^1 q_2(x(s))\,\mathrm{d}s,$$

while that of [6] with the conditions

$$x(0) = \Theta(x), \quad x(T) = x_T,$$

where $\Theta : C[0,T] \to \mathbb{R}$ is a continuous functional and $x_T \in \mathbb{R}$. The existence results are proved by: the Banach, Schauder, Krasnosel'skii and Leggett-Williams fixed point theorems, fixed point theorems on cones, a mixed monotone method, the Leray–Schauder nonlinear alternative, the lower and upper solution method and by fixed point index theory.

The aim of the present paper is to give the existence principle for solving the problem (1.2), (1.3) and to show its applications. We note that unlike the paper dealing with fractional differential equations for $1 < \alpha < 2$ (with the exception of [2, 16]), the nonlinearity f in (1.2) depends on the derivative of x. Due to this fact, we have to assume that f is an L^q-Carathéodory function with $q > \frac{1}{\alpha-1}$. The existence principle is proved by the Leray–Schauder degree theory (see, e.g., [9]). Note that our existence principle is closely related to that given in [24] for n-order differential equations, in [1, 20, 21, 22] for second-order differential equations and in [25] for second-order differential systems.

From now on, we assume that

$$(P) \qquad \mu \in (0,1), \quad \alpha \in (1,2), \quad q > \frac{1}{\alpha - 1} \quad \text{and} \quad p = \frac{q}{q-1}.$$

Then $\frac{1}{p} + \frac{1}{q} = 1$ and $(\alpha - 2)p + 1 > 0$.

The paper is organized as follows. Section 2 contains technical lemmas that are used in the subsequent sections. Section 3 presents the existence principle for solving the problem (1.2), (1.3). It is shown that the solvability of this problem is reduced to the existence of a fixed point of an integral operator. The existence of its fixed point is proved by the Leray–Schauder degree theory. In Section 4, we apply the existence principle for two sets of admissible boundary conditions. Examples demonstrate our results.

2. Preliminaries

In this section we state technical lemmas and results which are used in the subsequent sections. Lemmas 2.1, 2.2 and 2.4–2.6 are proved in [2]. Note that condition (P) holds in this and in the next sections.

Lemma 2.1. *Suppose $\gamma \in L^q[0,T]$. Then*

$$(a) \quad \int_0^t (t-s)^{\alpha-2}\gamma(s)\,\mathrm{d}s \quad \textit{is continuous on } [0,T],$$

$$(b) \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{\alpha-1}\gamma(s)\,\mathrm{d}s = (\alpha-1)\int_0^t (t-s)^{\alpha-2}\gamma(s)\,\mathrm{d}s \quad \textit{for } t \in [0,T].$$

Lemma 2.2. *Let $\{\rho_n\} \subset L^q[0,T]$ be L^q-convergent and let $\lim_{n\to\infty}\rho_n = \rho$. Then*

$$\lim_{n\to\infty}\int_0^t (t-s)^{\alpha-2}\rho_n(s)\,\mathrm{d}s = \int_0^t (t-s)^{\alpha-2}\rho(s)\,\mathrm{d}s \quad \textit{uniformly on } [0,T].$$

Corollary 2.3. *Suppose the assumptions of Lemma 2.2 are satisfied. Let $\{\lambda_n\} \subset [0,1]$ be convergent and $\lim_{n\to\infty}\lambda_n = \lambda$. Then*

$$\lim_{n\to\infty}\lambda_n\int_0^t (t-s)^{\alpha-2}\rho_n(s)\,\mathrm{d}s = \lambda\int_0^t (t-s)^{\alpha-2}\rho(s)\,\mathrm{d}s \quad \textit{uniformly on } [0,T].$$

Proof. The result follows from Lemma 2.2, where ρ_n is replaced by $\lambda_n\rho_n$ (note that $\lim_{n\to\infty}\lambda_n\rho_n = \lambda\rho$ in $L^q[0,T]$). $\square$

Lemma 2.4. *Let $\gamma \in L^q[0,T]$. Then solutions of the fractional differential equation*

$$^cD^\alpha x(t) = \gamma(t) \tag{2.1}$$

belong to the class $C^1[0,T]$, and

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\gamma(s) + x(0) + x'(0)t$$

are all solutions of (2.1).

Lemma 2.5. *Let $x \in C^1[0,T]$. Then*

$$^cD^\mu x(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t (t-s)^{-\mu} x'(s)\,\mathrm{d}s \quad \textit{for } t \in [0,T]$$

and $^cD^\mu x \in C[0,T]$.

Lemma 2.6. *Suppose that $\eta \in L^q[0,T]$ and $0 \le t_1 < t_2 \le T$. Then*

$$\left| \int_0^{t_2} (t_2-s)^{\alpha-2}\eta(s)\,\mathrm{d}s - \int_0^{t_1} (t_1-s)^{\alpha-2}\eta(s)\,\mathrm{d}s \right| \le$$
$$\le \left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d} \right)^{\frac{1}{p}} \|\eta\|_q + \left(\frac{(t_2-t_1)^d}{d} \right)^{\frac{1}{p}} \|\eta\|_q,$$

where $d = (\alpha-2)p+1$.

3. An Existence Principle

Suppose

$$f \text{ is a } L^q\text{-Carathéodory function on } [0,T]\times\mathbb{R}^3. \tag{3.1}$$

If $x \in C^1[0,T]$, then $^cD^\mu x \in C[0,T]$ by Lemma 2.5. Therefore the function $f(t,x(t),x'(t),(^cD^\mu x)(t))$ belongs to the set $L^q[0,T]$. Hence by Lemma 2.4, $x \in C^1[0,T]$ is a solution of (1.2) if and only if

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s),x'(s),(^cD^\mu x)(s))\,\mathrm{d}s + a + bt, \quad t \in [0,T], \tag{3.2}$$

where $a, b \in \mathbb{R}$. Let $\Phi, \Psi \in \mathcal{A}$. Define an operator $\mathcal{S} : C^1[0,T] \to C^1[0,T]$ by the formula

$$(\mathcal{S}x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s),x'(s),(^cD^\mu x)(s))\,\mathrm{d}s +$$
$$+ x(0) - \Phi(x) + (x'(0) - \Psi(x))t.$$

It is easy to check that if x is a fixed point of the operator $\mathcal{S}$, then equality (3.2) is fulfilled with $a = x(0) - \Phi(x)$, $b = x'(0) - \Psi(x)$, and $\Phi(x) = 0$, $\Psi(x) = 0$. Consequently, any fixed point x of $\mathcal{S}$ is a solution of problem (1.2), (1.3).

The following result is the existence principle for solving the problem (1.2), (1.3).

Theorem 3.1. *Let $\Phi, \Psi \in \mathcal{A}$ satisfy the admissible compatibility condition. Suppose that* (3.1) *holds and there exists a positive constant S such that*

$$\|x\| < S, \quad \|x'\| < S$$

for each $\lambda \in [0,1]$ and each solution x of the problem

$$\left.\begin{aligned} ({}^c\!D^\alpha x)(t) &= \lambda f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)), \\ \Phi(x) &= 0, \quad \Psi(x) = 0. \end{aligned}\right\} \tag{3.3}$$

Then problem (1.2), (1.3) *has a solution.*

Proof. We first note that since $\Phi, \Psi \in \mathcal{A}$ satisfy the admissible compatibility condition, system (1.1) has a solution for each $\nu \in [0,1]$ and there is a positive constant K such that $|a| \le K$ and $|b| \le K$ for each $\nu \in [0,1]$ and each solution $(a,b) \in \mathbb{R}^2$ of (1.1). Set

$$\Omega = \{x \in C^1[0,T] : \|x\| < S + (1+T)K,\ \|x'\| < S + K\}.$$

Then Ω is an open, bounded and symmetric with respect to $0 \in C^1[0,T]$ subset of the Banach space $C^1[0,T]$. We know that any fixed point of $\mathcal{S}$ is a solution of problem (1.2), (1.3). If

$$\mathrm{D}(\mathcal{I} - \mathcal{S}, \Omega, 0) \neq 0, \tag{3.4}$$

where "D" stands for the Leray–Schauder degree and $\mathcal{I}$ is the identical operator on $C^1[0,T]$, then $\mathcal{S}$ has a fixed point by the Leray–Schauder degree method. Hence to prove our theorem we need to show that (3.4) holds. To this end, define an operator $\mathcal{K} : [0,1] \times \overline{\Omega} \to C^1[0,T]$ by the formula

$$\begin{aligned} \mathcal{K}(\lambda, x)(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s + \\ + x(0) - \Phi(x) + (x'(0) - \Psi(x))t. \end{aligned}$$

Then $\mathcal{K}(1,\cdot) = \mathcal{S}$. We prove that $\mathcal{K}$ is a compact operator. We start with the proof that $\mathcal{K}$ is continuous. Let $\{\lambda_n\} \subset [0,1]$ and $\{x_n\} \subset \overline{\Omega}$ be convergent and let $\lim_{n\to\infty} \lambda_n = \lambda$, $\lim_{n\to\infty} x_n = x$. Let us put

$$\gamma_n(t) = f(t, x_n(t), x_n'(t), ({}^c\!D^\mu x_n)(t)), \quad \gamma(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)),$$

$$z_n(t) = \frac{\lambda_n}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma_n(s)\,\mathrm{d}s, \quad z(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(s)\,\mathrm{d}s.$$

We conclude from Lemma 2.5 that $\lim_{n\to\infty} {}^cD^\mu x_n = {}^cD^\mu x$ in $C[0,T]$ and

$$\|{}^cD^\mu x_n\| \le \frac{\|x_n'\|}{\Gamma(1-\mu)} \max\left\{ \int_0^t (t-s)^{-\mu}\,\mathrm{d}s : t\in[0,T]\right\} \le \le \frac{(S+K)T^{1-\mu}}{\Gamma(2-\mu)} \tag{3.5}$$

for $n\in\mathbb{N}$. Hence

$$\lim_{n\to\infty} \gamma_n(t) = \gamma(t) \quad \text{for a.e. } t\in[0,T], \tag{3.6}$$

and since f fulfils (3.1), $\{x_n\}$ is bounded in $C^1[0,T]$ and $\{{}^cD^\mu x_n\}$ is bounded in $C[0,T]$, there exists $w\in L^q[0,T]$ such that

$$|\gamma_n(t)| \le w(t) \quad \text{for a.e. } t\in[0,T] \text{ and all } n\in\mathbb{N}. \tag{3.7}$$

Therefore, $\lim_{n\to\infty} \|\gamma_n-\gamma\|_q = 0$ by the dominated convergence theorem in $L^q[0,T]$. Consequently, by Corollary 2.3 and Lemma 2.1(b),

$$\begin{aligned}\lim_{n\to\infty} z_n'(t) &= \lim_{n\to\infty} \frac{\lambda_n}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}\gamma_n(s)\,\mathrm{d}s = \\ &= \frac{\lambda}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}\gamma(s)\,\mathrm{d}s = \\ &= z'(t) \qquad \text{uniformly on } [0,T].\end{aligned}$$

As a result, $\lim_{n\to\infty} z_n = z$ in $C^1[0,T]$ since $z_n(0)=z(0)=0$. The continuity of $\mathcal{K}$ follows now from the equalities $\mathcal{K}(\lambda_n,x_n)(t) = z_n(t)+x_n(0)-\Phi(x_n)+(x_n'(0)-\Psi(x_n))t$, $\mathcal{K}(\lambda,x)(t) = z(t)+x(0)-\Phi(x)+(x'(0)-\Psi(x))t$ and from

$$\lim_{n\to\infty}(x_n(0)-\Phi(x_n)) = x(0)-\Phi(x), \quad \lim_{n\to\infty}(x_n(0)-\Psi(x_n)) = x(0)-\Psi(x).$$

We now prove that the set $\mathcal{K}([0,1]\times\overline{\Omega})$ is relatively compact in $C^1[0,T]$. Since the set $\{x(0)-\Phi(x)+(x'(0)-\Psi(x))t : x\in\overline{\Omega}\}$ is relatively compact in $\mathbb{R}$, which immediately follows from the properties of Φ and Ψ, it suffices to show that the set

$$\mathcal{B} = \left\{ \lambda \int_0^t (t-s)^{\alpha-1} f(s,x(s),x'(s),({}^cD^\mu x)(s))\,\mathrm{d}s : \lambda\in[0,1],\, x\in\overline{\Omega}\right\}$$

is relatively compact in $C^1[0,T]$. Since ${}^cD^\mu x\in C[0,T]$ for $x\in C^1[0,T]$ and (cf. (3.5)) $\|{}^cD^\mu x\| \le \frac{T^{1-\mu}\|x'\|}{\Gamma(2-\mu)}$ for $x\in\overline{\Omega}$, there exists $\rho\in L^q[0,T]$ such that

$$|f(t,x(t),x'(t),({}^cD^\mu x)(t))| \le \rho(t) \quad \text{for a.e. } t\in[0,T] \text{ and all } x\in\overline{\Omega}. \tag{3.8}$$

The boundedness of $\mathcal{B}$ in $C^1[0,T]$ follows from the relations (for $t \in [0,T]$ and $x \in \overline{\Omega}$)

$$\left| \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s \right| \le T^{\alpha-1} \|\rho\|_1$$

and

$$\begin{aligned}\left| \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s \right| &= \\ = (\alpha-1) \left| \int_0^t (t-s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s \right| &\le \\ \le (\alpha-1) \int_0^t (t-s)^{\alpha-2} \rho(s)\,\mathrm{d}s &\le \\ \le (\alpha-1) \left(\int_0^t (t-s)^{(\alpha-2)p}\,\mathrm{d}s \right)^{\frac{1}{p}} \left(\int_0^t \rho^q(s)\,\mathrm{d}s \right)^{\frac{1}{q}} &\le \\ \le (\alpha-1) \left(\frac{T^{(\alpha-2)p+1}}{(\alpha-2)p+1} \right)^{\frac{1}{p}} \|\rho\|_q,&\end{aligned}$$

where the Hölder inequality is used. Furthermore, for $0 \le t_1 < t_2 \le T$ and $x \in \overline{\Omega}$, Lemma 2.6 (for $\eta(t) = f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)))$ gives

$$\begin{aligned}&\left| \int_0^{t_2} (t_2-s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s \,- \right. \\ &\qquad \left. - \int_0^{t_1} (t_1-s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s \right| \le \\ &\qquad \le \left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d} \right)^{\frac{1}{p}} \|\rho\|_q + \left(\frac{(t_2-t_1)^d}{d} \right)^{\frac{1}{p}} \|\rho\|_q\end{aligned}$$

since $\|\eta\|_q \le \|\rho\|_q$. Here $d = (\alpha-2)p+1$. Hence the set $\{y' : y \in \mathcal{B}\}$ is equicontinuous on $[0,T]$, and thus $\mathcal{B}$ is relatively compact in $C^1[0,T]$ by the Arzelà-Ascoli theorem. To summarize, $\mathcal{K}$ is a compact operator.

