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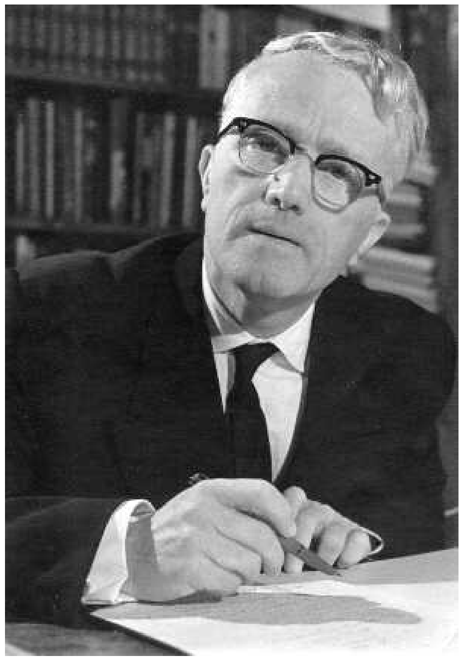

VIKTOR KUPRADZE 110

110 years have passed since the birthday of the outstanding Georgian scientist, a public figure and a statesman, academician Viktor Kupradze. Mathematicians and mechanicians throughout the world are well familiar with his name. Academician Viktor Kupradze made a tremendous contribution to the theory of differential and integral equations, problems of mathematical physics, the theory of elasticity and applied mathematics.

Viktor Kupradze was born on 2 November 1903 in village Kela in Georgia, in a railway worker's family. Little Viktor went to the specialized school in Kutaisi, where a comparatively extended course in mathematics was taught. Viktor's turn for mathematics attracted the attention of his teacher and, following his advice, in 1922 Kupradze became a student of the physico-mathematical faculty of the Tbilisi State University. In 1927 he graduated from the University with honours and as nominee of professors Andria Razmadze and Nikoloz Muskhelishvili, founders of the worldwide known Georgian mathematical school, was left at the University to be prepared for research work. He became an assistant of A. Razmadze in mathematical analysis and an assistant of N. Muskhelishvili in theoretical mechanics.

He also delivered lectures at the Tbilisi Polytechnical Institute. The scientific supervisor of Viktor Kupradze, professor N. Muskhelishvili wrote in the testimonial: “The post-graduate student has mastered quite well the main academic disciplines. He has invariably shown the ability to independent, creative and critical thinking. I can say with confidence that under proper conditions he will become an outstanding specialist in applied mathematics”.

In 1930–1933 he was a post-graduate student at the Academy of Sciences of the USSR in Leningrad (St. Petersburg), where his supervisors were the prominent Russian scientists Alexei Krilov and Vladimir Smirnov.

In the period from 1933 to 1935 Kupradze worked as scientific secretary at the Steklov Mathematical Institute of the Academy of Sciences of USSR. In 1935 he defended his doctor’s thesis (skipping the candidate thesis) on the topic: “Boundary Value Problems of the Electromagnetic Wave Theory”. In the same year Kupradze returned to Tbilisi where he was appointed director of the Tbilisi Mathematical Institute (now Andrea Razmadze Mathematical Institute). During the Great Patriotic War (the World War II) V. Kupradze served in the Soviet Army, participated in the cruel battles for Crimean Peninsula. Due to his fluent German, he was the Executive secretary of Editorial Board of the military newspaper “Zoldatenvaarheit” published in German. In 1943 he was demobilized and appointed pro-rector of the Tbilisi State University, responsible for research work.

From 1944 to 1953 Kupradze was the Minister of Education of Georgia.

In 1946 he was elected Full Member of the Academy of Sciences of Georgia.

In 1954–1958 he held the position of the rector of Tbilisi State University.

In 1962 the Georgian Mathematical Society was re-founded and V. Kupradze was elected its second president. The Society was first founded in 1923 by Andrea Razmadze, who was the president until he passed away in 1929.

In 1963 Kupradze was elected academician-secretary of the department of mathematics and physics of the Academy of Sciences of Georgia, where he worked fruitfully till 1981. At the same time he headed the chair of differential and integral equations of the Tbilisi State University. From 1947 to 1985 Kupradze was a member of Presidium of the Georgian Academy of Sciences.

V. Kupradze widely participated also in the public life of Georgia and the former USSR. In 1947 he took part in the Congress of Asiatic and African Peoples held in Delhi. From 1954 to 1963 he was Chairman of the Supreme Soviet (Parliament) of Georgia. In 1955 he was sent to the USA (New York) as a member of Soviet delegation to the 10-th Session of the UN General Assembly. V. Kupradze was actively involved in the international scientific cooperation. Being member of various reputable organizations such as the National Committee of Soviet Mathematicians, National Committee on Theoretical and Applied Mathematics, Bureau of the Scientific Council on Plasticity and Strength of the Academy of Sciences of the USSR. V. Kupradze played a significant role in strengthening scientific contacts between the scientists of different countries. He was a member of the editorial boards of domestic and international scientific journals, including “Uspekhi Matematicheskikh Nauk”, “Differentsial’nye Uravneniya”, “Journal of Thermal Stresses” etc.

Special tribute must be given to V. Kupradze as an excellent teacher, thesis adviser, and lecturer with a considerable personal charisma. For over 40 years he had been the head of the chair of differential and integral equations at Tbilisi State University and brought up several generations of Georgian mathematicians. He had many disciples and followers throughout the countries he visited. Attracted by Kupradze's charisma, many of his pupils became famous scientists and fruitfully continue mathematical scientific and academic activities both in Georgia and abroad.

V. Kupradze passed away on 25 April 1985, about 28 years ago, but all those people who knew him will cherish the memory of his warm, unforgettable personality and his profound intelligence.

The mathematical heritage of V. Kupradze is very rich. He began his scientific activities in the late twenties of the 20th century. His fruitful and tireless work actually has lasted about 55 years. V. Kupradze's contributions to mathematics and mechanics can be divided into six large groups:

- Problems related to the justification of Sommerfeld's Radiation Conditions and boundary value problems (BVP) for the Helmholtz equation;
- Diffraction and scattering of electro-magnetic waves;
- Mathematical problems of the theory of elasticity (BVPs of statics and steady state oscillations, and initial boundary value problems of general dynamics);
- Theory of one- and multi-dimensional singular integral equations and their applications;
- Investigation of refined models of the theory of elasticity (Thermoelasticity, Cosserat model etc.);
- Problems of numerical simulation and approximate solutions of BVPs of mathematical physics, Method of Fundamental Solutions.

A short account of V. Kupradze's contribution to the listed issues reads as follows.

1. Sommerfeld's radiation principle originally formulated in 1912 by the outstanding German physicist and mathematician A. Sommerfeld, concerns the existence and uniqueness of a solution to boundary value problems for the Helmholtz equation,

$$\Delta\, u(x) + k^2 u(x) = 0, \quad x \in \Omega, \tag{1}$$

where Δ is the Laplace operator, k^2 is a real valued constant, called the wavenumber, and Ω is an unbounded domain in the n-dimensional Euclidean space $\mathbb{R}^n$, $n = 2, 3, \dots$. The basic Dirichlet and Neumann boundary value problems, when either the traces of the solution itself or of its normal derivative are prescribed on the boundary $\Gamma := \partial\Omega$, have unique solutions only under special constraints on the growth of $u(x)$ at infinity

$$u(x) = \mathscr{O}\big(|x|^{\frac{1-n}{2}}\big) \ \text{ as } \ |x| \to \infty, \tag{2}$$

$$\frac{\partial u(x)}{\partial r} \pm ikr = \mathscr{o}\big(|x|^{\frac{1-n}{2}}\big) \ \text{ as } \ r := |x| \to \infty. \tag{3}$$

In 1934, V. Kupradze managed to substantiate this principle mathematically. He reduced these problems to Fredholm type boundary integral equations and showed the existence of a solution under sufficiently general conditions. Ten years later, the same result was obtained by H. Weyl. Moreover, Kupradze predicted and later I. Vekua and F. Rellich proved that the condition (2) is not independent and follows from the radiation condition (3).

2. Electromagnetic wave diffraction problems. A series of V. Kupradze's investigations are devoted to the diffraction of electromagnetic sinusoidal waves around an arbitrary plane contour, described by the Maxwell's equations

$$\begin{cases} \mathbf{curl}\, H + i\omega\varepsilon E = 0 \\ \mathbf{curl}\, E - i\omega\mu H = 0 \end{cases} \quad \text{in } \Omega \subset \mathbb{R}^2 \tag{4}$$

with corresponding boundary and transmission conditions.

These problems were previously solved by A. Sommerfeld, V. Sternberg, H. Freudental and other researchers for special domains with particular geometry. V. Kupradze made essential use of the method of integral equations.

He reduced the diffraction problems to equivalent boundary integral equations and proved their unique solvability.

For these results, in 1938 Viktor Kupradze was awarded the prize at the All-Union Competition of Young Scientists. It was included into the well-known V. Smirnov's university course on higher mathematics and translated into nearly all languages of the world.

3. Basic boundary value problems of statics and stationary oscillations of the elasticity theory. The approach developed for the Helmholtz equation, Viktor Kupradze generalized to investigate the system of stationary oscillation equations of elasticity

$$\mu\Delta\, U(x) + (\lambda+\mu)\,\mathbf{grad}\,\mathbf{div}\, U(x) + \varrho\omega^2 U(x) = 0, \quad x \in \Omega, \tag{5}$$

where $U(x) := (U_1(x), U_2(x)U_3(x))^\top$ is the displacement vector, λ and μ are Lamé constants, ϱ is the density of the elastic material, while ω is the oscillation frequency. On the boundary of the domain Ω (occupied by an elastic body) there is prescribed either the displacement vector

$$U^+(x) = F(x), \quad x \in \partial\Omega, \tag{6}$$

or the stress vector

$$(TU)^+(x) := 2\mu\,\frac{\partial U(x)}{\partial n} + \\ + \lambda n(x)\,\mathbf{div}\, U + \mu n(x) \times \mathbf{curl}\, U(x) = G(x), \quad x \in \partial\Omega. \tag{7}$$

For the system (5) endowed with the boundary condition either (6) or (7), V. Kupradze proved the uniqueness of a classical solution. Then he constructed solutions of three types, which he called a simple-, a double- and an antenna-layer potential. He investigated the fundamental properties of these potentials and derived jump formulas; Theorems analogous to the Lyapunov–Tauber theorem were proved, stating that the normal derivative of a regular harmonic double-layer potential is continuous up to the domain boundary (in contrast to the double layer potential and the normal derivative of a single-layer potential, which are discontinuous at the boundary). Furthermore, he proved an important fact that the above-mentioned boundary value problems are solvable under quite general conditions.

One of the first significant results obtained by V. Kupradze jointly with S. Sobolev concerns the wave propagation on the elastic body-fluid interface. The existence of a wave of a new type was established by mathematical means.

The basic boundary value problems of statics and steady-state oscillations of the elasticity theory with the first and second type boundary conditions, V. Kupradze reduced to equivalent systems of singular integral equations. In particular, he investigated the mentioned BVPs for homogeneous and piecewise-homogeneous elastic bodies showed that the corresponding boundary singular integral operators are of normal type.

From the 40s investigation of two- and three-dimensional problems of the elasticity theory held an ever growing place in the scientific activities of V. Kupradze and his followers. Building up a strong research team, he was extending, together with his disciples, the potential method to the basic boundary value and nonstandard transmission problems of the mathematical theory of elasticity. He constructed the matrix of fundamental solutions of the system of steady state elastic oscillations explicitly (now called “Kupradze’s matrix”) and formulated the radiation conditions in the elasticity theory (known as the “Sommerfeld–Kupradze principle”) which in the three-dimensional case read as follows,

$$\begin{cases} U = U^{(p)} + U^{(s)}, \\ \Delta U^{(p)} + k_1^2 U^{(p)} = 0, \quad \textbf{curl}\, U^{(p)} = 0, \quad k_1^2 = \dfrac{\varrho\,\omega^2}{\lambda + 2\mu}\,, \\ \Delta U^{(s)} + k_2^2 U^{(s)} = 0, \quad \textbf{div}\, U^{(s)} = 0, \quad k_2^2 = \dfrac{\varrho\,\omega^2}{\mu}\,, \\ \dfrac{\partial U^{(p)}(x)}{\partial r} - ik_1 U^{(p)}(x) = \mathscr{O}\big(|x|^{-1}\big) \ \text{ as } \ r = |x| \to \infty, \\ \dfrac{\partial U^{(s)}(x)}{\partial r} - ik_2 U^{(s)}(x) = \mathscr{O}\big(|x|^{-1}\big) \ \text{ as } \ r = |x| \to \infty, \end{cases} \tag{8}$$

where $U^{(p)}$ and $U^{(s)}$ are the so-called longitudinal (potential) and transverse (solenoidal) parts of the displacement vector U. These conditions have a crucial role in the proof of uniqueness theorems for exterior BVPs.

4. Multidimensional singular integral equations and their applications. In 1935, in his doctoral thesis V. Kupradze developed the method of potentials for three-dimensional problems of diffraction. During the subsequent 40 years V. Kupradze, and his collaborators developed and worked out the theory of singular integral equations on manifolds, generalizing results of S. Mikhlin and G. Giraud for multidimensional singular integral equations. They successfully applied the theory of singular potentials and newly created theory of singular integral equations to the analysis of boundary value problems of statics and steady state oscillations, as well as initial boundary value problems of general dynamics of the theory of elasticity. By the same approach, basic problems of some refined models of the theory of elasticity (anisotropic elasticity, thermoelasticity, couple-stress elasticity etc.) have been thoroughly studied. These results are exposed in the fundamental monograph "Three Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity" (V. Kupradze, T. G. Gegelia, M. O. Basheleishvili, T. V. Burchuladze; *North-Holland Publ. Comp., Amsterdam*, 1979). This monograph became a companion desk book for scientists working in the field.

5. Approximate solutions of boundary value problems of mathematical physics. In the early 1960s, by modifying and generalizing Picone's method V. Kupradze found new effective effective of constructing approximate solutions for a wide class of boundary value problems of mathematical physics. The method can be used for plane and spatial, basic and mixed boundary value problems of statics and oscillation theory in the case of homogeneous and piecewise-homogeneous, isotropic and anisotropic bodies. In the scientific literature this method is referred to as "Method of Fundamental Solutions" (MFS).

The main idea of the MFS is to distribute the singularity poles $\{y_k\}_{k=1}^{\infty}$ of the fundamental solution $\Gamma(x-y)$ of the differential operator outside the domain under consideration, construct the set of functions $\{\Gamma(x-y_k)\}_{k=1}^{\infty}$, prove its density properties in appropriate function spaces, and then approximate the solution by a linear combination of the fundamental solutions, $\sum\limits_{k=1}^{N} C_k\Gamma(x-y_k)$ with unknown coefficients C_k which are to be determined by satisfying the corresponding boundary conditions. Starting from 1970s, the MFS gradually became a useful technique for solving a large variety of physical and engineering problems.

The level of present-day computing facilities makes V. Kupradze's methods of constructing effective solutions even more important and enjoys ever growing popularity among mathematicians and engineers.

The theory and the methods developed by V. Kupradze are widely and successfully applied to many theoretical and practical spheres of mathematical physics and engineering even nowadays. That means that Viktor Kupradze as a celebrated scientist is still alive – as an intellectual and spiritual bridge from the 20th century to the 21st one.

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Viktor Kupradze's Scientific Works and Articles

1. On the definition of Green, Neumann and Klein functions for some simple domains. (Georgian) *Izv. Tbiliss. Univ.* **9** (1929), 219–240.
2. To the question on distribution of elastic waves at the boundary of two elastic media with different elastic properties (with S. L. Sobolev). (Russian) *Trudy Seismologich. Inst. Acad, Nauk SSSR*, 1930, No. 10, 1–23.
3. Über die Fortpflanzung der elastischen Schwingungen in Festen Medien. (German) *Trudy Mezhdunarodn. Seismicheskoi Conf. v Leningrade, L.*, 1930, 131–137.
4. On Mathie–Hankel's functions. (Russian) *Trudy Fiz.-Mat. Inst. im. V. A. Steklova Acad. Nauk SSSR* **4** (1933), 77–86.
5. Difrazione della onde elastiche sopra un contorno ellittico. *Atti della reale Accademia Nazionale dei Lincei*, ser. XI, **XVIII** (1933), fas. 3-4, 130–139.
6. Integral equations for electromagnetic waves. (Russian) *Doklady AN SSSR* **1** (1934), No. 4, 161–165; In german: Integralgleichunger für elektromagnetische Wellen. *Doklady AN SSSR* **1** (1934), No. 4, 166.
7. The method of integral equations in diffraction theory. (Russian) *Mat. Sbornik* **41** (1934), No. 4, 561–581 (German summary).
8. Some problems of diffraction in the theory of elasticity. (Russian) *Trudy Fiz.-Mat. Inst. im. V.A. Steklov AN SSSR* **5** (1934), 295–350.
9. On A. Zommerfeld's "radiation principle". (Russian) *Doklady AN SSSR* **1** (1934), No. 2, 52–58; In German: Über das "Ausstrahlungsprinzip" von A. Sommerfeld. *Doklady AN SSSR* **1** (1934), No. 2, 55–58.
10. Solution of boundary value problems of Helmholtz equations in extraordinary cases. (Russian) *Doklady AN SSSR* **2** (1934), No. 9, 521–526; In German: Lösung von Randwertproblemen für Helmholtz'sche Gleichungen in den ausgenommenen Fällen. *Doklady AN SSSR* **2** (1934), No. 9, 524–526.
11. The existence and uniqueness theorems in the diffraction theory. (Russian) *Doklady AN SSSR* **1** (1934), No. 5, 235–240; In German: Existenzbeweis und Eindeuting keitstheorem in der Diffraktionstheorie. *Doklady AN SSSR* **1** (1934), No. 5, 238–240.
12. The method of integral equations in the diffraction theory. (Russian) *Mat. Sbornik* **41** (1935), No. 4, 561–581.
13. Generalized "radiation principle" in the theory of elasticity. (Russian) *Doklady AN SSSR* **2** (1935), No. 1, 14–19; In German: Das verallde-menterte "Ausstrahlungsprinzip" in der Elastizitätstheorie. *Doklady AN SSSR* **2** (1935), No. 1, 17–19.

14. Basic problems of the mathematical theory of diffraction (Stationary processes). (Russian) *L.-M., Central Publishing Company of technical Literature*, 1935, pp. 111.
15. The uniqueness theorem in stationary boundary value problems of the theory of elasticity. (Russian) *Doklady AN SSSR* **2** (1935), No. 2, 100–102; In German: Das Eindeutigkeitstheorem in den Randwertaufgaben der stationarein Elastizitätstheorie. *Doklady AN SSSR* **2** (1935), No. 2, 102–104.
16. On Some singular integral equations of mathematical physics. (Russian) *Uspekhi Mat. Nauk*, 1936, No. 2, 196–237.
17. Distribution of electromagnetic waves in an inhomogeneous medium. (Russian) *Doklady AN SSSR* **1** (1936), No. 1, 6–8.
18. The uniqueness theorem in boundary value problems of the stationary theory of elasticity. (Russian) *Trudy Tbiliss. Univ.* **2** (1936), 256–272.
19. To the investigation of electromagnetic oscillations in a plane inhomogeneous field. (Russian) *Doklady AN SSSR* **XVI** (1937), No. 3, 173–176.
20. Distribution of electromagnetic waves in an inhomogeneous medium. (Russian) *Trudy Tbiliss. Mat. Inst.* **1** (1937), 115–123 (German summary).
21. The solution of general problem of diffraction of electromagnetic waves. (Russian) *Trudy Tbiliss. Mat. Inst.* **2** (1937), 143–162 (German summary); *Doklady AN SSSR* **XVI** (1937), No. 1, 31–34.
22. Zur Frage der Ausbreitung elektromagnetischer Wellen in einem inhomogenen ebenen Medium. *Composito Mathematica* **6** (1938), fac. 2, 228–234.
23. Some new applications of the resolvent theory to the boundary value problems of the potential theory. (Russian) *Doklady AN SSSR* **XXIII** (1939), No. 1, 7–14.
24. Zur Frage der Ausbreitung elektromagnetischer Wellen in einem inhomogenen ebenen Medium. *Compositio Math.* **6** (1939), 228–234.
25. On the solution of the Dirichlet problem for multiply-connected domains. (Russian) *Mitt. Georg. Abt. Akad. Wiss. USSR [Soobshchenia Gruzinskogo Filiala Akad. Nauk SSSR]* **1** (1940), 569–571.
26. Duality theorem in radiotelegraphy. (Russian) *Mitt. Georg. Abt. Akad. Wiss. USSR [Soobshchenia Gruzinskogo Filiala Akad. Nauk SSSR]* **1** (1940), No. 8, 573–576.
27. Eindeutigkeitssatz in der Theorie der Fortpflanzung elektromagnetischer harmonischer Schwingungen im inhomogenen dreidimensionalen Raum (with D. Awazaschwili). (Russian) With complete German translation *Mitt. Georg. Abt. Akad. Wiss. USSR [Soobshchenia Gruzinskogo Filiala Akad. Nauk SSSR]* **1** (1940), 35–41.

28. On the theory of integral equations with the principal Cauchy value integrals. (Russian) *Mitt. Akad. Wiss. Georgischen SSR [Soobshch. Akad. Nauk Gruzin. SSR]* **2** (1941), 23–28, 227–232.
29. On the theory of integral equations with integrals in the sense of Cauchy's principal values. (Russian) *Bull. Acad. Sci. Georgian SSR [Soobshch. Akad. Nauk Gruzin. SSR]* **2** (1941), 587–596.
30. On a problem of equivalence in the theory of singular integral equations. (Russian) *Bull. Acad. Sci. Georgian SSR [Soobshch. Akad. Nauk Gruzin. SSR]* **2** (1941), 793–798.
31. Zur Theorie der Integralgleichungen mit dem Integral im Sinne des Cauchyschen Hauptwertes. (Russian) *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **5** (1941), 255–262.
32. Duality theorem in wireless telegraphy. (Russian) *Izv. Gruz. Industr. Inst. im. S. M. Kirova*, 1941, Book 14, 103–108.
33. On one composition formula of singular integrals. (Russian) *Trudy Tbiliss. Univ.* **XXIII** (1942), 159–164.
34. To the theory of systems of singular integral equations. (Russian) *Izv. Gruz. Industr. Inst. im. S. M. Kirova*, 1943, No. 15, 2–13.
35. A Letter to Editorial Board [dealing with the paper On some fundamental properties of metaharmonic functions. (Russian) *Bull. Acad. Sci. Georgian SSR [Soobshch. Akad. Nauk Gruzin. SSR]* **4** (1943), 281–288]. *Bull. Acad. Sci. Georgian SSR [Soobshch. Akad. Nauk Gruzin. SSR]* **4** (1943), No. 7, 733–734.
36. The Noether theorems for systems of singular integral equations. (Russian) *Izv. Gruz. Industr. Inst. im. S. M. Kirova*, 1943, No. 1(15), 315–321.
37. Certain new theorems on the equation of vibrations and their applications to boundary problems. (Russian) *Trav. Univ. Tbilissi* **26A**, (1945), 1–11.
38. Solution of a fundamental boundary problem in the displacements for vibrations of an elastic medium. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR.* **9** (1948), 99–106.
39. Proof of theorems on the existence for the basic dynamical boundary value problems of the theory of elasticity. (Russian) *Scientific Session of Tbilisi State University (May 31 – June 5, 1948)*. Program of work and theses of reports. *Tbilisi Univ. Press, Tbilsi*, 1948, p. 5.
40. A space problem on the oscillation of an elastic body with given displacements on the boundary. (Russian) *Doklady Akad. Nauk SSSR (N.S.)* **67** (1949) 233–236.
41. The spatial dynamical problem of the theory of elasticity with given displacements on the boundary. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR.* **10** (1949), 3–8.

42. The spatial dynamical problem of the theory of elasticity with given stresses on the boundary. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR.* **10** (1949), 257–262.

43. Boundary problems of the theory of vibrations and integral equations. (Russian) *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow–Leningrad*, 1950, 280 pp.

44. On the boundary value problems of the steady vibrations of elastic bodies. (Russian) *Uspehi Matem. Nauk (N.S.)* **5** (1950), No. 3(37), 190–193.

45. Efficient solution of some problems of the theory of elasticity. (Russian) *Enlarged Scientific Session of A. Razmadze Mathematical Institute (January 26–28, 1950).* Program of the Work and Theses of Reports. *Izdat. Akad. Nauk Gruz. SSR, Tbilisi*, 1950, p. 14.

46. Some new remarks to the theory of singular integral equations. (Russian) *Trudy Tbil. Univ.* **42** (1951), 1–23.

47. Fundamental problems in the mathematical theory of diffraction (steady state processes). Translated by C. D. Benster. NBS Rep. 2008, *U. S. Department of Commerce, National Bureau of Standards, Los Angeles, Calif.*, 1952.

48. Boundary problems of the theory of steady elastic vibrations. (Russian) *Uspehi Matem. Nauk (N.S.)* **8** (1953), No. 3(55), 21–74.

49. New integral equations of the theory of elasticity of anisotropic bodies (with M. O. Basheleishvili). (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **15** (1954), 327–334.

50. New integral equations of the anisotropic theory of elasticity and their application to the solution of boundary problems (with M. O. Basheleishvili). (Russian) *Soobsc. Akad. Nauk Gruzin. SSR* **15** (1954), 415–422.

51. Randwertaufgaben der Schwingungstheorie und integralgleichongen. *Veb Deutscher verlag der Wissenschaften, Berlin*, 1956, 239 pp.

52. On some new works in the mathematical theory of elasticity at the Tbilisi University. (Russian) *Trudy III Vsesojuznogo Mat. s'ezda, Moscow*, 1956. Review reports. *M., Izd. AN SSSR* **3** (1958), 453–462.

53. On boundary problems in the theory of elasticity for piece-wise inhomogeneous bodies. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **22** (1959), 129–136.

54. Boundary problems in the theory of elasticity for piecewise non-homogeneous bodies. *Soobshch. Akad. Nauk Gruzin. SSR* **22** (1959), 265–271.

55. Theory of boundary problems for non-homogeneous elastic bodies. Fundamental theorem of equivalence. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **22** (1959), 401–408.

56. On boundary problems in the theory of elasticity for piecewise-inhomogeneous bodies. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **22** (1959), 521–528.

57. New boundary value problems of elasticity. (Russian) *Trudy Tbil. Univ.* **76** (1959), 1–41.

58. On the boundary value problems of the theory of elasticity for piecewise inhomogeneous media. (Russian). *Trudy Vsesojuznogo Soveshchania po Differents. Uravn. (Yerevan, November, 1958)*, pp. 102–106, *Izd. Akad. Nauk Arm. SSR, Yerevan*, 1960.

59. Singular integral equations and boundary-value problems of elasticity theory. (Russian) *Tbiliss. Gos. Univ. Trudy Ser. Meh.-Mat. Nauk* **84** (1962) 63–75.

60. The method of singular integral equations in the spatial theory of elasticity. (Russian) *Trudy Vsesojuzn. s'ezda po teoretich. i prikl. mekhanike*, 1960. *M.-L., Izd. AN SSSR*, 1962, 374–383.

61. On one method of an approximate solution of some problems of diffraction. (Russian). *The 2nd All-Union Symposium in Wave Diffraction Vtoroy Vsesojuzn. Simpoz. po difraktsii voln (Gorkii, June 7–13, 1962)*, pp. 5–7, *Annotatsii dokladov. Izd. AN SSSR, Moscow*, 1962.

62. An approximate method of solving certain boundary-value problems (with M. A. Aleksidze). (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **30** (1963), 529–536.

63. Potential-theoretic methods in the theory of elasticity. (Russian) *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow*, 1963, 472 pp.

64. A general mixed boundary-value problem in elasticity theory and potential theory (with T. V. Burchuladze). (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **32** (1963), 27–34.

65. Progress in Solid Mechanics. Vol.III. Dynamical problems in elasticity by V. D. Kupradze. Edited by I. N. Sneddon and R. Hill *North-Holland Publishing Co., Amsterdam; Interscience Publishers John Wiley & Sons, Inc. New York*, 1963.

66. The method of functional equations for the approximate solution of certain boundary-value problems (with M. A. Aleksidze). (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **4** (1964), 683–715.

67. On an approximate solution of limiting problems of mathematical physics. (Russian). *The 3rd All-Union Symposium in Wave Diffraction Tretii Vsesojuzn. Symp. po difraktsii voln (Tbilisi, September, 24–30, 1964)*, pp. 110–113, *Referati dokladov, Nauka, Moscow*, 1964.

68. On a method of solving approximately the limiting problems of mathematical physics. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **4** (1964), 1118–1121.

69. Potential methods in the theory of elasticity. (Russian) In: *Prilozhenia teorii funktsii v mekhanike sploshnoi sredy. Trudy Mezhdunarodn. Simposiuma* (17–23 *September, Tbilisi*, 1963). *Solid mechanics, Nauka, Moscow* **1** (1965), 211–216.

70. On the completeness of certain classes of functions. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **37** (1965), 257–258.

71. Potential methods in the theory of elasticity. Translated from the Russian by H. Gutfreund. Translation edited by I. Meroz Israel Program for Scientific Translations, *Jerusalem; Daniel Davey & Co., Inc., New York*, 1965, 339 pp.

72. The potential methods in the theory of elasticity (with T. V. Burchuladze). (Russian) *Lectures at the course-conference on dynamical problems of the theory of elasticity in Warszawa in Polish Academy of Sciences, 1966*, (*Lektsii na curs-conferentsii po dinamicheskim problemam teorii uprugosti v Varshave v Pol'skoi Akademii Nauk, 1966*) Tb., 1966, 174 p.

73. The potential theory methods in the theory of elasticity. (Polish) *Zaklad Narodowy imienia ossolinskich wydawnictwo. Polskiej Akademii nauk, Wroclaw–Warszawa–Krakow*, 1966.

74. Approximate solution of problems of mathematical physics. (Russian) *Uspehi Mat. Nauk* **22** (1967), No. 2(134), 59–107.

75. The basic existence theorems of space elasticity theory. (Russian) *Differencial'nye Uravnenija* **3** (1967), 707–721.

76. In the basic direction: [At the Chair of Differential and Integral Equations of Tbilisi University]. (Russian) *Vechernii Tbilisi*, 1968, October 24, p. 3.

77. Three-dimensional problems in the mathematical theory of elasticity (with T. G. Gegelia, M. O. Baseleisvili, and T. V. Burchuladze). (Russian) *Izdat. Tbilis. Univ., Tbilisi*, 1968, 627 pp. (loose errata).

78. Boundary value problems of thermoelasticity (with T. V. Burchuladze). (Russian) *Differencial'nye Uravnenija* **5** (1969), 3–43.

79. Certain thermoelasticity problems solvable in quadratures, I (with T. V. Burchuladze). (Russian) *Differencial'nye Uravnenija* **5** (1969), No. 10, 1735–1761;

80. Certain thermoelasticity problems solvable in quadratures, II (with T. V. Burchuladze). (Russian) *Differencial'nye Uravnenija* **5** (1969), No. 11, 1923–1939.

81. Über numerische Verfahren der Potential- und der Elastizitätstheorie. *Z. Angew. Math. Mech.* **49** (1969), 1–9.

82. Solution of the dynamical problems of elasticity theory (with T. V. Burchuladze). (Russian) *Gamoqeneb. Math. Inst. Sem. Mohsen. Anotacie. Vyp.* **3** (1970), 67–77.

83. Wybrane zagadnienia teorii spresystosci i termosprezystosci. Zaklad Narodowy imienia ossolinskich wydawnictwo (with T. G. Gegelia, M. O. Baszeleiszwvili, and T. V. Burczuladze). *Polskiej Akademii nauk, Wroclaw–Warszawa–Krakow*, 1970, 348 pp.

84. A proof of the existence and computation of the solutions of the fundamental mixed problems of the dynamics of a three-dimensional elastic body of arbitrary form (with T. V. Burchuladze). (Russian) *Sakharth. SSR Mecn. Akad. Math. Inst. Shrom.* 39 (1971), 23–42.

85. On dynamic problems of the theory of elasticity (with T. V. Burchuladze). *Trends in elasticity and thermoelasticity*, pp. 135–149. *Wolters–Noordhoff, Groningen*, 1971.

86. On the solution of a three-dimensional mixed boundary value problem of elasticity theory. (Russian) *Continuum mechanics and related problems of analysis (on the occasion of the eightieth birthday of Academician N. I. Mushelisvili) (Russian)*, pp. 267–271. *Izdat. "Nauka", Moscow*, 1972.

87. Lectures in modern problems of the mathematical theory of elasticity. *Tbilisi University Press, Tbilisi*, 1974, 116 pp.

88. The dynamical problems of the theory of elasticity and thermoelasticity (with T. V. Burchuladze). (Russian) *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat.* **7** (1975), 163–294; translation in *J. Sov. Math.* **7** (1977), 415–500.

89. Über die Losung einiger nichtklassischer Probleme der Elastizitatstheorie. *Function theoretic methods for partial differential equations (Proc. Internat. Sympos., Darmstadt, 1976)*, pp. 68–72. *Lecture Notes in Math.*, Vol. 561, *Springer, Berlin*, 1976.

90. Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity (with T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze). (Russian) Second edition, revised and supplemented. Edited by V. D. Kupradze. *Izdat. "Nauka", Moscow*, 1976, 663 pp.

91. A direct method for the solution of boundary value problems of mathematical physics. (Russian) *Problems in mathematical physics and numerical mathematics (Russian)*, pp. 195–199, 326, *"Nauka", Moscow*, 1977.

92. On the question of the solution of exterior problems of electrodynamics. (Russian) *Complex analysis and its applications (Russian)*, pp. 284–294, 668, *"Nauka", Moscow*, 1978.

93. On dynamic and related problems of elasticity. *Trends in applications of pure mathematics to mechanics, Vol. II (Second Sympos., Kozubnik,*

1977), pp. 179–186, *Monographs Stud. Math.*, 5, *Pitman, Boston, Mass.–London*, 1979.

94. On contact problems of the theory of elasticity. (Russian) *All-Union Conference in the Theory of Elasticity (Vsesojuzn. Conferentsia po teorii uprugosti)*, Tezisy dokladov., pp. 6-12, *Izd. AN ASSR, Yerevan*, 1979.

95. Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity (with T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze). Translated from the second Russian edition. Edited by V. D. Kupradze. *North-Holland Series in Applied Mathematics and Mechanics*, 25. *North-Holland Publishing Co., Amsterdam-New York*, 1979, 929 pp.; Rev.: Sneddon I. N., *Bulletin (New Series) of the American Mathematical Society* **3** (1980), No. 2, 870–879.

96. Contact problems of the theory of elasticity. (Russian) *Differentsial'nye Uravneniya* **16** (1980), No. 2, 293–310, 381.

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BOUNDARY VALUE PROBLEMS IN WEIGHTED SOBOLEV SPACES ON LIPSCHITZ MANIFOLDS

Dedicated to Victor Kupradze on his 110-th birthday anniversary

Abstract. We explore the extent to which well-posedness results for the Poisson problem with a Dirichlet boundary condition hold in the setting of weighted Sobolev spaces in rough settings. The latter includes both the case of (strongly and weakly) Lipschitz domains in an Euclidean ambient, as well as compact Lipschitz manifolds with boundary.

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რეზიუმე. ჩვენ ვიკვლევთ პუასონის განტოლებისათვის დირიხლეს სასაზღვრო ამოცანის განზოგადებას სობოლევის წონიან სივრცეებში არაგლუვი ზედაპირების შემთხვევაში. განხილულია ამოცანები (სუსტი და ძლიერი ამონახსნების აზრით) ლიპშიცის არეებისათვის ევკლიდეს სივრცეში და ლიპშიცის მრავალსახეობაზე საზღვრით.

1. Introduction

One of the fundamental issues in analysis is that of correlating the regularity of a geometric ambient to the well-posedness of boundary value problems arising naturally in that setting. For example, the treatment of elliptic boundary value problems formulated on scales of Sobolev/Besov spaces for differential operators with smooth coefficients is rather complete in the setting of $\mathscr{C}^\infty$ manifolds. See, e.g., [7], [10], [17]. By way of contrast, there are many interesting open questions formulated in the presence of less regular structures (see [8]).

Very often, a basic result which is used to jump-start the theory is the classical Lax–Milgram lemma. However, while this requires very little regularity for the objects involved, one is forced to stay within the constraints of Hilbert space structures, which enter typically through the considerations of L^2 (and various L^2-based) spaces.

In this paper we explore the extent to which it is possible to depart from this basic case and consider L^p-based Sobolev spaces with p not necessarily equal to 2. We do so without having to strengthen the original assumptions pertaining to the nature of the coefficients (which are assumed to be only bounded and measurable), and this naturally imposes limitations on the parameters intervening in the spaces involved. On the geometric side, the main novelty is the fact that we succeed in formulating our main well-posedness results in the rather general setting of Lipschitz manifolds. Not only does this category of manifolds encompass many particular cases of great interest for applications, but this also constitutes the minimally smooth setting where our problems may be formulated and solved. As such, our results are sharp from a multitude of perspectives.

The organization of the paper is as follows. In Section 2 we consider weighted Sobolev spaces of arbitrary smoothness in Euclidean Lipschitz domains and prove that Stein's extension operator continues to work in this setting. In turn, this is used to establish a very useful interpolation result (cf. Theorem 2.6). In Section 3 we study the trace theorem for such weighted Sobolev spaces, while in Section 4 we construct a boundary extension operator (which serves as an inverse from the right for the trace mapping). In Section 5 we treat boundary value problems for elliptic systems with bounded measurable coefficients in Euclidean Lipschitz domains. Our main well-posedness result in this regard is contained in Theorem 5.1. By means of counterexamples this is shown to be sharp. The scope of the theory developed up to this point is enlarged in Section 6 through the consideration of the class of weakly Lipschitz domains. Finally, in Section 7, we further generalize these results to the setting of compact Lipschitz manifolds with boundary. This portion of our paper may be regarded as a natural continuation of the work initiated in [4].

2. Weighted sobolev spaces and Stein's Extension Operator

We shall also work with the following weighted version of classical Sobolev spaces, which have been previously considered in [12].

Definition 2.1. If $p \in [1, \infty]$, $a \in (-1/p, 1-1/p)$ and $m \in \mathbb{N}_0$ are given and Ω is a nonempty, proper, open subset of $\mathbb{R}^n$, consider the weighted Sobolev space $W^{m,p}_a(\Omega)$, defined as the space of locally integrable functions u in Ω for which $\partial^\alpha u \in L^1_{loc}(\Omega)$ (with derivatives taken in the sense of distributions) whenever $\alpha \in \mathbb{N}^n_0$ has $|\alpha| \leq m$, and

$$\|u\|_{W^{m,p}_a(\Omega)} := \Bigg(\sum_{|\alpha|\leq m} \int_\Omega |(\partial^\alpha u)(x)|^p \operatorname{dist}(x, \partial\Omega)^{ap}\, dx \Bigg)^{1/p} < \infty. \tag{2.1}$$

Finally, in the case when Ω is understood from the context, we shall employ the notation

$$W^{m,p}_a(\mathbb{R}^n) := \Bigg\{ u \in L^1_{loc}(\mathbb{R}^n) :\ \partial^\alpha u \in L^1_{loc}(\mathbb{R}^n) \text{ whenever } |\alpha| \leq m, \text{ and}$$

$$\|u\|_{W^{m,p}_a(\mathbb{R}^n)} := \sum_{|\alpha|\leq m} \Bigg(\int_{\mathbb{R}^n} |(\partial^\alpha u)(x)|^p \operatorname{dist}(x, \partial\Omega)^{ap}\, dx \Bigg)^{1/p} < \infty \Bigg\}. \tag{2.2}$$

We wish to stress that $W^{m,p}_a(\mathbb{R}^n)$ is *not* $W^{m,p}_a(\Omega)$ corresponding to $\Omega = \mathbb{R}^n$ (which, incidentally, is not a permissible choice since Ω is assumed to be a proper subset of $\mathbb{R}^n$). Instead, the named space should always be understood in the sense of (2.2).

Hence, the case when $a = 0$ in Definition 2.1 describes the standard Sobolev spaces (L^p-based, of order m) defined intrinsically in the open set Ω. In such a scenario, we omit including $a(=0)$ in the notation for these spaces and simply write $W^{m,p}(\Omega)$.

Fix a Lipschitz domain Ω in $\mathbb{R}^n$ and recall from [1, Theorem 3.22, p. 68] that, since Ω satisfies the so-called segment condition, the inclusion operator

$$\mathscr{C}^\infty_b(\overline{\Omega}) \hookrightarrow W^{m,p}(\Omega) \text{ has dense range, if } p \in [1, \infty),\ m \in \mathbb{N}_0. \tag{2.3}$$

On the other hand, in the weighted case, given any Lipschitz domain Ω,

$$\begin{gathered} \mathscr{C}^\infty_b(\overline{\Omega}) \hookrightarrow W^{m,p}_a(\Omega) \text{ has dense range,} \\ \text{if } p \in (1, \infty),\ m \in \mathbb{N}_0, \text{ and } a \in (-1/p, 1-1/p). \end{gathered} \tag{2.4}$$

This is proved much as in (2.3), the new key technical ingredient being the fact that, given any Lipschitz domain $\Omega \subseteq \mathbb{R}^n$,

$$\begin{gathered} \operatorname{dist}(\,\cdot\,, \partial\Omega)^{ap} \text{ is a Muckenhoupt } A_p\text{-weight in } \mathbb{R}^n \\ \text{whenever } p \in (1, \infty) \text{ and } a \in (-1/p, 1-1/p). \end{gathered} \tag{2.5}$$

See [15] for more details in somewhat similar circumstances.

Let $\mathscr{L}^n$ denote the Lebesgue measure in $\mathbb{R}^n$.

Definition 2.2. Assume that $p \in (1,\infty)$ and $a \in (-1/p, 1-1/p)$ are given, and that Ω is a nonempty, proper, open subset of $\mathbb{R}^n$. In this context, let $L^p(\Omega, \operatorname{dist}(\cdot\,,\partial\Omega)^{ap}\,\mathscr{L}^n)$ denote the weighted Lebesgue space consisting of $\mathscr{L}^n$-measurable functions whose p-th power is absolutely integrable with respect to the weighted measure $\operatorname{dist}(\cdot\,,\partial\Omega)^{ap}\,\mathscr{L}^n$. Also, for each $m \in \mathbb{N}_0$, define the weighted Sobolev space of negative order $W_a^{-m,p}(\Omega)$ as the subspace of the space of distributions $\mathscr{D}'(\Omega)$ given by

$$W_a^{-m,p}(\Omega) := \Big\{ u \in \mathscr{D}'(\Omega) : \text{ there exist } \{f_\alpha\}_{|\alpha|\leq m} \subset L^p\big(\Omega, \operatorname{dist}(\cdot\,,\partial\Omega)^{ap}\,\mathscr{L}^n\big) \text{ such that } u = \sum_{|\alpha|\leq m} \partial^\alpha f_\alpha \text{ in } \mathscr{D}'(\Omega) \Big\}. \tag{2.6}$$

Equip this space with the norm

$$\|u\|_{W_a^{-m,p}(\Omega)} := \inf_{u=\sum\limits_{|\alpha|\leq m}\partial^\alpha f_\alpha} \Bigg(\sum_{|\alpha|\leq m} \int_\Omega |f_\alpha(x)|^p \operatorname{dist}(x,\partial\Omega)^{ap}\,dx \Bigg)^{1/p}. \tag{2.7}$$

Finally, introduce

$$\mathring{W}_a^{m,p}(\Omega) := \text{the completion of } \mathscr{C}_c^\infty(\Omega) \text{ in } W_a^{m,p}(\Omega), \tag{2.8}$$

and endow this space with the norm inherited from $W_a^{m,p}(\Omega)$.

The scales of spaces introduced above enjoy a number of useful properties, some of which are discussed in the proposition below.

Proposition 2.3. *Let $p \in (1,\infty)$, $a \in (-1/p, 1-1/p)$, and $m \in \mathbb{N}_0$ be given, and suppose Ω is a nonempty open subset of $\mathbb{R}^n$. Then $W_a^{m,p}(\Omega)$, $\mathring{W}_a^{m,p}(\Omega)$, $W_a^{-m,p}(\Omega)$ are reflexive Banach spaces and*

$$\big(\mathring{W}_a^{m,p}(\Omega)\big)^* = W_{-a}^{-m,p'}(\Omega), \tag{2.9}$$

where $1/p + 1/p' = 1$.

Proof. Fix a, p as in the statement and let N be the number of multi-indices $\alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| \leq m$. Define the injection $j : W_a^{m,p}(\Omega) \to [L^p\big(\Omega, \operatorname{dist}(\cdot\,,\partial\Omega)^{ap}\,\mathscr{L}^n\big)]^N$ by setting $j(u) := \{\partial^\alpha u\}_{|\alpha|\leq m}$. Then j is an isometry identifying $W_a^{m,p}(\Omega)$ with a closed subspace of $[L^p(\Omega, \operatorname{dist}(\cdot\,,\partial\Omega)^{ap}\,\mathscr{L}^n)]^N$. Since the latter is a reflexive Banach space, it follows that so is $W_a^{m,p}(\Omega)$. Having established this, it follows from (2.8) that $\mathring{W}_a^{m,p}(\Omega)$ is also a reflexive Banach space. Finally, that $W_a^{-m,p}(\Omega)$ is a reflexive Banach space will follow from what we have just established, once we justify the duality formula (2.9). This, in turn, is a consequence of the aforementioned isometric embedding of $W_a^{m,p}(\Omega)$ into a direct sum of weighted Lebesgue spaces, the Hahn–Banach theorem, and Riesz representation formula. □

Our next goal is to discuss the action of Stein's extension operator in the context of weighted Sobolev spaces. This requires some preparations and we begin by recalling that the function $\psi : [1,\infty) \to \mathbb{R}$ given by

$$\psi(\lambda) := \frac{e}{\pi\lambda} \cdot \mathrm{Im}\big\{e^{-e^{-i\pi/4}\cdot(\lambda-1)^{1/4}}\big\}, \quad \forall\, \lambda \geq 1, \tag{2.10}$$

has, according to [16, Lemma 1, p. 182], the following properties:

$$\psi \in \mathscr{C}^0([1,\infty)), \tag{2.11}$$

$$\int_1^\infty \psi(\lambda)\, d\lambda = 1, \tag{2.12}$$

$$\int_1^\infty \lambda^k \psi(\lambda)\, d\lambda = 0, \quad \forall\, k \in \mathbb{N}, \tag{2.13}$$

$$\psi(\lambda) = \mathscr{O}(\lambda^{-N}), \quad \forall\, N \in \mathbb{N} \text{ as } \lambda \to \infty. \tag{2.14}$$

In particular, (2.14) guarantees that $|\psi|$ decays at infinity faster than the reciprocal of any polynomial.

On a different topic, recall from [16, Theorem 2, p. 171] that for any closed set $F \subseteq \mathbb{R}^n$ there exists a function $\rho_{reg} : \mathbb{R}^n \to [0,\infty)$ such that

$$\rho_{reg} \in \mathscr{C}^\infty(\mathbb{R}^n \setminus F), \quad \rho_{reg} \approx \mathrm{dist}(\,\cdot\,, F) \text{ on } \mathbb{R}^n, \tag{2.15}$$

and, with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$|\partial^\alpha \rho_{reg}(x)| \leq C_\alpha \big[\mathrm{dist}(x,F)\big]^{1-|\alpha|}, \quad \forall\, \alpha \in \mathbb{N}_0^n \text{ and } \forall\, x \in \mathbb{R}^n \setminus F. \tag{2.16}$$

To proceed, let Ω be a graph Lipschitz domain in $\mathbb{R}^n$ and denote by $\mathscr{C}_b^\infty(\overline{\Omega})$ the vector space of restrictions to Ω of functions from $\mathscr{C}_c^\infty(\mathbb{R}^n)$. Also, if ρ_{reg} stands for the regularized distance function associated with $\overline{\Omega}$, we set $\rho := C\rho_{reg}$, where $C > 0$ is a fixed constant chosen large enough so that

$$\rho(z - s\mathbf{e}_n) > 2s, \quad \forall\, z \in \partial\Omega \text{ and } \forall\, s > 0, \tag{2.17}$$

where $\{\mathbf{e}_j\}_{1\leq j\leq n}$ denotes the standard orthonormal basis in $\mathbb{R}^n$ (hence, $\mathbf{e}_n := (0,\dots,0,1) \in \mathbb{R}^n$). The above normalization condition on ρ ensures that

$$x + \lambda\rho(x)\mathbf{e}_n \in \Omega, \quad \forall\, x \in \mathbb{R}^n \setminus \overline{\Omega} \text{ and } \forall\, \lambda \geq 1. \tag{2.18}$$

Let us also note that in the current case (i.e., when $F := \overline{\Omega}$ where Ω is a graph Lipschitz domain in $\mathbb{R}^n$), there holds

$$\rho \in \mathrm{Lip}(\mathbb{R}^n), \tag{2.19}$$

where $\mathrm{Lip}(\mathbb{R}^n)$ stands for the set of Lipschitz functions in $\mathbb{R}^n$.

The role of ρ is to permit us to define Stein's extension operator (cf. [16, (24), p. 182]) acting on $u \in \mathscr{C}_b^\infty(\overline{\Omega})$ according to

$$(\mathscr{E}_{\Omega\to\mathbb{R}^n}\, u)(x) := \int_1^\infty u\big(x + \lambda\rho(x)\mathbf{e}_n\big)\psi(\lambda)\, d\lambda, \quad \forall\, x \in \mathbb{R}^n. \tag{2.20}$$

Incidentally, the fact that

$$\mathscr{E}_{\Omega\to\mathbb{R}^n}\, u \in \mathrm{Lip}(\mathbb{R}^n) \ \text{ and } \ (\mathscr{E}_{\Omega\to\mathbb{R}^n}\, u)\big|_{\Omega} = u, \ \ \forall\, u \in \mathscr{C}^\infty_b(\overline{\Omega}), \tag{2.21}$$

is a direct consequence of (2.19), (2.20) and (2.12).

We are now in a position to state the following extension result.

Theorem 2.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists a linear mapping*

$$\mathscr{E}_{\Omega\to\mathbb{R}^n} : \mathscr{C}^\infty(\overline{\Omega}) \longrightarrow \mathrm{Lip}_c\,(\mathbb{R}^n) \tag{2.22}$$

with the property that for each $m \in \mathbb{N}_0$ the mapping $\mathscr{E}_{\Omega\to\mathbb{R}^n}$ extends to a bounded linear operator

$$\begin{gathered} \mathscr{E}_{\Omega\to\mathbb{R}^n} : W^{m,p}_a(\Omega) \longrightarrow W^{m,p}_a(\mathbb{R}^n) \\ \textit{such that } \ (\mathscr{E}_{\Omega\to\mathbb{R}^n}\, u)\big|_{\Omega} = u, \ \ \forall\, u \in W^{m,p}_a(\Omega), \end{gathered} \tag{2.23}$$

provided

$$\begin{gathered} \textit{either } p \in (1,\infty) \textit{ and } a \in (-1/p, 1-1/p), \\ \textit{or } p = 1 \textit{ and } a = 0. \end{gathered} \tag{2.24}$$

Proof. In the case when Ω is a graph Lipschitz domain, it has been proved in [3] that Stein's extension operator (2.20) does the job. This result may then be adjusted to the case when Ω is an arbitrary bounded Lipschitz domain. One way to see this is to glue together the extension operators constructed for various graph Lipschitz domains via arguments very similar to those in [16, Section 3.3, p. 189–192]. Another, perhaps more elegant argument is to change formula (2.20) to

$$(\mathscr{E}_{\Omega\to\mathbb{R}^n} u)(x) := \int_1^\infty u\big(x + \lambda\rho(x)h(x)\big)\psi(\lambda)\, d\lambda, \ \ \forall\, x \in \mathbb{R}^n, \tag{2.25}$$

where $h \in \mathscr{C}^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ is a suitably chosen vector field. In particular, it is assumed that h is transversal to $\partial\Omega$ in a uniform fashion, i.e., that for some constant $\kappa > 0$ there holds

$$\nu \cdot h \geq \kappa \ \ \mathscr{H}^{n-1}\text{-a.e. on } \ \partial\Omega, \tag{2.26}$$

where ν is the outward unit normal to Ω, and $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. The vector field h is a replacement of $\mathbf{e}_n$ and this permits us to avoid considering a multitude of special local systems of coordinates. □

We conclude this section by discussing an important interpolation formula for weighted Sobolev spaces of arbitrary order in Lipschitz domains in Theorem 2.6 below. As a preamble, we first record the following folklore interpolation result. Here and elsewhere $[\,\cdot\,,\,\cdot\,]_\theta$ denotes the usual complex interpolation bracket.

Lemma 2.5. *Assume that* X_0, X_1 *and* Y_0, Y_1 *are two compatible pairs of Banach spaces such that* $\{Y_0, Y_1\}$ *is a retract of* $\{X_0, X_1\}$ (*here and elsewhere the "extension" and "restriction" operators are denoted by* E *and* R, *respectively*)*. Then for each* $\theta \in (0,1)$ *one has*

$$[Y_0, Y_1]_\theta = R\big([X_0, X_1]_\theta\big). \tag{2.27}$$

Here is the theorem advertised earlier, asserting that our class of weighted Sobolev spaces is stable under complex interpolation. In this regard, we wish to stress that the extension result from Theorem 2.4 plays a key role.

Theorem 2.6. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^n$ *and assume that, for* $i \in \{0,1\}$, *we have* $1 < p_i < \infty$ *and* $-1/p_i < a_i < 1 - 1/p_i$. *Fix* $\theta \in (0,1)$ *and suppose that* $p \in (0,\infty)$ *and* $a \in \mathbb{R}$ *are such that* $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $a = (1-\theta)a_0 + \theta a_1$. *Then for each* $m \in \mathbb{N}_0$ *there holds*

$$\big[W^{m,p_0}_{a_0}(\Omega), W^{m,p_1}_{a_1}(\Omega)\big]_\theta = W^{m,p}_a(\Omega). \tag{2.28}$$

Proof. The outline of the proof is as follows. First, from the well-known interpolation results for Lebesgue spaces with change of measure (cf. [2, Theorem 5.5.3, p. 120]) it follows that formula (2.28) holds in the particular case when $\Omega = \mathbb{R}^n$ and $m = 0$. Making use of [14, Theorem 3.3] we then allow $m \in \mathbb{N}_0$ arbitrary via convolution with an appropriate Bessel potential. With this in hand, (2.28) follows from (2.23) in Theorem 2.4 and the abstract retract-type result from Lemma 2.5. □

3. The Trace Theorem for weighted Sobolev Spaces

For each $k \in \mathbb{N}_0 \cup \{\infty\}$, we denote by $\mathscr{C}^k_b(\overline{\mathbb{R}^n_+})$ the restrictions to $\overline{\mathbb{R}^n_+}$ of compactly supported functions of class $\mathscr{C}^k$ in $\mathbb{R}^n$. Recall that $\mathscr{L}^n$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$ and, for each $x \in \mathbb{R}^n_+$, abbreviate $\delta(x) := \operatorname{dist}(x, \partial\mathbb{R}^n_+)$. Next, for each $p \in (1,\infty)$ and each $a \in \big(-\frac{1}{p}, 1 - \frac{1}{p}\big)$, define the weighted Lebesgue space

$$L^p(\mathbb{R}^n_+, \delta^{ap}\mathscr{L}^n) = L^p(\mathbb{R}^n_+, \delta^{ap}dx) = L^p(\mathbb{R}^n_+, x_n^{ap}dx) \tag{3.1}$$

as the space of $\mathscr{L}^n$-measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that

$$\|f\|_{L^p(\mathbb{R}^n_+, \delta^{ap}\mathscr{L}^n)} := \bigg(\int\limits_{\mathbb{R}^n_+} |f|^p \delta^{ap}\, d\mathscr{L}^n\bigg)^{1/p} < \infty. \tag{3.2}$$

Moving on, given $p \in (1,\infty)$ and $a \in (-\frac{1}{p}, 1 - \frac{1}{p})$, define the homogeneous weighted Sobolev space (of order one) in $\mathbb{R}^n_+$ by setting

$$\dot{W}^{1,p}_a(\mathbb{R}^n_+) := \Big\{u \in L^1_{loc}(\mathbb{R}^n_+) :\ \partial_j u \in L^p(\mathbb{R}^n_+, \delta^{ap}dx),\ 1 \le j \le n\Big\}, \tag{3.3}$$

where each $\partial_j u$ above is understood in the sense of distributions.

Finally, for $p \in [1,\infty]$ and $s \in (0,1)$, define the homogeneous Besov norm $\|\cdot\|_{\dot{B}^{p,p}_s(\mathbb{R}^{n-1})}$ as

$$\|f\|_{\dot{B}^{p,p}_s(\mathbb{R}^{n-1})} := \Big(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x') - f(y')|^p}{|x'-y'|^{n-1+sp}}\, dx'\, dy' \Big)^{1/p}. \tag{3.4}$$

After this preamble, we are ready to deal with the main technical step in establishing the well-definiteness and boundedness of the trace operator for weighted Sobolev spaces in the upper half-space.

Proposition 3.1. *Let $p \in (1,\infty)$, pick $a \in (-\frac{1}{p}, 1-\frac{1}{p})$, and set $s := 1-a-1/p \in (0,1)$. Then for every $u \in \mathscr{C}^1_b(\overline{\mathbb{R}^n_+})$ there holds*

$$\begin{aligned} &\big\|u|_{\partial\mathbb{R}^n_+}\big\|_{\dot{B}^{p,p}_s(\mathbb{R}^{n-1})} \leq \\ &\leq C_{p,a,n}\, \big\|\partial_n u\big\|^{a+1/p}_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)} \big\|\nabla_{n-1} u\big\|^{1-a-1/p}_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)}, \end{aligned} \tag{3.5}$$

where $\nabla_{n-1}u := (\partial_1 u, \dots, \partial_{n-1}u)$, and the constant $C_{p,a,n} \in (0,\infty)$ is given by

$$\begin{aligned} C_{p,a,n} = \Big[&2^{2p+a-2+1/p} \cdot p^{ap+2} \cdot (ap+1)^{-a-1/p} \times \\ &\times (p(1-a)-1)^{a-2-ap+1/p} \cdot \omega_{n-2} \Big]^{1/p}. \end{aligned} \tag{3.6}$$

In particular, $C_{p,a,n}$ satisfies

$$a \in (-1,0] \Longrightarrow C_{p,a,n} \longrightarrow (-a)^{-1} \Big(\frac{2}{a+1}\Big)^{a+1} \omega_{n-2} \;\; \text{as} \;\; p \to 1^+, \tag{3.7}$$

and

$$a \in [0,1) \Longrightarrow C_{p,a,n} \to \infty \;\; \text{as} \;\; p \to \infty. \tag{3.8}$$

As a consequence of (3.5), *for every $u \in \mathscr{C}^1_b(\overline{\mathbb{R}^n_+})$ there holds*

$$\begin{aligned} &\|u|_{\partial\mathbb{R}^n_+}\|_{\dot{B}^{p,p}_s(\mathbb{R}^{n-1})} \leq \\ &\leq C_{p,a,n}\, \|\nabla u\|_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)} = C_{p,a,n}\, \|u\|_{\dot{W}^{1,p}_a(\mathbb{R}^n_+)}. \end{aligned} \tag{3.9}$$

Proof. Identifying $\partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$, by definition we have

$$\big\|u|_{\partial\mathbb{R}^n_+}\big\|^p_{\dot{B}^{p,p}_s(\mathbb{R}^{n-1})} = \int_{x'\in\mathbb{R}^{n-1}} \int_{y'\in\mathbb{R}^{n-1}} \frac{|u(x',0)-u(y',0)|^p}{|x'-y'|^{n-1+sp}}\, dy'\, dx'. \tag{3.10}$$

Fix $x', y' \in \mathbb{R}^{n-1}$ and let $\lambda \in (0,\infty)$ be a fixed constant to be determined later. By the triangle inequality and the fact that $p \in (1,\infty)$, we write

$$|u(x',0)-u(y',0)|^p \leq 2^{2(p-1)}(I_1+I_2+I_3), \tag{3.11}$$

where

$$\begin{aligned} I_1 &:= \Big|u(x',0) - u\big(x',\lambda|x'-y'|\big)\Big|^p, \\ I_2 &:= \Big|u\big(x',\lambda|x'-y'|\big) - u\big(y',\lambda|x'-y'|\big)\Big|^p, \\ I_3 &:= \Big|u\big(y',\lambda|x'-y'|\big) - u(y',0)\Big|^p. \end{aligned} \tag{3.12}$$

Using this notation, we now have

$$\big\|u|_{\partial\mathbb{R}^n_+}\big\|^p_{\dot{B}^{p,p}_s(\mathbb{R}^{n-1})} \leq$$
$$\leq 2^{2(p-1)} \sum_{j=1}^{3} \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{I_j}{|x'-y'|^{n-1+sp}}\,dy'\,dx'. \tag{3.13}$$

From here, we wish to estimate the individual contributions from I_1, I_2, and I_3. In this vein, consider first

$$\int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{I_1}{|x'-y'|^{n-1+sp}}\,dy'\,dx' =$$
$$= \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{|u(x',0)-u(x',\lambda|x'-y'|)|^p}{|x'-y'|^{n-1+sp}}\,dy'\,dx'. \tag{3.14}$$

Invoking the integral version of the (one-dimensional) mean value theorem in the n^{th} component then gives

$$\int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{|u(x',0)-u\,(x',\lambda|x'-y'|)|^p}{|x'-y'|^{n-1+sp}}\,dy'\,dx' =$$
$$= \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{1}{|x'-y'|^{n-1+sp}} \times$$
$$\times \left| \int_0^1 \lambda|x'-y'|\,(\partial_n u)\big(x',(1-t)\lambda|x'-y'|\big)\,dt\right|^p\,dy'\,dx' \leq$$
$$\leq \lambda^p \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{1}{|x'-y'|^{n-1+p(s-1)}} \times$$
$$\times \left(\int_0^1 \big|(\partial_n u)\big(x',t\lambda|x'-y'|\big)\big|\,dt\right)^p\,dy'\,dx', \tag{3.15}$$

after changing $t \mapsto 1-t$ and bringing the absolute value inside the integral. For each fixed $x' \in \mathbb{R}^{n-1}$, we will use polar coordinates to write $y' = x' + \rho\omega$, where $\omega \in S^{n-2}$ and $\rho \in (0,+\infty)$. Then, since $y' \in \mathbb{R}^{n-1}$, this implies

$dy' = \rho^{n-2} d\rho\, d\omega$. Thus,

$$\int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{I_1}{|x'-y'|^{n-1+sp}}\, dy'\, dx' \leq$$

$$\leq \lambda^p \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{\omega\in S^{n-2}} \int\limits_0^\infty \frac{\rho^{n-2}}{\rho^{n-1+p(s-1)}} \Big(\int\limits_0^1 \big|(\partial_n u)(x',\lambda\rho t)\big|\, dt \Big)^p d\rho\, d\omega\, dx' =$$

$$= \lambda^p\, \omega_{n-2} \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_0^\infty \frac{1}{\rho^{1+p(s-1)}} \Big(\int\limits_0^1 \big|(\partial_n u)(x',\lambda\rho t)\big|\, dt \Big)^p d\rho\, dx', \qquad (3.16)$$

where ω_{n-2} represents the area of the unit sphere in $\mathbb{R}^{n-1}$. Let us make the change of variables $\theta := (\lambda\rho)t$. This entails $d\theta = (\lambda\rho)\, dt$ and the interval of integration changes from $[0,1]$ to $[0,\lambda\rho]$. Therefore, the last integral in (3.16) may be written as

$$\lambda^p\, \omega_{n-2} \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_0^\infty \rho^{-1+p(1-s)} \Big(\int\limits_0^{\lambda\rho} \big|(\partial_n u)(x',\theta)\big| \frac{1}{\lambda\rho}\, d\theta \Big)^p d\rho\, dx' =$$

$$= \omega_{n-2} \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_0^\infty \rho^{-1-sp} \Big(\int\limits_0^{\lambda\rho} \big|(\partial_n u)(x',\theta)\big|\, d\theta \Big)^p d\rho\, dx'. \qquad (3.17)$$

Make another change of variables by letting $\eta := \lambda\rho$. This yields $d\eta = \lambda\, d\rho$ and the interval of integration changes from $[0,\lambda\rho]$ to $[0,\eta]$. Consequently, the last integral above becomes

$$\omega_{n-2} \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_0^\infty \Big(\frac{\eta}{\lambda}\Big)^{-1-sp} \Big(\int\limits_0^{\eta} \big|(\partial_n u)(x',\theta)\big|\, d\theta \Big)^p \frac{1}{\lambda}\, d\eta\, dx' =$$

$$= \lambda^{sp}\, \omega_{n-2} \int\limits_{x'\in\mathbb{R}^{n-1}} \Big\{ \int\limits_0^\infty \eta^{-1-sp} \Big(\int\limits_0^{\eta} \big|(\partial_n u)(x',\theta)\big|\, d\theta \Big)^p d\eta \Big\} dx'. \qquad (3.18)$$

At this point we wish to apply Hardy's inequality inside the curly brackets. Recall (cf., e.g., [16, p. 272, A.4]) that this states that for $q \in [1,\infty)$, $r \in (0,\infty)$, and $f : [0,\infty] \longrightarrow [0,\infty]$ measurable,

$$\int\limits_0^\infty \eta^{-1-r} \Big(\int\limits_0^{\eta} f(\theta)\, d\theta \Big)^q d\eta \leq \Big(\frac{q}{r}\Big)^q \int\limits_0^\infty f(\theta)^q\, \theta^{q-r-1}\, d\theta. \qquad (3.19)$$

Since $u \in \mathscr{C}^1_b(\overline{\mathbb{R}^n_+})$ it follows that $|(\partial_n u)(x',\cdot)|$ is measurable and non-negative. Moreover, $s \in (0,1)$ hence $r := sp \in (0,\infty)$. Thus, we are indeed in a position to use Hardy's inequality with $q := p \in (1, \infty)$. Doing so gives

$$\lambda^{sp}\,\omega_{n-2} \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_0^\infty \eta^{-1-sp}\Big(\int\limits_0^\eta \big|(\partial_n u)(x',\theta)\big|\,d\theta\Big)^p\,d\eta\,dx' \leq$$

$$\leq \lambda^{sp}\,\frac{\omega_{n-2}}{s^p} \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_0^\infty |(\partial_n u)(x',\theta)|^p\,\theta^{pa}\,d\theta\,dx' =$$

$$= \lambda^{sp}\,\frac{\omega_{n-2}}{s^p} \int\limits_{\mathbb{R}^n_+} \big|(\partial_n u)(x)\big|^p\,\delta(x)^{ap}\,dx, \quad (3.20)$$

where the last equality is due to Fubini. Putting everything together, we have established

$$\int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{I_1}{|x'-y'|^{n-1+sp}}\,dy'\,dx' \leq$$

$$\leq \lambda^{sp}\,\frac{\omega_{n-2}}{s^p} \int\limits_{\mathbb{R}^n_+} \big|(\partial_n u)(x)\big|^p\,\delta(x)^{ap}\,dx. \quad (3.21)$$

By interchanging the roles of x' and y', a similar argument shows

$$\int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{I_3}{|x'-y'|^{n-1+sp}}\,dy'\,dx' \leq$$

$$\leq \lambda^{sp}\,\frac{\omega_{n-2}}{s^p} \int\limits_{\mathbb{R}^n_+} \big|(\partial_n u)(x)\big|^p\,\delta^{ap}\,dx. \quad (3.22)$$

At this stage, we are left with estimating the contribution from I_2. With this goal in mind, apply the integral version of the mean value theorem in $\mathbb{R}^{n-1}$ in order to write

$$\int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{I_2}{|x'-y'|^{n-1+sp}}\,dy'\,dx' =$$

$$= \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{|u(x',\lambda|x'-y'|) - u(y',\lambda|x'-y'|)|^p}{|x'-y'|^{n-1+sp}}\,dy'\,dx' =$$

$$= \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{1}{|x'-y'|^{n-1+sp}}\Big|\int\limits_0^1 \Big(\big(x',\lambda|x'-y'|\big) - \big(y',\lambda|x'-y'|\big)\Big)\times$$

$$\times\,(\nabla u)\Big(t\big(x',\lambda|x'-y'|\big) + (1-t)\big(y',\lambda|x'-y'|\big)\Big)\,dt\Big|^p\,dy'\,dx' =$$

$$
= \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{1}{|x'-y'|^{n-1+sp}} \times
$$

$$
\times \Big| \int_0^1 (x'-y',0)\cdot(\nabla u)\big(tx'+(1-t)y',\lambda|x'-y'|\big)\,dt\Big|^p \,dy'\,dx' \leq
$$

$$
\leq \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{1}{|x'-y'|^{n-1+sp}} \times
$$

$$
\times \Big(\int_0^1 |x'-y'|\,\Big|(\nabla_{n-1}u)\big(tx'+(1-t)y',\lambda|x'-y'|\big)\Big|\,dt\Big)^p \,dy'\,dx', \quad (3.23)
$$

where the last step is based on the Cauchy–Schwarz inequality. In turn, the last expression in (3.23) may be dominated by

$$
\int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{1}{|x'-y'|^{n-1+p(s-1)}} \times
$$

$$
\times \Big[\int_0^1 \Big|(\nabla_{n-1}u)\big(tx'+(1-t)y',\lambda|x'-y'|\big)\Big|\,dt\Big]^p \,dy'\,dx' =
$$

$$
= \int\limits_{x'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \Big[\int_0^1 \Big(\frac{1}{|x'-y'|^{n-1+p(s-1)}}\Big)^{1/p} \times
$$

$$
\times \Big|(\nabla_{n-1}u)\big(tx'+(1-t)y',\lambda|x'-y'|\big)\Big|\,dt\Big]^p \,dy'\,dx'. \quad (3.24)
$$

We proceed by invoking the generalized Minkowski inequality which permits us to estimate the last expression above by

$$
\Big[\int_0^1 \Big(\int\limits_{y'\in\mathbb{R}^{n-1}} \int\limits_{x'\in\mathbb{R}^{n-1}} \frac{1}{|x'-y'|^{n-1+p(s-1)}} \times
$$

$$
\times \Big|(\nabla_{n-1}u)\big(y'+t(x'-y'),\lambda|x'-y'|\big)\Big|^p \,dx'\,dy'\Big)^{1/p} dt\Big]^p. \quad (3.25)
$$

Introducing $z' := x'-y'$, for each fixed $y'\in\mathbb{R}^{n-1}$, and then using Fubini further transforms this expression into

$$
\Big[\int_0^1 \Big(\int\limits_{z'\in\mathbb{R}^{n-1}} \int\limits_{y'\in\mathbb{R}^{n-1}} \frac{1}{|z'|^{n-1+p(s-1)}} \times
$$

$$
\times \Big|(\nabla_{n-1}u)\big(y'+tz',\lambda|z'|\big)\Big|^p \,dy'\,dz'\Big)^{1/p} dt\Big]^p. \quad (3.26)
$$

Let us perform another change of variables by letting $\xi' := y' + tz'$ for fixed $t \in [0,1]$ and fixed $z' \in \mathbb{R}^{n-1}$. This implies $d\xi' = dy'$ and (3.26) now becomes

$$\Big[\int_0^1 \Big(\int_{z'\in\mathbb{R}^{n-1}} \int_{\xi'\in\mathbb{R}^{n-1}} \frac{1}{|z'|^{n-1+p(s-1)}} \times \\ \times \Big|(\nabla_{n-1}u)\big(\xi', \lambda|z'|\big)\Big|^p \, d\xi'\, dz'\Big)^{1/p} dt\Big]^p = \\ = \int_{z'\in\mathbb{R}^{n-1}} \int_{\xi'\in\mathbb{R}^{n-1}} \frac{1}{|z'|^{n-1+p(s-1)}} \Big|(\nabla_{n-1}u)\big(\xi', \lambda|z'|\big)\Big|^p \, d\xi'\, dz'. \tag{3.27}$$

From here, pass to polar coordinates in the variable z'. Specifically, set $z' := (\rho\omega)/\lambda$ where $\rho \in (0,\infty)$ and $\omega \in S^{n-2}$. This entails $dz' = \rho^{n-2}/\lambda^{n-1}\, d\rho\, d\omega$, so we may write (3.27) as

$$\int_{z'\in\mathbb{R}^{n-1}} \int_{\xi'\in\mathbb{R}^{n-1}} \frac{1}{|z'|^{n-1+p(s-1)}} \big|(\nabla_{n-1}u)\big(\xi', \lambda|z'|\big)\big|^p \, d\xi'\, dz' = \\ = \lambda^{1-n}\lambda^{n-1+p(s-1)} \int_0^\infty \int_{S^{n-2}} \int_{\xi'\in\mathbb{R}^{n-1}} \frac{\rho^{n-2}}{\rho^{n-1+p(s-1)}} \big|(\nabla_{n-1}u)\big(\xi', \rho\big)\big|^p \, d\xi'\, d\omega\, d\rho = \\ = \lambda^{p(s-1)}\,\omega_{n-2} \int_0^\infty \int_{\xi'\in\mathbb{R}^{n-1}} \big|(\nabla_{n-1}u)\big(\xi', \rho\big)\big|^p \rho^{ap}\, d\xi'\, d\rho = \\ = \lambda^{p(s-1)}\,\omega_{n-2} \int_{\mathbb{R}^n_+} \big|(\nabla_{n-1}u)(x)\big|^p \delta(x)^{ap}\, dx, \tag{3.28}$$

where the last equality uses Fubini.

