E ISSN 1512-3391

A. RAZMADZE MATHEMATICAL INSTITUTE

of I. Javakhishvili Tbilisi State University

International Workshop on the Qualitative Theory of Differential Equations

QUALITDE – 2021

December 18 – 20, 2021

Tbilisi, Georgia

ABSTRACTS

Dedicated to the 130th birthday anniversary
of Academician N. Muskhelishvili

16.02.1891 – 16.07.1976

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On the Well-Posedness Question of the Cauchy Problem with Weight for Systems of Linear Generalized Ordinary Differential Equations and Impulsive Equations with Singularities

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Let $I=[a,b]\subset\mathbb{R}$ be a finite and closed interval non-degenerated in the point, $t_0\in\,]a,b[$ and $I_{t_0}=[a,b]\setminus\{t_0\}$, $I^-_{t_0}=[a,t_0[\,$, $I^+_{t_0}=\,]t_0,b]$.

Consider the Cauchy problem for the linear system of generalized ordinary differential equations (GODE) with singularities

$$dx=dA(t)\cdot x+df(t)\ \text{ for }\ t\in I_{t_0}, \tag{1}$$

$$\lim_{t\to t_0-}(H^{-1}(t)\,x(t))=0\ \text{ and }\ \lim_{t\to t_0+}(H^{-1}(t)\,x(t))=0, \tag{2}$$

where $A=(a_{ik})_{i,k=1}^n$ is an $n\times n$-matrix valued function and $f=(f_k)_{k=1}^n$ is an n-vector valued function, both of them have a locally bounded variation on $[a,b]\setminus\{t_0\}$; $H=\operatorname{diag}(h_1,\dots,h_n)$ is a continue diagonal matrix function, continuous and having an inverse $H^{-1}(t)$ for $t\in[a,b]\setminus\{t_0\}$.

Along with system (1) consider the perturbed system

$$dy=d\widetilde{A}(t)\cdot y+d\widetilde{f}(t)\ \text{ for }\ t\in I_{t_0} \tag{3}$$

under condition (2), where $\widetilde{A}$, $\widetilde{f}$ are, as above, a matrix- and vector-functions.

We are interested in the question whether the unique solvability of problem $(1),(2)$ guarantees the unique solvability of problem $(3),(2)$ and nearness of its solutions in the definite sense if matrix-functions A and $\widetilde{A}$ and vector-functions f and $\widetilde{f}$ are nearly among themselves.

The same and related problems for singular linear ordinary differential systems have been investigated in [3] (see also the references therein).

The singularity of system (1) consists in the fact that both A and f need not have bounded variations on any interval containing the point t_0.

The solvability of the singular problem $(1),(2)$ is investigated in [2]. To our knowledge, the well-posedness of $(1),(2)$ has not been considered up to now.

The theory of GODE has been introduced by J. Kurzweil [4]. The interest to the theory has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1, 2, 4] and the references therein).

We present sufficient conditions for the so called H-well-posedness of problem $(1),(2)$. We realize the presented results for systems of impulsive differential equations with fixed points of impulses actions.

We use the following notation and definitions.

$\mathbb{N}=\{1,2,\dots\}$, $\mathbb{R}=]-\infty,+\infty[\,$, $\mathbb{R}_+=[0,+\infty[\,$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ik})_{i,k=1}^{n,m}$ with the standard norm $\|X\|$.

$|X|=(|x_{ik}|)_{i,k=1}^{n,m}$,

$[X]_\pm=\frac{1}{2}\,(|X|\pm X)$; $r(X)$ is the spectral radius of $X\in\mathbb{R}^{n\times n}$. $\mathbb{R}^n=\mathbb{R}^{n\times 1}$.

$O_{n\times m}$ is the zero $n\times m$-matrix, 0_n is the zero n-vector. I_n is the identity $n\times n$-matrix.

$\bigvee\limits_a^b(X)$ is the sum of variations on $[a,b]$ of components of matrix-function X.

$\bigvee\limits_a^{b-}(X)=\lim\limits_{t\to b-}\bigvee\limits_a^t(X)$; $X(t-)$ and $X(t+)$ are the left and the right limits of X at the point t.

$d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions.

$\mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times m})$ is the set of all $X:I\to\mathbb{R}^{n\times m}$ for which the restriction to $[a,b]$ belong to $\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ for every closed interval $[a,b]$ from I_{t_0}.

If $X(t)=(x_{ik}(t))_{i,k=1}^{n,m}$, then $V(X)(t)=(\bigvee\limits_{a_j}^t(x_{ik}))_{i,k=1}^{n,m}$ for $(t-t_0)(a_j-t_0)>0$ $(j=1,2)$, where $a_1=a$, $a_2=b$.

$[X(t)]_\pm^v\equiv\frac{1}{2}\,(V(X)(t)\pm X(t))$.

$L_{loc}(I_{t_0};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:I_{t_0}\to\mathbb{R}^{n\times m}$ whose restrictions to every closed interval $[a,b]$ from I_{t_0} is integrable.

s_1,s_2 and s_c are the operators defined, respectively, by

$$s_1(x)(a_j)=s_2(x)(a_j)=0,\ \ s_c(x)(a_j)=x(a_j)\ \ (j=1,2),$$
$$s_1(x)(t)=s_1(x)(s)+\sum_{s<\tau\le t}d_1x(\tau),\ \ s_2(x)(t)=s_2(x)(s)+\sum_{s\le\tau<t}d_2x(\tau),$$
$$s_c(x)(t)=s_c(x)(s)+x(t)-x(s)-\sum_{j=1}^2(s_j(x)(t)-s_j(x)(s))$$
$$\text{for }\ a_1\le s<t<t_0\ \text{ or }\ t_0<s<t\le a_2.$$

If $X\in\mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times n})$,

$$\det\big(I_n+(-1)^jd_jX(t)\big)\neq 0\ \text{ for }\ t\in I_{t_0}\ \ (j=1,2),$$

and $Y\in\mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X,Y)(a)=O_{n\times m}\ \ \big(\mathcal{A}(X,Y)(b)=O_{n\times m}\big),$$
$$\mathcal{A}(X,Y)(t)-\mathcal{A}(X,Y)(s)=Y(t)-Y(s)+\sum_{s<\tau\le t}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)$$
$$-\sum_{s\le\tau<t}d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \text{ if }\ s<t<t_0\ \ (t_0<s<t).$$

If $g:[a,b]\to\mathbb{R}$ has bounded variation and $x:[a,b]\to\mathbb{R}$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,ds_c(g)(\tau)+\sum_{s<\tau\le t}x(\tau)\,d_1g(\tau)+\sum_{s\le\tau<t}x(\tau)\,d_2g(\tau),$$

where $\int_{]s,t[}x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $s_c(g)$. So $\int_s^t x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral [4].

Let
$$\int\limits_{s}^{t\pm} x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int\limits_{s}^{t\pm\delta} x(\tau)\,dg(\tau),\quad \int\limits_{s\pm}^{t} x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int\limits_{s\pm\delta}^{t} x(\tau).$$

If $G=(g_{ik})_{i,k=1}^{l,n}$ and $X=(x_{kj})_{k,j=1}^{n,m}$ are matrix-functions on $[a,b]$, then

$$\int\limits_{a}^{t} dG(\tau)\,X(\tau)\equiv\bigg(\sum_{k=1}^{n}\int\limits_{a}^{t} x_{kj}(\tau)dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m},\quad S_j(G)=\big(s_j(g_{ik})\big)_{i,k=1}^{l,n}.$$

A vector-function $x:I_{t_0}\to\mathbb{R}^n$ is said to be a solution of system (1) if $x\in\mathrm{BV}([c,d],\mathbb{R}^n)$ for every closed interval $[c,d]$ from I_{t_0} and

$$x(t)=x(s)+\int\limits_{s}^{t} dA(\tau)\cdot x(\tau)+f(t)-f(s)\ \text{ for }\ c\le s<t\le d.$$

We assume that $\det(I_n+(-1)^jd_jA(t))\neq 0$ for $t\in I_{t_0}$ $(j=1,2)$. The inequalities guarantee the unique solvability of the Cauchy problem for the case when $A\in\mathrm{BV}_{loc}(I,\mathbb{R}^{n\times n})$ and $f\in\mathrm{BV}_{loc}(I,\mathbb{R}^n)$ (see, [4]).

Let $A_0\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ is a matrix-function such that

$$\det\big(I_n+(-1)^jd_jA_0(t)\big)\neq 0\ \text{ for }\ t\in I_{t_0}\ \ (j=1,2). \tag{4}$$

Then a matrix-function $C_0:I_{t_0}\times I_{t_0}\to\mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous system $dx=dA_0(t)\cdot x$ if, for every interval $J\subset I$ and $\tau\in J$, the restriction of $C_0(\,\cdot\,,\tau)$ to J is the fundamental matrix of this system satisfying the condition $C_0(\tau,\tau)=I_n$.

Let
$$I_{t_0}^-(\delta)=[t_0-\delta,t_0[\,,\quad I_{t_0}^+(\delta)=\,]t_0,t_0+\delta],\quad I_{t_0}(\delta)=I_{t_0}^-(\delta)\cup I_{t_0}^+(\delta)\ \ (\delta>0).$$

Definition 1. Problem (1), (2) is said to be H-well-posed with respect to the pair of the matrix-functions $(S_1(A_0),S_2(A_0))$ if it has a unique solution x and for every $\varepsilon>0$ there exists $\eta>0$ such that problem (3), (2) has a unique solution y and the estimate

$$\big\|H(t)\,(x(t)-y(t))\big\|<\varepsilon\ \text{ for }\ t\in I \tag{5}$$

holds for every $\widetilde A\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ and $\widetilde f\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^n)$ such that

$$\det\big(I_n+(-1)^jd_j\widetilde A(t)\big)\neq 0\ \text{ for }\ t\in I_{t_0}\ \ (j=1,2),$$
$$\bigg\|\int\limits_{t_0\pm}^{t} H^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_0,\widetilde A-A))(s)\cdot H(s)\bigg\|<\eta\ \text{ for }\ t\in I_{t_0}^{\pm},\ \text{ respectively},$$
$$\bigg\|\int\limits_{t_0\pm}^{t} H^{-1}(s)d\,\mathrm{V}(\mathcal{A}(A_0,\widetilde f-f))(s)\bigg\|<\eta\ \text{ for }\ t\in I_{t_0}^{\pm},\ \text{ respectively}.$$

We note that the matrix-functions $S_1(A_0)$ and $S_2(A_0)$ are including in the definition of the operators $V(\mathcal{A}(A_0,\widetilde A-A))$ and $V(\mathcal{A}(A_0,\widetilde f-f))$.

Theorem 1. *Let there exist a matrix-function $A_0 \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ and constant matrices $B_0. B \in \mathbb{R}_+^{n\times n}$ such that the conditions $r(B) < 1$,*

$$|C_0(t,\tau)| \le H(t)\, B_0\, H^{-1}(\tau) \text{ for } t \in I_{t_0}(\delta), \ \ (t-t_0)(\tau-t_0) > 0, \ \ \frac{|\tau-t_0|}{|t-t_0|} \le 1,$$

$$\bigg| \int\limits_{t_0\pm}^{t} |C_0(t,s)|\, d\,\mathrm{V}(\mathcal{A}(A_0, A-A_0))(s)\, H(s) \bigg| \le H(t)B, \ \ \text{respectively, on } I_{t_0}^{\pm}(\delta)$$

hold for some $\delta > 0$. Let, moreover,

$$\lim_{t\to t_0\pm} \bigg\| \int\limits_{t_0\pm}^{t} H^{-1}(t)\, C_0(t,\tau)\, d\mathcal{A}(A_0, f)(\tau) \bigg\| = 0, \ \ \text{respectively.}$$

Then problem (1), (2) is H-well-posed with respect to $(S_1(A_0), S_2(A_0))$.

Theorem 2. *Let there exist a constant matrix $B = (b_{ik})_{i,k=1}^{n} \in \mathbb{R}_+^{n\times n}$ such that the conditions $r(B) < 1$,*

$$\big[(-1)^j d_j a_{ii}(t)\big]_+ > -1 \ \text{ for } \ t < t_0, \quad [(-1)^j d_j a_{ii}(t)]_- < 1 \ \text{ for } \ t > t_0 \ \ (j=1,2),$$

$$c_i(t,\tau) \le b_0 \frac{h_i(t)}{h_i(\tau)} \ \text{ for } \ t \in I_{t_0}(\delta), \quad (t-t_0)(\tau-t_0) > 0, \ \ |\tau-t_0| \le |t-t_0|,$$

$$\bigg| \int\limits_{t_0\pm}^{t} c_i(t,\tau) h_i(\tau)\, d\big[a_{ii}(\tau)\,\mathrm{sgn}(\tau-t_0)\big]_+^{v} \bigg| \le b_{ii}\, h_i(t), \ \ \text{respectively, on } I_{t_0}^{\pm}(\delta),$$

$$\bigg| \int\limits_{t_0\pm}^{t} c_i(t,\tau) h_k(\tau)\, d\,\mathrm{V}(\mathcal{A}(a_{0ii}, a_{ik}))(\tau) \bigg| \le b_{ik}\, h_i(t), \ \ \text{respectively, on } I_{t_0}^{\pm}(\delta)$$

$(i \ne k;\ i,k = 1,\dots,n)$ hold for some $b_0 > 0$ and $\delta > 0$. Let, moreover,

$$\lim_{t\to t_0\pm} \int\limits_{t_0\pm}^{t} \frac{c_i(t,\tau)}{h_i(t)}\, d\,\mathrm{V}(\mathcal{A}(a_{0ii}, f_i))(\tau) = 0, \ \ \text{respectively} \ \ (i=1,\dots,n),$$

where

$$a_{0ii}(t) \equiv -\big[a_{ii}(t)\,\mathrm{sgn}(t-t_0)\big]_-^{v}\,\mathrm{sgn}(t-t_0) \ \ (i=1,\dots,n),$$

and c_i is the Cauchy function of the equation $dx = x\, da_{0ii}(t)$ for $i \in \{1,\dots,n\}$. Then problem (1), (2) is H-well-posed with respect to the pair $(S_1(A_0), S_2(A_0))$, where

$$A_0(t) \equiv \mathrm{diag}(a_{011}(t), \dots, a_{0nn}(t)).$$

The Cauchy functions $c_i(t,\tau)$ $(i = 1,\dots,n)$, mentioned in the theorem, have the well known form (see, for example, [1]).

Now we apply the previous results for the Cauchy problem with weight for the singular impulsive differential system

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_l+) - x(\tau_l-) = G(l)x(\tau_l) + g(l) \ \ (l \in \mathbb{N}); \tag{6}$$

$$\lim_{t\to b-} (H^{-1}(t)x(t)) = 0, \tag{7}$$

where $\tau_l \in [a,b[$ $(l \in \mathbb{N})$, $\lim\limits_{l\to+\infty} \tau_l = b$, are points of fixed impulses actions, $P = (p_{ik})_{i,k=1}^n \in L_{loc}([a,b[;\mathbb{R}^{n\times n})$, $q = (q_k)_{k=1}^n \in L_{loc}([a,b[;\mathbb{R}^n)$, and $G = (g_{ik})_{i,k=1}^n \in E(\mathbb{N};\mathbb{R}^{n\times n})$, $g = (g_k)_{k=1}^n \in E(\mathbb{N};\mathbb{R}^n)$.

We assume that $T = \{\tau_1, \tau_2, \dots\}$, $\mathrm{AC}_{loc}([a,b[\setminus T;\mathbb{R}^{n\times m})$ is the matrix-function whose restrictions to every $[c,d] \subset [a,b[\setminus T$ is absolutely continuous. $E(\mathbb{N};\mathbb{R}^{n\times m})$ is the set of all discrete matrix-functions from $\mathbb{N}$ into $\mathbb{R}^{n\times m}$. $N_{\alpha,\beta} = \{l \in \mathbb{N} : \alpha \le \tau_l < \beta\}$.

Let $G_0 \in E(\mathbb{N};\mathbb{R}^{n\times n})$ be such that

$$\det(I_n + G_0(l)) \neq 0 \ \ (l \in \mathbb{N}).$$

Then for every $X \in L_{loc}([a,b[;\mathbb{R}^{n\times n})$ and $Y \in E(\mathbb{N};\mathbb{R}^{n\times n})$ we put

$$\mathcal{A}_\iota(G_0;X,Y)(t) \equiv \int\limits_a^t X(\tau)\,d\tau + \sum_{l\in N_{a,t}} (I_n + G_0(l))^{-1} Y(\tau_l).$$

A vector-function $x \in \mathrm{AC}_{loc}([a,b[\setminus T;\mathbb{R}^n)$ is said to be a solution of system (6) if $x'(t) = P(t)x(t) + q(t)$ for a.a. $t \in [a,b[\setminus T$ and there exist one-sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1,2,\dots)$ satisfying (8). In addition, x is a solution of system (6) if and only if it is a solution of (1), where

$$A(t) \equiv \int\limits_a^t P(\tau)\,d\tau + \sum_{l\in N_{a,t}} G(\tau_l), \quad f(t) \equiv \int\limits_a^t q(\tau)\,d\tau + \sum_{l\in N_{a,t}} u(\tau_l).$$

We assume that $\det(I_n + G(l)) \neq 0$ $(l = 1,2,\dots)$. Due to the conditions imposed on P, G, q and u, we have $A \in \mathrm{BV}_{loc}([a,b[,\mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}_{loc}([a,b[,\mathbb{R}^n)$. So system (6) is a particular case of system (1), and the impulsive problem (6), (7) to problem (1), (2) for $t_0 = b$.

Along with system (6) consider the perturbed singular system

$$\frac{dx}{dt} = \widetilde{P}(t)x + \widetilde{q}(t), \quad x(\tau_l+) - x(\tau_l-) = \widetilde{G}(l)x(\tau_l) + \widetilde{g}(l) \ \ (l \in \mathbb{N}). \tag{8}$$

Definition 2. Problem (6), (7) is said to be H-well-posed with respect to the matrix-function G_0 if it has a unique solution x and for every $\varepsilon > 0$ there exists $\eta > 0$ such that problem (8), (7) has a unique solution y and estimate (5) holds for every matrix-functions $\widetilde{P}$, $\widetilde{G}$ and vector-functions $\widetilde{q}$, $\widetilde{g}$ such that

$$\det(I_n + \widetilde{G}(l)) \neq 0 \ \ (l \in \mathbb{N}),$$
$$\bigg\| \int\limits_t^{b-} H^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}_\iota(G_0;\widetilde{P}-P,\widetilde{G}-G))(s)\cdot H(s) \bigg\| < \eta \ \text{ and}$$
$$\bigg\| \int\limits_t^{b-} H^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}_\iota(G_0;\widetilde{q}-q,\widetilde{g}-g))(s) \bigg\| < \eta \ \text{ for } \ t \in [a,b[.$$

Theorem 3. *Let there exist a constant matrix $B = (b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ such that the conditions*

$$r(B) < 1, \quad [g_{ii}(l)]_+ > -1 \ \ (i = 1,\dots,n; \ \ l \in \mathbb{N}),$$
$$c_i(t,\tau) \le b_0\,\frac{h_i(t)}{h_i(\tau)} \ \ \textit{for} \ \ b-\delta \le t \le \tau < b \ \ (i = 1,\dots,n),$$

$$\bigg|\int\limits_t^{b-} c_i(t,\tau)h_i(\tau)[p_{ii}(\tau)]_-\,d\tau+\sum_{l\in N_{t,b}} c_i(t_l,\tau_l)h_i(\tau_l)[g_{ii}(l)]_-\bigg|\le b_{ii}\,h_i(t)\ \ \textit{and}$$

$$\bigg|\int\limits_t^{b-} c_i(t,\tau)h_k(\tau)\,d\,\mathrm{V}\big(\mathcal{A}_\iota([g_{ii}]_+;p_{ik},g_{ik})\big)(\tau)\bigg|\le b_{ik}\,h_i(t)\ \ \textit{for}\ \ t\in\,]b-\delta,b[$$

$(i\neq k;\ i,k=1,\dots,n)$ hold for some $b_0>0$ and $\delta>0$. Let, moreover,

$$\lim_{t\to b-}\int\limits_t^{b-}\frac{c_i(t,\tau)}{h_i(t)}\,d\,\mathrm{V}\big(\mathcal{A}_\iota([g_{ii}]_+;q_i,g_i)\big)(\tau)=0\ \ (i=1,\dots,n),$$

where c_i is the Cauchy function of the impulsive equation

$$\frac{dx}{dt}=[p_{ii}(t)]_+\,x,\quad x(\tau_l+)-x(\tau_l-)=[g_{ii}(l)]_+\,x(\tau_l)\ \ (l\in\mathbb{N}).$$

Then problem $(6),(7)$ is H-well-posed with respect to G_0, where

$$G_0(l)\equiv\operatorname{diag}\big([g_{11}(l)]_+,\dots,[g_{nn}(l)]_+\big).$$

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A Problem for a Family of Partial Integro-Differential Equations with Weakly Singular Kernels

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On the domain $\Omega=[0,T]\times[0,\omega]$, we consider the family of problems for the system of partial integro-differential equations with weakly singular kernels

$$\frac{\partial v}{\partial t}=A(t,x)v+\int\limits_0^T K(t,s,x)v(s,x)\,ds+f(t,x), \tag{1}$$

$$P(x)v(0,x)+S(x)v(T,x)=\varphi(x), \quad x\in[0,\omega], \tag{2}$$

where $v(t,x)=col(v_1(t,x),v_2(t,x),\dots,v_n(t,x))$ is an unknown vector function, the $(n\times n)$ matrix $A(t,x)$, and n vector function $f(t,x)$ are continuous on Ω, the $(n\times n)$ matrix $K(t,s,x)$ has the form $K(t,s,x)=\frac{1}{|t-s|^\alpha}\,H(t,s,x)$, and the $(n\times n)$ matrix $H(t,s,x)$ is continuous on $[0,T]\times[0,T]\times[0,\omega]$, $0<\alpha<1$, the $(n\times n)$ matrices $P(x)$, $S(x)$ and n vector function $\varphi(x)$ are continuous on $[0,\omega]$.

A continuous function $v:\Omega\to\mathbb{R}^n$ that has a continuous derivative with respect to t on Ω is called a solution to the family problems for the system of integro-differential equations (1), (2) if it satisfies system (1) and condition (2) for all $(t,x)\in\Omega$ and $x\in[0,\omega]$, respectively.

Partial integro-differential equations and various problems for them are arisen as mathematical models of various physical processes [2, 3, 25]. Boundary value problems for ordinary integro-differential equations with continuous kernels and weakly singular or other nonsmooth kernels were researched in [1, 4–11, 13, 14, 16–24] by the different methods. Some problems for partial integro-differential equations with singular kernels were considered in [26–33].

Nevertheless, the establishment of conditions for the solvability of the family problems for partial integro-differential equations with weakly singular kernels is an actual problem.

The aim of the present communication is to apply the Dzhumabaev parametrization method [12] and the results of article [4] to the family of partial integro-differential equations with weakly singular kernels.

For this we construct a homogeneous family of partial integral equations with weakly singular kernels of the second kind and introduce an analog of regular partition for Ω by the initial data of the partial integro-differential equation (1).

For fixed $x\in[0,\omega]$ problem (1), (2) is a linear problem for the system of integro-differential equations with weakly kernels. Suppose a variable x is changed on $[0,\omega]$; then we obtain a family of problems for partial integro-differential equations with weakly kernels.

Let us divide domain Ω equally into N parts and denote this partition by Δ_N:

$$\Delta_N=\big\{t_0=0<t_1<\dots<t_N=T,\ 0\le x\le\omega\big\},$$

where $t_s=sT/N$.

By $v_r(t,x)$ we denote the restriction of the function $v(t,x)$ to the r-th domain $\Omega_r=[t_{r-1},t_r)\times[0,\omega]$, i.e. $v_r(t,x)=v(t,x)$, $(t,x)\in\Omega_r$, $r=1:N$.

Inputting the functional parameters $\lambda_r(x) \widehat{=} v_r(t_{r-1},x)$ and performing the substitution of functions $z_r(t,x) = v_r(t,x) - \lambda_r(x)$ in each of the r-th domain, we obtain the following problem with parameters:

$$\frac{\partial z_r}{\partial t} = A(t,x)(z_r + \lambda_r(x)) + \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} K(t,s,x)(z_j(s,x) + \lambda_j(x))\,ds + f(t,x), \ \ (t,x) \in \Omega_r, \tag{3}$$

$$z_r(t_{r-1},x) = 0, \ \ x \in [0,\omega], \ \ r = 1:N, \tag{4}$$

$$P(x)\lambda_1(x) + S(x)\lambda_N(x) + S(x) \lim_{t \to T-0} z_N(t,x) = \varphi(x), \ \ x \in [0,\omega], \tag{5}$$

$$\lambda_p(x) + \lim_{t \to t_p - 0} z_p(t,x) - \lambda_{p+1}(x) = 0, \ \ x \in [0,\omega], \ \ p = 1:(N-1), \tag{6}$$

where (6) are the conditions of continuity for the solution at the inner lines of the partition Δ_N.

Introduction of additional parameters [4–11] lets us obtain the initial data (5). Thus, it is possible to determine the system of functions $z([t],x)$ from the family of special Cauchy problems for the systems of integro-differential equations with weakly singular kernels (5), (6) for fixed values of the parameters $\lambda(x) \in C([0,\omega],\mathbb{R}^{nN})$. By using the fundamental matrix $U(t,x)$ of the differential equation $\frac{\partial v}{\partial t} = A(t,x)v$, we reduce problem (5), (6) to the equivalent system of integral equations

$$\begin{aligned} z_r(t,x) = U(t,x) \int\limits_{t_{r-1}}^{t} U^{-1}(\tau_1,x) \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} K(\tau_1,s,x)(z_j(s,x) + \lambda_j(x))\,ds\,d\tau_1 \\ + U(t,x) \int\limits_{t_{r-1}}^{t} U^{-1}(\tau_1,x)\big[A(\tau_1,x)\lambda_r(x) + f(\tau_1,x)\big]\,d\tau_1, \ \ (t,x) \in \Omega_r, \ \ r = 1:N. \end{aligned} \tag{7}$$

Introduce the notations

$$\Phi(\Delta_N,t,x,\alpha) = \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} K(t,s,x) z_j(s,x)\,ds,$$

$$M(\Delta_N,t,x,\tau,\alpha) = \int\limits_{\tau}^{t_j} K(t,\tau_1,x)U(\tau_1,x)\,d\tau_1 U^{-1}(\tau,x), \quad (t,x) \in \Omega, \ \ \tau \in [t_{j-1},t_j), \ \ j = 1:N,$$

$$M(\Delta_N,t,x,T,\alpha) = 0.$$

Consider the following family of integral equations of the second kind with weakly singular kernel

$$\Phi(\Delta_N,t,x,\alpha) = \int\limits_{0}^{T} M(\Delta_N,t,x,\tau,\alpha)\Phi(\Delta_N,\tau,x,\alpha)\,d\tau + D(\Delta_N,t,x,\alpha)\lambda + F(\Delta_N,t,x,\alpha), \tag{8}$$

and the corresponding homogeneous family of integral equations

$$\Phi(\Delta_N,t,x,\alpha) = \int\limits_{0}^{T} M(\Delta_N,t,x,\tau,\alpha)\Phi(\Delta_N,\tau,x,\alpha)\,d\tau, \ \ (t,x) \in \Omega. \tag{9}$$

Definition. A partition Δ_N is called regular if the family of integral equations with weakly singular kernel (9) has only a trivial solution.

