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Professor Roland Duduchava (on the occasion of 70th Birthday)

This year marks the 70th birthday of Professor Roland Duduchava, an eminent Georgian mathematician whose contribution to the theory of integral equations of convolution type with discontinuous presymbols is recognized worldwide. He is an author and co-author of 4 monographs and 109 research papers. His results are successfully used by researchers working on singular integral equations, pseudodifferential equations, boundary value problems for elliptic partial differential equations, and on many other problems of mathematics and its applications.

Roland Duduchava was born on November 12, 1945 in Tbilisi. He graduated from a secondary school at Sokhumi in 1962 and enrolled the faculty of Mechanics and Mathematics of Tbilisi State University, from which he graduated with honors in January 1968. He then became a PhD student at A. Razmadze Mathematical Institute of the Georgian Academy of Sciences, Tbilisi, Georgia. In 1971, Roland Duduchava finished his PhD study in Kishinev, Moldova, at the Institute of Mathematics and Computing Center of the Academy of Sciences of Moldova and in the same year defended his Candidate Thesis (PhD degree) under the supervision of Professor I. Gohberg.

Since 1971, Roland Duduchava had worked as a junior, senior, leading and principal researcher at A. Razmadze Mathematical Institute of the Georgian Academy of Sciences, and since 1995 he headed the Department of Mathematical Physics.

In 1983, he defended his higher doctoral thesis (Habilitation) at the M. Lomonosov Moscow State University. In 1989, he was granted the title of Professor by the Supreme Certifying Commission of the USSR.

At various times, Roland Duduchava worked as a professor at I. Javakhishvili Tbilisi State University, IB Euro-Caucasian University, Humboldt University in Berlin, Saarland University in Saarbrucken, and Stuttgart University.

Roland Duduchava is a fellow of the Alexander von Humboldt Foundation (1981–1989) and Professor Merkator of the German Research Council DFG (2001–2002).

He has received 10 international (Soros, AMS, INTAS, DFG and other) and 4 national (GNSF - Shota Rustaveli National Science Foundation) research grants as a head of a research group.

Roland Duduchava is a member of editorial boards of 6 international mathematical journals. He has successfully supervised 7 PhD students and has served as a consultant for one higher doctoral thesis.

He is the president of the Georgian Mathematical Union (since 2009) and the organizer of many international conferences including International Conference “1Continuum Mechanics and Related Problems of Analysis” dedicated to the 120-th birthday anniversary of academician N. Muskhelishvili (2011), Caucasian Mathematics Conference (2014), International Workshop on Operator Theory and Applications, IWOTA 2015, and others.

Professor Roland Duduchava is an outstanding scientist, whose life is full of great achievements in mathematics.

We congratulate Roland Duduchava on his birthday and wish him every joy, happiness and great fulfillment in the years to come.

D. Kapanadze, D. Natroshvili, E. Shargorodsky

List of Publications of Roland Duduchava

(i) Monographs

1. Convolution integral equations with discontinuous presymbols, singular integral equations with fixed singularities, and their applications to problems in mechanics. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR* **60** (1979), 136 pp.
2. Integral equations in convolution with discontinuous presymbols, singular integral equations with fixed singularities, and their applications to some problems of mechanics. With German, French and Russian summaries. Teubner-Texte zur Mathematik. [Teubner Texts on Mathematics] *BSB B. G. Teubner Verlagsgesellschaft, Leipzig*, 1979.
3. Boundary value problems in domains with peaks (with B. Silbermann). *Mem. Differential Equations Math. Phys.* **21** (2000), 1–122.
4. Interface crack problems for metallic-piezoelectric composite structures (with T. Buchukuri, O. Chkadua, and D. Natroshvili). *Mem. Differential Equations Math. Phys.* **55** (2012), 1–150; `http://rmi.tsu.ge/jeomj/memoirs/vol55/contents.htm`.

(ii) Papers

1. Singular integral operators in a Hölder space with weight. (Russian) *Dokl. Akad. Nauk SSSR* **191** (1970), 16–19.
2. The boundedness of the singular integration operator in Hölder spaces with weight. (Russian) *Mat. Issled.* **5** (1970), vyp. 1 (15), 56–76.
3. Singular integral equations in Hölder spaces with weight. I. Hölder coefficients. (Russian) *Mat. Issled.* **5** (1970), No. 2(16), 104–124.
4. Singular integral equations in Hölder spaces with weight. II. Partial Hölder coefficients. (Russian) *Mat. Issled.* **5** (1970), No. 3(17), 58–82.
5. The boundary value problem for systems of discrete Wiener–Hopf equations. (Russian) *Mat. Issled.* **7** (1972), No. 2(24), 234–240, 292.
6. Discrete Wiener–Hopf equations that are composed of the Fourier coefficients of piecewise Wiener functions. (Russian) *Dokl. Akad. Nauk SSSR* **207** (1972), 1273–1276; translation in *Sov. Math., Dokl.* **13** (1972), 1903–1907.
7. Discrete Wiener-Hopf equations in l_p spaces with weight. (Russian) *Sakharth. SSR Mecn. Akad. Moambe* **67** (1972), 17–20.
8. The algebras of singular integral operators in spaces of Hölder functions with weight. (Russian) *Sakharth. SSR Mecn. Akad. Moambe* **65** (1972), 25–28.
9. Wiener–Hopf integral operators with discontinuous symbols. (Russian) *Dokl. Akad. Nauk SSSR* **211** (1973), 277–280; translation in *Sov. Math., Dokl.* **14** (1973), 1001–1005.

10. Algebras of one-dimensional singular integral operators in space of Hölder functions with weight. (Russian) A collection of articles on the theory of functions, 5. *Sakharth. SSR Mecn. Akad. Math. Inst. Shrom.* **43** (1973), 19–52. (errata insert).
11. On Noether theorems for singular integral equations. (Russian) In *Proceedings of Symposium on Mechanics and Related Problems of Analysis*, vol. 1, Metsniereba, Tbilisi, 19–52, 1973.
12. Singular integral operators on piecewise smooth curves. (Russian) *Sakharth. SSR Mecn. Akad. Moambe* **71** (1973), 553–556.
13. Multidimensional convolution equations formed from the Fourier coefficients of discontinuous functions. (Russian) *Sakharth. SSR Mecn. Akad. Moambe* **74** (1974), 277–280.
14. Discrete convolution operators on symmetric spaces of sequences with weights. In *Theses of the Conference of Young Scientists and Postgraduates, I. Javakhishvili State University, I. Vekua Institute of Applied Mathematics*, pp. 63–64, Tbilisi University Press, Tbilisi, 1974.
15. Singular integral equations with unbounded coefficients. (Russian) A collection of articles on the equations of mathematical physics, 4. *Sakharth. SSR Mecn. Akad. Math. Inst. Shrom.* **44** (1974), 72–78.
16. Convolution integral operators with discontinuous coefficients. (Russian) *Dokl. Akad. Nauk SSSR* **218** (1974), 264–267; translation in *Sov. Math., Dokl.* **15** (1975), 1302-1306.
17. Bisingular integral operators and convolution operators in a quadrant. (Russian) *Dokl. Akad. Nauk SSSR* **221** (1975), No. 2, 279–282; translation in *Sov. Math., Dokl.* **16** (1975), 330–334.
18. Wiener–Hopf integral operators. (Russian) *Math. Nachr.* **65** (1975), 59–82.
19. Convolution integral operators with discontinuous symbols. (Russian) Collection of articles on functional analysis, 2. *Sakharth. SSR Mecn. Akad. Math. Inst. Shrom.* **50** (1975), 34–41.
20. The discrete Wiener–Hopf equations. (Russian) Collection of articles on functional analysis, 2. *Sakharth. SSR Mecn. Akad. Math. Inst. Shrom.* **50** (1975), 42–59.
21. Bisingular integral operators, and boundary value problems of the theory of analytic functions in spaces of generalized functions. (Russian) *Dokl. Akad. Nauk SSSR* **224** (1975), No. 5, 996–999; translation in *Sov. Math., Dokl.* **17** (1976), 1324–1328.
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Mikhail S. Agranovich

SPECTRAL PROBLEMS IN LIPSCHITZ DOMAINS IN SOBOLEV-TYPE BANACH SPACES

Dedicated to Roland Duduchava
with best wishes in connection with his jubilee

Abstract. This paper contains a short presentation of author's results on spectral properties of main boundary value problems for strongly elliptic second-order systems in bounded Lipschitz domains. We consider the questions on the completeness of root functions, on the summability of Fourier series with respect to them and on their basis property in spaces H_p^s with indices s, p close to $\pm 1, 2$. The complete presentation will be published elsewhere.

2010 Mathematics Subject Classification. 35J57, 35P05, 35P10.

Key words and phrases. Strongly elliptic system, Lipschitz domain, spectral problem, discrete spectrum, completeness of root functions, Abel–Lidskii summability.

რეზიუმე. ამ სტატიაში მოკლედ არის გადმოცემული ავტორის შედეგები, რომლებიც ეხება ძლიერ ელიფსური მეორე რიგის სისტემებისთვის შემოსაზღვრულ ლიფშიცის არეებში დასმული მთავარი სასაზღვრო ამოცანების სპექტრალურ თვისებებს. ჩვენ განვიხილავთ საკითხებს რომლებიც ეხება ფესვი ფუნქციების სისრულეს, მათ მიმართ ფურიეს მწკრივების კრებადობასა და მათ ძირითადი თვისებებს H_p^s

1. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary Γ. Assume that we have a matrix strongly elliptic [16] second-order operator

$$Lu := -\sum_{j,k=1}^{n} \partial_j a_{j,k} \partial_k u + \sum_{j=1}^{n} b_j \partial_j u + cu$$

in Ω with complex-valued coefficients of small smoothness (in particular, with Lipschitz higher-order coefficients). The form

$$\Phi(u, v) = \int\limits_{\Omega} \Big[\sum a_{j,k} \partial_k u \cdot \partial_j \overline{v} + \sum b_j \partial_j u \cdot \overline{v} + cu \cdot \overline{v} \Big]\, dx$$

is associated with L. We first consider the Dirichlet and Neumann problems in a weak sense for the equation $Lu = f$ with homogeneous boundary conditions. Solutions are defined by the Green formula

$$(Lu, v)_\Omega = \Phi(u, v). \tag{1}$$

In the simplest setting, in the Dirichlet problem

$$u, v \in \overset{\circ}{H}{}^1(\Omega) = \widetilde{H}^1(\Omega), \quad Lu = f \in H^{-1}(\Omega),$$

and in the Neumann problem

$$u, v \in H^1(\Omega) = W_2^1(\Omega), \quad Lu = f \in \widetilde{H}^{-1}(\Omega).$$

(The definitions of more general spaces can be seen in Section 2 below.) In such a generality, the Green formula is postulated. The functions f and u, v belong to spaces dual with respect to a continuation of the standard inner product in $L_2(\Omega)$

$$(u, v)_\Omega = \int\limits_{\Omega} u \cdot \overline{v}\, dx.$$

The bounded operators

$$L_D : \widetilde{H}^1(\Omega) \longrightarrow H^{-1}(\Omega) \ \text{ and } \ L_N : H^1(\Omega) \longrightarrow \widetilde{H}^{-1}(\Omega)$$

correspond to these problems. The domains of these operators are compactly and densely embedded in the right-hand spaces. We wish to consider spectral properties of these operators. We assume that the form Φ is *coercive*:

$$\|u\|^2_{H^1(\Omega)} \leq C_1 \operatorname{Re} \Phi(u, u) + C_2 \|u\|^2_{L_2(\Omega)}. \tag{2}$$

In the Dirichlet problem, the coerciveness is needed only on $\overset{\circ}{H}{}^1(\Omega)$ and follows from the strong ellipticity, For the Neumann problem, the simple sufficient conditions are known, fulfilled, in particular, for elasticity systems (see e.g. [2, Section 11]).

The last term in (2) can be removed by using a shift of the spectral parameter. After this, we have the *strong coercivity of* Φ. Below *it is assumed*. From it, the *invertibility* of the operators L_D and L_N follows by *the Lax–Milgram theorem* (see e.g. [2, Section 18]). The same is true for the adjoint operators L_D^* and L_N^* defined by the operator L^* formally adjoint to L (in Ω or $\overline{\Omega}$, respectively, see [2, Section 11]) and the Green formula

$$\Phi(u, v) = (u, L^* v)$$

with the same Φ.

The inverse operators are compact. Hence L_D and L_N are the operators with a discrete spectrum in their ranges. Our main question is: when their root functions are *complete*, i.e. their finite linear combinations are dense (in the ranges and hence in the domains), or are "better".

For the problems in the simplest setting indicated above, there are simple tools for the investigation of the completeness since only Hilbert spaces are used in this setting. In particular, L can be a formally self-adjoint operator in Ω or $\overline{\Omega}$:

$$\Phi(u, v) = \overline{\Phi(v, u)}$$

for u, v in $\widetilde{H}^1(\Omega)$ or $H^1(\Omega)$, respectively. Then we take the form $\Phi(u, v)$ for the inner product in the domain of L_D or L_N, respectively. In the ranges, we introduce the corresponding inner product e.g. $\Phi(L_D^{-1} f, L_D^{-1} g)$ in the case of the Dirichlet problem. The operators become self-adjoint, and a unique orthogonal basis of eigenfunctions exists in the both spaces.

Here, elementary, but very important remark consists in the fact that we need *the inner product defined by the operator*.

The asymptotics of the eigenvalues λ_k of self-adjoint operators L_D and L_N in a Lipschitz domain is known [12]. Namely, if λ_k are enumerated in the non-decreasing order taking multiplicities into account, then, as for the smooth problems,

$$\lambda_k \sim c k^{\frac{n}{2}}$$

(even with a fairly good remainder estimate). For non-self-adjoint compact operators L_D^{-1} and L_N^{-1}, this implies the estimate of "s-numbers" (see [7, Chapter 2])

$$s_k \leq C k^{-\frac{n}{2}}. \tag{3}$$

We have also the completeness if L is a *weak perturbation of a formally self-adjoint operator* (i.e. a perturbation in terms of order not greater than 1).

A more general condition, sufficient for the completeness, gives the *Dunford–Schwartz theorem* which is formulated in terms of angles between rays on the complex plane from the origin with power estimate for the norm of the resolvent (see [9, Chapter XI]). We only formulate a corollary for our problems in the simplest spaces.

Denote by Λ_θ the closed sector on the complex plane of opening 2θ with bisector $\mathbb{R}_+$. By M_θ we denote the closure of the complement to Λ_θ. Let θ_0 be such that the values of $\Phi(u, u)$ (with zero boundary values for u in the case of the Dirichlet problem) are contained in Λ_{θ_0}. Obviously, it contains all eigenvalues of L_D or L_N.

Note that $\theta_0 < \frac{\pi}{2}$ and that $e^{i\alpha}\Phi$ is strongly coercive if $0 < \alpha < \frac{\pi}{2} - \theta_0$.

Proposition 1. *The root functions of the operators L_D and L_N are complete in their ranges and domains if*

$$\theta_0 < \frac{\pi}{n}. \tag{4}$$

The proof uses (3) and the optimal resolvent estimate in M_θ with θ a little greater than θ_0 (see (8) below), it is easily obtained in our simplest spaces, see [2, Section 11].

2. However, our problems can be considered in more general spaces H_p^s of *Bessel potentials*. (For $p = 2$, they are H^s.) We remind definitions and some facts from their theory (cf. [2, Sections 14]).

1. $H_p^s(\mathbb{R}^n) = \Lambda^{-s} L_p(\mathbb{R}^n)$ for $1 < p < \infty$, $s \in \mathbb{R}$, where $\Lambda^{-s} = F^{-1}(1 + |\xi|^2)^{-s/2} F$ and F is the Fourier transform in the sense of distributions.
2. $H_p^s(\Omega)$ is the space of restrictions of elements in $H_p^s(\mathbb{R}^n)$ to Ω with inf-norm. For integers $s > 0$, they are the Sobolev spaces $W_p^s(\Omega)$.
3. $\widetilde{H}^s(\Omega)$ is the subspace in $H^s(\mathbb{R}^n)$ of elements supported in $\overline{\Omega}$.

We need to mention the following facts.

These spaces are separable and reflexive Banach spaces.

There is a universal bounded operator of continuation from $H_p^s(\Omega)$ to $H_p^s(\mathbb{R}^n)$ [13].

There is an operator of passage to the trace on Γ acting boundedly from $H_p^{s+\frac{1}{p}}(\Omega)$ to the Besov–Slobodetskii space $B_p^s(\Gamma) = W_p^s(\Omega)$ for $0 < s < 1$ (only) with a bounded right inverse.

The spaces $\widetilde{H}_p^s(\Omega)$ can be identified with $H_p^s(\Omega)$ for small $|s|$.

The spaces $H_p^s(\Omega)$ and $\widetilde{H}_{p'}^{-s}(\Omega)$ are dual. Here and below $\frac{1}{p} + \frac{1}{p'} = 1$.

We agree not to write Ω.

Now, in the Dirichlet problem

$$u \in \widetilde{H}_p^{\frac{1}{2}+s+\frac{1}{p}}, \quad f \in H_p^{-\frac{1}{2}+s-\frac{1}{p'}}, \quad v \in \widetilde{H}_{p'}^{\frac{1}{2}-s+\frac{1}{p'}},$$

and in the Neumann problem

$$u \in H_p^{\frac{1}{2}+s+\frac{1}{p}}, \quad f \in \widetilde{H}_p^{-\frac{1}{2}+s-\frac{1}{p'}}, \quad v \in H_{p'}^{\frac{1}{2}-s+\frac{1}{p'}}.$$

The solutions are defined by the same Green formula (1). The domains of the operators L_D and L_N: $u \longmapsto f$ are again compactly and densely embedded in their ranges. The functions u and f belong to the spaces with difference of superscripts equal 2. The functions f and v belong to the dual spaces. But $|s| < \frac{1}{2}$ in view of the trace theorem, and the functions f and u are generally not in dual spaces; because of this fact, the Lax–Milgram theorem cannot be applied.

Instead, the remarkable *Shneiberg's theorem* from the interpolation theory of operators is applicable. See [14] or [2, Section 13]. This is a theorem on the extrapolation of the invertibility of operators. According to it, there exist some numbers $\varepsilon \in (0, \frac{1}{2}]$ and (small) $\delta > 0$ such that our problem (Dirichlet or Neumann) is uniquely solvable for $|s| < \varepsilon$, $|r - \frac{1}{2}| < \delta$, where $r = \frac{1}{p}$. Simultaneously, this is a statement on the smoothness of solutions. If L has a formally self-adjoint principal part, then, under an easy additional condition at the points near Γ, $\varepsilon = \frac{1}{2}$.

Let $Q_{\varepsilon,\delta}$ be the rectangle of corresponding points $(s, \frac{1}{p})$. For convenience, we assume that it is common for the Dirichlet and Neumann problem and that $\varepsilon > \delta$. Below, we will consider only $(s,t) \in Q_{\varepsilon,\delta}$.

What can be said about spectral properties of our operators in these Banach spaces? Spectral properties of problems in abstract Banach spaces were investigated by many mathematicians (Grothendieck, Pietsch, König, Edmunds, Evans, Triebel, Markus, Matsaev, and many others). In particular, there are extensions of Dunford–Schwartz theorem ([6], [1]). But to apply them, one needs to have an extension of the resolvent estimate.

However, it turned out that *for our problems special theorems on the completeness in Banach spaces are non-necessary at all.* Let us explain this.

For a fixed p with $|\frac{1}{p} - \frac{1}{2}| < \delta$, denote by I_p the interval

$$\Big(-\frac{3}{2} - \varepsilon + \frac{1}{p},\ \frac{1}{2} + \varepsilon + \frac{1}{p}\Big).$$

This is the union of superscripts of "the most right" domain of our operator, "the most left" range of it and intermediate points. These spaces form a unique scale. When the superscript decreases, the space is expanded. The embedding is dense since smooth functions are dense in all spaces. Since L_D and L_N are invertible, their root functions belong to the domain and to the range simultaneously. If we have the completeness in one of these spaces, then this is true in the other one as well.

We obtain the following

Proposition 2. *The root functions of the operator L_D belong to all spaces corresponding to points of I_p, and if they are complete in one of them, they are complete in all other. The same is true for the operator L_N.*

This is useful in obtaining the following result.

Theorem 3. *The root functions belong to all spaces corresponding to points of the union of intervals I_p with $|\frac{1}{p} - \frac{1}{2}| < \delta$, and if they are complete for $p = 2$, then the same is true for all p.*

The proof uses, besides isomorphisms defined by our operator, the known embeddings for our spaces. For $p < 2$, the obvious embeddings are used for $s = \frac{1}{2} - \frac{1}{p}$. For $p > 2$, we use a less simple result (see [15, Section 4.6.1]):

Let

$$1 < p \le q < \infty, \quad \sigma - \tau \ge n\Big(\frac{1}{p} - \frac{1}{q}\Big).$$

Then there is a continuous and dense embedding $H_p^\sigma \subset H_q^\tau$. A similar statement is true for the spaces $\widetilde{H}_p^\sigma$.

It follows that for our operators the domain with the subscript p and superscript $\frac{1}{2} + \frac{1}{p}$ is embedded into the range with the subscript $q > p$ and superscript $-\frac{1}{2} - \frac{1}{q'}$ if

$$\frac{2}{n-1} \geq \frac{1}{p} - \frac{1}{q}.$$

We increase p by small steps and obtain the result in a finite number of steps. □

In a simpler case of smooth elliptic problems in Sobolev spaces, such approach was used by Agmon in his classical paper [4].

Remark. In the case of a formally self-adjoint L, in the spaces corresponding to the points of the interval I_2, it is possible to introduce inner products by using powers of the operator L_D or L_N, and then we have the same orthogonal basis of eigenfunctions in these spaces.

3. For our spectral problems, there exists *a second realization.* The corresponding operators can be considered as acting in $L_p(\Omega)$ (in particular, in $L_2(\Omega)$, which is especially popular in the literature, see e.g. [12]) instead of spaces with negative superscripts. We consider the Neumann problem for definiteness.

Let p be fixed with $|\frac{1}{p} - \frac{1}{2}| < \delta$. Denote by $\widehat{H}_p(\Omega)$ the space of such u that the form $\Phi(u, v)$ defines *a continuous anti-linear functional on* $L_{p'}(\Omega)$. Of course, it is continuous on $H_{p'}^{\frac{1}{2}-s+\frac{1}{p'}}(\Omega)$ for $|s| < \varepsilon$ (since the superscript is positive here). Hence formula

$$(L_N u, v) = \Phi(u, v) \tag{5}$$

defines a solution u of the equation $L_N u = f$ belonging to all $H_p^{\frac{1}{2}+s+\frac{1}{p}}(\Omega)$ with $|s| < \varepsilon$. In $\widehat{H}_p(\Omega)$, we introduce the graph norm by the equality

$$\|u\|^p_{\widehat{H}_p}(\Omega) = \|u\|^p_{L_p(\Omega)} + \|f\|^p_{L_p(\Omega)}.$$

For $p = 2$, it corresponds to the natural inner product in $\widehat{H}_2(\Omega)$. The first term in the right-hand side can be omitted.

Theorem 4. *The $\widehat{H}_p(\Omega)$ is a Banach space continuously embedded into the spaces $H_p^{\frac{1}{2}+s+\frac{1}{p}}(\Omega)$ for $|s| < \varepsilon$. The operator L_N defined by* (5) *maps the space $\widehat{H}_p(\Omega)$ onto $L_p(\Omega)$ isomorphically. Its spectrum and root functions remain the same, and the root functions are complete in $\widehat{H}_p(\Omega)$ if they are complete in $\widetilde{H}^{-1}(\Omega)$. In $L_2(\Omega)$, this operator is self-adjoint if it is self-adjoint in $\widetilde{H}^{-1}(\Omega)$, and then the orthonormal basis of eigenfunctions in $\widetilde{H}^{-1}(\Omega)$ remains an orthogonal basis in $\widehat{H}_2(\Omega)$.*

Remark. If the boundary Γ and the coefficients in L are smooth, then $\widehat{H}_p(\Omega)$ coincides with the subspace in $W_p^2(\Omega)$ of functions satisfying the homogeneous Neumann boundary conditions in the usual sense. Otherwise, $\widehat{H}_p(\Omega)$ can contain less smooth functions. The exact description of $\widehat{H}_p(\Omega)$ in a general Lipschitz domain is unavailable including $p = 2$.

The situation with the Dirichlet problem is similar.

4. Now we discuss *the summability of Fourier series with respect to root functions by the Abel–Lidskii method*. This is an intermediate property between the completeness and the basis property.

First, we define *the formal Fourier series with respect to the root vectors*. Let X and Y be separable Banach spaces with a compact and dense embedding $Y \subset X$, and let A be a bounded and invertible operator $Y \to X$. Assume that A has a complete minimal system $\{x_j\}_1^\infty$ of root vectors in X. Then the biorthogonal to it system $\{z_j\}_1^\infty$ is uniquely constructed from the root vectors of A^*, and to each vector $x \in X$ its formal Fourier series with respect to $\{x_j\}_1^\infty$ is associated:

$$x \sim \sum_1^\infty c_k x_k, \quad \text{where} \;\; c_k = (x, z_k), \tag{6}$$

$(\,\cdot\,,\,\cdot\,)$ is the duality between X and X^*. We enumerate the corresponding eigenvalues λ_k of A in order of increasing moduli taking multiplicities into account.

Let now A be one of our operators L_D and L_N, X and Y be its range and domain. Under some conditions (discussed below), it is possible to represent each vector $x \in X$ in the form

$$x = \frac{1}{2\pi i} \lim_{t \to 0} \int\limits_{\partial\Lambda^\theta} e^{-t\lambda^\gamma} R_A(\lambda)\, d\lambda\, x. \tag{7}$$

Here, the number γ and the parameter t are positive, the contour $\partial\Lambda_\theta$ is the boundary of Λ_θ with negative direction, and $R_A(\lambda)$ is the resolvent of A:

$$R_A(\lambda) = (A - \lambda I)^{-1}.$$

Moreover, assume that the domain Λ_θ can be divided into subdomains by arcs of radii $R_l \uparrow \infty$ not containing eigenvalues and that the integral (7) can be represented as the sum of integrals along the boundaries of these subdomains. Each integral is calculated via the residues of the integrand at the eigenvalues λ_k lying in the subdomain.

This is a summability method of order γ of the series (6) to the original vector x. This method was proposed by Lidskii in the case of a Hilbert space under the name Abel's method. Lidskii has found the conditions sufficient for the realization of this method [11]; see also [3, Chapter 5].

For our problems, it suffices to have (4). The key tool is the optimal resolvent estimate

$$\|R_A(\lambda)\| \leq C(1 + |\lambda|)^{-1} \tag{8}$$

in M_θ for $\theta > \theta_0$. For our operators in the simplest spaces, it is easily verified, and thus a deep strengthening of Proposition 1 is obtained.

To generalize this result to the spaces H_p^s, first, it is necessary to generalize the Lidskii theorem for the operators in Banach spaces. This was done in [1]. Here the abstract theorem is required. Secondly, it is necessary to generalize estimate (8) to these spaces. It turned out that this is not easy.

How to obtain the estimate, the paper by Gröger–Rehberg [8] suggested to the author. In this and some subsequent papers, the aim was to estimate the resolvent of the mixed problem in a very general statement, with domain of the corresponding operator contained in $W_p^1(\Omega)$, which is the diagonal direction $s + \frac{1}{p} = \frac{1}{2}$ in our notation. To obtain the estimate, they used Agmon's idea from the same paper [4].

Following this idea, we introduce the additional variable t and consider the Lipschitz cylinder $\Omega' = \Omega \times [-1, 1]$. In Ω', we consider the operator

$$L - \eta\partial_t^2$$

with the form

$$\int_0^1 \Phi(U, V)\, dt + \eta \int_\Omega \int_{-1}^1 \partial_t U \cdot \partial_t \overline{V}\, dt\, dx,$$

where $|\eta| = 1$, $|\arg\eta| < \frac{\pi}{2}$. *This form is strongly coercive on functions from* $H^1(\Omega')$*, equal to zero at* $t = \pm 1$. We apply the estimate that follows from Shneiberg's theorem to functions depending on the parameter μ:

$$U(x, t) = u(x)v(t), \ \text{ where } \ v(t) = \varphi(t)e^{i\mu t}, \ \ \mu = |\lambda|, \ \ \lambda = \eta\mu,$$

and $\varphi(t)$ is a function from $C_0^\infty[-1, 1]$ equal to 1 on $[-\frac{1}{2}, \frac{1}{2}]$.

Theorem 5. *Let* $\theta > \theta_0$*. Then for the resolvents of the operators* L_D *and* L_N *in the spaces corresponding to the points of some neighborhood of the centrum of the rectangle* $Q_{\varepsilon,\delta}$ *the uniform estimate* (8) *is valid for* $\lambda \in M_\theta$*.*

The proof is carried out first in two convenient directions $s + \frac{1}{p} = \frac{1}{2}$ (of Gröger–Rehberg) and $\frac{1}{p} = \frac{1}{2}$, on which the usual Sobolev–Slobodetskii norms can be used, and then the interpolation is applied.

Theorem 6. *Let condition* (4) *be fulfilled. Then the Fourier series with respect to the root functions of the operators* L_D *and* L_N *in the spaces corresponding to the points of some neighborhood of the centrum of the rectangle* $Q_{\varepsilon,\delta}$*, are summed to the corresponding vectors by the Abel–Lidskii method of order* $\gamma \in (\frac{n}{\pi}, \theta_0^{-1})$*.*

Remark. The estimate in Theorem 5 allows one to construct analytic semigroups e^{-tL_D} and e^{-tL_N} to solve "parabolic" problems in a Lipschitz cylinder in our Banach spaces. See [2, Section 17]. An essential additional remark: *the strong coerciveness of the form* Φ *is sufficient for this aim*, no additional assumptions on the coerciveness are needed.

5. A similar approach can be applied to other spectral problems. We indicate some of them. Cf. [2].

The mixed problem (with homogeneous Dirichlet and Neumann boundary conditions on two parts of Γ with common Lipschitz boundary of dimension $n-2$).

The Robin problem with boundary condition $T^+u + \beta u^+ = 0$, where u^+ is the boundary value of a solution and T^+u is its conormal derivative, $\operatorname{Re}\beta(x) \geq 0$.

The Dirichlet and Neumann problems for high-order strongly elliptic systems.

Of special interest is the Poincaré-Steklov spectral problem

$$Lu = 0 \text{ in } \Omega, \quad T^+u = \lambda u^+.$$

To it, *the Dirichlet-to-Neumann operator* is associated:

$$D : u^+ \longrightarrow T^+u.$$

Originally, it is considered as a bounded operator from $H^{\frac{1}{2}}(\Gamma) = B_2^{\frac{1}{2}}(\Gamma)$ to $H^{-\frac{1}{2}}(\Gamma) = B_2^{-\frac{1}{2}}(\Gamma)$. Its form (Du^+, u^+) coincides with $\Phi(u,u)$, which implies its strong coerciveness and the invertibility of the operator. By Shneiberg's theorem, for small $|s|$ and $|p - \frac{1}{2}|$ it has a bounded and invertible extension

$$B_p^{\frac{1}{2}+s}(\Gamma) \longrightarrow B_p^{-\frac{1}{2}+s}(\Gamma)$$

in Besov spaces on Γ, and we can investigate its spectral properties in these spaces. Cf. [5].

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Author's address:

MIEM, National Research University Higher School of Economics, 20 Myasnitskaya Str., Moscow 101000, Russia.

E-mail: magran@orc.ru

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Givi Berikelashvili and Bidzina Midodashvili

ON THE IMPROVEMENT OF CONVERGENCE RATE OF DIFFERENCE SCHEME FOR ONE MIXED BOUNDARY VALUE PROBLEM

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. A mixed problem with the third kind condition on one part of boundary and with the Dirichlet condition on the rest part of the boundary formulated for the Poisson equation, is considered in a unit square. To obtain an approximate solution, we suggest the two-stage finite-difference correction method. It is proved that the solution of the corrected scheme converges at the rate $O(h^m)$ in the discrete L_2-norm, when the solution of the initial problem belongs to the Sobolev space $W_2^m(\Omega)$ with exponent $m \in (2, 4]$.

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რეზიუმე. ერთეულოვან კვადრატში განხილულია პუასონის განტოლებისათვის დასმული შერეული ამოცანა, მესამე გვარის პირობით საზღვრის ერთ ნაწილზე და დირიხლეს პირობით საზღვრის დარჩენილ ნაწილზე. მიახლოებითი ამოხსნისათვის შემოთავაზებულია სასრულ-სხვაობიანი ორსაფეხურიანი კორექციის მეთოდი. დამტკიცებულია კორექტირებული სქემის ამონახსნის კრებადობა $O(h^m)$ სიჩქარით დისკრეტული L_2 ნორმის მიმართ, თუ გამოსავალი სასაზღვრო ამოცანის ამონახსნი მიეკუთვნება $m \in (2, 4]$ მაჩვენებლიან $W_2^m(\Omega)$ სობოლევის სივრცეს.

1. Introduction

For finite-difference schemes, just as for any numerical method, the question of accuracy is significant. One of the approaches for obtaining high accuracy solutions is the method of corrections by differences of higher order, offered empirically by L. Fox [4]. This idea is simple, but its theoretical foundation is connected with significant difficulties. This is evidenced in the works due to Volkov, in which the grounding of the method is given for the Laplace and Poisson equations (see e.g. [10, 11]); besides, the problem data are chosen in such a way that an exact solution belongs to the Holder class of functions $C_{6,\lambda}$.

When investigating difference schemes by the energetic method, it is desirable to take into account two points:

- the use of Taylor's formula for determination of an approximation error increases the requirement for the smoothness of an unknown solution;
- an unimprovable rate of convergence on the class W_2^m can be reached only by appropriate a priori estimates.

To overcome such difficulties in the last 30 years A. A. Samarskii and other authors (see e.g. [7, 5, 9]) worked out the methodology allowing one to obtain the estimates of convergence rate of difference schemes, in which the convergence rate is consistent with the smoothness of the solution sought for. For the elliptic problems such estimates have the form

$$\|U_h - u\|_{W_2^s(\omega)} \leq ch^{m-s}\|u\|_{W_2^m(\Omega)}.$$

In the present work we consider the Poisson's equation under the third kind boundary condition on one part of boundary and with the Dirichlet condition on the rest part of the boundary. As the first approximation, the solution of the difference scheme $\Lambda U = \varphi$ is considered which has the second order of approximation. Using the basic solution U of the first approximation, the correcting addend R for the right-hand side of the difference scheme is constructed. By means of the methodology for obtaining the consistent estimates, it is proved that the solution $\overline{U}$ of the corrected difference scheme $\Lambda\overline{U} = \varphi + R$ converges at rate $O(h^m)$ in the discrete L_2-norm, when the exact solution belongs to the Sobolev space $W_2^m(\Omega)$, $m \in (2, 4]$.

For determination of the convergence of the offered method we essentially use the convergence estimates obtained in the first and second stages with discrete W_2^2 and L_2-norms, respectively.

2. Statement of the Problem

Let $\Omega = \{x = (x_1, x_2) :\ 0 < x_\alpha < 1\}$ be a unit square with boundary Γ. Let $\Gamma_{-1} = \{(0, x_2) :\ 0 < x_2 < 1\}$, $\Gamma_0 = \Gamma \setminus \Gamma_{-1}$. Let D^ν denote the differential operator $D^\nu = \partial^{|\nu|}/(\partial x_1^{\nu_1}\partial x_2^{\nu_2})$, where $\nu = (\nu_1, \nu_2)$ are multiindices with nonnegative integer components, and $|\nu| = \nu_1 + \nu_2$. By $W_2^s(\Omega)$, $s \geq 0$, we denote the Sobolev space with the norm defined by

$$\|u\|^2_{W_2^s(\Omega)} = \sum_{k=1}^{s} |u|^2_{W_2^k(\Omega)}, \quad |u|^2_{W_2^k(\Omega)} = \sum_{|\nu|=k} \|D^\nu u\|^2_{L_2(\Omega)},$$

when s is an integer. If s is a noninteger, let $s = \overline{s} + \varepsilon$, where $\overline{s}$ is the integer part of s, and $0 < \varepsilon < 1$. In this case, the norm is defined by

$$\|u\|^2_{W_2^s(\Omega)} = \|u\|^2_{W_2^{\overline{s}}(\Omega)} + |u|^2_{W_2^s(\Omega)},$$

where

$$|u|^2_{W_2^s(\Omega)} = \int\limits_{\Omega} \int\limits_{\Omega} \frac{|D^\nu u(x) - D^\nu u(t)|^2}{|x-t|^{2+2\varepsilon}}\, dx\, dt.$$

In particular, for $s = 0$, we have $W_2^0 = L_2$.

In this paper, we investigate certain two-stage finite difference method for the following mixed boundary value problem:

$$\Delta u = -f, \quad x \in \Omega, \tag{2.1}$$

$$u = 0, \ x \in \Gamma_0, \quad \frac{\partial u}{\partial x_1} = \sigma u - g(x_2), \quad x \in \Gamma_{-1}. \tag{2.2}$$

We assume that the solution of the problem (2.1), (2.2) belongs to the space $W_2^m(\Omega),\ m > 2$.

Let $h = 1/n$; $\hbar = h/2$ if $x_1 = 0$, $\hbar = h$ if $x_1 \neq 0$.

We introduce the mesh domains $\omega_\alpha = \{x_\alpha = i_\alpha :\ i_\alpha = 1, \dots, n-1\}$, $\omega = \omega_1 \times \omega_2$, $\omega_\alpha^- = \omega_\alpha \cup \{0\}$, $\omega_\alpha^+ = \omega_\alpha \cup \{1\}$, $\overline{\omega}_\alpha = \omega_\alpha \cup \{0; 1\}$, $\gamma_{-1} = \{(0, x_2) :\ x_2 \in \omega_2\}$, $\gamma_0 = \gamma \setminus \gamma_{-1}$, $\overline{\omega} = \overline{\omega}_1 \times \overline{\omega}_2$, $\gamma = \Gamma \cap \overline{\omega}$.

We define the difference quotients in x_α direction as follows:

$$v_{x_\alpha} = \frac{(I^{(+\alpha)} - I)v}{h}, \quad v_{\overline{x}_\alpha} = \frac{(I - I^{(-\alpha)})v}{h},$$

where $Iv := v$, $I^{(\pm\alpha)} = v(x \pm h r_\alpha)$ and r_α is the unit vector on the x_α axis.

On the set of mesh functions given on the mesh $\overline{\omega}$ and vanishing on γ_0, we define the inner product

$$(y, v) = \sum_{\omega \cup \gamma_{-1}} \hbar h y(x) v(x).$$

The norm $\|y\| = (y, y)^{1/2}$ turns this set into normalized space which we denote by $\mathcal{H}_h$.

Let

$$(y, v)_{\widetilde{\omega}} = \sum_{\widetilde{\omega}} h^2 y(x) v(x), \quad \|y\|_{\widetilde{\omega}} = (y, y)^{1/2}_{\widetilde{\omega}}, \quad \widetilde{\omega} \subseteq \overline{\omega}.$$

Denote

$$\|y\|^2_{W_2^2(\omega)} = \|y_{\overline{x}_1 x_1}\|^2 + \|y_{\overline{x}_2 x_2}\|^2 + 2\|y_{\overline{x}_1 \overline{x}_2}\|^2_{\omega_1^+ \times \omega_2^+}.$$

3. Finite Difference Method

We need the following averaging operators for functions defined on Ω:

$$T_1 v(x) = \frac{1}{h^2} \int\limits_{x_1-h}^{x_1+h} \big(h - |x_1 - \xi_1|\big) v(\xi_1, x_2)\, d\xi_1, \quad x \in \omega,$$

$$T_1 v(x) = \frac{2}{h^2} \int\limits_{x_1}^{x_1+h} (h + x_1 - \xi_1) v(\xi_1, x_2)\, d\xi_1, \quad x \in \gamma_{-1},$$

$$T_2 v(x) = \frac{1}{h^2} \int\limits_{x_2-h}^{x_2+h} \big(h - |x_2 - \xi_2|\big) v(x_1, \xi_2)\, d\xi_2, \quad x \in \omega \cup \gamma_{-1}.$$

In the Hilbert space $\mathcal{H}_h$ we define the difference operators:

$$\partial_{x_1} y = y_{x_1}, \quad \Lambda_1 y = \begin{cases} y_{\overline{x}_1 x_1}, & x \in \omega \\ \dfrac{2}{h}\,(y_{x_1} - \sigma y), & x \in \gamma_{-1}, \end{cases}$$

$$\Lambda_2 y = \Big(1 + \sigma\, \frac{h}{3}\Big) y_{\overline{x}_2 x_2}, \quad \mathring{\Lambda}_2 y = y_{\overline{x}_2 x_2}.$$

We approximate problem (2.1), (2.2) by the following finite-difference scheme

$$\Lambda U := \Lambda_1 U + \Lambda_2 U = -\varphi, \quad x \in \omega \cup \gamma_{-1}, \tag{3.1}$$

where

$$\varphi := T_1 T_2 f + \delta(x_1) T_2 g - \frac{h^2}{4}\, \delta(x_1) g_{\overline{x}_2 x_2},$$

$$\delta(x_1) = \begin{cases} \dfrac{2}{h}, & x_1 = 0, \\ 0, & x_1 \neq 0. \end{cases}$$

Using obtained solution U on the second stage of the method we correct the right-hand side of the scheme and then we solve on the same mesh the following difference scheme

$$\Lambda \overline{U} = -\overline{\varphi}, \quad x \in \omega \cup \gamma_{-1}, \tag{3.2}$$

where

$$\overline{\varphi} = \varphi + \frac{h^2}{6}\, \big(\Lambda_1 \mathring{\Lambda}_2 U + \delta(x_1) g_{\overline{x}_2 x_2}\big).$$

The following theorem represents the main result of this paper.

Theorem 3.1. *Let the solution of problem* (2.2) *belong to the space* $W_2^m(\Omega)$, $m > 2$. *Then the convergence rate of the corrected difference scheme* (3.2) *in the discrete* L_2*-norm is defined by the estimate*

$$\|\overline{U} - u\|_{L_2(\omega)} \le c h^m \|u\|_{W_2^m(\Omega)}, \quad 2 < m \le 4, \tag{3.3}$$

where the positive constant c *does not depend on* u *and* h.

4. Auxiliary Results

Let $Z = U - u$, where U is a solution of the difference scheme (3.1), while u is a solution of the differential problem (2.1), (2.2).

Lemma 4.1. *The error of the difference scheme* (3.1) $Z = U - u$ *represents a solution of the following problem*

$$\Lambda Z = \eta_1 + \eta_2, \ \ Z \in \mathcal{H}_h, \tag{4.1}$$

where

$$\eta_1 = \begin{cases} \Lambda_1(T_2 u - u), & x \in \omega, \\ \Lambda_1\Big(T_2 u - u - \dfrac{h^2}{12}\, u_{\overline{x}_2 x_2}\Big), & x \in \gamma_{-1}, \end{cases}$$

$$\eta_2 = \begin{cases} (T_1 u - u)_{\overline{x}_2 x_2}, & x \in \omega, \\ \Big(T_1 u - u - \dfrac{h}{2}\, \dfrac{\partial u}{\partial x_1} + \dfrac{h}{6}\, u_{x_1}\Big)_{\overline{x}_2 x_2}, & x \in \gamma_{-1}. \end{cases}$$

Proof. From equation (2.1) we have:

$$(T_2 u)_{\overline{x}_1 x_1} + (T_1 u)_{\overline{x}_2 x_2} = -T_1 T_2 f, \ \ x \in \omega, \tag{4.2}$$

or, the same,

$$u_{\overline{x}_1 x_1} + u_{\overline{x}_2 x_2} + \eta_1 + \eta_2 = -T_1 T_2 f, \ \ x \in \omega. \tag{4.3}$$

Acting on the equation (2.1) by operator $T_1 T_2$ we obtain

$$\frac{2}{h}\, T_2\Big(u_{x_1} - \frac{\partial u}{\partial x_1}\Big) + (T_1 u)_{\overline{x}_2 x_2} = -T_1 T_2 f, \ \ x \in \gamma_{-1}. \tag{4.4}$$

Rewriting the addend of the left-hand side of this equality we get

$$\begin{aligned} \frac{2}{h}\, T_2\Big(u_{x_1} - \frac{\partial u}{\partial x_1}\Big) &= \frac{2}{h}\, T_2\Big(u_{x_1} - \sigma u\Big) + \frac{2}{h}\, T_2 g = \Lambda_1 T_2 u + \frac{2}{h}\, T_2 g \\ &= \Lambda_1 u + \eta_1 + \frac{h}{6}\, (u_{x_1 \overline{x}_2 x_2} - \sigma u_{\overline{x}_2 x_2}) + \frac{2}{h}\, T_2 g, \end{aligned} \tag{4.5}$$

$$\begin{aligned} (T_1 u)_{\overline{x}_2 x_2} &= \Big(1 + \sigma\, \frac{h}{3}\Big) u_{\overline{x}_2 x_2} - \frac{\sigma h}{3}\, u_{\overline{x}_2 x_2} \\ &\quad + \Big(T_1 u - u - \frac{h}{2}\, \frac{\partial u}{\partial x_1} + \frac{h}{6}\, u_{x_1}\Big)_{\overline{x}_2 x_2} \\ &\quad + \Big(\frac{h}{2}\, \frac{\partial u}{\partial x_1} - \frac{h}{6}\, u_{x_1}\Big)_{\overline{x}_2 x_2} \\ &= \Lambda_2 u + \eta_2 - \frac{\sigma h}{3}\, u_{\overline{x}_2 x_2} + \Big(\frac{h}{2}\, \frac{\partial u}{\partial x_1} - \frac{h}{6}\, u_{x_1}\Big)_{\overline{x}_2 x_2}. \end{aligned} \tag{4.6}$$

Summing up equalities (4.5), (4.6) we find

$$\begin{aligned} &\frac{2}{h}\, T_2\Big(u_{x_1} - \frac{\partial u}{\partial x_1}\Big) + (T_1 u)_{\overline{x}_2 x_2} \\ &\qquad = \Lambda_1 u + \Lambda_2 u + \eta_1 + \eta_2 + \frac{2}{h}\, T_2 g + \frac{h}{2}\, \Big(\frac{\partial u}{\partial x_1} - \sigma u\Big)_{\overline{x}_2 x_2} \end{aligned}$$

and according to (4.4) we have

$$\Lambda_1 u + \Lambda_2 u + \eta_1 + \eta_2 + \frac{2}{h}\, T_2 g - \frac{h}{2}\, g_{\overline{x}_2 x_2} = -T_1 T_2 f, \quad x \in \gamma_{-1}. \tag{4.7}$$

The equalities (4.3), (4.7) can be rewritten as follows

$$\Lambda_1 u + \Lambda_2 u + \eta_1 + \eta_2 = -\varphi, \quad x \in \omega \cup \gamma_{-1}. \tag{4.8}$$

Subtraction of (4.8) from (3.1) proves (4.1). □

Let $\overline{Z} = \overline{U} - u$, where U is a solution of the problem (3.2), and u is a solution of the differential problem (2.1), (2.2).

Lemma 4.2. *The error of the solution of difference scheme* (3.2) $\overline{Z} = \overline{U} - u$ *represents a solution of the following problem*

$$\Lambda \overline{Z} = \Lambda_1 \zeta_1 + \Lambda_2 \zeta_2 + \frac{h^2}{6}\, \Lambda_1 \mathring{\Lambda}_2 (u - U), \tag{4.9}$$

where

$$\zeta_1 = T_2 u - u - \frac{h^2}{12}\, u_{\overline{x}_2 x_2} + \frac{h^5}{720}\, \delta(x_1) \Lambda_2 \Big(\frac{\partial u}{\partial x_1}\Big)_{x_1}, \quad x \in \omega \cup \gamma_{-1},$$

$$\zeta_2 = \begin{cases} T_1 u - u - \dfrac{h^2}{12}\, u_{\overline{x}_1 x_1}, & x \in \omega, \\ T_1 u - u - \dfrac{h}{6}\, \dfrac{\partial u}{\partial x_1} - \dfrac{h}{6}\, u_{x_1} - \dfrac{h^3}{180}\, \Big(\dfrac{\partial u}{\partial x_1}\Big)_{x_1 x_1}, & x \in \gamma_{-1}. \end{cases}$$

Proof. (4.2) can be easily rewritten as follows

$$u_{\overline{x}_1 x_1} + u_{\overline{x}_2 x_2} + \frac{h^2}{6}\, u_{\overline{x}_1 x_1 \overline{x}_2 x_2} + \Lambda_1 \zeta_1 + \Lambda_2 \zeta_2 = -T_1 T_2 f, \quad x \in \omega. \tag{4.10}$$

Summing up (4.7) and identity

$$\mathring{\Lambda}_2 \Big(\frac{2h}{6}\, \frac{\partial u}{\partial x_1} - \frac{2h}{6}\, u_{x_1}\Big) + \frac{h^2}{6}\, \Lambda_1 \mathring{\Lambda}_2 u = -\frac{2h}{6}\, g_{\overline{x}_2 x_2}$$

we obtain

$$\begin{aligned} \Lambda_1 u + \Lambda_1 \zeta_1 + \Lambda_2 u + \mathring{\Lambda}_2 \zeta_2 + \frac{h^2}{6}\, \Lambda_1 \mathring{\Lambda}_2 u \\ = -T_1 T_2 f - \frac{2}{h}\, T_2 g + \frac{h}{6}\, g_{\overline{x}_2 x_2}, \quad x \in \gamma_{-1}. \end{aligned} \tag{4.11}$$

Then (4.10), (4.11) can be rewritten as follows

$$\begin{aligned} \Lambda_1 u + \Lambda_2 u + \frac{h^2}{6}\, \Lambda_1 u_{\overline{x}_2 x_2} + \Lambda_1 \zeta_1 + \Lambda_2 \zeta_2 \\ = -T_1 T_2 f - \delta(x_1) T_2 g + \frac{h^2}{12}\, \delta(x_1) g_{\overline{x}_2 x_2}, \quad x \in \omega \cup \gamma_{-1}. \end{aligned} \tag{4.12}$$

Subtracting (4.12) from (3.2) we conclude that the lemma is valid.

Lemma 4.3. *For solutions of the problems* (4.1) *and* (4.9) *the following a priori estimates*

$$\|Z\|_{W_2^2(\omega)} \leq c\big(\|\eta_1\| + \|\eta_2\|\big), \tag{4.13}$$

$$\|\overline{Z}\| \leq c\big(\|\zeta_1\| + \|\zeta_2\| + \|Z_{\overline{x}_2 x_2}\|\big) \tag{4.14}$$

are valid.

The proof follows from the facts that Λ_1, Λ_2 and, therefore, Λ are self-adjoint and negative definite (see e.g. [8, Ch. IV, § 2]):

$$\|\Lambda Z\| \geq c\|Z\|_{W_2^2(\omega)},$$
$$\|\Lambda^{-1}\Lambda_1\| \leq 1, \quad \|\Lambda^{-1}\Lambda_2\| \leq 1.$$

To determine the rate of convergence of the two-stage finite difference method with the help of Lemma 4.3, it is sufficient to estimate the terms on the right-hand sides of (4.13), (4.14).

