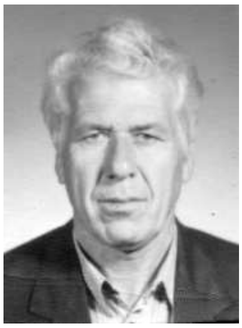
Memoirs on Differential Equations and Mathematical Physics
VOLUME 61, 2014, 1–3



Revaz Bantsuri

Georgian science has suffered a grievous loss. Revaz Bantsuri, a prominent Georgian mathematician, corresponding member of the Georgian National Academy of Sciences, Doctor of physical and mathematical sciences, Professor, passed away.

He was born on June 10, 1936 in the village of Bantsurtkari (Dusheti region). Upon graduation from I. Javakhishvili Tbilisi State University, starting from 1960 up to the end of his life he worked at A. Razmadze Mathematical Institute holding different positions. In 1966 he defended his Candidate's thesis and in 1982 Doctoral thesis at the Institute of Problems of Mechanics of the Russian Academy of Sciences. From 1983 he headed the department of mathematical theory of elasticity.

In 1997, Revaz Bantsuri was elected a corresponding member of the Georgian National Academy of Sciences. He was a member of Russian National Committee in Theoretical and Applied Mechanics.

Revaz Bantsuri was Niko Muskhelishvili's pupil and worthy successor of his scientific ideas.

All his works he devoted to: boundary and contact problems of the plane theory of elasticity, mixed boundary value problems of the theory of analytic functions, problems of elasticity for domains with partially unknown boundaries, systems of convolution type integral equations and infinite algebraic equations. He essentially developed the well-known Muskhelishvili's research area, having appreciably enriched with new trends a range of application of methods of the theory of analytic functions.

Using integral transformations, R. Bantsuri reduced contact problems of certain classes to new type boundary value problems of the theory of analytic functions and called them the Carleman type problems for a strip.

He elaborated a new type method of factorization and solved the Carleman type problem in a rather general case. Applying this method, he solved very important contact problems of various types for isotropic and anisotropic bodies.

This method, besides the theory of elasticity, can be used in the theory of convolution type integral equations and in the theory of systems of the same type infinite algebraic equations, in problems of heat distribution with third kind boundary conditions, in problems of electromagnetic wave diffraction, etc. The method for the above-mentioned problems is of the same importance as that developed by Muskhelishvili in the 40th of the past century for investigation of classical contact problems. The method is known as R. Bantsuri's method of canonical solutions, and presently is a unique general method successfully used for effective solution of the above-mentioned contact problems.

The problems for domains with partially unknown boundaries deal with optimal distribution of stresses in a body. They belong to mathematically complicated and very important problems of optimal projecting. In a general case, these problems are reduced to nonlinear problems.

Revaz Bantsuri formulated the problems of the plane theory of elasticity and plate bending for some classes of problems with partially unknown boundaries and reduced them first to linear problems and then to the problems of the theory of analytic functions with shifts and called them the Carleman type problems for a circular ring. He elaborated the second method of factorization whose application allowed us to get a completed theory of solvability for that class of problems.

Applying the methods of Muskhelishvili and Wiener-Hopf, R. Bantsuri reduced statical problems of cracks, when the crack comes to the boundary or to the interface of a piecewise homogeneous medium, to the problem of linear conjugation with a Wiener class coefficient. He constructed effective solutions and studied the question on the stress concentration at the crack ends. Thus he has obtained significant results in fracture mechanics. The above-mentioned R. Bantsuri's result is recognized by specialists as one of the best results.

The problems of crack distribution in a body with constant or varying velocity belong to such a class of mixed problems when the points of change of boundary conditions displace in time. R. Bantsuri considered the problems when semi-infinite cracks in a plane spread linearly with constant or varying velocity. The problems of crack distribution with constant velocity were reduced by means of variable transformations to the problem of classical dynamics, while in the problem of crack distribution with varying velocity we get by means of Fourier-Laplace transformation the generalized Wiener-Hopf problem. An effective solution of that problem is obtained. The above method is used in contact problems when a semi-infinite rigid punch moves with varying velocity at the boundary of a half-plane or a strip. Very interesting and significant results were obtained in this group of problems, as well.

The apparatus of the Cauchy type integral turned out to be insufficient for solving the Carleman type problems for a strip and a circular ring, hence Revaz Bantsuri constructed new integral representations which in this case have played the same role as the Cauchy type integrals in problems of linear conjugation. Using the obtained results, R. Bantsuri constructed for a circular ring a solution for the Riemann-Hilbert problem and for the mixed problem of the theory of analytic functions, he obtained effective solutions of a system of infinite convolution type algebraic equations.

R. Bantsuri together with G. Janashiya proved the invariance of Wiener functions algebra on the axis with respect to Hilbert transformations. This allowed him to reduce a solution of convolution type integral equations on the semi-axis for a summable kernel to the problem of linear conjugation in a class of Wiener functions.

Relying on the above-said, we can conclude that Revaz Bantsuri has made an internationally recognized contribution to the development of the theory of elasticity. He improved N. Muskhelishvili's method and largely extended an area of application of methods of the theory of analytic functions in the plane theory of elasticity.

A special mention should be made of Revaz Bantsuri's contribution to the cause of education of the young generation. For many years he worked at the Chair of Theoretical Mechanics of Tbilisi State University, delivered lectures in the theory of elasticity and brought up many candidates and doctors of sciences.

Revaz Bantsuri, a great researcher, remarkable citizen, excellent family man, modest and responsive, has passed away. He made a major contribution to the science, but there remained a lot of unrealizable thoughts and ideas. Editorial Board of our journal expresses sincerest condolences in connection with the death of a prominent scientist and dear colleague. His bright personality will leave the trace in our memory forever.

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Memoirs on Differential Equations and Mathematical Physics
Volume 61, 2013, 5–20

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SUFFICIENCY CONDITIONS FOR ASYMPTOTIC STABILITY OF SOLUTIONS OF A LINEAR HOMOGENEOUS NONAUTONOMOUS DIFFERENTIAL EQUATION OF SECOND ORDER

Abstract. The problem on the stability of second order linear homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0$$

is investigated in the case where the roots $\lambda_i(t)$ $(i = 1, 2)$ of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

are such that

$$\lambda_i(t) < 0 \ \text{ for } \ t \geq t_0, \qquad \int\limits_{t_0}^{+\infty} \lambda_i(t)\, dt = -\infty \ \ (i = 1, 2)$$

and there exist finite or infinite limits $\lim\limits_{t \to +\infty} \lambda_i(t)$ $(i = 1, 2)$.

2010 Mathematics Subject Classification. 34D05, 34E10.

Key words and phrases. Second order differential equation, linea, stability, characteristic equation.

რეზიუმე. გამოკვლეულია მეორე რიგის წრფივი ერთგვაროვანი დიფერენციალური განტოლების

$$y'' + p(t)y' + q(t)y = 0$$

მდგრადობის საკითხი იმ შემთხვევაში, როცა მახასიათებელი განტოლების

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

ფესვებს $\lambda_i(t)$ $(i = 1, 2)$ გააჩნიათ სასრული ან უსასრულო ზღვრები, როცა $t \to +\infty$, ამასთან

$$\lambda_i(t) < 0, \ \text{ როცა } \ t \geq t_0, \qquad \int\limits_{t_0}^{+\infty} \lambda_i(t)\, dt = -\infty \ \ (i = 1, 2).$$

.

1. Introduction

In the theory of stability of linear homogeneous on-line systems (LHS) of ordinary differential equations

$$\frac{dY}{dt} = P(t)Y, \quad t \in [t_0; +\infty) = I,$$

where $P(t)$ is, in general, complex matrix, the interest is focused on the investigation of stability of LHS depending on the roots $\lambda_i(t)$ $(i = \overline{1,n})$ of the characteristic equation

$$\det(P(t) - \lambda E) = 0.$$

L. Cesàro [1] considered a system of n-th order differential equations

$$\frac{dY}{dt} = [A + B(t) + C(t)]Y,$$

where A is a constant matrix, the roots λ_i $(i = \overline{1,n})$ of characteristic equation are different and satisfy the condition

$$\operatorname{Re}\lambda_i \le 0 \quad (i = \overline{1,n});$$

$$B(t) \to 0 \text{ as } t \to +\infty, \qquad \int\limits_{t_0}^{+\infty} \Big\| \frac{dB(t)}{dt} \Big\| \, dt < +\infty;$$

$$\int\limits_{t_0}^{+\infty} \|C(t)\| \, dt < +\infty;$$

the roots of characteristic equation of the matrix $A + B(t)$ have nonpositive real parts.

C. P. Persidsky's article [2] deals with the case, where elements of the matrix $P(t)$ are the functions with weak variation, that is, each such function can be represented as

$$f(t) = f_1(t) + f_2(t),$$

where $f_1(t) \in C_I$ and there exists $\lim\limits_{t\to+\infty} f_1(t) \in \mathbb{R}$, but $f_2(t)$ is such that

$$\sup_{t\in I} |f_2(t)| < +\infty, \qquad \lim_{t\to+\infty} f_2'(t) = 0,$$

and the condition $\operatorname{Re}\lambda_i(t) \le a \in \mathbb{R}_-$ $(i = \overline{1,n})$ is fulfilled.

N. Y. Lyashchenko [3] considered a case $\operatorname{Re}\lambda_i(t) < a \in \mathbb{R}_-$ $(i = \overline{1,n})$, $t \in I$,

$$\sup_{t\in I} \|A'(t)\| \le \varepsilon.$$

The case $n = 2$ is thoroughly studied by N. I. Izobov.

I. K. Hale [4] studied the asymptotic behavior of LHS comparing the roots of the characteristic equation with exponential functions

$$\operatorname{Re}\lambda_i(t) \le -gt^{\beta}, \quad g > 0, \quad \beta > -1 \quad (i = \overline{1,n}).$$

Then there exist the constants $K > 0$ and $0 < \rho < 1$ such that for solving the system

$$\frac{dy}{dt} = A(t)y$$

the estimate

$$\|y(t)\| \le Ke^{-\frac{\rho g}{1+\beta}t^{1+\beta}}\|y(0)\|$$

is fulfilled.

The present paper considers the problem of stability of a second order real linear homogeneous differential equation (LHDE)

$$y'' + p(t)y' + q(t)y = 0 \quad (t \in I) \tag{1}$$

provided that the roots $\lambda_i(t)$ $(i = 1, 2)$ of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

are such that

$$\lambda_i(t) < 0 \quad (t \in I), \qquad \int\limits_{t_0}^{+\infty} \lambda_i(t)\, dt = -\infty \quad (i = 1, 2) \tag{2}$$

and there are finite or infinite limits $\lim\limits_{t\to+\infty} \lambda_i(t)$ $(i = 1, 2)$. We have not encountered with such a statement of the problem even in the well-known works. The case where at least one of the roots satisfies the condition

$$0 < \int\limits_{t_0}^{+\infty} |\lambda_i(t)|\, dt < +\infty \quad (i = 1, 2)$$

should be considered separately.

Under the term "almost triangular LHS" we understand each LHS

$$\frac{dy_i(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t)y_k \quad (i = \overline{1, n}), \tag{3}$$

where $p_{ik}(t) \in C_I$ $(i, k = \overline{1, n})$, which differs little from a linear triangular system

$$\frac{dy_i^*(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t)y_k^* \quad (i = \overline{1, n}), \tag{4}$$

and the conditions of either Theorem 0.1 or Theorem 0.2 due to A. V. Kostin [5] are fulfilled.

Theorem 1. *Let the following conditions hold:*

1) *LHS* (4) *is stable for* $t \in I$*;*
2) *for a particular solution* $\sigma_i(t)$ $(i = \overline{1, n})$ *of a linear inhomogeneous triangular system*

$$\frac{d\sigma_i(t)}{dt} = \sum_{k=1}^{i-1} |p_{ik}(t)| + p_{ii}(t)\sigma_i(t) + \sum_{k=i+1}^{n} |p_{ik}(t)|\sigma_k(t) \quad (i = \overline{1, n}) \tag{5}$$

with the initial conditions $\sigma_i(t_0) = 0$ $(i = \overline{1,n})$ *the estimate of the form*

$$0 < \sigma_i(t) < 1 - \gamma \ \ (i = \overline{1,n}), \ \ \gamma = const, \ \ \gamma \in (0,1)$$

holds for all $t \in I$.

Then the zero solution of the system (3) *is a fortiori stable for* $t \in I$.

Theorem 2. *Suppose the system* (3) *satisfies all conditions of Theorem* 1 *and, moreover,*

1) *the triangular linear system* (4) *is asymptotically stable for* $t \in I$*;*
2) $\lim\limits_{t \to +\infty} \sigma_i(t) = 0$ $(i = \overline{1,n})$.

Then the zero solution of the system (3) *is asymptotically stable for* $t \in I$.

Theorem 3. *Suppose the system* (3) *satisfies all conditions of Theorem* 1 *and, moreover,*

1) *none of the functions*

$$\psi_i(t) = \sum_{k=1}^{i-1} |p_{ik}(t)| \ \ (i = \overline{2,n}) \not\equiv 0 \ \ \textit{for} \ \ t \in I;$$

2) $\lim\limits_{t \to +\infty} \sigma_i(t) = 0$ $(i = \overline{1,n})$.

Then the zero solution of the system (3) *is stable for* $t \in I$.

We will also use the following lemma [5]:

Lemma 1. *If the functions* $p(t), q(t) \in C_I$, $p(t) < 0$, $t \in I$,

$$\int\limits_{t_0}^{+\infty} p(\tau)\, d\tau = -\infty, \qquad \lim_{t \to +\infty} \frac{q(t)}{\operatorname{Re} p(t)} = 0,$$

then

$$e^{\int\limits_{t_0}^{t} p(\tau)\, d\tau} \int\limits_{t_0}^{t} q(\tau) e^{-\int\limits_{\tau_0}^{\tau} p(\tau_1)\, d\tau_1} d\tau = o(1), \ \ t \to +\infty.$$

Further, all limits and symbols o, O are assumed to be considered when $t \to +\infty$.

2. The Main Results

2.1. **Reduction of equation (1) to the system of the type (5).** Consider the real second order LHDE (1)

$$y'' + p(t)y' + q(t)y = 0 \ \ (t \in I)$$

where $p(t), q(t) \in C_I^1$. Let $y = y_1$, $y' = y_2$. We reduce the equation to an equivalent system

$$\begin{cases} y_1' = 0 \cdot y_1 + 1 \cdot y_2, \\ y_2' = -q \cdot y_1 - p \cdot y_2. \end{cases} \tag{6}$$

Consider the characteristic equation of LHS (6):

$$\begin{vmatrix} 0-\lambda & 1 \\ -q & -p-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 + p\lambda + q = 0, \tag{7}$$

and assume that $\frac{p^2}{2} - q > 0$ in I or $\frac{p^2}{2} - q \equiv 0$ in I. Then this equation has two roots: $\lambda_1(t)$ and $\lambda_2(t)$, $\lambda_i(t) \in C_I^1$ $(i = 1, 2)$, $\lambda_i(t)$ are real functions $(i = 1, 2)$.

There arises the question on the sufficient conditions for stability of a trivial solution of system (6).

We write the system (6) in vector form

$$Y' = A(t)Y,$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

To reduce this system to almost triangular form, we use a linear transformation of the form

$$Y = B(t)Z,$$

$$B(t) = \begin{pmatrix} 1 & 0 \\ \lambda_1(t) & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

where $z_i(t)$ are new unknown functions $(i = 1, 2)$. We obtain

$$B'Z + BZ' = ABZ$$

or, after obvious transformations,

$$Z' = (B^{-1}AB - B^{-1}B')Z,$$

$$\det B(t) = 1, \quad B^{-1}(t) = \begin{pmatrix} 1 & 0 \\ -\lambda_1(t) & 1 \end{pmatrix},$$

$$B'(t) = \begin{pmatrix} 0 & 0 \\ \lambda_1'(t) & 0 \end{pmatrix}, \quad B^{-1}B' = \begin{pmatrix} 0 & 0 \\ \lambda_1'(t) & 0 \end{pmatrix},$$

$$B^{-1}AB = \begin{pmatrix} 1 & 0 \\ -\lambda_1(t) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda_1(t) & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1(t) & 1 \\ 0 & \lambda_2(t) \end{pmatrix}.$$

The system with respect to new unknowns $z_i(t)$ $(i = 1, 2)$ in scalar form looks as

$$\begin{cases} z_1'(t) = \lambda_1(t) z_1(t) + z_2(t), \\ z_2'(t) = -\lambda_1'(t) z_1(t) + \lambda_2(t) z_2(t). \end{cases} \tag{8}$$

In accordance with Theorem 1, let us write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = \lambda_1(t)\sigma_1(t) + \sigma_2(t), \\ \sigma_2'(t) = |\lambda_1'(t)| + \lambda_2(t)\sigma_2(t) \end{cases} \tag{9}$$

and consider its particular solution with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$. This solution has the form

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t} \lambda_2(\tau)\,d\tau} \displaystyle\int\limits_{t_0}^{t} |\lambda_1'(t)| e^{-\int\limits_{\tau_0}^{\tau} \lambda_2(\tau_1)\,d\tau_1}\, d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t} \lambda_1(\tau)\,d\tau} \displaystyle\int\limits_{t_0}^{t} \widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau} \lambda_1(\tau_1)\,d\tau_1}\, d\tau. \end{cases} \tag{10}$$

2.2. **Various cases of behavior of the roots $\lambda_i(t)$ $(i = 1, 2)$.** Consider the following cases of behavior of the roots of the characteristic equation, assuming that the condition (2) is satisfied:

1) $\lambda_i(+\infty) \in \mathbb{R}_-$ $(i = 1, 2)$;
2) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) = o(1)$;
3) $\lambda_i(t) = o(1)$ $(i = 1, 2)$;
4) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(+\infty) = -\infty$;
5) $\lambda_1(t) = o(1)$, $\lambda_2(+\infty) = -\infty$;
6) $\lambda_i(+\infty) = -\infty$ $(i = 1, 2)$.

Theorems 4–9 correspond to the above-indicated cases 1)–6).

Theorem 4. *In case* 1), *a trivial solution of the equation* (1) *is asymptotically stable. Here it is sufficient to assume that $p(t), q(t) \in C_I$.*

This case is well-known. The validity of this theorem follows from the results obtained by A. M. Lyapunov.

Theorem 5. *Let the condition* (2) *for $i = 2$ and the conditions*

1) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) = o(1)$;
2) $\frac{\lambda_1'(t)}{\lambda_2(t)} = o(1)$

be fulfilled. Then a trivial solution of equation (1) *is asymptotically stable.*

Proof. We apply Theorem 3. Condition 1) of Theorem 3 is obviously satisfied: $\psi(t) = |\lambda_1'(t)| \not\equiv 0$ for $t \in I$. Therefore, it suffices to show that condition 2) of Theorem 3 also holds. By assumption 2)

$$\frac{|\lambda_1'(t)|}{\lambda_2(t)} = o(1).$$

Therefore, by virtue of Lemma 1, $\widetilde{\sigma}_2(t) = o(1)$. By condition 1) of this theorem,

$$\frac{\widetilde{\sigma}_2(t)}{\lambda_1(t)} = o(1), \quad \int\limits_{t_0}^{+\infty} \lambda_1(t)\, dt = -\infty,$$

and hence $\widetilde{\sigma}_1(t) = o(1)$ by Lemma 1. This implies that Theorem 5 is valid if we take into consideration that the transformation $B(t)$ is restricted in I. To obtain the estimate of solutions $y_i(t)$ $(i = 1, 2)$, we make in the system (8) the following change:

$$z_i(t) = e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau} \eta_i(t) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then the system (8) takes the form

$$\begin{cases} \eta_1'(t) = (\lambda_1(t) - \delta\lambda_2(t))\eta_1(t) + \eta_2(t), \\ \eta_2'(t) = -\lambda_1'(t)\eta_1(t) + (1 - \delta)\lambda_2(t)\eta_2(t) \end{cases}$$

and the system (9) takes the form

$$\begin{cases} \sigma_1'(t) = (\lambda_1(t) - \delta\lambda_2(t))\sigma_1(t) + \sigma_2(t), \\ \sigma_2'(t) = |\lambda_1'(t)| + (1 - \delta)\lambda_2(t)\sigma_2(t). \end{cases}$$

Next, consider a particular solution of this system with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t} (1-\delta)\lambda_2(\tau)\, d\tau} \int\limits_{t_0}^{t} |\lambda_1'(t)| e^{-\int\limits_{\tau_0}^{\tau} (1-\delta)\lambda_2(\tau_1)\, d\tau_1} d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t} (\lambda_1(\tau)-\delta\lambda_2(\tau))\, d\tau} \int\limits_{t_0}^{t} \widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau} (\lambda_1(\tau_1)-\delta\lambda_2(\tau_1))\, d\tau_1} d\tau. \end{cases}$$

In our case,

$$\lim_{t\to+\infty} \frac{|\lambda_1'(t)|}{(1-\delta)\lambda_2(t)} = 0.$$

Thus, by Lemma 1, $\widetilde{\sigma}_2(t) = o(1)$. Further,

$$\lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{\lambda_1(t) - \delta\lambda_2(t)} = \lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{\lambda_1(t)(1 - \delta\frac{\lambda_2(t)}{\lambda_1(t)})} = \lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{\lambda_1(t)} = 0$$

and hence $\widetilde{\sigma}_1(t) = o(1)$, by Lemma 1. Thus the validity of Theorem 5 is not violated. So,

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big) \ \ (i = 1, 2).$$

Taking into account the transformation $B(t)$,

$$\begin{cases} y_1(t) = z_1(t), \\ y_2(t) = \lambda_1(t) z_1(t) + z_2(t); \end{cases}$$

$$\begin{cases} y_1(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big), \\ y_2(t) = o\Big(e^{\int\limits_{t_0}^{t} (\delta\lambda_2(\tau) + \frac{\lambda_1'(t)}{\lambda_1(t)})\, d\tau}\Big) + o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big); \end{cases}$$

$$y_2(t) = o\Big(e^{\int\limits_{t_0}^{t} \lambda_2(\tau)(\delta + \frac{\lambda_1'(\tau)}{\lambda_1(\tau)\lambda_2(\tau)})\, d\tau}\Big) + o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big),$$

$$y_2(t) = o\Big(e^{\int\limits_{t_0}^{t} \lambda_2(\tau)(\delta + o(1))\, d\tau}\Big) + o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big).$$

Therefore,

$$y_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big) \;\; (i = 1, 2), \;\; \delta \in (0, 1). \qquad \square$$

Theorem 6. *Let the condition* (2) *and the conditions*

1) $\lambda_i(t) = o(1)$ $(i = 1, 2)$;
2) $\frac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$ *(or* $\frac{\lambda_2'(t)}{\lambda_2^2(t)} = o(1)$*),* $\frac{\lambda_1(t)}{\lambda_2(t)} = O(1)$

be fulfilled. Then a trivial solution of equation (1) *is asymptotically stable.*

Proof. We apply Theorems 3 and 2. We make in the system (8) the following change:

$$z_1(t) = \xi_1(t), \quad \frac{z_2(t)}{\lambda_1(t)} = \xi_2(t). \tag{11}$$

Then

$$z_1'(t) = \xi_1'(t), \quad z_2'(t) = \lambda_1'(t)\xi_2(t) + \lambda_1(t)\xi_2'(t).$$

Substituting these expressions into the system (8), we have

$$\begin{cases} \xi_1'(t) = \lambda_1(t)\xi_1(t) + \lambda_1(t)\xi_2(t), \\ \xi_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_1(t)}\xi_1(t) + \Big(\lambda_2(t) - \dfrac{\lambda_1'(t)}{\lambda_1(t)}\Big)\xi_2(t). \end{cases} \tag{12}$$

To obtain the estimate of solutions $y_i(t)$ $(i = 1, 2)$ we make in system (12) the following change:

$$\xi_i(t) = e^{\delta \int\limits_{t_0}^{t} \lambda_1(\tau)\, d\tau} \eta_i(t) \;\; (i = 1, 2), \;\; \delta \in (0, 1).$$

Then the system (12) takes the form

$$\begin{cases} \xi_1'(t) = (1 - \delta)\lambda_1(t)\xi_1(t) + \lambda_1(t)\xi_2(t), \\ \xi_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_1(t)}\,\xi_1(t) + \Big(\lambda_2(t) - \delta\lambda_1(t) - \dfrac{\lambda_1'(t)}{\lambda_1(t)}\Big)\xi_2(t). \end{cases}$$

Let us denote

$$\mu(t) = \frac{\lambda_1'(t)}{\lambda_1(t)}\,.$$

In accordance with Theorem 1, we write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = (1-\delta)\lambda_1(t)\sigma_1(t) + |\lambda_1(t)|\sigma_2(t), \\ \sigma_2'(t) = |\mu(t)| + \big(\lambda_2(t) - \delta\lambda_1(t) - \mu(t)\big)\sigma_2(t). \end{cases} \tag{13}$$

Let us consider its particular solution with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int_{t_0}^{t} (\lambda_2(\tau) - \delta\lambda_1(\tau) - \mu(\tau))\,d\tau} \displaystyle\int_{t_0}^{t} |\mu(\tau)| e^{-\int_{\tau_0}^{\tau} \lambda_2(\tau_1) - \delta\lambda_1(\tau_1) - \mu(\tau_1))\,d\tau_1}\,d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int_{t_0}^{t} (1-\delta)\lambda_1(\tau)\,d\tau} \displaystyle\int_{t_0}^{t} |\lambda_1(\tau)|\widetilde{\sigma}_2(\tau) e^{-\int_{\tau_0}^{\tau} (1-\delta)\lambda_1(\tau_1)\,d\tau_1}\,d\tau. \end{cases}$$

Condition 1) of Theorem 3 is obviously satisfied:

$$\psi(t) = |\mu(t)| \not\equiv 0 \ \text{ for } \ t \in I.$$

In our case,

$$\lim_{t\to+\infty} \frac{|\mu(t)|}{\lambda_2(t) - \delta\lambda_1(t) - \mu(t)} = -\lim_{t\to+\infty} \frac{\left|\frac{\lambda_1'(t)}{\lambda_1^2(t)}\right|}{\frac{\lambda_2(t)}{\lambda_1(t)} - \delta - \frac{\lambda_1'(t)}{\lambda_1^2(t)}} = 0.$$

If

$$\frac{\lambda_1'(t)}{\lambda_1^2(t)} \neq o(1),$$

then interchanging the elements $\lambda_1(t)$ and $\lambda_2(t)$, we obtain

$$\lim_{t\to+\infty} \frac{\left|\frac{\lambda_2'(t)}{\lambda_2(t)}\right|}{\lambda_1(t) - \delta\lambda_2(t) - \frac{\lambda_2'(t)}{\lambda_2(t)}} = -\lim_{t\to+\infty} \frac{\left|\frac{\lambda_2'(t)}{\lambda_2^2(t)}\right|}{\frac{\lambda_1(t)}{\lambda_2(t)} - \delta - \frac{\lambda_2'(t)}{\lambda_2^2(t)}} = 0.$$

Consequently, by Lemma 1, $\widetilde{\sigma}_2(t) = o(1)$. Then

$$\lim_{t\to+\infty} \frac{|\lambda_1(t)|}{(1-\delta)\lambda_1(t)}\widetilde{\sigma}_2(t) = -\lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{1-\delta} = 0.$$

Hence $\widetilde{\sigma}_1(t) = o(1)$, by Lemma 1. This implies that Theorem 6 is valid. Then, taking into account the change (11), we have

$$\begin{cases} z_1(t) = o\Big(e^{\delta\int_{t_0}^{t} \lambda_1(\tau)\,d\tau}\Big), \\ z_2(t) = o\Big(e^{\int_{t_0}^{t} (\delta\lambda_1(\tau) + \mu(\tau))\,d\tau}\Big). \end{cases}$$

$$z_2(t) = o\Big(e^{\int_{t_0}^{t} \lambda_1(\tau)(\delta + \frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\,d\tau}\Big) \Longrightarrow z_2(t) = o\Big(e^{\delta\int_{t_0}^{t} \lambda_1(\tau)\,d\tau}\Big).$$

Then,

$$\begin{cases} y_1(t) = o\Big(e^{\delta\int_{t_0}^{t} \lambda_1(\tau)\,d\tau}\Big), \\ y_2(t) = o\Big(e^{\int_{t_0}^{t} (\delta\lambda_1(\tau)+\mu(\tau))\,d\tau}\Big) + o\Big(e^{\int_{t_0}^{t} o(\lambda_1(\tau))\,d\tau}\Big); \end{cases}$$

$$y_2(t) = o\Big(e^{\int_{t_0}^{t} \lambda_1(\tau)(\delta + \frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\,d\tau}\Big) \Longrightarrow y_2(t) = o\Big(e^{\delta\int_{t_0}^{t} \lambda_1(\tau)\,d\tau}\Big).$$

Consequently,

$$y_i(t) = o\Big(e^{\delta\int_{t_0}^{t} \lambda_1(\tau)\,d\tau}\Big) \ \ (i = 1,2), \ \ \delta \in (0,1). \qquad \square$$

Theorem 7. *Let the condition* (2) *for* $i = 2$ *and the conditions*

1) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) \to -\infty$, $\lambda_2(t) < 0$ *at* I*;*
2) $\lambda_1'(t)$ *is bounded at* $t \to +\infty$

be fulfilled. Then a trivial solution of the equation (1) *is asymptotically stable.*

Proof. In system (8) we make the following change:

$$z_i(t) = e^{\delta\int_{t_0}^{t} \lambda_1(\tau)\,d\tau}\eta_i(t) \ \ (i = 1,2), \ \ \delta \in (0,1).$$

Then the system (8) takes the form

$$\begin{cases} \eta_1'(t) = (1-\delta)\lambda_1(t)\eta_1(t) + \eta_2(t), \\ \eta_2'(t) = -\lambda_1'(t)\eta_1(t) + \big(\lambda_2(t) - \delta\lambda_1(t)\big)\eta_2(t). \end{cases}$$

In accordance with Theorem 1, we write an auxiliary system of differential equations

$$\begin{cases} \sigma_1'(t) = (1-\delta)\lambda_1(t)\sigma_1(t) + \sigma_2(t), \\ \sigma_2'(t) = |\lambda_1'(t)| + \big(\lambda_2(t) - \delta\lambda_1(t)\big)\sigma_2(t). \end{cases}$$

Its particular solution with the initial conditions $\sigma_i(t_0) = 0 \quad (i = 1,2)$ has the form

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int_{t_0}^{t} (\lambda_2(\tau)-\delta\lambda_1(\tau))\,d\tau} \displaystyle\int_{t_0}^{t} |\lambda_1'(t)| e^{-\int_{\tau_0}^{\tau} (\lambda_2(\tau_1)-\delta\lambda_1(\tau_1))\,d\tau_1}\,d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int_{t_0}^{t} (1-\delta)\lambda_1(\tau)\,d\tau} \displaystyle\int_{t_0}^{t} \widetilde{\sigma}_2(\tau) e^{-\int_{\tau_0}^{\tau} (1-\delta)\lambda_1(\tau_1)\,d\tau_1}\,d\tau. \end{cases}$$

Since

$$\lim_{t\to+\infty}\frac{|\lambda_1'(t)|}{\lambda_2(t)-\delta\lambda_1(t)}=\lim_{t\to+\infty}\frac{|\lambda_1'(t)|}{\lambda_2(t)(1-\delta\frac{\lambda_1(t)}{\lambda_2(t)})}\lim_{t\to+\infty}\frac{|\lambda_1'(t)|}{\lambda_2(t)}=0,$$

by Lemma 1, $\widetilde{\sigma}_2(t)=o(1)$. As

$$\frac{\widetilde{\sigma}_2(t)}{(1-\delta)\lambda_1(t)}=o(1),$$

it is obvious that $\widetilde{\sigma}_1(t)=o(1)$. This implies that Theorem 7 is valid. Thus, $\eta_i(t)=o(1)$ $(i=1,2)$ and

$$z_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big)\ \ (i=1,2),\ \ \delta\in(0,1).$$

Moreover,

$$\begin{cases} y_1(t)=o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big),\\ y_2(t)=o\Big(e^{\int_{t_0}^{t}(\delta\lambda_1(\tau)+\mu(\tau))\,d\tau}\Big)+o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big). \end{cases}$$

and hence

$$\begin{cases} y_1(t)=o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big),\\ y_2(t)=o\Big(e^{\int_{t_0}^{t}\lambda_1(\tau)\delta+\frac{\lambda_1'(t)}{\lambda_1^2(t)})\,d\tau}\Big)+o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big), \end{cases}\qquad \delta\in(0,1).\qquad\square$$

Theorem 8. *Let the condition* (2) *for* $i=1$ *and the conditions*

1) $\lambda_1(t)=o(1)$, $\lambda_2(+\infty)=-\infty$;
2) $\frac{\lambda_1'(t)}{\lambda_1^2(t)}=o(1)$

be fulfilled. Then a trivial solution of the equation (1) *is asymptotically stable.*

Proof. In system (8) we make the following change:

$$\lambda_1(t)z_1(t)=\xi_1(t),\quad z_2(t)=\xi_2(t). \tag{14}$$

Then

$$z_1'(t)=\frac{\xi_1'(t)\lambda_1(t)-\xi_1(t)\lambda_1'(t)}{\lambda_1^2(t)}=\frac{1}{\lambda_1(t)}\,\xi_1'(t)-\frac{\lambda_1'(t)}{\lambda_1^2(t)}\,\xi_1(t),$$
$$z_2'(t)=\xi_2'(t).$$

After such a change, system (8) takes the form

$$\begin{cases}\xi_1'(t)=(\lambda_1(t)+\mu(t))\xi_1(t)+\lambda_1(t)\xi_2(t),\\ \xi_2'(t)=-\mu(t)\xi_1(t)+\lambda_2(t)\xi_2(t).\end{cases} \tag{15}$$

Now we make change in the system (15):

$$\xi_i(t) = e^{\delta \int\limits_{t_0}^{t} \lambda_1(\tau)\, d\tau} \eta_i(t) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

After that the system (15) takes the form

$$\begin{cases} \eta_1'(t) = \Big((1-\delta)\lambda_1(t) + \dfrac{\lambda_1'(t)}{\lambda_1(t)}\Big)\eta_1(t) + \lambda_1(t)\eta_2(t), \\ \eta_2'(t) = -\mu(t)\eta_1(t) + (\lambda_2(t) - \delta\lambda_1(t))\eta_2(t). \end{cases}$$

According to Theorem 1, for the obtained system we write an auxiliary system of differential equations

$$\begin{cases} \sigma_1'(t) = \big((1-\delta)\lambda_1(t) + \mu(t)\big)\sigma_1(t) + |\lambda_1(t)|\sigma_2(t), \\ \sigma_2'(t) = |\mu(t)| + (\lambda_2(t) - \delta\lambda_1(t))\sigma_2(t). \end{cases}$$

Condition 1) of Theorem 3 is obviously satisfied: $\psi(t) = |\mu(t)| \not\equiv 0$ for $t \in I$. Consider a particular solution of that system with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t} (\lambda_2(\tau) - \delta\lambda_1(\tau))\, d\tau} \displaystyle\int\limits_{t_0}^{t} |\mu(\tau)| e^{-\int\limits_{\tau_0}^{\tau} (\lambda_2(\tau_1) - \delta\lambda_1(\tau_1))\, d\tau_1}\, d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t} ((1-\delta)\lambda_1(\tau) + \mu(\tau))\, d\tau} \displaystyle\int\limits_{t_0}^{t} |\lambda_1(\tau)| \widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau} ((1-\delta)\lambda_1(\tau_1) + \mu(\tau_1))\, d\tau_1}\, d\tau. \end{cases}$$

According to condition 2) of the above theorem,

$$\mu(t) = o(\lambda_1(t)),$$

and, all the more,

$$\mu(t) = o(\lambda_2(t)).$$

Then

$$\lim_{t\to+\infty} \frac{|\mu(t)|}{\lambda_2(t) - \delta\lambda_1(t)} = \lim_{t\to+\infty} \frac{|\mu(t)|\, \frac{1}{\lambda_2(t)}}{1 - \delta\, \frac{\lambda_1(t)}{\lambda_2(t)}} = o(1).$$

Consequently, by Lemma 1, $\widetilde{\sigma}_2(t) = o(1)$. Further, we have

$$\lim_{t\to+\infty} \frac{|\lambda_1(t)|\widetilde{\sigma}_2(t)}{(1-\delta)\lambda_1(t) + \mu(t)} = \lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{\delta - 1 - \frac{\lambda_1'(t)}{\lambda_1^2(t)}} = 0$$

and thus, $\widetilde{\sigma}_1(t) = o(1)$.

Then

$$\lim_{t\to+\infty} z_1(t) = \lim_{t\to+\infty} \frac{\xi_1(t)}{\lambda_1(t)} = \lim_{t\to+\infty} \frac{e^{\delta\int\limits_{t_0}^{t}\lambda_1(\tau)\,d\tau}\eta_1(t)}{\lambda_1(t)} =$$

$$= \lim_{t\to+\infty} e^{\int\limits_{t_0}^{t}(\delta\lambda_1(\tau)-\mu(\tau))\,d\tau}\eta_1(t) = \lim_{t\to+\infty} e^{\int\limits_{t_0}^{t}\lambda_1(\tau)(\delta-\frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\,d\tau}\eta_1(t) =$$

$$= \lim_{t\to+\infty} e^{\delta\int\limits_{t_0}^{t}\lambda_1(\tau)\,d\tau}\eta_1(t) = \lim_{t\to+\infty}\xi_1(t)\eta_1(t) = 0.$$

This implies that Theorem 8 is valid. Moreover,

$$\begin{cases} z_1(t) = \dfrac{\xi_1(t)}{\lambda_1(t)}, \\ z_2(t) = \xi_2(t) \end{cases} \Longrightarrow \begin{cases} z_1(t) = o\Big(e^{\int\limits_{t_0}^{t}(\delta\lambda_1(\tau)-\mu(\tau))\,d\tau}\Big), \\ z_2(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big) \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} z_1(t) = o\Big(e^{\int\limits_{t_0}^{t}\lambda_1(\tau)(\delta-\frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\,d\tau}\Big), \\ z_2(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big) \end{cases} \Longrightarrow$$

$$\Longrightarrow z_i(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big) \ \ (i = 1, 2);$$

$$\begin{cases} y_1(t) = z_1(t), \\ y_2(t) = \lambda_1(t)z_1(t) + z_2(t) \end{cases} \Longrightarrow \begin{cases} y_1(t) = z_1(t), \\ y_2(t) = \xi_1(t) + z_2(t) \end{cases} \Longrightarrow$$

$$\Longrightarrow y_i(t) = o\Big(e^{\delta\int\limits_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \quad \square$$

Theorem 9. *Let the conditions*

1) $\lambda_i(+\infty) = -\infty$ $(i = 1, 2)$*;*
2) $\frac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$ *(or* $\frac{\lambda_2'(t)}{\lambda_2^2(t)} = o(1)$*),* $\frac{\lambda_1(t)}{\lambda_2(t)} = O(1)$

be fulfilled. Then a trivial solution of the equation (1) *is asymptotically stable.*

Proof. The condition (2) is obviously fulfilled. In the system (8) we make the substitution (14) and obtain the system (15). Next, we make the substitution

$$\xi_i(t) = e^{\int\limits_{t_0}^{t}\nu(\tau)\,d\tau}\eta_i(t), \ \ \nu(t) = o(\lambda_i(t)) \ \ (i = 1, 2).$$

Then the system (15) takes the form

$$\begin{cases} \eta_1'(t) = \big((\lambda_1(t) - \nu(t) + \mu(t)\big)\eta_1(t) + \lambda_1(t)\eta_2(t), \\ \eta_2'(t) = -\mu(t)\eta_1(t) + (\lambda_2(t) - \nu(t))\eta_2(t). \end{cases}$$

In accordance with Theorem 1, we write an auxiliary system

$$\begin{cases} \sigma_1'(t) = \big((\lambda_1(t) - \nu(t) + \mu(t)\big)\sigma_1(t) + \lambda_1(t)\sigma_2(t), \\ \sigma_2'(t) = |\mu(t)| + (\lambda_2(t) - \nu(t))\sigma_2(t). \end{cases}$$

According to conditions 1) and 2) of the given theorem,

$$\lim_{t\to+\infty} \frac{|\mu(t)|}{\lambda_2(t) - \nu(t)} = -\lim_{t\to+\infty} \frac{|\frac{\lambda_1'(t)}{\lambda_1^2(t)}|\,\frac{\lambda_1(t)}{\lambda_2(t)}}{1 - \frac{\nu(t)}{\lambda_2(t)}} = 0.$$

If $\frac{\lambda_1(t)}{\lambda_2(t)}$ is unbounded as $t \to +\infty$, then we interchange $\lambda_1(t)$ and $\lambda_2(t)$ and get

$$\lim_{t\to+\infty} \frac{|\frac{\lambda_2'(t)}{\lambda_2(t)}|}{\lambda_1(t) - \nu(t)} = -\lim_{t\to+\infty} \frac{|\frac{\lambda_2'(t)}{\lambda_2^2(t)}|\,\frac{\lambda_2(t)}{\lambda_1(t)}}{1 - \frac{\nu(t)}{\lambda_1(t)}} = 0.$$

Consequently, by Lemma 1, $\widetilde{\sigma}_2(t) = o(1)$. Further, we have

$$\lim_{t\to+\infty} \frac{|\lambda_1(t)|\widetilde{\sigma}_2(t)}{\lambda_1(t) - \nu(t) + \mu(t)} = -\lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{1 - \frac{\nu(t)}{\lambda_1(t)} + \frac{\lambda_1'(t)}{\lambda_1^2(t)}} = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 9 is valid. Moreover,

$$\begin{cases} z_1(t) = \dfrac{\xi_1(t)}{\lambda_1(t)} \\ z_2(t) = \xi_2(t) \end{cases} \Longrightarrow \begin{cases} z_1(t) = o\Big(e^{\int_{t_0}^{t} (\nu(\tau) - \frac{\lambda_1'(\tau)}{\lambda_1(\tau)})\, d\tau}\Big), \\ z_2(t) = o\Big(e^{\int_{t_0}^{t} \nu(\tau)\, d\tau}\Big); \end{cases}$$

$$z_1(t) = o\Big(e^{\int_{t_0}^{t} \lambda_1(\tau)(\frac{\nu(\tau)}{\lambda_1(\tau)} - \frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\, d\tau}\Big) \Longrightarrow$$

$$\Longrightarrow z_i(t) = o\Big(e^{\int_{t_0}^{t} \nu(\tau)\, d\tau}\Big), \ \ \nu(t) = o(\lambda_i(t)) \ \ (i = 1, 2);$$

$$\begin{cases} y_1(t) = z_1(t), \\ y_2(t) = \lambda_1(t)z_1(t) + z_2(t). \end{cases} \Longrightarrow$$

$$\Longrightarrow y_i(t) = o\Big(e^{\int_{t_0}^{t} \nu(\tau)\, d\tau}\Big), \ \ \nu(t) = o(\lambda_i(t)) \ \ (i = 1, 2).$$

Note that the condition $\frac{\lambda'(t)}{\lambda^2(t)} = o(1)$ is fulfilled for a sufficiently wide class of functions for which $\int_{t_0}^{+\infty} \lambda(t)\, dt = -\infty$. □

Conclusion

The paper reveals the sufficient conditions for asymptotic stability and gives evaluation of solutions of the homogeneous linear nonautonomous second order differential equation depending on the behavior of roots of the corresponding characteristic equation in the case of real roots. The results of the work allow one to proceed to considering higher order equations and the questions connected with a simple stability and instability. The case of complex-conjugate roots has been considered by us and will be published in a separate article.

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(Received May 22, 2013)

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Memoirs on Differential Equations and Mathematical Physics
Volume 61, 2014, 21–36

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FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY IN FRÉCHET SPACES

Abstract. Sufficient conditions for the existence and uniqueness of a mild solution on a semi-infinite interval for functional integro-differential equations with state dependent delay are obtained.

2010 Mathematics Subject Classification. 34G20, 34K30.
Key words and phrases. Functional integro-differential equations, state-dependent delay, mild solution, fixed point, Fréchet space, contraction.

რეზიუმე. ფუნქციონალური ინტეგრო-დიფერენციალური განტოლებებისათვის მდგომარეობისაგან დამოკიდებული დაგვიანებით დადგენილია სუსტი ამონახსნის არსებობისა და ერთადერთობის საკმარისი პირობები ნახევრად უსასრულო შუალედში.

1. Introduction

The purpose of this paper is to prove the existence of mild solutions defined on the positive semi-infinite real interval $J := [0, +\infty)$, for functional integro-differential equations with state-dependent delay of the form

$$y'(t) = Ay(t) + f\left(t, y_{\rho(t,y_t)}, \int_0^t e(t, s, y_{\rho(s,y_s)})\, ds\right), \quad \text{a.e. } t \in J, \tag{1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{2}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(T(t)_{t\geq 0})$ on a Banach space $(E, |\cdot|)$ and $f : J \times \mathcal{B} \times E \to E$, $e : J \times J \times \mathcal{B} \to E$, $\rho : J \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are the given function. For any continuous function y defined on $(-\infty, +\infty)$ and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. Here $y_t(\,\cdot\,)$ represents the history of the state from each time $\theta \in (-\infty, 0]$ up to the present time t. We assume that the histories y_t belong to some abstract phase space $\mathcal{B}$ to be specified later.

Integro-differential equations have attracted great interest due to their applications in characterizing many problems in physics, fluid dynamics, biological models and chemical kinetics. Qualitative properties such as the existence, uniqueness and stability for various functional integro-differential equations have been extensively studied by many researchers (see, for instance, [3, 4, 7, 18, 21, 23, 25]. Likewise, the functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equation has received a significant amount of attention in the last years (we refer to [2,5,6,8,13–15] and the references therein).

In the literature, the problem (1)–(2) has been studied by several authors without delay or with delay depending only on time. A method to reduce integro-differential equations with unbounded memory to systems of functional differential equations with bounded memory without integrals and analysis of stability of partial functional integro-differential equations on this basis was presented in [1]. An important study of functional differential equations with state dependent delay was presented in [11]. Hernández [12] has discussed the existence of mild solutions of partial neutral integro-differential equations with an infinite delay. Ravichandran and Mallika [21] investigated the fractional problem. Gunasekar *et al.* [19] have discussed the existence of mild solutions for an impulsive semilinear neutral functional integro-differential equations with infinite delay in Banach spaces by using the Hausdorff measure of noncompactness. When A depends on time, Marcos *et al.* [22] have discussed the case of the existence of solutions for a class of impulsive differential equations by using the fixed point theory of compact and condensing operators. Yan [26] investigated the existence of solutions for semilinear evolution integro-differential equations with nonlocal conditions. Recently, Hong-Kun [17] studied the existence of strong solutions of a nonlinear neutral integro-differential problem on an unbounded interval.

The main purpose of the paper is to establish a global uniqueness of solutions for the problem (1)–(2). Our approach here is based on a recent Frigon–Granas nonlinear alternative of Leray–Schauder type in Fréchet spaces [9] combined with the semigroup theory.

2. Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C([0,+\infty);E)$ be the space of continuous functions from $[0,+\infty)$ into E and $B(E)$ be the space of all bounded linear operators from E into E, with the usual supremum norm

$$N \in B(E), \quad \|N\|_{B(E)} = \sup\big\{|N(y)| : \ |y| = 1\big\}.$$

A measurable function $y : [0,+\infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For the Bochner integral properties, see the classical monograph of Yosida [24].

Let $L^1([0,+\infty), E)$ denote the Banach space of measurable functions $y : [0,+\infty) \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)|\, dt.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [10] and follow the terminology used in [16]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms:

- (A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on $[0,b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0,b)$ the following conditions hold:
  - (i) $y_t \in \mathcal{B}$;
  - (ii) there exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$;
  - (iii) there exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that

  $$\|y_t\|_{\mathcal{B}} \le K(t) \sup\big\{|y(s)| : \ 0 \le s \le t\big\} + M(t)\|y_0\|_{\mathcal{B}}.$$
- (A_2) For the function y in (A_1), y_t is a $\mathcal{B}$-valued continuous function on $[0,b]$.
- (A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : \ t \in [0,b]\}$ and $M_b = \sup\{M(t) : \ t \in [0,b]\}$.

Remark 2.1.

1. (ii) is equivalent to $|\phi(0)| \le H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.
2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \le 0$.
3. From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.

We now indicate some examples of phase spaces. For other details we refer, for instance, to the book due to Hino *et al.* [16].

Example 2.2. Let:

BC be the space of bounded continuous functions defined from $(-\infty, 0]$ to E;

BUC be the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E;

$$C^{\infty} := \Big\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E\Big\};$$
$$C^{0} := \Big\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\Big\},$$

be endowed with the uniform norm

$$\|\phi\| = \sup\big\{|\phi(\theta)| : \ \theta \le 0\big\}.$$

We have that the spaces BUC, C^{∞} and C^{0} satisfy conditions (A_1)–(A_3). However, BC satisfies (A_1), (A_3) but does not satisfy (A_2).

Example 2.3. The spaces C_g, UC_g, C_g^{∞} and C_g^{0}.

Let g be a positive continuous function on $(-\infty, 0]$. We define:

$$C_g := \Big\{\phi \in C((-\infty, 0], E) : \ \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0]\Big\};$$
$$C_g^{0} := \Big\{\phi \in C_g : \ \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0\Big\},$$

endowed with the uniform norm

$$\|\phi\| = \sup\Big\{\frac{|\phi(\theta)|}{g(\theta)} : \ \theta \le 0\Big\}.$$

Then we have that the spaces C_g and C_g^{0} satisfy conditions (A_3). We consider the following condition on the function g.

(g_1) For all $a > 0$,

$$\sup_{0 \le t \le a} \sup\Big\{\frac{g(t+\theta)}{g(\theta)} : \ -\infty < \theta \le -t\Big\} < \infty.$$

They satisfy conditions (A_1) and (A_2) if (g_1) holds.

Example 2.4. The space C_γ.

For any real positive constant γ, we define the functional space C_γ by

$$C_\gamma := \Big\{\phi \in C\big((-\infty,0],E\big) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\Big\}$$

endowed with the norm

$$\|\phi\| = \sup\Big\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \le 0\Big\}.$$

Then in the space C_γ the axioms $(A_1) - (A_3)$ are satisfied.

Definition 2.5. A function $f : J \times \mathcal{B} \times E \to E$ is said to be an L^1 -Carathéodory function if it satisfies:

(i) for each $t \in J$, the function $f(t,\cdot,\cdot) : \mathcal{B} \times E \to E$ is continuous;

(ii) for each $(y,z) \in \mathcal{B} \times E$, the function $f(\cdot,y,z) : J \to E$ is measurable;

(iii) for every positive integer k, there exists $h_k \in L^1(J;\mathbb{R}^+)$ such that

$$|f(t,y,z)| \le h_k(t)$$
$$\text{for all } \ \|y\|_{\mathcal{B}} \le k, \ \ |z| \le k \ \text{ and almost every } t \in J.$$

Let E be a Banach space and $B(E)$ be the Banach space of linear bounded operators.

Definition 2.6. A one parameter family $\{T(t) \mid t \ge 0\} \subset B(E)$ of bounded linear operators from $E \to E$ is a semigroup of bounded linear operator on E if satisfying the conditions:

(i) $T(t)T(s) = T(t+s)$, for $t,s \ge 0$;

(ii) $T(0) = I$.

Definition 2.7. Let $T(t)$ be a semigroup defined on E. A linear operator A defined by

$$D(A) = \Big\{x \in E| \ \lim_{h\to0^+} \frac{T(h)(x) - x}{h} \ \text{ exists in } E\Big\},$$

and

$$A(x) = \lim_{h\to0^+} \frac{T(h)x - x}{h} \ \text{ for } \ x \in D(A),$$

is the infinitesimal generator of the semigroup $T(t)$. $D(A)$ is called the domain of A.

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \cdots \ \text{ for every } \ x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n \ \text{ for all } \ y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for $x, y \in X$. We denote

$$X^n = \big(X|_{\sim_n}, \|\cdot\|_n\big)$$

the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote by $[x]_n$ the equivalence class of x of the subset X^n and we define $Y^n = \{[x]_n : x \in Y\}$. We denote by $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n.

The following definition is the appropriate concept of contraction in X.

Definition 2.8 ([9])**.** A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in [0,1)$ such that

$$\|f(x) - f(y)\|_n \le k_n \|x-y\|_n \text{ for all } x, y \in X.$$

The corresponding nonlinear alternative result is the following

Theorem 2.9 (Nonlinear Alternative of Granas–Frigon, [9])**.** *Let X be a Fréchet space and $Y \subset X$ a closed subset and let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

($C1$) *N has a unique fixed point;*

($C2$) *there exists $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $x \in \partial_n Y^n$ such that $\|x - \lambda N(x)\|_n = 0$.*

3. Existence results

3.1. Mild solutions.

Definition 3.1. We say that the function $y : (-\infty, +\infty) \to E$ is a mild solution of (1)–(2) if $y(t) = \phi(t)$ for all $t \le 0$ and y satisfies the following integral equation:

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s) f\left(s, y_{\rho(s,y_s)}, \int_0^s e(s,\tau, y_{\rho(\tau,y_\tau)})\, d\tau\right) ds \tag{3}$$

for each $t \ge 0$.