Suppose now that $\mathcal{K}(\lambda_*, x_*) = x_*$ for some $\lambda_* \in [0,T]$ and some $x_* \in \overline{\Omega}$. Let $\gamma_*(t) = f(t, x_*(t), x_*'(t), ({}^c\!D^\mu x_*)(t))$ for a.e. $t \in [0,T]$. Then $\gamma_* \in L^q[0,T]$ and the equality

$$x_*(t) = \frac{\lambda_*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma_*(s)\,\mathrm{d}s + x_*(0) - \Phi(x_*) + (x_*'(0) - \Psi(x_*))t \quad (3.9)$$

holds for $t \in [0,T]$. Hence $\Phi(x_*) = 0$, $\Psi(x_*) = 0$ and, by Lemma 2.4,

$$({}^c\!D^\alpha x_*)(t) = \lambda_* \gamma_*(t) \quad \text{for a.e. } t \in [0,T].$$

Hence x_* is a solution of problem (3.3) with $\lambda = \lambda_*$, and $\|x_*\| < S$, $\|x_*'\| < S$ by the assumptions. As a result, $\mathcal{K}(\lambda, x) \neq x$ for each $\lambda \in [0,1]$ and each $x \in \partial\Omega$. Therefore, by the homotopy property,

$$\mathrm{D}(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) = \mathrm{D}(\mathcal{I} - \mathcal{K}(1,\cdot), \Omega, 0). \tag{3.10}$$

We will now proceed to showing that

$$\mathrm{D}(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) \neq 0. \tag{3.11}$$

Let us define a compact operator $\mathcal{L} : [0,1] \times \overline{\Omega} \to C^1[0,T]$ as

$$\mathcal{L}(\nu, x) = x(0) + \Phi(x) - \nu\Phi(-x) + (x'(0) + \Psi(x) - \nu\Psi(-x))t.$$

Then $\mathcal{L}(1,\cdot)$ is odd (i.e., $\mathcal{L}(1,-x) = -\mathcal{L}(1,x)$ for $x \in \overline{\Omega}$) and

$$\mathcal{L}(0,\cdot) = \mathcal{K}(0,\cdot). \tag{3.12}$$

If $\mathcal{L}(\nu_1, x_1) = x_1$ for some $(\nu_1, x_1) \in [0,1] \times \overline{\Omega}$, then

$$x_1(t) = x_1(0) + \Phi(x_1) - \nu_1\Phi(-x_1) + (x_1'(0) + \Psi(x_1) - \nu_1\Psi(-x_1))t, \tag{3.13}$$

and therefore $x_1(t) = a + bt$ for $t \in [0,T]$, where $a = x_1(0) + \Phi(x_1) - \nu_1\Phi(-x_1)$ and $b = (x_1'(0) + \Psi(x_1) - \nu_1\Psi(-x_1))t$. Let $t = 0$ in $x_1(t)$ and $x_1'(t)$, where x_1 is given in (3.13), and have

$$\Phi(x_1) - \nu_1\Phi(-x_1) = 0, \quad \Psi(x_1) - \nu_1\Psi(-x_1) = 0,$$

which is system (1.1) with $\nu = \nu_1$. Hence due to the first part of the proof, the inequalities $|a| \le K$ and $|b| \le K$ are fulfilled. Consequently, $\|x_1\| \le (1+T)K$ and $\|x_1'\| \le K$, and thus $x_1 \notin \partial\Omega$. Next, by the homotopy property and the Borsuk antipodal theorem,

$$\mathrm{D}(\mathcal{I} - \mathcal{L}(0,\cdot), \Omega, 0) = \mathrm{D}(\mathcal{I} - \mathcal{L}(1,\cdot), \Omega, 0) \text{ and } \mathrm{D}(\mathcal{I} - \mathcal{L}(1,\cdot), \Omega, 0) \neq 0.$$

The last relations together with (3.12) give that (3.11) holds. Finally, we conclude that from (3.10) and (3.11) follows (3.4). □

4. Applications of the Existence Principle

4.1. **Functionals satisfying the admissible complementary condition.** We give two sets of nonlinear functionals $\Lambda, \Phi \in \mathcal{A}$ satisfying the admissible complementary condition. Such functionals in the nonlocal boundary conditions (1.3) will be used in the next subsection for solving problem (1.2), (1.3) by means of our existence principle.

For $j = 0, 1$, let $\mathcal{B}_j$ be the set of functionals $\Lambda \in \mathcal{A}$ for which there exists a positive constant $K = K(\Lambda)$ such that

$$x \in C^1[0,T],\ |x^{(j)}| \ge K \text{ on } [0,T] \Rightarrow \Lambda(x)\,\mathrm{sign}(x^{(j)}) > 0$$

Remark 4.1. The functionals from the set $\mathcal{B}_j$ have the following important property: If $\Lambda \in \mathcal{B}_j$ and $\Lambda(x) = 0$ for some $x \in C^1[0,T]$ and $j \in \{0,1\}$, then there exists $\xi \in [0,T]$ such that $|x^{(j)}(\xi)| < K(\Lambda)$.

Example 4.2. Let $0 \le a < b \le T$, $j \in \{0,1\}$, $\Theta : C^1[0,T] \to \mathbb{R}$ be continuous and $\sup\{|\Theta(x)| : x \in C^1[0,T]\} < \infty$. Then the functionals

$$\Lambda_1(x) = \min\{x^{(j)}(t) : a \le t \le b\} + \Theta(x),$$
$$\Lambda_2(x) = \max\{x^{(j)}(t) : a \le t \le b\} + \Theta(x),$$
$$\Lambda_3(x) = \int_a^b \max\{x^{(j)}(s) : a \le s \le t\}\,\mathrm{d}t + \Theta(x)$$

belong to the set $\mathcal{B}_j$. If $0 \le t_1 < t_2 < \cdots < t_n \le T$, $b_i > 0$, $g, f_i \in C(\mathbb{R})$ and $\lim_{u\to\pm\infty} g(u) = \lim_{u\to\pm\infty} f_i(u) = \pm\infty$, $i = 1,2,\dots,n$, then the functionals

$$\Lambda_4(x) = \sum_{i=1}^n b_i g(x^{(j)}(t_i)) + \Theta(x),$$
$$\Lambda_5(x) = \int_a^b \left(\sum_{i=1}^n b_i f_i(x^{(j)}(s))\right) \mathrm{d}s + \Theta(x),$$
$$\Lambda_6(x) = \int_a^b \left(\int_a^s \left(\sum_{i=1}^n b_i f_i(x^{(j)}(\xi))\right) \mathrm{d}\xi\right) \mathrm{d}s + \Theta(x)$$

also belong to $\mathcal{B}_j$.

Lemma 4.3. *Let $\Phi \in \mathcal{B}_0$ and $\Psi \in \mathcal{B}_1$. Then Φ, Ψ satisfy the admissible compatibility condition.*

Proof. Since $\Phi \in \mathcal{B}_0$ and $\Psi \in \mathcal{B}_1$, there exists a positive constant K such that for each $\nu \in [0,1]$ we have $[\Phi(a+bt) - \nu\Phi(-a-bt)]\mathrm{sign}(a+bt) > 0$ if $|a+bt| \ge K$ for $t \in [0,T]$ and $[\Psi(a+bt) - \nu\Psi(-a-bt)]\mathrm{sign}(b) > 0$ if $|b| \ge K$. Hence if $(a_0,b_0) \in \mathbb{R}^2$ is a solution of system (1.1) for some $\nu \in [0,1]$, then (see Remark 4.1) $|b_0| < K$ and $|a_0 + b_0\xi| < K$ for some $\xi \in [0,T]$. From the inequality $|a_0| \le |a_0 + b_0\xi| + |b_0\xi| < (1+K)T$ we see that for each $\nu \in [0,1]$, any solution $(a,b) \in \mathbb{R}^2$ of (1.1) satisfies the estimate

$$|a| < (1+K)T, \quad |b| < K. \tag{4.1}$$

Let $M = \{(a,b) \in \mathbb{R}^2 : |a| < (1+K)T,\ |b| < K\}$ and $\mathcal{F} : [0,1] \times \overline{M} \to \mathbb{R}^2$ be defined as

$$\mathcal{F}(\nu,a,b) = \left(\Phi(a+bt) - \nu\Phi(-a-bt),\ \Psi(a+bt) - \nu\Psi(-a-bt)\right).$$

Then $\mathcal{F}$ is a continuous operator and M is an open, bounded and symmetric with respect to $(0,0) \in \mathbb{R}^2$ subset of $\mathbb{R}^2$. We have also $\mathcal{F}(\nu,a,b) \ne (0,0)$ for $\nu \in [0,1]$ and $(a,b) \in \partial M$, and $\mathcal{F}(1,\cdot,\cdot)$ is an odd operator (that is, $\mathcal{F}(1,-a,-b) = -\mathcal{F}(1,a,b)$ for $(a,b) \in \overline{M}$). Hence by the Borsuk antipodal theorem and the homotopy property,

$$\deg(\mathcal{F}(1,\cdot,\cdot), M, 0) \ne 0,$$
$$\deg(\mathcal{F}(1,\cdot,\cdot), M, 0) = \deg(\mathcal{F}(\nu,\cdot,\cdot), M, 0) \quad \text{for } \nu \in [0,1],$$

where "deg" stands for the Brower degree. Consequently, the operator equation $\mathcal{F}(\nu, a, b) = (0, 0)$ has a solution for each $\nu \in [0, 1]$. Hence for each $\nu \in [0, 1]$ system (1.1) has a solution and any its solution (a, b) satisfies (4.1), and therefore Φ, Ψ satisfy the admissible complementary condition. $\square$

Remark 4.4. The special cases of the boundary conditions (1.3) are:

(a) the Dirichlet conditions $x(0) = A$, $x(T) = B$ (for $\Phi(x) = x(0) - A$, $\Psi(x) = \int_0^T x'(s)\,\mathrm{d}s + A - B$),

(b) the mixed conditions $x(0) = A$, $x'(T) = B$ (for $\Phi(x) = x(0) - A$, $\Psi(x) = x'(T) - B$) and $x'(0) = A$, $x(T) = B$ (for $\Phi(x) = x(T) - B$, $\Psi(x) = x'(0) - A$),

(c) the antiperiodic conditions $x(0) + x(T) = 0$, $x'(0) + x'(T) = 0$ (for $\Phi(x) = x(0) + x(T)$, $\Psi(x) = x'(0) + x'(T)$),

(d) the initial conditions $x(\xi) = A$, $x'(\xi) = B$, where $\xi \in [0, T]$ (for $\Phi(x) = x(\xi) - A$, $\Psi(x) = x'(\xi) - B$),

(e) the multipoint conditions $\sum\limits_{j=0}^{n} a_j x^{2l_j - 1}(t_j) = A$, $\sum\limits_{i=0}^{m} b_i \left(x'(s_i)\right)^{2k_i - 1} = B$, where $a_j, b_i \in (0, \infty)$, $l_j, k_i \in \mathbb{N}$ $(j = 0, \dots, n,\ i = 0, \dots, m)$, $0 \le t_0 < t_1 < \cdots < t_n \le T$, $0 \le s_0 < s_1 < \cdots < s_m \le T$ (for $\Phi(x) = \sum\limits_{j=0}^{n} a_j x^{2l_j - 1}(t_j) - A$, $\Psi(x) = \sum\limits_{i=0}^{m} b_i (x'(s_i))^{2k_i - 1} - B$).

Let $\mathcal{C}$ be the set of functionals $\Lambda \in \mathcal{A}$ such that $\sup\{|\Lambda(x)| : x \in C^1[0, T]\} < \infty$.

Lemma 4.5. *Let $0 \le \xi < \eta \le T$, $\Lambda_1, \Lambda_2 \in \mathcal{C}$ and*

$$\Phi(x) = x(\xi) + \Lambda_1(x),\ \ \Psi(x) = x(\eta) + \Lambda_2(x) \quad \textit{for } x \in C^1[0, T].$$

Then Φ, Ψ satisfy the admissible compatibility condition.

Proof. Since $\Lambda_1, \Lambda_2 \in \mathcal{C}$, there is a positive constant S such that $|\Lambda_1(x)| < S$ and $|\Lambda_1(x)| < S$ for $x \in C^1[0, T]$. System (1.1) has the form

$$\begin{aligned} &(1 + \nu)(a + b\xi) + \Lambda_1(a + bt) - \nu\Lambda_1(-a - bt) = 0, \\ &(1 + \nu)(a + b\eta) + \Lambda_2(a + bt) - \nu\Lambda_2(-a - bt) = 0. \end{aligned} \tag{4.2}$$

Suppose that $(a_0, b_0) \in \mathbb{R}^2$ is a solution of (4.2) for some $\nu \in [0, 1]$. Then

$$(1+\nu)(\eta-\xi)b_0 = \Lambda_1(a_0+b_0t)-\nu\Lambda_1(-a_0-b_0t)-\Lambda_2(a_0+b_0t)+\nu\Lambda_2(-a_0-b_0t),$$

and consequently, $|b_0| < \frac{2S}{\eta - \xi}$. Since

$$a_0 = -b_0\xi + \frac{1}{1 + \nu}[\nu\Lambda_1(-a_0 - b_0t) - \Lambda_1(a_0 + b_0t)],$$

we have

$$|a_0| < \frac{2ST}{\eta - \xi} + S = S\left(1 + \frac{2T}{\eta - \xi}\right).$$

As a result, for each $\nu \in [0,1]$, any solution $(a,b) \in \mathbb{R}^2$ of (4.2) satisfies the estimate

$$|a| < S\left(1 + \frac{2T}{\eta - \xi}\right), \ |b| < \frac{2S}{\eta - \xi}. \tag{4.3}$$

Put $M = \left\{(a,b) \in \mathbb{R}^2 : |a| < S\left(1 + \frac{2T}{\eta-\xi}\right), \ |b| < \frac{2S}{\eta-\xi}\right\}$. In order to prove that Φ, Ψ satisfy the admissible compatibility condition we define a continuous operator $\mathcal{F} : [0,1] \times \overline{M} \to \mathbb{R}^2$ by

$$\mathcal{F}(\nu, a, b) = \Big((1+\nu)(a + b\xi) + \Lambda_1(a + bt) - \nu\Lambda_1(-a - bt),$$
$$(1+\nu)(a + b\eta) + \Lambda_2(a + bt) - \nu\Lambda_2(-a - bt)\Big).$$

Then $\mathcal{F}(1,\cdot,\cdot)$ is an odd operator and $\mathcal{F}(\nu,a,b) \neq (0,0)$ for all $\nu \in [0,1]$ and $(a,b) \in \partial\mathcal{M}$. By the Borsuk antipodal theorem and by the homotopy property, we can prove just as in the proof of Lemma 4.3 that for each $\nu \in [0,1]$, the equation $\mathcal{F}(\nu,a,b) = (0,0)$ has a solution. Consequently, system (4.2) has a solution for each $\nu \in [0,1]$ and all its solutions (a,b) satisfy (4.3). Hence Φ, Ψ satisfy the admissible compatibility condition. □

4.2. **Existence results for nonlocal fractional BVPs.** Bearing in mind Section 4.1, we work with the boundary conditions

$$\Phi(x) = 0, \ \Psi(x) = 0, \quad \Phi \in \mathcal{B}_0, \ \Psi \in \mathcal{B}_1, \tag{4.4}$$

and

$$x(\xi) + \Lambda_1(x) = 0, \ x(\eta) + \Lambda_2(x) = 0, \quad 0 \le \xi < \eta \le T, \ \Lambda_1, \Lambda_2 \in \mathcal{C}. \tag{4.5}$$

Lemmas 4.3 and 4.5 show that the functionals Φ, Ψ in (4.4) and the functionals $x(\xi) + \Lambda_1(x)$, $x(\eta) + \Lambda_2(x)$ in (4.5) satisfy the admissible compatibility condition. We discuss the solvability of problems (1.2), (4.4) and (1.2), (4.5) by the existence principle (Theorem 3.1).