At this stage, combining (3.28), (3.27), (3.26), (3.25), (3.24), and (3.23) establishes

$$\int_{x'\in\mathbb{R}^{n-1}} \int_{y'\in\mathbb{R}^{n-1}} \frac{I_2}{|x'-y'|^{n-1+sp}}\, dy'\, dx' \leq \\ \leq \lambda^{p(s-1)}\,\omega_{n-2} \int_{\mathbb{R}^n_+} \big|(\nabla_{n-1}u)(x)\big|^p \delta(x)^{ap}\, dx. \tag{3.29}$$

In concert, (3.29), (3.22), (3.21), and (3.13), then yield

$$\big\|u|_{\partial\mathbb{R}^n_+}\big\|^p_{\dot B^{p,p}_s(\mathbb{R}^{n-1})} \leq 2^{2(p-1)}\Big(\lambda^{sp}\,\frac{2\,\omega_{n-2}}{s^p} \times \\ \times \int_{\mathbb{R}^n_+} \big|(\partial_n u)(x)\big|^p \delta(x)^{ap}\, dx + \lambda^{p(s-1)}\,\omega_{n-2} \int_{\mathbb{R}^n_+} \big|(\nabla_{n-1}u)(x)\big|^p \delta(x)^{ap}\, dx\Big) =$$

$$= \frac{2^{2p-1}\,\omega_{n-2}}{s^p}\,\|\partial_n u\|^p_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)}\,\lambda^{sp}+$$
$$+\,2^{2p-2}\,\omega_{n-2}\,\|\nabla_{n-1}u\|^p_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)}\,\lambda^{p(s-1)} = A\,\lambda^{sp} + B\,\lambda^{p(s-1)}, \quad (3.30)$$

where we have set

$$A := \frac{2^{2p-1}\,\omega_{n-2}}{s^p}\,\|\partial_n u\|^p_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)} \in [0,\infty) \quad (3.31)$$

and

$$B := 2^{2p-2}\,\omega_{n-2}\,\|\nabla_{n-1}u\|^p_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)} \in [0,\infty). \quad (3.32)$$

We need to consider several cases for the constants A and B. If $A=0$ and $B\in[0,\infty)$, then $\|\partial_n u\|_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)}=0$ which forces u to be constant in the last component; i.e, for each fixed $x'\in\mathbb{R}^{n-1}$, there exists $C_{x'}\in\mathbb{R}$ such that $u(x',t)=C_{x'}$ for every $t\in(0,\infty)$. Since $u\in\mathscr{C}^1_b(\overline{\mathbb{R}^n_+})$ (in particular, u has compact support), this implies that $C_{x'}=0$ for every $x'\in\mathbb{R}^{n-1}$. Hence, $u\equiv 0$ on the closure of the upper half-space and (3.5) is trivially valid in this case. The case when $B=0$ and $A\in[0,\infty)$ is handled in a similar fashion. Finally, when $A\in(0,\infty)$ and $B\in(0,\infty)$ define $f:(0,\infty)\to\mathbb{R}$ by setting

$$f(x) := A\,x^{sp} + B\,x^{p(s-1)} = A\,x^{p(1-a)-1} + B\,x^{-ap-1}, \quad \forall\, x\in(0,\infty).$$

We wish to minimize f. To this end, we begin by noting that $f\in\mathscr{C}^\infty((0,\infty))$ and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\big(A\,x^{p(1-a)-1} + B\,x^{-ap-1}\big) = \infty,$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\big(A\,x^{p(1-a)-1} + B\,x^{-ap-1}\big) = \infty. \quad (3.33)$$

Moreover, since $-2-ap\in(-p-1,-1)$ implies $-2-ap<0$, we have

$$f'(x) = 0 \Longleftrightarrow x^{-ap-2}\big[(p(1-a)-1)\,A\,x^p - (ap+1)\,B\big] = 0 \Longleftrightarrow$$
$$\Longleftrightarrow (p(1-a)-1)\,A\,x^p - (ap+1)\,B = 0. \quad (3.34)$$

Solving the latter equation for x and denoting this solution as λ gives

$$\lambda = \Big[\frac{(ap+1)B}{(p(1-a)-1)A}\Big]^{1/p} \in (0,\infty) \quad (3.35)$$

is the only local extreme point of f. To determine whether λ is a local maximum or local minumum for f, consider the second derivative of f, i.e.,

$$f''(x) = \big(p(1-a)-1\big)\big(p(1-a)-2\big)\,A\,x^{p(1-a)-3}+$$
$$+\,(ap+1)(ap+2)\,B\,x^{-ap-3}. \quad (3.36)$$

Evaluating f'' at λ then gives (after some elementary algebra)

$$f''(\lambda) = B^{1-a-3/p}\,A^{a+3/p}\,\big(p(1-a)-1\big)^{a+3/p}\,(ap+1)^{1-a-3/p}\,p > 0. \quad (3.37)$$

As such, by the second derivative test, λ is a local minimum for f. Combining (3.33) with the fact that λ is the only local extreme point for f gives that λ is a global minimum for f. Recall that $\|u|_{\partial\mathbb{R}^n_+}\|_{\dot{B}^{p,p}_s(\mathbb{R}^{n-1})}$ does not depend on λ. Therefore, we may minimize the right-hand side of (3.30) by choosing

λ as in (3.35). After a somewhat lengthy but elementary computation, this choice yields

$$\begin{aligned}\big\|u|_{\partial\mathbb{R}^n_+}\big\|^p_{\dot{B}^{p,p}_s(\mathbb{R}^{n-1})} &\leq \\ &\leq 2^{2p-2+a+1/p}\,\omega_{n-2}\,\frac{p^{ap+2}}{(ap+1)^{a+1/p}}\,\big(p(1-a)-1\big)^{a-2-ap+1/p}\times \\ &\quad\times\big\|\partial_n u\big\|^{p(a+1/p)}_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)}\big\|\nabla_{n-1}u\big\|^{p(1-a-1/p)}_{L^p(\mathbb{R}^n_+,\delta^{ap}dx)}, \qquad (3.38)\end{aligned}$$

as desired. □

We are now ready to state and prove the main result in this section.

Theorem 3.2. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and abbreviate $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ for each $x \in \mathbb{R}^n$. Also, let $p \in (1,\infty)$, pick $a \in (-\frac{1}{p}, 1-\frac{1}{p})$, and set $s := 1-a-1/p \in (0,1)$. Then the restriction to the boundary operator*

$$\mathscr{C}^\infty(\overline{\Omega}) \ni u \longmapsto u\big|_{\partial\Omega} \in \mathscr{C}^0(\partial\Omega) \qquad (3.39)$$

extends to a mapping, henceforth called the trace operator,

$$\operatorname{Tr} : W^{1,p}_a(\Omega) \longrightarrow B^{p,p}_s(\partial\Omega) \qquad (3.40)$$

which is well-defined, linear, and bounded. Concretely, Tr *satisfies the estimate*

$$\|\operatorname{Tr}\, u\|_{B^{p,p}_s(\partial\Omega)} \leq C\,\|u\|_{W^{1,p}_a(\Omega)}, \quad \forall\, u \in W^{1,p}_a(\Omega), \qquad (3.41)$$

where the constant $C \in (0,\infty)$ depends only on Ω, n, p, and a.

Furthermore, the kernel of the trace operator (3.40) *may be described as*

$$\Big\{u \in W^{1,p}_a(\Omega) : \ \operatorname{Tr} u = 0 \ \textit{in}\ B^{p,p}_s(\partial\Omega)\Big\} = \mathring{W}^{1,p}_a(\Omega). \qquad (3.42)$$

Proof. Via a localization argument (involving a partition of unity consisting of smooth, compactly supported functions), and by locally flattening the boundary of Ω via bi-Lipschitz maps (which preserve both the category of Besov spaces and the class of weighted Sobolev spaces presently considered), matters may be reduced to treating the case when $\Omega = \mathbb{R}^n_+$ and when the Besov and Sobolev spaces in question are homogeneous. In such a scenario, the desired conclusions in the first part of the statement follow from (3.9) and a density argument (cf. (2.4)).

The right-to-left inclusion in (3.42) is clear, so we focus on the opposite one. Specifically, pick $u \in W^{1,p}_a(\Omega)$ such that $\operatorname{Tr} u = 0$ in $B^{p,p}_s(\partial\Omega)$, with the goal of showing that $u \in \mathring{W}^{1,p}_a(\Omega)$. Let $\widetilde{u}$ be the extension of u to $\mathbb{R}^n$ taken to be zero outside Ω. Then $\widetilde{u} \in L^p(\mathbb{R}^n, \delta^{ap}\,dx)$ and we claim that

$$\partial_j(\widetilde{u}) = \widetilde{\partial_j u} \ \text{ in } \ \mathscr{D}'(\mathbb{R}^n), \ \ \forall\, j \in \{1,\dots,n\}. \qquad (3.43)$$

To this end, fix an arbitrary $j \in \{1, \dots, n\}$ and arbitrary $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then,

$$\langle \partial_j(\widetilde{u}), \varphi \rangle = -\langle \widetilde{u}, \partial_j \varphi \rangle = \\ = -\int_{\mathbb{R}^n} \widetilde{u}(x)(\partial_j\varphi)(x)\, dx = -\int_{\Omega} u(x)(\partial_j\varphi)(x)\, dx. \tag{3.44}$$

From (2.4) we know that $\mathscr{C}_b^\infty(\overline{\Omega}) \subseteq W_a^{1,p}(\Omega)$ densely. Hence, there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathscr{C}_b^\infty(\overline{\Omega})$ convergent to u in $W_a^{1,p}(\Omega)$. This makes it possible to write

$$\int_\Omega u(x)(\partial_j\varphi)(x)\, dx = \lim_{k\to\infty} \int_\Omega u_k(x)(\partial_j\varphi)(x)\, dx, \tag{3.45}$$

hence, with σ denoting the surface measure on $\partial\Omega$, and $\nu = (\nu_j)_{1\le j\le n}$ standing for the outward unit normal to Ω, we have

$$\begin{aligned} \langle \partial_j(\widetilde{u}), \varphi \rangle &= -\lim_{k\to\infty} \int_\Omega u_k(x)(\partial_j\varphi)(x)\, dx = \\ &= \lim_{k\to\infty} \Big[\int_\Omega (\partial_j u_k)(x)\varphi(x)\, dx - \int_{\partial\Omega} u_k \varphi \nu_j\, d\sigma \Big] = \\ &= \int_\Omega (\partial_j u)(x)\varphi(x)\, dx - \lim_{k\to\infty} \int_{\partial\Omega} u_k\varphi\nu_j\, d\sigma = \\ &= \int_{\mathbb{R}^n} (\widetilde{\partial_j u})(x)\varphi(x)\, dx - \lim_{k\to\infty} \int_{\partial\Omega} \big(u_k|_{\partial\Omega}\big)\varphi\nu_j\, d\sigma = \\ &= \langle \widetilde{\partial_j u}, \varphi \rangle - \lim_{k\to\infty} \int_{\partial\Omega} \operatorname{Tr} u_k\, \varphi\nu_j\, d\sigma. \end{aligned} \tag{3.46}$$

As far as the last limit above is concerned, note that

$$\Big| \int_{\partial\Omega} \operatorname{Tr} u_k\, \varphi\nu_j\, d\sigma \Big| \le \|\varphi\|_{L^{p'}(\partial\Omega)} \|\operatorname{Tr} u_k\|_{L^p(\partial\Omega)} \le \\ \le \|\varphi\|_{L^{p'}(\partial\Omega)} \|\operatorname{Tr} u_k\|_{B_s^{p,p}(\partial\Omega)} \longrightarrow 0 \text{ as } k \to \infty, \tag{3.47}$$

since, by the continuity of the trace operator, $\operatorname{Tr} u_k \to \operatorname{Tr} u = 0$ in $B_s^{p,p}(\partial\Omega)$ as $k \to \infty$. Now, (3.43) follows from (3.46). In turn, (3.43) proves that

$$\widetilde{u} \in W_a^{1,p}(\mathbb{R}^n). \tag{3.48}$$

Moreover, using a partition of unity there is no loss of generality in assuming that

$$\begin{gathered} \operatorname{supp} \widetilde{u} \text{ is contained in a neighborhood } \mathscr{O} \text{ of a point } x_* \in \partial\Omega, \\ \text{near which } \partial\Omega \text{ coincides with a Lipschitz graph.} \end{gathered} \tag{3.49}$$

In particular, we may assume that there is a truncated circular cone Γ with vertex at the origin with the property that

$$x + \Gamma \subseteq \Omega, \quad \forall\, x \in \mathscr{O} \cap \partial\Omega. \tag{3.50}$$

To proceed, select $\eta \in C_c^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp} \eta \subseteq \Gamma, \quad 0 \leq \eta \leq 1, \quad \int_{\mathbb{R}^n} \eta \, d\mathscr{L}^n = 1, \tag{3.51}$$

and, for each $\varepsilon > 0$, define $\eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by $\eta_\varepsilon(x) := \varepsilon^{-n} \eta(x/\varepsilon)$ for all $x \in \mathbb{R}^n$. Finally, for every $\varepsilon \in (0, 1/2)$, define

$$u_\varepsilon := \big[\widetilde{u} * \eta_\varepsilon\big]\Big|_{\Omega}.$$

Then, clearly, $u_\varepsilon \in \mathscr{C}_b^\infty(\overline{\Omega})$, and we claim that

$$\exists\, \varepsilon_* > 0 \ \text{ such that } \ \operatorname{supp} u_\varepsilon \subseteq \Omega, \quad \forall\, \varepsilon \in (0, \varepsilon_*). \tag{3.52}$$

Indeed,

$$\operatorname{supp} u_\varepsilon = \operatorname{supp}(\widetilde{u} * \eta_\varepsilon) \subseteq \operatorname{supp}(\widetilde{u}) + \operatorname{supp} \eta_\varepsilon \subseteq (\mathscr{O} \cap \overline{\Omega}) + \varepsilon \operatorname{supp} \eta \subseteq \Omega, \tag{3.53}$$

where the last inclusion (which uses the fact that $\operatorname{supp} \eta \subseteq \Gamma$) is valid for $\varepsilon > 0$ small enough.

From (3.52) we may therefore conclude that $u_\varepsilon \in \mathscr{C}_c^\infty(\Omega)$ for $\varepsilon > 0$ small, and the proof of the membership $u \in \mathring{W}_a^{1,p}(\Omega)$ is finished once we show that

$$u_\varepsilon \to u \ \text{ in } \ W_a^{1,p}(\Omega) \ \text{ as } \ \varepsilon \to 0^+. \tag{3.54}$$

Since distributional derivatives commute with restrictions to Ω, the claim in (3.54) follows from the usual approximation to the identity argument bearing in mind (3.43), (2.5), and the fact that the Hardy–Littlewood maximal operator is bounded on weighted L^p spaces when the weight in question belongs to the Muckenhoupt A_p class. □

4. The Boundary Extension Theorem for weighted Sobolev Spaces

The bulk of this section is devoted to proving the extension result stated in Theorem 4.1 below. In the last part we make use of this theorem in order to establish an interpolation formula which plays a basic role.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, let $p \in (1, \infty)$, $a \in \big(-\frac{1}{p}, 1 - \frac{1}{p}\big)$, and set $s := 1 - a - 1/p \in (0, 1)$. Then there exists a mapping*

$$\operatorname{Ex} : B_s^{p,p}(\partial\Omega) \longrightarrow W_a^{1,p}(\Omega) \tag{4.1}$$

that is linear, bounded, and satisfies

$$\operatorname{Tr}(\operatorname{Ex}(f)) = f, \quad \forall\, f \in B_s^{p,p}(\partial\Omega). \tag{4.2}$$

Proof. We first focus on the case when $\Omega = \mathbb{R}^n_+$. To this end, let $\eta \in \mathscr{C}^\infty(\mathbb{R}^n)$ be a function such that $\operatorname{supp}\eta \subseteq B(0,4)$, $\eta \equiv 1$ on $B(0,2)$, and $0 \leq \eta \leq 1$ on $\mathbb{R}^n$. Next, define the kernel

$$k : \mathbb{R}^n_+ \times \overline{\mathbb{R}^n_+} \longrightarrow \mathbb{R} \tag{4.3}$$

by setting

$$k(x,y) := \eta\Big(\frac{x-y}{x_n}\Big)\Bigg[\int\limits_{\mathbb{R}^{n-1}} \eta\Big(\frac{x-(z',0)}{x_n}\Big)\,dz'\Bigg]^{-1}, \tag{4.4}$$
$$\forall\, x = (x_1,\dots,x_n) \in \mathbb{R}^n_+,\quad \forall\, y \in \overline{\mathbb{R}^n_+}.$$

We claim that k is a well-defined, non-negative function belonging to $\mathscr{C}^\infty(\mathbb{R}^n_+ \times \overline{\mathbb{R}^n_+})$. Indeed, for each fixed point $x = (x', x_n) \in \mathbb{R}^n_+$, we have

$$\frac{x-(z',0)}{x_n} \in (0,2) \Longleftrightarrow |x-(z',0)| < 2x_n \Longleftrightarrow z' \in B_{n-1}(x', \sqrt{3}\,x_n). \tag{4.5}$$

Since $\mathscr{L}^{n-1}\big(B_{n-1}(x',\sqrt{3}\,x_n)\big) = c_n x_n^{n-1}$ (where B_{n-1} is an $(n-1)$-dimensional ball) and $\eta \equiv 1$ on $B(0,2)$, we have a strictly positive lower bound for the integral in the right-hand side of (4.4), namely

$$\int\limits_{\mathbb{R}^{n-1}} \eta\Big(\frac{x-(z',0)}{x_n}\Big)\,dz' \geq c_n x_n^{n-1}. \tag{4.6}$$

In particular, it is meaningful to discuss the reciprocal of this number, for which we have

$$\Bigg[\int\limits_{\mathbb{R}^{n-1}} \eta\Big(\frac{x-(z',0)}{x_n}\Big)\,dz'\Bigg]^{-1} \leq c_n x_n^{1-n}. \tag{4.7}$$

Having established this, the well-definedness and non-negativity of k follow immediately. Also, by design,

$$\int\limits_{\mathbb{R}^{n-1}} k\big(x,(y',0)\big)\,dy' = 1,\quad \forall\, x \in \mathbb{R}^n_+. \tag{4.8}$$

Concerning the regularity of k, this follows from the regularity of η and the Leibniz rule, which give that for every multi-index α

$$\partial_x^\alpha k(x,y) =$$
$$= \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}\,\partial_x^\beta\Big\{\eta\Big(\frac{x-y}{x_n}\Big)\Big\}\,\partial_x^\gamma\Bigg\{\Bigg[\int\limits_{\mathbb{R}^{n-1}} \eta\Big(\frac{x-(z',0)}{x_n}\Big)\,dz'\Bigg]^{-1}\Bigg\}, \tag{4.9}$$

then, finally, invoking the chain rule. For the last step, it helps to notice that

$$\int_{\mathbb{R}^{n-1}} \eta\Big(\frac{x-(z',0)}{x_n}\Big)\,dz' = \int_{\mathbb{R}^{n-1}} \eta\Big(\frac{x'-z'}{x_n},1\Big)\,dz' = \\ = (-x_n)^{n-1}\int_{\mathbb{R}^{n-1}} \eta(w',1)\,dw' = c\,x_n^{n-1}, \quad (4.10)$$

where $c := (-1)^{n-1}\int_{\mathbb{R}^{n-1}} \eta(w',1)\,dw'$ is a real constant. Hence, on the one hand,

$$\partial_x^\gamma\Big\{\Big[\int_{\mathbb{R}^{n-1}} \eta\Big(\frac{x-(z',0)}{x_n}\Big)\,dz'\Big]^{-1}\Big\} = c\,\partial_x^\gamma(x_n^{1-n}) = \\ = \begin{cases} c\Big(\prod_{j=0}^{|\gamma|-1}(1-n-j)\Big)x_n^{1-n-|\gamma|}, & \text{if } \gamma=(0,\dots,0,\gamma_n), \\ 0, & \text{otherwise.}\end{cases} \quad (4.11)$$

On the other hand, we have

$$\partial_x^\beta\Big[\eta\Big(\frac{x-y}{x_n}\Big)\Big] = \\ = \sum_{|\delta|\leq|\beta|}(\partial^\delta\eta)\Big(\frac{x-y}{x_n}\Big)\,\frac{P^{\beta,\delta}_{2|\beta|-|\beta|}(x_1-y_1,\dots,x_n-y_n,x_n)}{x_n^{2|\beta|}}\,, \quad (4.12)$$

where, generally speaking, $P_r^{\beta,\delta}(t_1,\dots,t_n,t_{n+1})$ is a homogeneous polynomial of degree r in the variables $t_1,\dots,t_{n+1}$; that is,

$$P_r^{\beta,\delta}(t) = \sum_{|\gamma|=r} a_\gamma^{\beta,\delta}\,t^\gamma, \quad t=(t_1,\dots,t_n,t_{n+1})\in\mathbb{R}^{n+1}\,, \quad (4.13)$$

where the $a_\gamma^{\beta,\delta}$'s are real-coefficients. Indeed, staring from the observation that, for each $j\in\{1,\dots,n\}$ and for each differentiable function F, there holds

$$\partial_{x_j}\Big[F\Big(\frac{x-y}{x_n}\Big)\Big] = \sum_{k=1}^n(\partial_k F)\Big(\frac{x-y}{x_n}\Big)\,\frac{\delta_{jk}x_n-(x_k-y_k)\delta_{jn}}{x_n^2}\,, \quad (4.14)$$

formula (4.12) may be justified by induction on the length of the multi-index $\beta\in\mathbb{N}_0^n$.

In particular, from (4.12) we see that for each $x = (x', x_n) \in \mathbb{R}^n_+$ and $y \in \overline{\mathbb{R}^n_+}$ we have

$$
\begin{aligned}
\frac{x-y}{x_n} \in \operatorname{supp}\big(\partial^\delta \eta\big) \Longrightarrow & \\
&\Longrightarrow |x-y| \leq 4x_n \\
&\Longrightarrow \Big| P^{\beta,\delta}_{2^{|\beta|}-|\beta|}(x_1-y_1,\dots,x_n-y_n,x_n)\Big| \leq C_{n,\beta,\delta}\; x_n^{2^{|\beta|}-|\beta|} \\
&\Longrightarrow \left|\partial_x^\beta\Big[\eta\Big(\frac{x-y}{x_n}\Big)\Big]\right| \leq C\, x_n^{-|\beta|}\chi_{|x-y|<4x_n}. \qquad (4.15)
\end{aligned}
$$

Collectively, (4.9), (4.11), and (4.15) imply that the function k satisfies

$$
\begin{gathered}
\big|(\partial_x^\alpha k)(x,y)\big| \leq C_{n,\alpha}\; x_n^{1-n-|\alpha|}\chi_{|x-y|<4x_n}, \qquad (4.16)\\
\forall\, x = (x', x_n) \in \mathbb{R}^n_+,\ \ \forall\, y \in \overline{\mathbb{R}^n_+},\ \ \forall\, \alpha \in \mathbb{N}_0^n.
\end{gathered}
$$

As a consequence,

$$
\big|k(x,y)\big| \leq c_n\, x_n^{1-n}\chi_{|x-y|<4x_n},\ \ \forall\,(x,y) \in \mathbb{R}^n_+ \times \overline{\mathbb{R}^n_+} \qquad (4.17)
$$

and

$$
\big|(\nabla_x k)(x,y)\big| \leq c_n\, x_n^{-n}\chi_{|x-y|<4x_n},\ \ \forall\, x = (x', x_n) \in \mathbb{R}^n_+,\ \ \forall\, y \in \overline{\mathbb{R}^n_+}. \quad (4.18)
$$

Moving on, consider the mapping $\mathscr{E}$ taking functions defined on $\partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$ to functions defined in $\mathbb{R}^n_+$ according to the formula

$$
(\mathscr{E} f)(x) := \int_{\mathbb{R}^{n-1}} k\big(x,(y',0)\big)\, f(y')\, dy',\ \ \forall\, x\in\mathbb{R}^n_+,\ \ \forall\, f\in\mathscr{C}_c^0(\mathbb{R}^{n-1}). \quad (4.19)
$$

Then, for each $f \in \mathscr{C}_c^0(\mathbb{R}^{n-1})$, we may employ (4.17) to conclude that $\mathscr{E} f$ is well-defined. Also, thanks to (4.16), we have that $\mathscr{E} f$ inherits the regularity of k, i.e., $\mathscr{E} f \in \mathscr{C}^\infty(\mathbb{R}^n_+)$.

We claim that for each $p \in (1, +\infty)$ and $a \in \big(-\frac{1}{p}, 1-\frac{1}{p}\big)$, there exists $C_{n,p,a} \in (0, +\infty)$ such that for each $f \in \mathscr{C}_c^0(\mathbb{R}^{n-1})$

$$
\int_{\mathbb{R}^n_+} \big|[\nabla(\mathscr{E} f)](x)\big|^p\, x_n^{ap}\, dx \leq C_{n,p,a} \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|f(y') - f(z')|^p}{|y'-z'|^{n-1+sp}}\, dy' dz', \quad (4.20)
$$

where, as usual, $s := 1 - a - 1/p \in (0,1)$.

To justify (4.20), fix an arbitrary $f \in \mathscr{C}_c^0(\mathbb{R}^{n-1})$ and observe that (4.19) implies that for each fixed $z' \in \mathbb{R}^{n-1}$

$$
\big|[\nabla(\mathscr{E} f)](x)\big| \leq \int_{\mathbb{R}^{n-1}} \big|(\nabla_x k)(x,(y',0))\big|\, \big|f(y') - f(z')\big|\, dy',\ \ \forall\, x \in \mathbb{R}^n_+. \quad (4.21)
$$

In turn, from (4.21), (4.18), and Hölder's inequality we obtain that for each $x = (x', x_n) \in \mathbb{R}^n_+$ and each $z' \in \mathbb{R}^{n-1}$

$$\begin{aligned} \big|[\nabla(\mathscr{E}f)](x)\big| &\leq C\Big(x_n^{-n} \int\limits_{|x-(y',0)|<4x_n} |f(y') - f(z')|\,dy'\Big)^p \\ &\leq C x_n^{-np}\cdot x_n^{(p-1)(n-1)} \int\limits_{|x-(y',0)|<4x_n} \big|f(y') - f(z')\big|^p\,dy'. \end{aligned} \tag{4.22}$$

At this stage, average the most extreme sides of (4.22) in $z' \in B_{n-1}(x, 4x_n) \subseteq \mathbb{R}^{n-1}$ in order to obtain

$$\begin{aligned} &\big|[\nabla(\mathscr{E}f)](x)\big|^p \leq \\ &\leq C x_n^{2-2n-p} \int\limits_{|x-(z',0)|<4x_n} \int\limits_{|x-(y',0)|<4x_n} \big|f(y') - f(z')\big|^p\,dy'\,dz' \end{aligned} \tag{4.23}$$

for each $x = (x', x_n) \in \mathbb{R}^n_+$. Consequently,

$$\begin{aligned} &\int\limits_{\mathbb{R}^n_+} \big|[\nabla(\mathscr{E}f)](x)\big|^p\, x_n^{ap}\,dx \leq \\ &\leq C \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}^{n-1}} \big|f(y') - f(z')\big|^p \Bigg[\int\limits_{\substack{|x-(z',0)|<4x_n \\ |x-(y',0)|<4x_n}} x_n^{ap-p-2n+2}\,dx\Bigg]\,dy'\,dz'. \end{aligned} \tag{4.24}$$

Observe that on the domain of integration of the inner-most integral we have $|x' - z'| < \sqrt{15}\,x_n$ and $|x' - y'| < \sqrt{15}\,x_n$, hence also $|y' - z'| < 2\sqrt{15}\,x_n$ by the triangle inequality. Bearing this in mind and using Fubini's theorem, we may estimate this inner-most integral by writing

$$\begin{aligned} &\int\limits_{\substack{|x-(z',0)|<4x_n \\ |x-(y',0)|<4x_n}} x_n^{ap-p-2n+2}\,dx \leq \\ &\leq \int\limits_{|y'-z'|/(2\sqrt{15})}^{\infty} \Bigg(\int\limits_{|x'-z'|<\sqrt{15}\,x_n} 1\,dx'\Bigg) x_n^{ap-p-2n+2}\,dx_n \leq \\ &\leq C_n \int\limits_{|y'-z'|/(2\sqrt{15})}^{\infty} x_n^{ap-p-n+1}\,dx_n = \frac{C_{n,a,p}}{|y' - z'|^{n+p-ap-2}}\,, \end{aligned} \tag{4.25}$$

where $C_{n,a,p} > 0$ is a finite constant, given that $ap - p - n + 1 < -1$. At this stage, (4.20) follows from (4.24) and (4.25).

Moving on, we claim that for each radius $R \in (0, +\infty)$ there exists a constant $C_{n,p,a,R} \in (0, +\infty)$ with the property that

$$\int_{\mathbb{R}^n_+ \cap B(0,R)} |(\mathscr{E} f)(x)|^p \, x_n^{ap} \, dx \leq$$

$$\leq C_{n,p,a,R} \int_{\mathbb{R}^{n-1}} |f(y')|^p \, dy', \quad \forall \, f \in \mathscr{C}_c^{\,0}(\mathbb{R}^{n-1}). \tag{4.26}$$

This estimate follows from a similar argument to that used in the verification of (4.20) (making use of (4.17) in place of (4.18)).

The final property of the operator $\mathscr{E}$ we wish to establish is that for each $f \in \mathscr{C}_c^{\,0}(\mathbb{R}^{n-1})$

$$\begin{aligned} &\mathscr{E} f \text{ extends continuously to } \overline{\mathbb{R}^n_+} \text{ and} \\ &\big[(\mathscr{E} f)|_{\partial\mathbb{R}^n_+}\big](x') = f(x'), \quad \forall \, x' \in \mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+. \end{aligned} \tag{4.27}$$

To this end, fix $f \in \mathscr{C}_c^{\,0}(\mathbb{R}^{n-1})$ along with some $x'_* \in \mathbb{R}^{n-1}$. Also, let some arbitrary $\varepsilon > 0$ be fixed. Since f is continuous at $(x'_*, 0)$, there exists $\delta > 0$ such that if $y' \in \mathbb{R}^{n-1}$ satisfies $|x'_* - y'| < \delta$ then $|f(x'_*) - f(y')| < \varepsilon$. Then for each $x = (x', x_n) \in \mathbb{R}^n_+$ we may estimate

$$\begin{aligned} \big|(\mathscr{E} f)(x) - f(x'_*)\big| &= \Big| \int_{\mathbb{R}^{n-1}} k\big(x, (y', 0)\big)\big(f(y') - f(x'_*)\big) \, dy' \Big| \\ &\leq \int_{\mathbb{R}^{n-1}} \big|k(x, (y', 0))\big| \, |f(y') - f(x'_*)| \, dy' \\ &\leq C_n \fint_{|x'-y'| < \sqrt{15}\, x_n} \big|f(y') - f(x'_*)\big| \, dy' \, , \end{aligned} \tag{4.28}$$

where the equality is based on (4.8), while for the last inequality we have used (4.17) and that the set $\{y' \in \mathbb{R}^{n-1} : |x - (y', 0)| < 4x_n\}$ is contained in the set $\{y' \in \mathbb{R}^{n-1} : \, |x' - y'| < \sqrt{15}\, x_n\}$. Thus,

$$\big|(\mathscr{E} f)(x) - f(x'_*)\big| \leq \varepsilon \;\text{ if }\; |x' - x'_*| < \delta/2 \;\text{ and }\; x_n < \delta/(2\sqrt{15}), \tag{4.29}$$

and the claims in (4.27) readily follow from this. In particular, $\mathrm{Tr}\, \mathscr{E} f = f$. This completes the discussion in the case when $\Omega = \mathbb{R}^n_+$.

The general situation when Ω is an arbitrary bounded Lipschitz domain may then be reduced to the case just treated via a smooth localization and by locally flatenning the boundary via bi-Lipschitz maps (as we have done in the past). Given that $\big(\mathscr{C}_c^\infty(\mathbb{R}^n)\big)\big|_{\partial\Omega}$ is dense in $B_s^{p,p}(\partial\Omega)$, the a priori bounds established in the first part of the proof may be used to conclude that all desired properties of the extension operator hold in this degree of generality. $\square$

In the last part of this section we once again revisit the issue of how weighted Sobolev spaces behave under complex interpolation. Our first result in this regard reads as follows.

Theorem 4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$\begin{aligned}&\big\{\mathring{W}^{1,p}_a(\Omega)\big\}_{1<p<\infty,\,-1/p<a<1-1/p},\\&\big\{W^{-1,p}_a(\Omega)\big\}_{1<p<\infty,\,-1/p<a<1-1/p}\end{aligned}\tag{4.30}$$

are complex interpolation scales, in the following precise sense. Suppose that, for $j \in \{0,1\}$, we have $1 < p_j < \infty$ and $-1/p_j < a_j < 1 - 1/p_j$. Also, fix $\theta \in (0,1)$ and assume that $p \in (0,\infty)$ and $a \in \mathbb{R}$ are such that $1/p = (1-\theta)/p_0 + \theta/p_1$ and $a = (1-\theta)a_0 + \theta a_1$. Then

$$\big[\mathring{W}^{1,p_0}_{a_0}(\Omega), \mathring{W}^{1,p_1}_{a_1}(\Omega)\big]_\theta = \mathring{W}^{1,p}_a(\Omega),\tag{4.31}$$

$$\big[W^{-1,p_0}_{a_0}(\Omega), W^{-1,p_1}_{a_1}(\Omega)\big]_\theta = W^{-1,p}_a(\Omega).\tag{4.32}$$

In the proof of the above theorem the following abstract interpolation result with constraints is going to be useful. For a proof, see [10, Theorem 14.3, p. 97] (cf. also [8]).

Lemma 4.3. *Let X_j, Y_j, Z_j, $j = 0,1$, be Banach spaces such that $X_0 \cap X_1$ is dense in both X_0 and X_1, and similarly for Z_0, Z_1. Suppose that $Y_j \hookrightarrow Z_j$, $j = 0,1$ and there exists a linear operator D such that $D : X_j \to Z_j$ boundedly for $j = 0,1$. Define the spaces*

$$X_j(D) := \{u \in X_j : \ Du \in Y_j\}, \quad j = 0,1,\tag{4.33}$$

equipped with the graph norm, i.e. $\|u\|_{X_j(D)} := \|u\|_{X_j} + \|Du\|_{Y_j}$, $j = 0,1$. Finally, suppose that there exist continuous linear mappings $K : Z_j \to X_j$ and $R : Z_j \to Y_j$ with the property $D \circ K = I + R$ on the spaces Z_j for $j = 0,1$. Then

$$\big[X_0(D), X_1(D)\big]_\theta = \Big\{u \in [X_0, X_1]_\theta : \ Du \in [Y_0, Y_1]_\theta\Big\}, \quad \theta \in (0,1).\tag{4.34}$$

We shall also need the well-known duality formula for the complex method of interpolation (see, for instance, [2]).

Lemma 4.4. *Let X_0, X_1 be a compatible couple of reflexive Banach spaces and let $\theta \in (0,1)$. Then*

$$\big([X_0, X_1]_\theta\big)^* = [X_0^*, X_1^*]_\theta.\tag{4.35}$$

We are prepared to present the

Proof of Theorem 4.2. Formula (4.31) follows from Theorem 4.1 and Lemma 4.3, used with

$$X_j := W^{1,p_j}_{a_j}(\Omega), \quad Y_j := 0, \ \text{ and } \ Z_j := B^{p_j,p_j}_{s_j}(\partial\Omega)\tag{4.36}$$

(as usual, $s_j := 1 - a_j - 1/p_j$), for $j = 0,1$, and where

$$D := \mathrm{Tr}, \quad K := \mathrm{Ex}, \ \text{ and } \ R := 0.\tag{4.37}$$

That $D \circ K = I + R$ on Z_j for $j = 0, 1$ makes the object of (4.2), and (4.34) becomes precisely (4.31), in light of (3.42). Finally, (4.32) is a consequence of (4.31), Lemma 4.4, and Proposition 2.3. □

5. Boundary Problems for Elliptic Systems with Bounded Measurable Coefficients in Euclidean Lipschitz Domains

The goal here is to prove the following sharp well-posedness result.

Theorem 5.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, connected, Lipschitz domain and assume that*

$$\mathbf{A} = \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\le j,k\le n\\ 1\le\alpha,\beta\le M}}, \quad a_{jk}^{\alpha\beta} \in L^\infty(\Omega), \tag{5.1}$$

is a coefficient tensor satisfying the strong Legendre-Hadamard ellipticity condition

$$\begin{gathered}\mathrm{Re}\Big[\sum_{j,\,k=1}^{n}\sum_{\alpha,\,\beta=1}^{M} a_{jk}^{\alpha\beta}(x)\,\zeta_j^\alpha\,\overline{\zeta_k^\beta}\,\Big] \ge c\,|\zeta|^2, \\ \forall\,\zeta = \big(\zeta_j^\alpha\big)_{\substack{1\le j\le n\\ 1\le\alpha\le M}} \in \mathbb{C}^{nM}, \ \text{ for a.e. } x \in \Omega,\end{gathered} \tag{5.2}$$

for some $c \in (0,\infty)$. Associated with the coefficient tensor $\mathbf{A}$ consider the $M \times M$ second order system in divergence form

$$L\mathbf{u} := \bigg(\sum_{j=1}^{n}\partial_j\Big(\sum_{k=1}^{n}\sum_{\beta=1}^{M} a_{jk}^{\alpha\beta}\,\partial_k u_\beta\Big)\bigg)_{1\le\alpha\le M}, \quad \mathbf{u} = (u_\beta)_{1\le\beta\le M}. \tag{5.3}$$

Then there exists some $\varepsilon > 0$ such that whenever

$$p \in (2-\varepsilon, 2+\varepsilon), \quad a \in (-1/p, 1-1/p) \cap (-\varepsilon,\varepsilon), \quad s := 1 - a - 1/p, \tag{5.4}$$

the Poisson boundary value problem with Dirichlet boundary data,

$$\begin{cases}\mathbf{u} \in W_a^{1,p}(\Omega),\\ L\mathbf{u} = \mathbf{f} \in W_a^{-1,p}(\Omega),\\ \mathrm{Tr}\,\mathbf{u} = \mathbf{g} \in B_s^{p,p}(\partial\Omega),\end{cases} \tag{5.5}$$

is well-posed. That is, assuming p, a, s are as in (5.4), for each $\mathbf{f} \in W_a^{-1,p}(\Omega)$ and $\mathbf{g} \in B_s^{p,p}(\partial\Omega)$ there exists a unique solution $\mathbf{u}$ of (5.5), which also satisfies the estimate

$$\|\mathbf{u}\|_{W_a^{1,p}(\Omega)} \le C\Big(\|\mathbf{f}\|_{W_a^{-1,p}(\Omega)} + \|\mathbf{g}\|_{B_s^{p,p}(\partial\Omega)}\Big), \tag{5.6}$$

where $C \in (0,+\infty)$ is independent of $\mathbf{f}$ and $\mathbf{g}$.

To set the stage, we first record a useful preliminary result in the proposition below. General abstract stability results of this type have been established in [9].

Proposition 5.2. *Suppose I is a convex Euclidean set and $(X_q)_{q\in I}$, $(Y_q)_{q\in I}$ are two complex interpolation scales of Banach spaces. In addition, assume that T is an operator such that*

$$\begin{aligned} &T : X_q \longrightarrow Y_q \text{ linearly and boundedly for each } q \in I, \text{ and} \\ &\exists\, q_* \in I \text{ such that } T : X_{q_*} \longrightarrow Y_{q_*} \text{ is an isomorphism.} \end{aligned} \tag{5.7}$$

Then there exists a neighborhood $\mathscr{O}$ of q_* such that $T : X_q \to Y_q$ is an isomorphism for every $q \in \mathscr{O}$.

We may now turn our attention to presenting the

Proof of Theorem 5.1. For starters, from the discussion in Section 2 we know that

$$W_a^{-1,p}(\Omega) = \Big\{ u \in \mathscr{D}'(\Omega) : \exists\, h_0, h_1, \dots, h_n \in L^p(\Omega, \delta^{ap} dx) \\ \text{such that } u = h_0 + \sum_{j=1}^n \partial_j h_j \text{ in } \mathscr{D}'(\Omega) \Big\}, \tag{5.8}$$

and the norm on this space is equivalent to

$$\|u\|_{W_a^{-1,p}(\Omega)} = \inf \Big\{ \sum_{j=0}^n \|h_j\|_{L^p(\Omega,\delta^{ap}dx)} : h_0, h_1, \dots, h_n \in L^p(\Omega, \delta^{ap} dx) \\ \text{such that } u = h_0 + \sum_{j=1}^n \partial_j h_j \text{ in } \mathscr{D}'(\Omega) \Big\}.$$

Granted these, it follows that

$$\begin{aligned} &L : W_a^{1,p}(\Omega) \longrightarrow W_a^{-1,p}(\Omega) \text{ linearly and boundedly,} \\ &\text{whenever } p \in (1, \infty) \text{ and } a \in (-1/p, 1 - 1/p). \end{aligned} \tag{5.9}$$

In addition, from the Lax–Milgram Lemma (which, in turn, makes use of the strong ellipticity condition on L) we deduce that

$$L : \mathring{W}^{1,2}(\Omega) \longrightarrow W^{-1,2}(\Omega) \text{ isomorphically.} \tag{5.10}$$

Our next claim is that there exists $\varepsilon > 0$ such that

$$\begin{aligned} &L : \mathring{W}_a^{1,p}(\Omega) \longrightarrow W_a^{-1,p}(\Omega) \text{ isomorphically} \\ &\text{whenever } p \in (2 - \varepsilon, 2 + \varepsilon) \text{ and } a \in (-1/p, 1 - 1/p) \cap (-\varepsilon, \varepsilon). \end{aligned} \tag{5.11}$$

This follows from (5.9), (5.10), and Proposition 5.2.

Having proved (5.11), the final step is to show that, for p, a as above and with $s := 1 - a - 1/p$, the boundary value problem (5.5) is well-posed. Uniqueness is clear from (5.11) and (3.42). For existence, let $\mathbf{f} \in W_a^{-1,p}(\Omega)$ and $\mathbf{g} \in B_s^{p,p}(\partial\Omega)$ be given. From Theorem 4.1, we know that $\mathbf{v} := \operatorname{Ex} \mathbf{g} \in W_a^{1,p}(\Omega)$ satisfies $\operatorname{Tr} \mathbf{v} = \mathbf{g}$. Moreover, since the operator Ex is bounded, we have

$$\|\mathbf{v}\|_{W_a^{1,p}(\Omega)} \le C\, \|\mathbf{g}\|_{B_s^{p,p}(\partial\Omega)}, \tag{5.12}$$

where $C \in (0,\infty)$ is independent of $\mathbf{g}$. Consider the function $\widetilde{\mathbf{f}} := \mathbf{f} - L\mathbf{v} \in W_a^{-1,p}(\Omega)$ and note that

$$\|\widetilde{\mathbf{f}}\|_{W_a^{-1,p}(\Omega)} \leq C\left(\|\mathbf{f}\|_{W_a^{-1,p}(\Omega)} + \|\mathbf{g}\|_{B_s^{p,p}(\partial\Omega)}\right), \tag{5.13}$$

where $C \in (0,\infty)$ is independent of $\mathbf{f}$ and $\mathbf{g}$. Since $L : \mathring{W}_a^{1,p}(\Omega) \to W_a^{-1,p}(\Omega)$ is an isomorphism and $\widetilde{\mathbf{f}} \in W_a^{-1,p}(\Omega)$, it follows that $\mathbf{w} := L^{-1}(\widetilde{\mathbf{f}}) \in \mathring{W}_a^{1,p}(\Omega)$ and $L\mathbf{w} = \widetilde{\mathbf{f}}$. Finally, take $\mathbf{u} := \mathbf{v} + \mathbf{w} \in W_a^{1,p}(\Omega)$ and compute

$$L\mathbf{u} = L\mathbf{v} + \widetilde{\mathbf{f}} = L\mathbf{v} + (\mathbf{f} - L\mathbf{v}) = \mathbf{f} \tag{5.14}$$

and

$$\operatorname{Tr}\mathbf{u} = \operatorname{Tr}(\operatorname{Ex}\mathbf{g}) + \operatorname{Tr}\left(L^{-1}(\widetilde{\mathbf{f}})\right) = \mathbf{g} + \operatorname{Tr}\mathbf{w} = \mathbf{g} + \mathbf{0} = \mathbf{g}. \tag{5.15}$$

This finishes the existence of a function $\mathbf{u}$ satisfying the boundary value problem. □

Theorem 5.1 is sharp, in the sense that the membership of p to a small neighborhood of 2 is a necessary condition, even when $\Omega \subseteq \mathbb{R}^n$ is a bounded $\mathscr{C}^\infty$ domain, and when $a = 0$ (i.e., in the unweighted case), if the coefficients of the system L are merely bounded and measurable.

When $n \geq 3$, $M = n$, a counterexample may be produced by altering a construction of E. De Giorgi from [5]. Specifically, consider $\Omega := \{x \in \mathbb{R}^n : |x| < 1\}$ and, for each $\gamma \in [0, \frac{n}{2})$ and $\alpha, \beta \in \{1, \dots, n\}$, let $A_{\alpha\beta}$ be the $n \times n$ matrix whose (i,j)-entry is

$$\begin{aligned} a_{ij}^{\alpha\beta}(x) &:= \delta_{\alpha\beta}\delta_{ij} + \\ &+ \frac{\gamma(n-\gamma)(n-2)^2}{(n-2\gamma)^2(n-1)^2}\left[\delta_{i\alpha} + \frac{n}{n-2}\frac{x_i x_\alpha}{|x|^2}\right]\left[\delta_{j\beta} + \frac{n}{n-2}\frac{x_j x_\beta}{|x|^2}\right] \end{aligned} \tag{5.16}$$

for each $x \in \Omega \setminus \{0\}$. Obviously, $a_{ij}^{\alpha\beta} \in L^\infty(\Omega, \mathscr{L}^n)$ and a straightforward calculation shows that

$$\begin{aligned} &\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x)\zeta_i^\alpha\zeta_j^\beta = \\ &= |\zeta|^2 + \frac{\gamma(n-\gamma)(n-2)^2}{(n-2\gamma)^2(n-1)^2}\Big(\sum_{i=1}^{n}\zeta_i^i + \frac{n}{n-2}\sum_{i,\alpha=1}^{n}\zeta_i^\alpha\frac{x_i x_\alpha}{|x|^2}\Big)^2 \end{aligned} \tag{5.17}$$

for each $\zeta = \left(\zeta_i^\alpha\right)_{1\leq\alpha,i\leq n} \in \mathbb{R}^{n^2}$ and $x \in \Omega \setminus \{0\}$. Given our assumptions on γ, it follows that the strong ellipticity condition holds:

$$\sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x)\zeta_i^\alpha\zeta_j^\beta \geq |\zeta|^2 \;\; \mathscr{L}^n\text{-a.e. in } \Omega, \qquad \forall\,\zeta = \left(\zeta_i^\alpha\right)_{1\leq\alpha,i\leq n} \in \mathbb{R}^{n^2}. \tag{5.18}$$

Now, the fact that $\gamma < n/2$ ensures that the function

$$u(x) := \frac{x}{|x|^\gamma} - x, \ \ \forall\, x \in \Omega \setminus \{0\}, \tag{5.19}$$

belongs to $W^{1,2}(\Omega)$. Since by design $u\big|_{\partial\Omega} = 0$, we deduce that actually $u \in \mathring{W}^{1,2}(\Omega)$. Furthermore, if

$$f := (f_1, \dots, f_n) \ \text{ with } \ f_i := -\sum_{\alpha=1}^{n}\sum_{j=1}^{n} \partial_\alpha a_{ij}^{\alpha j} \ \text{ for } \ 1 \leq i \leq n, \tag{5.20}$$

then clearly

$$f \in \bigcap_{1<p<\infty} W^{-1,p}(\Omega), \tag{5.21}$$

while a direct computation shows that

$$\sum_{\alpha,\beta=1}^{n} \partial_\alpha \big(A_{\alpha\beta}(x)\partial_\beta u\big) = f \ \text{ in } \ \mathscr{D}'(\Omega). \tag{5.22}$$

However, on the one hand $u \in W^{1,p}(\Omega)$ if and only if $p < n/\gamma$, while on the other hand $n/\gamma \searrow 2$ as $\gamma \nearrow n/2$. By duality, (note that L is formally self-adjoint), the same type of conclusion holds for $p < 2$.

6. The Setting of Weakly Lipschitz Domains

A careful inspection of the arguments in the proof of Theorem 5.1 reveals that we may relax the assumption on the domain Ω, originally assumed to be a Lipschitz domain. Specifically, it suffices to ask that $\Omega \subset \mathbb{R}^n$ is a a bounded, open set, with the property that for every $x_0 \in \partial\Omega$ there exist an open neighborhood U of x_0 in $\mathbb{R}^n$ and a mapping $F = (F_1, \dots, F_n) : U \to \mathbb{R}^n$ with the following properties:

(i) $F(U)$ is open and $F : U \to F(U)$ is a bi-Lipschitz map;

(ii) $\Omega \cap U = \{x \in U : \ F_n(x) > 0\}$.

In the sequel, we shall refer to such a set Ω as being a *weakly Lipschitz domain*. This is done in order to distinguish the latter from the more familiar category of "strongly" Lipschitz domains considered so far.

Note that if the bi-Lipschitzianity assumption for F is strengthened by demanding that F is a $\mathscr{C}^1$-diffeomorphism, then the resulting class becomes precisely the category of bounded $\mathscr{C}^1$ domains in $\mathbb{R}^n$. This is easily seen by invoking the standard Implicit Function Theorem for $\mathscr{C}^1$ functions. However, when dealing with the case when F is only bi-Lipschitz, the nature of the Implicit Function Theorem changes drastically and, as a result, the class of weakly Lipschitz domains is much larger than that of strongly Lipschitz domains. To shed light on this issue, we next discuss some concrete examples. In fact, since the bi-Lipschitz image of a strongly Lipschitz domain is a weakly Lipschitz domain, it suffices to show that the class of strongly Lipschitz domains is not stable under bi-Lipschitz homeomorphisms.

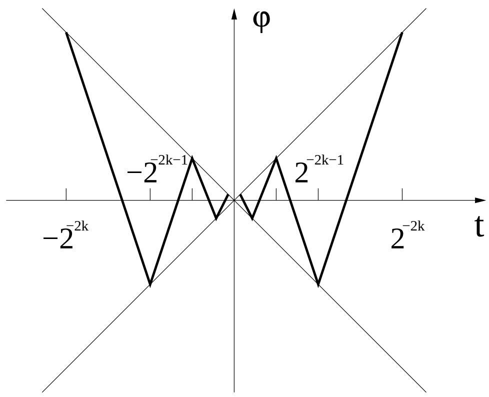
We start with an interesting example from (pp. 7–9 in) [6], where this is attributed to Zerner. Concretely, consider the bi-Lipschitz homeomorphism

$$F:\mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \ F(x_1,x_2) := (x_1, \varphi(x_1)+x_2), \tag{6.1}$$

where $\varphi:\mathbb{R}\to\mathbb{R}$ is the Lipschitz function

$$\varphi(t) := \begin{cases} 3|t| - \dfrac{1}{2^{2k-1}} & \text{for } \dfrac{1}{2^{2k+1}} \leq |t| \leq \dfrac{1}{2^{2k}}\,, \\ -3|t| + \dfrac{1}{2^{2k}} & \text{for } \dfrac{1}{2^{2k+2}} \leq |t| \leq \dfrac{1}{2^{2k+1}}\,. \end{cases} \tag{6.2}$$

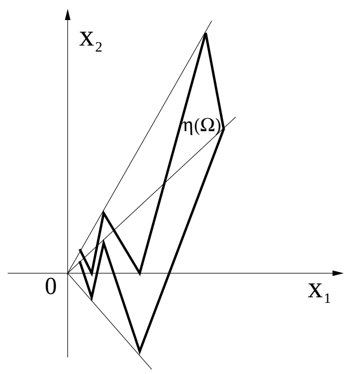
As is also visible from the picture below, the graph of φ is a zigzagged of lines of slopes ± 3:



If one now considers the bounded Lipschitz domain $\Omega\subset\mathbb{R}^2$,

$$\Omega := \Big\{(x_1,x_2):\ 0<x_1<1,\ 0<x_2<x_1\Big\}, \tag{6.3}$$

then $F(\Omega)$, depicted below



fails to be a strongly Lipschitz domain, since the cone property is violated at the origin.

In fact, the construction described above can be refined to show that *bi-Lipschitz functions may fail to map even bounded C^∞ planar domains into strongly Lipschitz domains.* Concretely, pick $x_0 \in \Omega$ and let $\varphi : S^1 \to (0,\infty)$ be the Lipschitz function uniquely determined by the requirement that $G : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $G(x) := \varphi((x-x_0)/|x-x_0|)(x-x_0)$ if $x \neq x_0$ and $G(x_0) := 0$, maps $\partial B(x_0, r)$ onto $\partial\Omega$ (for some fixed, sufficiently small $r > 0$). Then $F \circ G$ maps the bounded, C^∞ domain $B(x_0, r)$ onto the domain shown in the picture above. There are many other interesting examples of strongly Lipschitz domains $\Omega \subset \mathbb{R}^n$ and bi-Lipschitz maps $F : \mathbb{R}^n \to \mathbb{R}^n$ with the property that $F(\Omega)$ fails to be strongly Lipschitz. A large category of such examples can be found within the class of *conical domains*. In order to be more specific, let S^{n-1} stand for the unit sphere in $\mathbb{R}^n$ and denote by S^{n-1}_+ its upper hemisphere. Pick a bi-Lipschitz homeomorphism $\psi : S^{n-1} \to S^{n-1}$ along with an arbitrary Lipschitz function $\varphi : S^{n-1} \to (0,\infty)$, and set

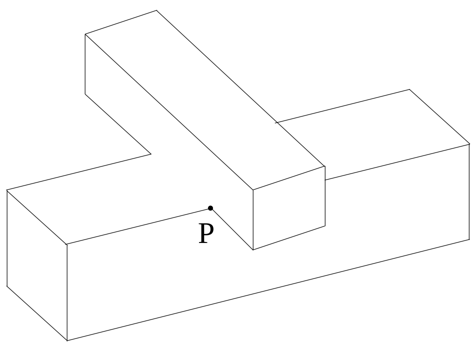
$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad F(r\omega) := r\varphi(\omega)\psi^{-1}(\omega), \;\; r \geq 0, \;\; \omega \in S^{n-1}, \tag{6.4}$$

$$\Omega := \Big\{ r\omega : \; \omega \in S^{n-1}_+, \; 0 < r < \varphi(\omega) \Big\}. \tag{6.5}$$

Using $|r_1\omega_1 - r_2\omega_2|^2 = |r_1 - r_2|^2 + r_1 r_2 |\omega_1 - \omega_2|^2$ for every $\omega_1, \omega_2 \in S^{n-1}$, $r_1, r_2 \geq 0$, and the fact that the inverse of (6.4) is $F^{-1}(r\omega) = r\varphi(\omega)^{-1}\psi(\omega)$, it can be easily checked that F above is bi-Lipschitz. However, while $\Omega \subset \mathbb{R}^n$ is clearly a strongly Lipschitz domain in $\mathbb{R}^n$,

$$F(\Omega) = \Big\{ \rho w : \; w \in \psi(S^{n-1}_+), \; 0 < \rho < \varphi(\omega) \Big\}, \tag{6.6}$$

may fail to be a strongly Lipschitz domain. In fact, near $0 \in \partial F(\Omega)$, the surface $\partial F(\Omega)$ may fail to be the graph of *any* real-valued function of $n-1$ variables, in any system of coordinates which is a rigid motion of the standard one (i.e., $\partial F(\Omega)$ is a non-Lipschitz cone). A concrete example, which can be produced using the above recipe, is Maz'ya's so-called *two-brick domain*:



A moment's reflection shows that, indeed, near the point P, the boundary of the above domain is not the graph of any function (as it fails the vertical line test) in any system of coordinates isometric to the original one.

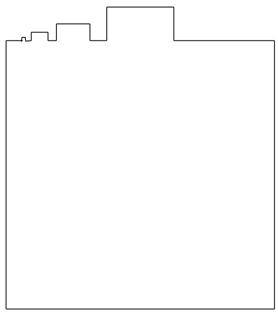
Moreover, images of bounded strongly Lipschitz domains via bi-Lipschitz maps can also develop spiral-like singularities, such as

$$\begin{aligned} F(\Omega) &= \Big\{re^{i(\theta-\ln r)} :\ 0<\theta<\pi/4,\ 0<r<1\Big\} \subset \mathbb{R}^2 \equiv \mathbb{C}, \\ \Omega &:= \Big\{re^{i\theta} :\ 0<r<1,\ 0<\theta<\pi/4\Big\}, \quad F(re^{i\theta}) := re^{i(\theta-\ln r)}. \end{aligned} \tag{6.7}$$

Another interesting example of the phenomenon described above is as follows. Let

$$\widetilde{\Omega} := \big[(0,1)\times(-1,0)\big] \cup \Bigg[\bigcup_{k=1}^{\infty}(3\cdot 2^{-k-2}, 5\cdot 2^{-k-2})\times[0,2^{-k-2})\Bigg] \tag{6.8}$$

be the planar domain in the picture below:



It is not difficult to see that the uniformity of the cone condition is violated in any neighborhood of the origin, so $\widetilde{\Omega}$ is not a strongly Lipschitz domain. Nonetheless, on p. 19 of [11], Maz'ya has constructed a bi-Lipschitz map $F:\mathbb{R}^2\to\mathbb{R}^2$ with the property that $\widetilde{\Omega} = F((0,1)\times(0,1))$.

In the next section we shall actually take this analysis a step further and indicate that well-posedness results in the spirit of those established so far continue to hold in the setting of Lipschitz manifolds with boundary, which is even more general (as all weakly Lipschitz domains in $\mathbb{R}^n$ fall into the latter category).

7. The Setting of Lipschitz Manifolds with Boundary

For the convenience of the reader, here we collect some basic rudiments of analysis on Lipschitz manifolds.

A compact topological manifold with boundary $\mathscr{M}$ of dimension n is a compact, Hausdorff topological space $\mathscr{M}$ with the property that for every $x\in\mathscr{M}$ there exists an open set U in $\mathscr{M}$, $x\in U$, and a mapping $\phi: U\to\mathbb{R}^n$ such that $\phi(U)$ is a relatively open subset of $\overline{\mathbb{R}^n_+}$ and $\phi: U\to\phi(U)$ is a homeomorphism. We shall call (U,ϕ) a *coordinate chart* (about x). An *atlas* on $\mathscr{M}$ is a finite family $\mathscr{A} = \{U_i, \phi_i\}_{i\in I}$ such that $\mathscr{M} = \bigcup_{i\in I} U_i$ and (U_i, ϕ_i) is a coordinate chart for each $i \in I$.

Define the interior Ω of $\mathscr{M}$ as the collection of points x for which there is a coordinate chart (U, ϕ) about x with the property that $\phi(U)$ is an open subset of $\mathbb{R}^n_+$. Then set $\partial\Omega := \mathscr{M} \setminus \Omega$ and call it the *boundary of* $\mathscr{M}$.

A compact topological manifold with boundary $\mathscr{M}$ is called a *compact Lipschitz manifold with boundary* if there exists an atlas (called Lipschitz atlas) $\mathscr{A} = \{U_i, \phi_i\}_{i\in I}$ such that for any $i, j \in I$ the transition map $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ is by-Lipschitz (with respect to the usual metric in $\mathbb{R}^n$). Two atlases are called *equivalent* provided their union is an atlas. A *Lipschitz structure* on $\mathscr{M}$ is the equivalence class of a certain Lipschitz atlas, called *structural atlas*. In what follows, given a compact Lipschitz manifold with boundary $\mathscr{M}$, we shall always assume that a Lipschitz structure on $\mathscr{M}$ has been fixed. Any Lipschitz atlas compatible with this structure will be referred to as a structural atlas.

Given a compact Lipschitz manifold with boundary $\mathscr{M}$, equipped with a structural atlas $\mathscr{A} = \{U_i, \phi_i\}_{i\in I}$, call a set $S \subseteq \mathscr{M}$ of *zero measure* in $\mathscr{M}$ if $\phi_i(U_i \cap S)$ has measure zero in $\mathbb{R}^n$ with respect to the usual n-dimensional Lebesgue measure for every $(U_i, \phi_i) \in \mathscr{A}$. Accordingly, a property is said to hold *almost everywhere* (a.e.) on $\mathscr{M}$ provided the set of points where it fails has zero measure in $\mathscr{M}$.

A real-valued function defined a.e. on $\mathscr{M}$ is called *measurable* if it is so in any coordinate chart of a structural atlas. Furthermore, the class $L^p(\mathscr{M})$, $1 \le p \le \infty$, of real valued functions L^p-integrable on $\mathscr{M}$ is introduced in a similar fashion.

Next we introduce the *singular set* of $\mathscr{M}$ relative to a structural atlas $\mathscr{A} = \{U_i, \phi_i\}_{i\in I}$ as being

$$\begin{aligned}\mathrm{Sing}(\mathscr{M};\mathscr{A}) := \Big\{x \in \mathscr{M} :\ &\text{there exist } i, j \in I \text{ with } x \in U_i \cap U_j \\ \text{and such that } &\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \\ &\text{is not differentiable at } \phi_j(x)\Big\}. \qquad (7.1)\end{aligned}$$

A basic observation is that the singular set of a compact, boundaryless, Lipschitz manifold, relative to any structural atlas, has measure zero. In the sequel, points in $\mathrm{Sing}(\mathscr{M};\mathscr{A})$ will be called *singular points* (relative to $\mathscr{A}$), whereas points in $\mathrm{Reg}(\mathscr{M};\mathscr{A}) := \mathscr{M} \setminus \mathrm{Sing}(\mathscr{M};\mathscr{A})$ will be referred to as *regular points* (relative to $\mathscr{A}$).

Definition 7.1. Let $(\mathscr{M}_j, \mathscr{A}_j)$ be two compact Lipschitz manifolds with boundary, $j = 1, 2$. A continuous mapping $f : \mathscr{M}_1 \to \mathscr{M}_2$ will be called *differentiable* at $x \in \mathscr{M}_1$ provided the following properties are valid:

(i) x is a regular point of $\mathscr{M}_1$, relative to some structural atlas $\mathscr{A}_1$;

(ii) $f(x)$ is a regular point of $\mathscr{M}_2$ relative to some structural atlas $\mathscr{A}_2$;

(iii) there exist $(U_j, \phi_j) \in \mathscr{A}_j$, $j = 1, 2$, with $x \in U_1$, $f(x) \in U_2$, such that the function $\phi_2 \circ f \circ \phi_1^{-1} : \phi_1(U_1 \cap f^{-1}(U_2)) \to \phi_2(U_2)$ is differentiable at $\phi_1(x)$.