The set of regular partitions Δ_N is denoted by $\sigma([0,T],x,\alpha)$. As it is known from the theory of integral equations with singular kernels [15], if $\Delta_N \in \sigma([0,T],x,\alpha)$, then (8) has a unique solution for any $\lambda(x) \in C([0,\omega],\mathbb{R}^{nN})$, $F(\Delta_N,t,x,\alpha) \in C(\Omega,\mathbb{R}^n)$, and this solution can be presented in the form

$$\begin{aligned}\Phi(\Delta_N,t,x,\alpha) &= D(\Delta_N,t,x,\alpha)\lambda + F(\Delta_N,t,x,\alpha)\\ &\quad + \int_0^T \Gamma(\Delta_N,t,x,s,1)\big(D(\Delta_N,s,x,\alpha)\lambda(x) + F(\Delta_N,s,x,\alpha)\big)\,ds, \quad (t,x)\in\Omega, \end{aligned} \tag{10}$$

where $\Gamma(\Delta_N,t,x,s,1)$ is the resolvent of the family of integral equations with weakly singular kernel (8). The $n\times nN$ matrix $D(\Delta_N,t,x,\alpha) = (D_r(\Delta_N,t,x,\alpha))$, $r=1:N$, continuous on Ω, and vector $F(\Delta_N,t,x,\alpha)$ are constructed by the integral representation (7):

$$\begin{aligned}D_r(\Delta_N,t,x,\alpha) &= \int_{t_{r-1}}^{t_r} K(t,\tau,x)U(\tau,x)\int_{t_{r-1}}^{\tau} U^{-1}(\tau_1,x)A(\tau_1,x)\,d\tau_1\,d\tau\\ &\quad + \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} K(t,\tau,x)U(\tau,x)\int_{t_{j-1}}^{\tau} U^{-1}(\tau_1,x)\int_{t_{r-1}}^{t_r} K(\tau_1,s,x)\,ds\,d\tau_1\,d\tau,\\ F(\Delta_N,t,x,\alpha) &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} K(t,\tau,x)U(\tau,x)\int_{t_{j-1}}^{\tau} U^{-1}(\tau_1,x)f(\tau_1,x)\,d\tau_1\,d\tau.\end{aligned}$$

Substituting $\sum\limits_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j} K(t,s,x)z_j(s,x)\,ds$ in (7) with the right-hand side of (10), we get the representation of the function $z_r(t,x)$ in terms of $\lambda(x)\in C([0,\omega],\mathbb{R}^{nN})$, $f(t,x)\in C(\Omega,\mathbb{R}^n)$. Then, using this representation, we determine $\lim\limits_{t\to T-0} z_N(t,x)$, $\lim\limits_{t\to t_p-0} z_p(t,x)$, $p=1:(N-1)$. Substituting these expressions in (5), (6) and multiplying both sides of (5) by $h=\frac{T}{N}$, we get the following linear system of equations for the introduced parameters $\lambda_r(x)$, $r=1:N$:

$$Q^*(\Delta_N,x,\alpha)\lambda(x) = -F^*(\Delta_N,x,\alpha), \quad \lambda(x)\in C([0,\omega],\mathbb{R}^{nN}). \tag{11}$$

Theorem 1. *If the matrix $Q^*(\Delta_N,x,\alpha):\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ in the partition $\Delta_N\in\sigma([0,T],x,\alpha)$ is invertible for all $x\in[0,\omega]$, then the family of problems for the system of partial integro-differential equations with weakly singular kernels* (1), (2) *has a unique solution.*

Theorem 2. *For the unique solvability of family of problems for the system of partial integro-differential equations with weakly singular kernels* (1), (2) *it is necessary and sufficient that the matrix $Q^*(\Delta_N,x,\alpha):\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ be invertible for any $\Delta_N\in\sigma([0,T],x,\alpha)$ and for all $x\in[0,\omega]$.*

Acknowledgment

This research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant # AP09258829).

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Remark on Continuous Dependence of Solutions to the Riccati equation on its Righthand Side

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Abstract. The Riccati equation is considered to show some special features of continuous dependence on the right-hand side of the equation.

Consider the initial value problem for the Riccati equation

$$u' + u^2 = K(t), \quad K \in C[0;T], \tag{1}$$

with the initial condition $u(0) = u_0$. Suppose that the problem has a solution on $[0;T]$. We are interested whether the existence of solutions on $[0;T]$ still holds under small perturbation of the right-hand side. The formulation is not strict enough and admits opposite answers for its different treatments.

We have the following result detailing, for the case under consideration, the classical theorem on continuous dependence of solutions on the right-hand side and initial conditions (see, for example, [1, Chapter 7, Theorem 6]).

Theorem 1. *Let $u(t)$ defined on $[0;T]$ be a solution to equation (1) with $u(0) = u_0$. Then for each function $F(t) \in C[0,T]$, satisfying on $[0;T]$ the condition*

$$|K(t) - F(t)| < \varepsilon = \left(4 \int\limits_0^T \exp\left(-2 \int\limits_0^t u(\tau)\,d\tau\right) dt \cdot \int\limits_0^T \exp\left(2 \int\limits_0^t u(\tau)\,d\tau\right) dt\right)^{-1},$$

the initial value problem

$$v' + v^2 = F(t), \quad v(0) = u_0 \tag{2}$$

also has a solution defined on $[0;T]$.

Proof. We are looking for the solution v having the form

$$v(t) = u(t) + z(t) \exp\left(-2 \int\limits_0^t u(\tau)\,d\tau\right) \tag{3}$$

with $z(t)$ to be determined. Immediate calculations show that

$$v' = u' + z' \exp\left(-2 \int\limits_0^t u(\tau)\,d\tau\right) - 2zu \exp\left(-2 \int\limits_0^t u(\tau)\,d\tau\right)$$

$$= K - u^2 + z' \exp\Big(-2\int_0^t u(\tau)\,d\tau\Big) - 2zu\exp\Big(-2\int_0^t u(\tau)\,d\tau\Big)$$

$$= F - v^2 = F - u^2 - 2zu\exp\Big(-2\int_0^t u(\tau)\,d\tau\Big) - z^2\exp\Big(-4\int_0^t u(\tau)\,d\tau\Big),$$

whence

$$z' + z^2\exp\Big(-2\int_0^t u(\tau)\,d\tau\Big) + (K-F)\exp\Big(2\int_0^t u(\tau)\,d\tau\Big) = 0. \tag{4}$$

Hereafter we use the following notation:

$$2\int_0^t u(\tau)\,d\tau = U(t), \quad \exp U(t) = E(t).$$

So, equation (4) can be rewritten as

$$z' + z^2 E^{-1} + (F-K)E = 0.$$

To find such z, we use a contracting operator Φ acting on the space Z_δ of all continuous functions z satisfying $|z(t)| \le \delta$ on $[0;T]$ with some $\delta > 0$. We define Φ by

$$\Phi(z)(t) = \int_0^t \Big(E(\tau)(K(\tau)-F(\tau)) - z^2(\tau)E(\tau)^{-1}\Big)\,d\tau$$

and have to check that

(i) $\Phi(z) \in Z_\delta$ whenever $z \in Z_\delta$

and

(ii) Φ is contracting.

First we prove (ii).

$$\big|\Phi(z_1)(t) - \Phi(z_2(t)\big| = \int_0^t \big|E(\tau)^{-1}\big|\,\big|z_2^2(\tau) - z_1^2(\tau)\big|\,d\tau$$

$$= \int_0^t E(\tau)^{-1}\big|z_2(\tau)-z_1(\tau)\big|\,\big|z_2(\tau)+z_1(\tau)\big|\,d\tau \le 2\delta\int_0^t E(\tau)^{-1}\big|z_2(\tau)-z_1(\tau)\big|\,d\tau$$

$$\le 2\delta\int_0^t E(\tau)^{-1}\sup_{\tau\in[0,T]}\big|z_2(\tau)-z_1(\tau)\big|\,d\tau = 2\delta\int_0^t E(\tau)^{-1}\,d\tau\cdot\|z_1-z_2\|.$$

The operator Φ is contracting if $\delta\int_0^t E(\tau)^{-1}\,d\tau < 1$ on $[0;T]$, i.e.

$$\delta < \frac{1}{2}\Big(\int_0^T E(\tau)^{-1}\,d\tau\Big)^{-1}.$$

Now we take such δ and prove (i). Suppose $|K(\tau)-E(\tau)|<\varepsilon$ and $\|z\|\le\delta$. Then

$$|\Phi(z)(t)| \le \varepsilon \int_0^T E(\tau)\,d\tau + \delta^2 \int_0^T E(\tau)^{-1}\,d\tau < \varepsilon \int_0^T E(\tau)\,d\tau + \frac{\delta}{2}\,,$$

which is less or equal to δ if

$$\varepsilon \le \frac{\delta}{2}\Big(\int_0^T E(\tau)\,d\tau\Big)^{-1}.$$

So, if we have a solution u to equation (1) with $u(0)=u_0$, then we can obtain E and $\int_0^T E(\tau)\,d\tau$. Now we have to find δ such that

$$2\varepsilon \int_0^T E(\tau)\,d\tau \le \delta < \frac{1}{2}\Big(\int_0^T E(\tau)^{-1}\,d\tau\Big)^{-1}.$$

This is possible whenever

$$\varepsilon < \frac{1}{4}\Big(\int_0^T E(\tau)^{-1}\,d\tau\Big)^{-1}\Big(\int_0^T E(\tau)\,d\tau\Big)^{-1}.$$

If the estimates are valid, then there exists $z\in Z_\delta$ such that $\Phi(z)=z$, whence $z'=E(K-F)-z^2E^{-1}$ and $z(0)=0$, i.e. the function v defined by (3) is a solution to the related Riccati equation and $v(0)=u(0)$. $\square$

Note that the existence of a solution to (2) in the proof depends not only on the difference of the functions K and F but also on the solution u itself. This is not just a defect of the proof as shown in the following result.

Theorem 2. *If $T=\pi/2$ and $K(t)\equiv -1$ in* (1)*, then for each $\varepsilon>0$ there exist an initial value u_0 and a continuous function F on $[0;T]$ such that $\|F-K\|<\varepsilon$, and equation* (1) *has a solution $u\in C^1[0;T]$ with $u(0)=u_0$, while there is no solution to* (2) *on $[0;T]$.*

Proof. First we can solve the equation $u'+u^2=-A^2$ with arbitrary $A\ne 0$ to obtain the solution $u=A\tan(At_0-At)$.

In the case $A=1$ we have a solution $u(t)=\tan(t_0-t)$ to equation (1) defined in particular for all t satisfying $-T<t_0-t<T$. If $0<t_0<T$, then the solution u is defined, inter alia, on the segment $[0;T]$.

Now consider $A=1+\varepsilon$ and $F(t)=-(1+\varepsilon)^2$. The function

$$v=(1+\varepsilon)\,\tan\big((1+\varepsilon)t_1-(1+\varepsilon)t\big)$$

is a solution to (2) provided that

$$v(0)=u(0)=\tan t_0=(1+\varepsilon)\tan((1+\varepsilon)t_1).$$

Thus,

$$t_1=\frac{1}{1+\varepsilon}\,\arctan\Big(\frac{\tan t_0}{1+\varepsilon}\Big).$$

The solution is bounded on $[0;T]$ if $(1+\varepsilon)t_1-(1+\varepsilon)t\in(-T;T)$ whenever $t\in[0;T]$.

For $t=0$ we have $(1+\varepsilon)t_1\in(-T;T)$.

For $t=T$ we need

$$-T<(1+\varepsilon)t_1-(1+\varepsilon)T<T,$$

whence

$$\varepsilon T<(1+\varepsilon)t_1=\arctan\Big(\frac{\tan t_0}{1+\varepsilon}\Big),$$

and therefore $\tan t_0>(1+\varepsilon)\,\tan\varepsilon T$.

So, if the constant t_0 does not satisfy this condition, then v is not defined on the whole segment $[0;T]$. For arbitrary small $\varepsilon>0$ there exists sufficiently small $t_0>0$ making the last inequality false. $\square$

So, no estimate based just on the difference $K(t)-F(t)$ is possible to provide the existence of a solution to problem (2) for all u_0.

Note also that Theorem 1 becomes wrong if we replace $[0;T]$ with $[0;T)$.

Acknowledgement

This work was partially supported by RSF (Project # 20-11-20272.)

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Description of the Perron Exponent of a Linear Differential System with Unbounded Coefficients as a Function of the Initial Vector of a Solution

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1 Introduction

For a given $n \in \mathbb{N}$, let $\widetilde{\mathcal{M}}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1.1}$$

with piecewise continuous coefficients, which are matrix-valued functions $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, and by $\mathcal{M}_n$ its subclass that consists of systems with coefficients that are bounded on the semiaxis $\mathbb{R}_+$. In what follows, we identify system (1.1) with its defining function $A(\,\cdot\,)$ and therefore write $A \in \mathcal{M}_n$ and the like. The vector space of solutions of system (1.1) will be denoted by $\mathcal{S}(A)$ and the set of its nonzero solutions by $\mathcal{S}_*(A)$ (i.e. $\mathcal{S}_*(A) = \mathcal{S}(A) \setminus \{\mathbf{0}\}$).

The following definition is due to O. Perron [19].

The lower exponent of a vector-function $x : \mathbb{R}_+ \to \mathbb{R}^n_* \equiv \mathbb{R}^n \setminus \{\mathbf{0}\}$ is the quantity

$$\pi[x] = \varliminf_{t\to+\infty} \frac{1}{t} \,\ln \|x(t)\|. \tag{1.2}$$

Note that a lower exponent may well be infinite and hence is generally a point of the extended real line $\overline{\mathbb{R}} \equiv \mathbb{R} \sqcup \{-\infty, +\infty\}$, which we equip with the standard order and the order topology.

Assigning to each solution $x(\,\cdot\,) \in \mathcal{S}_*(A)$ of system (1.1) its lower exponent $\pi[x]$, we obtain the functional $\pi^A : \mathcal{S}_*(A) \to \overline{\mathbb{R}}$, which is called *the lower exponent* of system (1.1). Obviously, if $A \in \mathcal{M}_n$, then the range of the functional π^A is contained in a bounded interval. Such defined functionals π^A have different domains, which is not always convenient. In order to have a unified view of these functionals and to make it possible to compare them with one another, they are put in one-to-one correspondence with functions defined on $\mathbb{R}^n_*$. Namely, there is a natural isomorphism $\iota_A : \mathbb{R}^n \to \mathcal{S}(A)$ defined by $\xi \mapsto x(\,\cdot\,, \xi)$, where $x(\,\cdot\,, \xi)$ is the solution of the system A starting at the initial moment $t = 0$ from the vector $\xi \in \mathbb{R}^n$. Then the function $\pi_A : \mathbb{R}^n_* \to \overline{\mathbb{R}}$ defined by $\pi_A = \pi^A \circ i_A$ is called *the Perron exponent* of system (1.1). As said above, π_A takes finite values and is bounded wherever $A \in \mathcal{M}_n$.

The Perron exponent is one of a number of asymptotic characteristics, which are functionals defined on solutions of differential systems and reflecting one or another of their qualitative or asymptotic properties. Historically, the first in this series was the Lyapunov exponent λ_A [18], which is of fundamental importance in the stability theory. As is well known, it is defined similarly to the Perron exponent with the replacement in (1.2) of the lower limit by the upper one. Several different asymptotic characteristics are proposed by I. N. Sergeev (see, e.g., [20,21]).

Lower exponents are introduced by O. Perron [19] by analogy with Lyapunov characteristic exponents, and he was the first to observe that some of their properties differ from those of Lyapunov exponents. At the same time no serious research of qualitative properties of solutions that Perron exponents represent has been carried out until recently. Situation has changed since the publication of works [22,23], in which Perron stability was defined and some properties of this notion were treated.

The first problem that arises when studying an asymptotic characteristic is to completely describe it as a function of an initial vector for linear differential systems, that is, for example, for the Perron exponent it is required to obtain a complete description of the following function classes:

$$\mathcal{P}_n = \big\{\pi_A : \ A \in \mathcal{M}_n\big\} \ \text{and} \ \widetilde{\mathcal{P}}_n = \big\{\pi_A : \ A \in \widetilde{\mathcal{M}}_n\big\}. \tag{1.3}$$

To date solutions to these problems are only known for the Lyapunov exponent [18], [7, p. 25–26] and the lower and upper Bohl exponents [5,6] (definition of the two latter ones see in [8, p. 171–172], [24]). It is worth remarking that whereas a description of the Lyapunov exponent involves linear algebra concepts, that of the Bohl exponents requires the language of descriptive function theory, and so is the case for classes (1.3). A description of the second of these is given in the report, while a description of the first one is unknown so far.

Let us provide a number of properties of the Perron exponent that demonstrate its fundamental difference from the Lyapunov exponent, which seems outwardly similar.

A. M. Lyapunov [18], [7, p. 30] established that the number of different values that the Lyapunov exponent of a system $A \in \mathcal{M}_n$ takes does not exceed its dimension n. O. Perron discovered [19] that for his eponymous exponent this is not the case. He gave an example of a two-dimensional diagonal system with bounded coefficients whose Perron exponent takes exactly three different values. N. A. Izobov showed [15] that the Perron exponent of a diagonal system (1.1) takes no more than 2^n-1 values, and in the work [1] for every integer $m \in [1, 2^n-1]$ a diagonal system (1.1) is constructed such that its Perron exponent takes exactly m different values.

For non-diagonal systems the structure of the range of the Perron exponent may be much more complicated: in the work [16] a system is constructed such that the lower exponents of its solutions fill an entire interval, and in [2] it is proved that a set P is the range of the Perron exponent of a system $A \in \mathcal{M}_n$ if and only if P is a bounded Suslin set containing its sup.

Despite these differences in structure of the ranges of the Lyapunov and Perron exponents of system (1.1), Lebesgue sets of restrictions of these functionals to affine subspaces have some similarity. So, N. A. Izobov established [15,17] that for any $A \in \mathcal{M}_n$ and affine subspace $\Pi_k \subset \mathbb{R}^n$ of dimension k $(1 \le k \le n)$ the set

$$P(\Pi_k) \equiv \Big\{\xi \in \Pi_k \setminus \{\mathbf{0}\} : \ \ \pi(\xi) < \sup_{\zeta \in \Pi_k \setminus \{\mathbf{0}\}} \pi(\zeta)\Big\}$$

has zero k-dimensional Lebesgue measure, i.e.

$$\operatorname{mes} P(\Pi_k) = 0. \tag{1.4}$$

In other words, the set of lower exponents of solutions with the initial vectors from an affine plane Π_k contains its sup, and for almost all (with respect to Lebesgue measure) initial vectors from Π_k, the corresponding solutions starting from them have lower exponents equal to this sup. For one-dimensional affine subspaces the specified property can be strengthened [3]: for any affine line Π_1, the set $P(\Pi_1)$ has zero Hausdorf $\ln^\nu |\ln(\,\cdot\,)|$-measure for all $\nu < -1$.

It is easy to verify [4] that $P(\Pi_1)$ is a $G_{\delta\sigma}$-set and π_A is Baire 2. These statements and the above-cited statements from [3] are unimprovable [4]: for each $n \geq 2$ there exists a system $A \in \mathcal{M}_n$ such that for a certain line Π_1 the set $P(\Pi_1)$ is exactly $G_{\delta\sigma}$ with infinite Hausdorf $\ln^{-1}|\ln(\,\cdot\,)|$-measure and the function π_A is exactly Baire 2.

A. G. Gargyants [10] discovered that for systems in $\widetilde{\mathcal{M}}_n \setminus \mathcal{M}_n$ for $n \geqslant 2$ property (1.4) is generally not valid: he constructed a system $A \in \widetilde{\mathcal{M}}^n$ such that for any k-dimensional $(1 \leq k \leq n)$ affine subspace $\Pi_k \subset \mathbb{R}^n$, different from a line containing the origin, the set $\Pi_k \setminus P(\Pi_k)$ has zero k-dimensional Lebesgue measure and is of the first Baire category with respect to Π_k. He also proved [12] that property (1.4) holds for all systems $A \in \widetilde{\mathcal{M}}^n$ satisfying

$$\overline{\lim_{t\to+\infty}}\, t^{-1} \ln \|A(t)\| \leq 0.$$

A natural question arises: is the set $P(\Pi_k)$ of the first Baire category with respect to Π_k for any $n \geqslant 2$, $k = 1, \dots, n$, and system in $\mathcal{M}_n$? The answer (in the negative) was obtained by A. G. Gargyants [14]: for each $n \geq 2$ there exists a system $A \in \mathcal{M}_n$ such that the set $\mathbb{R}^n \setminus P(\mathbb{R}^n)$ is of the first Baire category.

2 The main result

The problem is to obtain for each $n \geq 2$ a set-theoretic description of Perron exponents of systems in $\widetilde{\mathcal{M}}_n$, i.e. of the function class $\widetilde{\mathcal{P}}_n$ defined in (1.3). Note that a description of the classes $\mathcal{P}_1$ and $\widetilde{\mathcal{P}}_1$ is trivial: they consist of all constant functions $\mathbb{R}^1_* \to \mathbb{R}$ and $\mathbb{R}^1_* \to \overline{\mathbb{R}}$, respectively.

A. G. Gargyants obtained [11, 13] progress in this problem: he proved that for any $n \geq 2$ the class $\widetilde{\mathcal{P}}_n$contains all continuous functions $f : \mathbb{R}^n_* \to \mathbb{R}$ satisfying the condition

$$f(c\xi) = f(\xi), \quad \xi \in \mathbb{R}^n_*, \quad c \in \mathbb{R}_*. \tag{2.1}$$

In [9], this result was extended to upper semicontinuous functions.

For every $n \geq 2$, a complete description of the class $\widetilde{\mathcal{P}}_n$ is provided by the following

Theorem. *A function $f : \mathbb{R}^n_* \to \overline{\mathbb{R}}$ belongs to the class $\widetilde{\mathcal{P}}_n$ if and only if it satisfies* (2.1) *and for all $r \in \mathbb{R}$, the inverse image $f^{-1}([-\infty, r])$ of the closed ray $[-\infty, r]$ is a G_δ-set in $\mathbb{R}^n_*$.*

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Oscillation Problems for Generalized Emden–Fowler Equation

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Consider the super-linear generalized Emden–Fowler differential equation for $t \in I = [1, \infty)$

$$\big(|x'|^{\alpha} \operatorname{sgn} x'\big)' + b(t)g(x)|x|^{\beta} \operatorname{sgn} x = 0, \ \ 0 < \alpha < \beta, \tag{1}$$

and its special case

$$x'' + b(t)\ |x|^{\beta} \operatorname{sgn} x = 0, \ \ t \in I, \ \ \beta > 1, \tag{2}$$

where b is a positive absolute continuous function on I and g is a positive continuous function on $\mathbb{R}$.

A solution x of (1) is said to be *proper* if it is defined for all large t and $\sup\limits_{t \in [\tau, \infty)} |x(t)| > 0$ for any large τ. A proper solution x of (1) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is said to be *nonoscillatory*.

It is well known, see, e.g., [8], that the coexistence of nontrivial oscillatory and nonoscillatory solutions is possible for (1). Further, in [10] the question as to whether oscillatory solutions of (2) may coexist with nonoscillatory ones having at least one zero has been posed and Kiguradze in [8] has negatively answered to this question.

Concerning the existence of nonoscillatory solutions, it is well known that the class $\mathbb{P}$ of all eventually positive solutions x of (1) can be divided into three subclasses, according to the asymptotic behavior of x as $t \to \infty$, namely

$$\begin{aligned}
\mathbb{M}^{+}_{\infty,\ell} &= \Big\{x \in \mathbb{P}: \ \lim_{t\to\infty} x(t) = \infty, \ \ \lim_{t\to\infty} x'(t) = \ell_x, \ 0 < \ell_x < \infty\Big\},\\
\mathbb{M}^{+}_{\infty,0} &= \Big\{x \in \mathbb{P}: \ \lim_{t\to\infty} x(t) = \infty, \ \ \lim_{t\to\infty} x'(t) = 0\Big\},\\
\mathbb{M}^{+}_{\ell,0} &= \Big\{x \in \mathbb{P}: \ \lim_{t\to\infty} x(t) = \ell_x, \ \ \lim_{t\to\infty} x'(t) = 0, \ 0 < \ell_x < \infty\Big\},
\end{aligned}$$

see, e.g., [4]. Solutions in $\mathbb{M}^{+}_{\infty,\ell}$, $\mathbb{M}^{+}_{\infty,0}$, $\mathbb{M}^{+}_{\ell,0}$ are called also *dominant solutions, intermediate solutions* and *subdominant solutions*, respectively. Such a terminology has been introduced by the Japanese mathematical school and it is due to the fact that, if $x \in \mathbb{M}^{+}_{\infty,\ell}$, $y \in \mathbb{M}^{+}_{\infty,0}$, $z \in \mathbb{M}^{+}_{\ell,0}$, then we have $x(t) > y(t) > z(t)$ for large t.

Necessary and sufficient conditions for the existence of subdominant and dominant solutions are easily available in the literature, see, e.g., [4] and the references therein. However, as far we know, until now no general necessary and sufficient conditions for existence of intermediate solutions of (2) are known; this fact mainly is due to the lack of sharp upper and lower bounds for intermediate solutions, see, e.g., [7, page 3], [9, page 2].

Another interesting problem which arises, is whether all three types of nonoscillatory solutions can simultaneously exist. This problem has a long history. For equation (2), it started sixty years ago by Moore–Nehari [10] in case $\beta > 1$ and Belohorec [3] in case $\beta < 1$. This study was continued in some papers by Kamo, Kusano, Naito, Tanigawa, Usami for the more general equation

$$\big(a(t)|x'|^{\alpha}\operatorname{sgn} x'\big)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0, \tag{3}$$

where a is a positive continuous function, in both cases $\alpha = \beta$ and $\alpha \neq \beta$. In particular, under additional assumptions, it is proved that this triple coexistence is impossible, see [6] for more details. Finally, in [4] the study has been completed with a negative answer.

A much more subtle question concerns the possible coexistence between oscillatory solutions and nonoscillatory solutions. For the special case of (2) with $b(t) = 1/4$, that is for the equation

$$x'' + \frac{1}{4}\,t^{-(\beta+3)/2}|x|^{\beta}\operatorname{sgn} x = 0, \ \ t \in I, \ \ \beta > 1, \tag{4}$$

it has been proved in [10] that (4) has both oscillatory solutions and nonoscillatory ones. These nonoscillatory solutions are either subdominant solutions or intermediate solutions and both types exist. Moreover, intermediate solutions of (4) intersect the intermediate solution $\sqrt{t}$ infinitely many times.

Observe that, in view of [8, Theorem 8.5], equation (4) can be considered, roughly speaking, as the border equation between oscillation of at least one solution and nonoscillation of all solutions.

Our aim here is to present some results concerning the existence of oscillatory solutions and intermediate solutions for (1) and its special case (2). Moreover, we show also how the results in [10] for (4) concerning the coexistence between oscillatory solutions and intermediate solutions can be extended to the perturbed equation (2). These results are taken from [1, 2] and we refer these papers for more details.