Lemma 4.4. *Assume that the linear functional $l(u)$ is bounded in $W_2^s(E)$, where $s = \overline{s} + \varepsilon$, $\overline{s}$ is an integer, $0 < \varepsilon \leq 1$, and $l(P) = 0$ for every polynomial P of degree $\leq \overline{s}$ in two variables. Then, there exists a constant c, independent of u, such that $|l(u)| \leq c|u|_{W_2^s(E)}$.*

This lemma is a particular case of the Dupont–Scott approximation theorem [3] and represents a generalization of the Bramble–Hilbert lemma [2] (see also [8]).

Proof of Theorem 3.1. Functionals η_α, ζ_α, $\alpha = 1, 2$, are bounded when $u \in W_2^m(\Omega)$, $m > 2$, and they vanish on polynomials up to the third order. Using the well-known methodology (see e.g. [8, 1]), which is based on the Lemma 4.4, we have for them the following estimates

$$|\eta_\alpha| \leq ch^{m-3}|u|_{W_2^m(e)}, \ \ 2 < m \leq 4,$$
$$|\zeta_\alpha| \leq ch^{m-1}|u|_{W_2^m(e)}, \ 2 < m \leq 4,$$

where symbol e denotes those elementary cells on which functionals η_α, ζ_α, are defined:

$$e = e(x) = \begin{cases} \big\{(\xi_1, \xi_2): \ |x_\alpha - \xi_\alpha| < h, \ \alpha = 1, 2\big\}, & \text{if } \ x \in \omega, \\ \big\{(\xi_1, \xi_2): \ 0 < \xi_1 < 2h, \ |x_2 - \xi_2| < h\big\}, & \text{if } \ x \in \gamma_{-1}. \end{cases}$$

As a result we have

$$\begin{aligned} \|\eta_\alpha\|^2 &= \sum_{\omega \cup \gamma_{-1}} \hbar h |\eta_\alpha|^2 \\ &\leq c \sum_{\omega \cup \gamma_{-1}} h^{2m-4}|u|^2_{W_2^m(e)} \leq ch^{2m-4}|u|^2_{W_2^m(\Omega)}, \ \ 2 < m \leq 4, \end{aligned}$$

$$\|\zeta_\alpha\|^2 = \sum_{\omega\cup\gamma_{-1}} \hbar h|\zeta_\alpha|^2$$
$$\leq c \sum_{\omega\cup\gamma_{-1}} h^{2m}|u|^2_{W_2^m(e)} \leq ch^{2m}|u|^2_{W_2^m(\Omega)}, \quad 2 < m \leq 4.$$

These estimates with the Lemma 4.3 accomplish the proof of the Theorem 3.1. □

5. Numerical Experiments

Now, we present some numerical results to demonstrate the convergence order of the proposed method. The experimental order of convergence in the discrete L_2 and maximum norms are computed by formulas

$$Ord(Y) = \log_2 \frac{\|Y_h - u\|}{\|Y_{h/2} - u\|}, \quad Ord(Y) = \log_2 \frac{\|Y_h - u\|_\infty}{\|Y_{h/2} - u\|_\infty},$$

where u is the exact solution of original problem, while Y_h denotes the solution of the difference scheme on the grid with step h.

Below, in the examples the symbols U , $\overline{U}$ denote solutions of the difference schemes (3.1), (3.2), respectively.

Let $\Omega = \{x = (x_1, x_2) : \ |x_1| < 1, \ 0 < x_2 < 1\}$ and Γ be its boundary; $\Gamma_{-1} = \{(-1, x_2) : \ 0 < x_2 < 1\}$, $\Gamma_0 = \Gamma \setminus \Gamma_{-1}$.

Consider the problem

$$\Delta u = -f, \quad x \in \Omega,$$
$$u = 0, \quad x \in \Gamma_0, \qquad \frac{\partial u}{\partial x_1} = 3u - g(x_2), \quad x \in \Gamma_{-1},$$

where

$$f(x) = \begin{cases} \big(\pi^2(x_1^3 - x_1 + 1) - 6x_1\big)\sin(\pi x_2), & x \in (-1,0)\times(0,1), \\ \pi^2(1 - x_1)\sin(\pi x_2), & x \in [0,1)\times(0,1), \end{cases}$$

$g(x_2) = \sin(\pi x_2)$.

The exact solution is

$$u(x) = \begin{cases} (x_1^3 - x_1 + 1)\sin(\pi x_2), & x \in [-1,0)\times[0,1], \\ (1 - x_1)\sin(\pi x_2), & x \in [0,1]\times[0,1]. \end{cases} \tag{5.1}$$

The right-hand side is calculated by the computer algebra system (CAS) MuPAD.

For $x_1 = 0$:

$$\varphi = T_1T_2f = \Big(\pi^2 - \frac{\pi^2 h^3}{20} + h\Big)\lambda^2 \sin(\pi x_2).$$

For $x_1 = h, 2h, 3h, \ldots$:

$$\varphi = T_1T_2f = \pi^2(1 - x_1)\lambda^2 \sin(\pi x_2).$$

For $x_1 = -h, -2h, -3h, \dots, -(n-1)h$:

$$T_1T_2f = [\pi^2(x_1^3 + 1 - x_1) - 6x_1 + \frac{\pi^2h^2}{2}\, x_1]\lambda^2 \sin(\pi x_2).$$

For $x = -1$:

$$T_1T_2f = \Big(\pi^2 h\Big(\frac{h^2}{10} - \frac{h}{2} + \frac{2}{3}\Big) - 2h + \pi^2 + 6\Big)\lambda^2 \sin(\pi x_2),$$
$$T_2 g = \lambda^2 \sin(\pi x_2), \quad g_{\overline{x}_2 x_2} = -\pi^2\lambda^2 \sin(\pi x_2).$$

The results of calculations are given by Tables 1, 2.

Table 1. Experimental order of convergence with respect to the norm of L_2.

h	$\|U_h - u\|$	$\|\widetilde{U}_h - u\|$	$Ord(U)$	$Ord(\widetilde{U})$
$\frac{1}{4}$	1.6881 e−02	9.2278 e−04		
			2.0151	4.0074
$\frac{1}{8}$	4.1762 e−03	5.7377 e−05		
			2.0140	4.0245
$\frac{1}{16}$	1.0340 e−03	3.5256 e−06		
			2.0087	4.0178
$\frac{1}{32}$	2.5695 e−04	2.1765 e−07		
			2.0048	4.0103
$\frac{1}{64}$	6.4024 e−05	1.3507 e−08		
			2.0025	4.0055
$\frac{1}{128}$	1.5978 e−05	8.4099 e−10		

Remark. The function defined by formula (5.1) belongs to the class $W_2^{3.5}(\Omega)$. The order of convergence obtained experimentally, and equaled 4, may point at the fact that condition $u \in W_2^m(\Omega)$ in the Theorem 3.1 is sufficient, not necessary.

6. Conclusion

We consider a mixed boundary-value problem for the 2D Poisson's equation in a square which is solved by the finite-difference scheme with approximation of order $O(h^2)$ based on a 5-point stencil. Using the obtained solution, we correct the right-hand side of the scheme and repeatedly solve the scheme on the same mesh with the same stencil. Using the methodology of obtaining the consistent estimates, worked by Samarskiĭ et al., it is

Table 2. Experimental order of convergence with respect to the maximum norm.

h	$\lVert U_h - u \rVert_\infty$	$\lVert \widetilde{U}_h - u \rVert_\infty$	$Ord(U)$	$Ord(\widetilde{U})$
$\frac{1}{4}$	$2.8838\,e-02$	$1.6708\,e-03$		
			1.9843	3.8432
$\frac{1}{8}$	$7.2884\,e-03$	$1.1641\,e-04$		
			1.9960	3.9825
$\frac{1}{16}$	$1.8271\,e-03$	$7.3647\,e-06$		
			1.9990	3.9902
$\frac{1}{32}$	$4.5710\,e-04$	$4.6344\,e-07$		
			1.9997	3.9989
$\frac{1}{64}$	$1.1430\,e-04$	$2.8988\,e-08$		
			1.9997	3.9997
$\frac{1}{128}$	$2.8579\,e-05$	$1.8121\,e-09$		

proved that the solution of the corrected difference scheme converges at rate $O(h^m)$ in the discrete $L_2(\omega)$-norm, when the exact solution belongs to the Sobolev space $W_2^m(\Omega)$, $m \in (2,4]$. For determination of the convergence of the offered method we essentially use the convergence estimates obtained in the first and second stages with discrete W_2^2 and L_2 - norms, respectively.

The method can be generalized for an elliptic differential equation with mixed derivatives and a system of equations, and also for the case of other type boundary conditions.

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Authors' addresses:

Givi Berikelashvili

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

2. Department of Mathematics, Georgian Technical University, 77 M. Kostava St., Tbilisi 0175, Georgia.

E-mail: bergi@rmi.ge

Bidzina Midodashvili

1. Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University, 13 University Str., 0186 Tbilisi, Georgia.

2. Faculty of Education, Exact and Natural Sciences, Gori Teaching University, 53 Chavchavadze Str., Gori, Georgia.

E-mail: bidmid@hotmail.com

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Evgenii Burlakov, Evgeny Zhukovskiy,
Arcady Ponosov, and John Wyller

EXISTENCE, UNIQUENESS AND CONTINUOUS DEPENDENCE ON PARAMETERS OF SOLUTIONS TO NEURAL FIELD EQUATIONS

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. We obtain conditions for the existence and uniqueness of solutions to generalized neural field equations involving parameterized measure. We study continuous dependence of these solutions on the spatiotemporal integration kernel, delay effects, firing rate, external input and measure. We also construct the connection between the delayed Amari and Hopfield network models.

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რეზიუმე. მიღებულია განზოგადებული ნეირონული ველის განტოლებების ამონახსნების არსებობის და ერთადერთობის პირობები, რომლებიც შეიცავს პარამეტრიზებულ ზომას. შესწავლილია ამ ამონახსნების უწყვეტად დამოკიდებულება სივრცე-დროის ინტეგრირების გულზე, დაგვიანების ეფექტებზე, სიგნალების გენერირების სიხშირეზე, გარედან შეტანილ მონაცემებზე და ზომაზე. ჩვენ ასევე ვამყარებთ კავშირს და დაგვიანებულ ამარისა და ჰოპფულის ქსელის მოდელებს შორის.

Introduction

The main object of our study is the following parameterized integral equation involving integration with respect to an arbitrary measure:

$$u(t,x,\lambda)$$
$$= \int_{-\infty}^{t} ds \int_{\Omega} W(t,s,x,y,\lambda) f\big(u(s-\tau(s,x,y,\lambda),y,\lambda),\lambda\big)\nu(dy,\lambda)$$
$$+I(t,x,\lambda), \quad t>a, \quad x\in\Omega, \quad \lambda\in\Lambda \tag{1}$$

with the initial (prehistory) condition

$$u(\xi,x,\lambda)=\varphi(\xi,x,\lambda), \quad \xi\le a, \quad x\in\Omega, \quad \lambda\in\Lambda. \tag{2}$$

Here, the function u represents the activity of a neural element at time t and position x. The generalized spatio-temporal connectivity kernel W determines the time-dependent coupling between elements at positions x and y. The non-negative activation function f gives the firing rate of a neuron with activity u. The non-negative function τ represents the time-dependent axonal delay effects in the neural field, which require a prehistory condition given by the function φ. The function $I(t,x)$ represents a variable external input. All the above functions involve a parametrization by the parameter λ which, as well as introducing of an arbitrary parameterized measure $\nu(\,\cdot\,,\lambda)$, gives us some investigation advantages.

The equation (1) covers a wide variety of neural field models:

The most well-known Amari model [1]

$$\partial_t u(t,x) = -u(t,x) + \int_R \omega(x-y) f(u(t,y))dy + I(t,x), \quad t\ge 0, \quad x\in R,$$

can be obtained from the equation (1) by taking

$$W(t,s,x,y,\lambda) = \exp\big(-(t-s)\big)\omega(x-y),$$
$$\tau(t,x,y,\lambda) = \varphi(\xi,x,\lambda) \equiv 0.$$

The two-population Amari model (see [2], [16])

$$\begin{pmatrix}\partial_t u_e \\ \alpha\partial_t u_i\end{pmatrix}(t,x) = -\begin{pmatrix}u_e\\ u_i\end{pmatrix}(t,x)$$
$$+\int_R \begin{pmatrix}\omega_{ee} & -\omega_{ei}\\ \omega_{ie} & -\omega_{ii}\end{pmatrix}(x-y)\begin{pmatrix}f_e(u_e(t,x))\\ f_i(u_i(t,x))\end{pmatrix} dy$$
$$+\begin{pmatrix}I_e\\ I_i\end{pmatrix}(t,x), \quad t\ge 0, \quad x\in R,$$

can be obtained from the equation (1) by taking

$$W(t,s,x,y,\lambda)$$
$$= \operatorname{diag}\Big(\exp(-(t-s)), \exp\big(-(t-s)/\alpha\big)/\alpha\Big)\begin{pmatrix}\omega_{ee} & -\omega_{ei}\\ \omega_{ie} & -\omega_{ii}\end{pmatrix}(x-y),$$
$$\tau(t,x,y,\lambda)=\varphi(\xi,x,\lambda)\equiv 0.$$

The delayed Amari model (see e.g. [5])

$$\partial_t u(t,x) = -Lu(t,x) + \int\limits_{\Omega} \omega(t,x,y) f\big(u(t-\tau(x,y),y)\big)\,dy + I(t,x),$$
$$t\in\Big[-\max_{x,y\in\overline{\Omega}}\tau(x,y),\infty\Big),\quad x\in\Omega\subset B_{R^m}(0,r),\quad L=\operatorname{diag}(l_1,\dots,l_n),\quad l_i>0$$

with a time-dependent connectivity kernel is also a special case of the model (1) with

$$W(t,s,x,y,\lambda)=\operatorname{diag}\Big(l_1\exp\big(-l_1(t-s)\big),\dots,l_n\exp\big(-l_n(t-s)\big)\Big)\omega(t,x,y),$$
$$\tau(t,x,y,\lambda)=\tau(x,y),\quad \varphi(\xi,x,\lambda)\equiv 0.$$

Another special case of the equation (1) arises in models that take into account the microstructure of the neural field (see [4, 9, 13])

$$\begin{gathered}\partial_t u(t,x) = -u(t,x) + \int\limits_{R^m} \omega^{\varepsilon}(x-y) f(u(t,y))\,dy,\\ \omega^{\varepsilon}(x)=\omega(x,x/\varepsilon),\quad 0<\varepsilon\ll 1,\\ t\geq 0,\quad x\in R^m.\end{gathered}\tag{3}$$

If the microstructure is periodic, then, as the heterogeneity parameter $\varepsilon\to 0$, the above model converges (see e.g. [12]) to the homogenized Amari model

$$\begin{gathered}\partial_t u(t,x_c,x_f)\\ = -u(t,x_c,x_f) + \int\limits_{R^m}\int\limits_{\mathcal{Y}} \omega(x_c-y_c,x_f-y_f) f\big(u(t,y_c,y_f)\big)\,dy_c\,dy_f,\\ t>0,\quad x_c\in R^m,\quad x_f\in\mathcal{Y}\subset R^k,\end{gathered}\tag{4}$$

where x_c and x_f are the coarse-scale and fine-scale spatial variables, respectively. Taking

$$\Omega = R^m\times\mathcal{Y}\ \ (\mathcal{Y}\ \text{is some } k\text{-dimensional torus [15]}),$$
$$x=(x_c,x_f),\quad y=(y_c,y_f),$$
$$W(t,s,x,y,\lambda)=\exp\big(-(t-s)\big)\omega(x_c-y_c,x_f-y_f)$$

in (1) with

$$\tau(t,x,y,\lambda)=\varphi(\xi,x,\lambda)\equiv 0,$$

we get the model (4). It should be pointed out here that the case of non-periodic microstructure in the model (3) that leads (see [12]) to non-Lebesgue measure in (4) is also covered by (1). It is more realistic to assume some small deviations from the periodicity in the neural networks structure reflected in the properties of the connectivity kernel with respect to the second argument. Hence, it is natural to ask whether the solution of the model (3) with a non-periodic perturbation of the periodic connectivity kernel in some sense is “close” to the solution in the non-perturbed case. One possible answer to this question is suggested in Appendix. The answer is based on the main result of the paper which is the existence, uniqueness and continuous dependence of solutions to (1) on the model parameters.

Another application of the main result is the possibility to connect the models in use in the neural field theory to the well-known Hopfield network model [8] utilizing the parameterized measure involved in (1). As the network models of the Hopfield type are used for numerical simulations of the neural fields, our results thus justify implementation of such numerical schemes.

The paper is organized in the following way. In Section 1 a special case (that is relevant in the neural field theory) of the general statement on the solvability and continuous dependence on a parameter of solutions to the Volterra operator equation from the paper [3] is given. Based on this theorem, analogous results are obtained in Section 2 for the generalized neural field model (1). Section 3 is devoted to the connection between the delayed Amari and Hopfield network models. In addition, a mathematical justification of the two known numerical schemes is offered, which illustrates a generality of the methods suggested in the paper. Finally, Appendix contains a short informal description of the homogenization procedure for the neural field equations with non-periodic microstructure based on the convergence of Banach algebras with mean value.

1. Preliminaries

In this section we provide an overview of the notation used in the paper, introduce the main definitions and formulate a fixed point theorem for locally contracting Volterra operators.

Let us introduce the following notations:

- R^m is the m-dimensional real vector space with the norm $|\cdot|$;
- Λ is some metric space;
- $B_\Lambda(\lambda_0, r)$ is the ball in the space Λ of the radius $r > 0$ centered at the point $\lambda_0 \in \Lambda$;
- Ω is a closed subset of R^m;
- $\partial\Omega$ is the boundary of the Ω;
- $\Omega_r = \Omega \cap B_{R^m}(0, r)$;

– $BC(\Omega, R^n)$ is the space of bounded continuous functions $\vartheta : \Omega \to R^n$ with the norm $\|\vartheta\|_{BC(\Omega,R^n)} = \sup\limits_{x\in\Omega} |\vartheta(x)|$;

– $C_{comp}(\Omega, R^n)$ is the locally convex space of continuous functions $\vartheta : \Omega \to R^n$, with a compact support, equipped with the topology of uniform convergence on compact subsets;

– $Y(\mathbb{I}) = C(\mathbb{I}, BC(\Omega, R^n))$ consists of all continuous functions $\upsilon : \mathbb{I} \to BC(\Omega, R^n)$, with the norm $\|\upsilon\|_{Y(\mathbb{I})} = \max\limits_{t\in\mathbb{I}} \|\upsilon(t)\|_{BC(\Omega,R^n)}$ if $\mathbb{I}$ is compact; if $\mathbb{I}$ is not compact, then $Y(\mathbb{I})$ is a locally convex linear space equipped with the topology of uniform convergence on compact subsets of $\mathbb{I}$;

Let $[a, b]$ be a compact subinterval of the real line. In the three forthcoming definitions we use the following notation: $Y = Y([a, b])$, $Y_\xi = Y([a, a + \xi])$ for any $\xi \in (0, b{-}a)$.

Definition 1. An operator $\Psi : Y \to Y$ is said to be a Volterra operator if for any $\xi \in (0, b{-}a)$ and any $y_1, y_2 \in Y$ the equality $y_1(t) = y_2(t)$ on $[a, a{+}\xi]$ implies that $(\Psi y_1)(t) = (\Psi y_2)(t)$ on $[a, a{+}\xi]$.

Choosing an arbitrary $\xi \in (0, b{-}a)$, we introduce the following three important operators. Let $E_\xi : Y \to Y_\xi$ be defined as $(E_\xi y)(t) = y_\xi(t)$, $t \in [a, a{+}\xi]$, where $y_\xi(t)$ is a restriction of the function $y(t)$ to the subinterval $[a, a + \xi]$; conversely, to each $y_\xi \in Y_\xi$ the operator $P_\xi : Y_\xi \to Y$ assigns one of the extensions $y \in Y$ of the element y_ξ (P_ξ may not be uniquely defined); the operator $\Psi_\xi : Y_\xi \to Y_\xi$ is given by $\Psi_\xi y_\xi = E_\xi \Psi P_\xi y_\xi$. Note that for any Volterra operator $\Psi : Y \to Y$ the operator $\Psi_\xi : Y_\xi \to Y_\xi$ is also a Volterra operator and is independent of the choice of P_ξ.

Definition 2. A Volterra operator $\Psi : Y \to Y$ is called locally contracting if there exist $q < 1$, $\theta > 0$, such that for all elements $y_1, y_2 \in Y$ the following two conditions are satisfied:

$\mathfrak{q}_1$) $\|E_\theta \Psi y_1 - E_\theta \Psi y_2\|_{Y_\theta} \le q \|E_\theta y_1 - E_\theta y_2\|_{Y_\theta}$,

$\mathfrak{q}_2$) for any $\gamma \in [0, b{-}a{-}\theta]$, the equality $E_\gamma y_1 = E_\gamma y_2$ implies that

$$\left\|E_{\gamma+\theta} \Psi y_1 - E_{\gamma+\theta} \Psi y_2\right\|_{Y_{\gamma+\theta}} \le q \left\|E_{\gamma+\theta} y_1 - E_{\gamma+\theta} y_1\right\|_{Y_{\gamma+\theta}}. \tag{5}$$

Definition 3. If there exists $\gamma \in (0, b{-}a]$ and a function $y_\gamma \in Y_\gamma$, which satisfies the equation $\Psi_\gamma y_\gamma = y_\gamma$, then we call y_γ a local solution of the Volterra equation

$$y(t) = (\Psi y)(t), \quad t \in [a, b]. \tag{6}$$

In the case if $\gamma = b - a$, we call this solution global (relative to the interval $[a, b]$).

To study continuous dependence on a parameter, we need some more definitions.

Definition 4. Let $F(\,\cdot\,,\,\cdot\,) : Y \times \Lambda \to Y$ be a family of Volterra operators depending on a parameter $\lambda \in \Lambda$. This family is called uniformly locally contracting if for each $\lambda \in \Lambda$ the operator $F(\,\cdot\,,\lambda)$ is locally contracting and the constants $q \geq 0$ and $\theta > 0$ from Definition 3, are independent of $\lambda \in \Lambda$.

The following theorem concerning the well-posedness of the operator equation

$$y(t) = (F(y,\lambda))(t), \quad t \in [a,b], \quad \lambda \in \Lambda, \tag{7}$$

is a special case of Theorem 1 in Burlakov, et al [3]. It represents the main theoretical tool for the problems to be studied in this paper.

Theorem 1. *Assume that for some $\lambda_0 \in \Lambda$ and $r_0 > 0$, the family of Volterra operators $F(\,\cdot\,,\lambda) : Y \to Y$ ($\lambda \in B_\Lambda(\lambda_0, r_0)$) is uniformly locally contracting and the mapping $F(\,\cdot\,,\,\cdot\,) : Y \times \Lambda \to Y$ is continuous at (y, λ_0) for all $y \in Y$.*

Then there exists $r > 0$, such that the equation (7) *has a unique global solution $y(t,\lambda)$ for all $\lambda \in B_\Lambda(\lambda_0, r)$, and*

$$\|y(\,\cdot\,,\lambda) - y(\,\cdot\,,\lambda_0)\|_Y \to 0 \quad as \quad \lambda \to \lambda_0.$$

Moreover, for each $\lambda \in B_\Lambda(\lambda_0, r)$, any local solution of the equation (7) *is also unique and is a restriction of the corresponding global solution.*

2. The Main Result

In this section we justify the property of well-posedness for the generalized neural field equation (1).

The following assumptions will be imposed on the functions involved:

- (**A1**) The function $f : R^n \times \Lambda \to R^n$ is continuous, bounded and Lipschitz one in the first variable uniformly with respect to $\lambda \in \Lambda$.
- (**A2**) For any $b \in R$ and $r > 0$, the delay function $\tau : (-\infty, b] \times \Omega \times \Omega_r \times \Lambda_c \to [0, \infty)$ is uniformly continuous, where Λ_c is some compact subset of Λ.
- (**A3**) The initial (prehistory) function $\varphi : (-\infty, a] \times \Omega \times \Lambda_c \to R^n$ is uniformly continuous.
- (**A4**) The external input function $I : [a, \infty) \times \Omega \times \Lambda \to R^n$ generates a continuous mapping $\lambda \mapsto I(\,\cdot\,,\,\cdot\,,\lambda)$ from Λ to the space $Y[a, \infty)$.
- (**A5**) For any $b > a$ and $r > 0$, the kernel function $W : [a,b] \times [-r,r] \times \Omega \times \Omega_r \times \Lambda_c \to R^n$ is uniformly continuous.
- (**A6**) The complete σ-additive measures $\nu(\,\cdot\,,\lambda)$ ($\lambda \in \Lambda$) are finite on compact subsets of Ω and weakly continuous with respect to $\lambda \in \Lambda$ i.e. the measures can be interpreted as linear functionals on the separable locally convex space $C_{comp}(\Omega, R^n)$.

(**A7**) For any $b > a$,

$$\max_{t\in[a,b]}\Bigg(\int\limits_{-\infty}^{t} ds \sup_{x\in\Omega,\lambda\in\Lambda}\int\limits_{\Omega}\big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda)\Bigg) < \infty.$$

(**A8**) For any $b > a$,

$$\lim_{r\to\infty}\sup_{t\in[a,b],\ x\in\Omega,\ \lambda\in\Lambda}\int\limits_{-\infty}^{t} ds \int\limits_{\Omega-\Omega_r}\big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda) = 0.$$

Definition 5. Let $\lambda \in \Lambda$. We define a local solution to the problem (1), (2) on $[a, a+\gamma]\times R^n$, $\gamma \in (0,\infty)$, to be a function $u_\gamma \in Y([a, a+\gamma])$ that satisfies the equation (1) on $[a, a+\gamma]$ and the prehistory condition (2). We define a global solution to the problem (1), (2) to be a function $u \in Y([a,\infty))$, whose restriction u_γ to $[a, a+\gamma]$ is its local solution for any $\gamma \in (0,\infty)$.

Theorem 2. *Suppose that the assumptions* (**A1**)–(**A8**) *are fulfilled. Then the initial value problem* (1), (2) *has a unique continuous solution* $u(\,\cdot\,,\,\cdot\,,\lambda)\in Y([a,\infty))$ *for any* $\lambda \in \Lambda$*, and the correspondence* $\lambda \mapsto u(\,\cdot\,,\,\cdot\,,\lambda)$ *is a continuous mapping from* Λ *to* $Y([a,\infty))$*. Moreover, for each* $\lambda \in \Lambda$*, any local solution of the problem* (1), (2) *is also unique and it is a restriction of the corresponding global solution.*

Proof. Due to the definition of the topology in $Y([a,\infty))$, it suffices to prove this result for the case of an arbitrary compact interval $[a,b] \subset [a,\infty)$. In what follows we therefore keep fixed an arbitrary $b > a$ and keep the notation Y for the space $Y([a,b])$.

For each $\lambda \in \Lambda$ and $\varphi(\xi,x,\lambda)$ satisfying the assumption (**A3**) we define the following integral operator

$$(F(u,\lambda))(t,x) = I_1(t,x,\lambda) + I_2(t,x,\lambda) + \int\limits_{a}^{t} ds \int\limits_{\Omega} W(t,s,x,y,\lambda) f\big((S(u,\lambda))(t,s,x,y,\lambda),\lambda\big)\nu(dy,\lambda), \quad (8)$$

where

$$(S(u,\lambda))(t,x,y,\lambda) = \begin{cases} \varphi(t-\tau(t,x,y,\lambda),x,\lambda) & \text{if } t-\tau(t,x,y,\lambda) < a, \\ u(t-\tau(t,x,y,\lambda),y,\lambda) & \text{if } t-\tau(t,x,y,\lambda) \geq a, \end{cases} \quad (9)$$

and

$$I_1(t,x,\lambda) = \varphi(a,x,\lambda) + I(t,x,\lambda),$$

$$I_2(t,x,\lambda) = \int\limits_{-\infty}^{a} ds \int\limits_{\Omega} W(t,s,x,y,\lambda) f\big(\varphi(s-\tau(s,x,y,\lambda),x,\lambda),\lambda\big)\nu(dy,\lambda).$$

Below we assume that $|f(u)| \le M$ for all $u \in R^n$.

We have to apply Theorem 1. Towards this end, we need to show that the operator family $F(\,\cdot\,,\lambda)$ $(\lambda \in \Lambda)$ satisfies the assumptions of this theorem.

At the first step of the proof we will show that $F(u,\lambda) \in Y$ for each $u \in Y$, $\lambda \in \Lambda$. Applying the assumption (**A8**) for the given $\varepsilon > 0$, we find $r > 0$ such that

$$\sup_{t\in[a,b],\ x\in\Omega,\ \lambda\in\Lambda} \int_{-\infty}^{t} ds \int_{\Omega-\Omega_r} \big|W(t,s,x,y,\lambda)\big|\nu(dy,\lambda) < \frac{\varepsilon}{M}\,. \tag{10}$$

For this r and a fixed $\lambda \in \Lambda$, we find a positive $\delta = \delta(\lambda)$ (u is kept fixed) such that

$$\begin{aligned}\Big|W(t,s,x,y,\lambda)&f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)\\ &- W(t_0,s_0,x_0,y_0,\lambda)f\big((S(u,\lambda))(s_0,x_0,y_0,\lambda),\lambda\big)\Big|\\ &\qquad\qquad < \frac{\varepsilon}{((b-a)\nu(\Omega_r,\lambda))}\end{aligned} \tag{11}$$

for all $t, t_0, s, s_0 \in [a,b]$, $x, x_0 \in \Omega$, $y, y_0 \in \Omega_r$, satisfying

$$|t-t_0| < \delta, \quad |s-s_0| < \delta, \quad |x-x_0| < \delta, \quad |y-y_0| < \delta.$$

We show first that $F(\,\cdot\,,\lambda) : Y \to Y$ for each $\lambda \in \Lambda$. In other words, we have to prove that the mapping $t \mapsto (F(u,\lambda))(t,\,\cdot\,)$ is a continuous function from $[a,b]$ to $BC(\Omega, R^n)$.

As the assumptions (**A3**), (**A4**) imply $\varphi(a,\,\cdot\,,\lambda) \in BC(\Omega,R^n)$ and $I(\,\cdot\,,\,\cdot\,,\lambda) \in Y$ $(\lambda \in \Lambda)$, we only need to check that $I_2(\,\cdot\,,\,\cdot\,,\lambda) \in Y$ and $F_0(u,\lambda) \in Y$ for all $u \in Y$ and $\lambda \in \Lambda$, where

$$(F_0(u,\lambda))(t,x) = \int_a^t ds \int_\Omega W(t,s,x,y,\lambda) f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)\nu(dy,\lambda).$$

The proofs are similar, so we concentrate on the more involved case of F_0.

For any $t \in [a,b]$, we have

$$\begin{aligned}\big|(F_0(u,\lambda))(t,x) &- (F_0(u,\lambda))(t,x_0)\big|\\ &\le \int_a^t ds \int_{\Omega_r} \Big|W(t,s,x,y,\lambda)f\big((S(u,\lambda))(s,x,y,\lambda),\lambda\big)\\ &\quad - W(t,s,x_0,y,\lambda)f\big((S(u,\lambda))(s,x_0,y,\lambda),\lambda\big)\Big|\nu(dy,\lambda)\\ &+ \int_a^b ds \int_{\Omega-\Omega_r} \Big(\big|W(t,s,x,y,\lambda)\big| + \big|W(t,s,x_0,y,\lambda)\big|\Big)\nu(dy,\lambda) < 3\varepsilon\end{aligned}$$

as long as $|x-x_0| < \delta = \delta(\lambda)$ due to the estimates (10) and (11). This proves the continuity of $(F_0(u,\lambda))(t,x)$ in x.

The boundedness of this function for each $t \in [a, b]$ follows from the assumption (**A7**) and boundedness of the function $f : R^n \to R^n$.

Finally, we check that $t \mapsto (F_0(u, \lambda))(t, \cdot)$ is a continuous mapping from $[a, b]$ to $BC(\Omega, R^n)$ if $u \in Y$:

$$\sup_{x\in\Omega} \big|(F_0(u, \lambda))(t, x) - (F_0(u, \lambda))(t_0, x)\big|$$

$$\leq \sup_{x\in\Omega} \Big| \int_a^t ds \int_\Omega W(t, s, x, y, \lambda) f\big((S(u, \lambda))(s, x, y, \lambda), \lambda\big)$$

$$- \int_a^{t_0} ds \int_\Omega W(t_0, s, x, y, \lambda) f\big((S(u, \lambda))(s, x, y, \lambda), \lambda\big) \Big| \nu(dy, \lambda)$$

$$\leq \int_{t_0}^t ds \sup_{x\in\Omega} \int_\Omega \big|W(t, s, x, y, \lambda)\big| M \nu(dy, \lambda) < \varepsilon$$

as long as $t - t_0 < \delta$. (Here we have assumed that $t > t_0$ and again used the assumption (**A7**).) We have therefore proved that $F_0(\cdot, \lambda), F(\cdot, \lambda) : Y \to Y$ for each $\lambda \in \Lambda$.

At the second step of the proof we show that the Volterra operator (8) is a local contraction in the first variable, uniformly with respect to the parameter λ.

We choose arbitrary constants $q < 1$, $\gamma \in [0, b - a)$ and $\lambda \in \Lambda$. Let $\widetilde{f}$ be the Lipschitz constant for the function f. Since the space Y consists of continuous functions, we can unify the two properties from Definition 2 into a single one and prove that $u_1(t, \cdot) = u_2(t, \cdot)$, $t \in [a, a+\gamma)$, where $u_1, u_2 \in Y$, implies the inequality (5) for the chosen $q < 1$ and some $\theta > 0$. Indeed,

$$\big\|F(u_1, \lambda) - F(u_2, \lambda)\big\|_Y$$

$$= \sup_{t\in[a,a+\gamma+\theta],\ x\in\Omega} \Big| \int_a^t ds \int_\Omega W(t, s, x, y, \lambda) f\big((S(u_1, \lambda))(s, x, y, \lambda)\big) \nu(dy, \lambda)$$

$$- \int_a^t ds \int_\Omega W(t, s, x, y, \lambda) f\big((S(u_2, \lambda))(s, x, y, \lambda)\big) \nu(dy, \lambda) \Big|$$

$$\leq \sup_{t\in[a+\gamma,a+\gamma+\theta],\ x\in\Omega} \Big| \int_{a+\gamma}^t ds \int_\Omega W(t, s, x, y, \lambda) \Big(f\big((S(u_1, \lambda))(s, x, y, \lambda)\big)$$

$$- f\big((S(u_2, \lambda))(s, x, y, \lambda)\big) \Big) \nu(dy, \lambda) \Big|$$

$$\leq \sup_{t\in[a+\gamma,a+\gamma+\theta],\ x\in\Omega} \int_{a+\gamma}^{t} ds \int_{\Omega} \bigl|W(t,s,x,y,\lambda)\bigr|\widetilde{f}\nu(dy,\lambda)\|u_1-u_2\|_Y$$
$$\leq \widetilde{q}\|u_1-u_2\|_Y,$$

where

$$\widetilde{q} = \widetilde{f} \sup_{t\in[a+\gamma,a+\gamma+\theta],\ x\in\Omega} \int_{a+\gamma}^{t} ds \int_{\Omega} \bigl|W(t,s,x,y,\lambda)\bigr|\nu(dy,\lambda).$$

Using the assumption (**A7**), we can always find a $\theta > 0$ such that $\widetilde{q} \leq q < 1$. This proves the property of local contractivity of the operator $F(\,\cdot\,,\lambda) : Y \to Y$ for any $\lambda \in \Lambda$. Moreover, we easily obtain from $\gamma \in [0, b-a)$ the estimate on $\widetilde{q}$ that this property is uniform with respect to γ and λ, i.e. $\theta > 0$ and $q < 1$ can be chosen to be independent of $\gamma \in [0, b-a)$ and $\lambda \in \Lambda$.

At the third and final step of the proof we show the continuity of the mapping $F : Y \times \Lambda \to Y$. We pick arbitrary $\lambda_0 \in \Lambda$, $u_0 \in Y$, where continuity will be examined, and arbitrary sequences $\lambda_N \to \lambda_0$, $u_N \to u_0$ $(N \to \infty)$.

We start with estimation of the following difference:

$$\begin{aligned}
&\bigl|(S(u_N,\lambda_N))(s,x,y,\lambda_N) - (S(u_0,\lambda_0))(s,x,y,\lambda_0)\bigr| \\
&\qquad\leq \bigl|(S(u_N,\lambda_N))(s,x,y,\lambda_N) - \bigl(S(u_0,\lambda_N)\bigr)(s,x,y,\lambda_0)\bigr| \\
&\qquad\qquad + \bigl|\bigl(S(u_0,\lambda_N)\bigr)(s,x,y,\lambda_0) - \bigl(S(u_0,\lambda_0)\bigr)(s,x,y,\lambda_0)\bigr|.
\end{aligned}$$

The first term on the right-hand side of this inequality is less than $\varepsilon/2$ for all $s \in (-\infty, b]$, $x, y \in \Omega$, $N \geq N_1$ as $u_N \to u_0$ $(N \to \infty)$. By virtue of the assumptions (**A2**) and (**A3**), the second term on the right-hand side is less than $\varepsilon/2$ for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_r$, $N \geq N_2(r)$. Thus, for any $r > 0$, we have

$$\bigl|(S(u_N,\lambda_N))(s,x,y,\lambda_N) - (S(u_0,\lambda_0))(s,x,y,\lambda_0)\bigr| \leq \varepsilon \tag{12}$$

for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_r$, $N \geq N_3(r)$.

Then, choosing $\varepsilon > 0$, we find a number $r_0 > 0$ such that the estimate (10) holds true. Increasing, if necessary, the value of r_0, we may, in addition, assume without loss of generality that $\nu(\Omega_{r_0}, \lambda_0) > 0$ and $\nu(\partial\Omega_{r_0}, \lambda_0) = 0$, so that

$$\lim_{N\to\infty} \nu(\Omega_{r_0}, \lambda_N) = \nu(\Omega_{r_0}, \lambda_0)$$

(see e.g. [7, Chapter VI, Theorem 2]).

Using this r_0, we estimate the following difference:

$$\begin{aligned}
&\Bigl|f\bigl((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\bigr) - f\bigl((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\bigr)\Bigr| \\
&\leq \Bigl|f\bigl((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\bigr) - f\bigl((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\bigr)\Bigr| \\
&\quad + \Bigl|f\bigl((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_0\bigr) - f\bigl((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\bigr)\Bigr|.
\end{aligned}$$

By virtue of the assumption (**A1**), the first term on the right-hand side of the inequality is less than ε for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_4(r_0)$. Using the assumption (**A1**) and the estimate (12), we get that the second term on the right-hand side of the inequality is less than ε for all $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_3(r_0)$. Thus, taking into account (**A1**) and (**A7**), we obtain the inequality

$$\Big| \int\limits_{-\infty}^{t} ds \int\limits_{\Omega_{r_0}} W(t,s,x,y,\lambda_N)\Big(f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) \\ - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big)\nu(dy,\lambda_N)\Big| < \varepsilon \tag{13}$$

for all $t \in [a,b]$, $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_5(r_0)$.

The assumption (**A5**) yields

$$\big|W(t,s,x,y,\lambda_N) - W(t,s,x,y,\lambda_0)\big| < \frac{\varepsilon}{M((b-a)\nu(\Omega_r,\lambda))} \tag{14}$$

for all $t \in [a,b]$, $s \in (-\infty, b]$, $x \in \Omega$, $y \in \Omega_{r_0}$, $N \geq N_6(r_0)$.

Using the assumptions (**A3**), (**A4**), and (**A6**), we find a natural number $N_7(r_0)$ such that

$$\begin{gathered} \sup_{t\in[a,b],\ x\in\Omega} \Big| \int\limits_{\Omega_{r_0}} \Phi(t,x,y)\big(\nu(dy,\lambda_N) - \nu(dy,\lambda_0)\big)\Big| < \varepsilon, \\ \nu(\Omega_{r_0},\lambda_N) \leq 2\nu(\Omega_{r_0},\lambda_0), \\ \sup_{x\in\Omega} \big|\varphi(a,x,\lambda_N) - \varphi(a,x,\lambda_0)\big| < \varepsilon, \\ \sup_{t\in[a,b],\ x\in\Omega} \big|I(t,x,\lambda_N) - I(t,x,\lambda_0)\big| < \varepsilon, \ \ |\lambda_N - \lambda_0| < \delta \end{gathered} \tag{15}$$

for all $N \geq N_7(r_0)$. Here, the function

$$\Phi(t,x,y) = \int\limits_{-\infty}^{t} W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\, ds$$

is uniformly continuous on the set $[a,b] \times \Omega \times \Omega_{r_0}$, so that

$$\int\limits_{\Omega_{r_0}} \Phi(t,x,y)\nu(dy,\lambda_N) \longrightarrow \int\limits_{\Omega_{r_0}} \Phi(t,x,y)\nu(dy,\lambda_0)$$

as $n \to \infty$ uniformly with respect to the variables $t \in [a,b]$, $x \in \Omega$.

Next, we estimate

$$
\begin{aligned}
&\sup_{t\in[a,b],\ x\in\Omega} \big| I_2(t,x,\lambda_N) - I_2(t,x,\lambda_0) \big| \\
&\qquad \le \sup_{t\in[a,b],\ x\in\Omega} \bigg| \int\limits_{-\infty}^{t} ds \int\limits_{\Omega} W(t,s,x,y,\lambda_N) \\
&\qquad\quad \times f\Big(\varphi\big(s-\tau(s,x,y,\lambda_N),x,\lambda_N\big),\lambda_N\Big)\nu(dy,\lambda_N) \\
&\quad - \int\limits_{-\infty}^{t} ds \int\limits_{\Omega} W(t,s,x,y,\lambda_0) f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0) \bigg| \\
&\qquad \le \sup_{t\in[a,b],\ x\in\Omega} \bigg| \int\limits_{-\infty}^{t} ds \int\limits_{\Omega-\Omega_{r_0}} W(t,s,x,y,\lambda_N) \\
&\qquad\quad \times f\big(\varphi(s-\tau(s,x,y,\lambda_N),x,\lambda_N),\lambda_N\big)\nu(dy,\lambda_N) \\
&\quad - \int\limits_{-\infty}^{t} ds \int\limits_{\Omega-\Omega_{r_0}} W(t,s,x,y,\lambda_0) f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0) \bigg| \\
&\quad + \sup_{t\in[a,b],\ x\in\Omega} \bigg| \int\limits_{-\infty}^{t} ds \int\limits_{\Omega_{r_0}} W(t,s,x,y,\lambda_N) \Big(f\big(\varphi(s-\tau(s,x,y,\lambda_N),x,\lambda_N\big),\lambda_N\big) \\
&\qquad\quad - f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\Big)\nu(dy,\lambda_N) \bigg| \\
&\qquad + \sup_{t\in[a,b],\ x\in\Omega} \bigg| \int\limits_{-\infty}^{t} ds \int\limits_{\Omega_{r_0}} \big(W(t,s,x,y,\lambda_N) - W(t,s,x,y,\lambda_0) \big) \\
&\qquad\quad \times f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N) \bigg| \\
&\qquad\quad + \sup_{t\in[a,b],\ x\in\Omega} \bigg| \int\limits_{-\infty}^{t} ds \int\limits_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) \\
&\qquad\quad \times f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N) \\
&\quad - \int\limits_{-\infty}^{t} ds \int\limits_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) f\big(\varphi(s-\tau(s,x,y,\lambda_0),x,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0) \bigg|.
\end{aligned}
$$

The first term on the right-hand side of the inequality is less than 2ε as the estimate (10) and the assumption (**A1**) hold true. Each of the second and the third terms on the right-hand side of the inequality is less than

ε due to (13) and (**A1**), (**A7**), (14), respectively, for all $N > N_8(r_0) = \max\{N_5(r_0), N_6(r_0)\}$. The estimate (15) yields the last term on the right-hand side of the inequality is less than ε for all $N > N_7(r_0)$.

Thus, we get that

$$\sup_{t\in[a,b],\ x\in\Omega} \big| I_2(t,x,\lambda_N) - I_2(t,x,\lambda_0)\big| < 5\varepsilon \tag{16}$$

for all $N \geq N_9(r_0) = \max\{N_7(r_0), N_8(r_0)\}$.

Finally, taking into account the estimates (10), (11), (13)–(16) and the assumption (**A7**), we obtain

$$\begin{aligned}
&\big\|F(u_N,\lambda_N) - F(u_0,\lambda_0)\big\|_Y \leq \sup_{x\in\Omega}\big|\varphi(a,x,\lambda_N) - \varphi(a,x,\lambda_0)\big| \\
&\quad + \sup_{t\in[a,b],\ x\in\Omega}\big|I(t,x,\lambda_N) - I(t,x,\lambda_0)\big| \\
&\qquad\qquad + \sup_{t\in[a,b],\ x\in\Omega}\big|I_2(t,x,\lambda_N) - I_2(t,x,\lambda_0)\big| \\
&\quad + \sup_{t\in[a,b],\ x\in\Omega}\bigg|\int_a^t ds\int_\Omega W(t,s,x,y,\lambda_N) \\
&\qquad\qquad \times f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big)\nu(dy,\lambda_N) \\
&\quad - \int_a^t ds\int_\Omega W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\bigg| \\
&\quad \leq 7\varepsilon + \sup_{t\in[a,b],\ x\in\Omega}\bigg|\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_N) \\
&\qquad\qquad \times f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big)\nu(dy,\lambda_N) \\
&\quad - \int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\bigg| + 2\varepsilon \\
&\quad \leq 9\varepsilon + \sup_{t\in[a,b],\ x\in\Omega}\bigg|\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_N) \\
&\qquad \times \Big(f\big((S(u_N,\lambda_N))(s,x,y,\lambda_N),\lambda_N\big) \\
&\qquad\qquad - f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\Big)\nu(dy,\lambda_N)\bigg| \\
&+ \sup_{t\in[a,b],\ x\in\Omega}\bigg|\int_a^t ds\int_{\Omega_{r_0}} \big(W(t,s,x,y,\lambda_N) - W(t,s,x,y,\lambda_0)\big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N)\Bigg| \\
&+\sup_{t\in[a,b],x\in\Omega}\Bigg|\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) \\
&\qquad\qquad \times f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_N) \\
&-\int_a^t ds\int_{\Omega_{r_0}} W(t,s,x,y,\lambda_0) f\big((S(u_0,\lambda_0))(s,x,y,\lambda_0),\lambda_0\big)\nu(dy,\lambda_0)\Bigg| \\
&\qquad \le 10\varepsilon+(b-a)\nu(\Omega_{r_0},\lambda_N)\,\frac{\varepsilon}{((b-a)\nu(\Omega_{r_0},\lambda_0))} \\
&\qquad\quad +\sup_{t\in[a,b],\ x\in\Omega}\Bigg|\int_{\Omega_{r_0}}\Phi(t,x,y)(\nu(dy,\lambda_N)-\nu(dy,\lambda_0))\Bigg|<13\varepsilon
\end{aligned}
$$

for all $N\ge N_9(r_0)$.

The proof is complete. □

Remark 1. If Ω is compact, then the assumption (**A8**) is fulfilled automatically and can therefore be omitted, while the assumptions (**A2**)–(**A5**) only require continuity of the corresponding functions instead of their uniform continuity in the variable x.

3. The Hopfield Model with Delay

In this section we prove convergence of the generalized Hopfield network to the Amari neural field equation.

Consider the following delayed Hopfield network model (see e.g. [14])

$$
\dot z_i(t,N)=-\alpha z_i(t,N)+\sum_{j=1}^{N}\omega_{ij}(N)f\big(z_j(t-\tau_{ij}(t,N),N)\big)+\mathrm{J}_i(t,N),\quad t>a,\ \ i=1,\dots,N, \tag{17}
$$

parameterized by a natural parameter N. Here at each natural N, $z_i(\,\cdot\,,N)$ are n-dimensional vector functions, $\omega_{ij}(N)$ are real $n\times n$-matrices (connectivities), $\tau_{ij}(\,\cdot\,,N)$ are nonnegative real-valued continuous functions (axonal delays), $f : R^n\to R^n$ are firing rate functions which are Lipschitz and bounded and $\mathrm{J}_i(\,\cdot\,,N)$ are continuous external input n-dimensional vector functions.

The initial conditions for (17) are given as

$$
z_i(\xi,N)=\varphi_i(\xi,N),\ \ \xi\le a,\ \ i=1,\dots,N. \tag{18}
$$

We use the general well-posedness result from the previous section to justify the convergence of a sequence of the delayed Hopfield equations (17) (with the initial conditions (18)) to the Amari equation involving a spatio-temporal delay

$$\partial_t u(t,x) = -\alpha u(t,x) + \int\limits_{\Omega} \omega(x,y) f(u(s-\tau(t,x,y),y))\nu(dy) + J(t,x), \quad t > a, \ \ x \in \Omega, \tag{19}$$

with the initial (prehistory) condition

$$u(\xi,x) = \varphi(\xi,x), \ \ \xi \le a, \ \ x \in \Omega. \tag{20}$$

On the above functions we impose the following assumptions:

(**B1**) The function $f : R^n \to R^n$ is continuous, bounded and Lipschitz one.

(**B2**) The spatio-temporal delay $\tau : R \times \Omega \times \Omega \to [0,\infty)$ is continuous.

(**B3**) The initial (prehistory) function $\varphi : (-\infty, a] \times \Omega \to R^n$ is continuous.

(**B4**) For any $b > a$, the external input function $J : [a,b] \times \Omega \to R^n$ is uniformly continuous and bounded with respect to the second variable.

(**B5**) The kernel function $\omega : \Omega \times \Omega \to R^n$ is continuous.

(**B6**) $\nu(\,\cdot\,)$ is the Lebesgue measure on Ω.

(**B7**) For any $b > a$,

$$\sup_{x\in\Omega} \int\limits_{\Omega} |\omega(x,y)|\nu(dy) < \infty.$$

(**B8**) For any $b > a$,

$$\lim_{r\to\infty} \sup_{x\in\Omega} \int\limits_{\Omega-\Omega_r} |\omega(x,y)|\nu(dy) = 0.$$

The following theorem represents the main result of this section.