Set

$$\mathcal{R}(\rho^-) = \Big\{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B},\ \rho(s,\phi) \le 0\Big\}.$$

For each $b \in (0,\infty)$, we assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \le L^\phi(t) \|\phi\|_{\mathcal{B}} \text{ for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.2. The condition (H_ϕ) is frequently verified by the functions continuous and bounded. For more details, see for instance, [16].

Lemma 3.3 ([15, Lemma 2.4])**.** *If* $y:(-\infty,b]\to E$ *is a function such that* $y_0=\phi$, *then*

$$\|y_s\|_{\mathcal{B}}\le (M_b+L^\phi)\|\phi\|_{\mathcal{B}}+K_b\sup\Big\{|y(\theta)|:\ \theta\in[0,\max\{0,s\}]\Big\},\qquad s\in\mathcal{R}(\rho^-)\cup J,$$

where $L^\phi=\sup_{t\in\mathcal{R}(\rho^-)}L^\phi(t)$.

We introduce the following hypotheses:

$(H1)$ There exists a constant $\widehat{M}\ge 1$ such that

$$\|T(t)\|_{B(E)}\le \widehat{M}\ \text{ for every }\ t\ge 0.$$

(H_f) (i) There exist a function $p\in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ such that:

$$|f(t,\delta,w)|\le p(t)\psi\big(\|\delta\|_{\mathcal{B}}+\|w\|\big)\ \text{ for every }\ (t,\delta,x)\in J\times\mathcal{B}\times E.$$

(ii) For all $R>0$, there exists $l_R\in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$\big|f(t,\delta_1,w_1)-f(t,\delta_2,w_2)\big|\le l_R(t)\Big(\|\delta_1-\delta_2\|_{\mathcal{B}}+\|w_1-w_2\|\Big)$$

where $(t,\delta_i,w_i)\in J\times\mathcal{B}\times E$, $i=1,2$.

(H_e) (i) There exist a function $m\in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\Omega:\mathbb{R}_+\to(0,\infty)$ such that:

$$|e(t,s,\delta)|\le m(s)\Omega\big(\|\delta\|_{\mathcal{B}}\big)\ \text{ for all }\ (t,s,\delta)\in J\times J\times\mathcal{B}.$$

(ii) There exists a constant $C_1>0$ such that

$$\left|\int_0^t\big[e(t,s,x)-e(t,s,y)\big]\,ds\right|\le C_1\|x-y\|_{\mathcal{B}}\qquad \text{for }\ (t,s)\in J,\ \ (x,y)\in\mathcal{B}.$$

Consider the space

$$B_{+\infty}=\Big\{y:\ \mathbb{R}\to E:\ y\big|_{[0,T]}\text{ continuous for }T>0\text{ and }y_0\in\mathcal{B}\Big\},$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0,T]$.

Let us fix $\tau>1$. For every $n\in\mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by

$$\|y\|_n:=\sup\Big\{e^{-\tau\ L_n^*(t)}|y(t)|:\ t\in[0,n]\Big\},$$

where

$$L_n^*(t)=\int_0^t\bar{l}_n(s)\,ds,\quad \bar{l}_n(t)=(1+C_1)K_n\widehat{M}l_n(t)$$

and l_n is the function from $(H_f)(ii)$.

Then $B_{+\infty}$ is a Fréchet space with this family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$.

Theorem 3.4. *Assume that* $(H1), (H_f)$, (H_e) *and* (H_ϕ) *hold, and suppose that for* $n \in \mathbb{N}$,

$$\int\limits_{w(0)}^{+\infty} \frac{ds}{\psi(s)+\Omega(s)} > \int\limits_0^n \vartheta(s)\,ds. \tag{4}$$

Then the problem (1)–(2) *has a unique mild solution on* $(-\infty, +\infty)$.

Proof. We transform the problem (1)–(2) into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) =$$

$$= \begin{cases} \phi(t), \ \text{ if } \ t \le 0, \\ T(t)\phi(0) + \int\limits_0^t T(t-s) f\left(s, y_{\rho(s,y_s)}, \int\limits_0^s e(s,\tau,y_{\rho(\tau,y_\tau)})\,d\tau\right) ds, \\ \hfill \text{if } \ t \in J. \end{cases} \tag{5}$$

Clearly, fixed points of the operator N are mild solutions of the problem (1)–(2).

For $\phi \in \mathcal{B}$, we define the function $x(\,\cdot\,) : (-\infty, +\infty) \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \le 0, \\ T(t)\phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$ with $z_0 = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } \ t \le 0, \\ z(t), & \text{if } \ t \in J. \end{cases}$$

If $y(\,\cdot\,)$ satisfies (3), we can decompose it as $y(t) = \overline{z}(t) + x(t)$, $t \ge 0$, which implies that $y_t = \overline{z}_t + x_t$, for every $t \in J$ and the function $z(\,\cdot\,)$ satisfies

$$z(t) = \int\limits_0^t T(t-s) f\Big(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)},$$
$$\int\limits_0^s e\Big(s,\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\Big)\,d\tau\Big)\,ds \ \text{ for } \ t \in J.$$

Let

$$B^0_{+\infty} = \big\{ z \in B_{+\infty} : \ z_0 = 0 \in \mathcal{B} \big\}.$$

For any $z \in B^0_{+\infty}$, we have

$$\begin{aligned}\|z\|_{+\infty} &= \|z_0\|_{\mathcal{B}} + \sup\big\{|z(s)| :\ 0 \le s < +\infty\big\} = \\ &= \sup\big\{|z(s)| :\ 0 \le s < +\infty\big\}.\end{aligned}$$

Thus $(B^0_{+\infty}, \|\cdot\|_{+\infty})$ is a Banach space. We define the operator $F : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t T(t-s) f\Bigg(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \int_0^s e\Big(s,\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\Big)\, d\tau\Bigg)\, ds \ \text{ for } \ t \in J.$$

Obviously, the operator N has a fixed point is equivalent to F has one, so it turns to prove that F has a fixed point. Let $z \in B^0_{+\infty}$ be such that $z = \lambda F(z)$ for some $\lambda \in [0,1)$. By the hypotheses $(H1)$, $(H_f(i))$ and $(H_e(i))$, for each $t \in [0,n]$, we have

$$\begin{aligned}|z(t)| \le& \int_0^t \|T(t-s)\|_{B(E)} \Bigg| f\Bigg(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \\ &\int_0^s e\Big(s,\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\Big)\, d\tau\Bigg)\Bigg|\, ds \le \\ \le& \widehat{M} \int_0^t p(s)\psi\Bigg(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \\ &+ \int_0^s m(\tau)\Omega\Big(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\Big)\, d\tau\Bigg)\, ds \le \\ \le& \widehat{M} \int_0^t p(s)\psi\Bigg(K_n|z(s)| + \big(M_n + L^\phi + K_n MH\big)\|\phi\|_{\mathcal{B}} + \\ &+ \int_0^s m(\tau)\Omega\Big(K_n|z(s)| + \big(M_n + L^\phi + K_n MH\big)\|\phi\|_{\mathcal{B}}\Big)\, d\tau\Bigg)\, ds.\end{aligned}$$

Set

$$c_n := \big(M_n + K_n + L^\phi + K_n MH\big)\|\phi\|_{\mathcal{B}}.$$

Then we have

$$|z(t)| \le M \int_0^t p(s)\psi\Bigg(K_n|z(s)| + c_n + \int_0^s m(\tau)\Omega(K_n|z(s)| + c_n)\, d\tau\Bigg)\, ds.$$

Thus

$$K_n|z(t)| + c_n \leq$$

$$\leq c_n + K_n\widehat{M}\int_0^t p(s)\psi\left(K_n|z(s)| + c_n + \int_0^s m(\tau)\Omega(K_n|z(s)| + c_n)\,d\tau\right)ds.$$

We consider the function μ defined by

$$\mu(t) := \sup\left\{K_n|z(s)| + c_n :\ 0 \leq s \leq t\right\}, \quad 0 \leq t < +\infty.$$

Let $t^\star \in [0,t]$ be such that $\mu(t) = K_n|z(t^\star)| + c_n\|\phi\|_{\mathcal{B}}$. By the previous inequality, we have

$$\mu(t) \leq c_n + K_n\widehat{M}\int_0^t p(s)\psi\left(\mu(s) + \int_0^s m(\tau)\Omega(\mu(\tau))\,d\tau\right)ds$$

$$\text{for } t \in [0,n].$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have $\mu(t) \leq v(t)$ for all $t \in [0,n]$. This leads us to the following inequality:

$$v(t) \leq c_n + K_n\widehat{M}\int_0^t p(s)\psi\left(v(s) + \int_0^s m(\tau)\Omega\big(v(\tau)\,d\tau\big)\,ds\right) \quad \text{for } t \in [0,n],$$

whence

$$v'(t) \leq \widehat{M}\ K_n\ p(t)\psi\left(v(t) + \int_0^t m(\tau)\Omega\big(v(\tau)\,d\tau\big)\right).$$

Next, we consider the function

$$w(t) = v(t) + \int_0^t m(\tau)\Omega(v(\tau))\,d\tau,$$

thus we have that $v(0) = w(0)$ and $v(t) \leq w(t)$ for all $t \in [0,n]$.

Using the nondecreasing character of ψ, we get

$$w'(t) = v'(t) + p(t)\Omega(v(t)) \leq$$

$$\leq \widehat{M}K_n p(t)\psi(w(t)) + m(t)\Omega(w(t)) \ \text{ a.e. } t \in [0,n].$$

We define the function $\vartheta(t) = \max\big\{\widehat{M}K_n p(t), m(t)\big\}$, $t \in [0,n]$, which implies that

$$\frac{w'(t)}{\psi(w'(t)) + \Omega(w(t))} \leq \vartheta(t).$$

From condition (4), we have

$$\int_{w(0)}^{w(t)} \frac{ds}{\psi(s)+\Omega(s)} \le \int_0^t \vartheta(s)\,ds \le \int_{w(0)}^{+\infty} \frac{ds}{\psi(s)+\Omega(s)}\,.$$

Thus, for every $t \in [0,n]$, there exists a constant Λ_n such that $w(t) \le \Lambda_n$ and hence, $\mu(t) \le \Lambda_n$. Since $\|z\|_n \le \mu(t)$, we have $\|z\|_n \le \Lambda_n$.

Set

$$Z = \Big\{ z \in B^0_{+\infty} : \sup_{0\le t\le n} |z(t)| \le \Lambda_n + 1,\ \forall\, n \in \mathbb{N} \Big\}.$$

Clearly, Z is a closed subset of $B^0_{+\infty}$.

We shall show that $F : Z \to B^0_{+\infty}$ is a contraction operator. Indeed, consider $z, \overline{z} \in Z$, thus using $(H1)$ and $(H3)$ for each $t \in [0,n]$ and $n \in \mathbb{N}$,

$$\big|F(z)(t) - F(\overline{z})(t)\big| \le \int_0^t \big\|T(t-s)\big\|_{B(E)} \times$$

$$\times \bigg| f\bigg(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \int_0^s e\big(s,\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\big)\,d\tau\bigg) -$$

$$- f\bigg(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)},$$

$$\int_0^s e\big(s,\tau, \overline{z}_{\rho(\tau,\overline{z}_\tau+x_\tau)} + x_{\rho(\tau,\overline{z}_\tau+x_\tau)}\big)\,d\tau\bigg)\bigg|\,ds \le$$

$$\le \int_0^t \widehat{M} l_n(s)\Big(\big\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,\overline{z}_s+x_s)}\big\|_{\mathcal{B}} +$$

$$+ C_1 \big\|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,\overline{z}_s+x_s)}\big\|_{\mathcal{B}}\Big)\,ds.$$

Using (H_ϕ) and Lemma 3.3, we obtain

$$\big|F(z)(t) - F(\overline{z})(t)\big| \le$$

$$\le \int_0^t \widehat{M} l_n(s)\Big(K_n|z(s)-\overline{z}(s)| + C_1\big(K_n|z(s)-\overline{z}(s)|\big)\Big)\,ds \le$$

$$\le \int_0^t \widehat{M} l_n(s)[1+C_1]K_n|z(s)-\overline{z}(s)|\,ds \le$$

$$\le \int_0^t \Big[\overline{l}_n(s)\ e^{\tau L_n^*(s)}\Big]\Big[e^{-\tau L_n^*(s)}\big|z(s)-\overline{z}(s)\big|\Big]\,ds \le$$

$$\leq \int_0^t \left[\frac{e^{\tau L_n^*(s)}}{\tau}\right]' ds \, \|z-\overline{z}\|_n \leq \frac{1}{\tau}\, e^{\tau L_n^*(t)} \|z-\overline{z}\|_n.$$

Therefore,

$$\|F(z)-F(\overline{z})\|_n \leq \frac{1}{\tau}\, \|z-\overline{z}\|_n.$$

So, the operator F is a contraction for all $n \in \mathbb{N}$. By the choice of Z, there is no $z \in \partial Z^n$ such that $z = \lambda F(z)$, $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 2.9 does not hold. The nonlinear alternative due to Frigon and Granas shows that $(C1)$ holds. Thus, we conclude that the operator F has a unique fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, +\infty)$ is a fixed point of the operator N, which is the unique mild solution of the problem (1)–(2). □

4. An Example

To apply our results, we consider the following partial differential equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\xi) = \dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) + \\ \quad + m\Big(t, v\big(t-\sigma(v(t,0)),\xi\big), \displaystyle\int_0^t \eta\big(t,s,v\big(s-\sigma(v(s,0)),\xi\big)\big)\, ds\Big), \\ \qquad\qquad\qquad\qquad t \in [0,\infty), \quad \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \quad t \in [0,\infty), \\ v(\theta,\xi) = v_0(\theta,\xi), \quad \theta \in (-\infty, 0], \quad \xi \in [0,\pi], \end{cases} \tag{6}$$

where v_0 and $\sigma \in C(\mathbb{R}, [0,\infty))$ are continuous. Take $E = L^2[0,\pi]$ and define $A : D(A) \subset E \to E$ by $Aw = w''$ with the domain

$$D(A) = \Big\{ w \in E,\ w, w' \text{ are absolutely continuous,} \\ w'' \in E,\ w(0) = w(\pi) = 0 \Big\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} -n^2 (w, w_n) w_n, \quad w \in D(A),$$

where $w_n(s) = \sqrt{\frac{2}{\pi}}\, \sin ns$, $n = 1, 2, \ldots$, is the orthogonal set of eigenvalues of A. It is well known (see [20]) that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ in E and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} (w, w_n) w_n, \quad w \in E.$$

Since the analytic semigroup $T(t)$ is compact for $t > 0$, there exists a constant $M \geq 1$ such that $\|T(t)\| \leq M$.

Theorem 4.1. *Let $\mathcal{B} = BUC(\mathbb{R}_-, E)$ and $\phi \in \mathcal{B}$. Assume that condition (H_ϕ) holds. The function $m : J \times J \times [0,\pi] \to [0,\pi]$, $\sigma : \mathbb{R} \to \mathbb{R}^+$, $\eta : J \times J \times [0,\pi] \to [0,\pi]$ are continuous. Then there exists a unique mild solution of* (6)*.*

Proof. From the above assumptions, we have that the functions

$$\begin{aligned} f(t,\varphi,x)(\xi) &= m\Big(t, \varphi(0,\xi), \int_0^t \eta\big(t,s,\varphi(0,\xi)\big)\,ds\Big),\\ e(t,s,\varphi)(\xi) &= \eta\big(t,s,\varphi(0,\xi)\big),\\ \rho(t,\varphi) &= t - \sigma(\varphi(0,0)) \end{aligned}$$

are well defined, permitting to transform system (6) into the abstract system (1)–(2). Moreover, the function f is a bounded linear operator. Now the existence of mild solutions can be deduced from a direct application of Theorem 3.4. From Remark 3.2, we have the following result. □

Corollary 4.2. *Let $\varphi \in \mathcal{B}$ be continuous and bounded. Then there exists a unique mild solution of* (6) *on* $(-\infty, +\infty)$*.*

Acknowledgements

The authors are grateful to the referee for the helpful remarks and suggestions.

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(Received May 17, 2013; revised February 5, 2014)

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Memoirs on Differential Equations and Mathematical Physics
Volume 61, 2014, 37–61

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF n-TH ORDER WITH REGULARLY VARYING NONLINEARITIES

Abstract. The conditions of existence and asymptotic for $t \uparrow \omega$ ($\omega \leq +\infty$) representations of one class of monotonic solutions of n-th order differential equations containing in the right-hand side a sum of terms with regularly varying nonlinearities, are established.

2010 Mathematics Subject Classification. 34D05, 34C11.
Key words and phrases. Ordinary differential equations, regularly varying nonlinearities, asymptotics of solutions.

რეზიუმე. n-ური რიგის ჩვეულებრივი დიფერენციალური განტოლებებისათვის, რომელთა მარჯვენა მხარეები წარმოადგენენ რეგულარულად ცვალებადი არაწრფივობის მქონე წევრთა ჯამს, დადგენილია გარკვეული კლასის მონოტონურ ამონახსნთა არსებობის საკმარისი პირობები და მიღებული მათი ასიმპტოტური წარმოდგენები.

1. Introduction

The theory of regularly varying functions created by J. Karamata in 1930 has been later (see, for example, monographs [1], [2]) extensively developed and widely used in various mathematical researches. Particularly, the last decades of the past century is mentioned by a great interest in studying regularly and slowly varying solutions of various differential equations and in equations of the type

$$y'' = \alpha_0 p(t)\varphi(y),$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, +\infty[\to]0, +\infty[$ is a continuous function and $\varphi : \Delta_{Y_0} \to]0, +\infty[$ is a regularly varying continuous function of order $\sigma \neq 1$ as $y \to Y_0$; here Y_0 equals either zero or $\pm\infty$, and Δ_{Y_0} is a one-sided neighborhood of Y_0. Among the researches carried out within that period and dedicated to determination of asymptotics as $t \to +\infty$ of monotonic solutions for such equations, of special mention are the works [3], [4] and the monograph [5].

Here, according to the definition of regularly varying function (see E. Seneta [1, Ch. 1, Sect. 1.1, pp. 9–10]),

$$\varphi(y) = |y|^\sigma L(y),$$

where L is slowly varying as $y \to Y_0$ function, i.e., the condition

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{L(\lambda y)}{L(y)} = 1 \ \text{ with any } \ \lambda > 0$$

is satisfied. Considering such representation for φ, such class of equations is a natural extension of the class of generalized second order Emden–Fowler equations

$$y'' = \alpha_0 p(t)|y|^\sigma \operatorname{sign} y.$$

The basic results dealing with asymptotic properties of solutions for the second- and n-th order Emden–Fowler equations, obtained before 1990, can be found in the monograph due to I.T. Kiguradze and T.A. Chanturiya [6, Ch. IV, V, pp. 309–401]. The works [7]–[16], dedicated to the determination of asymptotics of monotonic differential equations of second and higher orders with power nonlinearities are also worth mentioning.

For the last decade, the results obtained in [17]–[22] and also those obtained in [12]–[16] were applied to differential equations

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'), \quad y'' = \sum_{k=1}^{m} \alpha_k p_k(t)\varphi_{k0}(y)\varphi_{k1}(y'),$$
$$y^{(n)} = \alpha_0 p(t)\varphi(y) \ \ (n \geq 2)$$

with nonlinearities, regularly varying as $y \to Y_0$ and $y' \to Y_1$, where $Y_i \in \{0; \pm\infty\}$ $(i = 0, 1)$, and with some additional restrictions to nonlinearity for the first two equations.

In the present paper we consider the following differential equation:

$$y^{(n)} = \sum_{k=1}^{m} \alpha_k p_k(t) \prod_{j=0}^{n-1} \varphi_{kj}(y^{(j)}), \tag{1.1}$$

where $n \geq 2$, $\alpha_k \in \{-1; 1\}$ $(k = \overline{1,m})$, $p_k : [a, \omega[\to]0, +\infty[$ $(k = \overline{1,m})$ are continuous functions, $\varphi_{kj} : \triangle_{Y_j} \to]0, +\infty[$ $(k = \overline{1,m};\ j = \overline{0,n-1})$ are continuous and regularly varying as $y^{(j)} \to Y_j$ functions of orders σ_{kj}, $-\infty < a < \omega \leq +\infty$,* $\triangle_{Y_j}$ is one-sided neighborhood of Y_j, Y_j equal either to 0 or to $\pm\infty$. It is assumed that numbers ν_j $(j = \overline{0,n-1})$ determined by

$$\nu_j = \begin{cases} 1, & \text{if either } Y_j = +\infty, \text{ or } Y_j = 0 \\ & \quad \text{and } \Delta_{Y_j}\text{-right neighborhood of } 0, \\ -1, & \text{if either } Y_j = -\infty, \text{ or } Y_j = 0 \\ & \quad \text{and } \Delta_{Y_j}\text{-left neighborhood of } 0, \end{cases} \tag{1.2}$$

are such that

$$\nu_j \nu_{j+1} > 0 \quad \text{with} \quad Y_j = \pm\infty \ \text{ and} \\ \nu_j \nu_{j+1} < 0 \ \text{ with } \ Y_j = 0 \ \ (j = \overline{0, n-2}). \tag{1.3}$$

Such conditions for ν_j $(j = \overline{0,n-1})$ are necessary for the equation (1.1) to have solutions defined in the left neighborhood of ω, each of which satisfying the conditions

$$y^{(j)}(t) \in \Delta_{Y_j} \ \text{ with } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y^{(j)}(t) = Y_j \ \ (j = \overline{0, n-1}). \tag{1.4}$$

Among strictly monotonic, with derivatives up to the $n-1$ order inclusive, in some left neighborhood of ω, solutions of equation (1.1) these ones are of special academic interest, because each of the rest ones admits only one representation of the type

$$y(t) = \pi_\omega^{k-1}(t)[c_{k-1} + o(1)] \ \ (k = \overline{1,n}),$$

where c_{k-1} $(k = \overline{1,n})$ are the non-zero real constants and

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t - \omega, & \text{if } \omega < +\infty. \end{cases} \tag{1.5}$$

The question on the existence of solutions of (1.1) with similar representations may be solved, in a whole, in a rather simple way by applying, for example, Corollary 8.2 for $\omega = +\infty$ from the monograph of I. T. Kiguradze and T. A. Chanturiya [1, Ch. II, p. 8, p. 207] and the schemes from the works [10], [12] as $\omega \leq +\infty$. As for the solutions with properties (1.4), for lack of particular representations for them, there arises the necessity to single out a class of solutions admitting one to get such representations.

*if $a > 1$, then $\omega = +\infty$, and $\omega - 1 < a < \omega$ if $\omega < +\infty$.

One of such rather wide classes of solutions has been introduced in [14]–[16] dedicated to generalized Emden–Fowler type equations of n-th order,

$$y^{(n)} = \alpha_0 p(t) \prod_{j=0}^{n-1} |y^{(j)}|^{\sigma_j}.$$

For the equation (1.1), this class is determined as follows.

Definition 1.1. A solution y of the equation (1.1) defined on the interval $[t_0, \omega[\subset [a, \omega[$, is called a $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if along with (1.4) the condition

$$\lim_{t\uparrow\omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}} = \lambda_0 \tag{1.6}$$

is satisfied.

If y is a solution of the differential equation (1.1) with properties (1.4) and the functions $\ln|y^{(n-1)}(t)|$ and $\ln|\pi_\omega(t)|$ are comparable with order one (see [23, Ch. 5, Sect. 4,5, pp. 296–301]) as $t \uparrow \omega$, then it is easy to check that this solution is the $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution for some λ_0 depending on the value of $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)}$.

Moreover, using assertions 1, 2, 5 and 9 (on the properties of regularly varying functions) from the monograph [5, Appendix, pp. 115–117], it can be verified that in the case of regularly varying as $t \uparrow \omega$ coefficients p_k ($k = \overline{1,m}$) of the equation (1.1), each of its regularly varying as $t \uparrow \omega$ solutions with properties (1.4) is a $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution for some final or equal to $\pm\infty$ value λ_0.

The aim of this note is to determine the conditions for existence of $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions of (1.1) in special cases, where $\lambda_0 = \frac{n-i-1}{n-i}$ as $i \in \{1, \dots, n-1\}$, and also asymptotic representations as $t \uparrow \omega$ for such solutions and their derivatives up to and including $n-1$ order.

By virtues of the results from [16], these solutions of the equation (1.1) possess the following a priori asymptotic properties.

Lemma 1.1. *Let $y : [t_0, \omega[\to \Delta_{Y_0}$ be an arbitrary $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution of the equation* (1.1)*. Then:*

(1) *if $n > 2$ and $\lambda_0 = \frac{n-i-1}{n-i}$ for some $i \in \{1, \dots, n-2\}$, then for $t \uparrow \omega$,*

$$\begin{aligned} y^{(k-1)}(t) &\sim \frac{[\pi_\omega(t)]^{i-k}}{(i-k)!}\, y^{(i-1)}(t) \quad (k = 1, \dots, i-1)^*, \\ y^{(i)}(t) &= o\Big(\frac{y^{(i-1)}(t)}{\pi_\omega(t)}\Big), \end{aligned} \tag{1.7}$$

$$y^{(k)}(t) \sim (-1)^{k-i} \frac{(k-i)!}{[\pi_\omega(t)]^{k-i}}\, y^{(i)}(t) \quad (k = i+1, \dots, n); \tag{1.8}$$

---

*At $i = 1$ these relationships do not exist.

(2) *if* $n \geq 2$ *and* $\lambda_0 = 0$, *then for* $t \uparrow \omega$,

$$\begin{aligned} y^{(k-1)}(t) &\sim \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\, y^{(n-2)}(t) \ \ (k = 1, \dots, n-2)^*, \\ y^{(n-1)}(t) &= o\Big(\frac{y^{(n-2)}(t)}{\pi_\omega(t)}\Big) \end{aligned} \tag{1.9}$$

and, in the case of existence of (finite or equal to $\pm\infty$*) limit* $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)}$,

$$y^{(n)}(t) \sim \frac{-1}{\pi_\omega(t)\, y^{(n-1)}(t)} \quad \text{with } t \uparrow \omega. \tag{1.10}$$

2. Statement of the Main Results

In order to formulate the theorems, we will need some auxiliary notation and one definition.

By virtue of the definition of regularly varying function, the nonlinearity in (1.1) is representable in the form

$$\varphi_{kj}(y^{(j)}) = |y^{(j)}|^{\sigma_{kj}} L_{kj}(y^{(j)}) \ \ (k = \overline{1, m}; \ \ j = \overline{0, n-1}), \tag{2.1}$$

where $L_{kj} : \Delta_{Y_j} \to]0, +\infty[$ are continuous and slowly varying as $y^j \to Y_j$ functions, for which with any $\lambda > 0$

$$\lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta_{Y_j}}} \frac{L_{kj}(\lambda y^{(j)})}{L_{kj}(y^{(j)})} = 1 \ \ (k = \overline{1, m}; \ \ j = \overline{0, n-1}). \tag{2.2}$$

It is also known (see [1, Ch. 1, Sect. 1.2, pp. 10–15]) that the limits (2.2) are uniformly fulfilled with respect to λ on any interval $[c, d] \subset]0, +\infty[$ (property M_1) and there exist continuously differentiable slowly varying as $y^{(j)} \to Y_j$ functions $L_{0kj} : \Delta_{Y_j} \to]0, +\infty[$ (property M_2) such that

$$\lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta_{Y_j}}} \frac{L_{kj}(y^{(j)})}{L_{0kj}(y^{(j)})} = 1 \ \text{ and } \ \lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta_{Y_j}}} \frac{y^{(j)} L'_{0kj}(y^{(j)})}{L_{0kj}(y^{(j)})} = 0 \tag{2.3}$$

$$(k = \overline{1, m}; \ \ j = \overline{0, n-1}).$$

Definition 2.1. We say that a slowly varying as $z \to Z_0$ function $L : \Delta_{Z_0} \to]0, +\infty[$, where Z_0 either equals zero, or $\pm\infty$, and Δ_{Z_0} is one-sided neighborhood of Z_0, satisfies condition S_0, if

$$L\big(\nu e^{[1+o(1)] \ln |z|}\big) = L(z)[1 + o(1)] \ \text{ with } \ z \to Z_0 \ \ (z \in \Delta_{Z_0}),$$

where $\nu = \operatorname{sign} z$.

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*At $n = 2$ these relationships do not exist.

Remark 2.1. If the slowly varying as $z \to Z_0$ function $L : \Delta_{Z_0} \to]0, +\infty[$ satisfies the condition S_0, then for every slowly varying as $z \to Z_0$ function $l : \Delta_{Z_0} \to]0, +\infty[$,

$$L(zl(z)) = L(z)[1 + o(1)] \text{ when } z \to Z_0 \ \ (z \in \Delta_{Z_0}).$$

The validity of this statement follows from the theorem of representation (see [1, Ch. 1, Sect. 1.2, p. 10]) of slowly varying function l and property M_1 of function L.

Remark 2.2 (see [22]). If slowly varying as $z \to Z_0$ function $L : \Delta_{Z_0} \to]0, +\infty[$ satisfies condition S_0, then the function $y : [t_0, \omega[\to \Delta_{Y_0}$ is continuously differentiable and such that

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \frac{y'(t)}{y(t)} = \frac{\xi'(t)}{\xi(t)} [r + o(1)] \text{ when } t \uparrow \omega,$$

where r is the non-zero real constant, ξ is continuously differentiable in some left neighborhood of ω real function, for which $\xi'(t) \neq 0$, then

$$L(y(t)) = L\big(\nu|\xi(t)|^r\big)[1 + o(1)] \text{ when } t \uparrow \omega,$$

where $\nu = \operatorname{sign} y(t)$ in the left neighborhood of ω.

Remark 2.3. If slowly varying as $z \to Z_0$ function $L : \Delta_{Z_0} \to]0, +\infty[$ satisfies condition S_0 and the function $r : \Delta_{Z_0} \times K \to \mathbb{R}$, where K is compact in $\mathbb{R}^m$, is such that

$$\lim_{\substack{z \to Z_0 \\ z \in \Delta_{Z_0}}} r(z, v) = 0 \text{ unifornly with respect to } v \in K,$$

then

$$\lim_{\substack{z \to Z_0 \\ z \in \Delta_{Z_0}}} \frac{L(\nu e^{[1 + r(z,v)] \ln |z|})}{L(z)} = 1$$

$$\text{uniformly with respect to } v \in K, \text{ where } \nu = \operatorname{sign} z.$$

Indeed, if it shouldn't be true, then there would exist a sequence $\{v_n\} \in K$ and a sequence $\{z_n\} \in \Delta_{Z_0}$ converging to Z_0 such that the inequality

$$\liminf_{n \to +\infty} \Big| \frac{L(\nu e^{[1 + r(z_n, v_n)] \ln |z_n|})}{L(z_n)} - 1 \Big| > 0 \tag{2.5}$$

is fulfilled.

Thus it is clear that there is the function $v : \Delta_{Z_0} \to K$ such that $v(z_n) = v_n$. For this function it is obvious that $\lim_{\substack{z \to Z_0 \\ z \in \Delta_{Z_0}}} r(z, v(z)) = 0$ and hence

$$\lim_{\substack{z \to Z_0 \\ z \in \Delta_{Z_0}}} \frac{L(\nu e^{[1 + r(z, v(z))] \ln |z|})}{L(z)} = 1,$$

which contradicts the inequality (2.5).

Finally, let us introduce auxiliary definitions assuming

$$\mu_{ki}=n-i-1+\sum_{j=0}^{i-2}\sigma_{kj}(i-j-1)-\sum_{j=i+1}^{n-1}\sigma_{kj}(j-i)\ \ (k=\overline{1,m};\ \ i=\overline{1,n}),$$

$$\gamma_k = 1-\sum_{j=0}^{n-1}\sigma_{kj},\quad \gamma_{ki} = 1-\sum_{j=i}^{n-1}\sigma_{kj}\ \ (k=\overline{1,m};\ \ i=\overline{1,n-1}),$$

$$C_{ki}=\frac{1}{(n-i)!}\prod_{j=0}^{i-1}[(i-j-1)!]^{-\sigma_{kj}}\prod_{j=i+1}^{n-1}[(j-i)!]^{\sigma_{kj}}\ \ (k=\overline{1,m};\ i=\overline{1,n-1}),$$

$$J_{ki}(t)=\int_{A_{ki}}^{t}p_k(s)|\pi_\omega(s)|^{\mu_{ki}}\prod_{\substack{j=0\\ j\neq i-1}}^{n-1}L_{kj}\big(\nu_j|\pi_\omega(s)|^{i-j-1}\big)\,ds\ \ (k=\overline{1,m};\ i=\overline{1,n}),$$

$$J_{kii}(t)=\int_{A_{kii}}^{t}|J_{ki}(s)|^{\frac{1}{\gamma_{ki}}}\,ds\ \ (k=\overline{1,m};\ \ i=\overline{1,n}),$$

where each of the limits of integration A_{km}, A_{kmm} ($m\in\{0,1\}$) is chosen equal to the point $a_0\in[a,\omega[$ (on the right of which, i.e., as $t\in[a_0,\omega[$, the integrand function is continuous) if under this value of limits of integration the corresponding integral tends to $\pm\infty$ as $t\uparrow\omega$, and equal to ω if at such value of limits of integration it tends to zero as $t\uparrow\omega$.

Theorem 2.1. *Let $n>2$, $i\in\{1,\dots,n-2\}$ and for some $s\in\{1,\dots,m\}$ the inequalities*

$$\limsup_{t\uparrow\omega}\frac{\ln p_k(t)-\ln p_s(t)}{\beta\ln|\pi_\omega(t)|}<$$
$$<\beta\sum_{\substack{j=0\\ j\neq i-1}}^{n-1}(\sigma_{sj}-\sigma_{kj})(i-j-1)\ \ \text{at all}\ \ k\in\{1,\dots,m\}\setminus\{s\},\qquad (2.6_i)$$

be fulfilled, where $\beta=\operatorname{sign}\pi_\omega(t)$ for $t\in[a,\omega[$. Moreover, let $\gamma_s\gamma_{si}\neq 0$ and the functions L_{sj} for all $j\in\{0,\dots,n-1\}\setminus\{i-1\}$ satisfy condition S_0. Then for the existence of $P_\omega(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i})$-solutions of the equation (1.1) *it is necessary, and if algebraic equation*

$$\sum_{j=i+1}^{n-1}\frac{\sigma_{sj}}{(j-i)!}\prod_{m=1}^{j-i}(m-\rho)+\sigma_{si}=\frac{1}{(n-i)!}\prod_{m=1}^{n-i}(m-\rho)\qquad(2.7)$$

has no roots with zero real part it is sufficient that (*along with* (1.3)) *the inequalities*

$$\begin{gathered}\nu_j\nu_{j-1}(i-j)\pi_\omega(t)>0\ \ \text{at all}\ \ j\in\{1,\dots,n-1\}\setminus\{i\},\\ \nu_i\nu_{i-1}\gamma_s\gamma_{si}J_{sii}(t)>0,\end{gathered}\qquad(2.8_i)$$

$$\nu_i\alpha_s(-1)^{n-i-1}\pi_\omega^{n-i-1}(t)\gamma_{si}J_{si}(t)>0\qquad(2.9_i)$$

be fulfilled in some left neighborhood of ω*, as well as the conditions*

$$\begin{gathered}\nu_{j-1}\lim_{t\uparrow\omega}|\pi_\omega(t)|^{i-j}=Y_{j-1}\ \ \text{at all}\ \ j\in\{1,\dots,n\}\setminus\{i\},\\ \nu_{i-1}\lim_{t\uparrow\omega}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}=Y_{i-1},\end{gathered}\tag{2.10$_i$}$$

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_{si}(t)}{J_{si}(t)}=-\gamma_{si},\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}=0.\tag{2.11$_i$}$$

Moreover, each solution of that kind admits as $t\uparrow\omega$ *the asymptotic representations*

$$y^{(j-1)}(t)=\frac{[\pi_\omega(t)]^{i-j}}{(i-j)!}\,y^{(i-1)}(t)[1+o(1)]\ \ (j=1,\dots,i-1),\tag{2.12$_i$}$$

$$y^{(j)}(t)=(-1)^{j-i}\frac{(j-i)!}{[\pi_\omega(t)]^{j-i}}\cdot\frac{\gamma_{si}J'_{sii}(t)}{\gamma_s J_{sii}(t)}\,y^{(i-1)}(t)[1+o(1)]\qquad (j=i,\dots,n-1),\tag{2.13$_i$}$$

$$\frac{|y^{(i-1)}(t)|^{\gamma_s}}{L_{si-1}(y^{(i-1)}(t))}=|\gamma_{si}C_{si}|\left|\frac{\gamma_s}{\gamma_{si}}J_{sii}(t)\right|^{\gamma_{si}}[1+o(1)]\ \ \text{with}\ \ t\uparrow\omega,\tag{2.14$_i$}$$

and in case $\omega=+\infty$ *there is* $i+1$*-parameter family of solutions if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si}>0$ *is valid, and* $i-1+l$*-parameter family if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si}<0$ *is valid, in case* $\omega<+\infty$*, there is* $r+1$*-parameter family if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si}>0$ *is valid, and* r*-parameter family if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si}<0$ *is valid, where* l *is a number of roots of the equation* (2.7) *with negative real part and* r *is a number of its roots with positive real part.*

Remark 2.4. Algebraic equation (2.7) has a fortiori no roots with zero real part, if $\sum\limits_{j=i}^{n-2}|\sigma_{sj}|<|1-\sigma_{sn-1}|$.

In Theorem 2.1, asymptotic representation for $y^{(i-1)}$ is written implicitly. The following theorem shows an additional restriction under which this representation may be presented explicitly.

Theorem 2.2. *If the conditions of Theorem* 2.1 *are fulfilled and a slowly varying at* $y^{(i-1)}\to Y_{i-1}$ *function* L_{si-1} *satisfies condition* S_0*, then for each* $P_\omega(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i})$*-solution of the equation* (1.1)*, asymptotic representations* (2.12$_i$)*,* (2.13$_i$) *and*

$$y^{(i-1)}(t)=\nu_{i-1}\left|\gamma_{si}C_{si}L_{si-1}\Big(\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}\Big)\right|^{\frac{1}{\gamma_s}}\times\\ \times\left|\frac{\gamma_s}{\gamma_{si}}J_{sii}(t)\right|^{\frac{\gamma_{si}}{\gamma_s}}[1+o(1)]\tag{2.15$_i$}$$

hold when $t\uparrow\omega$*.*

3. Proof of Theorems

Proof of Theorem 2.1. *Necessity.* Let $y : [t_0, \omega[\to \Delta_{Y_0}$ be an arbitrary $P_\omega(Y_0, \dots, Y_{n-1}, \frac{n-i-1}{n-i}$-solution of the equation (1.1). Then the conditions (1.4) are satisfied, there is $t_1 \in [a, \omega[$ such that $\nu_j y^{(j)}(t) > 0$ $(j = \overline{0, n-1})$ for $t \in [t_1, \omega[$ and by Lemma 1.1, the asymptotic relations (1.7), (1.8) hold. From (1.7) and (1.8) we obtain the relations

$$\frac{y^{(j)}(t)}{y^{(j-1)}(t)} = \frac{i-j+o(1)}{\pi_\omega(t)} \quad (j = \overline{1, n}) \ \text{ when } \ t \uparrow \omega \tag{3.1_i}$$

and therefore

$$\ln |y^{(j-1)}(t)| = \big[i-j+o(1)\big] \ln |\pi_\omega(t)| \quad (j = \overline{1, n}) \ \text{ when } \ t \uparrow \omega. \tag{3.2_i}$$

By virtue of (3.1_i), the first of inequalities (2.8_i) are fulfilled, and by virtue of (3.2_i), the first of conditions (2.10_i) are satisfied.

Taking into account (3.2_i), the representations (2.1) and the conditions

$$\lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta_{y_j}}} \frac{\ln L_{kj}(y^{(j)})}{\ln |y^{(j)}|} = 0 \quad (k = \overline{1, m}, \ \ j = \overline{0, n-1}), \tag{3.3}$$

which are satisfied due to the properties of slowly varying functions (see [1, Ch. 1, p. 1.5, p. 24]), we find that

$$\begin{gathered} \ln \varphi_{kj}(y^{(j)}(t)) = \sigma_{kj} \ln |y^{(j)}(t)| + \ln L_{kj}(y^{(j)}(t)) = \\ = \big[\sigma_{kj} + o(1)\big] \ln |y^{(j)}(t)| = \big[\sigma_{kj}(i-j-1) + o(1)\big] \ln |\pi_\omega(t)| \\ (k = \overline{1, m}, \ \ j = \overline{0, n-1}) \ \text{ when } \ t \uparrow \omega. \end{gathered}$$

That is why for each $k \in \{1, \dots, m\} \setminus \{s\}$,

$$\ln \left[\frac{p_k(t) \prod\limits_{j=0}^{n-1} \varphi_{kj}(y^{(j)}(t))}{p_s(t) \prod\limits_{j=0}^{n-1} \varphi_{sj}(y^{(j)}(t))} \right] = \ln \frac{p_k(t)}{p_s(t)} + \sum_{j=0}^{n-1} \Big[\ln \varphi_{kj}(y^{j)}(t) - \ln \varphi_{sj}(y^{(j)}(t) \Big] =$$

$$= \ln \frac{p_k(t)}{p_s(t)} + \ln |\pi_\omega(t)| \sum_{j=0}^{n-1} \big[(\sigma_{kj} - \sigma_{sj})(i-j-1) + o(1) \big] =$$

$$= \beta \ln |\pi_\omega(t)| \left[\frac{\ln p_k(t) - \ln p_s(t)}{\beta \ln |\pi_\omega(t)|} + \beta \sum_{\substack{j=0 \\ j \neq i-1}}^{n-1} (\sigma_{kj} - \sigma_{sj})(i-j-1) + o(1) \right]$$
$$\text{as} \quad t \uparrow \omega.$$

Since the expression, appearing on the right of this correlation, by virtue of (2.6_i) and the type of the function π_ω from (1.5), tends to $-\infty$ when $t \uparrow \omega$, therefore

$$\lim_{t\uparrow\omega}\frac{p_k(t)\prod\limits_{j=0}^{n-1}\varphi_{kj}(y^{(j)}(t))}{p_s(t)\prod\limits_{j=0}^{n-1}\varphi_{sj}(y^{(j)}(t))}=0 \ \text{ at all } \ k\in\{1,\dots,m\}\setminus\{s\}. \tag{3.4}$$

Then from (1.1) it follows that this solution implies asymptotic relation

$$y^{(n)}(t)=\alpha_s p_s(t)[1+o(1)]\prod_{j=0}^{n-1}\varphi_{sj}(y^{(j)}(t)) \ \text{ when } \ t\uparrow\omega. \tag{3.5}$$

Here, for all $j\in\{0,\dots,n-1\}\setminus\{i-1\}$, the functions L_{sj} in the representations (2.1) of functions φ_{sj} satisfy the condition S_0. Therefore, by virtue of (3.1_i) and Remark 2.2, for them we have

$$L_{sj}(y^{(j)}(t))=L_{sj}\big(\nu_j|\pi_\omega(t)|^{i-j-1}\big)[1+o(1)] \ \text{ when } \ t\uparrow\omega.$$

Taking into account (2.1) and the above representations, we can rewrite (3.5) in the form

$$\begin{aligned}y^{(n)}(t)=\alpha_s p_s(t)y^{(i-1)}(t)|^{\sigma_{si-1}}L_{si-1}(y^{(i-1)}(t))\times\\ \times\bigg(\prod_{\substack{j=0\\ j\neq i-1}}^{n-1}|y^{(j)}(t)|^{\sigma_{sj}}L_{sj}\big(\nu_j|\pi_\omega(t)|^{i-j-1}\big)\bigg)[1+o(1)] \ \text{ at } \ t\uparrow\omega.\end{aligned}$$

Hence, using (1.7), (1.8) and bearing in mind the fact that according to (3.1_i),

$$\begin{aligned}y^{(n)}(t)=\frac{y^{(n)}(t)}{y^{(n-1)}(t)}\cdots\frac{y^{(i+2)}(t)}{y^{(i+1)}(t)}y^{(i+1)}(t)\sim\\ \sim\frac{(-1)^{n-i-1}(n-i)!}{\pi_\omega^{n-i-1}(t)}y^{(i+1)}(t) \ \text{ at } \ t\uparrow\omega,\end{aligned}$$

and the notation introduced before formulation of theorems, we get the following relation:

$$\begin{aligned}\frac{y^{(i+1)}(t)|y^{(i)}(t)|^{\gamma_{si}-1}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{si-1}(y^{(i-1)}(t))}=\\ =\alpha_s(-1)^{n-i-1}\big(\operatorname{sign}[\pi_\omega(t)]^{n-i-1}\big)C_{si}p(t)|\pi_\omega(t)|^{\mu_{si}}\times\\ \times\prod_{\substack{j=0\\ j\neq i-1}}^{n-1}L_{sj}\big(\nu_j|\pi_\omega(t)|^{i-j-1}\big)[1+o(1)] \ \text{ at } \ t\uparrow\omega.\end{aligned} \tag{3.6}$$

By virtue of property M_2 of slowly varying functions, there is a continuously differentiated function $L_{0si-1}:\Delta_{Y_{i-1}}\to]0,+\infty[$ satisfying the conditions (2.3) for $k = s$ and $j = i-1$. Using these conditions and (3.1_i), we find that

$$\Big(\frac{|y^{(i)}(t)|^{\gamma_{si}}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))}\Big)' = \frac{\nu_i y^{(i+1)}(t)|y^{(i)}(t)|^{\gamma_{si}-1}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))}\times$$
$$\times\Bigg(\gamma_{si}-(\gamma_s-\gamma_{si})\frac{y^{(i)}(t)}{y^{(i+1)}(t)}\cdot\frac{y^{(i)}(t)}{y^{(i-1)}(t)}-$$
$$-\frac{y^{(i)}(t)}{y^{(i+1)}(t)}\cdot\frac{y^{(i)}(t)}{y^{(i-1)}(t)}\cdot\frac{y^{(i-1)}(t)L'_{0si-1}(y^{(i-1)}(t))}{L_{0si-1}(y^{(i-1)}(t))}\Bigg)=$$
$$=\frac{y^{(i+1)}(t)|y^{(i)}(t)|^{\gamma_{si}-1}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))}\big[\nu_i\gamma_{si}+o(1)\big]\ \text{ at }\ t\uparrow\omega.$$

Therefore (3.6) can be rewritten in the form

$$\Big(\frac{|y^{(i)}(t)|^{\gamma_{si}}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))}\Big)' =$$
$$=\nu_i\alpha_s(-1)^{n-i-1}\gamma_{si}\big(\operatorname{sign}[\pi_\omega(t)]^{n-i-1}\big)C_{si}p(t)|\pi_\omega(t)|^{\mu_{si}}\times$$
$$\times\prod_{\substack{j=0\\ j\neq i-1}}^{n-1}L_{sj}\big(\nu_j|\pi_\omega(t)|^{i-j-1}\big)[1+o(1)]\ \text{ at }\ t\uparrow\omega.$$

Integrating this relation on the interval between t_1 and t and taking into account that the fraction under the derivative sign due to the condition $\gamma_{si}\neq 0$ tends either to zero, or to $\pm\infty$ as $t\uparrow\omega$, we get

$$\frac{|y^{(i)}(t)|^{\gamma_{si}}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))}=$$
$$=\nu_i\alpha_s(-1)^{n-i-1}\gamma_{si}\big(\operatorname{sign}[\pi_\omega(t)]^{n-i-1}\big)C_{si}J_{si}(t)[1+o(1)]\ \text{ at }\ t\uparrow\omega.$$

From here first of all follows that the inequality (2.9_i) is fulfilled. Moreover, from this and (3.6), due to the equivalence of functions L_{si-1} and L_{0si-1} as $y^{(i-1)}\to Y_{i-1}$, we have

$$\frac{y^{(i+1)}(t)}{y^{(i)}(t)}=\frac{J'_{si}(t)}{\gamma_{si}J_{si}(t)}[1+o(1)]\ \text{ at }\ t\uparrow\omega,$$

whence, according to (3.1_i) for $j=i+1$, it follows that the first condition of (2.11_i) is valid.

From the obtained relation we also have

$$\frac{y^{(i)}(t)}{|y^{(i-1)}(t)|^{\frac{\gamma_{si}-\gamma_s}{\gamma_{si}}}L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}=$$
$$=\nu_i\big|C_{si}\gamma_{si}J_{si}(t)\big|^{\frac{1}{\gamma_{si}}}[1+o(1)]\ \text{ at }\ t\uparrow\omega. \qquad (3.7)$$

By virtue of the fact that

$$\left(\frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}\right)' =$$

$$= \frac{\nu_{i-1} y^{(i)}(t)|y^{(i-1)}(t)|^{\frac{\gamma_s-\gamma_{si}}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}\left[\frac{\gamma_s}{\gamma_{si}} - \frac{1}{\gamma_{si}}\frac{y^{(i-1)}(t)L'_{0si}(y^{(i-1)}(t))}{L_{0si}(y^{(i-1)}(t))}\right] =$$

$$= \frac{\nu_{i-1} y^{(i)}(t)|y^{(i-1)}(t)|^{\frac{\gamma_s-\gamma_{si}}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}\left[\frac{\gamma_s}{\gamma_{si}} + o(1)\right] \text{ at } t \uparrow \omega,$$

from (3.7) it follows

$$\left(\frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}\right)' = \frac{\nu_i\nu_{i-1}\gamma_s}{\gamma_{si}}\left|C_{si}\gamma_{si}J_{si}(t)\right|^{\frac{1}{\gamma_{si}}}[1+o(1)] \text{ when } t \uparrow \omega.$$

Here the fraction appearing under the derivative sign tends either to zero or to $\pm\infty$ as $t \uparrow \omega$, since by virtue of (1.4) and properties of slowly varying functions (see (3.3)),

$$\ln \frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))} = \ln|y^{(i-1)}(t)|\left[\frac{\gamma_s}{\gamma_{si}} - \frac{1}{\gamma_{si}}\frac{\ln L_{0si-1}(y^{(i-1)}(t))}{\ln|y^{(i-1)}(t)|}\right] =$$

$$= \ln|y^{(i-1)}(t)|\left[\frac{\gamma_s}{\gamma_{si}} + o(1)\right] \to \pm\infty \text{ at } t \uparrow \omega.$$

That is why, by integrating this correlation on the interval from t_1 to t, we get

$$\frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))} = \frac{\nu_i\nu_{i-1}\gamma_s}{\gamma_{si}}|\gamma_{si}C_{si}|^{\frac{1}{\gamma_{si}}}J_{sii}(t)[1+o(1)] \text{ at } t \uparrow \omega. \tag{3.8}$$

From here it follows the validity of the second inequality of (2.8_i) and also, in view of the equivalence of functions L_{si-1} and L_{0si-1} as $y^{(i-1)} \to Y_{i-1}$, the validity of the asymptotic representation (2.14_i). Besides, (3.7) and (3.8) yield

$$\frac{y^{(i)}(t)}{y^{(i-1)}(t)} = \frac{\gamma_{si}J'_{sii}(t)}{\gamma_s J_{sii}(t)}[1+o(1)] \text{ at } t \uparrow \omega. \tag{3.9$_i$}$$

By virtue of the last relation and Lemma 1.1, the second conditions of (2.10_i) and (2.11_i) are fulfilled, and asymptotic representations (2.12_i) and (2.13_i) hold.

Sufficiency. Let the conditions (2.8_i)–(2.11_i) be satisfied, and the algebraic equation (2.7) have no roots with zero real part. Let us show that in this case the equation (1.1) has solutions admitting asymptotic relations (2.12_i)–(2.14_i) as $t \uparrow \omega$.

Towards this end, we consider first the relation

$$\frac{|Y|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(Y)}=|\gamma_{si}C_{si}|^{\frac{1}{\gamma_{si}}}\left|\frac{\gamma_s}{\gamma_{si}}J_{sii}(t)\right|[1+v_n], \tag{3.10}$$

where $L_{0si}:\Delta_{Y_i}\to]0,+\infty[$ are continuously differentiated slowly varying as $Y\to Y_{i-1}$ functions, satisfying the conditions (2.3) (for $k=s$ and $j=i-1$) and existing due to the property M_2 of slowly varying functions.