We continue to assume that $(\mathscr{M}_j, \mathscr{A}_j)$, $j = 1, 2$, are two compact Lipschitz manifolds with boundary. A continuous map $f : \mathscr{M}_1 \to \mathscr{M}_2$ will be called *Lipschitz* if for any two coordinate charts $(U_j, \phi_j) \in \mathscr{A}_j$, $j = 1, 2$, the composition $\phi_2 \circ f \circ \phi_1^{-1} : \phi_1(U_1 \cap f^{-1}(U_2)) \longrightarrow \phi_2(U_2)$ is a Lipschitz function. Also, call f *bi-Lipschitz*, if f is a homeomorphism and both f and f^{-1} are Lipschitz. Is important to observe that a Lipschitz function maps sets of zero measure into sets of zero measure.

As a consequence of definitions and the celebrated theorem of Rademacher, according to which Lipschitz functions between Euclidean spaces are differentiable almost everywhere, we have the following result.

Proposition 7.2. *Assume that $\mathscr{M}_j$, $j = 1, 2$, are compact Lipschitz manifolds with boundary and that $f : \mathscr{M}_1 \to \mathscr{M}_2$ is a Lipschitz function. In addition, assume that*

$$f^{-1}(\mathrm{Sing}(\mathscr{M}_2; \mathscr{A}_2)) \text{ has zero measure in } \mathscr{M}_1, \quad \text{for any structural atlas } \mathscr{A}_2 \text{ of } \mathscr{M}_2. \tag{7.2}$$

(*We note that this condition is automatically verified if f is bi-Lipschitz, or if $\mathscr{M}_2$ is a $\mathscr{C}^1$ manifold.*) *Then f is differentiable almost everywhere in $\mathscr{M}_1$.*

Moving on, if $x \in \mathscr{M}$, two mappings f, g from a neighborhood of x into $\mathbb{R}$ are called *equivalent at x* (and we denote this by $f \overset{x}{\sim} g$) if there exists V open small neighborhood of x such that $f|_V = g|_V$. Classes of equivalence modulo $\overset{x}{\sim}$ will be called *germs at x*. We shall pay special attention to germs of differentiable functions at a regular point x, relative to a structural atlas $\mathscr{A}$, which will be denoted by $\mathrm{Diff}_x(\mathscr{M}; \mathscr{A})$. A continuous mapping $\gamma : (-\varepsilon, \varepsilon) \to \mathscr{M}$, $\varepsilon > 0$, with $\gamma(0) = x$ and such that there exists $(U, \phi) \in \mathscr{A}$ for which $x \in U$ and $\phi \circ \gamma$ is differentiable at 0, will be called *path* (through x). For such a path γ we define a linear mapping $\frac{d}{d\gamma} : \mathrm{Diff}_x(\mathscr{M}; \mathscr{A}) \to \mathbb{R}$ called *derivation along γ* (at x) by

$$\frac{d}{d\gamma}\,([f]) := \frac{d}{dt}\,(f \circ \gamma)(t)\Big|_{t=0},$$

for any $[f] \in \mathrm{Diff}_x(\mathscr{M}; \mathscr{A})$. Let $\{e_k\}_{1 \le k \le n}$ denote the standard orthonormal basis in $\mathbb{R}^n$. If $(U, \phi) \in \mathscr{A}$ is such that $x \in U$ then, for each $k = 1, 2, \dots, n$, the derivation along $\phi^{-1}(\phi(x) + te_k)$ at $x \in \mathrm{Reg}(\mathscr{M}; \mathscr{A})$ is denoted by $\frac{d}{d\phi_k}$. Note that

$$\frac{d}{d\phi_k}\,([f]) = \frac{\partial(f \circ \phi^{-1})}{\partial x_k}\,(\phi(x)), \quad k = 1, 2, \dots, n. \tag{7.3}$$

Once a structural atlas $\mathscr{A}$ has been fixed, we can define the tangent space at $x \in \mathscr{M}$ to the manifold $\mathscr{M}$ by setting

$$T_x\mathscr{M} := \Big\{\frac{d}{d\gamma} : \gamma \text{ path through } x\Big\} \text{ if } x \in \operatorname{Reg}(\mathscr{M};\mathscr{A}), \tag{7.4}$$

and

$$T_x\mathscr{M} := \{0\} \text{ if } x \in \operatorname{Sing}(\mathscr{M};\mathscr{A}). \tag{7.5}$$

It is not difficult to check that $T_x\mathscr{M}$ is a vector space and that in fact $\dim(T_x\mathscr{M}) = n$ (i.e., the same as the dimension of $\mathscr{M}$) at any regular point x, relative to $\mathscr{A}$. In fact, for $(U,\phi) \in \mathscr{A}$ a basis in $T_x\mathscr{M}$ at any regular point $x \in U$ is given by $\{\frac{d}{d\phi_k}\}_{k=1}^n$. Now, the *tangent bundle* is

$$T\mathscr{M} := \bigsqcup_{x\in\mathscr{M}} T_x\mathscr{M}. \tag{7.6}$$

We wish to emphasize that the tangent bundle $T\mathscr{M}$ depends on the choice of a structural atlas only up to a set of zero measure in $\mathscr{M}$.

Going further, let $f : \mathscr{M}_1 \to \mathscr{M}_2$ be a continuous function between two compact Lipschitz manifolds with boundary $\mathscr{M}_1$ and $\mathscr{M}_2$ which is differentiable almost everywhere. We then define the *gradient* of f as the mapping $\operatorname{Grad} f : T\mathscr{M}_1 \to T\mathscr{M}_2$ defined almost everywhere in the following way. At almost every differentiability point $x \in \mathscr{M}_1$ of f, $\operatorname{Grad}_{\mathscr{M}} f_x$ is defined as the mapping of $T_x\mathscr{M}_1$ into $T_{f(x)}\mathscr{M}_2$ given by

$$\operatorname{Grad}_{\mathscr{M}}\, f_x\Big(\frac{d}{d\gamma}\Big) := \frac{d}{d(f\circ\gamma)}\,, \tag{7.7}$$

for any path γ through x (note that $f \circ \gamma$ is a path through $f(x)$ for almost every x). Let us also note that if $(U, \phi = (\phi_1,\dots,\phi_n)) \in \mathscr{A}$ then $\operatorname{Grad}_{cM}\, \phi_j(\frac{d}{d\phi_k}) = \delta_{jk}\,\frac{d}{dt}$ for every $1 \le j,k \le n$, where we have denoted by $\frac{d}{dt}$ the standard derivation on $\mathbb{R}$ and, as before, δ_{jk} stands for the Kronecker symbol.

Assume next that the compact Lipschitz manifold with boundary $\mathscr{M}$ is *oriented* and equipped with a *(Lipschitz) Riemannian metric*. Being oriented is defined essentially as in the smooth case. That is, an orientation has been specified in $T_x\mathscr{M}$ for a.e. $x \in \mathscr{M}$ such that there exists a structural atlas $\mathscr{A}$ which contains only positive coordinate charts. Recall that a chart $(U,\phi) \in \mathscr{A}$ is called *positive* if the ordered n-tuple $(\frac{d}{d\phi_1},\dots,\frac{d}{d\phi_n})$ is a positively oriented basis of $T_x\mathscr{M}$ for a.e. $x \in U$. Also, by a *Lipschitz Riemannian structure*, we mean that at almost any point $x \in \mathscr{M}$ some inner product $\langle\,\cdot\,,\,\cdot\,\rangle_x$ has been specified on the tangent space $T_x\mathscr{M}$ with the following properties:

(i) $\langle\,\cdot\,,\,\cdot\,\rangle_x$ varies measurably with x, that is, if $\mathscr{A}$ is a structural atlas consisting of positive charts and $(U,\phi) \in \mathscr{A}$, then the functions

$$g_{ij}^U(x) := \Big\langle \frac{d}{d\phi_i}\,,\frac{d}{d\phi_j}\Big\rangle_x, \quad x \in U, \quad 1 \le i,j \le n, \tag{7.8}$$

are measurable on U;

(ii) there exist a structural atlas $\mathscr{A}$ and two finite constants $C_1, C_2 > 0$ such that for any $(U, \phi) \in \mathscr{A}$, for a.e. $x \in U$, and any path γ through x such that $\phi \circ \gamma$ is differentiable at 0, there holds

$$C_1 \big\|(\phi \circ \gamma)'(0)\big\|^2_{\mathbb{R}^n} \leq \Big\langle \frac{d}{d\gamma}, \frac{d}{d\gamma} \Big\rangle_x \leq C_2 \big\|(\phi \circ \gamma)'(0)\big\|^2_{\mathbb{R}^n} \tag{7.9}$$

(here and elsewhere, $\|\cdot\|_{\mathbb{R}^n}$ refers to the Euclidean norm in $\mathbb{R}^n$).

This latter condition implies that the matrix $G^U(x) := (g^U_{ij}(x))_{1 \leq i,j \leq n}$ is symmetric, bounded and positive definite in an uniform manner, for a.e. $x \in U$. In fact,

$$C_1 \|v\|^2_{\mathbb{R}^n} \leq \langle G^U(x)v, v\rangle_{\mathbb{R}^n} \leq C_2 \|v\|^2_{\mathbb{R}^n}, \tag{7.10}$$
$$\text{for any } \; v \in \mathbb{R}^n \; \text{ and a.e. } \; x \in U.$$

Proposition 7.3. *Any compact Lipschitz manifold with boundary $\mathscr{M}$ has a Lipschitz Riemannian metric.*

Proof. A Lipschitz Riemannian metric on $\mathscr{M}$ can be constructed by locally transferring the Euclidean metric from $\mathbb{R}^n$ in a standard fashion, and then gluing everything together via a Lipschitz partition of unity. □

The inner product on the tangent space $T_x\mathscr{M}$ induces a natural pointwise inner product $\langle \cdot\,, \cdot \rangle_{\Lambda^\ell T_x\mathscr{M}}$ on $\Lambda^\ell T_x\mathscr{M}$, the ℓ-th exterior power of the tangent bundle for each $0 \leq \ell \leq n$, at a.e. $x \in \mathscr{M}$. In particular, there exists a unique form, denoted by $dV_{\mathscr{M}}$, of maximal degree, normalized to one (in the norm $|\cdot|_{\Lambda^\ell T_x\mathscr{M}}$ associated with the above inner product) a.e. on $\mathscr{M}$ and which is positively oriented. We shall refer to this n-form as the volume element on $\mathscr{M}$. In turn, this gives rise to a Borel regular measure $\mathscr{L}_{\mathscr{M}}$ on $\mathscr{M}$, uniquely determined by the requirement that if f is a scalar-valued continuous function on $\mathscr{M}$ which is supported an open subset $\mathscr{O}$ of $\mathscr{M}$ then

$$\int_{\mathscr{O}} f \, d\mathscr{L}_{\mathscr{M}} = \sum_j \int_{\phi_j(U_j \cap \mathscr{O})} (\phi_j^{-1})^*(\theta_j f \, dV_{\mathscr{M}}), \tag{7.11}$$

where $\{\theta_j\}_j$ is a Lipschitz partition of unity on $\mathscr{M}$ subordinated to (a finite) open cover $(U_j)_j$ of $\mathscr{M}$, with the property that $(U_j, \phi_j) \in \mathscr{A}$ for each j.

Proposition 7.4. *Consider a compact, oriented Lipschitz manifold with boundary $\mathscr{M}$ equipped with a Lipschitz Riemannian metric. Also, fix a positive structural atlas $\mathscr{A}$ and denote by $dV_{\mathscr{M}}$ the volume element on $\mathscr{M}$. Then for every $(U, \phi) \in \mathscr{A}$ one has*

$$(\phi^{-1})^*(dV_{\mathscr{M}}) =$$
$$= \left[\det \left(\Big\langle \frac{d}{d\phi_i}, \frac{d}{d\phi_j} \Big\rangle_{\phi^{-1}(\cdot)} \right)_{i,j} \right]^{1/2} dV_{\mathbb{R}^n} \;\; \textit{a.e. on} \;\; \phi(U), \tag{7.12}$$

where $dV_{\mathbb{R}^n} = dx_1 \wedge \cdots \wedge dx_n$ *is the volume element in* $\mathbb{R}^n$*, and if* C_1, C_2 *are as in* (7.9) *then*

$$C_1^{n/2} \leq \Big[\det\Big(\Big\langle\frac{d}{d\phi_i}, \frac{d}{d\phi_j}\Big\rangle_x\Big)_{i,j}\Big] \leq C_2^{n/2}, \quad \textit{for a.e. } x \in U. \tag{7.13}$$

Proof. Formula (7.4) is a consequence of definitions and straightforward linear algebra, whereas (7.13) follows from (7.9). □

Recall that Ω denotes the interior of the compact Lipschitz manifold with boundary $\mathscr{M}$, and that $\partial\Omega := \mathscr{M} \setminus \Omega$. Fix an atlas $\{(U_i, \phi_i)\}_{i\in I}$ for $\mathscr{M}$ and pick a Lipschitz partition of unity $\{\xi_i\}_{i\in I}$ subordinate to the open cover $\{U_i\}_{i\in I}$ of $\mathscr{M}$. For $1 < p < \infty$ and $a \in (-1/p, 1-1/p)$, we then define the weighted Sobolev space $W^{1,p}_a(\Omega)$ as the collection of all locally integrable functions $u : \Omega \to \mathbb{C}$ such that

$$\|u\|_{W^{1,p}_a(\Omega)} := \sum_{i\in I} \big\|(\xi_i u)\circ\phi_i^{-1}\big\|_{L^p(\mathbb{R}^n_+,\, x_n^{ap}\,dx)} + \\ + \sum_{i\in I} \big\|\nabla\big[(\xi_i u)\circ\phi_i^{-1}\big]\big\|_{L^p(\mathbb{R}^n_+,\, x_n^{ap}\,dx)} < +\infty. \tag{7.14}$$

Assuming that $1 < p' < \infty$ is such that $1/p + 1/p' = 1$, we also define

$$W^{-1,p}_a(\Omega) := \big(\mathring{W}^{1,p'}_{-a}(\Omega)\big)^*. \tag{7.15}$$

Moving on, recall that for the range of indices $1 < p < \infty$ and $0 < s < 1$, the membership to the Besov space $B^{p,p}_s(\mathbb{R}^{n-1})$ is defined via the requirement

$$\|f\|_{B^{p,p}_s(\mathbb{R}^{n-1})} := \|f\|_{L^p(\mathbb{R}^{n-1})} + \\ + \Big(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|f(x') - f(y')|^p}{|x'-y'|^{n-1+sp}}\,dx'dy'\Big)^{1/p} < +\infty. \tag{7.16}$$

One natural and convenient way of defining Besov spaces $B^{p,p}_s(\partial\Omega)$, for $1 < p < \infty$ and $s \in (0,1)$, on the boundary $\partial\Omega$ of the Lipschitz manifold $\mathscr{M}$ is to transport the corresponding scale from $\mathbb{R}^{n-1}$ to $\partial\Omega$ via a partition of unity and bi-Lipschitz pull-back in local coordinate charts.

Some of the most useful properties for these weighted Sobolev spaces for us in this paper are collected in the theorem below. We agree to let Lip denote Lipschitz functions and Lip_0 Lipschitz functions with compact support.

Theorem 7.5. *Let* Ω *denote the interior of the compact Lipschitz manifold with boundary* $\mathscr{M}$*, and set* $\partial\Omega := \mathscr{M} \setminus \Omega$*. Also, assume that*

$$1 < p < \infty, \quad -1/p < a < 1 - 1/p, \quad s := 1 - a - 1/p \in (0,1). \tag{7.17}$$

Then the following assertions are true.

(i) *When equipped with the norm* (7.14), *the space* $W_a^{1,p}(\Omega)$ *becomes complete (hence Banach). Also,* $W_a^{1,p}(\Omega)$ *is a module over* $\mathrm{Lip}(\Omega)$ *and*

$$\mathrm{Lip}(\Omega) \hookrightarrow W_a^{1,p}(\Omega) \ \textit{densely.} \tag{7.18}$$

(ii) *The restriction to the boundary operator,* $\mathrm{Lip}(\mathscr{M}) \ni u \mapsto u|_{\partial\Omega} \in \mathrm{Lip}(\partial\Omega)$ *extends to a well-defined, linear, bounded mapping*

$$\mathrm{Tr} : W_a^{1,p}(\Omega) \longrightarrow B_s^{p,p}(\partial\Omega) \tag{7.19}$$

referred to in the sequel as the trace operator. Furthermore, this trace operator has a continuous right inverse, that is, there exists an extension operator

$$\mathrm{Ext} : B_s^{p,p}(\partial\Omega) \longrightarrow W_a^{1,p}(\Omega) \tag{7.20}$$

which is linear and bounded, and such that $\mathrm{Tr} \circ \mathrm{Ext} = \mathrm{I}$, *the identity.*

(iii) *There holds*

$$\mathrm{Lip}_0\,(\Omega) \hookrightarrow \big\{u \in W_a^{1,p}(\Omega) : \mathrm{Tr}\, u = 0\big\} \ \textit{densely.} \tag{7.21}$$

(iv) *If we define*

$$\mathring{W}_a^{1,p}(\Omega) := \textit{the closure of } \mathrm{Lip}_0\,(\Omega) \textit{ in } W_a^{1,p}(\Omega) \tag{7.22}$$

then

$$\mathring{W}_a^{1,p}(\Omega) = \big\{u \in W_a^{1,p}(\Omega) : \ \mathrm{Tr}\, u = 0\big\}. \tag{7.23}$$

(v) *The spaces* $W_a^{1,p}(\Omega)$, $\mathring{W}_a^{1,p}(\Omega)$, *and* $W_a^{-1,p}(\Omega)$, *are all reflexive.*

(vi) *Assume that* $1 < p' < \infty$ *is such that* $1/p + 1/p' = 1$. *Then every functional* $\Lambda \in \big(W_{-a}^{1,p'}(\Omega,)\big)^*$ *can be described as follows. For each* $u \in W_{-a}^{1,p'}(\Omega)$

$$\langle \Lambda, u\rangle = \sum_{i\in I}\Bigg(\int\limits_{\phi_i(U_i)} f_0^i(x)\big((\xi_i u)\circ\phi_i^{-1}\big)(x)\sqrt{g(x)}\,dx + \\ + \sum_{j=1}^{n}\int\limits_{\phi_i(U_i)} f_j^i(x)\partial_{x_j}\big((\xi_i u)\circ\phi_i^{-1}\big)(x)\sqrt{g(x)}\,dx\Bigg), \tag{7.24}$$

where $\{(U_i,\phi_i)\}_{i\in I}$ *is a finite atlas for* $\mathscr{M}$, *and* $\{\xi_i\}_{i\in I} \subset \mathrm{Lip}(\mathscr{M})$ *is a partition of unity subordinate to the open cover* $\{U_i\}_{i\in I}$ *of* $\mathscr{M}$.

Furthermore, for each $i \in I$, *the functions* f_j^i, $0 \le j \le n$, *appearing in* (7.24) *belong to* $L^p(\phi_i(U_i), x_n^{ap}\,dx)$ *and the norm* $\|\Lambda\|_{(W_{-a}^{1,p'}(\Omega))^*}$ *is equivalent to the infimum of the sum of the norms of* f_j^i*'s over all possible choices of the atlas, local charts, and partitions of unity.*

(vii) *The scales $W^{1,p}_a(\Omega)$, $\mathring{W}^{1,p}_a(\Omega)$, $W^{-1,p}_a(\Omega)$, are stable under complex interpolation. More specifically, if $1 < p_i < \infty$, $-1/p_i < a_i < 1 - 1/p_i$, $i \in \{0,1\}$, and $\theta \in (0,1)$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $a = (1-\theta)a_0 + \theta a_1$, then*

$$\big[W^{1,p_0}_{a_0}(\Omega), W^{1,p_1}_{a_1}(\Omega)\big]_\theta = W^{1,p}_a(\Omega), \tag{7.25}$$

$$\big[\mathring{W}^{1,p_0}_{a_0}(\Omega), \mathring{W}^{1,p_1}_{a_1}(\Omega)\big]_\theta = \mathring{W}^{1,p}_a(\Omega), \tag{7.26}$$

$$\big[W^{-1,p_0}_{a_0}(\Omega), W^{-1,p_1}_{a_1}(\Omega)\big]_\theta = W^{-1,p}_a(\Omega), \tag{7.27}$$

where $[\,\cdot\,,\,\cdot\,]_\theta$ denotes the usual complex interpolation bracket.

Proof. All the claims can then be deduced from their Euclidean counterpart (dealt with in earlier sections), via a standard localization argument and by making bi-Lipschitz changes of coordinates in local coordinate charts. □

Recall that Ω denotes the interior of $\mathscr{M}$ and that $\partial\Omega := \mathscr{M} \setminus \Omega$. Unraveling definitions to the point that well-known Euclidean results can be invoked, it is not difficult to show that the gradient induces a well-defined and bounded operator

$$\mathrm{Grad}_{\mathscr{M}} : W^{1,p}_a(\Omega) \longrightarrow L^p(\Omega, \delta^{ap}\mathscr{L}_{\mathscr{M}}) \otimes T\mathscr{M} \tag{7.28}$$

whenever $p \in (1,\infty)$ and $a \in (-1/p, 1-1/p)$. We denote the (sign) opposite of the adjoint of this operator by $\mathrm{Div}_{\mathscr{M}}$, and refer to it as the divergence operator on the Lipschitz manifold $\mathscr{M}$. Hence,

$$\mathrm{Div}_{\mathscr{M}} : L^p(\Omega, \delta^{ap}\mathscr{L}_{\mathscr{M}}) \otimes T\mathscr{M} \longrightarrow W^{-1,p}_a(\Omega) \tag{7.29}$$

is a bounded operator if $p \in (1,\infty)$ and $a \in (-1/p, 1-1/p)$. Finally, we define the Laplace–Beltrami operator $\Delta_{\mathscr{M}}$ on the Lipschitz manifold $\mathscr{M}$ as the composition

$$\Delta_{\mathscr{M}} := \mathrm{Div}_{\mathscr{M}} \circ \mathrm{Grad}_{\mathscr{M}}. \tag{7.30}$$

Hence, whenever $p \in (1,\infty)$ and $a \in (-1/p, 1-1/p)$, this induces a linear and bounded mapping

$$\Delta_{\mathscr{M}} : W^{1,p}_a(\Omega) \longrightarrow W^{-1,p}_a(\Omega). \tag{7.31}$$

Moreover, the adjoint of (7.31) is

$$\Delta_{\mathscr{M}} : W^{1,p'}_{-a}(\Omega) \longrightarrow W^{-1,p'}_{-a}(\Omega), \tag{7.32}$$

where $1/p' + 1/p = 1$, and $\Delta_{\mathscr{M}}$ in (7.31) is an isomorphism when $p = 2$ and $a = 0$.

One final comment pertains to the nature of the Laplace–Beltrami operator $\Delta_{\mathscr{M}}$ in local coordinates. Specifically, for each $(U,\phi) \in \mathscr{A}$, organize the functions introduced in (7.8) as a matrix $G_U := (g^U_{ij})_{1\le i,j\le n}$ and denote by $(g^{jk}_U)_{1\le j,k\le n}$ the inverse of the matrix G_U. Also, set $g_U := \det G_U$ so that, according to Proposition 7.4, the volume element in $dV_{\mathscr{M}}$ has the property that

$$(\phi^{-1})^*(dV_{\mathscr{M}}) = \sqrt{g_U}\, dx_1 \cdots dx_n \ \text{ in } \ \phi(U). \tag{7.33}$$

Then, in the local coordinates associated with the chart (U, ϕ), the Laplace–Beltrami operator $\Delta_{\mathscr{M}}$ can be described as

$$\Delta_{\mathscr{M}} = \frac{1}{\sqrt{g_U}} \sum_{j,k=1}^{n} \partial_j \big(g_U^{j,k} \sqrt{g_U}\, \partial_k \cdot \big), \tag{7.34}$$

where, as customary, we have identified $d/d\phi_i$ with ∂_i for each $i \in \{1, \dots, n\}$.

We are now ready to discuss the following sharp well-posedness result in the setting of compact Lipschitz manifolds with boundary.

Theorem 7.6. *Let Ω denote the interior of the compact Lipschitz manifold with boundary $\mathscr{M}$, and set $\partial\Omega := \mathscr{M} \setminus \Omega$. Then there exists $\varepsilon > 0$ such that whenever*

$$p \in (2-\varepsilon, 2+\varepsilon), \quad a \in (-1/p, 1-1/p) \cap (-\varepsilon, \varepsilon), \quad s := 1 - a - 1/p, \tag{7.35}$$

the Poisson boundary value problem with Dirichlet boundary data for the Laplace–Beltrami operator

$$\begin{cases} u \in W^{1,p}_a(\Omega), \\ \Delta_{\mathscr{M}} u = f \in W^{-1,p}_a(\Omega), \\ \operatorname{Tr} u = g \in B^{p,p}_s(\partial\Omega) \end{cases} \tag{7.36}$$

is well-posed.

Proof. This follows by arguing as in the proof of Theorem 5.1, making use of the functional analytic theory for weighted Sobolev spaces from Theorem 7.5. □

Theorem 7.6 is, once again, sharp (in that having p near 2 is a necessary condition). This follows from an example given by N. Meyers in [13, Section 5]. Specifically, take

$$\Omega := \big\{ x = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 < 1 \big\} \tag{7.37}$$

and consider the coefficient matrix given by

$$\begin{aligned} a_{11}(x_1,x_2) &= 1 - (1-\mu^2) x_2^2 (x_1^2 + x_2^2)^{-1}, \\ a_{12}(x_1,x_2) &= A_{21}(x_1,x_2) = (1-\mu^2) x_1 x_2 (x_1^2 + x_2^2)^{-1}, \\ a_{22}(x_1,x_2) &= 1 - (1-\mu^2) x_1^2 (x_1^2 + x_2^2)^{-1}, \\ &\qquad\qquad \forall\, (x,y) \in \Omega \setminus \{(0,0)\}, \end{aligned} \tag{7.38}$$

where $\mu \in (0,1)$ is a fixed parameter. Define the scalar operator $Lu := \sum_{j,k=1,2} \partial_j \big(a_{jk}(x_1,x_2) \partial_k u \big)$ in Ω. Note that the a_{jk}'s belong to $L^\infty(\Omega, \mathscr{L}^2)$ and a direct calculation shows that

$$\sum_{j,k=1,2} a_{jk}(x_1,x_2) \xi_j \xi_k = |\xi|^2 - (1-\mu^2) \frac{(x_1\xi_2 - x_2\xi_1)^2}{|x|^2} \geq \mu^2 |\xi|^2, \tag{7.39}$$

for each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $x = (x_1, x_2) \in \Omega \setminus \{0\}$. Hence, L is elliptic. To proceed, introduce the function

$$v(x_1, x_2) := x_1(x_2^2 + x_2^2)^{(\mu-1)/2} \in L^\infty(\Omega, \mathscr{L}^2) \cap \mathscr{C}^\infty\big(\overline{\Omega} \setminus \{0\}\big). \tag{7.40}$$

A straightforward calculation shows that $Lv = 0$ near the origin. Also, fix $\phi \in \mathscr{C}_c^\infty(\Omega)$ so that $\phi \equiv 1$ near the origin, and set $u := \phi\, v$. It follows that

$$\begin{aligned} &u \in \mathring{W}^{1,2}(\Omega), \quad f := Lu \in \mathscr{C}_c^\infty(\Omega), \\ &\big|(\nabla u)(x)\big| \approx |x|^{\mu-1} \ \text{ near } \ 0 \in \Omega. \end{aligned} \tag{7.41}$$

Consequently,

$$u \in W^{1,p}(\Omega) \Longleftrightarrow p < \frac{2}{1-\mu}. \tag{7.42}$$

In particular, the fact that $2/(1-\mu) \searrow 2$ as $\mu \searrow 0$ shows that that for each $p > 2$ there exists $\mu \in (0,1)$ with the property that the operator $L : \mathring{W}^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ fails to be an isomorphism. By duality, (note that L is formally self-adjoint), the same type of conclusion holds for $p < 2$.

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References

1. R. A. Adams and J. J. F. Fournier, Sobolev spaces. Second edition. *Pure and Applied Mathematics (Amsterdam)*, 140. *Elsevier/Academic Press, Amsterdam*, 2003.
2. J. Bergh and J. Löfström, Interpolation spaces. An introduction. *Grundlehren der Mathematischen Wissenschaften*, No. 223. *Springer-Verlag, Berlin–New York*, 1976.
3. K. Brewster, I. Mitrea, and M. Mitrea, Stein's extension operator on weighted Sobolev spaces on Lipschitz domains and applications to interpolation. *Recent advances in harmonic analysis and partial differential equations*, 13–38, *Contemp. Math.*, 581, *Amer. Math. Soc., Providence, RI*, 2012.
4. R. Duduchava, D. Mitrea, and M. Mitrea, Differential operators and boundary value problems on hypersurfaces. *Math. Nachr.* **279** (2006), No. 9-10, 996–1023.
5. E. De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. (Italian) *Boll. Un. Mat. Ital. (4)* **1** (1968), 135–137.
6. P. Grisvard, Elliptic problems in nonsmooth domains. *Monographs and Studies in Mathematics*, 24. *Pitman (Advanced Publishing Program), Boston, MA*, 1985.
7. L. Hörmander, The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. Second edition. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 256. *Springer-Verlag, Berlin*, 1990.
8. D. Jerison and C. E. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.* **130** (1995), No. 1, 161–219.
9. N. Kalton and M. Mitrea, Stability results on interpolation scales of quasi-Banach spaces and applications. *Trans. Amer. Math. Soc.* **350** (1998), No. 10, 3903–3922.
10. J.-L. Lions and E. Magenes, Problèmes aux limites non homogènes et applications. Vol. 1. (French) *Travaux et Recherches Mathématiques*, No. 17 *Dunod, Paris*, 1968.
11. V. G. Maz'ja, Sobolev spaces. Translated from the Russian by T. O. Shaposhnikova. *Springer Series in Soviet Mathematics. Springer-Verlag, Berlin*, 1985.

12. V. Maz'ya, M. Mitrea, and T. Shaposhnikova, The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients. *J. Anal. Math.* **110** (2010), 167–239.
13. N. G. Meyers, An L^pe-estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa (3)* **17** (1963), 189–206.
14. N. Miller, Weighted Sobolev spaces and pseudodifferential operators with smooth symbols. *Trans. Amer. Math. Soc.* **269** (1982), No. 1, 91–109.
15. M. Mitrea and M. Taylor, The Poisson problem in weighted Sobolev spaces on Lipschitz domains. *Indiana Univ. Math. J.* **55** (2006), No. 3, 1063–1089.
16. E. M. Stein, Singular integrals and differentiability properties of functions. *Princeton Mathematical Series*, No. 30 *Princeton University Press, Princeton, N.J.*, 1970.
17. M. E. Taylor, Partial differential equations. I-III. *Springer-Verlag, New York*, 1996.

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THE WEIERSTRASS–WHITTAKER INTEGRAL TRANSFORM

Dedicated to the memory of
Professor Viktor Kupradze (1903–1985)
on the 110th anniversary of his birthday

Abstract. We introduce a Weierstrass type transform associated with the Whittaker integral transform, which we refer to as *Weierstrass–Whittaker integral transform*. We examine some properties of the transform and show, in particular, that it is helpful in solving of a generalized non-stationary heat equation with an initial condition.

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1. Introduction

The Whittaker functions $M_{\mu,\nu}$ and $W_{\mu,\nu}$ of first and second order have acquired an increasing significance due to their frequent use in applications of mathematics to physical and technical problems (cf., e.g., [2]). Moreover, they are closely related to the confluent hypergeometric functions which play an important role in various branches of applied mathematics and theoretical physics. For instance, this is the case in fluid mechanics, electromagnetic diffraction theory and atomic structure theory. This justifies a continuous effort in studying properties of these functions and in gathering information about them, as well as the integral equations and transforms generated by them.

For a somehow much more detailed account of several significant results on the Whittaker and Weierstrass type transforms, over the last half-century, we refer to [1, 3–7, 11–14].

Let us consider the integral transform

$$[Wf](\tau) = \int_0^{+\infty} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) f(x) e^{-(x+\frac{1}{x})} x^{\alpha}\, dx, \quad \tau > 0, \tag{1.1}$$

where $\alpha > 0$. The main purpose of this work is to define an integral transform associated with the Whittaker integral transform (1.1) – which will be called *Weierstrass–Whittaker transform* – and to study some of its properties and possible applications. We define such integral transform by

$$[\mathcal{W}_t f](x) = \int_0^{+\infty} \mathcal{K}_t(x,y) f(y) e^{-(y+\frac{1}{y})} y^{\alpha}\, dy, \tag{1.2}$$

where $\mathcal{K}_t(x,y)$ is the heat kernel associated with the Whittaker transform (to be also studied later) and which is defined as

$$\mathcal{K}_t(x,y) = \int_0^{+\infty} e^{-4\nu^2\tau t} e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^{\alpha}\, d\tau$$

for $t, x, y > 0$.

The integral transform $\mathcal{W}_t f$ is a variant of the usual Weierstrass transform [9] and solves the heat type problem

$$\begin{cases} \partial_t [\mathcal{W}_t f](x) = -L_x [\mathcal{W}_t f](x), \\ \lim\limits_{t\to 0} [\mathcal{W}_t f](x) = f(x), \end{cases} \qquad t, x > 0,$$

where

$$L_x = 4\tau^3 x^2 \frac{d^2}{dx^2} + 4\tau^4 x^2 \frac{d}{dx} + \tau^3 x^2(\tau^2 - 1) + 4\mu\tau^2 x + \tau.$$

2. The Whittaker Integral Transform

In this section, we study some of the mapping properties of the integral transform (1.1) which may, in fact, be viewed as an operator acting from $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\,dx)$ into $L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\,d\tau)$.

So, we consider the weighted Hilbert spaces $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\,dx)$ endowed with the inner product

$$\langle f,g\rangle_{L^2(\mathbb{R}^+,e^{-(x+\frac{1}{x})}x^\alpha\,dx)} = \int_0^{+\infty} f(x)\overline{g(x)}e^{-(x+\frac{1}{x})}x^\alpha\,dx \tag{2.1}$$

which generates the associated norm

$$\|f\|_{L^2(\mathbb{R}^+,e^{-(x+\frac{1}{x})}x^\alpha\,dx)} = \left(\int_0^{+\infty} |f(x)|^2 e^{-(x+\frac{1}{x})}x^\alpha\,dx\right)^{1/2}. \tag{2.2}$$

In order to prove the convergence of the integral transform (1.1), we have the following auxiliary result.

Theorem 2.1. *Let* $f \in L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\,dx)$ *and*

$$\alpha > \max\{2|\nu| - 2, 0\}.$$

The integral transform (1.1) *is absolutely convergent and the following uniform estimate*

$$\big|[Wf](\tau)\big| \le C_{\mu,\nu}(\tau)\|f\|_{L^2(\mathbb{R}^+,e^{-(x+\frac{1}{x})}x^\alpha\,dx)}. \tag{2.3}$$

holds.

Proof. Invoking the Cauchy–Schwarz inequality and relation (2.19.24.7) in [8], we have

$$\begin{aligned}
\big|[Wf](\tau)\big| &\le \int_0^{+\infty} \left|e^{-\frac{x\tau}{2}}W_{\mu,\nu}(x\tau)f(x)e^{-(x+\frac{1}{x})}x^\alpha\right|dx \le \\
&\le \left(\int_0^{+\infty} e^{-\frac{x\tau}{2}}W_{\mu,\nu}(x\tau)e^{-\frac{x\tau}{2}}W_{\mu,\nu}(x\tau)e^{-(x+\frac{1}{x})}x^\alpha\,dx\right)^{1/2} \times \\
&\qquad \times \left(\int_0^{+\infty} |f(x)|^2 e^{-(x+\frac{1}{x})}x^\alpha\,dx\right)^{1/2} \le \\
&\le \left(\int_0^{+\infty} e^{-\frac{x\tau}{2}}W_{\mu,\nu}(x\tau)e^{-\frac{x\tau}{2}}W_{\mu,\nu}(x\tau)x^\alpha\,dx\right)^{1/2} \|f\|_{L^2(\mathbb{R}^+,e^{-(x+\frac{1}{x})}x^\alpha\,dx)} = \\
&= C_{\mu,\nu}(\tau)\,\|f\|_{L^2(\mathbb{R}^+,e^{-(x+\frac{1}{x})}x^\alpha\,dx)},
\end{aligned} \tag{2.4}$$

where

$$C_{\mu,\nu}(\tau) = \tau^{-\frac{\alpha+1}{2}}\Bigg(\frac{\Gamma(-2\nu)\Gamma(\alpha+2\nu+2)\Gamma(2+\alpha)}{\Gamma(\frac{1}{2}-\mu-\nu)\Gamma(\frac{5}{2}-\mu+\alpha+\nu)}\times$$
$$\times\, 3^F2\Big(\frac{1}{2}+\mu+\nu, 2+\alpha+2\nu, 2+\alpha; 1+2\nu, \frac{5}{2}+\alpha+\nu-\mu; 1\Big)+$$
$$+\frac{\Gamma(2\nu)\Gamma(\alpha-2\nu+2)\Gamma(2+\alpha)}{\Gamma(\frac{1}{2}-\mu+\nu)\Gamma(\frac{5}{2}-\mu+\alpha+\nu)}\times$$
$$\times\, 3^F2\Big(\frac{1}{2}-\mu+\nu, 2+\alpha, 2+\alpha-2\nu; 1-2\nu, \frac{5}{2}+\alpha-\nu-\mu; -1\Big)\Bigg)^{1/2},$$

with $\tau > 0$, and where 3^F2 denotes the generalized hypergeometric function. Hence, besides the estimation in question, the convergence of the integral transform (1.1) is also obtained. □

We now concentrate on the image of the integral transform for the elements considered above. Namely, for that elements, in the next result we obtain that $Wf \in L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$.

Theorem 2.2. *Let* $\alpha > \max\{2|\nu|-2, 0\}$.
If $f \in L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)$, *then the Whittaker integral transform* $[Wf](\tau)$ *belongs to the space* $L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$.

Proof. From the definition of the norm in $L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$, taking into account that $f \in L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)$ and using (2.4), we obtain

$$\|Wf\|^2_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)} = \int_0^{+\infty} \big|[Wf](\tau)\big|^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau \le$$
$$\le \int_0^{+\infty} (C_{\mu,\nu}(\tau))^2 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha)} e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau =$$
$$= C^*_{\mu,\nu}\|f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)} \int_0^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau \le$$
$$\le \Big(\Gamma(0,1)+\frac{1}{e}\Big) C^*_{\mu,\nu}\|f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)}, \tag{2.5}$$

where

$$C^*_{\mu,\nu} = \frac{\Gamma(-2\nu)\Gamma(\alpha+2\nu+2)\Gamma(2+\alpha)}{\Gamma(\frac{1}{2}-\mu-\nu)\Gamma(\frac{5}{2}-\mu+\alpha+\nu)}\times$$
$$\times\, 3^F2\Big(\frac{1}{2}+\mu+\nu, 2+\alpha+2\nu, 2+\alpha; 1+2\nu, \frac{5}{2}+\alpha+\nu-\mu; 1\Big)+$$
$$+\frac{\Gamma(2\nu)\Gamma(\alpha-2\nu+2)\Gamma(2+\alpha)}{\Gamma(\frac{1}{2}-\mu+\nu)\Gamma(\frac{5}{2}-\mu+\alpha+\nu)}\times$$

$$\times\, {}_3F_2\Big(\frac{1}{2}-\mu+\nu, 2+\alpha, 2+\alpha-2\nu; 1-2\nu, \frac{5}{2}+\alpha-\nu-\mu; -1\Big), \quad (2.6)$$

and

$$\int_0^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau =$$

$$= \int_0^1 \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau + \int_1^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \le$$

$$\le \int_0^1 \tau^{-1} e^{-\frac{1}{\tau}} e^{-\tau} \, d\tau + \int_1^{+\infty} \tau^{-\alpha} e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \le$$

$$\le \int_0^1 \tau^{-1} e^{-\frac{1}{\tau}} \, d\tau + \int_1^{+\infty} e^{-(\tau+\frac{1}{\tau})} \, d\tau \le$$

$$\le \int_0^1 \tau^{-1} e^{-\frac{1}{\tau}} \, d\tau + \int_1^{+\infty} e^{-\tau} \, d\tau = \Gamma(0,1) + \frac{1}{e}, \quad (2.7)$$

with $\Gamma(a,x)$ denoting the incomplete Gamma function. □

3. The Heat Kernel Related to the Whittaker Integral Transform

In order to introduce in a formal way the Weierstrass–Whittaker transform (1.2), we need first to study the heat kernel associated with the Whittaker transform. Therefore, we will introduce in this section the heat kernel associated with the Whittaker integral transform. Moreover, we will define and examine some of its properties.

Let us introduce the Hilbert space $H_K(\mathbb{R}^+)$, defined as the subspace of $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)$ formed by all functions f such that

$$Wf \in L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau).$$

$H_K(\mathbb{R}^+)$ is endowed with the inner product

$$\langle f, g\rangle_{H_K} = \int_0^{+\infty} [Wf](\tau) \overline{[Wg](\tau)} e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \quad (3.1)$$

and, consequently, the norm of $H_K(\mathbb{R}^+)$ is given by

$$\|f\|_{H_K} = \sqrt{\langle f, f\rangle_{H_K}} = \Big(\int_0^{+\infty} \big|[Wf](\tau)\big|^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \Big)^{1/2}. \quad (3.2)$$

Proposition 3.1. *Let* $\alpha > \max\{2|\nu| - 2, 0\}$*. For* $t > 0$*, we introduce* $\mathcal{K}_t(x,y)$ *defined on* $]0,+\infty[\times]0,+\infty[$ *by*

$$\mathcal{K}_t(x,y) = \int_0^{+\infty} e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau) e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau. \quad (3.3)$$

For all $y \in]0,+\infty[$*, the function*

$$x \mapsto \mathcal{K}_t(x,y)$$

belongs to $H_K(\mathbb{R}^+)$*.*

Proof. Invoking the Cauchy–Schwarz inequality and the relation (2.19.24.7) in [8], we will be able to prove first the fact that the kernel belongs to $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha \, dx)$. Indeed,

$$\|\mathcal{K}_t\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)} = \int_0^{+\infty} |\mathcal{K}_t(x,y)|^2 e^{-(x+\frac{1}{x})}x^\alpha \, dx =$$

$$= \int_0^{+\infty}\left(\int_0^{+\infty} e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau) e^{-(\tau+\frac{1}{\tau})}\tau^\alpha d\tau\right)^2 e^{-(x+\frac{1}{x})}x^\alpha dx \le$$

$$\le \int_0^{+\infty}\left(\int_0^{+\infty} (e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau))^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau\right)\times$$

$$\times\left(\int_0^{+\infty} \left(e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau)\right)^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau\right) e^{-(x+\frac{1}{x})}x^\alpha \, dx \le$$

$$\le \int_0^{+\infty}\left(\int_0^{+\infty} (e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau))^2 \tau^\alpha \, d\tau\right) e^{-(x+\frac{1}{x})}x^\alpha \, dx \times$$

$$\times\left(\int_0^{+\infty} \left(e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau)\right)^2 \tau^\alpha \, d\tau\right) =$$

$$= (C^*_{\mu,\nu})^2 y^{-(\alpha+1)} \int_0^{+\infty} x^{-(\alpha+1)} e^{-(x+\frac{1}{x})}x^\alpha \, dx \le$$

$$\le \left(\Gamma(0,1) + \frac{1}{e}\right)(C^*_{\mu,\nu})^2 y^{-(\alpha+1)}, \quad (3.4)$$

where $C^*_{\mu,\nu}$ is given by (2.6).

In order to prove that $\mathcal{K}_t \in H_K(\mathbb{R}^+)$, we still need to prove that $W\mathcal{K}_t \in L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)$.

For $\alpha > \max\{2|\nu| - 2, 0\}$, we obtain the following estimate by using the Cauchy–Schwarz inequality:

$$
\begin{aligned}
|W\mathcal{K}_t| &= \left| \int_0^{+\infty} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\mathcal{K}_t(x,y) e^{-(x+\frac{1}{x})} x^\alpha \, dx \right| \leq \\
&\leq \left(\int_0^{+\infty} (e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau))^2 e^{-(x+\frac{1}{x})} x^\alpha \, dx \right)^{1/2} \times \\
&\quad \times \left(\int_0^{+\infty} |\mathcal{K}_t(x,y)|^2 e^{-(x+\frac{1}{x})} x^\alpha \, dx \right)^{1/2} \leq \\
&\leq \left(\int_0^{+\infty} \left(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\right)^2 x^\alpha \, dx \right)^{1/2} \|\mathcal{K}_t\|_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)} = \\
&= (C^*_{\mu,\nu})^{1/2} \tau^{-\frac{\alpha+1}{2}} \|\mathcal{K}_t\|_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)}.
\end{aligned}
$$

Taking into account the previous inequality, we have

$$
\begin{aligned}
\|W\mathcal{K}_t\|^2_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)} &= \int_0^{+\infty} \left| W\mathcal{K}_t(x,y) \right|^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \leq \\
&\leq C^*_{\mu,\nu} \|\mathcal{K}_t\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)} \int_0^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \leq \\
&\leq \left(\Gamma(0,1) + \frac{1}{e} \right) C^*_{\mu,\nu} \|\mathcal{K}_t\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)}. \qquad (3.5)
\end{aligned}
$$

Therefore, we have just proved that, for $y > 0$, the function $x \mapsto \mathcal{K}_t(x,y)$ belongs to $H_K(\mathbb{R}^+)$. □

In order to obtain some important results related to the heat kernel and the Weierstrass transform, we need to introduce a new Hilbert space which we denote by $H^*_K(\mathbb{R}^+)$. Towards this end, we need first to guarantee the following result (which will ensure that the above-mentioned new space definition will be coherent with our purposes).

Lemma 3.2. *If $f \in H_K(\mathbb{R}^+)$, then*

$$
\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \qquad (3.6)
$$

belongs to $H_K(\mathbb{R}^+)$.

Proof. Having in mind the definition of $H_K(\mathbb{R}^+)$, under the above hypothesis, we realize that we have to prove that both the element in (3.6) and its image under W must belong to $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx)$.

For start, we will directly prove that for all elements $f \in H_K(\mathbb{R}^+)$ we have

$$\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \in L^2\big(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha \, dx\big).$$

Indeed,

$$\int_0^{+\infty} \Big| \int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \Big|^2 e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le \int_0^{+\infty} \bigg(\int_0^{+\infty} \big([Wf](\tau)\big)^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg) \times$$

$$\times \bigg(\int_0^{+\infty} \big(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\big)^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg) e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le \int_0^{+\infty} \bigg(\int_0^{+\infty} \big([Wf](\tau)\big)^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg) \times$$

$$\times \bigg(\int_0^{+\infty} \big(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\big)^2 \tau^\alpha \, d\tau \bigg) e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le C^*_{\mu,\nu} \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)} \int_0^{+\infty} x^{-\alpha-1} e^{-(x+\frac{1}{x})} x^\alpha \, dx \le$$

$$\le C^*_{\mu,\nu} \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)} \int_0^{+\infty} x^{-\alpha-1} e^{-x} x^\alpha \, dx \le$$

$$\le C^*_{\mu,\nu} \Big(\Gamma(0,1) + \frac{1}{e} \Big) \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)}. \tag{3.7}$$

From the previous inequality, taking into account the definition of the Whittaker integral transform (1.1), we have the following inequality related with the Whittaker transform:

$$\bigg| W\bigg[\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg] \bigg|^2 =$$

$$= \bigg| \int_0^{+\infty} e^{-\frac{x\tau'}{2}} W_{\mu,\nu}(x\tau') \bigg(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) \times$$

$$\times e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \bigg) e^{-(x+\frac{1}{x})} x^\alpha \, dx \bigg|^2 \le$$

$$\leq \int_0^{+\infty} \left(e^{-\frac{x\tau'}{2}} W_{\mu,\nu}(x\tau') e^{-(x+\frac{1}{x})} x^\alpha \right)^2 \times$$

$$\times \left(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \right)^2 dx \leq$$

$$\leq C^*_{\mu,\nu} \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)} \int_0^{+\infty} \big(e^{-\frac{x\tau'}{2}} W_{\mu,\nu}(x\tau') \big)^2 x^{2\alpha} x^{-\alpha-1} \, dx \leq$$

$$\leq (C^*_{\mu,\nu})^2 (\tau')^{-\alpha} \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)}. \tag{3.8}$$

Therefore, for $f \in H_K$, we have

$$W\left(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \right) \in L^2\big(\mathbb{R}^+, e^{-(\tau'+\frac{1}{\tau'})} (\tau')^\alpha \, d\tau'\big)$$

i.e.,

$$\int_0^{+\infty} e^{-\frac{x\tau'}{2}} W_{\mu,\nu}(x\tau') \left(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \right) e^{-(x+\frac{1}{x})} x^\alpha dx$$
$$\in L^2\big(\mathbb{R}^+, e^{-(\tau'+\frac{1}{\tau'})} (\tau')^\alpha \, d\tau'\big).$$

Indeed, from (3.8), we get

$$\int_0^{+\infty} \left| \left[W\left(\int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau \right) \right] (\tau') \right|^2 \times$$
$$\times e^{-(\tau'+\frac{1}{\tau'})} (\tau')^\alpha \, d\tau' \leq$$
$$\leq (C^*_{\mu,\nu})^2 \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)} \int_0^{+\infty} e^{-(\tau'+\frac{1}{\tau'})} (\tau')^\alpha (\tau')^{-\alpha} \, d\tau' \leq$$
$$\leq (C^*_{\mu,\nu})^2 \|Wf\|_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha \, d\tau)}.$$
□

Having in mind Lemma 3.2, we are now in a position to define $H^*_K(\mathbb{R}^+)$ as the space of elements $f \in H_K(\mathbb{R}^+)$ which admit the integral representation

$$f(x) = \int_0^{+\infty} [Wf](\tau) e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau) e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau. \tag{3.9}$$

We will now exhibit a significative result based on the representation of the elements of the space $H^*_K(\mathbb{R}^+)$ and the definition of the heat kernel.

Lemma 3.3. *Let $\mathcal{K}_t \in H_K^*(\mathbb{R}^+)$. Then, the Whittaker type transform* (1.1) *of the heat kernel is given by*

$$[W\mathcal{K}_t](\tau, x) = e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau). \tag{3.10}$$

Proof. From Proposition 3.1, we find that $\mathcal{K}_t \in H_K(\mathbb{R}^+)$. Taking into account the definition of heat kernel (3.3) and since $\mathcal{K}_t \in H_K^*(\mathbb{R}^+)$, we get $[W\mathcal{K}_t](\tau, x) = e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)$. □

4. Properties of the Weierstrass–Whittaker Transform

In this section, we shall define the above-mentioned Weierstrass–Whittaker transform in a formal way, and derive some of its properties.

Definition 4.1. The Weierstrass transform associated with the Whittaker integral transform and called *Weierstrass–Whittaker transform*, is defined in $L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha\, dy)$ by

$$[\mathcal{W}_t f](x) = \int_0^{+\infty} \mathcal{K}_t(x, y) f(y) e^{-(y+\frac{1}{y})} y^\alpha\, dy. \tag{4.1}$$

For the classical Weierstrass transform, one can see [9].

Proposition 4.2. *Let $\alpha > \max\{0, 2\nu - 2\}$. For all $t > 0$, the Weierstrass type transform $\mathcal{W}_t f$ is a bounded operator from $L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha\, dy)$ into $L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha\, dx)$ and, for all $f \in L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha\, dy)$, we have*

$$\|\mathcal{W}_t f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha\, dx)} \le \le (C^*_{\mu,\nu})^2 \Big(\Gamma(0,1) + \frac{1}{e}\Big)^2 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha\, dy)}. \tag{4.2}$$

Proof. The absolutely convergence of the integral (4.1) follows from the Cauchy–Schwarz inequality and Proposition 3.1. Indeed,

$$\begin{aligned}
\big|[\mathcal{W}_t f](x)\big| &\le \int_0^{+\infty} |\mathcal{K}_t(x, y)|\, |f(y)| e^{-(y+\frac{1}{y})} y^\alpha\, dy \le \\
&\le \Big(\int_0^{+\infty} |\mathcal{K}_t(x, y)|^2 e^{-(y+\frac{1}{y})} y^\alpha\, dy\Big)^{1/2} \Big(\int_0^{+\infty} |f(y)|^2 e^{-(y+\frac{1}{y})} y^\alpha\, dy\Big)^{1/2} \le \\
&\le \Big(\int_0^{+\infty} (C^*_{\mu,\nu})^2 x^{-(\alpha+1)} y^{-(\alpha+1)} e^{-(y+\frac{1}{y})} y^\alpha\, dy\Big)^{1/2} \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha\, dy)} \le \\
&\le C^*_{\mu,\nu} \Big(\Gamma(0,1) + \frac{1}{e}\Big)^{\frac{1}{2}} x^{-\frac{\alpha+1}{2}} \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})} y^\alpha\, dy)}.
\end{aligned} \tag{4.3}$$

Then, for all $f \in L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)$ and using the relation (4.3), we have

$$\|\mathcal{W}_t f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})}x^\alpha\, dx)} = \int_0^{+\infty} |[\mathcal{W}_t f](x)|^2 e^{-(x+\frac{1}{x})} x^\alpha\, dx \leq$$

$$\leq (C^*_{\mu,\nu})^2 \Big(\Gamma(0,1) + \frac{1}{e}\Big) \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)} \int_0^{+\infty} x^{-(\alpha+1)} e^{-(x+\frac{1}{x})} x^\alpha\, dx \leq$$

$$\leq (C^*_{\mu,\nu})^2 \Big(\Gamma(0,1) + \frac{1}{e}\Big)^2 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)}. \quad \square$$

Proposition 4.3. *Let $\alpha > \max\{0, 2\nu - 2\}$. For all $t > 0$, the Weierstrass–Whittaker transform $\mathcal{W}_t f$ belongs to the space $H_K(\mathbb{R}^+)$.*

Proof. From the previous proposition we have

$$\mathcal{W}_t f \in L^2\big(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha\, dx\big).$$

Now, in order to prove that $\mathcal{W}_t f$ belongs to the space $H_K(\mathbb{R}^+)$, we need to show that $W[\mathcal{W}_t f] \in L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$.

From the definition of the Whittaker type transform, we obtain

$$\big|\big[W[\mathcal{W}_t f]\big](\tau)\big| \leq \int_0^{+\infty} e^{-\frac{x\tau}{2}} |W_{\mu,\nu}(x\tau)|\, |\mathcal{W}_t f(x)| e^{-(x+\frac{1}{x})} x^\alpha\, dx$$

and by using (4.3) and taking into account the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\big|[W\big[\mathcal{W}_t f\big]](\tau)\big| &\leq \Big(\Gamma(0,1) + \frac{1}{e}\Big)^{\frac{1}{2}} C^*_{\mu,\nu} \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)} \times \\
&\quad \times \int_0^{+\infty} e^{-\frac{x\tau}{2}} |W_{\mu,\nu}(x\tau)| x^{-\frac{\alpha+1}{2}} e^{-(x+\frac{1}{x})} x^\alpha\, dx \leq \\
&\leq \Big(\Gamma(0,1) + \frac{1}{e}\Big)^{\frac{1}{2}} C^*_{\mu,\nu} \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)} \times \\
&\quad \times \Big(\int_0^{+\infty} \big(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\big)^2 e^{-(x+\frac{1}{x})} x^\alpha\, dx \Big)^{1/2} \times \\
&\quad \times \Big(\int_0^{+\infty} x^{-(\alpha+1)} e^{-(x+\frac{1}{x})} x^\alpha\, dx \Big)^{1/2} \leq \\
&\leq \tau^{-\frac{\alpha+1}{2}} \Big(\Gamma(0,1) + \frac{1}{e}\Big) (C^*_{\mu,\nu})^{\frac{3}{2}}\, \|f\|_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)}.
\end{aligned}$$

Having in mind the previous inequality, we obtain the following estimate:

$$\big\|W[\mathcal{W}_t f]\big\|^2_{L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)} = \int_0^{+\infty} \big|W[\mathcal{W}_t f](\tau)\big|^2 e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau \le$$

$$\le \Big(\Gamma(0,1) + \frac{1}{e}\Big)^2 (C^*_{\mu,\nu})^3 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)} \int_0^{+\infty} \tau^{-(\alpha+1)} e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau \le$$

$$\le \Big(\Gamma(0,1) + \frac{1}{e}\Big)^3 (C^*_{\mu,\nu})^3 \|f\|^2_{L^2(\mathbb{R}^+, e^{-(y+\frac{1}{y})}y^\alpha\, dy)}. \tag{4.4}$$

Hence, it follows that the composition of the Whittaker type transform (1.1) with the Weierstrass–Whittaker transform (4.1) belongs to the space $L^2(\mathbb{R}^+, e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau)$ and therefore $\mathcal{W}_t f \in H_K(\mathbb{R}^+)$. □

The just used composition of integral transformations can be described in an even more detailed way if we invoke the representation of the elements of the space $H^*_K(\mathbb{R}^+)$ and the definition of the Weierstrass–Whittaker transform, as we shall see in the next result.

Lemma 4.4. *Let $\mathcal{W}_t f \in H^*_K(\mathbb{R}^+)$. For all $t > 0$, we have*

$$\big[W[\mathcal{W}_t f]\big](\tau) = e^{-4\nu^2\tau t}[Wf](\tau). \tag{4.5}$$

Proof. From the definition of Weierstrass–Whittaker transform, the definition of inner product in $H_K(\mathbb{R}^+)$, Proposition 3.1, Proposition 4.3 and Lemma 3.3, we deduce

$$\begin{aligned}
[\mathcal{W}_t f](x) &= \int_0^{+\infty} \mathcal{K}_t(x,y) f(y) e^{-(y+\frac{1}{y})} y^\alpha\, dy = \\
&= \int_0^{+\infty} [W\mathcal{K}_t](\tau) W[f](\tau) e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau = \\
&= \int_0^{+\infty} e^{-4\nu^2\tau t} e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)[Wf](\tau) e^{-(\tau+\frac{1}{\tau})}\tau^\alpha\, d\tau.
\end{aligned}$$

Since $\mathcal{W}_t f \in H^*_K(\mathbb{R}^+)$, invoking (3.9), we find

$$\big[W[\mathcal{W}_t f]\big](\tau) = e^{-4\nu^2\tau t}[Wf](\tau). \tag{4.6}$$

□

5. The Weierstrass–Whittaker Transform as a Solution of a Heat Type Equation

In this last section we will show that the Weierstrass–Whittaker transform $\mathcal{W}_t f$ solves a non-stationary heat type equation (cf. (5.2)). To this end, first of all, we need to prove that the kernel $\mathcal{K}_t(x,y)$ is a solution of a variant of the heat equation.

We start by recalling that the Whittaker function is an eigenfunction of a second order differential operator. More precisely,

$$A_z W_{\mu,\nu}(z) = 4\nu^2 W_{\mu,\nu}(z),$$

where

$$A_z = 4z^2 \frac{d^2}{dz^2} - z^2 + 4\mu z + 1. \tag{5.1}$$

From the differential properties of the Whittaker function, the absolute and uniform convergence of the integral (1.3) and its derivatives with respect to t and x, we directly arrive at the following result.

Corollary 5.1. *The kernel $\mathcal{K}_t(x,y)$ satisfies the non-stationary heat type equation*

$$\partial_t u(t,x,y) = -L_x u(t,x,y), \quad t,x,y > 0, \tag{5.2}$$

where

$$L_x = 4\tau^3 x^2 \frac{d^2}{dx^2} + 4\tau^4 x^2 \frac{d}{dx} + \tau^3 x^2(\tau^2-1) + 4\mu\tau^2 x + \tau. \tag{5.3}$$

is a second order differential operator which satisfies

$$L_x\big(e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau)\big) = 4\nu^2 \tau e^{-\frac{x\tau}{2}} W_{\mu,\nu}(x\tau). \tag{5.4}$$

Furthermore, the kernel $\mathcal{K}_t(x,y)$ is also a solution of the non-stationary heat type equation

$$\partial_t u(t,x,y) = -L_y u(t,x,y), \quad t,x,y > 0, \tag{5.5}$$

where

$$L_y = 4\tau^3 y^2 \frac{d^2}{dy^2} + 4\tau^4 y^2 \frac{d}{dy} + \tau^3 y^2(\tau^2-1) + 4\mu\tau^2 y + \tau \tag{5.6}$$

is a second order differential operator which satisfies

$$L_y\big(e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau)\big) = 4\nu^2 \tau e^{-\frac{y\tau}{2}} W_{\mu,\nu}(y\tau). \tag{5.7}$$

Theorem 5.2. *Let $f \in H_K(\mathbb{R}^+)$. For all $t > 0$ and for all $\mathcal{W}_t f \in H_K^*(\mathbb{R}^+)$, the function $\mathcal{W}_t f$ solves the generalized heat equation* (5.2)*, with the initial condition $\lim_{t\to 0} [\mathcal{W}_t f](x) = f(x)$ in $H_K(\mathbb{R}^+)$.*

Proof. Propositions 3.1 and 4.2 guarantee the necessary differential properties of $\mathcal{W}_t f$, and from the differential properties of the Whittaker function we deduce that the function $\mathcal{W}_t f$ is a solution of (5.2).

We will now prove the initial condition. From the definition of the norm of $H_K(\mathbb{R}^+)$ (cf. (3.2)) and using Lemma 4.4, we have

$$\begin{aligned}\|\mathcal{W}_t f - f\|^2_{L^2(\mathbb{R}^+, e^{-(x+\frac{1}{x})} x^\alpha\, dx)} &= \\ &= \int_0^{+\infty} \Big| \big[W[\mathcal{W}_t f]\big](\tau) - [Wf](\tau) \Big|^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau = \\ &= \int_0^{+\infty} \big| e^{-4\nu^2 \tau t} - 1 \big|^2 \big| [Wf](\tau) \big|^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau. \qquad (5.8)\end{aligned}$$

Since $4\nu^2\tau t > 0$, we realize that the right-hand side of (5.8) is estimated by $\int\limits_0^{+\infty} |[Wf](\tau)|^2 e^{-(\tau+\frac{1}{\tau})} \tau^\alpha \, d\tau$. Then, we can pass to the limit $\to 0$ through equation (5.8) and the desired result is obtained. □

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References

1. Yu. A. Brychkov and A. P. Prudnikov, Integral transforms of generalized functions. Translated and revised from the second Russian edition. *Gordon and Breach Science Publishers, New York*, 1989.
2. H. Buchholz, The confluent hypergeometric function with special emphasis on its applications. Translated from the German by H. Lichtblau and K. Wetzel. *Springer Tracts in Natural Philosophy*, Vol. 15 *Springer-Verlag New York Inc., New York*, 1969.
3. Z. Ditzian, Representation theorems for the Weierstrass transform. *J. Austral. Math. Soc.* **14** (1972), 91–104.
4. I. I. Hirschman and D. V. Widder, The convolution transform. *Princeton University Press, Princeton, N. J.*, 1955.
5. V. Karunakaran and T. Venugopal, The Weierstrass transform for a class of generalized functions. *J. Math. Anal. Appl.* **220** (1998), No. 2, 508–527.
6. R. V. Kene and A. S. Gudadhe, Some properties of generalized Mellin–Whittaker transform. *Int. J. Contemp. Math. Sci.* **7** (2012), No. 9-12, 477–488.
7. S. Omri and L. T. Rachdi, Weierstrass transform associated with the Hankel operator. *Bull. Math. Anal. Appl.* **1** (2009), No. 2, 1–16.
8. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and series. Vol. 3. More special functions. Translated from the Russian by G. G. Gould. *Gordon and Breach Science Publishers, New York*, 1990.
9. S. Saitoh, The Weierstrass transform and an isometry in the heat equation. *Applicable Anal.* **16** (1983), No. 1, 1–6

10. S. Saitoh, Integral transforms, reproducing kernels and their applications. *Pitman Research Notes in Mathematics Series*, 369. *Longman, Harlow*, 1997.
11. R. K. Saxena, Relation between Whittaker transform and modified $\chi_{v,k,m}$-transform. *Math. Ann.* **154** (1964) 301–306.
12. A. I. Zayed, Handbook of function and generalized function transformations. *Mathematical Sciences Reference Series. CRC Press, Boca Raton, FL*, 1996.
13. A. H. Zemanian, A generalized Weierstrass transformation. *SIAM J. Appl. Math.* **15** (1967), 1088–1105.
14. A. H. Zemanian, Generalized integral transformations. *Pure and Applied Mathematics*, Vol. XVIII. Interscience Publishers [John Wiley & Sons, Inc.], *New York–London–Sydney*, 1968.

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LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS APPROACH FOR DIRICHLET PROBLEM OF THE THEORY OF PIEZO-ELASTICITY FOR INHOMOGENEOUS SOLIDS

Dedicated to the 110-th birthday anniversary of academician V. Kupradze

Abstract. The paper deals with the three-dimensional Dirichlet boundary-value problem (BVP) of piezo-elasticity theory for anisotropic inhomogeneous solids and develops the generalized potential method based on the localized parametrix method. Using Green's integral representation formula and properties of the localized layer and volume potentials we reduce the Dirichlet BVP to the localized boundary-domain integral equations (LBDIE) system. The equivalence between the Dirichlet BVP and the corresponding LBDIE system is studied. We establish that the obtained localized boundary-domain integral operator belongs to the Boutet de Monvel algebra and with the help of the Wiener–Hopf factorization method we investigate corresponding Fredholm properties and prove invertibility of the localized operator in appropriate function spaces.

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Key words and phrases. Piezo-elasticity, strongly elliptic systems, variable coefficients, boundary value problem, localized parametrix, localized boundary-domain integral equations, pseudodifferential operators.

რეზიუმე. ნაშრომი ეძღვნება ლოკალიზებული პარამეტრიქსის მეთოდის განვითარებას პიეზო-დრეკადობის თეორიის დირიხლეს სამგანზომილებიანი ამოცანისთვის არაერთგვაროვანი ანიზოტროპული სხეულების შემთხვევაში. გრინის ინტეგრალური წარმოდგენის ფორმულისა და ლოკალიზებული პოტენციალების გამოყენებით დირიხლეს ამოცანა დაიყვანება ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემაზე. შესწავლილია დირიხლეს სასაზღვრო ამოცანისა და მიღებულ ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემის ეკვივალენტობა. ვინერ–ჰოფის ფაქტორიზაციის მეთოდის გამოყენებით ნაჩვენებია, რომ ლოკალიზებული სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა ოპერატორი, რომელიც ეკუთვნის ბუტე დე მონველის ალგებრას, არის ფრედჰოლმური და დადგენილია მისი შებრუნებადობა შესაბამის სივრცეებში.

1. Introduction

We consider the three-dimensional Dirichlet boundary-value problem (BVP) of piezo-elasticity for anisotropic inhomogeneous solids and develop the generalized potential method based on the *localized parametrix method.*

Due to great theoretical and practical importance, problems of piezo-elasticity became very popular among mathematicians and engineers (for details see, e.g., [26]–[34], [42], [50]).

The BVPs and various type interface problems of piezo-elasticity for *homogeneous anisotropic solids*, i.e., when the material parameters are constants and the corresponding fundamental solution is available in explicit form, by the usual classical potential methods are investigated in [4]–[9], [41]. Unfortunately this classical potential method is not applicable in the case of inhomogeneous solids since for the corresponding system of differential equations with variable coefficients a fundamental solution is not available in explicit form in general.

Therefore, in our analysis we apply the so-called *localized parametrix method* which leads to the localized boundary-domain integral equations system.

Our main goal here is to show that solutions of the boundary value problem can be represented by *localized potentials* and that the corresponding *localized boundary-domain integral operator* (LBDIO) is invertible, which seems very important from the point of view of numerical analysis, since they lead to very convenient numerical schemes in applications (for details see [37], [43], [46]–[49]).