Theorem 3. *For each natural number N let $\{\Delta_i(N),\ i = 1,\dots,N\}$ be a finite family of open subsets of Ω satisfying the conditions*

$$\bigcup_{i=1}^{N} \overline{\Delta}_i(N) = \Omega_N \ \ \textit{and} \ \ \lim_{N\to\infty} \textit{mesh}\big\{\Delta_i(N),\ i = 1,\dots,N\big\} = 0.$$

Let $y_i(N)$ $(i = 1,\dots,N)$ be arbitrary points in $\Delta_i(N)$. Finally, let the assumptions (**B1**)–(**B8**) *be fulfilled. Then the sequence of the solutions $z_i(t,N)$ $(t \in R)$ of the initial value problem* (17), (18), *where the coefficients are defined by*

$$\begin{aligned} \omega_{ij}(N) &= \beta_i(N)\omega(y_i(N), y_j(N)), \ \ \textit{where} \ \ \beta_i(N) = \nu(\Delta_i(N)), \\ \tau_{ij}(t,N) &= \tau(t, y_i(N), y_j(N)), \quad \mathrm{J}_i(t,N) = J(t, y_i(N)), \end{aligned} \tag{21}$$

converges for any $b > a$ *to the solution* $u(t,x)$ $(t \in R,\ x \in \Omega)$ *of the initial value problem* (19), (20) *as* $N \to \infty$*, in the following sense:*

$$\lim_{N\to\infty} \sup_{t\in[a,b]} \Big(\sup_{1\le i\le N} \Big(\sup_{x\in\Delta_i(N)} |u(t,x) - z_i(t,N)| \Big)\Big) = 0. \tag{22}$$

In order to prove this theorem, we will need to use the following statement.

Lemma 1. *Assume that for each natural number* N *we have a finite family of open subsets* $\{\Delta_i(N),\ i = 1,\dots,N\}$ *of* Ω *satisfying the conditions*

$$\bigcup_{i=1}^{N} \overline{\Delta}_i(N) = \Omega_N \quad \textit{and} \quad \lim_{N\to\infty} \textit{mesh}\big\{\Delta_i(N),\ i = 1,\dots,N\big\} = 0.$$

Let $y_i(N)$ $(i = 1,\dots,N)$ *be arbitrary points in* $\Delta_i(N)$*,* $\mathfrak{D}_i(N)$ *be the Dirac measures at* $y_i(N)$ *and* $\beta_i(N) = \nu(\Delta_i(N))$*. Then the sequence of the discrete weighted measures*

$$\nu_N = \sum_{i=1}^{N} \beta_i(N)\mathfrak{D}_i(N) \tag{23}$$

weakly converges (in the sense of the weak topology on the dual space to $C_{comp}(\Omega)$*) to the Lebesgue measure on* Ω*.*

Proof. We simply observe that for any continuous and compactly supported function $\Phi(x)$, $x \in \Omega$, we get

$$\begin{aligned}\int_\Omega \Phi(x)\nu_N(dx) &= \sum_{i=1}^{N} \Phi\big(y_i(N)\big)\beta_i(N) \\ &= \sum_{i=1}^{N} \Phi\big(y_i(N)\big)\nu\big(\Delta_i(N)\big) \longrightarrow \int_\Omega \Phi(x)\nu(dx),\end{aligned} \tag{24}$$

as $N \to \infty$, due to the properties of the Riemann–Stiltjes integrals (see e.g. Chapter 2 in [11]). $\square$

Proof of the Theorem 3. In order to apply Theorem 2, we first of all define the metric space $\Lambda = \{\lambda_N,\ N = 0,1,2,\dots\}$, where $\lambda_0 = \infty$, $\lambda_N = N$ for natural numbers N, and the distance is given by $d(\lambda_N,\lambda_M) = |1/N - 1/M|$ $(N,M \ne 0)$ and $d(\lambda_N,\lambda_0) = 1/N$ $(N \ne 0)$, so that $\lambda_N \to \lambda_0$ simply means that $N \to \infty$. Multiplication by the function $\eta(t-s)$, where $\eta(\sigma) = \exp(-\alpha\sigma)$, followed by integration, converts the equation (19) into the equation (1), where f, τ,

$$W(t,s,x,y) = \exp(-\alpha(t-s))\omega(x,y),$$

$$I(t,x) = \int_a^t \exp(-\alpha(t-s))J(s,x)\,ds$$

are all independent of λ, and the measures are defined as $\nu(\,\cdot\,,\lambda_N) = \nu_N$ (see (23)) and $\nu(\,\cdot\,,\lambda_0) = \nu$, respectively.

The assumptions (**A1**)–(**A5**) of Theorem 2 are trivial, the assumption (**A6**) is fulfilled due to Lemma 1 and the above definition of convergence in Λ.

Taking into account that

$$\max_{t\in[a,b]} \int\limits_{-\infty}^{t} \exp(-\alpha(t-s))\,ds = \frac{1}{\alpha}\,,$$

it is straightforward to check the assumptions (**A7**) and (**A8**).

From Theorem 2 it now follows that the solutions $u(t,x,N)$ of the initial boundary value problems

$$\partial_t u(t,x,N) = -\alpha u(t,x,N)$$
$$+\int\limits_{\Omega} \omega(x,y) f\big(u(s-\tau(t,x,y),y,N)\big)\nu_N(dy) + J(t,x),\ \ t>a,\ \ x\in\Omega, \quad (25)$$

with the initial (prehistory) condition

$$u(\xi,x,N) = \varphi(\xi,x),\ \ \xi\le a,\ \ x\in\Omega, \tag{26}$$

converge to the solution $u(t,x)$ ($t\in R$, $x\in\Omega$) of the initial value problem (19), (20), as $N\to\infty$, uniformly on $[a,b]\times\Omega$ for any $b>a$. Evidently, replacing x by $y_i(N)$ in the equation (25) and in the initial condition (26) yields the initial value problem (17), (18). It remains therefore to notice that the set $z_i(t,N) = u(t,y_i(N),N)$ ($i=1,\dots,N$) is a (unique) solution of the latter problem. □

The theoretical results of this section can be applied to justify numerical integration schemes. For example, Faye et al [5] considered discretization of the following delayed Amari model

$$\partial_t u(t,x) = -\alpha u(t,x) + \int\limits_{\Omega} \omega\big(|x-y|\big) f\Big(u\Big(t-\frac{|x-y|}{v},y\Big)\Big)\,dy \tag{27}$$

in the cases

I. $u(t,x)\in R$, $\Omega=[-L,L]$,

II. $u(t,x)\in R^2$, $\Omega=[-L,L]$,

III. $u(t,x)\in R$, $\Omega=[-L,L]^2$.

Faye et al have justified their numerical schemes using convergence of the trapezoidal integration rule and the rectangular method to the corresponding integrals. We will show how our results can be applied for the more involved case III:

$$\partial_t u_{ij}(t) = -\alpha u_{ij}(t) + \sum_{k=1}^{M}\sum_{l=1}^{M} \omega\big(|(x_i^1, x_j^2) - (x_k^1, x_l^2)|\big) \times f\Big(u_{kl}\Big(t - \frac{|(x_i^1, x_j^2) - (x_k^1, x_l^2)|}{v}\Big)\Big)\, dy. \quad (28)$$

Here,

$$x = (x^1, x^2), \;\; u_{ij}(t) = u\big(t, (x_i^1, x_j^2)\big), \;\; i, j = 1, \ldots, M.$$

Denoting

$$z_i(t) = u_{ij}(t), \quad \omega_{ij} = \omega\big(|(x_i^1, x_j^2) - (x_k^1, x_l^2)|\big),$$
$$\tau_{ij}(t) = \frac{|(x_i^1, x_j^2) - (x_k^1, x_l^2)|}{v},$$
$$i = iM + j, \;\; j = kM + l, \;\; N = M^2,$$

in (28), we get the Hopfield network model (17). Applying Theorem 3, we prove convergence of the numerical scheme (28) to the equation (27).

Rankin et al [10] discretize the Amari model (27) for

$$u(t, x) \in R, \quad \Omega = [-L, L]^2, \quad v = \infty,$$

also by substituting Ω with the grid $\{(x_i^1, x_j^2), i, j = 1, \ldots, M\}$ and then use a combination of the Fourier transform and the inverse Fourier transform to obtain the solution numerically. Discretization of the Amari model on a hyperbolic disc $\Omega = \{x = (r, \theta),\, r \in [0, r_0],\, r_0 \in R,\, \theta \in [0, 2\pi)\}$ using the rectangular rule for the quadrature $\{(r_i, \theta_j),\, i = 1, \ldots, M,\, j = i = 1, \ldots, N\}$ was implemented in [6] to study of the localized solutions. As it easy to conclude from Theorem 3, the solutions obtained in both these cases converge to the corresponding analytical solutions as $M \to \infty$ and $N \to \infty$.

We emphasize here that Theorem 3 also allows one to justify discretization schemes on unbounded domains for equations involving spatio-temporal-dependent delay as well.

Appendix

In this section we consider the following neural field model with a general (i.e. non-periodic) microstructure:

$$\begin{aligned} \partial_t u(t, x) &= -u(t, x) + \int_{R^m} \omega_i^\varepsilon(x - y) f(u(t, y))\, dy, \\ \omega_i^\varepsilon(x) &= \omega_i(x, x/\varepsilon), \;\; 0 < \varepsilon \ll 1, \\ & t \geq 0, \;\; x \in R^m. \end{aligned} \quad (29)$$

which is a parametrized version of (3).

Question: What can we say about behavior of the solutions u_n to the equation (29) as $\omega_i^\varepsilon \to \omega_0^\varepsilon$ uniformly $(i \to \infty)$, where ω_0^ε is periodic with respect to the second argument?

Following the idea of homogenization of the equation (3) (see [12])), we first look at the family of homogenized problems

$$\partial_t u(t,x_c,x_f) = -u(t,x_c,x_f)$$
$$+\int\limits_{R^m}\int\limits_{K_n} \omega_i(x_c-y_c,x_f-y_f) f(u(t,y_c,y_f))\, dy_c\, \nu_n(dy_f), \tag{30}$$
$$t>0,\ \ x_c\in R^m,\ \ x_f\in K_i\subset R^k$$

and the corresponding limit problem as $i\to\infty$

$$\partial_t u(t,x_c,x_f) = -u(t,x_c,x_f)$$
$$+\int\limits_{R^m}\int\limits_{K_0} \omega_0(x_c-y_c,x_f-y_f) f(u(t,y_c,y_f))\, dy_c\, \nu_0(dy_f), \tag{31}$$
$$t>0,\ \ x_c\in R^m,\ \ x_f\in K_0\subset R^k.$$

As in [12], we assume that for each $i=0,1,2,\dots$, the connectivity kernel $\omega_i(x,\,\cdot\,)$ $(x\in R^m)$ belongs to A_i, where $A_i = C(K_i)$ are some Banach algebras of continuous functions defined on the compact sets $K_i\subset R^k$ and equipped with the mean values $\mathfrak{M}_i$ (which give rise to the finite measure ν_i defined on K_i). Further, we assume that there is a compact $\overline{K}$ such that $\bigcup\limits_{i=0}^{\infty} K_i\subseteq\overline{K}$, so we can extend the measures ν_i corresponding to the mean values $\mathfrak{M}_i$ $(i=0,1,2,\dots)$, to the compact $\overline{K}$ by putting $\nu_i(\overline{K}\setminus K_i)=0$. Finally, we assume that convergence of the connectivity kernels is a consequence of a convergence of the associated Banach algebras with mean. More precisely, we suppose that:

1) the compacts K_i converge to the compact K_0 in the Hausdorff metric;
2) $\mathfrak{M}_n(\chi\big|_{K_n})\to\mathfrak{M}_0(\chi\big|_{K_0})$ for any function $\chi\in C(\overline{K})$ (here $\chi\big|_{K_i}$ denotes the restriction of the function $\chi\in C(\overline{K})$ to the set K_i).

Thus, we get

$$\int\limits_{K_n}\chi(x)\nu_n(dx)\longrightarrow\int\limits_{K_0}\chi(x)\nu_0(dx)$$

for any $\chi\in C(\overline{K})$, which means that the sequence of measures ν_n weakly converges to the measure ν_0. Hence, we can apply Theorem 2 to the problems (30) and (31) and get uniform convergence of the corresponding solutions. This approach can serve as a possible answer to the above-formulated question.

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Authors' addresses:

Evgenii Burlakov, **John Wyller**

Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, P.O. Box 5003, Ås, 1432, Norway.

E-mail: evgenii.burlakov@nmbu.no; john.wyller@nmbu.no

Evgeny Zhukovskiy

Department of Mathematics, Physics and Computer Sciences, Tambov State University, 31 Internatsionalnaya St., Tambov, Russia

E-mail: zukovskys@mail.ru

Arcady Ponosov

Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, P.O. Box 5003, Ås, 1432, Norway.

E-mail: arkadi.ponossov@nmbu.no

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Otar Chkadua and David Natroshvili

LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS APPROACH FOR ROBIN TYPE PROBLEM OF THE THEORY OF PIEZO-ELASTICITY FOR INHOMOGENEOUS SOLIDS

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. The paper deals with the three-dimensional Robin type boundary value problem (BVP) of piezoelasticity for anisotropic inhomogeneous solids and develops the generalized potential method based on the use of localized parametrix. Using Green's integral representation formula and properties of the localized layer and volume potentials, we reduce the Robin type BVP to the localized boundary-domain integral equations (LBDIE) system. First we establish the equivalence between the original boundary value problem and the corresponding LBDIE system. We establish that the obtained localized boundary-domain integral operator belongs to the Boutet de Monvel algebra and by means of the Vishik-Eskin theory based on the Wiener-Hopf factorization method, we derive explicit conditions under which the localized operator possesses Fredholm properties and prove its invertibility in appropriate Sobolev-Slobodetskii and Bessel potential spaces.

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Key words and phrases. Piezoelasticity, partial differential equations with variable coefficients, boundary value problems, localized parametrix, localized boundary-domain integral equations, pseudo-differential operators.

რეზიუმე. ნაშრომი ეძღვნება ლოკალიზებული პარამეტრიქსის მეთოდის განვითარებას პიეზო-დრეკადობის თეორიის რობენის ტიპის სამგანზომილებიანი ამოცანისთვის არაერთგვაროვანი ანიზოტროპული სხეულების შემთხვევაში. გრინის ინტეგრალური წარმოდგენის ფორმულისა და ლოკალიზებული პოტენციალების თვისებების გამოყენებით რობენის ტიპის ამოცანა დაიყვანება ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემაზე, რომლის შესაბამისი ოპერატორი ეკუთვნის ბუტე დე მონველის ალგებრას. შესწავლილია რობენის ტიპის ამოცანისა და მიღებულ ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემის ექვივალენტობა. შემდეგ, ვიშიკ-ესკინის თეორიის გამოყენებით, რომელიც ეფუძნება ვინერ-ჰოფის ფაქტორიზაციის მეთოდს, დადგენილია პირობები, რომლის დროსაც ლოკალიზებულ სასაზღვრო-სივრცული ინტეგრალური ოპერატორი არის ფრედჰოლმური და ნაჩვენებია მისი შებრუნებადობა შესაბამის სობოლევ-სლობოდეცკისა და ბესელის პოტენციალთა სივრცეებში.

1. Introduction

In the present paper, we consider the three-dimensional Robin type boundary value problem (BVP) of piezoelasticity for anisotropic inhomogeneous solids and develop the generalized potential method based on the use of *localized parametrix.*

Note that the operator, generated by the system of piezoelasticity for inhomogeneous anisotropic solids, is the second order non-self-adjoint strongly elliptic partial differential operator with variable coefficients. In the reference [22] the Dirichlet problem of piezoelasticity theory was analyzed by the LBDIE approach. The same method for the case of scalar elliptic second order partial differential equations with variable coefficients is justified in [13]–[21], [39].

Due to a great theoretical and practical importance, the problems of piezoelasticity became very popular among mathematicians and engineers (for details see, e.g., [51], [43], [27]–[35]). The BVPs and various types of interface problems of piezoelasticity for *homogeneous anisotropic solids*, when the material parameters are constants and the corresponding fundamental solution is available in explicit form, have been investigated in [5], [6], [7], [8], [9], [42], [10] by means of the conventional classical potential methods.

Unfortunately, this classical potential method is not applicable in the case of inhomogeneous solids since for the corresponding system of differential equations with variable coefficients a fundamental solution is not available in explicit form, in general. Therefore, in our analysis we apply the so-called localized parametrix method which leads to the localized boundary-domain integral equations system.

Our main goal here is to show that solutions of the boundary value problem can be represented by *localized potentials* and that the corresponding *localized boundary-domain integral operator* (LBDIO) is invertible, which seems to be very important from the numerical analysis viewpoint, since they lead to very convenient numerical schemes in applications (for details see [38], [46], [47], [49], [50]).

Towards this end, using Green's representation formula and properties of the localized layer and volume potentials, we reduce the Robin type BVP of piezoelasticity to the *localized boundary-domain integral equations (LBDIE) system.* First, we establish the equivalence between the original boundary value problem and the corresponding LBDIE system which proved to be a quite nontrivial problem playing a crucial role in our analysis. Afterwards, we state that the localized boundary-domain integral operator associated with the Robin type BVP belongs to the Boutet de Monvel algebra of pseudo-differential operators. Finally, with the help of the Vishik–Eskin theory based on the factorization Wiener–Hopf method, we investigate the Fredholm properties of the localized boundary-domain integral operator and prove its invertibility in the appropriate Sobolev–Slobodetskii and Bessel potential spaces.

2. Reduction to LBDIE System and the Equivalence Theorems

2.1. **Formulation of the boundary value problem and localized Green's third formula.** Consider the system of statics of piezoelasticity for an inhomogeneous anisotropic medium [43]:

$$A(x,\partial_x)\,U+X=0, \tag{2.1}$$

where $U:=(u_1,u_2,u_3,u_4)^\top$, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $u_4=\varphi$ is the electric potential, $X=(X_1,X_2,X_3,X_4)^\top$, $(X_1,X_2,X_3)^\top$ is a given mass force density, X_4 is a given charge density, $A(x,\partial_x)$ is a formally non-self-adjoint matrix differential operator

$$\begin{aligned} A(x,\partial_x) &= \big[A_{jk}(x,\partial_x)\big]_{4\times4}\\ &:= \begin{bmatrix} [\partial_i\,(c_{ijlk}(x)\,\partial_l)]_{3\times3} & [\partial_i(e_{lij}(x)\partial_l)]_{3\times1}\\ [-\partial_i(e_{ikl}(x)\,\partial_l)]_{1\times3} & \partial_i(\varepsilon_{il}(x)\,\partial_l) \end{bmatrix}_{4\times4}, \end{aligned}$$

where $\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial_{x_j}=\partial/\partial x_j$. Here and in what follows, the Einstein summation by repeated indices from 1 to 3 is assumed if not otherwise stated.

The variable coefficients involved in the above equations satisfy the symmetry conditions:

$$c_{ijkl}=c_{jikl}=c_{klij}\in C^\infty,\ \ e_{ijk}=e_{ikj}\in C^\infty,\ \ \varepsilon_{ij}=\varepsilon_{ji}\in C^\infty,\\ i,j,k,l=1,2,3.$$

In view of these symmetry relations, the formally adjoint differential operator $A^*(x,\partial_x)$ reads as

$$\begin{aligned} A^*(x,\partial_x) &= \big[A^*_{jk}(x,\partial_x)\big]_{4\times4}\\ &:= \begin{bmatrix} [\partial_i\,(c_{ijlk}(x)\,\partial_l)]_{3\times3} & [-\partial_i(e_{lij}(x)\partial_l)]_{3\times1}\\ [\partial_i(e_{ikl}(x)\,\partial_l)]_{1\times3} & \partial_i(\varepsilon_{il}(x)\,\partial_l) \end{bmatrix}_{4\times4}. \end{aligned}$$

Moreover, from physical considerations it follows that (see, e.g., [43]):

$$c_{ijkl}(x)\,\xi_{ij}\,\xi_{kl}\geqslant c_0\,\xi_{ij}\,\xi_{ij}\ \text{ for all }\ \xi_{ij}=\xi_{ji}\in\mathbb{R}, \tag{2.2}$$

$$\varepsilon_{ij}(x)\,\eta_i\,\eta_j\geqslant c_1\,\eta_i\,\eta_i\ \text{ for all }\ \eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^3, \tag{2.3}$$

with some positive constants c_0 and c_1.

By virtue of inequalities (2.2) and (2.3) it can easily be shown that the operator $A(x,\partial_x)$ is uniformly strongly elliptic, that is, there is a constant $c>0$ such that

$$\operatorname{Re}A(x,\xi)\,\zeta\cdot\zeta\geqslant c\,|\xi|^2\,|\zeta|^2\ \text{ for all }\ \xi\in\mathbb{R}^3\ \text{ and for all }\ \zeta\in\mathbb{C}^4, \tag{2.4}$$

where $A(x,\xi)$ is the principal homogeneous symbol matrix of the operator $A(x,\partial_x)$ with opposite sign,

$$\begin{aligned} A(x,\xi) &= \big[A_{jk}(x,\xi)\big]_{4\times 4} \\ &:= \begin{bmatrix} [c_{ijlk}(x)\,\xi_i\,\xi_l]_{3\times 3} & [e_{lij}(x)\,\xi_i\,\xi_l]_{3\times 1} \\ [-e_{ikl}(x)\,\xi_i\,\xi_l]_{1\times 3} & \varepsilon_{il}(x)\,\xi_i\,\xi_l \end{bmatrix}_{4\times 4}. \end{aligned} \tag{2.5}$$

Here and in the sequel, the symbol $a\cdot b$ for $a,b\in\mathbb{C}^4$ denotes the scalar product of two vectors, $a\cdot b=\sum\limits_{j=1}^{4} a_j\overline{b_j}$, where the overbar denotes complex conjugation.

In the theory of piezoelasticity, the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n=(n_1,n_2,n_3)$ have the form

$$\sigma_{ij}\,n_i = c_{ijlk}\,n_i\,\partial_l u_k + e_{lij}\,n_i\,\partial_l\varphi \ \text{ for } \ j=1,2,3,$$

while the normal component of the electric displacement vector (with opposite sign) reads as

$$-D_i\,n_i = -e_{ikl}\,n_i\,\partial_l u_k + \varepsilon_{il}\,n_i\,\partial_l\varphi.$$

Let us introduce the following matrix differential operator:

$$\begin{aligned} \mathcal{T} = \mathcal{T}(x,\partial_x) &= \big[\mathcal{T}_{jk}(x,\partial_x)\big]_{4\times 4} \\ &:= \begin{bmatrix} [c_{ijlk}(x)\,n_i\,\partial_l]_{3\times 3} & [e_{lij}(x)\,n_i\,\partial_l]_{3\times 1} \\ [-e_{ikl}(x)\,n_i\,\partial_l]_{1\times 3} & \varepsilon_{il}(x)\,n_i\,\partial_l \end{bmatrix}_{4\times 4}. \end{aligned} \tag{2.6}$$

For a four–vector $U=(u,\varphi)^\top$, we have

$$\mathcal{T}\,U = \big(\sigma_{i1}\,n_i,\ \sigma_{i2}\,n_i,\ \sigma_{i3}\,n_i,\ -D_i\,n_i\big)^\top. \tag{2.7}$$

Clearly, the components of the vector $\mathcal{T}\,U$ given by (2.7) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of electro-elasticity and the forth one is the normal component of the electric displacement vector (with opposite sign). In Green's formulae there also appear the following boundary operator associated with the adjoint differential operator $A^*(x,\partial_x)$:

$$\begin{aligned} \mathcal{M} = \mathcal{M}(x,\partial_x) &= \big[\mathcal{M}_{jk}(x,\partial_x)\big]_{4\times 4} \\ &:= \begin{bmatrix} [c_{ijlk}(x)\,n_i\,\partial_l]_{3\times 3} & [-e_{lij}(x)\,n_i\,\partial_l]_{3\times 1} \\ [e_{ikl}(x)\,n_i\,\partial_l]_{1\times 3} & \varepsilon_{il}(x)\,n_i\,\partial_l \end{bmatrix}_{4\times 4}. \end{aligned} \tag{2.8}$$

Introduce the following matrices associated with the boundary operators (2.6) and (2.8)

$$\mathcal{T}(x,\xi) = \big[\mathcal{T}_{jk}(x,\xi)\big]_{4\times 4} := \begin{bmatrix} [c_{ijlk}(x)\, n_i\, \xi_l]_{3\times 3} & [e_{lij}(x)\, n_i\, \xi_l]_{3\times 1} \\ [-e_{ikl}(x)\, n_i\, \xi_l]_{1\times 3} & \varepsilon_{il}(x)\, n_i\, \xi_l \end{bmatrix}_{4\times 4}, \tag{2.9}$$

$$\mathcal{M}(x,\xi) = \big[\mathcal{M}_{jk}(x,\xi)\big]_{4\times 4} := \begin{bmatrix} [c_{ijlk}(x)\, n_i\, \xi_l]_{3\times 3} & [-e_{lij}(x)\, n_i\, \xi_l]_{3\times 1} \\ [e_{ikl}(x)\, n_i\, \xi_l]_{1\times 3} & \varepsilon_{il}(x)\, n_i\, \xi_l \end{bmatrix}_{4\times 4}. \tag{2.10}$$

Further, let $\Omega = \Omega^+$ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega = S \in C^\infty$, $\overline{\Omega} = \Omega \cup S$. Throughout the paper, $n = (n_1, n_2, n_3)$ denotes the unit normal vector to S directed outward with respect to the domain Ω. Set $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$.

By $H^r(\Omega) = H^r_2(\Omega)$ and $H^r(S) = H^r_2(S)$, $r \in \mathbb{R}$, we denote the Bessel potential spaces on a domain Ω and on a closed manifold S without boundary, while $\mathcal{D}(\mathbb{R}^3)$ and $\mathcal{D}(\Omega)$ denote classes of infinitely differentiable functions in $\mathbb{R}^3$ with a compact support in $\mathbb{R}^3$ and Ω respectively, and $\mathcal{S}(\mathbb{R}^3)$ stands for the Schwartz space of rapidly decreasing functions in $\mathbb{R}^3$. Recall that $H^0(\Omega) = L_2(\Omega)$ is a space of square integrable functions in Ω.

For the vector $U = (u_1, u_2, u_3, u_4)^\top$ the inclusion $U = (u_1, u_2, u_3, u_4)^\top \in H^r$ means that all components u_j, $j = \overline{1,4}$, belong to the space H^r.

Let us denote by $U^+ \equiv \{U\}^+$ and $U^- \equiv \{U\}^-$ the traces of U on S from the interior and exterior of Ω, respectively.

We also need the following subspace of $H^1(\Omega)$:

$$H^{1,0}(\Omega; A) := \Big\{ U = (u_1, u_2, u_3, u_4)^\top \in H^1(\Omega):\ A(x,\partial_x) U \in L_2(\Omega) \Big\}. \tag{2.11}$$

For arbitrary complex-valued vector-functions $U = (u_1, u_2, u_3, u_4)^\top$ and $V = (v_1, v_2, v_3, v_4)^\top$ from the space $H^2(\Omega)$, we have the following Green's formulae [9]:

$$\int\limits_{\Omega} \Big[A(x,\partial_x)\, U \cdot V + E(U,V) \Big]\, dx = \int\limits_{S} \big\{ \mathcal{T} U \big\}^+ \cdot \{V\}^+\, dS, \tag{2.12}$$

$$\int\limits_{\Omega} \Big[A(x,\partial_x)\, U \cdot V - U \cdot A^*(x,\partial_x)\, V \Big]\, dx = \int\limits_{S} \Big[\big\{ \mathcal{T} U \big\}^+ \cdot \{V\}^+ - \{U\}^+ \cdot \big\{ \mathcal{M} V \big\}^+ \Big]\, dS, \tag{2.13}$$

where

$$E(U,V) = c_{ijlk}\, \partial_i u_j\, \overline{\partial_l v_k} + e_{lij}\, (\partial_l u_4\, \overline{\partial_i v_j} - \partial_i u_j\, \overline{\partial_l v_4}) + \varepsilon_{jl}\, \partial_j u_4\, \overline{\partial_l v_4}. \tag{2.14}$$

Note that by means a standard limiting procedure the above Green's formulae can be generalized to Lipschitz domains and to vector–functions $U \in H^1(\Omega)$ and $V \in H^1(\Omega)$ with $A(x,\partial_x)U \in L_2(\Omega)$ and $A^*(x,\partial_x)V \in L_2(\Omega)$. By virtue of Green's formula (2.12), we can determine a *generalized trace vector* $\mathcal{T}^+U \equiv \{\mathcal{T}U\}^+ \in H^{-1/2}(\partial\Omega)$ for a function $U \in H^{1,0}(\Omega;A)$,

$$\langle \mathcal{T}^+U, V^+\rangle_{\partial\Omega} := \int\limits_{\Omega} A(x,\partial_x)\, U\cdot V\,dx + \int\limits_{\Omega} E(U,V)\,dx, \tag{2.15}$$

where $V \in H^1(\Omega)$ is an arbitrary vector-function.

Here, the symbol $\langle\,\cdot\,,\cdot\,\rangle_S$ denotes the duality between the spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ which extends the usual L_2 inner product

$$\langle f,g\rangle_S = \int\limits_S \sum_{j=1}^N f_j\,\overline{g_j}\,dS \quad\text{for}\quad f,g \in L_2(S).$$

Assume that the domain Ω is filled with an anisotropic inhomogeneous piezoelectric material and let us formulate the Robin type boundary value problem:

Find a vector-function $U = (u_1,u_2,u_3,u_4)^\top \in H^{1,0}(\Omega,A)$ satisfying the differential equation

$$A(x,\partial_x)U = f \quad\text{in}\quad \Omega \tag{2.16}$$

and the Robin type boundary condition

$$\mathcal{T}^+U + \beta U^+ = \Psi_0 \quad\text{on}\quad S, \tag{2.17}$$

where $\Psi_0 = (\Psi_{01},\Psi_{02},\Psi_{03},\Psi_{03})^\top \in H^{-1/2}(S)$, $f = (f_1,f_2,f_3,f_4)^\top \in H^0(\Omega)$ and $\beta = [\beta_{jk}]_{4\times4}$ is a positive definite constant matrix.

Equation (2.16) is understood in the distributional sense, while the Robin type boundary condition (2.17) is understood in the functional sense defined in (2.15).

Remark 2.1. From the conditions (2.2) and (2.3) it follows that for complex-valued vector-functions the sesquilinear form $E(U,V)$ defined by (2.14) satisfies the inequality

$$\operatorname{Re} E(U,U) \ge c\,(s_{ij}\,\overline{s}_{ij} + \eta_j\overline{\eta}_j) \ \ \forall\, U = (u_1,u_2,u_3,u_4)^\top \in H^1(\Omega)$$

with $s_{ij} = 2^{-1}(\partial_i u_j(x) + \partial_j u_i(x))$ and $\eta_j = \partial_j u_4(x)$, where c is some positive constant. Therefore, the first Green's formula (2.12) along with the Lax–Milgram lemma imply that the above-formulated Robin type BVP is uniquely solvable in the space $H^{1,0}(\Omega;A)$ (see, e.g., [36], [26], [37]).

As it has already been mentioned, our goal here is to develop the LBDIE method for the Robin type boundary value problem.

To this end, we define a *localized matrix parametrix* associated with the fundamental solution $F_1(x) := -[\,4\,\pi\,|x|\,]^{-1}$ of the Laplace operator

$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$,

$$P(x) \equiv P_\chi(x) := F_\chi(x)\, I$$
$$= \chi(x)\, F_1(x)\, I = -\frac{\chi(x)}{4\,\pi\,|x|}\, I \quad \text{with} \quad \chi(0) = 1, \tag{2.18}$$

where $F_\chi(x) = \chi(x) F_1(x)$, I is the unit 4×4 matrix and χ is a localizing function (see Appendix A),

$$\chi \in X^k_{1+}, \quad k \geq 4. \tag{2.19}$$

Throughout the paper, we assume that the condition (2.19) is satisfied and χ has a compact support if not otherwise stated.

Denote by $B(y,\varepsilon)$ a ball centered at the point y, of radius $\varepsilon > 0$ and let $\Sigma(y,\varepsilon) := \partial B(y,\varepsilon)$.

In Green's second formula (2.13), let us take in the place of $V(x)$ successively the columns of the matrix $P(x-y)$, where y is an arbitrarily fixed interior point in Ω, and write the identity (2.13) for the region $\Omega_\varepsilon := \Omega \setminus B(y,\varepsilon)$ with $\varepsilon > 0$ such that $\overline{B(y,\varepsilon)} \subset \Omega$. Keeping in mind that $P^\top(x-y) = P(x-y)$, we arrive at the equality

$$\int\limits_{\Omega_\varepsilon} \Big[P(x-y)\, A(x,\partial_x) U(x) - \big[A^*(x,\partial_x) P(x-y) \big]^\top U(x) \Big]\, dx$$
$$= \int\limits_S \Big[P(x-y) \big\{ \mathcal{T}(x,\partial_x) U(x) \big\}^+ - \big\{ \mathcal{M}(x,\partial_x) P(x-y) \big\}^\top \{U(x)\}^+ \Big]\, dS$$
$$- \int\limits_{\Sigma(y,\varepsilon)} \Big[P(x-y)\, \mathcal{T}(x,\partial_x) U(x) - \big\{ \mathcal{M}(x,\partial_x) P(x-y) \big\}^\top U(x) \Big]\, d\Sigma(y,\varepsilon). \tag{2.20}$$

The direction of the normal vector on $\Sigma(y,\varepsilon)$ is chosen as outward with respect to $B(y,\varepsilon)$.

It is evident that the operator

$$\mathcal{A}\, U(y) : = \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} \big[A^*(x,\partial_x) P(x-y) \big]^\top U(x)\, dx$$
$$= \text{v.p.} \int\limits_{\Omega} \big[A^*(x,\partial_x) P(x-y) \big]^\top U(x)\, dx \tag{2.21}$$

is a singular integral operator; here and in the sequel, "v.p." denotes the Cauchy principal value integral. If the domain of integration in (2.21) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{A}\, U \equiv \mathbf{A}\, U$, i.e.,

$$\mathbf{A}\, U(y) := \text{v.p.} \int\limits_{\mathbb{R}^3} \big[A^*(x,\partial_x) P(x-y) \big]^\top U(x)\, dx. \tag{2.22}$$

Note that

$$\frac{\partial^2}{\partial x_i\, \partial x_l} \frac{1}{|x-y|} = -\frac{4\,\pi\,\delta_{il}}{3}\, \delta(x-y) + \text{v.p.}\, \frac{\partial^2}{\partial x_i\, \partial x_l} \frac{1}{|x-y|}\,, \tag{2.23}$$

where δ_{il} is the Kronecker delta, while $\delta(\,\cdot\,)$ is the Dirac distribution. The derivatives in the left-hand side of (2.23) are understood in the distributional sense. In view of (2.18) and taking into account that $\chi(0)=1$, we can write the following equality in the distributional sense:

$$\big[A^*(x,\partial_x)P(x-y)\big]^\top$$
$$=\begin{bmatrix} \Big[\dfrac{\partial}{\partial x_i}\Big(c_{ijlk}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big)\Big]_{3\times 3} & \Big[\dfrac{\partial}{\partial x_i}\Big(e_{ikl}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big)\Big]_{3\times 1} \\ \Big[-\dfrac{\partial}{\partial x_i}\Big(e_{lij}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big)\Big]_{1\times 3} & \dfrac{\partial}{\partial x_i}\Big(\varepsilon_{il}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big) \end{bmatrix}_{4\times 4}$$
$$=\begin{bmatrix} \Big[c_{ijlk}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,F_\chi(x-y)\Big]_{3\times 3} & \Big[e_{ikl}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,F_\chi(x-y)\Big]_{3\times 1} \\ \Big[-e_{lij}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,F_\chi(x-y)\Big]_{1\times 3} & \varepsilon_{il}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,F_\chi(x-y) \end{bmatrix}_{4\times 4}$$
$$+\begin{bmatrix} \Big[\dfrac{\partial}{\partial x_i}\,c_{ijlk}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{3\times 3} & \Big[\dfrac{\partial}{\partial x_i}e_{ikl}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{3\times 1} \\ \Big[-\dfrac{\partial}{\partial x_i}\,e_{lij}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{1\times 3} & \dfrac{\partial}{\partial x_i}\varepsilon_{il}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y) \end{bmatrix}_{4\times 4}$$
$$=\begin{bmatrix} \big[c_{ijlk}(x)\,k_{il}(x,y)\big]_{3\times 3} & \big[e_{ikl}(x)\,k_{il}(x,y)\big]_{3\times 1} \\ \big[-e_{lij}(x)\,k_{il}(x,y)\big]_{1\times 3} & \varepsilon_{il}(x)\,k_{il}(x,y) \end{bmatrix}_{4\times 4}$$
$$+\begin{bmatrix} \Big[\dfrac{\partial}{\partial x_i}\,c_{ijlk}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{3\times 3} & \Big[\dfrac{\partial}{\partial x_i}\,e_{ikl}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{3\times 1} \\ \Big[-\dfrac{\partial}{\partial x_i}\,e_{lij}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{1\times 3} & \dfrac{\partial}{\partial x_i}\,\varepsilon_{il}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y) \end{bmatrix}_{4\times 4},$$

where

$$\begin{aligned} k_{il}(x,y) &:= \frac{\delta_{il}}{3}\,\delta(x-y) + \text{v.p.}\,\frac{\partial^2 F_\chi(x-y)}{\partial x_i\partial x_l} \\ &= \frac{\delta_{il}}{3}\,\delta(x-y) - \frac{1}{4\pi}\,\text{v.p.}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|} + m_{il}(x,y), \\ m_{il}(x,y) &:= -\frac{1}{4\pi}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{\chi(x-y)-1}{|x-y|}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} &\big[A^*(x,\partial_x)P(x-y)\big]^\top \\ &\quad= \mathbf{b}(x)\,\delta(x-y) + \text{v.p.}\,\big[A^*(x,\partial_x)P(x-y)\big]^\top \\ &\quad= \mathbf{b}(x)\,\delta(x-y) + R(x,y) \\ &\qquad\quad - \text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix} \big[c_{ijlk}(x)\vartheta_{il}(x,y)\big]_{3\times 3} & \big[e_{ikl}(x)\vartheta_{il}(x,y)\big]_{3\times 1} \\ \big[-e_{lij}(x)\vartheta_{il}(x,y)\big]_{1\times 3} & \varepsilon_{il}(x)\vartheta_{il}(x,y) \end{bmatrix}_{4\times 4} \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{b}(x)\,\delta(x-y) + R^{(1)}(x,y)\\
&\qquad - \text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix} \big[c_{ijlk}(y)\,\vartheta_{il}(x,y)\big]_{3\times 3} & \big[e_{ikl}(y)\vartheta_{il}(x,y)\big]_{3\times 1}\\ \big[-e_{lij}(y)\vartheta_{il}(x,y)\big]_{1\times 3} & \varepsilon_{il}(y)\vartheta_{il}(x,y)\end{bmatrix}_{4\times 4},
\end{aligned}
$$

where

$$
\mathbf{b}(x) = \frac{1}{3}\begin{bmatrix} [c_{ljlk}(x)]_{3\times 3} & [e_{lkl}(x)]_{3\times 1}\\ [-e_{llj}(x)]_{1\times 3} & \varepsilon_{ll}(x)\end{bmatrix}_{4\times 4}, \tag{2.24}
$$

$$
\vartheta_{il}(x,y) = \frac{\partial^2}{\partial x_i \partial x_l}\,\frac{1}{|x-y|}\,, \quad i,l = 1,2,3, \tag{2.25}
$$

$$
\begin{aligned}
R(x,y) &= \begin{bmatrix} \big[c_{ijlk}(x)\,m_{il}(x,y)\big]_{3\times 3} & \big[e_{ikl}(x)\,m_{il}(x,y)\big]_{3\times 1}\\ \big[-e_{lij}(x)\,m_{il}(x,y)\big]_{1\times 3} & \varepsilon_{il}(x)\,m_{il}(x,y)\end{bmatrix}_{4\times 4}\\
&+ \begin{bmatrix} \Big[\dfrac{\partial}{\partial x_i}\,c_{ijlk}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 3} & \Big[\dfrac{\partial}{\partial x_i}\,e_{ikl}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 1}\\ \Big[-\dfrac{\partial}{\partial x_i}\,e_{lij}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{1\times 3} & \dfrac{\partial}{\partial x_i}\,\varepsilon_{il}(x)\,\dfrac{\partial F_\chi(x-y)}{\partial x_l}\end{bmatrix}_{4\times 4},
\end{aligned}
$$

$$
\begin{gathered}
R^{(1)}(x,y) = R(x,y)\\
-\frac{1}{4\pi}\begin{bmatrix} \big[c_{ijlk}(x,y))\,\vartheta_{il}(x,y)\big]_{3\times 3} & \big[e_{lij}(x,y)\,\vartheta_{il}(x,y)\big]_{3\times 1}\\ \big[-e_{ikl}(x,y)\,\vartheta_{il}(x,y)\big]_{1\times 3} & \varepsilon_{il}(x,y)\,\vartheta_{il}(x,y)\end{bmatrix}_{4\times 4},\\
c_{ijlk}(x,y) := c_{ijlk}(x) - c_{ijlk}(y), \quad e_{lij}(x,y) := e_{lij}(x) - e_{ikl}(y),\\
\varepsilon_{il}(x,y) := \varepsilon_{il}(x) - \varepsilon_{il}(y).
\end{gathered}
$$

Evidently, the entries of the matrix-functions $R(x,y)$ and $R^{(1)}(x,y)$ possess weak singularities of type $\mathcal{O}(|x-y|^{-2})$ as $x \to y$. Therefore, we get

$$
\begin{aligned}
&\text{v.p.}\,[A^*(x,\partial_x)P(x-y)]^\top = R(x,y)\\
&+ \text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix} -\big[c_{ijlk}(x)\,\vartheta_{il}(x,y)\big]_{3\times 3} & -\big[e_{lij}(x)\,\vartheta_{il}(x,y)\big]_{3\times 1}\\ \big[e_{ikl}(x)\,\vartheta_{il}(x,y)\big]_{1\times 3} & -\varepsilon_{il}(x)\,\vartheta_{il}(x,y)\end{bmatrix}_{4\times 4},
\end{aligned} \tag{2.26}
$$

$$
\begin{aligned}
&\text{v.p.}\,[A^*(x,\partial_x)P(x-y)]^\top = R^{(1)}(x,y)\\
&+ \text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix} -\big[c_{ijlk}(y)\,\vartheta_{il}(x,y)\big]_{3\times 3} & -\big[e_{lij}(y)\,\vartheta_{il}(x,y)\big]_{3\times 1}\\ \big[e_{ikl}(y)\,\vartheta_{il}(x,y)\big]_{1\times 3} & -\varepsilon_{il}(y)\,\vartheta_{il}(x,y)\end{bmatrix}_{4\times 4}.
\end{aligned} \tag{2.27}
$$

Further, by direct calculations one can easily verify that

$$
\lim_{\varepsilon\to 0}\int\limits_{\Sigma(y,\varepsilon)} P(x-y)\,\mathcal{T}(x,\partial_x)U(x)\,d\Sigma(y,\varepsilon) = 0, \tag{2.28}
$$

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int\limits_{\Sigma(y,\varepsilon)} \big\{\mathcal{M}(x,\partial_x)P(x-y)\big\}^\top U(x)\,d\Sigma(y,\varepsilon)\\
&=\frac{1}{4\pi}\begin{bmatrix} \Big[c_{ijlk}(y)\int\limits_{\Sigma_1}\eta_i\eta_l\,d\Sigma_1\Big]_{3\times 3} & \Big[e_{ikl}(y)\int\limits_{\Sigma_1}\eta_l\eta_i\,d\Sigma_1\Big]_{3\times 1}\\ \Big[-e_{lij}(y)\int\limits_{\Sigma_1}\eta_i\eta_l\,d\Sigma_1\Big]_{1\times 3} & \varepsilon_{il}(y)\int\limits_{\Sigma_1}\eta_i\eta_l\,d\Sigma_1 \end{bmatrix}_{4\times 4} U(y)\\
&=\frac{1}{4\pi}\begin{bmatrix} \Big[c_{ijlk}(y)\,\dfrac{4\,\pi\,\delta_{il}}{3}\Big]_{3\times 3} & \Big[e_{ikl}(y)\,\dfrac{4\,\pi\,\delta_{li}}{3}\Big]_{3\times 1}\\ \Big[-e_{lij}(y)\,\dfrac{4\,\pi\,\delta_{il}}{3}\Big]_{1\times 3} & \varepsilon_{il}(y)\,\dfrac{4\,\pi\,\delta_{il}}{3} \end{bmatrix}_{4\times 4} U(y)\\
&=\mathbf{b}(y)\,U(y), \qquad (2.29)
\end{aligned}
$$

where Σ_1 is a unit sphere, $\eta=(\eta_1,\eta_2,\eta_3)\in\Sigma_1$ and $\mathbf{b}$ is defined by (2.24).

Passing to the limit in (2.20) as $\varepsilon\to 0$ and using the relations (2.21), (2.28) and (2.29), we obtain

$$
\begin{aligned}
\mathbf{b}(y)\,U(y)+\mathcal{A}\,U(y)-V(\mathcal{T}^+U)(y)+W(U^+)(y)&\\
=\mathcal{P}\big(A(x,\partial_x)U\big)(y),\ \ y\in\Omega,& \qquad (2.30)
\end{aligned}
$$

where $\mathcal{A}$ is a *localized singular integral operator* given by (2.21), while V, W and $\mathcal{P}$ are the *localized single layer, double layer and Newtonian volume potentials*,

$$
V(g)(y):=-\int\limits_S P(x-y)\,g(x)\,dS_x, \qquad (2.31)
$$

$$
W(g)(y):=-\int\limits_S \big[\mathcal{M}(x,\partial_x)\,P(x-y)\big]^\top g(x)\,dS_x, \qquad (2.32)
$$

$$
\mathcal{P}(h)(y):=\int\limits_\Omega P(x-y)\,h(x)\,dx. \qquad (2.33)
$$

Let us also introduce the scalar volume potential

$$
\mathbb{P}(\mu)(y):=\int\limits_\Omega F_\chi(x-y)\,\mu(x)\,dx \qquad (2.34)
$$

with μ being a scalar density function.

If the domain of integration in the Newtonian volume potential (2.33) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{P}\,h\equiv\mathbf{P}\,h$, i.e.,

$$
\mathbf{P}(h)(y):=\int\limits_{\mathbb{R}^3} P(x-y)\,h(x)\,dx. \qquad (2.35)
$$

Mapping properties of the above potentials are investigated in [16].

We refer the relation (2.30) as *Green's third formula*. By a standard limiting procedure we can extend Green's third formula (2.30) to the functions from the space $H^{1,0}(\Omega, A)$. In particular, it holds true for solutions of the above formulated Robin type BVP. In this case, the generalized trace vector $\mathcal{T}^+U$ is understood in the sense of definition (2.15).

For $U = (u_1, \dots, u_4)^\top \in H^1(\Omega)$, one can also derive the following relation:

$$\mathcal{A}\,U(y) = -\mathbf{b}(y)\,U(y) - W(U^+)(y) + \mathcal{Q}\,U(y), \;\; \forall\, y \in \Omega, \tag{2.36}$$

where

$$\mathcal{Q}\,U(y) := \begin{bmatrix} \Big[\dfrac{\partial}{\partial y_i}\,\mathbb{P}(c_{ijlk}\,\partial_l u_k)(y) + \dfrac{\partial}{\partial y_i}\,\mathbb{P}(e_{lij}\partial_l u_4)(y)\Big]_{3\times 1} \\ -\dfrac{\partial}{\partial y_i}\,\mathbb{P}(e_{ikl}\partial_l u_k)(y) + \dfrac{\partial}{\partial y_i}\,\mathbb{P}(\varepsilon_{il}\partial_l u_4)(y) \end{bmatrix}_{4\times 4} . \tag{2.37}$$

and $\mathbb{P}$ is defined in (2.34).

In what follows, for our analysis we need the explicit expression of the principal homogeneous symbol matrix $\mathfrak{S}\,(\mathcal{A})(y,\xi)$ of the singular integral operator $\mathcal{A}$. This matrix coincides with the Fourier transform of the singular matrix kernel defined by (2.26). Let $\mathcal{F}$ denote the Fourier transform operator,

$$\mathcal{F}_{z\to\xi}[g] = \int\limits_{\mathbb{R}^3} g(z)\, e^{i\,z\cdot\xi}\, dz,$$

and set

$$\begin{aligned} h_{il}(z) &: = \text{v.p.}\vartheta_{il}(x,t) = \text{v.p.}\,\frac{\partial^2}{\partial z_i \partial z_l}\,\frac{1}{|z|}\,, \\ \widehat{h}_{il}(\xi) &: = \mathcal{F}_{z\to\xi}(h_{il}(z)), \;\; i,l = 1,2,3. \end{aligned}$$

In view of (2.23) and taking into account the relations $\mathcal{F}_{z\to\xi}\delta(z) = 1$ and $\mathcal{F}_{z\to\xi}(|z|^{-1}) = 4\pi|\xi|^{-2}$ (see, e.g., [25]), we easily derive

$$\begin{aligned} \widehat{h}_{il}(\xi) := \mathcal{F}_{z\to\xi}(h_{il}(z)) &= \mathcal{F}_{z\to\xi}\Big(\frac{4\pi\delta_{il}}{3}\delta(z) + \frac{\partial^2}{\partial z_i\,\partial z_l}\,\frac{1}{|z|}\Big) \\ &= \frac{4\pi\delta_{il}}{3} + (-i\xi_i)(-i\xi_l)\mathcal{F}_{z\to\xi}\Big(\frac{1}{|z|}\Big) = \frac{4\pi\delta_{il}}{3} - \frac{4\pi\xi_i\xi_l}{|\xi|^2}\,. \end{aligned}$$

Now, for arbitrary $y \in \overline{\Omega}$ and $\xi \in \mathbb{R}^3 \setminus \{0\}$, due to (2.27), we get

$$\begin{aligned} \mathfrak{S}\,(\mathcal{A})(y,\xi) &= -\frac{1}{4\pi}\,\mathcal{F}_{z\to\xi} \begin{bmatrix} \big[c_{ijlk}(y)\,h_{il}(z)\big]_{3\times 3} & \big[e_{ikl}(y)\,h_{il}(z)\big]_{3\times 1} \\ \big[-e_{lij}(y)\,h_{il}(z)\big]_{1\times 3} & \varepsilon_{il}(y)\,h_{il}(z) \end{bmatrix}_{4\times 4} \\ &= -\frac{1}{4\pi} \begin{bmatrix} \big[c_{ijlk}(y)\,\widehat{h}_{il}(z)\big]_{3\times 3} & \big[e_{ikl}(y)\,\widehat{h}_{il}(z)\big]_{3\times 1} \\ \big[-e_{lij}(y)\,\widehat{h}_{il}(z)\big]_{1\times 3} & \varepsilon_{il}(y)\,\widehat{h}_{il}(z) \end{bmatrix}_{4\times 4} \end{aligned}$$

$$= -\mathbf{b}(y) + \frac{1}{|\xi|^2} \begin{bmatrix} \big[c_{ijlk}(y)\,\xi_i\xi_l\big]_{3\times 3} & \big[e_{lij}(y)\,\xi_l\xi_i\big]_{3\times 1} \\ \big[-e_{ikl}(y)\,\xi_i\xi_l\big]_{1\times 3} & \varepsilon_{il}(y)\,\xi_i\xi_l \end{bmatrix}_{4\times 4}$$
$$= \frac{1}{|\xi|^2}\,A(y,\xi) - \mathbf{b}(y). \tag{2.38}$$

As we can see, the entries of the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{A})(y,\xi)$ of the operator $\mathcal{A}$ are even rational homogeneous functions in ξ of order 0. It can easily be verified that both the characteristic function of the singular kernel in (2.27) and the Fourier transform (2.38) satisfy the Tricomi condition, i.e., their integral averages over the unit sphere vanish (cf. [40]).