Having chosen an arbitrary number $d\in]0,|\frac{\gamma_{si}}{\gamma_s}|[$, let us show that for some $t_0\in]a,\omega[$ the relation (3.10) defined uniquely, on the set $[t_0,\omega[\times\mathbb{R}_{\frac{1}{2}}$, where $\mathbb{R}_{\frac{1}{2}}=\{v\in\mathbb{R}:\ |v|\le\frac{1}{2}\}$, a continuously differentiated implicit function $Y=Y(t,v_n)$ of the type

$$Y(t,v_n)=\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}+z(t,v_n)}, \tag{3.11}$$

where z is the function such that

$$|z(t,v_n)|\le d\ \text{ for }\ (t,v_n)\in[t_0,\omega[\times\mathbb{R}_{\frac{1}{2}}\ \text{ and }\ \lim_{t\uparrow\omega}z(t,v_n)=0$$
$$\text{uniformly with respect to }\ v_n\in\mathbb{R}_{\frac{1}{2}}. \tag{3.12}$$

Assuming in (3.10)

$$Y=\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}+z} \tag{3.13}$$

and then taking the logarithm of the obtained relation, after elementary manipulations, we find that

$$z=a(t)+b(t,v_n)+Z(t,z), \tag{3.14}$$

where

$$a(t)=\frac{\gamma_{si}}{\gamma_s}\cdot\frac{\ln|\frac{\gamma_s}{\gamma_{si}}|+\frac{1}{\gamma_{si}}\ln|\gamma_{si}C_{si}|}{\ln|J_{sii}(t)|},\quad b(t,v_n)=\frac{\gamma_{si}}{\gamma_s}\cdot\frac{\ln[1+v_n]}{\ln|J_{sii}(t)|},$$
$$Z(t,z)=\frac{1}{\gamma_s}\cdot\frac{\ln L_{0si-1}(\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}+z})}{\ln|J_{sii}(t)|}.$$

Here, by virtue of the second condition of (2.10_i), by the choice of the limit of integration in J_{sii} and by the property (3.3) of slowly varying functions,

$$\nu_{i-1}\lim_{t\uparrow\omega}|J_{sii}(t)|^{\frac{\gamma_{is}}{\gamma_s}+z}=Y_{i-1}$$
$$\text{uniformly with respect to }\ z\in[-d,d],\quad \lim_{t\uparrow\omega}a(t)=0, \tag{3.15}$$

$$\begin{aligned}&\lim_{t\uparrow\omega}b(t,v_n)=0\quad\text{uniformly with respect to}\quad v_n\in\mathbb{R}_{\frac{1}{2}},\\&\lim_{t\uparrow\omega}Z(t,z)=0\quad\text{uniformly with respect to}\quad z\in[-d,d].\end{aligned} \tag{3.16}$$

Since

$$\frac{\partial Z(t,z)}{\partial z}=\frac{1}{\gamma_s}\cdot\frac{\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}+z}L'_{osi-1}(\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}+z})}{L_{0si-1}(\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}+z})},$$

by virtue of (2.3) and the first of the above-stated conditions, we likewise have

$$\lim_{t\uparrow\omega}\frac{\partial Z(t,z)}{\partial z}=0 \ \text{ uniformly with respect to } \ z\in[-d,d].$$

According to these conditions, there is a number $t_1 \in [a,\omega[$ such that

$$\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{is}}{\gamma_s}+z}\in\Delta_{Y_{i-1}} \ \text{ at } \ (t,z)\in[t_1,\omega[\,\times\mathbb{R}_d,$$
$$\text{where } \ \mathbb{R}_d=\big\{z\in\mathbb{R}:\ |z|\le d\big\}, \tag{3.17}$$
$$\big|a(t)+b(t,v_1,v_2)+Z(t,z)\big|\le d \ \text{ at } \ (t,v_n,z)\in[t_1,\omega[\,\times\mathbb{R}_{\frac12}\times\mathbb{R}_d$$

and

$$\big|Z(t,z_1)-Z(t,z_2)\big|\le\frac12\,|z_1-z_2| \ \text{ at } \ t\in[t_1,\omega[\ \text{ and } \ z_1,z_2\in\mathbb{R}_d. \tag{3.18}$$

Having chosen in this way the number t_1, we denote by $\mathbf{B}$ the Banach space of continuous and bounded on set $\Omega=[t_1,\omega[\,\times\mathbb{R}_{\frac12}$ functions $z:\Omega\to\mathbb{R}$ with the norm

$$\|z\|=\sup\big\{|z(t,v_n)|:\ (t,v_n)\in\Omega\big\}.$$

We distinguish from it the subspace $\mathbf{B}_0$ of those functions from $\mathbf{B}$, for which $\|z\|\le d$, and consider on $\mathbf{B}_0$, choosing a fortiori an arbitrary number $\nu\in(0,1)$, the operator

$$\Phi(z)(t,v_n)=z(t,v_n)-\nu\big[z(t,v_n)-a(t)-b(t,v_n)-Z(t,z(t,v_n))\big]. \tag{3.19}$$

By virtue of (3.17) and (3.18), for any $z\in\mathbf{B}_0$ and $z_1,z_2\in\mathbf{B}_0$, we have

$$|\Phi(z)(t,v_n)|\le(1-\nu)|z(t,v_n)|+\nu d\le d \ \text{ and } \ (t,v_n)\in\Omega$$

and

$$\begin{aligned}&|\Phi(z_1)(t,v_n)-\Phi(z_2)(t,v_n)|\le\\&\quad\le(1-\nu)|z_1(t,v_n)-z_2(t,v_n)|+\nu|Z(t,z_1(t,v_n))-Z(t,z_2(t,v_n)|\le\\&\qquad\le(1-\nu)|z_1(t,v_n)-z_2(t,v_n)|+\frac{\nu}{2}|z_1(t,v_n)-z_2(t,v_n)|\le\\&\qquad\qquad\le\Big(1-\frac{\nu}{2}\Big)\|z_1-z_2\| \ \textit{at} \ (t,v_n)\in\Omega.\end{aligned}$$

This implies that $\Phi(\mathbf{B}_0)\subset\mathbf{B}_0$ and $\|\Phi(z_1)-\Phi(z_2)\|\le(1-\frac{\nu}{2})\|z_1-z_2\|$.

It means that the operator Φ maps the space $\mathbf{B}_0$ into itself and is a contractor operator on it. Then, by the contraction mapping principle, there is a unique function $z\in\mathbf{B}_0$ such that $z=\Phi(z)$. By virtue of (3.19), this continuous on set Ω function is a unique solution of the equation (3.14) satisfying the condition $\|z\|\le d$. From (3.14), with regard for (3.15), (3.16), it follows that the given solution tends to zero as $t\uparrow\omega$ uniformly with respect to $v_n\in\mathbb{R}_{\frac12}$. Continuous differentiability of this solution on the set $[t_0,\omega[\,\times\mathbb{R}_{\frac12}$, where t_0 is some number from $[t_1,\omega[$, follows directly from the well-known local theorem on the existence of an implicit function defined by the relation (3.14). In virtue of replacement (3.13), the obtained function z corresponds to a continuously differentiated on set $[t_0,\omega[\,\times\mathbb{R}_{\frac12}$ function Y of type (3.11), where z possesses the properties (3.12) and which is a solution of the equation (3.10) and satisfies the conditions

$$\begin{gathered} Y(t,v_n)\in\Delta_{Y_{i-1}} \ \text{ for } \ (t,v_n)\in[t_0,\omega[\,\times\mathbb{R}_{\frac{1}{2}}, \\ \lim_{t\uparrow\omega} Y(t,v_n)=Y_{i-1} \ \text{ uniformly with respect to } \ v_n\in\mathbb{R}_{\frac{1}{2}}. \end{gathered} \tag{3.20}$$

Now, applying to differential equation (1.1) the transformation

$$\begin{gathered} y^{(j-1)}(t)=\frac{[\pi_\omega(t)]^{i-j}}{(i-j)!}\,y^{(i-1)}(t)[1+v_j(\tau)] \ \ (j=1,\dots,i-1), \\ y^{(j)}(t)=(-1)^{j-i}\frac{(j-i)!}{[\pi_\omega(t)]^{j-i}}\cdot\frac{\gamma_{si}J'_{sii}(t)}{\gamma_s J_{sii}(t)}\,y^{(i-1)}(t)[1+v_j(\tau)] \\ (j=i,\dots,n-1), \\ y^{(i-1)}(t)=Y(t,v_n(\tau)), \ \ \tau(t)=\beta\ln|\pi_\omega(t)|, \end{gathered} \tag{3.21$_i$}$$

where β is defined in (2.6_i), and bearing in mind that the function $y^{(i-1)}(t)=Y(t,v_n(\tau))$ for $t\in[t_0,\omega[$ and $v_n(\tau)\in\mathbb{R}^{\frac{1}{2}}$ satisfies equation

$$\frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}=|\gamma_{si}C_{si}|^{\frac{1}{\gamma_{si}}}\Big|\frac{\gamma_s}{\gamma_{si}}\,J_{sii}(t)\Big|[1+v_n(\tau)],$$

with the use of sign conditions (2.8_i), (2.9_i), we get a system of differential equations of the form

$$\begin{cases} v'_j=\beta\Big[(i-j)(v_{j+1}-v_j)-\dfrac{\gamma_{si}}{\gamma_s}h_1(\tau)(1+v_j)(1+v_i)\Big] \ \ (j=1,\dots,i-2), \\ v'_{i-1}=\beta\Big[-v_{i-1}-\dfrac{\gamma_{si}}{\gamma_s}\,h_1(\tau)(1+v_{i-1})(1+v_i)\Big], \\ v'_j=\beta\Big[(j-i)(1+v_j)-(j+1-i)(1+v_{j+1})-\dfrac{1}{\gamma_{si}}\,h_2(\tau)(1+v_j)+ \\ \qquad +\dfrac{1}{\gamma_s}h_1(\tau)(1+v_j)(\gamma_s-\gamma_{si}-\gamma_{si}v_i)\Big] \ \ (j=i,\dots,n-2), \\ v'_{n-1}=\beta\Bigg[\dfrac{n-i}{\gamma_{si}}\,h_2(\tau)\,\dfrac{\prod\limits_{j=0}^{i-2}|1+v_{j+1}|^{\sigma_{sj}}\prod\limits_{j=i}^{n-1}|1+v_j|^{\sigma_{sj}}}{|1+v_n|^{\gamma_{si}}}\,G(\tau,v_1,\dots,v_n)+ \\ \qquad +(n-i-1)(1+v_{n-1})-\dfrac{1}{\gamma_{si}}\,h_2(\tau)(1+v_{n-1})+ \\ \qquad +\dfrac{1}{\gamma_s}\,h_1(\tau)(1+v_{n-1})(\gamma_s-\gamma_{si}-\gamma_{si}v_i)\Bigg], \\ v'_n=\beta h_1(\tau)\Big[(1+v_n)(1+v_i)-(1+v_n)-\dfrac{1}{\gamma_s}\,H(\tau,v_n)(1+v_n)(1+v_i)\Big], \end{cases}$$

in which

$$h_1(\tau)=h_1(\tau(t))=\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}\,, \quad h_2(\tau)=h_2(\tau(t))=\frac{\pi_\omega(t)J'_{si}(t)}{J_{si}(t)}\,,$$

$$G(\tau(t),v_1,\dots,v_n)=\frac{L_{si-1}(Y(t,v_n))}{L_{0si-1}(Y(t,v_n))}\cdot\frac{\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}L_{sj}(Y^{[j]}(t,v_j,v_{j+1},v_n))}{\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}L_{sj}(\nu_j|\pi_\omega(t)|^{i-j-1})}\times$$

$$\times\frac{\sum\limits_{k=1}^{m}\alpha_k p_k(t)\varphi_{ki-1}(Y(t,v_n))\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\varphi_{kj}(Y^{[j]}(t,v_j,v_{j+1},v_n)}{\alpha_s p_s(t)\varphi_{si-1}(Y(t,v_n))\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\varphi_{sj}(Y^{[j]}(t,v_j,v_{j+1},v_n))},$$

$$H(\tau(t),v_n)=\frac{Y(t,v_n)L'_{0si-1}(Y(t,v_n))}{L_{0si-1}(Y(t,v_n))},$$

$$Y^{[j]}(t,v_j,v_{j+1},v_n)=$$

$$=\begin{cases}\dfrac{\pi_\omega^{i-j-1}(t)}{(i-j-1)!}\,Y(t,v_n)(1+v_{j+1}) & \text{when } j=\overline{0,i-2},\\[2mm] \dfrac{(j-i)!}{\pi_\omega^{j-i}(t)}\,\dfrac{\gamma_{si}}{\gamma_s}\,\dfrac{J'_{sii}(t)}{J_{sii}(t)}\,Y(t,v_n)(1+v_j) & \text{when } j=\overline{i,n-1}.\end{cases}$$

Here, the function $\tau(t)=\beta\ln|\pi_\omega(t)|$ possesses the properties

$$\tau'(t)>0 \ \text{ at } \ t\in[t_0,\omega[\,,\quad \lim_{t\uparrow\omega}\tau(t)=+\infty$$

and that is why, according to conditions (2.11_i),

$$\begin{aligned}&\lim_{\tau\to+\infty}h_1(\tau)=\lim_{t\uparrow\omega}h_1(\tau(t))=0,\\ &\lim_{\tau\to+\infty}h_2(\tau)=\lim_{t\uparrow\omega}h_2(\tau(t))=-\gamma_{si}.\end{aligned}\tag{3.22}$$

By virtue of (3.20) and (2.3) (for $k=s$ and $j=i-1$) the function H tends to zero as $\tau\to+\infty$ uniformly with respect to $v_n\in\mathbb{R}_{\frac{1}{2}}$, and first fraction in the representation of the function G tends to unity as $\tau\to+\infty$ uniformly with respect to $v_n\in\mathbb{R}_{\frac{1}{2}}$.

Let us show that the second and third fractions in the representation of function G likewise tends to unity as $\tau\to+\infty$ uniformly with respect to $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}$.

By virtue of (2.11_i) and using the l'Hospital's rule, we have

$$\lim_{t\uparrow\omega}\frac{\ln|J_{sii}(t)|}{\ln|\pi_\omega(t)|}=\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}=0,$$

$$\lim_{t\uparrow\omega}\frac{\ln\left|\frac{J'_{sii}(t)}{J_{sii}(t)}\right|}{\ln|\pi_\omega(t)|}=\lim_{t\uparrow\omega}\Big[\frac{\pi_\omega(t)J'_{si}(t)}{\gamma_{si}J_{si}(t)}-\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}\Big]=-1.$$

Taking into account the type of functions Y and $Y^{[j]}$ ($j=\overline{0,n-1}$, $j\neq i-1$), we find

$$\lim_{t\uparrow\omega}\frac{\ln|Y(t,v_n)|}{\ln|\pi_\omega(t)|}=\lim_{t\uparrow\omega}\Big[\frac{\gamma_s}{\gamma_{si}}+z(t,v_n)\Big]\lim_{t\uparrow\omega}\frac{\ln|J_{sii}(t)|}{\ln|\pi_\omega(t)|}=0$$

uniformly with respect to $v_n\in\mathbb{R}_{\frac{1}{2}}$,

$$\lim_{t\uparrow\omega}\frac{\ln|Y^{[j]}(t,v_j,v_{j+1},v_n)|}{\ln|\pi_\omega(t)|}=$$

$$=i-j-1+\lim_{t\uparrow\omega}\frac{\ln|Y(t,v_n)|}{\ln|\pi_\omega(t)|}+\lim_{t\uparrow\omega}\frac{\ln\frac{|1+v_{j+1}|}{(i-j-1)!}}{\ln|\pi_\omega(t)|}=i-j-1$$

uniformly with respect to $(v_{j+1},v_n)\in\mathbb{R}^2_{\frac{1}{2}}$ for $j=\overline{0,i-2}$

and

$$\lim_{t\uparrow\omega}\frac{\ln|Y^{[j]}(t,v_j,v_{j+1},v_n)|}{\ln|\pi_\omega(t)|}=i-j+\lim_{t\uparrow\omega}\frac{\ln|Y(t,v_n)|}{\ln|\pi_\omega(t)|}+$$

$$+\lim_{t\uparrow\omega}\frac{\ln|\frac{J'_{sii}(t)}{J_{sii}(t)}|}{\ln|\pi_\omega(t)|}+\lim_{t\uparrow\omega}\frac{\ln\frac{(j-i)!|\gamma_{si}(1+v_j)|}{|\gamma_s|}}{\ln|\pi_\omega(t)|}=i-j-1$$

uniformly with respect to $(v_j,v_n)\in\mathbb{R}^2_{\frac{1}{2}}$ for $j=\overline{i,n-1}$.

In view of these marginal ratios and using inequalities (2.6_i) we find, repeating the reasoning in proving the necessity, that for any $k\in\{1,\dots,m\}\setminus\{s\}$

$$\lim_{t\uparrow\omega}\frac{p_k(t)\varphi_{ki-1}(Y(t,v_n))\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\varphi_{kj}(Y^{[j]}(t,v_j,v_{j+1},v_n))}{p_s(t)\varphi_{si-1}(Y(t,v_n))\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\varphi_{sj}(Y^{[j]}(t,v_j,v_{j+1},v_n))}=0$$

uniformly with respect to $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}$.

Owing to these conditions, the last fraction in the representation of function G tends to unity as $\tau\to+\infty$ uniformly with respect to $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}$.

Moreover, taking into account marginal ratios stated above, we obtain the following representations:

$$Y^{[j]}(t,v_j,v_{j+1},v_n)=\nu_j e^{\ln|Y^{[j]}(t,v_j,v_{j+1},v_n)|}=$$

$$=\nu_j e^{[1+r_j(t,v_j,v_{j+1},v_n)]\ln|\pi_\omega(t)|^{i-j-1}}\quad\text{as } j\in\{0,\dots,n-1\}\setminus\{i-1\},$$

where

$$\lim_{t\uparrow\omega}r_j(t,v_j,v_{j+1},v_n)=0\ \text{ uniformly with respect to } (v_j,v_{j+1},v_n)\in\mathbb{R}^3_{\frac{1}{2}}$$

for all $j\in\{0,\dots,n-1\}\setminus\{i-1\}$.

Since the functions L_{sj} ($j=\overline{1,n-1}$, $j\neq i-1$) satisfy the condition S_0, by Remark 2.3, it follows that

$$\lim_{t\uparrow\omega}\frac{\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} L_{sj}(Y^{[j]}(t,v_j,v_{j+1},v_n))}{\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} L_{sj}(\nu_j|\pi_\omega(t)|^{i-j-1})}=1$$

$$\text{uniformly with respect to } (v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}.$$

Therefore, the second fraction in the representation of function G tends to unity as $\tau\to+\infty$ uniformly with respect to $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}$.

Due to above stated, the obtained system of differential equations can be written in form

$$\begin{cases} v_j' = \beta\Big[f_i(\tau,v_1,\dots,v_n)+\sum\limits_{k=1}^{n}p_{jk}v_k\Big] \ \ (k=\overline{1,n-2}),\\ v_{n-1}'=\beta\Big[f_{n-1}(\tau,v_1,\dots,v_n)+\sum\limits_{k=1}^{n}p_{n-1k}v_k+V_{n-1}(v_1,\dots,v_n)\Big],\\ v_n'=\beta h_1(\tau)\Big[f_n(\tau,v_1,\dots,v_n)+\sum\limits_{k=1}^{n}p_{nk}v_k+V_n(v_1,\dots,v_n)\Big], \end{cases} \qquad (3.23_i)$$

where the functions f_i $(i=\overline{1,n})$ are continuous on a set $[\tau_1,+\infty[\,\times\mathbb{R}^n_{\frac{1}{2}}$ for some $\tau_1\geq\beta\ln|\pi_\omega(t_0)|$ and are such that

$$\lim_{\tau\to+\infty} f_i(\tau,v_1,\dots,v_n)=0 \ \ (i=\overline{1,n})$$

$$\text{uniformly with respect to } (v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}, \qquad (3.24)$$

$$p_{jj}=j-i,\quad p_{jj+1}=i-j,$$

$${}_{jk}=0 \ \text{ at } \ k\in\{1,\dots,n\}\setminus\{j,j+1\} \ \ (j=\overline{1,i-2}),^{*}$$

$$p_{i-1i-1}=-1,\quad p_{i-1k}=0 \ \text{ at } \ k\in\{1,\dots,n\}\setminus\{i-1\},$$

$$p_{jj}=j-i+1,\quad p_{jj+1}=i-j-1,$$

$$p_{jk}=0 \ \text{ at } \ k\in\{1,\dots,n\}\setminus\{j,j+1\} \ \ (j=\overline{i,n-2}),$$

$$p_{n-1k}=-(n-i)\sigma_{sk-1} \ \ (k=\overline{1,i-1}),$$

$$p_{n-1k}=-(n-i)\sigma_{sk} \ \ (k=\overline{i,n-2}),\quad p_{n-1n-1}=(n-i)(1-\sigma_{sn-1}),$$

$$p_{n-1n}=(n-i)\gamma_{si},\quad p_{ni}=i,\quad p_{nk}=0 \ \text{ at } \ k\in\{1,\dots,n\}\setminus\{i\},$$

$$V_n(v_1,\dots,v_n)=v_iv_n,$$

$$V_{n-1}(v_1,\dots,v_n)=(i-n)\frac{\prod\limits_{j=0}^{i-2}|1+v_{j+1}|^{\sigma_{sj}}\prod\limits_{j=i}^{n-1}|1+v_j|^{\sigma_{sj}}}{|1+v_n|^{\gamma_{si}}}+$$

$$+(n-i)\Big[1+\sum_{k=1}^{i-1}\sigma_{sk-1}v_k+\sum_{k=i}^{n-1}\sigma_{sk}v_k-\gamma_{si}v_n\Big].$$

Since conditions (3.24) are satisfied and

$$\lim_{|v_1|+\dots+|v_n|\to 0}\frac{\partial V_j(v_1,\dots,v_n)}{\partial v_k}=0\ \ (j=n-1,n;\ \ k=\overline{1,n}),$$

this system belongs to the class of systems of differential equations, for which the criteria for the existence of vanishing at infinity solutions were obtained in [24]. Let us show that for this system the conditions of Theorem 2.6 are fulfilled (based on this paper).

First of all, taking into account the conditions (3.22) and the type of integral $J_{sii}(t)$, we notice that the function h_1 possesses the properties

$$\lim_{\tau\to+\infty}h_1(\tau)=0,\qquad \int\limits_{\tau_1}^{+\infty}h_1(\tau)\,d\tau=\beta\int\limits_{t_1}^{\omega}\frac{J'_{sii}(t)}{J_{sii}(t)}\,dt=$$
$$=\beta\ln|J_{sii}(t)|_{t_1}^{\omega}=\pm\infty\ \ (\tau_1=\beta\ln|\pi_\omega(t_1)|),$$
$$\lim_{\tau\to+\infty}\frac{h_1'(\tau)}{h_1(\tau)}=\lim_{t\uparrow\omega}\frac{(h_1(\tau(t)))_t'}{\tau'(t)h_1(\tau(t))}=$$
$$=\beta\lim_{t\uparrow\omega}\left[\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}+\frac{1}{\gamma_{si}}\,\frac{\pi_\omega(t)J'_{si}(t)}{J_{si}(t)}\,\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}-\Big(\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}\Big)^2\right]=0.$$

Next, consider the matrices $P_n=(p_{jk})_{j,k=1}^{n}$ and $P_{n-1}=(p_{jk})_{j,k=1}^{n-1}$, for which we have

$$\det P_{n-1}=(-1)^{i-1}(i-1)!(n-i)!\gamma_{si},\quad \det P_n=(-1)^{i}(i-1)!(n-i)!\gamma_{si},$$
$$\det\big[P_{n-1}-\rho E_{n-1}\big]=(-1)^{i-1}\prod_{k=1}^{i-1}(k+\rho)\Bigg[\prod_{m=1}^{n-i}(m-\rho)-$$
$$-(n-i)!\sum_{j=i+1}^{n-1}\frac{\sigma_{sj}}{(j-i)!}\prod_{m=1}^{j-i}(m-\rho)-(n-i)!\sigma_{si}\Bigg],$$

where E_{n-1} is the unit matrix of dimension $(n-1)\times(n-1)$.

Since algebraic equation (2.7), according to the conditions of Theorem, has no roots with zero real part, the characteristic equation of the matrix P_{n-1} has likewise no such roots, and the given characteristic equation has $i-1$ roots (if $i>1$) of the type $\rho_k=-k$ $(k=\overline{1,i-1})$.

Thus, for the system (3.23_i), all the conditions of Theorem 2.6 of [24] are satisfied. According to this theorem, the system (3.23_i) has at least one solution $(v_j)_{j=1}^n:[\tau_2,+\infty[\to\mathbb{R}^n$ $(\tau_2\ge\tau_1)$ tending to zero as $\tau\to+\infty$.

Moreover, if l is a number of roots of the equation (2.7) with negative real part, and r is a number of roots with positive real part, then according to the same Theorem, in case $\beta=1$, this system has $i+1$ - parametric family of such solutions if the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si}>0$ is fulfilled, and has $i-1+l$- parametric family if the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si}<0$ is fulfilled, whereas, in case $\beta=-1$, there is $r+1$ - parameter family of such solutions if there is the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si} > 0$ and r - parametric family if there is the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si} < 0$.

To every such solution of the system (3.23_i) there corresponds, due to the replacements (3.21_i) and the first condition of (2.3), the solution $y : [t_2,\omega[\to \mathbb{R}$ ($t_2 \in [a,\omega[$) of the equation (1.1) admitting as $t \uparrow \omega$ asymptotic representations (2.12_i)–(2.14_i). Using these representations and conditions (2.6_i), (2.8_i)–(2.11_i), it can be easily seen that it is a $P_\omega(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i})$-solution. □

Proof of Theorem 2.2. Let the equation (1.1) have $P_\omega(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i})$-solution $y : [t_0,\omega[\to \Delta_{Y_0}$. Then, according to Theorem 2.1, the conditions (2.8_i)–(2.11_i) are satisfied and for this solution the asymptotic representations (2.12_i)–(2.14_i) hold as $t \uparrow \omega$. Furthermore, from the proof of necessity of that theorem it is clear that the condition (3.9_i) is satisfied. Since the functions L_{si-1} satisfy the condition S_0, by virtue of (3.9_i) and Remark 2.2,

$$L_{si-1}(y^{(i-1)}(t)) = L_{sj}\big(\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}\big)[1+o(1)] \ \text{ at } \ t \uparrow \omega.$$

Therefore it follows from (2.14_i) that

$$|y^{(i-1)}(t)|^{\gamma_s} =$$
$$= |\gamma_{si}C_{si}|L_{si-1}\big(\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}\big)\bigg|\frac{\gamma_s}{\gamma_{si}}\,J_{sii}(t)\bigg|^{\gamma_{si}}[1+o(1)] \quad \text{at} \quad t \uparrow \omega,$$

which results in the presentation (2.15_i). □

4. Example of Equation with Regularly Varying as $t \uparrow \omega$ Coefficients

Suppose that in the differential equation (1.1), the continuous functions $p_k : [a,\omega[\to]0,+\infty[$ ($k = \overline{1,m}$) are regularly varying, as $t \uparrow \omega$, of orders ϱ_k ($k = \overline{1,m}$), and, moreover, the conditions of Theorem 2.1 as $i \in \{1,\dots,n-2\}$ are satisfied. In this case

$$\lim_{t\uparrow\omega}\frac{\ln p_k(t)}{\ln|\pi_\omega(t)|} = \varrho_k \tag{4.1}$$

and the conditions (2.6_i) take the form

$$\beta(\varrho_k - \varrho_s) < \beta \sum_{\substack{j=0\\ j\neq i-1}}^{n-1}(\sigma_{sj}-\sigma_{kj})(i-j-1)$$
$$\text{at all } \ k \in \{1,\dots,m\}\setminus\{s\}. \tag{4.2_i}$$

Since as $t \uparrow \omega$ the functions $L_{sj}(\nu_j|\pi_\omega(t)|^{i-j-1})$ ($j \in \{0,\dots,n-1\}\setminus\{i-1\}$) are slowly varying, and the function p_s is regularly varying of order ϱ_s, therefore the function J_{si} is regularly varying of order $1+\varrho_s+\mu_{si}$, and the function $|J_{sii}(t)|$ is regularly varying of order $1+\frac{1}{\gamma_{si}}(1+\varrho_s+\mu_{si})$ as $t\uparrow\omega$. This implies that

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_{si}(t)}{J_{si}(t)}=1+\varrho_s+\mu_{si},$$
$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}=1+\frac{1}{\gamma_{si}}\,(1+\varrho_s+\mu_{si}).$$

Therefore the conditions (2.11_i) will be of the following form:

$$1+\varrho_s+\gamma_{si}+\mu_{si}=0. \tag{4.3_i}$$

Taking into account this condition, the function $J_{sii}(t)$ should be slowly varying as $t\uparrow\omega$. In order to get asymptotic representation for this integral we have to know the type of a slowly varying component of the integrand equation.

Suppose that the functions p_s and φ_{sj} $(j=\overline{0,n-1})$ are of the form

$$\begin{aligned} p_s(t)&=|\pi_\omega(t)|^{\varrho_s}\big|\ln|\pi_\omega(t)|\big|^{r_s},\\ \varphi_{sj}(y^{(j)})&=|y^{(j)}|^{\sigma_{sj}}\big|\ln|y^{(j)}|\big|^{\lambda_{sj}}\quad (j=\overline{0,n-1}).\end{aligned} \tag{4.4}$$

In this case, $L_{sj}(y^{(j)})=\big|\ln|y^{(j)}|\big|^{\lambda_{sj}}$ $(j=\overline{0,n-1})$ and hence all of them satisfy the conditions of S_0. Additionally, we get as $t\uparrow\omega$ the following asymptotic relations

$$J_{si}(t)\sim-\frac{\beta}{\gamma_{si}}\prod_{\substack{j=0\\ j\neq i-1}}^{n-1}|i-j-1|^{\lambda_{sj}}|\pi_\omega(t)|^{-\gamma_{si}}\big|\ln|\pi_\omega(t)|\big|^{r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}}, \tag{4.5_i}$$

$$J_{sii}(t)\sim\begin{cases}\dfrac{\gamma_{si}\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}|i-j-1|^{\lambda_{sj}}}{|\gamma_{si}|^{\frac{1}{\gamma_{si}}}\Big(r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}+\gamma_{si}\Big)}\big|\ln|\pi_\omega(t)|\big|^{1+\frac{1}{\gamma_{si}}\Big(r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}\Big)}, & \\ \qquad\qquad \text{if } r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}\neq-\gamma_{si}, & \\ \dfrac{\beta}{|\gamma_{si}|^{\frac{1}{\gamma_{si}}}}\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}|j-i-1|^{\lambda_{sj}}\ln\big|\ln|\pi_\omega(t)|\big|, & \\ \qquad\qquad \text{if } r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}=-\gamma_{si}, & \end{cases} \tag{4.6_i}$$

$$\frac{J'_{sii}(t)}{J_{sii}(t)} \sim \begin{cases} \dfrac{r_s + \sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj} + \gamma_{si}}{\gamma_{si}\pi_\omega(t)\ln|\pi_\omega(t)|}, & \text{if } r_s + \sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj} \neq -\gamma_{si}, \\[2ex] \dfrac{1}{\pi_\omega(t)\ln|\pi_\omega(t)|\ln|\ln|\pi_\omega(t)||}, & \text{if } r_s + \sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj} = -\gamma_{si}. \end{cases} \tag{4.7$_i$}$$

From the above relations it, in particular, follows that the inequalities (2.8_i), (2.9_i) and the conditions (2.10_i) take the form

$$\begin{aligned} &\nu_j\nu_{j-1}(i-j)\pi_\omega(t) > 0 \quad \text{at all } j \in \{1,\dots,n-1\}\setminus\{i\}, \\ &\nu_i\alpha_s(-1)^{n-i}\pi_\omega^{n-i}(t) > 0, \end{aligned} \tag{4.8$_i$}$$

$$\nu_i\nu_{i-1}\gamma_s\gamma_{si} > 0\ (<0), \ \text{ if } 1 + \frac{1}{\gamma_{si}}\Big(r_s + \sum_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj}\Big) \ge 0\ (<0), \tag{4.9$_i$}$$

$$\nu_{j-1}\lim_{t\uparrow\omega}|\pi_\omega(t)|^{i-j} = Y_{j-1} \ \text{ at } j \in \{1,\dots,n\}\setminus\{i\}, \tag{4.10$_i$}$$

$$\nu_{i-1}Y_{i-1} = \infty\ (=0), \ \text{ if } \gamma_s\Big(r_s + \sum_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj} + \gamma_{si}\Big) \ge 0\ (<0). \tag{4.11$_i$}$$

By virtue of above-said, from Theorem 2.2 follows the following statement.

Corollary 4.1. *Let in the equation* (1.1) $n > 2$, *the functions* p_k $(k = \overline{1,m})$ *be regularly varying of orders* ϱ_k *at* $t \uparrow \omega$, $i \in \{1,\dots,n-2\}$ *and for some* $s \in \{1,\dots,m\}$, *the inequalities* (4.2_i) *be fulfilled. Let, moreover, the equation* $\gamma_s\gamma_{si} \neq 0$ *be fulfilled and the representations* (4.4) *hold. Then for the equation* (1.1) *to have* $P_\omega(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i})$*-solutions, it is necessary, and if algebraic equation* (2.7) *has no roots with zero real part, then it is sufficient that the conditions* (4.3_i), (4.8_i)–(4.11_i) (*along with* (1.3)) *are satisfied. Moreover, for each such solution there exist, as* $t \uparrow \omega$, *the following asymptotic representations:*

$$y^{(j-1)}(t) = \frac{[\pi_\omega(t)]^{i-j}}{(i-j)!}\, y^{(i-1)}(t)[1+o(1)] \ \ (j = 1,\dots,i-1), \tag{4.12$_i$}$$

$$y^{(j)}(t) = (-1)^{j-i}\,\frac{(j-i)!}{[\pi_\omega(t)]^{j-i}}\cdot\frac{\gamma_{si}J'_{sii}(t)}{\gamma_s J_{sii}(t)}\, y^{(i-1)}(t)[1+o(1)] \tag{4.13$_i$}$$

$$(j = i,\dots,n-1),$$

$$y^{(i-1)}(t) = \nu_{i-1}\Big|\gamma_{si}C_{si}\Big|\frac{\gamma_{si}}{\gamma_s}\Big|^{\lambda_{si-1}-\gamma_{si}}\Big|^{\frac{1}{\gamma_s}} \times$$

$$\times |J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}\big|\ln|J_{sii}(t)|\big|^{\frac{\lambda_{si-1}}{\gamma_s}}[1+o(1)], \tag{2.15$_i$}$$

where the functions $J_{sii}(t)$ *and* $\frac{J'_{sii}(t)}{J_{sii}(t)}$ *are defined by* (4.6_i) *and* (4.7_i)*, respectively, and for such solutions in case* $\omega = +\infty$ *there exists an* $i+1$*-parametric family if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si} > 0$ *is fulfilled, and an* $i-1+l$*-parameter family if there is the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si} < 0$*, while in case* $\omega < +\infty$ *there exists an* $r+1$*-parametric family of such solutions if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si} > 0$ *is fulfilled, and an* r*- parametric family if there is the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si} < 0$*, where* l *is a number of roots of the equation* (2.7) *with negative real part and* r *is a number of its roots with positive real part.*

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(Received December 4, 2014)

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Memoirs on Differential Equations and Mathematical Physics
Volume 61, 2014, 63–82

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BOUNDARY VALUE PROBLEMS OF THE THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR DOMAINS BOUNDED BY A SPHERICAL SURFACE

Abstract. We consider the stationary oscillation case of the theory of linear thermoelasticity of materials with microtemperatures. The representation formula of a general solution of the homogeneous system of differential equations obtained in the paper is expressed by means of seven metaharmonic functions. This formula is very convenient and useful in many particular problems for domains with concrete geometry. Here we demonstrate an application of this formulas to the Dirichlet and Neumann type boundary value problem for a ball. The uniqueness theorems are proved. An explicit solutions in the form of absolutely and uniformly convergent series are constructed.

2010 Mathematics Subject Classification. 74A15, 74B10, 74F20.
Key words and phrases. Microtemperature, thermoelasticity, Fourier-Laplace series, stationary oscillation.

რეზიუმე. ნაშრომში განხილულია თერმოდრეკადობის წრფივი თეორიის სტაციონარული რხევის ამოცანები მიკროტემპერატურის გათვალისწინებით. მიღებულია ერთგვაროვან დიფერენციალურ განტოლებათა სისტემის ამონახსნის ზოგადი წარმოდგენის ფორმულა გამოსახული შვიდი მეტაჰარმონიული ფუნქციის საშუალებით. მიღებული წარმოდგენა არის მეტად მოხერხებული კონკრეტული გეომეტრიის მქონე არეების შემთხვევაში სასაზღვრო ამოცანების ამოსახსნელად. ამ ნაშრომში შესწავლილია დირიხლესა და ნეიმანის ტიპის სასაზღვრო ამოცანები ბირთვისათვის. დამტკიცებულია ერთადერთობის თეორემები. ამოცანების ამოხსნები მიღებულია აბსოლუტურად და თანაბრად კრებადი მწკრივების სახით.

1. Introduction

Mathematical model describing the chiral properties of the linear thermoelasticity of materials with microtemperatures have been proposed by Iesan [6], [8] and recently it has been extended to a more general case, when the material points admit micropolar structure [7].

The Dirichlet, Neumann and mixed type boundary value problems corresponding to this model are well investigated for general domains of arbitrary shape, the uniqueness and existence theorems are proved, and regularity results for solutions are established by potential and variational methods (see [1, 10, 14, 15] and the references therein).

The main goal of this paper is to derive general representation formulas for the displacement vector of microtemperatures and temperature function by means of metaharmonic functions. That is, we can represent solutions to a very complicated coupled system of simultaneous differential equations of thermoelasticity with the help of solutions of simpler canonical equations.

In particular, here we apply these representation formulas to construct explicit solutions to the Dirichlet and Neumann type boundary value problems for a ball. We represent the solution in the form of Fourier–Laplace series and show their absolute and uniform convergence along with their derivatives of the first order if the boundary data satisfy appropriate smoothness conditions. One of the methods to satisfy the boundary conditions is given in A. Ulitko [17], F. Mors and G Feshbah [12], L. Giorgashvili [2,3], L. Giorgashvili, D. Natroshvili [4], L. Giorgashvili, A. Jaghmaidze, K. Skhvitaridze [5], D. Natroshvili, L. Giorgashvili, I. Stratis [13] and other papers.

2. Basic Equations and Auxiliary Theorems

A system of homogeneous differential equations of the stationary oscillation of the thermoelasticity with microtemperatures is written in the form [7]

$$\mu\Delta u(x) + (\lambda+\mu)\operatorname{grad}\operatorname{div} u(x) - \gamma \operatorname{grad}\theta(x) + \rho\sigma^2 u(x) = 0, \qquad (2.1)$$

$$\varkappa_6\Delta w(x) + (\varkappa_5+\varkappa_4)\operatorname{grad}\operatorname{div} w(x) - \varkappa_3 \operatorname{grad}\theta(x) + \tau w(x) = 0, \qquad (2.2)$$

$$\varkappa\Delta\theta(x) + i\sigma\gamma T_0 \operatorname{div} u(x) + \varkappa_1 \operatorname{div} w(x) + i\sigma a T_0\theta(x) = 0, \qquad (2.3)$$

where Δ is the three-dimensional Laplace operator, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $w=(w_1,w_2,w_3)^\top$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 ($T_0>0$), $\top$ is the transposition symbol, λ, μ, γ, $\varkappa$, $\varkappa_j$, $j=1,2,\dots,6$, are constitutive coefficients, satisfying the conditions [7]

$$\mu>0,\ \ 3\lambda+2\mu>0,\ \ \varkappa>0,\ \ 3\varkappa_4+\varkappa_5+\varkappa_6>0,\ \ \varkappa_6+\varkappa_5>0,$$
$$\varkappa_6-\varkappa_5>0,\ \ (\varkappa_1+T_0\varkappa_3)^2<4T_0\varkappa\varkappa_2,\ \ \gamma>0,\ \ a>0,$$

$\tau=-\varkappa_2+i\sigma\delta$, $\delta>0$, $\rho>0$ is the mass density of the elastic material. In the sequel we assume that $\sigma=\sigma_1+i\sigma_2$, $\sigma_2>0$, $\sigma_1\in\mathbb{R}$.

Let $U=(u,w,\theta)^\top$. The stress vector, which we denote by the symbol $P(\partial,n)U$, has the form

$$P(\partial,n)U=\Big(P^{(1)}(\partial,n)U',\;\; P^{(2)}(\partial,n)U'',\;\; P^{(3)}(\partial,n)U''\Big)^\top,$$

where $U'=(u,\theta)^\top$, $U''=(w,\theta)^\top$, $n=(n_1,n_2,n_3)^\top$ is the unit vector,

$$\begin{aligned}
P^{(1)}(\partial,n)U'&=T^{(1)}(\partial,n)u-\gamma n\theta,\\
P^{(2)}(\partial,n)U''&=T^{(2)}(\partial,n)w-\varkappa_3 n\theta,\\
P^{(3)}(\partial,n)U''&=\varkappa\frac{\partial\theta}{\partial n}+(\varkappa_1+\varkappa_3)(n\cdot w),\\
T^{(1)}(\partial,n)u&=2\mu\frac{\partial u}{\partial n}+\lambda n\operatorname{div}u+\mu[n\times\operatorname{rot}u],\\
T^{(2)}(\partial,n)w&=(\varkappa_6+\varkappa_5)\frac{\partial w}{\partial n}+\varkappa_4 n\operatorname{div}w+\varkappa_5[n\times\operatorname{rot}w].
\end{aligned}\tag{2.4}$$

Definition. The vector $U=(u,w,\theta)^\top$ is said to be regular in a domain $\Omega\subset\mathbb{R}^3$ if $U\in C^2(\Omega)\cap C^1(\overline{\Omega})$.

Theorem 2.1. *A vector $U=(u,w,\theta)^\top$ is a regular solution of system (2.1)–(2.3) in a domain $\Omega\subset\mathbb{R}^3$, if and only if it is represented in the form*

$$\begin{aligned}
u(x)&=\sum_{j=1}^{3}\operatorname{grad}\Phi_j(x)+\operatorname{rot}\operatorname{rot}(x\Phi_4(x))+\operatorname{rot}(x\Phi_5(x)),\\
w(x)&=\sum_{j=1}^{3}\alpha_j\operatorname{grad}\Phi_j(x)+\operatorname{rot}\operatorname{rot}(x\Phi_6(x))+\operatorname{rot}(x\Phi_7(x)),\\
\theta(x)&=-\sum_{j=1}^{3}\beta_j k_j^2\Phi_j(x),
\end{aligned}\tag{2.5}$$

where

$$(\Delta+k_j^2)\Phi_j(x)=0,\;\; j=1,2,3,\quad (\Delta+k_4^2)\Phi_j(x)=0,\;\; j=4,5,$$
$$(\Delta+k_5^2)\Phi_j(x)=0,\;\; j=6,7,$$

$k_4^2=\rho\sigma^2/\mu$, $k_5^2=\tau/\varkappa_6$, $-k_j^2$, $j=1,2,3$, *are the roots of the equation*

$$z^3+a_1z^2+a_2z+a_3=0\tag{2.6}$$

with

$$
\begin{aligned}
a_1 &= \frac{1}{\Delta_1}\Big\{l\big[i\sigma T_0(a(\lambda+2\mu)+\gamma^2)+\varkappa\rho\sigma^2\big]+(\lambda+2\mu)(i\sigma aT_0\varkappa\tau+\varkappa_1\varkappa_3)\Big\},\\
a_2 &= \frac{1}{\Delta_1}\Big\{\rho\sigma^2(\varkappa_1\varkappa_3+i\sigma aT_0 l+\varkappa\tau)+\tau\big[i\sigma T_0\gamma^2+i\sigma aT_0(\lambda+2\mu)\big]\Big\},\\
a_3 &= \frac{i}{\Delta_1}\,aT_0\rho\sigma^3\tau, \quad \Delta_1=\varkappa(\lambda+2\mu)l>0, \quad l=\varkappa_4+\varkappa_5+\varkappa_6>0,\\
\alpha_j &= \frac{\varkappa_3[\rho\sigma^2-(\lambda+2\mu)k_j^2]}{\gamma(\tau-lk_j^2)}, \quad \beta_j=\frac{i\sigma\gamma T_0+\varkappa_1\alpha_j}{\varkappa k_j^2-i\sigma aT_0}, \quad j=1,2,3.
\end{aligned} \tag{2.7}
$$

Proof. Assume that a vector $U=(u,w,\theta)^\top$ is a solution of system (2.1)–(2.3). From equations (2.1)–(2.2) we have

$$u(x)=u'(x)+u''(x), \quad w(x)=w'(x)+w''(x),$$

where

$$
\begin{aligned}
u'(x) &= \frac{1}{\rho\sigma^2}\ \mathrm{grad}\,\big[-(\lambda+2\mu)\,\mathrm{div}\,u(x)+\gamma\theta(x)\big],\\
w'(x) &= \frac{1}{\tau}\ \mathrm{grad}\,\big[-l\,\mathrm{div}\,w(x)+\varkappa_3\theta(x)\big];
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
u''(x) &= \frac{\mu}{\rho\sigma^2}\,\mathrm{rot}\,\mathrm{rot}\,u(x),\\
w''(x) &= \frac{\varkappa_6}{\rho}\,\mathrm{rot}\,\mathrm{rot}\,w(x).
\end{aligned} \tag{2.9}
$$

If we apply the operator div to both parts of equalities (2.1) and (2.2), and take into account equalities (2.3), then we obtain

$$
\begin{aligned}
\big[(\lambda+2\mu)\Delta+\rho\sigma^2\big]\,\mathrm{div}\,u(x)-\gamma\Delta\theta(x)&=0,\\
(l\Delta+\tau)\,\mathrm{div}\,w(x)-\varkappa_3\Delta\theta(x)&=0,\\
i\sigma\gamma T_0\,\mathrm{div}\,u(x)+\varkappa_1\,\mathrm{div}\,w(x)+(\varkappa\Delta+i\sigma aT_0)\theta(x)&=0.
\end{aligned}
$$

From these equations we get

$$(\Delta+k_1^2)(\Delta+k_2^2)(\Delta+k_3^2)(\mathrm{div}\,u,\mathrm{div}\,w,\theta)^\top=0, \tag{2.10}$$

where $-k_j^2$, $j=1,2,3$, are the roots of equation (2.6).

In view of equalities (2.8) and (2.10), we obtain

$$(\Delta+k_1^2)(\Delta+k_2^2)(\Delta+k_3^2)(u',w')^\top=0, \quad \mathrm{rot}\,u'=0, \quad \mathrm{rot}\,w'=0. \tag{2.11}$$

We represent the vectors $u'(x)$, $w'(x)$ and the function $\theta(x)$ as:

$$u'(x)=\sum_{j=1}^{3}u^{(j)}(x), \quad w'(x)=\sum_{j=1}^{3}w^{(j)}(x), \quad \theta(x)=\sum_{j=1}^{3}\theta^{(j)}(x). \tag{2.12}$$

Naturally,

$$(u^{(j)},\ w^{(j)},\ \theta^{(j)})^\top=\Big[\prod_{j\neq q=1}^{3}\frac{\Delta+k_q^2}{k_q^2-k_j^2}\Big](u',\ w',\ \theta)^\top, \quad j=1,2,3. \tag{2.13}$$

From (2.10)–(2.11) and (2.13) we derive

$$\begin{gathered}(\Delta+k_j^2)u^{(j)}(x)=0,\ \ \operatorname{rot}u^{(j)}(x)=0,\ \ j=1,2,3,\\ (\Delta+k_j^2)w^{(j)}(x)=0,\ \ \operatorname{rot}w^{(j)}(x)=0,\ \ j=1,2,3,\\ (\Delta+k_j^2)\theta^{(j)}(x)=0,\ \ j=1,2,3.\end{gathered}\tag{2.14}$$

Since $\operatorname{div}u=\operatorname{div}u'$, $\operatorname{div}w=\operatorname{div}w'$, $\operatorname{rot}u'=0$, $\operatorname{rot}w'=0$, with the help of (2.14) and the identity

$$\operatorname{grad}\operatorname{div}u'=\Delta u'+\operatorname{rot}\operatorname{rot}u'=\Delta u',\ \ \operatorname{grad}\operatorname{div}w'=\Delta w',$$

from (2.8) and (2.3) we get

$$\big[\rho\sigma^2-(\lambda+2\mu)k_j^2\big]u^{(j)}(x)-\gamma\operatorname{grad}\theta^{(j)}(x)=0,\tag{2.15}$$

$$(\tau-lk_j^2)w^{(j)}(x)-\varkappa_3\operatorname{grad}\theta^{(j)}(x)=0,\tag{2.16}$$

$$i\sigma\gamma T_0\operatorname{div}u^{(j)}(x)+\varkappa_1\operatorname{div}w^{(j)}(x)+(i\sigma aT_0-\varkappa k_j^2)\theta^{(j)}(x)=0,\quad j=1,2,3.\tag{2.17}$$

From (2.15) and (2.16) we have

$$w^{(j)}(x)=\alpha_j u^{(j)}(x),\ \ j=1,2,3,\tag{2.18}$$

where

$$\alpha_j=\frac{\varkappa_3[\rho\sigma^2-(\lambda+2\mu)k_j^2]}{\gamma(\tau-lk_j^2)},\ \ j=1,2,3.$$

If we substitute the expressions of $w^{(j)}(x)$ from (2.18) into (2.17), we get

$$\theta^{(j)}(x)=\beta_j\operatorname{div}u^{(j)}(x),\ \ j=1,2,3,\tag{2.19}$$

where

$$\beta_j=\frac{i\sigma\gamma T_0+\varkappa_1\alpha_j}{\varkappa k_j^2-i\sigma aT_0},\ \ j=1,2,3.$$

Substitute the expressions of $w^{(j)}(x)$ and $\theta^{(j)}(x)$, $j=1,2,3$, given by (2.18)–(2.19) into (2.12) to obtain

$$\begin{gathered}u'(x)=\sum_{j=1}^3u^{(j)}(x),\ \ w'(x)=\sum_{j=1}^3\alpha_ju^{(j)}(x),\\ \theta(x)=\sum_{j=1}^3\beta_j\operatorname{div}u^{(j)}(x),\ \ \operatorname{rot}u^{(j)}(x)=0,\ \ j=1,2,3.\end{gathered}\tag{2.20}$$

On the other hand, since $\operatorname{rot}u=\operatorname{rot}u''$, $\operatorname{rot}w=\operatorname{rot}w''$, $\operatorname{div}u''=0$, $\operatorname{div}w''=0$ and $\operatorname{rot}\operatorname{rot}u''=-\Delta u''$, $\operatorname{rot}\operatorname{rot}w''=-\Delta w''$, from (2.9) we get

$$\begin{gathered}(\Delta+k_4^2)u''(x)=0,\quad \operatorname{div}u''(x)=0,\\ (\Delta+k_5^2)w''(x)=0,\quad \operatorname{div}w''(x)=0,\end{gathered}\tag{2.21}$$

where $k_4^2=\rho\sigma^2/\mu$, $k_5^2=\tau/\varkappa_6$.

The following lemmas are valid [3, 12].

Lemma 2.2. *If a vector $v = (v_1, v_2, v_3)^\top$ in the domain $\Omega \subset \mathbb{R}^3$ satisfies the following system of differential equations*

$$(\Delta + k^2)v(x) = 0, \quad \operatorname{rot} v(x) = 0,$$

then v can be represented as

$$v(x) = \operatorname{grad} \Phi(x),$$

where $\Phi(x)$ is a solution of the Helmholtz equation $(\Delta + k^2)\Phi(x) = 0$; here k is an arbitrary constant.

Lemma 2.3. *If a vector $v = (v_1, v_2, v_3)^\top$ in the domain $\Omega \subset \mathbb{R}^3$ satisfies the following system of differential equations*

$$(\Delta + k^2)v(x) = 0, \quad \operatorname{div} v(x) = 0,$$

then v can be represented as

$$v(x) = \operatorname{rot}\operatorname{rot}(x\Psi_1(x)) + \operatorname{rot}(x\Psi_2(x)),$$

where $\Psi_j(x)$, $j = 1, 2$, are solutions of the Helmholtz equation $(\Delta + k^2)\Psi_j(x) = 0$, $j = 1, 2$; here k is an arbitrary constant.

Due to Lemma 2.2 and Lemma 2.3, a solution of systems (2.14) and (2.21) can be represented as

$$\begin{aligned} u'(x) &= \operatorname{grad} \Phi_j(x), \quad j = 1, 2, 3, \\ u''(x) &= \operatorname{rot}\operatorname{rot}(x\Phi_4(x)) + \operatorname{rot}(x\Phi_5(x)), \\ w''(x) &= \operatorname{rot}\operatorname{rot}(x\Phi_6(x)) + \operatorname{rot}(x\Phi_7(x)), \end{aligned} \tag{2.22}$$

where

$$(\Delta + k_j^2)\Phi_j(x) = 0, \ \ j = 1, 2, 3, \quad (\Delta + k_4^2)\Phi_j(x) = 0, \ \ j = 4, 5,$$
$$(\Delta + k_5^2)\Phi_j(x) = 0, \ \ j = 6, 7.$$

Substitution of the expressions (2.22) into (2.20) proves the first part of the theorem. As to the second part, it is proved by a straightforward verification that the vector $U = (u, w, \theta)^\top$ represented in the form (2.5) is a solution of system (2.1)–(2.3). □

Remark 2.4. Hereinafter, we will assume that $k_j \neq k_p$, $j \neq p$, $\Im k_j > 0$, $j = 1, 2, 3, 4, 5$.

Let $\Omega^+ = B(R) \subset \mathbb{R}^3$ be a ball with center at the origin, of radius R, and $\Sigma_R = \partial\Omega$. We denote $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

Theorem 2.5. *A vector $U = (u, w, \theta)^\top$ represented by (2.5) will be uniquely defined in the area Ω^+ by the functions $\Phi_j(x)$, $j = 1, 2, \dots, 7$, if the following conditions are fulfilled:*

$$\int\limits_{\Sigma_r} \Phi_j(x)\, d\Sigma_r = 0, \ \ j = 4, 5, 6, 7, \ \ r = |x| < R. \tag{2.23}$$

Proof. From formulas (2.5) we get

$$\sum_{j=1}^{3} k_j^2 \Phi_j(x) = -\operatorname{div} u, \quad \sum_{j=1}^{3} \alpha_j k_j^2 \Phi_j(x) = -\operatorname{div} w,$$

$$\sum_{j=1}^{3} \beta_j k_j^2 \Phi_j(x) = -\theta(x),$$

$$r^2\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} + k_{3+j}^2\Big)\Phi_{3+2j}(x) = x\cdot(\delta_{1j}\operatorname{rot} u + \delta_{2j}\operatorname{rot} w), \quad j=1,2,$$

$$r^2\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} + k_{3+j}^2\Big)\Phi_{2+2j}(x) = \\ = -\frac{1}{k_{3+j}^2}\, x\cdot(\delta_{1j}\operatorname{rot}\operatorname{rot} u + \delta_{2j}\operatorname{rot}\operatorname{rot} w), \quad j=1,2,$$

δ_{lj} is the Kronecker function.