Theorem 4.6. *Let* (3.1) *hold. Suppose that the estimate*

$$|f(t,x,y,z)| \le \rho(t)p(|x|,|y|,|z|)$$
$$\text{for a.e. } t \in [0,T] \text{ and all } (x,y,z) \in \mathbb{R}^3 \tag{4.6}$$

is fulfilled, where $\rho \in L^q[0,T]$ and $p \in C(\mathbb{R}^3_+)$ are nonnegative, p is nondecreasing in all its arguments and

$$\lim_{u \to \infty} \frac{p(u,u,u)}{u} = 0. \tag{4.7}$$

Then problems (1.2), (4.4) *and* (1.2), (4.5) *are solvable.*

Proof. By Theorem 3.1, we have to prove that there exists a positive constant S such that

$$\|x\| < S, \quad \|x'\| < S \tag{4.8}$$

for each $\lambda \in [0,1]$ and each solution x of the problems

$$({}^c\!D^\alpha x)(t) = \lambda f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)), \quad (4.4), \tag{4.9}$$

and

$$({}^c\!D^\alpha x)(t) = \lambda f(t, x(t), x'(t), ({}^c\!D^\mu x)(t)), \quad (4.5). \tag{4.10}$$

We start with problem (4.9). Let $x \in C^1[0,T]$ be a solution of (4.9). Since $\Phi \in \mathcal{B}_0$ and $\Psi \in \mathcal{B}_1$, there exists a positive constant K such that (cf. Remark 4.1) $|x(\xi_0)| < K$, $|x'(\xi_1)| < K$ for some $\xi_0, \xi_1 \in [0,T]$. Furthermore, by Lemma 2.5, ${}^c\!D^\mu x \in C[0,T]$ and

$$\|{}^c\!D^\mu x\| \le \frac{\|x'\|}{\Gamma(1-\mu)} \left\| \int_0^t (t-s)^{-\mu}\,\mathrm{d}s \right\| \le V\|x'\|,$$

where $V = \frac{T^{1-\mu}}{\Gamma(1-\mu)}$. From the equality $x(t) = x(\xi_0) + \int_{\xi_0}^t x'(s)\,\mathrm{d}s$ we get

$$\|x\| < K + T\|x'\|. \tag{4.11}$$

From estimate (4.6) it follows now that

$$|f(t, x(t), x'(t), ({}^c\!D^\mu x)(t))| \le \rho(t) p(K + T\|x'\|, \|x'\|, V\|x'\|) \tag{4.12}$$

for a.e. $t \in [0,T]$. Since x is a solution of the equation in (4.9), we have (cf. (3.2) and Lemma 2.1)

$$x'(t) = \frac{\lambda}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s + b \quad \text{for } t \in [0,T], \tag{4.13}$$

where $b \in \mathbb{R}$. From $|x'(\xi_1)| < K$, we obtain

$$|b| < K + \left| \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{\xi_1} (\xi_1 - s)^{\alpha-2} f(s, x(s), x'(s), ({}^c\!D^\mu x)(s))\,\mathrm{d}s \right|. \tag{4.14}$$

By Lemma 2.6 (with $\eta = \rho, t_2 = t$ and $t_1 = 0$),

$$\int_0^t (t-s)^{\alpha-2} \rho(s)\,\mathrm{d}s \le \left(\frac{T^{(\alpha-2)p+1}}{(\alpha-2)p+1} \right)^{\frac{1}{p}} \|\rho\|_q =: W.$$

We conclude from the last inequality and from (4.12), (4.13) and (4.14) that

$$\|x'\| \le \frac{2W}{\Gamma(\alpha-1)} p(K + T\|x'\|, \|x'\|, V\|x'\|) + K. \tag{4.15}$$

In view of (4.7), there is a positive constant S_1 such that the inequality

$$\frac{2W}{\Gamma(\alpha-1)} p(K + Tv, v, Vv) + K < v$$

is fulfilled for all $v \ge S_1$. Hence (4.15) yields $\|x'\| < S_1$, and hence $\|x\| < K + TS_1$ by (4.11). Put $S = \max\{S_1, K + TS_1\}$. Then (4.8) holds for each $\lambda \in [0,1]$ and each solution x of problem (4.9).

We proceed now to discussing problem (4.10). Let x be a solution of(4.10). Due to $\Lambda_1, \Lambda_2 \in \mathcal{C}$ there is $L > 0$ such that $|\Lambda_1(x)| \le L$ and

$|\Lambda_1(x)| \le L$ for $x \in C^1[0,T]$. Therefore $|x(\xi)| \le L$ and $|x(\eta)| \le L$. It follows from $x(\xi) - x(\eta) = x'(\tau)(\eta - \xi)$, where $\tau \in (\xi, \eta)$ that $|x'(\tau)| \le \frac{2L}{\eta-\xi}$. We have proved that for each $\lambda \in [0,1]$ and any solution x of problem (4.10) there exists $\tau = \tau(\lambda, x) \in (\xi, \eta)$ such that $|x(\xi)| \le L$ and $|x'(\tau)| \le \frac{2L}{\eta-\xi}$. Essentially the same reasoning as in the first part of the proof $\Big($with $K > \max\Big\{L, \frac{2L}{\eta-\xi}\Big\}\Big)$ yields that there is a positive constant S such that (4.8) holds for each $\lambda \in [0,1]$ and each solution x of problem (4.10). □

Example 4.7. Let $\gamma_i \in L^q[0,T]$ $(i = 0,1,2,3)$, $h \in C([0,T] \times \mathbb{R}^3)$ be bounded and $g_j \in C(\mathbb{R})$, $\lim_{u \to \pm\infty} \frac{g_j(u)}{u} = 0$ $(j = 1,2,3)$. Then the function

$$f(t,x,y,z) = \gamma_0(t)h(t,x,y,z) + \gamma_1(t)g_1(x) + \gamma_2(t)g_2(y) + \gamma_2(t)g_3(z)$$

satisfies the conditions of Theorem 4.6 with $\rho(t) = \sum_{i=0}^{3} |\gamma_i(t)|$ and

$$p(u_1,u_2,u_3) = \sum_{j=1}^{3} \max\Big\{|g_j(s)| : |s| \le u_j\Big\} + K \quad \text{for } (u_1,u_2,u_3) \in \mathbb{R}_+^3,$$

where $K = \sup\{|h(t,x,y,z)| : (t,x,y,z) \in [0,T] \times \mathbb{R}^3\}$. Hence Theorem 4.6 can be applied to problems (1.2), (4.4) and (1.2), (4.5).

In particular, equation (1.2) has solutions u_1 and u_2, where u_1 satisfies the boundary conditions

$$\min\{u(t) : t \in [0,T]\} = A, \quad \max\{u'(t) : t \in [0,T]\} = B, \quad A, B \in \mathbb{R},$$

and u_2 satisfies the boundary conditions

$$u(\xi) = \arctan(\|u\| - \|u'\|) + A, \quad u(\eta) = \int_0^T \sin(u'(t))\,\mathrm{d}t + B, \quad A, B \in \mathbb{R}.$$

where $0 \le \xi < \eta \le 1$.

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Short Communications

Malkhaz Ashordia

ON THE FREDHOLM PROPERTY FOR GENERAL LINEAR BOUNDARY VALUE PROBLEMS FOR IMPULSIVE SYSTEMS WITH SINGULARITIES

Dedicated to the blessed memory of Professor T. Chanturia

Abstract. A general linear singular boundary value problem

$$\frac{dx_i}{dt} = P_i(t) \cdot x_{3-i} + q_i(t) \ \ (i = 1, 2),$$
$$x_i(\tau_k+) - x_i(\tau_k-) = G_i(k) \cdot x_{3-i}(\tau_k) + h_i(k) \ \ (i = 1, 2; \ k = 1, 2, \dots);$$
$$l_i(x_1, x_2) = c_i \ \ (i = 1, 2)$$

is considered, where $P_i \in L_{loc}(]a, b[\,, \mathbb{R}^{n_i \times n_{3-i}})$, $q_i \in L_{loc}(]a, b[\,, \mathbb{R}^{n_i})$, $G_i : \{1, 2, \dots\} \to \mathbb{R}^{n_i \times n_{3-i}}$, $h_i : \{1, 2, \dots\} \to \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^{n_i}$, and l_i is a linear bounded operator $(i = 1, 2)$.

The singularity is understood in the sense that $P_i \notin L([a, b], \mathbb{R}^{n_i \times n_{3-i}})$, $q_j \notin L([a, b], \mathbb{R}^{n_j})$ or $\sum\limits_{k=1}^{\infty} \big(\|G_i(k)\| + \|h_j(k)\|\big) = +\infty$ for some $i, j \in \{1, 2\}$.

The conditions are established under which this problem is uniquely solvable if and only if the corresponding homogeneous boundary value problem has only the trivial solution.

Analogous problems for similar impulsive systems with small parameters are also considered.

რეზიუმე. განხილულია ზოგადი სახის წრფივი იმპულსური სასაზღვრო ამოცანა

$$\frac{dx_i}{dt} = P_i(t) \cdot x_{3-i} + q_i(t) \ \ (i = 1, 2),$$
$$x_i(\tau_k+) - x_i(\tau_k-) = G_i(k) \cdot x_{3-i}(\tau_k) + h_i(k) \ \ (i=1, 2; \ k = 1, 2, \dots);$$
$$l_i(x_1, x_2) = c_i \ \ (i = 1, 2),$$

სადაც $P_i \in L_{loc}(]a, b[\,, \mathbb{R}^{n_i \times n_{3-i}})$, $q_i \in L_{loc}(]a, b[\,, \mathbb{R}^{n_i})$, $G_i : \{1, 2, \dots\} \to \mathbb{R}^{n_i \times n_{3-i}}$, $h_i : \{1, 2, \dots\} \to \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^{n_i}$, ხოლო l_i $(i = 1, 2)$ წრფივი შემოსაზღვრული ოპერატორია.

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სინგულარობა გაიგება იმ აზრით, რომ $P_i \notin L([a,b],\mathbb{R}^{n_i\times n_{3-i}})$, $q_j \notin L([a,b],\mathbb{R}^{n_j})$ ან $\sum\limits_{k=1}^{\infty}\big(\|G_i(k)\|+\|h_j(k)\|\big)=+\infty$ გარკვეული $i,j\in\{1,2\}$-თვის.

მიღებულია პირობები, რომელთა შესრულების შემთხვევაში აღნიშნული ამოცანა ცალსახად ამოხსნადია მაშინ და მხოლოდ მაშინ, როდესაც ერთგვაროვან სასაზღვრო ამოცანას გააჩნია მხოლოდ ტრივიალური ამონახსნი.

აგრეთვე განხილულია ანალოგიური ამოცანები პარამეტრიანი წრფივი იმპულსური სისტემებისათვის.

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1. Statement of the Problem and Basic Notation

Let n_1 and n_2 be natural numbers; $-\infty < a < b < +\infty$, $a < \tau_1 < \tau_2 < \cdots < b$ and $\lim\limits_{k\to\infty}\tau_k = b$.

On the interval $]a,b[$ we consider the linear system of impulsive systems with singularities

$$\frac{dx_i}{dt} = P_i(t)\cdot x_{3-i} + q_i(t) \quad (i=1,2), \tag{1}$$

$$x_i(\tau_k+) - x_i(\tau_k-) = G_i(k)\cdot x_{3-i}(\tau_k) + h_i(k) \quad (i=1,2;\ k=1,2,\dots) \tag{2}$$

under the following two-point boundary value conditions:

$$l_i(x_1,x_2) = c_i \quad (i=1,2), \tag{3}$$

where $P_i \in L_{loc}(]a,b[\,;\mathbb{R}^{n_i\times n_{3-i}})$, $q_i \in L_{loc}(]a,b[\,;\mathbb{R}^{n_i})$, $G_i : \{1,2,\dots\} \to \mathbb{R}^{n_i\times n_{3-i}}$, $h_i : \{1,2,\dots\} \to \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^{n_i}$ $(i=1,2)$, $l_i : \mathrm{BV}([a_1,b_1],\mathbb{R}^{n_1}) \times \mathrm{BV}([a_2,b_2],\mathbb{R}^{n_2}) \to \mathbb{R}^{n_i}$ $(i=1,2)$ are linear bounded operators and $[a_i,b_i]$ $(i=1,2)$ are some closed intervals from $[a,b]$.

In the case, where P_i $(i=1,2)$ and q_i $(i=1,2)$ are the integrable on $[a,b]$ matrix- and vector-functions and $\sum\limits_{k=1}^{\infty}\big(\|G_i(k)\|+\|h_i(k)\|\big) < \infty$ $(i=1,2)$, in [1, 5, 11, 12], the conditions are established for as wether the problem (1), (2); (3) is Fredholm, i.e., the conditions under which the problem (1), (2); (3) is uniquely solvable if and only if the corresponding homogeneous system

$$\frac{dx_i}{dt} = P_i(t)\cdot x_{3-i} \quad (i=1,2), \tag{1_0}$$

$$x_i(\tau_k+) - x_i(\tau_k-) = G_i(k)\cdot x_{3-i}(\tau_k) \quad (i=1,2;\ \ k=1,2,\dots) \tag{2_0}$$

under the conditions

$$l_i(x_1,x_2) = 0 \quad (i=1,2) \tag{3_0}$$

has only trivial solutions. In the case, where the system (1), (2) has singularities at the points a and b, i.e.,

$$\int_a^b \|P_i(t)\|\,dt + \sum_{k=1}^{\infty} \|G_i(k)\| = +\infty,$$

$$\int_a^b \|q_j(t)\|\,dt + \sum_{k=1}^{\infty} \|h_j(k)\| = +\infty$$

for some $i, j \in \{1,2\}$, the question as to wether the problem (1), (2); (3) is Fredholm remains open. The present paper fills in this gap.

The results obtained in the paper are improved for the case, where the boundary condition (3) has the form

$$\sum_{k=1}^{m} \big[B_{1ik}x_1(t_{1ik}) + B_{2ik}x_2(t_{2ik})\big] = c_i \quad (i = 1,2), \tag{4}$$

where $B_{jik} \in \mathbb{R}^{n_i \times n_j}$ $t_{jik} \in \mathbb{R}$ $(i, j = 1,2;\ k = 1,\dots,m)$.

The impulsive system (1), (2) is a particular case of the so-called generalized ordinary differential system (see, e.g., [1–5, 10, 11] and the references therein). The analogous questions and some singular boundary value problems are investigated in [2], [3] for the generalized ordinary differential systems, and in [6, 8, 9] for ordinary differential systems.

In the present paper, on the basis of the results presented in [2, 3], we obtain tests for the Fredholm property for the above impulsive problem. Similar tests are obtained for every of the two linear singular impulsive systems with a small parameter $\varepsilon > 0$,

$$\frac{dx_i(t)}{dt} = \varepsilon^{i-1} P_i(t) \cdot x_{3-i}(t) + q_i(t) \ \ (i = 1,2), \tag{5_ε}$$

$$x_i(\tau_k+) - x_i(\tau_k-) =$$
$$= \varepsilon^{i-1} G_i(k) \cdot x_{3-i}(\tau_k) + h_i(k) \ \ (i = 1,2;\ \ k = 1,2,\dots) \tag{6_ε}$$

and

$$\frac{dx_i(t)}{dt} = \varepsilon^{2-i} P_i(t) \cdot x_{3-i}(t) + q_i(t) \ \ (i = 1,2), \tag{7_ε}$$

$$x_i(\tau_k+) - x_i(\tau_k-) =$$
$$= \varepsilon^{2-i} G_i(k) \cdot x_{3-i}(\tau_k) + h_i(k) \ \ (i = 1,2;\ \ k = 1,2,\dots) \tag{8_ε}$$

under the condition (3).