Theorem 1. *Assume that $t^{\gamma}b(t)$ is nonincreasing on I, where $\gamma = (\alpha\beta + 2\alpha + 1)(\alpha + 1)^{-1}$ and*

$$\begin{aligned} &g(u)\operatorname{sgn} u \text{ is nonincreasing on } (-\infty,0) \text{ and } (0,\infty);\ \lim_{u\to\infty} g(u) = M > 0,\\ &\lim_{u\to\infty} g(u) = M > 0. \end{aligned} \tag{5}$$

If

$$\int\limits_1^{\infty} s^{\beta} b(s)\,ds = \infty,$$

then equation (1) has infinitely many intermediate solutions.

A necessary condition for the existence of intermediate solutions follows from the following oscillation result.

Theorem 2. *Assume (5). Then any solution of (1) is oscillatory if and only if*

$$\int\limits_1^{\infty}\bigg(\int\limits_t^{\infty} b(s)\,ds\bigg)^{1/\alpha}dt = \infty.$$

For equation (2) we have the coexistence of oscillatory solutions and intermediate solutions, as the following result shows.

Theorem 3. *Consider equation (2) with*

$$b(t) = t^{-(\beta+3)/2}c(t),$$

where c is a positive absolute continuous function on I. If

$$\lim_{t\to\infty} c(t) = c_0 > 0 \ \ \textit{and} \ \ \int\limits_1^\infty |c'(t)|\,dt < \infty, \tag{6}$$

then we have:

(i$_1$) *Equation (2) has infinitely many oscillatory solutions. In addition, if $c'(t) \ge 0$, then every solution with zero is oscillatory.*

(i$_2$) *Equation (2) has infinitely many intermediate solutions x defined on I such that*

$$C_0 t^{1/2} \le x(t) \le C_1 t^{1/2} \ \ \textit{for large } t, \tag{7}$$

where C_0 and C_1 are suitable positive constants which does not depend on the choice of x. Moreover, intermediate solutions intersect the function

$$(4c(t))^{\frac{1}{1-\beta}}\sqrt{t}$$

infinitely many times.

If

$$\int\limits_1^\infty a^{-1/\alpha}(s)\,ds = \infty,$$

Theorem 1 and Theorem 2 can be extended to the more general equation (3) using the change of the independent variable

$$s = A(t) + 1 - c, \quad X(s) = x(t), \ \ t \in [1,\infty), \ \ s \in [1,\infty),$$

see [1, Section 6] for more details. Moreover, Theorem 1 extends recent results in [4,5] and Theorem 2 shows that the oscillation property reads in the same way for (1) and the Emden–Fowler equation (1) with $g(t) \equiv 1$, see, e.g., [8, Chapter V].

For equation (2), Theorem 3(i$_3$) extends analogues results in [5, Theorem 2.1] and [1, Theorem 3.1], where b is required to be nonincreasing for $t \ge 1$.

The proof of Theorem 1 is mainly based on certain asymptotic property of a suitable associated energy function, see [1, Lemma 3.3 and Lemma 3.4]. The proof of Theorem 3 uses some auxiliary results, which concerns with the equation

$$\ddot{u} - \frac{u}{4} + c(e^s)|u(s)|^\beta \operatorname{sgn} u(s) = 0, \ \ s \in [0,\infty), \tag{8}$$

where "$\cdot$" denotes the derivative with respect to the variable s.

Lemma 1. *The change of variable*

$$x(t) = t^{1/2}u(s), \quad s = \log t, \ \ t \in [1,\infty), \tag{9}$$

transforms equation (2) into equation (8). Moreover, equation (8) has two types of nonoscillatory solutions. Namely:

Type (1): *solution u satisfies for large s*

$$0 < |u(s)| \le De^{-s/2} \tag{10}$$

where $|u|$ is decreasing and $D > 0$ is a suitable constant.

Type (2): *solution u intersects the function*

$$Z(s) = (4c(e^s))^{\frac{1}{1-\beta}} \tag{11}$$

infinitely many times, i.e., there exists a sequence $\{s_n\}_{n=1}^{\infty}$, $\lim\limits_n s_n = \infty$ such that

$$|u(s_n)| = Z(s_n).$$

Observe that solutions u of Type (1) in Lemma 1 correspond, via the transformation (9), to subdominant solutions of equation (2) because

$$x(t) = t^{1/2}u(s) \le t^{1/2}K_2 e^{-s/2} = K_2,$$

while solutions u of Type (2) correspond to intermediate solutions of (2).

Concluding remark. Consider equation (2) with

$$b(t) = t^{-(\beta+3)/2}t^{\lambda}, \ \ 0 < \lambda < \frac{\beta-1}{2}\,.$$

Then Theorem 3 is not applicable. However, it is possible to construct equation for which intermediate solutions exist. Observe that for $\lambda = (\beta-1)/2$ all solutions of such equation are oscillatory. How to relax conditions (6) in order to exist intermediate solutions?

Acknowledgement

The research of the first and second authors has been supported by Grant # GA 20-11846S of the Czech Science Foundation. The third author was partially supported by GNAMPA, National Institute for Advanced Mathematics (INdAM).

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Sixth Order Accuracy Difference Schemes for the Helmholtz-Type Equation

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The first difference schemes with sixth order accuracy for approximation of elliptic equations were offered by S. Mikeladze [6, 7], and further were being studied by a number of authors. Convergence of these schemes with rate $O(h^6)$ were stated under condition that the solution of the differential problem belongs to the class $C^8(\overline{\Omega})$.

One of the most frequently encountering equations of numerical weather prediction, and fluid dynamics generally, is the Helmholtz-type diagnostic equation [5]. Below, we propose and investigate difference schemes approximating the following problem

$$\Delta u-\lambda u=f(x),\ \ x\in\Omega,\quad u(x)=0,\ \ x\in\Gamma, \tag{1}$$

where $\lambda\geq 0$ is a constant and $\Omega=\{x=(x_1,x_2):\ 0<x_\alpha<l,\ \alpha=1,2\}$ is the square with boundary Γ.

In $\overline{\Omega}=\Omega\cup\Gamma$ we introduce a grid $\overline{\omega}=\overline{\omega}_1\times\overline{\omega}_2$, where

$$\overline{\omega}_\alpha=\Big\{x_\alpha=i_\alpha h:\ i_\alpha=0,1,\dots,N,\ h=\frac{l}{N}\Big\},\ \ \gamma=\overline{\omega}\setminus\omega.$$

Besides

$$\omega_\alpha=\overline{\omega}_\alpha\cap(0;l),\ \ \omega_\alpha^+=\overline{\omega}_\alpha\cap(0;l],\ \ \omega=\omega_1\times\omega_2,\ \ \omega^+=\omega_1^+\times\omega_2^+,$$
$$\omega_{(1)}=\omega_1^+\times\omega_2,\ \ \omega_{(2)}=\omega_1\times\omega_2^+,\ \ \gamma=\overline{\omega}\setminus\omega.$$

Let

$$(y,v)_{\widetilde{\omega}}=\sum_{x\in\widetilde{\omega}}h^2y(x)v(x),\ \ \|y\|^2_{\widetilde{\omega}}=(y,y)_{\widetilde{\omega}}\ \text{ for }\ \widetilde{\omega}\subseteq\overline{\omega}.$$

Let's denote by H the set of grid functions given on $\overline{\omega}$ and vanishing on γ, with the scalar production and norm $(y,v)=(y,v)_\omega$, $\|y\|=\|y\|_\omega$.

Also, in space H, we introduce the norms

$$\|y\|^2_{(\alpha)}=(y,y)_{\omega(\alpha)},\ \ \alpha=1,2,$$
$$\|y\|_1^2=\|y\|^2_{W_2^1(\omega)}=\|y_{\overline{x}_1}\|^2_{(1)}+\|y_{\overline{x}_2}\|^2_{(2)},$$
$$\|y\|_2^2=\|y\|^2_{W_2^2(\omega)}=\|y_{\overline{x}_1x_1}\|^2+\|y_{\overline{x}_2\overline{x}_2}\|^2+2\|y_{\overline{x}_1\overline{x}_2}\|^2_{\omega^+},$$

It is supposed that

$$\|y\|_{W_2^0(\omega)} = \|y\|.$$

For functions with continuous argument we will use the following averaging operators

$$T_\alpha u = \int_{-1}^{1} (1 - |t|) u\big(x_1 + (2-\alpha)th, x_2 + (\alpha - 1)th\big)\, dt, \quad \alpha = 1, 2.$$

We will approximate problem (1) with the help of family of difference schemes dependent on a parameter ε:

$$Ay = \varphi(x), \quad x \in \omega, \quad y(x) = 0, \quad x \in \gamma, \tag{2}$$

where

$$\begin{aligned} A \equiv \Big(1 - \frac{\lambda^2 h^4}{360} - (1-\varepsilon)\,\frac{\lambda h^2}{12}\,\Big(1 - \frac{\lambda h^2}{12}\Big)\Big)(A_1 + A_2)& \\ - \frac{h^2}{6}\,\Big(1 - \frac{\lambda h^2}{60}\,(7 - 5\varepsilon)\Big) A_1 A_2 + \lambda\Big(1 + \frac{\lambda h^2}{12}\,\varepsilon\Big)E,& \end{aligned}$$

$$A_\alpha y \equiv -y_{\overline{x}_\alpha x_\alpha}, \quad \alpha = 1, 2, \quad Ey \equiv y, \quad \varphi = \Big(1 + \frac{\lambda h^2}{12}\,\varepsilon\Big) T_1 T_2 f + \frac{h^2}{240}\,(A_1 A_2 f + \lambda (A_1 + A_2) f).$$

It can be proved that operator A is self-conjugate and positively defined in H; the following estimations

$$A \geq \delta(A_1 + A_2) + \frac{4}{9}\,\lambda E, \quad \delta \|y\|_1^2 \leq (Ay, y), \quad \delta \|y\|_2 \leq \|Ay\|,$$

where

$$\delta = \frac{2}{3}\,\Big(1 + \frac{\lambda h^2}{12}\,\varepsilon\Big),$$

are valid for it.

Positive definiteness of operator A ensures unique solvability of the difference scheme (2).

Substituting $y = z + u$ in (2), we get the problem

$$Az = \psi, \quad x \in \omega, \quad z(x) = 0, \quad x \in \gamma \tag{3}$$

for error z, where $\psi = \varphi - Au$ is an approximation error.

Using equation (1) and the identity $T_\alpha \frac{\partial^2 u}{\partial x_\alpha^2} = u_{\overline{x}_\alpha x_\alpha}$ we represent ψ in the form

$$\psi = \Big(1 + \frac{\lambda h^2}{12}\,\varepsilon\Big)(A_1 \eta_1 + A_2 \eta_2 + A_3 \eta_3) + \lambda \eta_4,$$

where

$$\begin{aligned} \eta_{3-\alpha} &= T_\alpha u - u + \frac{h^2}{12}\,A_\alpha u - \frac{h^2}{240}\,A_\alpha\,\frac{\partial^2 u}{\partial x_\alpha^2}\,, \quad \alpha = 1, 2, \\ \eta_3 &= T_1 T_2 u - u + \frac{h^2}{12}\,(A_1 + A_2) u - \frac{11}{720}\,h^4 A_1 A_2 u - \frac{h^4}{240}\,(A_1 + A_2)\Delta u, \\ \eta_4 &= \Big(A_1 A_2 \Delta u + \lambda (A_1 + A_2)\Delta u + \frac{11}{3}\,\lambda A_1 A_2 u\Big)\,\frac{\varepsilon h^6}{12 \cdot 240}\,. \end{aligned}$$

For the solution of problem (3) the following estimations are true:

$$\begin{aligned} \|z\| &\leq \frac{3}{2}\,\Big(\|\eta_1\| + \|\eta_2\| + \frac{\lambda l^2}{16}\,(\|\eta_3\| + \|\eta_4\|)\Big), \\ \|z\|_1 &\leq \frac{3}{2}\,\Big(\|\eta_{1\overline{x}_1}\|_{(1)} + \|\eta_{2\overline{x}_2}\|_{(2)} + \frac{\lambda l}{4}\,(\|\eta_3\| + \|\eta_4\|)\Big), \\ \|z\|_2 &\leq \frac{3}{2}\,\Big(\|\eta_{1\overline{x}_1 x_1}\|_{(1)} + \|\eta_{2\overline{x}_2 x_2}\|_{(2)} + \lambda\big(\|\eta_3\| + \|\eta_4\|\big)\Big). \end{aligned}$$

It can be checked that expansions of linear (with respect to $u(x)$) functionals η_1, η_2, η_3, η_4 in the class of sufficiently smooth functions start from sixth order derivatives.

With the help of technique of investigation [1, 4, 8], based on using of approximating lemma of Bramble–Hilbert [2, 3], we become convinced in validness of the following

Theorem 1. *Let the solution of problem* (1) *belong to the space* $W_2^m(\Omega)$, $m > 3$. *Then the convergence of the difference scheme* (2) *at* $\varepsilon \geq 0$ *is characterized by the estimation*

$$\|y-u\|_{W_2^s(\omega)} \leq Mh^{m-s}\|u\|_{W_2^m}(\Omega), \quad s=0,1,2, \quad m \in (3, 6+s].$$

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Existence of a Complete Unstable Differential System with Perron and Upper-Limit Partial Stability

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The present report deals with a recently introduced [7] concept of the Qualitative Theory of Differential Equations, namely the Perron stability. It continues the series of papers by the author [1] and [2], reinforcing their results. The first of these works corrected the defect stated in Remark 4 to Theorem 1 [8], but the differential system constructed there possessed a non-zero (though limited on the whole semi-axis of time) linear approximation at zero. In the second paper the system with the same properties, but already with a zero linear approximation at zero, was constructed.

The following reinforcement of the above results consists in constructing a system with both Perron and upper-limit complete instability (and thus also Lyapunov global instability) and at the same time not just partial (as in all the examples discussed above) but even massive partial instability.

For a number $n \in \mathbb{N}$ and for a region $G \ni 0$ of the Euclidean space $\mathbb{R}^n$, consider the system

$$\dot{x} = f(t,x), \quad t \in \mathbb{R}_+ \equiv [0,\infty), \ \ x \in G, \tag{1}$$

with the right hand side $f : \mathbb{R}_+ \times G \to \mathbb{R}^n$ satisfying the conditions

$$f, f'_x \in C(\mathbb{R}_+ \times G), \quad f(t,0) = 0, \ \ t \in \mathbb{R}_+, \tag{2}$$

and therefore admitting the *zero* solution. Let us denote by $\mathcal{S}_*(f)$ the set of all *non-continuable* non-zero solutions to system (1) and by $\mathcal{S}_\delta(f)$ – the subset in $\mathcal{S}_*(f)$ consisting of those and only those solutions x which satisfy the *initial condition* $|x(0)| < \delta$ (here $|\cdot|$ is the Euclidean norm in the space $\mathbb{R}^n$).

Definition 1. Let us say that for system (1) (more exactly, for its *zero* solution, which we will not mention further for brevity) the following *Perron property* takes place:

1) *Perron stability* if for any $\varepsilon > 0$ there exists such $\delta > 0$ that any solution $x \in \mathcal{S}_\delta(f)$ satisfies the condition
$$\varliminf_{t\to+\infty} |x(t)| < \varepsilon; \tag{3}$$

2) *Perron instability* if there is no Perron stability, namely, if there exists such $\varepsilon > 0$ that for any $\delta > 0$ some solution $x \in \mathcal{S}_\delta(f)$ does not satisfy condition (3);

3) *complete Perron instability* if for some $\varepsilon, \delta > 0$ no solution $x \in \mathcal{S}_\delta(f)$ satisfies condition (3);

4) *particular Perron stability* if there is no complete Perron instability, namely, if for any $\varepsilon, \delta > 0$ some solution $x \in \mathcal{S}_\delta(f)$ satisfies condition (3);

5) *asymptotic Perron stability* if for some $\delta > 0$ any solution $x \in \mathcal{S}_\delta(f)$ satisfies condition

$$\varliminf_{t\to+\infty} |x(t)| = 0; \tag{4}$$

6) *asymptotic Perron instability* if there is no asymptotic Perron stability, namely, if for any $\delta > 0$ some solution $x \in \mathcal{S}_\delta(f)$ does not satisfy condition (4).

The definition of the Perron properties essentially relies on the transition to the lower limit when $t \to +\infty$ (see conditions (3) and (4) in Definition 1). Therefore they could also be called lower-limit ones and it would be appropriate to consider also their natural analogues using the upper limit instead of the lower limit. To do this, let us formulate

Definition 2. Let us compare to each Perron property from Definition 1 its upper-limit analogue, namely: *stability, instability, complete instability, particular stability, asymptotic stability, asymptotic instability* are obtained by repeating respectively the descriptions from steps 1–6 of Definition 1 with replacement in them conditions (3) and (4) by conditions

$$\varlimsup_{t\to+\infty} |x(t)| < \varepsilon \tag{5}$$

and, respectively,

$$\varlimsup_{t\to+\infty} |x(t)| = 0. \tag{6}$$

Let us emphasize that in Definition 1 of the Perron properties conditions (3) and (4), as well as in Definition 2 of the upper-limit properties, conditions (5) and (6) are considered as not fulfilled, in particular already in the case when the solution x is simply not defined on the whole semi-axis $\mathbb{R}_+$, which takes place if and only if its corresponding phase curve reaches the limit of the phase region G in finite time (according to the solution continuity theorem; see, for example, Theorem 23 [9]). Each Perron and upper-limit property 2–5 of Definitions 1 and 2 according to Theorem 3 from [10] is

(a) *local in its initial value*, i.e. to establish it is sufficient for an arbitrary fixed value of $r > 0$ to consider those and only those solutions x which satisfy the condition $|x(0)| < r$;

(b) *local in the phase variable*, i.e. to establishit is sufficient for an arbitrary fixed value of $r > 0$ to know the values of each solution x at those and only those moments $t \in \mathbb{R}_+$ for which it satisfies the condition $|x(t)| < r$.

Therefore to complete the picture it seems appropriate also to consider the properties characterizing the behaviour not only of near-zero solutions but of all solutions in general, i.e. having, so to speak, a global character. For this purpose let us formulate

Definition 3. Let us consider that the following property for system (1) takes place:

1) *Perron* or *upper-limit global stability* if any of its solutions $x \in \mathcal{S}_*(f)$ satisfies conditions (4) or (6), respectively;

2) *Perron* or *upper-limit partial instability* if it does not possess Perron or upper-limit global stability, namely, there is at least one its solution $x \in \mathcal{S}_*(f)$ that does not satisfy conditions (4) or (6), respectively;

3) *Perron* or *upper-limit partial stability* if for any $\varepsilon > 0$ at least one of its solution $x \in \mathcal{S}_*(f)$ satisfies, conditions (3) or (5), respectively;

4) *Perron* or *upper-limit global instability* if it does not possess Perron or upper-limit partial stability, namely, for some $\varepsilon > 0$ none of its solutions $x \in \mathcal{S}_*(f)$ satisfies conditions (3) or (5), respectively.

The main result of this paper is to prove the existence of the differential system, all the near-zero solutions of which tend to infinity at $t \to +\infty$ (hence the system is completely unstable both Perron and upper-limit) and all other solutions tend to zero. This means that such system is not globally unstable (neither Perron nor upper-limit) but it does possess both types of partial stability and not just partial one, but even massive partial stability (i.e. the condition of satisfying one of conditions (3) or (5) is imposed on the set of solutions of system (1)).

Theorem. *When $n = 2$, there exists system* (1) *satisfying conditions* (2) *and posessing the following three properties:*

1) *the right hand side of systen* (1) *is infinitely differentiable and*

$$f'_x(t,0) = 0, \quad t \in \mathbb{R}_+;$$

2) *for each solution x to system* (1)*, satisfying initial conditions $0 < |x(0)| < 1$ or $x(0) = (1,0)^T$ and also $|x(0)| = 1$ and $x_2(0) < 0$, there exists the equality*

$$\lim_{t\to+\infty} |x(t)| = +\infty;$$

3) *for all other solutions x to system* (1)*, satisfying initial conditions $|x(0)| > 1$ or $x(0) = (-1,0)^T$ and also $|x(0)| = 1$ or $x_2(0) > 0$, there exists the equality*

$$\lim_{t\to+\infty} |x(t)| = 0.$$

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On the Solvability of a Three Point Boundary Value Problem for First Order Linear Functional Differential Equations

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We consider a boundary value problem

$$\dot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [0,1], \tag{1}$$

$$x(0) + x(1) = 2x(c), \tag{2}$$

where $c \in (0,1)$ is a given point, T^+ and T^- are linear *positive* operators, acting from the space of real continuous functions $\mathbf{C}[0,1]$ into the space of real integrable functions $\mathbf{L}[0,1]$ with the standard norms, $f \in \mathbf{L}[0,1]$ (here positive operators map continuous non-negative functions into non-negative integrable functions). An absolutely continuous function $x : [0,1] \to \mathbb{R}$ is called a solution of problem (1), (2) if it satisfies equation (1) for almost all $t \in [0,1]$ and satisfies three point boundary value condition (2).

If we put $c = 0$ or $c = 1$, then condition (2) coincides with the periodic boundary value condition. Integral necessary and sufficient conditions for the unique solvability of the periodic boundary value problem for equation (1) in terms of two quantities $\int\limits_0^1 (T^+\mathbf{1})(s)\,ds$ and $\int\limits_0^1 (T^-\mathbf{1})(s)\,ds$ are known [1] (here $\mathbf{1} : [0,1] \to \mathbb{R}$ is the unit function). We formulate these conditions in the following form.

Theorem 1 ([1]). *Let two nonnegative numbers $\mathcal{T}^+$ and $\mathcal{T}^-$ be given. The periodic boundary value problem for equation* (1) *is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities*

$$\int\limits_0^1 (T^+\mathbf{1})(s)\,ds = \mathcal{T}^+, \quad \int\limits_0^1 (T^-\mathbf{1})(s)\,ds = \mathcal{T}^-,$$

if and only if the inequalities

$$\frac{X}{1-X} < Y < 2\Big(1 + \sqrt{1-X}\Big), \quad X < 1,$$

are fulfilled, where

$$X = \min\{\mathcal{T}^+, \mathcal{T}^-\}, \quad Y = \max\{\mathcal{T}^+, \mathcal{T}^-\}.$$

Various three-point boundary value problems are also considered for functional differential equations (see, for example, [2]). Similar integral necessary and sufficient conditions for three point problems, in particular, problem (1), (2), as far as we know, have not yet been obtained. It is natural to consider the conditions for the unique solvability of this problem in terms of four parameters, namely, the integrals of $T^+\mathbf{1}$ and $T^-\mathbf{1}$ over intervals $[0,c]$ and $[c,1]$:

$$\int\limits_0^c (T^+\mathbf{1})(s)\,ds \equiv P_L, \quad \int\limits_c^1 (T^+\mathbf{1})(s)\,ds \equiv P_R, \tag{3}$$

$$\int\limits_0^c (T^-\mathbf{1})(s)\,ds \equiv M_L, \quad \int\limits_c^1 (T^-\mathbf{1})(s)\,ds \equiv M_R. \tag{4}$$

We are interested in the structure of the uniquely solvable set $\Omega \in \mathbb{R}^4$, that is, the set of all points (P_L, P_R, M_L, M_R) for which any problem (1), (2) with positive linear operators T^+, T^- satisfying equalities (3), (4) is uniquely solvable. These conditions of solvability (necessary and sufficient) turn out to be rather complicated.

Define

$$\Delta \equiv P_L + M_R - P_R - M_L, \quad P \equiv P_L + P_R, \quad M \equiv M_L + M_R.$$

Theorem 2. *Let four non-negative numbers P_L, P_R, M_L, M_R be given, $\Delta > 0$. Then the boundary value problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities (3), (4) if and only if the following conditions are fulfilled:*

$$\Delta > M_L P_L + M_R P_R + 2M_L M_R; \quad \Delta > M_L P_L + M_R P_R + 2P_L P_R;$$
$$P_R + M_L < 1; \quad \Delta > \frac{(P_L + M_R)^2}{4};$$
$$\Delta + s_p^2 - s_p(P + M_R) > 0,$$
$$\Delta + s_p^2 - s_p(P + 2M_R) + 2M_R P_R + M_R M_L > 0,$$
$$\Delta + s_p^2 - s_p(P + 2M_R) + 2M P_R > 0 \ \text{for all} \ s_p \in [P_R, P];$$
$$\Delta + s_m^2 - s_m(P_L + M) > 0,$$
$$\Delta + s_m^2 - s_m(2P_L + M) + 2P_L M_L + P_R P_L > 0,$$
$$\Delta + s_m^2 - s_m(2P_L + M) + 2P\, M_L > 0 \ \text{for all} \ s_m \in [M_L, M].$$

Some conditions of the previous theorem can be simplified. Define several values

$$r_1 \equiv (P_L + P_R) M_R + \frac{(P_L + P_R - M_R)_+^2}{4},$$
$$r_2 \equiv (M_L + M_R) P_L + \frac{(M_L + M_R - P_L)_+^2}{4},$$
$$r_3 \equiv 2P_L M_R + \frac{(P_L + P_R - 2M_R)_+^2}{4} - M_L \min\{M_R, 2P_R\},$$
$$r_4 \equiv 2P_L M_R + \frac{(M_L + M_R - 2P_L)_+^2}{4} - P_R \min\{P_L, 2M_L\},$$
$$r_5 \equiv M_L P_L + M_R P_R + 2\max\{M_L M_R, P_L P_R\},$$

where $(a)_+ = \max\{0, a\}$ for every $a \in \mathbb{R}$. The numbers r_i' can be obtained from the numbers r_i, respectively, when replacing P_L with M_R, P_R with M_L, M_L with P_R, M_R with P_L; $\Delta' \equiv -\Delta$.

Theorem 3. *Let four non-negative numbers P_L, P_R, M_L, M_R be given. Then the boundary value problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities (3), (4) if and only if*

$$\Delta > 0, \quad \Delta > \max\big\{r_1, r_2, r_3, r_4, r_5\big\},$$

or

$$\Delta' > 0, \quad \Delta' > \max\big\{r_1', r_2', r_3', r_4', r_5'\big\}.$$

Corollary 1. *If the conditions of Theorem 3 are satisfied, then*

$$P_R + M_L < 1, \quad P_L + M_R < 2\Big(1 + \sqrt{1 - P_R - M_L}\,\Big).$$

or

$$P_L + M_R < 1, \quad P_R + M_L < 2\Big(1 + \sqrt{1 - P_L - M_R}\,\Big).$$

Corollary 2. *The uniquely solvable set Ω is non-empty and consists of two path-connected components, Ω_+, for whose points $\Delta > 0$, and Ω_-, for whose points $\Delta < 0$. For each point of each component of the uniquely solvable set, the intersection of this component with each straight line that is parallel to one of the axes and passes through this point is an open interval or a half-open interval of the form $[0, d)$.*

Thus, Corollary 2 gives a good idea of the structure of the uniquely solvable set. It remains only to study the boundaries of the set in more detail.

If $P_L = 0$ or $M_R = 0$, then the conditions for the solvability turn out to be simple (and close to the conditions for the solvability of the periodic boundary value problem).