If $u(x)=0$, $w(x)=0$, $\theta(x)=0$, $x\in\Omega^+$, we have $\Phi_j(x)=0$, $j=1,2,3$, $x\in\Omega^+$,

$$\begin{aligned} &r^2\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} + k_4^2\Big)\Phi_j(x) = 0, \quad j=4,5, \quad x\in\Omega^+, \\ &r^2\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} + k_5^2\Big)\Phi_j(x) = 0, \quad j=6,7, \quad x\in\Omega^+. \end{aligned} \tag{2.24}$$

Thus it remains to show that $\Phi_j(x)=0$, $j=4,5,6,7$. Applying the well known representation of metaharmonic functions in the form of series, we can write

$$\Phi_j(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(k_l r) A_{mk}^{(j)} Y_k^{(m)}(\vartheta,\varphi), \quad j=4,5,6,7, \quad x\in\Omega^+,$$

where $A_{mk}^{(j)}$ are constants, $Y_k^{(m)}(\vartheta,\varphi)$ is a spherical function

$$Y_k^{(m)}(\vartheta,\varphi) = \sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\; P_k^{(m)}(\cos\vartheta)e^{im\varphi},$$

$P_k^{(m)}(\cos\vartheta)$ is the associated Legendre polinomial of the first kind of degree k and order m,

$$g_k(k_l r) = r^{-1/2}\mathcal{J}_{k+1/2}(k_l r), \quad k_l = \begin{cases} k_4, & j=4,5, \\ k_5, & j=6,7, \end{cases}$$

$\mathcal{J}_{k+1/2}(k_l r)$ are the Bessel functions.

With the help of the equality

$$\Big(\frac{d^2}{dr^2} + \frac{2}{r}\,\frac{d}{dr} + k_l^2\Big) g_k(k_l r) = \frac{k(k+1)}{r^2} g_k(k_l r),$$

from (2.24) we get

$$\sum_{k=0}^{\infty}\sum_{m=-k}^{k} k(k+1)g_k(k_l r)A_{mk}^{(j)}Y_k^{(m)}(\vartheta,\varphi)=0, \quad j=4,5,6,7,$$

whence the equations $A_{mk}^{(j)}=0$ follow for $k\geq 1$ and $j=4,5,6,7$. Therefore

$$\Phi_j(x)=\frac{1}{2\sqrt{\pi}}g_0(k_l r)A_{00}^{(j)}, \quad j=4,5,6,7.$$

Further, from (2.23) we easily conclude $A_{00}^{(j)}=0$ for $j=4,5,6,7$, which completes the proof. □

3. Orthonormal System of Spherical Vectors

Let r, ϑ, φ ($0\leq r<+\infty$, $0\leq\vartheta\leq\pi$, $0\leq\varphi<2\pi$) be the spherical coordinates of $x\in\mathbb{R}^3$. Denote by Σ_1 the unit sphere.

In the space $L_2(\Sigma_1)$ consider the following complete orthonormal vectors system (see [2,12,17])

$$\begin{aligned}
&X_{mk}(\vartheta,\varphi)=e_r Y_k^{(m)}(\vartheta,\varphi), \quad k\geq 0,\\
&Y_{mk}(\vartheta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\Big(e_\vartheta\frac{\partial}{\partial\vartheta}+\frac{e_\varphi}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}\Big)Y_k^{(m)}(\vartheta,\varphi), \quad k\geq 1,\\
&Z_{mk}(\vartheta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\Big(\frac{e_\vartheta}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}-e_\varphi\,\frac{\partial}{\partial\vartheta}\Big)Y_k^{(m)}(\vartheta,\varphi), \quad k\geq 1,
\end{aligned}\tag{3.1}$$

where $|m|\leq k$, e_r, e_ϑ, e_φ are the orthonormal vectors in $\mathbb{R}^3$,

$$\begin{aligned}
e_r&=(\cos\varphi\sin\vartheta,\sin\varphi\sin\vartheta,\cos\vartheta)^\top,\\
e_\vartheta&=(\cos\varphi\cos\vartheta,\sin\varphi\cos\vartheta,-\sin\vartheta)^\top,\\
e_\varphi&=(-\sin\varphi,\cos\varphi,0)^\top,\\
Y_k^{(m)}(\vartheta,\varphi)&=\sqrt{\frac{2k+1}{4\pi}\,\frac{(k-m)!}{(k+m)!}}P_k^{(m)}(\cos\theta)e^{im\varphi},
\end{aligned}$$

$P_k^{(m)}(\cos\vartheta)$ is the adjoint Legendre function.

Let us assume that a vector-function $f^{(j)}=(f_1^{(j)},\ f_2^{(j)},\ f_3^{(j)})^\top$ and a function f_4 are represented as

$$\begin{aligned}
f^{(j)}(\vartheta,\varphi)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}&\Big\{\alpha_{mk}X_{mk}(\vartheta,\varphi)+\\
&+\sqrt{k(k+1)}\Big[\beta_{mk}^{(j)}Y_{mk}(\vartheta,\varphi)+\gamma_{mk}^{(j)}Z_{mk}(\vartheta,\varphi)\Big]\Big\},
\end{aligned}\tag{3.2}$$

$$f_4(\vartheta,\varphi)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\alpha_{mk}Y_k^m(\vartheta,\varphi),\tag{3.3}$$

where

$$\begin{aligned}
\alpha_{mk}^{(j)} &= \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} f^{(j)}(\vartheta,\varphi)\cdot \overline{X}_{mk}(\vartheta,\varphi)\sin\vartheta\, d\vartheta, \quad k\geq 0,\\
\beta_{mk}^{(j)} &= \frac{1}{\sqrt{k(k+1)}}\int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} f^{(j)}(\vartheta,\varphi)\cdot \overline{Y}_{mk}(\vartheta,\varphi)\sin\vartheta\, d\vartheta, \quad k\geq 1,\\
\gamma_{mk}^{(j)} &= \frac{1}{\sqrt{k(k+1)}}\int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} f^{(j)}(\vartheta,\varphi)\cdot \overline{Z}_{mk}(\vartheta,\varphi)\sin\vartheta\, d\vartheta, \quad k\geq 1,\\
\alpha_{mk} &= \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} f_4(\vartheta,\varphi)\cdot \overline{Y}_k^{(m)}(\vartheta,\varphi)\sin\vartheta\, d\vartheta, \quad k\geq 0.
\end{aligned} \tag{3.4}$$

The symbol $a\cdot\overline{b}$ denotes the scalar product of two vectors, $\overline{b}$ is complex conjugate of b.

Note that in formula (3.2) and, in the sequel, in the summands of analogous series, which contain the vectors $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$, the summation index k varies from 1 to $+\infty$.

Let us introduce a few important lemmas [3, 11].

Lemma 3.1. *Let $f^{(j)}\in C^l(\Sigma_1)$, $l\geq 1$; then the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$ defined by* (3.4) *admit the following estimates*

$$\alpha_{mk}^{(j)} = O(k^{-l}), \quad \beta_{mk}^{(j)} = O(k^{-l-1}), \quad \gamma_{mk}^{(j)} = O(k^{-l-1}).$$

Lemma 3.2. *Let $f_4\in C^l(\Sigma_1)$, $l\geq 1$; then the coefficients α_{mk} defined by* (3.4) *admit the following estimates*

$$\alpha_{mk} = O(k^{-l}).$$

Lemma 3.3. *The vectors $X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$ defined by equalities* (3.1) *admit the estimates:*

$$\begin{aligned}
\big|X_{mk}(\vartheta,\varphi)\big| &\leq \sqrt{\frac{2k+1}{4\pi}}\,, \quad k\geq 0,\\
\big|Y_{mk}(\vartheta,\varphi)\big| &< \sqrt{\frac{k(k+1)}{2k+1}}\,, \quad k\geq 1,\\
\big|Z_{mk}(\vartheta,\varphi)\big| &< \sqrt{\frac{k(k+1)}{2k+1}}\,, \quad k\geq 1,
\end{aligned}$$

Hereinafter we make use the following equalities [6]

$$\begin{gathered}
e_r\cdot X_{mk}(\vartheta,\varphi) = Y_k^{(m)}(\vartheta,\varphi), \quad e_r\cdot Y_{mk}(\vartheta,\varphi)=0,\\
e_r\cdot Z_{mk}(\vartheta,\varphi)=0,\\
e_r\times X_{mk}(\vartheta,\varphi)=0, \quad e_r\times Y_{mk}(\vartheta,\varphi) = -Z_{mk}(\vartheta,\varphi),\\
e_r\times Z_{mk}(\vartheta,\varphi) = Y_{mk}(\vartheta,\varphi);
\end{gathered} \tag{3.5}$$

$$
\begin{gathered}
\operatorname{grad}\big[a(r)Y_k^{(m)}(\vartheta,\varphi)\big]=\\
=\frac{da(r)}{dr}\,X_{mk}(\vartheta,\varphi)+\frac{\sqrt{k(k+1)}}{r}\,a(r)Y_{mk}(\vartheta,\varphi),\\
\operatorname{rot}\big[xa(r)Y_k^{(m)}(\vartheta,\varphi)\big]=\sqrt{k(k+1)}\,a(r)Z_{mk}(\vartheta,\varphi),\\
\operatorname{rot}\operatorname{rot}\big[xa(r)Y_k^{(m)}(\vartheta,\varphi)\big]=\\
=\frac{k(k+1)}{r}\,a(r)X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi),
\end{gathered}
\tag{3.6}
$$

$$
\begin{aligned}
\operatorname{div}\big[a(r)X_{mk}(\vartheta,\varphi)\big]&=\Big(\frac{d}{dr}+\frac{2}{r}\Big)a(r)Y_k^{(m)}(\vartheta,\varphi),\\
\operatorname{div}\big[a(r)Y_{mk}(\vartheta,\varphi)\big]&=-\frac{\sqrt{k(k+1)}}{r}\,a(r)Y_k^{(m)}(\vartheta,\varphi),\\
\operatorname{div}\big[a(r)Z_{mk}(\vartheta,\varphi)\big]&=0,\\
\operatorname{rot}\big[a(r)X_{mk}(\vartheta,\varphi)\big]&=\frac{\sqrt{k(k+1)}}{r}\,a(r)Z_{mk}(\vartheta,\varphi),\\
\operatorname{rot}\big[a(r)Y_{mk}(\vartheta,\varphi)\big]&=-\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Z_{mk}(\vartheta,\varphi),\\
\operatorname{rot}\big[a(r)Z_{mk}(\vartheta,\varphi)\big]&=\frac{\sqrt{k(k+1)}}{r}\,a(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi).
\end{aligned}
\tag{3.7}
$$

4. Statement of the Problem. The Uniqueness Theorem

Problem. Find, in the domain Ω^+, a regular vector $U=(u,w,\theta)^\top$ satisfying in this domain the system of differential equations (2.1)–(2.3) and, on the boundary $\partial\Omega$, one of the following boundary conditions:

$(\mathrm{I}^{(\sigma)})^+$ (the Dirichlet problem)

$$
\{u(z)\}^+=f^{(1)}(z),\quad \{w(z)\}^+=f^{(2)}(z),\quad \{\theta(z)\}^+=f_4(z);
$$

$(\mathrm{II}^{(\sigma)})^+$ (the Neumann problem)

$$
\begin{gathered}
\big\{P^{(1)}(\partial,n)U'(z)\big\}^+=f^{(1)}(z),\quad \big\{P^{(2)}(\partial,n)U''(z)\big\}^+=f^{(2)}(z),\\
\big\{P^{(3)}(\partial,n)U''(z)\big\}^+=f_4(z),
\end{gathered}
\tag{4.1}
$$

where the vectors $f^{(j)}=(f_1^{(j)},f_2^{(j)},f_3^{(j)})$, $j=1,2$, and the function f_4 are given on the boundary $\partial\Omega$, $n(z)$ is the outward normal unit vector at the point $z\in\partial\Omega$.

Theorem 4.1. *Problems $(\mathrm{I}^{(\sigma)})^+$ and $(\hat{\mathrm{II}}^{(\sigma)})^+$ have, in the domain Ω^+, a unique solution in the class of regular functions.*

Proof. The theorem will be proved if we show that the homogeneous problems ($f^{(j)}=0$, $j=1,2$, $f_4=0$) have only the trivial solution.

Let the vector $U=(u,w,\theta)^\top$ be a solution of the homogeneous problem either $(\mathrm{I}^{(\sigma)})^+$ or $(\mathrm{II}^{(\sigma)})^+$. We multiply both sides of (2.1) by the vector $i\,\overline{\sigma}\,T_0\,\overline{u}$, (2.2) by $\overline{w}$ and the complex-conjugate of (2.3) by the function θ. The integration of these expressions over the domain Ω^+ and summation give

$$\int\limits_{\partial\Omega}\Big[i\overline{\sigma}T_0\overline{u}(z)\cdot P^{(1)}(\partial,n)U'(z)+ + \overline{w}(z)\cdot P^{(2)}(\partial,n)U''(z)+\theta(z)\cdot P^{(3)}(\partial,n)\overline{U}''(z)\Big]^+\,ds- - \int\limits_{\Omega^+}\Big[i\overline{\sigma}T_0E^{(1)}(u,\overline{u})-i\rho\sigma|\sigma|^2|u(x)|^2+E^{(2)}(w,\overline{w})-\tau|w(x)|^2+ + \varkappa|\operatorname{grad}\theta(x)|^2+(\varkappa_1+\varkappa_3)\overline{w}(x)\cdot\operatorname{grad}\theta(x)+i\overline{\sigma}aT_0|\theta(x)|^2\Big]\,dx=0, \tag{4.2}$$

where [9, 15]

$$E^{(1)}(u,\overline{u})=\frac{3\lambda+2\mu}{3}\,|\operatorname{div}u|^2+\frac{\mu}{2}\sum_{k\neq j=1}^{3}\Big|\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\Big|^2+ + \frac{\mu}{3}\sum_{k,j=1}^{3}\Big|\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}\Big|^2,$$

$$E^{(2)}(w,\overline{w})=\frac{3\varkappa_4+\varkappa_5+\varkappa_6}{3}\,|\operatorname{div}w|^2+\frac{\varkappa_6-\varkappa_5}{2}|\operatorname{rot}w|^2+ + \frac{\varkappa_5+\varkappa_6}{4}\sum_{k\neq j=1}^{3}\Big|\frac{\partial w_k}{\partial x_j}+\frac{\partial w_j}{\partial x_k}\Big|^2+\frac{\varkappa_5+\varkappa_6}{6}\sum_{k,j=1}^{3}\Big|\frac{\partial w_k}{\partial x_k}-\frac{\partial w_j}{\partial x_j}\Big|^2.$$

Since $U=(u,w,\theta)^\top$ is a solution of the homogeneous problem, equality (4.2) implies

$$\int\limits_{\Omega^+}\Big[i\overline{\sigma}T_0E^{(1)}(u,\overline{u})-i\rho\sigma|\sigma|^2|u(x)|^2+E^{(2)}(w,\overline{w})-\tau|w(x)|^2+ + \varkappa|\operatorname{grad}\theta(x)|^2+(\varkappa_1+\varkappa_3)\overline{w}(x)\cdot\operatorname{grad}\theta(x)+i\overline{\sigma}aT_0|\theta(x)|^2\Big]\,dx=0.$$

If in this equality we separate the real part, we will get

$$\int\limits_{\Omega^+}\Big[\sigma_2T_0E^{(1)}(u,\overline{u})+E^{(2)}(w,\overline{w})+\rho\sigma_2|\sigma|^2|u(x)|^2+\sigma_2\delta|w(x)|^2+ + aT_0\sigma_2|\theta(x)|^2+\frac{4\varkappa\varkappa_2-(\varkappa_1+\varkappa_3)^2}{4\varkappa}|w(x)|^2+ + \frac{1}{4\varkappa}\,\big|(\varkappa_1+\varkappa_3)w(x)+2\varkappa\operatorname{grad}\theta(x)\big|^2\Big]\,dx=0.$$

Hence it follows that $u(x)=0$, $w(x)=0$, $\theta(x)=0$, $x\in\Omega^+$. □

5. Solution of the Boundary Value Problems

We seek a solution of the Dirichlet and Neumann Problems by formulas (2.5), where

$$\begin{aligned}
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(k_j r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)}, \quad j=1,2,3,\\
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(k_4 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)}, \quad j=4,5, \qquad (5.1)\\
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(k_5 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)}, \quad j=6,7.
\end{aligned}$$

Here $A_{mk}^{(j)}$, $j=\overline{1,7}$, are the sought constants, $Y_k^{(m)}(\vartheta,\varphi)$ is a spherical function and

$$g_k(k_j r)=\sqrt{\frac{R}{r}}\,\frac{\mathcal{J}_{k+\frac12}(k_j r)}{\mathcal{J}_{k+\frac12}(k_j R)}\,,$$

$\mathcal{J}_{k+\frac12}(x)$ is a Bessel function.

Substituting the expressions of $\Phi_j(x)$ $j=4,5,6,7$, from (5.1), into (2.23), we get $A_{00}^{(j)}=0$, $j=4,5,6,7$. If we substitute the expressions of the functions $\Phi_j(x)$ $j=\overline{1,7}$, from (2.5) and take into account equalities (3.6), we obtain

$$\begin{aligned}
u(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(1)}(r)X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}\Big[v_{mk}^{(1)}(r)Y_{mk}(\vartheta,\varphi)+w_{mk}^{(1)}(r)Z_{mk}(\vartheta,\varphi)\Big]\Big\},\\
w(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(2)}(r)X_{mk}(\vartheta,\varphi)+ \qquad (5.2)\\
&\quad+\sqrt{k(k+1)}\Big[v_{mk}^{(2)}(r)Y_{mk}(\vartheta,\varphi)+w_{mk}^{(2)}(r)Z_{mk}(\vartheta,\varphi)\Big]\Big\},\\
\theta(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} u_{mk}(r)Y_k^{(m)}(\vartheta,\varphi),
\end{aligned}$$

where

$$\begin{aligned}
u_{mk}^{(1)}(r) &= \sum_{j=1}^{3}\frac{d}{dr}\,g_k(k_j r)A_{mk}^{(j)}+\frac{k(k+1)}{r}\,g_k(k_4 r)A_{mk}^{(4)}, \quad k\geq 0,\\
v_{mk}^{(1)}(r) &= \sum_{j=1}^{3}\frac{1}{r}\,g_k(k_j r)A_{mk}^{(j)}+\Big(\frac{d}{dr}+\frac{1}{r}\Big)g_k(k_4 r)A_{mk}^{(4)}, \quad k\geq 1,\\
w_{mk}^{(1)}(r) &= g_k(k_4 r)A_{mk}^{(5)}, \quad k\geq 1,
\end{aligned}$$

$$
\begin{aligned}
u^{(2)}_{mk}(r) &= \sum_{j=1}^{3} \alpha_j \frac{d}{dr}\, g_k(k_j r) A^{(j)}_{mk} + \frac{k(k+1)}{r}\, g_k(k_5 r) A^{(6)}_{mk}, \quad k \ge 0,\\
v^{(2)}_{mk}(r) &= \sum_{j=1}^{3} \alpha_j \frac{1}{r}\, g_k(k_j r) A^{(j)}_{mk} + \Big(\frac{d}{dr} + \frac{1}{r}\Big) g_k(k_5 r) A^{(6)}_{mk}, \quad k \ge 1,\\
w^{(2)}_{mk}(r) &= g_k(k_5 r) A^{(7)}_{mk}, \quad k \ge 1,\\
u_{mk}(r) &= -\sum_{j=1}^{3} \beta_j k_j^2 g_k(k_j r) A^{(j)}_{mk}, \quad k \ge 0.
\end{aligned}
$$

If we substitute the expressions of the vectors $u(x)$, $w(x)$ and the function $\theta(x)$ into (2.4) and use equalities (3.5) and (3.7), we get

$$
\begin{aligned}
P^{(1)}(\partial, n) U'(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ a^{(1)}_{mk}(r) X_{mk}(\vartheta, \varphi) + \sqrt{k(k+1)} \times\\
&\qquad \times \Big[b^{(1)}_{mk}(r) Y_{mk}(\vartheta, \varphi) + c^{(1)}_{mk}(r) Z_{mk}(\vartheta, \varphi) \Big] \Big\},\\
P^{(2)}(\partial, n) U''(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ a^{(2)}_{mk}(r) X_{mk}(\vartheta, \varphi) + \sqrt{k(k+1)} \times\\
&\qquad \times \Big[b^{(2)}_{mk}(r) Y_{mk}(\vartheta, \varphi) + c^{(2)}_{mk}(r) Z_{mk}(\vartheta, \varphi) \Big] \Big\},\\
P^{(3)}(\partial, n) U''(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} a_{mk}(r) Y^{(m)}_k(\vartheta, \varphi),
\end{aligned} \tag{5.3}
$$

where

$$
\begin{aligned}
a^{(1)}_{mk}(r) &= \sum_{j=1}^{3} \Big[2\mu \frac{d^2}{dr^2} + (\gamma\beta_j - \lambda) k_j^2 \Big] g_k(k_j r) A^{(j)}_{mk} +\\
&\qquad + \frac{2\mu k(k+1)}{r} \Big(\frac{d}{dr} - \frac{1}{r}\Big) g_k(k_4 r) A^{(4)}_{mk}, \quad k \ge 0,\\
b^{(1)}_{mk}(r) &= \sum_{j=1}^{3} 2\mu \frac{1}{r} \Big(\frac{d}{dr} - \frac{1}{r}\Big) g_k(k_j r) A^{(j)}_{mk} +\\
&\qquad + \mu \Big[2\frac{d}{r} \Big(\frac{d}{dr} + \frac{1}{r}\Big) + k_4^2 \Big] g_k(k_4 r) A^{(4)}_{mk}, \quad k \ge 1,\\
c^{(1)}_{mk}(r) &= \mu \Big(\frac{d}{dr} - \frac{1}{r}\Big) g_k(k_4 r) A^{(5)}_{mk}, \quad k \ge 1,\\
a^{(2)}_{mk}(r) &= \sum_{j=1}^{3} \Big[(\varkappa_5 + \varkappa_6)\alpha_j \frac{d^2}{dr^2} + (\varkappa_3\beta_j - \varkappa_4\alpha_j) k_j^2 \Big] g_k(k_j r) A^{(j)}_{mk} +\\
&\qquad + \frac{(\varkappa_5 + \varkappa_6) k(k+1)}{r} \Big(\frac{d}{dr} - \frac{1}{r}\Big) g_k(k_5 r) A^{(6)}_{mk}, \quad k \ge 0,
\end{aligned}
$$

$$b^{(2)}_{mk}(r)=\sum_{j=1}^{3}\alpha_j(\varkappa_5+\varkappa_6)\,\frac{1}{r}\Big(\frac{d}{dr}-\frac{1}{r}\Big)g_k(k_jr)A^{(j)}_{mk}+$$

$$+\Big[(\varkappa_5+\varkappa_6)\,\frac{d}{dr}\Big(\frac{d}{dr}+\frac{1}{r}\Big)+\varkappa_5k_5^2\Big]g_k(k_5r)A^{(6)}_{mk},\quad k\geq 1,$$

$$c^{(2)}_{mk}(r)=\Big(\varkappa_6\,\frac{d}{dr}-\varkappa_5\,\frac{1}{r}\Big)g_k(k_5r)A^{(7)}_{mk},\quad k\geq 1,$$

$$a_{mk}(r)=\sum_{j=1}^{3}\big(\alpha_j(\varkappa_1+\varkappa_3)-\varkappa\beta_jk_j^2\big)\,\frac{d}{dr}\,g_k(k_jr)A^{(j)}_{mk}+$$

$$+\frac{(\varkappa_1+\varkappa_3)k(k+1)}{r}\,g_k(k_5r)A^{(6)}_{mk},\quad k\geq 0.$$

Let us first consider the Neumann problem.

Assume that the vectors $f^{(j)}(\vartheta,\varphi)$, $j=1,2$, and the function $f_4(\vartheta,\varphi)$ can be represented in the form (3.2) and (3.3).

Passing to the limit on both sides of (5.3) as $x\to z\in\partial\Omega$ and using both the Neumann boundary conditions (4.1) and equalities (3.2)–(3.3), for the sought constants $A^{(j)}_{mk}$, $j=\overline{1,7}$, we obtain the following system of linear algebraic equations:

(1) for $k=0$, $m=0$ (three simultaneous equations with the three unknowns $A^{(j)}_{00}$, $j=1,2,3$),

$$a^{(1)}_{00}(R)=\alpha^{(1)}_{00},\quad a^{(2)}_{00}(R)=\alpha^{(2)}_{00},\quad a_{00}(R)=\alpha_{00};\tag{5.4}$$

(2) for $k\geq 1$, $-k\leq m\leq k$

(a)

$$\begin{gathered}\mu\Big(\frac{d}{dR}-\frac{1}{R}\Big)g_k(k_4R)A^{(5)}_{mk}=\gamma^{(1)}_{mk},\\ \Big(\varkappa_6\,\frac{d}{dR}-\varkappa_5\,\frac{1}{R}\Big)g_k(k_5R)A^{(7)}_{mk}=\gamma^{(2)}_{mk};\end{gathered}\tag{5.5}$$

(b) (five simultaneous equations with the five unknowns $A^{(j)}_{mk}$, $j=1,2,3,4,6$)

$$a^{(j)}_{mk}(R)=\alpha^{(1)}_{mk},\quad b^{(j)}_{mk}(R)=\beta^{(j)}_{mk},\quad j=1,2,\quad a_{mk}(R)=\alpha_{mk}.\tag{5.6}$$

Due to Theorems 4.1 and 2.5, system (5.4)–(5.6) is uniquely solvable with respect to the unknowns $A^{(j)}_{mk}$, $j=\overline{1,7}$. Thus we can construct explicitly a formal solution of the Neumann problem in the form of series. Further we have to investigate the convergence of these formal series and their derivatives.

The asymptotic representations

$$g_k(k_jr)\approx\Big(\frac{r}{R}\Big)^k,\quad g'_k(k_jr)\approx\frac{k}{r}\Big(\frac{r}{R}\Big)^k,\quad r<R\tag{5.7}$$

are valid for $k\to+\infty$ [16].

If $x \in \Omega^+$ $(r < R)$, then by asymptotics (5.7), the series (5.2)–(5.3) converge absolutely and uniformly.

If $x \in \partial\Omega$ $(r = R)$, then by Lemmas 3.1–3.3 and asymptotics (5.7), series (5.2)–(5.3) will be absolutely and uniformly convergent provided that the majorized series

$$\sum_{k=k_0}^{\infty} k^{3/2} \sum_{j=1}^{2} \Big(|\alpha_{mk}^{(j)}| + k|\beta_{mk}^{(j)}| + k|\gamma_{mk}^{(j)}| + |\alpha_{mk}| \Big) \tag{5.8}$$

are convergent. Series (5.8) will be convergent if the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, α_{mk}, $j = 1, 2$, admit the following estimates

$$\begin{gathered} \alpha_{mk}^{(j)} = O(k^{-3}), \quad \beta_{mk}^{(j)} = O(k^{-4}), \\ \gamma_{mk}^{(j)} = O(k^{-4}), \quad \alpha_{mk} = O(k^{-3}), \ \ j = 1, 2. \end{gathered} \tag{5.9}$$

According to Lemmas 3.1 and 3.2, estimates (5.9) will hold if we require that

$$f^{(j)}(z) \in C^3(\partial\Omega), \ \ j = 1, 2, \ \ f_4(z) \in C^3(\partial\Omega). \tag{5.10}$$

Therefore if the boundary vector-functions satisfy conditions (5.10), then the vector $U = (u, w, \theta)^\top$ represented by equalities (5.2) will be a regular solution of Problem $(II^{(\sigma)})^+$.

Problem $(I^{(\sigma)})^+$ can be treated analogously.

6. Appendix: Properties of the Characteristic Roots and Wave Numbers

Let us introduce the blockwise 7×7 matrix differential operator corresponding to system (2.1)–(2.3)

$$L(\partial, \sigma) := \begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(2)}(\partial,\sigma) & L^{(5)}(\partial,\sigma) \\ L^{(3)}(\partial,\sigma) & L^{(4)}(\partial,\sigma) & L^{(6)}(\partial,\sigma) \\ L^{(7)}(\partial,\sigma) & L^{(8)}(\partial,\sigma) & L^{(9)}(\partial,\sigma) \end{bmatrix}_{7\times 7},$$

where

$$\begin{gathered} L^{(1)}(\partial,\sigma) := [\mu\Delta + \rho\sigma^2] I_3 + (\lambda + \mu) Q(\partial), \\ L^{(2)}(\partial,\sigma) := L^{(3)}(\partial,\sigma) = [O]_{3\times 3}, \\ L^{(4)}(\partial,\sigma) := [\varkappa_6 \Delta + \tau] I_3 + (\varkappa_4 + \varkappa_5) Q(\partial), \\ L^{(5)}(\partial,\sigma) := -\gamma \nabla^\top, \quad L^{(6)}(\partial,\sigma) := -\varkappa_3 \nabla^\top, \quad L^{(7)}(\partial,\sigma) := i\sigma\gamma T_0 \nabla, \\ L^{(8)}(\partial,\sigma) := \varkappa_1 \nabla, \quad L^{(9)}(\partial,\sigma) := \varkappa \nabla + i\sigma a T_0, \quad Q(\partial) = [\partial_k \partial_j]_{3\times 3}, \end{gathered}$$

$\nabla = \nabla(\partial) = [\partial_1, \partial_2, \partial_3]$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$, I_3 stands for the 3×3 unit matrix.

Due to the above notation, system (2.1)–(2.3) can be rewritten in matrix form as

$$L(\partial, \sigma) U(x) = 0, \quad U = (u, w, \theta)^\top. \tag{6.1}$$

Denote by $\mathfrak{F}_{x\to\xi}$ the Fourier transforms

$$\mathfrak{F}_{x\to\xi}[f]=\int\limits_{R^3} f(x)e^{ix\cdot\xi}\,dx=\widehat{f}(\xi),$$

where $x=(x_1,x_2,x_3)$, $\xi=(\xi_1,\xi_2,\xi_3)$.

The Fourier transform has the following property:

$$L(\partial^\alpha f)=(-i\xi)^\alpha\mathfrak{F}[f], \tag{6.2}$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $\xi^\alpha=\xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\,\xi_3^{\alpha_3}$.

Let us perform Fourier transforms of (6.1) and take into consideration (6.2); we obtain

$$L(-i\xi,\sigma)\widehat{U}(\xi)=0, \tag{6.3}$$

where

$$\begin{gathered}
L^{(1)}(-i\xi,\sigma):=\big(-\mu|\xi|^2+\rho\sigma^2\big)I_3-(\lambda+\mu)Q(\xi),\\
L^{(2)}(-i\xi,\sigma):=L^{(3)}(-i\xi,\sigma)=[O]_{3\times3},\\
L^{(4)}(-i\xi,\sigma):=\big(-\varkappa_6|\xi|^2+\tau\big)I_3-(\varkappa_4+\varkappa_5)Q(\xi),\\
L^{(5)}(-i\xi,\sigma):=i\gamma\xi^\top,\ \ L^{(6)}(-i\xi,\sigma):=i\varkappa_3\xi^\top,\ \ L^{(7)}(-i\xi,\sigma):=\sigma\gamma T_0\xi,\\
L^{(8)}(-i\xi,\sigma):=-i\varkappa_1\xi,\ \ L^{(9)}(-i\xi,\sigma):=-\varkappa|\xi|^2+i\sigma aT_0,\\
Q(\xi)=[\xi_k\,\xi_j]_{3\times3}.
\end{gathered}$$

The determinant of system (6.3) reads as

$$\det L(-i\xi,\sigma)=$$

$$=\mu(\lambda+2\mu)l\varkappa_6\big(\mu|\xi|^2-\rho\sigma^2\big)^2(\varkappa_6|\xi|^2-\tau)^2\big(|\xi|^6-a_1|\xi|^4+a_2|\xi|^2-a_3\big),$$

where a_1, a_2, a_3 are given by (2.7), $l=\varkappa_4+\varkappa_5+\varkappa_6$.

The numbers k_j^2, $j=\overline{1,5}$, are the roots of the equation $\det L(-i\xi,\sigma)=0$ with respect to $|\xi|$.

Lemma 6.1. *Let us assume that $\sigma=\sigma_1+i\sigma_2$ is a complex parameter, where $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$. Then*

$$\det L(-i\xi,\sigma)\neq0$$

for arbitrary $\xi\in\mathbb{R}^3$.

Proof. We prove the lemma by contradiction. Let $\det L(-i\xi,\sigma)=0$, $\xi\in\mathbb{R}^3$. Then the system of equations $L(-i\xi,\sigma)X=0$ has a nontrivial solution. Denote this solution by $X=(X^{(1)},X^{(2)},X^{(3)})^\top$, where $X^{(j)}=(X_1^{(j)},X_2^{(j)},X_3^{(j)})^\top\in\mathbb{C}^3$ $j=1,2$, and $X^{(3)}\in\mathbb{C}$. Taking into consideration (6.3), the system $L(-i\xi,\sigma)=0$ can be rewritten as follows:

$$\Big[\big(\rho\sigma^2-\mu|\xi|^2\big)I_3-(\lambda+\mu)Q(\xi)\Big]X^{(1)}+i\gamma\xi^\top X^{(3)}=0, \tag{6.4}$$

$$\Big[\big(\tau-\varkappa_6|\xi|^2\big)I_3-(\varkappa_4+\varkappa_5)Q(\xi)\Big]X^{(2)}+i\varkappa_3\xi^\top X^{(3)}=0, \tag{6.5}$$

$$\sigma\gamma T_0(\xi\cdot X^{(1)})-i\varkappa_1(\xi\cdot X^{(2)})+(-\varkappa|\xi|^2+i\sigma aT_0)X^{(3)}=0. \qquad (6.6)$$

Assume that $|\xi|\neq 0$.

Let us multiply equation (6.4) by the vector $i\overline{\sigma}T_0\overline{X^{(1)}}$, equation (6.5) by $\overline{X}^{(2)}$ and the complex-conjugate of equation (6.6) by the function $X^{(3)}$ and add the obtained results. After simplification, we obtain

$$i\overline{\sigma}T_0\big(\rho\sigma^2-\mu|\xi|^2\big)|X^{(1)}|^2-i\overline{\sigma}T_0(\lambda+\mu)|\xi\cdot X^{(1)}|^2+$$
$$+\big(\tau-\varkappa_6|\xi|^2\big)|X^{(2)}|^2-(\varkappa_4+\varkappa_5)|\xi\cdot X^{(2)}|^2+$$
$$+i(\varkappa_1+\varkappa_3)(\xi\cdot\overline{X}^{(2)})X^{(3)}+\big(-\varkappa|\xi|^2-i\overline{\sigma}aT_0\big)|X^{(3)}|^2=0.$$

Recall that the central dot denotes the scalar product, $a\cdot b=\sum\limits_{j=1}^{3}a_jb_j$ for the vectors a and b. Let us separate the real part:

$$T_0\sigma_2\Big[\big(\rho|\sigma|^2+\mu|\xi|^2\big)|X^{(1)}|^2+(\lambda+\mu)|\xi\cdot X^{(1)}|^2\Big]+$$
$$+\big(\sigma_2\delta+\varkappa_6|\xi|^2\big)|X^{(2)}|^2+(\varkappa_4+\varkappa_5)|\xi\cdot X^{(2)}|^2+\sigma_2aT_0|X^{(3)}|^2+$$
$$+\frac{4\varkappa\varkappa_2-(\varkappa_1+\varkappa_3)^2}{4\varkappa}\,|X^{(2)}|^2+\frac{1}{4\varkappa}\Big|(\varkappa_1+\varkappa_3)X^{(2)}-2i\varkappa\xi X^{(3)}\Big|^2=0. \qquad (6.7)$$

Here we have used the following relation:

$$\varkappa|\xi|^2|X^{(3)}|^2-(\varkappa_1+\varkappa_3)\,\mathrm{Re}\,\big[i(\xi\cdot\overline{X}^{(2)})X^{(3)}\big]+\varkappa_2|X^{(2)}|^2=$$
$$=\frac{4\varkappa\varkappa_2-(\varkappa_1+\varkappa_3)^2}{4\varkappa}\,|X^{(2)}|^2+\frac{1}{4\varkappa}\Big|(\varkappa_1+\varkappa_3)X^{(2)}-2i\varkappa\xi X^{(3)}\Big|^2\geq 0.$$

From equation (6.7) we obtain that $X^{(j)}=0$, $j=1,2,3$. For $\xi=0$ equation (6.7) recasts as

$$\rho|\sigma|^2\sigma_2T_0|X^{(1)}|^2+(\varkappa_2+\sigma_2\delta)|X^{(2)}|^2+\sigma_2aT_0|X^{(3)}|^2=0,$$

hence, $X^{(j)}=0$, $j=1,2,3$.

Thus, we obtain that the system $L(-i\xi,\sigma)X=0$ has only the trivial solution for arbitrary $\xi\in\mathbb{R}^3$. This contradiction proves the lemma. □

Corollary 6.2. *Let $\sigma=\sigma_1+i\sigma_2$ be a complex parameter with $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$. Consider the equation*

$$\det L(-i\xi,\sigma)=0 \qquad (6.8)$$

with respect to $|\xi|$. The roots $\pm k_j$, $j=\overline{1,5}$, of equation (6.8) *are complex with $\Im k_j>0$, $j=\overline{1,5}$.*

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(Received April 29, 2013)

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Memoirs on Differential Equations and Mathematical Physics
Volume 61, 2014, 83–102

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THE EXISTENCE OF HOMOCLINIC ORBITS FOR A CLASS OF FIRST-ORDER SUPERQUADRATIC HAMILTONIAN SYSTEMS

Abstract. Using critical point theory, we study the existence of homoclinic orbits for the first-order superquadratic Hamiltonian system

$$\dot{z} = JH_z(t, z),$$

where $H(t, z)$ depends periodically on t and is superquadratic.

2010 Mathematics Subject Classification. 34K13, 34K18, 58E50.
Key words and phrases. Homoclinic orbit, Hamiltonian system, critical point theory.

რეზიუმე. პირველი რიგის სუპერკვადრატული ჰამილტონური სისტემისათვის

$$\dot{z} = JH_z(t, z),$$

სადაც $H(t, z)$ არის სუპერკვადრატული და t-ს მიმართ პერიოდული, კრიტიკული წერტილის თეორიის გამოყენებით, გამოკვლეულია ჰომოკლინიკური ორბიტების არსებობის საკითხი.

1. Introduction

This paper is devoted to the study of the existence of homoclinic orbits for the first-order time-dependent Hamiltonian system

$$\dot{z} = JH_z(t, z), \tag{1.1}$$

where $z = (p, q) \in \mathbf{R}^N \times \mathbf{R}^N$. Here H has the form

$$H(t, z) = \frac{1}{2} B(t)z \cdot z + G(t, z) + h(t)z, \tag{1.2}$$

where $G \in C(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is T-periodic in t, $B(t)$ is a continuous T-periodic and symmetric $2N \times 2N$ matrix function, $h : \mathbf{R} \to \mathbf{R}^{2N}$ is a continuous and bounded function and J is the standard $2N \times 2N$ symplectic matrix

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}.$$

In recent years several authors studied homoclinic orbits for Hamiltonian systems via the critical point theory. For the second order Hamiltonian systems we refer the reader to [1, 2, 7, 8, 10–13] and for for the first order to [3–6, 9, 14–17] (and the references therein).

Throughout this paper, we always assume the following:

- (H_1) $G(t, z) \geq 0$, for all $(t, z) \in \mathbf{R} \times \mathbf{R}^{2N}$;
- (H_2) $G(t, z) = o(|z|^2)$ as $|z| \longrightarrow 0$ uniformly in t;
- (H_3) $\frac{G(t,z)}{|z|^2} \longrightarrow +\infty$ as $|z| \longrightarrow +\infty$ uniformly in t;
- (H_4) There exist constants $\beta > 1$, $1 < \lambda < 1 + \frac{\beta-1}{\beta}$, $a_1 > 0$, $a_2 > 0$ and $\tau \in L^1(\mathbf{R}, \mathbf{R}^+)$ such that

$$z \cdot G_z(t, z) - 2G(t, z) \geq a_1|z|^\beta - \tau(t), \quad (t, z) \in \mathbf{R} \times \mathbf{R}^{2N} \tag{1.3}$$

  and

$$|G_z(t, z)| \leq a_2|z|^\lambda, \quad \forall\, (t, z) \in \mathbf{R} \times \mathbf{R}^{2N}; \tag{1.4}$$

- (H_5) there exist constants $a_3 > 0$ and $\eta > 0$ such that

$$\int\limits_{\mathbf{R}} |h(t)|\, dt \leq a_3, \quad \Bigg(\int\limits_{\mathbf{R}} |h(t)|^2\, dt \Bigg)^{\frac{1}{2}} \leq \frac{\eta}{2\varrho},$$

$$\frac{2(\eta + \varrho\|\tau\|_{L^1})}{\varrho\xi} \leq 1, \quad a_2 < \min\left\{ \frac{\xi}{2}, \frac{\xi}{2\varrho^{\lambda+1}} \right\},$$

  where ϱ and ξ are two positive constants which will be defined in Proposition 3.1 and in (3.13) later.

A solution $z(t)$ of (1.1) is said to be homoclinic (to 0) if $z(t) \to 0$ as $t \to \pm\infty$. In addition, if $z(t) \not\equiv 0$, then $z(t)$ is called a nontrivial homoclinic solution.

Theorem 1.1. *Let* $(H_1) - (H_5)$ *be satisfied. Then* (1.1) *possesses a nontrivial homoclinic solution such that* $z(t) \to 0$ *as* $t \to \pm\infty$.

This paper is motivated by the work of Rabinowitz [12] in which the existence of nontrivial homoclinic solutions for the second order Hamiltonian system

$$\ddot{q} + V_q(t, q) = 0$$

was established.

The paper is organized as follows. In Section 2, we establish a variational structure for (1.1) with a periodic boundary value condition. Our main result (Theorem 1.1) will be proved in Section 3.

2. Variational Structure

Let $A = -(J(d/dt+B(t))$ be a self-adjoint operator acting on $L^2(\mathbf{R}, \mathbf{R}^{2N})$ with the domain $\widetilde{D}(A) = H^1(\mathbf{R}, \mathbf{R}^{2N})$. If $E := \widetilde{D}(|A|^{\frac{1}{2}})$, then E is a Hilbert space with the inner product

$$\langle z, v\rangle = (z, v)_{L^2} + \big(|A|^{\frac{1}{2}} z, |A|^{\frac{1}{2}} v\big)_{L^2}, \quad z, v \in E,$$

and $E = H^{\frac{1}{2}}(\mathbf{R}, \mathbf{R}^{2N})$. Let $E_k := H^{\frac{1}{2}}_{2kT}(\mathbf{R}, \mathbf{R}^{2N})$ for each $k \in \mathbf{N}$. Then E_k is a Hilbert space with the norm $\|\cdot\|_{E_k}$ given by (here $z \in E_k$)

$$\|z\|_{E_k} = \Bigg(\int_{-kT}^{kT} \big(\big||A|^{\frac{1}{2}} z\big|^2 + |z|^2\big)\, dt \Bigg)^{1/2}. \tag{2.1}$$

Furthermore, let $L^\infty_{2kT}(\mathbf{R}, \mathbf{R}^{2N})$ denote a space of $2kT$-periodic essentially bounded (measurable) functions from $\mathbf{R}$ into $\mathbf{R}^{2N}$ equipped with the norm

$$\|z\|_{L^\infty_{2kT}} := \operatorname{ess\,sup} \big\{ |z(t)| : \ t \in [-kT, kT] \big\}.$$

As in [10], a homoclinic solution of (1.1) will be obtained as a limit, as $k \to \pm\infty$, of a certain sequence of functions $z_k \in E_k$. We consider a sequence of systems of differential equations

$$\dot{z} = J(B(t)z + G_z(t, z) + h_k(t)), \tag{2.2}$$

where for each $k \in \mathbf{N}$, $h_k : \mathbf{R} \to \mathbf{R}^N$ is a $2kT$-periodic extension of the restriction of h to the interval $[-kT, kT]$ and z_k, a $2kT$-periodic solution of (2.1), will be obtained via a linking theorem.

We define

$$\langle Au, v\rangle = \int_{-kT}^{kT} \Big(-\Big(J\,\frac{d}{dt} + B\Big) u, v\Big)\, dt, \quad \forall\, u, v \in E_k \tag{2.3}$$

and

$$I_k(z) = \frac{1}{2}\,\langle Az, z\rangle - \int_{-kT}^{kT} G(t, z)\, dt - \int_{-kT}^{kT} h_k(t)\cdot z(t)\, dt. \tag{2.4}$$

We have from (2.3) that A has a sequence of eigenvalues

$$\cdots \xi_k^{(-m)} \le \cdots \le \xi_k^{(-2)} \le \xi_k^{(-1)} < 0 < \xi_k^{(1)} \le \xi_k^{(2)} \le \cdots \le \xi_k^{(m)} \cdots$$

with $\xi_k^{(m)} \longrightarrow \infty$ and $\xi_k^{(-m)} \longrightarrow -\infty$ as $m \longrightarrow \infty$. Let φ_k^j be the eigenvector of A corresponding to $\xi_k^{(j)}$, $j = \pm 1, \pm 2, \dots, \pm m, \dots$. Set

$$E_k^0 = ker(A), \quad E_k^- = \text{the negative eigenspace of } A$$

and

$$E_k^+ = \text{the positive eigenspace of } A.$$

Hence there exists an orthogonal decomposition $E_k = E_k^0 \oplus E_k^- \oplus E_k^+$ with $\dim(E_k^0) < \infty$.

Lemma 2.1 ([11])**.** *Let E be a real Hilbert space with $E = E^{(1)} \oplus E^{(2)}$ and $E^{(1)} = (E^{(2)})^\perp$. Suppose $I \in C^1(E, \mathbf{R})$ satisfies the* (**PS**) *condition, and*

- (C_1) *$I(u) = \frac{1}{2}(Lu, u) + b(u)$, where $Lu = L_1P_1u + L_2P_2u$, $L_i : E^{(i)} \longmapsto E^{(i)}$ is bounded and self-adjoint, P_i is the projector of E onto $E^{(i)}$, $i = 1, 2$;*
- (C_2) *b' is compact;*
- (C_3) *there exist a subspace $\widetilde{E} \subset E$, the sets $S \subset E$, $Q \subset \widetilde{E}$ and constants $\widetilde{\alpha} > \omega$ such that*
  - (i) *$S \subset E^{(1)}$ and $I|_S \geq \widetilde{\alpha}$;*
  - (ii) *Q is bounded and $I|_{\partial Q} \leq \omega$;*
  - (iii) *S and ∂Q are linked .*

Then I possesses a critical value $c \geq \widetilde{\alpha}$ given by

$$c = \inf_{g \in \Gamma} \sup_{u \in Q} I(g(1, u)),$$

where

$$\Gamma \equiv \Big\{ g \in C([0,1] \times E, E) | g \text{ satisfies } (\Gamma_1) - (\Gamma_3) \Big\},$$

- (Γ_1) *$g(0, u) = u$;*
- (Γ_2) *$g(t, u) = u$ for $u \in \partial Q$;*
- (Γ_3) *$g(t, u) = e^{\theta(t,u)L}u + \chi(t, u)$, where $\theta(t, u) \in C([0, 1] \times E, \mathbf{R})$, and χ is compact.*

3. Proof of the Main Result

The following result in [11, p. 36, Proposition 6.6] will be used.

Proposition 3.1. *There is a positive constant c_μ such that for each $k \in \mathbf{N}$ and $z \in E_k$ the following inequality holds:*

$$\|z\|_{L^\mu_{2kT}} \leq c_\mu \|z\|_{E_k}, \tag{3.1}$$

where $\mu \in [1, +\infty)$. For notational purposes let $c_{\lambda+1} = \varrho$.

Lemma 3.1. *Under the conditions of Theorem* 1.1*, I_k satisfies the* (**PS**) *condition.*

Proof. Assume that $\{z_{k_n}\}_{n\in\mathbf{N}}$ in E_k is a sequence such that $\{I_k(z_{k_n})\}_{n\in\mathbf{N}}$ is bounded and $I_k'(z_{k_n}) \to 0$ as $n \to +\infty$. Then there exists a constant $d_1 > 0$ such that

$$|I_k(z_{k_n})| \le d_1, \quad I_k'(z_{k_n}) \to 0 \text{ as } n \to \infty. \tag{3.2}$$

We first prove that $\{z_{k_n}\}_{n\in\mathbf{N}}$ is bounded. Let $z_{k_n} = z_{k_n}^0 + z_{k_n}^+ + z_{k_n}^- \in E_k^0 \oplus E_k^+ \oplus E_k^-$. From (1.3) of (H_4), (H_5), (2.4) and (3.1), there exists a constant $\widetilde{c}_{\widehat{\beta}} > 0$ such that (here $\frac{1}{\widehat{\beta}} + \frac{1}{\beta} = 1$)

$$\begin{aligned}
2d_1 &\ge 2I_k(z_{k_n}) - \left\langle I_k'(z_{k_n}), z_{k_n} \right\rangle = \\
&= \int_{-kT}^{kT} \Big[z_{k_n} \cdot G_{z_{k_n}}(t, z_{k_n}) - 2G(t, z_{k_n}) \Big]\, dt - \int_{-kT}^{kT} h_k(t) \cdot z_{k_n}\, dt \ge \\
&\ge \int_{-kT}^{kT} a_1 |z_{k_n}|^\beta\, dt - \int_{-kT}^{kT} \tau_k(t)\, dt - \int_{-kT}^{kT} |h_k(t)|\, |z_{k_n}|\, dt \ge \\
&\ge a_1 \|z_{k_n}\|^\beta_{L^\beta_{2kT}} - \|\tau_k\|_{L^1_{2kT}} - c_\beta \|h_k\|_{L^{\widehat{\beta}}_{2kT}} \|z_{k_n}\|_{L^\beta_{2kT}} \ge \\
&\ge a_1 \|z_{k_n}\|^\beta_{L^\beta_{2kT}} - \|\tau\|_{L^1} - c_\beta \|h\|_{L^{\widehat{\beta}}} \|z_{k_n}\|_{L^\beta_{2kT}} \ge \\
&\ge a_1 \|z_{k_n}\|^\beta_{L^\beta_{2kT}} - \|\tau\|_{L^1} - c_\beta \widetilde{c}_{\widehat{\beta}} \|h\|_{L^1} \|z_{k_n}\|_{L^\beta_{2kT}} \ge \\
&\ge a_1 \|z_{k_n}\|^\beta_{L^\beta_{2kT}} - \|\tau\|_{L^1} - c_\beta \widetilde{c}_{\widehat{\beta}} a_3 \|z_{k_n}\|_{L^\beta_{2kT}},
\end{aligned} \tag{3.3}$$

where for each $k \in \mathbf{N}$, $\tau_k : \mathbf{R} \to \mathbf{R}^N$ is a $2kT$-periodic extension of the restriction of $\tau(t)$ to the interval $[-kT, kT]$.