Denote by ℓ_0 the extension operator by zero from $\Omega = \Omega^+$ onto $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$. It is evident that for the function $U \in H^1(\Omega)$ we have

$$(\mathcal{A}\,U)(y) = (\mathbf{A}\ell_0 U)(y) \ \text{ for } \ y \in \Omega.$$

Introduce the notation

$$(\mathbf{K}\ell_0 U)(y) := (\mathbf{b}(y) - \mathbf{I})\,U(y) + (\mathbf{A}\ell_0 U)(y) \ \text{ for } \ y \in \Omega, \tag{2.39}$$

and for our further purposes we rewrite the third Green's formula (2.30) in a more convenient form

$$[\mathbf{I} + \mathbf{K}]\ell_0 U\,(y) - V(\mathcal{T}^+U)(y) + W(U^+)(y)$$
$$= \mathcal{P}(A(x,\partial_x)U)(y), \ \ y \in \Omega, \tag{2.40}$$

where $\mathbf{I}$ is the identity operator.

The relation (2.38) implies that the principal homogeneous symbols of the singular integral operators $\mathbf{K}$ and $\mathbf{I} + \mathbf{K}$ read as

$$\mathfrak{S}(\mathbf{K})(y,\xi) = |\xi|^{-2}A(y,\xi) - I \ \ \forall\, y \in \overline{\Omega}, \ \ \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}, \tag{2.41}$$
$$\mathfrak{S}(\mathbf{I} + \mathbf{K})(y,\xi) = |\xi|^{-2}A(y,\xi) \ \ \forall\, y \in \overline{\Omega}, \ \ \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{2.42}$$

It is evident that the symbol matrix (2.42) is uniformly strongly elliptic due to (2.4)

$$\operatorname{Re}\big(\mathfrak{S}(\mathbf{I} + \mathbf{K})(y,\xi)\,\zeta,\zeta\big) = |\xi|^{-2}\operatorname{Re}\big(A(y,\xi)\,\zeta,\zeta\big) \geq c\,|\zeta|^2 \tag{2.43}$$
$$\forall\, y \in \overline{\Omega}, \ \ \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}, \ \ \forall\, \zeta \in \mathbb{C}^3,$$

where c is the same positive constant as in (2.4).

From (2.39) it follows that (see, e.g., [3], [26, Theorem 8.6.1]) if $\chi \in X^k$ with integer $k \geqslant r + 2$, then

$$r_{\Omega}\mathbf{K}\,\ell_0 : H^r(\Omega) \longrightarrow H^r(\Omega), \ \ r \geqslant 0, \tag{2.44}$$

since the symbol (2.41) is rational and the operator with the kernel function either $R(x,y)$ or $R^{(1)}(x,y)$ maps $H^r(\Omega)$ into $H^{r+1}(\Omega)$ (cf. [16, Theorem 5.6]). Here and throughout the paper, r_{Ω} denotes the restriction operator to Ω.

Assuming that $U \in H^2(\Omega)$ and applying the differential operator $\mathcal{T}(x, \partial_x)$ to Green's formula (2.40) and using the properties of localized potentials described in Appendix B (see Theorems B.1–B.4) we arrive at the relation:

$$\mathcal{T}^+\mathbf{K}\ell_0 U + (\mathbf{I} - \mathbf{d})(\mathcal{T}^+U) - \mathcal{W}'(\mathcal{T}^+U) + \mathcal{L}(U^+) \\ = \mathcal{T}^+\mathcal{P}(A(x, \partial_x)U) \;\text{ on }\; S, \qquad (2.45)$$

where the localized boundary integral operators $\mathcal{W}'$ and $\mathcal{L} := \mathcal{L}^+$ are generated by the localized single- and double-layer potentials and are defined in (B.3) and (B.4), the matrix $\mathbf{d}$ is defined by (B.17), while

$$\mathcal{T}^+\mathbf{K}\ell_0 U \equiv \big\{\mathcal{T}(\mathbf{K}\,\ell_0 U)\big\}^+ \;\text{ on }\; S, \qquad (2.46)$$

$$\mathcal{T}^+\mathcal{P}(A(x, \partial_x)U) \equiv \big\{\mathcal{T}\mathcal{P}(A(x, \partial_x)U)\big\}^+ \;\text{ on }\; S. \qquad (2.47)$$

2.2. **LBDIE formulation of the Robin type problem and the equivalence theorem.** Let $U \in H^2(\Omega)$ be a solution to the Robin type BVP (2.16), (2.17) with $\psi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega)$. As we have derived above, there hold the relations (2.40) and (2.45), which now can be rewritten in the form

$$[\mathbf{I} + \mathbf{K}]\ell_0 U + W(\Phi) + V(\beta\Phi) = \mathcal{P}(f) + V(\Psi_0) \;\text{ in }\; \Omega, \qquad (2.48)$$

$$\mathcal{T}^+\mathbf{K}\ell_0 U + \mathcal{L}(\Phi) + (\mathbf{d} - \mathbf{I})\,\beta\,\Phi + \mathcal{W}'\beta\Phi \\ = \mathcal{T}^+\mathcal{P}(f) + (\mathbf{d} - \mathbf{I})\Psi_0 + \mathcal{W}'(\Psi_0) \;\text{ on }\; S, \qquad (2.49)$$

where $\Phi := U^+ \in H^{\frac{3}{2}}(S)$.

One can consider these relations as a LBDIE system with respect to the unknown vector-functions U and Φ. Now we prove the following equivalence theorem.

Theorem 2.2. *Let $\chi \in X^4_{1+}$. The Robin type boundary value problem* (2.16), (2.17) *is equivalent to LBDIE system* (2.48), (2.49) *in the following sense:*

(i) *If a vector-function $U \in H^2(\Omega)$ solves the Robin type BVP* (2.16), (2.17)*, then it is unique and the pair $(U, \Phi) \in H^2(\Omega) \times H^{\frac{3}{2}}(S)$ with*

$$\Phi = U^+\,, \qquad (2.50)$$

solves the LBDIE system (2.48), (2.49) *and, vice versa;*

(ii) *If a pair $(U, \Phi) \in H^2(\Omega) \times H^{\frac{3}{2}}(S)$ solves the LBDIE system* (2.48), (2.49)*, then it is unique and the vector-function U solves the Robin type BVP* (2.16), (2.17)*, and relation* (2.50) *holds.*

Proof. (i) The first part of the theorem is trivial and directly follows form the relations (2.40), (2.45), (2.50) and Remark 2.1.

(ii) Now, let a pair $(U, \Phi) \in H^2(\Omega) \times H^{\frac{3}{2}}(S)$ solve the LBDIE system (2.48), (2.49). We apply the differential operator $\mathcal{T}$ to equation (2.48), take its trace on S and compare with (2.49) to obtain

$$\mathcal{T}^+U + \beta\Phi = \Psi_0 \;\text{ on }\; S. \qquad (2.51)$$

Further, since $U \in H^2(\Omega)$, we can write the third Green's formula (2.40) which in view of (2.51) can be rewritten as

$$[\mathbf{I}+\mathbf{K}]\,\ell_0 U + V(\beta\Phi) - V(\Psi_{_0}) + W(U^+) = \mathcal{P}(A(x,\partial_x)U) \ \text{ in } \ \Omega. \tag{2.52}$$

From (2.48) and (2.52) it follows that

$$W(U^+ - \Phi) - \mathcal{P}\big(A(x,\partial_x)U - f\big) = 0 \ \text{ in } \ \Omega, \tag{2.53}$$

whence by Lemma 6.4 in [16] we conclude

$$A(x,\partial_x)U = f \ \text{ in } \ \Omega \quad \text{and} \quad U^+ = \Phi \ \text{ on } \ S.$$

Therefore, from (2.51) we get

$$\mathcal{T}^+ U + \beta U^+ = \Psi_{_0} \ \text{ on } \ S. \tag{2.54}$$

Thus U solves the Robin type BVP (2.16), (2.17) and, in addition, equation (2.50) holds.

The uniqueness of a solution to the LBDIE system (2.48), (2.49) in the class $H^2(\Omega) \times H^{\frac{3}{2}}(S)$ directly follows from the above-proven equivalence result and the uniqueness theorem for the Robin type problem (2.16), (2.17) (see Remark 2.1). $\square$

3. Invertibility of the LBDIO Corresponding to the Robin Type BVP

From Theorem 2.2 it follows that the LBDIE system (2.48), (2.49) with a special right-hand side is uniquely solvable in the class $H^2(\Omega, A) \times H^{3/2}(S)$. Here, our main goal is to investigate Fredholm properties of the localized boundary-domain integral operator generated by the left-hand side expressions in (2.48), (2.49) in appropriate functional spaces.

To this end, let us consider the LBDIE system for the unknown pair $(U,\Phi) \in H^2(\Omega) \times H^{3/2}(S)$,

$$(\mathbf{I}+\mathbf{K})\ell_0 U \ + W(\Phi) + V(\beta\Phi) = F_1 \ \text{ in } \ \Omega, \tag{3.1}$$

$$\mathcal{T}^+\mathbf{K}\ell_0 U + \mathcal{L}(\Phi) + (\mathbf{d}-\mathbf{I})\beta\Phi + \mathcal{W}'(\beta\Phi) = F_2 \ \text{ on } \ S, \tag{3.2}$$

where $F_1 \in H^2(\Omega)$ and $F_2 \in H^{1/2}(S)$.

Introduce the notation

$$\mathbf{B} := \mathbf{I} + \mathbf{K}. \tag{3.3}$$

In view of (2.42), the principal homogeneous symbol matrix of the operator $\mathbf{B}$ reads as

$$\mathfrak{S}(\mathbf{B})(y,\xi) = |\xi|^{-2} A(y,\xi) \ \text{ for } \ y \in \overline{\Omega}, \ \ \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{3.4}$$

The entries of the matrix $\mathfrak{S}(\mathbf{B})(y,\xi)$ are even rational homogeneous functions of order 0 in ξ. Moreover, due to (2.4), the matrix $\mathfrak{S}(\mathbf{B})(y,\xi)$ is uniformly strongly elliptic,

$$\operatorname{Re}\big(\mathfrak{S}(\mathbf{B})(y,\xi)\zeta,\zeta\big) \geq c\,|\zeta|^2 \ \text{ for all } \ y \in \overline{\Omega}, \ \ \xi \in \mathbb{R}^3 \setminus \{0\} \ \text{ and } \ \zeta \in \mathbb{C}^3.$$

Consequently, $\mathbf{B}$ is a uniformly strongly elliptic pseudodifferential operator of zero order (i.e., a singular integral operator) and the partial indices of factorization of the symbol (3.4) are equal to zero (cf. Lemma 1.20 in [12]).

Now we present some auxiliary material needed for our further analysis. Let $\widetilde{y} \in \partial\Omega$ be some fixed point and consider the frozen symbol $\mathfrak{S}(\mathbf{B})(\widetilde{y},\xi) \equiv \mathfrak{S}(\widetilde{\mathbf{B}})(\xi)$, where $\widetilde{\mathbf{B}}$ denotes the operator $\mathbf{B}$ written in a chosen local coordinate system. Further, let $\widehat{\widetilde{\mathbf{B}}}$ denote the pseudodifferential operator with the symbol

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})(\xi',\xi_3) := \mathfrak{S}(\widetilde{\mathbf{B}})\big((1+|\xi'|)\omega,\xi_3\big)$$
$$\text{with } \omega = \frac{\xi'}{|\xi'|}\,, \ \ \xi=(\xi',\xi_3), \ \ \xi'=(\xi_1,\xi_2).$$

The principal homogeneous symbol matrix $\mathfrak{S}(\widetilde{\mathbf{B}})(\xi)$ of the operator $\widehat{\widetilde{\mathbf{B}}}$ can be factorized with respect to the variable ξ_3,

$$\mathfrak{S}(\widetilde{\mathbf{B}})(\xi) = \mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi)\,\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})(\xi), \tag{3.5}$$

where

$$\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi) = \frac{1}{\xi_3 \pm i\,|\xi'|}\,\widetilde{A}^{(\pm)}(\xi',\xi_3),$$

$\widetilde{A}^{(\pm)}(\xi',\xi_3)$ are the "plus" and "minus" polynomial matrix factors of the first order in ξ_3 of the positive definite polynomial symbol matrix $\widetilde{A}(\xi',\xi_3) \equiv \widetilde{A}(\widetilde{y},\xi',\xi_3)$ (see Theorem 1 in [23], Theorem 1.33 in [45], Theorem 1.4 in [24]), i.e.

$$\widetilde{A}(\xi^{'},\xi_3) = \widetilde{A}^{(-)}(\xi',\xi_3)\,\widetilde{A}^{(+)}(\xi',\xi_3) \tag{3.6}$$

with $\det\widetilde{A}^{(+)}(\xi',\tau) \neq 0$ for $\operatorname{Im}\tau > 0$ and $\det\widetilde{A}^{(-)}(\xi',\tau) \neq 0$ for $\operatorname{Im}\tau < 0$. Moreover, the entries of the matrices $\widetilde{A}^{(\pm)}(\xi',\xi_3)$ are homogeneous functions in $\xi=(\xi',\xi_3)$ of order 1.

Denote by $a^{(\pm)}(\xi')$ the coefficients of ξ_3^4 in the determinants $\det\widetilde{A}^{(\pm)}(\xi',\xi_3)$. Evidently,

$$a^{(-)}(\xi')\,a^{(+)}(\xi') = \det\widetilde{A}(0,0,1) > 0 \ \text{ for } \ \xi' \neq 0. \tag{3.7}$$

It is easy to see that the inverse factor-matrices $[\widetilde{A}^{(\pm)}(\xi',\xi_3)]^{-1}$ have the following structure:

$$\big[\widetilde{A}^{(\pm)}(\xi',\xi_3)\big]^{-1} = \frac{1}{\det\widetilde{A}^{(\pm)}(\xi',\xi_3)}\,\big[p_{ij}^{(\pm)}(\xi',\xi_3)\big]_{4\times4}, \tag{3.8}$$

where $[p_{ij}^{(\pm)}(\xi',\xi_3)]_{4\times4}$ is the matrix of co-factors corresponding to the matrix $\widetilde{A}^{(\pm)}(\xi',\xi_3)$. They can be written in the form

$$p_{ij}^{(\pm)}(\xi',\xi_3) = c_{ij}^{(\pm)}(\xi')\,\xi_3^3 + b_{ij}^{(\pm)}(\xi')\,\xi_3^2 + d_{ij}^{(\pm)}(\xi')\,\xi_3 + e_{ij}^{(\pm)}(\xi') \tag{3.9}$$

with $c_{ij}^{(\pm)}$, $b_{ij}^{(\pm)}$, $d_{ij}^{(\pm)}$, and $e_{ij}^{(\pm)}$, $i,j=1,2,3,4$, being homogeneous functions in ξ' of order 0, 1, 2 and 3, respectively.

Denote by Π^+ the Cauchy type integral operator

$$\Pi^+(f)(\xi) = \frac{i}{2\pi} \lim_{t\to 0+} \int_{-\infty}^{+\infty} \frac{h(\xi', \eta_3)\, d\eta_3}{\xi_3 + i\,t - \eta_3}, \quad \xi = (\xi', \xi_3), \quad \xi' = (\xi_1, \xi_2), \tag{3.10}$$

which is well defined for any $\xi \in \mathbb{R}^3$ for a bounded smooth function $h(\xi', \,\cdot\,)$ satisfying the relation $h(\xi', \eta_3) = O(1 + |\eta_3|)^{-\nu}$ with some $\nu > 0$.

The following lemma holds (see [22]).

Lemma 3.1. *Let $\chi \in X^k_{1+}$ with integer $k \geqslant s+2$ and let ℓ_0 be the extension operator by zero from $\mathbb{R}^3_+$ onto the half-space $\mathbb{R}^3_-$. The operator*

$$r_{_{\mathbb{R}^3_+}} \widehat{\widetilde{\mathbf{B}}} \ell_0 : H^s(\mathbb{R}^3_+) \longrightarrow H^s(\mathbb{R}^3_+)$$

is invertible for all $s \geq 0$, where $r_{_{\mathbb{R}^3_+}}$ is the restriction operator to the half-space $\mathbb{R}^3_+$.

Moreover, for $f \in H^s(\mathbb{R}^3_+)$ with $s \geq 0$, the unique solution of the equation

$$r_{_{\mathbb{R}^3_+}} \widehat{\widetilde{\mathbf{B}}} \ell_0 U = f, \tag{3.11}$$

can be represented in the form

$$U_+ := \ell_0 u = \mathcal{F}^{-1} \Big\{ \big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^+ \Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1} \mathcal{F}(\ell f) \Big) \Big\}, \tag{3.12}$$

where $\ell f \in H^s(\mathbb{R}^3)$ is an arbitrary extension of f onto the whole space $\mathbb{R}^3$.

Lemma 3.2. *Let the factor matrix $\widetilde{A}^{(+)}(\xi', \tau)$ be as in* (3.6)*, and let $a^{(+)}$ and $c^{(+)}_{ij}$ be as in* (3.7) *and* (3.9)*, respectively. Then the following equality holds*

$$\frac{1}{2\pi i} \int_{\gamma^-} \big[\widetilde{A}^{(+)}(\xi', \tau)\big]^{-1} d\tau = \frac{1}{a^{(+)}(\xi')} \big[c^{(+)}_{ij}(\xi')\big]_{4\times 4}, \tag{3.13}$$

and

$$\det \big[c^{(+)}_{ij}(\xi')\big]_{4\times 4} \neq 0 \ \ \textit{for} \ \ \xi' \neq 0. \tag{3.14}$$

Here γ^- is a contour in the lower complex half-plane enclosing all roots of the polynomial $\det \widetilde{A}^{(+)}(\xi', \tau)$ *with respect to τ.*

It is well known that the differential operator $\mathcal{T}(x, \partial_x)$ covers the operator $A(x, \partial_x)$ on the boundary S (see, e.g., [1], [11], [41], [48]), i.e., the problem

$$\widetilde{A}\Big(\xi', i\,\frac{d}{dt}\Big)\, v(\xi', t) = 0, \quad t \in \mathbb{R}_+ = (0, +\infty), \tag{3.15}$$

$$\widetilde{\mathcal{T}}\Big(\xi', i\,\frac{d}{dt}\Big)\, v(\xi', t)\Big|_{t=0} = 0 \tag{3.16}$$

has only the trivial solution in the Schwartz space $\mathcal{S}(\mathbb{R}_+)$ of infinitely smooth, rapidly decreasing vector-functions at infinity. Here, $\widetilde{A}(\xi', \xi_3) := A(\widetilde{y}, \xi', \xi_3)$ and $\widetilde{\mathcal{T}}(\xi', \xi_3) := \mathcal{T}(\widetilde{y}, \xi', \xi_3)$ correspond, respectively, to the "frozen" differential and co-normal operators at the point $\widetilde{y} \in \partial\Omega$.

The above covering condition implies the following assertion.

Lemma 3.3. *Let* γ^- *be as in Lemma* 3.2. *The matrix*

$$\int\limits_{\gamma^-} \widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau \tag{3.17}$$

is non-singular for all $\xi' \neq 0$.

Proof. Let us consider the following matrix:

$$\int\limits_{\gamma^-} e^{-i\tau t}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau, \quad 0<t<\infty, \tag{3.18}$$

and denote by $v^{(1)}(\xi',t)$, $v^{(2)}(\xi',t)$, $v^{(3)}(\xi',t)$, and $v^{(4)}(\xi',t)$, the columns of the matrix (3.18).

Clearly, $v^{(k)}(\xi',\,\cdot\,)\in\mathcal{S}(\mathbb{R}_+)$, $k=1,2,3,4$.

First we show that $v^{(k)}(\xi',\,\cdot\,)$, $k=\overline{1,4}$, are linearly independent solutions of equation (3.15). Indeed, by direct differentiation it can be easily seen that the vector-functions $v^{(k)}(\xi',t)$, $k=\overline{1,4}$, solve the equation

$$\widetilde{A}^{(+)}\Big(\xi',i\,\frac{d}{dt}\Big)v(\xi',t)=0, \quad 0<t<\infty. \tag{3.19}$$

In view of the decomposition

$$\widetilde{A}\Big(\xi',i\,\frac{d}{dt}\Big)=\widetilde{A}^{(-)}\Big(\xi',i\,\frac{d}{dt}\Big)\widetilde{A}^{(+)}\Big(\xi',i\,\frac{d}{dt}\Big), \tag{3.20}$$

it follows that $v^{(k)}(\xi',t)$, $k=\overline{1,4}$, are solutions of equation (3.15).

Now let us show that the vector-functions $v^{(k)}(\xi',\,\cdot\,)$, $k=\overline{1,4}$, are linearly independent. Assume that for some scalar constants α_k, $k=\overline{1,4}$, the equality

$$\alpha_1\,v^{(1)}(\xi',t)+\alpha_2\,v^{(2)}(\xi',t)+\alpha_3\,v^{(3)}(\xi',t)+\alpha_4\,v^{(4)}(\xi',t)=0 \tag{3.21}$$

holds. Note that the matrix-function (3.18) is continuous at $t=0$. Therefore from (3.21) by passing to the limit, as $t\to 0$, we obtain the following linear algebraic system of equations with respect to $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^\top$,

$$\Big(\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)\alpha=0. \tag{3.22}$$

Due to Lemma 3.2,

$$\det\Big(\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)\neq 0 \ \text{ for all } \ \xi'\neq 0,$$

and consequently $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^\top=0$, implying that $v^{(k)}(\xi',\,\cdot\,)$, $k=\overline{1,4}$, are linearly independent solutions of equation (3.15).

Further, let us consider an arbitrary solution of equation (3.15) belonging to the class $\mathcal{S}(\mathbb{R}_+)$,

$$v(\xi',t)=\sum_{k=1}^{4}a_k\,v^{(k)}(\xi',t), \tag{3.23}$$

where a_1, a_2, a_3, a_4 are the scalar constants. If (3.23) satisfies in addition the condition (3.16), then due to the covering condition it should be identical zero. Substituting (3.23) into (3.16), we arrive at the following system of linear algebraic equations with respect to $a=(a_1,a_2,a_3,a_4)^\top$:

$$\bigg(\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\bigg)a=0. \tag{3.24}$$

Since this system should possess only the trivial solution, we conclude that

$$\det\bigg(\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\bigg)\neq 0 \ \text{ for all } \ \xi'\neq 0,$$

which completes the proof. □

Now, with the above auxiliary results in hand, we can investigate the invertibility of the localized boundary-domain integral operator generated by the left-hand side expressions in the system (3.1), (3.2). We denote this operator by $\mathfrak{R}$,

$$\mathfrak{R}:=\begin{bmatrix} r_{\Omega}\mathbf{B}\ell_0 & -r_{\Omega}W+r_{\Omega}V\beta\\ \mathcal{T}^{+}\mathbf{K}\ell_0 & \mathcal{L}+(\mathbf{d}-\mathbf{I})+\mathcal{W}'\beta\end{bmatrix}_{8\times 8}.$$

Let us introduce the following boundary operators depending on the parameter $t\in[0,1]$,

$$\begin{aligned}\mathcal{T}_t=\mathcal{T}_t(x,\partial_x)&:=(1-t)I\partial_n+t\mathcal{T}(x,\partial_x),\\ \mathcal{M}_t=\mathcal{M}_t(x,\partial_x)&:=(1-t)I\partial_n+t\mathcal{M}(x,\partial_x).\end{aligned} \tag{3.25}$$

Now we can prove the following assertion.

Theorem 3.4. *Let a localizing function $\chi\in X^{\infty}_{1+}$, $r\geq 1$, and the conditions*

$$\det\widetilde{\mathcal{T}}_t(\xi',-i\,|\xi'|)\neq 0,\quad \det\widetilde{\mathcal{M}}_t(\,\xi',-i\,|\xi'|)\neq 0, \tag{3.26}$$

be satisfied for all $\xi'\neq 0$ and for all $t\in(0,1]$, where the matrices $\widetilde{\mathcal{T}}_t(\xi',\xi_3)$ and $\widetilde{\mathcal{M}}_t(\xi',\xi_3)$ are defined as follows:

$$\begin{aligned}\widetilde{\mathcal{T}}_t(\xi',\xi_3)&:=(1-t)\xi_3 I+t\widetilde{\mathcal{T}}(\xi',\xi_3),\\ \widetilde{\mathcal{M}}_t(\xi',\xi_3)&:=(1-t)\xi_3 I+t\widetilde{\mathcal{M}}(\xi',\xi_3).\end{aligned} \tag{3.27}$$

Then the operator

$$\mathfrak{R}:H^{r+1}(\Omega)\times H^{r+1/2}(S)\longrightarrow H^{r+1}(\Omega)\times H^{r-1/2}(S) \tag{3.28}$$

is invertible.

Proof. We prove the theorem in four steps, where we show that

Step 1: the operator $r_{\Omega}\mathbf{B}\ell_0 : H^s(\Omega) \to H^s(\Omega)$ for $s \geq 0$ is Fredholm with zero index;
Step 2: the operator $\mathfrak{R}$ in (3.28) is Fredholm;
Step 3: $\operatorname{Ind}\mathfrak{R} = 0$;
Step 4: the operator $\mathfrak{R}$ is invertible.

Step 1. Since (3.4) is a rational function in ξ, we can apply the theory of pseudodifferential operators with the symbol satisfying the transmission conditions (see [25], [3], [44], [45], [4]). With the help of the local principal (see [2] and Lemma 23.9 in [25]) and the above Lemma 3.1 we can deduce that the operator

$$\mathcal{B} := r_{\Omega}\mathbf{B}\,\ell_0 : H^s(\Omega) \longrightarrow H^s(\Omega)$$

is Fredholm for all $s \geq 0$.

To show that $\operatorname{Ind}\mathcal{B} = 0$, we use the fact that the operators $\mathcal{B}$ and $\mathcal{B}_t = r_{\Omega}(\mathbf{I} + t\,\mathbf{K})\ell_0$, where $t \in [0,1]$, are homotopic. Note that $\mathcal{B} = \mathcal{B}_1$. The principal homogeneous symbol of the operator $\mathcal{B}_t$ has the form

$$\mathfrak{S}(\mathcal{B}_t)(y,\xi) = I + t\,\mathfrak{S}(\mathbf{K})(y,\xi) = (1-t)I + t\mathfrak{S}(\mathbf{B})(y,\xi).$$

It is easy to see that the operator $\mathcal{B}_t$ is uniformly strongly elliptic,

$$\operatorname{Re}\big(\mathfrak{S}(\mathcal{B}_t)(y,\xi)\zeta,\zeta\big) = (1-t)|\zeta|^2 + t\operatorname{Re}\big(\mathfrak{S}(\mathbf{B})(y,\xi)\zeta,\zeta\big) \geq c_1\,|\zeta|^2$$

for all $y \in \overline{\Omega}$, $\xi \neq 0$, $\zeta \in \mathbb{C}^3$, and $t \in [0,1]$, $c_1 = \min\{1, c\}$, where c is the constant involved in (2.4).

Since $\mathfrak{S}(\mathcal{B}_t)(y,\xi)$ is rational, even and homogeneous of order zero in ξ, as above, we again conclude that the operator

$$\mathcal{B}_t : H^s(\Omega) \longrightarrow H^s(\Omega)$$

is Fredholm for all $s \geq 0$ and for all $t \in [0,1]$. Therefore $\operatorname{Ind}\mathcal{B}_t$ is the same for all $t \in [0,1]$. On the other hand, due to the equality $\mathcal{B}_0 = r_{\Omega}\,I$, we get

$$\operatorname{Ind}\mathcal{B} = \operatorname{Ind}\mathcal{B}_1 = \operatorname{Ind}\mathcal{B}_t = \operatorname{Ind}\mathcal{B}_0 = 0.$$

Step 2. To investigate Fredholm properties of the operator $\mathfrak{R}$ we apply the local principle (cf. e.g., [25], § 19 and § 22). Due to this principle, we have to show that the operator $\mathfrak{R}$ is locally Fredholm at an arbitrary "frozen" interior point $\widetilde{y} \in S$, and secondly that the so-called generalized *Šapiro–Lopatinskiĭ condition* for the operator $\mathfrak{R}$ holds at an arbitrary "frozen" point $\widetilde{y} \in S$. To obtain the explicit form of this condition we proceed as follows. Let $\mathcal{U}$ be a neighborhood of a fixed point $\widetilde{y} \in \overline{\Omega}$ and let $\widetilde{\psi}_0, \widetilde{\varphi}_0 \in \mathcal{D}(\mathcal{U})$ be infinitely differentiable scalar functions such that

$$\operatorname{supp}\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0 \neq \varnothing, \quad \widetilde{y} \in \operatorname{supp}\,\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0,$$

and consider the operator $\widetilde{\psi}_0\mathfrak{R}\,\widetilde{\varphi}_0$. We consider separately two possible cases: $\widetilde{y} \in \Omega$ and $\widetilde{y} \in S$.

Case 1). Let $\widetilde{y} \in \Omega$. Then we can choose a neighborhood $\mathcal{U}_j$ of the point $\widetilde{y}$ such that $\overline{\mathcal{U}} \subset \Omega$. Therefore the operator $\widetilde{\psi}_0\mathfrak{R}\,\widetilde{\varphi}_0$ has the same Fredholm properties as the operator $\widetilde{\psi}_0\mathbf{B}\,\widetilde{\varphi}_0$ (see the similar arguments in the proof of Theorem 22.1 in [25]). Then owing to Step 1, we conclude that $\widetilde{\psi}_0\mathfrak{R}\,\widetilde{\varphi}_0$ is the locally Fredholm operator at interior points of Ω.

Case 2). Now let $\widetilde{y}\in S$. Then at this point we have to "froze" the operator $\widetilde{\psi}_0\,\mathfrak{R}\,\widetilde{\varphi}_0$, which means that we can choose a neighborhood $\mathcal{U}$ of the point $\widetilde{y}$ sufficiently small such that at the local co-ordinate system with the origin at the point $\widetilde{y}$ and the third axis coinciding with the normal vector at the point $\widetilde{y}\in S$, the following decomposition

$$\widetilde{\psi}_0\mathfrak{R}\,\widetilde{\varphi}_0=\widetilde{\psi}_0\Big(\widehat{\widehat{\mathfrak{R}}}+\widetilde{\mathbf{N}}+\widetilde{\mathbf{M}}\Big)\,\widetilde{\varphi}_0, \tag{3.29}$$

holds, where $\widetilde{\mathbf{N}}$ is a bounded operator with a small norm

$$\widetilde{\mathbf{N}}\ :H^{r+1}(\mathbb{R}^3_+)\times H^{r+1/2}(\mathbb{R}^2)\longrightarrow H^{r+1}(\mathbb{R}^3_+)\times H^{r-1/2}(\mathbb{R}^2),$$

while $\widetilde{\mathbf{M}}$ is a bounded operator

$$\widetilde{\mathbf{M}}\ :H^{r+1}(\mathbb{R}^3_+)\times H^{r+1/2}(\mathbb{R}^2)\longrightarrow H^{r+2}(\mathbb{R}^3_+)\times H^{r+1/2}(\mathbb{R}^2);$$

the operator $\widehat{\widehat{\mathfrak{R}}}$ is defined in the upper half-space $\mathbb{R}^3_+$ as follows

$$\widehat{\widehat{\mathfrak{R}}}:=\begin{bmatrix} r_+\widehat{\widehat{\mathbf{B}}}\ell_0 & r_+\widehat{\widetilde{\widetilde{W}}}\\ (\widehat{\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}})\ell_0 & \widehat{\widehat{\mathcal{L}}}\end{bmatrix}\ \text{ with }\ r_+=r_{\mathbb{R}^3_+}$$

and possesses the following mapping property

$$\widehat{\widehat{\mathfrak{R}}}\ :H^{r+1}(\mathbb{R}^3_+)\times H^{r+1/2}(\mathbb{R}^2)\longrightarrow H^{r+1}(\mathbb{R}^3_+)\times H^{r-1/2}(\mathbb{R}^2). \tag{3.30}$$

The operators with "hat" involved in the expression of $\widehat{\widehat{\mathfrak{R}}}$, are defined as follows: for the operator $\widetilde{G}$, the operator $\widehat{\widetilde{G}}$ denotes that in $\mathbb{R}^n$ $(n=2,3)$ constructed by the symbol

$$\widehat{\mathfrak{S}}(\widetilde{G})(\xi):=\mathfrak{S}(\widetilde{G})\big((1+|\xi'|)\omega,\xi_3\big)\ \text{ if }\ n=3$$

and

$$\widehat{\mathfrak{S}}(\widetilde{G})(\xi):=\mathfrak{S}(\widetilde{G})\big((1+|\xi'|)\omega\big)\ \text{ if }\ n=2,$$

where $\omega=\frac{\xi'}{|\xi'|}$, $\xi=(\xi',\xi_n)$, $\xi'=(\xi_1,\dots,\xi_{n-1})$.

The generalized Šapiro–Lopatinskiĭ condition is related to the invertibility of the operator (3.30). Indeed, let us write the system corresponding to the operator $\widehat{\widehat{\mathfrak{R}}}$:

$$r_+\widehat{\widehat{\mathbf{B}}}\ell_0\widetilde{U}+r_+\widehat{\widetilde{\widetilde{W}}}\widetilde{\Phi}=\widetilde{F}_1\ \text{ in }\ \mathbb{R}^3_+\,, \tag{3.31}$$

$$(\widehat{\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}})\ell_0\widetilde{U}+\widehat{\widehat{\mathcal{L}}}\,\widetilde{\Phi}=\widetilde{F}_2\ \text{ on }\ \mathbb{R}^2\,, \tag{3.32}$$

where $\widetilde{F}_1\in H^2(\mathbb{R}^3_+)$, $\widetilde{F}_2\in H^{1/2}(\mathbb{R}^2)$.

Note that the operator $r_+\widehat{\widetilde{\mathbf{B}}}\ell_0$ is a singular integral operator with even rational elliptic principal homogeneous symbol. Then due to Lemma 3.1, the operator

$$r_+\widehat{\widetilde{\mathbf{B}}}\ell_0 \ : H^{r+1}(\mathbb{R}^3_+) \longrightarrow H^{r+1}(\mathbb{R}^3_+)$$

is invertible. Therefore from equation (3.31) we can define $\widetilde{U}$. (3.31)

$$\begin{aligned}\ell_0\widetilde{U} = \ell_0\big[r_+\widehat{\widetilde{\mathbf{B}}}\ell_0\big]^{-1}\widetilde{f} = \\ = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell\widetilde{f})\Big)\Big\},\end{aligned} \tag{3.33}$$

where $\widetilde{f} = \widetilde{F}_1 - r_+\widehat{\widetilde{W}}\widetilde{\Phi}$, ℓ is an extension operator from $\mathbb{R}^3_+$ to $\mathbb{R}^3$ preserving the function space, while ℓ_0 is an extension operator $\mathbb{R}^3_+$ to $\mathbb{R}^3_-$ by zero, the operator Π^+ involved in (3.33) is defined in (3.10); here $\widehat{\mathfrak{S}}^{(\pm)}(\cdot)$ denote the so-called "plus" and "minus" factors in the factorization of the corresponding symbol $\widehat{\mathfrak{S}}(\cdot)$ with respect to the variable ξ_3. Note that the function $\ell_0\widetilde{U}$ in (3.33) does not depend on the extension operator ℓ.

Substituting (3.33) into (3.32), we get the following pseudodifferential equation with respect to the unknown function $\widetilde{\Phi}$:

$$\begin{aligned}-(\widehat{\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}})\mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\widehat{\widetilde{W}}\widetilde{\Phi})\Big)\Big\} + \widehat{\widetilde{\mathcal{L}}}\,\widetilde{\Phi} \\ = \widetilde{F} \ \text{ on } \ \mathbb{R}^2,\end{aligned} \tag{3.34}$$

where

$$\widetilde{F} = \widetilde{F}_2 - \widehat{\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}}\ell_0\,\big[r_+\widehat{\widetilde{\mathbf{B}}}\ell_0\big]^{-1}\widetilde{F}_1.$$

It can be shown that

$$\begin{aligned}\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}\,v\,(y') &= \Big[\mathcal{F}^{-1}_{\xi\to y}\big[\widetilde{\mathcal{T}}(-i\xi)\mathfrak{S}(\widetilde{\mathbf{K}})(\xi)\mathcal{F}(v)(\xi)\big]\Big]_{y_3=0+} \\ &= \mathcal{F}^{-1}_{\xi'\to y'}\Big[\Pi'\big[\widetilde{\mathcal{T}}(-i\xi)\mathfrak{S}(\widetilde{\mathbf{K}})(\xi)\mathcal{F}(v)(\xi)\big]\Big],\end{aligned} \tag{3.35}$$

where the operator Π' is defined as follows:

$$\Pi'(g)(\xi') = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} g(\xi',\xi_3)\,d\xi_3 \ \text{ for } \ g\in L_1(\mathbb{R}^3)$$

while (for details see [21], Appendix C)

$$\Pi'(g)(\xi') = \lim_{x_3\to 0+} r_+\mathcal{F}^{-1}_{\xi_3\to x_3}\,[g(\xi',\xi_3)] = -\frac{1}{2\pi}\int\limits_{\gamma^-} g(\xi',\zeta)\,d\zeta,$$

if the following conditions hold:

(i) $g(\xi',\xi_3)$ is rational in ξ_3 and the denominator does not vanish for nonzero real $\xi = (\xi',\xi_3)\in\mathbb{R}^3\setminus\{0\}$,

(ii) $g(\xi',\xi_3)$ is homogeneous of order $m\in\mathbb{Z} := \{0,\pm1,\pm2,\dots\}$ in $\xi = (\xi',\xi_3)$, and

(iii) $g(\xi',\xi_3)$ is infinitely differentiable with respect to real $\xi=(\xi',\xi_3)$ for $\xi'\neq 0$,

and γ^- is a contour in the lower complex half-plane orientated counter-clockwise and enclosing all the poles of the rational function g.

It is clear that if $g(\xi',\zeta)$ is analytic with respect to ζ in the lower half-plane ($\operatorname{Im}\zeta<0$), then

$$\Pi'(g)(\xi')=0 \ \text{ for all } \ \xi'.$$

Further, we can represent the double-layer potential as

$$W(\varphi)=\mathbf{P}\big(\mathcal{M}^\top(\Phi\otimes\delta_S)\big), \tag{3.36}$$

where the distribution $\mathcal{M}^\top(\Phi\otimes\delta_S)$ is supported on the boundary S and is defined by the relation

$$\big\langle\mathcal{M}^\top(\Phi\otimes\delta_S),\psi)\big\rangle_{\mathbb{R}^3}:=\langle\Phi,\mathcal{M}\psi\rangle_S \ \ \forall\,\psi\in\mathcal{D}(\mathbb{R}^3).$$

In the case if $S=\mathbb{R}^2$ is the boundary of the half-space, the distribution $\widetilde{\Phi}\otimes\delta_S$ is the direct product $\widetilde{\Phi}\otimes\delta_S=\widetilde{\Phi}(x_1,x_2)\times\delta(x_3)$ and in view of (3.35), we can write

$$\begin{aligned}(\widehat{\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}})\,\mathcal{F}^{-1}_{\xi\to\widetilde{x}}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\widehat{\widetilde{W}\widetilde{\Phi}})\Big)(\xi)\Big\}\,(\widetilde{y}')\\ =\mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\Big\{\Pi'\Big[\widehat{\widetilde{\mathcal{T}}}\,\widehat{\mathfrak{S}}(\widetilde{\mathbf{K}})\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\\ \times\Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\widehat{\mathfrak{S}}(\widetilde{\mathbf{P}})\widehat{\widetilde{\mathcal{M}}}^\top\Big)\Big](\xi')\mathcal{F}_{\widetilde{x}'\to\xi'}\widetilde{\Phi}\Big\}.\end{aligned} \tag{3.37}$$

By virtue of the above relations, equation (3.34) can be rewritten in the form

$$\mathcal{F}^{-1}_{\xi'\to y'}\big[\,\widehat{e}\,(\xi')\,\mathcal{F}(\widetilde{\Phi})(\xi')\big]=\widetilde{F}(y') \ \text{ on } \ \mathbb{R}^2, \tag{3.38}$$

where

$$\widehat{e}(\xi')=e\Big((1+|\xi'|)\,\omega\Big), \ \ \omega=\frac{\xi'}{|\xi'|} \tag{3.39}$$

with $e(\,\cdot\,)$ being a homogeneous matrix function of order 1 given by the equality

$$\begin{aligned}e(\xi')=-\,\Pi'\Big\{\widetilde{\mathcal{T}}\,\mathfrak{S}(\widetilde{\mathbf{K}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\widetilde{\mathcal{M}^\top}\Big)\Big\}(\xi')\\ +\,\mathfrak{S}(\widetilde{\mathcal{L}})(\xi') \ \ \forall\,\xi'\neq 0.\end{aligned} \tag{3.40}$$

If $\det e(\xi')$ is different from zero for all $\xi'\neq 0$, then $\det\widehat{e}(\xi')\neq 0$ for all $\xi'\in\mathbb{R}^2$, and the corresponding pseudodifferential operator

$$\widehat{\mathbf{E}}:H^s(\mathbb{R}^2)\longrightarrow H^{s-1}(\mathbb{R}^2),$$

generated by the left hand-side expression in (3.38), is invertible for all $s\in\mathbb{R}$. In particular, it follows that the system of equations (3.31), (3.32) is uniquely solvable with respect to $(\widetilde{U},\widetilde{\Phi})$ in the space $H^2(\mathbb{R}^3_+)\times H^{3/2}(\mathbb{R}^2)$ for arbitrary right-hand sides $(\widetilde{F}_1,\widetilde{F}_2)\in H^2(\mathbb{R}^3_+)\times H^{1/2}(\mathbb{R}^2)$. Consequently, the operator $\widehat{\widehat{\mathfrak{R}}}$ in (3.30) is invertible, which implies that the operator (3.29) possesses left and right regularizers. In turn, this yields that the operator (3.28) possesses left and right regularizers, as well. Thus the operator (3.28) is Fredholm if the matrix

$$\begin{aligned} e(\xi') = -\Pi'\Big\{ \widetilde{\mathcal{T}}\, \mathfrak{S}(\widetilde{\mathbf{K}}) \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1} \\ \times \Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}})\, \widetilde{\mathcal{M}}^{\top} \Big) \Big\}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}})(\xi') \end{aligned} \tag{3.41}$$

is non-singular for all $\xi' \neq 0$. This condition is called the *Šapiro–Lopatinskiĭ condition* (cf. [25], Theorems 12.2 and 23.1, and also formulas (12.27), (12.25)). Let us show that in our case the Šapiro–Lopatinskiĭ condition holds. To this end, let us note that the principal homogeneous symbols $\mathfrak{S}(\widetilde{\mathbf{K}})$, $\mathfrak{S}(\widetilde{\mathbf{B}})$, $\mathfrak{S}(\widetilde{\mathbf{P}})$, and $\mathfrak{S}(\widetilde{\mathcal{L}})$ of the operators $\mathbf{K}$, $\mathbf{B}$, $\mathbf{P}$, and $\mathcal{L}$ in the chosen local co-ordinate system involved in formula (3.41) read as:

$$\begin{gathered} \mathfrak{S}(\widetilde{\mathbf{K}})(\xi) = |\xi|^{-2} \widetilde{A}(\xi) - I, \\ \mathfrak{S}(\widetilde{\mathbf{B}})(\xi) = |\xi|^{-2} \widetilde{A}(\xi), \quad \mathfrak{S}(\widetilde{\mathbf{P}})(\xi) = -|\xi|^{-2} I, \\ \mathfrak{S}(\widetilde{\mathcal{L}})(\xi') = \frac{1}{2|\xi'|}\, \widetilde{\mathcal{T}}(\xi', -i|\xi'|)\, \widetilde{\mathcal{M}}^{\top}(\xi', -i|\xi'|), \\ \xi = (\xi', \xi_3), \ \ \xi' = (\xi_1, \xi_2). \end{gathered} \tag{3.42}$$

Recall that the matrices $\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})$ and $^{(-)}(\widetilde{\mathbf{B}})$ are the so-called "plus" and "minus" factors in the factorization of the symbol $\mathfrak{S}(\widetilde{\mathbf{B}})$ with respect to the variable ξ_3.

We rewrite (3.40) in the form

$$\begin{aligned} e(\xi') &= -\Pi'\Big\{ \widetilde{\mathcal{T}} \Big(\mathfrak{S}(\widetilde{\mathbf{B}}) - I \Big) \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1} \\ &\qquad \times \Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}})\, \widetilde{\mathcal{M}}^{\top} \Big) \Big\}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}})(\xi') \\ &= e_1(\xi') + e_2(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}})(\xi'), \end{aligned} \tag{3.43}$$

where $\mathfrak{S}(\widetilde{\mathcal{L}})(\xi')$ is defined in (3.42) and

$$\begin{aligned} e_1(\xi') = -\Pi'\Big\{ \widetilde{\mathcal{T}}\, \mathfrak{S}(\widetilde{\mathbf{B}}) \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1} \\ \times \Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}})\, \widetilde{\mathcal{M}}^{\top} \Big) \Big\}(\xi'), \end{aligned} \tag{3.44}$$

$$e_2(\xi') = \Pi'\Big\{ \widetilde{\mathcal{T}} \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1} \Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}})\, \widetilde{\mathcal{M}}^{\top} \Big) \Big\}(\xi'). \tag{3.45}$$

By direct calculations we get

$$\begin{aligned} &\Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}})\, \widetilde{\mathcal{M}}^{\top} \Big)(\xi') \\ &= \frac{i}{2\pi} \lim_{t \to 0+} \int_{-\infty}^{+\infty} \frac{\big([\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}}) \big)(\xi', \eta_3)\, \widetilde{\mathcal{M}}^{\top}(-i\xi', -i\eta_3)}{\xi_3 + i\,t - \eta_3}\, d\eta_3 \end{aligned}$$

$$
\begin{aligned}
&= -\frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{-\infty}^{+\infty}\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\eta_3)\widetilde{\mathcal{M}}^\top(-i\xi',-i\eta_3)}{(\xi_3+i\,t-\eta_3)(|\xi'|^2+\eta_3^2)}\,d\eta_3\\
&= \frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{\gamma^-}\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\tau)\widetilde{\mathcal{M}}^\top(-i\xi',-i\tau)}{(\xi_3+i\,t-\tau)(|\xi'|^2+\tau^2)}\,d\tau\\
&= \frac{1}{2\pi}\lim_{t\to 0+}\frac{2\pi i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{(\xi_3+i\,t+i|\xi'|)\,2\,(-i|\xi'|)}\\
&\quad = -\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|\,(\xi_3+i|\,\xi'|)}\,.
\end{aligned}
\tag{3.46}
$$

Now, from (3.44) by virtue of (3.46), we derive

$$
\begin{aligned}
e_1(\xi') &= -\Pi'\Big\{\widetilde{\mathcal{T}}\,\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\\
&\qquad\times\Pi^+\Big(\Big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\Big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^\top\Big)\Big\}(\xi')\\
&= -\Pi'\Big\{\widetilde{\mathcal{T}}\,\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\Pi^+\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^\top\Big)\Big\}(\xi')\\
&= \Pi'\bigg\{\widetilde{\mathcal{T}}(-i\xi',-i\xi_3)\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',\xi_3)\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|\,(\xi_3+i|\,\xi'|)}\Big)\bigg\}(\xi')\\
&= -i\,\Pi'\Big\{\frac{\widetilde{\mathcal{T}}(\xi',\xi_3)\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',\xi_3)}{\xi_3+i|\xi'|}\Big\}(\xi')\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\Big)\\
&= \frac{i}{2\,\pi}\int\limits_{\gamma^-}\frac{\widetilde{\mathcal{T}}(\xi',\tau)\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',\tau)}{\tau+i|\,\xi'|}\,d\tau\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\Big)\\
&= -\mathcal{T}(\xi',-i|\xi'|)\,\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',-i\,|\xi'|)\\
&\qquad\times\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\\
&= -\frac{1}{2\,|\xi'|}\,\widetilde{\mathcal{T}}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|).
\end{aligned}
\tag{3.47}
$$

Quite similarly, from (3.45), with the help of (3.46) and Lemma 3.2, we find

$$
\begin{aligned}
e_2(\xi') &= \Pi'\Big\{\widetilde{\mathcal{T}}\,[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}\Pi^+\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^\top\Big)\Big\}(\xi')\\
&= -\Pi'\Big\{\widetilde{\mathcal{T}}(-i\xi',-i\xi_3)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}(\xi',\xi_3)\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2|\xi'|\,(\xi_3+i|\xi'|)}\Big)\Big\}(\xi')\\
&= i\,\Pi'\Big\{\widetilde{\mathcal{T}}(\xi',\xi_3)\,\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\xi_3)}{\xi_3+i|\,\xi'|}\Big\}(\xi')\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\,\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\Big)\\
&= \frac{i}{2|\xi'|}\Big(-\frac{1}{2\pi}\int\limits_{\gamma^-}\frac{\widetilde{\mathcal{T}}(\xi',\tau)\,[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\tau)}{\tau+i|\,\xi'|}\,d\tau\Big)\\
&\qquad\times\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)\\
&= -\frac{i}{4\,\pi\,|\xi'|}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\\
&\qquad\times(-2\,i\,|\xi'|)\big[A^-(\xi',-i\,|\xi'|)\big]^{-1}\,\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)\\
&= -\Big(\frac{1}{2\,\pi}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)\\
&\qquad\times\big[\widetilde{A}^{(-)}(\xi',-i\,|\xi'|)\big]^{-1}\,\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|).
\end{aligned}\tag{3.48}
$$

Therefore, in view of relations (3.43), (3.42), (3.47), and (3.48) we finally obtain

$$
\begin{aligned}
e(\xi') = -\Big(\frac{1}{2\,\pi}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\,\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)&\\
\times\big[\widetilde{A}^{(-)}\big(\xi',-i\,|\xi'|\big)\big]^{-1}\,\widetilde{\mathcal{M}}^\top\big(\xi',-i|\xi'|\big)&.
\end{aligned}
$$

Since

$$
\det\Big(\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)\neq 0\ \text{ for all }\ \xi'\neq 0
$$

due to Lemma 3.3, and $\det\widetilde{A}^{(-)}(\xi',-i\,|\xi'|)\neq 0$ and $\det\widetilde{\mathcal{M}}(\xi',-i\,|\xi'|)\neq 0$ for all $\xi'\neq 0$ in accordance with (3.6) and (3.26), respectively, we deduce that

$$
\det e(\xi')\neq 0\ \text{ for all }\ \xi'\neq 0.
$$

Therefore for the operator $\mathfrak{R}$ the Šapiro–Lopatinskiĭ condition holds and the operator

$$
\mathfrak{R}: H^{r+1}(\Omega)\times H^{r+1/2}(S)\longrightarrow H^{r+1}(\Omega)\times H^{r-1/2}(S)
$$

is Fredholm for $r \geq 1$.