Theorem 4. *Let $M_R = 0$ and non-negative numbers P_L, P_R, M_L be given. Suppose*

$$P_L > P_R + M_L.$$

Then problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities (3), (4) if and only if

$$\frac{P_R + M_L}{1 - 2P_R - M_L} < P_L < 2 - P_R + 2\sqrt{1 - 2P_R - M_L}\,, \quad 2P_R + M_L < 1.$$

Let $P_L = 0$ and non-negative numbers M_R, P_R, M_L be given. Suppose

$$M_R > P_R + M_L.$$

Then problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities (3), (4) if and only if

$$\frac{P_R + M_L}{1 - P_R - 2M_L} < M_R < 2 - P_R + 2\sqrt{1 - P_R - 2M_L}\,, \quad P_R + 2M_L < 1.$$

It is also easy to construct the set of unique solvability for two zero parameters $M_L = 0$ and $P_R = 0$. This section plays an important role in the construction of the entire solvability set (one should pay attention to Corollary 3 below).

Theorem 5. *Let $M_L = 0$ and $P_R = 0$. Let non-negative numbers P_L, M_R be given. The boundary value problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying the equalities (3), (4) if and only if*

$$P_L \in (0, 4), \quad M_R = 0, \tag{5}$$

$$0 \le P_L < \begin{cases} 2\Big(1 - M_R + \sqrt{1 - M_R}\Big), & M_R \in \Big(0, \dfrac{3}{4}\Big), \\ \dfrac{M_R}{2M_R - 1}\,, & M_R \in \Big[\dfrac{3}{4}\,, \dfrac{3}{2}\Big), \\ \dfrac{1 - M_R + \sqrt{2M_R + 1}}{2}\,, & M_R \in \Big[\dfrac{3}{2}\,, 4\Big). \end{cases} \tag{6}$$

Let $\Omega^{0,0} = \{(t, s) :\ (P_L, 0, 0, M_R) \in \Omega_+\}$ be the set of all points (P_L, M_R) satisfying (5), (6). It is the section of the solvability set Ω_+ for $P_R = 0$, $M_L = 0$. Constructing a section of the solvability set when only one of the numbers P_R and M_L is zero is not such an easy task.

The components Ω_+ and Ω_- of the solvability set are symmetric. We investigate the set Ω_+, that is, the case when

$$\Delta = P_L + M_R - P_R - M_L > 0.$$

It turns out that the relations defining the boundaries of the solvability set in this case are easily resolved with respect to "small" parameters $P_R \equiv t$, $M_L \equiv s$, and have the form

$$\begin{aligned} s &< P_L + M_R - P_L M_R - t(1+M_R) - \frac{(P_L + t - M_R)_+^2}{4}\,, \\ \big(1 - \min\{M_R, 2t\}\big)s &< P_L + M_R - t - 2P_L M_R - \frac{(P_L + t - 2M_R)_+^2}{4}\,, \\ (1+P_L)s &< P_L + M_R - t(1 + M_R + 2P_L), \end{aligned} \tag{7}$$

and

$$\begin{aligned} t &< P_L + M_R - P_L M_R - s(1+P_L) - \frac{(M_R - P_L + s)_+^2}{4}\,, \\ \big(1 - \min\{P_L, 2s\}\big)t &< P_L + M_R - 2P_L M_R - s - \frac{(M_R - 2P_L + s)_+^2}{4}\,, \\ (1+M_R)t &< P_L + M_R - s(1 + 2M_R + P_L). \end{aligned} \tag{8}$$

Let $(P_L, P_R = t, M_L = s, M_R) \in \Omega_+$. Therefore, inequalities (7), (8) are fulfilled. Then it is easy to show that $(P_L, 0, 0, M_R) \in \Omega_+$.

Let sections of Ω_+ be defined by

$$\Omega_{P_L,M_R} = \big\{(t,s) :\ (P_L, t, s, M_R) \in \Omega_+\big\}.$$

Corollary 3. *The set Ω_{P_L,M_R} is not empty if and only if $(P_L, M_R) \in \Omega^{0,0}$.*

Corollary 4. *Let $(P_R, M_L) \in \Omega_{P_L,M_R}$. Then $(t,s) \in \Omega_{P_L,M_R}$ for all $t \in [0, P_R]$, $s \in [0, M_L]$.*

For some pairs of P_L and M_R, the border of Ω_{P_L,M_R} is relatively simple. Due to symmetry, it is sufficient to consider the case $P_L \le M_R$.

Theorem 6. *Let $M_R \in [0,1]$, $P_L \le \max\{M_R, 1/2\}$. Then*

$$\begin{aligned} \Omega_{P_L,M_R} = \bigg\{(t,s) :\ & t \in \Big[0, \frac{P_L + M_R - s(1+P_L)}{1 + M_R + 2P_L}\Big),\ s \in \Big[0, \frac{P_L}{1 + M_R + P_L}\Big]; \\ & t \in \Big[0, \frac{P_L + M_R - s(1 + 2M_R + P_L)}{1 + M_R}\Big),\ s \in \Big[\frac{P_L}{1 + M_R + P_L}, \frac{P_L + M_R}{1 + P_L}\Big]\bigg\}. \end{aligned}$$

Theorem 7. *Let $M_R \in [1, 1+\sqrt{2})$, $P_L \in [0, (1+\sqrt{2} - M_R)^2/4)$. Then*

$$\begin{aligned} \Omega_{P_L,M_R} = \bigg\{(t,s) :\ & t \in \Big[0, \frac{P_L + M_R - 2P_L M_R - s}{1 + M_R}\Big),\ s \in \Big[0, \frac{P_L M_R}{2M_R + P_L}\Big]; \\ & t \in \Big[0, \frac{P_L + M_R - s(1 + 2M_R + P_L)}{1 + M_R}\Big),\ s \in \Big[\frac{P_L M_R}{2M_R + P_L}, \frac{P_L + M_R}{1 + 2M_R + P_L}\Big]\bigg\}. \end{aligned}$$

Theorem 8. *Let $(M_L, P_R) \in \Omega^{0,0}$, $M_L \in [8/11, 4)$, $2(4 - M_R)/9 \le P_L \le M_R$. Then*

$$\begin{aligned} \Omega_{P_L,M_R} = \bigg\{(t,s) :\ & 0 \le t < \frac{P_L + M_R - 2P_L M_R - s - (M_R - 2P_L + s)^2/4}{1 - 2s}\,, \\ & s \in \Big[0, 2\sqrt{(1 - P_L)(1 + 2M_R)} - M_R - 2 + 2P_L\Big)\bigg\}. \end{aligned}$$

Theorem 9. *Let $M_L \in [3,4)$, $P_L \le (1 - M_R + \sqrt{2M_R + 1})/4$. Then*

$$\begin{aligned} \Omega_{P_L,M_R} = \bigg\{(t,s) :\ & t \in \Big[0, \frac{P_L + M_R - 2P_L M_R - s - (M_R - P_L + s)^2/4}{1 - \min\{P_L, 2s\}}\Big), \\ & s \in \Big[0, 2\sqrt{(1 - P_L)(1 + 2M_R)} - M_R - 2 + 2P_L\Big)\bigg\}. \end{aligned}$$

Acknowledgements

The work is performed as part of the State Task of the Ministry of Education and Science of the Russian Federation (project # 1.5336.2017/8.9).

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The Asymptotic Properties of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of Second Order Differential Equations with the Product of a Regularly Varying Function of Unknown Function and a Rapidly Varying Function of its First Derivative

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We consider the following differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y')\varphi_1(y). \tag{1}$$

In this equation $\alpha_0 \in \{-1; 1\}$, functions $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$ and $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i \in \{0, 1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[$ or the interval $]Y_i, y_i^0]$. If $Y_i = +\infty$ $(Y_i = -\infty)$, we put $y_i^0 > 0$ $(y_i^0 < 0)$.

We also suppose that function φ_1 is a regularly varying as $y \to Y_1$ function of index σ_1 [7, p. 10-15], function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_0(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{2}$$

It follows from the above conditions (2) that the function φ_0 and its derivative of the first order are rapidly varying functions as the argument tends to Y_0 [1]. Thus, the investigated differential equation contains the product of a regularly varying function of unknown function and a rapidly varying function of its first derivative in its right-hand side.

Previously we obtained results for this kind of equation containing a rapidly varying function of unknown function and a regularly varying function of its first derivative [2].

The main aim of the article is the investigation of conditions of the existence of following class of solutions of equation (1).

Definition 1. The solution y of equation (1), defined on the interval $[t_0, \omega[\subset [a, \omega[$, is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$, if the following conditions take place

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

This class of solutions was defined in the work by V. M. Evtukhov [3] for the n-th order differential equations of Emden–Fowler type and was concretized for the second-order equation. Due to the asymptotic properties of functions in the class of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions [4], every such solution belongs to one of four non-intersecting sets according to the value of λ_0 : $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, $\lambda_0 = 0$, $\lambda_0 = 1$, $\lambda_0 = \pm\infty$.

Now we consider the case $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ of such solutions, every $P_\omega(Y_0, Y_1, \lambda_0)$-solution and its derivative satisfy the following limit relations

$$\frac{y'(t)}{y(t)} = \frac{\lambda_0}{(\lambda_0 - 1)\pi_\omega(t)}\,[1 + o(1)], \quad \frac{y''(t)}{y'(t)} = \frac{1}{(\lambda_0 - 1)\pi_\omega(t)}\,[1 + o(1)] \ \text{ as } \ t \uparrow \omega, \tag{3}$$

From conditions (3) it follows that such $P_\omega(Y_0, Y_1, \lambda_0)$-solutions are regularly varying functions of index $\frac{\lambda_0}{\lambda_0 - 1}$, and their derivatives are regularly varying functions of index $\frac{1}{\lambda_0 - 1}$ as $t \uparrow \omega$ [7].

To formulate the main result, we introduce the following definitions.

Definition 2. Let $Y \in \{0, \infty\}$, Δ_Y is some one-sided neighborhood of Y. Continuous-differentiable function $L : \Delta_Y \to]0; +\infty[$ is called ([6], p.2-3) a normalized slowly varying function as $z \to Y$ $(z \in \Delta_Y)$ if the next statement is valid

$$\lim_{\substack{y \to Y\\ y \in \Delta_Y}} \frac{yL'(y)}{L(y)} = 0.$$

Definition 3. We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0; +\infty[$ satisfies the condition S as $z \to Y$, if for any continuous differentiable normalized slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to]0; +\infty[$ the next relation is valid

$$\theta(zL(z)) = \theta(z)(1 + o(1)) \ \text{ as } \ z \to Y, \ \ z \in \Delta_Y.$$

Definition 4. We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L_0 : \Delta_Y \to]0; +\infty[$ satisfies the condition S_1 as $z \to Y$ if for any finite segment $[a; b] \subset]0; +\infty[$ the next inequality is true

$$\limsup_{\substack{z \to Y\\ z \in \Delta_Y}} \Big| \ln|z| \cdot \Big(\frac{L(\lambda z)}{L(z)} - 1\Big) \Big| < +\infty \ \text{ for all } \ \lambda \in [a; b].$$

Conditions S and S_1 are satisfied by functions $\ln|y|$, $|\ln|y||^\mu$ $(\mu \in R)$, $\ln|\ln|y||$ and many others.

Introduce the necessary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \ \omega = +\infty, \\ t - \omega & \text{as } \ \omega < +\infty, \end{cases} \qquad \theta_1(y) = \varphi_1(y)|y|^{-\sigma_1},$$

$$\Phi_0(z) = \int\limits_{A_\omega}^{z} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)}\,, \quad A_\omega = \begin{cases} y_1^0 & \text{as } \displaystyle\int\limits_{y_1^0}^{Y_1} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)} = \pm\infty, \\[2ex] Y_1 & \text{as } \displaystyle\int\limits_{y_1^0}^{Y_1} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)} = const, \end{cases}$$

$$Z_0 = \lim_{\substack{z \to Y_1\\ z \in \Delta_{Y_1}}} \Phi_0(z), \quad \Phi_1(z) = \int\limits_{A_\omega}^{z} \Phi_0(s)\,ds, \quad Z_1 = \lim_{\substack{z \to Y_1\\ z \in \Delta_{Y_1}}} \Phi_1(z),$$

$$F(t) = \frac{\pi_\omega(t)I_1'(t)}{\Phi_1^{-1}(I_1(t))\Phi_1'(\Phi_1^{-1}(I_1(t)))}\,,$$

and in the case

$$y_0^0 \lim_{t \uparrow \omega} |\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0 - 1}} = Y_0,$$

we have

$$I(t)=\alpha_0 y_0^0\cdot\Big|\frac{\lambda_0-1}{\lambda_0}\Big|^{\sigma_1}\cdot\int\limits_{B_\omega^0}^{t}|\pi_\omega(\tau)|^{\sigma_1}p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0-1}}y_0^0\big)\,d\tau,$$

$$B_\omega^0=\begin{cases} b & \text{if } \displaystyle\int\limits_b^\omega|\pi_\omega(\tau)|^{\sigma_1}p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0-1}}y_0^0\big)\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_b^\omega|\pi_\omega(\tau)|^{\sigma_1}p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0-1}}y_0^0\big)\,d\tau<+\infty,\end{cases}$$

$$I_1(t)=\int\limits_{B_\omega^1}^{t}\frac{I(\tau)\Phi_0^{-1}(I(\tau))}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau,\quad B_\omega^1=\begin{cases} b & \text{if } \displaystyle\int\limits_b^\omega\frac{I(\tau)\Phi_0^{-1}(I(\tau))}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=\pm\infty,\\ \omega & \text{if } \displaystyle\int\limits_b^\omega\frac{I(\tau)\Phi_0^{-1}(I(\tau))}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau<+\infty,\end{cases}$$

where $b\in[a;\omega[$ is chosen so that

$$y_0^0\lim_{t\uparrow\omega}|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0-1}}\in\Delta_{Y_0}\ \text{ as }\ t\in[b;\omega].$$

Note 1. From conditions (3) of the function φ_0 it follows that $Z_0,Z_1\in\{0,+\infty\}$ and

$$\lim_{\substack{z\to Y_1\\ z\in\Delta_{Y_1}}}\frac{\Phi_0''(z)\cdot\Phi_0(z)}{(\Phi_0'(z))^2}=1,\quad \lim_{\substack{z\to Y_1\\ z\in\Delta_{Y_1}}}\frac{\Phi_1''(z)\cdot\Phi_1(z)}{(\Phi_1'(z))^2}=1. \tag{4}$$

Note 2. The following statements are true:

1)
$$\Phi_0(z)=(\sigma_1-1)\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(z)}{\varphi_0'(z)}\,[1+o(1)]\ \text{ as }\ z\to Y_1,\ \ y\in\Delta_{Y_1}.$$

Hence we have
$$\operatorname{sign}(\varphi_0'(z)\Phi_0(z))=\operatorname{sign}(\sigma_1-1)\ \text{ as }\ z\in\Delta_{Y_1}.$$

2)
$$\Phi_1(z)=\frac{\Phi_0^2(z)}{y\Phi_0'(z)}\,[1+o(1)]\ \text{ as }\ z\to Y_1,\ \ z\in\Delta_{Y_1}.$$

Hence we have
$$\operatorname{sign}(\Phi_1(z))=y_0^0\ \text{ as }\ z\in\Delta_{Y_1}.$$

3) The functions Φ_0^{-1} and Φ_1^{-1} exist and are slowly varying functions as inverse to rapidly varying functions as the arguments tend to Y_1 functions.

4) The function $\Phi_1'(\Phi_1^{-1})$ is a regularly varying function of the index 1 as the argument tends to Y_1.

Indeed, from (4) we have

$$\lim_{z\to Z_1}\frac{(\Phi_1'(\Phi_1^{-1}(z)))'z}{\Phi_1'(\Phi_1^{-1}(z))}=\lim_{z\to Z_1}\frac{\Phi_1''(\Phi_1^{-1}(z))z}{(\Phi_1'(\Phi_1^{-1}(z)))^2}$$
$$=\lim_{y\to Y_1}\frac{\Phi_1''(\Phi_1^{-1}(\Phi_1(y)))\Phi_1(y)}{(\Phi_1'(\Phi_1^{-1}(\Phi_1(y))))^2}=\lim_{y\to Y_1}\frac{\Phi_1''(y)\Phi_1(y)}{(\Phi_1'(y))^2}=1.$$

Let $Y\in\{0,\infty\}$, Δ_Y be some one-sided neighborhood of Y. A continuous-differentiable function $L:\Delta_Y\to]0;+\infty[$ is called [6, p. 2-3] a normalized slowly varying function as $z\to Y$ $(z\in\Delta_Y)$ if the next statement is valid

$$\lim_{\substack{y\to Y\\ y\in\Delta_Y}}\frac{yL'(y)}{L(y)}=0. \tag{5}$$

We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta:\Delta_Y\to]0;+\infty[$ satisfies the condition S as $z\to Y$, if for any normalized slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L:\Delta_{Y_i}\to$ $]0;+\infty[$ the following equality takes place as $z\to Y$ $(z\in\Delta_Y)$,

$$\theta(zL(z))=\theta(z)(1+o(1)).$$

We will consider that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L_0:\Delta_Y\to]0;+\infty[$ satisfies the condition S_1 as $z\to Y$ if for any finite segment $[a;b]\subset]0;+\infty[$ the next inequality is true

$$\limsup_{\substack{z\to Y\\ z\in\Delta_Y}}\Big|\ln|z|\cdot\Big(\frac{L(\lambda z)}{L(z)}-1\Big)\Big|<+\infty\ \text{ for all }\ \lambda\in[a;b].$$

Conditions S and S_1 are satisfied by functions $\ln|y|$, $|\ln|y||^{\mu}$ $(\mu\in R)$, $\ln|\ln|y||$ and many others.

The following theorem takes place.

Theorem. *Let $\sigma_1\in\mathbb{R}\setminus\{1\}$, the function θ_1 satisfy the condition S, and the functions θ_1 and $\Phi_1^{-1}\cdot\frac{\Phi_1'}{\Phi_1}(\Phi_1^{-1})$ satisfy the condition S_1. Then for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation* (1)*, in case $\lambda_0\in\mathbb{R}\setminus\{0,1\}$, it is necessary, and if the following condition takes place*

$$I(t)I_1(t)\sigma_1(\lambda_0-1)<0\ \text{ if }\ t\in]b,\omega[\,,$$

and there is a finite or infinite limit

$$\frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|}\,,$$

it is sufficient that the next conditions

$$\pi_\omega(t)y_1^0y_0^0\lambda_0(\lambda_0-1)>0,\quad y_1^0\alpha_0(\lambda_0-1)\pi_\omega(t)>0\ \text{ as }\ t\in[a;\omega[\,,$$
$$y_0^0\cdot\lim_{t\uparrow\omega}|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}=Y_0,\quad \lim_{t\uparrow\omega}I_1(t)=Z_1,$$
$$\lim_{t\uparrow\omega}\frac{I'(t)I_1(t)}{I_1'(t)I(t)}=1,\quad \lim_{t\uparrow\omega}\frac{\Phi(\Phi_1^{-1}(I_1(t)))}{I(t)}=1,\quad \lim_{t\uparrow\omega}F(t)=\frac{1}{\lambda_0-1}$$

are fulfilled. Moreover, for each such solution the next asymptotic representations as $t\uparrow\omega$ take place

$$y'(t)=\Phi_1^{-1}(I_1(t))[1+o(1)],\quad y(t)=\frac{(\lambda_0-1)\Phi_1^{-1}(I_1(t))\pi_\omega(t)}{\lambda_0}\,[1+o(1)].$$

For the equation under the investigation the question of the active existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions, in case $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, that have the received asymptotic representations, has been reduced to the question of the existence of infinitely small as arguments tend to ω solutions of the corresponding, equivalent to the investigated equation, systems of non-autonomous quasi-linear differential equations that admit applications of the known results from the works by V. M. Evtukhov and A. M. Samoilenko [5].

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Boundary Value Problems for Systems of Singular Integral-Differential Equations of Fredholm Type with Degenerate Kernel

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We investigate the problem of finding solutions [1]

$$y(t)\in\mathbb{D}^2[a;b],\ \ y'(t)\in\mathbb{L}^2[a;b]$$

of linear Noetherian ($n\neq v$) boundary value problem for a system of linear integral-differential equations of Fredholm type with degenerate kernel

$$A(t)y'(t)=B(t)y(t)+\Phi(t)\int\limits_a^b F(y(s),y'(s),s)\,ds+f(t),\ \ \ell y(\,\cdot\,)=\alpha,\ \ \alpha\in\mathbb{R}^p. \tag{1}$$

We seek a solution of the Noetherian boundary value problem (1) in a small neighborhood of the solution

$$y_0(t)\in\mathbb{D}^2[a;b],\ \ y_0'(t)\in\mathbb{L}^2[a;b]$$

of the generating problem

$$A(t)y_0'(t)=B(t)y_0(t)+f(t),\ \ \ell y_0(\,\cdot\,)=\alpha. \tag{2}$$

Here

$$A(t),B(t)\in\mathbb{L}^2_{m\times n}[a;b]:=\mathbb{L}^2[a;b]\otimes\mathbb{R}^{m\times n},\quad \Phi(t)\in\mathbb{L}^2_{m\times q}[a;b],\ \ f(t)\in\mathbb{L}^2[a;b].$$

We assume that the matrix $A(t)$ is, generally speaking, rectangular: $m\neq n$. It can be square, but singular. Assume that the function $F(y(t),y'(t),t)$ is linear with respect to unknown $y(t)$ in a small neighborhood of the generating solutions and with respect to the derivative $y'(t)$ in a small neighborhood of the function $y_0'(t)$. In addition, we assume that the function $F(y(t),y'(t),t)$ is continuous in the independent variable t on the segment $[a,b]$;

$$\ell y(\,\cdot\,):\mathbb{D}^2[a;b]\to\mathbb{R}^p$$

is a linear bounded vector functional defined on a space $\mathbb{D}^2[a;b]$. The problem of finding solutions of the boundary value problem (1) in case $A(t)=I_n$ was solved by A. M. Samoilenko and A. A. Boichuk [12]. Thus, the boundary value problem (1) is a generalization of the problem solved by A. M. Samoilenko and A. A. Boichuk.

Provided that the differential-algebraic system (2) with the constant-rank matrix $A(t)$ satisfies the conditions of the theorem from the paper [13, p. 15] in the case of p-order degeneration. Then, in the case of the p-order degeneration, the differential-algebraic system (2) has a solution which can be written the form

$$y_0(t,c_{\rho_{p-1}})=X_p(t)c_{\rho_{p-1}}+K[f(s),\nu_p(s)](t),\ \ c_{\rho_{p-1}}\in\mathbb{R}^{\rho_{p-1}}.$$

The solvability of the differential-algebraic boundary-value problem (2) substantially depends on the choice of the continuous vector function $\nu_p(t)$. If and only if condition

$$P_{Q^*}\big\{\alpha-\ell K[f(s),\nu_p(s)](\,\cdot\,)\big\}=0 \tag{3}$$

is satisfied, the solution of the differential-algebraic boundary-value problem (2), namely,

$$y_0(t,c_r)=X_r(t)c_r+G[f(s);\alpha](t),\ \ c_r\in\mathbb{R}^r$$

determines the generalized Green operator of the differential-algebraic boundary-value problem (2)

$$G[f(s);\alpha](t):=X_p(t)\,Q^+\big\{\alpha-\ell K[f(s),\nu_p(s)](\,\cdot\,)\big\}+K[f(s),\nu_p(s)](t),$$

where $K[f(s),\nu_p(s)](t)$ is the generalized Green operator of the Cauchy problem $z(a)=0$ for the differential-algebraic system (2). Here $X_r(t):=X_p(t)P_{Q_r}$, $X_p(t)$ is fundamental matrix of the differential-algebraic system (2),

$$P_{Q^*}:\mathbb{R}^{\upsilon}\to\mathbb{N}(Q^*),\ \ P_Q:\mathbb{R}^{\rho_{p-1}}\to\mathbb{N}(Q),\ \ Q:=\ell X_p(\,\cdot\,)\in\mathbb{R}^{\upsilon\times\rho_{p-1}}$$

are matrix-orthoprojectors [1,2,13], $P_{Q_r}\in\mathbb{R}^{\upsilon\times r}$ is an $(\rho_{p-1}\times r)$-matrix composed of r linearly independent columns of the orthoprojector P_Q. Thus, the following lemma is proved [13].

Lemma 1. *Provided that the differential-algebraic system* (2) *with the constant-rank matrix* $A(t)$ *satisfies the conditions of the theorem from the paper* [13, p. 15] *in the case of p-order degeneration. Then, in the case of the p-order degeneration, the differential-algebraic system* (2) *has a solution which can be written in the form*

$$y_0(t,c_{\rho_{p-1}})=X_p(t)c_{\rho_{p-1}}+K[f(s),\nu_p(s)](t),\ \ c_{\rho_{p-1}}\in\mathbb{R}^{\rho_{p-1}}.$$

In the critical case $P_{Q^*}\neq 0$ *the singular differential-algebraic boundary value problem* (2) *is solvable iff* (3) *holds. In the critical case the singular differential-algebraic boundary value problem* (2) *has a solution of the form*

$$y_0(t,c_r)=X_r(t)c_r+G[f(s);\alpha](t),\ \ X_r(t):=X(t)P_{Q_r},\ \ c_r\in\mathbb{R}^r,$$

which depends on the arbitrary vector-function $\nu_p(t)\in\mathbb{L}^2[a;b]$. *Here,* P_{Q_r} *is an* $(p\times r)$*-matrix composed of* r *linearly independent columns of the* $(p\times p)$*-matrix-orthoprojector:* $P_Q:\mathbb{R}^p\to\mathbb{N}(Q)$;

$$G[f(s);\alpha](t):=X(t)Q^+\big\{\alpha-\ell K[f(s)](\,\cdot\,)\big\}+K[f(s)](t)$$

is the generalized Green operator of the linear integral-differential problem (1);

$$K[f(s)](t):=X(t)\int\limits_a^t X^{-1}(s)\,\mathfrak{F}_0(s,\nu_0(s))\,ds$$

is the generalized Green operator of the Cauchy problem for the integral-differential system (2).

The results of the proved Lemma 1 were obtained making no use of the central canonical form, perfect pairs, and matrix triplets [1,2].

Denote the matrix

$$\Psi(t):=K[\Phi(s)](t)\in\mathbb{D}^2_{n\times q}[a;b]$$

and $P_{R_\omega}\in\mathbb{R}^{\rho_{p-1}\times\omega}$ matrix composed of ω linearly independent columns of the matrix-orthoprojector

$$P_R:\mathbb{R}^q\to\mathbb{N}(R),\ \ R:=\ell\,\Psi(\,\cdot\,).$$

Thus, the following theorem has been proved [10].