Since $\beta > 1$, this implies that there exists a constant $\widetilde{M_0} > 0$ with

$$\|z_{k_n}\|_{L^\beta_{2kT}} \le \widetilde{M_0}. \tag{3.4}$$

Consider $\{\|z_{k_n}^0\|_{E_k}\}_{n\in\mathbf{N}}$. Note $\dim(E_k^0) < +\infty$, and this implies that there are the constants b_1 and b_2 such that

$$b_1 \|z_{k_n}^0\|_{L^\beta_{2kT}} \le \|z_{k_n}^0||_{E_k} \le b_2 \|z_{k_n}^0\|_{L^\beta_{2kT}} \le b_2 \|z_{k_n}\|_{L^\beta_{2kT}}. \tag{3.5}$$

By (3.4) and (3.5), there exists a constant $\widetilde{M_1} > 0$ such that

$$\|z_{k_n}^0||_{E_k} \le \widetilde{M_1}. \tag{3.6}$$

Let $\alpha = \frac{\beta - 1}{\beta(\lambda - 1)}$, then

$$\begin{cases} 1 < \lambda < 1 + \dfrac{\beta - 1}{\beta}, & 0 < \dfrac{(\lambda\alpha - 1)}{\alpha} < 1, \\ \lambda\alpha - 1 = \alpha - \dfrac{1}{\beta}, & \alpha > 1. \end{cases} \tag{3.7}$$

If $0 < \|z\|_{L^{\infty}_{2kT}} \leq 1$ for $z \in E_k$, we have from (1.4) of (H_4) that

$$\int\limits_{-kT}^{kT} \big|G_z(t, z(t))\big|\, dt \leq a_2 \int\limits_{-kT}^{kT} |z(t)|\, dt. \tag{3.8}$$

By using (3.1) and (3.8), we have (here $\frac{1}{\alpha} + \frac{1}{\sigma} = 1$)

$$\begin{aligned}
&\|z^+_{k_n}\|_{E_k} \geq \big\langle I'_k(z_{k_n}), z^+_{k_n}\big\rangle = \\
&= \langle Az^+_{k_n}, z^+_{k_n}\rangle - \int\limits_{-kT}^{kT} \big[z^+_{k_n} \cdot G_{z_{k_n}}(t, z_{k_n})\big]\, dt - \int\limits_{-kT}^{kT} h_k(t) \cdot z^+_{k_n}\, dt = \\
&= \langle Az^+_{k_n}, z^+_{k_n}\rangle - \Bigg(\int\limits_{|z_{k_n}|\geq 1} + \int\limits_{|z_{k_n}|<1}\Bigg)\big[z^+_{k_n} \cdot G_{z_{k_n}}(t, z_{k_n})\big]\, dt - \int\limits_{-kT}^{kT} h_k(t) \cdot z^+_{k_n}\, dt \geq \\
&\geq \langle Az^+_{k_n}, z^+_{k_n}\rangle - \frac{\eta}{2\varrho}\|z_{k_n}\|_{E_k} - \int\limits_{|z_{k_n}|<1} a_2 |z_{k_n}|\, |z^+_{k_n}|\, dt - \\
&- \Bigg(\int\limits_{|z_{k_n}|\geq 1} \big|G_{z_{k_n}}(t, z_{k_n})\big|^{\alpha}\, dt\Bigg)^{\frac{1}{\alpha}} \Bigg(\int\limits_{-kT}^{kT} |z^+_{k_n}|^{\sigma}\, dt\Bigg)^{\frac{1}{\sigma}} \geq \\
&\geq \langle Az^+_{k_n}, z^+_{k_n}\rangle - \frac{\eta}{2\varrho}\|z_{k_n}\|_{E_k} - a_2\|z_{k_n}\|_{E_k}\|z^+_{k_n}\|_{E_k} - \\
&- \Bigg(\int\limits_{|z_{k_n}|\geq 1} \big|G_{z_{k_n}}(t, z_{k_n})\big|^{\alpha}\, dt\Bigg)^{\frac{1}{\alpha}} c_{\sigma}\|z_{k_n}\|_{E_k}
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\|z^-_{k_n}\|_{E_k} \geq -\big\langle I'_k(z_{k_n}), z^-_{k_n}\big\rangle = \\
&= -\langle Az^-_{k_n}, z^-_{k_n}\rangle + \int\limits_{-kT}^{kT} \big[z^-_{k_n} \cdot G_{z_{k_n}}(t, z_{k_n})\big]\, dt - \int\limits_{-kT}^{kT} h_k(t) \cdot z^-_{k_n}\, dt = \\
&= -\langle Az^-_{k_n}, z^-_{k_n}\rangle - \Bigg(\int\limits_{|z_{k_n}|\geq 1} + \int\limits_{|z_{k_n}|<1}\Bigg)\big[z^-_{k_n} \cdot G_{z_{k_n}}(t, z_{k_n})\big]\, dt - \\
&- \int\limits_{-kT}^{kT} h_k(t) \cdot z^-_{k_n}\, dt \geq \\
&\geq -\langle Az^-_{k_n}, z^-_{k_n}\rangle - \frac{\eta}{2\varrho}\|z_{k_n}\|_{E_k} - \int\limits_{|z_{k_n}|<1} a_2 |z_{k_n}|\, |z^-_{k_n}|\, dt -
\end{aligned}$$

$$-\Big(\int\limits_{|z_{k_n}|\geq 1}\big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt\Big)^{\frac{1}{\alpha}}\Big(\int\limits_{-kT}^{kT}|z_{k_n}^-|^{\sigma}\,dt\Big)^{\frac{1}{\sigma}}\geq$$

$$\geq -\langle Az_{k_n}^-, z_{k_n}^-\rangle - \frac{\eta}{2\varrho}\|z_{k_n}\|_{E_k} - a_2\|z_{k_n}\|_{E_k}\|z_{k_n}^-\|_{E_k}-$$

$$-\Big(\int\limits_{|z_{k_n}|\geq 1}\big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt\Big)^{\frac{1}{\alpha}}c_{\sigma}\|z_{k_n}\|_{E_k}. \qquad (3.10)$$

By using (1.4) of (H_4) and (3.1), there exists a constant $c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}}>0$ such that

$$\int\limits_{|z_{k_n}|\geq 1}\big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt \leq \int\limits_{|z_{k_n}|\geq 1} a_2^{\alpha}|z_{k_n}|^{\lambda\alpha}\,dt\leq$$

$$\leq a_2^{\alpha}\Big(\int\limits_{|z_{k_n}|\geq 1}|z_{k_n}|^{\beta}\,dt\Big)^{\frac{1}{\beta}}\Big(\int\limits_{|z_{k_n}|\geq 1}|z_{k_n}|^{(\lambda\alpha-1)\frac{\beta}{\beta-1}}\,dt\Big)^{1-\frac{1}{\beta}}\leq$$

$$\leq a_2^{\alpha}(c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}})^{\lambda\alpha-1}\Big(\int\limits_{|z_{k_n}|\geq 1}|z_{k_n}|^{\beta}\,dt\Big)^{\frac{1}{\beta}}\|z_{k_n}\|_{E_k}^{\lambda\alpha-1}. \qquad (3.11)$$

Combining (3.4) with (3.9)–(3.11), we find that

$$\|z_{k_n}^+\|_{E_k}+\|z_{k_n}^-\|_{E_k}\geq$$

$$\geq \langle Az_{k_n}^+, z_{k_n}^+\rangle - \langle Az_{k_n}^-, z_{k_n}^-\rangle - a_2\|z_{k_n}\|_{E_k}\big(\|z_{k_n}^+\|_{E_k}+\|z_{k_n}^-\|_{E_k}\big)-$$

$$-\frac{\eta}{\varrho}\|z_{k_n}\|_{E_k}-c_{\sigma}\Big(\int\limits_{|z_{k_n}|\geq 1}\big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt\Big)^{\frac{1}{\alpha}}\big(\|z_{k_n}\|_{E_k}+\|z_{k_n}\|_{E_k}\big)\geq$$

$$\geq \xi_1\|z_{k_n}^+\|_{E_k}^2-\xi_{-1}\|z_{k_n}^-\|_{E_k}^2-\frac{\eta}{\varrho}\|z_{k_n}\|_{E_k}-2a_2\|z_{k_n}\|_{E_k}^2-$$

$$-2c_{\sigma}(a_2^{\alpha}\Big[\big(c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}}\big)^{\lambda\alpha-1}\Big(\int\limits_{|z_{k_n}|\geq 1}|z_{k_n}|^{\beta}\,dt\Big)^{\frac{1}{\beta}}\Big]^{\frac{1}{\alpha}}\|z_{k_n}\|_{E_k}^{\frac{(\lambda\alpha-1)}{\alpha}}\|z_{k_n}\|_{E_k}\geq$$

$$\geq \xi_1\|z_{k_n}^+\|_{E_k}^2-\xi_{-1}\|z_{k_n}^-\|_{E_k}^2-\frac{\eta}{\varrho}\|z_{k_n}\|_{E_k}-$$

$$-2a_2\|z_{k_n}\|_{E_k}^2-2c_{\sigma}\widetilde{D}_0\big(\|z_{k_n}\|_{E_k}\big)^{\frac{(\lambda\alpha-1)+\alpha}{\alpha}}, \qquad (3.12)$$

where $\widetilde{D}_0=[a_2^{\alpha}((c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}})^{\lambda\alpha-1}\widetilde{M_0}]^{\frac{1}{\alpha}}$, and ξ_1 is the smallest positive eigenvalue, ξ_{-1} is the largest negative eigenvalue of the operator A, respectively. From (3.6) and (3.12), there exists a positive constant $\widetilde{D}_1>0$ such that

$$\widetilde{D}_1\Big(\|z_{k_n}^+\|_{E_k}+\|z_{k_n}^-\|_{E_k}+\|z_{k_n}^0\|_{E_k}\Big)\geq$$

$$\geq \|z_{k_n}^+\|_{E_k}+\|z_{k_n}^-\|_{E_k}+\xi\widetilde{M_1}\|z_{k_n}^0\|_{E_k}\geq\|z_{k_n}^+\|_{E_k}+\|z_{k_n}^-\|_{E_k}+\xi\|z_{k_n}^0\|_{E_k}^2\geq$$

$$\begin{aligned}
&\geq \xi_1\|z_{k_n}^+\|_{E_k}^2 - \xi_{-1}\|z_{k_n}^-\|_{E_k}^2 + \xi\|z_{k_n}^0\|_{E_k}^2 - \\
&\qquad -\frac{\eta}{\varrho}\|z_{k_n}\|_{E_k} - 2a_2\|z_{k_n}\|_{E_k}^2 - 2c_\sigma\widetilde{D}_0\big(\|z_{k_n}\|_{E_k}\big)^{\frac{(\lambda\alpha-1)+\alpha}{\alpha}} \geq \\
&\geq \xi\Big(\|z_{k_n}^+\|_{E_k}^2 + \|z_{k_n}^-\|_{E_k}^2 + \|z_{k_n}^0\|_{E_k}^2\Big) - \\
&\qquad -\frac{\eta}{\varrho}\|z_{k_n}\|_{E_k} - 2a_2\|z_{k_n}\|_{E_k}^2 - 2c_\sigma\widetilde{D}_0\big(\|z_{k_n}\|_{E_k}\big)^{\frac{(\lambda\alpha-1)+\alpha}{\alpha}}, \qquad (3.13)
\end{aligned}$$

where $\xi = \min\{\xi_1, -\xi_{-1}\}$. This implies that

$$\widetilde{D}_1 + \frac{\eta}{\varrho} \geq (\xi - 2a_2)\|z_{k_n}\|_{E_k} - 2c_\sigma\widetilde{D}_0\big(\|z_{k_n}\|_{E_k}\big)^{\frac{(\lambda\alpha-1)}{\alpha}}, \qquad (3.14)$$

where $0 < \frac{(\lambda\alpha-1)}{\alpha} < 1$. Since $\xi_1 - 2a_2 > 0$, we have that $\{\|z_{k_n}\|_{E_k}\}_{n\in\mathbf{N}}$ is bounded. Going if necessary to a subsequence, we can assume that there exists $z \in E_k$ such that $z_{k_n} \rightharpoonup z$, as $n \longrightarrow +\infty$, in E_k, which implies $z_{k_n} \longrightarrow z$ uniformly on $[-kT, kT]$. Hence $(I_k'(z_{k_n}) - I_k'(z))(z_{k_n} - z) \to 0$ and $\|z_{k_n} - z\|_{L^2_{[-kT,kT]}} \to 0$. Set

$$\Phi = \int_{-kT}^{kT} \Big(G_{z_{k_n}}(t, z_{k_n}(t)) - G_z(t, z(t)), z_{k_n} - z\Big)\, dt.$$

It is easy to check that $\Phi \to 0$ as $n \to +\infty$. Moreover, an easy computation shows that

$$\big(I_k'(z_{k_n}) - I_k'(z)\big)(z_{k_n} - z) = \big\langle A(z_{k_n} - z), (z_{k_n} - z)\big\rangle - \Phi.$$

This implies $\|z_{k_n} - z\|_{E_k} \to 0$. □

Lemma 3.2. *Under the conditions of Theorem* 1.1*, for every* $k \in \mathbf{N}$ *the system* (2.2) *possesses a* $2kT$*-periodic solution.*

Proof. The proof will be divided into three steps.

Step 1: Assume that $0 < \|z\|_{E_k} \leq 1$ for $z \in E_k^{(1)} = E_k^+$. From (1.3) of (H_4) and (3.1), we have

$$\begin{aligned}
&\int_{-kT}^{kT} G(t, z(t))\, dt \leq \frac{1}{2}\left[\int_{-kT}^{kT} z \cdot G_z(t, z(t))\, dt + \int_{-kT}^{kT} \tau_k(t)\, dt\right] \leq \\
&\leq \frac{1}{2}\left[a_2\int_{-kT}^{kT} |z(t)|^{\lambda+1}\, dt + \|\tau\|_{L^1}\right] \leq \frac{1}{2}\left[a_2\varrho^{\lambda+1}\|z\|_{E_k}^{\lambda+1} + \|\tau\|_{L^1}\right] \leq \\
&\qquad\qquad \leq \frac{1}{2}\left[a_2\varrho^{\lambda+1}\|z\|_{E_k}^2 + \|\tau\|_{L^1}\right]. \qquad (3.15)
\end{aligned}$$

From (2.4) and (3.15), for $z \in E_k^{(1)} = E_k^+$ and $0 < \|z\|_{E_k} \le 1$, we have

$$
\begin{aligned}
I_k(z) &= \frac{1}{2}\langle Az, z\rangle - \int_{-kT}^{kT} G(t,z)\,dt - \int_{-kT}^{kT} z\cdot h_k(t)\,dt \ge \\
&\ge \frac{\xi_1}{2}\|z\|_{E_k}^2 - \frac{1}{2}\Big[a_2\varrho^{\lambda+1}\|z\|_{E_k}^2 + \|\tau\|_{L^1}\Big] - \frac{\eta}{2\varrho}\|z\|_{E_k} \ge \\
&\ge \frac{1}{4}(\xi - 2a_2\varrho^{\lambda+1})\|z\|_{E_k}^2 + \frac{\xi}{4}\|z\|_{E_k}^2 - \frac{(\frac{\eta}{\varrho} + \|\tau\|_{L^1})}{2}\,. \qquad (3.16)
\end{aligned}
$$

Note from (H_5) that $\xi - 2a_2\varrho^{\lambda+1} > 0$. Set

$$
\rho = \Big(\frac{2(\frac{\eta}{\varrho} + \|\tau\|_{L^1})}{\xi}\Big)^{\frac{1}{2}} \quad \text{and} \quad \widetilde{\alpha} = \frac{\xi - 2a_2\varrho^{\lambda+1}}{4}\,.
$$

Let B_ρ denote the open ball in E_k with radius ρ about 0 and let ∂B_ρ denote its boundary. Let $S_k = \partial B_\rho \cap E_k^+$. If $z \in S_k$, then $\|z\|_{E_k} = \big(\frac{2(\frac{\eta}{\varrho}+\|\tau\|_{L^1})}{\xi}\big)^{\frac{1}{2}}$ (note that $||z||_{E_k} \le 1$ from (H_5)) and thus (3.16) gives

$$
I_k(z) \ge \widetilde{\alpha} \quad z \in S_k.
$$

Then $(C_3)(i)$ of Lemma 2.1 holds.

Step 2: Let $e \in E_k^+$ with $\|e\|_{E_k} = 1$ and $\widetilde{E}_k = E_k^- \oplus E_k^0 \oplus span\{e\}$. Let now

$$
\Theta_k = \big\{z \in \widetilde{E}_k : \ \|z\|_{\widetilde{E}_k} = 1\big\},
$$

$$
\mu = \inf_{z\in E_k^-, \|z^-\|_{E_k}=1} \big|\langle Az^-, z^-\rangle\big|, \quad \kappa = \Big(\frac{2\|A\|}{\mu}\Big)^{1/2}.
$$

For $z \in \Theta_k$, we write $z = z^- + z^0 + z^+$.

I) If $\|z^-\|_{E_k} > \kappa\|z^+ + z^0\|_{E_k}$, then for any $\gamma \ge \frac{2\eta(1+\kappa^2)}{\varrho\mu\kappa^2} > 0$, we have from (H_1) that

$$
\begin{aligned}
I_k(\gamma z) &= \frac{1}{2}\langle A\gamma z^-, \gamma z^-\rangle + \frac{1}{2}\langle A\gamma z^+, \gamma z^+\rangle - \\
&\quad - \int_{-kT}^{kT} G(t,\gamma z)\,dt - \int_{-kT}^{kT} \gamma z\cdot h_k(t)\,dt \le \\
&\le -\frac{\mu}{2}\gamma^2\|z^-\|_{E_k}^2 + \frac{\|A\|}{2}\gamma^2\|z^+\|_{E_k}^2 + \frac{\eta}{2\varrho}\gamma \le \\
&\le -\frac{\mu}{2}\gamma^2\|z^-\|_{E_k}^2 + \frac{\|A\|}{2}\gamma^2\|z^+ + z^0\|_{E_k}^2 + \frac{\eta}{2\varrho}\gamma \le \\
&\le -\frac{\mu}{2}\gamma^2\|z^-\|_{E_k}^2 + \frac{\|A\|}{2}\gamma^2\frac{1}{\kappa^2}\|z^-\|_{E_k}^2 + \frac{\eta}{2}\gamma = \\
&= -\frac{\mu}{4}\gamma^2\|z^-\|_{E_k}^2 + \frac{\eta}{2\varrho}\gamma \le 0; \qquad (3.17)
\end{aligned}
$$

note $\|z^-\|^2_{E_k} > \frac{\kappa^2}{1+\kappa^2}$, since

$$1 = \|z^-\|^2_{E_k} + \|z^+ + z^0\|^2_{E_k} < \frac{(1+\kappa^2)}{\kappa^2}\|z^-\|^2_{E_k}.$$

Let

$$\Delta_k = \Big\{z \in \Theta_k : \ \|z^-\|_{E_k} \le \kappa\|z^+ + z^0\|_{E_k}\Big\}.$$

II) If $\|z^-\|_{E_k} \le \kappa\|z^+ + z^0\|_{E_k}$, we have

$$1 = \|z\|^2_{E_k} = \|z^-\|^2_{E_k} + \|z^+ + z^0\|^2_{E_k} \le (1+\kappa^2)\|z^+ + z^0\|^2_{E_k}, \tag{3.18}$$

i.e.,

$$\|z^+ + z^0\|^2_{E_k} \ge \frac{1}{(1+\kappa^2)} > 0. \tag{3.19}$$

The argument in [6, pp. 6–7] guarantees that there exists $\varepsilon_1^k > 0$ such that, $\forall\, u \in \Delta_k$,

$$\text{meas}\Big\{t \in [0, 2kT] : \ |u(t)| \ge \varepsilon_1^k\Big\} \ge \varepsilon_1^k. \tag{3.20}$$

For $z = z^+ + z^0 + z^- \in \Delta_k$, let

$$\Omega_k^z = \big\{t \in [0, 2kT] : \ |z(t)| \ge \varepsilon_1^k\big\}.$$

By (H_3), for $M_k = \frac{\|A\|}{(\varepsilon_1^k)^3} > 0$, there exists L_k such that

$$G(t,z) \ge M_k|z|^2, \ \ \forall\, |z| \ge L_k, \ \text{ uniformly in } t. \tag{3.21}$$

Let

$$\gamma_k \ge \max\Big\{\frac{L_k}{\varepsilon_1^k}, \frac{\eta}{\varrho\|A\|}\Big\}.$$

For $\gamma \ge \gamma_k$, we have from (3.20) and (3.21) that

$$G(t, \gamma z) \ge M_k|\gamma z|^2 \ge M_k\gamma^2(\varepsilon_1^k)^2, \ \ \forall\, t \in \Omega_k^z. \tag{3.22}$$

From (H_1) and (3.22), for $\gamma \ge \gamma_k$ we have for $z \in \Delta_k$ that

$$\begin{aligned} I_k(\gamma z) &= \frac{1}{2}\gamma^2\langle Az^+, z^+\rangle + \frac{1}{2}\gamma^2\langle Az^-, z^-\rangle - \\ &- \int_{-kT}^{kT} G(t,\gamma z)\,dt - \int_{-kT}^{kT} \gamma z\cdot h_k(t)\,dt \le \frac{1}{2}\|A\|\gamma^2 - \int_{\Omega_k^z} G(t,\gamma z)\,dt + \frac{\eta}{2\varrho}\gamma \le \\ &\le \frac{1}{2}\|A\|\gamma^2 - M_k\gamma^2(\varepsilon_1^k)^3 + \frac{\eta}{2\varrho}\gamma = -\frac{1}{2}\|A\|\gamma^2 + \frac{\eta}{2\varrho}\gamma \le 0. \end{aligned} \tag{3.23}$$

Therefore we have shown that

$$I_k(\gamma z) \le 0 \ \text{ for any } \ z \in \Delta_k \ \text{ and } \ \gamma \ge \gamma_k. \tag{3.24}$$

Let

$$E_k^{(2)} = E_k^- \oplus E_k^0,$$

$$Q_k = \big\{\gamma e : \ 0 \le \gamma \le 2\gamma_k\big\} \oplus \big\{z \in E_k^{(2)} : \ \|z\|_{E_k} \le 2\gamma_k\big\}.$$

By (H_2), (3.16)–(3.17) and (3.24) we have $I_k|_{\partial Q_k} \le 0$, i.e., I_k satisfies $(C_2)(ii)$ of the Lemma 2.1.

Step 3: $(C_3)(iii)$ (i.e. S_k links ∂Q_k) holds from the definition of S_k and Q_k and [11, p. 32]. Thus $(C_3)(iii)$ holds.

From (H_2)–(H_5) and (2.3), (C_1) and (C_2) of Lemma 2.1 are true, so by Lemma 2.1, I_k possesses a critical value c_k given by

$$c_k = \inf_{g_k \in \Upsilon_k} \sup_{u_k \in Q_k} I_k(g_k(1, u_k)), \tag{3.25}$$

where Υ_k satisfies $(\Gamma_1) - (\Gamma_3)$. Hence, for every $k \in \mathbf{N}$, there is $z_k^* \in E_k$ such that

$$I_k(z_k^*) = c_k, \quad I_k'(z_k^*) = 0. \tag{3.26}$$

The function z_k^* is a desired classical $2kT$-periodic solution of (2.2). Since $c_k \ge \widetilde{\alpha} = \frac{\xi - 2a_2\varrho^{\lambda+1}}{4} > 0$, z_k^* is a nontrivial solution. □

Lemma 3.3. *Let* $\{z_k^*\}_{k\in\mathbf{N}}$ *be the sequence given by Lemma* 3.3. *There exists a* $z_0 \in C(\mathbf{R}, \mathbf{R}^{2N})$ *such that* $z_k^* \to z_0$ *in* $C_{loc}(\mathbf{R}, \mathbf{R}^{2N})$ *as* $k \to +\infty$.

Proof. The first step in the proof is to show that the sequences $\{c_k\}_{k\in\mathbf{N}}$ and $\{\|z_k^*\|_{E_k}\}_{k\in\mathbf{N}}$ are bounded. There exists $\widehat{z}_1^* \in E_1$ with $\widehat{z}_1^*(\pm T) = 0$ such that

$$c_1 \le I_1(\widehat{z}_1^*) = \inf_{g_1 \in \Upsilon_1} \sup_{u_1 \in Q_1, u_1(\pm T) = 0} I_1(g_1(1, u_1)). \tag{3.27}$$

For every $k \in \mathbf{N}$, let

$$\widehat{z}_k^*(t) = \begin{cases} \widehat{z}_1^*(t) & \text{for } |t| \le T \\ 0 & \text{for } T < |t| \le kT \end{cases} \tag{3.28}$$

and $\widetilde{g}_k : [0,1] \times E_k \longrightarrow E_k$ be a curve given by $\widetilde{g}_k(t, z) \equiv z$, where $z \in E_k$. Then $\widetilde{g}_k \in \Upsilon_k$ and $I_k(\widetilde{g}_k(1, \widehat{z}_k^*)) = I_1(\widetilde{g}_1(1, z_1^*)) = I_1(z_1^*)$ for all $k \in \mathbf{N}$. Therefore, from (3.25), (3.27) and (3.28),

$$c_k \le I_k\big(\widetilde{g}_k(1, \widehat{z}_k^*)\big) = I_1\big(\widetilde{g}_1(1, z_1^*)\big) = I_1(z_1^*) \equiv M_0. \tag{3.29}$$

We now prove that $\{z_k^*\}_{k\in\mathbf{N}}$ is bounded.

Let $z_k^* = (z_k^*)^0 + (z_k^*)^+ + (z_k^*)^- \in E_k^0 \oplus E_k^+ \oplus E_k^-$. From (1.3) of (H_4), (H_5), (2.4), (3.1) and (3.29), there exists a constant $\widehat{c}_{\widehat{\beta}} > 0$ such that (here $\frac{1}{\widehat{\beta}} + \frac{1}{\beta} = 1$)

$$\begin{aligned} 2M_0 &\ge 2I_k(z_k^*) - \langle I_k'(z_k^*), z_k^* \rangle \\ &= \int_{-kT}^{kT} \Big[z_k^* \cdot G_{z_k^*}(t, z_k^*) - 2G(t, z_k^*) \Big]\, dt - \int_{-kT}^{kT} h_k(t) \cdot z_k^*\, dt \ge \\ &\ge \int_{-kT}^{kT} a_1 |z_k^*|^\beta\, dt - \int_{-kT}^{kT} \tau_k(t)\, dt - \int_{-kT}^{kT} |h_k(t)|\, |z_k^*|\, dt \ge \end{aligned}$$

$$\geq a_1\|z_k^*\|^\beta_{L^\beta_{2kT}} - \|\tau_k\|_{L^1_{2kT}} - c_\beta\|h_k\|_{L^{\widehat{\beta}}_{2kT}}\|z_k^*\|_{L^\beta_{2kT}} \geq$$
$$\geq a_1\|z_k^*\|^\beta_{L^\beta_{2kT}} - \|\tau\|_{L^1} - c_\beta\widehat{c}_{\widehat{\beta}}\|h\|_{L^1}\|z_k^*\|_{L^\beta_{2kT}} \geq$$
$$\geq a_1\|z_k^*\|^\beta_{L^\beta_{2kT}} - \|\tau\|_{L^1} - c_\beta\widehat{c}_{\widehat{\beta}}a_3\|z_k^*\|_{L^\beta_{2kT}}. \tag{3.30}$$

Since $\beta > 1$, this implies that there exists a constant $\widetilde{M}_0^* >$ with

$$\|z_k^*\|_{L^\beta_{2kT}} \leq \widetilde{M}_0^*. \tag{3.31}$$

Note $\dim(E_k^0) < +\infty$, therefore there exists a constant $\widetilde{M}_1^* > 0$ such that

$$\|(z_k^*)^0\|_{E_k} \leq \widetilde{M}_1^*. \tag{3.32}$$

By using (3.1) and (3.8), we have (here $\frac{1}{\alpha} + \frac{1}{\sigma} = 1$)

$$\|(z_k^*)^+\|_{E_k} \geq \left\langle I_k'(z_k^*), (z_k^*)^+\right\rangle =$$
$$= \langle A(z_k^*)^+, (z_k^*)^+\rangle - \int_{-kT}^{kT}\Big[(z_k^*)^+ \cdot G_{z_k^*}(t, z_k^*)\Big]\,dt - \int_{-kT}^{kT} h_k(t)\cdot(z_k^*)^+\,dt =$$
$$= \langle A(z_k^*)^+, (z_k^*)^+\rangle - \left(\int_{|z_k^*|\geq 1} + \int_{|z_k^*|<1}\right)\big[(z_k^*)^+ \cdot G_{z_k^*}(t, z_k^*)\big]\,dt -$$
$$- \int_{-kT}^{kT} h_k(t)\cdot(z_k^*)^+\,dt \geq$$
$$\geq \langle A(z_k^*)^+, (z_k^*)^+\rangle - \frac{\eta}{2\varrho}\|z_k^*\|_{E_k} - \int_{|z_k^*|<1} a_2|z_k^*|\,|(z_k^*)^+|\,dt -$$
$$-\left(\int_{|z_k^*|\geq 1}|G_{z_k^*}(t, z_k^*)|^\alpha\,dt\right)^{\frac{1}{\alpha}}\left(\int_{-kT}^{kT}|z_{k_n}^+|^\sigma\,dt\right)^{\frac{1}{\sigma}} \geq$$
$$\geq \langle A(z_k^*)^+, (z_k^*)^+\rangle - \frac{\eta}{2\varrho}\|z_k^*\|_{E_k} - a_2\|z_k^*\|_{E_k}\|(z_k^*)^+\|_{E_k} -$$
$$-\left(\int_{|z_k^*|\geq 1}|G_{z_k^*}(t, z_k^*)|^\alpha\,dt\right)^{\frac{1}{\alpha}} c_\sigma\|z_k^*\|_{E_k} \tag{3.33}$$

and

$$\|(z_k^*)^-\|_{E_k} \geq \langle I_k'(z_k^*), (z_k^*)^-\rangle$$
$$= \langle A(z_k^*)^-, (z_k^*)^-\rangle - \int_{-kT}^{kT}\big[(z_k^*)^- \cdot G_{z_k^*}(t, z_k^*)\big]\,dt - \int_{-kT}^{kT} h_k(t)\cdot(z_k^*)^-\,dt =$$

$$= \langle A(z_k^*)^-, (z_k^*)^- \rangle - \Bigg(\int\limits_{|z_k^*|\geq 1} + \int\limits_{|z_k^*|<1} \Bigg) \big[(z_k^*)^- \cdot G_{z_k^*}(t, z_k^*) \big] \, dt -$$

$$- \int\limits_{-kT}^{kT} h_k(t) \cdot (z_k^*)^- \, dt \geq$$

$$\geq \langle A(z_k^*)^-, (z_k^*)^- \rangle - \frac{\eta}{2\varrho} \|z_k^*\|_{E_k} - \int\limits_{|z_k^*|<1} a_2 |z_k^*| \, |(z_k^*)^-| \, dt -$$

$$-\Bigg(\int\limits_{|z_k^*|\geq 1} |G_{z_k^*}(t, z_k^*)|^\alpha \, dt \Bigg)^{\frac{1}{\alpha}} \Bigg(\int\limits_{-kT}^{kT} |z_{k_n}^-|^\sigma \, dt \Bigg)^{\frac{1}{\sigma}} \geq$$

$$\geq \langle A(z_k^*)^-, (z_k^*)^- \rangle - \frac{\eta}{2\varrho} \|z_k^*\|_{E_k} - a_2 \|z_k^*\|_{E_k} \|(z_k^*)^-\|_{E_k} -$$

$$-\Bigg(\int\limits_{|z_k^*|\geq 1} |G_{z_k^*}(t, z_k^*)|^\alpha \, dt \Bigg)^{\frac{1}{\alpha}} c_\sigma \|z_k^*\|_{E_k}. \tag{3.34}$$

Combining (3.11), (3.31) with (3.33)–(3.34), we have

$$\|(z_k^*)^-\|_{E_k} + \|(z_k^*)^+\|_{E_k} \geq$$
$$\geq \xi_1 \|(z_k^*)^+\|_{E_k}^2 - \xi_{-1} \|(z_k^*)^-\|_{E_k}^2 - \frac{\eta}{\varrho} \|z_k^*\|_{E_k} -$$
$$- 2a_2 \|z_k^*\|_{E_k}^2 - 2c_\sigma \widetilde{D}_0^* \|z_k^*\|_{E_k}^{\frac{(\lambda\alpha-1)}{\alpha}} \|z_k^*\|_{E_k}, \tag{3.35}$$

where

$$\widetilde{D}_0^* = \Big[a_2^\alpha \big(c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}} \big)^{\lambda\alpha-1} \widetilde{M_0^*} \Big]^{\frac{1}{\alpha}}.$$

From (3.32) and (3.35), there exists a positive constant $\widetilde{D}_1^* > 0$ such that

$$\widetilde{D}_1^* (\|(z_k^*)^+\|_{E_k} + \|(z_k^*)^-\|_{E_k} + \|(z_k^*)^0\|_{E_k}) \geq$$
$$\geq \|(z_k^*)^+\|_{E_k} + \|(z_k^*)^-\|_{E_k} + \xi \widetilde{M_1^*} \|(z_k^*)^0\|_{E_k} \geq$$
$$\geq \|(z_k^*)^+\|_{E_k} + \|(z_k^*)^-\|_{E_k} + \xi \|(z_k^*)^0\|_{E_k}^2 \geq$$
$$\geq \xi \Big(\|(z_k^*)^+\|_{E_k}^2 + \|(z_k^*)^-\|_{E_k}^2 + |(z_k^*)^0\|_{E_k}^2 \Big) -$$
$$- \frac{\eta}{\varrho} \|z_k^*\|_{E_k} - 2a_2 \|z_k^*\|_{E_k}^2 - 2c_\sigma \widetilde{D}_0^* \big(\|z_k^*\|_{E_k} \big)^{\frac{(\lambda\alpha-1)+\alpha}{\alpha}}. \tag{3.36}$$

This implies that

$$\widetilde{D}_1^* + \frac{\eta}{\varrho} \geq (\xi - 2a_2) \|z_k^*\|_{E_k} - 2c_\sigma \widetilde{D}_0^* \big(\|z_k^*\|_{E_k} \big)^{\frac{(\lambda\alpha-1)}{\alpha}}, \tag{3.37}$$

where $0 < \frac{(\lambda\alpha-1)}{\alpha} < 1$. Since $\xi - 2a_2 > 0$, we have that $\{\|z_{k_n}\|_{E_k}\}_{n\in\mathbf{N}}$ is bounded. Hence (3.37) shows that there exists a constant $M_1 > 0$ such that

$$\|z_k^*\|_{E_k} \le M_1. \tag{3.38}$$

We now show that for a large enough k,

$$\|z_k^*\|_{L^\infty_{2kT}} \le M_2. \tag{3.39}$$

If not (note (2.1) and (3.38)), by passing to a subsequence, without loss of generality, for each $k \in N$, there exist z_k^*, ℓ_k and $\widetilde{\ell}_k$ such that $|z_k^*(\ell_k)| = M_k^*$, $|z_k^*(\widetilde{\ell}_k)| = 1$ and $1 \le |z_k^*(t)| \le M_k^*$ for $t \in (\widetilde{\ell}_k, \ell_k) \subseteq [-kT, kT]$ (and $M_k^* \to \infty$ as $k \to \infty$). Hence, we have from (1.3) of (H_4), (H_5) and (3.31) that

$$M_k^* - 1 = |z_k^*(\ell_k)| - |z_k^*(\widetilde{\ell}_k)| = \int_{\widetilde{\ell}_k}^{\ell_k} \frac{d}{ds}|z_k^*(s)|\,ds =$$

$$= \int_{\widetilde{\ell}_k}^{\ell_k} z_k^*(s)\cdot\frac{\dot{z}_k^*(s)}{|z_k^*(s)|}\,ds \le \int_{\widetilde{\ell}_k}^{\ell_k} |\dot{z}_k^*(s)|\,ds$$

$$\le \int_{\widetilde{\ell}_k}^{\ell_k} \left|G_{z_k^*}(t, z_k^*(s))\right|ds + \int_{\widetilde{\ell}_k}^{\ell_k} \left|B(s)z_k^*(s)\right|ds + \int_{\widetilde{\ell}_k}^{\ell_k} |h_k(s)|\,ds \le$$

$$\le \left(a_2 + \|B\|_{L^\infty_{2kT}}\right)\int_{\widetilde{\ell}_k}^{\ell_k} |z_k^*(s)|^\lambda\,ds + \|h_k\|_{L^1_{2kT}} \le$$

$$\le \left(a_2 + \|B\|_{L^\infty_{2kT}}\right)\int_{\widetilde{\ell}_k}^{\ell_k} |z_k^*(s)|^\beta\,ds + \|h\|_{L^1} \le \quad \left(\text{since } 1 < \lambda < 1 + \frac{\beta-1}{\beta} < \beta\right)$$

$$\le \left(a_2 + \|B\|_{L^\infty_{2kT}}\right)(\widetilde{M_0^*})^\beta + a_3, \tag{3.40}$$

where a_2, a_3, $\|B\|_{L^\infty_{2kT}}$ and $\widetilde{M_0^*}$ are k-independent constants. However, we have $M_k^* \to \infty$ as $k \to \infty$, which leads to a contradiction. Hence there exists a constant $M_2 > 0$ such that

$$\|z_k^*\|_{L^\infty_{2kT}} \le (a_2 + \|B\|_{L^\infty_{2kT}})(\widetilde{M_0^*})^\beta + a_3 + 1 = M_2. \tag{3.41}$$

This shows that (3.39) holds.

It remains now to show that $\{z_k^*\}_{k\in N}$ is equicontinuous. It suffices to prove that the sequence satisfies a Lipschitz condition with a constant, independent of k.

From (1.1) and (3.39), there exists a constant $M_3 > 0$, independent of k such that

$$|\dot{z}_k^*(t)| = |J(G_{z_k^*}(t, z_k^*(t)) + B(t)z_k^*(t) + h_k(t))| \le$$
$$\le M_3 \quad (\text{since } \|z_k^*\|_{L^\infty_{2kT}} \le M_2)$$

which implies

$$\|\dot{z}_k^*\|_{L^\infty_{2kT}} \le M_3. \tag{3.42}$$

Let $k \in \mathbf{N}$ and $t, t_0 \in R$, then

$$\left|z_k^*(t) - z_k^*(t_0)\right| = \left|\int_{t_0}^{t} \dot{z}_k^*(s)\, ds\right| \le \int_{t_0}^{t} |\dot{z}_k^*(s)|\, ds \le M_3(t - t_0).$$

Since $\{z_k^*\}_{k\in\mathbf{N}}$ is bounded in $L^\infty_{2kT}(\mathbf{R}, \mathbf{R}^{2N})$ and equicontinuous, we obtain that the sequence $\{z_k^*\}_{k\in\mathbf{N}}$ converges to a certain $z_0 \in C_{loc}(\mathbf{R}, \mathbf{R}^{2N})$ by using the Arzelà–Ascoli theorem. □

Lemma 3.4. *The function* z_0 *determined by Lemma* 3.4 *is the desired homoclinic solution of* (1.1)*.*

Proof. The proof will be divided into three steps.

Step 1: We prove that $z_0(t) \to 0$ as $t \to \pm\infty$.

We have

$$\int_{-\infty}^{+\infty} |z_0(t)|^2\, dt = \lim_{j\to+\infty} \int_{-jT}^{jT} |z_0(t)|^2\, dt = \lim_{j\to+\infty} \lim_{k\to+\infty} \int_{-jT}^{jT} |z_{n_k}^*(t)|^2\, dt.$$

Clearly, by (2.1) and (3.38), for every $j \in \mathbf{N}$ there exists $n_j \in \mathbf{N}$ such that for all $k \ge n_j$ we have

$$\int_{-jT}^{jT} |z_{n_k}^*(t)|^2\, dt \le \|z_{n_k}^*\|_{E_k}^2 \le M_1^2,$$

and now, letting $j \to +\infty$, we have

$$\int_{-\infty}^{+\infty} |z_0(t)|^2\, dt \le \widetilde{M}_1^2,$$

and hence

$$\int_{|t|\ge m} |z_0(t)|^2\, dt \to 0 \;\text{ as }\; m \to +\infty. \tag{3.43}$$

Then (3.43) shows that our claim holds.

Step 2: We show that $z_0 \not\equiv 0$ when $h(t) \equiv 0$.

Now, up to a subsequence, we have either

$$\int_{-\infty}^{+\infty} |z_0(t)|^2\, dt = \lim_{j\to+\infty} \int_{-jT}^{jT} |z_0(t)|^2\, dt =$$

$$= \lim_{j\to+\infty} \lim_{k\to+\infty} \int_{-jT}^{jT} |z_{n_k}^*(t)|^2\, dt = 0, \tag{3.44}$$

or there exist $\widehat{\alpha} > 0$ such that

$$\int_{-\infty}^{+\infty} |z_0(t)|^2\,dt = \lim_{j\to+\infty} \int_{-jT}^{jT} |z_0(t)|^2\,dt = $$

$$= \lim_{j\to+\infty} \lim_{k\to+\infty} \int_{-jT}^{jT} |z^*_{n_k}(t)|^2\,dt \geq \widehat{\alpha} > 0. \tag{3.45}$$

In the first case we shall say that z_0 is vanishing and in the second that z_0 is nonvanishing.

By assumptions (H_2), (H_3) and (1.4) of (H_4), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left|G(t, z^*_{n_k})\right| \leq \varepsilon |z^*_{n_k}|^2 + C_\varepsilon |z^*_{n_k}|^{\lambda+1}. \tag{3.46}$$

Hence, we have from (1.4) of (H_4) and (3.46) that

$$\begin{cases} \displaystyle\int_{-kT}^{kT} \left|(z^*_{n_k})^{\pm}\right| \left|G_{z^*_{n_k}}(t, z^*_{n_k})\right| dt \leq \\ \qquad\qquad \leq \varepsilon \|z^*_{n_k}\|_{L^2_{2kT}} \left\|(z^*_{n_k})^{\pm}\right\|_{L^2_{2kT}} + a_2 \|z^*_{n_k}\|^{\lambda+1}_{L^{\lambda+1}_{2kT}}, \\ \displaystyle\int_{-kT}^{kT} G(t, z^*_{n_k})\,dt \leq \varepsilon \|z^*_{n_k}\|^2_{L^2_{2kT}} + C_\varepsilon \|z^*_{n_k}\|^{\lambda+1}_{L^{\lambda+1}_{2kT}}. \end{cases} \tag{3.47}$$

Arguing indirectly, we suppose that $\{z^*_{n_k}\}_{k=1}^{\infty}$ is bounded and vanishing. We have from (3.44) and (3.47) that

$$\lim_{k\to\infty} \int_{-kT}^{kT} (z^*_k)^{\pm} \cdot G_{z^*_k}(t, z^*_k)\,dt = \lim_{k\to\infty} \int_{-kT}^{kT} G(t, z^*_k)\,dt = 0. \tag{3.48}$$

Since $\langle I'_k(z^*_{n_k}), (z^*_{n_k})^{\pm}\rangle = 0$, for some positive constant $\widetilde{C}$, using (3.1) and (3.47), we find that

$$\xi_1 \left\|(z^*_{n_k})^+\right\|^2_{E_k} \leq \left\langle A(z^*_{n_k})^+, (z^*_{n_k})^+\right\rangle = \int_{-kT}^{kT} (z^*_{n_k})^+ \cdot G_{z^*_{n_k}}(t, z^*_{n_k})\,dt \leq$$

$$\leq \varepsilon \|z^*_{n_k}\|_{E_k} \left\|(z^*_{n_k})^+\right\|_{E_k} + \widetilde{C} \|z^*_{n_k}\|^{\lambda+1}_{E_k} \leq \frac{\xi}{8} \|z^*_{n_k}\|^2_{E_k} + \widetilde{C} \|z^*_{n_k}\|^{\lambda+1}_{E_k} \tag{3.49}$$

and

$$-\xi_{-1} \left\|(z^*_{n_k})^-\right\|^2_{E_k} \leq -\left\langle A(z^*_{n_k})^-, (z^*_{n_k})^-\right\rangle = -\int_{-kT}^{kT} (z^*_{n_k})^-_k \cdot G_{z^*_{n_k}}(t, z^*_{n_k})\,dt \leq$$

$$\leq \varepsilon \|z^*_{n_k}\|_{E_k} \left\|(z^*_{n_k})^-\right\|_{E_k} + \widetilde{C} \|z^*_{n_k}\|^{\lambda+1}_{E_k} \leq \frac{\xi}{8} \left\|z^*_{n_k}\right\|^2_{E_k} + \widetilde{C} \|z^*_{n_k}\|^{\lambda+1}_{E_k}. \tag{3.50}$$

Note that $\dim(E_k^0) < +\infty$, there exist two positive constants $\widetilde{b}_1$, and $\widetilde{b}_2$ such that

$$\widetilde{b}_1\big\|(z_{n_k}^*)^0\big\|_{L^2_{2kT}} \le \big\|(z_{n_k}^*)^0\big\|_{E_k} \le \widetilde{b}_2\big\|(z_{n_k}^*)^0\big\|_{L^2_{2kT}} \le \widetilde{b}_2\|z_{n_k}^*\|_{L^2_{2kT}}. \tag{3.51}$$

From (3.44) and (3.51) we have

$$\xi\big\|(z_{n_k}^*)^0\big\|^2_{E_k} \le \xi\widetilde{b}_2\big\|(z_{n_k}^*)^0\big\|_{L^2_{2kT}} \longrightarrow 0 \ \text{ as } \ k \longrightarrow \infty. \tag{3.52}$$

Now (3.52) implies that there exists a positive constant $b_\varepsilon(0 < b_\varepsilon \le \frac{\xi}{4})$ such that

$$\xi\big\|(z_{n_k}^*)^0\big\|^2_{E_k} \le b_\varepsilon\|z_{n_k}^*\|^2_{E_k}. \tag{3.53}$$

Hence, from (3.49), (3.50) and (3.53) we obtain that

$$\begin{gathered}\xi\Big(\big\|(z_{n_k}^*)^+\big\|^2_{E_k} + \big\|(z_{n_k}^*)^-\big\|^2_{E_k} + \big\|(z_{n_k}^*)^0\big\|^2_{E_k}\Big) \le \\ \le \xi_1\big\|(z_{n_k}^*)^+\big\|^2_{E_k} + \xi_{-1}\big\|(z_{n_k}^*)^-\big\|^2_{E_k} + \xi\big\|(z_{n_k}^*)^0\big\|^2_{E_k} \le \\ \le \frac{\xi}{2}\,\|z_{n_k}^*\|^2_{E_k} + 2\widetilde{C}\|z_{n_k}^*\|^{\lambda+1}_{E_k},\end{gathered}$$

and $\|z_{n_k}^*\|_{E_k} \ge \widetilde{\zeta}$ for some $\widetilde{\zeta} > 0$.

On the other hand, from (3.44), (3.48) and (3.53), we have

$$\big\|(z_{n_k}^*)^\pm\big\|^2_{E_k} \to 0 \ \text{ and } \ \big\|(z_{n_k}^*)^0\big\|^2_{E_k} \to 0 \ \text{ as } \ k \to \infty.$$

This means that $\|z_{n_k}^*\|_{E_k} \to 0$ as $k \to \infty$, which leads to a contradiction. Hence $\{z_k^*\}$ is nonvanishing, so (3.45) holds, and this shows that our claim holds.

Step 3: We show that $z_0(t)$ is a nontrivial homoclinic solution of (1.1).
Proof. According to step 2, $z_0(t) \not\equiv 0$, it suffices to prove that for any $\varphi \in C_0^\infty(\mathbf{R}, \mathbf{R}^{2N})$,

$$\int_{-\infty}^{+\infty} (\dot{z}_0(t) - JH_{z_0}(t, z_0(t))) \cdot \varphi(t)\, dt = 0. \tag{3.54}$$

By step 1, we can choose k_0 such that $\operatorname{supp}\varphi \subseteq [-k_iT, k_iT]$ for all $k_i \ge k_0$, and we have for $k_i \ge k_0$

$$\int_{-\infty}^{+\infty} \Big\{\dot{z}_{k_i}^*(t) - J\Big[B(t)z_{k_i}^*(t) + G_{z_{k_i}^*}(t, z_{k_i}^*(t)) + h_{k_i}(t)\Big]\Big\} \cdot \varphi(t)\, dt = 0. \tag{3.55}$$

By (3.43) and (3.55), letting $k_i \to \infty$ we get (3.54), which shows $z_0(t)$ is a nontrivial homoclinic solution of (1.1). □

Proof of Theorem 1.1. The result follows from Lemma 3.4. □

Acknowledgements

This project is supported by National Natural Science Foundation of China (No. 51275094), by China postdoctoral science foundation (No. 20110490893) and by Natural Science Foundation of Guangdong Province (No. 10151009001000032).

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(October 29, 2013)

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Memoirs on Differential Equations and Mathematical Physics
Volume 61, 2014, 103–145

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THE RIEMANN PROBLEM AND LINEAR SINGULAR INTEGRAL EQUATIONS WITH MEASURABLE COEFFICIENTS IN LEBESGUE TYPE SPACES WITH A VARIABLE EXPONENT

Abstract. In the present work the Riemann problem for analysis functions $\phi^+(t) = G(t)\phi^-(t) + g(t)$ is considered in a class of Cauchy type integrals with density from $L^{p(t)}$ and a singular integral equation

$$a(t)\varphi(t) + \frac{b(t)}{\pi i} \int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t}\, d\tau = f(t)$$

in the space $\mathcal{L}^{p(t)}$ whose norm defined by the Lebesgue summation with a variable exponent. In both takes an integration curve is taken from a set containing non-smooth curves. The functions G and $(a-b)(a+b)^{-1}$ are take from a set of measurable functions $A(p(t),\Gamma)$ which is generalization of the class $A(p)$ of I. B. Simonenko. For the Riemann problem the necessary condition of solvability and the sufficient condition are pointed out, and solutions (if any) are constructed. For the singular integral equation the necessary Noetherity condition and one sufficient Noetherity condition are established; the index is calculated and solutions are constructed.

2010 Mathematics Subject Classification. 47B35, 30E20, 45P95, 47B38, 30E25.

Key words and phrases. Riemann's boundary value problem, measurable coefficient, factorization of functions, Lebesgue space with a variable exponent, Cauchy type integrals, Noetherian operator, Smirnov class of analytic functions with variable exponents, Cauchy singular integral equations.

რეზიუმე. ნაშრომში გამოკვლეულია ანალიზურ ფუნქციათა თეორიაში რიმანის სასაზღვრო ამოცანა $\phi^+(t) = G(t)\phi^-(t) + g(t)$ კოშის ტიპის ინტეგრალით წარმოდგენად იმ ფუნქციათა კლასში, რომელთა სიმკვრივე ეკუთვნის ლებეგის $L^{p(t)}$ სივრცეს და სინგულარული ინტეგრალური განტოლება

$$a(t)\varphi(t) + \frac{b(t)}{\pi i} \int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t}\, d\tau = f(t)$$

ბანახის $\mathcal{L}^{p(t)}$ სივრცეში, რომელიც შემოღებულია ცვლადი მაჩვენებლით ინტეგრების მოშველიებით. G და $(a-b)(a+b)^{-1}$ ფუნქციები აიღება ზომად ფუნქციათა $A(p(t),\Gamma)$ კლასიდან, რომელიც წარმოადგენს ი. სიმონენკოს $A(p)$ კლასის განზოგადებას. რიმანის ამოცანისთვის დადგენილია ამოხსნადობის აუცილებელი და საკმარისი პირობა და აგრეთვე ზოგი საკმარისი პირობა. აგებულია ამონახსნები კვადრატურებში. სინგულარული განტოლებისათვის კი გამოკვლეულია ნეტერისეულობის საკითხები და დათვლილია ინდექსი. აგებულია ამონახსნები.

1. Introduction

The boundary value problems of the theory of analytic functions and tightly connected with them linear singular integral equations with Cauchy kernel are well-studied (see, e.g., [1]–[8]).

If the domain D^+ is bounded by a simple, rectifiable, closed curve Γ, $D^- = \mathbb{C} \setminus \overline{D}^+$, $G(t)$, $g(t)$ are the given on Γ functions and we seek for a function ϕ representable by the Cauchy type integral with density from $L^p(\Gamma)$ whose angular boundary values ϕ^+ from D^+ and ϕ^- from D^- satisfy almost everywhere on Γ the condition

$$\phi^+(t) = G(t)\phi^-(t) + g(t), \tag{1}$$

then this problem is called the Riemann problem in the class $K^p(\Gamma)$.

When Γ is a Carleson curve, $\inf|G(t)| > 0$, $p > 1$, and

$$\phi(z) = (K_\Gamma\varphi)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - z}\,d\tau, \quad \varphi \in L^{p(\,\cdot\,)}(\Gamma), \quad p > 1,$$

$$S = S_\Gamma : \varphi \to S_\Gamma\varphi, \quad (S_\Gamma\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau,$$

then the problem (1) reduces equivalently to the equation

$$(1 - G(t))\varphi(t) + (1 + G(t))(S_\Gamma\varphi)(t) = g(t). \tag{2}$$

in $L^p(\Gamma)$ ([5, p. 134]).

Conversely, the considered in $L^p(\Gamma)$ equation

$$M\varphi := a(t)\varphi(t) + b(t)(S_\Gamma\varphi)(t) = f(t) \tag{3}$$

for

$$0 < \operatorname{ess\,inf}\left|a^2(t) - b^2(t)\right| \le \operatorname{ess\,sup}\left|a^2(t) - b^2(t)\right| < \infty$$

is equivalent to the problem

$$\phi^+(t) = \frac{a(t) - b(t)}{a(t) + b(t)}\,\phi^-(t) + \frac{f(t)}{a(t) + b(t)} \tag{4}$$

in $K^p(\Gamma)$.

The interest of researches in the Lebesgue spaces $L^{p(t)}(\Gamma)$ with a variable exponent and in their applications to the boundary value problems has appreciably increased in the recent years (see, e.g., [9]–[20]). A great number of problems of the theory of analytic functions have been investigated ([16]–[21]). Of importance are the works due to V. Kokilashvili and S. Samko in which they have revealed wide classes of curves for which the Cauchy singular operator is continuous in classes $L^{p(t)}(\Gamma)$, when $p(t)$ is Log–Hölder continuous and $\inf p(t) = \underline{p} > 1$. A more general result is presented in [10]. It is proved there that for the operator S to be continuous in $L^{p(t)}(\Gamma)$, it is necessary and sufficient that Γ is a Carleson curve. Further, in the case of the above-mentioned curves, it is stated that S is continuous in the space $L^{p(\cdot)}(\Gamma,\omega)$, $\omega=\prod\limits_{k=1}^{n}|t-t_k|^{\alpha_k}$, $t_k\in\Gamma$, $\alpha_k\in\mathbb{R}$, if and only if

$$-[p(t_k)]^{-1}<\alpha_k<[q(t_k)]^{-1},\ \ q(t)=p(t)[p(t)-1]^{-1}.$$

When $p(t)=const>1$, the problem (1) in the class $K^p(\Gamma)$ is thoroughly studied (see, e.g. [5]). The case, in which G is a measurable, oscillating function, has been investigated by I. Simonenko ([22]). He has introduced a class of functions $A(p)$ and showed that when Γ is the Lyapunov curve and $G\in A(p)$, then a picture of solvability inherent in such curves remains the same for continuous G. In [23], this result has been generalized to wider classes of coefficients and boundary curves.