Throughout the paper, the use will be made of the following notation and definitions.

$\mathbb{N} = \{1,2,\dots\}$, $\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$; $[a,b]$ and $]a,b[$ $(a,\, b \in \mathbb{R})$ are, respectively, the closed and open intervals.

$\mathbb{I}$ is an arbitrary closed or open interval from $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\|=\max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|.$$

$\mathbb{R}_+^{n\times m}=\big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\ge 0\ (i=1,\dots,n;\ j=1,\dots,m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix.

If $X=(x_{ij})_{i,j=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the inverse to X matrix, the determinant of X and the spectral radius of X; I_n is the identity $n\times n$-matrix.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

If $X:[a,b]\to\mathbb{R}^{n\times m}$ is a matrix-function, then $\overset{b}{\underset{a}{\vee}}(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i=1,\dots,n;\ j=1,\dots,m)$; $V(X)(t)=(V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a)=0$, $V(x_{ij})(t)=\overset{t}{\underset{a}{\vee}}(x_{ij})$ for $a<t\le b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(a-)=X(a),\ X(b+)=X(b))$.

$d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X)<\infty$).

$\mathrm{BV}_{loc}(\mathbb{I};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{I}\to\mathbb{R}^{n\times m}$ such that $\overset{b}{\underset{a}{\vee}}(X)<+\infty$ for $a,\ b\in\mathbb{I}$.

$L([a,b];\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$, measurable and integrable in the Lebesgue sense on the closed interval $[a,b]$.

$L_{loc}(\mathbb{I};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{I}\to\mathbb{R}^{n\times m}$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{I}$ belong to $L([a,b];\mathbb{R}^{n\times m})$.

$\widetilde{C}([a,b],\mathbb{R}^{n\times m})$ is the set of all absolutely continuous matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$.

$\widetilde{C}_{loc}(\mathbb{I},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{I}\to\mathbb{R}^{n\times m}$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{I}$ belong to $\widetilde{C}([a,b],\mathbb{R}^{n\times m})$.

$\widetilde{C}_{loc}(\mathbb{I}\setminus\{\tau_k\}_{k=1}^{\infty},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{I}\to\mathbb{R}^{n\times m}$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{I}\setminus\{\tau_k\}_{k=1}^{\infty}$ belong to $\widetilde{C}([a,b],\mathbb{R}^{n\times m})$.

If $X\in L_{loc}(]a,b[;\mathbb{R}^{l\times n})$, $G:\mathbb{N}\to\mathbb{R}^{l\times n}$, $Y\in L_{loc}(]a,b[;\mathbb{R}^{n\times m})$ and $Q:\mathbb{N}\to\mathbb{R}^{n\times m}$, then

$$\mathcal{F}_1(X,G;Y,Q)(s,t)=\int_s^t dA(X,G)(\tau)\cdot\big(A(Y,Q)(t)-A(Y,Q)(\tau)\big)$$

and

$$\mathcal{F}_2(X,G;Y,Q)(s,t)=\mathcal{F}_1(X,G;Y,Q)(t,s) \text{ for } s,t\in\,]a,b[\,,$$

where

$$A(Y,Q)(t)=\begin{cases}\int\limits_c^t Y(\tau)\,d\tau+\sum\limits_{c\le\tau_k<t}Q(k) & \text{for } c\le t<b,\\ \int\limits_c^t Y(\tau)\,d\tau-\sum\limits_{t<\tau_k\le c}Q(k) & \text{for } a<t<c,\\ O_{n\times m} & \text{for } t=c,\end{cases} \tag{9}$$

and $c=(a+\tau_1)/2$.

Using the formulae of integration-by-parts, formula I.4.33 and Lemma I.4.23 from [10], it is not difficult to verify that

$$\mathcal{F}_1(X,G;Y,Q)(s,t)=\int\limits_s^t\int\limits_s^\tau X(r)dr\cdot Y(\tau)\,d\tau+$$

$$+\sum_{s\le\tau_k<t}\Bigg(G(k)\int\limits_{\tau_k}^t Y(\tau)\,d\tau+\int\limits_s^{\tau_k}X(\tau)\,d\tau\cdot Q(k)+\sum_{l=1}^k G(l)\cdot Q(k)\Bigg) \tag{10}$$

$$\text{for } a<s<t<b.$$

Moreover, we introduce the operator

$$\mathcal{F}_0(X,G;Y,Q)(s,t)=$$

$$=\int\limits_s^t\Bigg(\int\limits_s^\tau X(r)\,dr+\sum_{s<\tau_k<\tau}G(k)\Bigg)\cdot\Bigg(\int\limits_\tau^t X(r)\,dr+\sum_{\tau<\tau_k<t}G(k)\Bigg)Y(\tau)\,d\tau+$$

$$+\sum_{s<\tau_k<t}\Bigg(\int\limits_s^{\tau_k}X(r)\,dr+\sum_{s<\tau_l<\tau_k}G(l)\Bigg)\cdot\Bigg(\int\limits_{\tau_k}^t X(r)\,dr+\sum_{\tau_k<\tau_l<t}G(l)\Bigg)\cdot Q(k) \tag{11}$$

$$\text{for } a<s<t<b.$$

Under a solution of the impulsive system $(1),(2)$ we understand a continuous from the left vector-function $(x_i)_{i=1}^2$, $x_i\in\widetilde{C}_{loc}(]a,b[\setminus\{\tau_k\}_{k=1}^\infty,\mathbb{R}^{n_i})\cap \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n_i})$ $(i=1,2)$, satisfying both the system

$$\frac{dx_i(t)}{dt}=P_i(t)x_{3-i}(t)+q_i(t) \text{ for a.e. } t\in]a,b[\setminus\{\tau_k\}_{k=1}^\infty \tag{12}$$

and the relation (2) for every $k\in\{1,2,\dots\}$. If the component x_i has a right (respectively, left) limit at the point a (respectively, at the point b), then this limit is assumed to be equal to $x_i(a)$ (respectively, to $x_i(b)$). Thus x_i is assumed to be continues at this point.

A solution of the impulsive system $(1),(2)$ is said to be a solution of the problem $(1),(2);(3)$ if there exist one-sided limits $x_i(a_i+)$ and $x_i(b_i-)$

$(i=1,2)$ and the function $x=(x_i)_{i=1}^2$ defined at the endpoints of the closed intervals $[a_i,b_i]$ $(i=1,2)$ by the continuity, satisfy the relation (3).

Consider now the general linear impulsive system

$$\frac{dz_i(t)}{dt}=P_{i1}(t)\cdot z_1(t)+P_{i2}(t)\cdot z_2(t)+\widetilde{q}_i(t)\ \ (i=1,2), \tag{13}$$

$$z_i(\tau_k+)-x_i(\tau_k-)=$$
$$=G_{i1}(k)\cdot z_1(\tau_k)+G_{i2}(k)\cdot z_2(\tau_k)+\widetilde{h}_i(k)\ \ (i=1,2;\ \ k=1,2,\dots), \tag{14}$$

for the boundary value problem

$$\widetilde{l}_i(z_1,z_2)=c_i\ \ (i=1,2), \tag{15}$$

where $P_{ij}\in L_{loc}(]a,b[\,;\mathbb{R}^{n_i\times n_j})$, $\widetilde{q}_i\in L_{loc}(]a,b[\,;\mathbb{R}^{n_i})$, $G_{i,j}:\mathbb{N}\to\mathbb{R}^{n_i\times n_j}$, $\widetilde{h}_i:\mathbb{N}\to\mathbb{R}^{n_i}$, and $\widetilde{l}_i$ is the linear bounded operator $(i=1,2)$.

For the general system (13), (14), we assume that

$$\det(I_n+G_{ii}(\tau_k))\neq 0\ \ (i=1,2;\ \ k=1,2,\dots).$$

Under this condition, there exists the fundamental matrix Y_i of the homogeneous system

$$\frac{dy_i(t)}{dt}=P_{i1}(t)\cdot y_1(t)+P_{i2}(t)\cdot y_2(t)\ \ (i=1,2),$$
$$y_i(\tau_k+)-y_i(\tau_k-)=$$
$$=G_{i1}(k)\cdot y_1(\tau_k)+G_{i2}(k)\cdot y_2(\tau_k)\ \ (i=1,2;\ \ k=1,2,\dots),$$

satisfying the condition $Y_i(c)=I_{n_i}$ for every $i\in\{1,2\}$ (see, for example, [10]).

Then it is not difficult to verify that the substitution $z_i(t)=Y_i(t)x_i(t)$ $(i=1,2)$ reduces the problem (13), (14); (15) to the problem (1), (2); (3), where

$$P_i(t)\equiv Y_i^{-1}(t)P_{i3-i}(t)Y_{3-i}(t),\ q_i(t)\equiv Y_i^{-1}(t)\widetilde{q}_i(t)\ \ (i=1,2);$$
$$G_i(k)\equiv Y_i^{-1}(\tau_k)(I_{n_i}+G_{ii}(k))^{-1}G_{i3-i}(k)Y_{3-i}(\tau_k)\ \ (i=1,2),$$
$$h_i(k)\equiv Y_i^{-1}(\tau_k)(I_{n_i}+G_{ii}(k))^{-1}\widetilde{h}_i(k)\ \ (i=1,2)$$

and

$$l_i(x_1,x_2)\equiv\widetilde{l}_i(Y_1x_1,Y_2x_2)\ \ (i=1,2).$$

2. Statement of the Main Results

Theorem 1. *Let $a_0\in]a,b[$ and $b_0\in]a_0,b[$, and let*

$$l_i:\mathrm{BV}([a,b],\mathbb{R}^{n_1})\times\mathrm{BV}([a_0,b_0],\mathbb{R}^{n_2})\to\mathbb{R}^{n_i}\ \ (i=1,2)$$

be linear bounded operators. (16)

In addition, suppose that

$$\int\limits_a^b \big(\|P_1(t)\| + \|q_1(t)\|\big)\,dt + \sum_{k=1}^{\infty} \big(\|G_1(k)\| + \|h_1(k)\|\big) < +\infty, \tag{17}$$

$$\|\mathcal{F}_0(|P_1|, |G_1|; |P_2|, |G_2|)(a+, b-)\| < +\infty, \tag{18}$$

$$\|\mathcal{F}_0(|P_1|, |G_1|; |q_2|, |h_2|)(a+, b-)\| < +\infty. \tag{19}$$

Then the problem (1), (2); (3) *is the Fredholm one, i.e., it is uniquely solvable if and only if the corresponding homogeneous problem* (1_0), (2_0); (3_0) *has only a trivial solution.*

Theorem 2. *Let* $b_0 \in]a, b[$ *and* $a_0 \in]a, b_0[$ *and let*

$$l_i : \mathrm{BV}([a, b_0], \mathbb{R}^{n_1}) \times \mathrm{BV}([a_0, b], \mathbb{R}^{n_2}) \to \mathbb{R}^{n_i} \quad (i = 1, 2)$$
be linear bounded operators. (20)

In addition, suppose that

$$\begin{gathered}\int\limits_a^{a_0} \big(\|P_1(t)\| + \|q_1(t)\|\big)\,dt + \sum_{a<\tau_k<a_0} \big(\|G_1(k)\| + \|h_1(k)\|\big) < +\infty,\\ \int\limits_{a_0}^{b} \big(\|P_2(t)\| + \|q_2(t)\|\big)\,dt + \sum_{a_0<\tau_k<b} \big(\|G_2(k)\| + \|h_2(k)\|\big) < +\infty;\end{gathered} \tag{21}$$

$$\begin{multlined}\Big\|\mathcal{F}_1(|P_1|, |G_1|; |P_2|, |G_2|)(a+, a_0)\|+ \\ +\|\mathcal{F}_1(|P_1|, |G_1|; |q_2|, |h_2|)(a+, a_0)\Big\| < \infty,\end{multlined} \tag{22}$$

$$\begin{multlined}\Big\|\mathcal{F}_2(|P_2|, |G_2|; |P_1|, |G_1|)(a_0, b-)\|+ \\ +\|\mathcal{F}_2(|P_2|, |G_2|; |q_1|, |h_1|)(a_0, b-)\Big\| < \infty.\end{multlined} \tag{23}$$

Then the assertion of Theorem 1 *is valid.*

Corollary 1. *Let either* $t_{1ik} \in [a, b]$, $t_{2ik} \in]a, b[$ $(i = 1, 2;\ k = 1, \dots, m)$ *and the conditions* (17)–(19) *be satisfied, or* $t_{1ik} \in [a, b[$, $t_{2ik} \in]a, b]$ $(i = 1, 2;\ k = 1, \dots, m)$ *and the conditions* (21)–(23) *be satisfied for some* $a_0 \in]a, b[$. *Then for the unique solvability of the problem* (1), (2); (4) *it is necessary and sufficient that the system* (1), (2) *under the homogeneous boundary condition*

$$\sum_{k=1}^{m} \big[B_{1ik} x_1(t_{1ik}) + B_{2ik} x_2(t_{2ik})\big] = 0 \quad (i = 1, 2) \tag{4_0}$$

has only a trivial solution.

Corollary 2. *Let* $P_i \in L([a,b];\mathbb{R}^{n_i\times n_{3-i}})$, $q_i \in L([a,b];\mathbb{R}^{n_i})$ *and*

$$\sum_{k=1}^{\infty}\big(\|G_i(k)\|+\|h_i(k)\|\big)<+\infty \ \ (i=1,2).$$

Let, moreover, either the condition (17) *or the condition* (21) *be fulfilled for, respectively, some* $a_0 \in]a,b[$ *and* $b_0 \in]a_0,b[$ *or for some* $b_0 \in]a,b[$ *and* $a_0 \in]a,b_0[$. *Then the problem* (1), (2); (3) *is the Fredholm one.*

Theorem 3. *Let the conditions* (16)–(19) *hold for some* $a_0 \in]a,b[$ *and* $b_0 \in]a_0,b[$. *Let, moreover,* $\Delta \neq 0$, *where* Δ *is the determinant of the system* $l_i(c_1+A(P_1,G_1)c_2,c_2)=0$ $(i=1,2)$, *and the matrix-function is defined by* (1.9). *Then there exists a positive number* ε_0, *independent of* P_i, G_i, q_i, h_i *and* c_i $(i=1,2)$, *such that the problem* (5_ε), (6_ε); (3) *has one and only one solution for each* $\varepsilon \in [0,\varepsilon_0]$.

Theorem 4. *Let the conditions* (20)–(23) *hold for some* $b_0 \in]a,b[$ *and* $a_0 \in]a,b_0[$. *Let, moreover,* $\Delta_0 \neq 0$, *where* Δ_0 *is the determinant of the system* $l_i(c_1,c_2)=0$ $(i=1,2)$. *Then there exists a positive number* ε_0 *independent of* P_i, G_i, q_i, h_i *and* c_i $(i=1,2)$ *such that the problem* (7_ε), (8_ε); (3) *has one and only one solution for each* $\varepsilon \in [0,\varepsilon_0]$.