Theorem 1. *Provided that the differential-algebraic system* (2) *with the constant-rank matrix* $A(t)$ *satisfies the conditions of the theorem from the paper [13, c. 15]. In the critical case* ($P_{Q^*} \neq 0$) *under condition* (3) *singular* ($P_{A^*}(t)) \neq 0$ *integral-differential boundary value problem* (2) *has a solution of the form*

$$y_0(t, c_r) = X_r(t)c_r + G[f(s); \alpha](t), \quad X_r(t) := X(t)P_{Q_r}, \ \ c_r \in \mathbb{R}^r,$$

which depends on the arbitrary vector-function $\nu_p(t) \in \mathbb{L}^2[a; b]$. *The singular integral-differential boundary value problem* (1) *has a solution of the form*

$$y(t) = y_0(t, c_r) + x(t), \quad x(t) = X_p(t)v + \Psi(t)u,$$
$$u(c_r) = P_{Q_r} c_r, \quad v(c_\omega) = P_{R_\omega} c_\omega, \quad \check{c} := \begin{pmatrix} c_r \\ c_r \\ c_1 \end{pmatrix},$$

where

$$\varphi(\check{c}) := u(c_0) - \int\limits_a^b F\Big(y_0(t, c_r) + x\big(t, u(c_r), v(c_\omega)\big), y_0'(t, c_r) + x'\big(t, u(c_r), v(c_\omega)\big), t\Big)\, dt = 0. \tag{4}$$

Suppose that for equation (4) *the following conditions are satisfied:*

1. *A non-linear vector-function* $\varphi(\check{c})$, *twice continuously differentiable with respect to* $\check{c}$ *in some region* $\Omega \subseteq \mathbb{R}^{2r+\omega}$, *in a neighborhood of the point* $\check{c}_0$ *has a root* $\check{c}$.
2. *In the neighborhood of the zeroth approximation* $\check{c}_0$ *there are inequalities*
$$\|J_k^+\| \leq \sigma_1(k), \ \ \|d^2\varphi(\xi_k; \check{c} - \check{c}_k)\| \leq \sigma_2(k) \cdot \|\check{c} - \check{c}_k\|.$$
3. *The following constant exists*
$$\theta := \sup_{k \in \mathbb{N}} \Big\{ \frac{\sigma_1(k)\sigma_2(k)}{2} \Big\}.$$

Then, under conditions

$$P_{J_k^*} = 0, \ \ J_k := \varphi'(\check{c}_k) \in \mathbb{R}^{n \times (2r+\omega)}, \ \ \theta \cdot \|\check{c} - \check{c}_0\| < 1 \tag{5}$$

to find the solution $\check{c}$ *of equation (4) the iteration scheme (6)*

$$\check{c}_{k+1} = \check{c}_k - J_k^+ \varphi(\check{c}_k), \ \ k \in \mathbb{N}, \tag{6}$$

is applicable, and the rate of convergence of the sequence $\check{c}$ *of equation (4) is quadratic.*

Here $P_{J_k^*} : \mathbb{R}^m \to \mathbb{N}(J_k^*)$ is an orthogonal projector of the matrix J_k and J_k^+ is the pseudoinverse Moore–Penrose matrix [1]. Note that condition (5) is equivalent [5, 7, 8] to the requirement of completeness of the rank matrix J_k and is possible only in case $m \leq n$.

The results of the proved Theorem 1 were obtained making no use of the central canonical form, perfect pairs, and matrix triplets [1, 2].

The proposed scheme of studies of the nonsingular integral-differential boundary value problem (1) can be transferred analogously to [1, 4, 6, 11] onto nonlinear singular integral-differential boundary value problem. On the other hand, in the case of nonsolvability, the nonsingular integral-differential boundary value problems can be regularized analogously [3, 14].

Conditions for the solvability of the linear boundary-value problem for systems of differential-algebraic equations with the variable rank of the leading-coefficient matrix and the corresponding solution construction procedure have been found in the paper [9].

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Linear Differential-Algebraic Boundary Value Problem with Pulse Influence

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The study of differential-algebraic boundary value problems was initiated in the works of K. Weierstrass, N. N. Luzin and F. R. Gantmacher. Systematic study of differential-algebraic boundary value problems is devoted to the works of S. Campbell, Yu. E. Boyarintsev, V. F. Chistyakov, A. M. Samoilenko, M. O. Perestyuk, V. P. Yakovets, O. A. Boichuk, A. Ilchmann and T. Reis [3]. The study of the differential-algebraic boundary value problems is associated with numerous applications of such problems in the theory of nonlinear oscillations, in mechanics, biology, radio engineering, theory of control, theory of motion stability. At the same time, the study of differential algebraic boundary value problems is closely related to the study of pulse boundary value problems for differential equations, initiated by M. O. Bogolybov, A. D. Myshkis, A. M. Samoilenko, M. O. Perestyk and O. A. Boichuk. Consequently, the actual problem is the transfer of the results obtained in the articles by S. Campbell, A. M. Samoilenko, M. O. Perestyuk and O. A. Boichuk on pulse linear boundary value problems for differential-algebraic equations, in particular finding the necessary and sufficient conditions for the existence of the desired solutions, and also the construction of the Green operator of the Cauchy problem and the generalized Green operator of a pulse linear boundary value problem for a differential-algebraic equation [2, 3, 5].

In this article we found the conditions of the existence and constructive scheme for finding the solutions

$$z(t) \in \mathbb{C}^1\big\{[a,b] \setminus \{\tau_i\}_I\big\}$$

of the linear Noetherian differential-algebraic boundary value problem for a differential-algebraic equation with the singular impulse action [2–5, 11]

$$A(t)z'(t) = B(t)z(t) + f(t), \ \ t \neq \tau_i, \tag{1}$$

$$\Delta z(\tau_i) = S_i z(\tau_i - 0) + a_i, \ \ i = 1, 2, \dots, p, \ \ \ell z(\,\cdot\,) = \alpha \in \mathbb{R}^q, \tag{2}$$

where $A(t), B(t) \in \mathbb{C}_{m\times n}[a,b]$, $f(t) \in \mathbb{C}[a,b]$, $\ell z(\,\cdot\,)$ is a linear bounded functional:

$$\ell z(\,\cdot\,) : \mathbb{C}^1\{[a,b] \setminus \{\tau_i\}_I\} \to \mathbb{R}^q.$$

We assume that the matrix $A(k)$ is, generally speaking, rectangular: $m = n$. It can be square, but singular. The proposed scheme of the research of the linear differential-algebraic boundary value problem for a differential-algebraic equation with impulse action in the critical case in this article can be transferred to the linear differential-algebraic boundary value problem for a differential-algebraic equation with a singular impulse action. The above scheme of the analysis of the seminonlinear differential-algebraic boundary value problems with an impulse action generalizes the results of S. Campbell, A. M. Samoilenko, M. O. Perestyuk and O. A. Boichuk and can be used for proving the solvability and constructing solutions of weakly nonlinear boundary value problems with a singular impulse action in the critical and noncritical cases [2–6, 8, 11]. For the case in which all

$$A(t) \equiv I_n, \ \ I_n + S_i, \ \ i = 1, 2, \dots, p$$

are non-degenerate matrices, we obtain the problem investigated in [2, 3]; in particular, if

$$A(t) \equiv I_n, \quad S_i = 0, \ \ i = 1, 2, \dots, p,$$

then we have the problem considered in [12]. If

$$S_i = 0, \ \ a_i = 0, \ \ \ell z(\,\cdot\,) := M_i(\tau_i - 0) + N_i z(\tau_i + 0), \ \ i = 1, 2, \dots, p,$$

then we obtain the problem analyzed in [10]. If $A(t) \equiv I_n$ and $I_n + S_i$ are degenerate matrices for some i, then we have a degenerate impulse action [5, 6].

The analysis of differential-algebraic equations with the help of the central canonical form, perfect pairs, and matrix triplets was made in the monographs [3, 4]. Sufficient conditions for the reducibility of a differential-algebraic linear system to the central canonical form were obtained by A. M. Samoilenko and V. P. Yakovets. In papers [8, 11], sufficient solvability conditions, as well as the procedure of constructing the generalized Green operator for the linear differential-algebraic boundary-value problem (1) without making use of the central canonical form, perfect pairs, and matrix triplets, were proposed.

Provided that the differential-algebraic system (1) with the constant-rank matrix $A(t)$ satisfies the condition

$$P_{A^*(t)} \equiv 0. \tag{3}$$

Then, in the case (3), the differential-algebraic system (1) has a solution, which can be written in the form [8, 11]

$$z(t, c) = X_0(t)c + K_0\big[f(s), \nu_0(s)\big](t), \ \ c \in \mathbb{R}^n,$$

where

$$K_0\big[f(s), \nu_0(s)\big](t) := X_0(t) \int\limits_a^t X_0^{-1}(s)\, \mathfrak{F}_0(s, \nu_0(s))\, ds, \ \ a \le t < \tau_1$$

is the generalized Green operator of the Cauchy problem $z(a) = 0$ for the differential-algebraic system (1), $X_0(t)$ is normal fundamental matrix:

$$X_0'(t) = A^+(t)B(t)X_0(t), \ \ X_0(a) = I_n$$

and

$$\mathfrak{F}_0(t, \nu_0(t)) := A^+(t)f(t) + P_{A_{\rho_0}}(t)\nu_0(t),$$

$A^+(t)$ is a pseudoinverse matrix, $P_{A^*(t)}$ is matrix-orthoprojectors [3]:

$$P_{A^*}(t) : \mathbb{R}^m \to \mathbb{N}(A^*(t)),$$

$P_{A_{\rho_0}}(t)$ is an $(n \times \rho_0)$-matrix composed of ρ_0 linearly independent columns of the $(n \times n)$-matrix-orthoprojector [3]:

$$P_A(t) : \mathbb{R}^n \to \mathbb{N}(A(t)).$$

The solvability of the differential-algebraic boundary-value problem (1), (2) substantially depends on the choice of the continuous vector function $\nu_0(t)$. In the case (3) the differential-algebraic system (1) has a solution which can be written in the form [5, 6, 9]

$$z(t,c) = X_1(t)c + K_1\big[f(s),\nu_0(s)\big](t), \;\; c \in \mathbb{R}^n,$$

where

$$X_1(t) = X_0(t)X_0^{-1}(\tau_1)(I_n + S_1)X_0(\tau_1), \;\; \tau_1 \le t < \tau_2,$$

and

$$K_1\big[f(s),\nu_0(s)\big](t) := X_0(t)X_0^{-1}(\tau_1)(I_n+S_1)K_0\big[f(s),\nu_0(s)\big](\tau_1) \\ + X_0(t)X_0^{-1}(\tau_1)\,a_1 + X_0(t)\int\limits_{\tau_1}^{t} X_0^{-1}(s)\,\mathfrak{F}_0(s,\nu_0(s))\,ds, \;\; \tau_1 \le t < \tau_2.$$

Denote the matrix

$$X_p(t) = X_0(t)X_0^{-1}(\tau_p)(I_n + S_p)X_{p-1}(\tau_p), \;\; \tau_p \le t \le b,$$

and

$$K_p\big[f(s),\nu_0(s)\big](t) := X_0(t)X_0^{-1}(\tau_p)(I_n+S_p)K_{p-1}\big[f(s),\nu_0(s)\big](\tau_p) \\ + X_0(t)X_0^{-1}(\tau_p)\,a_p + X_0(t)\int\limits_{\tau_p}^{t} X_0^{-1}(s)\,\mathfrak{F}_0(s,\nu_0(s))\,ds, \;\; \tau_p \le t \le b$$

is the generalized Green operator of the Cauchy problem for the differential-algebraic system (1) with the singular impulse action (2). Thus, the following lemma is proved.

Lemma 1. *In the case* (3) *the differential-algebraic system* (1) *with the singular impulse action* (2) *with the constant-rank matrix $A(t)$ has a solution which can be written in the form*

$$z(t,c) = X(t)\,c + K\big[f(s),\nu_0(s)\big](t), \;\; c \in \mathbb{R}^n,$$

where

$$X(t) = \begin{cases} X_0(t), & a \le t < \tau_1, \\ \dots\dots\dots\dots\dots\dots & \\ X_p(t), & \tau_p \le t \le b, \end{cases}$$

and

$$K\big[f(s),\nu_0(s)\big](t) = \begin{cases} K_0\big[f(s),\nu_0(s)\big](t), & a \le t < \tau_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ K_p\big[f(s),\nu_0(s)\big](t), & \tau_p \le t \le b \end{cases}$$

is the generalized Green operator of the Cauchy problem for the differential-algebraic system (1) *with the singular impulse action* (2)*.*

If and only if the condition

$$P_{Q_d^*}\{\alpha - \ell K\big[f(s),\nu_0(s)\big](\,\cdot\,)\} = 0 \tag{4}$$

is satisfied, the solution of the differential-algebraic boundary-value problem (1), (2) which can be written in the form

$$z(t,c_r)=X_r(t)c_r+X(t)Q^+\{\alpha-\ell K\big[f(s),\nu_0(s)\big](\,\cdot\,)\}+K\big[f(s),\nu_0(s)\big](t),\ \ c_r\in\mathbb{R}^r.$$

Here $X_r(t):=X(t)P_{Q_r}$ is a fundamental matrix of the boundary-value problem (1), (2)

$$P_{Q^*}:\mathbb{R}^q\to\mathbb{N}(Q^*),\ \ P_Q:\mathbb{R}^n\to\mathbb{N}(Q),\ \ Q:=\ell X(\,\cdot\,)\in\mathbb{R}^{q\times n}$$

are matrices-orthoprojectors [3], $P_{Q_r}\in\mathbb{R}^{q\times r}$ is an $(n\times r)$-matrix composed of r linearly independent columns of the orthoprojector P_Q. Thus, the following lemma is proved.

Lemma 2. *In the case* (3) *the differential-algebraic system* (1) *with the singular impulse action* (2) *with the constant-rank matrix* $A(t)$ *has a solution which can be written in the form*

$$z(t,c)=X(t)c+K\big[f(s),\nu_0(s)\big](t),\ \ c\in\mathbb{R}^n$$

is the generalized Green operator of the Cauchy problem for the system (1) *with the singular impulse action* (2)*. If and only if condition* (4) *holds, the solution of the differential-algebraic system* (1) *with the singular impulse action* (2)

$$z(t,c_r)=X_r(t)c_r+G\big[f(s);\nu_0(s);\alpha\big](t),\ \ c_r\in\mathbb{R}^r,$$

determines the generalized Green operator of the differential-algebraic system (1) *with the singular impulse action* (2)

$$G\big[f(s);\nu_0(s);\alpha\big](t):=X(t)Q^+\big\{\alpha-\ell K\big[f(s),\nu_0(s)\big](\,\cdot\,)\big\}+K\big[f(s),\nu_0(s)\big](t).$$

The above scheme of the analysis of the boundary value problems with the impulse action (1), (2) generalizes the results of [2–6,9] and can be used for proving the solvability and constructing solutions of weakly nonlinear boundary value problems with singular impulse action in the critical and noncritical cases [1, 3, 7]. The results of the proved Lemmas 1 and 2 were obtained making no use of the central canonical form, perfect pairs, and matrix triplets [3, 4].

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About the Reduction of the Linear Noetherian Difference Algebraic Boundary Value Problem to the Noncritical Case

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We investigate the problem of finding bounded solutions [2,3,6]

$$z(k) \in \mathbb{R}^n, \ \ k \in \Omega := \{0, , 2, \dots, \omega\}$$

of the linear Noetherian ($n \neq \upsilon$) boundary value problem for a system of linear difference-algebraic equations [2,6]

$$A(k)z(k+1) = B(k)z(k) + f(k), \ \ \ell z(\,\cdot\,) = \alpha, \ \ \alpha \in \mathbb{R}^{\upsilon}; \tag{1}$$

here $A(k), B(k) \in \mathbb{R}^{m\times n}$ are bounded matrices and $f(k)$ are real bounded column vectors,

$$\ell z(\,\cdot\,) : \mathbb{R}^n \to \mathbb{R}^{\upsilon}$$

is a linear bounded vector functional defined on a space of bounded functions. We assume that the matrix $A(k)$ is, generally speaking, rectangular: $m = n$. It can be square, but singular. The problem of finding bounded solutions $z(k)$ of a boundary value problem for a linear non-degenerate ($\det B(k) \neq 0$, $k \in \Omega$) system of first-order difference equations

$$z(k+1) = B(k)z(k) + f(k), \ \ \ell z(\,\cdot\,) = \alpha \in \mathbb{R}^{\upsilon}$$

was solved by A. A. Boichuk [2]. We investigate the problem of finding bounded solutions of the linear Noetherian boundary value problem for a system of linear difference-algebraic equations (1) in case

$$1 \leq \operatorname{rank} A(k) = \sigma_0, \ \ k \in \Omega.$$

As is known [1,14], any $(m \times n)$-matrix $A(k)$ can be represented in a definite basis in the form

$$A(k) = R_0(k) \cdot J_{\sigma_0} \cdot S_0(k), \quad J_{\sigma_0} := \begin{pmatrix} I_{\sigma_0} & O \\ O & O \end{pmatrix};$$

here, $R_0(k)$ and $S_0(k)$ are nonsingular matrices. The nonsingular change of the variable

$$y(k+1) = S_0(k)z(k+1)$$

reduces system (1) to the form [12]

$$A_1(k)\varphi(k+1) = B_1(k)\varphi(k) + f_1(k); \tag{2}$$

Under the condition [14], when

$$P_{A^*}(k) \neq 0, \ \ P_{A_1^*}(k) \equiv 0, \tag{3}$$

we arrive at the problem of construction of solutions of the linear difference-algebraic system

$$\varphi(k+1) = A_1^+(k)B_1(k)\varphi(k) + \mathfrak{F}_1(k, \nu_1(k)), \ \ \nu_1(k) \in \mathbb{R}^{\rho_1}; \tag{4}$$

here,

$$\mathfrak{F}_1(k,\nu_1(k)) := A_1^+(k)f_1(k) + P_{A_{\varrho_1}}(k)\nu_1(k),$$

$\nu_1(k)\in\mathbb{R}^{\rho_1}$ is an arbitrary bounded vector function, $A_1^+(k)$ is a pseudoinverse (by Moore–Penrose) matrix [3]. In addition, $P_{A_1^*(k)}$ is a matrix-orthoprojector [3]: $P_{A_1^*}(k):\mathbb{R}^{\sigma_0}\to\mathbb{N}(A_1^*(k))$, $P_{A_{\rho_1}}(k)$ is an $(\rho_0\times\rho_1)$-matrix composed of ρ_1 linearly independent columns of the $(\rho_0\times\rho_0)$-matrix-orthoprojector: $P_{A_1}(k):\mathbb{R}^{\rho_0}\to\mathbb{N}(A_1(k))$. By analogy with the classification of pulse boundary-value problems [3, 7, 8] we say in the (3), that, for the linear difference-algebraic system (1), the first-order degeneration holds. Thus, the following lemma is proved [12].

Lemma. *The first-order degeneration difference-algebraic system* (1) *has a solution of the form*

$$z(k,c_{\rho_0}) = X_1(k)\,c_{\rho_0} + K[f(j),\nu_1(j)](k),\ \ c_{\rho_0}\in\mathbb{R}^{\rho_0};$$

which depends on the arbitrary continuous vector-function $\nu_1(k)\in\mathbb{R}^{\rho_1}$, where $X_1(k)$ is fundamental matrix, $K[f(j),\nu_1(j)](k)$ is the generalized Green operator of the Cauchy problem for the linear difference-algebraic system (1)*.*

Denote the vector $\nu_1(k):=\Psi_1(k)\gamma$, $\gamma\in\mathbb{R}^{\theta}$; here, $\Psi_1(k)\in\mathbb{R}^{\rho_1\times\theta}$ is an arbitrary bounded full rank matrix. Generalized Green operator of the Cauchy problem for the linear difference-algebraic system (1) of the form

$$K[f(j),\nu_1(j)](k) = K[f(j)](k) + K[\Psi_1(j)](k)\,\gamma;$$

here,

$$K[\Psi_1(j)](k) := S_0^{-1}(k-1)P_{D_{\rho_0}}\mathcal{K}[\Psi_1(s))](k),$$

and

$$\begin{aligned}\mathcal{K}[\Psi_1(j)](0) := 0,\ \mathcal{K}[\Psi_1(j)](1) := P_{A_{\rho_1}}(0)\Psi_1(0),&\\ \mathcal{K}[\Psi_1(j)](2) := A_1^+(1)B_1(1)\mathcal{K}[\Psi_1(j)](1) + P_{A_{\rho_1}}(1)\Psi_1(1),\ \dots,&\\ \mathcal{K}[\Psi_1(j)](k+1) := A_1^+(k)B_1(k)\mathcal{K}[\Psi_1(j)](k) + P_{A_{\rho_1}}(k)\Psi_1(k).&\end{aligned}$$

Denote the matrix

$$\mathcal{D}_1 := \big\{Q_1;\ \ell K[\Psi_1(j)](\,\cdot\,)\big\}\in\mathbb{R}^{\upsilon\times(\rho_0+\theta)}.$$

Substituting the general solution of the system of the linear difference-algebraic equations (1) into the boundary condition (1), we arrive at the linear algebraic equation

$$\mathcal{D}_1\,\check{c} = \alpha - \ell K[A^+(j)f(j)](\,\cdot\,),\ \ \check{c} := \operatorname{col}(c_{\rho_0},\gamma)\in\mathbb{R}^{\rho_0+\theta}. \tag{5}$$

Equation (5) is solvable iff

$$P_{\mathcal{D}_1^*}\big\{\alpha - \ell K[f(j)](\,\cdot\,)\big\} = 0. \tag{6}$$

Here, $P_{\mathcal{D}_1^*}$ is a matrix-orthoprojector: $P_{\mathcal{D}_1^*}:\mathbb{R}^{\upsilon}\to\mathbb{N}(\mathcal{D}_1^*)$. In this case, the general solution of equation (5)

$$\check{c} = \mathcal{D}_1{}^+\big\{\alpha - \ell K[f(j)](\,\cdot\,)\big\} + P_{\mathcal{D}_1}\delta,\ \ \delta\in\mathbb{R}^{\rho_0+\theta}$$

determines the general solution of the boundary-value problem (1)

$$\begin{aligned}z(k,\delta) = \big\{X_1(k);K[\Psi_1(j)](k)\big\}\mathcal{D}_1^+\big\{\alpha - \ell K[f(j)](\,\cdot\,)\big\}&\\ + K[f(j)](k) + \big\{X_1(k);K[\Psi_1(j)](k)\big\}P_{\mathcal{D}_1}\delta.&\end{aligned}$$

Here, $P_{\mathcal{D}_1}$ is a matrix-orthoprojector: $P_{\mathcal{D}_1}:\mathbb{R}^{\rho_0+\theta}\to\mathbb{N}(\mathcal{D}_1)$. Thus, the following theorem is proved [12].

Corollary. *The problem of finding bounded solutions of a system of linear difference-algebraic equations* (1) *in the case of first-order degeneracy, under condition* (3)*, in the case of first-order degeneracy for a fixed full rank bounded matrix* $\Psi_1(k)$*, has a solution of the form*

$$z(k,c_{\rho_0}) = X_1(k)\,c_{\rho_0} + K\big[f(j),\nu_1(j)\big](k), \quad c_{\rho_0}\in\mathbb{R}^{\rho_0}.$$

Under condition $P_{Q_1^*}\neq 0$, $P_{\mathcal{D}_1^*}=0$*, the general solution of the difference-algebraic boundary value problem* (1)

$$z(k,c_r) = X_r(k)c_r + G\big[f(j);\Psi_1(j);\alpha\big](k), \quad c_r\in\mathbb{R}^r$$

is determined by the Green operator of a difference-algebraic boundary value problem (1)

$$G\big[f(j);\Psi_1(j);\alpha\big](k) := K[f(j)](k) + \big\{X_1(k);K[\Psi_1(j)](k)\big\}\mathcal{D}_1^+\big\{\alpha-\ell K[f(j)](\,\cdot\,)\big\}.$$

The matrix $X_r(k)$ *is composed of* r *linearly independent columns of the matrix*

$$\big\{X_1(k);K[\Psi_1(j)](k)\big\}P_{\mathcal{D}_1}.$$

Under condition $P_{\mathcal{D}_1^*}\neq 0$, we say that the difference-algebraic boundary-value problem (1) in the case of first-order degeneracy is a critical case, and vice versa: under condition $P_{Q_1^*}\neq 0$, $P_{\mathcal{D}_1^*}=0$, we say that the difference-algebraic boundary-value problem (1) is reduced to the non-critical case.

Example. The requirements of the proved Corollary 1 satisfy the problem of construction solutions of the difference-algebraic boundary-value problem

$$A\,z(k+1) = B(k)\,z(k) + f(k), \quad \ell z(\,\cdot\,) := M\,(z(0)-z(3)) = 0, \quad k=0,1,2,3, \tag{7}$$

here

$$A := \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & k+1 & 0 & 0 \end{pmatrix}, \quad f(k) := \begin{pmatrix} 1\\ k\\ 1 \end{pmatrix}, \quad M := \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first-order degeneration difference-algebraic system (7) has a solution of the form

$$z(k) = X_r(k)\,c_r + G\big[f(j),\nu_1(j),\alpha\big](k), \quad G\big[f(j),\nu_1(j),\alpha\big](k) = K\big[f(j),\nu_1(j)\big](k), \quad c_r\in\mathbb{R}^1,$$

where

$$X_r(k) := X_1(k)P_{Q_r} = P_{Q_r} = \begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix}, \quad \nu_1(k) := 0, \quad k\in\Omega := \{0,1,2,3\},$$

in addition,

$$K\big[f(j),\nu_1(j)\big](0) = -\begin{pmatrix} 0\\ 1\\ 0\\ 0 \end{pmatrix}, \quad K\big[f(j),\nu_1(j)\big](1) = \frac{1}{2}\begin{pmatrix} 0\\ -1\\ 2\\ 0 \end{pmatrix},$$

$$K\big[f(j),\nu_1(j)\big](2) = \frac{1}{3}\begin{pmatrix} 0\\ -1\\ 6\\ 3 \end{pmatrix}, \quad K\big[f(j),\nu_1(j)\big](0) = \frac{1}{4}\begin{pmatrix} 0\\ -1\\ 12\\ 8 \end{pmatrix}.$$

The proposed scheme of studies of difference-algebraic boundary-value problems can be transferred analogously to [2–4, 10] onto nonlinear difference-algebraic boundary-value problems. On the other hand, in the case of nonsolvability, the difference-algebraic boundary-value problems can be regularized analogously [9, 15]. The proposed scheme of studies of difference-algebraic boundary-value problems can be transferred analogously to [5, 11, 12] onto nonlinear difference-algebraic boundary-value problems with variable rank of leading coefficient matrix an analogously to [13] an inverse problem to the Cauchy problem for the difference-algebraic equation.

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A Condition for the Solvability of the Control Problem of Asynchronous Spectrum of Linear Almost Periodic Systems the Lower Triangular Representation of Mean Value of Coefficient Matrix

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In 1955 J. Kurzweil and O. Vejvoda proved that the system of almost periodic differential equations can have an almost periodic solution such that the intersection of the frequency modules of the solution and the right-hand side is trivial [3]. In what follows, such almost periodic solutions will be called strongly irregular, the frequency spectrum – asynchronous, and the described vibrations – asynchronous [1, 4]. Various aspects of control theory for ordinary differential systems of almost periodic equations were studied in a number of works (see, for example, [5] and others), the essential a feature of which is the consideration the regular case, when the frequency of the system itself and its solution coincide.

Now we will study the solvability of the control problem of the asynchronous spectrum of linear almost periodic systems for which the mean value of the coefficient matrix is lower triangular. Let's consider a linear non-stationary control system

$$\dot{x} = A(t)x + Bu, \ \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^n, \ \ n \geq 2, \tag{1}$$

where x is the phase vector, u is the input, B is the constant $n \times n$-matrix under control, $A(t)$ is a continuous almost periodic matrix with a modulus of frequencies $\mathrm{Mod}(A)$. Suppose that the control is specified in the form of a linear feedback in the phase variables

$$u = U(t)x \tag{2}$$

with a continuous almost periodic $n \times n$-matrix $U(t)$ (feedback coefficient), the frequency modulus of which is contained in the frequency modulus of the coefficient matrix, i.e.

$$\mathrm{Mod}(U) \subseteq \mathrm{Mod}(A).$$

It is required to obtain conditions on the right-hand side of system (1) such that for any choice of the feedback coefficient from the indicated admissible set, the closed-loop system

$$\dot{x} = (A(t) + BU(t))x, \tag{3}$$

has a strongly irregular almost periodic solution, the frequency spectrum of which contains a given subset (target set).