In Sections 3–7 of the present work we investigate the problem (1) in the class $K^{p(\cdot)}(\Gamma)$, when Γ belongs to a wide class of curves and $G(t)$ belongs to a class $A(p(t),\Gamma)$ introduced in Section 3. Sections 8–12 we consider equation (3) with measurable coefficients in the space $\mathcal{L}^{p(t)}(\Gamma)$ which is defined in Section 9. The norm of the element φ in that space is defined by equality

$$\|\varphi\|_{\mathcal{L}^{p(\cdot)}}=\|\varphi\|_{p(\cdot)}+\|T\varphi\|_{p(\cdot)}+\Big\|\frac{\varphi_1}{G}\Big\|_{p(\cdot)}+\Big\|\frac{\varphi_2}{G}\Big\|_{p(\cdot)},\tag{5}$$

where $T\varphi=X^+S\,\frac{\varphi}{X^+}$, $\varphi_1=\frac{1}{2}\,(\varphi+T\varphi)$, $\varphi_2=\frac{1}{2}\,(-\varphi+T\varphi)$, and X^+ is the function defined by means of G (see below (26)).

It should be noted that if Γ has singularities such, for example, as cusps, vorticities, or the coefficient G is "badly measurable", then all these facts should be taken into account on selecting the class of solutions. In [24], for instance, for a constant p, a space in which we are required to find a solution is chosen in such a way that the norm contains power weights of different growth on different sides from cusps. In our case, oscillation of the coefficient G has made a major contribution to that norm.

For investigation of the problem (1) we have used the method of factorization which this time met with an obstacle. The matter is that for the solvability of the problem (1) in $K^{p(t)}(\Gamma)$, it is necessary that the function Tg belong to $L^{p(t)}(\Gamma)$. When Γ has cusps and $G\in A(p(t),\Gamma)$, we have failed to prove or disprove that Tg satisfies this condition for any g from $L^{p(t)}(\Gamma)$. However, we have managed both to show that if $\operatorname{ind}G\geq 0$, then (1) has solutions from the set $\bigcap\limits_{0<\varepsilon<\underline{p}}K^{p(t)-\varepsilon}(\Gamma)$ and to construct all such solutions. If, in addition, $g\in\bigcup\limits_{\varepsilon>0}L^{p(t)+\varepsilon}(\Gamma)$, then the problem (1) is solvable in $K^{p(t)}(\Gamma)$, too. When $\operatorname{ind}G<0$, for the solvability of the problem there take place the conditions of orthogonality of the function g to solutions of a homogeneous conjugate problem (inherent in the problem (1) in classical assumptions).

We have succeeded in revealing such a picture of solvability (although not entirely complete, but rather informative) by reducing the problem (1) to a series of problems of the same type, but with a coefficient, different from a constant one in the neighborhood of some point. One of such methods, known for $p = const$ as the "local method" ([25]), or "local principle" ([4, pp. 353–363]) is valid for a variable p, as well (the proof is obtained by the method indicated in [4] with the use of results from [21]). Application of that method allows one in the best case to investigate the problem qualitatively, leaving the question of a solution construction in quadratures open.

Our approach is somewhat different from the "local method"; it provides us with opportunity to construct solutions (if any) in quadratures. But in this connection we have to require that $Tg \in L^{p(t)}(\Gamma)$. This circumstance did not allow us to get, on the basis of investigations of the Riemann problem, its traditional application, i.e., to prove the Noetherity of equation (3) in $L^{p(t)}(\Gamma)$.

However, our wish to possess Noether theorems for equation (3) is quite natural, if not in $L^{p(\cdot)}(\Gamma)$, but although for some space of type $L^{(p(t)}$, i.e., with the norm defined by the Lebesgue integration with a variable exponent.

Towards this end, we distinguish from $L^{p(t)}(\Gamma)$ a subset $\mathcal{L}^{p(t)}(\Gamma)$ and endow it with the norm (5) with respect to which this subset is the Banach space.

In the space $\mathcal{L}^{p(t)}(\Gamma)$, for equation (3) it is stated that: the operator M maps $\mathcal{L}^{p(t)}(\Gamma)$ into itself; the necessary and sufficient conditions of solvability are established; solutions (if any) are constructed; the space, conjugate to $\mathcal{L}^{(p(\cdot)}(\Gamma)$, is found; one necessary Noetherian condition is pointed out; the Noether theorems are proved and the index is calculated.

In this connection, of significance turned out to be the finding of properties of the operator T (in the spaces $L^{p(t)}(\Gamma)$ and $\mathcal{L}^{p(t)}(\Gamma)$).

In the final Section 13 we present a number of properties of the operator T which in the framework of the present paper are not applied, but have independent interest and will, in all probability, be applied to further investigations of the Riemann problem and singular integral equations of type (3).

2. Preliminaries

2.1. **Curves.** Throughout the paper, the use will be made of the following notation.

(a) C^1 is the set of Jordan smooth curves;

(b) $C^{1,L}$ is the set of the same Lyapunov curves;

(c) R is the set of regular (Carleson) simple, rectifiable, closed curves of Γ for which

$$\sup_{\rho>0,\ \zeta\in\Gamma} \rho^{-1}\ell(\zeta,\rho) < \infty,$$

where $\ell(\zeta,\rho)$ is a linear measure of some part of Γ falling into a circle with center ζ, of radius ρ;

(d) Λ is the set of Lavrentiev curves , i.e., curves Γ for which $s(t_1,t_2)|t_1-t_2|^{-1}<M<\infty$ for any $t_1,t_2\in\Gamma$, where $s(t_1,t_2)$ is the length of the smallest arc lying on Γ and connecting the points t_1 and t_2.

(e) J_0 is the set of curves with the equation $t=t(s)$, $0\le s\le l$, such that there exists a smooth curve γ with the equation $\mu=\mu(s)$, $0\le s\le l$, such that

$$\exp_{0\le s\le l}\left(\int\limits_0^l\left|\frac{t'(\sigma)}{t(\sigma)-t(s)}-\frac{\mu'(\sigma)}{\mu(\sigma)-\mu(s)}\right|d\sigma\right)<\infty.$$

.

(f) J^* is the set of those closed Jordan curves from Λ which are a union of a finite number of curves from J_0 having tangents at the ends.

(g) $C^1(A_1,\dots,A_n;\nu_1,\dots,\nu_n)$ is the set of piecewise-smooth curves Γ with angular points $A_1,\dots,A_n$ at which angle sizes, inner with respect to the domain bounded by Γ, are equal to $\pi\nu(A_k)$, $0\le\nu(A_k)\le 2$;

(h) $C^{1,L}(A_1,\dots,A_n;\nu_1,\dots,\nu_n)$ is the set of piecewise-Lyapunov curves for which the condition of item (g) is fulfilled.

Obviously, $C^1\subset J^*$. The class J^* contains curves of bounded variation (Radon's curves) ([6, pp. 20 and 146–7]), piecewise-smooth curves, free from cusps and, moreover, $J^*\subset R$ ([8, p. 23]).

2.2. **The class of functions $\mathcal{P}(\Gamma)$.** Let Γ be a simple rectifiable curve. We say that the given on Γ function $p=p(t)$ belongs to the class $\mathcal{P}(\Gamma)$ if:

(1) there exists a number $B(p)$ such that for any t_1 and t_2 from Γ we have

$$|p(t_1)-p(t_2)|<\frac{B(p)}{|\ln|t-t_0||}\ ;$$

(2) $1<\underline{p}=\inf|p(t)|\le\sup|p(t)|=\overline{p}<\infty$.

2.3. **Lebesgue spaces with a variable exponent.**

2.3.1. By $L^{p(t)}(\Gamma;\omega)$ we denote the weight Banach space of measurable on Γ function f such that $\|f\|_{p(\,\cdot\,),\omega}<\infty$, where

$$\|f\|_{p(\,\cdot\,),\omega}=\inf\left\{\lambda>0:\int\limits_0^l\left|\frac{f(t(s))\omega(t(s))}{\lambda}\right|^{p(t(s))}ds\le 1\right\}.$$

Here, $t=t(s)$, $0\le s\le l$, is the equation of the curve Γ with respect to the arc abscissa s.

Assume $L^{p(t)}(\Gamma):=L^{p(t)}(\Gamma,1)$.

2.3.2. For $p \in \mathcal{P}(\Gamma)$, a space, conjugate to $L^{p(\,\cdot\,)}(\Gamma;\omega)$, is $L^{q(t)}(\Gamma;\frac{1}{\omega})$, where $q(t)=\frac{p(t)}{p(t)-1}$. In particular,

$$[L^{p(t)}(\Gamma)]^* = L^{q(t)}(\Gamma),$$

(see [9]).

2.4. **Some properties of spaces** $L^{p(\,\cdot\,)}(\Gamma;\omega)$**.**

2.4.1. If $p \in \mathcal{P}(\Gamma)$, $u \in L^{p(\,\cdot\,)}(\Gamma;\omega)$, $v \in L^{q(\,\cdot\,)}(\Gamma;\frac{1}{\omega})$, then the inequality

$$\Big|\int_\Gamma u(\tau)v(\tau)\,d\tau\Big| \le K\|u\|_{p(\,\cdot\,),\omega}\|v\|_{q(\,\cdot\,),\frac{1}{\omega}}, \quad k = 1+\frac{1}{\underline{p}}+\frac{1}{\overline{p}} \tag{6}$$

is valid. Moreover,

$$\|f\|_{p(\,\cdot\,)} \sim \sup_{\|g\|_{q(\,\cdot\,)}\le 1}\Big|\int_\Gamma f(t)g(t)\,dt\Big|.$$

2.4.2. If $p(t)$ and $p_1(t)$ belong to $\mathcal{P}(\Gamma)$, and $p(t)\le p_1(t)$, then

$$\|f\|_{p(\,\cdot\,)} \le (1+\ell)\|f\|_{p_1(\,\cdot\,)}, \quad \ell = |\Gamma| = \operatorname{mes}\Gamma.. \tag{7}$$

2.4.3. If $p \in \mathcal{P}(\Gamma)$, then $L^{p(\,\cdot\,)}(\Gamma) \subset L^{\underline{p}}(\Gamma)$.

(For the proofs of statements 2.3.2, 2.4.1 and 2.4.2 see, e.g., [9]).

2.5. **Classes of functions** $\widetilde{K}^{p(\,\cdot\,)}(\Gamma)$ **and** $K^{p(\,\cdot\,)}(\Gamma)$**.** Assume

$$\widetilde{K}^{p(\,\cdot\,)}(\Gamma,\omega) = \bigg\{\phi(z) = (K_\Gamma\varphi)(z) + P_\phi(z) = \\ = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)}{\zeta - z}\,d\zeta + P_\phi(z),\ z\notin\Gamma,\ \varphi\in L^{p(\,\cdot\,)}(\Gamma;\omega)\bigg\},$$

where P_ϕ is a polynomial;

$$K^{p(\,\cdot\,)}(\Gamma,\omega) = \Big\{\phi:\ \phi\in\widetilde{K}^{p(\,\cdot\,)}(\Gamma,\omega),\ P_\phi = 0\Big\}.$$

Denote

$$\widetilde{K}^{p(\,\cdot\,)}(\Gamma) := \widetilde{K}^{p(\,\cdot\,)}(\Gamma,1), \quad K^{p(\,\cdot\,)}(\Gamma) := K^{p(\,\cdot\,)}(\Gamma,1).$$

Since $L^{p(\,\cdot\,)}(\Gamma) \subseteq L^{\underline{p}}(\Gamma) \subset L^1(\Gamma)$, the Cauchy type integral $\phi = (K_\Gamma\varphi)(z)$, when $\varphi \in L^{p(\,\cdot\,)}(\Gamma)$, $p \in \mathcal{P}(\Gamma)$, almost for all $t\in\Gamma$ has angular boundary value $\phi^+(t)$ ($\phi^-(t)$), as the point z tends nontangentially to the point t, lying to the left (to the right) from the chosen on Γ positive direction (see, e.g., [26]), and the Plemelj–Sokhotskii's equalities

$$\phi^\pm(t) = \pm\frac{1}{2}\,\varphi(t) + \frac{1}{2}\,(S\varphi)(t) \tag{8}$$

are valid.

2.6. **Classes of functions** $E^{p(t)}(D)$**.** Let D be a simply-connected domain with the boundary Γ. By $z = z(w)$ we denote conformal mapping of the circle $U = \{w : \ |w| < 1\}$ onto D.

We say that an analytic in D function ϕ belongs to the class $E^{p(t)}(D)$ if

$$\sup_{0<r<1} \int_0^{2\pi} \big|\phi(z(re^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))} |z'(re^{i\vartheta})|\, d\vartheta < \infty.$$

For $p = const$, this class coincides with Smirnov class $E^p(D)$. Some properties of functions from $E^{p(t)}(D)$ can be found in [16] and [20] (see also [21, Ch. 3]).

For the constant p, the classes $E^p(D)$ are defined for any $p > 0$. Their properties are treated in different books. We restrict ourselves to the reference [27].

If the operator S is continuous from $L^p(\Gamma)$ to $L^s(\Gamma)$, then the Cauchy type integral $(K_\Gamma\varphi)(z)$ belongs to $E^s(D)$ when $\varphi \in L^{p(\,\cdot\,)}(\Gamma)$ ([8, pp. 29–30]).

When $\Gamma \in R$, the operator S_Γ is continuous in the classes $L^p(\Gamma)$ for any $p \in (1, \infty)$ ([28]). Therefore, if $\Gamma \in R$, $\varphi \in L^p(\Gamma)$, $p > 1$, then $K_\Gamma\varphi \in E^p(D)$. Moreover, if $\varphi \in L^1(\Gamma)$, then $K_\Gamma\varphi \in \prod\limits_{\delta<1} E^\delta(D)$.

If $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$, then $E^{p(t)}(D) \subset K^{p(t)}(D)$ ([16], [20]). If, however, Γ is a piecewise-smooth curve without cusps, then $E^{p(t)}(D) = K^{p(t)}(D)$ ([21, Ch. 3]).

3. Classes of Functions $A(p(t), \Gamma)$

3.1. **Definition of the classes** $A(p(t), \Gamma)$**.**

Definition 1. Let Γ be a simple, closed, rectifiable curve, and $p \in \mathcal{P}(\Gamma)$. We say that the given on Γ function G belongs to the class $A(p(t), \Gamma)$ if:

(i) $0 < m = \operatorname{ess\,inf} |G(t)| = \operatorname{ess\,sup} |G(t)| = M < \infty$;

(ii) for every point $\tau \in \Gamma$, there exists the arc $\Gamma_\tau \subset \Gamma$ containing the point r at which almost all values of the function G lie inside the angle with vertex at the origin of coordinates and opening

$$\alpha = 2\pi\Big[\sup_{t\in\Gamma_\tau} \max(p(t), q(t))\Big]^{-1}.$$

It follows from the definition that

$$A(p(t), \Gamma) = A(q(t), \Gamma). \tag{9}$$

Let us consider the covering of the curve of Γ by the arcs Γ_τ. From that covering we can select a finite covering by the arcs $\Gamma_k = \Gamma_{\tau_k}$, $k = 1, \dots, \mu$. It follows from the definition of the class $A(p(t), \Gamma)$ that there exist numbers $\varepsilon_k > 0$ such that all values of $G(t)$ on Γ_k lie inside the angle of the opening

$$\alpha_{\varepsilon_k} = (2\pi - \varepsilon_k)\Big[\sup_{t\in\Gamma_k} \max(p(t), q(t))\Big]^{-1}.$$

Without loss of generality, we may reckon that no arc of Γ_k is contained in the union of two adjacent arcs. Thus, $\Gamma = \bigcup_{k=1}^{\mu} \Gamma_k$, and every arc of Γ_k intersects with two adjacent arcs. Suppose

$$\Gamma_k^{(1)} = \Gamma_k \cap \Gamma_{k-1}, \;\; \Gamma_k^{(3)} = \Gamma_k \cap \Gamma_{k+1}, \;\; \Gamma_k^{(2)} = \Gamma_k - (\Gamma_k^{(1)} \cup \Gamma_k^{(3)}),$$

then $\Gamma_k = \Gamma_k^{(1)} \cup \Gamma_k^{(2)} \cup \Gamma_k^{(3)}$. We renumerate the arcs $\Gamma_k^{(j)}$, denote them by $\gamma_1, \dots, \gamma_n$ and assume that they follow one after another. Let Γ_{j-1} and Γ_{j+1} be the arcs intersecting with γ_k; then there exists the number $m > 0$ such that if $\widetilde{\gamma}_k = \Gamma_{j-1} \cup \Gamma_{j+1} \cup \gamma_k$, then

$$\operatorname{dist}(\gamma_k, \Gamma \setminus \widetilde{\gamma}_k) > m > 0, \;\; k = 1, \dots, n. \tag{10}$$

Since every arc Γ_k is, in fact, a neighborhood of some point, therefore all values of $G(t)$ (on Γ_k) lie in the angle of size less than α_{ε_k}. Assume $\varepsilon = \min \varepsilon_k$. Then by this time, for every point $\tau \in \Gamma$, there exists the arc (denoted by Γ_τ) whose values $G(t)$ lie in the angle of size $\alpha_\varepsilon = (2\pi - \varepsilon)\big[\sup\limits_{t \in \Gamma_\tau} \max(p(t), q(t))^{-1}\big]$. Thus, when defining the class $A(p(\,\cdot\,), \Gamma)$, we can replace α in condition (ii) by the number α_ε.

3.2. **One property of functions of the class $A(p(t), \Gamma)$.** From the statement proven in Subsection 3.1, from the continuity of $p(t)$ and equality (9) it easily follows that for every function $G \in A(p(t), \Gamma)$ there exists the number $\eta_\varepsilon > 0$ such that $G(t) \in A(p(t) + \eta_\varepsilon, \Gamma)$. Consequently,

$$A(p(t), \Gamma) \subset \bigcup_{\eta > 0} A(p(t) + \eta, \Gamma). \tag{11}$$

3.3. **The class $A(p(t), \gamma)$ for $\gamma \subset \Gamma$, and one its property.** Let γ be the arc lying on the closed curve Γ, $\overline{\gamma}$ be its closure and, moreover, let a and b be end points of γ.

If neighborhoods of the points a and b are, respectively, the sets of the type $[a, c]$ and $[c, b]$, $c \in \gamma$, then the class $A(p(t), \overline{\gamma})$ is defined analogously to $A(p(\,\cdot\,), \Gamma)$.

Suppose

$$\underline{p}_\gamma = \inf_{t \in \gamma} p(t), \quad \widetilde{p}_\gamma = \max\big(\underline{p}_\gamma, (\underline{p}_\gamma)'\big).$$

Theorem 1. *Let $\Gamma \in R$, $\gamma \subset \Gamma$, $p \in \mathcal{P}(\Gamma)$ and $G \in A(p(t), \gamma)$. For every point $\tau \in \gamma$, there exists the arc neighborhood $\gamma_\tau \subset \gamma$ such that all values of G on γ_τ lie in the angle of size $(2\pi - \varepsilon)\big[\max(\underline{p}_\gamma, (\underline{p}_\gamma)')\big]^{-1}$. Thus,*

$$A(p(\,\cdot\,), \gamma) \subseteq A(\widetilde{p}_\gamma), \quad \widetilde{p}_\gamma = \max\big(\underline{p}_\gamma, (\underline{p}_\gamma)'\big). \tag{12}$$

Proof. We consider the cases: 1) $p(\tau) > 2$, 2) $p(\tau) < 2$, 3) $p(\tau) = 2$.

1) $p(\tau) > 2$. Owing to the continuity of $p(t)$ on γ_τ, there is the neighborhood of the point τ at which $p(t) > 2$. Then

$$\sup_{t \in \gamma_\tau} \max(p(t), q(t)) = \sup_{t \in \gamma_\tau} p(t) \geq \max\big(\underline{p}_\gamma, (\underline{p}_\gamma)'\big) = \underline{p}_\gamma$$

and hence,

$$\alpha \le \frac{2\pi - \varepsilon}{\max(\underline{p}_\gamma, (\underline{p}_\gamma)')} = \frac{2\pi - \varepsilon}{\widetilde{p}_\gamma} = \alpha_\gamma.$$

2) $p(\tau) < 2$. In this case, $q(\tau) > 2$, and there exists the arc γ_τ in which $q(t) > 2$; therefore,

$$\sup_{t\in\Gamma_\tau} \max(p(t), q(t)) = \sup_{t\in\gamma_\tau} q(t) = (\underline{p}_\gamma)' = \widetilde{p}_\gamma.$$

Consequently, $\alpha < \alpha_\gamma$.

3) $(\tau) = 2$. Having some small number $\eta > 0$, we find neighborhood γ_τ in which values $p(t)$ lie on the segment $(2-\eta, 2+\eta)$. Then

$$\max\big(\underline{p}_\gamma, (\underline{p}_\gamma)'\big) = \max\big(2+\eta, (2-\eta)'\big) = \max\Big(2+\eta, \frac{2-\eta}{1-\eta}\Big) = \\ = \max\Big(2+\eta, 2+\frac{\eta}{1-\eta}\Big) = 2+\frac{\eta}{1-\eta} = (\underline{p}_\gamma)' = \widetilde{p}_\gamma.$$

Hence, again, $\alpha < \alpha_\gamma$.

Thus, the point τ in all three cases possesses the neighborhood γ_τ with values $G(t)$ lying in the opening angle $\frac{2\pi-\varepsilon}{\widetilde{p}_\gamma}$. Since τ is arbitrary, this implies that the relations (12) are valid. □

3.4. **The index of the function of the class** $A(p(\cdot), \Gamma)$**. The class** $\widetilde{A}(p(\cdot), \Gamma)$**.** We choose the point $c \in \Gamma$ and fix the value of $\arg G(c) = [\arg G(c)]^-$ from the interval $[0, 2\pi]$. Following along Γ, we can define a branch of the function $\arg G(t)$ so as to have $|\arg G(t_1) - \arg G(t_2)| < \alpha$ for $t_1, t_2 \in \gamma_k$. Going around Γ, we reach the arc, containing c, with a new value $\arg G(c) = [\arg G(c)]^+$. The difference $[\arg G(c)]^+ - [\arg G(c)]^-$ does not depend on the covering choice and on the point c. The integer

$$\operatorname{ind} G = \varkappa(G) = \varkappa = \frac{1}{2\pi}\big[(\arg G(c))^+ - (\arg G(c))^-\big]$$

is called an index of the function G in the class $K^{p(\cdot)}(\Gamma)$.

A subset of the functions G from $A(p(\cdot), \Gamma)$ for which $\sup|\arg G(t)| < \pi/2$ we denote by $\widetilde{A}(p(\cdot), \Gamma)$. Obviously, if $G \in \widetilde{A}(p(\cdot), \Gamma)$, then $\operatorname{ind} G = 0$.

4. On Factorization of the Function from $A(p(t), \Gamma)$ in the Class $K^{p(t)}(\Gamma)$

4.1. Definition of factor-function.

Definition 2. Let Γ the closed, rectifiable Jordan curve bounding the domains D^+ and D^- $(z = \infty \in D^-)$.

We say that the function $X_G(z) = X(z)$, analytic on the plane, cut along Γ, is a factor-function of the function G in the class $K^{p(t)}(\Gamma)$, if the following conditions are fulfilled:

(1) $X \in \widetilde{\mathcal{K}}^{p(t)}(\Gamma)$;

(2) $[X(z)]^{-1} \in \widetilde{\mathcal{K}}^{q(t)}(\Gamma)$;

(3) almost everywhere on Γ, $X^{+}(t)[X^{-}(t)]^{-1} = G(t)$;

(4) $X^{+} \in W^{p(t)}(\Gamma)$, i.e., the operator

$$T = T_G : g(t) \to (Tg)(t), \quad (Tg)(t) = \frac{X_G^{+}(t)}{\pi i} \int\limits_{\Gamma} \frac{g(\zeta)}{X_G^{+}(\zeta)} \, \frac{d\zeta}{\zeta - t}\,, \quad t \in \Gamma, \quad (13)$$

is continuous in $L^{p(t)}(\Gamma)$.

4.2. **Some properties of factor-functions.**

4.2.1. *The Case of Constant p.* If $\Gamma \in C^{1,L}$ and $G \in A(p, \Gamma)$, then G is factorable in $K^{p}(\Gamma)$ ([22]). The same result is valid when $\Gamma \in J^{*}$, and G is taken from a wider than $A(p, \Gamma)$ class $\widetilde{A}$ which, in particular, contains all admissible piecewise-continuous functions, not fallen in $A(p, \Gamma)$ ([8, p. 192]).

4.2.2. *The Case when $G \in \widetilde{A}(p(t), \Gamma)$ and is equal to the constant on $\Gamma \setminus \gamma$, where $\gamma \subset \Gamma$.* Let $G \in \widetilde{A}(p(t), \Gamma)$, $\tau \in \Gamma$, and $\gamma = \gamma_{ab} = \gamma_{\tau}$ be the arc mentioned in Theorem 1. Assuming $p \in \mathcal{P}(\Gamma)$, we put $\underline{p}_{\gamma} = \inf\limits_{t \in \gamma} p(t)$ and $\widetilde{p}_{\gamma} = \max(\underline{p}_{\gamma}, (\underline{p}_{\gamma})')$.

Consider the function

$$G_{\gamma}(t) = \begin{cases} G(t), & t \in \gamma, \\ G(a), & t \in \Gamma \setminus \gamma. \end{cases} \quad (14)$$

By virtue of Theorem 1 we can easily conclude that $G_{\gamma} \in A(\widetilde{p}_{\gamma}, \Gamma)$. Therefore, assuming $\ln G_{\gamma}(\tau) = \ln |G_{\gamma}(t)| + i \arg G(\tau)$ and

$$X(z) = X_{G_{\gamma}}(z) = \exp\left\{ \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\ln G_{\gamma}(\zeta)}{\zeta - z} \, d\zeta \right\}, \quad (15)$$

$[X(z)]^{\pm 1}$ belongs to $\widetilde{K}^{\widetilde{p}_{\gamma}}(\Gamma)$, and the operator $T = T_G$ is continuous in $L^{\widetilde{p}_{\gamma}}(\Gamma)$, i.e.,

$$\|T_{G_{\gamma}} f\|_{\widetilde{p}_{\gamma}} \le \|T_{G_{\gamma}}\|_{\widetilde{p}_{\gamma}} \|f\|_{\widetilde{p}_{\gamma}}$$

([22]).

In the sequel, frequently, if it does not give rise to misunderstanding, the subscript in our writings X_G, $X_{G_{\gamma}}$, T_G, $T_{G_{\gamma}}$ will be omitted and we write $A(p(\cdot))$ instead of $A(p(\cdot), \Gamma)$.

4.2.3. *The class of functions $B(p(\cdot), \Gamma)$.* By $B(p(\cdot), \Gamma)$ we denote a set of those functions $G(t)$ with a finite number of points of discontinuity t_k for which $\operatorname{ess\,inf} |G| > 0$ and

$$-\big[p(t_k)\big]^{-1} < \alpha_k \pmod{2\pi} < \big[q(t_k)\big]^{-1}.$$

The branch of $\arg G(t)$ and index for the functions from $B(p(\cdot), \Gamma)$ are defined in the same manner as in [8, pp. 92–93]. For $p = const$, this class covers all those piecewise-continuous functions which are admissible in the condition (1) when its solutions are sought in the class $K^p(\Gamma)$.

The functions of the class $B(p,\Gamma)$ for $p>1$ and $\Gamma\subset J^*$ are factorable in $K^p(\Gamma)$. Moreover, there exists the number $\delta>0$ such that the factor-function X_G of the function $G\in B(p,\Gamma)$ possesses the property

$$X_G^{\pm}\in \widetilde{K}_\Gamma^{\mu+\delta}(\Gamma),\ \ \mu=\max(p,q), \tag{16}$$

([8, p. 115]).

4.2.4. *On the factorization of the function $G_\gamma(t)$ in the classes $K^{p(\,\cdot\,)}(\Gamma)$.* Let $G\in\widetilde{A}(p(\,\cdot\,),\Gamma)$ and $\gamma=\gamma_{ab}$ be the arc mentioned in Theorem 1. Without loss of generality, we may assume that $G(t)$ is defined at the point a and $G(a)$ lies in the corresponding to the point a angle of size α. Suppose

$$G_\gamma(t)=\begin{cases}\dfrac{G(t)}{G(a)}\,, & t\in\gamma,\\ 1, & t\in\Gamma\setminus\gamma.\end{cases} \tag{17}$$

Theorem 2. *Let $\Gamma\in J^*$ be a closed, simple, rectifiable curve bounding the domains D^+ and D^-, and $G\in\widetilde{A}(p(\,\cdot\,),\Gamma)$. Then the function G_γ defined by equality* (17) *is factorable in the class $K^{\widetilde{p}_\gamma}(\Gamma)$.*

Proof. Let us show that $G_\gamma\in A(\widetilde{p}_\gamma,\Gamma)$. By virtue of Theorem 1 and continuity of G_γ on $\Gamma\setminus\gamma$, only behavior of G_γ in the neighborhood of the points a and b needs testing. Let $\gamma_{1a}\subset\Gamma$ be the arc containing a point. By γ_{11} and γ_{12} we denote intersection of γ_{1a} with γ and $\Gamma\setminus\gamma$. Since γ_{11} lies on γ, all values of the function G_γ on it lie in the angle with vertex at the point $z=0$, of size $\beta=\frac{2\pi-\varepsilon}{\widetilde{p}_\gamma}$. As far as number 1 is in that angle, and $G_\gamma(t)$ on γ_{12} equals 1, therefore the values of G_γ on γ_{1a} lie in the above-mentioned angle.

Consider now the neighborhood of the point b. The point $G_\gamma(b)$ lies in the angle of size β together with the point $G_\gamma(a)=1$. Therefore the values of G_γ on the arc (c,a), where $e\subset\Gamma\setminus\gamma$, lie in the angle of size β. Thus it is proved that $G_\gamma\in A(\widetilde{p}_\gamma,\Gamma)$.

According to the statement in item 4.2.3, we can conclude that G_γ is factorable in $K^{\widetilde{p}_\gamma}(\Gamma)$, and its factor-function X_{G_γ} is given by the equality

$$X_{G_\gamma}(z)=\exp\left\{\frac{1}{2\pi i}\int\limits_\Gamma\frac{\ln G_\gamma(\zeta)}{\zeta-z}\,d\zeta\right\}. \tag{18}$$

The theorem is proved. □

Corollary. *If $G\in\widetilde{A}(p(\,\cdot\,),\Gamma)$, then the function*

$$\widetilde{G}(t)=\begin{cases}G(t), & t\in\gamma,\\ G(a), & t\in\Gamma\setminus\gamma,\end{cases}$$

where the arc γ *defined in Theorem* 1 *is factorable in* $K^{\widetilde{p}_\gamma}(\Gamma)$, *and its factor-function is*

$$\widetilde{X}(z) = \begin{cases} \dfrac{1}{M} X_{G_\gamma}(z), & z \in D^+, \\ X_{G_\gamma}(z), & z \in D^-, \end{cases}$$

where

$$M = \operatorname*{ess\,sup}_{t\in\Gamma} |G(t)| + \sup_{t\in\Gamma} |\arg G(t)|.$$

Proof. It suffices to show that $X^{\pm 1} \in \widetilde{K}^{\widetilde{p}_\gamma}(\Gamma)$. In view of Subsection 4.2.2 and (11), we have $(X_{G_\gamma})^{\pm 1} \in E^{\widetilde{p}_\gamma+\delta}(D^+)$, $(X_{G_\gamma})^{\pm 1} \in \widetilde{E}^{\widetilde{p}_\gamma}(D^-)$, where $E^\mu(D^+)$ is Smirnov class in D^+, and $\widetilde{E}^\mu(D^-) = \{\phi : \ F + const, \ F \in E^\mu(D^-)\}$. Therefore $X^{\pm 1}_{G_\gamma}$ and $X^{\pm 1}_{G_\gamma} - 1$ are representable by the Cauchy integral in D^+ and D^-, respectively. Consequently,

$$\widetilde{X}^{\pm 1}(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{(\widetilde{X}^+)^{\pm 1} - (\widetilde{X}^-)^{\pm 1}}{t-z}\, dt + 1. \qquad \square$$

4.2.5. *Auxiliary estimates.* Let $\Gamma \in J^*$, $G \in \widetilde{A}(p(\,\cdot\,), \Gamma)$, and let γ_k and $\widetilde{\gamma}_k$ be subsets of Γ defined in Subsection 3.1. Let, further, $\gamma_k = \gamma_{a_k b_k}$, $G_k(t) = G_{\gamma_k}(t)$ and

$$X_k(z) = \begin{cases} \dfrac{1}{G(a_k)} X_{G_k}(z), & z \in D^+, \\ X_{G_k}(z), & z \in D^-, \end{cases} , \tag{19}$$

where

$$X_{G_k}(z) = \exp\left\{ \frac{1}{2\pi i} \int\limits_\Gamma \frac{\ln G_{\gamma_k}(\zeta)}{\zeta - z}\, d\zeta \right\}.$$

Suppose

$$Y_k(t) = \prod_{j=1,\, j\neq k}^{n} X_j(t). \tag{20}$$

Lemma 1. *There exist the constants* $c_j > 0$, $j = 1, 2$, *such that for all* k, $k = 1, 2, \dots, n$, *we have*

$$\sup_{t\in\gamma_k} |Y_k(t)| < c_1, \quad \inf_{t\in\gamma_k} |Y_k(t)| > c_2. \tag{21}$$

Proof. We have

$$|Y_k(t)| \le \exp\left| \frac{1}{2\pi i} \int\limits_{\Gamma\setminus\gamma_k} \frac{\ln|G(\zeta)| + i\arg G(\zeta)}{\zeta - t}\, d\zeta \right| \le$$

$$\le \exp \frac{1}{2\pi} \int\limits_{\Gamma\setminus\gamma_k} \frac{\sup|\ln|G|| + \mu}{|\zeta - t|}\, |d\zeta|, \quad t \in \gamma_k,$$

where $\mu = \sup|\arg G(\zeta)|$. At the last step here we have taken in account that $G(\zeta) = 1$ for $\zeta \in \widetilde{\gamma}_k \setminus \gamma_k$.

The closed sets $\overline{\gamma}_k$ and $\overline{\Gamma \setminus \widetilde{\gamma}_k}$ do not intersect, hence according to (10), we have $\operatorname{dist}(\gamma_k; \Gamma \setminus \gamma_k) = m_k > 0$, whence it follows that

$$|Y_k(t)| \le \exp\Big(\frac{nM}{m}\,|\Gamma|\Big), \tag{22}$$

where

$$M = \sup_{\zeta\in\Gamma}\big|\ln|G(\zeta)|\big| + \mu, \quad m = \min_{k=1,2,\dots,n} m_k. \tag{23}$$

To estimate $|Y_k(t)|$, we note that if $Y_k(t) = \exp f_k(t)$, we have shown that $|\exp f_k| < \exp\frac{nM}{m}\,|\Gamma|$. But $|\exp f_k| \ge \exp(-\sup|f_k|)$, and therefore

$$|Y_k(t)| \ge \exp\Big(-\frac{nM}{m}\Big)\,|\Gamma|. \tag{24}$$

It follows from (22) and (24) that inequalities (21), where

$$c_1 = \exp\Big(\frac{nM}{m}\,|\Gamma|\Big), \quad c_2 = \exp\Big(-\frac{nM}{m}\,|\Gamma|\Big)$$

are valid, and the numbers M and m in these equalities are defined according to (23). □

5. Some Properties of the Function $X_G(z)$

As regards the data in the condition (1), we assume that either

$$\begin{gathered} \Gamma \in J^*, \ \ p \in \mathcal{P}(\Gamma), \ \ G \in A(p(\,\cdot\,), \Gamma), \\ \text{or} \\ \Gamma \text{ is a piecewise-smooth curve}, \ \ G \in B(p(\,\cdot\,), \Gamma), \ \ p \in \mathcal{P}(\Gamma). \end{gathered} \tag{25}$$

Let the conditions (25) are fulfilled, $\varkappa = \operatorname{ind} G(t)$ and $z_0 \in D^+$. Put

$$G_0(t) = (t - z_0)^{-\varkappa} G(t)$$

and

$$X(z) = \\ = \begin{cases} \exp\{K_\Gamma \ln G_0\}, & z \in D^+, \\ (z - z_0)^{-\varkappa} \exp(K_\Gamma \ln G_0)(z), & z \in D^-. \end{cases} \tag{26}$$

5.1. On the summability of the function $g|_{X^+}$.

Lemma 2. *If the conditions* (25) *are fulfilled, then there exists the number* $\eta > 0$ *such that*

$$g[X^+]^{-1} \in L^{1+\eta}(\Gamma), \ \ K_\Gamma\,\frac{g}{X^+} \in E^{1+\eta}(D^+), \ \ K_\Gamma\Big(\frac{g}{X^+}\Big) \in \widetilde{E}^{1+\eta}(D^-).$$

Proof. Let γ be that arc on Γ for which $G \in A(\widetilde{p}_\gamma, \gamma)$, then the function

$$G_\gamma(t) = \begin{cases} G(t), & t \in \gamma, \\ G(a), & t \in \Gamma \setminus \gamma, \end{cases}$$

belongs to $\widetilde{A}(\widetilde{p}_\gamma, \Gamma)$, and hence, $X_\gamma^{\pm 1} \in L^{\widetilde{p}_\gamma}(\Gamma)$ (see item 4.2.3). Assuming

$$g_\gamma(t) = \begin{cases} g(t), t \in \gamma, \\ 0, & t \in \Gamma \setminus \gamma, \end{cases}$$

we have $g_\gamma \in L^{\underline{p}_\gamma}(\Gamma)$, and hence, we obtain

$$\frac{g_\gamma}{X_\gamma^+} \in L^\alpha(\Gamma), \quad \alpha = \underline{p}_\gamma(\underline{p}_\gamma + \delta)(\underline{p}_\gamma + \underline{p}_\gamma + \delta)^{-1}.$$

Let us consider two possible cases: 1) $\widetilde{p}_\gamma = \underline{p}_\gamma$, 2) $\widetilde{p}_\gamma = (\underline{p}_\gamma)'$.

1) $\widetilde{p}_\gamma = \underline{p}_\gamma$. This is possible when $p_\gamma \geq 2$. Denote $\lambda = \underline{p}_\gamma$, then we have

$$\alpha = \lambda(\lambda + \delta)(2\lambda + \delta)^{-2} = \Big(\frac{\lambda}{2} + \frac{\delta}{2}\Big)\Big(1 + \frac{\delta}{2\lambda}\Big)^{-1}.$$

Since $\lambda \geq 2$, then $\alpha > 1$ and therefore

$$g_\gamma(X_\gamma^+)^{-1} \in L^{1+\eta}(\Gamma), \quad \eta < \alpha < 1.$$

2) $\widetilde{p}_\gamma = (\underline{p}_\gamma)'$, then

$$\alpha = \lambda(\lambda' + \delta)(\lambda + \lambda' + \delta)^{-1} = \Big(1 + \frac{\delta}{\lambda'}\Big)\Big(1 + \frac{\delta}{\lambda\lambda'}\Big)^{-1} > 1$$

and, hence, again $g/X^+ \in L^{1+\eta}(\Gamma)$.

Since $\Gamma = \cup\gamma_k$, and on γ_k we have $g_k/X^+ = g_k/(X_k^+ Y_k^+)$, $(g_k := g_{\gamma_k})$, taking into account Lemmas 1 and 2, we obtain

$$\int\limits_\Gamma \Big|\frac{g}{X^+}\Big|^{1+\eta} ds = \sum \int\limits_{\gamma_k} \Big|\frac{g_k}{X_k^+ Y_k^+}\Big|^{1+\eta} ds \leq$$
$$\leq \frac{1}{c_2^{1+\eta}} \sum \int\limits_{\gamma_k} \Big|\frac{g_k}{X_k^+}\Big|^{1+\eta} ds < \infty.$$

Statement of the lemma regarding $K_\Gamma \frac{g}{X^+}$ follows from the results given in Subsections 2.6 and in item 2.4.3. □

5.2. **On the summability of the function X_G.**

Theorem 3. *When the conditions* (25) *are fulfilled, we have* $X_G^+ \in L^{p(\cdot)}(\Gamma)$ *and* $(X_G^+)^{-1} \in L^{q(t)}(\Gamma)$.

Proof. Let γ be the arc mentioned in Theorem 1. Then $G \in A(\widetilde{p}_\gamma, \gamma)$, and the function G_γ belongs to $A(\widetilde{p}_\gamma, \Gamma)$. Since $\Gamma \in J^*$, therefore $X_G \in \widetilde{K}^{\widetilde{p}_\gamma + \delta}(\Gamma)$ ([8, p. 29]) and, hence $X_{G_\gamma}^+ \in L^{\widetilde{p}_\gamma}(\Gamma)$.

Represent now Γ in the form $\Gamma = \bigcup\limits_{k=1}^{n} \gamma_k$, where the curves γ_k satisfy the condition of Theorem 1. Then, according to the above-said,

$$X_k = X_{\gamma_k} \in L^{p_k+\delta}(\Gamma), \ \ p_k = \widetilde{p}_{\gamma_n}, \ \ X_k = \exp\big\{K_\Gamma(\ln(G_{\gamma_k})\big\}.$$

We have $X_G = \prod\limits_{k=1}^{n} X_k Y_k$. Then

$$\int\limits_\Gamma |X_G^+|^{p(t(s))}\,ds \le \sup_{t\in\gamma_k,\ k=1,2,\dots,n} \sum_{k=1}^{n} \int\limits_{\gamma_k} |X_k^+|^{p(t(s))}\,ds \le \\ \le c_1(1+\Gamma)\int\limits_\gamma |X^+|^{p(t(s))}\,ds. \tag{27}$$

On γ_k, we have $\underline{p}_{\gamma_k} \le p(t) \le \overline{p}_{\gamma_k}$, $k = 1,2,\dots,n$.

Due to the uniform continuity of $p(t)$ on Γ, there exists for $\delta > 0$ the number l_δ such that for any arc $\gamma_k \in \Gamma$ such that $|\gamma_k| < l_\delta$, we have

$$\overline{p}_{\gamma_k} - \underline{p}_{\gamma_k} < \delta, \ \ (\overline{p}_{\gamma_k})' = \frac{\overline{p}_{\gamma_k}}{\overline{p}_{\gamma_k} - 1}\,. \tag{28}$$

For some γ_k, the condition $|\gamma_k| < l_\delta$ may violate. In this case we consider a new covering of Γ reducing the arcs γ_k to those of lesser length than l_δ. For the sake of simplicity, we denote again the arcs forming a new covering by γ_k. Then, according to (28), on γ_k we have $\overline{p}_{\gamma_k} - \underline{p}_{\gamma_k} < \delta$. Moreover, on the above-mentioned arc,

$$\underline{p}_{\gamma_k} \le p(t) \le \overline{p}_{\gamma_k},$$

whence $p(t) - p_{\gamma_k} \le \overline{p}_{\gamma_k} - \underline{p}_{\gamma_k} < \delta$, i.e., on γ_k, we have $p(t) < \underline{p}_{\gamma_k} < \delta$. By virtue of inequalities (8) and (27), we now obtain

$$\int\limits_\Gamma |X^+(t)|^{p(t)}\,ds \le c_3 \sum_{k=1}^{n} \int\limits_{\gamma_k} |X_k^+|^{\underline{p}_{\gamma_k}+\delta}\,ds < \infty.$$

Thus, the first statement of the theorem is proved.

The second statement follows from Lemma 2 according to which for an arbitrary function $g \in L^{p(\,\cdot\,)}(\Gamma)$, we have $g(t)\cdot \frac{1}{X^+(t)} \in L^1(\Gamma)$. This means that $\frac{1}{X^+}$ belongs to the class $L^{q(\,\cdot\,)}$. □

Corollary. *The function X_G in the conditions* (25) *belongs to $L^{p(\,\cdot\,)+\delta}$ for some $\delta > 0$.*

This follows from the inclusions (12), (16) and Theorem 3.

6. On the Operator T_G for $G \in \widetilde{A}(p(t), \Gamma)$

6.1. **The operator T_G acts from $L^{p(\cdot)}$ to L^λ for some $\lambda > 0$.**

Lemma 3. *If the conditions* (25) *are fulfilled, then the operator T_G acts from $L^{p(\cdot)}$ to the space $L^\lambda(\Gamma)$, $\lambda \in (0, \frac{2+2\eta}{3+\eta})$, where η is the number defined in Lemma* 2*.*

Proof. From the condition $G \in \widetilde{A}(p(\cdot), \Gamma)$ it follows that $G(t) \in A(2, \Gamma)$, therefore $X_G^\pm \in E^2(D^\pm)$ and, hence, $X^+ \in L^2$ (see Subsection 4.2). Assuming $0 < \lambda < 2$, we have

$$I = \int_\Gamma |Tg|^\lambda \, ds = \int_\Gamma |X^+|^\lambda \Big| S \, \frac{g}{X^+} \Big|^\lambda ds \leq$$

$$\leq \Big(\int_\Gamma |X^+|^2 \, ds \Big)^{\frac{\lambda}{2}} \Big(\int_\Gamma \Big| S \, \frac{g}{X^+} \Big|^{\frac{2\lambda}{2-\lambda}} ds \Big)^{\frac{2-\lambda}{2}}.$$

from which it can be easily seen that $I < \infty$, if $2\lambda(2-\lambda)^{-1} < 1 + \eta$, i.e., $\lambda < \frac{2+2\eta}{\zeta+\eta}$. □

6.2. **On the operator $T_G^2 = T_G(T_G)$.**

Theorem 4. *Under the conditions* (25)*, we have*

$$T^2 g = g. \tag{29}$$

Proof. We have

$$T(Tg) = X^+ S_\Gamma \Big(\frac{1}{X^+} \cdot X^+ S_\Gamma \, \frac{g}{X^+} \Big) = X^+ S_\Gamma \Big(S_\Gamma \, \frac{g}{X^+} \Big). \tag{30}$$

Since $\Gamma \in R$, the operator S_Γ is continuous in the Lebesgue spaces $L^\lambda(\Gamma)$, $\lambda > 1$. Consequently, since $\frac{g}{X^+} \in L^{1+\eta}(\gamma)$ (see Lemma 2), we have $S_\Gamma \frac{g}{X^+} \in L^{1+\eta}(\Gamma)$, whence $(K_\Gamma \frac{g}{X^+})(z) \in E^{1+\eta}(D^+) \subset E^1(D^+)$ (see Subsection 2.6). But if $K_\Gamma \varphi \in E^1(D^+)$, then $S_\Gamma(S_\Gamma \varphi) = \varphi$ ([8, p. 30]).

In the case under consideration, $\varphi = \frac{g}{X^+}$, and hence, $S_\Gamma(S_\Gamma \frac{g}{X^+}) = \frac{g}{X^+}$. Substituting this value into (30), we get equality (29). □

6.3. **The continuity of the operator T_G from $L^{p(\cdot)}(\Gamma)$ to the space of convergence in measure.**

Definition 3. By $M(\Gamma)$ we denote the space of measurable on Γ functions with metric

$$\rho(f, \varphi) = \int_\Gamma \frac{|f - \varphi|}{1 + |f - \varphi|} \, ds.$$

The convergence of the sequence $\{f_n\}$ to f_0 in the space $M(\Gamma)$ is equivalent to the convergence of $\{f_n\}$ in measure to f_0.

Lemma 4. *If $g_n \in L^\lambda(\Gamma)$, $0 < \lambda < 1$, and*

$$I_n = \int_\Gamma |g_n - g_0|^\lambda \, ds \to 0, \tag{31}$$

then g_n converges to g_0 in $M(\Gamma)$, as well.

It is not difficult to get the proof by estimating the integral I_n for large n on the set $\ell_{n,\sigma} = \{s : \ |g_n - g_0| > \sigma\}$.
Lemmas 3 and 4 lead to

Statement 1. *The operator T_G is continuous from $L^{p(\,\cdot\,)}(\Gamma)$ to $M(\Gamma)$.*

6.4. **Closure of the operator T_G from $L^{p(\,\cdot\,)}(\Gamma)$ to $L^{p(\,\cdot\,)}(\Gamma)$.** Remind the notion of a closed operator. Let A be the linear operator defined in the Banach space X (i.e., the operator defined on some lineal from X and is linear in it) with the domain of definition $D(A)$ and acting to the Banach space Y. The operator A is called closed from X to Y if it possesses the following property:

if $\|x_n - x_0\|_X \to 0$ and $\|Ax_n - y_0\|_Y \to 0$, then $x_0 \in D(A)$ and $Ax_0 = y_0$.

Theorem 5. *If the conditions* (25) *are fulfilled, the operator $T = T_G$ is closed from $L^{p(\,\cdot\,)}(\Gamma)$ to $L^{p(\,\cdot\,)}(\Gamma)$.*

Proof. The domain of definition of the operator $T = T_G$ will be assumed to be a linear set

$$D(T) = \Big\{ g : \ g \in L^{p(\,\cdot\,)}(\Gamma), \ Tg \in L^{p(\,\cdot\,)}(\Gamma) \Big\}.$$

Let $g_n \in D(T)$, $n \in \mathbb{N}$, $\|g_n - g_0\|_{p(\,\cdot\,)} \to 0$, $\|Tg_n - f_0\|_{p(\,\cdot\,)} \to 0$. Then $g_0, f_0 \in L^{p(\,\cdot\,)}(\Gamma)$, and owing to Statement 1, $\|Tg_n - Tg_0\|_{M(\Gamma)} \to 0$. It follows from the condition $\|Tg_n - f_0\|_{p(\,\cdot\,)} \to 0$ that $\|Tg_n - f_0\|_{M(\Gamma)} \to 0$, whence we conclude that $f_0 = Tg_0$, by virtue of the limit uniqueness in measure. Thus, we have

$$g_0 \in L^{p(\,\cdot\,)}(\Gamma), \quad Tg_0 = f_0 \in L^{p(\,\cdot\,)}(\Gamma).$$

This implies that $g_0 \in D(T)$, and since $\|Tg_n - Tg_0\|_{p(\,\cdot\,)} \to 0$, the operator T is closed from $L^{p(\,\cdot\,)}(\Gamma)$ to $L^{p(\,\cdot\,)}(\Gamma)$. □

7. The Riemann Problem in the Class $K^{p(\,\cdot\,)}(\Gamma)$

7.1. **Statement of the problem.** Let Γ be the simple, rectifiable, closed curve, bounding the domains D^+ and D^- ($z = \infty \in D^-$), $g \in L^{p(\,\cdot\,)}(\Gamma)$ and the conditions (25) are fulfilled. We are required to find the functions $\phi \in K^{p(\,\cdot\,)}(\Gamma)$ whose angular boundary values $\phi^+(t)$ and $\phi^-(t)$ almost everywhere on Γ satisfy the boundary condition (1).

7.2. **Reduction of the problem** (1) **to the jump problem, when** $\varkappa(G)=0$**.** Let

$$X_G(z)=X(z)=\exp\left\{\frac{1}{2\pi i}\int\limits_\Gamma \frac{\ln G(\zeta)}{\zeta-z}\,d\zeta\right\}.$$

By Theorem 3, we have $\frac{1}{X(z)}\in \widetilde{K}^{p(\cdot)}(\Gamma)$ and $X(\infty)=1$. Since $G(t)=X^+(t)[X^-(t)]^{-1}$, the condition (1) can be written in the form

$$\Big(\frac{\phi}{X}\Big)^+-\Big(\frac{\phi}{X}\Big)^-=\frac{g}{X^+}\,.$$

Putting $\phi_1(z)=\phi(z)[X(z)]^{-1}$, we get $\phi_1\in K^1(\Gamma)$ and $\phi_1^+-\phi_1^-\in g[X^+]^{-1}$. By Lemma 2, we have $g[X^+]^{-1}\in L^{1+\eta}(\Gamma)$, $\eta>0$. Therefore, the solution of the last problem is unique, and $\phi_1(z)=(K_\Gamma\,\frac{g}{X^+})(z)$. Consequently, the solution of the problem (1) may be only the function

$$\phi(z)=X(z)\Big(K_\Gamma\,\frac{g}{X^+}\Big)(z), \tag{32}$$

and we have to elucidate the conditions under which this function belongs to the class $L^{p(\cdot)}$.

7.3. **Criterion of solvability of the problem** (1) **when** $G(t)\in A(p(\cdot),\Gamma)$ **and** $\operatorname{ind}G=0$**.** If the conditions (25) are fulfilled, then $K[g(X^+)^{-1}]\in E^{1+\eta}(D^\pm)$ (see Lemma 2). Therefore the function ϕ given by equality (32) is representable by the Cauchy type integral with density $\varphi=\phi^+-\phi^-$. Hence $\phi\in K^{p(\cdot)}(\Gamma)$, if and only if

$$\varphi(t)=\big[\phi^+(t)-\phi^-(t)\big]\in L^{p(\cdot)}. \tag{33}$$

Using formulas (8) and taking into account the fact that $G=\frac{X^+}{X^-}$, we obtain

$$\phi^+=\frac{1}{2}\,(g+Tg),\quad \phi^-=\frac{1}{2G}\,(-g+Tg).$$

It now follows from (33) that

$$\varphi(t)=\frac{G+1}{2G}\,g(t)+\frac{G-1}{2G}\,(Tg)(t).$$

Obviously, if $G\equiv 1$, then $\varphi\in L^{p(\cdot)}(\Gamma)$. However, if $G\neq 1$, then for the condition (33) to be fulfilled, it is necessary and sufficient that the function Tg belong to $L^{p(\cdot)}(\Gamma)$.