Finally, it should be noted that the vector-function $x=(x_i)_{i=1}^2$, with components $x_i \in \widetilde{C}_{loc}(]a,b[\setminus\{\tau_k\}_{k=1}^{\infty},\mathbb{R}_i^n)\cap \mathrm{BV}([a,b];\mathbb{R}_i^n)$, is a solution of the impulsive system (1), (2) if and only if it is a solution of the generalized ordinary differential system

$$dx_i(t)=dA_i(t)\cdot x_{3-i}(t)+df_i(t) \ \ (i=1,2),$$

where $A_i(t)\equiv A(P_i,G_i)(t)$ and $f_i(t)\equiv A(q_i,h_i)(t)$ $(i=1,2)$, and the matrix- and vector-functions $A(P_i,G_i)$ $(i=1,2)$ and $A(q_i,h_i)$ $(i=1,2)$ are defined by (9).

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Ivan Kiguradze

THE DIRICHLET AND FOCAL BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER QUASI-HALFLINEAR SINGULAR DIFFERENTIAL EQUATIONS

Dedicated to the blessed memory of
my dear friend, Professor T. Chanturia

Abstract. For higher order quasi-halflinear singular differential equations, the Dirichlet and focal boundary value problems are considered. Analogues of the Fredholm first theorem are proved and on the basis of these results optimal in some sense sufficient conditions of solvability of the above-mentioned problems are found.

რეზიუმე. მაღალი რიგის კვაზი ნახევრადწრფივი სინგულარული დიფერენციალური განტოლებებისათვის განხილულია დირიხლესა და ფოკალური სასაზღვრო ამოცანები. დამტკიცებულია ფრედჰოლმის პირველი თეორემის ანალოგები, რის საფუძველზეც დადგენილია აღნიშნული ამოცანების ამოხსნადობის ოპტიმალური საკმარისი პირობები.

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1. **Statement of the Problem and the Main Notation.** In the interval $]a,b[$, we consider the differential equation

$$u^{(2m)}=\sum_{i=1}^{k}p_i(t)\bigg(\prod_{j=1}^{m}|u^{(j-1)}|^{\alpha_{ij}}\bigg)\operatorname{sgn}u+q(t,u,\dots,u^{(m-1)}) \tag{1}$$

with the Dirichlet boundary conditions

$$u^{(i-1)}(a)=0,\quad u^{(i-1)}(b)=0\ \ (i=1,\dots,m) \tag{2}$$

and with the focal boundary conditions

$$u^{(i-1)}(a)=0,\quad u^{(m+i-1)}(b)=0\ \ (i=1,\dots,m). \tag{3}$$

Here, m and k are arbitrary natural numbers, α_{ij} $(i=1,\dots,k;\ j=1,\dots,m)$ are nonnegative constants such that

$$\alpha_{i1}>0,\quad \sum_{j=1}^{m}\alpha_{ij}=1\ \ (i=1,\dots,m), \tag{4}$$

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$-\infty < a < b < +\infty$, the functions $p_i :]a,b[\to R$ $(i = 1,\dots,n)$ are integrable on every compact interval contained in $]a,b[$, and $q :]a,b[\times R^m \to R$ is the function, satisfying the local Carathéodory conditions.

The equation (1) is said to be singular if at least one of the coefficients p_i $(i = 1,\dots,m)$, or the function $q(\cdot\,, x_1,\dots,x_n)$, is not integrable on $[a,b]$, having singularity at one or at both boundary points of that interval. We say that the equation (1) has a strong singularity at the point a (at the point b) if for some $i \in \{1,\dots,k\}$ the condition

$$\int_a^t (s-a)^{2m-\sum\limits_{j=1}^m j\alpha_{ij}} |p_i(s)|\,ds = +\infty$$

$$\Bigg(\int_t^b (b-s)^{2m-\sum\limits_{j=1}^m j\alpha_{ij}} |p_i(s)|\,ds = +\infty\Bigg) \ \text{ for } \ a < t < b$$

is fulfilled.

In the case, when if (1) is a linear equation with strong singularities at the points a and b (at the point a), i.e., when $k = 1$, $\alpha_{11} = 1$, $\alpha_{1i} = 0$ for $i > 1$ and

$$\int_a^t (s-a)^{2m-1}|p_1(s)|\,ds = +\infty,$$

$$\int_t^b (b-s)^{2m-1}|p_1(s)|\,ds = +\infty \ \text{ for } \ a < t < b$$

$$\Bigg(\int_a^t (s-a)^{2m-1}|p_1(s)|\,ds = +\infty, \quad \int_t^b |p_1(s)|\,ds < +\infty \ \text{ for } \ a < t < b\Bigg),$$

the problem (1), (2) (the problem (1), (3)) is thoroughly investigated in [1]. As for the case, when $\sum\limits_{j=2}^m \alpha_{ij} = 1 - \alpha_{ij} > 0$ $(i = 1,\dots,k)$ and the equation (1) has strong singularities at the points a and b, the above-mentioned problems remain $4n$ studied. The present paper is devoted to fill up this gap.

Throughout the paper, we use the following notation.

$L([a,b])$ is the space of Lebesgue integrable functions $y : [a,b] \to R$.

$L_{loc}(\,]a,b[)$ $(L_{loc}(\,]a,b]))$ is the space of functions $y :]a,b[\to R$ which are integrable on $[a+\varepsilon, b-\varepsilon]$ (on $[a+\varepsilon, b]$) for arbitrarily small $\varepsilon > 0$.

$\widetilde{C}^{2m-1}_{loc}(\,]a,b[)$ $(\widetilde{C}^{2m-1}_{loc}(\,]a,b]))$ is the space of functions $u :]a,b[\to R$ which are absolutely continuous together with $u',\dots,u^{2(m-1)}$ on $[a+\varepsilon, b-\varepsilon]$ (on $[a+\varepsilon, b]$) for arbitrarily small $\varepsilon > 0$.

$\widetilde{C}^{2m-1,m}(\,]a,b[)$ $(\widetilde{C}^{2m-1,m}(\,]a,b]))$ is the space of functions $u \in \widetilde{C}^{2m-1}_{loc}(\,]a,b[)$ $(\widetilde{C}^{2m-1}_{loc}(\,]a,b]))$ such that

$$\int_a^b |u^{(m)}(s)|^2\,ds < +\infty.$$

$$\alpha_i = 2m+1-\sum_{j=1}^{m} j\alpha_{ij} \quad (i=1,\dots,k); \tag{5}$$

$$\gamma_{1i} = \frac{2^m}{(2m-1)!!}\prod_{j=1}^{m}\Big(\frac{2^{m-j+1}}{(2m-2j+1)!!}\Big)^{\alpha_{ij}},$$

$$\gamma_{2i} = \frac{1}{(m-1)!\sqrt{2m-1}}\prod_{j=1}^{m}\Big(\frac{1}{(m-j)!\sqrt{2m-2j+1}}\Big)^{\alpha_{ij}} \tag{6}$$
$$(i=1,\dots,k);$$

$$\varphi_{1i}(t) =$$
$$= \big((t-a)^{-2m}+(b-t)^{-2m}\big)^{\frac{1}{2}}\prod_{j=1}^{m}\Big((t-a)^{2j-2m-2}+(b-t)^{2j-2m-2}\Big)^{\frac{\alpha_{ij}}{2}},$$

$$\varphi_{2i}(t) = \tag{7}$$
$$= \big((t-a)^{1-2m}+(b-t)^{1-2m}\big)^{\frac{1}{2}}\prod_{j=1}^{m}\Big((t-a)^{2j-2m-1}+(b-t)^{2j-2m-1}\Big)^{\frac{\alpha_{ij}}{2}}$$
$$(i=1,\dots,k).$$

As it has been said above, we assume that the function $q :]a,b[\times R^m \to R$ satisfies the local Carathéodory conditions, i.e., $q(t,\cdot,\dots,\cdot) : R^m \to R$ is continuous for almost all $t \in]a,b[$, $q(\cdot,x_1,\dots,x_m) :]a,b[\to R$ is measurable for any $(x_1,\dots,x_m)$, and

$$q^*(\cdot\,,y_1,\dots,y_m) \in L_{loc}(\,]a,b[) \ \text{ for } \ y_1>0,\dots,y_m>0,$$

where

$$q^*(t,y_1,\dots,y_m) = \max\Big\{\big|q(t,x_1,\dots,x_m)\big| : \ |x_1|\le y_1,\dots,|x_m|\le y_m\Big\}. \tag{8}$$

We investigate the problem (1), (2) in the case, when

$$p_i \in L_{loc}(\,]a,b[) \ \ (i=1,\dots,k),$$

$$\lim_{\rho\to\infty}\int_a^b \psi_1(t)\,\frac{q^*\big(t,\psi_1(t)\rho,\dots,\psi_m(t)\rho\big)}{\rho}\,dt = 0, \tag{9}$$

where $\psi_j(t) = (t-a)^{m-j+\frac{1}{2}}(b-t)^{m-j+\frac{1}{2}}$ $(j = 1, \dots, m)$, and the problem $(1), (3)$ in the case, when

$$p_i \in L_{loc}(\,]a,b])\ \ (i = 1, \dots, k),$$
$$\lim_{\rho \to +\infty} \int\limits_a^b (t-a)^{m-\frac{1}{2}} \,\frac{q^*\big(t, (t-a)^{m-\frac{1}{2}}\rho, \dots, (t-a)^{\frac{1}{2}}\rho\big)}{\rho}\, dt = 0. \tag{10}$$

In both cases the function q is sublinear with respect to the phase variables and, consequently, the equation (1) is quasi-halflinear.

In theorems on the existence of positive negative solutions of the problems $(1), (2)$ and $(1), (3)$, on the function q we impose either the restriction

$$\lim_{\rho>0,\, \rho\to 0} \int\limits_a^b (t-a)^m (b-t)^m \,\frac{q_*\big(t, \rho(t-a)^m(b-t)^m\big)}{\rho}\, dt = +\infty, \tag{11}$$

or the restriction

$$\lim_{\rho>0,\, \rho\to 0} \int\limits_a^b (t-a)^m \,\frac{q_*\big(t, \rho(t-a)^m\big)}{\rho}\, dt = +\infty, \tag{12}$$

where

$$q_*(t,y) = \inf\Big\{\big|q(t,x_1,\dots,x_m)\big| :\ (x_1,\dots,x_m) \in R^m,\ |x_1| \ge y\Big\}. \tag{13}$$

A function $u \in \widetilde{C}^{2m-1}_{loc}(\,]a,b[)$ is said to be a solution of the equation (1) if it satisfies this equation almost everywhere on $]a,b[$. A solution of the equation (1) is said to be a solution of the problem $(1), (2)$ (of the problem $(1), (3)$) if it satisfies the boundary conditions (2) (the boundary conditions (3)), where by $u^{(i-1)}(a)$ (by $u^{(j-1)}(b)$) it is understood the right (the left) limit of the function $u^{(i-1)}$ (of the function $u^{(j-1)}$) at the point a (at the point b).

For the problems $(1), (2)$ and $(1), (3)$ we have proved the analogues of Fredholm first theorem (see Theorems 1–4) on the basis of which the sufficient conditions of solvability of these problems are established in the spaces $\widetilde{C}^{2m-1,m}(\,]a,b[)$ and $\widetilde{C}^{2m-1,m}(\,]a,b])$ (Theorems 5 and 6). The conditions are also found under which the problem $(1), (2)$ (the problem $(1), (3)$) along with the trivial has a positive and negative on $]a,b[$ solutions (Theorems 7 and 8). All the above-mentioned theorems cover the case for which equation (1) has strong singularities at the points a and b. These theorems are new not only for a singular case, but also for a regular one, i.e for the case, where

$$p_i \in L([a,b])\ \ (i = 1, \dots, k),$$
$$q^*(\cdot, \rho_1, \dots, \rho_m) \in L([a,b]) \ \text{ for } \ \rho_1 > 0, \dots, \rho_m > 0$$

(see [2]–[11] and the references therein).

2. **Fredholm Type Theorems.** Along with (1), let us consider the half-linear homogeneous differential equation

$$u^{(n)} = \lambda \sum_{i=1}^{k} p_i(t) \bigg(\prod_{j=1}^{m} |u^{(j-1)}|^{\alpha_{ij}} \bigg) \operatorname{sgn} u, \tag{14}$$

depending on the parameter $\lambda \in [0,1]$.

Theorem 1. *Let the condition* (9) *be fulfilled and almost everywhere on* $]a,b[$ *the inequalities*

$$(-1)^m p_i(t) \le l_i \varphi_{1i}(t) + p_{0i}(t)\varphi_{2i}(t) \quad (i = 1,\dots,k) \tag{15}$$

be satisfied, where l_i $(i = 1,\dots,m)$ *are nonnegative constants, and* $p_{0i} : [a,b] \to [0,+\infty[$ $(i = 1,\dots,k)$ *are integrable functions. If, moreover,*

$$\sum_{i=1}^{k} \gamma_{1i} l_i < 1 \tag{16}$$

and for an arbitrary $\lambda \in [0,1]$ *the problem* (14), (2) *has only the trivial solution in the space* $\widetilde{C}^{2m-1,m}(\,]a,b[)$, *then the problem* (1), (2) *has at leat one solution in the same space.*

Theorem 2. *Let the conditions of Theorem* 1 *and the condition* (11) *be fulfilled. If, moreover,*

$$\begin{gathered} (-1)^m p_i(t) \ge 0 \quad (i = 1,\dots,m), \\ (-1)^m q(t,x_1,\dots,x_n)x_1 \ge 0 \;\; \textit{for} \;\; t \in]a,b[\,, \;\; (x_1,\dots,x_m) \in R^m, \end{gathered} \tag{17}$$

then the problem (1), (2) *in the space* $\widetilde{C}^{2m-1,m}(\,]a,b[)$ *along with the trivial solution has a positive and a negative on* $]a,b[$ *solutions.*

Theorem 3. *Let the condition* (10) *be fulfilled and almost everywhere on* $]a,b[$ *the inequalities*

$$(-1)^m p_i(t) \le l_i (t-a)^{-\alpha_i} + p_{0i}(t)(t-a)^{1-\alpha_i} \quad (i = 1,\dots,k) \tag{18}$$

be satisfied, where $p_{0i} : [a,b] \to [0,+\infty[$ $(i = 1,\dots,m)$ *are integrable functions, and* l_i $(i = 1,\dots,m)$ *are nonnegative constants, satisfying the inequality* (16). *If, moreover, for an arbitrary* $\lambda \in [0,1]$ *the problem* (14), (3) *has only the trivial solution in the space* $\widetilde{C}^{2m-1,m}(\,]a,b])$, *then the problem* (1), (3) *in the same space has at least one solution.*

Theorem 4. *If along with the conditions of Theorem* 3 *the conditions* (12) *and* (17) *are fulfilled, then the problem* (1), (3) *in the space* $\widetilde{C}^{2m-1,m}(\,]a,b])$ *along with trivial solution has a positive and a negative on* $]a,b[$ *solutions.*

Remark 1. The condition (16) in Theorems 1 and 3 is unimprovable and it cannot be replaced by the condition

$$\sum_{i=1}^{k} \gamma_{1i} l_i \le 1. \tag{16'}$$

Indeed, if

$$k=1,\ \ \alpha_{11}=1,\ \ \alpha_{1j}=0\ \text{ for }\ j\neq 1,\ \ l_1=(-1)^m 4^{-m}\big((2m-1)!!\big)^2,$$
$$p_1(t)\equiv l_1(t-a)^{-m},$$
$$q(t,x_1,\dots,x_m)\equiv\left(\prod_{i=1}^{2m}(\nu-i+1)-l_1\right)(t-a)^{\nu-2m},\ \ \nu>m,$$

then all the conditions of Theorems 1 and 3 are fulfilled, except (16), instead of which the condition (16$'$) is fulfilled, but nevertheless, as is shown in [1], the problem (1), (2) (the problem (1), (3)) in the case under consideration has no solution in the space $\widetilde{C}^{2m-1,m}(\,]a,b[)$ (in the space $\widetilde{C}^{2m-1,m}(\,]a,b])$).