Step 3. We can now show that $\operatorname{Ind} \mathfrak{R} = 0$. To this end, for $t \in [0,1]$ let us consider the operator

$$\mathfrak{R}_t := \begin{bmatrix} r_{\Omega} \mathbf{B}_t \ell_0 & r_{\Omega} W_t + r_{\Omega} V \beta \\ t\,(\mathcal{T}^+ \mathbf{K})\ell_0 & \mathcal{L}_t + (\mathbf{d} - \mathbf{I}) + \mathcal{W}' \beta \end{bmatrix} \tag{3.49}$$

with $\mathbf{B}_t = \mathbf{I} + t\,\mathbf{K}$ and prove that it is homotopic to the operator $\mathfrak{R} = \mathfrak{R}_1$, where

$$W_t(g)(y) := -\int\limits_{S} \big[\mathcal{M}_t(x, \partial_x)\, P(x-y)\big]^{\top} g(x)\, dS_x, \quad y \in S, \quad t \in [0,1], \tag{3.50}$$

and

$$\mathcal{L}_t g(y) := \big[\mathcal{T}_t(y, \partial_y) W_t\, g(y)\big]^+, \quad y \in S, \quad t \in [0,\, 1], \tag{3.51}$$

with $\mathcal{T}_t(y, \partial_y)$ and $\mathcal{M}_t(y, \partial_y)$ defined in (3.25). Clearly, $\mathcal{L} = \mathcal{L}_1$.

We have to check that for the operator $\mathfrak{R}_t$ the Šapiro–Lopatinskiĭ condition is satisfied for all $t \in [0,1]$. Indeed, in this case the matrix associated with the Šapiro–Lopatinskiĭ condition reads as (cf. (3.40))

$$\begin{aligned} e_t(\xi') &= -\Pi' \Big\{ \widetilde{\mathcal{T}}_t \Big(\mathfrak{S}(\widetilde{\mathbf{B}}_t) - I \Big) \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1} \\ &\qquad \times \Pi^+ \Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}})\, \widetilde{\mathcal{M}}_t^{\top} \Big) \Big\}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}}_t)(\xi') \\ &= e_t^{(1)}(\xi') + e_t^{(2)}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}}_t)(\xi'), \end{aligned} \tag{3.52}$$

where

$$\begin{aligned} e_t^{(1)}(\xi') &= -\Pi' \Big\{ \widetilde{\mathcal{T}}_t \mathfrak{S}(\widetilde{\mathbf{B}}_t) \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1} \\ &\qquad\qquad \times \Pi^+ \Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}})\, \widetilde{\mathcal{M}}_t^{\top} \Big) \Big\}(\xi') \\ &= -\frac{1}{2\,|\xi'|}\, \widetilde{\mathcal{T}}_t(\xi', -i|\xi'|)\, \widetilde{\mathcal{M}}_t^{\top}\big(\xi', -i|\xi'|\big), \\ e_t^{(2)}(\xi') &= \Pi' \Big\{ \widetilde{\mathcal{T}}_t \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1} \Pi^+ \Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}}) \widetilde{\mathcal{M}}_t \Big) \Big\}(\xi'), \\ \mathfrak{S}(\widetilde{\mathcal{L}}_t)(\xi') &= \frac{1}{2\,|\xi'|}\, \widetilde{\mathcal{T}}_t(\xi', -i|\xi'|)\, \widetilde{\mathcal{M}}_t^{\top}\big(\xi', -i|\xi'|\big). \end{aligned} \tag{3.53}$$

We have to show that $e_t(\xi')$ is non-singular for all $\xi' \neq 0$ and $t \in [0,1]$.

By direct calculations, we get

$$
\begin{aligned}
e_t^{(2)}(\xi') &= \Pi'\Big\{\widetilde{\mathcal{T}}_t\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}_t^\top\Big)\Big\}(\xi') \\
&= -\Pi'\Bigg\{\widetilde{\mathcal{T}}_t(-i\xi',-i\xi_3)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}(\xi',\xi_3) \\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',-i|\xi'|)\,\widetilde{\mathcal{M}}_t^\top(\xi',-i|\xi'|)}{2\,|\xi'|(\xi_3+i|\,\xi'|)}\Big)\Bigg\}(\xi') \\
&= i\Pi'\Big\{\frac{\widetilde{\mathcal{T}}_t(\xi',\xi_3)[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',\xi_3)}{\xi_3+i|\,\xi'|}\Big\}(\xi') \\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}_t^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\Big) \\
&= \frac{i}{2\,|\xi'|}\Big(-\frac{1}{2\,\pi}\int\limits_{\gamma^-}\frac{\widetilde{\mathcal{T}}_t(\xi',\tau)[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',\tau)}{\tau+i|\,\xi'|}\,d\tau\Big) \\
&\qquad\times\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}_t^\top\big(\xi',-i|\xi'|\big) \\
&= -\frac{i}{4\,\pi\,|\xi'|}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}_t(\xi',\tau)\big[\widetilde{A}_t^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\big(-2\,i\,|\xi'|\big) \\
&\qquad\times\big[\widetilde{A}_t^{(-)}(\xi',-i|\xi'|)\big]^{-1}\widetilde{\mathcal{M}}_t^\top\big(\xi',-i|\xi'|\big) \\
&= -\Big(\frac{1}{2\,\pi}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}_t(\xi',\tau)\big[\widetilde{A}_t^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big) \\
&\qquad\times\big[\widetilde{A}_t^{(-)}(\xi',-i|\xi'|)\big]^{-1}\widetilde{\mathcal{M}}_t^\top\big(\xi',-i|\xi'|\big),
\end{aligned}
\tag{3.54}
$$

where $\widetilde{A}_t(\xi)=(1-t)|\xi|^2I+t\widetilde{A}(\xi)$ and $\widetilde{A}_t(\xi',\xi_3)=\widetilde{A}_t^{(-)}(\xi',\xi_3)\widetilde{A}_t^{(+)}(\xi',\xi_3)$, $\widetilde{A}_t^{(\pm)}(\xi',\xi_3)$ are the "plus" and "minus" polynomial matrix factors in ξ_3 of the polynomial symbol matrix $\widetilde{A}_t(\xi',\xi_3)$.

Analogously to Lemma 3.3, we can prove that the matrix

$$\int\limits_{\gamma^-}\widetilde{\mathcal{T}}_t(\xi',\tau)\big[\widetilde{A}_t^{(+)}(\xi',\tau)\big]^{-1}\,d\tau$$

is non-singular for all $\xi'\neq 0$ and for all $t\in[0,1]$.

Therefore, by (3.52), (3.54) and (3.26) we have

$$\det e_t(\xi')=\det e_t^{(2)}(\xi')\neq 0\ \text{ for all }\ \xi'\neq 0\ \text{ and for all }\ t\in[0,1], \tag{3.55}$$

which implies that for the operator $\mathfrak{R}_t$ the Šapiro–Lopatinskiĭ condition is satisfied.

Hence the operator

$$\mathfrak{R}_t\ :H^{r+1}(\Omega)\times H^{r+1/2}(S)\longrightarrow H^{r+1}(\Omega)\times H^{r-1/2}(S)$$

is Fredholm for all $r\geq 1$ and for all $t\in[0,1]$.

Further, we prove that the index of the operator

$$\mathfrak{R}_0 = \begin{bmatrix} r_{\Omega}\mathbf{I}\,\ell_0 & r_{\Omega}W_0 + r_{\Omega}V\beta \\ 0 & \mathcal{L}_0 + (\mathbf{d} - \mathbf{I}) + \mathcal{W}'\beta \end{bmatrix} : H^{r+1}(\Omega) \times H^{r+1/2}(S) \\ \longrightarrow H^{r+1}(\Omega) \times H^{r-1/2}(S)$$

is zero. Towards this end, first we show that the index of the operator $\mathcal{L}_t$ equals zero for all $t \in [0,\, 1]$.

The principal homogeneous symbol matrix of the operator $\mathcal{L}_t$ reads as

$$\mathfrak{S}(\widetilde{\mathcal{L}}_t)(\xi') = \frac{1}{2\,|\xi'|}\,\widetilde{\mathcal{T}}_t\big(\xi', -i|\xi'|\big)\widetilde{\mathcal{M}}_t^{\top}\big(\xi', -i|\xi'|\big)$$

and is elliptic due to (3.26). Consequently, the operator $\mathcal{L}_t : H^{s+1/2}(S) \to H^{s-1/2}(S)$ with $s \in \mathbb{R}$ is Fredholm for all $t \in [0,1]$. Moreover, the principal part of the operator $\mathcal{L}_0 : H^{1/2}(S) \to H^{-1/2}(S)$ is selfadjoint due to the equality

$$\mathcal{L}_0 g = \mathcal{L}_{\Delta} g,$$

where $\mathcal{L}_{\Delta}$ stands for the trace of the normal derivative of the localized harmonic double-layer potential,

$$\mathcal{L}_{\Delta} g(y) = -\bigg\{\frac{\partial}{\partial n(y)} \int\limits_S \frac{\partial P(x-y)}{\partial n(x)}\, g(x)\, dS_x\bigg\}^{+}.$$

Therefore,

$$\operatorname{Ind}\mathcal{L} = \operatorname{Ind}\mathcal{L}_1 = \operatorname{Ind}\mathcal{L}_t = \operatorname{Ind}\mathcal{L}_0 = 0 \ \text{ for all } \ t \in [0,\, 1] \ \text{ and for all } \ s \in \mathbb{R},$$

implying that the index of the operator $\mathfrak{R}_0$ equals zero. Since the family of the operators $\mathfrak{R}_t$ for $t \in [0,\, 1]$ are homotopic, we conclude that

$$\operatorname{Ind}\mathfrak{R} = \operatorname{Ind}\mathfrak{R}_1 = \operatorname{Ind}\mathfrak{R}_t = \operatorname{Ind}\mathfrak{R}_0 = 0 \ \text{ for all } \ t \in [0,\, 1] \ \text{ and for all } \ r \geq 1.$$

Step 4. From the equivalence Theorem 2.2 it follows that $\operatorname{Ker}\mathfrak{R} = \{0\}$ in the space $H^{r+1}(\Omega) \times H^{r+1/2}(S)$ for all $r \geq 1$ and, consequently, the operator

$$\mathfrak{R} : H^{r+1}(\Omega) \times H^{r+1/2}(S) \longrightarrow H^{r+1}(\Omega) \times H^{r-1/2}(S)$$

is invertible for all $r \geq 1$. □

Corollary 3.5. *Let a localizing function* $\chi \in X^4_{1+}$ *and the condition* (3.26) *be fulfilled. Then the operator*

$$\mathfrak{R} \,:\, H^2(\Omega) \times H^{3/2}(S) \longrightarrow H^2(\Omega) \times H^{1/2}(S)$$

is invertible.

Proof. It is word for word repeats the above proof with $r = 1$. □

4. Appendix A: Classes of Localizing Functions

Here we introduce the classes of localizing functions used in the main text of the paper (for details see the reference [16]).

Definition A.1. We say $\chi \in X^k$ for integer $k \geq 0$ if $\chi(x) = \breve{\chi}(|x|)$, $\breve{\chi} \in W_1^k(0,\infty)$ and $\varrho\breve{\chi}(\varrho) \in L_1(0,\infty)$. We say $\chi \in X_+^k$ for integer $k \geq 1$ if $\chi \in X^k$, $\chi(0) = 1$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{\widehat{\chi}_s(\omega)}{\omega} > 0 & \text{for } \omega \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\int_0^\infty \varrho\breve{\chi}(\varrho)\, d\varrho & \text{for } \omega = 0, \end{cases} \tag{A.1}$$

and $\widehat{\chi}_s(\omega)$ denotes the sine-transform of the function $\breve{\chi}$

$$\widehat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}(\varrho) \sin(\varrho\omega)\, d\varrho. \tag{A.2}$$

We say $\chi \in X_{1+}^k$ for integer $k \geq 1$ if $\chi \in X_+^k$ and

$$\omega\widehat{\chi}_s(\omega) \leq 1, \ \ \forall\, \omega \in \mathbb{R}. \tag{A.3}$$

Evidently, we have the following embeddings: $X^{k_1} \subset X^{k_2}$ and $X_+^{k_1} \subset X_+^{k_2}$, $X_{1+}^{k_1} \subset X_{1+}^{k_2}$ for $k_1 > k_2$. The class X_+^k is defined in terms of the sine-transform. The following lemma provides us with an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class (for details see [16]).

Lemma A.2. *Let $k \geq 1$. If $\chi \in X^k$, $\breve{\chi}(0) = 1$, $\breve{\chi}(\varrho) \geq 0$ for all $\varrho \in (0,\infty)$, and $\breve{\chi}$ is a non-increasing function on $[0,+\infty)$, then $\chi \in X_+^k$.*

The following examples for χ are presented in [16],

$$\chi_1(x) = \begin{cases} \Big[1 - \dfrac{|x|}{\varepsilon}\Big]^k & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A.4}$$

$$\chi_2(x) = \begin{cases} \exp\Big[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\Big] & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A.5}$$

$$\chi_3(x) = \begin{cases} \Big(1 - \dfrac{|x|}{\varepsilon}\Big)^2\Big(1 - 2\dfrac{|x|}{\varepsilon}\Big) & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon. \end{cases} \tag{A.6}$$

One can notice that $\chi_1 \in X_+^k$, while $\chi_2 \in X_+^\infty$ due to Lemma A.2, and $\chi_3 \in X_+^2$. Moreover, $\chi_1 \in X_{1+}^k$ for $k = 2$ and $k = 3$, and $\chi_3 \in X_{1+}^2$, while $\chi_1 \notin X_{1+}^1$ and $\chi_2 \notin X_{1+}^\infty$ (for details see [16]).

5. Appendix B: Properties of Localized Potentials

Here we collect some theorems describing mapping properties of the localized potentials. The proofs can be found in [16] (see also [26], Chapter 8 and the references therein).

Here we employ the notation V, W, and $\mathcal{P}$ introduced in the main text for the localized layer and volume potentials, see (2.31)–(2.33). Further, let us introduce the boundary operators generated by the localized layer potentials associated with the localized parametrix $P(x-y) \equiv P_\chi(x-y)$,

$$\mathcal{V}g(y) := -\int\limits_S P(x-y)\, g(x)\, dS_x, \quad y \in S, \tag{B.1}$$

$$\mathcal{W}g(y) := -\int\limits_S \big[\mathcal{M}(x,\partial_x)P(x-y)\big]^\top g(x)\, dS_x, \;\; y \in S, \tag{B.2}$$

$$\mathcal{W}'g(y) := -\int\limits_S \big[\mathcal{T}(y,\partial_y)P(x-y)\big] g(x)\, dS_x, \;\; y \in S, \tag{B.3}$$

$$\mathcal{L}^\pm g(y) := \big[\mathcal{T}(y,\partial_y)Wg(y)\big]^\pm, \;\; y \in S, \tag{B.4}$$

where $\mathcal{T}(x,\partial_x)$ and $\mathcal{M}(x,\partial_x)$ are defined in (2.6) and (2.8).

Theorem B.1. *The following operators are continuous:*

$$\mathcal{P} : \widetilde{H}^s(\Omega) \longrightarrow H^{s+2,s}(\Omega;\Delta), \;\; -\frac{1}{2} < s < \frac{1}{2}\,, \;\; \chi \in X^1, \tag{B.5}$$

$$: H^s(\Omega) \longrightarrow H^{s+2,s}(\Omega;\Delta), \;\; -\frac{1}{2} < s < \frac{1}{2}\,, \;\; \chi \in X^1, \tag{B.6}$$

$$: H^s(\Omega) \longrightarrow H^{\frac{5}{2}-\varepsilon,\frac{1}{2}-\varepsilon}(\Omega;\Delta), \;\; \frac{1}{2} \le s < \frac{3}{2}\,, \;\; \forall\,\varepsilon \in (0,1), \;\; \chi \in X^2, \tag{B.7}$$

where Δ is the Laplace operator.

Theorem B.2. *The following operators are continuous:*

$$V : H^{s-\frac{3}{2}}(S) \longrightarrow H^s(\mathbb{R}^3), \;\; s < \frac{3}{2}\,, \;\; \textit{if } \chi \in X^1, \tag{B.8}$$

$$: H^{s-\frac{3}{2}}(S) \longrightarrow H^{s,s-1}(\Omega^\pm;\Delta), \;\; \frac{1}{2} < s < \frac{3}{2}\,, \;\; \textit{if } \chi \in X^2, \tag{B.9}$$

$$W : H^{s-\frac{1}{2}}(S) \longrightarrow H^s(\Omega^\pm), \;\; s < \frac{3}{2}\,, \;\; \textit{if } \chi \in X^2, \tag{B.10}$$

$$: H^{s-\frac{1}{2}}(S) \longrightarrow H^{s,s-1}(\Omega^\pm;\Delta), \;\; \frac{1}{2} < s < \frac{3}{2}\,, \;\; \textit{if } \chi \in X^3. \tag{B.11}$$

Theorem B.3. *If $\chi \in X^k$ has a compact support and $-\frac{1}{2} \le s \le \frac{1}{2}$, then the following localized operators are continuous:*

$$V : H^s(S) \longrightarrow H^{s+\frac{3}{2}}(\Omega^\pm) \;\; \textit{for } \; k=2, \tag{B.12}$$

$$W : H^{s+1}(S) \longrightarrow H^{s+\frac{3}{2}}(\Omega^\pm) \;\; \textit{for } \; k=3. \tag{B.13}$$

Theorem B.4. *Let $\psi \in H^{-\frac{1}{2}}(S)$ and $\varphi \in H^{\frac{1}{2}}(S)$. Then the following jump relations hold on S:*

$$V^{+}\psi = V^{-}\psi = \mathcal{V}\psi, \ \ \chi \in X^1, \tag{B.14}$$

$$W^{\pm}\varphi = \mp\,\mathbf{d}\varphi + \mathcal{W}\varphi, \ \ \chi \in X^2, \tag{B.15}$$

$$\mathcal{T}^{\pm}V\psi = \pm\,\mathbf{d}\psi + \mathcal{W}'\psi, \ \ \chi \in X^2, \tag{B.16}$$

where

$$\mathbf{d}(y) := \frac{1}{2}\begin{bmatrix} [c_{ijlk}(y)\,n_i\,n_l]_{3\times 3} & [e_{lij}(y)\,n_i\,n_l]_{3\times 1} \\ [-e_{ikl}(y)\,n_i\,n_l]_{1\times 3} & \varepsilon_{il}(y)\,n_i\,n_l \end{bmatrix}_{4\times 4}, \ \ y \in S, \tag{B.17}$$

and $\mathbf{d}(y)$ is strongly elliptic due to (2.4) .

Theorem B.5. *Let $-\frac{1}{2} \le s \le \frac{1}{2}$. The following operators*

$$\mathcal{V} : H^s(S) \longrightarrow H^{s+1}(S), \ \ \chi \in X^2, \tag{B.18}$$

$$\mathcal{W} : H^{s+1}(S) \longrightarrow H^{s+1}(S), \ \ \chi \in X^3, \tag{B.19}$$

$$\mathcal{W}' : H^s(S) \longrightarrow H^s(S), \ \ \chi \in X^3, \tag{B.20}$$

$$\mathcal{L}^{\pm} : H^{s+1}(S) \longrightarrow H^s(S), \ \ \chi \in X^3, \tag{B.21}$$

are continuous.

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Authors' addresses:

Otar Chkadua

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia.

2. Sokhumi State University, 9 Politkovskaya St., Tbilisi 0186, Georgia.

E-mail: chkadua@rmi.ge

David Natroshvili

1. Department of Mathematics, Georgian Technical University, 77 M. Kostava St., Tbilisi 0175, Georgia.

2. I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia.

E-mail: natrosh@hotmail.com

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Marvin Fleck, Richards Grzhibovskis, and Sergej Rjasanow

A NEW FUNDAMENTAL SOLUTION METHOD BASED ON THE ADAPTIVE CROSS APPROXIMATION

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. A new adaptive Fundamental Solution Method (FSM) for the approximate solution of scalar elliptic boundary value problems is presented. The construction of the basis functions is based on the Adaptive Cross Approximation (ACA) of the fundamental solutions of the corresponding elliptic operator. An algorithm for an immediate computer implementation of the method is formulated. A series of numerical examples for the Laplace and Helmholtz equations in three dimensions illustrates the efficiency of the method. Extensions of the method to elliptic systems are discussed.

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რეზიუმე. წარმოდგენილია ფუნდამენტური ამონახსნების ახალი ადაპტიური მეთოდის (FSM) გამოყენება სკალარული ელიფსური სასაზღვრო ამოცანების მიახლოებითი ამოხსნისთვის. საბაზისო ფუნქციების აგება ემყარება შესაბამისი ელიფსური ოპერატორის ფუნდამენტური ამონახსნების ადაპტიურ ჯვარედინ აპროქსიმაციას (ACA). ფორმულირებულია ამ მეთოდის უშუალო კომპიუტერული განხორციელების ალგორითმი. მეთოდის ეფექტურობა ილუსტრირებულია სამგანზომილებიანი ლაპლასისა და ჰელმჰოლცის განტოლებებისთვის მოყვანილი რიცხვითი მაგალითებით. განხილულია ამ მეთოდის განზოგადების შესაძლებლობა ელიფსური სისტემებისთვის.

1. Introduction

The Fundamental Solution Method (FSM) is also known as the Method of Fundamental Solutions, Charge Simulation Method or as a special version of the Boundary Collocation Method. It resembles a Trefftz method [7], which means that the solution to a Dirichlet boundary value problem in $\Omega \subset \mathbb{R}^3$, $\Gamma = \partial\Omega$

$$\begin{aligned} \mathcal{L}u(x) &= 0 \ \text{ for } \ x \in \Omega, \\ u(x) &= g(x) \ \text{ for } \ x \in \Gamma, \end{aligned}$$

is approximated by a linear combination of $\mathcal{L}$-harmonic functions. As the name indicates, the method uses fundamental solutions for basis functions, whose singularities are located outside Ω. It was introduced by Kupradze and Aleksidze [4] in 1963 for treating the Laplace equation. First investigations from a numerical point of view were performed by Mathon and Johnston [5] in 1977. Comprehensive summaries of the attributes of the FSM were written, among others, by Smyrlis [6] and Bogomolny [3].

Two peculiar aspects of the Fundamental Solution Method are an extremely fast convergence, but also a very high condition number of the system matrix, both with respect to a number of collocation points. We address the problem of high condition numbers by adaptively choosing a smaller number of collocation points while keeping the local error below a given threshold, but not necessarily equal to zero, for the remaining collocation points. Thus an approximation is obtained, while condition numbers are kept lower due to smaller system matrices. The quality of the approximation is comparable to that of classical FSM. By means of this approach the problems that are too big for classical FSM can be treated. The adaptive strategy features are new basis functions which vanish at collocation points already treated and thus do not alter the corresponding local approximation. The construction of these basis functions uses concepts from the Adaptive Cross Approximation (ACA) [2].

In Section 2 we formulate a model problem and present the classical (collocation-based) Fundamental Solution Method. Section 3 briefly summarizes the Adaptive Cross Approximation. The approximation algorithm presented therein leads directly to the construction of basis functions for the Adaptive Fundamental Solution Method in Section 4. In Section 5 we present numerical results for the adaptive method applied to the Laplace and Helmholtz equations, respectively.

2. Formulation of the Problem

We consider the following Dirichlet boundary value problem for an elliptic equation in $\mathbb{R}^3$

$$\begin{aligned} \mathcal{L}u(x) &= 0 \ \text{ for } \ x \in \Omega, \\ u(x) &= g(x) \ \text{ for } \ x \in \Gamma, \end{aligned}$$

where $\mathcal{L}$ is an elliptic second order differential operator and $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain with the boundary Γ. In the classical setting, the Dirichlet datum g is assumed to be continuous on Γ and the solution u is assumed to be smooth, i.e.

$$u \in C^2(\Omega) \cap C(\overline{\Omega})\,.$$

In this paper, we will consider the Laplace operator

$$\mathcal{L}u = -\Delta u$$

and the Helmholtz operator

$$\mathcal{L}u = -\Delta u - \kappa^2 u\,.$$

For these operators, the corresponding fundamental solution u^*, i.e. the solution of the equation

$$\mathcal{L}u^* = \delta \tag{2}$$

in the distributional sense is known and given by

$$u^*(x) = \frac{1}{4\pi}\frac{1}{|x|}$$

for the Laplace operator, and

$$u^*(x) = \frac{1}{4\pi}\frac{e^{\imath\kappa|x|}}{|x|}$$

for the Helmholtz operator. In (2) δ denotes the Dirac δ-distribution.

2.1. **Fundamental solution method.** Let $X \subset \Gamma$ be a discrete set of N pairwise different control (collocation) points on the boundary Γ and $Y \subset \mathbb{R}^3 \backslash \overline{\Omega}$ a discrete set of N pairwise different singularity points. Consider a system of basis functions

$$\Phi = \big\{\varphi_1, \ldots, \varphi_N\big\}, \quad \varphi_\ell(x) = u^*(x - y_\ell), \;\; y_\ell \in Y, \;\; \ell = 1, \ldots, N.$$

Since $y_\ell \notin \overline{\Omega}$, $\ell = 1, \ldots, N$, every basis function φ_ℓ is $\mathcal{L}$-harmonic in Ω and the function

$$u_N(x) = \sum_{\ell=1}^{N} \alpha_\ell \varphi_\ell(x) = \Phi(x)\underline{a}\,, \quad \underline{a} = (\alpha_1, \ldots, \alpha_N)^\top \in \mathbb{R}^N$$

can be considered as an approximation of the solution u of the boundary value problem (1). The most simple choice for the coefficients α_ℓ is the point collocation for the boundary condition

$$u_N(x) = g(x) \;\text{ for }\; x \in X.$$

This can be equivalently formulated as a linear system for obtaining N coefficients α_l:

$$\sum_{\ell=1}^{N} \alpha_\ell \varphi_\ell(x_k) = g(x_k) \;\text{ for }\; k = 1, \ldots, N$$

or, in a matrix form,

$$F\underline{a} = \underline{g}\,, \tag{3}$$

where

$$F = \big(\varphi_\ell(x_k)\big)_{k,\ell=1}^{N} \in \mathbb{R}^{N\times N}, \quad \underline{a}_{\,\ell} = \alpha_\ell, \;\; \underline{g}_{\,\ell} = g(x_\ell), \;\; 1 \le \ell \le N.$$

The main properties of the FSM can be summarised as follows.

1. Since no topology of the discrete point sets X and Y is required, the FSM can be considered as a meshfree numerical method.
2. The entries of the matrix F in (3) are easy to compute as opposed to matrix entries coming from Boundary Element Methods (BEM).
3. The dimension of the matrix F is comparable to those of the BEM (e.g. $N \sim 10^4 - 10^5$ for 3D problems).
4. The matrix F is fully populated as in the BEM and $Mem(F) = \mathcal{O}(N^2)$.
5. The condition number of the matrix F grows exponentially, i.e. $\operatorname{cond}(F) = \mathcal{O}(q^N)$ for some $q > 1$.

For large N the application of a direct solver to the system (3) is expensive, while an iterative solver does not converge due to the extremely high condition number of the matrix F.

However, the numerical results for small systems show an exponential convergence of the method not only for the solution u itself but also for its gradient

$$\operatorname{grad} u = \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\Big)^{\top}$$

and even for its Hessian matrix

$$\mathcal{H}u = \Big(\frac{\partial^2 u}{\partial x_k \partial x_\ell}\Big)_{k,\ell=1}^{3},$$

i.e.

$$\begin{aligned}
\mathcal{O}(|u(x) - u_N(x)|) &= \mathcal{O}\big(\big|\operatorname{grad}(u(x) - u_N(x))\big|\big) \\
&= \mathcal{O}\big(\big|\big|\mathcal{H}(u(x) - u_N(x))\big|\big|_F\big) \\
&= \mathcal{O}(q^{-N})
\end{aligned}$$

for $x \in \Omega$. Note that the derivatives of the approximate solution u_N can be easily computed analytically.

2.2. **Choice of pseudo boundary.** In the theoretical analysis of Fundamental Solution Methods one introduces the concept of pseudo-boundaries, i.e. surfaces where the singularity points are located. Pseudo-boundaries fulfilling the so-called embracing condition provide for the suitability of corresponding fundamental solutions as basis functions [6]. However, one still has great freedom in choosing an actual pseudo-boundary and in the subsequent choice of the location of singularity points.

Here we briefly present the definition and the central theorem for pseudo-boundaries. A thorough overview can be found in [6].

Definition 1 (Segment condition)**.** Let $\Omega \subset \mathbb{R}^d$ be an open set. Ω fulfills the segment condition, if for every $x \in \partial\Omega$ there exist a neighborhood $U(x)$ of x and a nonzero vector $\xi(x) \in \mathbb{R}^d$ such that if $y \in U(x) \cap \overline{\Omega}$, then $y + t\xi(x) \in \Omega$, $\forall t \in (0, 1)$.

Definition 2 (Embracing boundary)**.** Let $\Omega, \Omega' \subset \mathbb{R}^d$ be open and connected. Ω' embraces Ω, if:

1. $\overline{\Omega} \subset \Omega'$;
2. For each connected component V of $\mathbb{R}^d \setminus \overline{\Omega}$ there is an open connected component V' of $\mathbb{R}^d \setminus \overline{\Omega}\,'$ such that $\overline{V}\,' \subset V$.

Theorem 1. *If $\Omega \subset \mathbb{R}^d$ fulfills the segment condition and $\Omega' \subset \mathbb{R}^d$ embraces Ω, then for $d \geq 3$ and $l \geq 0$ the space $\mathcal{X}$ spanned by finite linear combinations of Fundamental solutions*

$$u_N(x) = \sum_{j=1}^{N} \alpha_j u^*(x - y_j)$$

with singularities $y_j \in \partial\Omega'$ is dense in

$$\mathcal{Y}_l = \left\{ v \in C^2(\Omega) : \ \Delta v = 0 \ in \ \Omega \right\} \cap C^l(\overline{\Omega})$$

with respect to the norm of $C^l(\overline{\Omega})$. For $d = 2$ the density result holds true for $\mathcal{X} \oplus \{c \cdot 1|_{\overline{\Omega}} : \ c \in \mathbb{R}\}$.

Proof. The proof can be found in [6]. □

Similar results exist for the operators Δ^m, $m > 1$, and $\Delta - \kappa^2$, $\kappa > 0$, [6, 3].

One can prove in the two-dimensional case that an increase in the distance between the boundary $\partial\Omega$ and the pseudo-boundary $\partial\Omega'$ leads both to a better approximation and to a larger condition number of F. This can also be observed in three-dimensional settings.

For simple domain shapes the common choice of the singularity points consists in shifting collocation points along the outer normal. This strategy may fail for more complex domains. On the other hand, the construction of pseudo boundaries by means of distance functions may be computationally expensive.

In what follows, we will introduce a new method with the same convergence properties but almost without disadvantages of the FSM, i.e. without necessity of numerical solving of big, dense and badly conditioned systems of linear equations. Our main tool is the Adaptive Cross Approximation.

3. Adaptive Cross Approximation

The initial analytical form of the ACA algorithm was been designed to interpolate and, hopefully, to approximate a given function $K : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ of two variables x and y by a degenerate function S_n, i.e.

$$K(x, y) \approx S_n\,(x, y) = \sum_{\ell=1}^{n} u_\ell(x) v_\ell(y),$$

where $u_l : \mathbb{X} \to \mathbb{R}$, $v_l : \mathbb{Y} \to \mathbb{R}$, $l = 1, \ldots, n$. The construction runs as follows. Let $X \subset \mathbb{X} \subset \mathbb{R}^3$ and $Y \subset \mathbb{Y} \subset \mathbb{R}^3$ be discrete point sets.

Algorithm 1.

1. `initialization`
   `1.1 set initial residual and initial approximation`
   $$R_0(x, y) = K(x, y), \quad S_0(x, y) = 0;$$
   `1.2 choose initial pivot position`
   $$x_0 \in X, \;\; y_0 \in Y, \;\; R_0(x_0, y_0) \neq 0.$$
2. `recursion for` $k = 0, 1, \ldots$.
   `2.1 new residual`
   $$R_{k+1}(x, y) = R_k(x, y) - \frac{R_k(x, y_k) R_k(x_k, y)}{R_k(x_k, y_k)}\,;$$
   `2.2 new approximation`
   $$S_{k+1}(x, y) = S_k(x, y) + \frac{R_k(x, y_k) R_k(x_k, y)}{R_k(x_k, y_k)}\,;$$
   `2.3 new pivot position`
   $$x_{k+1} \in X, \;\; y_{k+1} \in Y, \;\; R_{k+1}(x_{k+1}, y_{k+1}) \neq 0.$$

After $n \geq 1$ steps of the ACA-Algorithm 1, we obtain a sequence of residuals $R_0, \ldots, R_n$ and a sequence of approximations $S_0, \ldots, S_n$ with the following properties.

1. Approximation property for $k = 0, \ldots, n$
   $$R_k(x, y) + S_k(x, y) = K(x, y), \;\; x \in X, \;\; y \in Y; \tag{4}$$
2. Interpolation property for $k = 1, \ldots, n$ and $\ell = 0, \ldots, k-1$
   $$R_k(x, y_\ell) = R_k(x_\ell, y) = 0, \;\; x \in X, \;\; y \in Y$$
   or
   $$S_k(x, y_\ell) = K(x, y_\ell), \;\; x \in X, \quad S_k(x_\ell, y) = K(x_\ell, y), \;\; y \in Y;$$
3. Harmonicity property for $k = 0, \ldots, n$.
   If
   $$\mathcal{L}_x K(x, y) = 0, \;\; x \in \Omega,$$
   then
   $$\mathcal{L}_x R_k(x, y) = \mathcal{L}_x S_k(x, y) = 0, \;\; x \in \Omega;$$

4. Non-recursive representation for $k = 1, \ldots, n$

$$S_k(x, y) = u_k^\top(x) V_k^{-1} w_k(y), \quad V_k \in \mathbb{R}^{k \times k}, \quad u_k(x), w_k(y) \in \mathbb{R}^k \tag{5}$$

with

$$u_k(x) = \big(K(x, y_0), \ldots, K(x, y_{k-1})\big)^\top,$$
$$w_k(y) = \big(K(x_0, y), \ldots, K(x_{k-1}, y)\big)^\top$$

and

$$V_k = \big(K(x_i, y_j)\big)_{i,j=0}^{k-1}$$

The above properties, except the last one, can be easily seen. The proof of the non-recursive representation is more technical and can be found in [2].

4. Adaptive FSM

In this section, we formulate a new adaptive FSM for the boundary value problem (1). Let u^* be the fundamental solution of the differential operator $\mathcal{L}$, $X \subset \Gamma$ a discrete set of the control points, $Y \subset \mathbb{R}^3 \setminus \overline{\Omega}$ a discrete set of the singularity points and ε an upper threshold for the error in the collocation points.

Algorithm 2.

1. `initialization`
   - 1.1 `initial error and initial pivot position`
     $$Error_1 = \mathtt{Max}_{x \in X} |g(x)|, \quad x_1 = \mathtt{ArgMax}_{x \in X} |g(x)|;$$
   - 1.2 `initial residual`
     $$R_1(x, y) = u^*(x - y);$$
   - 1.3 `first basis function`
     $$\varphi_1(x) = \frac{R_1(x, y_1)}{R_1(x_1, y_1)};$$
   - 1.4 `first approximation`
     $$u_1(x) = \alpha_1 \varphi_1(x), \quad \alpha_1 = g(x_1).$$
2. `recursion for` $k = 1, 2, \ldots$
   - 2.1 `new error and new pivot position`
     $$Error_{k+1} = \mathtt{Max}_{x \in X} |g(x) - u_k(x)|, \quad x_{k+1} = \mathtt{ArgMax}_{x \in X} |g(x) - u_k(x)|;$$
   - 2.2 `stopping criteria`
     `Stop if` $Error_{k+1} \le \varepsilon$ `or` $k = \#$ of points in X;
   - 2.3 `next residual`
     $$R_{k+1}(x, y) = R_k(x, y) - \frac{R_k(x, y_k) R_k(x_k, y)}{R_k(x_k, y_k)}\,; \tag{6}$$
   - 2.4 `next basis function`
     $$\varphi_{k+1}(x) = \frac{R_{k+1}(x, y_{k+1})}{R_{k+1}(x_{k+1}, y_{k+1})}\,; \tag{7}$$

`2.5 next approximation`

$$u_{k+1}(x) = u_k(x) + \alpha_{k+1}\varphi_{k+1}(x), \quad \alpha_{k+1} = g(x_{k+1}) - u_k(x_{k+1}).$$

After n steps of the above algorithm, we obtain the following approximation:

$$u_n(x) = \sum_{k=1}^{n} \alpha_k \varphi_k(x). \tag{8}$$

The basis functions φ_k are $\mathcal{L}$-harmonic for all $k = 1, \dots, n$

$$\mathcal{L}\varphi_k(x) = 0 \ \text{ for } \ x \in \Omega.$$

Therefore, the function u_n is likewise $\mathcal{L}$-harmonic

$$\mathcal{L}u_n(x) = 0 \ \text{ for } \ x \in \Omega.$$

The function u_n fulfills the boundary condition pointwise at the pivot points

$$u_n(x_k) = g(x_k) \ \text{ for } \ k = 1, \dots, n$$

and approximates the boundary condition in the other points

$$|u_n(x) - g(x)| \le \varepsilon \ \text{ for } \ x \in X \setminus \{x_1, \dots, x_n\}.$$

Later on, our numerical examples will show that the number n of steps required to obtain a given accuracy is rather small compared to, and seems to be independent of, the number of control points N, i.e. $n \ll N$. Due to the ACA interpolation property of the residuals R_k, the basis functions φ_k , $k \ge 2$ vanish at all previous pivot points

$$\varphi_k(x_\ell) = 0, \ \ \ell = 1, \dots, k-1, \ \ k = 2, \dots, n$$

and due to the construction,

$$\varphi_k(x_k) = 1, \ \ k = 1, \dots, n.$$

Thus, the coefficients α_k in (8) can be easily computed as in `Step 2.5` of the Algorithm 2 without the need to solve a system of equations, or more precisely by solving a small system

$$F\underline{a} = \underline{g}\,, \quad F = \big(\varphi_\ell(x_k)\big)_{k,\ell=1}^{n} \in \mathbb{R}^{n\times n}, \ \ \underline{a}, \underline{g} \in \mathbb{R}^n$$

with the following triangular matrix

$$F = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \varphi_1(x_2) & 1 & 0 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_1(x_n) & \varphi_2(x_n) & \varphi_3(x_n) & \dots & 1 \end{pmatrix}.$$

However, the price for the above simple and efficient algorithm is a more complicated evaluation of the basis functions φ_k and hence of the approximation u_n at a given point $x \in \Omega$. We use the non-recursive representation (5) of the ACA approximation S_n, the approximation property (4), and the definition of the basis function in `Step 2.4` of the Algorithm 2 to obtain

$$\varphi_1(x) = \frac{u^*(x - y_1)}{u^*(x_1 - y_1)}$$

and for $k = 2, \ldots, n$

$$\varphi_k(x) = \frac{u^*(x - y_k) - u_k^\top(x) z_k}{u^*(x_k - y_k) - u_k^\top(x_k) z_k} \tag{9}$$

with

$$\begin{aligned} u_k(x) &= \big(u^*(x - y_1), \ldots, u^*(x - y_{k-1})\big)^\top, \\ z_k &= V_k^{-1} w_k(y_k)\,, \\ w_k(y_k) &= \big(u^*(x_1 - y_k), \ldots, u^*(x_{k-1} - y_k)\big)^\top \end{aligned} \tag{10}$$

and

$$V_k = \big(u^*(x_i - y_j)\big)_{i,j=1}^{k-1}.$$

The vectors $z_k \in \mathbb{R}^{k-1}$, as well as the normalizing constants $(u^*(x_1 - y_1))^{-1}$ and $(u^*(x_k - y_k) - u_k^\top(x_k) z_k)^{-1}$ can be precomputed during the algorithm as follows. Let

$$V_2 = L_2 U_2 = 1 \cdot u^*(x_1 - y_1)$$

be the LU-decomposition of the 1×1-matrix V_2. Then, making use of the LU-decomposition of the $(k-1) \times (k-1)$-matrix

$$V_k = L_k U_k, \quad k = 2, \ldots, n-1,$$

we get for the $k \times k$-matrix V_{k+1}

$$V_{k+1} = \begin{pmatrix} L_k & 0 \\ a^\top & 1 \end{pmatrix} \begin{pmatrix} U_k & b \\ 0 & c \end{pmatrix}$$

with

$$L_k b = w_k(y_k), \quad a^\top U_k = u_k^\top(x_k), \quad c = u^*(x_k - y_k) - a^\top b.$$

For the vectors z_k, we get

$$z_k = U_k^{-1} L_k^{-1} w_k(y_k) = U_k^{-1} b.$$

From equations (6) and (7), we can see that

$$\begin{aligned} R_{k+1}(x, y) &= R_k(x, y) - R_k(x_k, y) R_k^{-1}(x_k, y_k) R_k(x, y_k) \\ &= R_k(x, y) - R_k(x_k, y) \varphi_k(x) \\ &= u^*(x - y) - \sum_{j=1}^{k} R_j(x_j, y) \varphi_j(x) \quad \forall k > 0 \end{aligned}$$

and thus

$$u^*(x - y) = R_{k+1}(x, y) + \sum_{l=1}^{k} R_l(x_l, y) \varphi_l(x) \quad \forall k \geq 0.$$

This leads to

$$\begin{pmatrix} u^*(x_1-y_1) & \cdots & u^*(x_1-y_k) \\ \vdots & \ddots & \vdots \\ u^*(x_k-y_1) & \cdots & u^*(x_k-y_k) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \varphi_1(x_2) & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_1(x_k) & \varphi_2(x_k) & \cdots & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} R_1(x_1,y_1) & R_1(x_1,y_2) & \cdots & R_1(x_1,y_k) \\ 0 & R_2(x_2,y_2) & \cdots & R_2(x_2,y_k) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & R_k(x_k,y_k) \end{pmatrix}.$$

Due to the uniqueness of the LU decomposition with unit diagonal entries, we see that

$$L_n = F.$$

A numerical evaluation of the basis function φ_k in (9) requires the scalar product $u_k^\top(x)z_k$ and, therefore $\mathcal{O}(k)$ arithmetical operations. The approximate solution u_n will require $\mathcal{O}(n^2)$ arithmetical operations for every evaluation.

The adaptive Fundamental Solution Method allows us to elaborate alternative strategies for choosing singularity points y_i. Instead of introducing fixed pairs (x_i, y_i) of collocation and singularity points we may equip a simply shaped pseudo-boundary, e.g. an ellipsoid with a large number of uniformly distributed candidate points. The adaptive FSM can be tuned to pick from those candidates a singularity point that maximizes the current basis function's pivot element $R_{k+1}(x_{k+1}, y)$ in (7).

5. Numerical Examples

In order to investigate the features of the adaptive Fundamental Solution Method, we perform numerical experiments for the BVP (1) with Laplace or Helmholtz operators. We compare the results of classical FSM, adaptive FSM with the given thresholds as well as a threshold-free adaptive method. For the latter method we store the maximal local error of an iteration step and terminate, if no improvement is achieved after a given number of further iterations. By dropping the coefficients associated with these additional steps, we restore the currently best result (with respect to local errors).

The indicated condition numbers are calculated by using LAPACK routines [1]. Condition numbers of respective system matrices are labeled cond_{sys}, while those of matrices required for evaluation of basis function in the adaptive method are labeled cond_{LU}. In the latter case we only indicate condition numbers of the respective largest matrix, i.e. the matrix used in the evaluation of the basis function with highest index.

TABLE 1. Laplace equation in the unit ball, $N = 5120$.

threshold	max.err.	rel.err.	$\mathrm{cond}_{\mathrm{sys}}$	cond_{LU}	# nodes
classical	$8.80 \cdot 10^{-10}$	$1.01 \cdot 10^{-11}$	$1.42 \cdot 10^{21}$	-	5120
10^{-4}	$9.78 \cdot 10^{-5}$	$1.08 \cdot 10^{-6}$	$1.91 \cdot 10^{2}$	$4.53 \cdot 10^{10}$	415
10^{-6}	$1.01 \cdot 10^{-6}$	$1.10 \cdot 10^{-8}$	$3.26 \cdot 10^{2}$	$8.00 \cdot 10^{12}$	711
10^{-8}	$1.05 \cdot 10^{-8}$	$1.13 \cdot 10^{-10}$	$4.94 \cdot 10^{2}$	$7.26 \cdot 10^{14}$	1069
10^{-10}	$1.09 \cdot 10^{-10}$	$1.11 \cdot 10^{-12}$	$6.09 \cdot 10^{2}$	$9.81 \cdot 10^{16}$	1531

5.1. **Laplace equation.** We consider the model problem

$$\begin{aligned} -\Delta v(x) &= 0 \quad \text{for} \quad x \in \Omega, \\ v(x) &= g(x) \quad \text{for} \quad x \in \partial\Omega, \end{aligned}$$

with the known analytical solution

$$v(x) = \sin(2\pi x_1) x_2 e^{-2\pi x_3}, \quad g = v\big|_\Gamma,$$

in order to display some general observations regarding the adaptive FSM, which are also relevant for other equation types we have already considered.

Performance of the adaptive FSM. As one can see in Table 1, the adaptive method uses only a small number of collocation points which increases upon setting a lower threshold. The accuracy of the full FSM can be achieved even with a relatively small subset of collocation points. This reduction leads to condition numbers of the involved matrices which are significantly lower than those of the full method's system matrix. Since the approximation space of the adaptive method is always a subset of the full FSM approximation space, the outperformance in the last row in the table can only be explained by a loss of accuracy due to high condition numbers.

Evolution of maximal local error. The strategy of the adaptive method consists in eliminating the currently largest residual of all collocation points, while not altering those at collocation points already treated. However, there is no guarantee that after any elimination step the new maximal error is actually smaller than the previous one. In fact, as the maximal error asymptotically decreases during the elimination process, short-term increases are rather typical (cf. Figure 1 for a brick-shaped domain).

For tight thresholds the adaptive method uses a number of collocation points comparable to that of classical FSM. For very large problems this may lead to errors in the evaluation of basis functions (evaluation of z_k in (10)) and ultimately to an asymptotic increase of the maximal local error. For these cases it is handy to store information about the "best" step so far and restore the corresponding result. Thus, although the threshold is not met, the results in these extreme cases are far better than those of the classical method.

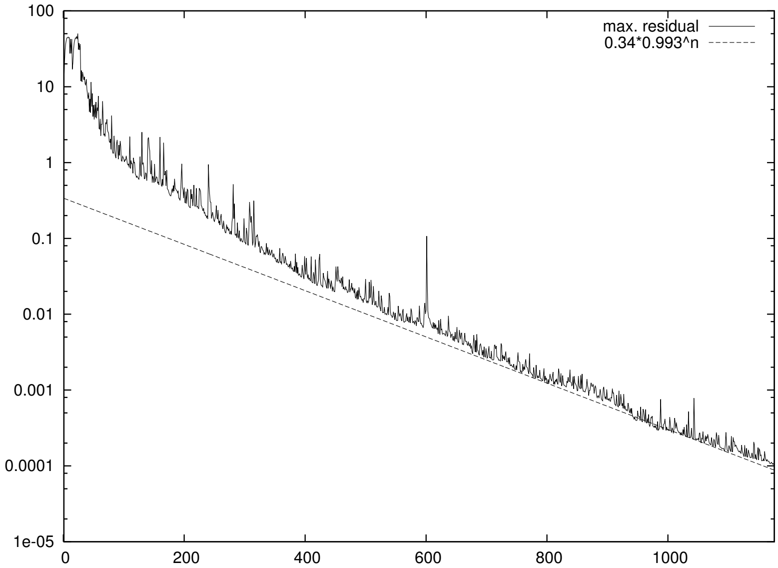

FIGURE 1. Evolution of maximal local error for a Laplace equation on a brick-shaped domain.

Distribution of errors. Both methods, classical and adaptive FSM, control errors at the collocation points only. Therefore there arises the question, how the errors behave inside the domain and on the boundary between the collocation points.

Fundamental solutions and the derived basis functions of the adaptive method are $\mathcal{L}$-harmonic. Due to the maximum principle for the Laplace equations the error assumes its maximum on the boundary of Ω. This can be illustrated in an error plot along a line segment through the domain (cf. Figure 2). The gradient and the Hessian errors show similar behavior.

Looking at the error on the boundary in case of the adaptive method, one observes a pattern of low error speckles (cf. Figure 3). These correspond to the collocation points where local errors have been eliminated. In this example, the singularity points were located on an ellipsoidal pseudo-boundary adapted to the domain's shape. One can observe a higher concentration of speckles in regions located closer to the pseudo-boundary. This is in agreement with the theory of classical FSM, where a lower distance between the boundaries leads to higher stability, but to slower convergence [6].

5.2. **Helmholtz equation.** We perform experiments for the Helmholtz equation

$$\begin{aligned} \Delta v(x) + \kappa^2 v(x) = 0, \quad \kappa = 2^n, \quad n = 1, \dots, 5 \quad \text{for} \quad x \in \Omega, \\ v(x) = g(x) \quad \text{for} \quad x \in \partial\Omega, \end{aligned}$$

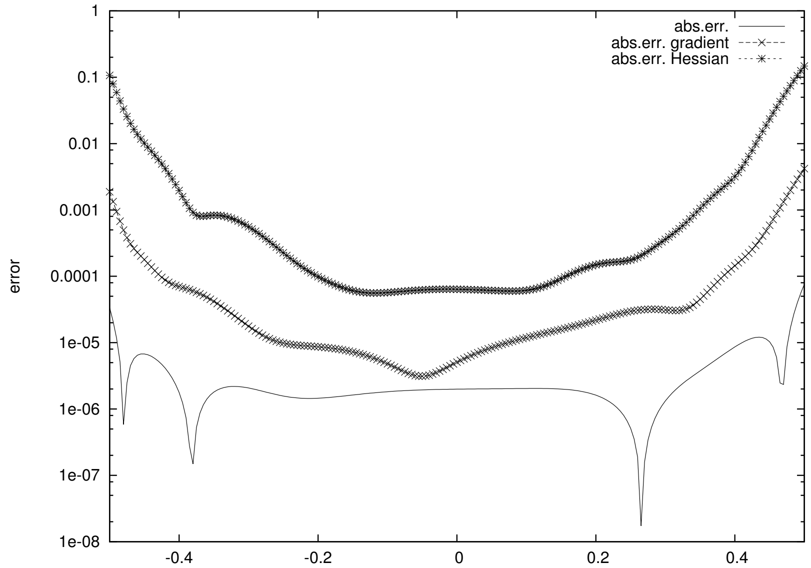

Figure 2. Absolute values of errors for Laplace problem on a brick-shaped domain measured on a line segment intersecting the surface at $\pm\frac{1}{2}$. Downward spikes correspond to the changes of sign.