To this end, using Green's representation formula and properties of the localized layer and volume potentials, we reduce the Dirichlet BVP of piezo-elasticity to the *localized boundary-domain integral equations (LBDIE) system.* First we establish the equivalence between the original boundary value problem and the corresponding LBDIE system which proved to be a quite nontrivial problem and plays a crucial role in our analysis. Afterwards we establish that the localized boundary domain matrix integral operator generated by the LBDIE belongs to the Boutet de Monvel algebra and with the help of the Vishik–Eskin theory, based on the factorization method (Wiener–Hopf factorization method), we investigate Fredholm properties and prove invertibility of the localized operator in appropriate function spaces.

Note that the operator, generated by the system of piezo-elasticity for inhomogeneous anisotropic solids, is second order nonself-adjoint strongly elliptic partial differential operator with variable coefficients. In [21], the LBDIE method has been developed for the Dirichlet problem in the case of self-adjoint second order strongly elliptic systems with variable coefficients, while the same method for the case of scalar elliptic second order partial differential equations with variable coefficients is justified in [11]–[20], [38].

2. Reduction to LBDIE System and the Equivalence Theorem

2.1. **Formulation of the boundary value problem and localized Green's third formula.** Consider the system of static equations of piezoelectricity for an inhomogeneous anisotropic medium [42]:

$$A(x,\partial_x)U+X=0,$$

where $U:=(u_1,u_2,u_3,u_4)^\top$, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $u_4=\varphi$ is the electric potential, $X=(X_1,X_2,X_3,X_4)^\top$, $(X_1,X_2,X_3)^\top$ is a given mass force density, X_4 is a given charge density, $A(x,\partial_x)$ is a formally nonself-adjoint matrix differential operator

$$A(x,\partial_x)=\big[A_{jk}(x,\partial_x)\big]_{4\times 4}:= \\ :=\begin{bmatrix} \big[\partial_i(c_{ijlk}(x)\partial_l)\big]_{3\times 3} & \big[\partial_i(e_{lij}(x)\partial_l)\big]_{3\times 1} \\ \big[-\partial_i(e_{ikl}(x)\partial_l)\big]_{1\times 3} & \partial_i(\varepsilon_{il}(x)\partial_l) \end{bmatrix}_{4\times 4},$$

where $\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial_{x_j}=\partial/\partial x_j$. Here and in what follows by repeated indices summation from 1 to 3 is meant if not otherwise stated.

The variable coefficients involved in the above equations satisfy the symmetry conditions:

$$c_{ijkl}=c_{jikl}=c_{klij}\in C^\infty,\ \ e_{ijk}=e_{ikj}\in C^\infty,\ \ \varepsilon_{ij}=\varepsilon_{ji}\in C^\infty, \\ i,j,k,l=1,2,3.$$

In view of these symmetry relations, the formally adjoint differential operator $A^*(x,\partial_x)$ reads as

$$A^*(x,\partial_x)=\big[A^*_{jk}(x,\partial_x)\big]_{4\times 4}:= \\ :=\begin{bmatrix} \big[\partial_i(c_{ijlk}(x)\partial_l)\big]_{3\times 3} & \big[-\partial_i(e_{lij}(x)\partial_l)\big]_{3\times 1} \\ \big[\partial_i(e_{ikl}(x)\partial_l)\big]_{1\times 3} & \partial_i(\varepsilon_{il}(x)\partial_l) \end{bmatrix}_{4\times 4}.$$

Moreover, from physical considerations it follows that (see, e.g., [42]):

$$c_{ijkl}(x)\xi_{ij}\xi_{kl}\geqslant c_0\xi_{ij}\xi_{ij}\ \text{ for all }\ \xi_{ij}=\xi_{ji}\in\mathbb{R}, \tag{2.1}$$

$$\varepsilon_{ij}(x)\eta_i\eta_j\geqslant c_1\eta_i\eta_i\ \text{ for all }\ \eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^3, \tag{2.2}$$

where c_0 and c_1 are positive constants.

With the help of the inequalities (2.1) and (2.2) it can easily be shown that the operator $A(x,\partial_x)$ is uniformly strongly elliptic, that is,

$$\operatorname{Re}A(x,\xi)\zeta\cdot\zeta\geqslant c|\xi|^2|\zeta|^2\ \text{ for all }\ \xi\in\mathbb{R}^3\ \text{ and for all }\ \zeta\in\mathbb{C}^4, \tag{2.3}$$

where $A(x,\xi)$ is the principal homogeneous symbol matrix of the operator $A(x,\partial_x)$ with opposite sign:

$$A(x,\xi)=\big[A_{jk}(x,\xi)\big]_{4\times 4}:= \\ :=\begin{bmatrix} \big[c_{ijlk}(x)\xi_i\xi_l\big]_{3\times 3} & \big[e_{lij}(x)\xi_i\xi_l\big]_{3\times 1} \\ \big[-e_{ikl}(x)\xi_i\xi_l\big]_{1\times 3} & \varepsilon_{il}(x)\xi_i\xi_l \end{bmatrix}_{4\times 4}. \tag{2.4}$$

Here and in what follows $a \cdot b$ denotes the scalar product of two vectors $a, b \in \mathbb{C}^4$, $a \cdot b = \sum_{j=1}^{4} a_j \overline{b_j}$.

In the theory of piezoelasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n = (n_1, n_2, n_3)$ have the form

$$\sigma_{ij} n_i = c_{ijlk} n_i \partial_l u_k + e_{lij} n_i \partial_l \varphi \ \text{ for } \ j = 1, 2, 3,$$

while the normal component of the electric displacement vector (with opposite sign) reads as

$$-D_i n_i = -e_{ikl} n_i \partial_l u_k + \varepsilon_{il} n_i \partial_l \varphi.$$

Let us introduce the following matrix differential operator

$$\mathcal{T} = \mathcal{T}(x, \partial_x) = \big[\mathcal{T}_{jk}(x, \partial_x)\big]_{4\times 4} := \\ := \begin{bmatrix} \big[c_{ijlk}(x) n_i \partial_l\big]_{3\times 3} & \big[e_{lij}(x) n_i \partial_l\big]_{3\times 1} \\ \big[-e_{ikl}(x) n_i \partial_l\big]_{1\times 3} & \varepsilon_{il}(x) n_i \partial_l \end{bmatrix}_{4\times 4}.$$

For a four–vector $U = (u, \varphi)^\top$ we have

$$\mathcal{T} U = \big(\sigma_{i1} n_i,\ \sigma_{i2} n_i,\ \sigma_{i3} n_i,\ -D_i n_i\big)^\top. \tag{2.5}$$

Clearly, the components of the vector $\mathcal{T}U$ given by (2.5) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of electro-elasticity, and the forth one is the normal component of the electric displacement vector (with opposite sign).

In Green's formulae there also appear the following boundary operator associated with the adjoint differential operator $A^*(x, \partial_x)$:

$$\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(x, \partial_x) = \big[\widetilde{\mathcal{T}}_{jk}(x, \partial_x)\big]_{4\times 4} := \\ := \begin{bmatrix} \big[c_{ijlk}(x) n_i \partial_l\big]_{3\times 3} & \big[-e_{lij}(x) n_i \partial_l\big]_{3\times 1} \\ \big[e_{ikl}(x) n_i \partial_l\big]_{1\times 3} & \varepsilon_{il}(x) n_i \partial_l \end{bmatrix}_{4\times 4}.$$

Further, let $\Omega = \Omega^+$ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega = S \in C^\infty$, $\overline{\Omega} = \Omega \cup S$. Throughout the paper $n = (n_1, n_2, n_3)$ denotes the unit normal vector to S directed outward with respect to the domain Ω. Set $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$.

By $H^r(\Omega) = H^r_2(\Omega)$ and $H^r(S) = H^r_2(S)$, $r \in \mathbb{R}$, we denote the Bessel potential spaces on a domain Ω and on a closed manifold S without boundary, while $\mathcal{D}(\mathbb{R}^3)$ stands for C^∞ functions in $\mathbb{R}^3$ with compact support and $\mathcal{S}(\mathbb{R}^3)$ denotes the Schwartz space of rapidly decreasing functions in $\mathbb{R}^3$. Recall that $H^0(\Omega) = L_2(\Omega)$ is a space of square integrable functions in Ω.

For a vector $U = (u_1, u_2, u_3, u_4)^\top$ the inclusion $U = (u_1, u_2, u_3, u_4)^\top \in H^r$ means that all components u_j, $j = \overline{1, 4}$, belong to H^r.

Let us denote by $U^+ \equiv \{U\}^+$ and $U^- \equiv \{U\}^-$ the traces of U on S from the interior and exterior of Ω, respectively.

We also need the following subspace of $H^1(\Omega)$:

$$H^{1,0}(\Omega;A) := \Big\{U=(u_1,u_2,u_3,u_4)^\top \in H^1(\Omega):\ A(x,\partial)U \in H^0(\Omega)\Big\}.$$

Assume that the domain Ω is filled with an anisotropic inhomogeneous piezoelectric material.

The Dirichlet boundary-value problem reads as follows:

Find a vector-function $U=(u,\varphi)^\top=(u_1,u_2,u_3,u_4)^\top \in H^{1,0}(\Omega,A)$ satisfying the differential equation

$$A(x,\partial_x)U = f \ \text{ in } \ \Omega \tag{2.6}$$

and the Dirichlet boundary condition

$$U^+ = \Phi_0 \ \text{ on } \ S, \tag{2.7}$$

where $\Phi_0=(\Phi_{01},\Phi_{02},\Phi_{03},\Phi_{04})^\top \in H^{1/2}(S)$ and $f=(f_1,f_2,f_3,f_4)^\top \in L_2(\Omega)$ are given vector-functions.

The equation (2.6) is understood in the distributional sense, while the Dirichlet-type boundary condition (2.7) is understood in the usual trace sense.

For arbitrary complex-valued vector-functions $U=(u_1,u_2,u_3,u_4)^\top \in H^2(\Omega)$ and $V=(v_1,v_2,v_3,v_4)^\top \in H^2(\Omega)$, we have the following Green's formulae [8]:

$$\int\limits_\Omega \Big[A(x,\partial_x)U\cdot V + E(U,V)\Big]\,dx = \int\limits_S \{\mathcal{T}U\}^+\cdot\{V\}^+\,dS, \tag{2.8}$$

$$\int\limits_\Omega \Big[A(x,\partial_x)U\cdot V - U\cdot A^*(x,\partial_x)V\Big]\,dx =$$
$$= \int\limits_S \Big[\{\mathcal{T}U\}^+\cdot\{V\}^+ - \{U\}^+\cdot\{\widetilde{\mathcal{T}}V\}^+\Big]\,dS, \tag{2.9}$$

where

$$E(U,V) = c_{ijlk}\partial_i u_j\overline{\partial_l v_k} + e_{lij}\big(\partial_i u_j\overline{\partial_l v_4} - \partial_l u_4\overline{\partial_i v_j}\,\big) + \varepsilon_{jl}\partial_j u_4\overline{\partial_l v_4} \tag{2.10}$$

with $u=(u_1,u_2,u_3)^\top$ and $v=(v_1,v_2,v_3)^\top$, and the overbar denotes complex conjugation.

Note that the above Green's formulae can be generalized, by a standard limiting procedure, to Lipschitz domains and to vector–functions $U\in H^1(\Omega)$ and $V\in H^1(\Omega)$ with $A(x,\partial_x)U\in L_2(\Omega)$ and $A^*(x,\partial_x)V\in L_2(\Omega)$.

With the help of Green's formula (2.8) we can determine a *generalized trace vector* $\mathcal{T}^+U \equiv \{\mathcal{T}U\}^+ \in H^{-1/2}(\partial\Omega)$ for a vector-function $U\in H^{1,0}(\Omega;A)$ (cf. [39])

$$\big\langle \mathcal{T}^+U, V^+\big\rangle_{\partial\Omega} := \int\limits_\Omega A(\partial,\tau)U\cdot V\,dx + \int\limits_\Omega E(U,V)\,dx, \tag{2.11}$$

where $V \in H^1(\Omega)$ is an arbitrary vector-function.

Here the symbol $\langle \cdot\, , \cdot \rangle_S$ denotes the duality between the function spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ which extends the usual L_2-scalar product

$$\langle f, g\rangle_S = \int\limits_S \sum_{j=1}^N f_j \overline{g_j}\, dS \ \text{ for } \ f, g \in [L_2(S)]^N.$$

Remark 2.1. From the conditions (2.1) and (2.2) it follows that for complex-valued vector-functions the sesquilinear form $E(U,V)$ defined by (2.10) satisfies the inequality

$$\operatorname{Re} E(U,U) \geq c(s_{ij}\overline{s_{ij}} + \eta_j \overline{\eta_j}) \ \ \forall\, U = (u_1, u_2, u_3, u_4)^\top \in H^1(\Omega)$$

with $s_{ij} = 2^{-1}\big(\partial_i u_j(x) + \partial_j u_i(x)\big)$, $\eta_j = \partial_j u_4(x)$, where c is a positive constant. Therefore Green's first formula (2.8) and the Lax–Milgram lemma imply that the above formulated Dirichlet BVP is uniquely solvable in the space $H^{1,0}(\Omega; A)$ (see, e.g., [25], [35], [36]).

As it has already been mentioned, our goal here is to develop a generalized potential method and justify the LBDIE approach for the Dirichlet boundary value problem.

Define a *localized matrix parametrix* corresponding to the fundamental solution function $F_1(x) := -[4\pi|x|\,]^{-1}$ of the Laplace operator, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$,

$$P(x) \equiv P_\chi(x) := F_\chi(x) I = \\ = \chi(x) F_1(x) I = -\frac{\chi(x)}{4\pi|x|}\, I \ \text{ with } \ \chi(0) = 1, \qquad (2.12)$$

where $F_\chi(x) := \chi(x) F_1(x)$, I is the unit 4×4 matrix, while χ is a localizing function (see Appendix A)

$$\chi \in X_+^k, \quad k \geq 3. \qquad (2.13)$$

Throughout the paper we assume that the condition (2.13) is satisfied and χ has a compact support if not otherwise stated.

Denote by $B(y,\varepsilon)$ a ball centered at the point y and radius $\varepsilon > 0$ and let $\Sigma(y,\varepsilon) := \partial B(y,\varepsilon)$.

In Green's second formula (2.9), let us take in the role of $V(x)$ successively the columns of the matrix $P(x-y)$, where y is an arbitrarily fixed interior point in Ω, and write the identity (2.9) for the region $\Omega_\varepsilon := \Omega \setminus \overline{B(y,\varepsilon)}$ with $\varepsilon > 0$ such that $\overline{B(y,\varepsilon)} \subset \Omega$. Keeping in mind that $P^\top(x-y) = P(x-y)$, we arrive at the equality

$$\int\limits_{\Omega_\varepsilon} \Big[P(x-y) A(x,\partial_x) U(x) - \big[A^*(x,\partial_x) P(x-y)\big]^\top U(x)\Big]\, dx = \\ = \int\limits_S \Big[P(x-y) \{\mathcal{T}(x,\partial_x) U(x)\}^+ - \big\{\widetilde{\mathcal{T}}(x,\partial_x) P(x-y)\big\}^\top \{U(x)\}^+\Big]\, dS -$$

$$-\int\limits_{\Sigma(y,\varepsilon)}\Big[P(x-y)\mathcal{T}(x,\partial_x)U(x)-\big\{\widetilde{\mathcal{T}}(x,\partial_x)P(x-y)\big\}^\top U(x)\Big]\,d\Sigma(y,\varepsilon). \quad (2.14)$$

The direction of the normal vector on $\Sigma(y,\varepsilon)$ is chosen as outward.

It is clear that the operator

$$\mathcal{A}U(y):=\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon}\big[A^*(x,\partial_x)P(x-y)\big]^\top U(x)\,dx=$$

$$=\text{v.p.}\int\limits_{\Omega}\big[A^*(x,\partial_x)P(x-y)\big]^\top U(x)\,dx \quad (2.15)$$

is a singular integral operator, "v.p." means the Cauchy principal value integral. If the domain of integration in (2.15) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{A}U\equiv\mathbf{A}U$, i.e.,

$$\mathbf{A}U(y):=\text{v.p.}\int\limits_{\mathbb{R}^3}\big[A^*(x,\partial_x)P(x-y)\big]^\top U(x)\,dx.$$

Note that

$$\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}=-\frac{4\pi\delta_{il}}{3}\,\delta(x-y)+\text{v.p.}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}\,, \quad (2.16)$$

where δ_{il} is the Kronecker delta, while $\delta(\,\cdot\,)$ is the Dirac distribution. The left-hand side in (2.16) is understood in the distributional sense. In view of (2.12) and (2.16), and taking into account that $\chi(0)=1$ we can write the following equality in the distributional sense

$$\big[A^*(x,\partial_x)P(x-y)\big]^\top=$$

$$=\begin{bmatrix}\Big[\dfrac{\partial}{\partial x_i}\Big(c_{ijlk}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big)\Big]_{3\times 3} & \Big[\dfrac{\partial}{\partial x_i}\Big(e_{ikl}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big)\Big]_{3\times 1}\\ \Big[-\dfrac{\partial}{\partial x_i}\Big(e_{lij}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big)\Big]_{1\times 3} & \dfrac{\partial}{\partial x_i}\Big(\varepsilon_{il}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big)\end{bmatrix}_{4\times 4}=$$

$$=\begin{bmatrix}\Big[c_{ijlk}(x)\,\dfrac{\partial^2 F_\chi(x-y)}{\partial x_i\partial x_l}\Big]_{3\times 3} & \Big[e_{ikl}(x)\,\dfrac{\partial^2 F_\chi(x-y}{\partial x_i\partial x_l}\Big]_{1\times 3}\\ \Big[-e_{lij}(x)\,\dfrac{\partial^2 F_\chi(x-y)}{\partial x_l\partial x_i}\Big]_{3\times 1} & \varepsilon_{il}(x)\,\dfrac{\partial^2 F_\chi(x-y)}{\partial x_i\partial x_l}\end{bmatrix}_{4\times 4}+$$

$$+\begin{bmatrix}\Big[\dfrac{\partial c_{ijlk}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 3} & \Big[\dfrac{\partial e_{ikl}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{1\times 3}\\ \Big[-\dfrac{\partial e_{lij}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 1} & \dfrac{\partial\varepsilon_{il}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\end{bmatrix}_{4\times 4}=$$

$$=\begin{bmatrix}\big[c_{ijlk}(x)k_{il}(x,y)\big]_{3\times 3} & \big[e_{ikl}(x)k_{il}(x,y)\big]_{1\times 3}\\ \big[-e_{lij}(x)k_{il}(x,y)\big]_{3\times 1} & \varepsilon_{il}(x)k_{il}(x,y)\end{bmatrix}_{4\times 4}+$$

$$+\begin{bmatrix} \Big[\dfrac{\partial c_{ijlk}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 3} & \Big[\dfrac{\partial e_{ikl}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{1\times 3} \\ \Big[-\dfrac{\partial e_{lij}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 1} & \dfrac{\partial \varepsilon_{il}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l} \end{bmatrix}_{4\times 4},$$

where

$$k_{il}(x,y) := \frac{\delta_{il}}{3}\,\delta(x-y) + \text{v.p.}\,\frac{\partial^2 F_\chi(x-y)}{\partial x_i\partial x_l} =$$

$$= \frac{\delta_{il}}{3}\,\delta(x-y) - \frac{1}{4\pi}\,\text{v.p.}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|} + m_{il}(x,y),$$

$$m_{il}(x,y) := -\frac{1}{4\pi}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{\chi(x-y)-1}{|x-y|}\,.$$

Therefore,

$$\big[A^*(x,\partial_x)P(x-y)\big]^\top =$$

$$= \mathbf{b}(x)\delta(x-y) + \text{v.p.}\big[A^*(x,\partial)P(x-y)\big]^\top =$$

$$= \mathbf{b}(x)\delta(x-y) + R(x,y) - \frac{1}{4\pi}\times$$

$$\times\text{v.p.}\begin{bmatrix} \Big[c_{ijlk}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{3\times 3} & \Big[e_{ikl}(x)\,\dfrac{\partial^2}{\partial x_l\partial x_i}\,\dfrac{1}{|x-y|}\Big]_{3\times 1} \\ \Big[-e_{lij}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{1\times 3} & \varepsilon_{il}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|} \end{bmatrix}_{4\times 4} =$$

$$= \mathbf{b}(x)\delta(x-y) + R^{(1)}(x,y) - \frac{1}{4\pi}\times$$

$$\times\text{v.p.}\begin{bmatrix} \Big[c_{ijlk}(y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{3\times 3} & \Big[e_{ikl}(y)\,\dfrac{\partial^2}{\partial x_l\partial x_i}\,\dfrac{1}{|x-y|}\Big]_{3\times 1} \\ \Big[-e_{lij}(y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{1\times 3} & \varepsilon_{il}(y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|} \end{bmatrix}_{4\times 4}, \quad (2.17)$$

where

$$\mathbf{b}(x) := \frac{1}{3}\begin{bmatrix} [c_{ljlk}(x)]_{3\times 3} & [e_{lkl}(x)]_{3\times 1} \\ [-e_{llj}(x)]_{1\times 3} & \varepsilon_{ll}(x) \end{bmatrix}_{4\times 4}, \quad (2.18)$$

$$R(x,y) = \begin{bmatrix} \big[c_{ijlk}(x)m_{il}(x,y)\big]_{3\times 3} & \big[e_{ikl}(x)m_{il}(x,y)\big]_{1\times 3} \\ \big[-e_{lij}(x)m_{il}(x,y)\big]_{3\times 1} & \varepsilon_{il}(x)m_{il}(x,y) \end{bmatrix}_{4\times 4} +$$

$$+\begin{bmatrix} \Big[\dfrac{\partial c_{ijlk}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 3} & \Big[\dfrac{\partial e_{ikl}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{1\times 3} \\ \Big[-\dfrac{\partial e_{lij}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 1} & \dfrac{\partial \varepsilon_{il}(x)}{\partial x_i}\,\dfrac{\partial F_\chi(x-y)}{\partial x_l} \end{bmatrix}_{4\times 4},$$

$$R^{(1)}(x,y)=R(x,y)-$$
$$-\frac{1}{4\pi}\begin{bmatrix}\Big[c_{ijlk}(x,y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{3\times 3} & \Big[-e_{lij}(x,y)\,\dfrac{\partial^2}{\partial x_l\partial x_i}\,\dfrac{1}{|x-y|}\Big]_{3\times 1}\\ \Big[e_{ikl}(x,y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{1\times 3} & \varepsilon_{il}(x,y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\end{bmatrix}_{4\times 4},$$
$$c_{ijlk}(x,y):=c_{ijlk}(x)-c_{ijlk}(y),$$
$$e_{lij}(x,y):=e_{lij}(x)-e_{lij}(y),$$
$$\varepsilon_{il}(x,y):=\varepsilon_{il}(x)-\varepsilon_{il}(y).$$

Clearly, the entries of the matrix-functions $R(x,y)$ and $R^{(1)}(x,y)$ possess weak singularities of type $\mathcal{O}(|x-y|^{-2})$ as $x\to y$. Therefore we get

$$\text{v.p.}A^{\top}(x,\partial_x)P(x-y)=R(x,y)+$$
$$+\text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix}\Big[-c_{ijlk}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{3\times 3} & \Big[e_{lij}(x)\,\dfrac{\partial^2}{\partial x_l\partial x_i}\,\dfrac{1}{|x-y|}\Big]_{3\times 1}\\ \Big[-e_{ikl}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{1\times 3} & -\varepsilon_{il}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\end{bmatrix}_{4\times 4},$$
$$\text{v.p.}A^{\top}(x,\partial_x)P(x-y)=R^{(1)}(x,y)+ \tag{2.19}$$
$$+\text{v.p.}\frac{1}{4\pi}\begin{bmatrix}\Big[-c_{ijlk}(y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{3\times 3} & \Big[e_{lij}(y)\,\dfrac{\partial^2}{\partial x_l\partial x_i}\,\dfrac{1}{|x-y|}\Big]_{3\times 1}\\ \Big[-e_{ikl}(y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\Big]_{1\times 3} & -\varepsilon_{il}(y)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,\dfrac{1}{|x-y|}\end{bmatrix}_{4\times 4}.$$

Further, by direct calculations one can easily verify that

$$\lim_{\varepsilon\to 0}\int\limits_{\Sigma(y,\varepsilon)}P(x-y)\mathcal{T}(x,\partial_x)U(x)\,d\Sigma(y,\varepsilon)=0, \tag{2.20}$$

$$\lim_{\varepsilon\to 0}\int\limits_{\Sigma(y,\varepsilon)}\big\{\widetilde{\mathcal{T}}(x,\partial_x)P(x-y)\big\}^{\top}U(x)\,d\Sigma(y,\varepsilon)=$$
$$=\frac{1}{4\pi}\int\limits_{\Sigma_1}\begin{bmatrix}\big[c_{ijlk}(y)\eta_i\eta_l\big]_{3\times 3} & \big[e_{ikl}(y)\eta_l\eta_i\big]_{3\times 1}\\ \big[-e_{lij}(y)\eta_i\eta_l\big]_{1\times 3} & \varepsilon_{il}(y)\eta_i\eta_l\end{bmatrix}_{4\times 4}d\Sigma_1\,U(y)=$$
$$=\frac{1}{4\pi}\begin{bmatrix}\Big[c_{ijlk}(y)\,\dfrac{4\pi\delta_{il}}{3}\Big]_{3\times 3} & \Big[e_{ikl}(y)\,\dfrac{4\pi\delta_{li}}{3}\Big]_{3\times 1}\\ \Big[-e_{lij}(y)\,\dfrac{4\pi\delta_{il}}{3}\Big]_{1\times 3} & \varepsilon_{il}(y)\,\dfrac{4\pi\delta_{il}}{3}\end{bmatrix}_{4\times 4}U(y)=$$
$$=\mathbf{b}(y)U(y), \tag{2.21}$$

where Σ_1 is a unit sphere, $\eta=(\eta_1,\eta_2,\eta_3)\in\Sigma_1$, and $\mathbf{b}$ is defined by (2.18).

Passing to the limit in (2.14) as $\varepsilon \to 0$ and using the relations (2.15), (2.20), and (2.21) we obtain

$$\begin{aligned}\mathbf{b}(y)U(y) + \mathcal{A}U(y) - V(\mathcal{T}^+U)(y) + W(U^+)(y) = \\ = \mathcal{P}\big(A(x,\partial_x)U\big)(y), \quad y \in \Omega, \quad (2.22)\end{aligned}$$

where $\mathcal{A}$ is the *localized singular integral operator* given by (2.15), while V, W, and $\mathcal{P}$ are the *localized single layer, double layer, and Newtonian volume vector-potentials*:

$$V(g)(y) := -\int\limits_S P(x-y)g(x)\,dS_x, \tag{2.23}$$

$$W(g)(y) := -\int\limits_S \big[\widetilde{\mathcal{T}}(x,\partial_x)P(x-y)\big]^\top g(x)\,dS_x,$$

$$\mathcal{P}(h)(y) := \int\limits_\Omega P(x-y)h(x)\,dx. \tag{2.24}$$

Here the densities g and h are four dimensional vector-functions.

Let us also introduce the scalar volume potential

$$\mathbb{P}(\mu)(y) := \int\limits_\Omega F_\chi(x-y)\mu(x)\,dx \tag{2.25}$$

with μ beeing a scalar density function.

If the domain of integration in the Newtonian volume potential (2.24) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{P}h \equiv \mathbf{P}h$, i.e.,

$$\mathbf{P}(h)(y) := \int\limits_{\mathbb{R}^3} P(x-y)h(x)\,dx.$$

Mapping properties of the above potentials are investigated in [14].

We refer to the relation (2.22) as *Green's third formula.* It is evident that by a standard limiting procedure we can extend Green's third formula to functions from the space $H^{1,0}(\Omega, A)$. In particular, it holds true for solutions of the above formulated Dirichlet BVP. In this case, the generalized trace vector $\mathcal{T}^+U$ is understood in the sense of the definition (2.11).

For $U = (u_1, \dots, u_4) \in H^1(\Omega)$ one can easily derive the following relation

$$\mathcal{A}U(y) = -\mathbf{b}(y)U(y) - W(U^+)(y) + \mathcal{Q}U(y), \quad \forall\, y \in \Omega, \tag{2.26}$$

where

$$\mathcal{Q}U(y) := \frac{\partial}{\partial y_l}\begin{bmatrix}\big[\mathbb{P}(c_{ijlk}\partial_i u_k)(y) + \mathbb{P}(e_{ikl}\partial_i u_4)(y)\big]_{3\times1} \\ -\mathbb{P}(e_{lij}\partial_i u_j)(y) + \mathbb{P}(\varepsilon_{il}\partial_i u_4)(y)\end{bmatrix}_{4\times1} \tag{2.27}$$

and $\mathbb{P}$ is defined in (2.25).

In what follows, in our analysis we need explicit expression of the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{A})(y,\xi)$ of the singular integral operator $\mathcal{A}$. This matrix coincides with the Fourier transform of the singular matrix kernel defined by (2.19). Let $\mathcal{F}$ denote the Fourier transform operator,

$$\mathcal{F}_{z\to\xi}[g]=\int\limits_{\mathbb{R}^3} g(z)e^{iz\cdot\xi}\,dz,$$

and set

$$\begin{aligned} h_{il}(z) &:= \text{v.p.}\,\frac{\partial^2}{\partial z_i\partial z_l}\,\frac{1}{|z|}\,,\\ \widehat{h}_{il}(\xi) &:= \mathcal{F}_{z\to\xi}(h_{il}(z)),\quad i,l=1,2,3. \end{aligned}$$

In view of (2.16) and taking into account the relations $\mathcal{F}_{z\to\xi}\delta(z)=1$ and $\mathcal{F}_{z\to\xi}(|z|^{-1})=4\pi|\xi|^{-2}$ (see, e.g., [23]), we easily derive

$$\begin{aligned} \widehat{h}_{il}(\xi)=\mathcal{F}_{z\to\xi}(h_{il}(z))=\mathcal{F}_{z\to\xi}\Big(\frac{4\pi\delta_{li}}{3}\,\delta(z)+\frac{\partial^2}{\partial z_i\partial z_l}\,\frac{1}{|z|}\Big)=\\ =\frac{4\pi\delta_{li}}{3}+(-i\xi_i)(-i\xi_l)\mathcal{F}_{z\to\xi}\Big(\frac{1}{|z|}\Big)=\frac{4\pi\delta_{il}}{3}-\frac{4\pi\xi_i\xi_l}{|\xi|^2}\,. \end{aligned}$$

Now, for arbitrary $y\in\overline{\Omega}$ and $\xi\in\mathbb{R}^3\setminus\{0\}$, due to (2.19) we get

$$\begin{aligned} \mathfrak{S}(\mathcal{A})(y,\xi) &= -\frac{1}{4\pi}\,\mathcal{F}_{z\to\xi}\begin{bmatrix} \big[c_{ijlk}(y)h_{il}(z)\big]_{3\times3} & \big[e_{ikl}(y)h_{il}(z)\big]_{3\times1}\\ \big[-e_{lij}(y)h_{il}(z)\big]_{1\times3} & \varepsilon_{il}(y)h_{il}(z)\end{bmatrix}_{4\times4}=\\ &= -\frac{1}{4\pi}\begin{bmatrix} \big[c_{ijlk}(y)\widehat{h}_{il}(\xi)\big]_{3\times3} & \big[e_{ikl}(y)\widehat{h}_{il}(\xi)\big]_{3\times1}\\ \big[-e_{lij}(y)\widehat{h}_{il}(\xi)\big]_{1\times3} & -\varepsilon_{il}(y)\widehat{h}_{il}(\xi)\end{bmatrix}_{4\times4}=\\ &= -\mathbf{b}(y)+\frac{1}{|\xi|^2}\begin{bmatrix} \big[c_{ijlk}(y)\xi_i\xi_l\big]_{3\times3} & \big[e_{ikl}(y)\xi_l\xi_i\big]_{3\times1}\\ \big[-e_{lij}(y)\xi_i\xi_l\big]_{1\times3} & \varepsilon_{il}(y)\xi_i\xi_l\end{bmatrix}_{4\times4}=\\ &= \frac{1}{|\xi|^2}\,A(y,\xi)-\mathbf{b}(y), \end{aligned}\tag{2.28}$$

where $A(y,\xi)$ is the matrix defined in (2.4), while $\mathbf{b}(y)$ is given by (2.18).

As we see the entries of the symbol matrix $\mathfrak{S}(\mathcal{A})(y,\xi)$ of the operator $\mathcal{A}$ are even rational homogeneous functions in ξ of order 0. It can be easily verified that both the characteristic function of the singular kernel in (2.17) and the Fourier transform (2.28) satisfy the Tricomi condition, i.e., their integral averages over the unit sphere vanish (cf. [40]).

Denote by ℓ_0 the extension operator by zero from Ω onto Ω^-. It is evident that for a function $U\in H^1(\Omega)$ we have

$$\big(\mathcal{A}U\big)(y)=\big(\mathbf{A}\ell_0U\big)(y)\ \text{ for }\ y\in\Omega.$$

Now we rewrite Green's third formula (2.22) in a more convenient form for our further purposes

$$[\mathbf{b}+\mathbf{A}]\ell_0U(y)-V(\mathcal{T}^+u)(y)+W(U^+)(y)=\mathcal{P}\big(A(x,\partial_x)U\big)(y),\quad y\in\Omega. \tag{2.29}$$

The relation (2.28) implies that the principal homogeneous symbols of the singular integral operators $\mathbf{A}$ and $\mathbf{b}+\mathbf{A}$ read as

$$\mathfrak{S}(\mathbf{A})(y,\xi) = |\xi|^{-2}A(y,\xi) - \mathbf{b}(y) \ \ \forall\, y \in \overline{\Omega}, \ \ \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}, \tag{2.30}$$

$$\mathfrak{S}(\mathbf{b}+\mathbf{A})(y,\xi) = |\xi|^{-2}A(y,\xi) \ \ \forall\, y \in \overline{\Omega}, \ \ \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{2.31}$$

It is evident that the symbol matrix (2.31) is strongly elliptic due to (2.3),

$$\operatorname{Re}\mathfrak{S}(\mathbf{b}+\mathbf{A})(y,\xi)\zeta\cdot\zeta = |\xi|^{-2}\operatorname{Re}A(y,\xi)\zeta\cdot\zeta \geq c|\zeta|^2$$
$$\forall\, y \in \overline{\Omega}, \ \ \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}, \ \ \forall\, \zeta \in \mathbb{C}^4,$$

where c is the same positive constant as in (2.3).

From the decomposition (2.17) and the equality (2.28) it follows that (see, e.g., [2], [25, Theorem 8.6.1])

$$r_{\Omega}\mathbf{A}\ell_0 : H^1(\Omega) \to H^1(\Omega),$$

since the symbol (2.30) is rational and the operators with the kernel functions either $R(x,y)$ or $R_1(x,y)$ maps $H^1(\Omega)$ into $H^2(\Omega)$ for $\chi \in X^2$ (cf. [14, Theorem 5.6]). Here and throughout the paper r_{Ω} denotes the restriction operator to Ω.

Using the properties of localized potentials described in the Appendix B (see Theorems B.1 and B.4) and taking the trace of the equation (2.29) on S we arrive at the relation:

$$\mathbf{A}^{+}\ell_0 U - \mathcal{V}(\mathcal{T}^{+}U) + (\mathbf{b}-\mathbf{d})U^{+} + \mathcal{W}(U^{+}) = \mathcal{P}^{+}(A(x,\partial_x)U) \ \text{ on } \ S, \tag{2.32}$$

where the localized boundary integral operators $\mathcal{V}$ and $\mathcal{W}$ are generated by the localized single and double layer potentials and are defined in (B.1) and (B.2), the matrix $\mathbf{d}$ is defined by (B.3), while

$$\mathbf{A}^{+}\ell_0 U \equiv \gamma^{+}\mathbf{A}\ell_0 U := \{\mathbf{A}\ell_0 U\}^{+} \ \text{ on } \ S,$$
$$\mathcal{P}^{+}(f) \equiv \gamma^{+}\mathcal{P}(f) := \{\mathcal{P}(f)\}^{+} \ \text{ on } \ S.$$

Now we prove the following technical lemma.

Lemma 2.2. *Let* $\chi \in X^3$ *and*

$$f = (f_1,f_2,f_3,f_4)^{\top} \in H^0(\Omega), \quad F = (F_1,F_2,F_3,F_4)^{\top} \in H^{1,0}(\Omega,\Delta),$$
$$\Psi = (\psi_1,\psi_2,\psi_3,\psi_4)^{\top} \in H^{-\frac{1}{2}}(S), \quad \Phi = (\varphi_1,\varphi_2,\varphi_3,\varphi_4)^{\top} \in H^{\frac{1}{2}}(S).$$

Moreover, let $U = (u_1,u_2,u_3,u_4)^{\top} \in H^1(\Omega)$ *and the following equation hold*

$$\mathbf{b}(y)U(y) + \mathcal{A}U(y) - V(\Psi)(y) + W(\Phi)(y) = F(y) + \mathcal{P}(f)(y), \ \ y \in \Omega. \tag{2.33}$$

Then $U \in H^{1,0}(\Omega, A)$.

Proof. Note that by Theorem B.1 $\mathcal{P}(f) \in H^2(\Omega)$ for arbitrary $f \in H^0(\Omega)$, while by Theorem B.2 the inclusions $V(\Psi), W(\Phi) \in H^{1,0}(\Omega,\Delta)$ hold for arbitrary $\Psi \in H^{-\frac{1}{2}}(S)$ and $\Phi \in H^{\frac{1}{2}}(S)$. Using the relations (2.26)–(2.27), the equation (2.33) can be rewritten as

$$\frac{\partial}{\partial y_l}\begin{bmatrix}\big[\mathbb{P}(c_{ijlk}\partial_i u_k)(y)+\mathbb{P}(e_{ikl}\partial_i u_4)(y)\big]_{3\times 1}\\ -\mathbb{P}(e_{lij}\partial_i u_j)(y)+\mathbb{P}(\varepsilon_{il}\partial_i u_4)(y)\end{bmatrix}_{4\times 1}= \\ = F(y)+\mathcal{P}(f)(y)+V(\Psi)(y)-W(\Phi-U^+)(y),\ \ y\in\Omega.$$

Due to Theorems B.1 and B.2 it follows that the right-hand side function in the above equality belongs to the space

$$H^{1,0}(\Omega,\Delta):=\Big\{v\in H^1(\Omega):\ \Delta v\in H^0(\Omega)\Big\},$$

since $U^+\in H^{\frac{1}{2}}(S)$, and therefore the same holds true for the left-hand side function,

$$\frac{\partial}{\partial y_l}\begin{bmatrix}\big[\mathbb{P}(c_{ijlk}\partial_i u_k)(y)+\mathbb{P}(e_{ikl}\partial_i u_4)(y)\big]_{3\times 1}\\ -\mathbb{P}(e_{lij}\partial_i u_j)(y)+\mathbb{P}(\varepsilon_{il}\partial_i u_4)(y)\end{bmatrix}_{4\times 1}\in H^{1,0}(\Omega,\Delta). \tag{2.34}$$

Note that

$$\Delta(\partial_x)P(x-y)=[\delta(x-y)+R_\Delta(x-y)]I, \tag{2.35}$$

where

$$R_\Delta(x-y):=-\frac{1}{4\pi}\left\{\frac{\Delta\chi(x-y)}{|x-y|}+2\,\frac{\partial\chi(x-y)}{\partial x_l}\,\frac{\partial}{\partial x_l}\,\frac{1}{|x-y|}\right\}. \tag{2.36}$$

Clearly, $R_\Delta(x-y)=\mathcal{O}(|x-y|^{-2})$ as $x\to y$ and with the help of (2.35) and (2.36) one can prove that for arbitrary scalar function $\phi\in\mathcal{D}(\Omega)$ there holds the relation (see, e.g., [40])

$$\Delta(\partial_y)\mathbb{P}(\phi)(y)=\phi(y)+\mathcal{R}_\Delta(\phi)(y),\ \ y\in\Omega, \tag{2.37}$$

where

$$\mathcal{R}_\Delta(\phi)(y):=\int\limits_\Omega R_\Delta(x-y)\phi(x)\,dx. \tag{2.38}$$

Evidently (2.38) remains true for $\phi\in H^0(\Omega)$, since $\mathcal{D}(\Omega)$ is dense in $H^0(\Omega)$. The operator $\mathcal{R}_\Delta$ has the following mapping property (see [14]):

$$\mathcal{R}_\Delta: H^0(\Omega)\to H^1(\Omega). \tag{2.39}$$

Applying the Laplace operator Δ to the vector-function (2.34) and keeping in mind the relation (2.37), we arrive at the following equation in Ω,

$$\Delta(\partial_y)\,\frac{\partial}{\partial y_l}\begin{bmatrix}\big[\mathbb{P}(c_{ijlk}\partial_i u_k)(y)+\mathbb{P}(e_{ikl}\partial_i u_4)(y)\big]_{3\times 1}\\ -\mathbb{P}(e_{lij}\partial_i u_j)(y)+\mathbb{P}(\varepsilon_{il}\partial_i u_4)(y)\end{bmatrix}_{4\times 1}= \\ =\begin{bmatrix}\Big[\frac{\partial}{\partial y_l}\big(\Delta(\partial_y)\mathbb{P}(c_{ijlk}\partial_i u_k)(y)\big)+\frac{\partial}{\partial y_l}\big(\Delta(\partial_y)\mathbb{P}(e_{ikl}\partial_i u_4)(y)\big)\Big]_{3\times 1}\\ -\frac{\partial}{\partial y_l}\big(\Delta(\partial_y)\mathbb{P}(e_{lij}\partial_i u_j)(y)\big)+\frac{\partial}{\partial y_l}\big(\Delta(\partial_y)\mathbb{P}(\varepsilon_{il}\partial_i u_4)(y)\big)\end{bmatrix}=$$

$$= \begin{bmatrix} \Big[\dfrac{\partial}{\partial y_l}\Big(c_{ijlk}(y)\,\dfrac{\partial u_k(y)}{\partial y_i}\Big) + \dfrac{\partial}{\partial y_l}\Big(e_{ikl}(y)\,\dfrac{\partial u_4(y)}{\partial y_i}\Big)\Big]_{3\times 1} \\ -\dfrac{\partial}{\partial y_l}\Big(\Big(e_{lij}(y)\,\dfrac{\partial u_j(y)}{\partial y_i}\Big)\Big) + \dfrac{\partial}{\partial y_l}\Big(\varepsilon_{il}(y)\,\dfrac{\partial u_4(y)}{\partial y_i}\Big) \end{bmatrix} +$$

$$+ \begin{bmatrix} \Big[\dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(c_{ijlk}\partial_i u_k)(y) + \dfrac{\partial}{\partial y_l}\mathcal{R}_\Delta(e_{ikl}\partial_i u_4)(y)\Big]_{3\times 1} \\ -\dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(e_{lij}\partial_i u_j)(y) + \dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(\varepsilon_{il}\partial_i u_4)(y) \end{bmatrix} =$$

$$= A(y,\partial_y)U + \begin{bmatrix} \Big[\dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(c_{ijlk}\partial_i u_k)(y) + \dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(e_{ikl}\partial_i u_4)(y)\Big]_{3\times 1} \\ -\dfrac{\partial}{\partial y_l}\,\mathcal{R}_\Delta(e_{lij}\partial_i u_j)(y) + \dfrac{\partial}{\partial y_l}\mathcal{R}_\Delta(\varepsilon_{il}\partial_i u_4)(y) \end{bmatrix}.$$

Whence the embedding $A(y,\partial_y)U \in H^0(\Omega)$ follows due to (2.34) and (2.39). □

Actually, in the proof of Lemma 2.2 we have shown the following assertion.

Corollary 2.3. *Let $\chi \in X^3$. The operator*

$$\mathbf{b} + \mathbf{A} : H^{1,0}(\Omega, A) \to H^{1,0}(\Omega, \Delta)$$

is bounded.

Now, we are in the position to reduce the above formulated Dirichlet boundary value problem to the LBDIEs system equivalently.

2.2. **LBDIE formulation of the Dirichlet problem and the equivalence theorem.** Let $U \in H^{1,0}(\Omega, A)$ be a solution to the Dirichlet BVP (2.6), (2.7) with $\Phi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega)$. As we have derived above, there hold the relations (2.29) and (2.32), which now can be rewritten in the form

$$[\mathbf{b} + \mathbf{A}]\ell_0 U - V(\Psi) = \mathcal{P}(f) - W(\Phi_0) \ \text{ in } \ \Omega, \tag{2.40}$$

$$\mathbf{A}^+\ell_0 U - \mathcal{V}(\Psi) = \mathcal{P}^+(f) - (\mathbf{b} - \mathbf{d})\Phi_0 - \mathcal{W}(\Phi_0) \ \text{ on } \ S, \tag{2.41}$$

where $\Psi := \mathcal{T}^+U \in H^{-\frac{1}{2}}(S)$ and $\mathbf{d}$ is defined by (B.3).

One can consider these relations as the LBDIE system with respect to the unknown vector-functions U and Ψ. Now we prove the following equivalence theorem.

Theorem 2.4. *Let $\chi \in X^3_+$, $\Phi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega)$.*

(i) *If a vector-function $U \in H^{1,0}(\Omega, A)$ solves the Dirichlet BVP* (2.6), (2.7)*, then it is unique and the pair $(U, \Psi) \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S)$ with*

$$\Psi = \mathcal{T}^+U, \tag{2.42}$$

solves the LBDIE system (2.40), (2.41) *and vice versa.*

(ii) *If a pair* $(U,\Psi)\in H^{1,0}(\Omega,A)\times H^{-\frac12}(S)$ *solves the LBDIE system* (2.40), (2.41)*, then it is unique and the vector-function* u *solves the Dirichlet BVP* (2.6), (2.7)*, and relation* (2.42) *holds.*

Proof. (i) The first part of the theorem is trivial and directly follows form the relations (2.29), (2.32), (2.42), and Remark 2.1.

(ii) Now, let a pair $(U,\Psi)\in H^{1,0}(\Omega,A)\times H^{-\frac12}(S)$ solve the LBDIE system (2.40), (2.41). Taking the trace of (2.40) on S and comparing with (2.41) we get

$$U^+=\Phi_0 \ \text{ on } \ S. \tag{2.43}$$

Further, since $U\in H^{1,0}(\Omega,A)$, we can write Green's third formula (2.29) which in view of (2.43) can be rewritten as

$$[\mathbf{b}+\mathbf{A}]\ell_0U-V(\mathcal{T}^+U)=\mathcal{P}\big(A(x,\partial_x)U\big)-W(\Phi_0) \ \text{ in } \ \Omega. \tag{2.44}$$

From (2.40) and (2.44) it follows that

$$V(\mathcal{T}^+U-\Psi)+\mathcal{P}\big(A(x,\partial_x)U-f\big)=0 \ \text{ in } \ \Omega.$$

Whence by Lemma 6.3 in [14] we have

$$A(x,\partial_x)U=f \ \text{ in } \ \Omega \ \text{ and } \ \mathcal{T}^+U=\psi \ \text{ on } \ S.$$

Thus U solves the Dirichlet BVP (2.6), (2.7) and equation (2.42) holds.

The uniqueness of solution to the LBDIE system (2.40), (2.41) in the class $H^{1,0}(\Omega,A)\times H^{-\frac12}(S)$ directly follows from the above proved equivalence result and the uniqueness theorem for the Dirichlet problem (2.6), (2.7) (see Remark 2.1). □

3. Invertibility of the Dirichlet LBDIO

From Theorem 2.4 it follows that the LBDIE system (2.40), (2.41) with the special right-hand sides is uniquely solvable in the class $H^{1,0}(\Omega,A)\times H^{-1/2}(S)$. We investigate Fredholm properties of the localized boundary-domain integral operator generated by the left-hand side expressions in (2.40), (2.41) and show the invertibility of the operator in appropriate functional spaces.

The LBDIE system (2.40), (2.41) with an arbitrary right-hand side vector-functions from the space $H^1(\Omega)\times H^{1/2}(S)$ can be written as

$$(\mathbf{b}+\mathbf{A})\ell_0U-V\Psi=F_1 \ \text{ in } \ \Omega, \tag{3.1}$$

$$\mathbf{A}^+\ell_0U-\mathcal{V}\Psi=F_2 \ \text{ on } \ S, \tag{3.2}$$

where $F_1\in H^1(\Omega)$ and $F_2\in H^{1/2}(S)$. Denote

$$\mathbf{B}:=\mathbf{b}+\mathbf{A}. \tag{3.3}$$

Evidently, the principal homogeneous symbol matrix of the operator $\mathbf{B}$ reads as (see (2.31))

$$\mathfrak{S}(\mathbf{B})(y,\xi)=|\xi|^{-2}A(y,\xi) \ \text{ for } \ y\in\overline{\Omega}, \ \ \xi\in\mathbb{R}^3\setminus\{0\}. \tag{3.4}$$

It is an even rational homogeneous matrix-function of order 0 in ξ and due to (2.3) it is uniformly strongly elliptic,

$$\operatorname{Re}\mathfrak{S}(\mathbf{B})(y,\xi)\zeta\cdot\zeta \geq c|\zeta|^2 \ \text{ for all } \ y\in\overline{\Omega}, \ \ \xi\in\mathbb{R}^3\setminus\{0\}, \ \ \zeta\in\mathbb{C}^4.$$

Consequently, $\mathbf{B}$ is a strongly elliptic pseudodifferential operator of zero order (i.e., singular integral operator) and the partial indices of factorization of the symbol (3.4) equal to zero (cf. [10, Lemma 1.20]).

In our further analysis we need some auxiliary assertions. To formulate them, let $\widetilde{y}\in\partial\Omega$ be some fixed point and consider the frozen symbol $\mathfrak{S}(\mathbf{B})(\widetilde{y},\xi)\equiv\mathfrak{S}(\widetilde{\mathbf{B}})(\xi)$, where $\widetilde{\mathbf{B}}$ denotes the operator $\mathbf{B}$ written in a chosen local coordinate system. Further, let $\widehat{\widetilde{\mathbf{B}}}$ denote the pseudodifferential operator with the symbol

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})(\xi',\xi_3) := \mathfrak{S}(\widetilde{\mathbf{B}})\big((1+|\xi'|)\omega,\xi_3\big),$$
$$\omega=\frac{\xi'}{|\xi'|},\ \xi=(\xi',\xi_3),\ \xi'=(\xi_1,\xi_2).$$

The principal homogeneous symbol matrix $\mathfrak{S}(\widetilde{\mathbf{B}})(\xi)$ of the operator $\widehat{\widetilde{\mathbf{B}}}$ can be factorized with respect to the variable ξ_3,

$$\mathfrak{S}(\widetilde{\mathbf{B}})(\xi)=\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi)\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})(\xi), \tag{3.5}$$

where

$$\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi)=\frac{1}{\Theta^{(\pm)}(\xi',\xi_3)}\,\widetilde{A}^{(\pm)}(\xi',\xi_3),$$

$\Theta^{(\pm)}(\xi',\xi_3):=\xi_3\pm i|\xi'|$ are the "plus" and "minus" factors of the symbol $\Theta(\xi):=|\xi|^2$, and $\widetilde{A}^{(\pm)}(\xi',\xi_3)$ are the "plus" and "minus" polynomial matrix factors of the first order in ξ_3 of the polynomial symbol matrix $\widetilde{A}(\xi',\xi_3)\equiv\widetilde{A}(\widetilde{y},\xi',\xi_3)$ (see [22, Theorem 1], [45, Theorem 1.33], [24, Theorem 1.4]), i.e.

$$\widetilde{A}(\xi',\xi_3)=\widetilde{A}^{(-)}(\xi',\xi_3)\widetilde{A}^{(+)}(\xi',\xi_3) \tag{3.6}$$

with $\det\widetilde{A}^{(+)}(\xi',\tau)\neq 0$ for $\operatorname{Im}\tau>0$ and $\det\widetilde{A}^{(-)}(\xi',\tau)\neq 0$ for $\operatorname{Im}\tau<0$. Moreover, the entries of the matrices $\widetilde{A}^{(\pm)}(\xi',\xi_3)$ are homogeneous functions in $\xi=(\xi',\xi_3)$ of order 1. Denote by $a^{(\pm)}(\xi')$ the coefficients at ξ_3^4 in the determinants $\det\widetilde{A}^{(\pm)}(\xi',\xi_3)$. Evidently,

$$a^{(-)}(\xi')a^{(+)}(\xi')=\det\widetilde{A}(0,0,1)>0 \quad\text{for}\quad \xi'\neq 0. \tag{3.7}$$

It is easy to see that the factor-matrices $\widetilde{A}^{(\pm)}(\xi',\xi_3)$ have the structure

$$\big[\widetilde{A}^{(\pm)}(\xi',\xi_3)\big]^{-1}=\frac{1}{\det\widetilde{A}^{(\pm)}(\xi',\xi_3)}\,\big[p_{ij}^{(\pm)}(\xi',\xi_3)\big]_{4\times 4},$$

where $p_{ij}^{(\pm)}(\xi',\xi_3)$ is the co-factor corresponding to the element $\widetilde{A}_{ji}^{(\pm)}(\xi',\xi_3)$ of the matrix $\widetilde{A}^{(\pm)}(\xi',\xi_3)$, which can be written in the form

$$p_{ij}^{(\pm)}(\xi',\xi_3)=c_{ij}^{(\pm)}(\xi')\xi_3^3+b_{ij}^{(\pm)}(\xi')\xi_3^2+d_{ij}^{(\pm)}(\xi')\xi_3+e_{ij}^{(\pm)}(\xi'). \tag{3.8}$$

Here $c_{ij}^{(\pm)}$, $b_{ij}^{(\pm)}$, $d_{ij}^{(\pm)}$, and $e_{ij}^{(\pm)}$, $i,j=1,2,3,4$, are homogeneous functions in ξ' of order 0, 1, 2, and 3, respectively. From the above mentioned it follows that the entries of the factor-symbol matrices

$$\mathfrak{B}^{(\pm)}(\omega,r,\xi_3)=\big[\mathfrak{b}_{kj}^{(\pm)}(\omega,r,\xi_3)\big]_{3\times 3}:=\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi',\xi_3)$$

with $\omega=\xi'/|\xi'|$ and $r=|\xi'|$ satisfy the following relations:

$$\frac{\partial^l\mathfrak{b}_{kj}^{(\pm)}(\omega,0,-1)}{\partial r^l}=(-1)^l\,\frac{\partial^l\mathfrak{b}_{kj}^{(\pm)}(\omega,0,+1)}{\partial r^l}\,,\quad l=0,1,2,\dots\,. \tag{3.9}$$

These relations imply that the entries of the matrices $\mathfrak{S}^{\pm}(\widetilde{\mathbf{B}})(\xi',\xi_3)$ belong to the class of symbols D_0 introduced in [23, Ch. III, § 10],

$$\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi',\xi_3)\in D_0. \tag{3.10}$$

Denote by $\Pi^{\pm}$ the Cauchy type integral operators

$$\Pi^{\pm}(h)(\xi):=\pm\frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{-\infty}^{+\infty}\frac{h(\xi',\eta_3)}{\xi_3\pm it-\eta_3}\,d\eta_3, \tag{3.11}$$

which are well defined for a bounded smooth function $h(\xi',\cdot)$ satisfying the relation $h(\xi',\eta_3)=\mathcal{O}(1+|\eta_3|)^{-\kappa}$ with some $\kappa>0$.

First we prove the following auxiliary lemma.

Lemma 3.1. *Let $\chi\in X_+^k$ with integer $k\geqslant s+2$ and let ℓ_0 be the extension operator by zero from $\mathbb{R}^3_+$ onto the half-space $\mathbb{R}^3_-$. The operator*

$$r_{\mathbb{R}^3_+}\widehat{\widetilde{\mathbf{B}}}\ell_0: H^s(\mathbb{R}^3_+)\to H^s(\mathbb{R}^3_+)$$

is invertible for all $s\geq 0$, where $r_{\mathbb{R}^3_+}$ is the restriction operator to the half-space $\mathbb{R}^3_+$. Moreover, for $f\in H^s(\mathbb{R}^3_+)$ with $s\geq 0$, the unique solution of the equation

$$r_{\mathbb{R}^3_+}\widehat{\widetilde{\mathbf{B}}}\ell_0 U=f \tag{3.12}$$

can be represented in the form

$$U_+:=\ell_0U=\mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\},$$

where $\ell f\in H^s(\mathbb{R}^3)$ is an arbitrary extension of f onto the whole space $\mathbb{R}^3$.

Proof. Since the right-hand side f of the equation (3.12) belongs to the space $H^s(\mathbb{R}^3_+)$ with $s\geq 0$, it follows that $f\in H^0(\mathbb{R}^3_+)$.

First we show that the equation (3.12) is uniquely solvable in the space $H^0(\mathbb{R}^3_+)$.

Let $U\in H^0(\mathbb{R}^3_+)$ be a solution of the equation (3.12) with $f\in H^0(\mathbb{R}^3_+)$ and let

$$U_-=\ell f-\widehat{\widetilde{\mathbf{B}}}U_+, \tag{3.13}$$

where $U_+ = \ell_0 U \in \widetilde{H}^0(\mathbb{R}^3_+)$ and $\ell f \in H^0(\mathbb{R}^3)$ is an arbitrary extension of $f \in H^0(\mathbb{R}^3_+)$ onto $\mathbb{R}^3_+$. We assume that

$$\|\ell f\|_{H^0(\mathbb{R}^3)} \leq 2\|f\|_{H^0(\mathbb{R}^3_+)}.$$

Since $\ell f \in H^0(\mathbb{R}^3)$ and $\widehat{\widetilde{\mathbf{B}}} U_+ \in H^0(\mathbb{R}^3)$, we have $U_- \in H^0(\mathbb{R}^3)$. In addition, $U_- \in \widetilde{H}^0(\mathbb{R}^3_-)$. Here and in what follows we employ the notation

$$\widetilde{H}^s(\Omega) := \Big\{ V \in H^s(\Omega) : \ \operatorname{supp} V \subset \overline{\Omega} \Big\}.$$

The Fourier transform of (3.13) gives the relation

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})(\xi)\mathcal{F}(U_+) + \mathcal{F}(U_-)(\xi) = \mathcal{F}(\ell f)(\xi). \tag{3.14}$$

Due to (3.5) we have the factorization

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})(\xi', \xi_3) = \widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})(\xi', \xi_3)\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})(\xi', \xi_3), \tag{3.15}$$

where $\widehat{\mathfrak{S}}^{(\pm)}(\widetilde{\mathbf{B}})(\xi', \xi_3) = \mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})((1+|\xi'|)\omega, \xi_3)$ with $\omega = \frac{\xi'}{|\xi'|}$. Substituting (3.15) into (3.14) and multiplying both sides by $[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})]^{-1}$, we get

$$\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})(\xi)\mathcal{F}(U_+)(\xi) + \big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\mathcal{F}(U_-)(\xi) = \\ = \big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\mathcal{F}(\ell f)(\xi). \tag{3.16}$$

Introduce the notation

$$v_+(x) = \mathcal{F}^{-1}_{\xi\to x}\Big(\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})(\xi)\mathcal{F}(U_+)(\xi)\Big), \tag{3.17}$$

$$v_-(x) = \mathcal{F}^{-1}_{\xi\to x}\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\mathcal{F}(U_-)(\xi)\Big),$$

$$g(x) = \mathcal{F}^{-1}_{\xi\to x}\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\mathcal{F}(\ell f)(\xi)\Big). \tag{3.18}$$

Then we can conclude that (see [23, Theorem 4.4 and Lemmas 20.2, 20.5])

$$v_+ \in \widetilde{H}^0(\mathbb{R}^3_+), \quad v_- \in \widetilde{H}^0(\mathbb{R}^3_-), \quad g \in H^0(\mathbb{R}^3), \tag{3.19}$$

since the degree of homogeneity of $\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})(\xi)$ and $\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi)$ equals to 0.

In view of the above notation, the equation (3.16) acquires the form

$$\mathcal{F}(v_+)(\xi) + \mathcal{F}(v_-)(\xi) = \mathcal{F}(g)(\xi). \tag{3.20}$$

In accordance with Lemma 5.4 in [23], we conclude that the representation of the vector-function $\mathcal{F}(g)(\xi)$ in the form (3.20) is unique in view of the inclusions (3.19) which in turn leads to the relations

$$\mathcal{F}(v_+) = \Pi^+\mathcal{F}(g), \quad \mathcal{F}(v_-) = \Pi^-\mathcal{F}(g). \tag{3.21}$$

Now, from (3.17), (3.18), and the first equation in (3.21) it follows that $U_+ \in \widetilde{H}^0(\mathbb{R}^3_+)$ is representable in the form

$$U_+ = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\}. \tag{3.22}$$

Evidently, for the solution $U \in H^0(\mathbb{R}^3_+)$ of the equation (3.12) we get the following representation

$$U = r_{\mathbb{R}^3_+} \mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\}. \tag{3.23}$$

Note that the representation (3.23) does not depend on the choice of the extension operator ℓ. Indeed, let $\ell_1 f \in H^0(\mathbb{R}^3)$ be another extension of $f \in H^0(\mathbb{R}^3_+)$, i.e., $r_{\mathbb{R}^3_+}\ell_1 f = f$. Since $f_- = \ell f - \ell_1 f \in \widetilde{H}^0(\mathbb{R}^3_-)$, it follows that (see [23, Theorem 4.4, Lemmas 20.2 and 20.5])

$$\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(f_-)\big) \in \widetilde{H}^0(\mathbb{R}^3_-),$$

while

$$\Pi^+\Big\{\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(f_-)\Big\} = \mathcal{F}\Big\{\theta^+\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(f_-)\big)\Big\} = 0$$

(see [23, Lemma 5.2]), where θ^+ denotes the multiplication operator by the Heaviside step function $\theta(x_3)$ which equals to 1 for $x_3 > 0$ and vanishes for $x_3 < 0$. Therefore

$$\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big) = \Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell_1 f)\big)$$

and the claim follows.

If, in particular, $f = 0$, then we can take $\ell f = 0$, and hence $U = 0$ by virtue of (3.22). Thus the equation (3.12) possesses at most one solution in the space $H^0(\mathbb{R}^3_+)$.

Further, we show that

$$U = r_{\mathbb{R}^3_+} \mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^+\Big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\Big)\Big\} \tag{3.24}$$

is a solution of the equation (3.12) for any $f \in H^0(\mathbb{R}^3_+)$.

To this and, let us first note that for the vector-function involved in (3.24) the following embedding holds

$$\mathcal{F}^{-1}\Big\{[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})]^{-1}\Pi^+\big([\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})]^{-1}\mathcal{F}(\ell f)\big)\Big\} \in \widetilde{H}^0(\mathbb{R}^3_+). \tag{3.25}$$

Indeed, we have

$$\begin{aligned}\mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\} &= \\ = \mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1} \mathcal{F}\Big[\theta^+\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big]\Big\}&\end{aligned}$$

and (3.25) follows from Theorem 4.4, Lemmas 20.2 and 20.5 in [23]. From (3.24) and (3.25) we then get

$$U_+ := \ell_0 U = \mathcal{F}^{-1}\Big\{ \big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\}. \tag{3.26}$$

With the help of the following relation (see Lemma 5.4 in [23])

$$\begin{aligned}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big) &= \\ = \big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f) - \Pi^-\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big),&\end{aligned}$$

from the equality (3.26) we derive

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})\mathcal{F}(U_+) = \widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\Pi^+\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big) = \\ = \mathcal{F}(\ell f) - \widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\Pi^-\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big).$$

Since

$$F^{-1}\Big\{\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\Pi^-\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}F(\ell f)\big)\Big\} \in \widetilde{H}^0(\mathbb{R}^3_-),$$

(see [23, Theorems 4.4, 5.1 and Lemmas 20.2, 20.5]), we easily obtain

$$r_{\mathbb{R}^3_+}\widehat{\widetilde{\mathbf{B}}}U_+ = r_{\mathbb{R}^3_+}(\ell f) - r_{\mathbb{R}^3_+}\mathcal{F}^{-1}\Big\{\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\Pi^-\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\} = \\ = r_{\mathbb{R}^3_+}(\ell f) = f,$$

i.e., the vector-function (3.24) solves the equation (3.12) and belongs to the space $H^0(\mathbb{R}^3_+)$ for $f \in H^0(\mathbb{R}^3_+)$.

In what follows, we prove that for $f \in H^s(\mathbb{R}^3_+)$ and $\ell f \in H^s(\mathbb{R}^3)$ with

$$\|\ell f\|_{H^s(\mathbb{R}^3)} \leq 2\|f\|_{H^s(\mathbb{R}^3_+)}, \quad s \geq 0, \tag{3.27}$$

the vector-function defined by (3.24) satisfies the inequality

$$\|U\|_{H^s(\mathbb{R}^3_+)} \leq C\|f\|_{H^s(\mathbb{R}^3_+)}, \tag{3.28}$$

and hence belongs to $H^s(\mathbb{R}^3_+)$. Indeed, since (see [23, Lemma 5.2 and Theorem 5.1])

$$\Pi^+(\mathcal{F}g) = \mathcal{F}(\theta^+ g) \ \text{ for all } \ g \in H^0(\mathbb{R}^3),$$

then the representation (3.26) of U_+ can be rewritten as

$$U_+ = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}\Big[\theta^+\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big]\Big\}.$$

Therefore, using (3.27) and in view of (3.10), from Theorem 10.1, Lemmas 4.4, 20.2, and 20.5 in [23] we finally derive

$$\|U\|_{H^s(\mathbb{R}^3_+)} \leq c_1\Big\|\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\|_{H^s(\mathbb{R}^3_+)} \leq \\ \leq c_1\Big\|\mathcal{F}^{-1}\big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\big)\Big\|_{H^s(\mathbb{R}^3)} \leq c\|\ell f\|_{H^s(\mathbb{R}^3)} \leq 2c\|f\|_{H^s(\mathbb{R}^3_+)}$$

with some positive constants c and c_1, whence (3.28) follows. This completes the proof. □

Lemma 3.2. *Let the factor matrix $\widetilde{A}^{(+)}(\xi',\tau)$ be as in (3.6), and $a^{(+)}$ and $c_{ij}^{(+)}$ be as in (3.7) and (3.8), respectively. Then the following equality holds*

$$\frac{1}{2\pi i}\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau = \frac{1}{a^{(+)}(\xi')}\,\big[c_{ij}^{(+)}(\xi')\big]_{4\times 4},$$

and

$$\det\big[c_{ij}^{(+)}(\xi')\big]_{4\times 4} \neq 0 \ \textit{ for } \ \xi' \neq 0.$$

Here γ^- is a contour in the lower complex half-plane enclosing all the roots of the polynomial $\det \widetilde{A}^{(+)}(\xi', \tau)$ *with respect to* τ.

Proof. Note that $\det \widetilde{A}^{(+)}(\xi', \tau)$ is a forth order polynomial in τ, while $p_{ij}^{(+)}(\xi', \tau)$ is a third order polynomial in τ defined in (3.8).