3. **Existence Theorems.** On the basis of Theorems 1 and 3 we prove Theorems 5 and 6 below which contain effective conditions for solvability of the problems (1), (2) and (1), (3).

Theorem 5. *Let the condition* (9) *be fulfilled and almost everywhere on* $]a,b[$ *the inequalities* (15) *be satisfied, where* l_i $(i=1,\dots,m)$ *and* $p_{0i}:\ ]a,b[\,\to[0,+\infty[$ $(i=1,\dots,m)$ *are, respectively, nonnegative numbers and integrable functions, satisfying the inequality*

$$\sum_{i=1}^{k}\left(\gamma_{1i}l_i+\gamma_{2i}\int_a^b p_{0i}(t)\,dt\right)<1. \tag{19}$$

Then the problem (1), (2) *in the space* $\widetilde{C}^{2m-1,m}(\,]a,b[)$ *has at least one solution.*

Theorem 6. *Let the condition* (10) *be fulfilled and almost everywhere on* $]a,b[$ *the inequalities* (18) *be satisfied, where* l_i $(i=1,\dots,m)$ *and* $p_{0i}:\ ]a,b[\,\to[0,+\infty[$ $(i=1,\dots,m)$ *are, respectively, nonnegative numbers and integrable functions, satisfying the inequality* (19). *Then the problem* (1), (3) *in the space* $\widetilde{C}^{2m-1,m}(\,]a,b[)$ *has at least one solution.*

Remark 2. If we use the example given in Remark 1, then it will become clear that the strict inequality (19) in Theorems 4 and 5 cannot be replaced by the nonstrict inequality

$$\sum_{i=1}^{k}\left(\gamma_{1i}l_i+\gamma_{2i}\int_a^b p_{0i}(t)\,dt\right)\leq 1.$$

4. **Theorems on the Non-Unique Solvability of the Problems** (1), (2) **and** (1), (3)**.**

Theorem 7. *If along with the conditions of Theorem* 5 *the conditions* (11) *and* (17) *are fulfilled, then the problem* (1), (2) *in the space* $\widetilde{C}^{2m-1,m}(\,]a,b[)$ *along with the trivial solution has a positive and a negative solutions on* $]a,b[$.

Theorem 8. *If along with the conditions of Theorem* 6 *are fulfilled the conditions* (12) *and* (17)*, then the problem* (1), (3) *in the space* $\widetilde{C}^{2m-1,m}(\,]a,b])$ *along with a trivial solution has a positive and a negative on* $]a,b[$ *solutions.*

As examples, we consider the differential equations

$$u^{(2m)} = (-1)^m \prod_{i=1}^{k} l_i \varphi_{1i}(t) \bigg(\prod_{j=1}^{m} |u^{(j-1)}|^{\alpha_{ij}} \bigg) \operatorname{sgn} u + \\ + (-1)^m (t-a)^{-\mu} (b-t)^{-\mu} q_0(t) |u|^{\lambda} \operatorname{sgn} u \quad (20)$$

and

$$u^{(2m)} = (-1)^m \prod_{i=1}^{k} l_i (t-a)^{-\alpha_i} \bigg(\prod_{j=1}^{m} |u^{(j-1)}|^{\alpha_{ij}} \bigg) \operatorname{sgn} u + \\ + (-1)^m (t-a)^{-\mu} q_0(t) |u|^{\lambda} \operatorname{sgn} u, \quad (21)$$

where $q_0 :]a,b[\to]0,+\infty[$ is an integrable function,

$$0 < \lambda < 1, \quad \mu = \Big(m - \frac{1}{2}\Big)(\lambda + 1),$$

and l_i $(i = 1, \dots, m)$ are the nonnegative constants, satisfying the inequality (16). According to Theorem 7 (Theorem 8), the problem (20), (2) (the problem (21), (2)) in the space $\widetilde{C}^{2m-1,m}(]a,b[)$ (in the space $\widetilde{C}^{2m-1,m}(]a,b])$) along with the trivial solution has a positive and a negative on $]a,b[$ solutions.

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Sulkhan Mukhigulashvili and Nino Partsvania

ON TWO–POINT BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STRONG SINGULARITIES

Dedicated to the blessed memory of Professor T. Chanturia

Abstract. For higher order linear singular functional differential equations, the Agarwal–Kiguradze type theorems on the unique solvability of two-point boundary value problems are proved.

რეზიუმე. მაღალი რიგის წრფივი სინგულარული ფუნქციონალურ-დიფერენციალური განტოლებებისათვის დამტკიცებულია აგარვალ-კიღურაძის ტიპის თეორემები ორწერტილოვანი სასაზღვრო ამოცანების ცალსახად ამოხსნადობის შესახებ.

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Key words and phrases: Functional differential equation, linear, higher order, strong singularity,two-point boundary value problem.

Consider the functional differential equation

$$u^{(2m)}(t) = p(t)u(\tau(t)) + q(t) \tag{1}$$

with the boundary conditions

$$u^{(i-1)}(a) = 0, \quad u^{(i-1)}(b) = 0 \ \ (i = 1, \dots, m), \quad \int\limits_a^b |u^{(m)}(s)|^2 ds < +\infty \tag{2}$$

or

$$u^{(i-1)}(a) = 0, \ \ u^{(m+i-1)}(b) = 0 \ \ (i = 1, \dots, m), \quad \int\limits_a^b |u^{(m)}(s)|^2 ds < +\infty. \tag{3}$$

Here m is a natural number, $-\infty < a < b < +\infty$, $\tau : [a, b] \to [a, b]$ is a measurable function, and the functions p and $q :]a, b[\to \mathbb{R}$ are Lebesgue integrable on $[a + \varepsilon, b - \varepsilon]$ for arbitrarily small $\varepsilon > 0$. However, these functions may be non-integrable on $[a, b]$, having singularities at the endpoints of that interval. In that sense, the equation (1) is singular.

For $\tau \equiv t$, the equation (1) has the form

$$u^{(n)}(t) = p(t)u(t) + q(t). \tag{4}$$

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From the results of the monographs [1, 4] and the papers [3, 5, 7–15] it follow rather delicate conditions guaranteeing the existence of a unique solution of the singular differential equation (4), satisfying the boundary conditions

$$u^{(i-1)}(a)=0, \quad u^{(i-1)}(b)=0 \ \ (i=1,\dots,m) \tag{5}$$

or

$$u^{(i-1)}(a)=0, \quad u^{(m+i-1)}(b)=0 \ \ (i=1,\dots,m). \tag{6}$$

However, all these results concern the cases, where the function p satisfies either the condition

$$\int\limits_a^b (t-a)^{2m-1}(b-t)^{2m-1}\big(|p(t)|+(-1)^m p(t)\big)dt<+\infty, \tag{7}$$

or the condition

$$\int\limits_a^b (t-a)^{2m-1}\big(|p(t)|+(-1)^m p(t)\big)dt<+\infty. \tag{8}$$

Note that if the condition (7) (the condition (8)) is satisfied, then (1), (2) and (4), (5) ((1), (3) and (4), (6)) are equivalent problems. However, if

$$\int\limits_a^b (t-a)^{2m-1}(b-t)^{2m-1}\big(|p(t)|+(-1)^m p(t)\big)dt=+\infty \tag{9}$$

or

$$\int\limits_a^b (t-a)^{2m-1}\big(|p(t)|+(-1)^m p(t)\big)dt=+\infty, \tag{10}$$

then the above-mentioned problems are not equivalent. More precisely, from the unique solvability of the problem (1), (2) (of the problem (1), (3)) it does not follow the unique solvability of the problem (4), (5) (of the problem (4), (6)). In that case we will say that the function p has strong singularities.

By I. Kiguradze and R. P. Agarwal [2, 6], unimprovable sufficient conditions are found for the unique solvability of the problem (4), (2) (of the problem (4), (3)), which cover the cases when the function p has strong singularities. In the present paper, the Agarwal–Kiguradze type results are established for the equation (1).

Throughout the paper we use the following notation.

$[x]_+$ is the positive part of a number x, i.e.,

$$[x]_+=\frac{x+|x|}{2}.$$

$L_{loc}(\,]a,b[)$ $(L_{loc}(\,]a,b]))$ is the space of functions $y:]a,b[\to\mathbb{R}$ which are integrable on $[a+\varepsilon,b-\varepsilon]$ (on $[a+\varepsilon,b]$) for arbitrarily small $\varepsilon>0$.

$L_{\alpha,\beta}(\,]a,b[)$ is the space of integrable with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y:]a,b[\to\mathbb{R}$ with the norm

$$\|y\|_{L_{\alpha,\beta}}=\int\limits_a^b (t-a)^\alpha(b-t)^\beta|y(t)|dt.$$

$$h_1(p)(t)=(2m-1)\Big|\int\limits_c^t [(-1)^m p(s)]_+ds\Big| \quad \text{for } \ a<t<b, \ \ c=\frac{a+b}{2},$$

$$h_2(p)(t)=(2m-1)\int\limits_t^b [(-1)^m p(s)]_+ds \quad \text{for } \ a<t<b,$$

$$(2m-1)!!=\prod_{i=1}^{2m}(2i-1), \ \ \mu_m=\Big(\frac{2^m}{(2m-1)!!}\Big)^2, \ \ \nu_m=2\big((m-1)!(2m-1)\big)^{-\frac12}.$$

Theorem 1. *Let $p\in L_{loc}(\,]a,b[)$ and let there exist a nonnegative constant ℓ such that*

$$(t-a)^{2m-1}h_1(p)(t)\le\ell \quad \textit{for } \ a<t<c$$

and

$$(b-t)^{2m-1}h_1(p)(t)\le\ell \quad \textit{for } \ c<t<b.$$

Let, moreover,

$$\mu_m\ell+\Big(\frac{b-a}{\pi}\Big)^{m-1}\nu_m\Bigg(\int\limits_a^c (s-a)^{m-\frac12}|\tau(s)-s|^{\frac12}|p(s)|ds+$$

$$+\int\limits_c^b (b-s)^{m-\frac12}|\tau(s)-s|^{\frac12}|p(s)|ds\Bigg)<1. \tag{11}$$

Then for every $q\in L_{m-\frac12,m-\frac12}(\,]a,b[)$ the problem (1), (2) *has one and only one solution.*

Corollary 1. *Let $p\in L_{loc}(\,]a,b[)$ and let there exist a nonnegative constant ℓ such that*

$$(-1)^m p(t)\le\ell(t-a)^{-2m} \quad \textit{for } \ a<t<c$$

and

$$(-1)^m p(t)\le\ell(b-t)^{-2m} \quad \textit{for } \ c<t<b.$$

If, moreover, the inequality (11) *holds, then for every $q\in L_{m-\frac12,m-\frac12}(\,]a,b[)$ the problem* (1), (2) *has one and only one solution.*

Theorem 2. *Let $p\in L_{loc}(\,]a,b])$ and let there exist a nonnegative constant ℓ such that*

$$(t-a)^{2m-1}h_2(p)(t)\le\ell \quad \textit{for } \ a<t<b.$$

Let, moreover,

$$\mu_m \ell + 2^{m-1}\Big(\frac{b-a}{\pi}\Big)^{m-1} \nu_m \int\limits_a^b (s-a)^{m-\frac{1}{2}} |\tau(s)-s|^{\frac{1}{2}} |p(s)| ds < 1. \tag{12}$$

Then for every $q \in L_{m-\frac{1}{2},0}(\,]a,b[)$ *the problem* $(1), (3)$ *has one and only one solution.*

Corollary 2. *Let* $p \in L_{loc}(\,]a,b])$ *and let there exist a nonnegative constant* ℓ *such that*

$$(-1)^m p(t) \leq \ell (t-a)^{-2m} \quad \textit{for } \ a < t < b.$$

If, moreover, the inequality (12) *holds, then for every* $q \in L_{m-\frac{1}{2},0}(\,]a,b[)$ *the problem* $(1), (3)$ *has one and only one solution.*

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ON SOLVABILITY AND WELL-POSEDNESS OF TWO-POINT WEIGHTED SINGULAR BOUNDARY VALUE PROBLEMS

Abstract. For second order nonlinear ordinary differential equations with strong singularities, unimprovable in a certain sense sufficient conditions for the solvability and well-posedness of two-point weighted boundary value problems are established.

რეზიუმე. მეორე რიგის არაწრფივი ჩვეულებრივი დიფერენციალური განტოლებებისათვის ძლიერი სინგულარობებით დადგენილია ორწერტილოვანი წონიანი სასაზღვრო ამოცანების ამოხსნადობისა და კორექტულობის არაგაუმჯობესებადი საკმარისი პირობები.

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In an open interval $]a,b[$, we consider the second order nonlinear differential equation

$$u'' = f(t,u) \tag{1}$$

with two-point weighted boundary conditions of one of the following two types:

$$\limsup_{t\to a} \frac{|u(t)|}{(t-a)^\alpha} < +\infty, \quad \limsup_{t\to b} \frac{|u(t)|}{(b-t)^\beta} < +\infty \tag{2}$$

and

$$\limsup_{t\to a} \frac{|u(t)|}{(t-a)^\alpha} < +\infty, \quad \lim_{t\to b} u'(t) = 0. \tag{3}$$

Here $f :]a,b[\times R \to R$ is a continuous function, $\alpha \in]0,1[$, and $\beta \in]0,1[$.

Eq. (1) is said to be regular if

$$\int\limits_a^b f^*(t,x)dt < +\infty \quad \text{for } x > 0,$$

where

$$f^*(t,x) = \max\big\{|f(t,y)| :\ 0 \le y \le x\big\} \quad \text{for } a < t < b, \ \ x \ge 0. \tag{4}$$

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And if

$$\int_a^{t_0} f^*(t,x)dt = +\infty \quad \left(\int_{t_0}^b f^*(t,x)dt = +\infty\right) \quad \text{for } a < t_0 < b, \ \ x > 0,$$

then it is said that Eq. (1) with respect to the time variable has a singularity at the point a (at the point b). In that case Eq. (1) is called singular, and boundary value problems for such equations are called singular boundary value problems.

Following R. P. Agarwal and I. Kiguradze [2, 8] we say that Eq. (1) with respect to the time variable has a strong singularity at the point a (at the point b) if for any $t_0 \in]a,b[$ and $x > 0$ the condition

$$\int_a^{t_0} (t-a)\big[|f(t,x)| - f(t,x)\,\mathrm{sgn}\,x\big]dt = +\infty$$

$$\left(\int_{t_0}^b (b-t)\big[|f(t,x)| - f(t,x)\,\mathrm{sgn}\,x\big]dt = +\infty\right)$$

is satisfied.

The boundary conditions (2) and (3), respectively, yield the conditions

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u(t) = 0, \tag{2_0}$$

and

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u'(t) = 0. \tag{3_0}$$

On the other hand, if $\alpha = \beta = \frac{1}{2}$, then the conditions

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u(t) = 0, \quad \int_a^b u'^2(t)dt < +\infty, \tag{$2'$}$$

and

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u'(t) = 0, \quad \int_a^b u'^2(t)dt < +\infty \tag{$3'$}$$

imply the conditions (2) and (3), respectively.