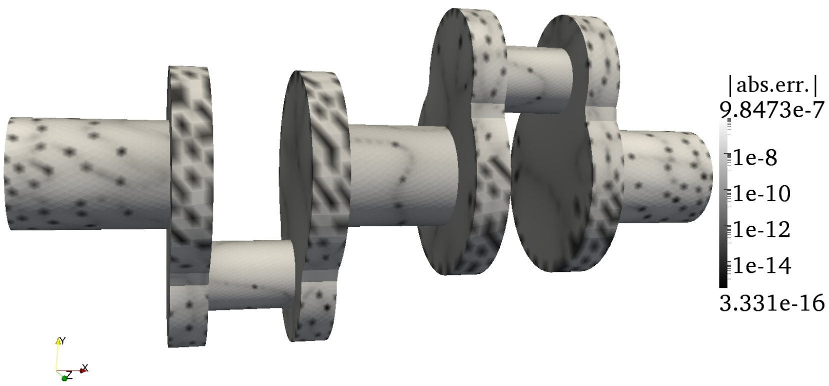

Figure 3. Absolute error for Laplace problem on the boundary of a crankshaft domain.

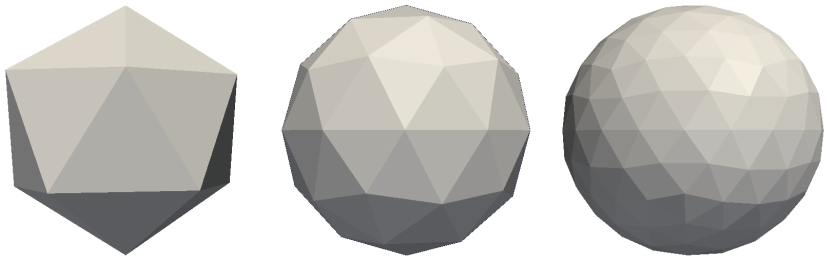

FIGURE 4. Approximations of unit sphere with 20, 80 and 320 triangles, respectively.

with the known analytical solution

$$v(x) = \exp\Big(-\frac{\kappa}{\sqrt{2}} i x_1 \Big) x_2 \sin\Big(\frac{\kappa}{\sqrt{2}} x_3 \Big), \quad g = v\big|_\Gamma.$$

Here Ω is the unit ball; its surface Γ is approximated by triangulated surface meshes. These meshes are obtained by a quasi-uniform refinement starting from an icosahedron (cf. Figure 4). The collocation points are derived as barycenters of the mesh triangles, singularity points are obtained by shifting the collocation points along the surface normal. Although Fundamental Solution Methods do not require an actual mesh, we will stick to this term, since the collocation points are derived from meshes.

TABLE 2. Helmholtz equation, varying κ, $N = 20480$.

κ	rel. err. classical FSM	rel. err. ada 10^{-11}	steps required (thres. 10^{-11})
1	$4.57 \cdot 10^{-13}$	$1.42 \cdot 10^{-11}$	976
2	$2.92 \cdot 10^{-13}$	$7.84 \cdot 10^{-12}$	1019
4	$6.46 \cdot 10^{-13}$	$6.18 \cdot 10^{-12}$	1102
8	$1.43 \cdot 10^{-12}$	$5.85 \cdot 10^{-12}$	1277
16	$3.00 \cdot 10^{-12}$	$5.94 \cdot 10^{-12}$	1792
32	$1.80 \cdot 10^{-11}$	$2.59 \cdot 10^{-11}$	3653 (no thres.)

Performance of the adaptive FSM. As one could expect, for both, classical and adaptive FSM, the quality of results gets worse with increasing κ

TABLE 3. Helmholtz equation, $\kappa = 8$, $N = 1280$.

threshold	max.err.	rel.err.	$\mathrm{cond}_{\mathrm{sys}}$	cond_{LU}	# nodes
classical	$3.96 \cdot 10^{-10}$	$1.43 \cdot 10^{-10}$	$3.11 \cdot 10^{13}$	-	1280
10^{-8}	$1.20 \cdot 10^{-8}$	$7.40 \cdot 10^{-9}$	$8.36 \cdot 10^{2}$	$2.67 \cdot 10^{10}$	721
10^{-9}	$1.43 \cdot 10^{-9}$	$8.69 \cdot 10^{-10}$	$9.18 \cdot 10^{2}$	$2.73 \cdot 10^{11}$	872
10^{-10}	$5.39 \cdot 10^{-10}$	$1.86 \cdot 10^{-10}$	$1.01 \cdot 10^{3}$	$4.94 \cdot 10^{12}$	1042
10^{-11}	$3.94 \cdot 10^{-10}$	$1.47 \cdot 10^{-10}$	$1.08 \cdot 10^{3}$	$1.44 \cdot 10^{14}$	1200
10^{-12}	$3.96 \cdot 10^{-10}$	$1.43 \cdot 10^{-10}$	$1.11 \cdot 10^{3}$	$2.64 \cdot 10^{14}$	1276
10^{-13}	$3.96 \cdot 10^{-10}$	$1.43 \cdot 10^{-10}$	$1.12 \cdot 10^{3}$	$2.66 \cdot 10^{14}$	1278

TABLE 4. Helmholtz equation, $\kappa = 8$, $N = 20480$.

threshold	max.err.	rel.err.	$\mathrm{cond}_{\mathrm{sys}}$	cond_{LU}	# nodes
classical	$2.08 \cdot 10^{-12}$	$1.43 \cdot 10^{-12}$	$1.26 \cdot 10^{21}$	-	20480
10^{-10}	$9.86 \cdot 10^{-11}$	$6.06 \cdot 10^{-11}$	$3.13 \cdot 10^{3}$	$2.72 \cdot 10^{12}$	1086
10^{-11}	$1.01 \cdot 10^{-11}$	$5.85 \cdot 10^{-12}$	$3.43 \cdot 10^{3}$	$1.70 \cdot 10^{13}$	1277
10^{-12}	$1.02 \cdot 10^{-12}$	$6.15 \cdot 10^{-13}$	$3.65 \cdot 10^{3}$	$1.80 \cdot 10^{14}$	1493
10^{-13}	$1.08 \cdot 10^{-13}$	$5.97 \cdot 10^{-14}$	$3.98 \cdot 10^{3}$	$2.36 \cdot 10^{15}$	1753
none	$7.83 \cdot 10^{-14}$	$4.80 \cdot 10^{-14}$	$4.00 \cdot 10^{3}$	$3.99 \cdot 10^{15}$	1781

(cf. Table 2). While classical FSM suffers from a loss of approximation quality, the adaptive method compensates for this by the use of a larger number of basis functions. For strict thresholds, the adaptive method achieves the accuracy of the classical method on coarser meshes (cf. Table 3) and even outperforms it on fine meshes (cf. Table 4). On such meshes, the classical FSM suffers from extremely high condition numbers $\mathrm{cond}_{\mathrm{sys}}$ of the system matrix leading to a loss of accuracy.

Number of required collocation points. Of interest is the observation when the number of available collocation points increases, the number of steps required in the adaptive method to reach a certain threshold does not seem to grow (cf. Figure 5). As is seen from Table 5 on smaller clusters, where the threshold is reached faster, the results are worse. This is due to the fact, that on finer meshes the adaptive FSM has more points to choose from during the error elimination steps. Nevertheless, any threshold can be achieved theoretically by eliminating all (or almost all) errors at the collocation points. In this case, the adaptive method is equivalent to the full FSM.

Effects of large condition numbers. Figure 6 shows the loss of accuracy in the classical Fundamental Solution Method. When the number of collocation points grows beyond a critical value, the error starts to grow slowly. While this growth does not necessarily lead to very large errors, if a better

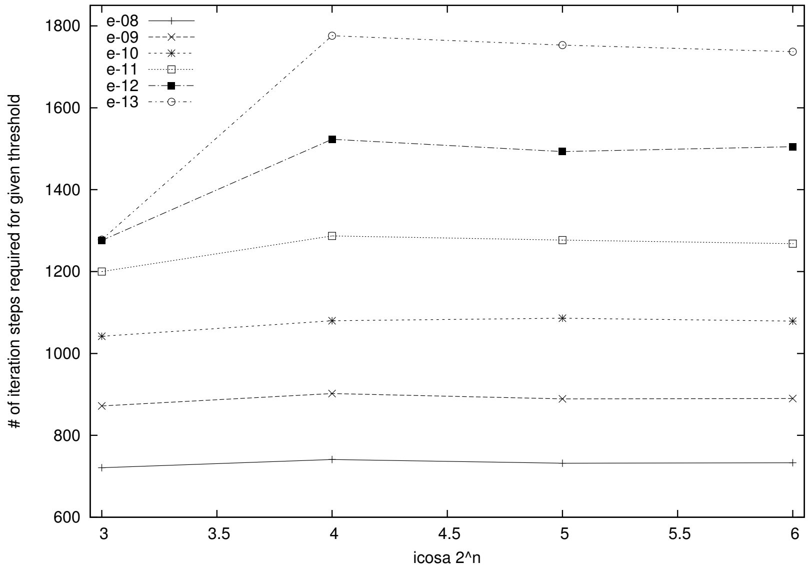

FIGURE 5. Number of iteration steps required to reach a certain threshold for a varying number of collocation points.

TABLE 5. Helmholtz equation, $\kappa = 8$, different geometries, threshold $= 10^{-12}$.

N	steps required	rel.err.	rel.err. classical FSM
1280	1276	$1.43 \cdot 10^{-10}$	$1.43 \cdot 10^{-10}$
5120	1523	$6.47 \cdot 10^{-13}$	$5.85 \cdot 10^{-13}$
20480	1493	$6.15 \cdot 10^{-13}$	$1.43 \cdot 10^{-12}$
81920	1505	$5.92 \cdot 10^{-13}$	-

approximation is desired, one has to repeat the calculation with fewer collocation points. In the same figure, the growth of the condition number is indicated. To the same problem the adaptive FSM is applied (cf. Figure 7). It can be seen that there exists a critical step after which the maximal local error will grow due to the loss of accuracy in floating point operations. However, we can still use stored data from the previous steps in order to obtain a better result.

6. Conclusion and Outlook

When applied to large problems, the Fundamental Solution Method features system matrices with extremely large condition numbers. We have

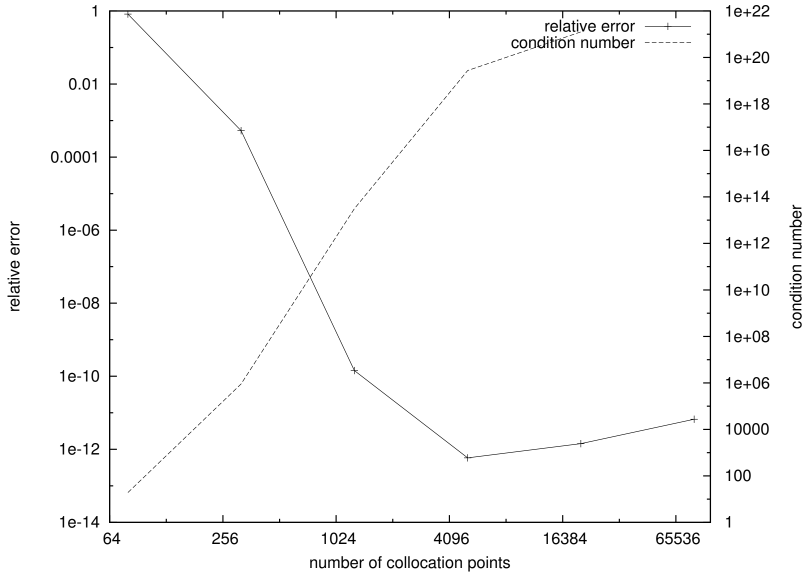

FIGURE 6. Full FSM: Condition number and loss of accuracy for large numbers of collocation points.

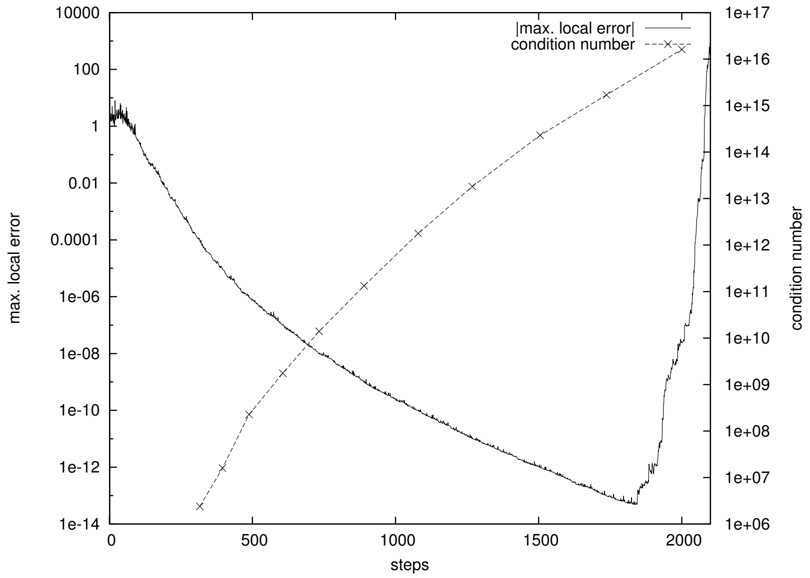

FIGURE 7. Adaptive FSM: Maximal local error, Helmholtz equation, $\kappa = 8$, $N = 81920$.

presented an adaptive method in which dimensions and the condition numbers of the matrices involved are reduced by several orders. Numerical results show that the quality of approximations is comparable to that of the classical method. Also, the new method leads to reasonable results even in scenarios, where the classical method fails.

Future work will include the extension of the adaptive method to vector-valued problems with a special focus on elastostatics. We are also planning to investigate the convergence of the method in a theoretical context.

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Authors' address:

Department of Mathematics, Saarland University, P.O. Box 15 11 50, 66041 Saarbrücken, Germany.

E-mails: `fleck@num.uni-sb.de`; `richards@num.uni-sb.de`; `rjasanow@num.uni-sb.de`

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Tony Hill

SHAMIR–DUDUCHAVA FACTORIZATION OF ELLIPTIC SYMBOLS

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. This paper considers the factorization of elliptic symbols which can be represented by matrix-valued functions. Our starting point is a *Fundamental Factorization Theorem*, due to Budjanu and Gohberg [2]. We critically examine the work of Shamir [15], together with some corrections and improvements as proposed by Duduchava [6]. As an integral part of this work, we give a new and detailed proof that certain sub-algebras of the Wiener algebra on the real line satisfy a sufficient condition for a right standard factorization. Moreover, assuming only the Fundamental Factorization Theorem, we provide a complete proof of an important result from Shargorodsky [16], on the factorization of an elliptic homogeneous matrix-valued function, useful in the context of the inversion of elliptic systems of multidimensional singular integral operators in a half-space.

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Key words and phrases. Matrix-valued elliptic symbols, factorization, rationally dense algebras of smooth functions, splitting algebras.

რეზიუმე. სტატიაში განხილულია ისეთი ელიფსური სიმბოლოების ფაქტორიზაცია, რომლებიც შეიძლება წარმოდგენილი იქნას მატრიცული მნიშვნელობების მქონე ფუნქციებით. ჩვენი ამოსავალი წერტილა ბიჯანუს და გოჰბერგის ფუნდამენტური ფაქტორიზაციის თეორემა (იხ. [2]). ჩვენ კრიტიკულად განვიხილავთ შამირის ნაშრომს [15] დუდუჩავას მიერ შეტანილი ცვლილებებისა და გაუმჯობესებების გათვალისწინებით. ამ ნაშრომის განუყოფელი ნაწილია ჩვენს მიერ მოყვანილი ახალი და დეტალური დამტკიცება იმ ფაქტისა, რომ ვინერის ალგებრის ზოგიერთი ქვეალგებრები ნამდვილ წრფეზე აკმაყოფილებენ ნამდვილი სტანდარტული ფაქტორიზაციის სტანდარტულ პირობას. ამის გარდა, მხოლოდ ფუნდამენტურ ფაქტორიზაციის თეორემაზე დაყრდნობით ჩვენ მოგვყავს სრული დამტკიცება შარგოროდსკის მნიშვნელოვანი შედეგისა [16] ელიფსური ერთგვაროვანი მატრიცული მნიშვნელობების მქონე ფუნქციების ფაქტორიზაციის შესახებ, რომელიც სასარგებლოა ნახევარსივრცეში მრავალგანზომილებიანი სინგულარული ინტეგრალური ოპერატორების შებრუნებისას.

1. Introduction

This paper considers the factorization of elliptic symbols which can be represented by matrix-valued functions. Our starting point is a *Fundamental Factorization Theorem* due to Budjanu and Gohberg [2]. We critically examine the work of Shamir [15], together with some corrections and improvements as proposed by Duduchava [6]. We shall call the combined efforts of these two latter authors the *Shamir–Duduchava* factorization method.

One important application of the Shamir–Duduchava factorization method has been given by Shargorodsky [16]. Our primary goal is to provide, in a single place, a complete proof of Shargorodsky's result on the factorization of a matrix-valued elliptic symbol, assuming only the Fundamental Factorization Theorem. As an integral part of this work, we will give a new and detailed proof that certain sub-algebras of the Wiener algebra on the real line satisfy a sufficient condition for the right standard factorization.

2. Background

Let Γ denote a simple closed smooth contour dividing the complex plane into two regions D_+ and D_-, where for a bounded contour we identify D_+ with the domain contained within Γ. We shall be especially interested in the case where $\Gamma = \dot{\mathbb{R}}$, the one point compactification of the real line. In this situation, of course, $D_\pm$ are simply the upper and lower half-planes, respectively. Let $G_\pm$ denote the union $D_\pm \cup \Gamma$.

2.1. **Factorization.** Suppose we are given a nonsingular matrix-valued function $A(\zeta) = \big(a_{jk}(\zeta)\big)_{j,k=1}^N$, then we define a *right standard factorization*, or simply the *factorization* as a representation of the form

$$A(\zeta) = A_-(\zeta)D(\zeta)A_+(\zeta) \quad (\zeta \in \Gamma), \tag{2.1}$$

where $D(\zeta)$ is strictly diagonal with non-zero elements $d_{jj} = ((\zeta - \lambda^+)/(\zeta - \lambda^-))^{\kappa_j}$ for $j = 1, \dots, N$. The exponents $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_N$ are integers and $\lambda^\pm$ are certain fixed points chosen in $D_\pm$, respectively. (In passing, we note that if $\Gamma = \dot{\mathbb{R}}$, it is customary to take $\lambda^\pm = \pm i$.) $A_\pm(\zeta)$ are square $N \times N$ matrices that are analytic in $D_\pm$ and continuous in $G_\pm$. Moreover, the determinant of $A_+(A_-)$ is nonzero on $G_+(G_-)$.

As one would expect, interchanging the matrices $A_-(\zeta)$ and $A_+(\zeta)$ in (2.1) gives rise to a *left standard factorization.* In either a right or a left factorization, the integers $\kappa_j = \kappa_j(A)$ are uniquely determined (see [9]) by the matrix $A(\zeta)$. Further, if the matrix $A(\zeta)$ admits a factorization for a pair of points $\lambda^\pm$, then it admits a factorization of the same type for any pair of points $\mu^\pm \in D_\pm$, in that the *right* or *left indices*, denoted by $\{\kappa_j(A),\ j = 1, \dots, N\}$, are independent of the points $\lambda^\pm$.

2.2. **Banach algebras of continuous functions.** Let $\mathcal{U}(\Gamma)$ denote a Banach algebra of continuous functions on Γ which includes the set of all rational functions $R(\Gamma)$ not having any poles on Γ. Further we insist that

$\mathcal{U}(\Gamma)$ is *inverse closed* in the sense that if $a(\zeta) \in \mathcal{U}(\Gamma)$ and $a(\zeta)$ does not vanish anywhere on Γ, then $a^{-1}(\zeta) \in \mathcal{U}(\Gamma)$. Of course, $\mathcal{U}(\Gamma) \subset C(\Gamma)$, where $C(\Gamma)$ is the Banach algebra of all continuous functions on Γ, with the usual supremum norm.

Consider the region G_+. By $R^+(\Gamma)$ we denote the set of all rational functions not having any poles in this domain and by $C^+(\Gamma)$ the closure of $R^+(\Gamma)$ in $C(\Gamma)$ with respect to the norm of $C(\Gamma)$. It is easy to see that $C^+(\Gamma)$ is a subalgebra of $C(\Gamma)$ consisting of those functions that have analytic continuation to D_+ and which are continuous on G_+. We can now define $\mathcal{U}^+(\Gamma) = \mathcal{U}(\Gamma) \cap C^+(\Gamma)$. Again, it is straightforward to show that $\mathcal{U}^+(\Gamma)$ is a subalgebra of $\mathcal{U}(\Gamma)$. (Similar definitions of $C^-(\Gamma)$ and $\mathcal{U}^-(\Gamma)$ follow by considering the region G_-.)

2.3. **Splitting algebras.** It turns out that the ability to factorize a given matrix is intimately linked to the ability to express $\mathcal{U}(\Gamma)$ as a direct sum of two subalgebras - one containing analytic functions defined on D_+ and the other analytic functions on D_-. To ensure the uniqueness of this partition we let $\mathring{\mathcal{U}}^-(\Gamma)$ denote the subalgebra of $\mathcal{U}^-(\Gamma)$ consisting of all functions that vanish at the chosen point $\lambda^- \in D_-$. We now say that a Banach algebra $\mathcal{U}(\Gamma)$ *splits* if we can write

$$\mathcal{U}(\Gamma) = \mathcal{U}^+(\Gamma) \oplus \mathring{\mathcal{U}}^-(\Gamma).$$

The prototypical example of a splitting algebra is the Wiener algebra, $W(\mathbb{T})$, of all functions defined on $\mathbb{T}$, the unit circle $|\zeta| = 1$, of the form

$$a(\zeta) = \sum_{j=-\infty}^{\infty} a_j \zeta^j \quad \left(\sum_{j=-\infty}^{\infty} |a_j| < \infty \right)$$

with the norm $\|a(\zeta)\| = \sum_{j=-\infty}^{\infty} |a_j|$. The Banach algebras $W^{\pm}(\mathbb{T})$ have a simple characterization. For example, $W^+(\mathbb{T})$ consists of all functions in $W(\mathbb{T})$ that can be expanded as an absolutely converging series in nonnegative powers of ζ. However, the algebra $C(\mathbb{T})$ does not split. (For more details see [2].)

2.4. ***R*-algebras.** We say that a Banach algebra $\mathcal{U}(\Gamma)$ of complex-valued functions continuous on Γ is an *R-algebra* if the set of all rational functions $R(\Gamma)$ with poles not lying on Γ is contained in $\mathcal{U}(\Gamma)$ and this set is dense, with respect to the norm of $\mathcal{U}(\Gamma)$. In passing, we note that any R-algebra of continuous functions is inverse closed. (See, for example, [4, Chapter 2, Section 3, p. 44].) Following Theorem 5.1, p. 20 [3], we have:

Theorem 2.1 (Fundamental Factorization Theorem)**.** *Let $\mathcal{U}(\Gamma)$ be an arbitrary splitting R-algebra. Then every nonsingular matrix-valued function $A(\zeta) \in \mathcal{U}_{N\times N}(\Gamma)$ admits a right standard factorization with factors $A_\pm(\zeta)$ in the subalgebras $\mathcal{U}^{\pm}_{N\times N}(\Gamma)$.*

2.5. **Wiener algebras on the real line.** Let $L_1(\mathbb{R})$ denote the usual convolution algebra of Lebesgue integrable functions on the real line. For any $g \in L_1(\mathbb{R})$, we define the *Fourier Transform* of g as the function $\mathcal{F}g$, or $\widehat{g}$, given by

$$(\mathcal{F}g)(t) = \widehat{g}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{ixt}\,dx.$$

We let $C_0^\infty(\mathbb{R})$ denote the algebra of continuous functions f on $\mathbb{R}$ which vanish at $\pm\infty$. It is well known (see, for example, [13, Chapter 9, Theorem 9.6, p. 182]) that if $g \in L_1(\mathbb{R})$, then

$$\widehat{g} \in C_0^\infty(\mathbb{R}), \quad \|\widehat{g}\|_\infty \le \|g\|_1. \tag{2.2}$$

The *Wiener algebra* $W(\mathbb{R})$ is the set of all functions of the form $f = \widehat{g} + c$, where $g \in L_1(\mathbb{R})$ and c is a constant. The norm on $W(\mathbb{R})$ is given by

$$\|f\|_{W(\mathbb{R})} = \|g\|_1 + |c|.$$

Suppose $f_1 = \widehat{g}_1 + c_1$, $f_2 = \widehat{g}_2 + c_2 \in W(\mathbb{R})$. Then since $\widehat{g}_1\widehat{g}_2 = \widehat{g_1 * g_2}$ (see, for example, [13, Chapter 9, Theorem 9.2, p. 179]), it is straightforward to show that $W(\mathbb{R})$ is a Banach algebra.

We will also consider certain subalgebras of the Wiener algebra $W(\mathbb{R})$. For $r = 0, 1, 2, \ldots$ we define $W^r(\mathbb{R})$ to be the set of functions f such that

$$(1 - it)^k D^k f(t) \in W(\mathbb{R}) \quad (k = 0, 1, \ldots, r),$$

where D^k is the kth order derivative. (Of course, $W^0(\mathbb{R})$ is simply $W(\mathbb{R})$.) We shall show that $W^r(\mathbb{R})$ is a Banach algebra and, moreover, is a splitting R-algebra.

2.6. **Homogeneity, differentiability and ellipticity.** Suppose $\xi = (\xi_1, \ldots \xi_n) \in \mathbb{R}^n$ for some integer $n \ge 2$. It will be convenient to write $\xi = (\xi', \xi_n)$, where $\xi' \in \mathbb{R}^{n-1}$. We assume that $\mathbb{R}^n$ has the usual Euclidean norm, and we let $\mathbb{S}^{n-1}$ denote the set $\{\xi \in \mathbb{R}^n \mid \ \xi_1^2 + \cdots + \xi_n^2 = 1\}$.

We further suppose that $A_0(\xi', \xi_n)$ is an $N \times N$ matrix-valued function defined on $\mathbb{R}^n$, which is homogeneous of degree 0. In addition, we will assume that the elements of the matrix $A_0(\xi', \xi_n)$ belong to $C^{r+2}(\mathbb{S}^{n-1})$, for some non-negative integer r, where $C^r(\mathbb{S}^{n-1})$ denotes the set of r times continuously differentiable functions on the domain $\mathbb{S}^{n-1}$. Finally, we assume that $A_0(\xi', \xi_n)$ is elliptic, in that

$$\inf_{\xi \in \mathbb{S}^{n-1}} \left| \det A_0(\xi) \right| > 0.$$

2.7. **The matrices $E_\pm$ and E.** We will be particularly interested in the behavior of $A_0(\xi', \xi_n)$ as $\xi_n \to \pm\infty$.

Our approach is effectively to fix ξ', and thereby consider factorization in the one-dimensional (scalar) variable ξ_n. Since $A_0(\xi', \xi_n)$ is homogeneous of degree zero,

$$\lim_{\xi_n \to \pm\infty} A_0(\xi', \xi_n) = A_0(0, \ldots, 0, \pm 1),$$

for fixed ξ'. We define

$$E_{\pm} := A_0(0, \dots, 0, \pm 1) \text{ and } E := E_+^{-1} E_- . \tag{2.3}$$

2.8. **The matrices $B_{\pm}$.** It is a standard result that any $E \in \mathbb{C}_{N \times N}$ can be expressed in *the Jordan Canonical Form*

$$h_1 E h_1^{-1} = J := \operatorname{diag}[J_1, \dots, J_l],$$

where the *Jordan block* $J_k = J_k(\lambda_k)$ is a matrix of order m_k with eigenvalue λ_k on every diagonal entry, 1 on the super-diagonal and 0 elsewhere. The matrix h_1 is invertible and

$$m_1 + \dots + m_l = N.$$

The Jordan matrix J is unique up to the ordering of the blocks J_k, $k = 1, \dots, l$.

Let $B^m(z)$ be the $m \times m$ matrix $(b_{jk}(z))_{j,k=1}^m$ given by

$$b_{jk}(z) := \begin{cases} 0, & j < k, \\ 1, & j = k, \\ \dfrac{z^{j-k}}{(j-k)!}, & j > k. \end{cases}$$

We now define

$$K := \operatorname{diag}[K_1, \dots, K_l], \tag{2.4}$$

where $K_k := \lambda_k B^{m_k}(1)$. By construction, K is a lower triangular matrix whose block structure *and* diagonal elements are identical to those of J.

A routine inspection of the equation

$$K_k u = \lambda_k u$$

shows that the eigenspace associated with the eigenvalue λ_k has dimension one. Therefore (see [5, p. 191]), the matrix K_k is similar to the Jordan block $J_k(\lambda_k)$ for $k = 1, \dots, l$. Thus K is similar to J, and we have

$$J = h_2 \, K \, h_2^{-1},$$

for some nonsingular matrix h_2. Hence we can write

$$E = h \, K \, h^{-1}, \text{ where } h := h_1^{-1} h_2. \tag{2.5}$$

For any $z_1, z_2 \in \mathbb{C}$ and positive integer m, it is easy to show that the matrix-valued functions $B^m(z)$ satisfy

$$B^m(z_1 + z_2) = B^m(z_1) B^m(z_2), \quad B^m(0) = I. \tag{2.6}$$

In particular, taking $z_2 = -z_1$, gives

$$B^m(-z_1) = \left[B^m(z_1)\right]^{-1}. \tag{2.7}$$

In the analysis that follows we will use the logarithm function on the complex plane. Unless specifically stated to the contrary, we will always take the principal branch of the logarithm Log z defined by

$$\operatorname{Log} z = \log|z| + i \arg z, \quad -\pi < \arg z \leq \pi,$$

for any non-zero $z \in \mathbb{C}$. In other words, we assume that the discontinuity in $\arg z$ occurs across the negative real axis.

For any $t \in \mathbb{R}$, we now define the complex-valued functions

$$\alpha_\pm(t) := (2\pi i)^{-1} \log(t \pm i). \tag{2.8}$$

Then

$$\lim_{t\to+\infty} \big[\alpha_+(t) - \alpha_-(t)\big] = 0, \quad \lim_{t\to-\infty} \big[\alpha_+(t) - \alpha_-(t)\big] = 1. \tag{2.9}$$

Corresponding to the block decomposition in (2.4), we set

$$B_\pm(t) = \operatorname{diag}\Big[B^{m_1}\big((2\pi i)^{-1}\log(t\pm i)\big), \dots, B^{m_l}\big((2\pi i)^{-1}\log(t\pm i)\big)\Big]. \tag{2.10}$$

We note, in passing, that in the special case that $l = N$, then $B_\pm(t) = I$.

Following [15], we now give a simple test for membership of $W^r(\mathbb{R})$ for continuously differentiable functions.

Lemma 2.2. *Let $r = 0, 1, 2, \dots$ and suppose the function $b(t) \in C^{r+1}(\mathbb{R})$ has the property that, for some $\delta > 0$,*

$$D^k b(t) = O\big(|t|^{-k-\delta}\big), \quad k = 0, 1, \dots, (r+1),$$

then $b(t) \in W^r(\mathbb{R})$.

Proof. We follow the approach given in [15]. For $0 \le k \le r$, we define

$$g_k(t) = (1 - it)^k b^{(k)}(t).$$

Our goal is to show that $g_k(t) \in W(\mathbb{R})$.

Differentiating with respect to t,

$$g_k'(t) = -ik(1 - it)^{k-1} b^{(k)}(t) + (1 - it)^k b^{(k+1)}(t).$$

Then, by hypothesis, g_k and g_k' are continuous. Moreover, as $|t| \to \infty$, we have $g_k(t) = O(|t|^{-\delta})$ and $g_k'(t) = O(|t|^{-1-\delta})$. Hence $g_k'(t) \in L^2(\mathbb{R})$.

On applying the Fourier transform ($\mathcal{F}_{t\to\xi}$) to the function $g_k'(t)$, we obtain $\xi\, \widehat{g}_k(\xi) \in L^2(\mathbb{R})$. But using the Cauchy–Schwarz inequality

$$\int\limits_{|\xi|\ge\epsilon} |\widehat{g}_k(\xi)|\, d\xi = \int\limits_{|\xi|\ge\epsilon} \frac{1}{|\xi|}\, |\xi\, \widehat{g}_k(\xi)|\, d\xi \le \Bigg(\int\limits_{|\xi|\ge\epsilon} \frac{1}{|\xi|^2}\, d\xi \Bigg)^{\frac{1}{2}} \|\xi\, \widehat{g}_k\|_{L^2} < \infty.$$

Hence, $\widehat{g}_k(\xi)$ is absolutely integrable everywhere outside a neighborhood $(-\epsilon, \epsilon)$ of zero. On the other hand, for small $|\xi|$, from [17, Theorem 127, p. 173], $\widehat{g}_k(\xi) = O(|\xi|^{\delta-1})$ and hence $\widehat{g}_k(\xi)$ is absolutely integrable inside $(-\epsilon, \epsilon)$.

Thus, $\widehat{g}_k(\xi) \in L^1(\mathbb{R})$. We now define a new function $h_k(x) = \widehat{g}_k(-x)$. Then, by construction, $h_k(x) \in L^1(\mathbb{R})$ and taking the Fourier transform

$(\mathcal{F}_{x\to t})$ of $h_k(x)$, we obtain

$$\begin{aligned}\widehat{h}_k(t) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\widehat{g}_k(-x)e^{ixt}dx\\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\widehat{g}_k(x)e^{-ixt}dx\\ &= g_k(t).\end{aligned}$$

Now $\widehat{h}_k(t) \in W(\mathbb{R})$, and hence, $g_k(t) \in W(\mathbb{R})$. This completes the proof of the lemma. □

2.9. **Key theorem from Shamir.** The next theorem (see [15, Appendix, pp. 122–123]) considers some properties of a certain matrix-valued function derived from an elliptic homogeneous matrix-valued function of degree zero. Together with Theorem 2.1, it will provide the starting point for proving our second result.

Theorem 2.3. *Suppose that $A_0(\xi', \xi_n) \in C^{r+3}_{N\times N}(\mathbb{S}^{n-1})$ is a matrix-valued function which is homogeneous of degree 0 and elliptic. Suppose that the Jordan form of $A_0^{-1}(0,\dots,0,1)A_0(0,\dots,0,-1)$ has blocks $J_k(\lambda_k)$ of size m_k for $k = 1,\dots,l$. Let the matrix $c := A_0^{-1}(0,\dots,0,1)$, and the constant invertible matrix h be as in equation (2.5). Then, for the fixed $\xi' \neq 0$,*

$$\begin{aligned}&\lim_{\xi_n\to+\infty} h^{-1}cA_0(\xi',\xi_n)h = I,\\ &\lim_{\xi_n\to-\infty} h^{-1}cA_0(\xi',\xi_n)h = \operatorname{diag}\left[\lambda_1 B^{m_1}(1),\dots,\lambda_l B^{m_l}(1)\right].\end{aligned}$$

Further, let $\zeta = (\zeta_1,\dots,\zeta_N)$, where

$$\zeta_q = -\frac{\log\lambda_j}{2\pi i} \quad \textit{for} \quad \sum_{k=1}^{j-1} m_k < q \le \sum_{k=1}^{j} m_k, \quad q = 1,\dots,N, \tag{2.11}$$

and define

$$(\xi_n \pm i)^{\zeta} := \operatorname{diag}\left[(\xi_n \pm i)^{\zeta_1},\dots,(\xi_n \pm i)^{\zeta_N}\right].$$

Then, for the fixed $\xi' \neq 0$,

$$\begin{aligned}A_0^*(\xi',\xi_n) := (\xi_n - i)^{-\zeta}B_-(\xi_n)h^{-1}&\\ \times cA_0(\xi',\xi_n)hB_+^{-1}(\xi_n)(\xi_n+i)^{\zeta} &\in W^{r+2}_{N\times N}(\mathbb{R}),\end{aligned}$$

and

$$\lim_{\xi_n\to\pm\infty} A_0^*(\xi',\xi_n) = I. \tag{2.12}$$

Proof. A detailed proof of this theorem is given in Appendix A. □

Remark 2.4. Note that in (2.11), the definition of ζ_q, $q = 1,\dots,N$ includes a multiplicative factor of (-1) not given in [15].

Remark 2.5. Since we are assuming that for every non-zero $z \in \mathbb{C}$ we have $-\pi < \arg z \le \pi$, it follows immediately that

$$-\frac{1}{2} \le \operatorname{Re}\zeta_j < \frac{1}{2}\,, \quad j = 1, \dots, N.$$

and hence

$$\delta_0 := \min_{1 \le j,k \le N} (1 - \operatorname{Re}\zeta_k + \operatorname{Re}\zeta_j) > 0. \tag{2.13}$$

3. Statement of results

Theorem 3.1. *For $r = 0, 1, 2, \dots, W^r(\mathbb{R})$ is a splitting R-algebra.*

Our second result considers the factorization of an elliptic matrix-valued function of degree μ, and it confirms the isotropic case of Lemma 1.9, p. 60 [16].

Theorem 3.2. *Let $r := [n/2]+1$. Suppose that $A \in C^{r+3}_{N\times N}(\mathbb{R}^n)$ is a matrix-valued function which is homogeneous of degree μ and elliptic. Then, for the fixed $\omega \in \mathbb{S}_{n-2}$,*

$$A_\omega(\xi) = A\big(|\xi'|\omega_1, \dots, |\xi'|\omega_{n-1}, \xi_n\big)$$

admits the factorization

$$A_\omega(\xi) = \big(\xi_n - i|\xi'|\big)^{\mu/2} A^-_\omega(\xi)\, D(\omega, \xi)\, A^+_\omega(\xi) \big(\xi_n + i|\xi'|\big)^{\mu/2},$$

where $(A^-_\omega(\xi))^{\pm 1}$ and $(A^+_\omega(\xi))^{\pm 1}$ are homogeneous matrix-valued functions of order 0 that, for the fixed $\xi' \neq 0$, satisfy estimates of the form

$$\sum_{0 \le q \le r} \operatorname*{ess\,sup}_{\xi_n \in \mathbb{R}} \left| \xi_n^q D^q_{\xi_n} \left(A^\pm_\omega(\xi', \xi_n)\right)_{j,k} \right| < +\infty, \quad 1 \le j, k \le N. \tag{3.1}$$

Further, they have analytic extensions with respect to ξ_n in the lower half-plane and the upper half-plane, respectively.

$D(\omega, \xi)$ is a lower triangular matrix with elements

$$\Big(\frac{\xi_n - i|\xi'|}{\xi_n + i|\xi'|}\Big)^{\kappa_k(\omega)+\zeta_k}$$

on its diagonal. Its off-diagonal terms are homogeneous of degree 0, and they satisfy an estimate of the form (3.1). The integer

$$\begin{aligned}\kappa(\omega) &:= \sum_{k=1}^N \kappa_k(\omega) \\ &= \frac{1}{2\pi} \Delta \arg\det \Big[\big(|\xi'|^2 + \xi_n^2\big)^{-\mu/2} A_\omega(\xi', \xi_n) \Big] \Big|_{\xi_n=-\infty}^{+\infty} - \sum_{k=1}^N \operatorname{Re}\zeta_k\end{aligned}$$

depends continuously on $\omega \in \mathbb{S}^{n-2}$. *The partial sums* $\sum\limits_{k=1}^{M} \kappa_j(\omega)$, $1 \le M < N$, *are upper semicontinuous,*

$$\zeta_k = -\frac{\log \lambda_j}{2\pi i} \quad \textit{for} \quad \sum_{\nu=1}^{j-1} m_\nu < k \le \sum_{\nu=1}^{j-1} m_\nu, \quad k = 1, \ldots, N,$$

λ_j *are eigenvalues of the matrix* $A^{-1}(0,\ldots,0,+1)A(0,\ldots,0,-1)$ *to which there correspond Jordan blocks of dimension* m_j.

4. Proof of the First Result

The objective of this section is to prove Theorem 3.1. Let $\theta^{\pm}$ denote the characteristic functions of $\mathbb{R}^{\pm}$, respectively.

Lemma 4.1. *The Wiener algebra* $W(\mathbb{R})$ *is an* R*-algebra.*

Proof. An abbreviated proof of this lemma is given in [4, Chapter 2, Section 4, pp. 62–63]. A more detailed proof is included here, both for completeness and to introduce some analysis that will be useful when considering the subalgebras $W^r(\mathbb{R})$ for $r \ge 1$.

We begin by showing that $W(\mathbb{R})$ contains all rational functions with poles off $\dot{\mathbb{R}}$. Firstly, we note the identities

$$(t - z_+)^{-1} = \mathcal{F}_{x\to t}\big(\sqrt{2\pi}\, i\, \theta^-(x)\, e^{-iz_+x}\big), \quad \operatorname{Im} z_+ > 0,$$
$$(t - z_-)^{-1} = -\mathcal{F}_{x\to t}\big(\sqrt{2\pi}\, i\, \theta^+(x)\, e^{-iz_-x}\big), \quad \operatorname{Im} z_- < 0,$$

where the functions $\theta^-(x)e^{-iz_+x}$ and $\theta^+(x)e^{-iz_-x} \in L_1(\mathbb{R})$. Secondly, since all functions in $W(\mathbb{R})$ are bounded at infinity, any rational function in $W(\mathbb{R})$ must be such that the degree of the numerator must be less than or equal to the degree of the denominator. (In particular, non-constant polynomial functions are not included in $W(\mathbb{R})$.) Finally, the fact that $W(\mathbb{R})$ contains all rational functions with poles off $\dot{\mathbb{R}}$ now follows directly, because $W(\mathbb{R})$ is an algebra, and we have the usual partial fraction decomposition over $\mathbb{C}$.

We now wish to show that rational functions with poles off $\dot{\mathbb{R}}$ are dense in $W(\mathbb{R})$. Suppose $f \in W(\mathbb{R})$ is arbitrary and $r \in W(\mathbb{R})$ is rational. By definition, we can write $f(t) = \widehat{g}(t) + c$ and $r(t) = \widehat{s}(t) + d$, where $g, s \in L_1(\mathbb{R})$ and $c, d \in \mathbb{C}$. Let $C_c^\infty(\mathbb{R})$ denote the set of smooth functions with compact support in $\mathbb{R}$. Then $C_c^\infty(\mathbb{R})$ is dense in $L_1(\mathbb{R})$ and

$$\begin{aligned}
\|f - r\|_W &:= \|g - s\|_{L_1} + |c - d| \\
&\le \|g - h\|_{L_1} + \|h - s\|_{L_1} + |c - d| \quad (\text{where } h \in C_c^\infty(\mathbb{R})) \\
&= \|g - h\|_{L_1} + \|\theta^+h + \theta^-h - \theta^+s - \theta^-s\|_{L_1} \quad (\text{taking } d = c) \\
&\le \|g - h\|_{L_1} + \|\theta^+h - \theta^+s\|_{L_1} + \|\,\theta^-h - \theta^-s\|_{L_1}.
\end{aligned}$$

Of course, the approximations to θ^+h and θ^-h, by θ^+s and θ^-s, respectively, are independent but similar. Hence, to prove that $W(\mathbb{R})$ is an R-algebra, it is enough for us to show that we can approximate $\theta^+(x)h(x)$, where $h \in C_c^\infty(\mathbb{R})$, arbitrarily closely in the $L_1(\mathbb{R})$ norm by a function $\theta^+(x)s(x)$ such that $\widehat{\theta^+ s}$ is rational and has no poles in the upper half-plane.

For $x \geq 0$, we let $y = e^{-x}$ and define

$$\psi(y) := \begin{cases} h(-\log(y))/y & \text{if } y \in (0,1], \\ 0 & \text{if } y = 0. \end{cases}$$

Since $h(x)$ has compact support, $\psi(y)$ is identically zero in some interval $[0,\nu)$, where $\nu > 0$. Thus, by construction, $\psi(y) \in C^\infty[0,1]$.

Hence, given any $\epsilon > 0$, we can choose a Bernstein polynomial (see [12]) $(B_M\psi)(y)$, of degree $M = M(\epsilon)$ such that

$$\begin{aligned} &\sup_{y\in[0,1]} \left|\psi(y) - (B_M\psi)(y)\right| < \epsilon \\ \Longrightarrow\ &\sup_{y\in[0,1]} \left|\psi(y) - \sum_{k=0}^{M} b_k y^k\right| < \epsilon \ \text{ for certain } \ b_k \in \mathbb{C}, \quad k = 0,1,2,\dots,M \\ \Longrightarrow\ &\sup_{x\in[0,\infty)} \left|h(x)e^x - \sum_{k=0}^{M} b_k e^{-kx}\right| < \epsilon. \end{aligned}$$

We let $S(x) = \sum\limits_{k=0}^{M} b_k e^{-kx}$ and observe, therefore, that our proposed approximant to $\theta^+ h(x)$ is $\theta^+ S(x)e^{-x}$.

Of course, the Fourier transform of $\theta^+ S(x)e^{-x}$ is a rational function with no poles in the upper half-plane, since for $k = 1,2,3,\dots$ we have

$$\widehat{\theta^+ e^{-kx}} = \frac{i}{\sqrt{2\pi}}\,\frac{1}{t + ik}\,.$$

Finally, we take $\theta^+ s(x) := \theta^+ S(x)e^{-x}$ and then

$$\begin{aligned} \|\theta^+ h - \theta^+ s(x)\|_{L_1} &= \int_0^\infty |h(x) - S(x)e^{-x}|\,dx \\ &= \int_0^\infty |h(x)e^x - S(x)|\,e^{-x}\,dx \\ &\leq \epsilon \int_0^\infty e^{-x}\,dx \\ &= \epsilon. \end{aligned}$$

This completes the proof that $W(\mathbb{R})$ is an R-algebra. □

Remark 4.2. Suppose now that $f = \widehat{g} \in W(\mathbb{R})$. From the proof of the above lemma, we can show that $\widehat{\theta^+ g} \in C^+(\dot{\mathbb{R}})$. (See section 2.2.) Indeed, applying inequality (2.2), we have

$$\left\|\widehat{\theta^+ g} - \widehat{\theta^+ s(x)}\right\|_\infty \le \|\theta^+ g - \theta^+ s(x)\|_{L_1}.$$

Since $\widehat{\theta^+ s(x)} \in R^+(\dot{\mathbb{R}})$, we immediately have $\widehat{\theta^+ g} \in C^+(\dot{\mathbb{R}})$, because $C^+(\dot{\mathbb{R}})$ is the closure of $R^+(\dot{\mathbb{R}})$ with respect to the supremum norm. It follows in an exactly similar way that $\widehat{\theta^- g} \in C^-(\dot{\mathbb{R}})$.

Lemma 4.3. *The Wiener algebra $W(\mathbb{R})$ splits.*

Proof. An abbreviated proof of this lemma is given in [4, Chapter 2, Section 4, p. 63]. A more detailed proof is included here for completeness.

Our method of proof is a direct construction. Suppose $f = \widehat{g} + c \in W(\mathbb{R})$ then, since $g = \theta^+ g + \theta^- g$, we have

$$\begin{aligned} f &= \widehat{\theta^+ g} + \widehat{\theta^- g} + c \\ &= \big(\widehat{\theta^+ g} + c_+\big) + \big(\widehat{\theta^- g} + c_-\big) \end{aligned}$$

where $c = c_+ + c_-$, and c_- is chosen such that

$$(\widehat{\theta^- g})(-i) + c_- = 0.$$

But since $g \in L_1(\mathbb{R})$, we have $\theta^\pm g \in L_1(\mathbb{R})$. Moreover, from Remark 4.2, we have $\widehat{\theta^\pm g} \in C^\pm(\dot{\mathbb{R}})$ and thus

$$\widehat{\theta^\pm g} \in W(\mathbb{R}) \cap C^\pm(\dot{\mathbb{R}}).$$

In other words, we have the required decomposition, and thus

$$W(\mathbb{R}) = W^+(\mathbb{R}) \oplus \mathring{W}^-(\mathbb{R})$$

where $\mathring{W}^-(\mathbb{R}) = \{h \in W^-(\mathbb{R}) :\ h(-i) = 0\}$. This completes the proof that $W(\mathbb{R})$ splits. $\square$

Remark 4.4. For any $\varphi \in \mathcal{S}(\mathbb{R})$, we now define three integral operators:

$$\Pi^\pm \varphi(t) = \frac{(\pm 1)}{2\pi i} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{\varphi(\tau)}{\tau - (t \pm i\epsilon)}\, d\tau, \quad S_{\mathbb{R}}\varphi(t) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(\tau)}{\tau - t}\, d\tau.$$

For more details see [7] and [8]. Each of these operators is bounded on $S(\mathbb{R})$. Moreover (see [7, Chapter II Section 5, pp. 70–71]),

$$\Pi^\pm \widehat{\varphi} = \widehat{\theta^\pm \varphi}.$$

But since $S(\mathbb{R})$ is dense in $W_0(\mathbb{R}) := \{f \in W(\mathbb{R}) :\ f = \widehat{g},\ g \in L_1(\mathbb{R})\}$, each of the singular integral operators can be extended, by continuity, to a bounded operator on $W_0(\mathbb{R})$.

Finally, we have the well-known formulae

$$\Pi^+ + \Pi^- = I, \quad \Pi^+ = \frac{1}{2}(I + S_{\mathbb{R}}), \quad \Pi^- = \frac{1}{2}(I - S_{\mathbb{R}}).$$

Lemma 4.5. *For $r = 1, 2, 3, \ldots$, $W^r(\mathbb{R})$ is a Banach algebra with a norm that is equivalent to the norm*

$$\|f\|_{W^r} = \|f\|_W + \sum_{k=1}^{r} \left\|(1-it)^k D^k f(t)\right\|_W.$$

Proof. The proof that $W^r(\mathbb{R})$ is a Banach algebra is straightforward. However, as an illustration, we will prove that given $f_1, f_2 \in W^r(\mathbb{R})$, the product $f_1 f_2 \in W^r(\mathbb{R})$ and $\|f_1 f_2\|_{W^r} \leq C_r \|f_1\|_{W^r} \|f_2\|_{W^r}$, for some constant C_r that depends only on r.

The existence of a norm $\|\cdot\|'_{W^r}$ equivalent to $\|\cdot\|_{W^r}$ and such that $\|f_1 f_2\|'_{W^r} \leq \|f_1\|'_{W^r}\|f_2\|'_{W^r}$ is then guaranteed by [14, Theorem 10.2, p. 246].

Suppose $f_1, f_2 \in W^r(\mathbb{R})$. Then, for any integer p satisfying $1 \leq p \leq r$,

$$(1-it)^p D_t^p [f_1(t) f_2(t)] = \sum_{k=0}^{p} \binom{p}{k} \left[(1-it)^k D^k f_1\right] \left[(1-it)^{p-k} D^{p-k} f_2\right].$$

We assume that $W(\mathbb{R})$ is a Banach algebra and therefore, $f_1 f_2 \in W(\mathbb{R})$ and $(1-it)^p D^p [f_1(t) f_2(t)] \in W(\mathbb{R})$. Hence, $f_1 f_2 \in W^r(\mathbb{R})$, as required.