Let γ_R be a circle with sufficiently large radius R and centered at the origin. Then by Cauchy theorem we derive

$$\frac{1}{2\pi i} \int\limits_{\gamma^-} \Big\{ \big[\widetilde{A}^{(+)}(\xi', \tau)\big]^{-1} \Big\}_{ij} d\tau =$$

$$= \frac{1}{2\pi i} \int\limits_{\gamma^-} \frac{p_{ij}^{(+)}(\xi', \tau)}{\det \widetilde{A}^{(+)}(\xi', \tau)} d\tau = \frac{1}{2\pi i} \int\limits_{\gamma_R} \frac{p_{ij}^{(+)}(\xi', \tau)}{\det \widetilde{A}^{(+)}(\xi', \tau)} d\tau =$$

$$= \frac{1}{2\pi i} \frac{c_{ij}^{(+)}(\xi')}{a^{(+)}(\xi')} \int\limits_{\gamma_R} \frac{1}{\tau} d\tau + \int\limits_{\gamma_R} Q_{ij}(\xi', \tau)\, d\tau =$$

$$= \frac{c_{ij}^{(+)}(\xi')}{a^{(+)}(\xi')} + \int\limits_{\gamma_R} Q_{ij}(\xi', \tau)\, d\tau, \quad (3.29)$$

where

$$Q_{ij}(\xi', \tau) = O(|\tau|^{-2}) \;\text{ as }\; |\tau| \to \infty.$$

It is clear that

$$\lim_{R\to\infty} \int\limits_{\gamma_R} Q_{ij}(\xi', \tau)\, d\tau = 0.$$

Therefore by passing to the limit in (3.29) as $R \to \infty$ we obtain

$$\frac{1}{2\pi i} \int\limits_{\gamma^-} \Big\{ \big[\widetilde{A}^{(+)}(\xi', \tau)\big]^{-1} \Big\}_{ij} d\tau = \frac{c_{ij}^{(+)}(\xi')}{a^{(+)}(\xi')}\,.$$

Now we show that $\det[c_{ij}^{(+)}]_{4\times 4} \neq 0$. We introduce the notation

$$P^{(+)}(\xi', \xi_3) = [p_{ij}^{(+)}(\xi', \xi_3)]_{4\times 4} =$$
$$= C^{(+)}(\xi')\xi_3^3 + B^{(+)}(\xi')\xi_3^2 + D^{(+)}(\xi')\xi_3 + E^{(+)}(\xi'),$$

where

$$C^{(+)}(\xi') = \big[c_{ij}^{(+)}(\xi')\big]_{4\times 4}, \quad B^{(+)}(\xi') = \big[b_{ij}^{(+)}(\xi')\big]_{4\times 4},$$
$$D^{(+)}(\xi') = \big[d_{ij}^{(+)}(\xi')\big]_{4\times 4}, \quad E^{(+)}(\xi') = \big[e_{ij}^{(+)}(\xi')\big]_{4\times 4}.$$

In accordance with the relation $\det[\widetilde{A}^{(+)}(\xi', \xi_3)]^{-1} \neq 0$ for $\xi = (\xi', \xi_3) \neq 0$, we conclude that $\det P^{(+)}(\xi', \xi_3) \neq 0$ for $\xi = (\xi', \xi_3) \neq 0$.

Let us introduce new coordinates $r = |\xi'|$ and $\omega = \xi'/|\xi'|$, and denote

$$\mathcal{P}^{(+)}(\omega, r, \xi_3) := P^{(+)}(\xi', \xi_3) = P^{(+)}(\omega r, \xi_3).$$

Then we have

$$\det \mathcal{P}^{(+)}(\omega, r, \xi_3) = \det P^{(+)}(\xi', \xi_3) =$$
$$= \det\Big(C^{(+)}(\omega)\xi_3^3 + B^{(+)}(\omega)\xi_3^2 r + D^{(+)}(\omega)\xi_3 r^2 + E^{(+)}(\omega) r^3\Big) \neq 0$$
$$\text{for all } \xi_3 \neq 0,$$

whence

$$\lim_{r\to 0} \det \mathcal{P}^{(+)}(\omega, r, \xi_3) = \xi_3^{12} \det C^{(+)}(\omega).$$

Consequently,

$$\det C^{(+)}(\omega) = \det[c_{ij}^{(+)}(\omega)]_{4\times 4} \neq 0$$

and Lemma 3.2 is proved. □

Let us introduce the operator Π' defined as

$$\Pi'(g)(\xi') := \lim_{x_3\to 0+} r_{\mathbb{R}^3_+} \mathcal{F}^{-1}_{\xi_3\to x_3}[g(\xi', \xi_3)] =$$
$$= \frac{1}{2\pi} \lim_{x_3\to 0+} \int\limits_{-\infty}^{+\infty} g(\xi', \xi_3) e^{-ix_3\xi_3}\, d\xi_3 =$$
$$= \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} g(\xi', \xi_3)\, d\xi_3 \;\text{ for }\; g(\xi', \,\cdot\,) \in L_1(\mathbb{R}).$$

The operator Π' can be extended to the class of functions $g(\xi', \xi_3)$ being rational in ξ_3 with the denominator not vanishing for real non-zero $\xi = (\xi', \xi_3) \in \mathbb{R}^3 \setminus \{0\}$, homogeneous of order $m \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ in ξ and infinitely differentiable with respect to ξ for $\xi' \neq 0$. Then one can show that (see [20, Appendix C])

$$\Pi'(g)(\xi') = \lim_{x_3\to 0+} r_{\mathbb{R}_+} \mathcal{F}^{-1}_{\xi_3\to x_3}[g(\xi', \xi_3)] = -\frac{1}{2\pi} \int\limits_{\gamma^-} g(\xi', \zeta)\, d\zeta,$$

where $r_{\mathbb{R}_+}$ denotes the restriction operator onto $\mathbb{R}_+ = (0, +\infty)$ with respect to x_3, γ^- is a contour in the lower complex half-plane $\operatorname{Im}\zeta < 0$, orientated anticlockwise and enclosing all the poles of the rational function $g(\xi', \cdot)$. It is clear that if $g(\xi', \zeta)$ is holomorphic in ζ in the lower complex half-plane ($\operatorname{Im}\zeta < 0$), then $\Pi'(g)(\xi') = 0$.

Denote by $\mathfrak{D}$ the localized boundary-domain integral operator generated by the left-hand side expressions in LBDIE system (3.1), (3.2),

$$\mathfrak{D} := \begin{bmatrix} r_{\Omega^+}\mathbf{B}\ell_0 & -r_{\Omega^+}V \\ \mathbf{A}^+\ell_0 & -\mathcal{V} \end{bmatrix}.$$

Now we prove the following assertion.

Theorem 3.3. *Let a cut-off function $\chi \in X^\infty_+$ and $r > -\frac{1}{2}$. Then the following operator*

$$\mathfrak{D} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S) \tag{3.30}$$

is invertible.

Proof. We prove the theorem into four steps where we show that

Step 1. The operator $r_{\Omega^+}\mathbf{B}\ell_0 : H^s(\Omega) \to H^s(\Omega)$ for $s \geq 0$ is Fredholm operator with zero index;

Step 2. The operator $\mathfrak{D}$ given as in (3.30) is Fredholm operator;

Step 3. $\operatorname{Ind}\mathfrak{D} = 0$;

Step 4. The operator $\mathfrak{D}$ is invertible.

Step 1. Since (3.4) is a rational function in ξ, we can apply the theory of pseudodifferential operators with symbol satisfying the transmission conditions (see [2], [3], [23], [44], [45]). Now with the help of the local principle (see, e.g., [1], [23, Lemma 23.9]) and the above Lemma 3.1 we deduce that the operator

$$\mathcal{B} := r_{\Omega+}\mathbf{B}\ell_0 : H^s(\Omega) \to H^s(\Omega)$$

is Fredholm operator for all $s \geq 0$.

To show that $\operatorname{Ind}\mathcal{B} = 0$, we use that the operators $\mathcal{B}$ and

$$\mathcal{B}_t = r_{\Omega^+}(\mathbf{b} + t\mathbf{A})\ell_0,$$

where $t \in [0,1]$, are homotopic. Note that $\mathcal{B} = \mathcal{B}_1$. The principal homogeneous symbol of the operator $\mathcal{B}_t$ has the form

$$\mathfrak{S}(\mathcal{B}_t)(y,\xi) = \mathbf{b}(y) + t\mathfrak{S}(\mathbf{A})(y,\xi) = (1-t)\mathbf{b}(y) + t\mathfrak{S}(\mathbf{B})(y,\xi).$$

It is easy to see that the operator $\mathcal{B}_t$ is uniformly strongly elliptic,

$$\operatorname{Re}\mathfrak{S}(\mathcal{B}_t)(y,\xi)\zeta\cdot\zeta = (1-t)\operatorname{Re}\mathbf{b}(y)\zeta\cdot\zeta + t\operatorname{Re}\mathfrak{S}(\mathbf{B})(y,\xi)\zeta\cdot\zeta \geq c|\zeta|^2$$

for all $y \in \overline{\Omega}$, $\xi \neq 0$, $\zeta \in \mathbb{C}^4$ and $t \in [0,1]$, where c is some positive number.

Since $\mathfrak{S}(\mathcal{B}_t)(y,\xi)$ is rational, even, and homogeneous of order zero in ξ, as above we conclude that the operator

$$\mathcal{B}_t : H^s(\Omega) \to H^s(\Omega)$$

is Fredholm operator for all $s \geq 0$ and for all $t \in [0,1]$. Therefore $\operatorname{Ind}\mathcal{B}_t$ is the same for all $t \in [0,1]$. On the other hand, due to the equality $\mathcal{B}_0 = r_{\Omega^+}I$, we get

$$\operatorname{Ind}\mathcal{B} = \operatorname{Ind}\mathcal{B}_1 = \operatorname{Ind}\mathcal{B}_t = \operatorname{Ind}\mathcal{B}_0 = 0.$$

Step 2. To investigate Fredholm properties of the operator $\mathfrak{D}$ we apply the local principle (cf. e.g., [1], [23, §§ 19, 22]). Due to this principle, we have to check that the so-called generalized *Šapiro–Lopatinskiĭ condition* for the operator $\mathfrak{D}$ holds at an arbitrary "frozen" point $\widetilde{y} \in S$. To obtain the explicit form of this condition we proceed as follows. Let $\widetilde{\mathcal{U}}$ be a neighbourhood of a fixed point $\widetilde{y} \in \overline{\Omega}$ and let $\widetilde{\psi}_0, \widetilde{\varphi}_0 \in \mathcal{D}(\widetilde{\mathcal{U}})$ such that

$$\operatorname{supp}\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0 \neq \varnothing, \quad \widetilde{y} \in \operatorname{supp}\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0,$$

and consider the operator $\widetilde{\psi}_0\mathfrak{D}\widetilde{\varphi}_0$. We separate the two possible cases 1) $\widetilde{y} \in \Omega$ and 2) $\widetilde{y} \in S$.

Case 1). If $\widetilde{y} \in \Omega$, then we can choose a neighbourhood $\overline{\widetilde{\mathcal{U}}}$ of the point $\widetilde{y}$ such that $\overline{\widetilde{\mathcal{U}}} \subset \Omega$. Then

$$\widetilde{\psi}_0 \mathfrak{D} \widetilde{\varphi}_0 = \widetilde{\psi}_0 \mathbf{B} \widetilde{\varphi}_0$$

where $\mathbf{B}$ is the operator defined by (3.3). As we have already shown above (see Step 1) this operator is Fredholm operator with zero index.

Case 2). If $\widetilde{y} \in S$, then at this point we have to "froze" the operator $\widetilde{\psi}_0 \mathfrak{D} \widetilde{\varphi}_0$, which means that we can choose a neighbourhood $\overline{\widetilde{\mathcal{U}}}$ of the point $\widetilde{y}$ sufficiently small such that at the local coordinate system with the origin at the point $\widetilde{y}$ and the third axis coinciding with the normal vector at the point $\widetilde{y} \in S$, the following decomposition holds

$$\widetilde{\psi}_0 \mathfrak{D} \widetilde{\varphi}_0 = \widetilde{\psi}_0 \Big(\widehat{\widehat{\mathfrak{D}}} + \widetilde{\mathbf{K}} + \widetilde{\mathbf{T}} \Big) \widetilde{\varphi}_0, \tag{3.31}$$

where $\widetilde{\mathbf{K}}$ is a bounded operator with small norm

$$\widetilde{\mathbf{K}} : H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2) \to H^{r+1}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2),$$

while $\widetilde{\mathbf{T}}$ is a bounded operator

$$\widetilde{\mathbf{T}} : H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2) \to H^{r+2}(\mathbb{R}^3_+) \times H^{r+3/2}(\mathbb{R}^2).$$

The operator $\widehat{\widehat{\mathfrak{D}}}$ is defined in the upper half-space $\mathbb{R}^3_+$ as follows

$$\widehat{\widehat{\mathfrak{D}}} := \begin{bmatrix} r_{\mathbb{R}^3_+} \widehat{\widehat{\mathbf{B}}} \ell_0 & -r_{\mathbb{R}^3_+} \widehat{\widehat{V}} \\ \widehat{\widehat{\mathbf{A}}}^{+} \ell_0 & -\widehat{\widehat{\mathcal{V}}} \end{bmatrix}$$

and possesses the following mapping property

$$\widehat{\widehat{\mathfrak{D}}} : H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2) \to H^{r+1}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2). \tag{3.32}$$

The operators involved in the expression of $\widehat{\widehat{\mathfrak{D}}}$ are defined as follows: for the operator $\widetilde{M}$, $\widehat{\widehat{M}}$ denotes the operator in $\mathbb{R}^n$ $(n = 2, 3)$ constructed by the symbol

$$\widehat{\mathfrak{S}}(\widetilde{M})(\xi) = \mathfrak{S}(\widetilde{M})\big((1 + |\xi'|)\omega, \xi_3\big) \ \text{ if } \ n = 3,$$

and

$$\widehat{\mathfrak{S}}(\widetilde{M})(\xi) = \mathfrak{S}(\widetilde{M})\big((1 + |\xi'|)\omega\big) \ \text{ if } \ n = 2,$$

where $\omega = \frac{\xi'}{|\xi'|}$, $\xi = (\xi', \xi_n)$, $\xi' = (\xi_1, \dots, \xi_{n-1})$.

The generalized Šapiro–Lopatinskiĭ condition is related to the invertibility of the operator (3.32). Indeed, let us write the system corresponding to the operator $\widehat{\widehat{\mathfrak{D}}}$:

$$r_{\mathbb{R}^3_+} \widehat{\widehat{\mathbf{B}}} \ell_0 \widetilde{U} - r_{\mathbb{R}^3_+} \widehat{\widehat{V}} \widetilde{\Psi} = \widetilde{F}_1 \ \text{ in } \ \mathbb{R}^3_+, \tag{3.33}$$

$$\widehat{\widehat{\mathbf{A}}}^{+} \ell_0 \widetilde{U} - \widehat{\widehat{\mathcal{V}}} \widetilde{\Psi} = \widetilde{F}_2 \ \text{ on } \ \mathbb{R}^2, \tag{3.34}$$

where $\widetilde{F}_1 \in H^1(\mathbb{R}^3_+)$, $\widetilde{F}_2 \in H^{1/2}(\mathbb{R}^2)$.

Note that the operator $r_{\mathbb{R}^3_+}\widehat{\widehat{\mathbf{B}}}\ell_0$ is a singular integral operator with even rational elliptic principal homogeneous symbol. Then due to Lemma 3.1 the operator

$$r_{\mathbb{R}^3_+}\widehat{\widehat{\mathbf{B}}}\ell_0 : H^{r+1}(\mathbb{R}^3_+) \to H^{r+1}(\mathbb{R}^3_+)$$

is invertible. Therefore we can define $\widetilde{U}$ from equation (3.33)

$$\ell_0\widetilde{U} = \big[r_{\mathbb{R}^3_+}\widehat{\widehat{\mathbf{B}}}\ell_0\big]^{-1}\widetilde{f} = \\ = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell\widetilde{f})\big)\Big\}, \quad (3.35)$$

where $\widetilde{f} = \widetilde{F}_1 + r_{\mathbb{R}^3_+}\widehat{\widehat{V}}\widetilde{\Psi}$, ℓ is an extension operator from $\mathbb{R}^3_+$ to $\mathbb{R}^3$ preserving the function space, while ℓ_0 is an extension operator $\mathbb{R}^3_+$ to $\mathbb{R}^3_-$ by zero; here $\widehat{\mathfrak{S}}^{\,(\pm)}(M)$ denote the so-called "plus" and "minus" factors in the factorization of the symbol $\widehat{\mathfrak{S}}(M)$ with respect to the variable ξ_3. The operator Π^+ involved in (3.35) is the Cauchy type integral (see (3.11)). Note that the function $\ell_0\widetilde{U}$ in (3.35) does not depend on the extension operator ℓ.

Substituting (3.35) into (3.34) leads to the following pseudodifferential equation with respect to the unknown function $\widetilde{\Psi}$:

$$\widehat{\widehat{\mathbf{A}}}^{+}\mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\widehat{\widehat{V}}\widetilde{\Psi})\big)\Big\} - \widehat{\widehat{\mathcal{V}}}\widetilde{\Psi} = \widetilde{F} \ \text{ on } \mathbb{R}^2, \quad (3.36)$$

where

$$\widetilde{F} = \widetilde{F}_2 - \widehat{\widehat{\mathbf{A}}}^{+}\ell_0\big[r_{\mathbb{R}^3_+}\widehat{\widehat{\mathbf{B}}}\ell_0\big]^{-1}\widetilde{F}_1.$$

It is easy to see that

$$\widetilde{\mathbf{A}}^{+}v(\widetilde{y}') = \Big[\mathcal{F}^{-1}_{\xi\to\widetilde{y}}\big[\mathfrak{S}(\widetilde{\mathbf{A}})(\xi)\mathcal{F}(v)(\xi)\big]\Big]_{\widetilde{y}_3=0+} = \\ = \mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\Big[\Pi'\big[(\mathfrak{S}\widetilde{\mathbf{A}})(\xi)\mathcal{F}(v)(\xi)\big]\Big],$$

and in view of the relation

$$V(\Psi) = -\mathbf{P}(\Psi\otimes\delta)$$

with $\delta = \delta(x_3)$ being the Dirac distribution, we arrive at the equality

$$\widehat{\widehat{\mathbf{A}}}^{+}\mathcal{F}^{-1}_{\xi\to\widetilde{x}}\Big\{\big[\widehat{\mathfrak{S}}^{\,(+)}(\mathbf{B})(\xi)\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\widehat{\widehat{V}}\Psi)\big)(\xi)\Big\}(\widetilde{y}') = \\ = -\mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\Big\{\Pi'\Big[\widehat{\mathfrak{S}}(\widetilde{\mathbf{A}})\big[\widehat{\mathfrak{S}}^{\,(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\widehat{\mathfrak{S}}^{\,(-)}(\widetilde{\mathbf{B}})\big]^{-1}\widehat{\mathfrak{S}}(\widetilde{\mathbf{P}})\big)\Big](\xi')\mathcal{F}_{\widetilde{x}'\to\xi'}\widetilde{\Psi}\Big\}.$$

With the help of these relations equation (3.36) can be rewritten in the following form

$$\mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\big[\widehat{e}(\xi')\mathcal{F}(\widetilde{\psi})(\xi')\big] = \widetilde{F}(\widetilde{y}') \ \text{ on } \mathbb{R}^2, \quad (3.37)$$

where

$$\widehat{e}(\xi') = e\big((1+|\xi'|)\omega\big), \ \ \omega = \frac{\xi'}{|\xi'|},$$

with e being a homogeneous function of order -1 given by the equality

$$e(\xi') = -\Pi'\Big\{\mathfrak{S}(\widetilde{\mathbf{A}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') - \\ -\mathfrak{S}(\widetilde{\mathcal{V}})(\xi') \ \ \forall\, \xi' \neq 0. \tag{3.38}$$

If the function $\det e(\xi')$ is different from zero for all $\xi' \neq 0$, then $\det \widehat{e}(\xi') \neq 0$ for all $\xi' \in \mathbb{R}^2$, and the corresponding pseudodifferential operator

$$\widehat{\mathbf{E}} : H^s(\mathbb{R}^2) \to H^{s+1}(\mathbb{R}^2) \ \text{ for all } \ s \in \mathbb{R}$$

generated by the left-hand side expression in (3.37) is invertible. In particular, it follows that the system of equation (3.33), (3.34) is uniquely solvable with respect to $(\widetilde{U}, \widetilde{\Psi})$ in the space $H^1(\mathbb{R}^3_+) \times H^{-1/2}(\mathbb{R}^2)$ for arbitrary right-hand sides $(\widetilde{F}_1, \widetilde{F}_2) \in H^1(\mathbb{R}^3_+) \times H^{1/2}(\mathbb{R}^2)$. Consequently, the operator $\widehat{\widetilde{\mathfrak{D}}}$ in (3.32) is invertible, which implies that the operator (3.31) possesses a left and right regularizer. In turn, this yields that the operator (3.30) possesses a left and right regularizer as well. Thus the operator (3.30) is Fredholm operator if

$$\det e(\xi') \neq 0 \ \ \forall\, \xi' \neq 0. \tag{3.39}$$

This condition is called the *Šapiro–Lopatinskiĭ condition* (cf. [23, Theorems 12.2 and 23.1 and also formulas (12.27), (12.25)]). Let us show that in our case the Šapiro–Lopatinskiĭ condition holds. To this end, let us note that the principal homogeneous symbols $\mathfrak{S}(\widetilde{\mathbf{A}})$, $\mathfrak{S}(\widetilde{\mathbf{B}})$, $\mathfrak{S}(\widetilde{\mathbf{P}})$, and $\mathfrak{S}(\widetilde{\mathcal{V}})$ of the operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{P}$, and $\mathcal{V}$ in the chosen local coordinate system involved in the formula (3.39) read as:

$$\begin{aligned}
&\mathfrak{S}(\widetilde{\mathbf{A}})(\xi) = |\xi|^{-2}\widetilde{A}(\xi) - \widetilde{\mathbf{b}},\\
&\mathfrak{S}(\widetilde{\mathbf{B}})(\xi) = |\xi|^{-2}\widetilde{A}(\xi),\\
&\mathfrak{S}(\widetilde{\mathbf{P}})(\xi) = -|\xi|^{-2}I,\\
&\mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|}\,I,\\
&\xi = (\xi', \xi_3), \ \ \xi' = (\xi_1, \xi_2),
\end{aligned}$$

where $\widetilde{\mathbf{b}}$ denotes the matrix $\mathbf{b}$ written in the chosen local co-ordinate system. Further, $\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})$ and $\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})$ are the so-called "plus" and "minus" factors in the factorization of the symbol $\mathfrak{S}(\widetilde{\mathbf{B}})$ with respect to the variable ξ_3, i.e.

$$\mathfrak{S}(\widetilde{\mathbf{B}}) = \mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}),$$

where

$$\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi) = \frac{1}{\Theta^{(\pm)}(\xi)}\widetilde{A}^{(\pm)}(\xi)$$

due to (3.4). Rewrite (3.38) in the form

$$e(\xi') = -\Pi'\Big\{\big(\mathfrak{S}(\widetilde{\mathbf{B}}) - \widetilde{\mathbf{b}}\big)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') - \\ - \mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = e_1(\xi') + e_2(\xi') - \mathfrak{S}(\widetilde{\mathcal{V}})(\xi'), \quad (3.40)$$

where

$$e_1(\xi') = -\Pi'\Big\{\mathfrak{S}(\widetilde{\mathbf{B}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi'), \quad (3.41)$$

$$e_2(\xi') = \widetilde{\mathbf{b}}\Pi'\Big\{\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi'), \quad (3.42)$$

$$\mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|}\, I. \quad (3.43)$$

Direct calculations give

$$\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)(\xi') = \\ = \frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{-\infty}^{+\infty}\frac{([\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}}))(\xi',\eta_3)}{\xi_3+it-\eta_3}\,d\eta_3 = \\ = -\frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{-\infty}^{+\infty}\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\eta_3)}{(\xi_3+it-\eta_3)(|\xi'|^2+\eta_3^2)}\,d\eta_3 = \\ = \frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{\gamma^-}\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\tau)}{(\xi_3+it-\tau)(|\xi'|^2+\tau^2)}\,d\tau = \\ = \frac{i}{2\pi}\lim_{t\to 0+}\frac{2\pi i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{(\xi_3+it+i|\xi'|)2(-i|\xi'|)} = \\ = -\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|\Theta^{(+)}(\xi',\xi_3)}\,. \quad (3.44)$$

Now from (3.41) with the help of (3.44) we derive

$$e_1(\xi') = \\ = -\Pi'\Big\{\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') = \\ = -\Pi'\Big\{\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') = \\ = \Pi'\Big\{\frac{\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})}{\Theta^{(+)}}\Big\}(\xi')\Big(\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|}\Big) = \\ = -\frac{1}{2\pi}\int\limits_{\gamma^-}\frac{\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',\tau)}{\tau+i|\xi'|}\,d\tau\,\Big(\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|}\Big) = \\ = -i\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big(\xi',-i|\xi'|\big)\,\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|} = \frac{1}{2|\xi'|}\,I. \quad (3.45)$$

Quite similarly, from (3.42) with the help of (3.44) we get

$$e_2(\xi') = \widetilde{\mathbf{b}}\Pi'\Big\{\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') =$$

$$= -\widetilde{\mathbf{b}}\Pi'\Big\{\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}}{\Theta^{(+)}}\Big\}(\xi')\Big(\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)}{2|\xi'|}\Big) =$$

$$= -\frac{i\widetilde{\mathbf{b}}}{2|\xi'|}\bigg(-\frac{1}{2\pi}\int\limits_{\gamma^-}\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\tau)}{\tau+i|\xi'|}\,d\tau\bigg)\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\big(\xi',-i|\xi'|\big) =$$

$$= \frac{i\widetilde{\mathbf{b}}}{4\pi|\xi'|}\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}d\tau(-2i|\xi'|)\,\big[\widetilde{A}^{(-)}\big(\xi',-i|\xi'|\big)\big]^{-1} =$$

$$= i\widetilde{\mathbf{b}}\bigg\{\frac{1}{2\pi i}\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}d\tau\bigg\}\big[\widetilde{A}^{(-)}\big(\xi',-i|\xi'|\big)\big]^{-1}.$$

Therefore, due to Lemma 3.2, we have

$$e_2(\xi') = i\widetilde{\mathbf{b}}\,\frac{[c_{ij}^{(+)}(\xi')]_{4\times4}}{a^{(+)}(\xi')}\,\big[\widetilde{A}^{(-)}(\xi',-i|\xi'|)\big]^{-1}. \tag{3.46}$$

In view of (3.40), (3.43), (3.45), and (3.46) we finally obtain

$$e(\xi') = e_2(\xi') = i\widetilde{\mathbf{b}}\,\frac{[c_{ij}^{(+)}(\xi')]_{4\times4}}{a^{(+)}(\xi')}\,\big[\widetilde{A}^{(-)}\big(\xi',-i|\xi'|\big)\big]^{-1},$$

where

$$\det\widetilde{\mathbf{b}} \neq 0, \quad \det[c_{ij}^{(+)}]_{4\times4} \neq 0$$

(see Lemma 3.2), and $\det\widetilde{A}^{(-)}(\xi',-i|\xi'|) \neq 0$ for all $\xi' \neq 0$.

Then it is clear that for all $\xi' \neq 0$ we have

$$\det e(\xi') = \frac{1}{(a^{(+)}(\xi'))^4}\,\det\widetilde{\mathbf{b}}\det[c_{ij}^{(+)}]_{4\times4}\det\big[\widetilde{A}^-\big(\xi',-i|\xi'|\big)\big]^{-1} \neq 0.$$

Thus, we have obtained that for the operator $\mathfrak{D}$ the Šapiro–Lopatinskiĭ condition holds. Therefore, the operator

$$\mathfrak{D} : H^{r+1}(\Omega)\times H^{r-1/2}(S) \to H^{r+1}(\Omega)\times H^{r+1/2}(S)$$

is Fredholm operator for $r > -\frac{1}{2}$.

Step 3. Here we will show that $\operatorname{Ind}\mathfrak{D} = 0$. To this end, for $t\in[0,1]$ let us consider the operator

$$\mathfrak{D}_t := \begin{bmatrix} r_{\Omega^+}\mathbf{B}_t\ell_0 & -r_{\Omega^+}V \\ t\mathbf{A}^+\ell_0 & -\mathcal{V} \end{bmatrix}$$

with $\mathbf{B}_t = \mathbf{b}+t\mathbf{A}$ and establish that it is homotopic to the operator $\mathfrak{D} = \mathfrak{D}_1$. We have to check that for the operator $\mathfrak{D}_t$ the Šapiro–Lopatinskiĭ condition is satisfied for all $t \in [0,1]$. Indeed, in this case the Šapiro–Lopatinskiĭ condition reads as (cf. (3.39))

$$\det e_t(\xi') \neq 0 \ \ \forall\, \xi' \neq 0,$$

where

$$e_t(\xi') = -\Pi'\Big\{\big(\mathfrak{S}(\widetilde{\mathbf{B}}_t) - \widetilde{\mathbf{b}}\big)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') - \\ -\mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = e_t^{(1)}(\xi') + e_t^{(2)}(\xi') - \mathfrak{S}(\widetilde{\mathcal{V}})(\xi'), \tag{3.47}$$

$$e_t^{(1)}(\xi') = -\Pi'\Big\{\mathfrak{S}(\widetilde{\mathbf{B}}_t)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') = \\ = \frac{1}{2|\xi'|}\, I, \tag{3.48}$$

$$e_t^{(2)}(\xi') = \widetilde{\mathbf{b}}\Pi'\Big\{\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi'), \\ \mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|}\, I. \tag{3.49}$$

By direct calculations we get

$$e_t^{(2)}(\xi') = \widetilde{\mathbf{b}}\Pi'\Big\{\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)\Big\}(\xi') = \\ = -\widetilde{\mathbf{b}}\Pi'\Big\{\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)]^{-1}}{\Theta^{(+)}}\Big\}(\xi')\Big(\frac{i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',-i|\xi'|)}{2|\xi'|}\Big) = \\ = -\frac{i\widetilde{\mathbf{b}}}{2|\xi'|}\bigg(-\frac{1}{2\pi}\int\limits_{\gamma^-}\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',\tau)}{\tau+i|\xi'|}\,d\tau\bigg)\big[\mathfrak{S}^{-}(\widetilde{\mathbf{B}}_t)\big]^{-1}(\xi',-i|\xi'|) = \\ = \frac{i\widetilde{\mathbf{b}}}{4\pi|\xi'|}\int\limits_{\gamma^-}\big[\widetilde{A}_t^{(+)}(\xi',\tau)\big]^{-1}d\tau\,(-2i|\xi'|)\big[\widetilde{A}_t^{(-)}\big(\xi',-i|\xi'|\big)\big]^{-1} = \\ = i\widetilde{\mathbf{b}}\Big\{\frac{1}{2\pi i}\int\limits_{\gamma^-}\big[\widetilde{A}_t^{(+)}(\xi',\tau)\big]^{-1}d\tau\Big\}\big[\widetilde{A}_t^{(-)}\big(\xi',-i|\xi'|\big)\big]^{-1}, \tag{3.50}$$

where $\widetilde{A}_t(\xi) = (1-t)|\xi|^2\widetilde{\mathbf{b}} + t\widetilde{A}(\xi)$ and $\widetilde{A}_t(\xi',\xi_3) = \widetilde{A}_t^{(-)}(\xi',\xi_3)\widetilde{A}_t^{(+)}(\xi',\xi_3)$, $\widetilde{A}_t^{(\pm)}(\xi',\xi_3)$ are the "plus" and "minus" polynomial matrix factors in ξ_3 of the polynomial symbol matrix $\widetilde{A}_t(\xi',\xi_3)$.

Due to Lemma 3.2 and the equality (3.50) we have

$$e_t^{(2)}(\xi') = i\widetilde{\mathbf{b}}\,\frac{[c_{ij,t}^{(+)}(\xi')]_{4\times 4}}{a_t^{(+)}(\xi')}\,\big[\widetilde{A}_t^{(-)}\big(\xi',-i|\xi'|\big)\big]^{-1}, \tag{3.51}$$

where $c_{ij,t}^{(+)}$, $i,j=\overline{1,4}$, are the main coefficients of the co-factors $p_{ij,t}^{(+)}(\xi',\tau)$ of the polynomial matrix $\widetilde{A}_t^{(+)}(\xi',\tau)$ and $a^{(+)}$ is the coefficient at τ^4 in the determinant $\det \widetilde{A}_t^{(+)}(\xi',\tau)$.

In view of (3.47), (3.48), (3.49), and (3.51), we finally obtain

$$e_t(\xi') = e_t^{(2)}(\xi') = i\widetilde{\mathbf{b}}\,\frac{[c_{ij,t}^{(+)}(\xi')]_{4\times 4}}{a_t^{(+)}(\xi')}\,\big[\widetilde{A}_t^{(-)}(\xi',-i|\xi'|)\big]^{-1},$$

where $\det\widetilde{\mathbf{b}} \neq 0$, $\det[c_{ij,t}^{(+)}]_{4\times 4} \neq 0$ (see Lemma 3.2), and $\det\widetilde{A}_t^{(-)}(\xi',-i|\xi'|) \neq 0$ for all $\xi' \neq 0$ and $t \in [0,1]$.

Then it follows that

$$\det e_t(\xi') = \frac{1}{[a_t^{(+)}(\xi')]^4}\,\det\mathbf{b}\,\det\big[c_{ij,t}^{(+)}(\xi')\big]_{4\times 4}\det\big[\widetilde{A}_t^{(-)}(\xi',-i|\xi'|)\big]^{-1} \neq 0$$
$$\text{for all } \ \xi' \neq 0 \ \text{ and for all } \ t \in [0,1],$$

which implies that for the operator $\mathfrak{D}_t$ the Šapiro–Lopatinskiĭ condition is satisfied.

Therefore the operator

$$\mathfrak{D}_t : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is Fredholm operator for all $r > -\frac{1}{2}$ and $t \in [0,1]$. Consequently,

$$\operatorname{Ind}\mathfrak{D} = \operatorname{Ind}\mathfrak{D}_1 = \operatorname{Ind}\mathfrak{D}_t = \operatorname{Ind}\mathfrak{D}_0 = 0.$$

Step 4. Since the operator $\mathfrak{D}$ is Fredholm operator with zero index, its injectivity implies the invertibility. Thus it remains to prove that the null space of the operator $\mathfrak{D}$ is trivial for $r > -\frac{1}{2}$. Assume that $\mathcal{U} = (U,\Psi)^\top \in H^{r+1}(\Omega) \times H^{r-1/2}(S)$ is a solution to the homogeneous equation

$$\mathfrak{D}\mathcal{U} = 0. \tag{3.52}$$

The operator

$$\mathfrak{D} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is Fredholm operator with index zero for all $r > -\frac{1}{2}$. It is well known that then there exists a left regularizer $\mathfrak{L}$ of the operator $\mathfrak{D}$,

$$\mathfrak{L} : H^{r+1}(\Omega) \times H^{r+1/2}(S) \to H^{r+1}(\Omega) \times H^{r-1/2}(S),$$

such that

$$\mathfrak{L}\mathfrak{D} = I + \mathfrak{T},$$

where $\mathfrak{T}$ is the operator of order -1 (cf. [23, Proofs of Theorems 22.1 and 23.1]), i.e.,

$$\mathfrak{T} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+2}(\Omega) \times H^{r+1/2}(S). \tag{3.53}$$

Therefore, from (3.52) we have

$$\mathfrak{L}\mathfrak{D}\mathcal{U} = U + \mathfrak{T}\mathcal{U} = 0. \tag{3.54}$$

In view of (3.53) we see that

$$\mathfrak{T}\mathcal{U} \in H^{r+2}(\Omega) \times H^{r+1/2}(S).$$

Consequently, in view of (3.54),

$$\mathcal{U} = (U, \Psi)^\top \in H^{r+2}(\Omega) \times H^{r+1/2}(S). \tag{3.55}$$

If $r \geq 0$, this implies $U \in H^{1,0}(\Omega, A)$. If $-\frac{1}{2} < r < 0$, we iterate the above reasoning for U satisfying (3.55) to obtain

$$\mathcal{U} = (U, \Psi)^\top \in H^{r+3}(\Omega) \times H^{r+3/2}(S)$$

which again implies $U \in H^{1,0}(\Omega, A)$. Then we can apply the equivalence Theorem 2.4 to conclude that a solution $\mathcal{U} = (U, \Psi)^\top$ to the homogeneous equation (3.52) is zero vector, i.e.,

$$U = 0 \text{ in } \Omega, \quad \Psi = 0 \text{ on } S.$$

Thus, $\operatorname{Ker} \mathfrak{D} = \{0\}$ in the class $H^{r+1}(\Omega) \times H^{r-1/2}(S)$ and therefore the operator

$$\mathfrak{D} \ : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is invertible for all $r > -\frac{1}{2}$. □

For localizing function χ of finite smoothness we have the following result.

Corollary 3.4. *Let a cut-off function $\chi \in X^3_+$. Then the operator*

$$\mathfrak{D} : H^1(\Omega) \times H^{-1/2}(S) \to H^1(\Omega) \times H^{1/2}(S)$$

is invertible.

Proof. We have to use mapping properties of the localized potentials with a localizing cut-off function of finite smoothness (see Appendix B) and repeat word for word the arguments of the above proof of Theorem 3.3 for $r = 0$. □

From Corollaries 2.3, 3.4, and Lemma 2.2 the following result follows directly.

Corollary 3.5. *Let a cut-off function $\chi \in X^3_+$. Then the operator*

$$\mathfrak{D} : H^{1,0}(\Omega, A) \times H^{-1/2}(S) \to H^{1,0}(\Omega, \Delta) \times H^{1/2}(S)$$

is invertible.

Appendix A: Classes of Cut-off Functions

Here we present the classes of localizing functions used in the main text (for details see the reference [14]).

Definition A.1. We say $\chi \in X^k$ for integer $k \geq 0$, if $\chi(x) = \breve{\chi}(|x|)$, $\breve{\chi} \in W^k_1(0, \infty)$ and $\varrho\breve{\chi}(\varrho) \in L_1(0, \infty)$. We say $\chi \in X^k_+$ for integer $k \geq 1$, if

$\chi \in X^k$, $\chi(0) = 1$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{\widehat{\chi}_s(\omega)}{\omega} > 0 & \text{for } \omega \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\int_0^\infty \varrho \breve{\chi}(\varrho)\, d\varrho & \text{for } \omega = 0, \end{cases}$$

and $\widehat{\chi}_s(\omega)$ denotes the sine-transform of the function $\breve{\chi}$

$$\widehat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}(\varrho) \sin(\varrho\omega)\, d\varrho.$$

Evidently, we have the following imbeddings: $X^{k_1} \subset X^{k_2}$ and $X_+^{k_1} \subset X_+^{k_2}$ for $k_1 > k_2$. The class X_+^k is defined in terms of the sine-transform. The following lemma provides an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class (for details see [14]).

Lemma A.2. *Let $k \geq 1$. If $\chi \in X^k$, $\breve{\chi}(0) = 1$, $\breve{\chi}(\varrho) \geq 0$ for all $\varrho \in (0, \infty)$, and $\breve{\chi}$ is a non-increasing function on $[0, +\infty)$, then $\chi \in X_+^k$.*

The following examples for χ are presented in [14],

$$\chi_{1k}(x) = \begin{cases} \left[1 - \dfrac{|x|}{\varepsilon}\right]^k & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases}$$

$$\chi_2(x) = \begin{cases} \exp\left[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\right] & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon. \end{cases}$$

One can observe that $\chi_{1k} \in X_+^k$, while $\chi_2 \in X_+^\infty$ due to Lemma A.2.

Appendix B: Properties of Localized Potentials

Here we collect some theorems describing mapping properties of the localized layered and volume potentials defined by the relations (2.23)–(2.24). The proofs can be found in [14] (see also [25], Chapter 8 and the references therein).

Let us introduce the boundary operators generated by the localized layer potentials associated with the localized parametrix $P(x - y) \equiv P_\chi(x - y)$

$$\mathcal{V}g(y) := -\int_S P(x - y) g(x)\, dS_x, \quad y \in S, \tag{B.1}$$

$$\mathcal{W}g(y) := -\int_S \left[\widetilde{T}(x, \partial_x) P(x - y)\right]^\top g(x)\, dS_x, \quad y \in S, \tag{B.2}$$

$$\mathcal{W}'g(y) := -\int\limits_S \big[\mathcal{T}(y,\partial_y)P(x-y)\big]g(x)\,dS_x, \;\; y\in S,$$
$$\mathcal{L}^{\pm}g(y) := \big[\mathcal{T}(y,\partial_y)Wg(y)\big]^{\pm}, \;\; y\in S.$$

Theorem B.1. *The following operators are continuous*

$$\begin{aligned}\mathcal{P} &: \widetilde{H}^s(\Omega)\to H^{s+2,s}(\Omega;\Delta), \;\; -\frac{1}{2}<s<\frac{1}{2}\,, \;\; \chi\in X^1,\\ &: H^s(\Omega)\to H^{s+2,s}(\Omega;\Delta), \;\; -\frac{1}{2}<s<\frac{1}{2}\,, \;\; \chi\in X^1,\\ &: H^s(\Omega)\to H^{\frac52-\varepsilon,\frac12-\varepsilon}(\Omega;\Delta), \;\; \frac{1}{2}\le s<\frac{3}{2}\,, \;\; \forall\,\varepsilon\in(0,1), \;\; \chi\in X^2,\end{aligned}$$

where $\mathcal{P}$ is the volume localized potential defined in (2.24) *and Δ is the Laplace operator.*

Theorem B.2. *The following localized single and double layer operators are continuous*

$$\begin{aligned}V &: H^{s-\frac32}(S)\to H^s(\mathbb{R}^3), \;\; s<\frac{3}{2}\,, \;\; \text{if } \chi\in X^1,\\ &: H^{s-\frac32}(S)\to H^{s,s-1}(\Omega^{\pm};\Delta), \;\; \frac12<s<\frac32\,, \;\; \text{if } \chi\in X^2,\\ W &: H^{s-\frac12}(S)\to H^s(\Omega^{\pm}), \;\; s<\frac32\,, \;\; \text{if } \chi\in X^2,\\ &: H^{s-\frac12}(S)\to H^{s,s-1}(\Omega^{\pm};\Delta), \;\; \frac12<s<\frac32\,, \;\; \text{if } \chi\in X^3.\end{aligned}$$

Theorem B.3. *If $\chi\in X^k$ has a compact support and $-\frac12\le s\le\frac12$, then the following localized operators are continuous:*

$$\begin{aligned}V &: H^s(S)\to H^{s+\frac32}(\Omega^{\pm}) \;\; \text{for } k=2,\\ W &: H^{s+1}(S)\to H^{s+\frac32}(\Omega^{\pm}) \;\; \text{for } k=3.\end{aligned}$$

Theorem B.4. *Let $\psi\in H^{-\frac12}(S)$ and $\varphi\in H^{\frac12}(S)$. Then the following jump relations hold on S:*

$$\begin{aligned}V^+\psi = V^-\psi &= \mathcal{V}\psi, \;\; \chi\in X^1,\\ W^{\pm}\varphi &= \mp\mathbf{d}\varphi+\mathcal{W}\varphi, \;\; \chi\in X^2,\\ \mathcal{T}^{\pm}V\psi &= \pm\mathbf{d}\psi+\mathcal{W}'\psi, \;\; \chi\in X^2,\end{aligned}$$

where

$$\mathbf{d}(y) := \frac12\begin{bmatrix}[c_{ijlk}(y)n_in_l]_{3\times3} & [e_{lij}(y)n_in_l]_{3\times1}\\ [-e_{ikl}(y)n_in_l]_{1\times3} & \varepsilon_{il}(y)n_in_l\end{bmatrix}_{4\times4}, \;\; y\in S, \tag{B.3}$$

and $\mathbf{d}(y)$ is strongly elliptic due to (2.3).

Theorem B.5. *Let $-\frac{1}{2} \le s \le \frac{1}{2}$. The following operators*

$$\mathcal{V} : H^s(S) \to H^{s+1}(S), \quad \chi \in X^2,$$
$$\mathcal{W} : H^{s+1}(S) \to H^{s+1}(S), \quad \chi \in X^3,$$
$$\mathcal{W}' : H^s(S) \to H^s(S), \quad \chi \in X^3,$$
$$\mathcal{L}^{\pm} : H^{s+1}(S) \to H^s(S), \quad \chi \in X^3,$$

are continuous.

References

1. M. S. Agranovic, Elliptic singular integro-differential operators. (Russian) *Uspehi Mat. Nauk* **20** (1965), No. 5 (125), 3–120.
2. L. Boutet de Monvel, Boundary problems for pseudo-differential operators. *Acta Math.* **126** (1971), No. 1-2, 11–51.
3. A. V. Brenner and E. M. Shargorodsky, Boundary value problems for elliptic pseudodifferential operators. Translated from the Russian by Brenner. *Encyclopaedia Math. Sci.*, 79, *Partial differential equations, IX*, 145–215, *Springer, Berlin*, 1997.
4. T. Buchukuri and T. Gegelia, Some dynamic problems of the theory of electroelasticity. *Mem. Differential Equations Math. Phys.* **10** (1997), 1–53.
5. T. Buchukuri and O. Chkadua, Boundary problems of thermopiezoelectricity in domains with cuspidal edges. *Georgian Math. J.* **7** (2000), No. 3, 441–460.
6. T. Buchukuri, O. Chkadua, D. Natroshvili, and A.-M. Sandig, Solvability and regularity results to boundary-transmission problems for metallic and piezoelectric elastic materials. *Math. Nachr.* **282** (2009), No. 8, 1079–1110.
7. T. Buchukuri, O. Chkadua, D. Natroshvili, and A.-M. Sandig, Interaction problems of metallic and piezoelectric materials with regard to thermal stresses. *Mem. Differential Equations Math. Phys.* **45** (2008), 7–74.
8. T. Buchukuri, O. Chkadua, and D. Natroshvili, Mixed boundary value problems of thermopiezoelectricity for solids with interior cracks. *Integral Equations Operator Theory* **64** (2009), No. 4, 495–537.
9. T. Buchukuri, O. Chkadua, R. Duduchava, and D. Natroshvili, Interface crack problems for metallic-piezoelectric composite structures. *Mem. Differential Equations Math. Phys.* **55** (2012), 1–150.
10. O. Chkadua and R. Duduchava, Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotic. *Math. Nachr.* **222** (2001), 79–139.
11. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. I. Equivalence and invertibility. *J. Integral Equations Appl.* **21** (2009), No. 4, 499–543.
12. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. II. Solution regularity and asymptotics. *J. Integral Equations Appl.* **22** (2010), No. 1, 19–37.
13. O. Chkadua, S. Mikhailov, and D. Natroshvili, About analysis of some localized boundary-domain integral equations for a variable-coefficient BVP. In: *Advances in Boundary Integral Methods. Proceedings of the 6th UK Conference on Boundary Integral Methods* (Edited by J. Trevelyan), 291–302. *Durham University Publ., UK,* 2007.
14. O. Chkadua, S. E. Mikhailov, and D. Natroshvilli, Analysis of some localized boundary-domain integral equations. *J. Integral Equations Appl.* **21** (2009), No. 3, 405–445.
15. O. Chkadua, S. Mikhailov, and D. Natroshvili, Analysis of some boundary-domain integral equations for variable-coefficient problems with cracks. In: *Advances*

in Boundary Integral Methods. Proceedings of the 7th UK Conference on Boundary Integral Methods, 37–51. *Nottingham University Publ., UK,* 2009.

16. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Localized direct segregated boundary-domain integral equations for variable coefficient transmission problems with interface crack. *Mem. Differential Equations Math. Phys.* **52** (2011), 17–64.
17. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Analysis of segregated boundary-domain integral equations for variable-coefficient problems with cracks. *Numer. Methods Partial Differential Equations* **27** (2011), No. 1, 121–140.
18. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Analysis of some localized boundary-domain integral equations for transmission problems with variable coefficients. *Integral methods in science and engineering*, 91–108, *Birkhäuser/Springer, New York*, 2011.
19. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Analysis of direct segregated boundary-domain integral equations for variable-coefficient mixed BVPs in exterior domains. *Anal. Appl. (Singap.)* **11** (2013), No. 4, 1350006, 33 pp.
20. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Localized boundary-domain singular integral equations based on harmonic parametrix for divergence-form elliptic PDEs with variable matrix coefficients. *Integral Equations Operator Theory* **76** (2013), No. 4, 509–547.
21. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Localized boundary-domain integral equations approach for Dirichlet problem for self-adjoint second order strongly elliptic systems with variable coefficients. (to appear)
22. L. Ephremidze and I. M. Spitkovsky, A remark on a polynomial matrix factorization theorem. *Georgian Math. J.* **19** (2012), No. 3, 489–495.
23. G. I. Eskin, Boundary value problems for elliptic pseudodifferential equations. Translated from the Russian by S. Smith. Translations of *Mathematical Monographs*, 52. *American Mathematical Society, Providence, R.I.*, 1981.
24. I. Gohberg, M. A. Kaashoek, and I. M. Spitkovsky, An overview of matrix factorization theory and operator applications. *Factorization and integrable systems (Faro, 2000)*, 1–102, *Oper. Theory Adv. Appl.*, 141, *Birkhäuser, Basel*, 2003.
25. G. C. Hsiao and W. L. Wendland, Boundary integral equations. *Applied Mathematical Sciences*, 164. *Springer-Verlag, Berlin*, 2008.
26. S. Lang, Guide to the Literature of Piezoelectricity and Pyroelectricity. 20. *Ferroelectrics* **297** (2003), 107–253.
27. S. Lang, Guide to the Literature of Piezoelectricity and Pyroelectricity. 22. *Ferroelectrics* **308** (2004), 193–304.
28. S. Lang, Guide to the Literature of Piezoelectricity and Pyroelectricity. 23. *Ferroelectrics* **321** (2005), 91–204.
29. S. Lang, Guide to the Literature of Piezoelectricity and Pyroelectricity. 24. *Ferroelectrics* **322** (2005), 115–210.
30. S. Lang, Guide to the Literature of Piezoelectricity and Pyroelectricity. 25. *Ferroelectrics* **330** (2006), 103–182.
31. S. Lang, Guide to the Literature of Piezoelectricity and Pyroelectricity. 26. *Ferroelectrics* **332** (2006), 227–321.
32. S. Lang, Guide to the Literature of Piezoelectricity and Pyroelectricity. 27. *Ferroelectrics* **350** (2007), 130–216.
33. S. Lang, Guide to the Literature of Piezoelectricity and Pyroelectricity. 28. *Ferroelectrics* **361** (2007), 124–239.
34. S. Lang, Guide to the Literature of Piezoelectricity and Pyroelectricity. 29. *Ferroelectrics* **366** (2008), 122–237.
35. J.-L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications. Vol. II. Translated from the French by P. Kenneth. *Die Grundlehren der mathematischen Wissenschaften*, Band 182. *Springer-Verlag, New York–Heidelberg*, 1972.

36. W. McLean, Strongly elliptic systems and boundary integral equations. *Cambridge University Press, Cambridge*, 2000.
37. S. E. Mikhailov, Localized direct boundary-domain integro-differential formulations for scalar nonlinear boundary-value problems with variable coefficients. *J. Engrg. Math.* **51** (2005), No. 3, 283–302.
38. S. E. Mikhailov, Analysis of united boundary-domain integro-differential and integral equations for a mixed BVP with variable coefficient. *Math. Methods Appl. Sci.* **29** (2006), No. 6, 715–739.
39. S. E. Mikhailov, Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. *J. Math. Anal. Appl.* **378** (2011), No. 1, 324–342.
40. *S. G. Mikhlin and S. Prössdorf*, Singular integral operators. Translated from the German by Albrecht Böttcher and Reinhard Lehmann. *Springer-Verlag, Berlin*, 1986.
41. D. Natroshvili, T. Buchukuri, and O. Chkadua, Mathematical modelling and analysis of interaction problems for piezoelastic composites. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)* **30** (2006), 159–190.
42. W. Nowacki, Efecty electromagnetyczne w stalych cialach odksztalcalnych. *Warszawa, Panstwowe Widawnictwo Naukowe*, 1983.
43. A. Pomp, The boundary-domain integral method for elliptic systems. With an application to shells. *Lecture Notes in Mathematics*, 1683. *Springer-Verlag, Berlin*, 1998.
44. S. Rempel and B.-W. Schulze, Index theory of elliptic boundary problems. *Akademie-Verlag, Berlin*, 1982.
45. E. Shargorodsky, An L_p-analogue of the Vishik–Eskin theory. *Mem. Differential Equations Math. Phys.* **2** (1994), 41–146.
46. J. Sladek, V. Sladek, and S. N. Atluri, Local boundary integral equation (LBIE) method for solving problems of elasticity with nonhomogeneous material properties. *Comput. Mech.* **24** (2000), No.6, 456–462.
47. A. E. Taigbenu, The Green element method. *Springer*, 1999.
48. T. Zhu, J.-D. Zhang, and S. N. Atluri, A local boundary integral equation (LBIE) method in computational mechanics, and a meshless discretization approach. *Comput. Mech.* **21** (1998), No. 3, 223–235.
49. T. Zhu, J. Zhang, and S. N. Atluri, A meshless numerical method based on the local boundary integral equation (LBIE) to solve linear and nonlinear boundary value problems. *Eng. Anal. Bound. Elem.* **23** (1999), No.5-6, 375–389.
50. W. Voigt, Lehrbuch der Kristallphysik. *B. G. Teubner, Leipzig*, 1911.

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L^p-DISSIPATIVITY OF THE LAMÉ OPERATOR

In Memory of Victor Kupradze

Abstract. We study conditions for the L^p-dissipativity of the classical linear elasticity operator. In the two-dimensional case we show that L^p-dissipativity is equivalent to the inequality

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,.$$

Previously [2] this result has been obtained as a consequence of general criteria for elliptic systems, but here we give a direct and simpler proof. We show that this inequality is necessary for the L^p-dissipativity of the three-dimensional elasticity operator with variable Poisson ratio. We give also a more strict sufficient condition for the L^p-dissipativity of this operator. Finally we find a criterion for the n-dimensional Lamé operator to be L^p-negative with respect to the weight $|x|^{-\alpha}$ in the class of rotationally invariant vector functions.

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Key words and phrases. Elasticity system, L^p-dissipativity.

რეზიუმე. შევისწავლით დრეკადობის კლასიკური თეორიის წრფივი ოპერატორის L^p-დისიპატიურობის პირობებს. ორგანზომილებიან შემთხვევაში ვაჩვენებთ, რომ L^p-დისიპატიურობის პირობა ეკვივალენტურია

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}$$

უტოლობის. ეს შედეგი [2] ნაშრომში მიღებული იყო ზოგადი ელიფსური შემთხვევის კრიტერიუმიდან გამომდინარე. ჩვენ აქ მოვიყვანთ პირდაპირ და უფრო მარტივ დამტკიცებას. ასევე ვაჩვენებთ, რომ დრეკადობის სამგანზომილებიანი ოპერატორის L^p-დისიპატიურობისთვის ეს უტოლობა არის აუცილებელი პირობა ცვლადი პუასონის კოეფიციენტების შემთხვევაში. ბოლოს, ჩამოვაყალიბებთ კრიტერიუმს, რომელიც უზრუნველყოფს n-განზომილებიანი ლამეს ოპერატორის L^p-უარყოფითობას ბრუნვის მიმართ ინვარიანტულ ვექტორ ფუნქციათა კლასში $|x|^{-\alpha}$ წონით.

1. Introduction

It is well known that Victor Kupradze has made seminal contributions to the theory of elasticity, in particular, to the study of BVPs of statics and steady state oscillations, as well as initial BVPs of general dynamics.

His monographs in the field of elasticity testify the great work he made (see, for instance, [6–9]). In particular, his book *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity* [10–12]) became a must for every mathematician working in this field.

The present paper concerning elasticity theory is dedicated to him.

Let us consider the classical operator of linear elasticity

$$Eu = \Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u, \tag{1}$$

where ν is the Poisson ratio. Throughout this paper, we assume that either $\nu > 1$ or $\nu < 1/2$. It is well known that E is strongly elliptic if and only if this condition is satisfied (see, for instance, Gurtin [5, p. 86]).

Let $\mathscr{L}$ be the bilinear form associated with operator (1), i.e.

$$\mathscr{L}(u,v) = -\int_\Omega (\langle \nabla u, \nabla v\rangle + (1-2\nu)^{-1} \operatorname{div} u \operatorname{div} v)\, dx, \tag{2}$$

where $\langle \cdot\,,\,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$. Here Ω is a domain of $\mathbb{R}^n$.

Following [1], we say that the form $\mathscr{L}$ is L^p-dissipative in Ω if

$$-\int_\Omega \Big(\big\langle \nabla u, \nabla(|u|^{p-2}u)\big\rangle + (1-2\nu)^{-1} \operatorname{div} u \operatorname{div}(|u|^{p-2}u)\Big)\, dx \leqslant 0 \quad \text{if } p \geqslant 2, \tag{3}$$

$$-\int_\Omega \Big(\big\langle \nabla u, \nabla(|u|^{p'-2}u)\big\rangle + (1-2\nu)^{-1} \operatorname{div} u \operatorname{div}(|u|^{p'-2}u)\Big)\, dx \leqslant 0 \quad \text{if } p < 2, \tag{4}$$

for all $u \in (C_0^1(\Omega))^2$ $(p' = p/(p-1))$. We use here that $|u|^{q-2}u \in C_0^1(\Omega)$ for $q \geqslant 2$ and $u \in C_0^1(\Omega)$.

In [1, 2] necessary and sufficient conditions for the L^p-dissipativity of the forms related to partial differential operators have been obtained. In particular, for the planar elasticity it was proved in [2] that the form $\mathscr{L}$ is L^p-dissipative if and only if

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,. \tag{5}$$

Let us now suppose that Ω is a sufficiently smooth bounded domain and consider the operator (1) defined on $D(E) = (W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega))^n$. As usual $W^{l,p}(\Omega)$ denotes the Sobolev space of functions which distributional derivatives of order l are in $L^p(\Omega)$. We also use the notation $\mathring{W}^{1,p}(\Omega)$ for the completion of $C_0^\infty(\Omega)$ in the Sobolev $W^{1,p}(\Omega)$ norm. The operator E is said to be L^p-dissipative ($1 < p < \infty$) in the domain $\Omega \subset \mathbb{R}^n$ if

$$\int_\Omega \big(\Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u\big)|u|^{p-2}u\, dx \leqslant 0 \tag{6}$$

for any real vector-valued function $u \in D(E)$. Here and in the sequel the integrand is extended by zero on the set where u vanishes.

The equivalence between the L^p-dissipativity of the form and the dissipativity of the operator was discussed in [1, Section 5, p. 1086–1093]. It turns out that, if $n = 2$ and a certain smoothness assumption on $\Omega \subset \mathbb{R}^2$ is fulfilled, the operator of planar elasticity is L^p-dissipative (i.e. (6) holds for any $u \in D(E)$) if and only if condition (5) is satisfied.

In [2] these facts have been established as a consequence of results concerning general systems of partial differential equations, but in the present paper we give a direct and simpler proof just for the Lamé system. The result is followed by two Corollaries (obtained for the first time in [2]) concerning the comparison between the Lamé operator and the Laplacian from the point of view of the L^p-dissipativity.

In Section 3 we show that condition (5) is necessary for the L^p-dissipativity of operator (1), even when the Poisson ratio is not constant. For the time being it is not known if condition (5) is also sufficient for the L^p-dissipativity of elasticity operator for $n > 2$, in particular, for $n = 3$. Nevertheless in the same section we give a more strict explicit condition which is sufficient for the L^p-dissipativity of (1).

In Section 4 we give necessary and sufficient conditions for a weighted L^p-negativity of the Dirichlet–Lamé operator, i.e. for the validity of the inequality

$$\int_\Omega \big(\Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u\big)|u|^{p-2}u\, \frac{dx}{|x|^\alpha} \leqslant 0 \tag{7}$$

under the condition that the vector u is rotationally invariant, i.e. u depends only on $\varrho = |x|$ and u_ϱ is the only nonzero spherical component of u. Namely we show that (7) holds if and only if

$$-(p-1)(n+p'-2) \leqslant \alpha \leqslant n+p-2.$$

2. L^p-dissipativity of planar elasticity

In this section we give a necessary and sufficient condition for the L^p-dissipativity of operator (1) in the case $n = 2$.

First we consider the L^p-dissipativity of form (2).

Lemma 1. *Let Ω be a domain of $\mathbb{R}^2$. Form (2) is L^p-dissipative if and only if*

$$\int_\Omega \Big[C_p \big|\nabla|v|\big|^2 - \sum_{j=1}^{2} |\nabla v_j|^2 + \gamma\, C_p\, |v|^{-2}\big|v_h \partial_h |v|\big|^2 - \gamma\, |\operatorname{div} v|^2\Big]\, dx \leqslant 0 \tag{8}$$

for any $v \in (C_0^1(\Omega))^2$, *where*

$$C_p = (1-2/p)^2, \quad \gamma = (1-2\nu)^{-1}. \tag{9}$$

Proof. Sufficiency. First suppose $p \geqslant 2$. Let $u \in (C_0^1(\Omega))^2$ and set $v = |u|^{p-2}u$. We have $v \in (C_0^1(\Omega))^2$ and $u = |v|^{(2-p)/p}v$. One checks directly that

$$\begin{aligned}\big\langle \nabla u, \nabla(|u|^{p-2}u)\big\rangle + (1-2\nu)^{-1} \operatorname{div} u \, \operatorname{div}(|u|^{p-2}u) = \\ = \sum_j |\nabla v_j|^2 - C_p \big|\nabla |v|\big|^2 - \gamma\, C_p \big|v_h \partial_h |v|\big|^2 + \gamma |\operatorname{div} v|^2.\end{aligned}$$

The left-hand side of (3) being equal to the left-hand side of (8), inequality (3) is satisfied for any $u \in C_0^1(\Omega)$.

If $1 < p < 2$ we find

$$\begin{aligned}\big\langle \nabla u, \nabla(|u|^{p'-2}u)\big\rangle + (1-2\nu)^{-1} \operatorname{div} u \, \operatorname{div}(|u|^{p'-2}u) = \\ = \sum_j |\nabla v_j|^2 - C_{p'} \big|\nabla |v|\big|^2 - \gamma\, C_{p'} \big|v_h \partial_h |v|\big|^2 + \gamma |\operatorname{div} v|^2\end{aligned}$$

and since $1 - 2/p' = -1 + 2/p$ (which implies $C_p = C_{p'}$), we get the result also in this case.

Necessity. Let $p \geqslant 2$ and set

$$g_\varepsilon = (|v|^2 + \varepsilon^2)^{1/2}, \quad u_\varepsilon = g_\varepsilon^{2/p-1} v,$$

where $v \in C_0^1(\Omega)$. We have

$$\begin{aligned}\big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle = \\ = |u_\varepsilon|^{p-2} \langle \partial_h u_\varepsilon, \partial_h u_\varepsilon\rangle + (p-2)|u_\varepsilon|^{p-3} \langle \partial_h u_\varepsilon, u_\varepsilon\rangle\, \partial_h |u_\varepsilon|.\end{aligned}$$

A direct computation shows that

$$\begin{aligned}\big\langle \nabla u_\varepsilon, \nabla(\,|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle = \Big[(1-2/p)^2 g_\varepsilon^{-(p+2)} |v|^p - \\ - 2(1-2/p) g_\varepsilon^{-p} |v|^{p-2}\Big] \sum_k |v_j \partial_k v_j|^2 + g_\varepsilon^{2-p} |v|^{p-2} \langle \partial_h v, \partial_h v\rangle,\end{aligned}$$

$$\begin{aligned}|u_\varepsilon|^{p-3} \langle \partial_h u_\varepsilon, u_\varepsilon\rangle\, \partial_h |u_\varepsilon| = \\ = \Big\{(1-2/p)\Big[(1-2/p) g_\varepsilon^{-(p+2)} |v|^p - g_\varepsilon^{-p} |v|^{p-2}\Big] + \\ + \big[g_\varepsilon^{2-p} |v|^{p-4} - (1-2/p) g_\varepsilon^{-p} |v|^{p-2}\big]\Big\} \sum_k |v_j \partial_k v_j|^2\end{aligned}$$

on the set $E = \{x \in \Omega \mid \; |v(x)| > 0\}$. The inequality $g_\varepsilon^a \leqslant |v|^a$ for $a \leqslant 0$, shows that the right-hand sides are dominated by L^1 functions. Since $g_\varepsilon \to$

$|v|$ pointwise as $\varepsilon \to 0^+$, we find

$$\lim_{\varepsilon\to 0^+} \big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle =$$

$$= \langle \partial_h v, \partial_h v\rangle + \Big[(1-2/p)^2 - 2(1-2/p) + 4(p-2)/p^2\Big]|v|^{-2}\sum_k |v_j\partial_k v_j|^2 =$$

$$= -(1-2/p)^2\big|\nabla|v|\big|^2 + \sum_j |\nabla v_j|^2$$

and dominated convergence gives

$$\lim_{\varepsilon\to 0^+}\int_E \big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle\, dx = \int_E \Big[-C_p\big|\nabla|v|\big|^2 + \sum_j|\nabla v_j|^2\Big]\, dx. \quad (10)$$

Similar arguments show that

$$\lim_{\varepsilon\to o^+}\int_E \operatorname{div} u_\varepsilon \operatorname{div}(|u_\varepsilon|^{p-2}u_\varepsilon)\, dx =$$

$$= \int_E \Big[-C_p|v|^{-2}\big|v_h\partial_h|v|\big|^2 + |\operatorname{div} v|^2\Big]\, dx. \quad (11)$$

Formulas (10) and (11) lead to

$$\lim_{\varepsilon\to o^+}\int_\Omega \big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\big\rangle + \gamma \operatorname{div}(|u_\varepsilon|^{p-2}u_\varepsilon)\, dx =$$

$$= \int_\Omega \Big(-C_p\big|\nabla|v|\big|^2 + \sum_j|\nabla v_j|^2 - \gamma\, C_p|v|^{-2}\big|v_h\partial_h|v|\big|^2 + \gamma\,|\operatorname{div} v|^2\Big)\, dx. \quad (12)$$

The function u_ε being in $(C_0^1(\Omega))^2$, the left-hand side is greater than or equal to zero and (8) follows.

If $1<p<2$, we can write, in view of (12),

$$\lim_{\varepsilon\to o^+}\int_\Omega \big\langle \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p'-2}u_\varepsilon)\big\rangle + \gamma \operatorname{div}(|u_\varepsilon|^{p'-2}u_\varepsilon)\, dx =$$

$$= \int_\Omega \Big(-C_{p'}\big|\nabla|v|\big|^2 + \sum_j|\nabla v_j|^2 - \gamma\, C_{p'}|v|^{-2}\big|v_h\partial_h|v|\big|^2 + \gamma\,|\operatorname{div} v|^2\big)\, dx.$$

Since $C_{p'} = C_p$, (4) implies (8). □

Remark 1. The previous Lemma holds in any dimension with the same proof.

The next Lemma provides a necessary algebraic condition for the L^p-dissipativity of form (2).

Lemma 2. *Let Ω be a domain of $\mathbb{R}^2$. If form* (2) *is L^p-dissipative, we have*

$$C_p\big[\,|\xi|^2 + \gamma\,\langle\xi,\omega\rangle^2\big]\langle\lambda,\omega\rangle^2 - |\xi|^2|\lambda|^2 - \gamma\,\langle\xi,\lambda\rangle^2 \leqslant 0 \quad (13)$$

for any $\xi, \lambda, \omega \in \mathbb{R}^2$, $|\omega| = 1$ (*the constants* C_p *and* γ *being given by* (9)).

Proof. Assume first that $\Omega = \mathbb{R}^2$. Let us fix $\omega \in \mathbb{R}^2$ with $|\omega| = 1$ and take $v(x) = w(x)\,\eta(\log|x|/\log R)$, where

$$w(x) = \mu\,\omega + \psi(x),$$

μ, $R \in \mathbb{R}^+$, $\psi \in (C_0^\infty(\mathbb{R}^2))^2$, $\eta \in C^\infty(\mathbb{R}^2)$, $\eta(t) = 1$ if $t \leqslant 1/2$ and $\eta(t) = 0$ if $t \geqslant 1$.