Thus we have proved

Theorem 6. *If the conditions* (25) *are fulfilled and* $G(t)\equiv 1$*, then the problem* (1) *is uniquely solvable in the class* $K^{p(\cdot)}(\Gamma)$*. If, however,* $G\not\equiv 1$ *and* $\operatorname{ind}G=0$*, then for its solvability it is necessary and sufficient that* $Tg\in L^{p(\cdot)}(\Gamma)$*. In case this condition is fulfilled, a solution is unique and given by the equality*

$$\phi(z)=K_\Gamma\Big[\frac{G+1}{2G}\,g+\frac{G-1}{2G}\,Tg\Big](z). \tag{34}$$

7.4. **Problem** (1) **in the class** $K^{p(\,\cdot\,)}(\Gamma)$ **when** $G \in A(p(\,\cdot\,),\Gamma)$ **and** $\varkappa(G)=\varkappa>0$**.** Let the conditions (25) be fulfilled and $Tg \in L^{p(\,\cdot\,)}(\Gamma)$.

As usually (see [5, p. 118]), we fix the point $z_0 \in D^+$ and write the condition (1) in the form

$$\phi^+(t) = \phi^-(t)(t-z_0)^{\varkappa} G(t)(t-z_0)^{-\varkappa} + g(t). \tag{35}$$

Assume

$$F(z) = \begin{cases} \phi(z), & z \in D^+, \\ \phi(z)(z-z_0)^{\varkappa}, & z \in D^-. \end{cases} \tag{36}$$

Then $F(z)$ has at the point $z=\infty$ the pole of order $\varkappa-1$. Hence, there is the polynomial $\Omega_{\varkappa-1}$ of order $\varkappa-1$ such that

$$\psi(z) = \big(F(z) - \Omega_{\varkappa-1}(z)\big] \in K^{p(\,\cdot\,)}(\Gamma). \tag{37}$$

The condition (35) yields

$$\psi^+(z) = G_0(t)\psi^-(t) + g_0(t), \tag{38}$$

where

$$\begin{aligned} G_0(t) &= |G(t)| e^{i[\arg G(t) - \varkappa \arg(t-z_0)]} |t-z_0|^{-\varkappa}, \\ g_0(t) &= g(t) - \Omega_{\varkappa-1}(t) + G_0(t)\Omega_{\varkappa-1}(t). \end{aligned}$$

It can be easily shown that $\psi \in K^{p(\,\cdot\,)}(\Gamma)$, and $G_0 \in \widetilde{A}(p(t),\Gamma)$. Using Theorem 6, we can conclude that the problem (38) is solvable if g_0 and Tg_0 belong to $L^{p(\,\cdot\,)}(\Gamma)$.

Since G_0 and $G_0\Omega_{\varkappa-1}$ are bounded functions, therefore $g_0 \in L^{p(\,\cdot\,)}(\Gamma)$.

Let us show that $Tg_0 = Tg - T\Omega_{\varkappa-1} + T(G_0\Omega_{\varkappa-1})$ belongs to $L^{p(\,\cdot\,)}(\Gamma)$. By our assumption, $Tg \in L^{p(\,\cdot\,)}(\Gamma)$. Putting $X_0(z) = X_{G_0}(z)$ for $T\Omega_{\varkappa-1}$, we have

$$T\Omega_{\varkappa-1} = X_0^+ S_\Gamma \,\frac{\Omega_{\varkappa-1}}{X_0^+}\,, \quad G_0 = \frac{X_0^+}{X_0^-}\,, \quad X_0(\infty) = a \neq 0.$$

Since $\Omega_{\varkappa-1}$ is polynomial and $\frac{1}{X(z)} \in E^{1+\eta}(D^-)$, it follows that $\frac{\Omega_{\varkappa-1}(z)}{X_0(z)} \in E^1(D^+)$, and consequently, $S_\Gamma \frac{\Omega_{\varkappa-1}}{X_0^+} = \frac{\Omega_{\varkappa-1}}{X_0^+}$. Therefore $T\Omega_{\varkappa-1} = \Omega_{\varkappa-1}$.

Next,

$$T(G_0\Omega_{\varkappa-1}) = X_0^+ S_\Gamma \,\frac{\Omega_{\varkappa-1}G_0}{X_0^+} = X_0^+ S\,\frac{\Omega_{\varkappa-1}}{X_0^-}\,.$$

The function $\Omega_{\varkappa-1}$ is constant if $\varkappa=1$; then assuming $\Omega_0 = b$, we have

$$S_\Gamma \,\frac{\Omega_0}{X_0^-} = S_\Gamma \,\frac{b}{X_0^-} = S_\Gamma\Big(\frac{\Omega_0}{X_0^-} - \frac{b}{a}\Big) + S_\Gamma \,\frac{b}{a} = -\frac{b}{X_0^-} + \frac{2b}{a}\,,$$

that is, for $\varkappa=1$, we have $T\,\frac{\Omega_{\varkappa-1}}{X_0^+} = -bG_0 + \frac{2b}{a}\,X_0^+$, and this function by Theorem 6 belongs to $L^{p(\,\cdot\,)}(\Gamma)$.

If $\varkappa - 1 \geq 1$, then there exists the polynomial $P_{\varkappa-2}$ of order $\varkappa - 2$ such that the function $\frac{\Omega_{\varkappa-1}}{X_0(z)} - P_{\varkappa-2}(z)$ in the domain D^- belongs to $E^1(D^-)$. Therefore

$$T(G_0\Omega_{\varkappa-1}) = X_0^+ S\Big[\frac{\Omega_{\varkappa-1}}{X_0^-} - P_{\varkappa-2}\Big] + X_0^+ SP_{\varkappa-2} =$$
$$= X_0^+\Big(-\Big(\frac{\Omega_{\varkappa-1}}{X_0^-} - P_{\varkappa-2}\Big)\Big) + X_0^+ P_{\varkappa-2} =$$
$$= -G_0\Omega_{\varkappa-1} + X_0^+ P_{\varkappa-1} + X_0^+ P_{\varkappa-2} = -G_0\Omega_{\varkappa-1} + 2X_0^+ P_{\varkappa-2}.$$

From the above, we can easily see that $T(G_0\Omega_{\varkappa-1})$ likewise belongs to $L^{p(\cdot)}(\Gamma)$. Thus g_0 and Tg_0 belong to $L^{p(\cdot)}(\Gamma)$, and the problem (38) is solvable in $K^{p(\cdot)}(\Gamma)$. Having solved it and getting back to $\phi(z)$, we successively get

$$\psi(z) = X_0(z)K_\Gamma\Big(\frac{g_0}{X_0^+}\Big)(z), \quad X_0(z) = \exp\Big\{K_\Gamma(\ln G_0)(z)\Big\},$$
$$K_\Gamma\,\frac{g_0}{X_0^+} = K_\Gamma\,\frac{g}{X_0^+} - \frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{X_0^+(t)}\,\frac{dt}{t-z} + \frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{X_0^-(t)}\,\frac{dt}{t-z}\,.$$

The last summands can be easily calculated:

$$\frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{X_0^+(t)}\,\frac{dt}{t-z} = \begin{cases} \dfrac{\Omega_{\varkappa-1}(z)}{X_0(z)}\,, z\in D^+, \\ 0, \qquad\qquad z\in D^-, \end{cases}$$
$$\frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{X_0^-(t)}\,\frac{dt}{t-z} =$$
$$= \frac{1}{2\pi i}\int\limits_\Gamma \Big[\frac{\Omega_{\varkappa-1}(t)}{X_0^-(t)} - \Omega_{\varkappa-1}(t)\Big]\frac{dt}{t-z} + \frac{1}{2\pi i}\int\limits_\Gamma \frac{\Omega_{\varkappa-1}(t)}{t-z}\,dt =$$
$$= \begin{cases} \Omega_{\varkappa-1}(z), & z\in D^+, \\ -\dfrac{\Omega_{\varkappa-1}(z)}{X_0(z)} + \Omega_{\varkappa-1}(z), & z\in D^-. \end{cases}$$

Putting

$$X(z) = \begin{cases} X_0(z), & z\in D^+, \\ (z-z_0)^{-\varkappa}X_0(z), & z\in D^-, \end{cases} \quad X_0(z) = \exp(K_\Gamma \ln G_0)(z), \tag{39}$$

and take into (37) and (38), we obtain

$$\phi(z) = \frac{X(z)}{2\pi i}\int\limits_\Gamma \frac{g(t)}{X^+(t)}\,\frac{dt}{t-z} + X(z)\Omega_{\varkappa-1}(z).$$

7.5. **The case for $\varkappa < 0$.** In this case, the function $F(z)$ given by equality (36) belongs to $K^{p(\cdot)}(\Gamma)$, and $F^+ = G_0F^- + g$. Consequently, $F(z) = X_0(z)K_\Gamma(\frac{g}{X_0^+})(z)$. For the function $\phi(z) = (z - z_0)^{-\varkappa}F(z)$ in the domain D^- to belong to $E^1(D^-)$ (the fulfilment of this condition is necessary for $\phi(z) \in K^{p(\cdot)}(\Gamma)$), it is necessary that

$$\int_\Gamma \frac{g(t)}{X_0^+(t)}\, t^k\, dt = 0, \quad k = 0, 1, \dots, |\varkappa| - 1. \tag{40}$$

If these conditions are fulfilled, then $\phi \in E^1(D^-)$, and since $\phi^- \in L^{p(\cdot)}(\Gamma)$, therefore

$$\phi(z) = -\frac{1}{2\pi i} \int_\Gamma \frac{\phi^-}{t - z}\, dt, \quad z \in D^-.$$

Hence

$$\phi(z) = K_\Gamma(\phi^+ - \phi^-)(z) \in K^{p(\cdot)}(\Gamma).$$

Now we are ready to state the theorem on the solvability of the problem (1) in the class $K^{p(\cdot)}(\Gamma)$ when

$$g \in L^{p(\cdot)}(\Gamma), \quad Tg \in L^{p(\cdot)}(\Gamma). \tag{41}$$

But first we present one simple sufficient condition with respect to g which ensures belonging of the function Tg to the class $L^{p(\cdot)}(\Gamma)$.

Theorem 7. *Let the conditions* (25) *be fulfilled and* $\operatorname{ind} G = 0$. *If* $g \in \bigcup_{\delta>0} L^{p(\cdot)+\delta}(\Gamma)$, *then* $Tg \in L^{p(\cdot)}(\Gamma)$.

Proof. Since $g \in \bigcup_{\delta>0} L^{p(\cdot)+\delta}(\Gamma)$, there exists the number $\eta > 0$ such that $g \in L^{p(\cdot)+\eta}(\Gamma)$.

We divide Γ into the arcs γ_k so as to fulfil simultaneously the condition of the theorem and

$$\overline{p}_k - \underline{p}_k < \eta, \quad \text{where } \overline{p}_k = \sup_{t\in\gamma_k} p(t), \quad \underline{p}_k = \inf_{t\in\gamma_k} p(t).$$

Then for $t \in \gamma_k$ we have $\overline{p}_k < \underline{p}_k + \eta$, and hence,

$$p(t) + \eta > \underline{p}_k + \eta > \overline{p}_k.$$

Consequently, $g \in L^{\overline{p}_k}(\gamma_k)$. In addition, since

$$\sup_{t\in\gamma_k} \max(p(t), q(t)) \ge \sup_{t\in\gamma_k} \max p(t) = \overline{p}_k,$$

we find that $G \in A(\overline{p}_k, \gamma_k)$. Owing to this fact, the functions $X_k(z)$ given by equalities (19) belong to $L^{p(\cdot)}(\Gamma)$ (see Subsection 5.2) and moreover, $\operatorname{ind} G$ in $K^{\overline{p}_k}(\Gamma)$ equals zero. Consequently, the function $\phi(z)$ given by equality (32) belongs to classes $L^{p(\cdot)}(\gamma_k)$ from which it follows that $\phi \in K^{p(\cdot)}(\Gamma)$, that is, $\phi^+ \in L^{p(\cdot)}(\Gamma)$. But $\phi^+ = \frac{1}{2}(g + Tg)$. Hence, $Tg \in L^{p(\cdot)}(\Gamma)$. $\square$

From the above theorem follows

Statement 2. *If $g \in L^{p(\cdot)}(\Gamma)$ and the conditions* (25) *are fulfilled, then the function*

$$\phi(z) = X_G(z) \int_\Gamma \frac{g(\tau)}{X_G^+(\tau)} \frac{d\tau}{\tau - z} \tag{42}$$

belongs to the class $K^{p(\cdot)-\delta}(\Gamma)$ for any $\delta \in (0, \underline{p})$.

To prove this, it suffices to notice that for $g \in L^{p(\cdot)}(\Gamma)$ we have $g \in L^{(p(\cdot)-\delta)+\delta}(\Gamma)$.

7.6. **The theorem below is a summation of results stated in Subsections 7.1–7.5.**

Theorem 8. *If the conditions* (25) *are fulfilled and $g \in L^{p(\cdot)}(\Gamma)$, then the Riemann problem has a solution ϕ (given by equality* (42)*), satisfying the condition $\phi \in \bigcap\limits_{\delta \in (0,\underline{p})} K^{p(\cdot)-\delta}(\Gamma)$.*

If, however, $G \in A(p(\cdot), \Gamma)$, then for the Riemann problem to be solvable in the class $K^{p(\cdot)}(\Gamma)$ for $\varkappa(G) \geq 0$, it is necessary and sufficient that the condition

$$Tg \in L^{p(\cdot)}(\Gamma) \tag{43}$$

is fulfilled.

When $\varkappa < 0$, for the solvability of the problem it is necessary and sufficient that the conditions (43) *and*

$$\int_\Gamma \frac{g(t)}{X^+(t)}\, t^k \, dt = 0, \quad k = 0, 1 \dots, |\varkappa - 1|$$

are fulfilled.

If the above-mentioned conditions are fulfilled, then the problem for $\varkappa \leq 0$ is uniquely solvable, but for $\varkappa > 0$ it is solvable unconditional. In all cases the solution is given by the equality

$$\phi(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{g(t)}{X^+(t)} \frac{dt}{t - z} + X(z)\Omega_{\varkappa-1}(z), \tag{44}$$

where $\Omega_{\varkappa-1}(z)$ is an arbitrary polynomial of order $\varkappa - 1$ ($\Omega_{\varkappa-1}(z) \equiv 0$ for $\varkappa \leq 1$), and $X(z)$ given by (26).

8. On the Noetherity of the Operator $M\varphi = a\varphi + bS_\Gamma\varphi$ in the Space $L^{p(\cdot)}(\Gamma)$

The results of Sections 3–7 do not allow us to establish Noetherity of the operator M in the space $L^{p(\cdot)}(\Gamma)$, when $G = (a-b)(a+b)^{-1} \in A(p(\cdot), \Gamma)$.

We intend to construct a space $\mathcal{L}^{p(t)}$ in which under sufficiently general assumptions with respect to Γ, p and G the operator M will be Noetherian.

As concerns the space $L^{p(\cdot)}(\Gamma)$, we can point out one necessary condition for the operator M to be Noetherian in $L^{p(\cdot)}(\Gamma)$. This condition for $p \in \mathcal{P}(\Gamma)$ will be the same as for the constant p. We start with this result.

Theorem 9. *Let $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$, $a(t)$, and $b(t)$ be measurable bounded on Γ functions. For the operator $M\varphi = a\varphi + bS_\Gamma\varphi$ to be Noetherian in $L^{p(\,\cdot\,)}(\Gamma)$, it is necessary that the conditions*

$$\operatorname*{ess\,inf}_{t\in\Gamma} |a(t)+b(t)| > 0, \quad \operatorname*{ess\,inf}_{t\in\Gamma} |a(t)-b(t)| > 0 \tag{45}$$

are fulfilled.

Proof. Let us consider in $L^{p(\,\cdot\,)}(\Gamma)$ the equation

$$M\varphi = f, \quad f \in L^{p(\,\cdot\,)}(\Gamma). \tag{46}$$

Let $\phi(z) = (K_\Gamma\varphi)(z)$, where φ is a solution of equation (46). By the Sokhotskii–Plemelj formulas, $\varphi = \phi^+ - \phi^-$, $S_\Gamma\varphi = \phi^+ + \phi^-$. Therefore, (46) can be written in the form

$$(a+b)\phi^+ + (b-a)\phi^- = f.$$

Assuming $c = a+b$, $d = b-a$, we obtain

$$c\phi^+ + d\phi^- = f. \tag{47}$$

Assume now to the contrary that M is Noetherian in $L^{p(\,\cdot\,)}(\Gamma)$ and, for example,

$$\operatorname{ess\,inf} |a+b| = \operatorname{ess\,inf} |c| = 0. \tag{48}$$

Since the operator under small perturbations preserves Noetherity ([4, p. 144]), there exists the number $\varepsilon > 0$ such that: if the operator $M_1\varphi = c_1\varphi + d_1S_\Gamma\varphi$ is Noetherian and $\|M - M_1\|_{p(\,\cdot\,)} < \varepsilon$, then M_1 is likewise Noetherian.

Let $\eta < \frac{\varepsilon}{1+\|S_\Gamma\|_{p(\,\cdot\,)}}$. Consider the functions

$$\begin{aligned} c_1(t) &= \begin{cases} c(t) & \text{if } |c(t)| \geq \eta, \\ 0 & \text{if } |c(t)| < \eta, \end{cases} \\ d_1(t) &= \begin{cases} d(t) & \text{if } |d(t)| \geq \eta, \\ 0 & \text{if } |c(t)| < \eta. \end{cases} \end{aligned} \tag{49}$$

Obviously,

$$\begin{aligned} \|M\varphi - M_1\varphi\|_{p(\,\cdot\,)} &\leq \eta\|\varphi\|_{p(\,\cdot\,)} + 2\eta\|S\varphi\|_{p(\,\cdot\,)} < \\ &< 2\eta\big(1 + \|S\|_{p(\,\cdot\,)}\big)\|\varphi\|_{p(\,\cdot\,)} < \varepsilon\|\varphi\|_{p(\,\cdot\,)}, \end{aligned}$$

therefore the operator M_1 is Noetherian in $L^{p(\,\cdot\,)}(\Gamma)$. Let us show that the equation

$$M_1\varphi = 0 \tag{50}$$

has only a zero solution. Towards this end, we notice that $|d_1| > 0$ on Γ, and $c_1 = 0$ on the set e of positive measure, where $\operatorname{mes} e < \operatorname{mes}\Gamma$. Indeed, if $\operatorname{mes} e = \operatorname{mes}\Gamma$, then $d_1\phi \equiv 0$ on Γ, hence $\phi^- \equiv 0$ on Γ. Then any function of the type $\int\limits_\Gamma \frac{F^+(\tau)}{\tau - t}\,d\tau$, where $F \in E^1(D^+)$ with a boundary value $F^+ \in L^{p(\,\cdot\,)}(\Gamma)$ will be a solution of equation (50). Sets of such functions are of infinite dimension, hence M_1 is not Noetherian. Thus $\operatorname{mes} e < \operatorname{mes}\Gamma$, and hence $\operatorname{mes}(\Gamma \setminus e) > 0$.

On e, we now have $d_1\phi^- = 0$, and then $\phi^- = 0$ on e. By the theorem on the uniqueness of analytic functions (see, e.g., [27, p. 232]), $\phi^- = 0$ on Γ. Consequently, on $\Gamma \setminus e$ we have $c_1 \neq 0$ and $c_1\phi^+ = 0$. Again, by the uniqueness theorem, we conclude that $\phi^+ = 0$ on Γ. Finally, we obtain that on Γ both ϕ^- and ϕ^+ are equal to zero. This implies that $\varphi = \phi^+ - \phi^- = 0$. Thereby, equation (50) has only a zero solution. Hence $M_1\varphi = 0$ has only a zero solution and the operator M_1 is Noetherian one. Since $|d_1| > \eta > 0$, the operator $\widetilde{M} = c_1(d_1)^{-1}\phi^+ + \phi^-$ together with $c_1\phi_1^+ + d_1\phi^-$ is likewise Noetherian, and $\widetilde{M}\varphi$ has only a zero solution. In addition, the coefficient c_1/d_1 on e equals zero and is different from zero on $\Gamma \setminus e$; both sets are of positive measure. Therefore, also for the operator $(\widetilde{M})^*$ we have $\dim N((\widetilde{M})^*) = 0$ (this case for a variable $p(t)$ is proved in the same way as Lemma 4.1 in [4] on pages 292-3 for a constant p). Since the operators $\widetilde{M}$ and $\widetilde{M}^*$ are Noetherian, this implies that they are invertible. Owing to this fact, the equation $\frac{c_1}{d_1}\,\phi^+ + \phi^- = g$ should have a solution in $L^{p(\cdot)}(\Gamma)$ for any function $g \in L^{p(\cdot)}(\Gamma)$.

Let us show that this is not true.

Let $f = 1$, then $c_1\phi^+ + d_1\phi^- = d_1$, $t \in \Gamma$. But for $t \in e$, we get $0 + d_1\phi^- = d_1$, i.e., $\phi^-(t) \equiv 1$. If $F(z) = \phi(z) - 1$, then $F \in K^{p(\cdot)}(\Gamma)$. Hence $F(z)$ belongs to $E^1(D^-)$, and $F^-(t) = 0$, $t \in e$, whence it follows that $\phi(z) = 1$, $z \in D^-$, and $\phi(\infty) = 1$, as well. But this is impossible due to $\phi \in K^{p(\cdot)}(\Gamma)$, and for such functions we have $\phi(\infty) = 0$.

The obtained contradiction shows that the assumption (48) is invalid, hence $\operatorname{ess\,inf}|a(t) + b(t)| > 0$.

The validity of the second inequality in (48) can be proved analogously. □

As a conclusion, it should be noted that in proving the lemma we have followed the method suggested in [4, pp. 256-8].

9. The Space $\mathcal{L}^{p(\cdot)}$

9.1. **Definition of $\mathcal{L}^{p(\cdot)}$; its Banachity.** Let

$$\Gamma \in R, \;\; p \in \mathcal{P}(\Gamma), \;\; G \in A(p(\cdot)). \tag{51}$$

Assume

$$g \in L^{p(\cdot)}, \;\; Tg \in L^{p(\cdot)}, \;\; T\Big(g_k\,\frac{1}{G}\Big) \in L^{p(\cdot)}, \;\; k = 1,2, \tag{52}$$

where

$$g_1 = \frac{1}{2}\,(g + Tg), \quad g_2 = \frac{1}{2}(-g + Tg). \tag{53}$$

It follows from (52) that

$$g_k \in L^{p(\cdot)}, \;\; k = 1,2. \tag{54}$$

Let

$$\mathcal{L}^{p(\cdot)} = \Big\{g :\ g \in L^{p(\cdot)},\ Tg \in L^{p(\cdot)},\ \ T\Big(g_k\,\frac{1}{G}\Big) \in L^{p(\cdot)}\Big\}. \tag{55}$$

For the elements from $\mathcal{L}^{p(\cdot)}$ we introduce the norm as follows:

$$\|g\|_{\mathcal{L}^{p(\cdot)}} = \|g\|_{p(\cdot)} + \|Tg\|_{p(\cdot)} + \Big\|Tg_1\,\frac{1}{G}\Big\|_{p(\cdot)} + \Big\|Tg_2\,\frac{1}{G}\Big\|_{p(\cdot)}. \tag{56}$$

The set $\mathcal{L}^{p(\cdot)}$ together with the above-introduced norm, i.e.,

$$\mathcal{L}^{p(\cdot)} = \big\{g :\ \|g\|_{\mathcal{L}^{p(\cdot)}} < \infty\big\}$$

turns into a linear normalized space.

Lemma 5. *If the conditions* (51) *are fulfilled, than $\mathcal{L}^{p(\cdot)}$ is a complete space.*

Proof. Let $\{g^n\}$ be the fundamental sequence in $\mathcal{L}^{p(\cdot)}$, then it follows from (55) that the sequences $\{g^n\}$, $\{Tg^n\}$, $\big\{T(g_k^n\,\frac{1}{G})\big\}$, $k = 1, 2$, are fundamental in $L^{p(\cdot)}$. Let μ, λ, e, ψ be the functions from $L^{p(\cdot)}$ to which these sequences converge, respectively, i.e.,

$$\begin{gathered} \|g^n - \mu\|_{p(\cdot)} \to 0, \ \ \|Tg^n - \lambda\|_{p(\cdot)} \to 0, \\ \Big\|T\Big(g_1^n\,\frac{1}{G}\Big) - e\Big\|_{p(\cdot)} \to 0, \ \ \Big\|T\Big(g_2^n\,\frac{1}{G}\Big) - \psi\Big\|_{p(\cdot)} \to 0. \end{gathered} \tag{57}$$

Since T is continuous from $L^{p(\cdot)}$ to the space $M(\Gamma)$, Tg^n converges in measure to $T\mu$, and hence

$$\lambda = T\mu \tag{58}$$

Next, since $g_1^n = \frac{1}{2}\,(g^n + Tg^n)$, $\{g_1^n\}$ converges in $L^{p(\cdot)}$ and in measure to $\frac{1}{2}\,(\mu+\lambda)$, and owing to the fact that $\frac{1}{G}$ is bounded, we conclude that the sequences $\big\{g_k^n\,\frac{1}{G}\big\}$, $k = 1, 2$, converge in $L^{p(\cdot)}$, respectively, to $\frac{1}{2}\,(\mu + \lambda)\,\frac{1}{G}$ and to $\frac{1}{2}(-\mu + \lambda)\,\frac{1}{G}$. This implies that

$$e = \frac{1}{2}\,\bigg(\mu + T\mu + T\Big(\mu_1\,\frac{1}{G}\Big) + T\Big(\mu_2\,\frac{1}{G}\Big)\bigg), \tag{59}$$

$$\mu_1 = \mu + \lambda, \ \ \mu_2 = -\mu + \lambda,$$

$$\psi = \frac{1}{2}\,\bigg(\mu + T\mu - T\Big(\mu_1\,\frac{1}{G}\Big) + T\Big(\mu_2\,\frac{1}{G}\Big)\bigg), \tag{60}$$

and from (56)–(59) we conclude that

$$\|g^n - \mu\|_{\mathcal{L}^{p(\cdot)}} \to 0.$$ □

9.2. **The necessary condition for the operator M to be Noetherian in $\mathcal{L}^{p(\cdot)}$.** Let us show that the analogue of Theorem 9 is valid for the operator M to be Noetherian in $\mathcal{L}^{p(\cdot)}$.

Theorem 10. *Let $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$, and let a and b be bounded measurable on Γ functions, then for the operator $M = a\varphi + bS\varphi$ to be Noetherian in $\mathcal{L}^{p(\,\cdot\,)}$, it is necessary that the conditions* (45) *or, what comes to the same thing, the condition*

$$\operatorname{ess\,inf} |a^2 - b^2| > 0$$

is fulfilled.

Proof. We proceed from the proof of Theorem 9. Tracing its proof, we conclude that we have used the following facts:

(1) $L^{p(\,\cdot\,)}$ is the Banach space;
(2) the set of Noetherian operators in the Banach space (and hence in $L^{p(\,\cdot\,)}(\Gamma)$), is open;
(3) equation (50) in $L^{p(\,\cdot\,)}$ has only a zero solution;
(4) if two analytic functions have in the domain G the same angular boundary values on the set of positive measure, then they are equal everywhere in G;
(5) the function $f \equiv 1$ belongs to $L^{p(\,\cdot\,)}$.

In the case under consideration:

(1′) $\mathcal{L}^{p(\,\cdot\,)}$ is the Banach space;
(2′) since $\mathcal{L}^{p(\,\cdot\,)}$ is the Banach space, the set of Noetherian operators is open;
(3′) equation (50) has in $\mathcal{L}^{p(\,\cdot\,)}$ only a zero solution, since in a wider space $L^{p(\,\cdot\,)}$ it has only a zero solution;
(4′) the theorem on the uniqueness of analytic functions is applicable;
(5′) the function $f \equiv 1$ belongs to $\mathcal{L}^{p(\,\cdot\,)}$;

By virtue of statements (1′)–(5′), repeating the same arguments as in proving Theorem 9, we find that Theorem 10 is likewise valid. □

10. Solution of Equation $M\varphi = f$ in the Space $\mathcal{L}^{p(\,\cdot\,)}$

10.1. **The case $\varkappa = 0$.** Assume that the conditions (51)–(52) with

$$G = \frac{a-b}{a+b} \in A(p(\,\cdot\,)) \tag{61}$$

and

$$\operatorname{ess\,inf} |a^2 - b^2| > 0 \tag{62}$$

are fulfilled, and consider the equation

$$M\varphi = a\varphi + bS\varphi = f, \quad f(a+b)^{-1} \in \mathcal{L}^{p(\,\cdot\,)}. \tag{63}$$

This equation is equivalent to the following Riemann problem:

$$\phi^+(t) = G(t)\phi^-(t) + g(t), \quad g(t) = \frac{f(t)}{a(t)+b(t)} \tag{64}$$

in the class $K\mathcal{L}^{p(\,\cdot\,)}$, i.e., in the class of Cauchy type integrals with density from $\mathcal{L}^{p(\,\cdot\,)}$.

Indeed, if $\phi = K_\Gamma\varphi$ (where $\varphi \in \mathcal{L}^{p(\cdot)}$) is a solution of the problem (64), then it can be easily verified that φ is a solution of equation (63) of the class $\mathcal{L}^{p(\cdot)}$.

Conversely, if φ is a solution of equation (63) of the class $\mathcal{L}^{p(\cdot)}$, then $\phi = K_\Gamma\varphi \in K\mathcal{L}^{p(\cdot)}$, and it satisfies the condition (64).

Lemma 6. *If for* Γ, p *and* G *the conditions* (25) *are fulfilled and the functions* g_1 *and* g_2 *are defined by equalities* (53)*, then the equalities*

$$Tg_1 = g_1, \quad Tg_2 = -g_2 \tag{65}$$

are valid.

Proof. Follows immediately from the equality $T(Tg) = g$, valid due to the conditions (25) (see Theorem 4). □

Lemma 7. *If there take place the inclusions* (51)–(52) *and* $\operatorname{ind} G = \operatorname{ind} \frac{a-b}{a+b} = 0$*, then equation* (63) *is uniquely solvable in the class* $\mathcal{L}^{p(\cdot)}$*, and a solution is given by the equality*

$$\varphi = g_1 - \frac{g_2}{G}\,,$$

where

$$g_1 = \frac{1}{2}\,(g + Tg), \quad g_2 = \frac{1}{2}\,(-g + Tg), \quad g = \frac{f}{a+b}\,. \tag{66}$$

Proof. By virtue of Theorem 8, the problem (64) in $L^{p(\cdot)}$ has a unique solution

$$\phi(z) = X(z)\Big[K_\Gamma\Big(\frac{g}{X^+}\Big)\Big](z). \tag{67}$$

By the Sokhotskii–Plemelj formulas, we obtain

$$\phi^+ = \frac{1}{2}\,(g + Tg) = g_1, \quad \phi^- = \frac{1}{2G}\,(-g + Tg) = \frac{g_2}{G}\,. \tag{68}$$

Since $\operatorname{ind} G = 0$, therefore $\phi \in E^1(D^\pm)$, and hence

$$\phi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\phi^+(t) - \phi^-(t)}{t - z}\,dt = \frac{1}{2\pi i}\int\limits_\Gamma \frac{g_1 - \frac{g_2}{G}}{t - z}\,dt. \tag{69}$$

Thereby, the only possible solution of equation (63) is the function

$$\varphi = g_1 - \frac{g_2}{G}\,. \tag{70}$$

Let us prove that $\varphi \in \mathcal{L}^{p(\cdot)}$, i.e., that

$$\varphi \in L^{p(\cdot)}, \;\; T\varphi \in L^{p(\cdot)}, \;\; T\Big(\frac{\varphi_k}{G}\Big) \in L^{p(\cdot)}, \;\; k = 1, 2. \tag{71}$$

From the assumptions $g \in L^{p(\cdot)}$, $Tg \in L^{p(\cdot)}$, $\frac{1}{G} \in L^\infty$, it follows that

$$\varphi \in L^{p(\cdot)}. \tag{72}$$

Further, due to (65) and (70),

$$T\varphi = Tg_1 - T\,\frac{g_2}{G} = \Big(g_1 - T\,\frac{g_2}{G}\Big) \in L^{p(\cdot)}. \tag{73}$$

To study $T\,\frac{\varphi_k}{G}$, we first note that

$$\begin{aligned}\varphi_1 &= \frac{1}{2}\,(\varphi+T\varphi)=\frac{1}{2}\Big(g_1-\frac{g_2}{G}+Tg_1-T\,\frac{g_2}{G}\Big)=\\ &=\frac{1}{2}\,(g_1+Tg_1)-\frac{1}{2}\Big(\frac{g_2}{G}+T\,\frac{g_2}{G}\Big)=g_1-\frac{1}{2}\Big(\frac{g_2}{G}+T\,\frac{g_2}{G}\Big), \qquad (74)\\ \varphi_2 &= \frac{1}{2}\,(-\varphi+T\varphi)=\frac{1}{2}\Big(-g_1+\frac{g_2}{G}+Tg_1+T\,\frac{g_2}{G}\Big)=\\ &=\frac{1}{2}\Big(\frac{g_2}{G}+T\,\frac{g_2}{G}\Big). \qquad (75)\end{aligned}$$

It follows from (52) and (70) that $\varphi_k\in L^{p(\,\cdot\,)}$.

Now, we have

$$\begin{aligned}T\,\frac{\varphi_1}{G} &= T\,\frac{g_1}{G}-\frac{1}{2}\bigg(T\,\frac{g_2}{G}\cdot\frac{1}{G}+T\Big(T\,\frac{g_2}{G}\Big)\,\frac{1}{G}\bigg)=\\ &=T\,\frac{g_1}{G}-\frac{1}{2}\Big(T\,\frac{g_1}{G}\cdot\frac{1}{G}+\frac{g_2}{G}\cdot\frac{1}{G}\Big), \qquad (76)\\ T\,\frac{\varphi_2}{G} &= \frac{1}{2}\bigg(T\,\frac{g_2}{G}\cdot\frac{1}{G}+\Big(T\,\frac{g_2}{G}\Big)\,\frac{1}{G}\bigg)=\\ &=\frac{1}{2}\Big(T\,\frac{g_2}{G}\cdot\frac{1}{G}+\frac{g_2}{G}\cdot\frac{1}{G}\Big). \qquad (77)\end{aligned}$$

Taking into account (70), relying on (76) and (77), we conclude that

$$T\,\frac{\varphi_1}{G},T\,\frac{\varphi_2}{G}\in L^{p(\,\cdot\,)}. \qquad (78)$$

The inclusions (72), (73) and (78) imply that the inclusion (71) is valid, and hence $\varphi\in\mathcal{L}^{p(\,\cdot\,)}$. □

10.2. **The case $\varkappa>0$.** Since $Tg\in L^{p(\,\cdot\,)}$, all possible solutions of the problem (64) lie in the set

$$\phi(z)=X(z)\Big(K_\Gamma\,\frac{g}{X^+}\Big)(z)+P_{\varkappa-1}(z)X(z)$$

(see item 7.4). The first summand here belongs to $K\mathcal{L}^{p(\,\cdot\,)}$ (see item 7.3). Let us show that the second summand likewise belongs to $K\mathcal{L}^{p(\,\cdot\,)}$.

Since $X(t)$ has at infinity zero of order $\varkappa$, $P_{\varkappa-1}(z)X(z)$ is representable by the Cauchy integral in the domains D^+ and D^-.

Lemma 8. *The function*

$$\varphi(t)=\big[X^+(t)-X^-\big]P_{\varkappa-1}(t)$$

satisfies the conditions (52)*, and hence* $\varphi\in\mathcal{L}^{p(\,\cdot\,)}$.

Proof. Since $X^+,X^-\in L^p$ (see Theorem 3), $\varphi\in L^{p(\,\cdot\,)}$.

Further,

$$T\varphi = TX^+P - TX^-P = X^+S_\Gamma \frac{X^+P}{X^+} - X^+S_\Gamma \frac{X^-P}{X^+} = \\ = X^+P - X^+S_\Gamma \frac{P}{G}\,. \tag{79}$$

Here, for the multiplier $S_\Gamma \frac{P}{G}$, we have

$$S_\Gamma \frac{P}{G} = \int\limits_\Gamma \frac{P(\tau)}{G(\tau)}\,\frac{d\tau}{\tau - t} = \int\limits_\Gamma \frac{1}{G(\tau)}\,\frac{P(\tau)-P(t)}{\tau - t}\,d\tau + P(t)S\,\frac{1}{G}\,. \tag{80}$$

By virtue of the inclusion (11), we find that $X^+ \in L^{p(\cdot)+\eta}$ (see Corollary of Theorem 3). Next, the first summand in equality (80) is a bounded function; moreover, since $\Gamma \in R$ and $\frac{1}{G} \in L^\infty$, we have $S\,\frac{1}{G} \in \bigcap\limits_{s>1} L^s$. Then $PS\,\frac{1}{G} \in L^{p(\cdot)}$, and since $X^+ \in L^{p(\cdot)+\eta}$, therefore $X^+S_\Gamma \frac{1}{G} \in L^{p(\cdot)}$, as well. By virtue of (80), we can conclude from (79) that $T\varphi \in L^{p(\cdot)}$.

Further,

$$2\varphi_1 = \varphi + T\varphi = 2X^+P - X^-P + X^+S_\Gamma\,\frac{P}{G}\,, \quad 2\varphi_2 = X^-P + X^+S_\Gamma\,\frac{P}{G}$$

and hence

$$\frac{\varphi_1}{G} = 2X^-P - \frac{X^-P}{G} + X^-S_\Gamma\,\frac{P}{G} = X^-\Big(2P - \frac{P}{G} + S_\Gamma\,\frac{P}{G}\Big), \\ \frac{\varphi_2}{G} = X^-\Big(\frac{P}{G} + S_\Gamma\,\frac{P}{G}\Big)$$

from which we get

$$T\,\frac{\varphi_1}{G} = X^+S_\Gamma\left[\frac{1}{G}\Big(2P - \frac{P}{G} + S_\Gamma\,\frac{P}{G}\Big)\right], \\ T\,\frac{\varphi_2}{G} = X^+S_\Gamma\left[\frac{1}{G}\Big(\frac{P}{G} + S_\Gamma\,\frac{P}{G}\Big)\right]. \tag{81}$$

It can be easily seen that $T\,\frac{\varphi_1}{G} \in L^{p(\cdot)}$ if the function $X^+S_\Gamma\,(\frac{2P}{G} - \frac{P}{G^2} + \frac{1}{G}\,S_\Gamma\,\frac{P}{G})$ belongs to $L^{p(\cdot)}$. Since $\frac{P}{G}$, $\frac{P}{G^2}$ belong to L^∞ and $\Gamma \in R$ we have $S_\Gamma\,\frac{P}{G}$ and $S_\Gamma\,\frac{P}{G^2}$ belong to the set $\bigcap\limits_{\nu>1} L^\nu$. Moreover, $X^+ \in L^{p(\cdot)+\varepsilon}$ (see Corollary of Theorem 3). These two facts allow us to conclude that

$$T\,\frac{\varphi_1}{G} \in L^{p(\cdot)}. \tag{82}$$

Analogously, we can prove that $T\,\frac{\varphi_2}{G} \in L^{p(\cdot)}$.

Thus we have proved that for φ the conditions (52) are fulfilled, and hence $\varphi \in \mathcal{L}^{p(\cdot)}$. □

10.3. **The case $\varkappa < 0$.**

Lemma 9. *If the conditions* (45), (51)–(52) *are fulfilled, and* $\varkappa < 0$, *then for equation* (63) *to be solvable in the class* $\mathcal{L}^{p(\cdot)}$, *it is necessary and sufficient that*

$$\int\limits_{\Gamma} \frac{f(\tau)}{a(\tau)+b(\tau)}\,\frac{\tau^k}{X^+(\tau)}\,d\tau = 0, \quad k = 0, 1, \dots, |-\varkappa| - 1. \tag{83}$$

Proof. In the case under consideration, $X(z)$ has at infinity a pole of order $|\varkappa|$, therefore the only possible solution of equation (62) may be only the function $\varphi(t) = \phi^+(t) - \phi^-(t)$, where $\phi(z) = X(z)(K_\Gamma \frac{g}{X^+})(z)$, $g(t) = \frac{f(t)}{a(t)+b(t)}$. But the function $\varphi(t)$ belongs to $K\mathcal{L}^{p(\cdot)}$, if and only if $\phi(z) \in E^1(D^\pm)$, i.e., when the function $(K_\Gamma \frac{f}{a+b})(z)$ at the point $z = \infty$ has zero of order $|\varkappa|$. Thus it is necessary and sufficient that equalities (83) are fulfilled. And if this condition is fulfilled, the solution is unique and given by the equality

$$\varphi = \frac{1}{2}\Big(\frac{f}{a+b} + T\,\frac{f}{a+b}\Big) - \frac{1}{2G}\Big(-\frac{f}{a+b} + T\,\frac{f}{a+b}\Big). \tag{84}$$

□

10.4. **Summation of results stated in items 10.1–10.3.**

Theorem 11. *Let* Γ *be a simple, closed, rectifiable curve* $p \in \mathcal{P}(\Gamma)$, *and let* $a(t)$ *and* $b(t)$ *be bounded measurable on* Γ *functions such that*

$$\operatorname{ess\,inf}\big|a^2(t) - b^2(t)\big| > 0$$

and $G(t) = (a(t) - b(t))(a(t) + b(t))^{-1}$. *If for* Γ, p *and* G *the conditions* (25) *are fulfilled.*

Then the equation

$$M\varphi = a(t)\varphi(t) + b(t)(S_\Gamma\varphi)(t) = f(t), \quad \frac{f(t)}{a(t)+b(t)} \in \mathcal{L}^{p(\cdot)}$$

for $\varkappa = \varkappa(G) \geq 0$ *is solvable in the class* $\mathcal{L}^{p(\cdot)}(\Gamma)$; *for* $\varkappa = 0$, *it is unique and for* $\varkappa > 0$, *the homogeneous equation has* $\varkappa$ *linearly independent solutions. If* $\varkappa < 0$, *for the equation* $M\varphi = f$ *to be solvable in the class* $\mathcal{L}^{p(\cdot)}(\Gamma)$, *it is necessary and sufficient that the conditions* (83) *are fulfilled.*

In all cases when a solution exists, it is given by the equality

$$\begin{aligned}\varphi(t) = \frac{1}{2}\Big(\frac{f}{a+b} + T\,\frac{f}{a+b}\Big) - \frac{1}{2G}\Big(-\frac{f}{a+b} + T\,\frac{f}{a+b}\Big) +\\ + (X^+ - X^-)P_{\varkappa-1}\end{aligned} \tag{85}$$

($P_\nu \equiv 0$, if $\nu < 0$).

11. The Spaces $\widetilde{\mathcal{L}}^{p(t)}$ and $(\mathcal{L}^{p(t)})^*$

11.1. **Definition and some properties of the space $\widetilde{\mathcal{L}}^{p(\cdot)}$.** Let $\psi \in L^{p(\cdot)}$, and X be the function given by equality (26). Assume

$$\begin{gathered}\widetilde{T}\psi = \frac{1}{X^+}\, S(X^+\psi), \\ \widetilde{\mathcal{L}}^{p(\cdot)} = \big\{\psi :\ \psi \in L^{p(\cdot)},\ \widetilde{T}\psi \in L^{p(\cdot)}\big\}.\end{gathered} \tag{86}$$

For the functions $\psi \in \widetilde{\mathcal{L}}^{p(\cdot)}$ we introduce the norm

$$\|\psi\|_{\widetilde{\mathcal{L}}^{p(\cdot)}} = \|\psi\|_{p(\cdot)} + \|\widetilde{T}\psi\|_{p(\cdot)}. \tag{87}$$

Due to the continuity of the operator T from $\mathcal{L}^{p(\cdot)}$ to the space of convergence in measure, we can easily prove

Lemma 10. *If $\Gamma \in J^*$, $p \in \mathcal{P}(\Gamma)$, $G \in \widetilde{A}(p(\cdot))$, then the operator $\widetilde{T}$ is continuous from $\widetilde{\mathcal{L}}^{p(\cdot)}$ to the space of convergence in measure.*

Lemma 11. *$\widetilde{\mathcal{L}}^{p(\cdot)}$ is the complete, linear, normalized space.*

Proof runs in the same way as that of Lemma 5.

11.2. **The spaces ℓ_1 and ℓ_2.** Assume

$$\begin{aligned}\ell_1 &= \big\{\psi :\ \psi \in \mathcal{L}^{p(\cdot)},\ T\psi = \psi\big\},\ \ \|\psi\|^1 = \|\psi\|_{p(\cdot)}, \\ \ell_2 &= \big\{\psi :\ \psi \in \mathcal{L}^{p(\cdot)},\ T\psi = -\psi\big\},\ \ \|\psi\|^2 = \|\psi\|_{p(\cdot)}.\end{aligned} \tag{88}$$

Lemma 12. *ℓ_k, $k = 1, 2$, are closed subspaces of the space $L^{p(\cdot)}$.*

Proof. Let $\psi_n \in \ell_k$ and $\{\psi_n\}$ be the fundamental sequence in $L^{p(\cdot)}$, then there exists $\psi_0 \in L^{p(\cdot)}$ such that $\|\psi_n - \psi_0\|_{p(\cdot)} \to 0$. Let us prove that $\psi_0 \in \ell_k$.

Assuming for the definiteness that $k = 1$, then $T\psi_k = \psi_k$, and hence $\{T\psi_k\}$ converges in $L^{p(\cdot)}$ to ψ_0. By statement 1, $\{T\psi_k\}$ converges in measure to $T\psi_0$. Hence $\psi_0 = T\psi_0 \in \ell_1$. Consequently, ℓ_1 is closed in $L^{p(\cdot)}$.

The closure of ℓ_2 in $L^{p(\cdot)}$ is proved analogously. □

Lemma 13.

$$\mathcal{L}^{p(\cdot)} = \ell_1 \oplus \ell_2. \tag{89}$$

Proof. Let $\psi \in \mathcal{L}^{p(\cdot)}$; obviously,

$$\psi = \frac{1}{2}\,(\psi + T\psi) + \frac{1}{2}\,(\psi - T\psi) = \psi_1 + \psi_2, \tag{90}$$

where $\psi_1 = \frac{1}{2}\,(\psi + T\psi)$ and $\psi_2 = \frac{1}{2}\,(-\psi + T\psi)$. We have

$$T\psi_1 = \frac{1}{2}\,(T\psi + \psi) = \psi_1, \quad T\psi_2 = \frac{1}{2}\,(T\psi - \psi) = -\psi_2.$$

This implies that $\psi_k \in \ell_k$.

If $\psi = \mu_1 + \mu_1$, $\mu_k \in \ell_k$, then $\psi_1 - \mu_1 = \psi_2 - \mu_2$, where $\psi_k - \mu_k \in \ell_k$. Thereby, $(\psi_k - \mu_k) \in \ell_1 \cap \ell_2$. But it can be easily verified that $\ell_1 \cap \ell_2 = \{0\}$.

Indeed, if $\psi \in \ell_1 \cap \ell_2$, then $T\psi = \psi$ and $T\psi = -\psi$, i.e., $\psi = -\psi$, and hence $\psi = 0$.

Thus, for any $\psi \in \mathcal{L}^{p(\cdot)}$, the unique representation of type (90) with $\psi_k \in \ell_k$ is valid. This means that equality (89) is valid. □

11.3. **The space $(\mathcal{L}^{p(\cdot)})^*$.** Since $\mathcal{L}^{p(\cdot)} = \ell_1 \oplus \ell_2$, then following [30, p. 103], we have

$$(\mathcal{L}^{p(\cdot)})^* = \ell_1^* \oplus \ell_2^*.$$

Lemma 14. *Every linear continuous functional $\Lambda \in (\mathcal{L}^{p(\cdot)})^*$ generates the linear continuous functional $\widehat{\Lambda}$ from $(L^{p(\cdot)})^*$.*

Proof. We denote the narrowing of the functional Λ on ℓ_k by Λ_k (i.e., $\Lambda_k f = \Lambda f$, when $f \in \ell_k$).

Since ℓ_k is the closed subspace of the space $L^{p(\cdot)}$, there exists the linear, continuous functional Λ_k on $L^{p(\cdot)}$ such that $\widehat{\Lambda}_k f = \Lambda f$ when $f \in \ell_k$ (see e.g., [31, p. 72]).

Assume

$$\widehat{\Lambda} = \widehat{\Lambda}_1 + \widehat{\Lambda}_2.$$

By the continuity of functionals $\widehat{\Lambda}_k$, we conclude that $\widehat{\Lambda}$ is the linear, continuous functional on $L^{p(\cdot)}$.

If $f \in \mathcal{L}^{p(\cdot)}$, then $f = f_1 + f_2$, $f_k \in \ell_k$, therefore

$$\begin{aligned} \widehat{\Lambda} f = \widehat{\Lambda}_1 f + \widehat{\Lambda}_2 f = \widehat{\Lambda}_1(f_1 + f_2) + \widehat{\Lambda}_2(f_1 + f_2) = \\ = \widehat{\Lambda}_1 f + \widehat{\Lambda}_1 f_2 + \widehat{\Lambda}_2 f_1 + \widehat{\Lambda}_2 f_2. \end{aligned} \tag{91}$$

Before going further, we need the following

Lemma 15. *The equalities*

$$\widehat{\Lambda}_1 f_2 = 0, \quad f_2 \in \ell_2, \quad \widehat{\Lambda}_2 f_1 =, \quad f_1 \in \ell_1, \tag{92}$$

are valid.

Proof. Let $f = f_1 + f_2$, then $\widehat{\Lambda} f_1 = \widehat{\Lambda}_1 f_1 + \widehat{\Lambda}_2 f_1$, $\widehat{\Lambda} f_2 = \widehat{\Lambda}_1 f_2 + \widehat{\Lambda}_2 f_2$. By the definition of functionals Λ_k, we have $\widehat{\Lambda}_1 f_1 = \Lambda f_1$ and $\widehat{\Lambda}_2 f_2 = \Lambda f_2$. By virtue of the above-said, from the last equalities we arrive at equalities (92). □

We can now complete the proof of Lemma 14. Equalities (91) yield

$$\widehat{\Lambda} f = \widehat{\Lambda}_1 f_1 + \widehat{\Lambda}_2 f_2 = \Lambda f_1 + \Lambda f_2 = \Lambda(f_1 + f_2) = \Lambda f,$$

i.e., $\widehat{\Lambda}$ is an extension of the functional Λ on $\mathcal{L}^{p(\cdot)}$ to the functional on $L^{p(\cdot)}$.

For the functional $\widehat{\Lambda}$ from Lemma 14, we have

$$\widehat{\Lambda} f = \int_\Gamma f\mu\, dt, \tag{93}$$

where $\mu \in L^{p'(\cdot)}$ (since $(L^{p(\cdot)})^* = L^{p'(\cdot)}$, (see item 2.3.2). □

Lemma 16. *The function μ in equality* (93) *belongs to $L^{p'(\cdot)}$.*

Proof. We have

$$\widehat{\Lambda}\psi = \int_\Gamma (\psi_1+\psi_2)\mu\,dt = \int_\Gamma \psi_1\mu\,dt + \int_\Gamma \psi_2\mu\,dt = I_1+I_2. \tag{94}$$

Here,

$$2I_1 = \int_\Gamma \psi_1\mu\,dt = \int_\Gamma (\psi+T\psi)\mu\,dt = \int_\Gamma \psi\mu\,dt + \int_\Gamma T\psi\mu\,dt. \tag{95}$$

Transforming the second summand in (95) and applying the Riesz equalities

$$\int_\Gamma fS_\Gamma g\,dt = -\int_\Gamma gS_\Gamma f\,dt, \quad f\in L^{p(\cdot)}, \quad g\in L^{p'(\cdot)} \tag{96}$$

([17]), we have

$$\int_\Gamma T\psi\mu\,dt = \int_\Gamma X^+S_\Gamma\,\frac{\psi}{X^+}\,\mu\,dt = \\ = \int_\Gamma \mu X^+S_\Gamma\,\frac{\psi}{X^+}\,dt = -\int_\Gamma \frac{\psi}{X^+}\,S_\Gamma X^+\,dt.$$

Assuming for the present that $\mu=\mu_n$ and $\psi=\psi_\nu$ are rational functions, we can apply formula (96). Thus we obtain

$$\int_\Gamma T\psi_\nu\mu_n\,dt = -\int_\Gamma \frac{\psi_\nu}{X^+}\,S_\Gamma X^+\mu_n\,dt = -\int_\Gamma \psi_\nu\widetilde{T}\mu_n\,dt. \tag{97}$$

For the fixed μ_n, in right-hand side of equality (97) we can pass to the limit with respect to ν. We get

$$\lim_{\nu\to\infty}\int_\Gamma T\psi_\nu\mu_n\,dt = -\int_\Gamma \psi\widetilde{T}\mu_n\,dt, \quad \psi\in L^{p(\cdot)}.$$

As far as $\{T\psi_\nu\}$ converges in measure to $T\psi$, we select a subsequence converging almost everywhere to $T\psi$ and, by the Fatou lemma, we find that

$$\int_\Gamma T\psi\mu_n\,dt = -\int_\Gamma \psi\widetilde{T}\mu_n\,dt.$$

In the above equality, we can pass to the limit in left-hand side and as a result, we have

$$\int_\Gamma T\psi\mu\,dt = \lim_{n\to\infty}\int_\Gamma \psi\widetilde{T}\mu_n\,dt.$$

According to Lemma 10, $\{\widetilde{T}\mu_n\}$ converges in measure to $\widetilde{T}\mu$. Just as above, we apply Fatou's lemma and obtain

$$\int\limits_{\Gamma} T\psi\mu\,dt = -\int\limits_{\Gamma} \psi\widetilde{T}\mu\,dt, \tag{98}$$

where $\mu \in L^{p'(\cdot)}$, $\psi \in L^{p(\cdot)}$. From (98) we can conclude that $\widetilde{T}\mu \in L^{p'(\cdot)}$. Consequently, $\mu \in L^{p'(\cdot)}$, $\widetilde{T}\mu \in L^{p'(\cdot)}$. $\square$

It follows from equalities (93), (95) and (98) that if $\mu \in \mathcal{L}^{p'(\cdot)}$, then

$$\Lambda\psi = \int\limits_{\Gamma} \psi\mu_1\,dt, \quad \mu_1 = -T\mu \in \widetilde{\mathcal{L}}^{p'(\cdot)}$$

is the linear continuous functional in $\mathcal{L}^{p(\cdot)}$. This and the statement of Lemma 14 allow us to conclude that the following theorem is valid.