Let L be the target frequency set. We will assume that

$$\operatorname{rank} B = r < n \ \ (n - r = d). \tag{4}$$

In this case, there is a constant non-singular real $(n\times n)$-matrix S such that in the matrices $D=SB$ the first d columns are zero, while the rest r columns are linearly independent.

Let us introduce the transformation of phase variables

$$y=Sx, \tag{5}$$

which transform system (3) to the system

$$\dot{y}=(C(t)+DV(t))y, \tag{6}$$

where

$$C(t)=SA(t)S^{-1}, \quad V(t)=U(t)S^{-1}, \quad D=QB.$$

System (6) has a strongly irregular almost periodic solution if and only if this solution satisfies the system

$$\dot{y}=(\widehat{C}+D\widehat{V})y, \quad (\widetilde{C}(t)+D\widetilde{V}(t))y=0, \tag{7}$$

where sign "$\widehat{\ }$" denotes a averaging, for example,

$$\widehat{C}=\lim_{T\to+\infty}\frac{1}{T}\int\limits_0^T C(s)\,ds, \quad \widetilde{C}(t)=C(t)-\widehat{C}, \quad \widetilde{V}(t)=V(t)-\widehat{V}.$$

Let us denote the matrix composed of the last r rows of the matrix D, by $D_{r,n}$. It follows from the construction of the matrix D that the following condition is fulfilled

$$\operatorname{rank} D_{r,n}=r. \tag{8}$$

The rank condition (8) means that the matrix $D_{r,n}$ rows are linearly independent. Since their number is less than the number of columns, then adding any columns to such a matrix does not change its rank.

Let us represent the matrix of coefficients $C(t)$ in the block form, corresponding to the structure of the matrix D. Let $C_{d,d}^{(11)}(t)$, $C_{r,d}^{(21)}(t)$ – its upper and lower left, and $C_{d,r}^{(12)}(t)$, $C_{r,r}^{(22)}(t)$ – its upper and lower right blocks (the lower indices indicate the dimensionality). According to this representation, the averaged matrix $\widehat{C}$ will be decomposed into four blocks of the same dimensions $\widehat{C}_{d,d}^{(11)}$, $\widehat{C}_{r,d}^{(21)}$, $\widehat{C}_{d,r}^{(12)}$, $\widehat{C}_{r,r}^{(22)}$.

Taking into account the structure of the matrix D and the block representation averaging of the matrix of coefficients $C(t)$, we write system (7) in the form

$$\begin{gathered}\dot{y}^{[d]}=\widehat{C}_{d,d}^{(11)}y^{[d]}+\widehat{C}_{d,r}^{(12)}y_{[r]}, \quad \dot{y}_{[r]}=\big(\widehat{C}_{r,d}^{(21)}+D_{r,n}\widehat{V}_{n,d}\big)y^{[d]}+\big(\widehat{C}_{r,r}^{(22)}+D_{r,n}\widehat{V}_{n,r}\big)y_{[r]},\\ \widetilde{C}_{d,d}^{(11)}(t)y^{[d]}+\widetilde{C}_{d,r}^{(12)}(t)y_{[r]}=0, \quad \big(\widetilde{C}_{r,d}^{(21)}(t)+D_{r,n}\widetilde{V}_{n,d}(t)\big)y^{[d]}+\big(\widetilde{C}_{r,r}^{(22)}(t)+D_{r,n}\widetilde{V}_{n,r}(t)\big)y_{[r]}=0,\end{gathered} \tag{9}$$

where

$$\begin{gathered}y=\operatorname{col}(y^{[d]},y_{[r]}), \quad y^{[d]}=\operatorname{col}(y_1,\dots,y_d), \quad y_{[r]}=\operatorname{col}(y_{d+1},\dots,y_n),\\ \widehat{V}=\{\widehat{V}_{n,d}\widehat{V}_{n,r}\}, \quad \widetilde{V}(t)=\{\widetilde{V}_{n,d}(t)\widetilde{V}_{n,r}(t)\}\end{gathered}$$

are the corresponding‘ representation of the stationary and oscillatory components of the matrix $V(t)$.

Thus, it is true

Lemma. *If conditions* (4), (8) *are fulfilled, systems* (3) *and* (9) *are equivalent in the sense of existence of strongly irregular almost periodic solutions.*

Suppose that the averaging of the matrix of coefficients of the original system with using the transforming matrix S is reduced to the lower-triangular form. In other words, this means that the matrix $\widehat{C}$ has the form

$$\widehat{C}=\begin{pmatrix} \widehat{c}_{11} & 0 & 0\dots & 0\\ \widehat{c}_{21} & \widehat{c}_{22} & 0\dots & 0\\ \dots & \dots & \dots & \dots\\ \widehat{c}_{n1} & \widehat{c}_{n2} & \widehat{c}_{n3}\dots & \widehat{c}_{nn} \end{pmatrix}. \tag{10}$$

Let's give the conditions to solve the posed problem. Taking into account the lemma, this problem is reduced to finding conditions for the existence of strongly irregular almost periodic solutions $y=y(t)=\operatorname{col}(y^{[d]}(t),y_{[r]}(t))$ with the frequencies L of system (9).

We have

Theorem. *The control problem of asynchronous spectrum of system* (1)*,* (4)*,* (10) *with the target set L is solvable if and only if the conditions*

$$\operatorname{rank}_{\mathrm{col}} C_{12}=r_1<r$$

and

$$|L|\leq\Big[\frac{r-r_1}{2}\Big]$$

are satisfied.

Acknowledgements

The work was carried out at the Institute of Mathematics of the National Academy of Sciences of Belarus within the framework of the Belarusian Republican Foundation for Fundamental Research, project Ф20Р – 005 *"Control problems on the first approximation for non-stationary systems under uncertainty"*.

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Asymptotic Behaviour of Solutions of Third-Order Differential Equations with Rapid Varying Nonlinearities

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Consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0,+\infty[$ is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} 0, \\ \text{or} \ \ +\infty, \end{cases} \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 is equal either to zero or to $\pm\infty$, Δ_{Y_0} is a one-sided neighborhood of the point Y_0.

It follows directly from conditions (2) that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ as } \ y\to Y_0, \ \ y\in\Delta_{Y_0} \ \text{ and } \ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

By virtue of these conditions, the function φ and its first-order derivative are (see the monograph by M. Maric [10, Chapter 3, § 3.4, Lemmas 3.2, 3.3, pp. 91-92]) rapidly varying as $y\to Y_0$.

For second-order differential equations with the right-hand side the same as in (1), the asymptotic behavior of solutions was studied in [2, 3, 5–7, 10].

In the work of V. M. Evtukhov, N. V. Sharay (see [9]) for the differential equation (1) the questions on the existence and asymptotics of so-called $P_\omega(Y_0,\lambda_0)$ – solutions for $\lambda_0 \in \mathbb{R}\setminus\{0;1;\frac{1}{2}\}$ were solved.

Definition. A solution y of the differential equation (1) is called $P_\omega(Y_0,\lambda_0)$ – solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0,\omega[\subset[a,\omega[$ and satisfies the following conditions

$$\lim_{t\uparrow\omega} y(t) = Y_0, \quad \lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} 0, \\ \text{or} \ \ \pm\infty \end{cases} \quad (k=1,2), \qquad \lim_{t\uparrow\omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The aim of the present report is to obtain the asymptotics of $P_\omega(Y_0,\lambda_0)$ – solutions of the differential equation (1) in the special case when $\lambda_0 = 1$. For each such solution, due to a priori asymptotic properties of $P_\omega(Y_0,1)$ – solutions (see [4, Chapter 3, § 10]), the following relations

$$\frac{y'(t)}{y(t)} \sim \frac{y''(t)}{y'(t)} \sim \frac{y'''(t)}{y''(t)} \ \text{ as } \ t\uparrow\omega, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \pm\infty, \tag{3}$$

hold, where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t-\omega & \text{if } \ \omega = -\infty. \end{cases}$$

Hence, in particular, it follows that $P_\omega(Y_0,1)$ – solution of equation (1) and its derivatives up to the second order inclusive are rapidly varying functions as $t\uparrow\omega$.

Moreover, here and in the sequel, without loss of generality, we assume that

$$\Delta_{Y_0}=\Delta_{Y_0}(y_0), \ \ \text{where} \ \ \Delta_{Y_0}(y_0)=\begin{cases}[y_0,Y_0[& \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ ]Y_0,y_0] & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0,\end{cases} \tag{4}$$

where $y_0\in\Delta_{Y_0}$ is such that $|y_0|<1$ as $Y_0=0$ and $y_0>1$ $(y_0<-1)$ as $Y_0=+\infty$ (as $Y_0=-\infty$).

Let us introduce the necessary auxiliary notations and assume that the definition area of the function φ in equation (1) is determined by formula (4). Further, we put

$$\mu_0=\operatorname{sign}\varphi'(y), \quad \nu_0=\operatorname{sign}y_0, \quad \nu_1=\begin{cases}1 & \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1 & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the following functions

$$J_0(t)=\int\limits_{A_0}^{t}p_0^{\frac13}(\tau)\,d\tau, \quad \Phi(y)=\int\limits_{B}^{y}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}\,,$$

where $p_0:[a,\omega[\,\to]0,+\infty[$ is a continuous or continuously differentiable function such that $p(t)\sim p_0(t)$ as $t\uparrow\omega$,

$$A_0=\begin{cases}\omega & \text{if } \displaystyle\int\limits_a^{\omega}p_0^{\frac13}(\tau)\,d\tau<+\infty,\\ a & \text{if } \displaystyle\int\limits_a^{\omega}p_0^{\frac13}(\tau)\,d\tau=+\infty,\end{cases} \qquad B=\begin{cases}Y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}=const,\\ y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}=\pm\infty.\end{cases}$$

It is clear that the conditions

$$\nu_0\nu_1<0 \ \text{ if } \ Y_0=0, \quad \nu_0\nu_1>0 \ \text{ if } \ Y_0=\pm\infty,$$

are necessary for the existence of $P_\omega(Y_0,1)$–solutions. Moreover, by virtue of (1), Definition and (3), it is also necessary that the inequalities

$$\alpha_0\nu_1>0, \quad \nu_0\operatorname{sign}y''(t)>0$$

hold.

The entered function Φ keeps a sign on Δ_{Y_0}, tends either to zero or to $\pm\infty$ as $y\to Y_0$ and is increasing on Δ_{Y_0}, since on this interval $\Phi'(y)=y^{-\frac23}\varphi^{-\frac13}(y)>0$. Therefore, there is an inverse function $\Phi^{-1}:\Delta_{Z_0}\longrightarrow\Delta_{Y_0}$, where, by virtue of the second of conditions (2) and the monotonic increase of Φ^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases}0,\\ \text{or } \ +\infty,\end{cases}$$

$$\Delta_{Z_0}=\begin{cases}[z_0,Z_0[& \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0] & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases} \qquad z_0=\Phi(y_0).$$

We also introduce auxiliary functions:

$$q(t)=\frac{(\Phi^{-1}(\alpha_0J_0(t)))'}{\alpha_0J_2(t)}\,,\quad H(t)=\frac{\Phi^{-1}(\alpha_0J_0(t))\varphi'(\Phi^{-1}(\alpha_0J_0(t)))}{\varphi(\Phi^{-1}(\alpha_0J_0(t)))}\,,$$

$$J_1(t)=\int\limits_{A_1}^{t}p_0(\tau)\varphi(\Phi^{-1}(\alpha_0J_0(\tau)))\,d\tau,\quad J_2(t)=\int\limits_{A_2}^{t}J_1(\tau)\,d\tau,$$

where

$$A_1=\begin{cases} t_1 & \text{if } \displaystyle\int\limits_{t_1}^{\omega}p_0(\tau)\varphi\big(\Phi^{-1}(\alpha_0J_0(\tau))\big)\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_{t_1}^{\omega}p_0(\tau)\varphi\big(\Phi^{-1}(\alpha_0J_0(\tau))\big)\,d\tau<+\infty,\end{cases}\qquad t_1\in[a,\omega],$$

$$A_2=\begin{cases} t_1 & \text{if } \displaystyle\int\limits_{t_1}^{\omega}J_1(\tau)d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_{t_1}^{\omega}J_1(\tau)d\tau<+\infty.\end{cases}$$

Note that with the implementation of the properties with regular varying and rapid varying functions [1,11], as well as the results of work [4,8] for equation (1) conditions for the existence of solutions are established.

Theorem 1. *For the existence of $P_\omega(Y_0,1)$ – solutions of the differential equation* (1)*, it is necessary that the inequalities*

$$\alpha_0\nu_1>0,\quad \alpha_0\mu_0J_0(t)<0 \ \ as\ \ t\in\,]a,\omega[\,,\tag{5}$$

$$\nu_0\alpha_0<0 \ \ if\ \ Y_0=0,\quad \nu_0\alpha_0>0 \ \ if\ \ Y_0=\pm\infty\tag{6}$$

and the conditions

$$\frac{\alpha_0J_2(t)}{\Phi^{-1}(\alpha_0J_0(t))}\sim\frac{J_1(t)}{J_2(t)}\sim\frac{J_1'(t)}{J_1(t)}\sim\frac{(\Phi^{-1}(\alpha_0J_0(t)))'}{\Phi^{-1}(\alpha_0J_0(t))}\ \ as\ \ t\uparrow\omega,\tag{7}$$

$$\alpha_0\lim_{t\uparrow\omega}J_0(t)=Z_0,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)(\Phi^{-1}(\alpha_0J_0(t)))'}{\Phi^{-1}(\alpha_0J_0(t))}=\pm\infty,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_0'(t)}{J_0(t)}=\pm\infty\tag{8}$$

hold. Moreover, each such solution of that kind admits the asymptotic, as $t\uparrow\omega$, representations

$$y(t)=\Phi^{-1}(\alpha_0J_0(t))\Big[1+\frac{o(1)n}{H(t)}\Big],\tag{9}$$

$$y'(t)=\alpha_0(t)J_2(t)[1+o(1)],\quad y''(t)=\alpha_0J_1(t)[1+o(1)].\tag{10}$$

Theorem 2. *Let $p_0:[a,\omega[\,\to\,]0,+\infty[$ be a continuously differentiable function and along with* (5)–(8) *the conditions*

$$\lim_{t\uparrow\omega}\frac{q'(t)H^{\frac13}(t)J_2(t)}{J_2'(t)}=0,\quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2}\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^{\frac23}=0$$

hold. Then the differential equation (1) *in case* $\alpha_0\mu_0 > 0$ *has a two-parameter and in case* $\alpha_0\mu_0 < 0$ *has a one-parameter family of* $P_\omega(Y_0, 1)$ *– solutions that admit asymptotic, as* $t \uparrow \omega$*, representations* (9) *and moreover, their derivatives of the first and second order satisfy the asymptotic, as* $t \uparrow \omega$*, relations*

$$y'(t) = \alpha_0 J_2(t)\big[q(t) + o((H(t))^{-\frac{2}{3}})\big], \quad y''(t) = \alpha_0 J_1(t)\big[q(t) + o((H(t))^{-\frac{1}{3}})\big].$$

It is possible to notice that in the asymptotic relations (7)

$$\frac{(\Phi^{-1}(\alpha_0 J_0(t)))'}{\Phi^{-1}(\alpha_0 J_0(t))} = \alpha_0\Big(\frac{p_0(t)\varphi(\Phi^{-1}(\alpha_0 J_0(t)))}{\Phi^{-1}(\alpha_0 J_0(t))}\Big)^{\frac{1}{3}}.$$

Therefore, it follows from (7) that

$$J_2(t) = \Big(p_0(t)(\Phi^{-1}(\alpha_0 J_0(t)))^2\varphi(\Phi^{-1}(\alpha_0 J_0(t)))\Big)^{\frac{1}{3}}[1 + o(1)] \ \text{ as } \ t \uparrow \omega,$$
$$J_1(t) = \alpha_0(\Phi^{-1}(\alpha_0 J_0(t)))^{\frac{1}{3}}\Big(p_0(t)\varphi(\Phi^{-1}(\alpha_0 J_0(t)))\Big)^{\frac{2}{3}}[1 + o(1)] \ \text{ as } \ t \uparrow \omega.$$

These relations allow to rewrite the asymptotic relations (10) without integrals.

Theorem 3. *Let* $p_0 : [a, \omega[\to]0, +\infty[$ *be a continuous function and, along with* (5)–(8)*, the conditions*

$$\lim_{t\uparrow\omega}[1 - q(t)]H^{\frac{2}{3}}(t) = 0, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2}\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^{\frac{2}{3}} = 0$$

hold. Then the differential equation (1) *in case* $\alpha_0\mu_0 > 0$ *has a two-parameter family, and in case* $\alpha_0\mu_0 < 0$ *has a one-parameter family of* $P_\omega(Y_0, 1)$*– solutions, admitting as* $t \uparrow \omega$ *the asymptotic representations* (9) *and*

$$y'(t) = \alpha_0 J_2(t)\Big[1 + \frac{o(1)}{H^{\frac{2}{3}}(t)}\Big], \quad y''(t) = \alpha_0 J_1(t)\Big[1 + \frac{o(1)}{H^{\frac{1}{3}}(t)}\Big].$$

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On Some Estimates for the First Eigenvalue of a Sturm–Liouville Problem

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1 Introduction

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \ \ x \in (0,1), \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all measurable locally integrable on $(0,1)$ functions with non–negative values such that the following integral conditions hold:

$$\int_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx = 1, \ \ \gamma \neq 0, \tag{1.3}$$

$$\int_0^1 x(1-x)Q(x)\,dx < \infty. \tag{1.4}$$

A function y is a *solution to problem* (1.1), (1.2) if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

This work gives estimates for

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q) \ \text{ and } \ M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

Some of these results were obtained using approaches and ideas applied in works [1, 4–6].

In Theorem 1 [3], it was proved that if condition (1.4) does not hold, then for any $0 \le p \le \infty$, there is no non-trivial solution y of equation (1.1) with the properties $y(0) = 0$, $y'(0) = p$.

From the results of [4, Chapter 1, § 2, Theorem 3] it follows that $T_{\alpha,\beta,\gamma}$ is empty provided $\gamma < 0$, $\alpha \le 2\gamma - 1$ or $\beta \le 2\gamma - 1$, for other values α, β, γ, $\gamma \neq 0$, the set $T_{\alpha,\beta,\gamma}$ is not empty. Thus, for $\gamma < 0$, $\alpha \le 2\gamma - 1$ or $\beta \le 2\gamma - 1$, there is no function Q satisfying (1.3) and (1.4) taken together and, as a consequence, the first eigenvalue of problem (1.1), (1.2) does not exist.

Consider the functional

$$R[Q,y] = \frac{\int\limits_0^1 y'^2\,dx - \int\limits_0^1 Q(x)y^2\,dx}{\int\limits_0^1 y^2\,dx}\,.$$

If condition (1.4) is satisfied, then the functional $R[Q,y]$ is bounded below in $H_0^1(0,1)$. In order to show it, let us consider the set Γ_* of functions $y \in H_0^1(0,1)$ such that

$$\int_0^1 y^2\,dx = 1$$

and the functional

$$I[Q,y] = \int_0^1 y'^2\,dx - \int_0^1 Q(x)y^2\,dx.$$

For any $y \in H_0^1(0,1)$ and $x \in (0,1)$, by the Hölder inequality, we have

$$y^2(x) = \Big(\int_0^x y'(t)\,dt\Big)^2 \le x\int_0^x y'^2(t)\,dt,$$

$$y^2(x) = \Big(-\int_x^1 y'(t)\,dt\Big)^2 \le (1-x)\int_x^1 y'^2(t)\,dt.$$

Then

$$\frac{y^2}{x(1-x)} = \frac{y^2}{x} + \frac{y^2}{1-x} \le \int_0^x y'^2(t)\,dt + \int_x^1 y'^2(t)\,dt = \int_0^1 y'^2(t)\,dt$$

and

$$\int_0^1 Q(x)y^2 dx \le \Big(\int_0^1 y'^2\,dx\Big)\int_0^1 x(1-x)Q(x)\,dx.$$

For some positive k, consider

$$E_k = \big\{x \in [0,1] \mid \ Q(x) \le k\big\}, \quad \overline{E}_k = \big\{x \in [0,1] \mid \ Q(x) > k\big\}.$$

We have

$$\int_0^1 Q(x)y^2\,dx = \int_{E_k} Q(x)y^2\,dx + \int_{\overline{E}_k} Q(x)y^2\,dx \le k\int_0^1 y^2 dx + \int_0^1 y'^2\,dx \int_{\overline{E}_k} x(1-x)Q(x)\,dx.$$

Since the integral $\int_0^1 x(1-x)Q(x)\,dx$ is finite and the measure of $\overline{E}_k$ tends to 0 as $k \to \infty$, then $\int_{\overline{E}_k} x(1-x)Q(x)\,dx$ tends to 0 as $k \to \infty$ and we can choose $k = k_*$ so that

$$\int_{\overline{E}_{k_*}} x(1-x)Q(x)\,dx \le \frac{1}{2}\,.$$

Then

$$\int_0^1 Q(x)y^2 dx \le k_* + \frac{1}{2}\int_0^1 y'^2\,dx$$

and

$$\int_0^1 y'^2\,dx - \int_0^1 Q(x)y^2\,dx \geq \frac{1}{2}\int_0^1 y'^2\,dx - k_* \geq -k_*.$$

Thus, if condition (1.4) is satisfied, then for any $Q \in T_{\alpha,\beta,\gamma}$, $I[Q,y]$ is bounded below in Γ_*, $R[Q,y]$ is bounded below in $H_0^1(0,1)$, and

$$\inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y] = \inf_{y\in\Gamma_*} I[Q,y].$$

It was proved [3] that for any $Q \in T_{\alpha,\beta,\gamma}$,

$$\lambda_1(Q) = \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y].$$

For any $Q \in T_{\alpha,\beta,\gamma}$, we have

$$M_{\alpha,\beta,\gamma} = \sup_{Q\in T_{\alpha,\beta,\gamma}} \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y] \leq \inf_{y\in H_0^1(0,1)\setminus\{0\}} \frac{\int_0^1 y'^2\,dx}{\int_0^1 y^2\,dx} = \pi^2.$$

2 Main results

Theorem 2.1. *If $\gamma > 1$, $\alpha,\beta < 2\gamma-1$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_0^1(0,1)$, $u > 0$ on $(0,1)$, such that $m_{\alpha,\beta,\gamma} = R[Q_*,u]$. Moreover, u satisfies the equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{\gamma+1}{\gamma-1}} \tag{2.1}$$

and the integral condition

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2\gamma}{\gamma-1}}\,dx = 1. \tag{2.2}$$

Theorem 2.2.

(1) *If $\gamma = 1$, $\alpha,\beta \leqslant 0$, then $m_{\alpha,\beta,\gamma} \geqslant \frac{3}{4}\pi^2$.*

(2) *If $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$ or $\alpha \leqslant 0 < \beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

(3) *If $\gamma = 1$, $0 < \alpha,\beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

(4) *If $\gamma > 1$, $\alpha,\beta \leqslant \gamma$, then $m_{\alpha,\beta,\gamma} = 0$.*

(5) *If $\gamma \geqslant 1$, $\alpha > \gamma$ or $\beta > \gamma$, then $m_{\alpha,\beta,\gamma} < 0$.*

(6) *If $\gamma < 0$, $\alpha,\beta > 2\gamma-1$ or $0 < \gamma < 1$, $-\infty < \alpha,\beta < \infty$, then $m_{\alpha,\beta,\gamma} = -\infty$.*

Theorem 2.3.

(1) *If $\gamma > 1$, $-\infty < \alpha,\beta < \infty$ or $0 < \gamma \leq 1$, $\alpha \leq 2\gamma-1$, $-\infty < \beta < \infty$ ($\beta \leq 2\gamma-1$, $-\infty < \alpha < \infty$), then $M_{\alpha,\beta,\gamma} = \pi^2$.*

(2) *If $\gamma < 0$ or $0 < \gamma < 1$, $\alpha,\beta > 2\gamma-1$, then $M_{\alpha,\beta,\gamma} < \pi^2$.*

(3) *If $\gamma < -1$, $\alpha,\beta > 2\gamma-1$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_0^1(0,1)$, $u > 0$ on $(0,1)$, such that $M_{\alpha,\beta,\gamma} = R[Q_*,u]$. Moreover, u satisfies equation (2.1) and the integral condition (2.2).*

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Positive Solutions to Boundary Value Problems for Nonlinear Functional Differential Equations

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Consider a boundary value problem for a functional differential equation

$$u'(t) = \ell(u)(t) + \lambda F(u)(t) \ \text{ for a.e. } \ t \in [a,b], \quad h(u) = 0. \tag{1}$$

Here, $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ and $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ are linear bounded operators, $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a continuous operator satisfying the Carathéodory conditions, and $\lambda \in \mathbb{R}$ is a parameter. By a solution to the problem (1) we understand an absolutely continuous function $u : [a,b] \to \mathbb{R}$ that satisfies the equation in (1) almost everywhere on $[a,b]$ and satisfies the boundary condition in (1). We say that a solution u to (1) is positive if $u(t) > 0$ for $t \in [a,b]$.

Although the assumptions of the main results do not exclude the case when $F(0) \not\equiv 0$, the main importance of our results is that they are applicable in the case when the problem (1) possesses a trivial solution, i.e., $F(0)(t) = 0$ for a.e. $t \in [a,b]$.

Notation 1.

$\mathbb{N}$ is the set of all natural numbers, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ =]0,+\infty[\,$, $\mathbb{R}_0^+ = [0,+\infty[\,$.

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v : [a,b] \to \mathbb{R}$ with the norm $\|v\|_C = \max\{|v(t)| : \ t \in [a,b]\}$.

If $D \subset \mathbb{R}$, then $C_h([a,b];D) = \{u \in C([a,b];\mathbb{R}) : \ u(t) \in D \text{ for } t \in [a,b], \ h(u) = 0\}$.

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$.

If $D \subset \mathbb{R}$, then $L([a,b];D) = \{p \in L([a,b];\mathbb{R}) : \ p(t) \in D \text{ for a.e. } t \in [a,b]\}$.

If $A : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ is a linear bounded operator, by $\|A\|$ we denote the norm of A.

Definition 1. We say that a pair of operators (ℓ,h) belongs to the set $\mathcal{V}^+$ if every nontrivial absolutely continuous function $u : [a,b] \to \mathbb{R}$, satisfying

$$u'(t) \geq \ell(u)(t) \ \text{ for a.e. } \ t \in [a,b], \quad h(u) = 0, \tag{2}$$

admits the inequality

$$u(t) > 0 \ \text{ for } \ t \in [a,b].$$

Definition 2. We say that a pair of operators (ℓ, h) belongs to the set $\mathcal{U}^+$ if there exists $c > 0$ such that every absolutely continuous function $u : [a, b] \to \mathbb{R}$ satisfying (2) admits the inequality

$$u(t) \geq c \int_a^b [u'(s) - \ell(u)(s)]\, ds \ \text{ for } \ t \in [a, b].$$

It can be easily seen that if $(\ell, h) \in \mathcal{V}^+$, resp. $(\ell, h) \in \mathcal{U}^+$, then the homogeneous problem

$$u'(t) = \ell(u)(t) \ \text{ for a.e. } \ t \in [a, b], \quad h(u) = 0 \tag{3}$$

has only the trivial solution. Note also that $\mathcal{U}^+ \subseteq \mathcal{V}^+$. However, $\mathcal{U}^+ \neq \mathcal{V}^+$ in general.