On the set where $v \neq 0$ one has

$$\big\langle \nabla|v|, \nabla|v|\big\rangle = \big\langle \nabla|w|, \nabla|w|\big\rangle\,\eta^2(\log|x|/\log R) + \\ +2\,(\log R)^{-1}|w|\,\big\langle \nabla|w|, x\big\rangle\,|x|^{-2}\eta(\log|x|/\log R)\,\eta'(\log|x|/\log R) + \\ +(\log R)^{-2}|w|^2|x|^{-2}\big(\eta'(\log|x|/\log R)\big)^2.$$

Choose δ such that $\operatorname{spt}\psi \subset B_\delta(0)$ and $R > \delta^2$. If $|x| > \delta$ one has $w(x) = \mu\,\omega$ and then $\nabla|w| = 0$, while if $|x| < \delta$, then $\eta(\log|x|/\log R) = 1$, $\eta'(\log|x|/\log R) = 0$. Therefore

$$\int_{\mathbb{R}^2} \big\langle \nabla|v|, \nabla|v|\big\rangle\,dx =$$

$$= \int_{B_\delta(0)} \big\langle \nabla|w|, \nabla|w|\big\rangle\,dx + \frac{1}{\log^2 R}\int_{B_R(0)\setminus B_{\sqrt{R}}(0)} \frac{|w|^2}{|x|^2}\big(\eta'(\log|x|/\log R)\big)^2\,dx.$$

Since

$$\lim_{R\to+\infty}\frac{1}{\log^2 R}\int_{B_R(0)\setminus B_{\sqrt{R}}(0)} \frac{dx}{|x|^2} = 0,$$

we find

$$\lim_{R\to+\infty}\int_{\mathbb{R}^2} \big\langle \nabla|v|, \nabla|v|\big\rangle\,dx = \int_{B_\delta(0)} \big\langle \nabla|w|, \nabla|w|\big\rangle\,dx.$$

By similar arguments we obtain

$$\lim_{R\to+\infty}\int_{\mathbb{R}^2}\Big[C_p\big|\nabla|v|\big|^2 - \sum_{j=1}^2|\nabla v_j|^2 + \gamma\,C_p\,|v|^{-2}\big|v_h\partial_h|v|\big|^2 - \gamma\,|\operatorname{div} v|^2\Big]\,dx =$$

$$= \int_{B_\delta(0)}\Big[C_p\big|\nabla|w|\big|^2 - \sum_{j=1}^2|\nabla w_j|^2 + \gamma\,C_p\,|w|^{-2}\big|w_h\partial_h|w|\big|^2 - \gamma\,|\operatorname{div} w|^2\Big]\,dx.$$

In view of Lemma 1, (8) holds. Putting v in this formula and letting $R \to +\infty$, we find

$$\int_{B_\delta(0)}\Big[C_p\big|\nabla|w|\big|^2 - \sum_{j=1}^2|\nabla w_j|^2 + \gamma\,C_p\,|w|^{-2}\big|w_h\partial_h|w|\big|^2 - \gamma\,|\operatorname{div} w|^2\Big]\,dx \leqslant 0. \quad (14)$$

From the identities

$$\partial_h w = \partial_h \psi, \quad \operatorname{div} w = \operatorname{div} \psi,$$

$$\big|\nabla |w|\big|^2 = |\mu\,\omega + \psi|^{-2} \sum_{h=1}^{2} \langle \mu\,\omega + \psi, \partial_h \psi\rangle^2,$$

$$|w|^{-2} |w_h \partial_h w|^2 = |\mu\,\omega + \psi|^{-4} \Big|(\mu\,\omega_h + \psi_h) \langle \mu\,\omega + \psi, \partial_h \psi\rangle\Big|^2$$

we infer, letting $\mu \to +\infty$ in (14),

$$\int_{\mathbb{R}^2} \Big[C_p \sum_{h=1}^{2} \langle \omega, \partial_h \psi\rangle^2 - \sum_{j=1}^{2} |\nabla \psi_j|^2 + \gamma\, C_p \big|\omega_h\, \langle \omega, \partial_h \psi\rangle\big|^2 - \gamma\, |\operatorname{div} \psi|^2\Big]\, dx \leqslant 0. \tag{15}$$

Putting in (15)

$$\psi(x) = \lambda\, \varphi(x)\, \cos(\mu\langle \xi, x\rangle) \;\text{ and }\; \psi(x) = \lambda\, \varphi(x)\, \sin(\mu\langle \xi, x\rangle),$$

where $\lambda \in \mathbb{R}^2$, $\varphi \in C_0^\infty(\mathbb{R}^2)$ and μ is a real parameter, by standard arguments (see, e.g, Fichera [4, p. 107–108]) we find (13).

If $\Omega \neq \mathbb{R}^2$, fix $x_0 \in \Omega$ and $0 < \varepsilon < \operatorname{dist}(x_0, \partial\Omega)$. Given $\psi \in (C_0^1(\Omega))^2$, put the function

$$v(x) = \psi((x - x_0)/\varepsilon)$$

in (8). By a change of variables we find

$$\int_{\mathbb{R}^2} \left[C_p \big|\nabla |\psi|\big|^2 - \sum_{j=1}^{2} |\nabla \psi_j|^2 + \gamma\, C_p\, |\psi|^{-2} \big|\psi_h \partial_h |\psi|\big|^2 - \gamma\, |\operatorname{div} \psi|^2\right] dx \leqslant 0.$$

The arbitrariness of $\psi \in (C_0^1(\Omega))^2$ and what we have proved for $\mathbb{R}^2$ gives the result. □

We are now in a position to give a necessary and sufficient condition for the L^p-dissipativity of form (2).

Theorem 1. *Form* (2) *is L^p-dissipative if and only if*

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)^2 \leqslant \frac{2(\nu - 1)(2\nu - 1)}{(3 - 4\nu)^2}\,. \tag{16}$$

Proof. Necessity. In view of Lemma 2, the L^p-dissipativity of $\mathscr{L}$ implies the algebraic inequality (13) for any $\xi, \lambda, \omega \in \mathbb{R}^2$, $|\omega| = 1$.

Without loss of generality we may suppose $\xi = (1, 0)$ and (13) can be written as

$$C_p(1 + \gamma\omega_1^2)(\lambda_j\omega_j)^2 - |\lambda|^2 - \gamma\lambda_1^2 \leqslant 0 \tag{17}$$

for any $\lambda, \omega \in \mathbb{R}^2$, $|\omega| = 1$.

Condition (17) holds if and only if

$$C_p(1 + \gamma\omega_1^2)\omega_1^2 - 1 - \gamma \leqslant 0,$$

$$\big[C_p(1 + \gamma\omega_1^2)\omega_1\omega_2\big]^2 \leqslant \big[-C_p(1 + \gamma\omega_1^2)\omega_1^2 + 1 + \gamma\big]\,\big[-C_p(1 + \gamma\omega_1^2)\omega_2^2 + 1\big]$$

for any $\omega \in \mathbb{R}^2$, $|\omega| = 1$.

In particular, the second condition has to be satisfied. This can be written in the form

$$1+\gamma-C_p(1+\gamma\omega_1^2)(1+\gamma\omega_2^2)\geqslant 0 \tag{18}$$

for any $\omega\in\mathbb{R}^2$, $|\omega|=1$. The minimum of the left-hand side of (18) on the unit sphere is given by

$$1+\gamma-C_p(1+\gamma/2)^2.$$

Hence (18) is satisfied if and only if $1+\gamma-C_p(1+\gamma/2)^2\geqslant 0$. The last inequality means

$$\frac{2(1-\nu)}{1-2\nu}-\Big(\frac{p-2}{p}\Big)^2\Big(\frac{3-4\nu}{2(1-2\nu)}\Big)^2\geqslant 0,$$

i.e. (16). From the identity $4/(p\,p')=1-(1-2/p)^2$ it follows that (16) can be written also as

$$\frac{4}{p\,p'}\geqslant\frac{1}{(3-4\nu)^2}\,. \tag{19}$$

Sufficiency. In view of Lemma 1, $\mathscr{L}$ is L^p-dissipative if and only if (8) holds for any $v\in(C_0^1(\Omega))^2$. Choose $v\in(C_0^1(\Omega))^2$ and define

$$X_1=|v|^{-1}(v_1\partial_1|v|+v_2\partial_2|v|),\quad X_2=|v|^{-1}(v_2\partial_1|v|-v_1\partial_2|v|),$$
$$Y_1=|v|\big[\partial_1(|v|^{-1}v_1)+\partial_2(|v|^{-1}v_2)\big],\quad Y_2=|v|\big[\partial_1(|v|^{-1}v_2)-\partial_2(|v|^{-1}v_1)\big]$$

on the set $E=\{x\in\Omega\mid v\neq 0\}$. From the identities

$$|\nabla|v||^2=X_1^2+X_2^2,$$
$$Y_1=(\partial_1v_1+\partial_2v_2)-X_1,\quad Y_2=(\partial_1v_2-\partial_2v_1)-X_2$$

it follows

$$Y_1^2+Y_2^2=\big|\nabla|v|\big|^2+(\partial_1v_1+\partial_2v_2)^2+(\partial_1v_2-\partial_2v_1)^2-$$
$$-2(\partial_1v_1+\partial_2v_2)X_1-2(\partial_1v_2-\partial_2v_1)X_2.$$

Keeping in mind that $\partial_h|v|=|v|^{-1}v_j\partial_hv_j$, one can check that

$$(\partial_1v_1+\partial_2v_2)\big(v_1\partial_1|v|+v_2\partial_2|v|\big)+(\partial_1v_2-\partial_2v_1)\big(v_2\partial_1|v|-v_1\partial_2|v|\big)=$$
$$=|v|\,\big|\nabla|v|\big|^2+|v|(\partial_1v_1\partial_2v_2-\partial_2v_1\partial_1v_2),$$

which implies

$$\sum_j|\nabla v_j|^2=X_1^2+X_2^2+Y_1^2+Y_2^2. \tag{20}$$

Thus (8) can be written as

$$\int_E\Big[\frac{4}{p\,p'}\,(X_1^2+X_2^2)+Y_1^2+Y_2^2-\gamma\,C_pX_1^2+\gamma\,(X_1+Y_1)^2\Big]\,dx\geqslant 0. \tag{21}$$

Let us prove that

$$\int_E X_1Y_1\,dx=-\int_E X_2Y_2\,dx. \tag{22}$$

Since $X_1 + Y_1 = \operatorname{div} v$ and $X_2 + Y_2 = \partial_1 v_2 - \partial_2 v_1$, keeping in mind (20), we may write

$$2\int_E (X_1Y_1 + X_2Y_2)\, dx =$$

$$= \int_E \Big[(X_1+Y_1)^2 + (X_2+Y_2)^2 - (X_1^2 + X_2^2 + Y_1^2 + Y_2^2)\Big]\, dx =$$

$$= \int_E \Big[(\operatorname{div} v)^2 + (\partial_1 v_2 - \partial_2 v_1)^2 - \sum_j |\nabla v_j|^2\Big]\, dx,$$

i.e.

$$\int_E (X_1Y_1 + X_2Y_2)\, dx = \int_E (\partial_1 v_1 \partial_2 v_2 - \partial_1 v_2 \partial_2 v_1)\, dx.$$

The set $\{x \in \Omega \setminus E | \nabla v(x) \neq 0\}$ has zero measure and then

$$\int_E (X_1Y_1 + X_2Y_2)\, dx = \int_\Omega (\partial_1 v_1 \partial_2 v_2 - \partial_1 v_2 \partial_2 v_1)\, dx.$$

There exists a sequence $\{v^{(n)}\} \subset C_0^\infty(\Omega)$ such that $v^{(n)} \to v$, $\nabla v^{(n)} \to \nabla v$ uniformly in Ω and hence

$$\int_\Omega \partial_1 v_1 \partial_2 v_2\, dx = \lim_{n\to\infty} \int_\Omega \partial_1 v_1^{(n)} \partial_2 v_2^{(n)}\, dx =$$

$$= \lim_{n\to\infty} \int_\Omega \partial_1 v_2^{(n)} \partial_2 v_1^{(n)}\, dx = \int_\Omega \partial_1 v_2 \partial_2 v_1\, dx$$

and (22) is proved. In view of this, (21) can be written as

$$\int_E \Big(\frac{4}{p\,p'}\,(1+\gamma)X_1^2 + 2\vartheta\gamma\, X_1Y_1 + (1+\gamma)Y_1^2\Big)\, dx +$$

$$+ \int_E \Big(\frac{4}{p\,p'}\, X_2^2 - 2(1-\vartheta)\gamma\, X_2Y_2 + Y_2^2\Big)\, dx \geqslant 0$$

for any fixed $\vartheta \in \mathbb{R}$.

If we choose

$$\vartheta = \frac{2(1-\nu)}{3-4\nu}$$

we find

$$(1-\vartheta)\gamma = \frac{1}{3-4\nu}\,, \quad \vartheta^2\gamma^2 = \frac{(1+\gamma)^2}{(3-4\nu)^2}\,.$$

Inequality (19) leads to

$$\vartheta^2\gamma^2 \leqslant \frac{4}{p\,p'}\,(1+\gamma)^2, \quad (1-\vartheta)^2\gamma^2 \leqslant \frac{4}{p\,p'}\,.$$

Observing that (16) implies $1+\gamma = 2(1-\nu)(1-2\nu)^{-1} \geqslant 0$, we get

$$\frac{4}{p\,p'}\,(1+\gamma)x_1^2 + 2\vartheta\gamma\, x_1 y_1 + (1+\gamma)y_1^2 \geqslant 0,$$

$$\frac{4}{p\,p'}\,x_2^2 - 2(1-\vartheta)\gamma\, x_2 y_2 + y_2^2 \geqslant 0$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$. This shows that (21) holds. Then (8) is true for any $v \in (C_0^1(\Omega))^2$ and the proof is complete. □

The results we have obtained so far hold for any domain Ω. For the rest of the present section we suppose that Ω is a bounded domain whose boundary is in the class C^2. We could consider more general domains, in the spirit of Maz'ya and Shaposhnikova [14, Ch. 14], but here we prefer to avoid the related technicalities.

Theorem 2. *Let E be the two-dimensional elasticity operator* (1) *with domain $(W^{2,p}(\Omega)\cap \mathring{W}^{1,p}(\Omega))^2$. The operator E is L^p-dissipative if and only if condition* (16) *holds.*

Proof. By means of the same arguments as in [1, Section 5, p. 1086–1093], we have the equivalence between the L^p-dissipativity of form (2) and the L^p-dissipativity of the elasticity operator (1). The result follows from Theorem 1. □

We shall now give two corollaries of this result. They concerns the comparison between E and Δ from the point of view of the L^p-dissipativity.

Corollary 1. *There exists $k > 0$ such that $E - k\Delta$ is L^p-dissipative if and only if*

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)^2 < \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,. \qquad (23)$$

Proof. Necessity. We remark that if $E - k\Delta$ is L^p-dissipative, then

$$\begin{cases} k \leqslant 1 & \text{if } \ p = 2, \\ k < 1 & \text{if } \ p \neq 2. \end{cases} \qquad (24)$$

In fact, in view of Theorem 1, we have the necessary condition

$$-(1-2/p)^2\big[(1-k)|\xi|^2 + (1-2\nu)^{-1}(\xi_j\omega_j)^2\big](\lambda_j\omega_j)^2 + \\ + (1-k)|\xi|^2|\lambda|^2 + (1-2\nu)^{-1}(\xi_j\lambda_j)^2 \geqslant 0 \qquad (25)$$

for any $\xi, \lambda, \omega \in \mathbb{R}^2$, $|\omega| = 1$. If we take $\xi = (1,0)$, $\lambda = \omega = (0,1)$ in (25) we find

$$\frac{4}{p\,p'}\,(1-k) \geqslant 0$$

and then $k \leqslant 1$ for any p. If $p \neq 2$ and $k = 1$, taking $\xi = (1,0)$, $\lambda = (0,1)$, $\omega = (1/\sqrt{2}, 1/\sqrt{2})$ in (25), we find $-(1-2/p)^2(1-2\nu)^{-1} \geqslant 0$. On the other hand, taking $\xi = \lambda = (1,0)$, $\omega = (0,1)$ we find $(1-2\nu)^{-1} \geqslant 0$. This is a contradiction and (24) is proved.

It is clear that if $E-k\Delta$ is L^p-dissipative, then $E-k'\Delta$ is L^p-dissipative for any $k'<k$. Therefore it is not restrictive to suppose that $E-k\Delta$ is L^p-dissipative for some $0<k<1$. Moreover, E is also L^p-dissipative.

The L^p-dissipativity of $E-k\Delta$ $(0<k<1)$ is equivalent to the L^p-dissipativity of the operator

$$E'u = \Delta u + (1-k)^{-1}(1-2\nu)^{-1}\nabla \operatorname{div} u. \tag{26}$$

Setting

$$\nu' = \nu(1-k)+k/2, \tag{27}$$

we have $(1-k)(1-2\nu)=1-2\nu'$. Theorem 1 shows that

$$\frac{4}{p\,p'} \geqslant \frac{1}{(3-4\nu')^2}\,. \tag{28}$$

Since $3-4\nu' = 3-4\nu-2k(1-2\nu)$, condition (28) means $|3-4\nu-2k(1-2\nu)| \geqslant \sqrt{p\,p'}/2$, i.e.

$$\Big|k - \frac{3-4\nu}{2(1-2\nu)}\Big| \geqslant \frac{\sqrt{p\,p'}}{4|1-2\nu|}\,. \tag{29}$$

Note that the L^p-dissipativity of E implies that (16) holds. In particular, we have $(3-4\nu)/(1-2\nu)>0$. Hence (29) is satisfied if either

$$k \leqslant \frac{1}{2|1-2\nu|}\Big(|3-4\nu| - \frac{\sqrt{p\,p'}}{2}\Big) \tag{30}$$

or

$$k \geqslant \frac{1}{2|1-2\nu|}\Big(|3-4\nu| + \frac{\sqrt{p\,p'}}{2}\Big). \tag{31}$$

Since

$$\frac{|3-4\nu|}{2|1-2\nu|} - 1 = \frac{3-4\nu}{2(1-2\nu)} - 1 = \frac{1}{2(1-2\nu)} \geqslant -\frac{\sqrt{p\,p'}}{4|1-2\nu|},$$

we have

$$\frac{1}{2|1-2\nu|}\Big(|3-4\nu| + \frac{\sqrt{p\,p'}}{2}\Big) \geqslant 1$$

and (31) is impossible. Then (30) holds. Since $k>0$, we have the strict inequality in (19) and (23) is proved.

Sufficiency. Suppose (23). Since

$$\frac{4}{p\,p'} > \frac{1}{(3-4\nu)^2}\,,$$

we can take k such that

$$0<k<\frac{1}{2|1-2\nu|}\Big(|3-4\nu| - \frac{\sqrt{p\,p'}}{2}\Big). \tag{32}$$

Note that

$$\frac{|3-4\nu|}{2|1-2\nu|} - 1 = \frac{3-4\nu}{2(1-2\nu)} - 1 = \frac{1}{2(1-2\nu)} \leqslant \frac{\sqrt{p\,p'}}{4|1-2\nu|}\,.$$

This means

$$\frac{1}{2|1-2\nu|}\Big(|3-4\nu| - \frac{\sqrt{p\,p'}}{2}\Big) \leqslant 1$$

and then $k < 1$. Let ν' be given by (27). The L^p-dissipativity of $E - k\Delta$ is equivalent to the L^p-dissipativity of the operator E' defined by (26).

Condition (29) (i.e. (28)) follows from (32) and Theorem 1 gives the result. □

Corollary 2. *There exists $k < 2$ such that $k\Delta - E$ is L^p-dissipative if and only if*

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)^2 < \frac{2\nu(2\nu-1)}{(1-4\nu)^2}\,. \tag{33}$$

Proof. We may write $k\Delta - E = \widetilde{E} - \widetilde{k}\Delta$, where $\widetilde{k} = 2-k$, $\widetilde{E} = \Delta + (1 - 2\widetilde{\nu})^{-1}\nabla\,\mathrm{div}$, $\widetilde{\nu} = 1-\nu$. Theorem 1 shows that $\widetilde{E} - \widetilde{k}\Delta$ is L^p-dissipative if and only if

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)^2 < \frac{2(\widetilde{\nu}-1)(2\widetilde{\nu}-1)}{(3-4\widetilde{\nu})^2}\,. \tag{34}$$

Condition (34) coincides with (33) and the corollary is proved. □

3. L^p-dissipativity of three-dimensional elasticity

As far as the three-dimensional Lamé system is concerned, necessary and sufficient conditions for the L^p-dissipativity are not known. The next Theorem shows that condition (16) is necessary, even in the case of a non-constant Poisson ratio. Here Ω is a bounded domain in $\mathbb{R}^3$ whose boundary is in the class C^2.

Theorem 3. *Suppose $\nu = \nu(x)$ is a continuos function defined in Ω such that*

$$\inf_{x\in\Omega} |2\nu(x) - 1| > 0.$$

If (1) is L^p-dissipative in Ω, then

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)^2 \leqslant \inf_{x\in\Omega} \frac{2(\nu(x)-1)(2\nu(x)-1)}{(3-4\nu(x))^2}\,. \tag{35}$$

Proof. We have

$$\int_\Omega \big(\Delta u + (1-2\nu(x)\big)^{-1}\nabla\,\mathrm{div}\, u|u|^{p-2}u\, dx \leqslant 0 \tag{36}$$

for any $u \in (W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega))^3$, in particular, for any $u \in (C_0^\infty(\Omega))^3$. Take $v \in (C_0^\infty(\mathbb{R}^2))^2$, $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \geqslant 0$ and $x^0 \in \Omega$; define $v_\varepsilon(x_1, x_2) = v((x_1 - x_1^0)/\varepsilon, (x_2 - x_2^0)/\varepsilon)$,

$$u(x_1, x_2, x_3) = \big(v_{\varepsilon,1}(x_1,x_2), v_{\varepsilon,2}(x_1,x_2), 0\big)\,\varphi(x_3).$$

We suppose that the support of v is contained in the unit ball, $0 < \varepsilon < \mathrm{dist}(x^0, \partial\Omega)$ and the support of φ is contained in $(-\varepsilon, \varepsilon)$. In this way the function u belongs to $(C_0^\infty(\Omega))^3$.

Setting $\gamma(x_1,x_2,x_3)=(1-2\,\nu(x_1,x_2,x_3))^{-1}$, we have

$$\Delta u+\gamma\,\nabla\,\mathrm{div}\,u=(\Delta v_\varepsilon+\gamma\,\nabla\,\mathrm{div}\,v_\varepsilon)\,\varphi+v_\varepsilon\varphi''$$

and then

$$(\Delta u+\gamma\,\nabla\,\mathrm{div}\,u)|u|^{p-2}u=(\Delta v_\varepsilon+\gamma\,\nabla\,\mathrm{div}\,v_\varepsilon)\,|v_\varepsilon|^{p-2}v_\varepsilon\varphi^p+v_\varepsilon^2\varphi''\varphi^{p-1}.$$

We can write, in view of (36),

$$\int\limits_{\mathbb{R}}\varphi^p\,dx_3\iint\limits_{\mathbb{R}^2}(\Delta v_\varepsilon+\gamma\,\nabla\,\mathrm{div}\,v_\varepsilon)\,|v_\varepsilon|^{p-2}v_\varepsilon\;dx_1\,dx_2+ \\ +\int\limits_{\mathbb{R}}\varphi^{p-1}\varphi''\,dx_3\iint\limits_{\mathbb{R}^2}|v_\varepsilon|^p\,dx_1\,dx_2\leqslant 0.$$

Noting that

$$\Delta v_\varepsilon+\gamma\,\nabla\,\mathrm{div}\,v_\varepsilon= \\ =\frac{1}{\varepsilon^2}\left[\Delta v\Big(\frac{x_1-x_1^0}{\varepsilon},\frac{x_1-x_1^0}{\varepsilon}\Big)+\gamma(x_1,x_2,x_3)\nabla\,\mathrm{div}\,v\Big(\frac{x_1-x_1^0}{\varepsilon},\frac{x_1-x_1^0}{\varepsilon}\Big)\right],$$

a change of variables in the double integral gives

$$\int\limits_{\mathbb{R}}\varphi^p(x_3)\,dx_3\iint\limits_{\mathbb{R}^2}\Big(\Delta v(t_1,t_2)+\gamma(x_1^0+\varepsilon\,t_1,x_2^0+\varepsilon\,t_2,x_3)\nabla\,\mathrm{div}\,v(t_1,t_2)\Big)\times \\ \times\big|v(t_1,t_2)\big|^{p-2}v(t_1,t_2)\;dt_1\,dt_2+ \\ +\varepsilon^2\int\limits_{\mathbb{R}}\varphi^{p-1}\varphi''\,dx_3\iint\limits_{\mathbb{R}^2}|v(t_1,t_2)|^p\;dt_1\,dt_2\leqslant 0.$$

Letting $\varepsilon\to 0^+$, we get

$$\int\limits_{\mathbb{R}}\varphi^p(x_3)\,dx_3\iint\limits_{\mathbb{R}^2}\Big(\Delta v(t_1,t_2)+\gamma(x_1^0,x_2^0,x_3)\nabla\,\mathrm{div}\,v(t_1,t_2)\Big)\times \\ \times\big|v(t_1,t_2)\big|^{p-2}v(t_1,t_2)\;dt_1\,dt_2\leqslant 0.$$

For the arbitrariness of φ, this implies

$$\iint\limits_{\mathbb{R}^2}\Big(\Delta v(t_1,t_2)+\gamma(x_1^0,x_2^0,x_3^0)\nabla\,\mathrm{div}\,v(t_1,t_2)\Big)\times \\ \times\big|v(t_1,t_2)\big|^{p-2}v(t_1,t_2)\;dt_1\,dt_2\leqslant 0$$

for any $v\in(C_0^\infty(B))^2$, B being the unit ball in $\mathbb{R}^2$.

Suppose $p\geqslant 2$. Integrating by parts, we get

$$\mathscr{L}(v,|v|^{p-2}v)\leqslant 0 \tag{37}$$

for any $v\in(C_0^\infty(B))^2$.

Given $v \in (C_0^\infty(B))^2$, define $u_\varepsilon = g_\varepsilon^{2/p-1} v$. Since $u_\varepsilon \in (C_0^\infty(B))^2$, in view of (37) we write

$$\mathscr{L}(u_\varepsilon, |u_\varepsilon|^{p-2} u_\varepsilon) \leqslant 0.$$

By means of the computations we made in the Necessity of Lemma 1, letting $\varepsilon \to 0^+$, we find inequality (8) for any $v \in (C_0^\infty(B))^2$. This implies that (8) holds for any $v \in (C_0^1(B))^2$.

In fact, let $v_m \in (C_0^\infty(B))^2$ such that $v_m \to v$ in C^1-norm. Let us show that

$$\chi_{E_n} |v_m|^{-1} v_m \nabla v_m \to \chi_E |v|^{-1} v \nabla v \ \text{ in } L^2(B), \tag{38}$$

where $E_n = \{x \in B \mid v_m(x) \neq 0\}$, $E = \{x \in \Omega \mid v(x) \neq 0\}$. We see that

$$\chi_{E_n} |v_m|^{-1} v_m \nabla v_m \to \chi_E |v|^{-1} v \nabla v \tag{39}$$

on the set $E \cup \{x \in B \mid \nabla v(x) = 0\}$. The set $\{x \in B \setminus E \mid \nabla v(x) \neq 0\}$ having zero measure, (39) holds almost everywhere. Moreover, since

$$\int\limits_G \chi_{E_n} |v_m|^{-2} |v_m \nabla v_m|^2 \, dx \leqslant \int\limits_G |\nabla v_m|^2 \, dx$$

for any measurable set $G \subset \Omega$ and $\{\nabla v_m\}$ is convergent in $L^2(\Omega)$, the sequence $\{|\chi_{E_n} |v_m|^{-1} v_m \nabla v_m - \chi_E |v|^{-1} v \nabla v|^2\}$ has uniformly absolutely continuous integrals. Now we may appeal to Vitali's Theorem to obtain (38).

Inequality (8) holding for any $v \in (C_0^1(B))^2$, the result follows from Theorem 1.

Let now $1 < p < 2$. From the L^p dissipativity of E it follows that the operator $E - \lambda I$ ($\lambda > 0$) is invertible on $L^p(\Omega)$. This means that for any $f \in L^p(\Omega)$ there exists one and only one $u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$ such that $(E - \lambda I) u = f$. Because of well known regularity results for solutions of elliptic systems [3], we have also that, if f belongs to $L^{p'}(\Omega)$, the solution u belongs to $W^{2,p'}(\Omega) \cap \mathring{W}^{1,p'}(\Omega)$ and there exists the bounded resolvent $(E^* - \lambda I)^{-1} : L^{p'}(\Omega) \to W^{2,p'}(\Omega) \cap \mathring{W}^{1,p'}(\Omega)$.

Since E is L^p-dissipative and $\|(E^* - \lambda I)^{-1}\| = \|(E - \lambda I)^{-1}\|$, we may write

$$\left\|(E^* - \lambda I)^{-1}\right\| \leqslant \frac{1}{\lambda}$$

for any $\lambda > 0$, i.e. we have the $L^{p'}$-dissipativity of E^*, $p' > 2$. We have reduced the proof to the previous case. Therefore (35) holds with p replaced by p'. Since

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 = \left(\frac{1}{2} - \frac{1}{p'}\right)^2,$$

the proof is complete. □

We do not know if condition (16) is sufficient for the L^p-dissipativity of the three-dimensional elasticity. The next theorem provides a more strict sufficient condition.

Theorem 4. *Let Ω be a domain in $\mathbb{R}^3$. If*

$$(1-2/p)^2 \leqslant \begin{cases} \dfrac{1-2\nu}{2(1-\nu)} & \textit{if } \nu < 1/2, \\ \dfrac{2(1-\nu)}{1-2\nu} & \textit{if } \nu > 1, \end{cases} \tag{40}$$

operator (1) *is L^p-dissipative.*

Proof. In view of Remark 1, the operator E is L^p-dissipative if and only if inequality (8) holds for any $v \in (C_0^1(\Omega))^3$. This can be written as

$$C_p \int_\Omega \Big[\big|\nabla |v|\big|^2 + \gamma\, |v|^{-2} \big| v_h \partial_h |v|\big|^2 \Big]\, dx \leqslant$$

$$\leqslant \int_\Omega \Big[\sum_{j=1}^3 |\nabla v_j|^2 + \gamma\, |\operatorname{div} v|^2 \Big]\, dx. \tag{41}$$

Note that the integral on the left-hand side of (41) is nonnegative. In fact, setting $\xi_{hj} = \partial_h v_j$, $\omega_j = |v|^{-1} v_j$, we have

$$\big|\nabla |v|\big|^2 + \gamma\, |v|^{-2} \big| v_h \partial_h |v|\big|^2 = \omega_i \omega_j (\delta_{hk} + \gamma \omega_h \omega_k) \xi_{hi} \xi_{kj}.$$

Then we can write

$$\big|\nabla |v|\big|^2 + \gamma\, |v|^{-2} \big| v_h \partial_h |v|\big|^2 = |\lambda|^2 + \gamma (\lambda \cdot \omega)^2, \tag{42}$$

where λ is the vector whose h-th component is $\omega_i \xi_{hi}$. Since ω is a unit vector and $\gamma > -1$ we have

$$\big|\nabla |v|\big|^2 + \gamma\, |v|^{-2} \big| v_h \partial_h |v|\big|^2 \geqslant 0.$$

Also the right-hand side of (41) is nonengative. In fact, denoting by $\widehat{v_j}$ the Fourier transform of v_j

$$\widehat{v_j}(y) = \int_{\mathbb{R}^3} v_j(x) e^{-iy\cdot x}\, dx,$$

we have

$$\int_\Omega \Big[\sum_{j=1}^3 |\nabla v_j|^2 + \gamma\, |\operatorname{div} v|^2 \Big]\, dx = \int_\Omega (\partial_h v_j \partial_h v_j + \gamma \partial_h v_h \partial_j v_j)\, dx =$$

$$= (2\pi)^{-3} \int_{\mathbb{R}^3} \big(\widehat{\partial_h v_j}\, \overline{\widehat{\partial_h v_j}} + \gamma \widehat{\partial_h v_h}\, \overline{\widehat{\partial_j v_j}} \big)\, dy = (2\pi)^{-3} \int_{\mathbb{R}^3} \big(|y|^2 |\widehat{v}|^2 + \gamma |y \cdot \widehat{v}|^2 \big)\, dy \geqslant$$

$$\geqslant \min\{1, 1+\gamma\} (2\pi)^{-3} \int_{\mathbb{R}^3} |y|^2 |\widehat{v}|^2\, dy =$$

$$= \min\{1, 1+\gamma\} \int_\Omega \sum_{j=1}^3 |\nabla v_j|^2\, dx. \tag{43}$$

This implies that (41) holds for any v such that the left-hand side vanishes and that E is L^p-dissipative if and only if

$$C_p \leqslant \inf \frac{\int\limits_\Omega \big[\sum\limits_{j=1}^{3} |\nabla v_j|^2 + \gamma \,|\operatorname{div} v|^2 \big]\, dx}{\int\limits_\Omega \big[\, |\nabla |v||^2 + \gamma\, |v|^{-2} |v_h \partial_h |v||^2 \big]\, dx}\,, \tag{44}$$

where the infimum is taken over all $v \in (C_0^1(\Omega))^3$ such that the denominator is positive.

From (42) we get

$$\big|\nabla |v|\big|^2 + \gamma\, |v|^{-2} \big|v_h \partial_h |v|\big|^2 \leqslant$$
$$\leqslant \max\{1, 1+\gamma\}\, |\lambda|^2 \leqslant \max\{1, 1+\gamma\} \sum_{j=1}^{3} |\nabla v_j|^2.$$

Keeping in mind also (43) we find that

$$\frac{\int\limits_\Omega \big[\sum_{j=1}^3 |\nabla v_j|^2 + \gamma \,|\operatorname{div} v|^2 \big]\, dx}{\int\limits_\Omega \big[\, |\nabla |v||^2 + \gamma\, |v|^{-2} |v_h \partial_h |v||^2 \big]\, dx} \geqslant \frac{\min\{1, 1+\gamma\}}{\max\{1, 1+\gamma\}}\,.$$

Therefore condition (44) is satisfied if

$$C_p \leqslant \frac{\min\{1, 1+\gamma\}}{\max\{1, 1+\gamma\}}\,.$$

This inequality being equivalent to (40), the proof is complete. □

Remark 2. The Theorems of this section hold in any dimension $n \geqslant 3$ with the same proof.

4. Weighted L^p-negativity of elasticity system defined on rotationally symmetric vector functions

Let Φ be a point on the $(n-2)$-dimensional unit sphere S^{n-2} with spherical coordinates $\{\vartheta_j\}_{j=1,\dots,n-3}$ and φ, where $\vartheta_j \in (0, \pi)$ and $\varphi \in [0, 2\pi)$. A point $x \in \mathbb{R}^n$ is represented as a triple $(\varrho, \vartheta, \Phi)$, where $\varrho > 0$ and $\vartheta \in [0, \pi]$. Correspondingly, a vector u can be written as $u = (u_\varrho, u_\vartheta, u_\Phi)$ with $u_\Phi = (u_{\vartheta_{n-3}}, \dots, u_{\vartheta_1}, u_\varphi)$. We call $u_\varrho, u_\vartheta, u_\Phi$ the spherical components of the vector u.

Theorem 5. *Let the spherical components u_ϑ and u_Φ of the vector u vanish, i.e. $u = (u_\varrho, 0, 0)$, and let u_ϱ depend only on the variable ϱ. Then, if $\alpha \geqslant n-2$, we have*

$$\int\limits_{\mathbb{R}^n} \Big(\Delta u + (1-2\nu)^{-1} \nabla \operatorname{div} u\Big) |u|^{p-2} u \,\frac{dx}{|x|^\alpha} \leqslant 0 \tag{45}$$

for any $u \in (C_0^\infty(\mathbb{R}^n \setminus \{0\}))^n$ *satisfying the aforesaid symmetric conditions, if and only if*

$$-(p-1)(n+p'-2) \leqslant \alpha \leqslant n+p-2. \tag{46}$$

If $\alpha < n-2$ *the same result holds replacing* $(C_0^\infty(\mathbb{R}^n \setminus \{0\}))^n$ *by* $(C_0^\infty(\mathbb{R}^n))^n$.

Proof. Setting

$$g_\varepsilon(s) = (s^2+\varepsilon^2)^{1/2},$$

and denoting by ω_{n-1} the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$, we have

$$\int\limits_{\mathbb{R}^n} \Delta u \, g_\varepsilon(|u|)^{p-2} u \, \frac{dx}{|x|^\alpha} =$$

$$= \omega_{n-1} \int\limits_0^{+\infty} \Big(\frac{1}{\varrho^{n-1}} \, \partial_\varrho(\varrho^{n-1}\partial_\varrho u_\varrho) - \frac{n-1}{\varrho^2} u_\varrho \Big) g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \varrho^{n-1-\alpha} \, d\varrho.$$

An integration by parts gives

$$\int\limits_0^{+\infty} \partial_\varrho(\varrho^{n-1}\partial_\varrho u_\varrho) g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \varrho^{-\alpha} \, d\varrho =$$

$$= - \int\limits_0^{+\infty} \varrho^{n-1} \partial_\varrho u_\varrho \partial_\varrho \big(g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \varrho^{-\alpha} \big) \, d\varrho =$$

$$= - \int\limits_0^{+\infty} \partial_\varrho u_\varrho \partial_\varrho (g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho) \varrho^{n-1-\alpha} \, d\varrho +$$

$$+ \alpha \int\limits_0^{+\infty} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \partial_\varrho u_\varrho \varrho^{n-\alpha-2} \, d\varrho. \tag{47}$$

Since

$$\partial_\varrho(g_\varepsilon\big(|u_\varrho|)^p\big) = p \, g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \partial_\varrho u_\varrho, \tag{48}$$

we have, by means of another integration by parts in the last integral of (47),

$$\alpha \int\limits_0^{+\infty} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho \partial_\varrho u_\varrho \varrho^{n-\alpha-2} \, d\varrho = \frac{\alpha}{p} \int\limits_0^{+\infty} \partial_\varrho \big(g_\varepsilon(|u_\varrho|)^p \big) \varrho^{n-\alpha-2} \, d\varrho =$$

$$= - \frac{\alpha(n-2-\alpha)}{p} \int\limits_K g_\varepsilon(|u_\varrho|)^p \varrho^{n-3-\alpha} \, d\varrho + \mathcal{O}(\varepsilon^p),$$

where K is the support of u_ϱ.

This proves the identity

$$\int_{\mathbb{R}^n} \Delta u\, g_\varepsilon(|u|)^{p-2} u \,\frac{dx}{|x|^\alpha} = -\omega_{n-1}\Big[(n-1)\int_K g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2 \varrho^{n-3-\alpha}\, d\varrho + \\ + \frac{\alpha(n-2-\alpha)}{p}\int_K g_\varepsilon(|u_\varrho|)^p \varrho^{n-3-\alpha}\, d\varrho + \\ + \int_K \partial_\varrho u_\varrho \partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big)\varrho^{n-1-\alpha}\, d\varrho\Big] + \mathcal{O}(\varepsilon^p). \quad (49)$$

We have also

$$\int_{\mathbb{R}^n} \nabla(\operatorname{div} u) g_\varepsilon(|u|)^{p-2} u \,\frac{dx}{|x|^\alpha} = -\int_{\mathbb{R}^n} \operatorname{div} u \ \operatorname{div}\big(g_\varepsilon(|u|)^{p-2} u |x|^{-\alpha}\big)\, dx = \\ = -\omega_{n-1}\int_0^{+\infty} \frac{1}{\varrho^{n-1}}\,\partial_\varrho(\varrho^{n-1}u_\varrho)\partial_\varrho\big(\varrho^{n-1-\alpha} g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big)\, d\varrho. \quad (50)$$

Moreover,

$$\frac{1}{\varrho^{n-1}}\,\partial_\varrho(\varrho^{n-1}u_\varrho)\,\partial_\varrho\big(\varrho^{n-1-\alpha} g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big) = \\ = (n-1)(n-1-\alpha)\varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho^2 + \\ + (n-1)\varrho^{n-2-\alpha}u_\varrho\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big) + (n-1-\alpha)\varrho^{n-2-\alpha} g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\partial_\varrho u_\varrho + \\ + \varrho^{n-1-\alpha}\partial_\varrho u_\varrho \partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big). \quad (51)$$

In view of (48) we may write

$$\int_0^{+\infty} \varrho^{n-2-\alpha} g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\partial_\varrho u_\varrho\, d\varrho = \frac{1}{p}\int_0^{+\infty} \varrho^{n-2-\alpha}\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^p\big)\, d\varrho = \\ = -\frac{n-2-\alpha}{p}\int_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^p\, d\varrho + \mathcal{O}(\varepsilon^p). \quad (52)$$

Since

$$\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho^2\big) = u_\varrho\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big) + g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\partial_\varrho u_\varrho$$

and using again (48), we find

$$\int_0^{+\infty} \varrho^{n-2-\alpha}u_\varrho\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\big)\, d\varrho = \\ = \int_0^{+\infty} \varrho^{n-2-\alpha}\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho^2\big)\, d\varrho - \int_0^{+\infty} \varrho^{n-2-\alpha} g_\varepsilon(|u_\varrho|)^{p-2}u_\varrho\partial_\varrho u_\varrho\, d\varrho =$$

$$= -(n-2-\alpha)\int_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2\, d\varrho -$$

$$-\frac{1}{p}\int_0^{+\infty} \varrho^{n-2-\alpha}\partial_\varrho\big(g_\varepsilon(|u_\varrho|)^p\big)\, d\varrho + \mathcal{O}(\varepsilon^p) =$$

$$= -(n-2-\alpha)\int_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2\, d\varrho +$$

$$+\frac{n-2-\alpha}{p}\int_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^p\, d\varrho + \mathcal{O}(\varepsilon^p). \tag{53}$$

By (50), (51), (52) and (53) we obtain

$$\int_{\mathbb{R}^n} \nabla(\operatorname{div} u) g_\varepsilon(|u|)^{p-2} u \frac{dx}{|x|^\alpha} =$$

$$= -\omega_{n-1}\Big[(n-1)\int_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2\, d\varrho +$$

$$+\frac{\alpha(n-2-\alpha)}{p}\int_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^p\, d\varrho +$$

$$+\int_K \partial_\varrho u_\varrho \partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho\big)\varrho^{n-1-\alpha}\, d\varrho\Big] + \mathcal{O}(\varepsilon^p). \tag{54}$$

From (49) and (54) it follows that

$$\int_{\mathbb{R}^n} \Big(\Delta u + \frac{1}{1-2\nu}\nabla \operatorname{div} u\Big) g_\varepsilon(|u|)^{p-2} u \frac{dx}{|x|^\alpha} =$$

$$= -\omega_{n-1}\frac{2(1-\nu)}{1-2\nu}\Big[(n-1)\int_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho^2\, d\varrho +$$

$$+\frac{\alpha(n-2-\alpha)}{p}\int_K \varrho^{n-3-\alpha} g_\varepsilon(|u_\varrho|)^p\, d\varrho +$$

$$+\int_K \partial_\varrho u_\varrho \partial_\varrho\big(g_\varepsilon(|u_\varrho|)^{p-2} u_\varrho\big)\varrho^{n-1-\alpha}\, d\varrho\Big] + \mathcal{O}(\varepsilon^p).$$

Seeing that, given $a \in \mathbb{R}$, there exists a constant C_α such that $(g_\varepsilon(s))^a \leqslant C_\alpha(s^a + \varepsilon^a)$ $(s \geqslant 0)$, we may apply the dominated convergence theorem and find

$$\int\limits_{\mathbb{R}^n} \Big(\Delta u + \frac{1}{1-2\nu}\nabla \operatorname{div} u\Big)|u|^{p-2}u\,\frac{dx}{|x|^\alpha} =$$

$$= -\omega_{n-1}\frac{2(1-\nu)}{1-2\nu}\bigg\{\Big[n-1+\frac{\alpha(n-2-\alpha)}{p}\Big]\int\limits_K \varrho^{n-3-\alpha}|u_\varrho|^p\,d\varrho +$$

$$+\int\limits_K \partial_\varrho u_\varrho \partial_\varrho(|u_\varrho|^{p-2}u_\varrho)\varrho^{n-1-\alpha}\,d\varrho\bigg\}.$$

Keeping in mind that either $\nu > 1$ or $\nu < 1/2$, the last equality shows that (45) holds if and only if

$$\Big[n-1+\frac{\alpha(n-2-\alpha)}{p}\Big]\int\limits_K \varrho^{n-3-\alpha}|u_\varrho|^p\,d\varrho +$$

$$+\int\limits_K \partial_\varrho u_\varrho \partial_\varrho(|u_\varrho|^{p-2})\varrho^{n-1-\alpha}\,d\varrho \geqslant 0. \quad (55)$$

Setting $v_\varrho = |u_\varrho|^{(p-2)/2}u_\varrho$, we see that (55) is equivalent to

$$\Big[n-1+\frac{\alpha(n-2-\alpha)}{p}\Big]\int\limits_0^{+\infty} |v_\varrho|^2\varrho^{n-3-\alpha}\,d\varrho +$$

$$+\frac{4}{p\,p'}\int\limits_0^{+\infty}(\partial_\varrho v_\varrho)^2\varrho^{n-1-\alpha}\,d\varrho \geqslant 0. \quad (56)$$

If $\alpha = n-2$ the inequality (56) is obviously satisfied. For $\alpha \neq n-2$, we recall the Hardy inequality (see, for instance, Maz'ya [13, p. 40])

$$\int\limits_0^{+\infty}\frac{v^2(\varrho)}{\varrho^{\alpha-n+3}}\,d\varrho \leqslant \frac{4}{(\alpha-n+2)^2}\int\limits_0^{+\infty}\frac{(\partial_\varrho v(\varrho))^2}{\varrho^{\alpha-n+1}}\,d\varrho, \quad (57)$$

which holds for any $v \in C_0^\infty(\mathbb{R})$ provided $\alpha \neq n-2$, under the condition $v(0) = 0$ when $\alpha > n-2$.

Inequality (56) can be written as

$$-\frac{p\,p'}{4}\Big[n-1+\frac{\alpha(n-2-\alpha)}{p}\Big]\int\limits_0^{+\infty}|v_\varrho|^2\varrho^{n-3-\alpha}\,d\varrho \leqslant$$

$$\leqslant \int\limits_0^{+\infty}(\partial_\varrho v_\varrho)^2\varrho^{n-1-\alpha}\,d\varrho. \quad (58)$$

Now we see that (58) holds if and only if

$$-\frac{p\,p'}{4}\left[n-1+\frac{\alpha(n-2-\alpha)}{p}\right]\leqslant\frac{(\alpha-n+2)^2}{4}\,. \tag{59}$$

In fact, if (59) holds, then (58) is true, because of (57). Viceversa, if (58) holds, thanks to the arbitrariness of v_ϱ and to the sharpness of the constant in (57), we get (59).

A simple manipulation shows that the latter inequality is equivalent to

$$-\frac{(\alpha-(n+p-2))(\frac{\alpha}{p-1}+(n+p'-2))}{p\,p'}\geqslant 0,$$

which in turn is equivalent to (46). The theorem is proved. □

We remark that the inequalities

$$-(p-1)(n+p'-2)<0<n+p-2$$

are always satisfied and therefore condition (46) is never empty.

References

1. A. Cialdea and V. Maz'ya, Criterion for the L^p-dissipativity of second order differential operators with complex coefficients. *J. Math. Pures Appl. (9)* **84** (2005), No. 8, 1067–1100.
2. A. Cialdea and V. Maz'ya, Criteria for the L^p-dissipativity of systems of second order differential equations. *Ric. Mat.* **55** (2006), No. 2, 233–265.
3. A. Douglis and L. Nirenberg, Interior estimates for elliptic systems of partial differential equations. *Comm. Pure Appl. Math.* **8** (1955), 503–538.
4. G. Fichera, Linear elliptic differential systems and eigenvalue problems. *Lecture Notes in Mathematics*, 8 *Springer-Verlag, Berlin–New York*, 1965.
5. M. E. Gurtin, The linear theory of elasticity. In *Mechanics of Solids*, Vol. II, C. Truesdell (ed.), *Springer-Verlag*, 1984.
6. V. D. Kupradze, Methods of a potential in the theory of elasticity. *Fizmatgiz, Moscow*, 1963.
7. V. D. Kupradze, Progress in Solid Mechanics: Dynamical Problems in Elasticity. Eds. Ian Naismith Sneddon, and Rodney Hill. *North-Holland Publishing Company*, 1963.
8. V. D. Kupradze, Potential methods in the theory of elasticity. Translated from the Russian by H. Gutfreund. Translation edited by I. Meroz Israel Program for Scientific Translations, Jerusalem; *Daniel Davey & Co., Inc., New York*, 1965.
9. V. D. Kupradze and T. V. Burchuladze, The dynamical problems of the theory of elasticity and thermoelasticity. (Russian) *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat.* **7** (1975), 163–294; translation in *J. Sov. Math.* **7** (1977), 415–500.
10. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems in the mathematical theory of elasticity. (Russian) *Izdat. Tbilis. Univ., Tbilisi*, 1968.
11. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. (Russian) Edited by V. D. Kupradze. *Izdat. "Nauka", Moscow*, 1976.

12. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. Translated from the second Russian edition. Edited by V. D. Kupradze. *North-Holland Series in Applied Mathematics and Mechanics*, 25. *North-Holland Publishing Co., Amsterdam–New York*, 1979.
13. V. Maz'ya, Sobolev spaces with applications to elliptic partial differential equations. Second, revised and augmented edition. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*], 342. *Springer, Heidelberg*, 2011.
14. V. G. Maz'ya, and T. O. Shaposhnikova, Theory of Sobolev multipliers. With applications to differential and integral operators. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*], 337. *Springer-Verlag, Berlin*, 2009.

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MELLIN CONVOLUTION OPERATORS IN BESSEL POTENTIAL SPACES WITH ADMISSIBLE MEROMORPHIC KERNELS

Dedicated to the memory of Academician Victor Kupradze on the occasion of his 110-th birthday anniversary

Abstract. The paper is devoted to Mellin convolution operators with meromorphic kernels in Bessel potential spaces. We encounter such operators while investigating boundary value problems for elliptic equations in planar 2D domains with angular points on the boundary.

Our study is based upon two results. The first concerns commutants of Mellin convolution and Bessel potential operators: Bessel potentials alter essentially after commutation with Mellin convolutions depending on the poles of the kernel (in contrast to commutants with Fourier convolution operatiors.) The second basic ingredient is the results on the Banach algebra $\mathfrak{A}_p$ generated by Mellin convolution and Fourier convolution operators in weighted $\mathbb{L}_p$-spaces obtained by the author in 1970's and 1980's. These results are modified by adding Hankel operators. Examples of Mellin convolution operators are considered.

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Key words and phrases. Fourier convolution, Mellin convolution, Bessel potentials, meromorphic kernel, Banach algebra, symbol, fixed singularity, Fredholm property, index.

რეზიუმე. ნაშრომი მიძღვნილია მელინის კონვოლუციის ოპერატორებისადმი მერომორფული ბირთვებით, რომლებიც მოქმედებენ ბესელის პოტენციალთა სივრცეებში. ასეთი ოპერატორები გვხვდება სასაზღვრო ამოცანების გამოკვლევებში ელიფსური დიფერენციალური განტოლებებისათვის ბრტყელ 2-განზომილებიან არეებში კუთხოვანი საზღვრით.

ჩვენი გამოკვლევა ეყრდნობა ორ შედეგს. პირველი ეხება მელინის კონვოლუციების და ბესელის პოტენციალების კომუტანტს: ბესელის პოტენციალის ოპერატორი განიცდის არსებით ცვლილებას მელინის კონვოლუციის ოპერატორთან გადასმის შედეგად და ეს ცვლილება დამოკიდებულია მერომორფული ბირთვის პოლუსებზე (ფურიეს კონვოლუციის ოპერატორისაგან განსხვავებით, რომელთანაც გადასმის შედეგად ბესელის პოტენციალი არ იცვლება). მეორე მნიშვნელოვანი შედეგი, რომელსაც ვეყრდნობით, წარმოადგენს ოპერატორების გამოკვლევის შედეგებს ბანახის ალგებრიდან $\mathfrak{A}_p$, რომელიც წარმოქმნილია მელინის კონვოლუციის და ფურიეს კონვოლუციის ოპერატორების მიერ წონიან $\mathbb{L}_p$-სივრცეებში, რომლებიც მიღებულია 1980-იან წლებში სტატიის ავტორის მიერ. ამ შედეგებზე დამატებულია ახალი შედეგი, რომელიც ეხება ჰანკელის ოპერატორებს. განხილულია მელინის კონვულუციის ოპერატორების მაგალითები.

Introduction

It is well-known that various boundary value problems for PDE in planar domains with angular points on the boundary, e.g. Lamé systems in elasticity (cracks in elastic media, reinforced plates), Maxwell's system and Helmholtz equation in electromagnetic scattering, Cauchy–Riemann systems, Carleman–Vekua systems in generalized analytic function theory etc. can be studied with the help of the Mellin convolution equations of the form

$$\mathbf{A}\varphi(t) := c_0\varphi(t) + \frac{c_1}{\pi i}\int\limits_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \int\limits_0^\infty \mathscr{K}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau} = f(t), \tag{1}$$

with the kernel $\mathscr{K}$ satisfying the condition

$$\int\limits_0^\infty t^{\beta-1}|\mathscr{K}(t)|\,dt < \infty, \quad 0 < \beta < 1, \tag{2}$$

which makes it a bounded operator in the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$, provided $1 \leqslant p \leqslant \infty$, $-1 < \gamma < p - 1$, $\beta := (1+\gamma)/p$ (cf. [17]).

In particular, integral equations with fixed singularities in the kernel

$$c_0(t)\varphi(t) + \frac{c_1(t)}{\pi i}\int\limits_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \\ + \sum_{k=0}^n \frac{c_{k+2}(t)t^{k-r}}{\pi i}\int\limits_0^\infty \frac{\tau^r\varphi(\tau)\,d\tau}{(\tau+t)^{k+1}} = f(t), \quad 0 \leqslant t \leqslant 1, \tag{3}$$

where $0 \leqslant r \leqslant k$ are of type (1) after localization, i.e. after "freezing" the coefficients.

The Fredholm theory and the unique solvability of equations (1) in the weighted Lebesgue spaces were accomplished in [17]. This investigation was based on the following observation: if $1 < p < \infty$, $-1 < \gamma < p - 1$, $\beta := (1+\gamma)/p$, the following mutually invertible exponential transformations

$$\begin{gathered} Z_\beta : \mathbb{L}_p([0,1], t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \\ Z_\beta\varphi(\xi) := e^{-\beta\xi}\varphi(e^{-\xi}), \quad \xi \in \mathbb{R} := (-\infty, \infty), \\ Z_\beta^{-1} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p([0,1], t^\gamma), \\ Z_\beta^{-1}\psi(t) := t^{-\beta}\psi(-\ln t), \quad t \in \mathbb{R}^+ := (0, \infty), \end{gathered} \tag{4}$$

transform the equation (1), treated in the weighted Lebesgue space $f, \varphi \in \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ into the Fourier convolution equation $W^0_{\mathscr{A}_\beta}\psi = g$, $\psi = Z_\beta\varphi, g = Z_\beta f \in \mathbb{L}_p(\mathbb{R})$ of the form

$$W^0_{\mathscr{A}_\beta}\psi(x) = c_0\psi(x) + \int\limits_{-\infty}^\infty \mathscr{K}_1(x-y)\varphi(y)\,dy,$$

$$\mathscr{K}_1(x) = e^{-\beta x}\Big[\frac{c_1}{1-e^{-x}} + \mathscr{K}(e^{-x})\Big].$$

Note that the symbol of the operator $W^0_{\mathscr{A}_\beta}$, viz. the Fourier transform of the kernel

$$\mathscr{A}_\beta(\xi) := c_0 + \int\limits_{-\infty}^{\infty} e^{i\xi x}\mathscr{K}_1(x)\,dx$$

$$:= c_0 - ic_1 \cot\pi(\beta - i\xi) + \int\limits_{-\infty}^{\infty} e^{(i\xi-\beta)x}\mathscr{K}(e^{-x})\,dx, \ \ \xi \in \mathbb{R} \qquad (5)$$

is a piecewise continuous function. Let us recall that the theory of Fourier convolution operators with discontinuous symbols is well developed, cf. [13, 14, 15, 16, 42]. This allows one to investigate various properties of the operators (1), (3). In particular, Fredholm criteria, index formula and conditions of unique solvability of the equations (1) and (3) have been established in [17].

Similar integral operators with fixed singularities in kernel arise in the theory of singular integral equations with the complex conjugation

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)\,dt}{\tau - t} + \frac{e(t)}{\pi i}\overline{\int\limits_\Gamma \frac{\overline{\varphi(\tau)}\,dt}{\tau - t}} = f(t), \ \ t \in \Gamma$$

and in more general R-linear equations

$$a(t)\varphi(t) + b(t)\overline{\varphi(t)} + \frac{c(t)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)\,dt}{\tau - t} + \frac{d(t)}{\pi i}\int\limits_\Gamma \frac{\overline{\varphi(\tau)}\,dt}{\tau - t} +$$

$$+ \frac{e(t)}{\pi i}\overline{\int\limits_\Gamma \frac{\varphi(\tau)\,dt}{\tau - t}} + \frac{g(t)}{\pi i}\overline{\int\limits_\Gamma \frac{\overline{\varphi(\tau)}\,dt}{\tau - t}} = f(t), \ \ t \in \Gamma,$$

if the contour Γ possesses corner points. Note that a complete theory of such equations is presented in [24, 25], whereas approximation methods have been studied in [10, 11].

Let $t_1, \dots, t_n \in \Gamma$ be the corner points of a piecewise-smooth contour Γ, and let $\mathbb{L}_p(\Gamma, \rho)$ denote the weighted $\mathbb{L}_p$-space with a power weight $\rho(t) := \prod\limits_{j=1}^{n} |t - t_j|^{\gamma_j}$. Assume that the parameters p and $\beta_j := (1 + \gamma_j)/p$ satisfy the conditions

$$1 < p < \infty, \ \ 0 < \beta_j < 1, \ \ j = 1, \dots, n.$$

If the coefficients of the above equations are piecewise-continuous matrix functions, one can construct a function $\mathscr{A}_{\vec{\beta}}(t, \xi)$, $t \in \Gamma$, $\xi \in \mathbb{R}$, $\vec{\beta} := (\beta_1, \dots, \beta_n)$, called the symbol of the equation (of the related operator). It is possible to express various properties of the equation in terms of $\mathscr{A}_{\vec{\beta}}$:

- The equation is Fredholm in $\mathbb{L}_p(\Gamma, \rho)$ if and only if its symbol is elliptic., i.e. iff $\inf_{(t,\xi)\in\Gamma\times\mathbb{R}} |\, \mathscr{A}_{\vec{\beta}}(t,\xi)| > 0$;
- To an elliptic symbol $\mathscr{A}_{\vec{\beta}}(t,\xi)$ there corresponds an integer valued index
  ind $\mathscr{A}_{\vec{\beta}}(t,\xi)$, the winding number, which coincides with the Fredholm index of the corresponding operator modulo a constant multiplier.

For more detailed survey of the theory and various applications to the problems of elasticity we refer the reader to [13, 14, 15, 17, 18, 19, 20, 21, 40].

Similar approach to boundary integral equations on curves with corner points based on Mellin transformation has been exploited by M. Costabel and E. Stephan [5, 6].

However, one of the main problems in boundary integral equations for elliptic partial differential equations is the absence of appropriate results for Mellin convolution operators in Bessel potential spaces, cf. [18, 20, 21] and recent publications on nano-photonics [1, 2, 32]. Such results are needed to obtain an equivalent reformulation of boundary value problems into boundary integral equations in Bessel potential spaces. Nevertheless, numerous works on Mellin convolution equations seem to pay almost no attention to the mentioned problem.

The first arising problem is the boundedness results for Mellin convolution operators in Bessel potential spaces. The conditions on kernels known so far are very restrictive. The following boundedness result for the Mellin convolution operator is proved in the yet unpublished paper by V. Didenko and R. Duduchava.

Proposition 0.1. *Let $1 < p < \infty$ and let $m = 1, 2, \ldots$ be an integer. If a function $\mathscr{K}$ satisfies the condition*

$$\int_0^1 t^{\frac{1}{p}-m-1}|\mathscr{K}(t)|\,dt + \int_1^\infty t^{\frac{1}{p}-1}|\mathscr{K}(t)|\,dt < \infty, \tag{6}$$

then the Mellin convolution operator (see (1)*)*

$$\boldsymbol{A} = \mathfrak{M}^0_{\mathscr{A}_{1/p}} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+) \tag{7}$$

with the symbol (*see* (5))

$$\mathscr{A}_{1/p}(\xi) := c_0 + c_1 \coth \pi\Big(\frac{i}{p} + \xi\Big) + \int_0^\infty t^{\frac{1}{p}-i\xi}\mathscr{K}(t)\,\frac{dt}{t}\,, \quad \xi \in \mathbb{R}, \tag{8}$$

is bounded for any $0 \leqslant s \leqslant m$.

Note that the condition

$$K_\beta := \int_0^\infty t^{\beta-1}|\mathscr{K}(t)|\,dt < \infty \tag{9}$$

and the constraints (16) ensure that the operator

$$\mathfrak{M}^0_a : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$$

is bounded and the norm of the Mellin convolution

$$\mathfrak{M}^0_{a_\beta}\varphi(t) := \int_0^\infty \mathscr{K}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau}\,. \tag{10}$$

admits the estimate $\|\mathfrak{M}^0_{a_\beta}\| \leqslant K_\beta$.

The above-formulated result has very restricted application. For example, the operators

$$\begin{aligned} \boldsymbol{N}_\alpha\varphi(t) &:= \frac{\sin\alpha}{\pi}\int_0^\infty \frac{t\,\varphi(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\alpha}\,,\\ \boldsymbol{N}^*_\alpha\varphi(t) &:= \frac{\sin\alpha}{\pi}\int_0^\infty \frac{\tau\,\psi_j(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\alpha}\,,\\ \boldsymbol{M}_\alpha\varphi(t) &:= \frac{1}{2\pi}\int_{\mathbb{R}^+} \frac{[\tau\,\cos\alpha-t]\varphi(\tau)\,d\tau}{t^2+\tau^2-2t\,\tau\cos\alpha}\,,\quad -\pi<\alpha<\pi, \end{aligned} \tag{11}$$

which we encounter in boundary integral equations for elliptic boundary value problems (see [4, 27]), as well as the operators

$$\boldsymbol{N}_{m,k}\varphi(t) := \frac{t^k}{\pi i}\int_0^\infty \frac{\tau^{m-k}\varphi(\tau)\,d\tau}{(\tau+t)^{m+1}}\,,\quad k=0,\dots,m, \tag{12}$$

represented in (3), do not satisfy the conditions (6). In particular, $\boldsymbol{N}_\alpha$ satisfies condition (6) only for $m=1$ and $\boldsymbol{N}_{m,k}$ only for $m=k$. Although, as we will see below in Theorem 2.5, all operators $\boldsymbol{N}_\alpha$, $\boldsymbol{N}^*_\alpha$ and $\boldsymbol{N}_{m,k}$ are bounded in Bessel potential spaces in the setting (17) for all $s\in\mathbb{R}$.

In the present paper we introduce *admissible kernels*, which are meromorphic functions on the complex plane $\mathbb{C}$, vanishing at the infinity

$$\begin{aligned} &\mathscr{K}(t) := \sum_{j=0}^{\ell}\frac{d_j}{t-c_j} + \sum_{j=\ell+1}^{\infty}\frac{d_j}{(t-c_j)^{m_j}}\,,\quad c_j\neq 0,\quad j=0,1,\dots,\\ &\qquad c_0,\dots,c_\ell\in\mathbb{R},\quad 0<\alpha_k:=|\arg c_k|\leqslant\pi,\quad k=\ell+1,\ell+2,\dots \end{aligned} \tag{13}$$

having poles at $c_0, c_1, \ldots \in \mathbb{C}\setminus\{0\}$ and complex coefficients $d_j\in\mathbb{C}$. The Mellin convolution operator

$$\boldsymbol{K}^m_c\varphi(t) := \int_0^\infty \frac{\tau^{m-1}\varphi(\tau)\,d\tau}{(t-c\,\tau)^m}. \tag{14}$$

corresponding to the kernel

$$\mathscr{K}(t) := \frac{1}{(t-c)^m},\quad c_j\neq 0$$

(see Definition 2.1) turns out to be bounded in the Bessel potential spaces (see Theorem 2.5).

In order to study Mellin convolution operators in Bessel potential spaces, we use the "lifting" procedure, performed with the help of the Bessel potential operators Λ^s_+ and Λ^{s-r}_-, which transform the initial operator $\mathfrak{M}^0_a$ into the lifted operator $\Lambda^{s-r}_-\mathfrak{M}^0_a\Lambda^{-s}_+$ acting already on a Lebesgue $\mathbb{L}_p$ spaces. However, the lifted operator is neither Mellin nor Fourier convolution and to describe its properties, one has to study the commutants of Bessel potential operators and Mellin convolutions with meromorphic kernels. It turns out that Bessel potentials alter after commutation with Mellin convolutions and the result depends essentially on poles of the meromorphic kernels. These results allows us to show that the lifted operator $\Lambda^{s-r}_-\mathfrak{M}_a\Lambda^{-s}_+$ belongs to the Banach algebra of operators generated by Mellin and Fourier convolution operators with discontinuous symbols. Since such algebras have been studied before [22], one can derive various information (Fredholm properties, index, the unique solvability) about the initial Mellin convolution equation $\mathfrak{M}^0_a\varphi = g$ in Bessel potential spaces in the settings $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $g \in \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+)$ and in the settings $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $g \in \mathbb{H}^{s-r}_p(\mathbb{R}^+)$.

The results of the present work will be applied to the investigation of some boundary value problems studied before by Lax–Milgram Lemma in [1, 2]. Note that the present approach is more flexible and provides better tools for analyzing the solvability of the boundary value problems and the asymptotic behavior of their solutions.

It is worth noting that the obtained results can also be used to study Schrödinger operator on combinatorial and quantum graphs. Such a problem has attracted a lot of attention recently, since the operator mentioned above possesses interesting properties and has various applications, in particular, in nano-structures (see [36, 37] and the references there). Another area for application of the present results are Mellin pseudodifferential operators on graphs. This problem has been studied in [39], but in the periodic case only. Moreover, some of the results can be applied in the study of stability of approximation methods for Mellin convolution equations in Bessel potential spaces.

The present paper is organized as follows. In the first section we observe Mellin and Fourier convolution operators with discontinuous symbols acting on Lebesgue spaces. Most of these results are well known and we recall them for convenience. In the second section we define Mellin convolutions with admissible meromorphic kernels and prove their boundedness in Bessel potential spaces. In Section 2 is proved the key result on commutants of the Mellin convolution operator (with admissible meromorphic kernel) and a Bessel potential. In Section 3 we enhance results on Banach algebra generated by Mellin and Fourier convolution operators by adding explicit definition of the symbol of a Hankel operator, which belong to this algebra. In Sections 4 the obtained results are applied to describe Fredholm properties and the index of Mellin convolution operators with admissible meromorphic kernels in Bessel potential spaces.

1. Mellin Convolution and Bessel Potential Operators

Let N be a positive integer. If there arises no confusion, we write $\mathfrak{A}$ for both scalar and matrix $N\times N$ algebras with entries from $\mathfrak{A}$. Similarly, the same notation $\mathfrak{B}$ is used for the set of N-dimensional vectors with entries from $\mathfrak{B}$. It will be usually clear from the context what kind of space or algebra is considered.