By definition,

$$\begin{aligned}
\|f_1 f_2\|_{W^r} &= \|f_1 f_2\|_W + \sum_{k=1}^{r} \left\|(1-it)^k D^k [f_1 f_2]\right\|_W \\
&= \|f_1 f_2\|_W \\
&\quad + \sum_{k=1}^{r} \left\| \sum_{j=0}^{k} \binom{k}{j} \left[(1-it)^j D^j f_1\right] \left[(1-it)^{k-j} D^{k-j} f_2\right] \right\|_W \\
&\leq \|f_1\|_W \|f_2\|_W \\
&\quad + \sum_{k=1}^{r} \sum_{j=0}^{k} \binom{k}{j} \left\|(1-it)^j D^j f_1\right\|_W \left\|(1-it)^{k-j} D^{k-j} f_2\right\|_W \\
&\leq C_r \|f_1\|_{W^r} \|f_2\|_{W^r},
\end{aligned}$$

where the strictly positive constant C_r depends only on the integer r. This completes the proof of the lemma. □

We now show that $W^r(\mathbb{R})$ splits. To do this, we will need two intermediate lemmas.

Lemma 4.6. *Suppose $f(t), Df(t) \in W(\mathbb{R})$ and $\lim_{t\to\pm\infty} f(t) = 0$. Then $\Pi^{\pm} Df(t) = D\Pi^{\pm} f(t)$.*

Proof. From [8, Chapter I, Section 4.4, p. 31], we have

$$DS_{\mathbb{R}} f(t) = S_{\mathbb{R}} Df(t)$$

But, from Remark 4.4 we have $\Pi^{\pm} = \frac{1}{2}(I \pm S_{\mathbb{R}})$, respectively, and so

$$D\Pi^{\pm} f(t) = \Pi^{\pm} Df(t).$$ □

Lemma 4.7. *Suppose $f(t), tf(t) \in W(\mathbb{R})$ and $\lim_{t\to\pm\infty} f(t) = 0$. Let $[tI, \Pi^{\pm}]$ denote the commutator of tI and $\Pi^{\pm}$. Then $[tI, \Pi^{\pm}]f \in \mathbb{C}$.*

Proof. Suppose $f(t), tf(t) \in W(\mathbb{R})$. Then

$$\begin{aligned}
[tI, \Pi^+]f &= (tI\Pi^+ - \Pi^+ tI)f \\
&= \frac{1}{2}\big(t(I+S_{\mathbb{R}}) - (I+S_{\mathbb{R}})tI\big)f \quad \text{(by Remark 4.4)} \\
&= \frac{1}{2}(tS_{\mathbb{R}} - S_{\mathbb{R}}t)f \\
&= \frac{t}{\pi i}\int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - t}\,d\tau - \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\tau f(\tau)}{\tau - t}\,d\tau \\
&= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{(t-\tau)f(\tau)}{\tau - t}\,d\tau \\
&= \frac{(-1)}{\pi i}\int_{-\infty}^{\infty} f(\tau)\,d\tau \\
&\in \mathbb{C}.
\end{aligned}$$

Finally, we note that $[tI, \Pi^-] = [tI, I - \Pi^+] = [tI, I] - [tI, \Pi^+] = 0 - [tI, \Pi^+]$ and, hence, $[tI, \Pi^-] \in \mathbb{C}$. This completes the proof of the lemma. ☐

Lemma 4.8. *For $r = 0, 1, 2, \ldots$ the algebra $W^r(\mathbb{R})$ splits.*

Proof. Suppose $f(t) \in W^r(\mathbb{R})$ for some nonnegative integer r. Since $f(t) \in W(\mathbb{R})$, it is enough to consider the case where $\lim_{t\to\pm\infty} f(t) = 0$. Moreover, by Remarks 4.2 and 4.4, we can write

$$f(t) = \Pi^+ f(t) + \Pi^- f(t), \quad \Pi^{\pm} f \in W(\mathbb{R}) \cap C^{\pm}(\dot{\mathbb{R}}).$$

Thus, to complete the proof, we have to show that $\Pi^{\pm} f(t) \in W^r(\mathbb{R})$. That is, we have prove that for $k = 0, 1, \ldots r$ we have $(1 - it)^k D^k \Pi^{\pm} f(t) = i^{-k}(t+i)^k D^k \Pi^{\pm} f(t) \in W(\mathbb{R})$.

We now proceed by induction on r. Our inductive hypothesis is that for any $f \in W^r(\mathbb{R})$, we have $(t+i)^r D^r \Pi^{\pm} f(t) = (\Pi^{\pm}(t+i)^r D^r f(t) + c) \in W(\mathbb{R})$. We have previously proved this result for $r = 0$. Suppose that the inductive hypothesis holds for $k = 0, \ldots, (r-1)$.

From Lemma 4.6,

$$\begin{aligned}
(t+i)^r D^r \Pi^{\pm} f &= t \cdot (t+i)^{r-1} D^r \Pi^{\pm} f + i \cdot (t+i)^{r-1} D^r \Pi^{\pm} f \\
&= t \cdot (t+i)^{r-1} D^{r-1} \Pi^{\pm}(Df) + i \cdot (t+i)^{r-1} D^{r-1} \Pi^{\pm}(Df).
\end{aligned}$$

But since $Df \in W^{r-1}(\mathbb{R})$, applying the inductive hypothesis, we get

$$\begin{aligned}&(t+i)^r D^r \Pi^{\pm} f\\ &\quad = t \cdot \Pi^{\pm}(t+i)^{r-1} D^{r-1}(Df) + i \cdot \Pi^{\pm}(t+i)^{r-1} D^{r-1}(Df) + c\\ &\quad = t \cdot \Pi^{\pm}(t+i)^{r-1} D^r f + i \cdot \Pi^{\pm}(t+i)^{r-1} D^r f + c.\end{aligned}$$

Hence, using Lemma 4.7 (applied to $(t+i)^{r-1}D^r f$), we obtain

$$\begin{aligned}(t+i)^r D^r \Pi^{\pm} f &= \Pi^{\pm} t(t+i)^{r-1} D^r f + \Pi^{\pm} i(t+i)^{r-1} D^r f + c'\\ &= \Pi^{\pm}(t+i)^r D^r f + c'\\ &\in W(\mathbb{R}).\end{aligned}$$

This completes the proof by induction. So, finally, for $k = 0, 1, \ldots, r$, we have $(1-it)^k D^k \Pi^{\pm} f(t) \in W(\mathbb{R})$ and thus, for $r = 0, 1, 2, \ldots$, the algebra $W^r(\mathbb{R})$ splits. □

Our final objective in this section is to show that $W^r(\mathbb{R})$ is an *R*-algebra for $r = 1, 2, 3, \ldots$, noting that in Lemma 4.1 we have proved this result for the special case $W(\mathbb{R})$, corresponding to $r = 0$.

In Appendix B we show that the Fourier transforms of smooth functions with a compact support and which are zero in a neighborhood of $x = 0$, are dense in the space $W^r(\mathbb{R})$. Then, proceeding analogously to Lemma 4.1, it is enough for us to show that we can approximate $\widehat{\theta^+ h}$, where $h \in C_c^\infty(\mathbb{R})$ and is zero near 0, arbitrarily closely in the $W^r(\mathbb{R})$ norm by the function $\widehat{\theta^+ s}$, that is rational and has no poles in the upper half- plane.

As previously, for $x \geq 0$, we set $y = e^{-x}$ and define

$$\psi(y) = \begin{cases} h(-\log(y))/y & \text{if } \ y \in (0,1],\\ 0 & \text{if } \ y = 0.\end{cases}$$

Since $h(x)$ has compact support, $\psi(y)$ is identically zero in some interval $[0, \nu)$, where $\nu > 0$. Thus, by construction, $\psi(y) \in C^\infty[0,1]$.

Remark 4.9. The motivation for choosing the Bernstein polynomial, $(B_M\psi)(y)$, can be found in [12], as the approximant to $\psi(y)$ in Lemma 4.1, is that we can *simultaneously* choose $M = M(\epsilon)$ such that for $1 \leq j \leq r$

$$\sup_{y\in[0,1]} \left|\psi(y) - (B_M\psi)(y)\right| < \epsilon \ \text{ and } \ \sup_{y\in[0,1]} \left|D_y^j \psi(y) - D_y^j (B_M\psi)(y)\right| < \epsilon.$$

Given $y = e^{-x}$, we can consider $\psi(y)$ in terms of x, as given by the equation $\psi(y) = e^x h(x)$. The following lemma expresses the derivatives of $\psi(y)$ in terms of the derivatives of $h(x)$.

Lemma 4.10.

$$D_y^j \psi(y) = (-1)^j e^{(j+1)x}(D_x + 1) \cdots (D_x + j) h(x) \ \ \textit{for} \ \ j = 1, 2, \ldots .$$

Proof. Note that by definition, $y = e^{-x}$ and $\psi(y) = e^x h(x)$. We use proof by induction on j.

Suppose $j = 1$. Then $D_x\psi(y) = D_y\psi(y) \cdot (dy/dx)$ and, hence.

$$D_y\psi(y) = -e^x D_x(e^x h) = -e^x(e^x h + e^x D_x h) = (-1)e^{2x}(D_x + 1)h,$$

completing the first step of the inductive proof.

Now suppose the result is true for $j = m$. Then, by the inductive hypothesis,

$$D_x[D_y^m\psi(y)] = D_x\Big[(-1)^m e^{(m+1)x}(D_x + 1)\cdots(D_x + m)h(x)\Big].$$

Hence,

$$\begin{aligned} D_y^{m+1}\psi(y)(dy/dx) &= (-1)^m(m+1)e^{(m+1)x}(D_x + 1)\cdots(D_x + m)h(x) \\ &\quad + (-1)^m e^{(m+1)x} D_x(D_x + 1)\cdots(D_x + m)h(x). \end{aligned}$$

Therefore,

$$\begin{aligned} D_y^{m+1}\psi(y) &= (-1)^{m+1}e^{(m+2)x}[m + 1 + D_x](D_x + 1)\cdots(D_x + m)h(x) \\ &= (-1)^{m+1}(m+1)e^{(m+2)x}(D_x + 1)\cdots(D_x + m)(D_x + m + 1)h(x), \end{aligned}$$

proving the result for $j = m+1$. This completes the proof by induction. □

Motivated by Lemma 4.10, for $j = 0, 1, 2, \ldots,$ we now define:

$$h_j(x) = \begin{cases} (D_x + 1)\cdots(D_x + j)h(x) & \text{if } j > 0, \\ h(x) & \text{if } j = 0. \end{cases}$$

Hence, we can write

$$D_y^j\psi(y) = (-1)^j e^{(j+1)x} h_j(x), \quad j = 0, 1, 2 \ldots . \tag{4.1}$$

In exactly the same way, given $y = e^{-x}$ and $(B_M\psi)(y) = S(x)$, we define $T(x) = S(x)e^{-x}$. Hence, $(B_M\psi)(y) = e^x T(x)$ and

$$D_y^j(B_M\psi)(y) = (-1)^j e^{(j+1)x}(D_x + 1)\cdots(D_x + j)T(x) \ \text{ for } \ j = 1, 2, \ldots .$$

Analogously, for $j = 0, 1, 2, \ldots$ we define:

$$T_j(x) = \begin{cases} (D_x + 1)\cdots(D_x + j)T(x) & \text{if } j > 0, \\ T(x) & \text{if } j = 0. \end{cases}$$

Hence, we can similarly write

$$D_y^j(B_M\psi)(y) = (-1)^j e^{(j+1)x} T_j(x), \quad j = 0, 1, 2 \ldots, \tag{4.2}$$

and we can now express our approximations in terms of the variable x.

Remark 4.11. Using equations (4.1) and (4.2), we can now reformulate the Bernstein polynomial, $(B_M\psi)(y)$, approximations to $\psi(y)$ and its derivatives as

$$\sup_{x\in[0,\infty)} \left| e^x h_0(x) - e^x T_0(x) \right| < \epsilon. \tag{4.3}$$

and for $1 \le j \le r$,

$$\sup_{x\in[0,\infty)} \left| e^{(j+1)x} h_j(x) - e^{(j+1)x} T_j(x) \right| < \epsilon.$$

Lemma 4.12. *For* $r = 1, 2, 3, \ldots$ $W^r(\mathbb{R})$ *is an* R*-algebra.*

Proof. Our proposed approximant to $\theta^+ h(x)$ is $\theta^+ S(x) e^{-x}$. From Appendix B, to show convergence to $\widehat{\theta^+ h}$ in $\|\cdot\|_{W^r}$, it suffices to show the convergence to $\theta^+ h, x^k(\theta^+ h)$ and $D_x^j(x^k(\theta^+ h))$ in $\|\cdot\|_{L_1}$ for all $1 \le j \le k \le r$.

Of course, one important consequence of the fact that our smooth function h is zero in the neighborhood of 0 is that it implies that $\theta^+ h$ is also smooth.

We have already seen in Lemma 4.1 that

$$\left\|\theta^+ h(x) - \theta^+ S(x) e^{-x}\right\|_{L_1} < \epsilon.$$

Similarly, for $1 \le k \le r$, we have

$$\begin{aligned}
\left\|\theta^+ x^k h(x) - \theta^+ x^k S(x) e^{-x}\right\|_{L_1} &= \int_0^\infty \left|x^k h(x) - x^k S(x) e^{-x}\right| dx \\
&= \int_0^\infty \left|e^x h(x) - S(x)\right| x^k e^{-x}\, dx \\
&= \int_0^\infty \left|e^x h_0(x) - e^x T_0(x)\right| x^k e^{-x}\, dx \\
&\le \epsilon \int_0^\infty x^k e^{-x}\, dx \quad \text{by (4.3)} \\
&= (k!)\,\epsilon, \quad \text{since} \quad \int_0^\infty x^k e^{-x} dx = k!\,.
\end{aligned}$$

Suppose that $j \ge 1$. Clearly, there exist constants $\{c_l : \ 0 \le l \le j\}$, *that depend only on* j such that

$$D_x^j h = \sum_{l=0}^{j} c_l h_l, \quad D_x^j T = \sum_{l=0}^{j} c_l T_l,$$

where $h_0 = h$, and $h_l = (D_x + 1)\cdots(D_x + l)h$ for $l > 0$, and $T_0 = T = S e^{-x}$, and $T_l = (D_x + 1)\cdots(D_x + l)T$ for $l > 0$.

Hence, for $1 \le j \le k \le r$,

$$\begin{aligned}
&\left\|\theta^+ x^k D_x^j h - \theta^+ x^k D_x^j (S e^{-x})\right\|_{L_1} \\
&\qquad = \left\|\theta^+ x^k \sum_{l=0}^{j} c_l (h_l - T_l)\right\|_{L_1} \\
&\qquad \le \sum_{l=0}^{j} |c_l| \cdot \left\|\theta^+ x^k (h_l - T_l)\right\|_{L_1}
\end{aligned}$$

$$= \sum_{l=0}^{j} |c_l| \int_0^\infty \left| x^k (h_l(x) - T_l(x)) \right| dx$$

$$= \sum_{l=0}^{j} |c_l| \int_0^\infty \left| e^{(l+1)x} h_l(x) - e^{(l+1)x} T_l(x) \right| e^{-(l+1)x} x^k \, dx$$

$$\leq \epsilon \sum_{l=0}^{j} |c_l| \int_0^\infty e^{-(l+1)x} x^k \, dx \quad \text{by (4.11)}$$

$$\leq \epsilon \sum_{l=0}^{j} |c_l| \frac{k!}{(l+1)^{k+1}} \, .$$

Therefore, $W^r(\mathbb{R})$ is an R-algebra, as required. □

5. Proof of the Second Result

The objective of this section is to prove Theorem 3.2. In determining certain asymptotic estimates for matrices arising during factorization, we follow the approach of Duduchava [6]. (For full details, see Appendix C.)

Proof. We begin by defining

$$A_0(\xi', \xi_n) := \left(|\xi'|^2 + |\xi_n|^2 \right)^{-\mu/2} A(\xi', \xi_n). \tag{5.1}$$

For the fixed $\xi' \neq 0$, we set

$$\omega := \frac{\xi'}{|\xi'|} \, ; \quad t := \frac{\xi_n}{|\xi'|} \, .$$

From Theorem 2.3, for the fixed $\omega \in S^{n-2}$,

$$A_0^*(\omega, t) = (t - i)^{-\zeta} B_-(t) h^{-1} c A_0(\omega, t) h B_+^{-1}(t) (t + i)^{\zeta} \in W_{N \times N}^{r+2}(\mathbb{R}).$$

Moreover, from Lemmas 4.8 and 4.12, $W^{r+2}(\mathbb{R})$ is a splitting R-algebra. Hence, by Theorem 2.1, the matrix $A_0^*(\omega, t)$ admits a right standard factorization.

Therefore, we can write,

$$\begin{aligned} cA_0(\omega, t) &= h B_-^{-1}(t) (t - i)^{\zeta} A_0^*(\omega, t) (t + i)^{-\zeta} B_+(t) h^{-1} \\ &= h B_-^{-1}(t) (t - i)^{\zeta} \left[(A_-^*(\omega, t))^{-1} \operatorname{diag} \left(\frac{t - i}{t + i} \right)^{\kappa(\omega)} A_+^*(\omega, t) \right] \\ &\qquad \times (t + i)^{-\zeta} B_+(t) h^{-1}, \end{aligned}$$

where the factors $(A_\pm^*)^{\pm 1} \in W_{N \times N}^{r+2}(\mathbb{R})$, and have analytic extensions with respect to ξ_n, to the lower half-plane and the upper half-plane, respectively.

Moreover (see [10, p. 37]), since $\lim_{t \to \pm\infty} A_0^*(\omega, t) = I$, there exist factors $A_\pm^* \in W_{N \times N}^{r+2}(\mathbb{R})$ such that

$$\lim_{t \to \pm\infty} A_\pm^*(\omega, t) = I. \tag{5.2}$$

We now define

$$A_1^{\pm}(\omega,t) := (t \pm i)^{\zeta}\, A_{\pm}^{*}(\omega,t)\,(t \pm i)^{-\zeta}. \tag{5.3}$$

Hence, as the diagonal matrices commute,

$$cA_0(\omega,t) = hB_-^{-1}(t)(A_1^-(\omega,t))^{-1}\operatorname{diag}\left(\frac{t-i}{t+i}\right)^{\kappa(\omega)+\zeta} A_1^+(\omega,t)B_+(t)h^{-1}.$$

From equation (5.3),

$$(A_1^{\pm})_{j,k} = (t+i)^{\zeta_j-\zeta_k}\,(A_{\pm}^{*})_{j,k}.$$

Suppose $j \neq k$. Then, from Lemma C.4,

$$\lim_{t\to\pm\infty}(A_1^{\pm}(w,t))_{j,k} = 0 \ \ (j \neq k).$$

Given this result for the off-diagonal terms of $A_1^{\pm}(w,t)$ and equations (5.2) and (5.3), we have

$$\lim_{t\to\pm\infty} A_1^{\pm}(w,t) = I.$$

Further, if we set

$$A_0^{\pm}(\omega,t) := A_1^{\pm}(\omega,t)\,B_{\pm}(t)h^{-1},$$

then we can write

$$cA_0(\omega,t) = (A_0^-(\omega,t))^{-1}\operatorname{diag}\left(\frac{t-i}{t+i}\right)^{\kappa(\omega)+\zeta} A_0^+(\omega,t).$$

Now, by definition,

$$\begin{aligned} A_0^{\pm} &= A_1^{\pm}B_{\pm}h^{-1} \\ &= \big[(A_1^{\pm}-I)+I\big]B_{\pm}h^{-1} \\ &= B_{\pm}\big[B_{\pm}^{-1}(A_1^{\pm}-I)B_{\pm}+I\big]h^{-1} \\ &= B_{\pm}A_2^{\pm}h^{-1}, \end{aligned}$$

where we now define

$$A_2^{\pm}(\omega,t) := B_{\pm}^{-1}(t)\,(A_1^{\pm}(\omega,t)-I)\,B_{\pm}(t)+I. \tag{5.4}$$

Remark 5.1. We have already noted that the factors $(A_{\pm}^{*})^{\pm1}$ have analytic extensions with respect to ξ_n, to the lower half-plane and to the upper half-plane, respectively. From definitions (5.3) and (5.4), it is clear that this property is likewise shared by the factors $(A_1^{\pm})^{\pm1}$ and $(A_2^{\pm})^{\pm1}$.

Remark 5.2. From Lemma C.6,

$$\big[(A_2^{\pm})^{\pm1}\big]_{j,k} \in W^r(\mathbb{R}) \ \text{ for } \ 1 \le j,k \le N.$$

In particular, each element of the matrices $(A_2^{\pm})^{\pm1}$ satisfies a condition of the form:

$$\sum_{0\le q\le r}\ \operatorname*{ess\,sup}_{\xi_n\in\mathbb{R}}\big|\xi_n^q D_{\xi_n}^q (A_2^{\pm}(\xi',\xi_n))_{j,k}\big| < +\infty.$$

Finally, we have the required factorization, namely,

$$\begin{aligned} cA_0(\omega,t) &= h(A_2^-)^{-1}B_-^{-1}\operatorname{diag}\left(\frac{t-i}{t+i}\right)^{\kappa(\omega)+\zeta}B_+A_2^+h^{-1} \\ &= h(A_2^-)^{-1}d(\omega,t)A_2^+h^{-1}, \end{aligned} \tag{5.5}$$

where

$$d(\omega,t) := B_-^{-1}(t)\operatorname{diag}\left(\frac{t-i}{t+i}\right)^{\kappa(\omega)+\zeta}B_+(t).$$

Remark 5.3. Note that, by construction, the matrix-valued functions $B_\pm(\xi_n)$ commute with the diagonal matrix $(\xi_n \pm i)^\zeta$. (To see this, choose an arbitrary block $J_k(\lambda_k)$. On this block, $(\xi_n \pm i)^\zeta$ acts like a scalar, since the relevant components of the vector ζ are all equal to $-(\log\lambda_k)/(2\pi i)$.)

By Remark 5.3, equation (2.10) which defines $B_\pm(t)$, together with the properties of the blocks (see equations (2.6) and (2.7)), we can write

$$\begin{aligned} d(\omega,t) &= \operatorname{diag}\left(\frac{t-i}{t+i}\right)^{\kappa(\omega)+\zeta}B_-^{-1}(t)B_+(t) \\ &= \operatorname{diag}\left(\frac{t-i}{t+i}\right)^{\kappa(\omega)+\zeta} \\ &\qquad\times\operatorname{diag}\left[B^{m_1}\Big(\frac{1}{2\pi i}\,\log\frac{t+i}{t-i}\Big),\ldots,B^{m_l}\Big(\frac{1}{2\pi i}\,\log\frac{t+i}{t-i}\Big)\right]. \end{aligned}$$

Remark 5.4. We notice that, by definition,

$$t = \frac{\xi_n}{|\xi'|} \quad\text{and}\quad \frac{t+i}{t-i} = \frac{\xi_n + i|\xi'|}{\xi_n - i|\xi'|}\,.$$

Hence, the functions of t or $(t+i)/(t-i)$ are homogeneous in the variable $\xi = (\xi', \xi_n)$.

It remains to consider the sum and partial sums of the factorization indices. For the fixed ξ', our final factorization (see equation (5.5)) is

$$cA_0(\omega,t) = h(A_2^-)^{-1}d(\omega,t)A_2^+h^{-1}.$$

Hence, since c, h are constant matrices and $\lim\limits_{t\to\pm\infty} A_2^\pm = I$, we have

$$\begin{aligned} &\Delta\arg\det\left[\left(|\xi'|^2+|\xi_n|^2\right)^{-\mu/2}A_\omega(\xi',\xi_n)\right]\Big|_{\xi_n=-\infty}^{\xi_n=+\infty} \\ &\qquad= \Delta\arg\det A_0(\xi',\xi_n)\Big|_{\xi_n=-\infty}^{\xi_n=+\infty} \quad \text{(see equation (5.1))} \\ &\qquad= \Delta\arg\det d(\omega,t)\Big|_{t=-\infty}^{t=+\infty}. \end{aligned} \tag{5.6}$$

Now, $d(\omega,t)$ is a lower triangular matrix, and hence its determinant is the product of the entries on its main diagonal. Thus

$$\det d(\omega,t) = \prod_{k=1}^{N}\left(\frac{t-i}{t+i}\right)^{\kappa_k(\omega)+\zeta_k}.$$

Therefore (see [7, Chapter II, Section 6, p. 88]),

$$\frac{1}{2\pi}\Delta \arg\det d(\omega,t)\Big|_{t=-\infty}^{t=+\infty} = \sum_{k=1}^{N}\kappa_k(\omega) + \sum_{k=1}^{N}\operatorname{Re}\zeta_k. \tag{5.7}$$

From equations (5.6) and (5.7), we have

$$\begin{aligned}\frac{1}{2\pi}\,\Delta \arg\det \Big[\big(|\xi'|^2+|\xi_n|^2\big)^{-\mu/2}A_\omega(\xi',\xi_n)\Big]\Big|_{\xi_n=-\infty}^{\xi_n=+\infty}&\\ = \sum_{k=1}^{N}\kappa_k(\omega) + \sum_{k=1}^{N}\operatorname{Re}\zeta_k.&\end{aligned}$$

The remaining assertions in Theorem 3.2 concerning the continuity of the sum and the semicontinuity of the partial sums of the factorization indices, can be found in [15, Theorem 3.1, p. 113]. □

Appendix A. Proof of the Key Theorem from Shamir

We shall give a proof of Theorem 2.3 in two steps:

(i) $E_+^{-1}E_-$ is similar to $\operatorname{diag}[\lambda_1,\dots,\lambda_N]$;

(ii) The general case.

Our overall approach will be to reduce the general case to a simpler case.

We begin by establishing some simple decay estimates.

Lemma A.1. *Suppose that* $A_0(\xi',\xi_n)\in C^{r+3}_{N\times N}(\mathbb{S}^{n-1})$ *is a matrix-valued function which is homogeneous of degree* 0. *Then, for the fixed* $\xi'\neq 0$,

$$D^k_{\xi_n}\big[A_0(\xi',\xi_n)-E_\pm\big] = O\big(|\xi_n|^{-k-1}\big),\ \ \xi_n\to\pm\infty,\ \ 0\le k\le (r+3), \tag{A.1}$$

where these estimates are uniform for $\xi'\in\mathbb{S}^{n-2}$.

Proof. Suppose that $\xi_n\to\infty$. Then since A_0 is homogeneous of degree 0,

$$\begin{aligned}A_0(\xi',\xi_n)-E_+ &= A_0(\xi'\xi_n^{-1},1)-A_0(0,1)\\ &= \sum_{j=1}^{n-1}\frac{\partial A_0}{\partial\xi_j}(0,1)\,\frac{\xi_j}{\xi_n} + O\big(|\xi_n|^{-2}\big)\\ &= O\big(|\xi_n|^{-1}\big),\ \ \xi_n\to\infty.\end{aligned}$$

This completes the proof for $k=0$.

For $1\le k\le (r+3)$, we can ignore the constant matrix E_+, and we readily obtain

$$D^k_{\xi_n}A_0(\xi',\xi_n) = D^k_{\xi_n}A_0(\xi'\xi_n^{-1},1) = O\big(|\xi_n|^{-k-1}\big),\ \ \xi_n\to\infty.$$

Of course, estimates for the case $\xi_n\to-\infty$ follow in exactly the same way. This completes the proof of the lemma. □

Let us consider the first step. We assume that the invertible matrix $E_+^{-1}E_-$ is similar to $\operatorname{diag}[\lambda_1,\dots,\lambda_N]$. In this formulation the eigenvalues λ_j, $j=1,\dots,N$ are listed according to their multiplicity and, of course, are all non-zero.

We now define $\zeta=(\zeta_1,\dots,\zeta_N)$ by

$$\zeta_j = -\frac{\log\lambda_j}{2\pi i}\,, \quad j=1,\dots,N. \tag{A.2}$$

Remark A.2. The definition of ζ, given by equation (A.2), includes a multiplicative factor (-1) is not shown in [15]. As will be seen, this modification allows us to correct an error in the treatment of the discontinuity across the negative real axis. (See [15, Lemma 4.2].)

Lemma A.3. *Suppose that $A_0(\xi',\xi_n)\in C^{r+3}_{N\times N}(\mathbb{S}^{n-1})$ is a matrix-valued function which is homogeneous of degree 0 and elliptic. Suppose further that for some invertible constant matrix h_1,*

$$E = E_+^{-1}E_- = h_1 \operatorname{diag}[\lambda_1,\dots,\lambda_N]h_1^{-1}. \tag{A.3}$$

If $\zeta_j=-\frac{\log\lambda_j}{2\pi i}$ for $j=1,\dots N$, $\zeta=(\zeta_1,\dots,\zeta_N)$ and $c:=A_0^{-1}(0,\dots,0,1)$, then for the fixed $\xi'\neq 0$,

$$A_0^*(\xi',\xi_n) := (\xi_n-i)^{-\zeta}h_1^{-1}cA_0(\xi',\xi_n)h_1(\xi_n+i)^{\zeta}\in W^{r+2}_{N\times N}(\mathbb{R}),$$

and

$$\lim_{\xi_n\to\pm\infty} A_0^*(\xi',\xi_n)=I.$$

Proof. By hypothesis, we have

$$E = E_+^{-1}E_- = h_1 \operatorname{diag}[\lambda_1,\dots,\lambda_N]h_1^{-1}.$$

If we define $\widetilde{A}_0(\xi',\xi_n):=h_1^{-1}cA_0(\xi',\xi_n)h_1$, we may assume, without loss of generality, that

$$E_+=I, \quad E_-=\operatorname{diag}[\lambda_1,\dots,\lambda_N].$$

We define a new matrix-valued function

$$A_0^*(\xi',\xi_n)=(\xi_n-i)^{-\zeta}\widetilde{A}_0(\xi',\xi_n)(\xi_n+i)^{\zeta}. \tag{A.4}$$

Then, for $\xi_n>0$, we can write

$$A_0^*(\xi',\xi_n)=(\xi_n-i)^{-\zeta}\big[\widetilde{A}_0(\xi',\xi_n)-E_+\big](\xi_n+i)^{\zeta}+(\xi_n-i)^{-\zeta}E_+(\xi_n+i)^{\zeta}$$

and similarly for $\xi_n<0$, we have

$$A_0^*(\xi',\xi_n)=(\xi_n-i)^{-\zeta}\big[\widetilde{A}_0(\xi',\xi_n)-E_-\big](\xi_n+i)^{\zeta}+(\xi_n-i)^{-\zeta}E_-(\xi_n+i)^{\zeta}.$$

Since the matrices $(\xi_n-i)^{-\zeta}$ and $(\xi_n+i)^{\zeta}$ are diagonal, we can write a typical element of the first summand as

$$\begin{aligned}(\xi_n-i)^{-\zeta_j}&\big[\widetilde{A}_0(\xi',\xi_n)-E_\pm\big]_{jl}(\xi_n+i)^{\zeta_l}\\ &= O\big(|\xi_n|^{-\operatorname{Re}\zeta_j-1+\operatorname{Re}\zeta_l}\big)=O\big(|\xi_n|^{-\delta_0}\big)\end{aligned} \tag{A.5}$$

using Lemma A.1 and equation (2.13).

Now suppose $1 \le k \le (r+3)$ and $\alpha \in \mathbb{C}$. Then

$$D^k_{\xi_n}(\xi_n \pm i)^\alpha = C_{\alpha,k}(\xi_n \pm i)^{\alpha-k},$$

where $C_{\alpha,k}$, $k = 1, 2, \ldots$ are certain constants. Moreover, from Lemma A.1,

$$D^k_{\xi_n}\big[\widetilde{A}_0(\xi', \xi_n) - E_\pm\big] = O\big(|\xi_n|^{-k-1}\big) \text{ as } \xi_n \to \pm\infty.$$

Hence, for $k = 0, 1, \ldots, r+3$, we can write

$$D^k_{\xi_n}\Big\{(\xi_n - i)^{-\zeta_j}\big[\widetilde{A}_0(\xi', \xi_n) - E_\pm\big]_{jl}(\xi_n + i)^{\zeta_l}\Big\} \\ = O\big(|\xi_n|^{-\operatorname{Re}\zeta_j + \operatorname{Re}\zeta_l - k - 1}\big) = O\big(|\xi_n|^{-\delta_0 - k}\big). \quad \text{(A.6)}$$

We now consider the second summand, which is diagonal as it is the product of diagonal matrices. For any $\alpha \in \mathbb{C}$ and $\xi_n \to \pm\infty$, it will be useful to factorize $(\xi_n \pm i)^\alpha$ using the following identity:

$$(\xi_n \pm i)^\alpha = (\xi_n \pm i0)^\alpha (1 \pm i\xi_n^{-1})^\alpha$$

noting, as expected, that the decomposition on the right-hand side preserves the modulus and argument of the left-hand side.

For $\xi_n > 0$, the (j, j) entry of the second summand is given by

$$(\xi_n - i)^{-\zeta_j}\, 1\, (\xi_n + i)^{\zeta_j} = (\xi_n - i0)^{-\zeta_j}(\xi_n + i0)^{\zeta_j}\, 1\, (1 - i\xi_n^{-1})^{-\zeta_j}(1 + i\xi_n^{-1})^{\zeta_j} \\ = (1 - i\xi_n^{-1})^{-\zeta_j}(1 + i\xi_n^{-1})^{\zeta_j},$$

since the product of the first two terms is 1.

Similarly, for $\xi_n < 0$, we have

$$\begin{aligned}
&(\xi_n - i)^{-\zeta_j}\lambda_j(\xi_n + i)^{\zeta_j} \\
&\quad = (\xi_n - i0)^{-\zeta_j}(\xi_n + i0)^{\zeta_j}\lambda_j(1 - i\xi_n^{-1})^{-\zeta_j}(1 + i\xi_n^{-1})^{\zeta_j} \\
&\quad = e^{-\zeta_j \log|\xi_n|}e^{i\zeta_j\pi}e^{\zeta_j \log|\xi_n|}e^{i\zeta_j\pi}\lambda_j(1 - i\xi_n^{-1})^{-\zeta_j}(1 + i\xi_n^{-1})^{\zeta_j} \\
&\quad = e^{2\pi i\zeta_j}\lambda_j(1 - i\xi_n^{-1})^{-\zeta_j}(1 + i\xi_n^{-1})^{\zeta_j} \\
&\quad = (1 - i\xi_n^{-1})^{-\zeta_j}(1 + i\xi_n^{-1})^{\zeta_j}
\end{aligned}$$

since $\lambda_j = e^{\log\lambda_j} = e^{-2\pi i\zeta_j}$ for $j = 1, \ldots, N$ from equation (A.2).

So, for the second summand, combining the results for $\xi_n \to \pm\infty$, for $|\xi_n| > 1$ we have

$$(\xi_n - i)^{-\zeta}E_\pm(\xi_n + i)^\zeta - I = (1 - i\xi_n^{-1})^{-\zeta}(1 + i\xi_n^{-1})^\zeta - I.$$

So, expanding the factors on the right-hand side in powers of ξ_n^{-1}, we have

$$(\xi_n - i)^{-\zeta}E_\pm(\xi_n + i)^\zeta - I = \sum_{l=1}^{\infty} A_l\xi_n^{-l} \text{ for } |\xi_n| > 1.$$

Thus, on differentiating k times with respect to ξ_n, we obtain

$$D^k_{\xi_n}\Big\{(\xi_n - i)^{-\zeta}E_\pm(\xi_n + i)^\zeta - I\Big\} = O\big(|\xi_n|^{-1-k}\big) \quad \text{(A.7)}$$

for $k = 0, 1, \ldots, (r+3)$. Combining estimates (A.6) and (A.7), we obtain

$$D^k_{\xi_n}\big\{A_0^*(\xi', \xi_n) - I\big\} = O\big(|\xi_n|^{-\delta_0 - k}\big), \quad |\xi_n| \to \infty. \tag{A.8}$$

From Lemma 2.2, we have $A_0^*(\xi', \xi_n) \in W^{r+2}_{N\times N}(\mathbb{R})$. This completes the proof of the first step. □

With these preparations complete, we now turn to the general case. For convenience, we now restate Theorem 2.3.

Lemma A.4. *Suppose that $A_0(\xi', \xi_n) \in C^{r+3}_{N\times N}(\mathbb{S}^{n-1})$ is a matrix-valued function which is homogeneous of degree 0 and elliptic. Suppose that the Jordan form of $A_0^{-1}(0,\ldots,0,1)A_0(0,\ldots,0,-1)$ has blocks $J_k(\lambda_k)$ of size m_k for $k = 1, \ldots, l$. Let $\zeta = (\zeta_1, \ldots \zeta_N)$, where*

$$\zeta_q = -\frac{\log\lambda_j}{2\pi i} \ \ \textit{for} \ \ \sum_{p=1}^{j-1} m_p < q \le \sum_{p=1}^{j} m_p, \ \ q = 1, \ldots, N.$$

Let $c := A_0^{-1}(0,\ldots,0,1)$. Then for the fixed $\xi' \neq 0$,

$$\begin{aligned} A_0^*(\xi', \xi_n) := {} & (\xi_n - i)^{-\zeta} B_-(\xi_n) h^{-1} \\ & \times c A_0(\xi', \xi_n) h B_+^{-1}(\xi_n)(\xi_n + i)^{\zeta} \in W^{r+2}_{N\times N}(\mathbb{R}), \end{aligned} \tag{A.9}$$

and

$$\lim_{\xi_n \to \pm\infty} A_0^*(\xi', \xi_n) = I.$$

Proof. By hypothesis, and using equation (2.5), we have

$$E = E_+^{-1}E_- = h\,\mathrm{diag}\,\big[\lambda_1 B^{m_1}(1), \ldots, \lambda_l B^{m_l}(1)\big]h^{-1},$$

for some invertible matrix h. If we define $\widetilde{A}_0(\xi', \xi_n) := h^{-1}cA_0(\xi', \xi_n)h$, we may assume, without loss of generality, that

$$E_+ = I, \quad E_- = \mathrm{diag}\,\big[\lambda_1 B^{m_1}(1), \ldots, \lambda_l B^{m_l}(1)\big].$$

Mimicing the approach in the first case, we define

$$\zeta_j' = -\frac{\log\lambda_j}{2\pi i}\,, \ \ -\frac{1}{2} \le \mathrm{Re}\,\zeta_j' < \frac{1}{2}\,, \ \ \text{where} \ \ j = 1, \ldots, l.$$

Moreover, we calculate

$$\min_{1\le j,k\le l}(1 - \mathrm{Re}\,\zeta_k' + \Re\zeta_j') = \delta_0 > 0. \tag{A.10}$$

We now define

$$\zeta_q = \zeta_j' \ \ \text{for} \ \ \sum_{p=1}^{j-1} m_p < q \le \sum_{p=1}^{j} m_p, \ \ q = 1, \ldots, N.$$

Now we can set $\zeta = (\zeta_1, \ldots, \zeta_N)$, exactly as in the first case.

Remark A.5. When regarded as a function of $z \in \mathbb{C}$, the matrix-valued functions $B_\pm(z)$ are analytic in the regions $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$, respectively. Note also that, by construction, the matrix-valued functions $B_\pm(\xi_n)$ commute with the diagonal matrix $(\xi_n \pm i)^\zeta$. (To see this, choose an arbitrary block J_k. On this block, $(\xi_n \pm i)^\zeta$ acts like a scalar, since the relevant components of the vector ζ are all equal to $-\frac{\log \lambda_k}{2\pi i}$.)

As previously (see equation (A.4)), we define a new matrix-valued function

$$A_0^*(\xi', \xi_n) = (\xi_n - i)^{-\zeta} B_-(\xi_n) \widetilde{A}_0(\xi', \xi_n) B_+^{-1}(\xi_n)(\xi_n + i)^\zeta. \tag{A.11}$$

Now, since $E_+ = I$, using the established properties of $B^m(\alpha_\pm)$, we have

$$\begin{aligned}
&\lim_{\xi_n \to \infty} B_-(\xi_n) \widetilde{A}_0(\xi', \xi_n) B_+^{-1}(\xi_n) \\
&\qquad = \lim_{\xi_n \to \infty} B_-(\xi_n) B_+^{-1}(\xi_n) \\
&\qquad = \lim_{\xi_n \to \infty} B_-(\xi_n) \operatorname{diag} \Big[B^{m_1}\big(- \alpha_+(\xi_n)\big), \ldots, B^{m_l}\big(- \alpha_+(\xi_n)\big) \Big] \\
&\qquad = \lim_{\xi_n \to \infty} \operatorname{diag} \Big[B^{m_1}\big(\alpha_-(\xi_n) - \alpha_+(\xi_n)\big), \ldots, B^{m_l}\big(\alpha_-(\xi_n) - \alpha_+(\xi_n)\big) \Big] \\
&\qquad = \operatorname{diag} \big[B^{m_1}(0), \ldots, B^{m_l}(0) \big] \\
&\qquad = I.
\end{aligned}$$

On the other hand, since $E_- = \operatorname{diag}[\lambda_1 B^{m_1}(1), \ldots, \lambda_l B^{m_l}(1)]$,

$$\begin{aligned}
&\lim_{\xi_n \to -\infty} B_-(\xi_n) \widetilde{A}_0(\xi', \xi_n) B_+^{-1}(\xi_n) \\
&\qquad = \lim_{\xi_n \to -\infty} B_-(\xi_n) \operatorname{diag} \big[\lambda_1 B^{m_1}(1), \ldots, \lambda_l B^{m_l}(1) \big] B_+^{-1}(\xi_n) \\
&\qquad = \lim_{\xi_n \to -\infty} B_-(\xi_n) \operatorname{diag} \Big[\lambda_1 B^{m_1}(1 - \alpha_+), \ldots, \lambda_l B^{m_l}(1 - \alpha_+) \Big] \\
&\qquad = \lim_{\xi_n \to -\infty} \operatorname{diag} \Big[\lambda_1 B^{m_1}(1 - \alpha_+ + \alpha_-), \ldots, \lambda_l B^{m_l}(1 - \alpha_+ + \alpha_-) \Big] \\
&\qquad = \operatorname{diag} \big[\lambda_1 B^{m_1}(0), \ldots, \lambda_l B^{m_l}(0) \big] \\
&\qquad = \operatorname{diag} \big[\lambda_1 I^{m_1}, \ldots, \lambda_l I^{m_l} \big],
\end{aligned}$$

where I^m is an $m \times m$ block identity matrix.

So, as in Lemma A.3, we can see that

$$\begin{aligned}
&\lim_{\xi_n \to \pm\infty} A_0^*(\xi', \xi_n) \\
&\qquad = \lim_{\xi_n \to \pm\infty} (\xi_n - i)^{-\zeta} B_-(\xi_n) \widetilde{A}_0(\xi', \xi_n) B_+^{-1}(\xi_n)(\xi_n + i)^\zeta = I.
\end{aligned} \tag{A.12}$$

To show that $D_{\xi_n}^k A_0^*(\xi', \xi_n)$, $k = 1, \ldots, (r + 3)$ satisfies estimates of the form given in equation (A.8), we follow exactly the approach taken in Lemma A.3

$$\begin{aligned} A_0^*(\xi',\xi_n) &= (\xi_n - i)^{-\zeta} B_-(\xi_n)\widetilde{A}_0(\xi',\xi_n)B_+^{-1}(\xi_n)(\xi_n+i)^{\zeta} \\ &= (\xi_n - i)^{-\zeta} B_-(\xi_n)(\widetilde{A}_0(\xi',\xi_n) - E_\pm)B_+^{-1}(\xi_n)(\xi_n+i)^{\zeta} \\ &\quad + (\xi_n - i)^{-\zeta} B_-(\xi_n)E_\pm B_+^{-1}(\xi_n)(\xi_n+i)^{\zeta}, \end{aligned}$$

where

$$\begin{aligned} & B_-(\xi_n)E_\pm B_+^{-1}(\xi_n) \\ &= E_\pm \operatorname{diag}\left[B^{m_j}\Big(\log\frac{\xi_n - i}{\xi_n + i}\Big)\right] = E_\pm \operatorname{diag}\left[B^{m_j}\Big(\log\frac{1 - i/\xi_n}{1 + i/\xi_n}\Big)\right]. \end{aligned}$$

The presence of the logarithmic terms in the matrices $B_\pm$ adds only a minor complication. For any fixed positive integer m, we have

$$D_{\xi_n}\big[\log(\xi_n \pm i)\big]^m = m\big[\log(\xi_n \pm i)\big]^{m-1}(\xi_n \pm i)^{-1}.$$

But since

$$\lim_{\xi_n\to\pm\infty}\frac{[\log(\xi_n \pm i)]^p}{(\xi_n \pm i)^{\epsilon}} = 0,$$

for any fixed integer p and any $\epsilon > 0$, we can effectively repeat the proof of Lemma A.3 with any δ' satisfying $0 < \delta' < \delta_0$.

From Lemma 2.2, we have $A_0^*(\xi',\xi_n) \in W^{r+2}_{N\times N}(\mathbb{R})$. This completes the proof of the general case. ☐

Appendix B. Function Approximation in $W^r(\mathbb{R})$

The goal in this appendix is to prove that the Fourier transforms of smooth functions which have compact support and are zero in a neighborhood of $x = 0$, are dense in the space $W^r(\mathbb{R})$. To show this, we use the standard approach of cut-off functions and convolution with a mollifier. In simple terms, this analysis is required because we are effectively working in a *weighted* Sobolev space. (See, for example, [1]).

Lemma B.1. *Suppose $f \in W^r(\mathbb{R})$ and $0 \le j \le k \le r$. Then*

$$D^k f,\ \ t^j D^k f \ \ \textit{and}\ \ D^k(t^j f) \in W(\mathbb{R}).$$

In addition, if $f = \widehat{g}$, then $\|x^k g\|_{L_1} = \|D^k f\|_W$,

$$\|D^j(x^k g)\|_{L_1} = \|t^j D^k f\|_W \ \ \textit{and}\ \ \|x^k D^j g\|_{L_1} = \|D^k(t^j f)\|_W.$$

Proof. Since $W(\mathbb{R})$ is an R-algebra and $0 \le j \le k$,

$$\frac{t^j}{(1 - it)^k} \in W(\mathbb{R}). \tag{B.1}$$

By definition, $(1 - it)^k D^k f \in W(\mathbb{R})$ and it follows from (B.1) that

$$t^j D^k f = \frac{t^j}{(1 - it)^k}\,(1 - it)^k D^k f \in W(\mathbb{R}). \tag{B.2}$$

Moreover, from (B.2),

$$D^k t^j f = \sum_{l=0}^{j} c_l t^{j-l} D^{k-l} f \in W(\mathbb{R}),$$

where c_l are some constants. (Note that $j \le k$ implies that $j-l \le k-l$.)

From [11, Proposition 2.2.11, p. 100],

$$\mathcal{F}_{x\to t}\big[(ix)^k g(x)\big] = D^k \widehat{g}.$$

Hence,

$$\|x^k g\|_{L_1} = \|D^k \widehat{g}\|_W = \|D^k f\|_W.$$

Let $h_k(x) := x^k g(x)$. Then, again from [11, Proposition 2.2.11, p. 100],

$$\mathcal{F}_{x\to t} D^j h_k = (-it)^j \widehat{h_k},$$

and thus

$$\|D^j(x^k g)\|_{L_1} = \|t^j \widehat{x^k g}\|_W = \|t^j D^k f\|_W.$$

Finally,

$$\|x^k D^j g\|_{L_1} = \|\widehat{x^k D^j g}\|_W = \|D^k \widehat{D^j g}\|_W = \|D^k(t^j f)\|_W.$$

This completes the proof of the lemma. □

Suppose $\widehat{g}(t) \in W^r(\mathbb{R})$. Then, by definition,

$$\|\widehat{g}\|_{W^r} = \|g\|_{L_1} + \sum_{k=1}^{r} \|(D_x + 1)^k (x^k g(x))\|_{L_1}.$$

Hence,

$$\|\widehat{g}\|_{W^r} \le \|g\|_{L_1} + \sum_{k=1}^{r} \left\{ \|x^k g\|_{L_1} + \sum_{j=1}^{k} \binom{k}{j} \|D_x^j (x^k g)\|_{L_1} \right\}.$$

Thus, to show the convergence to $\widehat{g}$ in $\|\cdot\|_{W^r}$, it suffices to show the convergence to $g, x^k g$ and $D_x^j(x^k g)$ in $\|\cdot\|_{L_1}$ for all $1 \le j \le k \le r$.

Note that for $j \ge 1$,

$$D_x^j(x^k g) = \sum_{l=0}^{j} \binom{j}{l} (D_x^l x^k)(D_x^{j-l} g).$$

Hence, we have

$$\|\widehat{g}\|_{W^r} \le C_r \sum_{0 \le j \le k \le r} \|x^k D_x^j g\|_{L_1} := C_r \|g\|_*, \tag{B.3}$$

where C_r is a constant that depends *only* on r. So, an alternative sufficient condition for the convergence to $\widehat{g}$ in $\|\cdot\|_{W^r}$ is the convergence to $x^k D_x^j g$ in $\|\cdot\|_{L_1}$ for all $0 \le j \le k \le r$.

We now define a cut-off function that is zero in a neighborhood of $x = 0$, and is also equal to zero when $|x|$ is sufficiently large. Firstly, we define two smooth functions

$$\alpha(x) = \begin{cases} 0 & \text{if } |x| \le 1/2, \\ 1 & \text{if } |x| \ge 1, \end{cases}$$

and

$$\beta(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

Then, for $0 < \epsilon < 1$, we define the smooth cut-off function ϕ_ϵ by

$$\phi_\epsilon(x) = \alpha(x/\epsilon)\beta(\epsilon x).$$

Notice that by construction,

$$\phi_\epsilon(x) = \begin{cases} 1 & \text{if } |x| \in \Big[\epsilon, \dfrac{1}{\epsilon}\Big], \\ 0 & \text{if } |x| \in \Big[0, \dfrac{\epsilon}{2}\Big) \cup \Big[\dfrac{2}{\epsilon}, \infty\Big). \end{cases}$$

In particular, for each $0 < \epsilon < 1$, the function ϕ_ϵ has compact support, and is identically zero in the neighborhood of 0.