Now we formulate some of the assumptions of the main results.

(H.1) F transforms $C_h([a, b]; \mathbb{R}_0^+)$ into $L([a, b]; \mathbb{R}_0^+)$ and it is not the zero operator, i.e., there exists $x_0 \in C_h([a, b]; \mathbb{R}_+)$ such that

$$\int_a^b F(x_0)(s)\, ds > 0.$$

(H.2) F is sublinear with respect to $C_h([a, b]; \mathbb{R}_0^+)$, i.e., there exists a Carathéodory function $\eta : [a, b] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ non-decreasing in the second variable such that

$$F(v)(t) \leq \eta(t, \|v\|_C) \ \text{ for a.e. } \ t \in [a, b], \ \ v \in C_h([a, b]; \mathbb{R}_0^+)$$

and

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b \eta(s, x)\, ds = 0.$$

(H.3) F is nondecreasing in the neighbourhood of zero, i.e., for every $\rho > 0$ there exists $m_\rho \in C_h([a, b]; \mathbb{R}_+)$ such that $m_\rho(t) \leq \rho$ for $t \in [a, b]$ and

$$F(y)(t) \leq F(x)(t) \ \text{ for a.e. } \ t \in [a, b]$$

whenever $x, y \in C_h([a, b]; \mathbb{R}_+)$,

$$y(t) \leq m_\rho(t) \ \text{ and } \ y(t) \leq x(t) \leq \rho \ \text{ for } \ t \in [a, b].$$

(H.4) F is concave in the neighbourhood of zero, i.e., for every $x \in C_h([a, b]; \mathbb{R}_+)$ there exists $\mu_x > 0$ such that

$$\mu F(x)(t) \leq F(\mu x)(t) \ \text{ for a.e. } \ t \in [a, b], \ \ \mu \in \,]0, \mu_x[\,.$$

Notation 2. Let $\lambda \in \mathbb{R}$. Then by $\mathcal{S}(\lambda)$ we denote the set of all positive solutions to (1) for corresponding λ.

Theorem 1. *Let* $(\ell, h) \in \mathcal{V}^+$ *and let* (H.1)–(H.4) *be fulfilled. Then there exists a critical parameter* $\lambda_c \geq 0$ *such that*

(i) *the problem* (1) *has a positive solution provided* $\lambda > \lambda_c$*;*

(ii) *the problem* (1) *has no positive solution provided* $\lambda < \lambda_c$*.*

Moreover,

$$\lim_{\lambda\to+\infty}\inf\big\{\|u\|_C:\ u\in\mathcal{S}(\lambda)\big\}=+\infty.$$

If, in addition, $(\ell,h)\in\mathcal{U}^+$, *then for every* $\rho>0$ *there exists* $\lambda(\rho)>\lambda_c$ *such that*

$$u(t)>\rho \ \text{ for } \ t\in[a,b], \ \ u\in\mathcal{S}(\lambda), \ \ \lambda>\lambda(\rho).$$

As for the critical case $\lambda=\lambda_c$, the existence or nonexistence of a positive solution to (1) depends on the properties of the operator F; both cases can occur. If we slightly strengthen the assumption (H.4), in particular, if we assume

(H.4′) For every $x\in C_h([a,b];\mathbb{R}_+)$ there exists $\mu_x>0$ such that

$$\mu F(x)(t)\le F(\mu x)(t) \ \text{ for a.e. } \ t\in[a,b], \ \ \mu\in\,]0,\mu_x[$$

and

$$\mu_0\int_a^b F(x)(s)\,ds<\int_a^b F(\mu_0x)(s)\,ds$$

for some $\mu_0\in\,]0,\mu_x[$

instead, we can establish a result about the nonexistence of a positive solution to (1) with $\lambda=\lambda_c$.

Theorem 2. *Let* $(\ell,h)\in\mathcal{V}^+$ *and let* (H.1)–(H.3), *and* (H.4′) *be fulfilled. Then* $\mathcal{S}(\lambda_c)=\varnothing$ *and*

$$\lim_{\lambda\to\lambda_c^+}\sup\big\{\|u\|_C:\ u\in\mathcal{S}(\lambda)\big\}=0.$$

Suppose that the operator F includes a linear part, i.e.,

$$F(v)(t)=\widetilde{F}(v,v)(t) \ \text{ for a.e. } \ t\in[a,b], \ \ v\in C([a,b];\mathbb{R}),$$

where $\widetilde{F}:C([a,b];\mathbb{R})\times C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ is a continuous operator satisfying the Carathéodory conditions and it is linear and nondecreasing in the first variable. Therefore, instead of (1) we consider the problem

$$u'(t)=\ell(u)(t)+\lambda\widetilde{F}(u,u)(t) \ \text{ for a.e. } \ t\in[a,b], \quad h(u)=0, \tag{4}$$

where ℓ and λ are the same as in (1) and $\widetilde{F}$ is described above. The set of all positive solutions to (4) we denote again by $\mathcal{S}(\lambda)$ as (4) is a particular case of (1).

Theorem 3. *Let* $(\ell,h)\in\mathcal{V}^+$ *and let* (H.1)–(H.3) *and* (H.4′) *be fulfilled. Then* $\lambda_c>0$, *the problem*

$$u'(t)=\ell(u)(t)+\lambda_c\widetilde{F}(u,0)(t) \ \text{ for a.e. } \ t\in[a,b], \quad h(u)=0 \tag{5}$$

has a positive solution u_c, *the set of solutions to* (5) *is one-dimensional (generated by* u_c*), and*

$$(T_\lambda,h)\in\mathcal{V}^+, \ \ \text{resp.} \ \ (T_\lambda,h)\in\mathcal{U}^+, \ \ \text{for} \ \ \lambda\in\,]0,\lambda_c[\,,$$

where

$$T_\lambda(v)(t)\overset{def}{=}\ell(v)(t)+\lambda\widetilde{F}(v,0)(t) \ \text{ for a.e. } \ t\in[a,b], \ \ v\in C([a,b];\mathbb{R}),$$

provided $(\ell,h)\in\mathcal{V}^+$, *resp.* $(\ell,h)\in\mathcal{U}^+$.

Theorem 3 gives us a method how to calculate the precise value of λ_c in the cases where F includes a linear part. Indeed, define an operator $A : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ by

$$A(x)(t) \stackrel{def}{=} \int_a^b G(t,s)\widetilde{F}(x,0)(s)\,ds \ \text{ for } \ t \in [a,b], \ \ x \in C([a,b];\mathbb{R}),$$

where G is Green's function to (3). Then

$$u_c(t) = \lambda_c A(u_c)(t) \ \text{ for } \ t \in [a,b],$$

i.e., $1/\lambda_c$ is the first eigenvalue to A corresponding to the positive eigenfunction u_c. Therefore, according to Krasnoselski's theory and Gelfand's formula,

$$\lambda_c = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{\|A^n\|}}\,.$$

Application

Most of population models with a delayed harvesting term can be represented as an equation

$$u'(t) = -\delta(t)u(t) - H(t)u(t-\sigma(t)) + \lambda \sum_{k=1}^{N} P_k(t)u(t-\tau_k(t))f_k(u(t-\tau_k(t))), \tag{6}$$

where $N \in \mathbb{N}$,

(A.1) (i) $\delta, H, P_k : \mathbb{R} \to \mathbb{R}_0^+$ $(k = 1,\dots,N)$ are T-periodic locally integrable functions,

$$\int_0^T \big[\delta(s) + H(s)\big]\,ds > 0, \quad \int_0^T \sum_{k=1}^{N} P_k(s)\,ds > 0,$$

(ii) $\sigma : \mathbb{R} \to [0,\sigma_*]$, $\tau_k : \mathbb{R} \to [0,\tau_*]$ $(k = 1,\dots,N)$ are T-periodic locally measurable functions (σ_* and τ_* are non-negative constants),

(iii) $f_k : \mathbb{R}_0^+ \to \mathbb{R}_+$ $(k = 1,\dots,N)$ are continuous decreasing functions that are continuously differentiable at some neighbourhood of zero, and

$$\lim_{x\to+\infty} f_k(x) = 0 \ \ (k = 1,\dots,N).$$

By a T-periodic solution to (6) we understand a T-periodic locally absolutely continuous function defined on $\mathbb{R}$ and satisfying the equality (6) for almost every $t \in \mathbb{R}$.

Theorem 4. *Let* (A.1) *be fulfilled, and let there exist $\gamma : \mathbb{R} \to \mathbb{R}_+$ that is locally absolutely continuous such that*

$$\gamma'(t) \leq -\delta(t)\gamma(t) - H(t)\gamma(t-\sigma(t)) \ \text{ for a.e. } \ t \in \mathbb{R}. \tag{7}$$

Then $(\ell, h) \in \mathcal{U}^+$ and F satisfies (H.1)–(H.3) *and* (H.4′) *with $\mu_x = 1$ for all $x \in C([0,T];\mathbb{R}_+)$, where the operators $\ell, F : C([0,T];\mathbb{R}) \to L([0,T];\mathbb{R})$ and $h : C([0,T];\mathbb{R}) \to \mathbb{R}$ are defined by*

$$\ell(v)(t) \stackrel{def}{=} -\delta(t)v(t) - H(t)v(\sigma_0(t)), \quad h(v) \stackrel{def}{=} v(0) - v(T) \ \text{ for } \ v \in C([0,T];\mathbb{R}),$$

$$F(v)(t) \overset{def}{=} \sum_{k=1}^{N} P_k(t)v(\tau_{0k}(t))f_k(v(\tau_{0k}(t))) \ \ for \ \ v \in C([0,T];\mathbb{R}),$$

$$\sigma_0(t) \overset{def}{=} t - \sigma(t) + \left\lfloor \frac{T-(t-\sigma(t))}{T} \right\rfloor T \ \ for \ a.e. \ \ t \in [0,T],$$

$$\tau_{0k}(t) \overset{def}{=} t - \tau_k(t) + \left\lfloor \frac{T-(t-\tau_k(t))}{T} \right\rfloor T \ \ for \ a.e. \ \ t \in [0,T] \ \ (k=1,\dots,N).$$

One of the efficient conditions guaranteeing the existence of a positive γ satisfying (7) is

$$\int\limits_{t-\sigma(t)}^{t} H(s)\exp\bigg(\int\limits_{s-\sigma(s)}^{s} \delta(\xi)\,d\xi\bigg)\,ds \le \frac{1}{e} \ \ \text{for a.e.} \ \ t \in [0,T].$$

Another conditions guaranteeing the inclusion $(\ell,h) \in \mathcal{U}^+$ are

$$\text{either} \ \int\limits_{0}^{T} \big[\delta(s) + H(s)\big]\,ds < 1 \ \ \text{or} \ \ \int\limits_{0}^{T} H(s)\exp\bigg(\int\limits_{s-\sigma(s)}^{s} \delta(\xi)\,d\xi\bigg)\,ds < 1$$

provided (A.1)(i) is fulfilled.

Anti-Perron's Effect of Changing Linear Exponentially Decreasing Perturbations of All Positive Characteristic Exponents of the First Linear Approximation to Negative Ones

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Consider the linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq t_0, \tag{1}$$

with bounded infinitely differentiable coefficients and positive characteristic exponents $\lambda_n(A) \geq \cdots \geq \lambda_1(A) > 0$, as well as the perturbed systems

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq t_0, \tag{2}$$

with infinitely differentiable exponentially decreasing perturbation $n \times n$-matrices Q satisfying the estimate

$$\|Q(t)\| \leq C_Q e^{-\sigma t}, \ \ \sigma > 0, \ \ C_Q = const, \ \ t \geq t_0. \tag{3}$$

There arises the question on the existence of such, for example, two-dimensional system (1) and perturbation (3) that the perturbed system (2) has a nontrivial solution with a negative Lyapunov exponent. The solution to this (first) problem may serve as a preliminary step in solving the more important (second) problem about the existence of nontrivial solutions with negative exponents of a nonlinear differential system

$$\dot{y} = A(t)y + f(t, y), \ \ y \in \mathbb{R}^n, \ \ t \geq t_0, \tag{4}$$

with an infinitely differentiable m-perturbation $f(t, y)$:

$$\|f(t, y)\| \leq C_f \|y\|^m, \ \ y \in \mathbb{R}^n, \ \ C_f = const, \ \ t \geq t_0$$

of $m > 1$ order of smallness in the nighbourhood of the origin $y = 0$ and admissible growth outside it in the "anti-Perron" case when all characteristic exponents of linear approximation (1) are positive. Indeed, according to the principle of linear inclusion [1, p. 159], any solution $y_0(t) \neq 0$ of system (4), infinitely extendably to the right, with a negative exponent, is likewise a solution of system (2) with exponentially decresing perturbation $Q_{y_0}(t)$, satisfying the condition

$$\|Q_{y_0}(t)\| \leq C_f \|y_0(t)\|^{m-1}, \ \ t \geq t_0.$$

Therefore, in the case of admissible negative solution of the first problem there follows the same solution of the second problem.

Note that in the Perron effect ([5], [4, pp. 50-51]) of changing the values of negative characteristic exponents of system (1) by positive exponents of solutions of system (4) we have obtained in [4] and [5] a finale complete description of sets of all positive and all negative (including those in the absence of the latter) exponents of solutions of system (4) for which all nontrivial solutions are infinitely extendable to the right and have bounded finite exponents.

The present paper is devoted to the positive solution of the first problem.

Theorem 1. *For any parameters* $\lambda_2 \geq \lambda_1 > 0$, $\theta > 1$ *and* $\sigma \in (0, \lambda_1 + \theta^{-1}\lambda_2)$ *there exist:*

1) *the two-dimensional linear system* (1) *with bounded infinitely differentiable coefficients and characteristic exponents* $\lambda_i(A) = \lambda_i$, $i = 1, 2$*;*

2) *the infinitely differentiable exponentially decreasing and satisfying estimate* (3) *perturbation* $Q(t)$

such that the perturbed linear system (2) *has a unique (among all its linear independent) solution* $y(t)$ *with a negative Lyapunov exponent, equal to*

$$\lambda_0 = \frac{\theta\sigma - \theta\lambda_1 - \lambda_2}{\theta - 1}.$$

There likewise arises the question on a possible number of linearly independent solutions with negative Lyapunov exponents for the n-dimensional linear perturbed system (2) in which the first approximation system (1) has all positive characteristic exponents, and the perturbation $Q(t)$ is exponentially decreasing.

The following theorem is valid.

Theorem 2. *For any parameters*

$$\lambda_n \geq \cdots \geq \lambda_2 \geq \lambda_1 > 0, \ \ n \geq 3, \ \ \theta > 1, \ \ 0 < \sigma < \lambda_1 + \theta^{-1}\lambda_2$$

there exist:

1) *the* n*-dimensional system* (1) *with bounded infinitely differentiable coefficients and characteristic exponents* $\lambda_i(A) = \lambda_i$, $i = 1, \dots, n$*;*

2) *the infinitely differentiable exponentially decreasing and satisfying estimate* (3) *perturbation* $Q(t)$

such that the n*-dimensional perturbed system* (2) *has exactly* $n-1$ *linear independent solutions*

$$Y_1(t), \dots, Y_{n-1}(t)$$

with negative exponents

$$\lambda[Y_i] = \frac{\sigma\theta - \theta\lambda_1 - \lambda_{i+1}}{\theta - 1} \equiv \Lambda_i, \ \ i = 1, \dots, n-1.$$

Proof of Theorem 2 is based on the statement of Theorem 1 and its proof.

Remark. Is the statement:

$$\text{if } \ \lambda_i(A) > 0, \ \ i = 1, \dots, n, \ \ \text{then } \ \lambda_n(A + Q) > 0$$

valid for any piecewise continuous bounded $n \times n$-matrix $A(t)$ and exponentially decreasing $n \times n$-perturbation $Q(t)$?

Acknowledgement

The present work is carried out under the financial support of Belarus Republican (project # Φ20P-005) and Russian (project # 20-57-00001Bel_a) Funds of Fundamental Researches.

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On One System of Nonlinear Degenerate Integro-Differential Equations of Parabolic Type

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The integro-differential equations are applied in many branches of science, such as physics, engineering, biochemistry, etc. A lot of scientific works are dedicated to the investigation and numerical resolution of integro-differential models (see, for example, [2, 7, 11, 13, 16, 18] and the references therein).

One type of nonlinear integro-differential parabolic model is obtained at the mathematical simulation of processes of electromagnetic field penetration into a substance. Based on Maxwell system [14], the mentioned model at first appeared in [3]. The integro-differential system obtained in [8] describes many other processes as well (see, for example, [7, 11] and the references therein). Equations and systems of such types still yield to the investigation for special cases. In this direction the latest and rather complete bibliography can be found in the following monographs [7, 11].

The purpose of this note is to analyze degenerate one-dimensional case of such type models. Unique solvability and convergence of the constructed semi-discrete scheme with respect to the spatial derivative and fully discrete finite difference scheme are studied.

The investigated problem has the following form. In the rectangle $Q=(0,1)\times(0,T]$, where T is a fixed positive constant, we consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\bigg\{\bigg[\int\limits_0^t\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau+\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\bigg]\,\frac{\partial U}{\partial x}\bigg\}=f(x,t), \tag{1}$$

$$\frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\bigg\{\bigg[\int\limits_0^t\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau+\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\bigg]\,\frac{\partial V}{\partial x}\bigg\}=g(x,t), \tag{2}$$

$$U(0,t)=U(1,t)=V(0,t)=V(1,t)=0,\ \ t\in[0,T], \tag{3}$$

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x),\ \ x\in[0,1]. \tag{4}$$

Here $f=f(x,t)$, $g=g(x,t)$, $U_0=U_0(x)$, $V_0=V_0(x)$ are given functions of their arguments and $U=U(x,t)$, $V=V(x,t)$ are unknown functions.

It is necessary to mention that (1), (2) is a degenerate type parabolic system with integro-differential and p-Laplacian ($p=4$) terms. Let us note that non-degenerate variants of (1)–(4) type problem for more general nonlinearities are studied in [6]. Many works are devoted to the investigation of multi-dimensional cases of such type equations and systems as well (see, for example, [1, 4, 7, 9–12, 15] and the references therein). We would also like to note that in recent years special attention has been paid to the construction and investigation of splitting models for this type and their generalized variants of multi-dimensional integro-differential equations (see, for example, [7, 9, 10] and the references therein).

As it was already mentioned, (1), (2) type models arise, on the one hand, when solving real applied problems, and on the other hand, as a natural generalization of some nonlinear parabolic equations and systems studied for example, in [16, 17] and in many other works as well.

Problems of (1)–(4) type at first were studied in [1], where the monotonicity of the considered operator is proved and the unique solvability is obtained.

Applying one modification of compactness method developed in [17] (see also [16]) the following uniqueness and existence statement takes place.

Theorem 1. *If $f, g \in W_2^1(Q)$, $f(x,0) = g(x,0) = 0$, $U_0, V_0 \in \overset{\circ}{W}{}_2^1(0,1)$, then there exists the unique solution U, V of problem* (1)–(4) *satisfying the following properties:*

$$U, V \in L_4(0,T; \overset{\circ}{W}{}_4^1(0,1) \cap W_2^2(0,1)), \quad \frac{\partial U}{\partial t}, \frac{\partial V}{\partial t} \in L_2(Q), \quad \sqrt{T-t}\,\frac{\partial^2 U}{\partial t}, \sqrt{T-t}\,\frac{\partial^2 V}{\partial t} \in L_2(Q).$$

Here usual well-known spaces are used.

In order to describe the space-discretization for problem (1)–(4), let us introduce nets: $\omega_h = \{x_i = ih,\ i = 1, 2, \dots, M-1\}$, $\overline{\omega}_h = \{x_i = ih,\ i = 0, 1, \dots, M\}$ with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at (x_i, t) is designed by $u_i = u_i(t)$, $v_i = v_i(t)$. The exact solution of problem (1)–(4) at point (x_i, t) is denoted by $U_i = U_i(t)$, $V_i = V_i(t)$.

Approximating the space derivatives by a forward and backward differences:

$$w_{x,i} = \frac{w_{i+1} - w_i}{h}, \quad w_{\overline{x},i} = \frac{u_i - w_{i-1}}{h},$$

let us correspond the following semi-discrete scheme to problem (1)–(4):

$$\frac{du_i}{dt} - \left\{ \left[\int_0^t \left[(u_{\overline{x},i})^2 + (v_{\overline{x},i})^2 \right] d\tau + (u_{\overline{x},i})^2 + (v_{\overline{x},i})^2 \right] u_{\overline{x},i} \right\}_{x,i} = f(x_i, t), \quad i = 1, \dots, M-1, \tag{5}$$

$$\frac{dv_i}{dt} - \left\{ \left[\int_0^t \left[(u_{\overline{x},i})^2 + (v_{\overline{x},i})^2 \right] d\tau + (u_{\overline{x},i})^2 + (v_{\overline{x},i})^2 \right] v_{\overline{x},i} \right\}_{x,i} = g(x_i, t), \quad i = 1, \dots, M-1, \tag{6}$$

$$u_0(t) = u_M(t) = v_0(t) = v_M(t) = 0, \quad t \in [0, T], \tag{7}$$

$$u_i(0) = U_{0,i}, \quad i = 0, 1, \dots, M, \tag{8}$$

which approximates problem (1)–(4) on smooth solutions with the first order of accuracy with respect to spatial step h.

The semi-discrete scheme (5)–(8) represents a Cauchy problem for nonlinear system of ordinary integro-differential equations. It is stable with respect to initial data and right-hand side of equations (5), (6) in the norm

$$\|w\|_h = (w, w)_h^{1/2}, \quad (w, z)_h = \sum_{i=1}^{M-1} w_i z_i h.$$

It is not difficult to obtain the following estimate for (5)–(8)

$$\|u\|_h^2 + \|v\|_h^2 + \int_0^t \left[||u_{\overline{x}}|_h^2 + ||v_{\overline{x}}|_h^2 \right] d\tau < C,$$

where the norm under the integral is defined as follows

$$||w]|_h^2 = (w,w]_h = \sum_{i=1}^{M} w_i w_i h.$$

Here C denotes the positive constant independent of the mesh parameter h. This estimate gives the above-mentioned stability as well as the global existence of a solution to problem (5)–(8).

In Theorems 2 and 3, using an approach of the work [5] for investigation the finite-difference scheme, the convergence of the approximate solutions are stated.

For earlier work on discretization in time or space, or both, of models such as (1), (2), see, e.g., [5–12].

The following statement takes place.

Theorem 2. *The solution*

$$u(t) = \big(u_1(t), u_2(t), \dots, u_{M-1}(t)\big), \quad v(t) = \big(v_1(t), v_2(t), \dots, v_{M-1}(t)\big)$$

of the semi-discrete scheme (5)–(8) *converges to the solution of problem* (1)–(4)

$$U(t) = \big(U_1(t), U_2(t), \dots, U_{M-1}(t)\big), \quad V(t) = \big(V_1(t), V_2(t), \dots, V_{M-1}(t)\big)$$

in the norm $\|\cdot\|_h$ *as* $h \to 0$.

In order to describe the fully discrete analog of problem (1)–(4), let us construct grid on the rectangle $\overline{Q}$. For using the time-discretization in equations (1), (2) the net is introduced as follows $\omega_\tau = \{t_j = j\tau,\ j = 0,1,\dots,J\}$, with $\tau = T/J$ and $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau$, $u_i^j = u(x_i, t_j)$.

Let us correspond the following implicit finite difference scheme to problem (1)–(4), where the terms with time derivatives in (5), (6) are approximated using the forward finite difference formula:

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \bigg\{\Big[\tau \sum_{k=1}^{j+1} \big[(u_{\overline{x},i}^k)^2 + (v_{\overline{x},i}^k)^2\big] + (u_{\overline{x},i}^{j+1})^2 + (v_{\overline{x},i}^{j+1})^2\Big] u_{\overline{x},i}^{j+1}\bigg\}_{x,i} = f_i^{j+1}, \tag{9}$$

$$\frac{v_i^{j+1} - v_i^j}{\tau} - \bigg\{\Big[\tau \sum_{k=1}^{j+1} \big[(u_{\overline{x},i}^k)^2 + (v_{\overline{x},i}^k)^2\big] + (u_{\overline{x},i}^{j+1})^2 + (v_{\overline{x},i}^{j+1})^2\Big] v_{\overline{x},i}^{j+1}\bigg\}_{x,i} = g_i^{j+1}, \tag{10}$$

$$i = 1,2,\dots,M-1, \quad j = 0,1,\dots,J-1;$$

$$u_0^j = u_M^j = v_0^j = v_M^j = 0, \quad j = 0,1,\dots,J, \tag{11}$$

$$u_i^0 = U_{0,i}, \quad v_i^0 = V_{0,i}, \quad i = 0,1,\dots,M. \tag{12}$$

Thus, the system of nonlinear algebraic equations (9)–(12) is obtained, which approximates problem (1)–(4) on sufficiently smooth solution with the first order of accuracy with respect to time and spatial steps τ and h.

The following estimate can be obtained easily for the solution of the finite difference scheme (9)–(12)

$$\max_{0 \le j\tau \le T} \big(\|u^j\|_h^2 + \|v^j\|_h^2\big) + \sum_{k=1}^{J} \big(\, ||u_{\overline{x}}^k]|_h^2 + ||v_{\overline{x}}^k]|_h^2\,\big)\tau < C,$$

which guarantees the stability and solvability of scheme (9)–(12). It is proved also that system (9)–(12) has a unique solution. Here C represents positive constant independent from time and spatial steps τ and h.

The following main conclusion is valid for scheme (9)–(12).

Theorem 3. *The solution*

$$u^j=\big(u_1^j,u_2^j,\dots,u_{M-1}^j\big),\quad v^j=\big(v_1^j,v_2^j,\dots,v_{M-1}^j\big),\ \ j=1,2,\dots,J$$

of the difference scheme (9)–(12) *converges to the solution*

$$U^j=\big(U_1^j,U_2^j,\dots,U_{M-1}^j\big),\quad V^j=\big(V_1^j,V_2^j,\dots,V_{M-1}^j\big),\ \ j=1,2,\dots,J$$

of problem (1)–(4) *in the norm* $\|\cdot\|_h$ *as* $\tau\to 0$ *and* $h\to 0$*.*

Note that for solving the difference scheme (9)–(12) Newton iterative process is used. Various numerical experiments are done. These experiments agree with theoretical research.

Statements such Theorems 1–3 for (1), (2) type equation are stated in [8]. As it was mentioned in [8], it is very interesting to looking for assumptions on the data of the considered problem (1)–(4) that provide the regularity for the solution $U(x,t)$, $V(x,t)$, which is required for obtaining rates of convergence in Theorems 2 and 3 as well as the optimal rates of convergence. It is important also to study more general nonlinearities for such kind degenerate and non-degenerate equations and systems.