In the case, where Eq. (1) is regular, the problems (1),(2); (1),(2_0), and (1),($2'$) (the problems (1),(3); (1),(3_0), and (1),($3'$)) are equivalent to each other. However, if Eq. (1) is singular, then the above-mentioned problems are not equivalent. Precisely, if Eq. (1) with respect to the time variable has singularities at the points a and b (has a singularity at the point a), then from the solvability of the problem (1),(2_0) (of the problem (1),(3_0)), generally speaking, it does not follow the solvability of the problem (1),(2) or the problem (1),($2'$) (of the problem (1),(3) or the problem (1),($3'$)). On the other hand, in the above-mentioned cases the unique solvability of the problem (1),(2) or the problem (1),(2′) (of the problem (1),(3) or the problem (1),(3′)) does not imply the unique solvability of the problem (1),(2_0) (of the problem (1),(3_0)).

The investigation of two-point boundary value problems for second order singular ordinary differential equations was initiated by I. Kiguradze [4,5]. Nowadays the singular problems (1),(2_0) and (1),(3_0) are studied in full detail (see, e.g., [1,3–7,10–17,19,20], and the references therein).

The problems (1),(2′) and (1),(3′) and the analogous problems for higher order differential equations with strong singularities are studied in [2, 8, 9, 18].

As for the singular problems (1),(2) and (1),(3), they remain still unstudied. In the present paper, an attempt is made to fill this gap. Theorems 1 and 2 (Theorems 3 and 4) below contain unimprovable in a certain sense sufficient conditions for the solvability and well-posedness of the problem (1),(2) (of the problem (1),(3)), at that these theorems, unlike the results from the above-mentioned works [1, 2–7, 10–17, 19, 20], cover the case, where Eq. (1) with respect to the time variable has strong singularities at the points a and b (has a strong singularity at the point a).

Before passing to the formulation of the main results, we introduce some definitions and notation.

By G_0 and G_1 we denote the Green functions of the problems

$$u'' = 0; \quad u(a) = u(b) = 0$$

and

$$u'' = 0; \quad u(a) = u'(b) = 0,$$

respectively, i.e.,

$$G_0(t,s) = \begin{cases} \dfrac{(s-a)(t-b)}{b-a} & \text{for } a \le s \le t \le b, \\ \dfrac{(t-a)(s-b)}{b-a} & \text{for } a \le t < s \le b, \end{cases}$$

and

$$G_1(t,s) = \begin{cases} a-s & \text{for } a \le s \le t \le b, \\ a-t & \text{for } a \le t < s \le b. \end{cases}$$

For any continuous function $h :]a,b[\to R$, we assume

$$\nu_{\alpha,\beta}(h) = \sup\Big\{(t-a)^{-\alpha}(b-t)^{-\beta}\int_a^b |G_0(t,s)h(s)|ds \ : a < t < b\Big\},$$

$$\nu_{\alpha}(h) = \sup\Big\{(t-a)^{-\alpha}\int_a^b |G_1(t,s)h(s)|ds \ : a < t < b\Big\}.$$

Definition 1. A function $u :]a,b[\to R$ is said to be **a solution of Eq. (1)** if it is twice continuously differentiable and satisfies that equation at each point of the interval $]a,b[$. A solution of Eq. (1), satisfying the boundary conditions (2) (the boundary conditions (3)), is said to be **a solution of the problem (1),(2)** (**of the problem (1),(3)**).

Definition 2. The problem (1),(2) (the problem (1),(3)) is said to be **well-posed** if for any continuous function $h :]a,b[\to R$, satisfying the condition

$$\nu_{\alpha,\beta}(h) < +\infty \quad \big(\nu_\alpha(h) < +\infty\big), \tag{5}$$

the perturbed differential equation

$$v'' = f(t,v) + h(t) \tag{6}$$

has a unique solution, satisfying the boundary conditions (2) (the boundary conditions (3)), and there exists a positive constant r, independent of the function h, such that in the interval $]a,b[$ the inequality

$$|u(t) - v(t)| \le r\nu_{\alpha,\beta}(h)(t-a)^\alpha(b-t)^\beta \quad \Big(|u(t) - v(t)| \le r\nu_\alpha(h)(t-a)^\alpha\Big)$$

is satisfied, where u and v are the solutions of the problems (1),(2) and (6),(2) (of the problems (1),(3) and (6),(3)), respectively.

It is clear that

$$\nu_{\alpha,\beta}(h) \le (b-a)^{-1}\int_a^b (s-a)^{1-\alpha}(b-s)^{1-\beta}|h(s)|ds,$$

$$\nu_\alpha(h) \le \int_a^b (s-a)^{1-\alpha}|h(s)|ds.$$

Thus for the condition (5) to be fulfilled it is sufficient that

$$\int_a^b (s-a)^{1-\alpha}(b-s)^{1-\beta}|h(s)|ds < +\infty \quad \left(\int_a^b (s-a)^{1-\alpha}|h(s)|ds < +\infty\right).$$

Now we formulate the main results. First we consider the problem (1),(2).

Theorem 1. *Let there exist continuous functions p and $q :]a,b[\to [0,+\infty[$ such that*

$$f(t,x)\operatorname{sgn} x \ge -(t-a)^{-\alpha}(b-t)^{-\beta}p(t)|x| - q(t) \quad \textit{for } a<t<b, \;\; x\in R,$$
$$\nu_{\alpha,\beta}(p) < 1, \quad \nu_{\alpha,\beta}(q) < +\infty. \tag{7}$$

Then the problem (1),(2) *has at least one solution.*

Corollary 1. *Let there exist a constant $\ell \in [0,1[$ and a continuous function $q :]a,b[\to R$ such that*

$$f(t,x)\operatorname{sgn} x \ge -\ell\left(\frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2}\right)|x| - q(t)$$
$$\textit{for } a<t<b, \;\; x\in R,$$

and $\nu_{\alpha,\beta}(q) < +\infty$. *Then the problem* (1),(2) *has at least one solution.*

Theorem 2. *Let there exist a continuous function* $p :]a,b[\to [0,+\infty[$ *such that*

$$f(t,x) - f(t,y) \geq -(t-a)^{-\alpha}(b-t)^{-\beta}p(t)(x-y) \quad \text{for } a < t < b, \ \ x > y.$$

If, moreover, the condition (7) *holds, where* $q(t) \equiv f(t,0)$, *then the problem* (1),(2) *is well-posed.*

Corollary 2. *Let there exist a constant* $\ell \in [0,1[$ *such that*

$$f(t,x) - f(t,y) \geq -\ell\left(\frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2}\right)(x-y)$$
$$\text{for } a < t < b, \ \ x > y,$$

and $\nu_{\alpha,\beta}(f(\cdot,0)) < +\infty$. *Then the problem* (1),(2) *is well-posed.*

A particular case of (1) is the differential equation

$$u'' = f_1(t)u + f_2(t)|u|^{\mu}\operatorname{sgn} u + f_0(t), \tag{8}$$

where $f_i :]a,b[\to R$ $(i = 0,1,2)$ are continuous functions, and $\mu > 0$.
Corollary 2 yields

Corollary 3. *Let there exist a constant* $\ell \in [0,1[$ *such that*

$$f_1(t) \geq -\ell\left(\frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2}\right) \quad \text{for } a < t < b.$$

If, moreover, $f_2(t) \geq 0$ *for* $a < t < b$, *and* $\nu_{\alpha,\beta}(f_0) < +\infty$, *then the problem* (8),(2) *is well-posed.*

Example 1. Let us consider the differential equation

$$u'' = -\left(\frac{\alpha(1-\alpha)}{(t-a)^2} + \frac{2\alpha\beta}{(t-a)(b-t)} + \frac{\beta(1-\beta)}{(b-t)^2}\right)\Big(\ell|u| + (s-a)^{\alpha}(b-s)^{\beta}\Big), \tag{9}$$

where ℓ is a nonnegative constant. If $\ell < 1$, then by virtue of Corollary 2 the problem (9),(2) is well-posed. Let us show that if $\ell \geq 1$, then that problem has no solution. Assume the contrary that the problem (9),(2) has a solution u. If we suppose

$$\delta = \inf\left\{\frac{|u(t)|}{(t-a)^{\alpha}(b-t)^{\beta}} \ : a < t < b\right\},$$

then from the representation

$$u(t) =$$
$$= \int_a^b |G_0(t,s)|\left(\frac{\alpha(1-\alpha)}{(s-a)^2} + \frac{2\alpha\beta}{(s-a)(b-s)} + \frac{\beta(1-\beta)}{(b-s)^2}\right)\Big(\ell|u(s)| + (s-a)^{\alpha}(b-s)^{\beta}\Big)ds$$

we get

$$u(t) \geq (1+\delta)\times$$

$$\times \int\limits_a^b |G_0(t,s)| \left(\frac{\alpha(1-\alpha)}{(s-a)^2} + \frac{2\alpha\beta}{(s-a)(b-s)} + \frac{\beta(1-\beta)}{(b-s)^2} \right) (s-a)^\alpha (b-s)^\beta ds =$$

$$= (1+\delta)(t-a)^\alpha (b-t)^\beta \quad \text{for } \ a<t<b.$$

Hence we obtain the contradiction $\delta \geq 1+\delta$. Thus we have proved that the problem (9),(2) has no solution.

The above-constructed example shows that the condition $\nu_{\alpha,\beta}(p) < 1$ in Theorems 1 and 2 is unimprovable and it cannot be replaced by the condition $\nu_{\alpha,\beta}(p) \leq 1$. Moreover, the strict inequality $\ell < 1$ in Corollaries 1–3 cannot be replaced by the non-strict one $\ell \leq 1$.

Now we consider the problem (1),(3).

Theorem 3. *Let*

$$\int\limits_t^b f^*(s,x)ds < +\infty \quad \textit{for } \ a<t<b, \ \ x>0, \tag{10}$$

and let the condition

$$f(t,x)\operatorname{sgn} x \geq -(t-a)^{-\alpha} p(t)|x| - q(t) \quad \textit{for } \ a<t<b, \ \ x \in R$$

be fulfilled, where f^* is a function, given by the equality (4)*, and $p,q:]a,b[\to [0,+\infty[$ are continuous functions such that*

$$\nu_\alpha(p) < 1, \quad \nu_\alpha(q) < +\infty. \tag{11}$$

Then the problem (1),(3) *has at least one solution.*

Corollary 4. *Let there exist a constant $\ell < \alpha(1-\alpha)$ and a continuous function $q:]a,b[\to [0,+\infty[$ such that*

$$f(t,x)\operatorname{sgn} x \geq -\frac{\ell}{(t-a)^2}|x| - q(t) \quad \textit{for } \ a<t<b, \ \ x \in R$$

and $\nu_\alpha(q) < +\infty$. If, moreover, the condition (10) *holds, then the problem* (1),(3) *has at least one solution.*

Theorem 4. *Let there exist a continuous function $p:]a,b[\to [0,+\infty[$ such that*

$$f(t,x) - f(t,y) \geq -(t-a)^{-\alpha} p(t)(x-y) \quad \textit{for } \ a<t<b, \ \ x>y,$$

and the conditions (11) *are satisfied, where $q(t) \equiv f(t,0)$. If, moreover, the condition* (10) *holds, then the problem* (1),(3) *is well-posed.*

Corollary 5. *Let there exist a constant $\ell < \alpha(1-\alpha)$ such that*

$$f(t,x) - f(t,y) \geq -\frac{\ell}{(t-a)^2}(x-y) \quad \textit{for } \ a<t<b, \ \ x>y.$$

If, moreover, $\nu_\alpha(f(\cdot,0)) < +\infty$ *and the condition* (10) *holds, then the problem* (1),(3) *is well-posed.*

For the Eq. (8), Corollary 5 yields

Corollary 6. *Let there exist a constant* $\ell < \alpha(1-\alpha)$ *such that*

$$f_1(t) \geq -\frac{\ell}{(t-a)^2} \quad \text{for } a < t < b.$$

If, moreover, $f_2(t) \geq 0$ *for* $a < t < b$*, and* $\nu_\alpha(f_0) < +\infty$*, then the problem* (8),(3) *is well-posed.*

Example 2. Let us consider the differential equation

$$u'' = -\frac{\ell}{(t-a)^2}|u| - (t-a)^{\alpha-2}, \tag{12}$$

where $\alpha \in \,]0,1[$ and ℓ is a nonnegative constant. If $\ell < \alpha(1-\alpha)$, then according to Corollary 5 the problem (12),(3) is well-posed. On the other hand, it is easy to show that if $\ell \geq \alpha/(1-\alpha)$, then the problem (12),(3) has no solution.

The above-constructed example shows that the condition $\nu_\alpha(p) < 1$ in Theorems 3 and 4 is unimprovable and it cannot be replaced by the condition $\nu_\alpha(p) = 1 + \varepsilon$ no matter how small $\varepsilon > 0$ would be. Analogously, the condition $\ell < \alpha(1-\alpha)$ in Corollaries 4–6 cannot be replaced by the condition $\ell = \alpha(1-\alpha)(1+\varepsilon)$.

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OPTIMAL SOLVABILITY CONDITIONS OF THE CAUCHY–NICOLETTI PROBLEM FOR SINGULAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Dedicated to the blessed memory of Professor T. Chanturia

Abstract. For the systems of singular functional differential equations the unimprovable sufficient conditions of solvability of the Cauchy–Nicoletti problem are established.

რეზიუმე. სინგულარული ფუნქციონალურ-დიფერენციალურ განტოლებათა სისტემებისათვის დადგენილია კოში–ნიკოლეტის ამოცანის ამოხსნადობის არაგაუმჯობესებადი საკმარისი პირობები.

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Let $-\infty < a < b < +\infty$,

$$I = [a,b], \;\; t_i \in I, \quad I_i = I \setminus \{t_i\} \;\; (i = 1,\dots,n).$$

In the interval I we consider a system of functional differential equations

$$\frac{dx_i(t)}{dt} = f_i(x_1,\dots,x_n)(t) \;\; (i = 1,\dots,n) \tag{1}$$

with the boundary conditions

$$x_i(t_i) = 0 \;\; (i = 1,\dots,n). \tag{2}$$

Here every f_i is the operator acting from the space of continuous on I n-dimensional vector functions to the space of functions, Lebesgue integrable on every closed interval contained in I_i. We are, in the main, interested in a singular case, in which there exist $i \in \{1,\dots,n\}$ and continuous functions $x_k : I \to R$ $(k = 1,\dots,n)$, such that

$$\int_a^b \big|f_i(x_1,\dots,x_n)(t)\big|\,dt = +\infty.$$

(2) are called the boundary conditions of Cauchy–Nicoletti. In the case, where $t_1 = \dots = t_n$, these conditions represent the initial, i.e. the Cauchy conditions.

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I. Kiguradze [4]–[8] has developed technique for a priori estimates of solutions of one-sided differential inequalities allowing one to investigate the Cauchy and Cauchy–Nicoletti problems for a singular differential system

$$\frac{dx_i(t)}{dt} = f_{0i}\big(t, x_1(t), \dots, x_n(t)\big) \ \ (i = 1, \dots, n) \tag{3}$$

which is a particular case of system (1). The singular problem (3), (2) is investigated also in [19].

I. Kiguradze and Z. Sokhadze [12], [13], [21] have found the sufficient conditions of local and global solvability of the Cauchy problem for evolution singular functional differential systems of type (1) and proved Kneser type theorem on the structure of a set of solutions of the above-mentioned problem [14].