Therefore, for $j = 1, 2, \dots$, the support of $D_x^j \phi_\epsilon(x)$ is contained in E_ϵ, where

$$E_\epsilon := \Big[-\frac{2}{\epsilon}, -\frac{1}{\epsilon}\Big] \cup \Big[-\epsilon, -\frac{\epsilon}{2}\Big] \cup \Big[\frac{\epsilon}{2}, \epsilon\Big] \cup \Big[\frac{1}{\epsilon}, \frac{2}{\epsilon}\Big].$$

For any positive integer k, we have

$$D_x^k \alpha\Big(\frac{x}{\epsilon}\Big) = \Big(\frac{1}{\epsilon}\Big)^k \alpha^{(k)}\Big(\frac{x}{\epsilon}\Big)$$

and

$$D_x^k \beta(\epsilon x) = \epsilon^k \beta^{(k)}(\epsilon x).$$

Hence, for $l = 1, 2, \dots,$

$$x^l D_x^l \phi_\epsilon = \sum_{k=0}^{l} c_k \Big[\Big(\frac{x}{\epsilon}\Big)^k \alpha^{(k)}\Big(\frac{x}{\epsilon}\Big)\Big]\big[(\epsilon x)^{l-k}\beta^{(l-k)}(\epsilon x)\big],$$

for certain constants c_k that depend only on k. Moreover, $\alpha^{(k)}(y) = 0$ unless $\frac{1}{2} \le |y| \le 1$ and $\beta^{(k)}(y) = 0$ unless $1 \le |y| \le 2$.

Hence, for $l = 1, 2, \dots,$

$$\sup_{x\in\mathbb{R},\, 0<\epsilon<1} |x^l D_x^l \phi_\epsilon| \le C_{\alpha,\beta,l}, \tag{B.4}$$

where $C_{\alpha,\beta,l}$ is a (finite) constant that depends *only* on the smooth functions α, β and the index l.

Lemma B.2. *Suppose* $\widehat{g} \in W^r(\mathbb{R})$. *Then* $\|\widehat{\phi_\epsilon g} - \widehat{g}\|_{W^r} \to 0$ *as* $\epsilon \searrow 0$.

Proof. Suppose $\widehat{g} \in W(\mathbb{R})$. Then, by definition, $g(x) \in L_1(\mathbb{R})$ and $\|\widehat{g}\|_W = \|g\|_{L_1}$. It is immediately clear from the definition of the L_1 norm that $\phi_\epsilon g \in L_1(\mathbb{R})$ and $\|\phi_\epsilon g - g\|_{L_1} \to 0$ as $\epsilon \searrow 0$. That is, $\|\widehat{\phi_\epsilon g} - \widehat{g}\|_W \to 0$ as $\epsilon \searrow 0$, as required.

Now suppose $\widehat{g} \in W^r(\mathbb{R})$ and $1 \le k \le r$. Then

$$\left\|x^k(\phi_\epsilon g) - x^k g\right\|_{L_1} = \left\|\phi_\epsilon(x^k g) - x^k g\right\|_{L_1} \longrightarrow 0 \text{ as } \epsilon \searrow 0.$$

Further, suppose that $\widehat{g} \in W^r(\mathbb{R})$, and $1 \le j \le k \le r$. Then

$$D_x^j(\phi_\epsilon g) = \phi_\epsilon(D_x^j g) + \sum_{l=1}^{j} \binom{j}{l} D_x^l \phi_\epsilon \cdot D_x^{j-l} g.$$

We now show that for $1 \le l \le j \le k \le r$,

$$\begin{aligned}
\left\|x^k D_x^l \phi_\epsilon \cdot D_x^{j-l} g\right\|_{L_1} &= \int_{\mathbb{R}} \left|x^k D_x^l \phi_\epsilon \cdot D_x^{j-l} g\right| dx \\
&= \int_{E_\epsilon} \left|x^k D_x^l \phi_\epsilon \cdot D_x^{j-l} g\right| dx \text{ since } D_x^l \phi_\epsilon = 0 \text{ outside } E_\epsilon \\
&\le C_{\alpha,\beta,l} \int_{E_\epsilon} |x^{k-l} D_x^{j-l} g| \, dx \text{ from (B.4)} \\
&\longrightarrow 0 \text{ as } \epsilon \searrow 0,
\end{aligned}$$

since $x^{k-l} D_x^{j-l} g \in L_1(\mathbb{R})$. Hence,

$$\begin{aligned}
&\left\|x^k D_x^j(\phi_\epsilon g) - x^k D_x^j g\right\|_{L_1} \\
&\qquad \le \left\|x^k \phi_\epsilon(D_x^j g) - x^k D_x^j g\right\|_{L_1} + \sum_{l=1}^{j} \binom{j}{l} \left\|x^k D_x^l \phi_\epsilon \cdot D_x^{j-l} g\right\|_{L_1} \\
&\qquad = \left\|\phi_\epsilon(x^k D_x^j g) - x^k D_x^j g\right\|_{L_1} + \sum_{l=1}^{j} \binom{j}{l} \left\|x^k D_x^l \phi_\epsilon \cdot D_x^{j-l} g\right\|_{L_1} \\
&\qquad \longrightarrow 0 \text{ as } \epsilon \searrow 0.
\end{aligned}$$

That is, $\|\widehat{\phi_\epsilon g} - \widehat{g}\|_{W^r} \to 0$, as $\epsilon \searrow 0$, as required. □

Remark B.3. The significance of Lemma B.2 is that we can effectively assume for the ensuing density arguments that any function in $L_1(\mathbb{R})$ has both compact support and is also identically zero in the neighborhood of the origin. (To see this, we simply approximate $g \in L_1(\mathbb{R})$ by $h = \phi_\epsilon g$.)

Following, for example [1], we now introduce the concept of a *mollifier.* Let J be a nonnegative, real-valued function in $C_0^\infty(\mathbb{R})$ satisfying the two conditions, $J(x) = 0$ if $|x| \ge 1$, and $\int_{\mathbb{R}} J(x)\, dx = 1$.

For $\delta > 0$, we define $J_\delta(x) = \delta^{-1} J(x/\delta)$. Then $J_\delta(x) \in C_0^\infty(\mathbb{R})$ and:

(a) $J_\delta(x) = 0$ if $|x| \ge \delta$ and

(b) $\int\limits_{\mathbb{R}} J_\delta(x)\,dx = 1$.

As $\delta \searrow 0$, the mollifier $J_\delta(x)$ approaches the delta-function supported on $x = 0$. Formally, we define the convolution

$$(J_\delta * u)(x) = \int\limits_{\mathbb{R}} J_\delta(x-y)u(y)\,dy.$$

Suppose $v \in L_1(\mathbb{R})$ and has compact support. Then:

(i) $(J_\delta * v) \in C_0^\infty(\mathbb{R})$;

(ii) $(J_\delta * v) \in L_1(\mathbb{R})$ and $\|J_\delta * v\|_{L_1} \leq \|v\|_{L_1}$;

(iii) if $Dv \in L_1(\mathbb{R})$, then $D(J_\delta * v) = J_\delta * Dv$;

(iv) $\lim\limits_{\delta \searrow 0} \|J_\delta * v - v\|_{L_1} = 0$.

As a simple consequence of the above, we observe that if $D^j v \in L_1(\mathbb{R})$, then

$$\begin{aligned}\left\|D^j(J_\delta * v) - D^j v\right\|_{L_1} &= \left\|(J_\delta * D^j v) - D^j v\right\|_{L_1} \\ &\longrightarrow 0 \ \text{ as } \ \delta \searrow 0,\end{aligned}$$

and, thus, $\|J_\delta * v - v\|_{W^{r,1}} \to 0$ as $\delta \searrow 0$ in the (unweighted) Sobolev space $W^{r,1}(\mathbb{R})$.

Lemma B.4. *Suppose $\widehat{h}(t) \in W^r(\mathbb{R})$, and further that $h(x)$ has a compact support and is identically zero in a neighborhood of $x = 0$. Then*

$$\left\|\widehat{J_\delta * h} - \widehat{h}\right\|_{W^r} \to 0 \ \text{ as } \ \delta \searrow 0.$$

Proof. Suppose that $h(x)$ has compact support and is identically zero in a neighborhood of $x = 0$. Then there exist positive real numbers ϵ and R such that

$$\operatorname{supp} h \subseteq \big\{x \in \mathbb{R} : \ \epsilon \leq |x| \leq R\big\}.$$

Suppose $\delta \leq \min\{\epsilon/2, 1\}$. Then

$$\operatorname{supp} J_\delta * h \subseteq \left\{x \in \mathbb{R} : \ \frac{\epsilon}{2} \leq |x| \leq (R+1)\right\}.$$

Now let $H(x)$ be any function with

$$\operatorname{supp} H \subseteq \left\{x \in \mathbb{R} : \frac{\epsilon}{2} \leq |x| \leq (R+1)\right\},$$

and $\widehat{H}(t) \in W^r(\mathbb{R})$. Then for any integers j, k such that $0 \leq j \leq k \leq r$, we have

$$\left(\frac{\epsilon}{2}\right)^k \|D^j H\|_{L_1} \leq \|x^k D^j H\|_{L_1} \leq (R+1)^k \|D^j H\|_{L_1}.$$

Therefore,

$$\left(\frac{\epsilon}{2}\right)^r \|D^j H\|_{L_1} \leq \|x^k D^j H\|_{L_1} \leq (R+1)^r \|D^j H\|_{L_1},$$

and summing over all $0 \leq j \leq k \leq r$, we have

$$\left(\frac{\epsilon}{2}\right)^r \|H\|_{W^{r,1}} \leq \|H\|_* \leq (r+1)(R+1)^r \|H\|_{W^{r,1}},$$

where $\|\cdot\|_{W^{r,1}}$ denotes the (unweighted) Sobolev norm, and $\|\cdot\|_*$ is defined by equation (B.3).

Thus, since

$$\|J_\delta * h - h\|_{W^{r,1}} \to 0 \ \text{ as } \ \delta \searrow 0,$$

we have

$$\|J_\delta * h - h\|_* \ \text{ as } \ \delta \searrow 0.$$

Finally, from equation (B.3),

$$\|\widehat{J_\delta * h} - \widehat{h}\|_{W^r} \to 0 \ \text{ as } \ \delta \searrow 0,$$

as required. □

Appendix C. Matrix Factor Estimates from Duduchava

In this appendix we follow the approach taken by Duduchava [6], and derive some asymptotic estimates for certain matrices arising during factorization.

Given $A_0^*(\xi', \xi_n) \in W_{N\times N}^{r+2}(\mathbb{R})$, we have the factorization

$$A_0^*(\omega, t) = \big(A_-^*(\omega, t)\big)^{-1} \operatorname{diag}\left(\frac{t-i}{t+i}\right)^{\kappa(\omega)} A_+^*(\omega, t), \tag{C.1}$$

where $A_\pm^* \in W_{N\times N}^{r+2}(\mathbb{R})$, and have analytic extensions with respect to ξ_n, to the upper half-plane and to the lower half-plane, respectively. Moreover (see [10, p. 37]), since $\lim\limits_{t\to\pm\infty} A_0^*(\omega, t) = I$, there exist factors $A_\pm^* \in W_{N\times N}^{r+2}(\mathbb{R})$ such that

$$\lim_{t\to\pm\infty} A_\pm^*(\omega, t) = I. \tag{C.2}$$

We now define

$$A_1^\pm(\omega, t) = (t \pm i)^\zeta A_\pm^*(\omega, t)(t \pm i)^{-\zeta}. \tag{C.3}$$

We begin with two technical lemmas that will be useful later. Let $\mathbb{T}$ denote the unit circle in the complex plane.

Lemma C.1. *Let $0 < \nu < 1$. Suppose $\phi(t) \in W^{r+2}(\mathbb{R})$ and $\phi_k(t) := t^k D_t^k \phi(t) = O(|t|^{-\nu})$ as $|t| \to \infty$, for $k = 0, 1, \ldots, r+2$. Define*

$$\Phi_k(z) := \phi_k\Big(i\,\frac{1+z}{1-z}\Big), \ \ z \in \mathbb{T} \setminus \{1\}, \quad \Phi_k(1) := \lim_{z\to 1} \Phi_k(z).$$

Then for $j = 0, 1, \ldots, r+1$, $\Phi_j \in H_\nu(\mathbb{T})$, where $H_\nu(\mathbb{T})$ denotes the Hölder space of order ν. Moreover, $\Phi_j(1) = 0$.

Proof. Choose any $z \in \mathbb{T}$. Then $z = e^{i\theta}$ for some $\theta \in [-\pi, \pi)$, and it is straightforward to show that

$$i\,\frac{1+z}{1-z} = -\cot\left(\frac{\theta}{2}\right).$$

Hence, for $j = 0, 1, \ldots, r+1$, we can write

$$\Phi_j(z) = \phi_j\Big(-\cot\left(\frac{\theta}{2}\right)\Big) := \psi_j(\theta).$$

In particular, as $\theta \to 0$, so, $z \to 1$, and we obtain $\Phi_j(1) = 0$. Now,

$$\frac{d\psi_j}{d\theta} = \frac{d\phi_j}{d\tau}\,\frac{d\tau}{d\theta}, \quad \text{where } \tau := -\cot\left(\frac{\theta}{2}\right).$$

By hypothesis,

$$\frac{d\phi_j}{d\tau} = O\big(|\tau|^{-\nu-1}\big) = O\big(|\theta|^{\nu+1}\big) \ \text{ as } \ |\theta| \to 0.$$

Moreover, by the direct calculation,

$$\frac{d\tau}{d\theta} = \frac{1}{2\sin^2(\frac{\theta}{2})} = O\big(|\theta|^{-2}\big) \ \text{ as } \ |\theta| \to 0.$$

Combining these results,

$$\frac{d\psi_j}{d\theta} = O\big(|\theta|^{\nu-1}\big) \ \text{ as } \ |\theta| \to 0. \tag{C.4}$$

Suppose now $z_1, z_2 \in \mathbb{T}$. Then, by relabeling, if necessary, we can suppose

$$|z_1 - 1| \le |z_2 - 1|,$$

and we consider three cases:

Case 1: $|z_1 - z_2| < |z_1 - 1| \le |z_2 - 1|$;

Case 2: $|z_1 - 1| \le |z_1 - z_2| \le |z_2 - 1|$;

Case 3: $|z_1 - 1| \le |z_2 - 1| < |z_1 - z_2|$.

We begin with Case 1 and apply the Mean Value Theorem to ψ_j:

$$\begin{aligned}\big|\Phi_j(z_1) - \Phi_j(z_2)\big| &= \big|\psi_j(\theta_1) - \psi(\theta_2)\big| \\ &= \left|\frac{d\psi_j}{d\theta}(\theta^*)\right| \cdot |\theta_1 - \theta_2| \quad \big(|\theta_1| \le |\theta^*| \le |\theta_2|\big),\end{aligned}$$

where, due to the constraints applicable in this case, θ_1 and θ_2 must have the same sign. Hence, from (C.4),

$$\begin{aligned}|\Phi_j(z_1) - \Phi_j(z_2)| &\le C'|z^* - 1|^{\nu-1} \cdot |z_1 - z_2| \ \text{ for some constant } C' \\ &= C'\Big(\frac{|z_1 - z_2|}{|z^* - 1|}\Big)^{1-\nu} |z_1 - z_2|^{\nu} \\ &\le C'\Big(\frac{|z_1 - z_2|}{|z_1 - 1|}\Big)^{1-\nu} |z_1 - z_2|^{\nu} \\ &\le C'|z_1 - z_2|^{\nu}.\end{aligned}$$

For Case 2,

$$\begin{aligned}
\big|\Phi_j(z_1)-\Phi_j(z_2)\big| &\le |\Phi_j(z_1)|+|\Phi_j(z_2)|\\
&= \Big|\phi_j\Big(-\cot\Big(\frac{\theta_1}{2}\Big)\Big)\Big| + \Big|\phi_j\Big(-\cot\Big(\frac{\theta_2}{2}\Big)\Big)\Big|\\
&\le C\Big|\cot\Big(\frac{\theta_1}{2}\Big)\Big|^{-\nu} + C\Big|\cot\Big(\frac{\theta_2}{2}\Big)\Big|^{-\nu}\\
&\le 2\,C|\theta_1|^{\nu}+2\,C|\theta_2|^{\nu}\\
&\le 4\,C|z_1-1|^{\nu}+4\,C|z_2-1|^{\nu}
\end{aligned}$$

But in this case, $|z_1-1|\le|z_1-z_2|$ and, moreover,

$$|z_2-1|\le|z_2-z_1|+|z_1-1|\le 2|z_1-z_2|.$$

Therefore,

$$\big|\Phi_j(z_1)-\Phi_j(z_2)\big|\le 12\,C|z_1-z_2|^{\nu}.$$

Finally, turning to Case 3,

$$\begin{aligned}
\big|\Phi_j(z_1)-\Phi_j(z_2)\big| &\le |\Phi_j(z_1)|+|\Phi_j(z_2)|\\
&= \Big|\phi_j\Big(-\cot\Big(\frac{\theta_1}{2}\Big)\Big)\Big| + \Big|\phi_j\Big(-\cot\Big(\frac{\theta_2}{2}\Big)\Big)\Big|\\
&\le C\Big|\cot\Big(\frac{\theta_1}{2}\Big)\Big|^{-\nu} + C\Big|\cot\Big(\frac{\theta_2}{2}\Big)\Big|^{-\nu}\\
&\le 2\,C|\theta_1|^{\nu}+2\,C|\theta_2|^{\nu}\\
&\le 4\,C|z_1-1|^{\nu}+4\,C|z_2-1|^{\nu}\\
&\le 8\,C|z_1-z_2|^{\nu}.
\end{aligned}$$

□

Lemma C.2. *Let* $0<\nu<1$ *and* $\phi(t)\in W^{r+2}(\mathbb{R})$*. Suppose*

$$\phi_k(t):=t^kD_t^k\phi(t)=O\big(|t|^{-\nu}\big)\ \ \textit{as}\ \ |t|\to\infty,\ \ \textit{for}\ \ k=0,1,\dots,r+2.$$

Then

$$t^kD_t^kS_{\mathbb{R}}\phi(t)=O\big(|t|^{-\nu}\big)\ \ \textit{as}\ \ |t|\to\infty,\ \ \textit{for}\ \ k=0,1,\dots,r+1.$$

Proof. By definition, for $k=0,1,\dots,r+2$,

$$\begin{aligned}
t^kD_t^kS_{\mathbb{R}}\phi(t) &:= \frac{t^k}{\pi i}D_t^k\int_{-\infty}^{\infty}\frac{\phi(\tau)}{\tau-t}\,d\tau\\
&= \frac{t^k}{\pi i}\int_{-\infty}^{\infty}\frac{(D_\tau^k\phi)(\tau)}{\tau-t}\,d\tau \ \ \text{(see [8, Chapter I, Section 4.4 p. 31])}\\
&= \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\phi_k(\tau)}{\tau-t}\,d\tau,
\end{aligned}$$

where, in the last step, we use the identity

$$t^k=\tau^k+(t-\tau)(t^{k-1}+t^{k-2}\tau+\cdots+t\tau^{k-2}+\tau^{k-1}),$$

and repeat integration by parts.

For any $t \in \mathbb{R}$, we define the change of variable

$$z := \frac{t-i}{t+i} \quad \Big(\text{or equivalently} \ \ t = i\,\frac{1+z}{1-z}\Big),$$

where, of course, $z \in \mathbb{T}$. Note that, as $t \to \pm\infty$, we have $z \to 1$. As previously, we define

$$\Phi_k(z) := \phi_k\Big(i\,\frac{1+z}{1-z}\Big).$$

By Lemma C.1, $\Phi_k \in H_\nu(\mathbb{T})$ with $\Phi_k(1) = 0$, for $k = 0, 1, \ldots, r+1$.

With this change of variable,

$$\begin{aligned} t^k D_t^k S_{\mathbb{R}}\phi(t) &= \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{\phi_k(\tau)}{\tau - t}\, d\tau \\ &= \frac{1}{\pi i} \int\limits_{|w|=1} \frac{1-z}{1-w}\, \frac{\Phi_k(w)}{w-z}\, dw \\ &= \frac{1}{\pi i} \int\limits_{|w|=1} \frac{\Phi_k(w)}{w-z}\, dw - \frac{1}{\pi i} \int\limits_{|w|=1} \frac{\Phi_k(w)}{w-1}\, dw \\ &= (S_{\mathbb{T}}\Phi_k)(z) - (S_{\mathbb{T}}\Phi_k)(1). \end{aligned}$$

But the operator $S_{\mathbb{T}}$ is bounded on $H_\nu(\mathbb{T})$, and hence

$$t^k D_t^k S_{\mathbb{R}}\phi(t) = O\big(|z-1|^\nu\big) = O\big(|t|^{-\nu}\big),$$

as $t \to \pm\infty$ ($z \to 1$). This completes the proof of the lemma. $\square$

Our first task is to obtain some asymptotic estimates for the non-diagonal elements of $A_1^\pm$. Due to the similarity of calculations, it is enough to prove this result for the matrix A_1^+. For brevity, we will ignore any constant terms that do affect the proof.

Lemma C.3. *Suppose $1 \le j, k \le N$ with $j \ne k$. Then*

$$D_t^q (A_1^\pm)_{j,k}(\omega, t) = O\big(|t|^{-\sigma - q}\big),$$

for $q = 0, 1, \ldots, r+2$, and some $\sigma > 0$.

Proof. We begin by noting that from the definition of A_1^+,

$$(A_1^+)_{j,k} = (t+i)^{\zeta_j - \zeta_k} (A_+^*)_{j,k}.$$

Firstly, we suppose that $\mathrm{Re}(\zeta_j - \zeta_k) < 0$. Then, in this case we can simply take

$$\sigma = -\,\mathrm{Re}(\zeta_j - \zeta_k),$$

so that $\sigma > 0$. Since $(A_+^*)_{j,k} \in W^{r+2}(\mathbb{R})$, the required result follows immediately. Note that, taking $q = 0$,

$$\lim_{t \to \pm\infty} (A_1^+)_{j,k}(\omega, t) = 0. \tag{C.5}$$

Secondly, suppose that $j \neq k$ and $\mathrm{Re}(\zeta_j - \zeta_k) \geq 0$. From equation (C.1),

$$\begin{aligned} A_+^* - A_-^* &= \operatorname{diag}\Big(\frac{t+i}{t-i}\Big)^{\kappa(\omega)} A_-^* A_0^* - A_-^* \\ &= \Big[\operatorname{diag}\Big(\frac{t+i}{t-i}\Big)^{\kappa(\omega)} - I\Big] A_-^* + \operatorname{diag}\Big(\frac{t+i}{t-i}\Big)^{\kappa(\omega)} (A_-^* A_0^* - A_-^*) \\ &= \Big[\operatorname{diag}\Big(\frac{t+i}{t-i}\Big)^{\kappa(\omega)} - I\Big] A_-^* + \operatorname{diag}\Big(\frac{t+i}{t-i}\Big)^{\kappa(\omega)} A_-^* (A_0^* - I) \\ &:= b_1(t) + b_2(t)(A_0^* - I). \end{aligned}$$

Consider now

$$b_1(t) := \Big[\operatorname{diag}\Big(\frac{t+i}{t-i}\Big)^{\kappa(\omega)} - I\Big] A_-^*(\omega, t).$$

We note that, as $t \to \pm\infty$,

$$\begin{aligned} D_t^q\Big[\operatorname{diag}\Big(\frac{t+i}{t-i}\Big)^{\kappa(\omega)} - I\Big] &= O\big(|t|^{-q-1}\big), \\ D_t^q\big[A_-^*(\omega, t)\big] &= O\big(|t|^{-q}\big) \ \ (\text{since } \ A_-^* \in W_{N\times N}^{r+2}(\mathbb{R})), \end{aligned}$$

for $q = 0, 1, \ldots, r+2$. Hence, as $t \to \pm\infty$,

$$D_t^q b_1(t) = O\big(|t|^{-q-1}\big). \tag{C.6}$$

Consider now the second term,

$$\big[b_2(A_0^* - I)\big]_{j,k} = \sum_{s=1}^{N} (b_2)_{j,s}(t)\big(A_0^*(\omega, t) - I\big)_{s,k}(t)$$

where, by definition,

$$b_2(t) := \Big(\frac{t+i}{t-i}\Big)^{\kappa(\omega)} A_-^*(\omega, t).$$

Since $A_-^* \in W_{N\times N}^{r+2}(\mathbb{R})$ we immediately have

$$D_t^q(b_2)(t) = O\big(|t|^{-q}\big). \tag{C.7}$$

Moreover, from estimates (A.6) and (A.7),

$$D_t^q(A_0^* - I)_{s,k} = O\big(|t|^{-q-\mathrm{Re}\,\zeta_s + \mathrm{Re}\,\zeta_k + \epsilon - 1}\big). \tag{C.8}$$

where ϵ is an arbitrarily small positive number that takes account of the logarithmic terms in the matrices $B_\pm(t)$ used in the construction of A_0^*. (See (A.12).) Using estimates (C.7) and (C.8),

$$\begin{aligned} D_t^q\big[b_2(A_0^* - I)\big]\big]_{j,k} &= \sum_{s=1}^{N} O\big(|t|^{-q-\mathrm{Re}\,\zeta_s + \mathrm{Re}\,\zeta_k + \epsilon - 1}\big) \\ &= \sum_{s=1}^{N} O\big(|t|^{RRe(\zeta_k - \zeta_j) - q + \epsilon - \{\mathrm{Re}(\zeta_s - \zeta_j) + 1\}}\big) \\ &= O\big(|t|^{-\,\mathrm{Re}(\zeta_j - \zeta_k) - q + \epsilon - \delta_0}\big). \end{aligned} \tag{C.9}$$

Let $\nu = \mathrm{Re}(\zeta_j - \zeta_k) + \delta_0 - \epsilon$. By assumption, $\mathrm{Re}(\zeta_j - \zeta_k) \geq 0$ and hence we can choose any ϵ such that $0 < \epsilon < \delta_0$ to ensure that $\nu > 0$. Moreover, $\mathrm{Re}(\zeta_j - \zeta_k) + \delta_0 \leq 1$, and hence $\nu < 1$.

Combining estimates (C.6) and (C.9),

$$D_t^q (A_+^* - A_-^*)_{j,k}(\omega, t) = O\big(|t|^{-\nu - q}\big),$$

for $q = 0, 1, \ldots, r + 2$, where $0 < \nu < 1$.

For the fixed ω, we can now apply Lemma C.2 with

$$\phi(t) = (A_+^* - A_-^*)_{j,k}(\omega, t).$$

Let $\sigma := \delta_0 - \epsilon$. Then

$$\begin{aligned} D_t^q (A_+^*)_{j,k}(\omega, t) &= D_t^q \frac{1}{2}(I + S_{\mathbb{R}})(A_+^* - A_-^*)_{j,k}(\omega, t) \\ &= O\big(|t|^{-\,\mathrm{Re}(\zeta_j - \zeta_k) - \sigma - q}\big). \end{aligned}$$

Finally,

$$D_t^q (A_1^+)_{j,k}(\omega, t) = O\big(|t|^{-\sigma - q}\big), \tag{C.10}$$

for $q = 0, 1, \ldots, r + 1$, and $\sigma > 0$. ☐

Lemma C.4. *Suppose* $1 \leq j, k \leq N$ *with* $j \neq k$. *Then*

$$\lim_{t \to \pm\infty} \big(A_1^{\pm}(\omega, t)\big)_{j,k} = 0.$$

Proof. The proof of the lemma follows directly from the estimates (C.5) and (C.10). (Of course, using estimate (C.10), we take $q = 0$.) ☐

Remark C.5. From equation (C.2), $\lim_{t \to \pm\infty} (A_{\pm}^*)_{j,j} = 1$, and hence,

$$\lim_{t \to \pm\infty} \big(A_1^{\pm}(\omega, t)\big)_{j,j} = 1, \quad 1 \leq j \leq N.$$

Thus, the proof of Lemma C.3 can readily be extended to obtain (c.f. (A.8))

$$D_t^q (A_1^{\pm}(\omega, t) - I) = O\big(|t|^{-\sigma - q}\big)$$

for $q = 0, 1, \ldots, r + 1$ and $\sigma > 0$.

Lemma C.6. *Suppose* $1 \leq j, k \leq N$. *Let*

$$A_2^{\pm}(\omega, t) = B_{\pm}^{-1}(t)\big(A_1^{\pm}(\omega, t) - I\big)B_{\pm}(t) + I$$

Then

$$\big(A_2^{\pm}(\omega, t)\big)_{j,k} \in W^r(\mathbb{R}).$$

Proof. From Remark C.5,

$$D_t^q \big(A_1^{\pm}(\omega, t) - I\big) = O\big(|t|^{-\sigma - q}\big),$$

for $\sigma > 0$. Hence, from the definition of $A_2^{\pm}$,

$$D_t^q \big(A_2^{\pm}(\omega, t) - I\big) = O\big(|t|^{-\sigma' - q}\big),$$

for $q = 0, 1, \ldots, r + 1$ and any σ' such that $0 < \sigma' < \sigma$. The required result now follows from Lemma 2.2. ☐

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Author's address:

Department of Natural and Mathematical Sciences, King's College, London, UK.

E-mail: tony.hill@kcl.ac.uk

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Short Communication

Malkhaz Ashordia and Goderdzi Ekhvaia

ON THE SOLVABILITY OF MULTIPOINT BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR DIFFERENCE EQUATIONS

Abstract. The effective sufficient conditions are given for the solvability of the multipoint boundary value problems for systems of nonlinear difference equations.

რეზიუმე. არაწრფივ სხვაობიან განტოლებათა სისტემებისათვის მოყვანილია მრავალწერტილოვანი სასაზღვრო ამოცანების ამოხსნადობის ეფექტური საკმარისი პირობები.

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Let m_0 be a fixed natural number, $\mathbb{N}_{m_0} = \{1,\dots,m_0\}$ and $\widetilde{\mathbb{N}}_{m_0} = \{1,\dots,m_0\}$. Consider the problem of finding a vector-function $y=(y_i)_{i=1}^n : \widetilde{\mathbb{N}}_{m_0} \to \mathbb{R}^n$ satisfying the system of difference equations

$$\Delta y_i(k-1) = g_i\big(k, y_1(k),\dots,y_n(k), y_1(k-1),\dots,y_n(k-1)\big) \quad \text{for } k\in\mathbb{N}_{m_0} \ \ (i=1,\dots,n) \tag{1}$$

and the multipoint boundary value problem of the Caucy–Nicoletti's type

$$y_i(k_i) = \xi_i(y_1,\dots,y_n) \ \ (i=1,\dots,n), \tag{2}$$

where $k_i\in\widetilde{\mathbb{N}}_{m_0}$ $(i=1,\dots,n)$, $g_i(k,\,\cdot\,)\in C(\mathbb{R}^{2n},\mathbb{R})$ $(k=1,\dots,m_0)$, and $\xi_i : E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$ $(i=1,\dots,n)$ are continuous functional, in general nonlinear.

In the paper some effective sufficient conditions are given for the solvability and unique solvability of the boundary value problem (1), (2). Some results of the same type, among them necessary and sufficient condition, are given in [1]. The general nonlinear boundary problems for the difference system (1) is considered in [5], where the Conti–Opial's type existence and uniqueness theorems are given for the problem.

The various question for the linear and nonlinear boundary value problems for the systems of difference equations are considered in $[1,2,5,7,11]$

(see also the references therein). The questions, analogous to considered in the paper and in [1], are studied sufficiently well (see, for example, [8, 9]) for the boundary value problems for the ordinary differential systems, and in [3, 4, 6, 10, 12] for the impulsive systems (see also the references therein).

We realize the results for the boundary condition

$$y_i(t_i) = c_i \ \ (i = 1, \dots, n), \tag{3}$$

where $c_i \in \mathbb{R}^n$ $(i = 1, \dots, n)$ are constant vectors.

Throughout the paper the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ $(a, b \in \mathbb{R})$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

$\mathbb{R}_+^{n\times m} = \big\{(x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m)\big\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

If $l \in \mathbb{N}$, then $\mathbb{N}_l = \{1, \dots, l\}$, $\widetilde{\mathbb{N}}_l = \{0, 1, \dots, l\}$.

$E(J, \mathbb{R}^{n\times m})$, where $J \subset \mathbb{Z}$, is the space of all matrix-functions $Y = (y_{ij})_{i,j=1}^{n,m} : J \to \mathbb{R}^{n\times m}$ with the norm $\|Y\|_J = \max\{\|Y(k)\| : \ k \in J\}$, $\|Y\|_{\nu,J} = \big(\sum\limits_{k\in J} \|Y(k)\|^\nu\big)^{\frac{1}{\nu}}$ if $1 \leq \nu < +\infty$, and$\|Y\|_{+\infty,\alpha} = \|Y\|_J$.

Δ is the difference operator of the first order, i.e.

$$\Delta Y(k-1) = Y(k) - Y(k-1) \quad \text{for } \ Y \in E(\widetilde{\mathbb{N}}_l, \mathbb{R}^{n\times m}), \ \ k \in \mathbb{N}_l.$$

If a function Y is defined on $\mathbb{N}_l$ or $\widetilde{\mathbb{N}}_{l-1}$, then we assume $Y(0) = O_{n\times m}$, or $Y(l) = O_{n\times m}$, respectively, if it is necessary.

If B_1 and B_2 are normed spaces, then an operator $\xi : B_1 \to B_2$ (nonlinear, in general) is called positive homogeneous if $\xi(\lambda x) = \lambda\xi(x)$ for every $\lambda \in \mathbb{R}_+$ and $x \in B_1$. If the spaces B_1 and B_2 are partial ordered then the operator ξ is called nondecreasing if the inequality $\xi(x) \leq \xi(y)$ holds for every $x, y \in B_1$ such that $x \leq y$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

Definition 1. Let $k_1, \dots, k_n \in \widetilde{\mathbb{N}}_{m_0}$. We say that the triplet $(Q_1, Q_2; \xi_0)$, consisting of matrix-functions $Q_j = (q_{jil})_{i,l=1}^n \in E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$ and a positive homogeneous nondecreasing continuous vector-functional $\xi_0 = (\xi_{0i})_{i=1}^n : E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}_+^n) \to \mathbb{R}_+^n$, belongs to the set $U(k_1, \dots, k_n)$ if $q_{jil}(t) \geq 0$ $(j = 1, 2; \ i \neq l; \ i, l = 1, \dots, n)$ and the system of difference inequalities

$$\begin{aligned}\Delta y_i(k-1)\operatorname{sgn}\Big(k - k_i - \frac{1}{2}\Big)& \\ \leq \sum_{l=1}^{n} &\big(q_{1il}y_l(k) + q_{2il}y_l(k-1)\big) \ \ (i = 1, \dots, n),\end{aligned}$$

has no nontrivial nonnegative solution satisfying the condition

$$y_i(k_i) \le \xi_{0i}(|y_1|, \dots, |y_n|) \quad (i = 1, \dots, n).$$

The set analogous to $U(k_1, \dots, k_{m_0})$ has been introduced by I. Kiguradze for ordinary differential equations (see [8, 9]).

Theorem 1. *Let the inequalities*

$$\begin{aligned} g_i(t, y_1, \dots, y_{2n}) \operatorname{sgn}\Big[\Big(k - k_i - \frac{1}{2}\Big) y_{j\,n+i}\Big] &\\ \le \sum_{l=1}^{n} \big(p_{1il}(k)|y_l| + p_{2il}(k)|y_{n+l}|\big)&\\ + q_i\Big(k, \sum_{l=1}^{2n} |y_l|\Big) \quad (j = 0, 1; \ \ k = 1, \dots, m_0; \ \ i = 1, \dots, n)& \end{aligned} \tag{4}$$

and

$$|\xi_i(y_1, \dots, y_n)| \le \xi_{0i}\big(|y_1|, \dots, |y_n|\big) + \gamma_i\Big(\sum_{l=1}^{n} |y_l|\Big) \quad (i = 1, \dots, n)$$

be fulfilled on the sets $\mathbb{R}^{2n}$ and $E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$, respectively, where $(p_{jil})_{i,l=1}^n \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$, and $q_i(k, \cdot\,) \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\gamma_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(i = 1, \dots, n;\ k = 1, \dots, m_0)$ are nondecreasing functions satisfying the conditions

$$\lim_{\rho \to +\infty} \frac{q_i(k, \rho)}{\rho} = \lim_{\rho \to +\infty} \frac{\gamma_i(\rho)}{\rho} = 0 \ \ \text{for} \ \ k \in \mathbb{N}_{m_0} \quad (i = 1, \dots, n), \tag{5}$$

$\xi_{0i} : E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}_+) \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are positive homogeneous nondecreasing functionals. Moreover, let there exist a matrix-functions $Q_j = (q_{jil})_{i,l=1}^n \in E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$ such that

$$(Q_1, Q_2; \xi) \in U(k_1, \dots, k_{m_0}), \tag{6}$$

here $\xi = (\xi_{0i})_{i=1}^n$, and

$$p_{jil}(k) \le q_{jil}(k) \ \ \text{for} \ \ k \in \mathbb{N}_{m_0} \ \ (j = 1, 2; i, l = 1, \dots, n). \tag{7}$$

Then the problem (1), (2) *is solvable.*

Corollary 1. *Let the inequalities* (4) *and*

$$|\xi_i(y_1, \dots, y_n)| \le \sum_{m=1}^{n} l_{im} \|y_m\|_{\nu, \mathbb{N}_{m_0}} + \gamma_i\Big(\sum_{l=1}^{n} |y_l|\Big) \quad (i = 1, \dots, n)$$

be fulfilled on the sets $\mathbb{R}^{2n}$ and $E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$, respectively, where $(p_{jil})_{i,l=1}^n \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$, and $q_i(k, \cdot\,) \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\gamma_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(i = 1, \dots, n;\ k = 1, \dots, m_0)$ are nondecreasing functions satisfying the condition (5), *$l_{im} \in \mathbb{R}_+$ $(i, m = 1, \dots, n)$, $2 \le \nu \le +\infty$. Moreover, let* *the module of every characteristic value of the matrices* $\mathcal{H} = (h_{jim})_{i,j=1}^{n}$ $(j = 1, 2)$ *be less then* 1*, where*

$$h_{jim} = (2-j)m_0^{\frac{1}{\nu}} l_{im} + \Big(\frac{1}{2}\sin^{-1}\frac{\pi}{4m_0+2}\Big)^{\frac{2}{\nu}} \|q_{jim}\|_{\mu,\mathbb{N}_{m_0}}$$
$$(j = 1, 2;\ \ i, m = 1, \dots, n),$$

where $\frac{1}{\mu} + \frac{2}{\nu} = 1$*. Then the problem* $(1), (2)$ *is solvable.*

Corollary 2. *Let the inequalities*

$$g_i(t, y_1, \dots, y_{2n}) \operatorname{sgn}\Big[\Big(k - k_i - \frac{1}{2}\Big) y_{j\,n+i}\Big] \le \sum_{l=1}^{n} \Big(\eta_{1il}(k)|y_l| + \eta_{2il}(k)|y_{n+l}|\Big) + q_i\Big(k, \sum_{l=1}^{2n} |y_l|\Big) \ \ (j = 0, 1;\ \ k = 1, \dots, m_0;\ \ i = 1, \dots, n) \quad (8)$$

and

$$\big|\xi_i(y_1, \dots, y_n)\big| \le \mu_i |y_i(l_i)| + \gamma_i\Big(\sum_{l=1}^{n} |y_i|\Big) \ \ (i = 1, \dots, n)$$

be fulfilled on the sets $\mathbb{R}^{2n}$ *and* $E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$*, respectively, where* $\eta_{jil} \in \mathbb{R}_+$ $(j = 1, 2;\ i \ne l;\ i, l = 1, \dots, n)$, $-1 < \eta_{jii} < 0$ $(j = 1, 2;\ i = 1, \dots, n)$, $q_i(k, \cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and* $\gamma_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(i = 1, \dots, n;\ k = 1, \dots, m_0)$ *are nondecreasing functions satisfying the condition* (5)*, and* $\mu_i \in \mathbb{R}_+$ *and* $l_i \in \{1, \dots, m_0\}$ $l_i \ne k_i$ $(i = 1, \dots, n)$*. Moreover, let*

$$\mu_i \max\big\{\gamma_{1i}(l_i), \gamma_{2i}(l_i)\big\} < 1 \ \ (i = 1, \dots, n), \quad (9)$$

and the real part of every characteristic value of the matrix $(\xi_{il})_{i,l=1}^{n}$ *be negative, where*

$$\gamma_{ji}(k) \equiv \big(1 + (-1)^j \eta_{jii} \operatorname{sgn}(k - k_i)\big)^{(-1)^j (k-k_i)} \ \ (j = 1, 2; i = 1, \dots, n),$$
$$\xi_{ii} = \eta_{1ii} + \eta_{2ii}), \ \ \xi_{il} = \eta_{1il} h_{1i} + \eta_{2i} h_{2il} \ \ (i \ne l;\ \ i, l = 1, \dots, n),$$

and

$$h_{1i} = h_{2i} = 1 \ \ \text{if} \ \ 0 \le \mu_i \le 1,$$
$$h_{ji} = 1 + (\mu_i - 1)\big(1 - \mu_i \gamma_{ji}(l_i)\big)^{-1} \ \ \text{if} \ \ \mu_i > 1 \ \ (i = 1, \dots, n).$$

Then the problem $(1), (2)$ *is solvable.*

Theorem 2. *Let the inequalities*

$$\big[g_i(t,y_1,\dots,y_{2n})-g_i(t,z_1,\dots,z_{2n})\big]\operatorname{sgn}\Big[\Big(k-k_i-\frac{1}{2}\Big)(y_{j\,n+i}-z_{j\,n+i})\Big]$$
$$\leq \sum_{l=1}^{n}\Big(p_{1il}(k)|y_l-z_l|+p_{2il}(k)|y_{n+l}-z_{n+l}|\Big) \tag{10}$$
$$(j=0,1;\ \ k=1,\dots,m_0;\ \ i=1,\dots,n)$$

and

$$\big|\xi_i(y_1,\dots,y_n)-\xi_i(z_1,\dots,z_n)\big|$$
$$\leq \xi_{0i}\big(|y_1-z_l|,\dots,|y_n-z_n|\big)\ \ (i=1,\dots,n) \tag{11}$$

be fulfilled on the sets $\mathbb{R}^{2n}$ *and* $E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$, *respectively, where* $(p_{jil})_{i,l=1}^{n}\in E(\mathbb{N}_{m_0},\mathbb{R}^{n\times n})$ $(j=1,2)$. *Moreover, let there exists a matrix-functions* $Q_j=(q_{jil})_{i,l=1}^{n}\in E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ $(j=1,2)$ *such that the conditions* (6) *and* (7) *hold, where* $\xi=(\xi_{0i})_{i=1}^{n}$. *Then the problem* (1), (2) *has one and only one solution.*

Corollary 3. *Let the inequalities* (10) *and*

$$\big|\xi_i(y_1,\dots,y_n)-\xi_i(z_1,\dots,z_n)\big|\leq \sum_{m=1}^{n} l_{im}\|y_m-z_m\|_{\nu,\mathbb{N}_{m_0}}\ \ (i=1,\dots,n)$$

be fulfilled on the sets $\mathbb{R}^{2n}$ *and* $E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$, *respectively, where* $(p_{jil})_{i,l=1}^{n}\in E(\mathbb{N}_{m_0},\mathbb{R}^{n\times n})$ $(j=1,2)$, *and* $l_{im}\in\mathbb{R}_+$ $(i,m=1,\dots,n)$, $2\leq\nu\leq+\infty$. *Moreover, let the module of every characteristic value of the matrices* $\mathcal{H}=(h_{jim})_{i,j=1}^{n}$ $(j=1,2)$, *appearing in the Corollary* 1, *be less then* 1, *where where* $\frac{1}{\mu}+\frac{2}{\nu}=1$. *Then the problem* (1), (2) *has one and only one solution.*

Corollary 4. *Let the inequalities*

$$\big[g_i(t,y_1,\dots,y_{2n})-g_i(t,z_1,\dots,z_{2n})\big]\operatorname{sgn}\Big[\Big(k-k_i-\frac{1}{2}\Big)(y_{j\,n+i}-z_{j\,n+i})\Big]$$
$$\leq \sum_{l=1}^{n}\Big(\eta_{1il}|y_l-z_l|+\eta_{2il}|y_{n+l}-z_{n+l}|\Big)$$
$$(j=0,1;\ \ k=1,\dots,m_0;\ \ i=1,\dots,n)$$

be fulfilled on the sets $\mathbb{R}^{2n}$, *where* $\eta_{jil}\in\mathbb{R}_+$ $(j=1,2;\ i\neq l;\ i,l=1,\dots,n)$, $-1<\eta_{jii}<0$ $(j=1,2;\ i=1,\dots,n)$. *Moreover, let* $\mu_i\in\mathbb{R}_+$ *and* $l_i\in\{1,\dots,m_0\}$, $l_i\neq k_i$ $(i=1,\dots,n)$, *be such the condition* (9) *hold and the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^{n}$ *be negative, where*

$$\gamma_{ji}(k)\equiv\big(1+(-1)^j\eta_{jii}\operatorname{sgn}(k-k_i)\big)^{(-1)^j(k-k_i)}\ \ (j=1,2;i=1,\dots,n),$$
$$\xi_{ii}=\eta_{1ii}+\eta_{2ii},\quad \xi_{il}=\eta_{1il}h_{1il}+\eta_{2il}h_{2il}\ \ (i\neq l;\ \ i,l=1,\dots,n),$$

and

$$h_{1i} = h_{2i} = 1 \ \text{ if } \ 0 \le \mu_i \le 1,$$
$$h_{ji} = 1 + (\mu_i - 1)\big(1 - \mu_i \gamma_{ji}(l_i)\big)^{-1} \ \text{ if } \ \mu_i > 1 \ \ (i = 1, \dots, n),$$

Then the system (1) *has one and only one solution under the condition*

$$y_i(k_i) = \lambda_i y_i(l_i) + \beta_i \ \ (i = 1, \dots, n)$$

for every $\lambda_i \in [-\mu_i, \mu_i]$ *and* $\beta_i \in \mathbb{R}$ $(i = 1, \dots, n)$.

Theorem 3. *Let the matrix functions* $Q_j = (q_{jil})_{i,l=1}^n \in E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n \times n})$ $(j = 1, 2)$ *and the linear continuous vector-functional* $\xi_0 = (\xi_{0i})_{i=1}^n : E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}_+^n) \to \mathbb{R}_+^n$ *be such that* $q_{jil}(t) \ge 0$ $(j = 1, 2;\ i \ne l;\ i, l = 1, \dots, n)$ *but the condition* (6) *be violated. Then there exist matrix-functions* $(p_{jil})_{i,l=1}^n \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n \times n})$ $(j = 1, 2)$, *functions* $g_i(k, \cdot) \in C(\mathbb{R}^{2n}, \mathbb{R})$ $(k = 1, \dots, m_0)$, *and continuous functionals* $\xi_i : E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$ $(i = 1, \dots, n)$ *such that the condition* (7) *hold, the inequalities* (10) *and* (11) *are fulfilled on the sets* $\mathbb{R}^{2n}$ *and* $E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$, *respectively, but the problem* (1), (2) *is not solvable.*

The conditions for the solvability of the problem (1), (3) follows from the theorems and corollaries given above if we assume $\xi_i(y_1, \dots, y_n) \equiv c_i$ $(i = 1, \dots, n)$.

We have the following results for the solvability of the problem (1), (3).

Theorem 4. *Let the inequalities* (4) *be fulfilled on the set* $\mathbb{R}^{2n}$, *where* $(p_{jil})_{i,l=1}^n \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n \times n})$ $(j = 1, 2)$, *and* $q_i(k, \cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(i = 1, \dots, n;\ k = 1, \dots, m_0)$ *are nondecreasing functions satisfying the condition* (5). *Moreover, let there exist a matrix-functions* $Q_j = (q_{jil})_{i,l=1}^n \in E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n \times n})$ $(j = 1, 2)$ *such that the condition* (7) *hold, and the system of difference inequalities appearing in the Definition* 1 *has no nontrivial nonnegative solution satisfying the condition*

$$y_i(k_i) = 0 \ \ (i = 1, \dots, n).$$

Then the problem (1), (3) *is solvable.*

Corollary 5. *Let the inequalities* (4) *be fulfilled on the set* $\mathbb{R}^{2n}$, *respectively, where* $(p_{jil})_{i,l=1}^n \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n \times n})$ $(j = 1, 2)$, *and* $q_i(k, \cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(i = 1, \dots, n;\ k = 1, \dots, m_0)$ *are nondecreasing functions satisfying the condition* (5). *Moreover, let the module of every characteristic value of the matrices* $\mathcal{H}_j = (h_{jim})_{i,j=1}^n$ $(j = 1, 2)$ *be less then* 1, *where*

$$h_{jim} = \Big(\frac{1}{2} \sin^{-1} \frac{\pi}{4m_0 + 2}\Big)^{\frac{2}{\nu}} \|q_{jim}\|_{\mathbb{N}_{m_0}} \ \ (j = 1, 2; i, m = 1, \dots, n).$$

Then the problem (1), (3) *is solvable.*

Corollary 6. *Let the inequalities* (8) *be fulfilled on the set* $\mathbb{R}^{2n}$*, where* $\eta_{jil} \in \mathbb{R}_+$ $(j = 1, 2;\ i \neq l;\ i, l = 1, \dots, n)$, $-1 < \eta_{jii} < 0$ $(j = 1, 2;\ i = 1, \dots, n)$, $q_i(k, \cdot) \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(i = 1, \dots, n;\ k = 1, \dots, m_0)$ *are nondecreasing functions satisfying the condition* (5). *Moreover, let the real part of every characteristic value of the matrix* $(\eta_{1il} + \eta_{2il})_{i,l=1}^n$ *be negative. Then the problem* (1), (3) *is solvable.*

Theorem 5. *Let the inequalities* (9) *be fulfilled on the set* $\mathbb{R}^{2n}$*, where* $(p_{jil})_{i,l=1}^n \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$. *Moreover, let there exists a matrix-functions* $Q_j = (q_{jil})_{i,l=1}^n \in E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$ *such that the condition* (7) *hold, and the system of difference inequalities, appearing in the Definition* 1*, has no nontrivial nonnegative solution satisfying the condition* (13)*, where* $\xi = (\xi_{0i})_{i=1}^n$. *Then the problem* (1), (3) *has one and only one solution.*

Corollary 7. *Let the inequalities* (10) *be fulfilled on the set* $\mathbb{R}^{2n}$*, where* $(p_{jil})_{i,l=1}^n \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$. *Moreover, let the module of every characteristic value of the matrices* $\mathcal{H} = (h_{jim})_{i,j=1}^n$ $(j = 1, 2)$*, appearing in the Corollary* 5*, be less then* 1. *Then the problem* (1), (3) *has one and only one solution.*

Corollary 8. *Let the inequalities* (8) *be fulfilled on the set* $\mathbb{R}^{2n}$*, where* $\eta_{jil} \in \mathbb{R}_+$ $(j = 1, 2;\ i \neq l;\ i, l = 1, \dots, n)$, $-1 < \eta_{jii} < 0$ $(j = 1, 2;\ i = 1, \dots, n)$. *Moreover, let the real part of every characteristic value of the matrix* $(\eta_{1il} + \eta_{2il})_{i,l=1}^n$ *be negative. Then the problem* (1), (3) *has one and only one solution.*

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Authors' addresses:

Malkhaz Ashordia

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia.

2. Sokhumi State University, 12 Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: ashord@rmi.ge

Goderdzi Ekhvaia

Sokhumi State University, 12 Politkovskaia St., Tbilisi 0186, Georgia.

E-mail: goderdzi.ekhvaia@mail.ru

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