The integral operator (1) is called Mellin convolution. More generally, if $a\in\mathbb{L}_\infty(\mathbb{R})$ is an essentially bounded measurable $N\times N$ matrix function, the Mellin convolution operator $\mathfrak{M}^0_a$ is defined by

$$\mathfrak{M}^0_a\varphi(t):=\mathscr{M}^{-1}_\beta a\mathscr{M}_\beta\varphi(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}a(\xi)\int\limits_0^\infty\Big(\frac{t}{\tau}\Big)^{i\xi-\beta}\varphi(\tau)\,\frac{d\tau}{\tau}\,d\xi,\ \ \varphi\in\mathbb{S}(\mathbb{R}^+),$$

where $\mathbb{S}(\mathbb{R}^+)$ is the Schwartz space of fast decaying functions on $\mathbb{R}^+$, whereas $\mathscr{M}_\beta$ and $\mathscr{M}^{-1}_\beta$ are the Mellin transform and its inverse, i.e.

$$\mathscr{M}_\beta\psi(\xi):=\int\limits_0^\infty t^{\beta-i\xi}\psi(t)\,\frac{dt}{t},\ \ \xi\in\mathbb{R},$$
$$\mathscr{M}^{-1}_\beta\varphi(t):=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}t^{i\xi-\beta}\varphi(\xi)\,d\xi,\ \ t\in\mathbb{R}^+.$$

The function $a(\xi)$ is usually referred to as a symbol of the Mellin operator $\mathfrak{M}^0_a$. Further, if the corresponding Mellin convolution operator $\mathfrak{M}^0_a$ is bounded on the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+,t^\gamma)$ of N-vector functions endowed with the norm

$$\big\|\varphi\mid\mathbb{L}_p(\mathbb{R}^+,t^\gamma)\big\|:=\bigg[\int\limits_0^\infty t^\gamma|\varphi(t)|^p\,dt\bigg]^{1/p},$$

then the symbol $a(\xi)$ is called an $\mathbb{L}_p(\mathbb{R}^+,t^\gamma)$ *Mellin multiplier*. The transformations

$$\mathbf{Z}_\beta:\mathbb{L}_p(\mathbb{R}^+,t^\gamma)\longrightarrow\mathbb{L}_p(\mathbb{R}),\quad \mathbf{Z}_\beta\varphi(\xi):=e^{-\beta t}\varphi(e^{-\xi}),\ \ \xi\in\mathbb{R},$$
$$\mathbf{Z}^{-1}_\beta:\mathbb{L}_p(\mathbb{R})\longrightarrow\mathbb{L}_p(\mathbb{R}^+,t^\gamma),\quad \mathbf{Z}^{-1}_\beta\psi(t):=t^{-\beta}\psi(-\ln t),\ \ t\in\mathbb{R}^+,$$

generate an isometrical isomorphism between the corresponding $\mathbb{L}_p$-spaces. Moreover, the relations

$$\begin{gathered}\mathscr{M}_\beta=\mathscr{F}\mathbf{Z}_\beta,\quad \mathscr{M}^{-1}_\beta=\mathbf{Z}^{-1}_\beta\mathscr{F}^{-1},\\ \mathfrak{M}^0_a=\mathscr{M}^{-1}_\beta a\mathscr{M}_\beta=\mathbf{Z}^{-1}_\beta\mathscr{F}^{-1}a\mathscr{F}\mathbf{Z}_\beta=\mathbf{Z}^{-1}_\beta W^0_a\mathbf{Z}_\beta,\end{gathered}\tag{15}$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transform and its inverse,

$$\mathscr{F}\varphi(\xi) := \int_{-\infty}^{\infty} e^{i\xi x}\varphi(x)\,dx, \quad \mathscr{F}^{-1}\psi(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\psi(\xi)\,d\xi, \ \ x \in \mathbb{R},$$

show a close connection between Mellin $\mathfrak{M}^0_a$ and Fourier

$$W^0_a\varphi := \mathscr{F}^{-1}a\mathscr{F}\varphi, \ \ \varphi \in \mathbb{S}(\mathbb{R}),$$

convolution operators, as well as between the corresponding transforms. Here $\mathbb{S}(\mathbb{R})$ denotes the Schwartz class of infinitely smooth functions, decaying fast at the infinity.

An $N \times N$ matrix function $a(\xi)$, $\xi \in \mathbb{R}$ is called a *Fourier* $\mathbb{L}_p$*-multiplier* if the operator $W^0_a : \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R})$ is bounded. The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$.

From (15) immediately follows the following

Proposition 1.1. *The class $\mathfrak{M}_p(\mathbb{R})$ of Fourier $\mathbb{L}_p$-multipliers coincides with the class of Mellin $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ multiplier.*

It is known, see, e.g. [17], that $\mathfrak{M}_p(\mathbb{R})$ is a Banach algebra which contains the algebra $\mathbf{V}_1(\mathbb{R})$ of all functions with finite variation provided that

$$\beta := \frac{1+\gamma}{p}, \ \ 1 < p < \infty, \ \ -1 < \gamma < p-1. \tag{16}$$

As it was already mentioned, the primary aim of the present paper is to study Mellin convolution operators $\mathfrak{M}^0_a$ acting in Bessel potential spaces,

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+). \tag{17}$$

The symbols of these operators are $N \times N$ matrix functions $a \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, continuous on the real axis $\mathbb{R}$ with the only one possible jump at infinity. We commence with the definition of the Besseel potential spaces and Bessel potentials, arranging isometrical isomorphisms between these spaces and enabling the lifting procedure, writing a Fredholm equivalent operator (equation) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ for the operator $\mathfrak{M}^0_a$ in (17).

For $s \in \mathbb{R}$ and $1 < p < \infty$, the Bessel potential space, known also as a fractional Sobolev space, is the subspace of the Schwartz space $\mathbb{S}'(\mathbb{R})$ of distributions having the finite norm

$$\big\|\varphi \mid \mathbb{H}^s_p(\mathbb{R})\big\| := \bigg[\int_{-\infty}^{\infty} \big|\mathscr{F}^{-1}\big(1+|\xi|^2\big)^{s/2}(\mathscr{F}\varphi)(t)\big|^p\,dt\bigg]^{1/p} < \infty.$$

For an integer parameter $s = m = 1, 2, \dots$, the space $\mathbb{H}^s_p(\mathbb{R})$ coincides with the usual Sobolev space endowed with an equivalent norm

$$\big\|\varphi \mid \mathbb{W}^m_p(\mathbb{R})\big\| := \bigg[\sum_{k=0}^{m}\int_{-\infty}^{\infty} \Big|\frac{d^k\varphi(t)}{dt^k}\Big|^p\,dt\bigg]^{1/p}. \tag{18}$$

If $s<0$, one gets the space of distributions. Moreover, $\mathbb{H}^{-s}_{p'}(\mathbb{R})$ is the dual to the space $\mathbb{H}^s_p(\mathbb{R}^+)$, provided $p' := \frac{p}{p-1}$, $1<p<\infty$. Note that $\mathbb{H}^s_2(\mathbb{R})$ is a Hilbert space with the inner product

$$\langle\varphi,\psi\rangle_s = \int\limits_{\mathbb{R}} (\mathscr{F}\varphi)(\xi)\overline{(\mathscr{F}\psi)(\xi)}(1+\xi^2)^s\,d\xi, \quad \varphi,\psi\in\mathbb{H}^s(\mathbb{R}).$$

By r_Σ we denote the operator restricting functions or distributions defined on $\mathbb{R}$ to the subset $\Sigma\subset\mathbb{R}$. Thus $\mathbb{H}^s_p(\mathbb{R}^+) = r_+(\mathbb{H}^s_p(\mathbb{R}))$, and the norm in $\mathbb{H}^s_p(\mathbb{R}^+)$ is defined by

$$\big\| f \mid \mathbb{H}^s_p(\mathbb{R}^+)\big\| = \inf_{\ell} \big\|\ell f \mid \mathbb{H}^s_p(\mathbb{R})\big\|,$$

where ℓf stands for any extension of f to a distribution in $\mathbb{H}^s_p(\mathbb{R})$.

Further, we denote by $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ the (closed) subspace of $\mathbb{H}^s_p(\mathbb{R})$ which consists of all distributions supported in the closure of $\mathbb{R}^+$.

Notice that $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ is always continuously embedded in $\mathbb{H}^s_p(\mathbb{R}^+)$, and if $s\in(1/p-1,1/p)$, these two spaces coincide. Moreover, $\mathbb{H}^s_p(\mathbb{R}^+)$ may be viewed as the quotient-space $\mathbb{H}^s_p(\mathbb{R}^+) := \mathbb{H}^s_p(\mathbb{R})/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^-)$, $\mathbb{R}^- := (-\infty,0)$.

Let $a\in\mathbb{L}_{\infty,loc}(\mathbb{R})$ be a locally bounded $m\times m$ matrix function. The Fourier convolution operator (FCO) with the symbol a is defined by

$$W^0_a := \mathscr{F}^{-1}a\mathscr{F}. \tag{19}$$

If the operator

$$W^0_a : \mathbb{H}^s_p(\mathbb{R}) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}) \tag{20}$$

is bounded, we say that a is an $\mathbb{L}_p$-multiplier (of order 0). The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$.

The Fourier convolution operator (FCO) on the semi-axis $\mathbb{R}^+$ with the symbol a is defined by $W_a = r_+W^0_a$ where $r_+ := r_{\mathbb{R}^+} : \mathbb{H}^s_p(\mathbb{R}) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$ is the restriction operator.

Consider FCO

$$W_a = r_+W^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+), \tag{21}$$

and Hankel operators

$$H_a = r_+\boldsymbol{V}W^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+), \quad \boldsymbol{V}\psi(t) := \psi(-t), \tag{22}$$

where r_+ is the restriction operator to the semi-axes $\mathbb{R}^+$. Note that the generalized Hoermander's kernel of a FCO W_a depends on the difference of arguments $\mathscr{K}_a(t-\tau)$, while the Hoermander's kernel ä of a Hankel operator $r_+\boldsymbol{V}W^0_a$ depends of the sum of the arguments $\mathscr{K}_a(t+\tau)$.

If W_a in (22) is bounded, we say that W_a has order r and a is an $\mathbb{L}_p$ multiplier of order r. The set of all $\mathbb{L}_p$ multipliers of order r is denoted by $\mathfrak{M}^r_p(\mathbb{R})$. We did not use in the definition of the class of multipliers $\mathfrak{M}^r_p(\mathbb{R})$ the parameter $s\in\mathbb{R}$. This is due to the fact that $\mathfrak{M}^r_p(\mathbb{R})$ is independent of s: if the operator W_a in (22) is bounded for some $s\in\mathbb{R}$, it is bounded for all other values of s. Another definition of the multiplier class $\mathfrak{M}^r_p(\mathbb{R})$

is written as follows: $a \in \mathfrak{M}_p^r(\mathbb{R})$ if and only if $\lambda^{-r}a \in \mathfrak{M}_p(\overline{\mathbb{R}}) = \mathfrak{M}_p^0(\overline{\mathbb{R}})$, where $\lambda^r(\xi) := (1+|\xi|^2)^{r/2}$. This assertion is one of the consequences of the following theorem.

Theorem 1.2. *Let $1 < p < \infty$. Then*

(1) *For any $r, s \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\operatorname{Im}\gamma > 0$ the convolution operators (ΨDOs)*

$$\begin{aligned} \Lambda^r_\gamma = W^0_{\lambda^r_\gamma} &: \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+), \\ \Lambda^r_{-\gamma} = r_+ W^0_{\lambda^r_{-\gamma}}\ell &: \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+), \\ \lambda^r_{\pm\gamma}(\xi) &:= (\xi \pm \gamma)^r, \quad \xi \in \mathbb{R}, \quad \operatorname{Im}\gamma > 0, \end{aligned} \tag{23}$$

which arrange isomorphisms of the corresponding spaces (*see* [17, 28]). *Here $\ell : \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R})$ is some extension operator, define an isomorphism between the corresponding spaces. The final result is independent of the choice of an extension ℓ. r_+ is the restriction from the axes $\mathbb{R}$ to the semi-axes $\mathbb{R}^+$.*

(2) *For any operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ of the order r, the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) & \xrightarrow{\ \mathbf{A}\ } & \mathbb{H}^{s-r}_p(\mathbb{R}^+) \\ \Lambda^{-s}_+ \big\uparrow & & \big\downarrow \Lambda^{s-r}_- \\ \mathbb{L}_p(\mathbb{R}^+) & \xrightarrow[\Lambda^{s-r}_- \mathbf{A} \Lambda^{-s}_+]{} & \mathbb{L}_p(\mathbb{R}^+) \end{array} . \tag{24}$$

The diagram (23) *provides an equivalent lifting of the operator $\mathbf{A}$ of order r to the operator $\Lambda^{s-r}_- \mathbf{A} \Lambda^{-s}_+ : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ of order* 0.

(3) *If $\mathbf{A} = W_a : \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ is a bounded convolution operator of order r, then the lifted operator $\Lambda^{s-r}_- \mathbf{A} \Lambda^{-s}_+ : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is also a convolution operator W_{a_0}, with the symbol*

$$a_0(\xi) = \lambda^{s-r}_{-\gamma}(\xi) a(\xi) \lambda^{-s}_\gamma(\xi) = \left(\frac{\xi - \gamma}{\xi + \gamma}\right)^{s-r} \frac{a(\xi)}{(\xi + i)^r} .$$

Proof. For the proof we refer the reader to [17, Lemma 5.1] and [26, 28]. □

Remark 1.3. The class of Fourier convolution operators is a subclass of pseudodifferential operators (ΨDOs). Moreover, for integer parameters $m = 1, 2, \dots$ the Bessel potentials $\Lambda^m_\pm = W_{\lambda^m_{\pm\gamma}}$, which are the Fourier convolutions of order m, are ordinary differential operators of the same order m:

$$\Lambda^m_{\pm\gamma} = W_{\lambda^m_{\pm\gamma}} = \left(i\,\frac{d}{dt} \pm \gamma\right)^m = \sum_{k=0}^m \binom{m}{k} i^k (\pm\gamma)^{m-k} \frac{d^k}{dt^k} . \tag{25}$$

These potentials map both spaces (cf. (23))

$$\begin{aligned} \Lambda^m_{\pm\gamma} &: \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+), \\ &: \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-m}_p(\mathbb{R}^+), \end{aligned} \tag{26}$$

but the mappings are not isomorphisms because the inverses $\Lambda^{-m}_{\pm\gamma}$ do not map both spaces, only those indicated in (23).

2. Mellin Convolutions with Admissible Meromorphic Kernels

Now we consider kernels $\mathscr{K}(t)$, exposed in (13), (14), which are meromorphic functions on the complex plane $\mathbb{C}$, vanishing at infinity, having poles at $c_0, c_1, \ldots \in \mathbb{C} \setminus \{0\}$ and complex coefficients $d_j \in \mathbb{C}$.

Definition 2.1. We call a kernel $\mathscr{K}(t)$ in (13) admissible iff:

(i) $\mathscr{K}(t)$ has only a finite number of poles $c_0, \ldots, c_\ell$ which belong to the positive semi-axes, i.e., $\arg c_0 = \cdots = \arg c_\ell = 0$;

(ii) The corresponding multiplicities are one $m_0 = \cdots = m_\ell = 1$;

(iii) The points $c_{\ell+1}, c_{\ell+2}, \ldots$ do not condense to the positive semi-axes except a finite number of points $c_0 > 0, \ldots, c_\ell > 0$ from conditions (i)–(ii) and their real parts are uniformly bounded

$$\varliminf_{j\longrightarrow\infty} c_j \notin [0,\infty), \qquad \sup_{j=\ell+1,\ell+2,\ldots} \operatorname{Re} c_j \leqslant K < \infty. \tag{27}$$

(iv) If $\mathscr{K}(t)$ emerges as a kernel of the operator, a superposition of finite number of operators with admissible kernels.

Example 2.2. The function

$$\mathscr{K}(t) = \exp\left(\frac{1}{t-c}\right), \quad \operatorname{Re} c < 0 \ \text{ or } \ \operatorname{Im} c \neq 0$$

is an example of the admissible kernel which also satisfies the condition of the next Theorem 2.5. More trivial examples of operators with admissible kernels (which also satisfies the condition of the next Theorem 2.5) are operators which we encounter in (3), in (11) and in (21) and, in general, any finite sum in (13).

Example 2.3. The function

$$\mathscr{K}(t) = \frac{\ln\tau - c_1c_2\ln t}{t - c_1c_2\tau}\,, \quad \operatorname{Im} c_1 \neq 0, \quad \operatorname{Im} c_2 \neq 0,$$

is another example of the admissible kernel, which is the composition of operators $c_2\boldsymbol{K}^1_{c_1}\boldsymbol{K}^1_{c_2}$ (see (14)) with admissible kernels which also satisfies the condition of the next Theorem 2.5. More trivial examples of operators with admissible kernels (which also satisfies the condition of the next Theorem 2.5) are operators which we encounter in (3), in (11) and in (21) and, in general, any finite sum in (13).

Theorem 2.4. *Let conditions* (16) *hold,* $\mathscr{K}(t)$ *in* (13) *be an admissible kernel and*

$$K_\beta := \frac{\pi}{|\sin\pi\beta|}\sum_{j=0}^{\infty} 2^{m_j}|d_j|\,|c_j|^{\beta-m_j} < \infty. \tag{28}$$

Then the Mellin convolution $\mathfrak{M}^0_{a_\beta}$ in (10) *with the admissible meromorphic kernel $\mathscr{K}(t)$ in* (13) *is bounded in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ and its norm is estimated by the constant $\|\mathfrak{M}^0_{a_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\| \leqslant MK_\beta$ with some $M > 0$.*

We can drop the constant M and replace 2^{m_j} by $2^{\frac{m_j}{2}}$ in the estimate (28) *provided $\operatorname{Re} c_j < 0$ for all $j = 0, 1, \dots$.*

Proof. The first $\ell + 1$ summands in the definition of the admissible kernel (13) correspond to the Cauchy operators

$$A_0\varphi(t) = \sum_{j=0}^{\ell} \frac{d_j}{\pi i} \int_0^\infty \frac{\varphi(\tau)\, d\tau}{t - c_j\tau}, \quad c_j > 0, \quad j = 0, 1, \dots, \ell,$$

and their boundedness property in the weighted Lebesgue space

$$A_0 : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \tag{29}$$

under constraints (16) is well known (see [35] and also [30]). Therefore we can ignore the first ℓ summands in the expansion of the kernel $\mathscr{K}(t)$ in (13). To the boundedness of the operator $\mathfrak{M}^0_{a^\ell_\beta}$ with the remainder kernel

$$\mathscr{K}^\ell(t) := \sum_{j=\ell+1}^{\infty} \frac{d_j}{(t - c_j)^{m_j}}, \quad c_j \neq 0, \quad j = 0, 1, \dots,$$
$$0 < \alpha_k := |\arg c_k| \leqslant \pi, \quad k = \ell + 1, \ell + 2, \dots$$

(see (13)), we apply the estimate (9)

$$\big\|\mathfrak{M}^0_{a^\ell_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\big\| \leqslant$$
$$\leqslant \int_0^\infty t^{\beta-1}|\mathscr{K}^\ell(t)|\, dt \leqslant \sum_{j=\ell+1}^{\infty} |d_j| \int_0^\infty \frac{t^{\beta-1} dt}{|t - c_j|^{m_j}}\,. \tag{30}$$

Note now that

$$|t - c_j|^{-m_j} = \big(t^2 + |c_j|^2 - 2\operatorname{Re} c_j t\big)^{-\frac{m_j}{2}} \leqslant \Big(\frac{t^2 + |c_j|^2}{2}\Big)^{-\frac{m_j}{2}} \leqslant$$
$$\leqslant 2^{m_j}(t + |c_j|)^{-m_j} \ \text{ for all } \ t \geqslant 2K = 2\sup|\operatorname{Re} c_j| > 0.$$

due to the constraints (27). On the other hand,

$$|t - c_j|^{-m_j} \leqslant M(t + |c_j|)^{-m_j} \ \text{ for all } \ 0 \leqslant t \leqslant 2K$$

and a certain constant $M > 0$. Therefore

$$|t - c_j|^{-m_j} \leqslant M2^{m_j}(t + |c_j|)^{-m_j} \ \text{ for all } \ 0 < t < \infty. \tag{31}$$

Next we recall the formula from [31, Formula 3.194.4]

$$\int_0^\infty \frac{t^{\beta-1}\,dt}{(t+c)^m} = (-1)^{m-1}\binom{\beta-1}{m-1}\frac{\pi c^{\,\beta-m}}{\sin\pi\beta}, \quad -\pi<\arg c<\pi, \quad \operatorname{Re}\beta<1, \qquad (32)$$

$$\binom{\beta-1}{m-1} := \frac{(\beta-1)\cdots(\beta-m+1)}{(m-1)!}, \quad \binom{\beta-1}{0} := 1$$

to calculate the integrals. By inserting the estimate (31) into (30) and applying (32), we get

$$\begin{aligned}\left\|\mathfrak{M}^0_{a^\ell_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\right\| &\leqslant \\ \leqslant \sum_{j=\ell+1}^\infty |d_j| \int_0^\infty \frac{t^{\beta-1}dt}{|t-c_j|^{m_j}} &\leqslant M \sum_{j=\ell+1}^\infty 2^{m_j}|d_j| \int_0^\infty \frac{t^{\beta-1}dt}{(t+|c_j|)^{m_j}} \leqslant \\ &\leqslant \frac{\pi M}{\sin\pi\beta}\sum_{j=\ell+1}^\infty 2^{m_j}|d_j| \left|\binom{\beta-1}{m_j-1}\right| c_j^{\beta-m_j} \leqslant \\ &\leqslant \frac{\pi M}{\sin\pi\beta}\sum_{j=\ell+1}^\infty 2^{m_j}|d_j| c_j^{\beta-m_j} = MK_\beta, \qquad (33)\end{aligned}$$

since (see (32))

$$\left|\binom{\beta-1}{m_j-1}\right| \leqslant 1,$$

where K_β is from (28). The boundedness (29) and the estimate (33) imply the claimed estimate

$$\left\|\mathfrak{M}^0_{a_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\right\| \leqslant MK_\beta.$$

If $\operatorname{Re} c_j < 0$ for all $j = 0, 1, \ldots$, we have

$$\begin{aligned}\frac{1}{|t-c_j|^{m_j}} &= \left(t^2+|c|^2-2\operatorname{Re} c_j t\right)^{-\frac{m_j}{2}} \leqslant \\ &\leqslant \left(t^2+|c|^2\right)^{-\frac{m_j}{2}} \leqslant 2^{\frac{m_j}{2}}\left(t+|c_j|\right)^{-m_j}\end{aligned}$$

valid for all $t > 0$ and a constant M does not emerge in the estimate. □

Let us find the symbol (the Mellin transform of the kernel) of the operator (14) for $0 < |\arg c| < \pi$, $m = 1, 2, \ldots$ (see (42), (14)). For this we apply formula (32):

$$\begin{aligned}\mathscr{M}_\beta \mathscr{K}^m_c(\xi) &= \int_0^\infty t^{\beta-i\xi-1}\mathscr{K}^m_c(t)\,dt = \int_0^\infty \frac{t^{\beta-i\xi-1}}{(t+e^{\mp\pi i}c)^m}\,dt = \\ &= \binom{\beta-i\xi-1}{m-1}\frac{\pi(-1)^{m-1}e^{\mp\pi(\beta-i\xi-m)i}}{\sin\pi(\beta-i\xi)}\,c^{\beta-i\xi-m} =\end{aligned}$$

$$= -\binom{\beta - i\xi - 1}{m-1} \frac{\pi e^{\mp\pi(\beta-i\xi)i}}{\sin\pi(\beta-i\xi)}\, c^{\beta-i\xi-m}, \quad 0 < \pm\arg c < \pi \tag{34}$$

and

$$\mathscr{M}_\beta \mathscr{K}^m_{-d}(\xi) = \int_0^\infty \frac{t^{\beta-i\xi-1}dt}{(t+d)^m} = \binom{\beta - i\xi - 1}{m-1} \frac{(-1)^{m-1}\pi d^{\beta-i\xi-m}}{\sin\pi(\beta-i\xi)} \tag{35}$$

$$\text{for } \ 0 < |\arg d| < \pi, \ \ \xi\in\mathbb{R}.$$

In particular,

$$\mathscr{M}_\beta \mathscr{K}^1_c(\xi) = -\frac{\pi e^{\mp\pi(\beta-i\xi)i} c^{\beta-i\xi-1}}{\sin\pi(\beta-i\xi)}\,, \quad 0 < \pm\arg c < \pi, \tag{36}$$

$$\mathscr{M}_\beta \mathscr{K}^1_{-d}(\xi) = \frac{\pi d^{\,\beta-i\xi-1}}{\sin\pi(\beta-i\xi)}\,, \quad 0 < |\arg d| < \pi, \tag{37}$$

$$\mathscr{M}_\beta \mathscr{K}^1_{-1}(\xi) = \frac{\pi}{\sin\pi(\beta-i\xi)}\,, \quad \xi\in\mathbb{R}. \tag{38}$$

Now let us find the symbol of the Cauchy singular integral operator $K^1_1 = -\pi i S_{\mathbb{R}^+}$ (see (43), (44)). For this we apply Plemeli formula and formula (32):

$$\begin{aligned}\mathscr{M}_\beta \mathscr{K}^1_1(t) &:= \int_0^\infty t^{\beta-i\xi-1}\mathscr{K}^1_1(t)\,dt = -\int_0^\infty \frac{t^{\beta-i\xi-1}\,dt}{t-1} = \\ &= \lim_{\varepsilon\longrightarrow 0} \frac{1}{2}\int_0^\infty \Big[\frac{t^{\beta-i\xi-1}}{t+e^{i(\pi-\varepsilon)}} + \frac{t^{\beta-i\xi-1}}{t+e^{-i(\pi-\varepsilon)}}\Big]\,dt = \\ &= \lim_{\varepsilon\longrightarrow 0} \pi\, \frac{e^{i(\pi-\varepsilon)(\beta-i\xi-1)} + e^{-i(\pi-\varepsilon)(\beta-i\xi-1)}}{2\sin\pi(\beta-i\xi)} = \\ &= \pi\cot\pi(\beta-i\xi). \end{aligned} \tag{39}$$

For an admissible kernel with simple (non-multiple) poles $m_0 = m_1 = \cdots = 1$ and $\arg c_0 = \arg c_\ell = 0$ and $0 < \pm\arg c_j < \pi$, $j = \ell+1, \dots$ we get

$$\begin{aligned}\mathscr{M}_\beta \mathscr{K}(\xi) &= \pi\cot\pi(\beta-i\xi)\sum_{j=0}^{\ell} d_j c_j^{\beta-i\xi-1} - \\ &- \frac{\pi}{\sin\pi(\beta-i\xi)} \sum_{j=\ell+1}^{\infty} d_j \binom{\beta-i\xi-1}{m-1} \pi e^{\mp\pi(\beta-i\xi)i} c^{\beta-i\xi-m}. \end{aligned} \tag{40}$$

Theorem 2.5. *Let $1 < p < \infty$ and $s\in\mathbb{R}$. The Mellin convolution operator $\mathfrak{M}^0_{a_\beta}$ in* (10) *with an admissible kernel $\mathscr{K}$* (*see* (13)) *is bounded in Bessel potential spaces*

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+), \tag{41}$$

provided the condition (28) *holds and* $m^0 := \sup\limits_{j=0,1,\dots} m_j < \infty$.

The condition on the parameter p *can be relaxed to* $1 \leqslant p \leqslant \infty$*, provided the admissible kernel* $\mathscr{K}$ *in* (13) *has no poles on positive semi-axes* $\alpha_j = \arg c_j \neq 0$ *for all* $j = 0, 1, \dots$.

Proof. Due to the representation (13), we have to prove the theorem only for a model kernel

$$\mathscr{K}^m_c(t) := \frac{1}{(t-c)^m}, \quad c \neq 0, \quad 0 \leqslant |\arg c| < \pi, \quad m = 1, 2, \dots . \tag{42}$$

The corresponding Mellin convolution operator $\boldsymbol{K}^m_c$ (see (14)) is bounded in $\mathbb{L}_p(\mathbb{R}^+)$ for all $1 \leqslant p \leqslant \infty$ for arbitrary $0 < |\arg c| < \pi$ (cf. (2)).

For $\arg c = 0$ (i.e., $c \in (0, \infty)$,) by the definition of an admissible kernel $m = 1$ and the corresponding operator coincides with the Cauchy singular integral operator $S_{\mathbb{R}^+}$

$$S_{\mathbb{R}^+}\varphi(t) := \frac{1}{\pi i}\int_0^\infty \frac{\varphi(\tau)\, d\tau}{\tau - t} \tag{43}$$

modulo compact multiplier

$$\boldsymbol{K}^1_c\varphi(t) := \int_0^\infty \frac{\varphi(\tau)d\tau}{t - c\,\tau} = -\frac{\pi i}{c}\,(S_{\mathbb{R}^+}\varphi)\Big(\frac{t}{c}\Big) \tag{44}$$

and is bounded in $\mathbb{L}_p(\mathbb{R}^+)$ for all $1 < p < \infty$ (cf., e.g., [17, 30]).

Now let $0 < \arg c < 2\pi$ and $m = 1$. Then, if $\varphi \in C^\infty_0(\mathbb{R}^+)$ is a smooth function with compact support and $k = 1, 2, \dots$, integrating by parts we get

$$\begin{aligned}\frac{d^k}{dt^k}\,\boldsymbol{K}^1_c\varphi(t) &= \int_0^\infty \frac{d^k}{dt^k}\,\frac{1}{t - c\,\tau}\varphi(\tau)\, d\tau = (-c)^{-k}\int_0^\infty \frac{d^k}{d\tau^k}\,\frac{1}{t - c\,\tau}\varphi(\tau)\, d\tau = \\ &= c^{-k}\int_0^\infty \frac{1}{t - c\,\tau}\,\frac{d^k\varphi(\tau)}{d\tau^k}\, d\tau = c^{-k}\Big(\boldsymbol{K}^1_c\,\frac{d^k}{dt^k}\varphi\Big)(t).\end{aligned} \tag{45}$$

For $m = 2, 3, \dots$, we similarly get

$$\begin{aligned}\frac{d}{dt}\,\boldsymbol{K}^m_c\varphi(t) &= \int_0^\infty \frac{d}{dt}\,\frac{\tau^{m-1}}{(t - c\,\tau)^m}\,\varphi(\tau)\, d\tau = \\ &= \sum_{j=0}^{m-1}(-c)^{-1-j}\int_0^\infty \frac{d}{d\tau}\,\frac{\tau^{m-1-j}}{(t - c\,\tau)^{m-j}}\,\varphi(\tau)\, d\tau = \\ &= -\sum_{j=0}^{m-1}(-c)^{-1-j}\int_0^\infty \frac{\tau^{m-1-j}}{(t - c\,\tau)^{m-j}}\,\frac{d}{d\tau}\,\varphi(\tau)\, d\tau = \\ &= -\sum_{j=0}^{m-1}(-c)^{-1-j}\Big(\boldsymbol{K}^{m-j}_c\,\frac{d}{dt}\,\varphi\Big)(t)\end{aligned}$$

and, recurrently,

$$\frac{d^k}{dt^k}\,\boldsymbol{K}^m_c\varphi(t)=(-1)^k\sum_{j=0}^{m-1}(-c)^{-k-j}\gamma^k_j\Big(\boldsymbol{K}^{m-j}_c\,\frac{d^k}{dt^k}\varphi\Big)(t),\quad k=1,2,\dots,\tag{46}$$

$$\gamma^1_j=j+1,\quad \gamma^k_0=1,\quad \gamma^k_j:=\sum_{r=0}^{j}\gamma^{k-1}_r,\quad j=0,1,\dots,m,\quad k=1,2,\dots.$$

Recall now that for an integer $s=n$ the spaces $\mathbb{H}^n_p(\mathbb{R}^+)$, $\widetilde{\mathbb{H}}^n_p(\mathbb{R}^+)$ coincide with the Sobolev spaces $\mathbb{W}^n_p(\mathbb{R}^+)$, $\widetilde{\mathbb{W}}^n_p(\mathbb{R}^+)$, respectively (these spaces are isomorphic and the norms are equivalent) and $C^\infty_0(\mathbb{R}^+)$ is a dense subset in $\widetilde{\mathbb{W}}^n_p(\mathbb{R}^+)=\widetilde{\mathbb{H}}^n_p(\mathbb{R}^+)$. Then, using the equalities (45), (46) and the boundedness results of the operators $\boldsymbol{K}^{m-j}_c$ (see (14) and (43)), we proceed as follows:

$$\begin{aligned}\big\|\boldsymbol{K}^m_c\varphi\mid\mathbb{H}^n_p(\mathbb{R}^+)\big\|&=\sum_{k=0}^{n}\Big\|\frac{d^k}{dt^k}\,\boldsymbol{K}^m_\alpha\varphi\mid\mathbb{L}_p(\mathbb{R}^+)\Big\|=\\&=\sum_{k=0}^{m}\sum_{j=0}^{m-1}|c|^{-k-j}\gamma^k_j\Big\|\boldsymbol{K}^{m-j}_c\,\frac{d^k}{dt^k}\,\varphi\mid\mathbb{L}_p(\mathbb{R}^+)\Big\|\leqslant\\&\leqslant M\sum_{k=0}^{m}\Big\|\frac{d^k}{dt^k}\,\varphi\mid\mathbb{L}_p(\mathbb{R}^+)\Big\|=M\big\|\varphi\mid\mathbb{H}^m_p(\mathbb{R}^+)\big\|,\end{aligned}\tag{47}$$

where $M>0$ is a constant, and there follows the boundedness result (41) for $s=0,1,2,\dots$. The case of an arbitrary $s>0$ follows by the interpolation between the spaces $\mathbb{H}^m_p(\mathbb{R}^+)$ and $\mathbb{H}^0_p(\mathbb{R}^+)=\mathbb{L}_p(\mathbb{R}^+)$, also between the spaces $\widetilde{\mathbb{H}}^m_p(\mathbb{R}^+)$ and $\widetilde{\mathbb{H}}^0_p(\mathbb{R}^+)=\mathbb{L}_p(\mathbb{R}^+)$.

The boundedness result (41) for $s<0$ follows by duality: the adjoint operator to $\boldsymbol{K}^m_c$ is

$$\boldsymbol{K}^{m,*}_c\varphi(t):=\int_0^\infty\frac{t^{m-1}\varphi(\tau)\,d\tau}{(\tau-c\,t)^m}=\sum_{j=1}^{m}\omega_j\boldsymbol{K}^j_{c^{-1}}\varphi(t),\tag{48}$$

for some constant coefficients $\omega_1,\dots,\omega_m$. The operator $\boldsymbol{K}^{m,*}_c$ has the admissible kernel and, due to the proved part of the theorem is bounded in the space setting $\boldsymbol{K}^{m,*}_c:\widetilde{\mathbb{H}}^{-s}_{p'}(\mathbb{R}^+)\longrightarrow\mathbb{H}^{-s}_{p'}(\mathbb{R}^+)$, $p':=p/(p-1)$, since $-s>0$. The initial operator $\boldsymbol{K}^m_c:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is dual to $\boldsymbol{K}^{m,*}_c$ and, therefore, is bounded as well □

Corollary 2.6. *Let $1<p<\infty$ and $s\in\mathbb{R}$. A Mellin convolution operator $\mathfrak{M}^0_a$ with an admissible kernel described in Definition* 2.1 (*also see Example* 2.3) *and Theorem* 2.5 *is bounded in Bessel potential spaces, see* (41).

With the help of formulae (25) and (45) for an integer $m = 1, 2\dots$ and arbitrary complex parameters $\gamma, c \in \mathbb{C}$ it follows that

$$\Lambda^m_{-\gamma}\boldsymbol{K}^1_c\varphi = \Big(i\,\frac{d}{dt} \pm \gamma\Big)^m \boldsymbol{K}^1_c\varphi = \sum_{k=0}^m \binom{m}{k} i^k(\pm\gamma)^{m-k}\,\frac{d^k}{dt^k}\,\boldsymbol{K}^1_c\varphi =$$
$$= \sum_{k=0}^m \binom{m}{k} i^k(\pm\gamma)^{m-k}c^{-k}\Big(\boldsymbol{K}^1_c\,\frac{d^k}{dt^k}\,\varphi\Big)(t) =$$
$$= c^{-m}\boldsymbol{K}^1_c\Big(\sum_{k=0}^m \binom{m}{k} i^k(\pm c\,\gamma)^{m-k}\,\frac{d^k}{dt^k}\,\varphi\Big)(t) =$$
$$= c^{-m}\boldsymbol{K}^1_c\Lambda^m_{-c\,\gamma}\varphi, \;\; \varphi\in\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+), \;\; 0<|\arg\gamma|<\pi. \tag{49}$$

Next, we will generalize formula (49).

Theorem 2.7. *Let $0<|\arg c|<\pi$, $0<|\arg\gamma|<\pi$, $0<|\arg(c\,\gamma)|<\pi$, $r, s\in\mathbb{R}$, $m = 1, 2, \dots$, $1<p<\infty$. Then*

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\varphi =$$
$$= \begin{cases} e^{\sigma(c,\gamma)\pi s i}c^{-s}\boldsymbol{K}^m_c\Lambda^s_{-c\,\gamma}\varphi & \text{if } -\pi<\arg c\,\gamma<0,\\ e^{\sigma(c,\gamma)\pi s i}c^{-s}\widetilde{\boldsymbol{K}}^m_c\Lambda^s_{-c\,\gamma}\varphi & \text{if } 0<\arg c\,\gamma<\pi, \;\; \varphi\in\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+),\end{cases} \tag{50}$$

where

$$\sigma(c,\gamma) := \begin{cases} 0 & \text{if } 0<\arg c<\pi,\\ \operatorname{sign}\arg(c\,\gamma)-\operatorname{sign}\arg\gamma & \text{if } -\pi<\arg c<0,\end{cases} \tag{51}$$

$$\widetilde{\boldsymbol{K}}^m_c\psi(t) = \boldsymbol{K}^m_c\psi_+(t) + (-1)^{m-1}\boldsymbol{K}^m_{-c}\psi_-(t), \;\; \psi\in\mathbb{L}_p(\mathbb{R}), \;\; \psi_\pm\in\mathbb{L}_p(\mathbb{R}^+), \tag{52}$$

$\psi_\pm(t) := r_+\psi(\pm t)$ and r_+ is the restriction from $\mathbb{R}$ to $\mathbb{R}^+$.

Proof. First we consider the case $m = 1$ (a simple pole). Let $\Lambda^s_{-\gamma,t}\psi(t,\tau)$ denote the action of the Bessel potential operator $\Lambda^s_{-\gamma}$ (see (23)) on a function $\psi(t,\tau)$ with respect to the variable t (see (14)):

$$\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t) := r_+\int_0^\infty \Big[\Lambda^s_{-\gamma,t}\,\frac{1}{t-c\,\tau}\Big]\varphi(\tau)\,d\tau =$$
$$= \frac{1}{2\pi}\,r_+\int_0^\infty \varphi(\tau)\int_{-\infty}^\infty e^{-i\xi t}(\xi-\gamma)^s\int_{-\infty}^\infty \frac{e^{i\xi y}}{y-c\tau}\,dy\,d\xi\,d\tau, \tag{53}$$

where r_+ is the restriction to $\mathbb{R}^+$. The integrand in the last integral in (53) is a meromorphic function with a single pole at $c\,\tau$ and the function vanishes as $|y|\longrightarrow\infty$, provided $\xi<0$ for $0<\arg c<\pi$ and for $\xi>0$ for $-\pi<\arg c<0$, respectively. Therefore, by the Cauchy theorem, the integral vanishes for $\xi<0$ in the first and for $\xi>0$ in the second case, respectively. Since $\tau>0$, the integral is found with the help of the residue theorem:

$$\int_{-\infty}^{\infty} \frac{e^{i\xi y}}{y-c\tau}\,dy = \begin{cases} 0 & \text{for } \ \xi \arg c < 0, \\ 2\pi i e^{ic\xi\tau} & \text{for } \ \xi > 0 \ \text{ and } \ 0 < \arg c < \pi, \\ -2\pi i e^{ic\xi\tau} & \text{for } \ \xi < 0 \ \text{ and } \ -\pi < \arg c < 0. \end{cases} \tag{54}$$

Then

$$\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t) = ir_+\int_0^\infty \varphi(\tau)\int_0^\infty e^{-i\xi(t-c\tau)}(\xi-\gamma)^s\,d\xi\,d\tau, \quad 0<\arg c<\pi, \tag{55a}$$

$$\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t) = -ir_+\int_0^\infty \varphi(\tau)\int_{-\infty}^0 e^{-i\xi(t-c\tau)}(\xi-\gamma)^s\,d\xi\,d\tau =$$

$$= -ie^{-\sigma(\gamma)\pi si}r_+\int_0^\infty \varphi(\tau)\int_0^\infty e^{i\xi(t-c\tau)}(\xi+\gamma)^s\,d\xi\,d\tau \tag{55b}$$

$$\text{for } \ \sigma(\gamma) := \operatorname{sign}\arg\gamma, \quad -\pi < \arg c < 0$$

because $\arg(-\xi-\gamma) = \arg(\xi+\gamma) \pm \pi \in (-\pi,\pi)$ for $0 < \mp\arg\gamma < \pi$. To (55a) and (55b) we apply the formula (see [31, Formula 3.382.4])

$$\int_0^\infty e^{-\mu\xi}(\xi+\nu)^s\,d\xi = \mu^{-s-1}e^{\nu\mu}\Gamma(s+1,\nu\mu), \tag{56}$$

$$s\in\mathbb{R}, \quad -\pi < \arg\nu < \pi, \quad \operatorname{Re}\mu > 0.$$

To comply with the constraint $-\pi < \arg\nu < \pi$ for $\nu = -\gamma$, we choose $\arg(-\gamma) = \arg\gamma \pm \pi$ for $0 < \mp\arg\gamma < \pi$. From $0 < \arg c < \pi$ follows the constraint $\operatorname{Re}\mu > 0$ for $\mu = i(t-c\tau)$ and from (55a) with the help of (56) we get

$$\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t) = ir_+\int_0^\infty (it-ic\tau)^{-s-1}e^{-i\gamma(t-c\tau)}\Gamma(s+1,-i\gamma(t-c\tau))\varphi(\tau)\,d\tau =$$

$$= e^{-\frac{\pi}{2}si}r_+\int_0^\infty \frac{e^{-i\gamma(t-c\tau)}\Gamma(s+1,-i\gamma(t-c\tau))}{(t-c\tau)^{s+1}}\,\varphi(\tau)\,d\tau, \tag{57a}$$

since $\arg(it-ic\tau) = \arg(t-c\tau)+\pi/2 \in (-\pi,\pi)$ and, therefore, $i(it-ic\tau)^{-s-1} = e^{-\frac{\pi}{2}si}(t-c\tau)^{-s-1}$.

Similarly, from $-\pi < \arg c < 0$ follows the constraint $\operatorname{Re}\mu > 0$ for $\mu = -i(t-c\tau)$ and from (55b) with the help of (56) we get

$$\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t) =$$

$$= -ie^{-\sigma(\gamma)\pi si}r_+\int_0^\infty (-it+ic\tau)^{-s-1}e^{-i\gamma(t-c\tau)}\Gamma(s+1,-i\gamma(t-c\tau))\varphi(\tau)\,d\tau =$$

$$= e^{(-\sigma(\gamma)\pi+\frac{\pi}{2})si}\, r_+ \int_0^\infty \frac{e^{-i\gamma(t-c\tau)}\Gamma(s+1,-i\gamma(t-c\tau))}{(t-c\tau)^{s+1}}\,\varphi(\tau)\,d\tau, \tag{57b}$$

$$\text{for }\ \sigma(\gamma) := \operatorname{sign}\arg\gamma, \quad -\pi < \arg c < 0,$$

since $\arg(-it+ic\tau) = \arg(t-c\tau) - \pi/2 \in (-\pi,\pi)$ and, therefore, $i(-it+ic\tau)^{-s-1} = -e^{\frac{\pi}{2}\,si}\,(t-c\tau)^{-s-1}$.

Next, we check what are the results if the Bessel potential $\Lambda^s_{c\,\gamma,\tau}$ is applied to the kernel $\frac{1}{t-c\tau}$ of the operator $\boldsymbol{K}^1_c$ with respect to the variable τ:

$$\boldsymbol{A}_\gamma\varphi(t) := r_+ \int_0^\infty \Big[\Lambda^s_{c\,\gamma,y}\,\frac{1}{t-c\,y}\Big]\varphi(\tau)\,d\tau =$$

$$= \frac{1}{2\pi}\, r_+ \int_0^\infty \varphi(\tau) \int_{-\infty}^\infty e^{-i\xi\tau}(\xi+c\,\gamma)^s \int_{-\infty}^\infty \frac{e^{i\xi y}\,dy}{t-cy}\,d\xi\,d\tau =$$

$$= -\frac{1}{2\pi c}\, r_+ \int_0^\infty \varphi(\tau) \int_{-\infty}^\infty e^{-i\xi\tau}(\xi+c\,\gamma)^s \int_{-\infty}^\infty \frac{e^{i\xi y}\,dy}{y-c^{-1}t}\,d\xi\,d\tau. \tag{58}$$

The last integral in (58) is found with the help of the residue theorem, by taking into account that $\tau > 0$ (cf. (54)):

$$\int_{-\infty}^\infty \frac{e^{i\xi y}}{y-c^{-1}t}\,dy = \begin{cases} 0 & \text{for }\ \xi\arg c > 0,\\ -2\pi i e^{ic^{-1}\xi\,t} & \text{for }\ \xi < 0 \ \text{ and }\ 0 < \arg c < \pi,\\ 2\pi i e^{ic^{-1}\xi\,t} & \text{for }\ \xi > 0 \ \text{ and }\ -\pi < \arg c < 0.\end{cases} \tag{59}$$

Applying formula (59), we proceed as follows:

$$\boldsymbol{A}_\gamma\varphi(t) = \frac{i}{c}\, r_+ \int_0^\infty \varphi(\tau) \int_{-\infty}^0 e^{-i\xi(\tau-c^{-1}t)}(\xi+c\,\gamma)^s\,d\xi\,d\tau =$$

$$= \frac{ie^{\sigma(c\gamma)\pi si}}{c}\, r_+ \int_0^\infty \varphi(\tau) \int_0^\infty e^{-ic^{-1}\xi(t-c\,\tau)}(\xi-c\,\gamma)^s\,d\xi\,d\tau, \tag{60a}$$

$$\sigma(\gamma) := \operatorname{sign}\arg\gamma \ \text{ for }\ 0 < \arg c < \pi,$$

because $\arg(-\xi+c\,\gamma) = \arg(\xi-c\,\gamma) \pm \pi \in (-\pi,\pi)$. Similarly,

$$\boldsymbol{A}_\gamma\varphi(t) = -\frac{i}{c}\, r_+ \int_0^\infty \varphi(\tau) \int_0^\infty e^{-i\xi(\tau-c^{-1}t)}(\xi+c\,\gamma)^s\,d\xi\,d\tau =$$

$$= -\frac{i}{c}\, r_+ \int_0^\infty \varphi(\tau) \int_0^\infty e^{ic^{-1}\xi(t-c\,\tau)}(\xi+c\,\gamma)^s\,d\xi\,d\tau, \quad -\pi < \arg c < 0. \tag{60b}$$

To (60a) and (60b) we apply the formula (56) with $\mu = \pm ic^{-1}(t-c\tau)$ and $\nu = \mp c\,\gamma$, which yields $\nu\mu = -i\gamma(t-c\tau)$. The constraint $0 < |\arg(c\,\gamma)| < \pi$, imposed in the theorem, allows us to comply with the condition $-\pi < \nu < \pi$ by choosing $\arg(-c\,\gamma) = \arg(c\,\gamma) \mp \pi$ for $\pm \arg(c\,\gamma) > 0$. Another constraint $0 < |\arg c| < \pi$ allows to comply with the condition $\operatorname{Re}\mu > 0$ in (56): $\operatorname{Re}(\pm i c^{-1} t \mp i\tau) = \mp \operatorname{Im} c^{-1} t = \pm t\, \frac{\operatorname{Im} c}{|c|^2} > 0$ for $0 < \pm \arg c < \pi$. We get the following:

$$\boldsymbol{A}_\gamma \varphi(t) = \frac{i e^{-\sigma(c\gamma)\pi s i}}{c}\, r_+ \times$$

$$\times \int_0^\infty (i c^{-1})^{-s-1} (t - c\,\tau)^{-s-1} e^{-i\gamma(t-c\,\tau)} \Gamma(s+1, -i\gamma(t - c\,\tau)) \varphi(\tau)\, d\tau =$$

$$= c^s e^{(\sigma(c\gamma)\pi - \frac{\pi}{2}) s i} r_+ \int_0^\infty \frac{e^{-i\gamma(t-c\,\tau)} \Gamma(s+1, -i\gamma(t-c\tau))}{(t - c\,\tau)^{s+1}}\, \varphi(\tau)\, d\tau \tag{61a}$$

$$\text{for } \ \sigma(c\,\gamma) := \operatorname{sign} \arg(c\gamma), \ \ 0 < \arg c < \pi,$$

since $i^{-s-1} = i^{-1} e^{-\frac{\pi}{2}\, si}$, and

$$\boldsymbol{A}_\gamma \varphi(t) = -\frac{i}{c}\, r_+ \times$$

$$\times \int_0^\infty (-i c^{-1})^{-s-1} (t - c\,\tau)^{-s-1} e^{-i\gamma(t-c\,\tau)} \Gamma(s+1, -i\gamma(t - c\,\tau)) \varphi(\tau)\, d\tau =$$

$$= c^s e^{\frac{\pi}{2}\, si} r_+ \int_0^\infty \frac{e^{-i\gamma(t-c\,\tau)} \Gamma(s+1, -i\gamma(t - c\,\tau))}{(t - c\,\tau)^{s+1}}\, \varphi(\tau)\, d\tau \tag{61b}$$

for $-\pi < \arg c < 0$, since $(-i)^{-s-1} = i e^{\frac{\pi}{2}\, si}$.

From (57a)–(57b), (58) and (61)–(61) we derive the following equality:

$$\Lambda^s_{-\gamma} \boldsymbol{K}^1_c \varphi(t) = \int_0^\infty \Big[\Lambda^s_{-\gamma,t}\, \frac{1}{t - c\tau}\Big] \varphi(\tau)\, d\tau =$$

$$= c^{-s} e^{-\sigma_0(c,\gamma)\pi\, si} \int_{-\infty}^\infty \Big[\Lambda^s_{c\,\gamma,\tau}\, \frac{1}{t - c\tau}\Big] \varphi_0(\tau)\, d\tau, \tag{62}$$

where

$$\sigma_0(c,\gamma) := \begin{cases} \sigma(c\,\gamma) & \text{if } \ 0 < \arg c < \pi, \\ \sigma(\gamma) & \text{if } \ -\pi < \arg c < 0 \end{cases} \tag{63}$$

and $\varphi_0 \in \mathbb{H}^1_2(\mathbb{R})$ is the extension of $\varphi_0 \in \widetilde{\mathbb{H}}^1_2(\mathbb{R}^+)$ by 0 to the semi-axes $\mathbb{R}^- := \mathbb{R} \setminus \overline{\mathbb{R}^+}$. Now note, that the operator $\Lambda^s_{c\,\gamma,\tau}$ is the dual (adjoint) to the operator $e^{\sigma(c\,\gamma)\pi si}\Lambda^s_{-c\,\gamma,\tau}$ i.e,

$$\int\limits_{-\infty}^{\infty}(\Lambda^s_{c\,\gamma,\tau}u)(\tau)v(\tau)\,d\tau=e^{\sigma(c\,\gamma)\pi si}\int\limits_{-\infty}^{\infty}u(\tau)(\Lambda^s_{-c\,\gamma,\tau}v)(\tau)\,d\tau,\qquad \forall\,u,v\in C^\infty_0(\mathbb{R}),$$

the equality can easily be verified by changing the orders of integration and change the Fourier transform variable ξ to $-\xi$. We continue the equality (62) as follows:

$$\begin{aligned}\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t)&=c^{-s}e^{-\sigma_0(c,\gamma)\pi\,si}\int\limits_{-\infty}^{\infty}\Big[\Lambda^s_{c\,\gamma,\tau}\,\frac{1}{t-c\tau}\Big]\varphi_0(\tau)\,d\tau=\\&=e^{\sigma(c,\gamma)\pi\,si}c^{-s}\int\limits_{-\infty}^{\infty}\frac{\Lambda^s_{-c\,\gamma}\varphi(\tau)\,d\tau}{t-c\tau}\,,\end{aligned}$$

where $\sigma(c,\gamma)$ is defined in (51). By the properties of the Bessel potential $\Lambda^m_{-c\,\gamma}$, it maintains the support of a function $\operatorname{supp}\varphi\subset\overline{\mathbb{R}^+}$ for $-\pi<\arg c\,\gamma<0$ but not for $0<\arg c\,\gamma<\pi$. Therefore,

$$\begin{aligned}\Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t)&=e^{\sigma(c,\gamma)\pi\,si}c^{-s}\boldsymbol{K}^1_c\Lambda^s_{-c\,\gamma}\varphi(t)\ \text{ for }\ -\pi<\arg c\,\gamma<0,\\ \Lambda^s_{-\gamma}\boldsymbol{K}^1_c\varphi(t)&=e^{\sigma(c,\gamma)\pi\,si}c^{-s}\widetilde{\boldsymbol{K}}{}^1_c\Lambda^s_{-c\,\gamma}\varphi(t)\ \text{ for }\ 0<\arg c\,\gamma<\pi.\end{aligned}\tag{64}$$

Formula (64) accomplishes the proof of formula (50) for an operator $\boldsymbol{K}^1_c$ (case $m=1$) and under the additional constraint $\arg c\neq0$. For an operator $\boldsymbol{K}^1_c$ (case $m=1$) but $\arg c=0$ and a case of an operator $\boldsymbol{K}^m_c$, $m=2,3,\dots$ we can deal with a perturbation:

$$\begin{gathered}\frac{1}{(t-c)^m}=\lim_{ve\longrightarrow0}\mathscr{K}_\varepsilon(t),\\ \mathscr{K}_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}(t):=\frac{1}{(t-c_{1,\varepsilon})\cdots(t-c_{m,\varepsilon})}=\sum_{j=1}^m\frac{d_j(\varepsilon)}{t-c_{j,\varepsilon}}\,,\\ c_{j,\varepsilon}=c(1+\varepsilon e^{i\omega_j}),\ \ \omega_j\in(-\pi,\pi),\ \ \arg c_{j,\varepsilon},\ \ \arg c_{j,\varepsilon}\,\gamma_j\neq0,\ \ j=1,\dots,m.\end{gathered}\tag{65}$$

the points and $\omega_1,\dots,\omega_m\in(-\pi,\pi]$ are distinct $\omega_j\neq\omega_k$ for $j\neq k$. The case $\arg c=0$ is covered for $m=1$. By equating the numerators in the formula (65)

$$\begin{aligned}&\sum_{j=1}^m d_j(\varepsilon)t^m-(m-1)\sum_{j=1}^m d_j(\varepsilon)c_{j,\varepsilon}t^{m-1}+\cdots=\\&\qquad=\sum_{j=1}^m d_j(\varepsilon)(t^m-ct^{m-1})-(m-1)\varepsilon\sum_{j=1}^m e^{\omega_j}d_j(\varepsilon)t^{m-1}+\mathscr{O}(\varepsilon)=1,\end{aligned}$$

we derive the last two equalities

$$d_j(\varepsilon)=\mathscr{O}(\varepsilon^{-1}),\quad \sum_{j=1}^m d_j(\varepsilon)=0,\quad \sum_{j=1}^m e^{\omega_j}d_j(\varepsilon)=0, \tag{66}$$

while the first one is well known. The claimed equality (64) holds for each operator $\boldsymbol{K}^1_{c_{j,\varepsilon}}$ and

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}\varphi=\sum_{j=1}^m d_j(\varepsilon)\Lambda^s_{-\gamma}\boldsymbol{K}^1_{c_{j,\varepsilon}}\varphi=\sum_{j=1}^m c^s_{j,\varepsilon}d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma_{j,\varepsilon}}\varphi, \tag{67}$$

where

$$\boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}\varphi(t)=\int_0^\infty\mathscr{K}_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau}=\int_0^\infty\frac{\tau^{m-1}\varphi(\tau)\,d\tau}{(t-c_{1,\varepsilon}\tau)\cdots(t-c_{m,\varepsilon}\tau)}\,.$$

Further, we assume that $-\pi<\arg c\,\gamma<0$. The case $0<\arg c\,\gamma<\pi$ is considered similarly and we drop its proof.

Using the Bessel potentials (see (23)), we get

$$\Lambda^{-s}_{-c\,\gamma}\big[\Lambda^s_{-c\,\gamma_{j,\varepsilon}}-\Lambda^s_{-c\,\gamma}\big]=W_{a_{j,\varepsilon}}-I=W_{a_{j,\varepsilon}-1},\quad \sigma:=\sigma(c,\gamma)=\sigma(c,\gamma),$$

$$a_{j,\varepsilon}(\xi)-1=\Big(\frac{\xi-c\,\gamma_{j,\varepsilon}}{\xi-c\,\gamma}\Big)^s-1=\Big(1-\frac{\varepsilon e^{i\omega_j}}{\frac{\xi}{c\,\gamma}-1}\Big)^s-1=$$

$$=-\frac{se^{i\omega_j}}{\frac{\xi}{c\,\gamma}-1}\varepsilon+a^0_{j,\varepsilon}(\xi)\varepsilon^2=\frac{s\,c\,\gamma e^{i\omega_j}}{\xi-c\,\gamma}\,\varepsilon+a^0_{j,\varepsilon}(\xi)\varepsilon^2,\ \ a^0_{j,\varepsilon}=\mathscr{O}(1), \tag{68}$$

$$c^{-s}_{j,\varepsilon}=c^{-s}(1+\varepsilon e^{i\omega_j})^{-s}=c^{-s}-c^{-s}se^{i\omega_j}\varepsilon+b_{j,\varepsilon}\varepsilon^2,\ \ b_{j,\varepsilon}=\mathscr{O}(1) \tag{69}$$

as $\varepsilon\longrightarrow 0$. For ε sufficiently small, the value $\sigma(c_{j,\varepsilon},\gamma)$ becomes independent of $j=1,\dots,m$ and ε, and we use the notation $\sigma(c,\gamma):=\sigma(c_{j,\varepsilon},\gamma)$. Then, by virtue of the equality (66) and asymptotic (68), (69), we get the following equalities:

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}}\varphi:=\sum_{j=1}^m d_j(\varepsilon)\Lambda^s_{-\gamma}\boldsymbol{K}^1_{c_{j,\varepsilon}}\varphi=$$

$$=\sum_{j=1}^m e^{\sigma(c,\gamma)\pi\,si}c^{-s}_{j,\varepsilon}d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma_{j,\varepsilon}}\varphi=$$

$$=\sum_{j=1}^m e^{\sigma(c,\gamma)\pi\,si}\Big[c^{-s}-c^{-s}se^{i\omega_j}\varepsilon+b_{j,\varepsilon}\varepsilon^2\Big]d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma_{j,\varepsilon}}\varphi=$$

$$=\sum_{j=1}^m e^{\sigma(c,\gamma)\pi\,si}\big[c^{-s}+b_{j,\varepsilon}\varepsilon^2\big]d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma_{j,\varepsilon}}\varphi=$$

$$=e^{\sigma(c,\gamma)\pi\,si}c^{-s}\sum_{j=1}^m d_j(\varepsilon)\boldsymbol{K}^1_{c_{j,\varepsilon}}\Lambda^s_{-c\,\gamma}\varphi+$$

$$+ e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} d_j(\varepsilon) \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-c\,\gamma} \Lambda^{-s}_{-c\,\gamma} \Big[\Lambda^s_{-c\,\gamma_{j,\varepsilon}} - \Lambda^s_{-c\,\gamma}\Big]\varphi +$$
$$+ \varepsilon^2 e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} d_j(\varepsilon) b_{j,\varepsilon} \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-c\,\gamma_{j,\varepsilon}} \varphi =$$
$$= e^{\sigma(c,\gamma)\pi\, si} c^{-s} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \Lambda^s_{-c\,\gamma}\varphi +$$
$$+ e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} \boldsymbol{K}^1_{c_{j,\varepsilon}} d_j(\varepsilon) \Big[- s\, c\, \gamma e^{i\omega_j} W_{\frac{1}{\xi + c\,\gamma}} \varepsilon + W_{a^0_{j,\varepsilon}(\xi)} \varepsilon^2 \Big] \Lambda^s_{-c\,\gamma}\varphi +$$
$$+ e^{\sigma(c,\gamma)\pi\, si} c^{-s} \varepsilon^2 \sum_{j=1}^{m} d_j(\varepsilon) b_{j\varepsilon} \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-\gamma\, c_{j,\varepsilon}} \varphi =$$
$$= e^{\sigma(c,\gamma)\pi\, si} c^{-s} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \Lambda^s_{-c\,\gamma}\varphi +$$
$$+ \varepsilon^2 e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} d_j(\varepsilon) \big[b_{j\varepsilon} + W_{a^0_{j,\varepsilon}}\big] \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-c\,\gamma_{j,\varepsilon}} \varphi. \quad (70)$$

By using the boundedness result proved in Theorem 2.5, we get

$$\lim_{\varepsilon \longrightarrow 0} \big\| \boldsymbol{K}^m_c - \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \varphi \mid \mathbb{H}^\nu_2(\mathbb{R}^+) \big\| \leqslant$$
$$\leqslant \lim_{\varepsilon \longrightarrow 0} \varepsilon \sum_{j=1}^{m} \big\| \boldsymbol{K}^m_{c,\dots,c,c_{j,\varepsilon},\dots,c_{m,\varepsilon}} \varphi \mid \mathbb{H}^\nu_2(\mathbb{R}^+) \big\| = 0. \quad (71)$$

Further, invoking the well known formula for the norm of a convolution operator in the Hilbert-Bessel spaces $\mathbb{L}_p(\mathbb{R}^+)$

$$\big\| W_a \mid \mathscr{L}(\mathbb{H}^\mu_2(\mathbb{R}^+)) \big\| = \big\| W_a \mid \mathscr{L}(\mathbb{L}_2(\mathbb{R}^+)) \big\| = \sup_{\xi \in \mathbb{R}} |a(\xi)| \quad (72)$$

(cf., e.g., [17]) and using the property $\lim\limits_{\varepsilon \longrightarrow 0} \varepsilon^2 d_j(\varepsilon) = 0$ (see (66)), from (70)–(72) we derive

$$\Lambda^s_{-\gamma} \boldsymbol{K}^m_c \varphi = \lim_{\varepsilon \longrightarrow 0} \Lambda^s_{-\gamma} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \varphi =$$
$$= \lim_{\varepsilon \longrightarrow 0} \Bigg[e^{\sigma(c,\gamma)\pi\, si} c^{-s} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \Lambda^s_{-c\,\gamma}\varphi +$$
$$+ \varepsilon^2 e^{\sigma(c,\gamma)\pi\, si} c^{-s} \sum_{j=1}^{m} d_j(\varepsilon) \big[b_{j\varepsilon} + W_{a^0_{j,\varepsilon}}\big] \boldsymbol{K}^1_{c_{j,\varepsilon}} \Lambda^s_{-c\,\gamma_{j,\varepsilon}} \varphi \Bigg] =$$
$$= e^{\sigma(c,\gamma)\pi\, si} c^{-s} \lim_{\varepsilon \longrightarrow 0} \boldsymbol{K}^m_{c_{1,\varepsilon},\dots,c_{m,\varepsilon}} \Lambda^s_{-c\,\gamma}\varphi =$$
$$= e^{\sigma(c,\gamma)\pi\, si} c^{-s} \boldsymbol{K}^m_c \Lambda^s_{-c\,\gamma}\varphi$$

which accomplishes the proof. □

3. Algebra Generated by Mellin and Fourier Convolution Operators

Unlike the operators W^0_a and $\mathfrak{M}^0_a$ (see Section 1), possessing the property

$$W^0_a W^0_b = W^0_{ab}, \quad \mathfrak{M}^0_a \mathfrak{M}^0_b = \mathfrak{M}^0_{ab} \ \text{ for all } \ a, b \in \mathfrak{M}_p(\mathbb{R}), \tag{73}$$

the composition of the convolution operators on the semi-axes W_a and W_b (see (73)) cannot be computed by the rules similar to (73). Nevertheless, the following propositions hold.

Proposition 3.1 ([17, Section 2])**.** *Assume that* $1 < p < \infty$, *and let* $[W_a, W_b] := W_a W_b - W_b W_a$ *be the commutant of the operators* W_a *and* W_b. *If* $a, b \in \mathfrak{M}_p(\overline{\mathbb{R}}^+) \cap PC(\dot{\mathbb{R}})$ *are piecewise-continuous scalar* $\mathbb{L}_p$*-multipliers, then the commutant* $[W_a, W_b] : \mathbb{L}_p(\mathbb{R}^+) \longmapsto \mathbb{L}_p(\mathbb{R}^+)$ *is compact.*

Moreover, if, in addition, the symbols $a(\xi)$ *and* $b(\xi)$ *of the operators* W_a *and* W_b *have no common discontinuity points, i.e., if*

$$\big[a(\xi + 0) - a(\xi + 0)\big]\big[b(\xi + 0) - b(\xi + 0)\big] = 0 \ \textit{ for all } \ \xi \in \dot{\mathbb{R}},$$

then $T = W_a W_b - W_{ab}$ *is a compact operator in* $\mathbb{L}_p(\mathbb{R}^+)$.

Note that the algebra of $N \times N$ matrix multipliers $\mathfrak{M}_2(\mathbb{R})$ coincides with the algebra of $N \times N$ matrix functions essentially bounded on $\mathbb{R}$. For $p \neq 2$, the algebra $\mathfrak{M}_p(\mathbb{R})$ is rather complicated. There are multipliers $g \in \mathfrak{M}_p(\mathbb{R})$ which are elliptic, i.e. $\operatorname{ess\,inf} |g(x)| > 0$, but $1/g \notin \mathfrak{M}_p(\mathbb{R})$. In connection with this, let us consider the subalgebra $PC\mathfrak{M}_p(\mathbb{R})$ which is the closure of the algebra of piecewise-constant functions on $\mathbb{R}$ in the norm of multipliers $\mathfrak{M}_p(\mathbb{R})$

$$\big\|a \mid \mathfrak{M}_p(\mathbb{R})\big\| := \big\|W^0_a \mid \mathbb{L}_p(\mathbb{R})\big\|.$$

Note that any function $g \in PC\mathfrak{M}_p(\mathbb{R}) \subset PC(\mathbb{R})$ has limits $g(x \pm 0)$ for all $x \in \overline{\mathbb{R}}$, including the infinity. Let

$$C\mathfrak{M}_p(\overline{\mathbb{R}}) := C(\overline{\mathbb{R}}) \cap PC\mathfrak{M}^0_p(\mathbb{R}), \quad C\mathfrak{M}^0_p(\dot{\mathbb{R}}) := C(\dot{\mathbb{R}}) \cap PC\mathfrak{M}_p(\mathbb{R}),$$

where functions $g \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ (functions $h \in C(\dot{\mathbb{R}})$) might have jump only at the infinity $g(-\infty) \neq g(+\infty)$ (are continuous at the infinity $h(-\infty) = h(+\infty)$).

$PC\mathfrak{M}_p(\mathbb{R})$ is a Banach algebra and contains all functions of bounded variation as a subset for all $1 < p < \infty$ (Stechkin's theorem, see [17, Section 2]). Therefore, $\coth \pi(i\beta + \xi) \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ for all $p \in (1, \infty)$.

Proposition 3.2 ([17, Section 2])**.** *If* $g \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$ *is an* $N \times N$ *matrix multiplier, then its inverse* $g^{-1} \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$ *if and only if it is elliptic, i.e.* $\det g(x \pm 0) \neq 0$ *for all* $x \in \overline{\mathbb{R}}$. *If this is the case, the corresponding Mellin convolution operator* $\mathfrak{M}^0_g : \mathbb{L}_p(\mathbb{R}^+) \longmapsto \mathbb{L}_p(\mathbb{R}^+)$ *is invertible and* $(\mathfrak{M}^0_g)^{-1} = \mathfrak{M}^0_{g^{-1}}$.

Moreover, any $N\times N$ matrix multiplier $b\in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ can be approximated by polynomials

$$r_n(\xi):=\sum_{j=-m}^{m} c_m\Big(\frac{\xi-i}{\xi+i}\Big)^m,\quad r_m\in C\mathfrak{M}^0_p(\overline{\mathbb{R}}),$$

with constant $N\times N$ matrix coefficients, whereas any $N\times N$ matrix multiplier $g\in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$ having a jump discontinuity at infinity can be approximated by $N\times N$ matrix functions $d\coth\pi(i\beta+\xi)+r_m(\xi)$, $0<\beta<1$.

Due to the connection between the Fourier and Mellin convolution operators (see Introduction, (4)), the following is a direct consequence of Proposition 3.2.

Corollary 3.3. *The Mellin convolution operator*

$$\boldsymbol{A}=\mathfrak{M}^0_{\mathscr{A}_\beta}:\mathbb{L}_p(\mathbb{R},t^\gamma),$$

in (1) *with the symbol $\mathscr{A}_\beta(\xi)$ in* (5) *is invertible if and only if the symbol is elliptic,*

$$\inf_{\xi\in\mathbb{R}}\big|\det\mathscr{A}_\beta(\xi)\big|>0 \tag{74}$$

and the inverse is then written as $\mathbf{A}^{-1}=\mathfrak{M}^0_{\mathscr{A}^{-1}_{1/p}}$.