Theorem 12. *If the conditions of Theorem* 3 *are fulfilled, then*

$$(\mathcal{L}^{p(\cdot)})^* = \widetilde{\mathcal{L}}^{q(\cdot)}, \quad q(t) = \frac{p(t)}{p(t)-1}\,.$$

12. On the Noetherity of Operator M in the Space $\mathcal{L}^{p(\cdot)}$

12.1. **The operator, conjugate to the operator M.** If the operator M acts from the Banach space X to Y, then the operator M^* acts from Y^* to X^* which to the linear functional Λ from Y^* to $\mathbb{C}$ puts into correspondence the functional Λ^* defined by the equality $\Lambda^* x = \Lambda(Mx)$, $x \in X$.

In the case under consideration, $X = Y = \mathcal{L}^{p(\cdot)}$ and $Y^* = X^* = \widetilde{\mathcal{L}}^{q(\cdot)}$.

Let $f \in \mathcal{L}^{p(\cdot)}$, then

$$\Lambda f = \int\limits_{\Gamma} f\psi\,dt, \quad \psi \in \widetilde{\mathcal{L}}^{q(\cdot)},$$

$$\begin{aligned}\Lambda^* f &= \int\limits_{\Gamma} \psi M f\,dt = \int\limits_{\Gamma} \psi(t)\big(a(t)f(t) + b(t)(Sf)(t)\big)\,dt = \\ &= \int\limits_{\Gamma} a(t)\psi(t)f(t)\,dt + \int\limits_{\Gamma} \psi(t)b(t)(Sf)(t)\,dt = \\ &= \int\limits_{\Gamma} a(t)\psi(t)f(t)\,dt - \int\limits_{\Gamma} f(t)(Sb\psi)(t)\,dt = \\ &= \int\limits_{\Gamma} f(t)\big(a(t)\psi(t) - (Sb\psi)(t)\big)\,dt.\end{aligned}$$

Consequently, the conjugate to the operator $M : \mathcal{L}^{p(\cdot)} \to \mathcal{L}^{p(\cdot)}$ is the operator $M^* : \widetilde{\mathcal{L}}^{q(\cdot)} \to \widetilde{\mathcal{L}}^{q(\cdot)}$ given by the equality

$$M^*\psi = a\psi - Sb\psi. \tag{99}$$

12.2. **About the equation $M^*\psi = \mu$.** The equation

$$M^*\psi = \mu. \tag{100}$$

considered in $\widetilde{\mathcal{L}}^{q(\,\cdot\,)}$ is equivalent to the problem of conjugation

$$\Psi^+ = \frac{1}{G}\,\Psi^- + \frac{\mu}{a-b}\,, \tag{101}$$

considered in the class $K\widetilde{\mathcal{L}}^{q(\,\cdot\,)}$. In addition,

$$\Psi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{b(\tau)\psi(\tau)}{\tau - z}\,d\tau. \tag{102}$$

Since

$$\Psi^+ = \frac{1}{2}\,(b\psi + Sb\psi), \quad \Psi^- = \frac{1}{2}\,(-b\psi + Sb\psi),$$

therefore

$$\Psi^+ - \Psi^- = b\psi, \quad \Psi^+ + \Psi^- = Sb\psi.$$

If $\mu = 0$, then $\psi = Sb\psi = \Psi^+ - \Psi^-$, and hence $a\psi = \Psi^+ + \Psi^-$, $b\psi = \Psi^+ - \Psi^-$. This implies that $(a+b)\psi = 2\Psi^+$, i.e.,

$$\psi(z) = \frac{2\Psi^+}{a+b}\,. \tag{103}$$

Since $\frac{1}{G} \in A(q(\,\cdot\,))$, for $\varkappa = \varkappa(G) \geq 0$ we have ind $\frac{1}{G} \leq 0$, therefore the equation

$$a\psi - Sb\psi = 0 \tag{104}$$

has only a zero solution.

When $\varkappa(G) < 0$, it is not difficult to verify that a general solution of the problem (101) for $\mu \equiv 0$ will have the form

$$\Psi = \frac{1}{2}\,X(z)P_{|\varkappa|-1}(z)$$

and from (73) we find that the set of functions

$$\Psi = \frac{P_{|\varkappa|-1}(z)}{X^+(a+b)}$$

provides us with a general solution of equation $M^*\psi = 0$. The base of a general solution for that equation is

$$\frac{1}{X^+(a+b)}\,, \frac{\tau}{X^+(a+b)}\,, \dots, \frac{\tau^{|\varkappa|-1}}{X^+(a+b)}\,.$$

12.3. **On the Noetherity of the operator** M**.** The conditions (54) designate that equation (63) for $\varkappa < 0$ is normal solvability.

If $\varkappa \geq 0$, then $N(M^*) = \{0\}$, and the equation $M\varphi = f$ is solvable for any $\frac{f}{a+b} \in \mathcal{L}^{p(\,\cdot\,)}$, i.e., the condition of normal solvability is fulfilled again.

This and the fact that $\ell = N(M) = \max(0, \varkappa)$ and $\ell' = N(M^*) = \max(0, -\varkappa)$ allow us to conclude that the theorem below is valid.

Theorem 13. *Let* Γ *be the simple, closed, rectifiable curve and let* $a(t)$ *and* $b(t)$ *be measurable bounded functions such that*

$$\operatorname{ess\,inf} \big|a^2(t) - b^2(t)\big| > 0,$$

and $G(t) = (a(t) - b(t))(a(t) + b(t))^{-1}$ *. If the conditions* (25) *are fulfilled, then the equation*

$$M\varphi := a(t)\varphi(t) + b(t)(S\varphi)(t) = f(t)$$

is Noetherian in the space $\mathcal{L}^{p(\,\cdot\,)}$*, where*

$$M^*\psi = a\psi - Sb\psi, M^* : \widetilde{\mathcal{L}}^{q(\,\cdot\,)} \to \widetilde{\mathcal{L}}^{q(\,\cdot\,)},$$
$$\operatorname{ind}(M; \mathcal{L}^{p(\,\cdot\,)}) = \varkappa(G) = \varkappa = \operatorname{ind}\big((a-b)(a+b)^{-1}\big).$$

In all cases where a solution exists, it is given by equality (85)*.*

Corollary. *If* V *is a compact operator from* $\mathcal{L}^{p(\,\cdot\,)}$ *to* $\mathcal{L}^{p(\,\cdot\,)}$ *and the conditions* (25) *are fulfilled, then the operator* $M + V$ *is Noetherian in* $\mathcal{L}^{p(\,\cdot\,)}$*, and* $\operatorname{ind}(M + V, \mathcal{L}^{p(\,\cdot\,)}) = \operatorname{ind} M = \operatorname{ind} \frac{a-b}{a+b}$.

This statement is a consequence of the result obtained in [29] according to which it follows that the addition of a compact operator to the Noetherian one does not change its Noetherity and index.

13. Some Properties of the Operator $T = T_G$, when $G \in A(p(\,\cdot\,))$

Above we frequently applied properties of the operator T_G proven in Section 6. Remind these properties.

(1) Under the assumptions (25), we have $T(Tg) = g$.
(2) The operator T is continuous from $L^{p(t)}$ to the space of convergence in measure.
(3) The operator T is closed from $L^{p(\,\cdot\,)}$ to $L^{p(\,\cdot\,)}$.

Moreover, when proving Lemma 6, we have used equality (66) which will be proved in Subsection 13.1.

Below, we will present some other properties of the operator T. We start with Lemma 17 which will be highly useful in establishing operator properties which will be treated in Subsections 13.3–13.5.

All curves considered in Section 13 are assumed (except requirements made by the theorem) to be simple, rectifiable and closed.

13.1. **Lemma about** $S(ab)$**.**

Lemma 17. *If* $\Gamma \in R$*,* $p \in \mathcal{P}(\Gamma)$*,* $a \in L^{p(\,\cdot\,)}$*,* $b \in L^{q(\,\cdot\,)}$*, then almost everywhere on* Γ *the equality*

$$S(ab) = bSa + aSb - S(Sa \cdot Sb) \tag{105}$$

is valid.

Proof. Assume that the point $z = 0$ lies in the inner domain bounded by Γ. Then rational functions of the type

$$\sum_{k=-m}^{-1} a_k t^k + \sum_{k=0}^{n} a_k t^k = m(t) + h(t)$$

form a complete set both in $L^{p(\,\cdot\,)}$ and in $L^{q(\,\cdot\,)}$. We denote it by Q.

Let us show that if $a(t) = m(t) + h(t)$, $b(t) = r(t) + s(t)$, then equality (105) is valid.

We have

$$\begin{aligned} S(ab) &= S\big((m+h)(r+s)\big) = S(mr + hr + ms + hs) = \\ &= S(mr + hs) + S(ms + hr) = mr - hs + S(ms + hs). \end{aligned} \tag{106}$$

Here we have used the equalities

$$(SP)(t) = P(t), \quad S\Big(P\Big(\frac{1}{t}\Big)\Big) = -P\Big(\frac{1}{t}\Big),$$

where P is the polynomial of its own argument.

Further,

$$\begin{aligned} & bSa + aSb - S(Sa \cdot Sb) = \\ &= (r+s)(m-h) + (m+h)(r-s) - S(mr - ms - hr + hs) = \\ &= rm - rh + sm - sh + mr - ms + hr - hs - S(mr - ms - hr + hs) = \\ &= 2rm - 2h - (mr - hs) - S(ms + hr) = \\ &= mr - hs + S(ms + hr). \end{aligned} \tag{107}$$

From equalities (106) and (107) we obtain (105) in the form

$$S(R_n Q_m) = SR_n \cdot Q_m + R_n SQ_m - S(SR_n \cdot SQ_m), \tag{108}$$

where R_n and Q_m belong to Q.

Let now $a \in L^{p(\,\cdot\,)}$ and $b \in L^{q(\,\cdot\,)}$ be arbitrary functions, and let $\|R_n - a\|_{p(\,\cdot\,)} \to 0$, $\|Q_m - b\|_{q(\,\cdot\,)} \to 0$.

Since $\Gamma \in R$ and $p \in \mathcal{P}(\Gamma)$, by the boundedness of the operator S in $L^{p(\,\cdot\,)}$ ([10]), we admit in equality (108) the passage to the limit which allows us to conclude that equality (105) is valid in a general case. □

Corollary. *If* $\Gamma \in R$*,* $p \in \mathcal{P}(\Gamma)$*,* $m \in L^{p(\,\cdot\,)}$*,* $n \in L^{q(\,\cdot\,)}$*, then*

$$T(mn) = Tm \cdot n + m \cdot Sn - T(Tm \cdot Sn).$$

Proof. According to (105), we get

$$T(mn) = X^+S\Big(\frac{m}{X^+}\,n\Big) = X^+\Big(nS\,\frac{m}{X^+} + \frac{m}{X^+}\,Sn\Big) - X^+S\Big(S\,\frac{m}{X^+}\,Sn\Big) =$$
$$= Tm\cdot n + m\cdot Sn + X^+ - X^+S\Big(\frac{1}{X^+}\,X^+S\,\frac{m}{X^+}\,Sn\Big) =$$
$$= Tm\cdot n + m\cdot Sn - T(Tm\cdot Sn). \qquad \square$$

13.2. **Value of** $\sup\|T\|_\alpha$ **when** $\alpha\in[\underline{p},\overline{p}]$**.**

Lemma 18. *If* $\Gamma\in R$, $p\in\mathcal{P}(\Gamma)$, $\underline{p}=\inf\limits_{t\in\Gamma}p(t)$, $\overline{p}=\sup\limits_{t\in\Gamma}p(t)$, *and for any* $\alpha\in I=[\underline{p},\overline{p}]$ *we have* $\|T\|_\alpha<\infty$, *then*

$$\sup_\alpha\|T\|_\alpha<\infty.$$

Proof. Assume the contrary; then there exists the sequence $\{\alpha_n\}$, $\alpha_n\in I$, such that

$$\|T\|_{\alpha_n}\to\infty.$$

Note that if p and p_1 belong to $\mathcal{P}(\Gamma)$, and $p(t)\le p_1(t)$, then

$$\|f\|_{p(\,\cdot\,)}\le(1+\operatorname{mes}\Gamma)\|f\|_{p_1(\,\cdot\,)}$$

(see item 2.4.2).

Let $\alpha_0=\sup\alpha_n$, then $\alpha_0\in I$. Taking into account the last inequality, we obtain

$$\|T\|_{\alpha_0}=\sup_{\|\varphi\|_{\alpha_0}\le1}\|T\varphi\|_{\alpha_0}\ge\sup_{\|\varphi\|_{\alpha_0}\le1}\|T\varphi\|_{\alpha_n}\cdot\frac{1}{1+\operatorname{mes}\Gamma}\,. \tag{109}$$

But $\|\varphi\|_{\alpha_0}\ge\frac{1}{1+\operatorname{mes}\Gamma}\,\|\varphi\|_{\alpha_n}$, hence $\|\varphi\|_{\alpha_n}\le(1+\operatorname{mes}\Gamma)\|\varphi\|_{\alpha_0}$.

Consequently,

$$\sup_{\|\varphi\|_{\alpha_0}\le1}\|T\varphi\|_{\alpha_n}=(1+\operatorname{mes}\Gamma)\sup_{\|\varphi\|_{\alpha_n}\le1}\|T\varphi\|_{\alpha_n}=(1+\operatorname{mes}\Gamma)\|T\|_{\alpha_n}.$$

This together with the estimate (109) result in $\|T\|_{\alpha_0}=\infty$. But this contradicts the assumptions of the lemma by which $\|T\|_{\alpha_0}$ should be finite, since $\alpha_0\in I$. $\square$

13.3. **On the operator** $T_{1/G}$**, when** $G\in A(p(\,\cdot\,))$**.**

Lemma 19. *If* $\Gamma\in R$, $p\in\mathcal{P}(\Gamma)$ *and the operator* T_G, $G\in A(p(\,\cdot\,))$, *is continuous in* $L^{p(\,\cdot\,)}$, *then the operator*

$$T_{1/G}:f\to T_{1/G}f,\quad (T_{1/G}f)(t)=\frac{1}{2\pi iX^+(t)}\int\limits_\Gamma\frac{X^+(\tau)f(\tau)}{\tau-t}\,dt$$

is continuous in $L^{q(\,\cdot\,)}$.

Conversely, if $T_{1/G}$ *is continuous in* $L^{q(\,\cdot\,)}$, *then* T_G *is continuous in* $L^{p(\,\cdot\,)}$.

Moreover,

$$\|T_G\|_{p(\,\cdot\,)}\le k\|T_{1/G}\|_{q(\,\cdot\,)},\quad \|T_{1/G}\|\le k\|T_G\|_{p(\,\cdot\,)}, \tag{110}$$

where $k = 1 + \frac{1}{\underline{p}} + \frac{1}{\overline{p}}$ *is the constant from inequality* (6).

Proof. We proceed from the relation

$$\|f\|_{p(\,\cdot\,)} \sim \sup_{\|g\|_{q(\,\cdot\,)} \le 1} \bigg| \int_\Gamma f g \, dt \bigg|$$

(see item 2.4.1).

Assuming for the present that f and g are rational functions of the class Q, we get

$$\|T_{1/G} g\|_{q(\,\cdot\,)} \sim \sup_{\|f\|_{p(\,\cdot\,)} \le 1} \bigg| \int_\Gamma f T_{1/G} \, dt \bigg| = \sup_{\|f\|_{p(\,\cdot\,)} \le 1} \bigg| \int_\Gamma f \, \frac{1}{X^+} \, S X^+ g \, dt \bigg|.$$

Using the Riesz equality (see formula (66)), we obtain

$$\|T_{1/G} g\|_{q(\,\cdot\,)} = \sup_{\|f\|_{p(\,\cdot\,)} \le 1} \bigg| \int_\Gamma g X^+ S \, \frac{f}{X^+} \, dt \bigg| = \sup_{\|f\|_{p(\,\cdot\,)} \le 1} \bigg| \int_\Gamma g T_G f \, dt \bigg| \le$$
$$\le k \cdot \sup_{\|f\|_{p(\,\cdot\,)} \le 1} \|g\|_{q(\,\cdot\,)} \|T_G f\|_{p(\,\cdot\,)} \le k \|g\|_{q(\,\cdot\,)} \|T_G\|_{p(\,\cdot\,)} \|f\|_{p(\,\cdot\,)} =$$
$$= k \|T_G\|_{p(\,\cdot\,)} \|g\|_{q(\,\cdot\,)}.$$

Thus

$$\|T_{1/G} g\|_{q(\,\cdot\,)} \le k \|T_G\|_{p(\,\cdot\,)} \|g\|_{q(\,\cdot\,)}, \quad f, g \in Q. \tag{111}$$

Analogously we can prove that

$$\|T_G f\|_{p(\,\cdot\,)} \le k \|T_{1/G}\|_{q(\,\cdot\,)} \|f\|_{p(\,\cdot\,)}. \tag{112}$$

By the passage to the limit (which is admissible due to $\Gamma \in R$), we find that inequalities (111) and (112) are valid for any $f \in L^{p(\,\cdot\,)}$, $g \in L^{q(\,\cdot\,)}$, i.e., inequalities (110) are valid in a general case. □

13.4. **On the operator** ST**.**

Theorem 14. *Let* $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$, $G \in A(p(\,\cdot\,))$, $g \in L^{p(\,\cdot\,)}$, *then*

$$S(Tg) = g + Tg - Sg.$$

Proof. First of all, we note that $Tg \in L^{p(\,\cdot\,)-\varepsilon} \in L^1$ (see Theorem 7).

Since $Tg \in L^1$, almost everywhere on Γ there exists the integral $S_\Gamma \frac{g}{X^+}$, and hence $g(X^+)^{-1} \in L^1$. This implies that $(K_\Gamma \frac{g}{X^+})(z)$ belongs to the set $\bigcap_{\delta<1} E^\delta(D^+)$ (see Subsection 2.6). Since

$$\frac{1}{X(z)} = \frac{1}{X_G(z)} = \exp\bigg(- \frac{1}{2\pi i} \int_\Gamma \frac{\ln|G(\tau)| + i \arg G(t)}{\tau - z} \, d\tau \bigg).$$

$\Gamma \in R$ and $G \in \widetilde{A}(p(\,\cdot\,))$, and hence $\ln G$ is the bounded function, therefore $X(z)$ and $1/X(z)$ belong to $E^\nu(D^+)$ for some $\nu > 0$ ([8, pp. 96–98]). Thus the function

$$F(z) = \frac{X(z)}{2\pi i} \int\limits_{\Gamma} \frac{g(\tau)}{X^+(\tau)} \, \frac{d\tau}{\tau - z} = X(z)\Big(K_\Gamma \, \frac{g}{X^+}\Big)(z), \tag{113}$$

being a product of two Smirnov class functions, belongs to some class $E^\eta(D^+)$, $\eta > 0$. Moreover, $F^+ = \frac{1}{2}(g + Tg)$. Here, $g \in L^{p(\cdot)}$, while $Tg \in L(\Gamma)$. Thereby, $F^+ \in L(\Gamma)$. Thus, according to Smirnov's theorem (see, e.g., [27, p. 254]), we find that $F \in E^1(D^+)$. But then $S_\Gamma F^+ = F^+$. This results in

$$\frac{1}{2}\,(g + Tg) = \frac{1}{2}(Sg + STg)$$

from which we obtain the provable equality. □

13.5. **On the operator TS.** As it has been shown in proving Theorem 14, the function $F(z)$ given by equality (113) belongs to $E^1(D^+)$. This fact allows us to prove that the following theorem is valid.

Theorem 15. *In the assumptions of Theorem* 14*, the equality*

$$(TS)(g) = Sg + g - Tg \tag{114}$$

is valid.

Proof. Let

$$\Psi(z) = \frac{(K_\Gamma g)(z)}{X(z)}\,,$$

then $\Psi(z) \in E^1(D^+)$, therefore

$$S\big[(K_\Gamma g)X^{-1}\big]^+ = (K_\Gamma g)^+(X^+)^{-1},$$

that is,

$$S\,\frac{g + Sg}{X^+} = \frac{g + Sg}{X^+}$$

from which we successively obtain

$$\begin{aligned} X^+ S\,\frac{g + Sg}{X^+} &= g + Sg, \\ T(g + Sg) &= g + Sg, \\ Tg + TSg &= g + Sg. \end{aligned}$$

Indeed, the last equalities show that equality (114) is valid. □

Acknowledgement

The author expresses his sincere thanks to Professor Vakhtang Kokilashvili for valuable suggestions and comments.

The present work was supported by Shota Rustaveli National Science Foundation (Grant D13/23).

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(Received November 4, 2013)

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Mem. Differential Equations Math. Phys. 61 (2013), 147–159

Short Communication

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ON THE WELL-POSSEDNESS OF GENERAL NONLINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH FINITE AND FIXED POINTS OF IMPULSES

Abstract. The general nonlocal boundary value problems are considered for systems of differential equations with finite and fixed points of impulses. The sufficient conditions, among which are effective spectral ones, are established for the well-posedness of these problems.

რეზიუმე. განხილულია ზოგადი სახის არაწრფივი სასაზღვრო ამოცანები დიფერენციალურ განტოლებათა სისტემებისთვის სასრული და ფიქსირებული იმპულსების წერტილებით. დადგენილია ამ ამოცანების კორექტულობის საკმარისი, მათ შორის ეფექტური სპექტრალური პირობები.

2000 Mathematics Subject Classification: 34K10, 34K45.
Key words and phrases: Nonlocal boundary value problems, nonlinear systems, impulsive equations, solvability, unique solvability, effective conditions.

1. Statement of the Problem and Basic Notation

In the present paper, we consider the system of nonlinear impulsive equations with a finite number of impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \tag{1.1}$$

$$x(\tau_l+) - x(\tau_l-) = I_l(x(\tau_l)) \ \ (l = 1,\dots,m_0) \tag{1.2}$$

with the general boundary value problem

$$h(x) = 0, \tag{1.3}$$

where $a < \tau_1 < \cdots < \tau_{m_0} \le b$ (we will assume $\tau_0 = a$ and $\tau_{m_0+1} = b$, if necessary), $-\infty < a < b < +\infty$, m_0 is a natural number, $f = (f_i)_{i=1}^n$ belongs to Carathéodory class $\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$, $I_l = (I_{li})_{i=1}^n : R^n \to \mathbb{R}^n$ $(l = 1,\dots,m_0)$ are continuous operators, and $h : C([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ is a continuous, nonlinear in general, vector-functional.

Let x_0 be a solution of the problem $(1.1),(1.2);(1.3)$.

Consider a sequence of vector-functions $f_k \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$ $(k=1,2,\dots)$, the sequences of continuous operators $I_{lk}: R^n \to \mathbb{R}^n$ $(k=1,2,\dots;\ l=1,\dots,m_0)$, the sequences τ_{lk} $(k=1,2,\dots;\ l=1,\dots,m_0)$ of points $a<\tau_{1k}<\dots<\tau_{m_0k}\le b$ and the sequence of continuous vector-functionals $h_k: C([a,b],\mathbb{R}^n;\tau_{1k},\dots,\tau_{m_0k})\to\mathbb{R}^n$ $(k=1,2,\dots)$.

In this paper the sufficient conditions are given guaranteing both the solvability of the impulsive boundary value problems

$$\frac{dx}{dt}=f_k(t,x) \ \text{ almost everywhere on } \ [a,b]\setminus\{\tau_{1k},\dots,\tau_{m_0k}\}, \qquad (1.1_k)$$

$$x(\tau_{lk}+)-x(\tau_{lk}-)=I_{lk}(x(\tau_{lk})) \ \ (l=1,\dots,m_0); \qquad (1.2_k)$$

$$h_k(x)=0 \qquad (1.3_k)$$

$(k=1,2,\dots)$ for any sufficient large k and the convergence of its solutions to a solution of the problem $(1.1),(1.2);(1.3)$ as $k\to+\infty$.

As is known, the question of the well-possedness for the nonlinear impulsive boundary value problems was not investigated in earlier works. So the statement of the problems under consideration is actual.

The obtained results are analogous to ones given in [12] (see also the references therein) for the general nonlinear boundary value problems for systems of ordinary differential equations. Some results obtained in the paper are more general than already known ones even for ordinary differential case.

The analogous question is investigated in [4], [8] for linear boundary value problems for systems with impulses, and in [1]–[3], [12]–[15] for linear and nonlinear boundary value problems for systems of ordinary differential equations. Notice that the necessary and sufficient conditions are obtained for the well-possedness of the linear boundary value problem in [8] for impulsive, and in [1]–[3] for ordinary differential systems.

Quite a number of issues of the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for a survey of the results on impulsive systems see, e.g., [5]–[7], [9]–[11], [16], [17] and the references therein).

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R}=\,]-\infty,+\infty[\,$, $\mathbb{R}_+=[0,+\infty[\,$; $[a,b]$ $(a,b\in R)$ is a closed segment.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norms

$$\|X\|=\max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|;$$

$$|X|=(|x_{ij}|)_{i,j=1}^{n,m}, \quad [X]_+=\frac{|X|+X}{2};$$

$$\mathbb{R}_+^{n\times m}=\Big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\ge 0\ (i=1,\dots,n;\ j=1,\dots,m)\Big\};$$

$$\mathbb{R}^{(n\times n)\times m}=\mathbb{R}^{n\times n}\times\dots\times\mathbb{R}^{n\times n} \ \ (m\text{-times}).$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = R_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; $I_{n\times n}$ is the identity $n \times n$-matrix.

$\overset{b}{\underset{a}{\vee}}(X)$ is the total variation of the matrix-function $X : [a,b] \to R^{n\times m}$, i.e., the sum of total variations of the latters' components; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = \overset{t}{\underset{a}{\vee}}(x_{ij})$ for $a < t \leq b$;

$X(t-)$ and $X(t+)$ are the left and the right limit of the matrix-function $X : [a,b] \to R^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \leq a$ and $X(t) = X(b)$ for $t \geq b$, if necessary);

$$\|X\|_s = \sup\big\{\|X(t)\| : \ t \in [a,b]\big\}.$$

$\mathrm{BV}([a,b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to R^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X) < +\infty$);

$C([a,b], D)$, where $D \subset R^{n\times m}$, is the set of all continuous matrix-functions $X : [a,b] \to D$;

$C([a,b], D; \tau_1, \dots, \tau_{m_0})$ is the set of all matrix-functions $X : [a,b] \to D$, having the one-sided limits $X(\tau_l-)$ $(l = 1, \dots, m_0)$ and $X(\tau_l+)$ $(l = 1, \dots, m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\}$ belong to $C([c,d], D)$;

$C_s([a,b], \mathbb{R}^{n\times m}; \tau_1, \dots, \tau_{m_0})$ is the Banach space of all $X \in C([a,b], \mathbb{R}^{n\times m}; \tau_1, \dots, \tau_{m_0})$ with the norm $\|X\|_s$.

If $y \in C_s([a,b], \mathbb{R}; \tau_1, \dots, \tau_{m_0})$ and $r \in \,]0, +\infty[\,$, then

$$U(y;r) = \Big\{x \in C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) : \ \|x - y\|_s < r\Big\};$$

$D(y,r)$ is the set of all $x \in \mathbb{R}^n$ such that $\inf\{\|x - y(t)\| : \ t \in [a,b]\} < r$.

$\widetilde{C}([a,b], D)$, where $D \subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$;

$\widetilde{C}([a,b], D; \tau_1, \dots, \tau_{m_0})$ is the set of all matrix-functions $X : [a,b] \to D$, having the one-sided limits $X(\tau_l-)$ $(l = 1, \dots, m_0)$ and $X(\tau_l+)$ $(l = 1, \dots, m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b] \setminus \{\tau_k\}_{k=1}^m$ belong to $\widetilde{C}([c,d], D)$.

If B_1 and B_2 are the normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x) = \lambda g(x)$ for every $\lambda \in R_+$ and $x \in B_1$.

The inequalities between the matrices are understood component wise.

An operator $\varphi : C([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to R^n$ is called nondecreasing if for every $x, y \in C([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0})$ such that $x(t) \leq y(t)$ for $t \in [a,b]$ the inequality $\varphi(x)(t) \leq \varphi(y)(t)$ holds for $t \in [a,b]$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$L([a,b],D)$, where $D \subset R^{n\times m}$, is the set of all measurable and integrable matrix-functions $X:[a,b]\to D$.

If $D_1 \subset R^n$ and $D_2 \subset R^{n\times m}$, then $\mathrm{Car}([a,b]\times D_1, D_2)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$:

(a) the function $f_{kj}(\cdot,x):[a,b]\to D_2$ is measurable for every $x\in D_1$;

(b) the function $f_{kj}(t,\cdot):D_1\to D_2$ is continuous for almost every $t\in[a,b]$, and

$$\sup\big\{|f_{kj}(\cdot,x)|:\ x\in D_0\big\}\in L([a,b],R;g_{ik})\ \text{ for every compact } D_0\subset D_1.$$

$\mathrm{Car}^0([a,b]\times D_1,D_2)$ is the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that the functions $f_{kj}(\,\cdot\,,x(\,\cdot\,))$ $(j=1,\dots,m;\ k=1,\dots,n)$ are measurable for every vector-function $x:[a,b]\to\mathbb{R}^n$ with the bounded total variation.

$M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is the set of all functions $\omega\in\mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ such that the function $\omega(t,\,\cdot\,)$ is nondecreasing, and $\omega(t,0)\equiv 0$.

By a solution of the impulsive system (1.1), (1.2) we understand a continuous from the left vector-function $x\in\widetilde{C}([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})$ satisfying both the system (1.1) for a.e. on $[a,b]\setminus\{\tau_1\dots,\tau_{m_0}\}$ and the relation (1.2) for every $l\in\{1,\dots,m_0\}$.

Definition 1.1. Let $l: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$ be a linear continuous operator, and let $l_0: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n_+$ be a positive homogeneous operator. We say that a pair $(P,\{J_l\}_{l=1}^{m_0})$, consisting of a matrix-function $P\in\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a finite sequence of continuous operators $J_l=(J_{li})_{i=1}^n:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,\dots,m_0)$, satisfy the Opial condition with respect to the pair (l,l_0) if:

(a) there exist a matrix-function $\Phi\in L([a,b],\mathbb{R}^n_+)$ and constant matrices $\Psi_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that

$$|P(t,x)|\le\Phi(t)\ \text{ a.e. on } [a,b],\ \ x\in\mathbb{R}^n$$

and

$$|J_l(x)|\le\Psi_l\ \text{ for } x\in\mathbb{R}^n\ \ (l=1,\dots,m_0);$$

(b)

$$\det(I_{n\times n}+G_l)\ne 0\ \ (l=1,\dots,m_0) \tag{1.4}$$

and the problem

$$\frac{dx}{dt}=A(t)x\ \text{ a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \tag{1.5}$$

$$x(\tau_l+)-x(\tau_l-)=G_lx(\tau_l)\ \ (l=1,\dots,m_0); \tag{1.6}$$

$$|l(x)|\le l_0(x) \tag{1.7}$$

has only the trivial solution for every matrix-function $A \in L([a,b],\mathbb{R}^{n\times n})$ and constant matrices G_l $(l=1,\dots,m_0)$ for which there exists a sequence $y_k \in \widetilde{C}([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\int_a^t P(\tau,y_k(\tau))d\tau = \int_a^t A(\tau)d\tau \ \text{ uniformly on } [a,b]$$

and

$$\lim_{k\to+\infty} J_l(y_k(\tau_l)) = G_l \ \ (l=1,\dots,m_0).$$

Remark 1.1. In particular, the condition (1.4) holds if

$$\|\Psi_l\| < 1 \ \ (l=1,\dots,m_0).$$

Below we will assume that $f = (f_i)_{i=1}^n \in \operatorname{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$ and, in addition, $f(\tau_l,x)$ is arbitrary for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$.

Let x^0 be a solution of the problem (1.1), (1.2); (1.3), and r be a positive number. Let us introduce the following definition.

Definition 1.2. The solution x^0 is said to be strongly isolated in the radius r if there exist, respectively, the matrix– and the vector-functions $P \in \operatorname{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and $q \in \operatorname{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$, a finite sequence of continuous operators $J_l:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,\dots,m_0)$ and $h_l:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,\dots,m_0)$, a linear continuous operator $l: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$, a positive homogeneous operator $l_0: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n_+$, and a continuous operator $\widetilde{l}: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$ such that

(a) the equalities

$$f(t,x) = P(t,x)x + q(t,x) \ \text{ for } \ t\in[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ \|x-x^0(t)\|<r,$$
$$I_l(x) = J_l(x)x + h_l(x) \ \text{ for } \ \|x-x^0(\tau_l)\|<r \ \ (l=1,\dots,m_0);$$

and

$$h(x) = l(x) + \widetilde{l}(x) \ \text{ for } \ x\in U(x^0;r)$$

are valid;

(b) the functions

$$\alpha(t,\rho) = \max\big\{\|q(t,x)\|: \ \|x\|\le\rho\big\},$$
$$\beta_l(\rho) = \max\big\{\|h_l(x)\|: \ \|x\|\le\rho\big\} \ \ (l=1,\dots,m_0)$$

and

$$\gamma(\rho) = \sup\Big\{\big[|\widetilde{l}(x)| - l_0(x)\big]_+ : \ \|x\|_s\le\rho\Big\}$$

satisfy the condition

$$\limsup_{\rho\to+\infty}\frac{1}{\rho}\left(\gamma(\rho)+\int_a^b\alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta_l(\rho)\right)<1;$$

(c) the problem

$$\frac{dx}{dt}=P(t,x)x+q(t,x)\ \text{ a.e. on }\ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\},$$
$$x(\tau_l+)-x(\tau_l-)=J_l(x(\tau_l))+h_l(x(\tau_l))\ \ (l=1,\dots,m_0);$$
$$l(x)+\widetilde{l}(x)=0$$

has no solution different from x^0.

(d) the pair $(P,\{J_l\}_{l=1}^{m_0})$ satisfy the Opial condition with respect to the pair (l,l_0).

Definition 1.3. We say that a sequence $(f_k,\{I_{lk}\}_{l=1}^{m_0};h_k)$ $(k=1,2,\dots)$ belongs to the set $W_r(f,\{I_l\}_{l=1}^{m_0},h;x^0)$ if

(a) the equalities

$$\lim_{k\to+\infty}\int_a^t f_k(\tau,x)\,d\tau=\int_a^t f(\tau,x)d\tau\ \text{ uniformly on }\ [a,b]$$

and

$$\lim_{k\to+\infty}I_{lk}(x)=I_l(x)\ \ (l=1,\dots,m_0)$$

are valid for every $x\in D(x^0;r)$;

(b)

$$\lim_{k\to+\infty}h_k(x)=h(x)\ \text{ uniformly on }\ U(x^0;r);$$

(c) there exist a sequence of functions $\omega_k\in M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ $(k=1,2,\dots)$ and sequences of nondecreasing functions $\omega_{lk}\in C(\mathbb{R}_+,\mathbb{R}_+)$, $\omega_{lk}(0)=0$, $(k=1,2,\dots;l=1,\dots,m_0)$ such that

$$\sup\left\{\int_a^b\omega_k(t,r)\,dt:\ k=1,2,\dots\right\}<+\infty,\tag{1.8}$$

$$\sup\left\{\sum_{l=1}^{m_0}\omega_{lk}(r):\ k=1,2,\dots\right\}<+\infty;\tag{1.9}$$

$$\lim_{s\to0+}\sup\left\{\int_a^b\omega_k(t,s)\,dt:\ k=1,2,\dots\right\}=0,\tag{1.10}$$

$$\lim_{s\to 0+} \sup \left\{ \sum_{l=1}^{m_0} \omega_{lk}(s): \ k=1,2,\dots \right\} = 0; \tag{1.11}$$

and

$$\big\|f_k(t,x)-f_k(t,y)\big\| \le \omega_k\big(t,\|x-y\|\big)$$
$$\text{for } \ t\in[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x,y\in D(x^0;r) \ \ (k=1,2,\dots),$$
$$\big\|I_{lk}(x)-I_{lk}(y)\big\| \le \omega_{lk}\big(\|x-y\|\big)$$
$$\text{for } \ x,y\in D(x^0;r) \ \ (l=1,\dots,m_0; \ \ k=1,2,\dots).$$

Remark 1.2. If for every natural m there exists a positive number ν_m such that

$$\omega_k(t,m\delta) \le \nu_m \omega_k(t,\delta)$$
$$\text{for } \ \delta>0, \ \ t\in[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\} \ \ (k=1,2,\dots),$$

then the estimate (1.8) follows from the condition (1.10); analogously, if

$$\omega_{lk}(m\delta) \le \nu_m \omega_{lk}(\delta) \ \text{ for } \ \delta>0 \ \ (l=1,\dots,m_0; \ \ k=1,2,\dots),$$

then the estimate (1.9) follows from the condition (1.11). In particular, the sequences of the functions

$$\omega_k(t,\delta)=\max\Big\{\big\|f_k(t,x)-f_k(t,y)\big\|: \ x,y\in U(0,\|x^0\|+r), \ \|x-y\|\le\delta\Big\}$$
$$\text{for } \ t\in[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\} \ (k=1,2,\dots)$$

and

$$\omega_{lk}(\delta)=\max\Big\{\big\|I_{lk}(x)-I_{lk}(y)\big\|: \ x,y\in U(0,\|x^0\|+r), \ \|x-y\|\le\delta\Big\}$$
$$(l=1,\dots,m_0; \ \ k=1,2,\dots)$$

have, respectively, the latters' properties.

Definition 1.4. The problem $(1.1), (1.2); (1.3)$ is said to be $(x^0;r)$-correct if for every $\varepsilon\in\,]0,r[$ and $\big((f_k,\{I_{lk}\}_{l=1}^{m_0};h_k)\big)_{k=1}^{+\infty}\in W_r(f,\{I_l\}_{l=1}^{m_0},h;x^0)$ there exists a natural number k_0 such that the problem $(1,1_k)$, (1.2_k); (1.3_k) has at least a solution contained in $U(x^0;r)$ and any such solution belongs to the ball $U(x^0;)$ for every $k\ge k_0$.

Definition 1.5. The problem $(1.1), (1.2); (1.3)$ is said to be correct if it has the unique solution x^0 and it is $(x^0;r)$-correct for every $r>0$.

Theorem 1.1. *If the problem* $(1.1), (1.2)$*;* (1.3) *has a solution* x^0*, strongly isolated in the radius* $r>0$*, then it is* $(x^0;r)$*-correct.*

Theorem 1.2. *Let the conditions*

$$\big\|f(t,x)-P(t,x)x\big\| \le \alpha\big(t,\|x\|\big)$$
$$a.e. \ on \ \ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \quad x\in\mathbb{R}^n, \tag{1.12}$$

$$\big\|I_l(x)-J_l(x)x\big\| \le \beta_l\big(\|x\|\big) \ \ for \ \ x\in\mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{1.13}$$

and

$$|h(x) - l(x)| \le l_0(x) + l_1\big(\|x\|_s\big) \ \ for \ \ x \in \mathrm{BV}([a,b],\mathbb{R}^n) \tag{1.14}$$

hold, where $\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *are, respectively, linear continuous and positive homogeneous continuous operators, the pair* $(P,\{J_l\}_{l=1}^{m_0})$ *satisfies the Opial condition with respect to the pair* (ℓ,ℓ_0)*;* $\alpha \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ *is a function nondecreasing in the second variable, and* $\beta_l \in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are, respectively, nondecreasing functions and vector-function such that*

$$\limsup_{\rho\to+\infty} \frac{1}{\rho}\bigg(\|l_1(\rho)\| + \int\limits_a^b \alpha(t,\rho)\,dt + \sum_{l=1}^{m_0}\beta_l(\rho)\bigg) < 1. \tag{1.15}$$

Then one-valued solvability of the problem (1.1), (1.2)*;* (1.3) *guarantees its correctness.*

Theorem 1.3. *Let the conditions* (1.12)–(1.14),

$$P_1(t) \le P(t,x) \le P_2(t) \ \ a.e. \ on \ \ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x\in\mathbb{R}^n,$$

and

$$J_{1l} \le I_k(x) \le J_{2l} \ \ for \ \ x\in\mathbb{R}^n \ \ (l=1,\dots,m_0)$$

hold, where $P \in \mathrm{Car}^0([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$*,* $P_i \in L([a,b],\mathbb{R}^{n\times n})$ $(i=1,2)$*,* $J_{il} \in \mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$*,* $l : C_s([a,b],\mathbb{R}^{n\times n};\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ *and* $l_0 : C_s([a,b],\mathbb{R}^{n\times m};\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *are, respectively, the linear continuous and the positive homogeneous continuous operators;* $\alpha \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ *is a function nondecreasing in the second variable, and* $\beta_l \in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are, respectively, nondecreasing functions and vector-function such that the condition* (1.15) *holds. Let, moreover, the condition* (1.4) *hold and the problem* (1.5), (1.6)*;* (1.7) *have only the trivial solution for every matrix-function* $A \in L([a,b],\mathbb{R}^{n\times n})$ *and the constant matrices* $G_l \in \mathbb{R}^{n\times n}(l=1,\dots,m_0)$ *such that*

$$P_1(t) \le A(t) \le P_2(t) \ \ a.e. \ on \ \ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x\in\mathbb{R}^n$$

and

$$J_{1l} \le G_l \le J_{2l} \ \ for \ \ x\in\mathbb{R}^n \ \ (l=1,\dots,m_0).$$

Then one-valued solvability of the problem (1.1), (1.2)*;* (1.3) *guarantees its correctness.*

Remark 1.3. Theorem 1.3 is interesting only in the case where $P \notin \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, because the theorem immediately follows from Theorem 1.2 in the case where $P \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$.

Theorem 1.4. *Let the conditions* (1.14),

$$\big|f(t,x)-P_0(t)x\big| \le Q(t)|x| + q\big(t,\|x\|\big) \ \ \text{a.e. on} \ \ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \quad x\in\mathbb{R}^n,$$

and

$$\big|I_l(x)-J_{0l}x\big| \le H_l|x| + h_l\big(\|x\|\big) \ \ \text{for} \ \ x\in\mathbb{R}^n \ \ (l=1,\dots,m_0)$$

hold, where $P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $Q \in L([a,b],\mathbb{R}_+^{n\times n})$, J_{0l} *and* $H_l \in \mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are constant matrices,* $l : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ *and* $l_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}_+^n$ *are, respectively, the linear continuous and positive homogeneous continuous operators;* $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *is a vector-function nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}_+^n)$ *are, respectively, the nondecreasing functions and vector-function such that the condition* (1.15) *holds. Let, moreover, the conditions*

$$\det(I_{n\times n}+J_{0l}) \ne 0 \ \ (l=1,\dots,m_0) \tag{1.16}$$

and

$$\|H_l\|\cdot\big\|(I_{n\times n}+J_{0l})^{-1}\big\| < 1 \ \ (l=1,\dots,m_0) \tag{1.17}$$

hold, and the system of impulsive inequalities

$$\Big|\frac{dx}{dt}-P_0(t)x\Big| \le Q(t)x \ \ \text{a.e. on} \ \ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \tag{1.18}$$

$$\big|x(\tau_l+)-x(\tau_l-)-J_{0l}x(\tau_l)\big| \le H_l x(\tau_l) \ \ (l=1,\dots,m_0) \tag{1.19}$$

have only the trivial solution under the condition (1.7). *Then one-valued solvability of the problem* (1.1), (1.2)*;* (1.3) *guarantees its correctness.*

Corollary 1.1. *Let the conditions* (1.16)

$$\big|f(t,x)-P_0(t)x\big| \le q\big(t,\|x\|\big) \ \ \text{a.e. on} \ \ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ x\in\mathbb{R}^n, \tag{1.20}$$

and

$$\big|I_l(x)-J_{0l}x\big| \le h_l\big(\|x\|\big) \ \ \text{for} \ \ x\in\mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{1.21}$$

hold, where $P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $J_{0l} \in \mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices,* $\ell : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}_+^n$ *are, respectively, the linear continuous and positive homogeneous continuous operators;* $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *is a vector-function nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}_+^n)$ *are, respectively, the nondecreasing functions and a vector-function such that the condition* (1.15) *holds. Let, moreover,*

$$|h(x)-\ell(x)| \le \ell_1\big(\|x\|_s\big) \ \ \text{for} \ \ x\in \mathrm{BV}([a,b],\mathbb{R}^n) \tag{1.22}$$

and the impulsive system

$$\frac{dx}{dt} = P_0(t)x \;\; a.e. \;\; on \;\; [a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\},$$
$$x(\tau_l+) - x(\tau_l-) = J_{0l}x(\tau_l) \;\; (l = 1, \dots, m_0)$$

have only the trivial solution under the condition (1.7). *Then one-valued solvability of the problem* (1.1), (1.2); (1.3) *guarantees its correctness.*

For every matrix-function $X \in L([a,b], \mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k = 1, \dots, m_0)$ we introduce the operators

$$\big[(X, Y_1, \dots, Y_{m_0})(t)\big]_0 = I_n \;\text{ for }\; a \le t \le b,$$
$$\big[(X, Y_1, \dots, Y_{m_0})(a)\big]_i = O_{n\times n} \;\; (i = 1, 2, \dots),$$
$$\big[(X, Y_1, \dots, Y_{m_0})(t)\big]_{i+1} = \int_a^t X(\tau) \cdot \big[(X, Y_1, \dots, Y_{m_0})(\tau)\big]_i \, d\tau +$$
$$+ \sum_{a \le \tau_l < t} Y_l \cdot \big[(X, Y_1, \dots, Y_{m_0})(\tau_l)\big]_i \;\text{ for }\; a < t \le b \;\; (i = 1, 2, \dots). \qquad (1.23)$$

Corollary 1.2. *Let the conditions* (1.16), (1.20)–(1.22) *hold, where*

$$\ell(x) \equiv \int_a^b dK(t) \cdot x(t),$$

$P_0 \in L([a,b], \mathbb{R}^{n\times n})$, $J_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m_0)$ *are constant matrices,* $K \in L([a,b], \mathbb{R}^{n\times n})$, $\ell_0 : C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n_+$ *are the positive homogeneous continuous operators;* $q \in \mathrm{Car}([a,b] \times \mathbb{R}_+, \mathbb{R}^n_+)$ *is a vector-function nondecreasing in the second variable, and* $h_l \in C([a,b], \mathbb{R}_+)$ $(l = 1, \dots, m_0)$ *and* $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *are, respectively, the nondecreasing functions and a vector-function such that the condition* (1.15) *holds. Let, moreover, there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1} \int_a^b dK(t) \cdot \big[(P_0, G_l, \dots, G_{m_0})(t)\big]_i$$

is nonsingular and

$$r(M_{k,m}) < 1,$$

where the operators $[(P_0,G_1,\dots,G_{m_0})(t)]_i$ $(i=0,1,\dots)$ *are defined by* (1.23)*, and*

$$M_{k,m}=\Big[\big(|P_0|,|G_1|,\dots,|G_{m_0}|\big)(b)\Big]_m+\\ +\sum_{i=0}^{m-1}\Big[\big(|P_0|,|G_1|,\dots,|G_{m_0}|\big)(b)\Big]_i\times\\ \times\int_a^b dV(M_k^{-1}K)(t)\cdot\Big[\big(|P_0|,|G_1|,\dots,|G_{m_0}|\big)(t)\Big]_k.$$

Then one-valued solvability of the problem (1.1)*,* (1.2)*;* (1.3) *guarantees its correctness.*

Theorem 1.5. *Let the conditions* (1.16)*,* (1.17)*,*

$$\big|f(t,x)-f(t,y)-P_0(t)(x-y)\big|\leq Q(t)|x-y|\\ \text{a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\},\ \ x,y\in\mathbb{R}^n,\\ \big|I_l(x)-I_l(y)-J_{0l}\cdot(x-y)\big|\leq H_k\cdot|x-y|\\ \text{for } x,y\in\mathbb{R}^n\ \ (k=l,\dots,m_0)$$

and

$$\big|h(x)-h(y)-\ell(x-y)\big|\leq\ell_0(x-y)\ \text{ for } x,y\in\mathrm{BV}([a,b],\mathbb{R}^n)$$

hold, where $P_0\in L([a,b],\mathbb{R}^{n\times n})$*,* $Q\in L([a,b],\mathbb{R}_+^{n\times n})$*,* J_{0k} *and* $H_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices,* $\ell:C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$ *and* $\ell_0:C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}_+^n$ *are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the system of impulsive inequalities* (1.18)*,* (1.19) *have only the trivial solution under the condition* (1.7)*. Then the problem* (1.1)*,* (1.2)*;* (1.3) *is correct.*

Corollary 1.3. *Let the system* (1.1)*,* (1.2)*;* (1.3) *have a unique solution* x^0 *defined on the whole closed interval* $[a,b]$*, where* $h(x)\equiv x(t_0)-c_0$*, and* $t_0\in[a,b]$ *and* $c_0\in\mathbb{R}^n$ *are such that* $I_l(c_0)=0$ *if* $t_0=\tau_l$ *for some* $l\in\{1,\dots,m_0\}$*. Let, moreover,*

$$\limsup_{\rho\to+\infty}\Big(\inf\big\{\|x+I_l(y)\|:\ \|x\|\geq\rho,\ \|y\|=\rho\big\}\Big)>\|x^0\|_s \tag{1.24}$$

for every $l\in\{1,\dots,m_0\}$ *such that* $\tau_l>t_0$*. Then the problem* (1.1)*,* (1.2)*;* (1.24) *is correct.*

Remark 1.4. It is evident that the condition (1.24) is valid if $I_l(y)\equiv 0$ for every $l\in\{1,\dots,m_0\}$ such that $\tau_l>t_0$. If the last condition is not fulfilled, i.e., $I_l(y)\not\equiv 0$ for some $l\in\{1,\dots,m_0\}$, then the strict inequality (1.24) cannot be replaced by a non-strict one for this l. Below, we will give the corresponding example.

Example. Let $n = 1$, $m_0 > 2$ be an arbitrary natural number, $\tau_l \in]a, b[$ $(l = 1, \dots, m_0)$, $h(x) \equiv x(t_0) - c_0$, $t_0 = b$, $c_0 = 0$; $h_k(x) \equiv x(t_k) - c_k$ $(k = 1, 2, \dots)$, $t_k \to b$ $(k \to +\infty)$ and $c_k \to c_0$ $(k \to +\infty)$; $f(t, x) = f_k(t, x) \equiv 0$ $(k = 1, 2, \dots)$; $I_l(x) = I_{kl}(x) \equiv 0$ $(l = 1, \dots, m_0 - 1;\ k = 1, 2, \dots)$;

$$I_{m_0}(x) = \begin{cases} 0 & \text{for } x < 0, \\ (1 + c_{i+1} - c_i)(i - x) - i - c_i & \text{for } x \in [i, i+1[\ \ (i = 0, 1, \dots) \end{cases}$$

and

$$I_{km_0}(x) =$$

$$= \begin{cases} I_{m_0}(x) & \text{for } x \in]-\infty, k-1[\cup]k+1, +\infty[\,, \\ (1 - c_{k-1} - c_k)(k - x) + c_k - k & \text{for } x \in [k-1, k[\,, \\ (1 + c_{k+1} + c_k)(k - x) + c_k - k & \text{for } x \in [k, k+1[\end{cases}$$

$$(k = 1, 2, \dots).$$

Then $x^0(t) \equiv 0$, $\big((f_k, \{I_{lk}\}_{l=1}^{m_0}; h_k)\big)_{k=1}^{+\infty} \in W_r(f, \{I_l\}_{l=1}^{m_0}, h; x^0)$. Moreover, the problem $(1.1_k), (1.2_k); (1.3_k)$ has the unique solution

$$x_k(t) = \begin{cases} c_k & \text{for } a \le t \le \tau_{m_0}, \\ k & \text{for } \tau_{m_0} < t \le b \end{cases}$$

for every natural k. As to the condition (1.24), it is transformed into the equality for $t = \tau_{m_0}$ only.

Acknowledgement

The present work was supported by the Shota Rustaveli National Science Foundation (Grant No. FR/182/5-101/11).

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(Received 21.03.2013)

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Mem. Differential Equations Math. Phys. 61 (2014)

CONTENTS

Short Communication

მემუარები დიფერენციალურ განტოლებებსა და მათემატიკურ ფიზიკაში
61 (2014)

შ ი ნ ა ა რ ს ი

მოკლე წერილი