Optimal sufficient conditions of solvability of two-point problems of Cauchy–Nicoletti type for singular differential equations of second and higher orders and for linear singular differential systems can be found in [1]–[3], [9], [11], [15]–[18], [20].

In the case, where f_i $(i = 1, \dots, n)$ are not evolution operators, for the singular functional differential system (1) not only the Cauchy–Nicoletti problem, but also the Cauchy problem remain little studied. Just that very case our work is devored to.

Throughout the paper, we adopt the following notation:

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$.

$\mathbb{R}^n$ is n-dimensional real Euclidean space.

$x = (x_i)_{i=1}^n$ and $X = (x_{ik})_{i,k=1}^n$ are the n-dimensional column vector and $n \times n$-matrix with elements x_i and $x_{ik} \in \mathbb{R}$ $(i = 1, \dots, n)$.

$r(X)$ is the spectral radius of the matrix X.

$C(I; \mathbb{R}^n)$ is the Banach space of the n-dimensional continuous vector functions $x = (x_i)_{i=1}^n : I \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\Big\{ \sum_{i=1}^n |x_i(t)| : \ t \in I \Big\}.$$

$L(I; \mathbb{R})$ is the Banach space of the Lebesgue integrable functions $y : I \to \mathbb{R}$ with the norm

$$\|y\|_L = \int_a^b |y(s)|\, ds.$$

$L_{loc}(I_i; \mathbb{R})$ is the space of functions $y : I_i \to \mathbb{R}$, Lebesgue integrable on every closed interval contained in I_i.

$\mathcal{K}_{loc}(I \times \mathbb{R}^m; \mathbb{R})$ is the set of functions $g : I \times \mathbb{R}^m \to \mathbb{R}$ satisfying the local Carathéodory conditions, i.e., such that $g(\cdot, x_1, \dots, x_n) : I \to \mathbb{R}$ is measurable for any $(x_k)_{k=1}^m \in \mathbb{R}^m$, $g(t, \cdot, \dots, \cdot) : \mathbb{R}^m \to \mathbb{R}$, continuous almost for all $t \in I$ and

$$g^*_\rho \in L(I; \mathbb{R}) \ \text{ for } \ \rho \in \mathbb{R}_+,$$

where

$$g^*_\rho(t) = \max\Big\{\big|g(t,x_1,\dots,x_n)\big| : \sum_{k=1}^m |x_k| \le \rho\Big\}. \tag{4}$$

$\mathcal{K}_{loc}(I_i \times \mathbb{R}^m;\mathbb{R})$ is the set of functions $g : I_i \times \mathbb{R}^m \to \mathbb{R}$, such that $g(\cdot,x_1,\dots,x_m) : I \to \mathbb{R}$ is measurable for any $(x_k)_{k=1}^m \in \mathbb{R}^m$, $g(t,\cdot,\dots,\cdot) : \mathbb{R}^m \to \mathbb{R}$, continuous for almost all $t \in I$ and

$$g^*_\rho \in L_{loc}(I_i;\mathbb{R}) \ \text{ for } \ \rho \in \mathbb{R}_+,$$

where g^*_ρ is the function defined by the equality (4).

$\mathcal{K}_{loc}(C(I;\mathbb{R}^n);L(I;\mathbb{R}))$ is the set of continuous operators $f : C(I;\mathbb{R}^n) \to L(I;\mathbb{R})$, such that

$$g^*_\rho \in L(I;\mathbb{R}) \ \text{ for } \ \rho \in \mathbb{R}_+,$$

where

$$g^*_\rho(t) = \sup\Big\{\big|f(x_1,\dots,x_n)(t)\big| : \sum_{k=1}^n \big\|x_k\big\|_C \le \rho\Big\}.$$

$\mathcal{K}_{loc}(C(I;\mathbb{R}^n);L_{loc}(I_i;\mathbb{R}))$ is the set of operators $f : C(I;\mathbb{R}^n) \to L_{loc}(I_i;\mathbb{R})$, such that

$$f \in \mathcal{K}_{loc}(C(I;\mathbb{R}^n);L_{loc}(J;\mathbb{R}))$$

for an arbitrary closed interval J contained in I_i.

We investigate the problem (1), (2) in the case, where

$$f_i \in \mathcal{K}_{loc}(C(I;\mathbb{R}^n);L_{loc}(I_i;\mathbb{R})) \ \ (i=1,\dots,n). \tag{5}$$

A vector function $(x_k)_{k=1}^n : I \to \mathbb{R}^n$ with absolutely continuous components $x_k : I \to \mathbb{R}$ $(k=1,\dots,n)$ is said to be a solution of the system (1) if it satisfies this system almost everywhere on I. The solution of the system (1), satisfying the boundary conditions (2), is said to be a solution of the problem (1), (2).

For an arbitrary $\delta > 0$, we put

$$\chi_i(t,\delta) = \begin{cases} 0 & \text{for } \ t \in [t_i-\delta, t_i+\delta] \\ 1 & \text{for } \ t \notin [t_i-\delta, t_i+\delta] \end{cases}$$

and along with (1) consider the functional differential system

$$\frac{dx_i(t)}{dt} = \lambda\chi_i(t,\delta) f_i(x_1,\dots,x_n)(t) \ \ (i=1,\dots,n) \tag{6}$$

depending on the parameters $\lambda \in\,]0,1]$ and $\delta > 0$.

The following propositions hold.

Theorem 1 (Principle of a Priori Boundedness)**.** *Let the condition* (5) *be fulfilled and there exist a positive number* δ_0 *and continuous functions* $\rho_i : I \to \mathbb{R}_+$ $(i=1,\dots,n)$*, such that*

$$\rho_i(t_i) = 0 \ \ (i=1,\dots,n)$$

and for arbitrary $\delta \in]0, \delta_0[$ *and* $\lambda \in]0, 1]$ *every solution* $(x_i)_{i=1}^n$ *of the problem* $(6), (2)$ *admits the estimates*

$$|x_i(t)| \le \rho_i(t) \ \ \textit{for} \ \ a \le t \le b \ \ (i = 1, \dots, n).$$

Then the problem $(1), (2)$ *has at least one solution.*

Theorem 2. *Let the condition* (5) *be fulfilled and there exist nonnegative operators*

$$p_i \in \mathcal{K}_{loc}(C(I; \mathbb{R}^n); L_{loc}(I; \mathbb{R})) \ \ (i = 1, \dots, n),$$

nonnegative numbers h_{ik}, h_i $(i, k = 1, \dots, n)$ *and nonnegative functions* $q_{ik} \in L(I; \mathbb{R})$, $q_i \in L(I; \mathbb{R})$ $(i, k = 1, \dots, n)$, *such that for any* $(x_k)_{k=1}^n \in C(I; \mathbb{R}^n)$, *almost everywhere on* I *the inequalities*

$$\begin{aligned} f_i(x_1, \dots, x_n)(t) \operatorname{sgn}\big((t - t_i)x_i(t)\big) \le \\ \le p_i(x_1, \dots, x_n)(t)\Big(- |x_i(t)| + \sum_{k=1}^n h_{ik}\|x_k\|_C + h_i\Big) + \\ + \sum_{k=1}^n q_{ik}(t)\|x_k\|_C + q_i(t) \ \ (i = 1, \dots, n) \end{aligned}$$

hold. If, moreover, the matrix $H = (h_{ik} + \|q_{ik}\|_L)_{i,k=1}^n$ *satisfies the condition*

$$r(H) < 1, \tag{7}$$

then the problem $(1), (2)$ *has at least one solution.*

For regular systems (1) and (3), the results analogous to Theorem 2 are contained in [10] and [22].

An important particular case (1) is the differential system with deviating arguments

$$\frac{dx_i(t)}{dt} = g_i\Big(t, x_1(\tau_1(t)), \dots, x_n(\tau_n(t)), x_i(t)\Big) \ \ (i = 1, \dots, n), \tag{8}$$

where

$$g_i \in \mathcal{K}_{loc}(I_i \times \mathbb{R}^{n+1}; \mathbb{R}) \ \ (i = 1, \dots, n), \tag{9}$$

and $\tau_i : I \to I$ $(i = 1, \dots, n)$ are measurable functions.

If

$$f_i(x_1, \dots, x_n)(t) \equiv g_i\Big(t, x_1(\tau_1(t)), \dots, x_n(\tau_n(t)), x_i(t)\Big) \ \ (i = 1, \dots, n),$$

then the condition (9) ensures the fulfilment of the condition (5). Thus from Theorem 2 we arrive at the following proposition.

Corollary 1. *Let the condition* (9) *be fulfilled and there exist nonnegative numbers* h_{ik}, h_i $(i, k = 1, \dots, n)$ *and nonnegative functions* $q_{ik} \in L(I; \mathbb{R})$,

$q_i \in L(I;\mathbb{R})$, $g_{0i} \in \mathcal{K}_{loc}(I \times \mathbb{R}^{n+1};\mathbb{R})$ $(i,k = 1,\dots,n)$, such that for every $i \in \{1,\dots,n\}$ the inequality

$$\begin{aligned} & g_i(t,y_1,\dots,y_{n+1}) \operatorname{sgn}\big((t-t_i)y_{n+1}\big) \le \\ & \le g_{0i}(t,y_1,\dots,y_{n+1})\Big(-|y_{n+1}| + \sum_{k=1}^n h_{ik}|y_k| + h_i\Big) + \sum_{k=1}^n q_{ik}(t)|y_k| + q_i(t) \end{aligned} \tag{10}$$

holds on the set $I_i \times \mathbb{R}^n$. If, moreover, the matrix $H = (h_{ik} + \|q_{ik}\|_L)_{i,k=1}^n$ satisfies the condition (7), then the problem (8), (2) has at least one solution.

Example 1. Let

$$\mu_i = \max\big\{t_i - a, b - t_i\big\}, \quad \tau_i = \begin{cases} a & \text{for } \ \mu_i = t_i - a \\ b & \text{for } \ \mu_i = t_i - b \end{cases}, \tag{11}$$

$$h_{ik} = 0 \ \text{ for } \ k < i, \quad h_{ik} > 0 \ \text{ for } \ k \ge i. \tag{12}$$

Consider the differential system

$$\begin{aligned} \frac{dx_i(t)}{dt} &= \frac{1 + |x_i(\tau_i)|}{\mu_i} \times \\ \times \bigg(-\frac{\mu_i x_i(t)}{|t - t_i|} &+ \sum_{k=1}^n h_{ik}|x_k(\tau_i)| + 2\bigg) \operatorname{sgn}(t - t_i) \ \ (i = 1,\dots,n). \end{aligned} \tag{13}$$

Clearly, for every $i \in \{1,\dots,n\}$ the function

$$g_i(t,y_1,\dots,y_{n+1}) \equiv \frac{1 + |y_i|}{\mu_i}\bigg(-\frac{\mu_i y_{n+1}}{|t - t_i|} + \sum_{k=1}^n h_{ik}|y_k| + 2\bigg) \operatorname{sgn}(t - t_i)$$

on $I_i \times \mathbb{R}^{n+1}$ satisfies the inequality (10), where

$$g_{0i}(t,y_1,\dots,y_{n+1}) \equiv \frac{1 + |y_i|}{\mu_i}, \quad h_i = 2,$$
$$q_{ik}(t) \equiv 0, \quad q_i(t) \equiv 0 \ \ (i,k = 1,\dots,n).$$

Moreover, taking into account (12), we have

$$H = \big(h_{ik} + \|q_{ik}\|_L\big)_{i,k=1}^n = (h_{ik})_{i,k=1}^n, \quad r(H) = \max\{h_{11},\dots,h_{nn}\}. \tag{14}$$

If the inequality (7) is fulfilled, then according to Corollary 1, the problem (13), (2) has at least one solution. Consider now the case, where inequality (7) is violated. Then in view of (14), there exists $i \in \{1,\dots,n\}$, such that

$$r(H) = h_{ii} \ge 1. \tag{15}$$

Assume that the problem (13), (2) has in this case a solution $(x_k)_{k=1}^n$, as well. Then

$$x_i(t) = \frac{1 + |x_i(\tau_i)|}{\mu_i(2 + |x_i(\tau_i)|)}\Big(2 + \sum_{k=1}^n h_{ik}|x_k(\tau_i)|\Big)|t - t_i| \ \text{ for } \ a \le t \le b.$$

Taking into account (11), (12) and (15), the above equality results in

$$x_i(\tau_i) = \frac{1+|x_i(\tau_i)|}{2+|x_i(\tau_i)|}\Big(2+\sum_{k=1}^{n} h_{ik}|x_k(\tau_i)|\Big) \geq 1+|x_i(\tau_i)|.$$

The obtained contradiction proves that the problem (13), (2) is solvable iff the inequality (7) is fulfilled. Consequently, the condition (7) in Theorem 2 and in Corollary 1 is optimal and it cannot be replaced by the condition

$$r(H) \leq 1.$$

Example 2. Consider the differential system

$$\frac{dx_i(t)}{dt} = -\Bigg[\exp\bigg(|t-t_i|^{-1}+\sum_{k=1}^{n}|x_k(\tau_k(t))|+|x_i(t)|\bigg)\operatorname{sgn}(t-t_i)\Bigg]x_i(t)+$$
$$+\, g_{1i}\Big(t, x_1(\tau_1(t)),\dots,x_n(\tau_n(t)),x_i(t)\Big), \tag{16}$$

where $\tau_k : I \to I$ $(k=1,\dots,n)$ are measurable functions, and $g_{1i} \in \mathcal{K}_{loc}(I\times \mathbb{R}^n;\mathbb{R})$ $(i=1,\dots,n)$ are the functions satisfying the inequality

$$\sum_{i=1}^{n}\big|g_{1i}(t,y_1,\dots,y_{n+1})\big| \leq \ell\exp\Big(\sum_{k=1}^{n+1}|y_k|\Big),$$

where $l = \text{const} > 0$. Then for any $i \in \{1,\dots,n\}$, the function

$$g_i(t,y_1,\dots,y_{n+1}) \equiv$$
$$\equiv -\Bigg[\exp\Big(|t-t_i|^{-1}+\sum_{k=1}^{n+1}|y_k|\Big)\operatorname{sgn}(t-t_i)\Bigg]y_{n+1} + g_{1i}(t,y_1,\dots,y_{n+1})$$

on the set $I_i \times \mathbb{R}^n$ admits the estimate (10), where

$$g_{0i}(t,y_1,\dots,y_{n+1}) \equiv \exp\Big(\sum_{k=1}^{n+1}|y_k|\Big), \quad h_{ik}=0, \quad h_i = \ell,$$
$$q_{ik}(t) \equiv 0, \quad q_i(t) \equiv 0 \quad (i=1,\dots,n).$$

Moreover, $H = (h_{ik} + \|q_{ik}\|_L)_{i,k=1}^{n}$ is a zero matrix, and hence $r(H) = 0$. Thus according to Corollary 1, it follows that the problem (16), (2) is solvable. On the other hand, it is evident that the system (16) in the case under consideration is superlinear, and the order of singularity for every function $g_i(\cdot,y_1,\dots,y_n) : I \to \mathbb{R}$ at the point t_i is equal to infinity, or more exactly, for an arbitrary natural m we have

$$\int_a^b |t-t_i|^m\big|g_i(t,y_1,\dots,y_{n+1})\big|\,dt = +\infty$$

if only $y_{n+1} \neq 0$.

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Mem. Differential Equations Math. Phys. 54 (2011)

CONTENTS

Short Communications

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