The Hilbert transform on the semi-axis

$$S_{\mathbb{R}^+}\varphi(x):=\frac{1}{\pi i}\int_0^\infty\frac{\varphi(y)\,dy}{y-x} \tag{75}$$

is the Fourier convolution $S_{\mathbb{R}^+}=W_{-\mathrm{sign}}$ on the semi-axis $\mathbb{R}^+$ with the discontinuous symbol $-\operatorname{sign}\xi$ (see [17, Lemma 1.35]), and it is also the Mellin convolution

$$S_{\mathbb{R}^+}=\mathfrak{M}^0_{s_\beta}=\mathbf{Z}_\beta W^0_{s_\beta}\mathbf{Z}^{-1}_\beta, \tag{76}$$

$$s_\beta(\xi):=\coth\pi(i\beta+\xi)=\frac{e^{\pi(i\beta+\xi)}+e^{-\pi(i\beta+\xi)}}{e^{\pi(i\beta+\xi)}-e^{-\pi(i\beta+\xi)}}=-i\cot\pi(\beta-i\xi),\ \xi\in\mathbb{R}$$

(cf. (5) and (8)). Indeed, to verify (76) rewrite $S_{\mathbb{R}^+}$ in the following form

$$S_{\mathbb{R}^+}\varphi(x):=\frac{1}{\pi i}\int_0^\infty\frac{\varphi(y)}{1-\frac{x}{y}}\,\frac{dy}{y}=\int_0^\infty K\Big(\frac{x}{y}\Big)\varphi(y)\,\frac{dy}{y}\,,$$

where $K(t):=(1/\pi i)(1-t)^{-1}$. Further, using the formula

$$\int_0^\infty\frac{t^{z-1}}{1-t}\,dt=\pi\cot\pi z,\quad \operatorname{Re}z<1,$$

cf. [31, formula 3.241.3], one shows that the Mellin transform $\mathscr{M}_\beta K(\xi)$ coincides with the function $s_\beta(\xi)$ from (76).

For our aim we will need certain results concerning the compactness of Mellin and Fourier convolutions in $\mathbb{L}_p$-spaces. These results are scattered in literature. For the convenience of the reader, we reformulate them here as Propositions 3.4–3.8. For more details, the reader can consult [8, 17, 22].

Proposition 3.4 ([22, Proposition 1.6]). *Let* $1 < p < \infty$, $a \in C(\dot{\overline{\mathbb{R}}}^+)$, $b \in C\mathfrak{M}_p^0(\dot{\overline{\mathbb{R}}})$ *and* $a(0) = b(\infty) = 0$. *Then the operators* $a\mathfrak{M}_b^0, \mathfrak{M}_b^0 aI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proposition 3.5 ([17, Lemma 7.1] and [22, Proposition 1.2]). *Let* $1 < p < \infty$, $a \in C(\dot{\overline{\mathbb{R}}}^+)$, $b \in C\mathfrak{M}_p^0(\dot{\overline{\mathbb{R}}})$ *and* $a(\infty) = b(\infty) = 0$. *Then the operators* $aW_b, W_b\, aI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proposition 3.6 ([22, Lemma 2.5, Lemma 2.6] and [8]). *Assume that* $1 < p < \infty$. *Then*

(1) *If* $g \in C\mathfrak{M}_p^0(\dot{\overline{\mathbb{R}}})$ *and* $g(\infty) = 0$, *the Hankel operator* $H_g : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *is compact;*

(2) *If the functions* $a \in C(\dot{\overline{\mathbb{R}}})$, $b \in C\mathfrak{M}_p^0(\overline{\mathbb{R}})$, $c \in C(\overline{\mathbb{R}}^+)$ *and satisfy at least one of the conditions*

  (i) $c(0) = b(+\infty) = 0$ *and* $a(\xi) = 0$ *for all* $\xi > 0$,

  (ii) $c(0) = b(-\infty) = 0$ *and* $a(\xi) = 0$ *for all* $\xi < 0$,

  then the operators $cW_a\mathfrak{M}_b^0$, $c\mathfrak{M}_b^0 W_a$, $W_a\mathfrak{M}_b^0\, cI$, $\mathfrak{M}_b^0 W_a\, cI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proof. Let us comment only on item 2 in Proposition 3.6, which is not proved in [22], although is well known. The kernel $k(x+y)$ of the operator H_a is approximated by the Laguerre polynomials $k_m(x+y) = e^{-x-y}p_m(x+y)$, $m = 1, 2, \dots$, where $p_m(x+y)$ are polynomials of order m so that the corresponding Hankel operators converge in norm $\|H_a - H_{a_m} \,|\, \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+))\| \longrightarrow 0$, where $a_m = \mathscr{F}k_m$ are the Fourier transforms of the Laguerre polynomials (see, e.g. [29]). Since

$$|k_m(x+y)| = \left|e^{-x-y}p_m(x+y)\right| \leqslant C_m e^{-x}e^{-y}x^m y^m, \quad m = 1, 2, \dots,$$

for some constant C_m, the condition on the kernel

$$\int_0^\infty \left[\int_0^\infty |k_m(x+y)|^{p'}\, dy\right]^{p/p'} dx < \infty, \quad p' := \frac{p}{p-1},$$

holds and ensures the compactness of the operator $H_{a_m} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$. Then the limit operator $H_a = \lim\limits_{m \longrightarrow \infty} H_{a_m}$ is compact as well. □

Proposition 3.7 ([17, Lemma 7.4] and [22, Lemma 1.2]). *Let* $1 < p < \infty$ *and let* a *and* b *satisfy at least one of the conditions*

(i) $a \in C(\overline{\mathbb{R}}^+)$, $b \in \mathfrak{M}_p^0(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$,

(ii) $a \in PC(\overline{\mathbb{R}}^+)$, $b \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$.

Then the commutants $[aI, W_b]$ *and* $[aI, \mathfrak{M}^0_b]$ *are compact operators in the space* $\mathbb{L}_p(\mathbb{R}^+)$.

Proposition 3.8 ([22]). *The Banach algebra, generated by the Cauchy singular integral operator* $S_{\mathbb{R}^+}$ *and by the identity operator* I *on the semi-axis* $\mathbb{R}^+$, *contains all Mellin and Fourier convolution operators on the semi-axis with symbols from* $C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, *having discontinuity of the jump type only at the infinity.*

Moreover, the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ *generated by the Cauchy singular integral operators with "shifts"*

$$S^c_{\mathbb{R}^+}\varphi(x) := \frac{1}{\pi i}\int_0^\infty \frac{e^{-ic(x-y)}\varphi(y)\,dy}{y-x} = W_{-\operatorname{sign}(\xi-c)}\varphi(x) \ \textit{ for all } \ c \in \mathbb{R}$$

and by the identity operator I *on the semi-axis* $\mathbb{R}^+$ *over the field of* $N \times N$ *complex valued matrices coincides with the Banach algebra generated by Fourier convolution operators with piecewise-constant* $N \times N$ *matrix symbols contains all Fourier convolution* W_a *and hankel* H_b *operators with* $N \times N$ *matrix symbols* (*multipliers*) $a, b \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$.

Let us consider the Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$ generated by Mellin convolution and Fourier convolution operators in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$

$$\mathbf{A} := \sum_{j=1}^m \mathfrak{M}^0_{a_j} W_{b_j}, \tag{77}$$

where $\mathfrak{M}^0_{a_j}$ are Mellin convolution operators with continuous $N \times N$ matrix symbols $a_j \in C\mathfrak{M}_p(\overline{\mathbb{R}})$, W_{b_j} are Fourier convolution operators with $N \times N$ matrix symbols $b_j \in C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^- \cup \overline{\mathbb{R}}^+)$ in the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, x^\alpha)$. The algebra of $N \times N$ matrix $\mathbb{L}_p$-multipliers $C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\})$ consists of those piecewise-continuous $N \times N$ matrix multipliers $b \in \mathfrak{M}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$ which are continuous on the semi-axis $\mathbb{R}^-$ and $\mathbb{R}^+$ but might have finite jump discontinuities at 0 and at the infinity.

This and more general algebras (see Remark 3.14) were studied in [22] and also in earlier works [12, 21, 42].

In order to keep the exposition self-contained, to improve formulations from [22] and to add Hankel operators as generators of the algebra, the results concerning the Banach algebra generated by the operators (77) are presented here with some modification and the proofs.

Note that the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ is actually a subalgebra of the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ generated by the Fourier convolution operators W_a acting on the space $\mathbb{L}_p(\mathbb{R}^+)$ and having piecewise-constant symbols $a(\xi)$, cf. Proposition 3.8. Let $\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denote the ideal of all compact operators in $\mathbb{L}_p(\mathbb{R}^+)$. Since the quotient algebra $\mathfrak{F}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative in the scalar case $N = 1$, the following is true.

Corollary 3.9. *The quotient algebra* $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ *is commutative in the scalar case* $N=1$.

To describe the symbol of the operator (77), consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3$, where (cf. Figure 1)

$$\Gamma_1 := \overline{\mathbb{R}} \times \{+\infty\}, \ \ \Gamma_2^{\pm} := \{\pm\infty\} \times \overline{\mathbb{R}}^+, \ \ \Gamma_3 := \overline{\mathbb{R}} \times \{0\}.$$

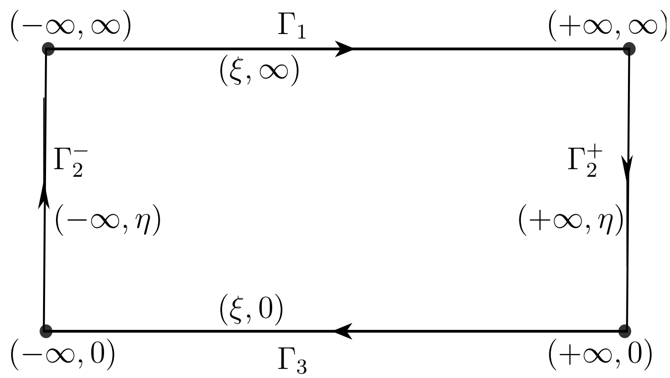

FIGURE 1. The domain $\mathfrak{R}$ of definition of the symbol $\mathscr{A}_p(\xi,\eta)$.

The symbol $\mathscr{A}_p(\omega)$ of the operator $\boldsymbol{A}$ in (77) is a function on the set $\mathfrak{R}$, viz.

$$\mathscr{A}_p(\omega) := \begin{cases} \displaystyle\sum_{j=1}^{m} a_j(\xi)(b_j)_p(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ \displaystyle\sum_{j=1}^{m} a_j(+\infty)b_j(-\eta), & \omega = (+\infty,\eta) \in \Gamma_2^+, \\ \displaystyle\sum_{j=1}^{m} a_j(-\infty)b_j(\eta), & \omega = (-\infty,\eta) \in \Gamma_2^-, \\ \displaystyle\sum_{j=1}^{m} a_j(\xi)(b_j)_p(0,\xi), & \omega = (\xi,0) \in \overline{\Gamma_3}. \end{cases} \tag{78}$$

In (78) for a piecewise continuous function $g \in PC(\overline{\mathbb{R}})$ we use the notation

$$\begin{aligned} g_p(\infty,\xi) &:= \frac{1}{2}\big[g(+\infty)+g(-\infty)\big] - \\ &\qquad - \frac{1}{2}\big[g(+\infty)-g(-\infty)\big]\cot\pi\Big(\frac{1}{p}-i\xi\Big), \\ g_p(t,\xi) &:= \frac{1}{2}\big[g(t+0)+g(t-0)\big] - \\ &\qquad - \frac{1}{2}\big[g(t+0)-g(t-0)\big]\coth\pi\Big(\frac{1}{p}-i\xi\Big), \end{aligned} \tag{79}$$

where $t,\xi \in \mathbb{R}$.

To make the symbol $\mathscr{A}_p(\omega)$ continuous, we endow the rectangle $\mathfrak{R}$ with a special topology. Thus let us define the distance on the curves Γ_1, $\Gamma_2^{\pm}$, Γ_3 and on $\overline{\mathbb{R}}$ by

$$\rho(x,y) := \left|\arg\frac{x-i}{x+i} - \arg\frac{y-i}{y+i}\right| \quad \text{for arbitrary} \ \ x,y\in\overline{\mathbb{R}}.$$

In this topology, the length $|\mathfrak{R}|$ of $\mathfrak{R}$ is 6π, and the symbol $\mathscr{A}_p(\omega)$ is continuous everywhere on $\mathfrak{R}$. The image of the function $\det\mathscr{A}_p(\omega)$, $\omega\in\mathfrak{R}$ $(\det\mathscr{B}_p(\omega))$ is a closed curve in the complex plane. It follows from the continuity of the symbol at the angular points of the rectangle $\mathfrak{R}$ where the one-sided limits coincide. Thus

$$\mathscr{A}_p(\pm\infty,\infty) = \sum_{j=1}^m [a_j(\pm\infty)b_j(\mp\infty),$$
$$\mathscr{A}_p(\pm\infty,0) = \sum_{j=1}^m [a_j(\pm\infty)b_j(0\mp 0).$$

Hence, if the symbol of the corresponding operator is elliptic, i.e. if

$$\inf_{\omega\in\mathfrak{R}} \left|\det\mathscr{A}_p(\omega)\right| > 0, \tag{80}$$

the increment of the argument $(1/2\pi)\arg\mathscr{A}_p(\omega)$ when ω ranges through $\mathfrak{R}$ in the positive direction is an integer, is called the winding number or the index and it is denoted by $\operatorname{ind}\det\mathscr{A}_p$.

Theorem 3.10. *Let* $1<p<\infty$ *and let* $\mathbf{A}$ *be defined by* (77). *The operator* $\mathbf{A}:\mathbb{L}_p(\mathbb{R}^+)\longrightarrow\mathbb{L}_p(\mathbb{R}^+)$ *is Fredholm if and only if its symbol* $\mathscr{A}_p(\omega)$ *is elliptic. If* $\mathbf{A}$ *is Fredholm, the index of the operator has the value*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\det\mathscr{A}_p. \tag{81}$$

Proof. Note that our study is based on a localization technique. For more details concerning this approach we refer the reader to [17, 19, 9, 30, 41].

Let us apply the Gohberg–Krupnik local principle to the operator $\mathbf{A}$ in (79), "freezing" the symbol of $\mathbf{A}$ at a point $x\in\overline{\mathbb{R}}:=\mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$. For $x\in\mathbb{R}$ and $\ell\in\mathbb{N}$, $\ell\geq 1$, let $C_x^\ell(\overline{\mathbb{R}})$ denote the set of all ℓ-times differentiable non-negative functions which are supported in a neighborhood of $x\in\mathbb{R}$ and are identically one everywhere in a smaller neighborhood of x. For $x\in\{-\infty\}\cup\{+\infty\}\cup\{\infty\}$, the functions from the corresponding classes $C_{+\infty}^\ell(\overline{\mathbb{R}})$ and $C_{-\infty}^\ell(\overline{\mathbb{R}})$ vanish on semi-infinite intervals $[-\infty,c)$ and $(-c,\infty]$, respectively, for certain $c>0$ and are identically one in smaller neighborhoods. It is easily seen that the system of localizing classes $\{C_x^\ell(\overline{\mathbb{R}})\}_{x\in\overline{\mathbb{R}}}$ is covering in the algebras $C(\overline{\mathbb{R}})$, $\mathfrak{M}_p(\overline{\mathbb{R}})$, respectively (cf. [17, 19, 9, 30]).

Let us now consider a system of localizing classes $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. These localizing classes depend on two variables, viz. on $\omega\in\mathfrak{R}$ and $x\in\overline{\mathbb{R}^+}$. In particular, the class $\mathfrak{L}_{\omega,x}$

contains the operator $\Lambda_{\omega,x}$,

$$\Lambda_{\omega,x} := \begin{cases} \big[h_0\mathfrak{M}^0_{v_\xi}W_{g_\infty}\big] = \big[h_0\mathfrak{M}^0_{v_\xi}\big] \\ \qquad\qquad \text{if } \ \omega=(\xi,\infty)\in\Gamma_1, \ \ x=0; \\ \big[h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_\infty}\big] = \big[h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_{\mp\infty}}\big] \\ \qquad\qquad \text{if } \ \omega=(\pm\infty,\infty)\in\Gamma^\pm_2\cap\Gamma_1, \ \ x\in\mathbb{R}^+; \\ \big[h_\infty\mathfrak{M}^0_{v_{\pm\infty}}W_{g_\eta}\big] = \big[h_\infty\mathfrak{M}^0_{v_{\pm\infty}}W_{g_{\mp\,\eta}}\big] \\ \qquad\qquad \text{if } \ \omega=(\pm\infty,\eta)\in\Gamma^\pm_2, \ \ x=\infty; \\ \big[h_\infty\mathfrak{M}^0_{v_\xi}W_{g_0}\big] = \big[\mathfrak{M}^0_{v_\xi}W_{g_0}\big] \\ \qquad\qquad \text{if } \ \omega=(\xi,0)\in\overline{\Gamma}_3, \ \ x=\infty, \end{cases} \tag{82}$$

where $h_x \in C^1_x(\overline{\mathbb{R}}^+)$, $v_\xi \in C^1_\xi(\overline{\mathbb{R}}^+)$, $g_\eta \in C^1_\eta(\overline{\mathbb{R}}^+)$, and $[\mathbf{A}] \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denotes the coset containing the operator $\mathbf{A}\in\mathfrak{A}_p(\mathbb{R}^+)$.

To verify the equalities in (82), one has to show that the difference between the operators in the square brackets is compact.

Consider the first equality in (82): The operator

$$h_0W_{g_\infty} - h_0I = h_0W_{(g_\infty-1)} = h_0W_{g_0}$$

is compact, since both functions h_0 and $1-g_\infty = g_0$ have compact supports, so Proposition 3.4 applies.

To check the second equality in (82), let us note that $h_x(0)=0$, $v_{\pm\infty}(\mp\infty) = 0$ and $g_{\pm\infty}(\xi)=0$ for all $\mp\,\xi>0$. From the fourth part of Proposition 3.6 we derive that for any $x\in\mathbb{R}^+$ the operator $h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_{\pm\infty}}$ is compact. This leads to the claimed equality since

$$\big[h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_\infty}\big] = \big[h_x\mathfrak{M}^0_{v_{\pm\infty}}\{W_{g_{-\infty}}+W_{g_{+\infty}}\}\big] = \big[h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_{\mp\infty}}\big].$$

The third identity in (82) can be verified analogously. As far as the fourth identity in (82) is concerned, one can replace h_∞ by 1 because the difference $h_\infty W_{g_0} - W_{g_0} = (1-h_\infty)W_{g_0} = h_0W_{g_0}$ is compact due to Proposition 3.4.

Consider now other properties of the system $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$. Propositions 3.4–3.6 imply that

$$\big[h_x\mathfrak{M}^0_{v_\xi}W_{g_\infty}\big] = 0 \ \text{ for all } \ (\xi,\eta,x)\in\overline{\mathbb{R}}\times\overline{\mathbb{R}}\times\overline{\mathbb{R}}^+\setminus\mathfrak{R}\times\overline{\mathbb{R}}^+.$$

Therefore, the system of localizing classes $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ is covering: for a given system $\{\Lambda_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ of localizing operators one can select a finite number of points $(\omega_1,x_1)=(\xi_1,\eta_1,x_1),\ldots,(\omega_s,x_s)=(\xi_s,\eta_s,x_s)\in\mathfrak{R}$ and add appropriately chosen terms $[h_{x_{s+j}}\mathfrak{M}^0_{v_{\xi_{s+j}}}W_{g_{s+j}}] = 0$ with $(\xi_{s+j},\eta_{s+j},x_{s+j}))\in\overline{\mathbb{R}}\times\overline{\mathbb{R}}\times\overline{\mathbb{R}}^+\setminus(\mathfrak{R}\times\overline{\mathbb{R}}^+)$, $j=1,2,\ldots,r$ so, that the equality

$$\sum_{j=1}^{r}\sum_{k=1}^{s}\big[c_{x_j}\mathfrak{M}^0_{a_{\xi_j}}W_{b_{\eta_k}}\big] = \big[c\mathfrak{M}^0_aW_b\big] \tag{83}$$

holds and the functions $c \in C(\overline{\mathbb{R}}^+)$, $a \in C\mathfrak{M}_p(\overline{\mathbb{R}})$, $b \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ are all elliptic. This implies the invertibility of the coset $[c\mathfrak{M}^0_a W_b]$ in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ and the inverse coset is $[c\mathfrak{M}^0_a W_b]^{-1} = [c^{-1}\mathfrak{M}^0_{a^{-1}} W_{b^{-1}}]$.

Note that the choice of a finite number of terms in (83) is possible due to Borel–Lebesgue lemma and the compactness of the sets $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^+$ (two point and one point compactification of $\mathbb{R}$ and of $\mathbb{R}^+$, respectively).

Moreover, localization in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ leads to the following local representatives of the cosets containing Mellin and Fourier convolution operators with symbols $a, b \in C\mathfrak{M}_p(\overline{\mathbb{R}})$:

$$[\mathfrak{M}^0_a] \overset{\mathfrak{M}^0_{v_{\xi_0}}}{\sim} [\mathfrak{M}^0_{a(\xi_0)}] = [a(\xi_0)I] \ \text{ if } \ \xi_0 \in \overline{\mathbb{R}}, \tag{84a}$$

$$[\mathfrak{M}^0_a] \overset{v_{x_0}I}{\sim} [\mathfrak{M}^0_{a_\infty}] \ \text{ if } \ \xi_0 \in \overline{\mathbb{R}^+}, \ \ x_0 \neq 0, \tag{84b}$$

$$[\mathfrak{M}^0_a] \overset{v_0 I}{\sim} [\mathfrak{M}^0_a] \ \text{ if } \ \xi_0 = 0, \tag{84c}$$

$$[W_b] \overset{W_{b_{\eta_0}}}{\sim} [W_{b(\eta_0)}] = [b(\eta_0)I] \ \text{ if } \ \eta_0 \in \mathbb{R} \setminus \{0\}, \tag{84d}$$

$$[W_b] \overset{W_{b_0}}{\sim} [W_{b^0}] = [\mathfrak{M}^0_{b_p(0,\cdot)}] \ \text{ if } \ \eta = 0, \tag{84e}$$

$$[W_b] \overset{W_{g_\infty}}{\sim} [W_{b^\infty(\infty,\cdot)}] = [\mathfrak{M}^0_{b_p(\infty,\cdot)}] \ \text{ if } \ \eta_0 = \pm\infty, \tag{84f}$$

$$[W_b] \overset{v_{x_0}I}{\sim} [W_{b^\infty}] = [\mathfrak{M}^0_{b_p(\infty,\cdot)}] \ \text{ if } \ x_0 \in \mathbb{R}^+, \tag{84g}$$

$$[W_b] \overset{v_\infty I}{\sim} [W_b] \ \text{ if } \ x_0 = \infty, \tag{84h}$$

where

$$\begin{aligned} g^\infty(\xi) &:= \frac{1}{2}\big[g(+\infty) + g(-\infty)\big] + \frac{1}{2}\big[g(+\infty) - g(-\infty)\big]\operatorname{sign}\xi = \\ &\qquad = g(-\infty)\chi_-(\xi) + g(+\infty)\chi_+(\xi), \\ g^0(\xi) &:= \frac{1}{2}\big[g(0+0) + g(0-0)\big] + \frac{1}{2}\big[g(0+0) + g(0-0)\big]\operatorname{sign}\xi = \\ &\qquad = g(0-0)\chi_-(\xi) + g(0+0)\chi_+(\xi), \end{aligned} \tag{85}$$

and $\chi_\pm(\xi) := (1/2)(1 \pm \operatorname{sign}\xi)$. Note that in the equivalency relations (84e)–(84g) we used the identities, cf. (75) and (79),

$$\begin{aligned} W_{g^\infty} &= \frac{1}{2}\big[g(-\infty) - g(+\infty)\big] - \frac{1}{2}\big[g(-\infty) - g(+\infty)\big]S_{\mathbb{R}^+} = \mathfrak{M}_{g_p(\infty,\cdot)}, \\ W_{g^0} &= \frac{1}{2}\big[g(0+0) + g(0-0)\big] - \frac{1}{2}\big[g(0+0) - g(0-0)\big]S_{\mathbb{R}^+} = \mathfrak{M}_{g_p(0,\cdot)}, \end{aligned}$$

which means that the Fourier convolution operators with homogeneous of order 0 symbols $g^\infty(\xi)$ and $g^0(\xi)$ are, simultaneously, Mellin convolutions with the symbols $g_p(\infty, \xi)$, $g_p(0, \xi)$.

Using the equivalence relations (84a)–(84h) and the compactness of the corresponding operators, cf. Propositions 3.4–3.6, one finds easily the following local representatives of the operator (coset) $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}\mathbb{L}_p(\mathbb{R}^+)$

(see (79) for the operator $\mathbf{A}$):

$$[\mathbf{A}] \overset{\Lambda_{(\xi_0,\infty),0}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\xi_0)} W_{(b_j)_\infty}\Big] =$$

$$= \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\xi_0)(b_j)_p(\infty,\cdot)}\Big] \overset{\Lambda_{(\xi_0,\infty),0}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\xi_0)(b_j)_p(\infty,\xi_0)}\Big] =$$

$$= \big[\mathscr{A}_p(\xi_0,\infty)I\big] \;\text{ if }\; \omega=(\xi_0,\infty)\in\Gamma_1,\;\; x_0=0, \tag{86a}$$

$$[\mathbf{A}] \overset{\Lambda_{(\pm\infty,\infty),x_0}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)} W_{(b_j)_\infty}\Big] = \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)(b_j)_p(\infty,\cdot)}\Big] =$$

$$= \big[\mathfrak{M}^0_{\mathscr{A}_p(\pm\infty,\cdot)}\big] \overset{\Lambda_{(\pm\infty,\infty),x_0}}{\sim} \big[\mathscr{A}_p(\pm\infty,\infty)I\big] \tag{86b}$$

$$\text{if }\; \omega=(\pm\infty,\infty)\in\overline{\Gamma_2^{\pm}}\cap\overline{\Gamma_1},\;\; 0<x_0<\infty;$$

$$[\mathbf{A}] \overset{\Lambda_{(\pm\infty,\mp\eta_0),\infty}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)} W_{b_j(\mp\eta_0)}\Big] = \Big[\sum_{j=1}^{m} a_j(\pm\infty) b_j(\mp\eta_0) I\Big] =$$

$$= \big[\mathscr{A}_p(\pm\infty,\mp\eta_0)I\big] \;\text{ if }\; \eta_0>0,\; \omega=(\pm\infty,\mp\eta_0)\in\Gamma_2^{\pm},\; x_0=\infty; \tag{86c}$$

$$[\mathbf{A}] \overset{\Lambda_{(\xi_0,0),\infty}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j} W_{b_j^0}\Big] =$$

$$= \Big[\sum_{j=1}^{m} a_j(\xi_0)\mathfrak{M}_{(b_j)_p(0,\cdot)}\Big] \overset{\Lambda_{(\xi_0,0),\infty}}{\sim} \Big[\sum_{j=1}^{m} a_j(\xi_0)(b_j)_p(0,\xi_0)\Big] =$$

$$= \big[\mathscr{A}_p(\xi_0,0)I\big] \;\text{ if }\; \omega=(\xi_0,0)\in\overline{\Gamma}_3,\;\; x_0=\infty; \tag{86d}$$

$$[\mathbf{A}] \overset{\Lambda_{(\pm\infty,\eta),\infty}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)} W_{b_j(0)}\Big] = \Big[\sum_{j=1}^{m} a_j(\pm\infty) b_j(0) I\Big] =$$

$$= \big[\mathscr{A}_p(\pm\infty,0)I\big] \;\text{ if }\; \omega=(\pm\infty,0)\in\overline{\Gamma}_3,\;\; x_0=\infty. \tag{86e}$$

It is remarkable that the local representatives (86a)–(86e) are just the quotient classes of multiplication operators by constant $N\times N$ matrices $[\mathscr{A}_p(\xi_0,\eta_0)I]$. If $\det\mathscr{A}_p(\xi_0,\eta_0)=0$, these representatives are not invertible, both locally and globally. On the other hand, they are globally invertible if $\det\mathscr{A}_p(\xi_0,\eta_0)\neq 0$. Thus, the conditions of the local invertibility for all points $\omega_0=(\xi_0,\eta_0)\in\mathfrak{R}$ and the global invertibility of the operators under consideration coincide with the ellipticity condition for the symbol $\inf\limits_{(\xi_0,\eta_0)\in\mathfrak{R}} \det\mathscr{A}_p(\xi_0,\eta_0)\neq 0$.

The index $\operatorname{Ind}\mathbf{A}$ is a continuous integer-valued multiplicative function $\operatorname{Ind}\mathbf{AB}=\operatorname{Ind}\mathbf{A}+\operatorname{Ind}\mathbf{B}$ defined on the group of Fredholm operators of $\mathfrak{A}_p(\mathbb{R}^+)$. On the other hand, the index function $\operatorname{ind}\det\mathscr{A}_p$ defined on L_p-symbols $\mathscr{A}_p$ possesses the same property $\operatorname{ind}\det\mathscr{A}_p\mathscr{B}_p=\operatorname{ind}\det\mathscr{A}_p+\operatorname{ind}\det\mathscr{B}_p$, see explanations after (80). Moreover, the set of operators (79) is dense in the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ and the corresponding set of their symbols is dense in the algebra $C(\mathfrak{R})$ of all continuous functions on $\mathfrak{R}$. For $p = 2$ these algebras even coincide. Therefore, there is an algebraic homeomorphism between the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ and the algebra of their symbols which is a dense subalgebra of $C(\mathfrak{R})$. Hence, two various index functions can be only connected by the relation $\operatorname{Ind}\mathbf{A} = M_0 \operatorname{ind}\det\mathscr{A}_p$ with an integer constant M_0 independent of $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Since for any Fourier convolution operator $\mathbf{A} = W_a$ the index formula is $\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\det\mathscr{A}_p$ [12, 13, 17], the constant $M_0 = -1$, and the index formula (81) is proved. □

Remark 3.11. Let us emphasize that the formula (81) does not contradict the invertibility of "pure Mellin convolution" operators $\mathfrak{M}^0_a : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ with an elliptic matrix symbol $a \in C\mathfrak{M}^0_p(\mathbb{R})$, $\inf_{\xi\in\mathbb{R}} |a(\xi)| > 0$, stated in Proposition 0.1, even if $\operatorname{ind} a \neq 0$.

In fact, computing the symbol of $\mathfrak{M}^0_a$ by formula (78), one obtains

$$(\mathfrak{M}^0_a)_p(\omega) := \begin{cases} a(\xi), & \omega = (\xi, \infty) \in \overline{\Gamma_1}, \\ a(+\infty), & \omega = (+\infty, \eta) \in \Gamma^+_2, \\ a(-\infty), & \omega = (-\infty, \eta) \in \Gamma^-_2, \\ a(\xi), & \omega = (\xi, 0) \in \overline{\Gamma_3}. \end{cases}$$

Noting that on the sets Γ_1 and Γ_3 the variable ω runs in opposite direction, the increment of the argument $[\arg\det(\mathfrak{M}^0_a)_p(\omega)]_{\mathfrak{R}} = 0$ is zero, implying $\operatorname{Ind}\mathfrak{M}^0_a = 0$.

In contrast to the above, the pure Fourier convolution operators $W_b : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ with elliptic matrix symbol $b \in C\mathfrak{M}^0_p(\mathbb{R})$, $\inf_{\xi\in\mathbb{R}} |b_p(\xi, \eta)| > 0$ can possess non-zero indices. Since

$$b_p(\omega) := \begin{cases} b_p(\infty, \xi), & \omega = (\xi, \infty) \in \overline{\Gamma_1}, \\ b(-\eta), & \omega = (+\infty, \eta) \in \Gamma^+_2, \\ b(\eta), & \omega = (-\infty, \eta) \in \Gamma^-_2, \\ b(0), & \omega = (\xi, 0) \in \overline{\Gamma_3}, \end{cases}$$

one arrives at the well-known formula

$$\operatorname{Ind} W_b = -\operatorname{ind} b_p.$$

Moreover, in the case where the symbol $b(-\infty) = b(+\infty)$ is continuous, one has $b_p(\xi, \eta) = b(\xi)$. Thus the ellipticity of the corresponding operator leads to the formula

$$\operatorname{ind} b_p = \operatorname{ind}\det b.$$

If $\mathscr{A}_p(\omega)$ is the symbol of an operator $\mathbf{A}$ of (77), the set $\mathscr{R}(\mathscr{A}_p) := \{\mathscr{A}_p(\omega) \in \mathbb{C} : \omega \in \mathfrak{R}\}$ coincides with the essential spectrum of $\mathbf{A}$. Recall that the essential spectrum $\sigma_{ess}(\mathbf{A})$ of a bounded operator $\mathbf{A}$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $\mathbf{A} - \lambda I$ is not Fredholm in $\mathbb{L}_p(\mathbb{R}^+)$ or, equivalently, the coset $[\mathbf{A} - \lambda I]$ is not invertible in the quotient algebra

$\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Then, due to Banach theorem, the essential norm $\|\|\mathbf{A}\|\|$ of the operator $\mathbf{A}$ can be estimated as follows

$$\sup_{\omega\in\omega}|\mathscr{A}_p(\omega)| \leqslant \|\|\mathbf{A}\|\| := \inf_{T\in\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))}\big\|(\mathbf{A}+T)\mid\mathscr{L}(\mathbb{L}_p(\mathbb{R}^+))\big\|. \tag{87}$$

The inequality (87) enables one to extend continuously the symbol map (78)

$$[\mathbf{A}] \longrightarrow \mathscr{A}_p(\omega), \quad [\mathbf{A}]\in\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+)) \tag{88}$$

on the whole Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$. Now, using Theorem 3.10 and conventional methods, cf. [22, Theorem 3.2], one can derive the following result.

Corollary 3.12. *Let $1 < p < \infty$ and $\mathbf{A}\in\mathfrak{A}_p(\mathbb{R}^+)$. The operator $\mathbf{A}:\mathbb{L}_p(\mathbb{R}^+)\longrightarrow\mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if it's symbol $\mathscr{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, then*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\mathscr{A}_p.$$

Theorem 3.10 and Corollary 3.12 lead to the assertion.

Corollary 3.13. *The set of maximal ideals of the commutative Banach quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ generated by scalar $N=1$ operators in* (77)*, is homeomorphic to $\mathfrak{R}$, and the symbol map in* (78), (88) *is a Gelfand homeomorphism of the corresponding Banach algebras.*

The proof of this result is similar to [22, Theorem 3.1] and is left to the reader.

Remark 3.14. All the above results are valid in a more general setting viz., for the Banach algebra $\mathfrak{P}\mathfrak{A}^{N\times N}_{p,\alpha}(\mathbb{R}^+)$ generated in the weighted Lebesgue space of N-vector-functions $\mathbb{L}^N_p(\mathbb{R}^+,x^\alpha)$ by the operators

$$\mathbf{A} := \sum_{j=1}^{m}\Big[d^1_j\mathfrak{M}^0_{a^1_j}W_{b^1_j} + d^2_j\mathfrak{M}^0_{a^2_j}H_{c^1_j} + d^3_jW^0_{b^2_j}H_{c^2_j}\Big] \tag{89}$$

when coefficients $d^1_j, d^2_j, d^3_j \in PC^{N\times N}(\overline{\mathbb{R}})$ are piecewise-continuous $N\times N$ matrix functions, symbols of Mellin convolution operators $\mathfrak{M}^0_{a^1_j}$, $\mathfrak{M}^0_{a^2_j}$, Winer–Hopf (Fourier convolution) operators $W_{b^1_j}$, $W_{b^2_j}$ and Hankel operators $H_{c^1_j}$, $H_{c^2_j}$ are $N\times N$ piecewise-continuous matrix $\mathbb{L}_p$-multipliers $a^k_j, b^k_j, c^k_j \in PC^{N\times N}\mathfrak{M}_p(\mathbb{R})$.

The spectral set $\Sigma(\mathfrak{P}\mathfrak{A}^{N\times N}_{p,\alpha}(\mathbb{R}^+))$ of such Banach algebra (viz., the set where the symbols are defined, e.g. $\mathfrak{R}$ for the Banach algebra $\mathfrak{A}^{N\times N}_p(\mathbb{R}^+)$ investigated above) is more sophisticated and described in the papers [15, 16, 22, 42]. Let $\mathfrak{C}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+)\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ be the sub-algebra of $\mathfrak{P}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+) = \mathfrak{P}\mathfrak{A}^{1\times1}_{p,\alpha}(\mathbb{R}^+)$ generated by scalar operators (89) with continuous coefficients $c_j, h_j\in C(\overline{\mathbb{R}})$ and scalar piecewise-continuous $\mathbb{L}_p$-multipliers) $a_j, b_j, d_j, g_j \in PC\mathfrak{M}_p(\mathbb{R})$. The quotient-algebra $\mathfrak{C}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+)\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ with respect to the ideal of all compact operators is a commutative algebra and the spectral set $\Sigma(\mathfrak{P}\mathfrak{A}_{p,\alpha}(\mathbb{R}^+))$ is homeomorphic to the set of maximal ideals.

We drop further details about the Banach algebra $\mathfrak{PA}^{N\times N}_{p,\alpha}(\mathbb{R}^+)$, because the result formulated above are sufficient for the purpose of this and subsequent papers dealing with the BVPs in domains with corners at the boundary.

4. Mellin Convolution Operators in Bessel Potential Spaces

As it was already mentioned, the primary aim of the present paper is to study Mellin convolution operators $\mathfrak{M}^0_a$ acting in Bessel potential spaces,

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+). \tag{90}$$

The symbols of these operators are $N \times N$ matrix functions $a \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, continuous on the real axis $\mathbb{R}$ with the only possible jump at infinity.

Theorem 4.1. *Let $0 < |\arg\gamma| < \pi$, $0 < |\arg c| < \pi$, $0 < |\arg(c\,\gamma)| < \pi$, $r, s \in \mathbb{R}$, $m = 1, 2, \ldots$, $1 < p < \infty$. Then the operator $K^m_c : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$ is lifted equivalently to the operator*

$$\boldsymbol{A}^{m,s}_c := \Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \tag{91a}$$

where

$$\boldsymbol{A}^{m,s}_c = \begin{cases} e^{\sigma(c,\gamma)\pi s i}c^{-s}\boldsymbol{K}^m_c W_{g^s_{\gamma,c}} \\ \qquad\qquad \text{if } \ -\pi < \arg c\,\gamma < 0, \\ e^{\sigma(c,\gamma)\pi s i}c^{-s}\Big[\boldsymbol{K}^m_c W_{g^s_{\gamma,c}} + (-1)^{m-1}\boldsymbol{K}^m_{-c}H_{g^s_{\gamma,c}}\Big] \\ \qquad\qquad \text{if } \ 0 < \arg c\,\gamma < \pi, \end{cases} \tag{91b}$$

$$H_{g^s_{\gamma,c}} = \begin{cases} I + T \\ \qquad\qquad \text{if } \ \sigma(c,\gamma) \neq 0, \\ H_{g^s_\infty} + T = e^{\sigma(\gamma)\pi s i}\Big[\cos\pi s I - \sigma(\gamma)\,\dfrac{\sin\pi s}{\pi}\boldsymbol{K}^1_{-1}\Big] + T \\ \qquad\qquad \text{if } \ \sigma(c,\gamma) = 0, \end{cases} \tag{91c}$$

$$\begin{aligned} g^s_{\gamma,c}(\xi) &:= \Big(\frac{\xi - c\,\gamma}{\xi+\gamma}\Big)^s, \\ g^s_\infty(\xi) &:= \frac{1}{2}\left[e^{\sigma(\gamma)2\pi s i} + 1\right] + \frac{1}{2}\left[e^{\sigma(\gamma)2\pi s i} - 1\right]\operatorname{sign}\xi, \end{aligned} \tag{91d}$$

T is a compact operator in $\mathbb{L}_p(\mathbb{R}^+)$, $\sigma(\gamma) := \operatorname{sign}\arg\gamma$ and $\sigma(c,\gamma)$ is defined in (51)

$$\sigma(c,\gamma) := \begin{cases} 0 & \text{if } \ 0 < \arg c < \pi, \\ \operatorname{sign}\arg(c\,\gamma) - \operatorname{sign}\arg\gamma & \text{if } \ -\pi < \arg c < 0. \end{cases}$$

Proof. Let $a_\pm \in \mathbb{L}_\infty(\mathbb{R})$ be $\mathbb{L}_p$-multipliers, which have analytic extensions $a_-(\xi)$ in the lower $\operatorname{Im}\xi < 0$ and $a_+(\xi)$ in the upper $\operatorname{Im}\xi > 0$ complex half planes. Then

$$W_{a_-}W_gW_{a_+} = W_{a_-ga_+}, \ \ \forall\, g \in \mathbb{L}_\infty(\mathbb{R}) \tag{92}$$

(cf., e.g., [17]).

Let $-\pi < \arg c\,\gamma < 0$. Theorem 2.7 and the property 92 yield the equalities

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_{\gamma} = e^{\sigma(c,\gamma)\pi si}c^{-s}\boldsymbol{K}^m_c\Lambda^s_{-c\,\gamma}\Lambda^{-s}_{\gamma} = \\ = e^{\sigma(c,\gamma)\pi si}c^{-s}\boldsymbol{K}^m_c W_{\lambda^s_{-c\,\gamma}} W_{\lambda^{-s}_{\gamma}} = e^{\sigma(c,\gamma)\pi si}c^{-s}\boldsymbol{K}^m_c W_{g^s_{\gamma,c}}.$$

For $0 < \arg c\,\gamma < \pi$ we have similarly to (92)

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_{\gamma} = e^{\sigma(c,\gamma)\pi si}c^{-s}\widetilde{\boldsymbol{K}}^m_c\Lambda^s_{-c\,\gamma}\Lambda^{-s}_{\gamma} = \\ = e^{\sigma(c,\gamma)\pi si}c^{-s}\widetilde{\boldsymbol{K}}^m_c W^0_{\lambda^s_{-c\,\gamma}} W^0_{\lambda^{-s}_{\gamma}} = e^{\sigma(c,\gamma)\pi si}c^{-s}\widetilde{\boldsymbol{K}}^m_c W^0_{g^s_{\gamma,c}}. \quad (93)$$

On the other hand,

$$\widetilde{\boldsymbol{K}}^m_c W^0_{g^s_{\gamma,c}}\varphi(t) = \boldsymbol{K}^m_c W_{g^s_{\gamma,c}}\varphi(t) + \int_{-\infty}^{0} \frac{\tau^{m-1}W^0_{g^s_{\gamma,c}}\varphi(\tau)\,d\tau}{(t-c\,\tau)^m}\,\varphi(t) = \\ = \boldsymbol{K}^m_c W_{g^s_{\gamma,c}}\varphi(t) + \int_0^{\infty} \frac{(-\tau)^{m-1}r_+\boldsymbol{V}W^0_{g^s_{\gamma,c}}\varphi(\tau)\,d\tau}{(t+c\,\tau)^m}\,\varphi(t) = \\ = \boldsymbol{K}^m_c W_{g^s_{\gamma,c}}\varphi(t) + (-1)^{m-1}\boldsymbol{K}^m_{-c}r_+\boldsymbol{V}W^0_{g^s_{\gamma,c}}\varphi(t) = \\ = \boldsymbol{K}^m_c W_{g^s_{\gamma,c}}\varphi(t) + (-1)^{m-1}\boldsymbol{K}^m_{-c}H_{g^s_{\gamma,c}}\varphi(t). \quad (94)$$

The proved equalities justify formula (91b) for $\boldsymbol{A}^{m,s}_c$.

To justify the remainder formulae (91c) and (91d) note that if $\sigma(c,\gamma)\neq 0$, the meromorphic function $g_{\gamma,c}(\xi)$ in (91d) has one pole and one zero in the same half-plane $\operatorname{Im}\xi < 0$ or $\operatorname{Im}\xi > 0$ and, therefore, has equal limits at the infinity: $\lim\limits_{\xi\pm\infty} g^s_{\gamma,c}(\xi) = 1$. Then $g^s_{\gamma,c}(\xi) := 1 + g^s_0(\xi)$ where $g^s_0(\xi)$ is continuous (is $C^\infty(\mathbb{R})$-smooth) and vanishes at the infinity: $g^s_0(\pm\infty) = 0$. By virtue of Proposition 3.6 the operator $T := H_{g^s_0}$ is compact in $\mathbb{L}_p(\mathbb{R}^+)$.

In contrast to the foregoing case, where $\sigma(c,\gamma) = 0$, the meromorphic function $g_{\gamma,c}(\xi)$ in (91c) has the pole and the zero in different half-planes and, therefore, the function has different limits at the infinity:

$$g^s_{\gamma,c}(-\infty) = \lim_{\xi\longrightarrow-\infty} g^s_{\gamma,c}(\xi) = 1, \\ g^s_{\gamma,c}(+\infty) = \lim_{\xi\longrightarrow+\infty} g_{\gamma,c}(\xi) = e^{\sigma(\gamma)2\pi si},$$

where $\sigma(\gamma) = \sigma(c\,\gamma) = \operatorname{sign}\arg\gamma = \operatorname{sign}\operatorname{Im}\gamma$. Consider the representation

$$g^s_{\gamma,c}(\xi) := g^s_\infty(\xi) + g^s_0(\xi), \quad (95)$$

where $g^s_\infty(\xi)$ is defined in (91c) and the function h^s_0 is, as above, continuous and $g^s_0(\pm\infty) = 0$. The operator $T := H_{g^s_0}$ is compact in $\mathbb{L}_p(\mathbb{R}^+)$.

On the other hand,

$$\begin{aligned} H_{g^s_\infty} &= \frac{1}{2}\left[e^{\sigma(\gamma)2\pi si}+1\right]I-\frac{1}{2}\left[e^{\sigma(\gamma)2\pi si}-1\right]H_{-\,\mathrm{sign}}=\\ &=\frac{1}{2}\left[e^{\sigma(\gamma)2\pi si}+1\right]I-\frac{1}{2}\left[e^{\sigma(\gamma)2\pi si}-1\right]r_+\boldsymbol{V}S_{\mathbb{R}^+}=\\ &=e^{\sigma(\gamma)\pi si}\Big[\cos\pi sI-\sigma(\gamma)\,\frac{\sin\pi s}{\pi}\,\boldsymbol{K}^1_{-1}\Big]. \end{aligned} \tag{96}$$

From (94)–(96) follows the representation (91b), (91d) in the case $0 < \arg c\,\gamma < \pi$, and the proof is complete. □

Let us consider a combined convolution operator

$$\mathbf{A} := d_0 I + W_a + \sum_{j=1}^{n} d_j \boldsymbol{K}^{m_j}_{c_j},\quad c_1,\dots,c_n\in\mathbb{C},\quad a\in C\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\}) \tag{97}$$

with constant coefficients $d_0, d_1,\dots,d_n\in\mathbb{C}$ in Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^+)$.

For a complex number $\gamma\in\mathbb{C}$, with the positive imaginary part $0 < \arg\gamma < \pi$, we assume the following:

$$\begin{aligned} -\pi < \arg c_j\gamma < 0 \ \text{ for } \ j=1,\dots,m,\\ 0<\arg c_j\gamma<\pi \ \text{ for } \ j=m+1,\dots,n. \end{aligned} \tag{98}$$

Then, due to the imposed constraint (97), the lifting property (91b) of the Mellin convolution operator and the lifting property (24) of the Fourier convolution operator, the lifted operator

$$\mathbf{A}^s := \Lambda^s_{-\gamma}\boldsymbol{A}\Lambda^{-s}_{\gamma} : \mathbb{L}_p(\mathbb{R}^+)\longrightarrow \mathbb{L}_p(\mathbb{R}^+) \tag{99}$$

has the form

$$\begin{aligned} \mathbf{A}^s &:= W_{d_0 g^s_\gamma} + W_{a g^s_\gamma} + \sum_{j=1}^{m} d_j c_j^{-s}\boldsymbol{K}^{m_j}_{c_j} W_{g^s_{\gamma,c_j}} + \\ &+ \sum_{j=m+1}^{n} d_j e^{\sigma(c_j,\gamma)\pi si} c_j^{-s}\Big[\boldsymbol{K}^{m_j}_{c_j} W_{g^s_{\gamma,c_j}} - (-1)^{m_j}\boldsymbol{K}^{m_j}_{-c_j} H_{g^s_{\gamma,c_j}}\Big] + T, \end{aligned} \tag{100}$$

where (see (51))

$$\sigma(c_j,\gamma) := \begin{cases} 0 & \text{if } \ 0<\arg c_j<\pi,\\ 0 & \text{if } \ -\pi<\arg c_j<0,\ \ 0<\arg c_j\gamma<\pi,\\ -2 & \text{if } \ -\pi<\arg c_j<0,\ \ -\pi<\arg c_j\gamma<0, \end{cases} \tag{101}$$

the functions $g^s_{\gamma,c_j}\in C(\dot{\mathbb{R}})$ are defined in (91d) and, due to the conditions (98), have the following limits at the infinity:

$$\begin{gathered} g^s_{\gamma,c_j}(-\infty)=1,\ \ g^s_{\gamma,c_j}(0)=e^{-\sigma(c_j)\pi si}c_j^s,\ \ g^s_{\gamma,c_j}(+\infty)=1,\ \ j=1,\dots,m,\\ g^s_{\gamma,c_j}(-\infty)=1,\ \ g^s_{\gamma}(0)=e^{-\sigma(c_j)\pi si}c_j^s,\ \ g^s_{\gamma,c_j}(+\infty)=e^{2\pi si},\ \ j=m+1,\dots,n,\\ \sigma(c_j):=\operatorname{sign}\arg c_j. \end{gathered}$$

The function $g^s_\gamma \in C(\mathbb{R})$ is continuous on $\mathbb{R}$, but has different limits at the infinity

$$g^s_\gamma(-\infty) = 1, \quad g^s_\gamma(+\infty) = e^{2\pi si}.$$

And, finally, the symbols

$$\mathscr{K}^{m_j}_{c_j,p}(\xi) := \mathscr{M}_{1/p}\mathscr{K}^{m_j}_{c_j}(\xi), \quad \mathscr{K}^1_{-1,p}(\xi) := \mathscr{M}_{1/p}\mathscr{K}^1_{-1}(\xi)$$

of the operators $\boldsymbol{K}^{m_j}_{c_j}$ and $\boldsymbol{K}^1_{-1} = \pi i S_{\mathbb{R}^+}$ are defined in (34)–(38) and have the following limits at the infinity

$$\mathscr{K}^{m_j}_{c_j}(\pm\infty) = 0, \ \ j = 1, \dots, n, \quad \mathscr{K}^1_{-1,p}(\pm\infty) = \pm 1.$$

Using the equality (100), we announce the symbol $\mathscr{A}^s_p(\omega)$, $\omega \in \mathfrak{R}$, of the lifted operator $\boldsymbol{A}^s$ in $\mathbb{L}_p(\mathbb{R}^+)$ as the symbol of $\boldsymbol{A}$ in Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^+)$ (cf. the definition (78))

$$\mathscr{A}^s_p(\omega) :=$$

$$:= \begin{cases} d_0 g^s_p(\xi) + a^s_p(\infty,\xi) + \\ \quad + \displaystyle\sum_{j=1}^{m} d_j c_j^{-s} \mathscr{K}^{m_j}_{c_j,p}(\xi) + \sum_{j=m+1}^{n} d_j e^{\sigma(c_j,\gamma)\pi si} c_j^{-s} \times \\ \quad \times \Big[\mathscr{K}^{m_j}_{c_j,p}(\xi) \mathscr{W}_{g^s_{\gamma,c_j,p}}(\infty,\xi) - (-1)^{m_j} \mathscr{K}^{m_j}_{-c_j,p}(\xi) \mathscr{H}_{g^s_{\gamma,c_j,p}}(\infty,\xi)\Big], \\ \qquad\qquad \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ \big\{d_0 + a(-\eta)\big\}\Big(\dfrac{\eta+\gamma}{\eta-\gamma}\Big)^s, \ \ \omega = (+\infty,\eta) \in \Gamma_2^+, \\ \big\{d_0 + a(\eta)\big\}\Big(\dfrac{\eta-\gamma}{\eta+\gamma}\Big)^s \omega = (-\infty,\eta) \in \Gamma_2^-, \\ e^{\pi si}\big\{d_0 + a_p(0,\xi)\big\} + \\ \quad + \displaystyle\sum_{j=1}^{m} d_j e^{-\sigma(c_j)\pi si} \mathscr{K}^{m_j}_{c_j,p}(\xi) + \sum_{j=m+1}^{n} d_j e^{\sigma(c_j,\gamma)\pi si} \times \\ \quad \times \Big[e^{-\sigma(c_j)\pi si} \mathscr{K}^{m_j}_{c_j,p}(\xi) - (-1)^{m_j} c_j^{-s} \mathscr{K}^{m_j}_{-c_j,p}(\xi) \mathscr{H}_{g^s_{\gamma,c_j,p}}(\infty,\xi)\Big], \\ \qquad\qquad \omega = (\xi,0) \in \overline{\Gamma_3}, \end{cases} \tag{102}$$

where, since $\sigma(\gamma) = \operatorname{sign}\arg\gamma = 1$,

$$\mathscr{W}_{g^s_{\gamma,c_j,p}}(\infty,\xi) := e^{\pi si}\Big[\cos\pi s - \sin\pi s\, \cot\pi\Big(\frac{1}{p} - i\xi\Big)\Big], \tag{103}$$

$$\mathscr{H}_{g^s_{\gamma,c_j,p}}(\infty,\xi) := e^{\pi si}\Big[\cos\pi s - \frac{\sin\pi s}{\sin\pi(1/p - i\xi)}\Big], \ \ j = m+1, \dots, n, \tag{104}$$

$$\begin{aligned} a^s_p(\infty,\xi) &:= \frac{1}{2}\left[e^{2\pi si}a(+\infty)+a(-\infty)\right]-\\ &\qquad -\frac{1}{2}\left[e^{2\pi si}a(+\infty)-a(-\infty)\right]\cot\pi\Big(\frac{1}{p}-i\xi\Big),\\ a_p(t,\xi) &:= \frac{1}{2}\left[a(t+0)+a(t-0)\right]-\\ &\qquad -\frac{1}{2}\left[a(t+0)-a(t-0)\right]\cot\pi\Big(\frac{1}{p}-i\xi\Big). \end{aligned}$$

Theorem 4.2. *Let $1<p<\infty$, $s\in\mathbb{R}$ and let $\mathbf{A}$ be defined by* (97)*. The operator $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathscr{A}^s_p(\omega)$, defined in* (102)*, is elliptic. If $\mathbf{A}$ is Fredholm, the index of the operator has the value*

$$\operatorname{Ind}\mathbf{A}=-\operatorname{ind}\det\mathscr{A}^s_p. \tag{105}$$

Proof. The proof follows if we apply to the lifted operator $\boldsymbol{A}^s$ (see (99)) having the form (100), Theorem 3.10. □

For the definition of the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p(\Omega)=\mathbb{B}^s_{p,p}(\Omega)$, $\widetilde{\mathbb{W}}^s_p(\Omega)=\widetilde{\mathbb{B}}^s_{p,p}(\Omega)$ we for arbitrary domain $\Omega\subset\mathbb{R}^n$, including the half axes $\mathbb{R}^+$ refer, e.g., to the monograph [43].

Corollary 4.3. *Let $1<p<\infty$, $s\in\mathbb{R}$ and let $\mathbf{A}$ be defined by* (87)*. If the operator $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm* (*is invertible*) *for all $a\in(s_0,s_1)$ and $p\in(p_0,p_1)$, where $-\infty<s_0<s_1<\infty$, $1<p_o<p_1<\infty$, then*

$$\mathbf{A}:\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{W}^s_p(\mathbb{R}^+),\quad s\in(s_0,s_1),\ \ p\in(p_0,p_1) \tag{106}$$

is Fredholm and has the equal index

$$\operatorname{Ind}\mathbf{A}=-\operatorname{ind}\det\mathscr{A}^s_p. \tag{107}$$

(*is invertible, respectively*) *in the Sobolev–Slobodeckij* (*Besov*) *spaces $\mathbb{W}^s_p=\mathbb{B}^s_{p,p}$.*

Proof. First of all recall that the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p=\mathbb{B}^s_{p,p}$ emerge as the result of interpolation with the real interpolation method between Bessel potential spaces

$$\begin{aligned} &\big(\mathbb{H}^{s_0}_{p_0}(\Omega),\mathbb{H}^{s_1}_{p_1}(\Omega)\big)_{\theta,p}=\mathbb{W}^s_p(\Omega),\ \ s:=s_0(1-\theta)+s_1\theta,\\ &\big(\widetilde{\mathbb{H}}^{s_0}_{p_0}(\Omega),\widetilde{\mathbb{H}}^{s_1}_{p_1}(\Omega)\big)_{\theta,p}=\widetilde{\mathbb{W}}^s_p(\Omega),\ \ p:=\frac{1}{p_0}\,(1-\theta)+\frac{1}{p_1}\,\theta,\ \ 0<\theta<1. \end{aligned} \tag{108}$$

If $\mathbf{A}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm (or is invertible) for all $s\in(s_0,s_1)$ and $p\in(p_0,p_1)$, it has a regularizer $\mathbf{R}$ (has the inverse $\mathbf{A}^{-1}=\mathbf{R}$, respectively), which is bounded in the setting

$$\mathbf{R}:\mathbb{W}^s_p(\mathbb{R}^+)\longrightarrow\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$$

due to the interpolation (108) and

$$\mathbf{RA} = I + \mathbf{T}_1, \quad \mathbf{AR} = I + \mathbf{T}_2,$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and in $\longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$, respectively ($\mathbf{T}_1 = \mathbf{T}_2 = 0$ if $\mathbf{A}$ is invertible).

Due to the Krasnoselskij interpolation theorem (see [43]), $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$ and in $\mathbb{W}^s_p(\mathbb{R}^+)$, respectively for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$ and, therefore, $\mathbf{A}$ in (106) is Fredholm (is invertible, respectively).

The index formulae (107) follows from the embedding properties of the Sobolev–Slobodeckij and Bessel potential spaces by standard well-known arguments. □

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References

1. A.-S. Bonnet-Ben Dhia, L. Chesnel, and P, Ciarlet, Jr. T-coercivity for scalar interface problems between dielectrics and metamaterials. *ESAIM Math. Model. Numer. Anal.* **46** (2012), No. 6, 1363–1387.
2. A.-S. Bonnet-Ben Dhia, L. Chesnel, and X. Claeys, Radiation condition for a non-smooth interface between a dielectric and a metamaterial. *Math. Models Methods Appl. Sci.* **23** (2013), No. 9, 1629–1662; `http://hal.inria.fr/hal-00651008/`
3. T. Buchukuri, O. Chkadua, R. Duduchava, and D. Natroshvili, Interface crack problems for metallic-piezoelectric composite structures. *Mem. Differential Equations Math. Phys.* **55** (2012), 1–150.
4. T. Buchukuri, R. Duduchava, D. Kapanadze, and M. Tsaava, The boundary value problems for the Helmholtz equation in arbitrary $2D$-sectors. *Manuscript.*
5. M. Costabel, Boundary integral operators on curved polygons. *Ann. Mat. Pura Appl. (4)* **133** (1983), 305–326.
6. M. Costabel and E. Stephan, The method of Mellin transformation for boundary integral equations on curves with corners. In: A. Gerasoulis, R. Vichnevetsky (Eds.), *Numerical Solutions of Singular Integral Equations, IMACS, New Brunswick,* 1984, pp. 95–102.
7. D. Colton and R. Kress, Integral equation methods in scattering theory. *Pure and Applied Mathematics. A Wiley-Interscience Publication, John Wiley & Sons. New York etc.*, 1983.
8. H. O. Cordes, Pseudo-differential operators on a half-line. *J. Math. Mech.* **18** (1968/69), 893–908.
9. V. D. Didenko and B. Silbermann, Approximation of additive convolution-like operators. Real C^*-algebra approach. *Frontiers in Mathematics. Birkhäuser Verlag, Basel*, 2008.
10. V. D. Didenko, S. Roch, and B. Silbermann, Approximation methods for singular integral equations with conjugation on curves with corners. *SIAM J. Numer. Anal.* **32** (1995), No. 6, 1910–1939.
11. V. Didenko and E. Venturino, Approximate solutions of some Mellin equations with conjugation. *Integral Equations Operator Theory* **25** (1996), No. 2, 163–181.
12. R. Duduchava, Convolution integral operators with discontinuous coefficients. (Russian) *Dokl. Akad. Nauk SSSR* **218** (1974), 264–267; translation in *Sov. Math. Doklady* **15** (1974), 1302–1306.

13. R. V. Duduchava, On Wiener–Hopf integral operators. *Math. Nachr.* **65** (1975), No. 1, 59–82.
14. R. V. Duduchava, Convolution integral operators with discontinuous symbols. (Russian) *Tr. Tbilis. Mat. Inst. Razmadze* **50** (1975), 34–41.
15. R. V. Duduchava, Integral operators of convolution type with disconnected coefficients. *Math. Nachr.* **79** (1977), 75–98.
16. R. V. Duduchava, Integral equations of convolution type with discontinuous coefficients. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **92** (1978), No. 2, 281–284.
17. R. Duduchava, Integral equations in convolution with discontinuous presymbols, singular integral equations with fixed singularities, and their applications to some problems of mechanics. *BSB B. G. Teubner Verlagsgesellschaft, Leipzig*, 1979.
18. R. Duduchava, An application of singular integral equations to some problems of elasticity. *Integral Equations Operator Theory* **5** (1982), No. 4, 475–489.
19. R. Duduchava, On multidimensional singular integral operators. I. The half-space case. *J. Operator Theory* **11** (1984), Nno. 1, 41–76; II. The case of compact manifolds. *J. Operator Theory* **11** (1984), No. 2, 199–214.
20. R. Duduchava, On general singular integral operators of the plane theory of elasticity. *Rend. Sem. Mat. Univ. Politec. Torino* **42** (1984), No. 3, 15–41.
21. R. V. Duduchava, General singular integral equations and fundamental problems of the plane theory of elasticity. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR* **82** (1986), 45–89.
22. R. Duduchava, On algebras generated by convolutions and discontinuous functions. *Special issue: Wiener–Hopf problems and applications (Oberwolfach, 1986). Integral Equations Operator Theory* **10** (1987), No. 4, 505–530.
23. R. Duduchava, The Green formula and layer potentials. *Integral Equations Operator Theory* **41** (2001), No. 2, 127–178.
24. R. V. Duduchava and T. I. Latsabidze, The index of singular integral equations with complex-conjugate functions on piecewise-smooth lines. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR* **76** (1985), 40–59.
25. R. Duduchava, T. Latsabidze, and A. Saginashvili, Singular integral operators with the complex conjugation on curves with cusps. Integral Equations Operator Theory **22** (1995), No. 1, 1–36.
26. R. Duduchava and F.-O. Speck, Pseudodifferential operators on compact manifolds with Lipschitz boundary. *Math. Nachr.* **160** (1993), 149–191.
27. R. Duduchava and M. Tsaava, Mixed value problems for the Helmholtz equation in arbitrary $2D$-sectors. *Manuscript*.
28. G. I. Eskin, Boundary value problems for elliptic pseudodifferential equations. Translated from the Russian by S. Smith. *Translations of Mathematical Monographs*, 52. *American Mathematical Society, Providence, R.I.*, 1981.
29. I. Z. Gochberg and I. A. Feldman, Faltungsgleichungen und Projektionsverfahren zu ihrer Losung. (German) Übersetzung aus dem Russischen von Reinhard Lehmann und Jürgen Leiterer. *Mathematische Reihe*, Band 49. *Birkhäuser Verlag, Basel–Stuttgart*, 1974.
30. I. Gohberg and N. Krupnik, One-dimensional linear singular integral equations. I, II. *Operator Theory: Advances and Applications* 53–54. *Birkhäuser Verlag, Basel etc.*, 1992.
31. I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products. *Academic Press, Inc. Boston, MA*, 1994.
32. D. K. Gramotnev and S. I. Bozhevolnyi, Plasmonics beyond the diffraction limit. *Nature Photonics* **4** (2010), No. 2, 83–91.
33. L. Hörmander, Estimates for translation invariant operators in L^p spaces. *Acta Math.* **104** (1960), 93–140.
34. A. I. Kalandiya, Mathematical methods of two-dimensiobnal elasticity. *Nauka, Moscow*, 1973.

35. B. V. Khvedelidze, Linear discontinuous boundary problems in the theory of functions, singular integral equations and some of their applications. (Russian) *Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze* **23** (1956), 3–158.
36. P. Kuchment, Quantum graphs. I. Some basic structures. *Waves Random Media* **14** (2004), No. 1, S107–S128.
37. P. Kuchment, Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs. *J. Phys. A* **38** (2005), No. 22, 4887–4900.
38. W. McLean, Strongly elliptic systems and boundary integral equations. *Cambridge University Press, Cambridge*, 2000.
39. V. S. Rabinovich and S. Roch, Pseudodifferential operators on periodic graphs. *Integral Equations Operator Theory* **72** (2012), No. 2, 197–217.
40. R. Schneider, Integral equations with piecewise continuous coefficients in L^p-spaces with weight. *J. Integral Equations* **9** (1985), No. 2, 135–152.
41. I. B. Simonenko, A new general method of investigating linear operator equations of singular integral equation type. I. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 567–586.
42. G. Thelen, Zur Fredholmtheorie singulärer Integrodifferentialoperatoren auf der Halbachse. *Dissertation Dr. rer. nat. Darmstadt*, 1985.
43. H. Triebel, Interpolation theory, function spaces, differential operators. *Johann Ambrosius Barth, Heidelberg*, 1995.

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IV International Conference of the Georgian Mathematical Union dedicated to the 110-th birthday anniversary of academician Victor Kupradze and the 90-th year anniversary of Georgian Mathematical Union

September 9–15, 2013, Tbilisi and Batumi, Georgia

The opening ceremony of the conference and V. Kupradze memorial evening took place in Tbilisi, at Georgian National Academy of Sciences on September 9. David Natroshvili (Georgia) gave a talk about V. Kupradze's life and scientific heritage.

At the memorial evening their speeches were delivered by Roland Duduchava, Vakhtang Kokilashvili, George Kvesitadze, Jumber Lominadze, Roin Metreveli and Guram Kekelia.

The scientific part of the conference took place in Batumi at Shota Rustaveli State University from September 11 to September 15.

The conference was organized by:

- Georgian Mathematical Union;
- Georgian National Academy of Sciences;
- Shota Rustaveli State University, Batumi

The conference covered the following topics:

- Real Analysis;
- Complex Analysis;
- Topology;
- Algebra and Number Theory;
- Differential Equations and Applications;
- Probability & Statistics, Financial Mathematics;
- Mathematical Logic, Applied Logic and Programming;
- Mathematical Modelling;
- Mathematical Physics;
- Numerical Analysis;
- Mathematical Education and History;
- Continuum Mechanics.

In the conference participated over 150 scientists from 18 countries, among them 40 from abroad and about 110 from Georgia. The participants contributed 114 reports (30 min. each) on sections, 5 plenary and 13 semi-plenary talks (50 minute). The plenary talks were delivered by Chkareuli Juansher (Georgia), Kokilashvili Vakhtang (Georgia), Sloan Ian (Australia), Böttcher Albrecht (Germany), Kaashoek Marinus (Netherlands).

The semi-plenary talks were delivered by Bojarsky Bogdan (Poland), Duduchava Roland (Georgia), Kadeishvili Tornike (Georgia), Kapanadze David (Georgia), Karkashadze David & Zaridze Revaz (Georgia), Lanza de Cristoforis Massimo (Italy), Lashkhi Alexander (Georgia), Meskhi Alexander (Georgia), Mikhailov Sergey (UK), Ovchinnikov Vladimir (Russia), Pkhakadze Konstantine (Georgia) and Vasilevski Nikolai (Mexico).

More detailed information about the conference, posters, program, abstracts, the list of plenary speakers and participants are available on the WEB: `http://www.gmu.ge/`

Prof. Roland Duduchava

Chairman of the Organizing Committee,
President of the Georgian Mathematical Union

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