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Oscillatory Properties of Solutions of Second Order Half-Linear Differential Equations

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1 Introduction

Consider the second order half-linear differential equation

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{HL}$$

where α is a positive constant, $p(t)$ and $q(t)$ are positive, continuously differentiable functions on $[a,\infty)$, $a \geqq 0$, and $\varphi_\gamma : \mathbb{R} \to \mathbb{R}$ denotes the odd function defined by

$$\varphi_\gamma(u) = |u|^\gamma \operatorname{sgn} u = |u|^{\gamma-1}u, \quad u \in \mathbb{R}, \ \ \gamma > 0.$$

It is known that all proper solutions of (HL) are either oscillatory, in which case equation (HL) itself is called oscillatory, or else nonoscillatory, in which case (HL) itself is called nonoscillatory. Our attention will be focused on oscillatory equations of the form (HL).

Let $x(t)$ be an oscillatory solution of (HL) existing on $[a,\infty)$. We denote by $\{\sigma_k\}_{k=1}^\infty$ $(\sigma_k < \sigma_{k+1})$ the sequence of zeros of $x(t)$, and by $\{\tau_k\}_{k=1}^\infty$ $(\tau_k < \tau_{k+1})$ the sequence of points at which $x(t)$ takes on extrema (i.e. local maxima or minima). Naturally, $x(\sigma_k) = 0$ and $x'(\tau_k) = 0$ for all k. The values $|x'(\sigma_k)|$ and $|x(\tau_k)|$ are referred to as the *slope* and *amplitude*, respectively, of the k-th wave of $x(t)$. We use the following notations:

$$\mathcal{A}^*[x] = \sup_k |x(\tau_k)|, \quad \mathcal{A}_*[x] = \inf_k |x(\tau_k)|, \quad \mathcal{S}^*[x] = \sup_k |x'(\sigma_k)|, \quad \mathcal{S}_*[x] = \inf_k |x'(\sigma_k)|.$$

An oscillatory solution $x(t)$ of (HL) is bounded if $\mathcal{A}^*[x] < \infty$, and unbounded if $\mathcal{A}^*[x] = \infty$. Two cases are possible for a bounded oscillatory solution: either $\lim\limits_{k\to\infty} |x(\tau_k)| = 0$ which is equivalent to $\lim\limits_{t\to\infty} x(t) = 0$, or $\liminf\limits_{k\to\infty} |x(\tau_k)| > 0$ which amounts to $\mathcal{A}_*[x] > 0$. In the former case $x(t)$ is called a *decaying oscillatory solution*, while in the latter case $x(t)$ is called an *non-decaying oscillatory solution* of (HL).

Recently, Kusano and Yoshida [1] have shown the existence and the qualitative properties, i.e., "amplitudes" and "slopes", of oscillatory solutions $x(t)$ of the linear differential equation

$$(p(t)x')' + q(t)x = 0, \ \ t \geqq a. \tag{L}$$

The purpose of this paper is to report to the QUALITDE – 2021 that some of their results can be extended to half-linear differential equations of the form (HL).

2 Main results

Our first result concerns the estimation of $\mathcal{A}^*[x]$ and $\mathcal{A}_*[x]$.

Theorem 2.1. *Let* (HL) *be oscillatory and let $x(t)$ be a solution of it satisfying the initial condition*

$$x(a) = l, \quad x'(a) = m, \tag{2.1}$$

where l and m are any given constants such that $(l,m) \neq (0,0)$.

(i) *Suppose that $p'(t) \geqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(\infty)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ q(\infty) > 0, \tag{2.2}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ p(\infty) < \infty. \tag{2.3}$$

(ii) *Suppose that $p'(t) \leqq 0$ and $q'(t) \geqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ p(\infty) > 0, \tag{2.4}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(\infty)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ q(\infty) < \infty. \tag{2.5}$$

(iii) *Suppose that $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(a)}\Big]^{\frac{1}{\alpha+1}}, \tag{2.6}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(\infty)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ p(\infty)^{\frac{1}{\alpha}} q(\infty) < \infty. \tag{2.7}$$

(iv) *Suppose that $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(\infty)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0, \tag{2.8}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(a)}\Big]^{\frac{1}{\alpha+1}}. \tag{2.9}$$

Since the constants l and m in (2.1) are arbitrary, the above inequalities (2.2)–(2.9) guarantee under the indicated conditions on $p(\infty)$ and/or $q(\infty)$ that $\mathcal{A}^*[x] < \infty$ and/or $\mathcal{A}_*[x] > 0$ for all solutions $x(t)$ of (HL). Then, $\mathcal{A}^*[x] < \infty$ gives the boundedness of $x(t)$ on $[a,\infty)$ and $\mathcal{A}^*[x] < \infty$ and $\mathcal{A}_*[x] > 0$ imply the non-decaying boundedness of $x(t)$ on $[a,\infty)$.

Corollary 2.1. *Suppose that* (HL) *is oscillatory. All of its solutions are bounded on* $[a,\infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions:*

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$ *for* $t \geqq a$ *and* $q(\infty) > 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$ *for* $t \geqq a$ *and* $p(\infty) > 0$;

(iii) $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ *for* $t \geqq a$;

(iv) $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0$.

Corollary 2.2. *Supposet that* (HL) *is oscillatory. All of its solutions are non-decaying bounded on* $[a,\infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions:*

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$ *for* $t \geqq a$ *and* $p(\infty) < \infty$, $q(\infty) > 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$ *for* $t \geqq a$ *and* $p(\infty) > 0$, $q(\infty) < \infty$;

(iii) $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) < \infty$;

(iv) $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0$.

The estimation of $\mathcal{S}^*[x]$ and $\mathcal{S}_*[x]$ are given in the following

Theorem 2.2. *Let* (HL) *be oscillatory and let* $x(t)$ *be a solution of it satisfying* (2.1).

(i) *Suppose that* $p'(t) \geqq 0$ *and* $q'(t) \leqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}},$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) < \infty \text{ and } q(\infty) > 0.$$

(ii) *Suppose that* $p'(t) \leqq 0$ *and* $q'(t) \geqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) > 0 \text{ and } q(\infty) < \infty,$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}}.$$

(iii) *Suppose that* $p'(t) \geqq 0$ *and* $q'(t) \geqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(a) q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } q(\infty) < \infty,$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}} \Big\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\Big\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) < \infty.$$

(iv) *Suppose that* $p'(t) \leqq 0$ *and* $q'(t) \leqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}} \{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) > 0,$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(a) q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } q(\infty) > 0.$$

Corollary 2.3. *Let (HL) be oscillatory. If* $p(t)$ *and* $q(t)$ *are monotone functions such that* $0 < p(\infty) < \infty$ *and* $0 < q(\infty) < \infty$, *then* $\mathcal{S}^*[x] < \infty$ *and* $\mathcal{S}_*[x] > 0$ *for all solutions* $x(t)$ *of* (HL).

3 Example

Example. Consider the half-linear differential equation

$$\big((\coth(t+\tau))^\alpha \varphi_\alpha(x')\big)' + k\tanh(t+\tau)\varphi_\alpha(x) = 0 \tag{3.1}$$

on $[0,\infty)$, where $\tau \geqq 0$ and $k > 0$ are constants. Equation (3.1) is oscillatory since the functions $p(t) = (\coth(t+\tau))^\alpha$ and $q(t) = k\tanh(t+\tau)$ are not integrable on $[0,\infty)$. It is clear that $p(t)$ and $q(t)$ satisfy $p'(t) \leqq 0$, $q'(t) \geqq 0$, $(p(t)^{\frac{1}{\alpha}} q(t))' = 0$, $p(0) = (\coth\tau)^\alpha$, $p(\infty) = 1$, $q(0) = k\tanh\tau$ and $q(\infty) = k$, all nontrivial solutions of equation (3.1) are bounded and non-decaying by (ii) and (iii) of Corollary 2.2. As regards the estimates for upper and lower amplitudes and upper and lower slopes of solutions of (3.1), we obtain, for example,

$$\mathcal{A}^*[x] \leqq \Big[\coth\tau\,|l|^{\alpha+1} + \frac{\alpha}{k}\,(\coth\tau)^{\alpha+2}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}},$$
$$\mathcal{A}_*[x] \geqq \Big[\tanh\tau\,|l|^{\alpha+1} + \frac{\alpha}{k}\,(\coth\tau)^{\alpha}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}}$$

from (ii) of Theorem 2.1, and

$$\mathcal{S}^*[x] \leqq \Big[\frac{k}{\alpha}\,\coth\tau|l|^{\alpha+1} + (\coth\tau)^{\alpha+2}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}},$$
$$\mathcal{S}_*[x] \geqq \Big[\frac{k}{\alpha}\,(\tanh\tau)^{\alpha+1}|l|^{\alpha+1} + |m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}}$$

from (ii) of Theorem 2.2. If in particular $\tau = 0$ and $k = \alpha$, then the upper and lower amplitudes and slopes coincide, that is,

$$\mathcal{A}^*[x] = \mathcal{A}_*[x] = \mathcal{S}^*[x] = \mathcal{S}_*[x] = \big[|l|^{\alpha+1} + |m|^{\alpha+1}\big]^{\frac{1}{\alpha+1}}.$$

This value may well be called the amplitude $\mathcal{A}[x]$ and the slope $\mathcal{S}[x]$ of the solution $x(t)$ of the equation

$$\big((\coth t)^\alpha \varphi_\alpha(x')\big)' + \alpha\tanh t\,\varphi_\alpha(x) = 0. \tag{3.2}$$

Notice that (3.2) is reduced to the generalized harmonic oscillator

$$(\varphi_\alpha(\dot z))^{\cdot} + \alpha\varphi_\alpha(z) = 0, \quad \cdot = \frac{d}{d\sigma}\,, \tag{3.3}$$

by means of the change of variables $(t,x) \to (\sigma,z)$ given by $\sigma = \log(\cosh t)$, $z(\sigma) = x(t)$. Equation (3.3) is known as a differential equation generating a generalized trigonometic function. Its solution $z(\sigma)$ determined by the initial condition $z(0) = 0$, $\dot z(0) = 1$ is the generalized sine function $z = S(\sigma)$ which exists on $\mathbb{R}$, is periodic with period $2\pi_\alpha$, $\pi_\alpha = \frac{2\pi}{\alpha+1}/\sin(\frac{\pi}{\alpha+1})$, and vanishes at $\sigma = k\pi_\alpha$, $k \in \mathbb{Z}$. It follows that (3.2) has an oscillatory solution $x(t) = S(\log(\cosh t))$ on $[0,\infty)$ whose zeros are located at $t_n = \cosh^{-1}(e^{n\pi_\alpha})$, $n = 0,1,2,\dots$, and whose amplitude and slope are given by $\mathcal{A}[x] = 1$ and $\mathcal{S}[x] = 1$, respectively.

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On the Inequality of Characteristics in the Statement of the First Darboux Problem for Second Order Hyperbolic Systems with Non-Split Principal Parts

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On the plane of independent variables x and y we consider the general system of second-order linear homogeneous differential equations

$$Au_{xx} + Bu_{xy} + Cu_{yy} + au_x + bu_y + cu = 0, \tag{1}$$

where A, B, C, a, b and c are given real $N \times N$ matrices and $u = (u_1, \dots, u_N)$ is the unknown N-dimensional real vector. We assume that $\det C \neq 0$ and $N > 1$ is a natural number.

By $P(x, y; \xi, \eta)$ we denote the characteristic determinant of system (1), i.e.

$$P(x, y; \xi, \eta) := \det Q(x, y; \xi, \eta),$$

where

$$Q(x, y; \xi, \eta) := A(x, y)\xi^2 + B(x, y)\xi\eta + C(x, y)\eta^2$$

and ξ, η are arbitrary real parameters.

Since $\det C \neq 0$, we have the following representation

$$P(x, y; 1, \lambda) = \det C \prod_{i=1}^{l} (\lambda - \lambda_i(x, y))^{k_i},$$

$$\sum_{i=1}^{l} k_i = 2N, \ \ l = l(x, y), \ \ k_i = k_i(x, y), \ \ i = 1, \dots, l.$$

System (1) is said to be hyperbolic at the point (x, y) if $l > 1$ and all roots $\lambda_1(x, y), \dots, \lambda_l(x, y)$ of the polynomial $P(x, y; 1, \lambda)$ are real (see, e.g., [1,6]).

One can readily show that [1,6]

$$k_i(x, y) \geq N - rank\, Q(x, y; 1, \lambda_i(x, y)), \ \ i = 1, \dots, l.$$

The hyperbolic system (1) is said to be normally hyperbolic at the point (x, y) if

$$k_i(x, y) = N - rank\, Q(x, y; 1, \lambda_i(x, y)), \ \ i = 1, \dots, l.$$

In the formulation of the characteristic Goursat problem for system (1), in contrast to scalar hyperbolic equations, generally speaking as was shown in [1–4], it can be ill-posed and one should be careful. In these works there are considered linear second order hyperbolic systems, for which the corresponding homogeneous characteristic Goursat problem has infinite number of linearly independent solutions. In the works [1, 3–5, 7] there are considered the question of the influence of lower terms on the correctness of the statement of the Goursat characteristic problem for second order hyperbolic systems with non-split principal part. As it is investigated in [6], the Goursat and Darboux first and second type problems for normally hyperbolic systems are well-posed. For the author until this day it is not known what will happen when in the case of Darboux first problem the condition of normally hyperbolicity is violated. The presented note is devoted to this question.

Suppose in system (1) that

$$N=2,\quad A=\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix},\quad B=\begin{Vmatrix}-1 & 1\\ 1 & -1\end{Vmatrix},\quad C=\begin{Vmatrix}0 & -1\\ -1 & 0\end{Vmatrix},\quad a=b=c=0,$$

and thus consider the following system

$$\begin{cases} u_{1xx}-u_{1xy}+u_{2xy}-u_{2yy}=0,\\ u_{2xx}+u_{1xy}-u_{2xy}-u_{1yy}=0. \end{cases} \tag{2}$$

System (2) is hyperbolic, since its characteristic determinate

$$D(\lambda):=(\lambda-1)^3(\lambda+1)$$

has the real roots $\lambda=1,\ \lambda=-1$.

By D_1 (D_2) we denote the domain on the plane of independent variables x and y bounded by the characteristic $x-y=0$ ($x+y=0$), $x\geq 0$ of system (2) and by the non-characteristic $y=0$, $x\geq 0$.

The Darboux first problem: in the domain $D_1(D_2)$ find a regular solution u of system (2) under the conditions

$$u\big|_{y=x}=f_1(x)\quad \big(u\big|_{y=-x}=f_2(x)\big),\ \ x\geq 0, \tag{3}$$

and

$$u\big|_{y=0}=f_3,\ \ x\geq 0, \tag{4}$$

where the functions f_1, $i=1,\dots,3$ are given twice continuously differentiable functions with respect to their arguments, satisfying the matching conditions: $f_1(0)=f_2(0)=f_3(0)$.

System (2) is rewritten in the following form

$$\begin{cases} \widetilde{w}_{\xi\eta}=0,\\ \widetilde{v}_{\eta\eta}=0, \end{cases} \tag{5}$$

where

$$\xi=x+y,\ \ \eta=x-y,\ \ \widetilde{w}(\xi,\eta):=w\Big(\frac{\xi+\eta}{2}\,,\frac{\xi-\eta}{2}\Big),\ \ \widetilde{v}(\xi,\eta):=v\Big(\frac{\xi+\eta}{2}\,,\frac{\xi-\eta}{2}\Big),$$
$$w:=u_1+u_2,\ \ v:=u_1-u_2.$$

By integrating system (5), we obtain

$$\widetilde{w}(\xi,\eta)=2\varphi_1(\xi)+2\varphi_2(\eta),\quad \widetilde{v}(\xi,\eta)=2\eta\varphi_3(\xi)+2\varphi_4(\xi),$$

where φ_i, $i=1,\dots,4$ are arbitrary twice continuously differentiable functions with respect to their arguments.

Returning to the previous variables, we obtain that the general classical solution of system (2) has the following form

$$\begin{cases} u_1(x,y)=\varphi_1(x+y)+\varphi_2(x-y)+(x-y)\varphi_3(x+y)+\varphi_4(x+y),\\ u_2(x,y)=\varphi_1(x+y)+\varphi_2(x-y)-(x-y)\varphi_3(x+y)-\varphi_4(x+y). \end{cases} \tag{6}$$

Based on formulas (6), we conclude that:

1) The unique solution of problem (2)–(4) in the domain D_1 is given by the formulas

$$\begin{aligned}
u_1(x,y) = {} & \frac{y-x}{2(x+y)}\Big[f_1^1\Big(\frac{x+y}{2}\Big) - f_1^2\Big(\frac{x+y}{2}\Big) - f_2^1(x+y) + f_2^2(x+y)\Big] \\
& + f_1^1\Big(\frac{x+y}{2}\Big) - \frac{1}{2}\Big[f_1^1\Big(\frac{x-y}{2}\Big) + f_1^2\Big(\frac{x-y}{2}\Big) - f_2^1(x-y) - f_2^2(x-y)\Big], \\
u_2(x,y) = {} & \frac{x-y}{2(x+y)}\Big[f_1^1\Big(\frac{x+y}{2}\Big) - f_1^2\Big(\frac{x+y}{2}\Big) - f_2^1(x+y) + f_2^2(x+y)\Big] \\
& + f_1^2\Big(\frac{x+y}{2}\Big) - \frac{1}{2}\Big[f_1^1\Big(\frac{x-y}{2}\Big) + f_1^2\Big(\frac{x-y}{2}\Big) - f_2^1(x-y) - f_2^2(x-y)\Big].
\end{aligned}$$

2) The corresponding to (2)–(4) homogeneous problem in the domain D_2 has infinitely many linearly independent solutions given by the formulas

$$u_1(x,y) = -y\varphi_0(x+y), \quad u_2(x,y) = y\varphi_0(x+y), \quad \varphi_0(0) = 0,$$

where φ_0 is an arbitrary twice continuously differentiable function with respect to its arguments.

3) The inhomogeneous problem (2)–(4) in the domain D_2 is not always solvable for an arbitrary right-hand side.

Remark. The question of finding the well-posed problems for system (2) is certainly of scientific interest and will be the subject of further research by the author.

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Stability of Solutions of Semidiscrete Stochastic Systems Using N. V. Azbelev's W-Method

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Semidiscrete systems of equations constitute an important subclass of so-called "hybrid systems" characterized by the presence of two components in the state space: discrete and continuous. Intuitively, this means that the dynamics is mostly continuous, but at certain instants is exposed to abrupt influences. Such systems naturally appear in applications, for example, in biological and ecological models [10, 12] as well as in the control theory [11]. Some models with impulsive actions [9] are also an important example of semidiscrete problems.

Finally, accounting for stochastic effects is an important part of any realistic approach to modeling. For example, in the population dynamics, demographic and ecological stochasticity arises due to a change in time of factors external to the system, but affecting the survival of the population, and in control theory, random coefficients can simulate, for example, inaccuracies in measurements. Therefore, the study of hybrid stochastic systems has recently attracted the attention of many specialists (see e.g. [7] and the references therein).

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\ge0}$ of complete σ-subalgebras of $\mathcal{F}$. By E we denote the expectation on this probability space.

To describe semidiscrete systems, we fix a natural number l $(1\le l<n)$, for which $x_1(t),\dots,x_l(t)$ $(t\ge0)$ will be the continuous components of the state vector of the system, while $x_{l+1}(s),\dots,x_n(s)$ $(s\in N_+\equiv\{0,1,2,\dots\})$ will be its discrete components. In the vector notation it will look as follows:

$$\widehat{x}(t)=\operatorname{col}(x_1(t),\dots,x_l(t))\ \ (t\ge0),\quad \widetilde{x}(s)=\operatorname{col}(x_{l+1}(s),\dots,x_n(s))\ \ (s\in N_+),$$
$$x(t)=\operatorname{col}\big(x_1(t),\dots,x_l(t),x_{l+1}([t]),\dots,x_n([t])\big)\ \ (t\ge0),$$

where $[t]$ is the integer part of the number t.

We study the moment exponential stability of solutions of the following system of linear differential and difference Itô equations with aftereffect:

$$\begin{gathered}
d\widehat{x}(t)=-\sum_{j=1}^{m_1}A_{1j}(t)x(h_{1j}(t))\,dt+\sum_{i=2}^{m}\sum_{j=1}^{m_i}A_{ij}(t)x(h_{ij}(t))\,d\mathcal{B}_i(t)\ \ (t\ge0),\\
\begin{aligned}
\widetilde{x}(s+1)=\widetilde{x}(s)-\sum_{j=-\infty}^{s}A_1(s,j)x(j)h\\
+\sum_{i=2}^{m}\sum_{j=-\infty}^{s}A_i(s,j)x(j)\big(\mathcal{B}_i((s+1)h)-\mathcal{B}_i(sh)\big)\ \ (s\in N_+)
\end{aligned}
\end{gathered}\tag{0.1}$$

with respect to the initial conditions

$$x(\varsigma) = \varphi(\varsigma) \ \ (\varsigma < 0), \tag{0.1a}$$

$$x(0) = b. \tag{0.1b}$$

Here

- $x(t) = \operatorname{col}(x_1(t), \dots, x_l(t), x_{l+1}([t]), \dots, x_n([t]))$ $(t \geq 0)$ is a n-dimensional unknown stochastic process;
- $A_{ij}(t)$ are $l \times n$ - matrices $(i = 1, \dots, m,\ j = 1, \dots, m_i)$, where the entries of the matrices $A_{1j}(t),\ j = 1, \dots, m_1$ are progressively measurable scalar stochastic processes on interval $[0, \infty)$ with almost surely (a.s.) locally integrable trajectories, and the entries of the matrices $A_{ij}(t),\ i = 2, \dots, m,\ j = 1, \dots, m_i$ are progressively measurable scalar stochastic processes on $[0, \infty)$, whose trajectories a.s. locally square-integrable;
- $h_{ij}(t)$, $i = 1, \dots, m$, $j = 1, \dots, m_i$ are Borel measurable functions defined on $[0, \infty)$ and such that $h_{ij}(t) \leq t$ $(t \geq 0)$ are almost everywhere Lebesgue measurable for all $i = 1, \dots, m$, $j = 1, \dots, m_i$;
- h is some positive real number;
- $A_i(s, j) - (n - l) \times n$ are matrices whose entries are $\mathcal{F}_s$-measurable scalar random variables for all $i = 1, \dots, m$, $s \in N_+$, $j = -\infty, \dots, s$;
- $\varphi(\varsigma) = \operatorname{col}(\varphi_1(\varsigma), \dots, \varphi_l(\varsigma), \varphi_{l+1}([\varsigma]), \dots, \varphi_n([\varsigma]))$ $(\varsigma < 0)$ is a $\mathcal{F}_0$-measurable, n-dimensional stochastic process with a.s. essentially bounded trajectories;
- $b = \operatorname{col}(b_1, \dots, b_n)$ is a $\mathcal{F}_0$-measurable n-dimensional random variable.

Under these assumptions, the problem (0.1)–(0.1b) has a unique global solution.

The moment exponential stability is defined in

Definition 0.1. System (0.1) is called exponentially q-stable with respect to the initial data if there are positive numbers c, λ such that all solutions $x(t, b, \varphi)$ $(t \in (-\infty, \infty))$ of the initial value problem (0.1), (0.1a), (0.1b) satisfy the estimate

$$\big(E|x(t, b, \varphi)|^q\big)^{1/q} \leq c \exp\{-\lambda t\} \Big(\big(E|b|^q\big)^{1/q} + \operatorname{ess\,sup}_{\varsigma < 0} \big(E|\varphi(\varsigma)|^q\big)^{1/q}\Big) \ \ (t \geq 0).$$

The next definition is used in the main result of the paper.

Definition 0.2. An invertible matrix $B = (b_{ij})_{i,j=1}^m$ is called positive invertible if all elements of the matrix B^{-1} are positive.

According to [3], the matrix B will be positive invertible if $b_{ij} \leq 0$ for $i, j = 1, \dots, m$, $i \neq j$ and all diagonal minors of the matrix B are positive. In particular, matrices with strict diagonal dominance and non-positive off-diagonal elements are positive invertible.

1 Sufficient stability conditions

In this section we use a special constant c_p, which is defined in

Lemma. *For any scalar, progressive measurable stochastic process $f(\varsigma)$*

$$\bigg(E\bigg|\int\limits_0^t f(\varsigma)\,d\mathcal{B}(\varsigma)\bigg|^{2p}\bigg)^{\frac{1}{2p}} \le c_p\bigg(E\bigg(\int\limits_0^t |f(\varsigma)|^2 d\varsigma\bigg)^p\bigg)^{\frac{1}{2p}}, \tag{1.1}$$

where c_p is some number depending on $p \ge 1$. Here $\mathcal{B}(\varsigma)$ is the scalar Wiener process.

Estimate (1.1) follows from the inequality given in the monograph [8, p. 65], where the formulas for c_p can also be found.

Let μ be the Lebesgue measure on $[0,\infty)$. Consider three groups of conditions on the coefficients of System (0.1).

Assume that

- there exist numbers $\tau_{ij} \ge 0$, $i = 1,\dots,m$, $j = 1,\dots,m_i$ such that $0 \le t - h_{ij}(t) \le \tau_{ij}$ $(t \ge 0)$ μ-almost everywhere for all these indices;
- there exist numbers $\overline{a}^{ij}_{kr} \ge 0$, $i = 1,\dots,m$, $j = 1,\dots,m_i$, $k = 1,\dots,l$, $r = 1,\dots,n$ such that $|a^{ij}_{kr}(t)| \le \overline{a}^{ij}_{kr}$ $(t \ge 0)$ $P \times \mu$-almost everywhere for all these indices.

In addition, assume that there exist $\lambda_k \ge 0$, $k = 1,\dots,n$, for which

- the diagonal entries of the matrices $A_1(s,s)$ $(s \in N_+)$ can be represented as $a^1_{kk}(s,s) + \lambda_k$ $(s \in N_+)$, $k = l+1,\dots,n$;
- $\sum\limits_{j\in I_k} a^{1j}_{kk}(t) \ge \lambda_k$ $(t \ge 0)$ $P \times \mu$-almost everywhere $(k = 1,\dots,l)$ and some subsets $I_k \subset \{1,\dots,m_1\}$, $k = 1,\dots,l$;
- $0 < \lambda_k h < 1$ if $k = l+1,\dots,n$.

Finally, assume that there exist numbers $d_i \in N_+$, $i = 1,\dots,m$, for which

- the entries of the matrices $A_i(s,j)$ are equal to 0 P-almost everywhere for all $s \in N_+$, $j = -\infty,\dots,s-d_i-1$, $i = 1,\dots,m$;
- $|a^i_{kr}(s,j)| \le \overline{a}^i_{kr}(s,j)$ P-almost everywhere for all $i = 1,\dots,m$, $k = l+1,\dots,n$, $r = 1,\dots,n$, $s \in N_+$, $j = s-d_i,\dots,s$, and, in addition,

  $$\sup_{\tau\in N_+} \sum_{j=\nu_i(\tau)}^{\tau} \overline{a}^1_{kr}(\tau,j) < \infty \ \text{ for all } \ i = 1,\dots,m, \ \ k = l+1,\dots,n, \ \ r = 1,\dots,n,$$

  where $\nu_i(\tau) = 0$ if $0 \le \tau \le d_i$ and $\nu_i(\tau) = \tau - d_i$ if $\tau > d_i$.

The entries of the $n \times n$-matrix C are defined by

$$c_{kk} = \frac{1}{\lambda_k}\,\bigg(\sum_{j\in I_k} \overline{a}^{1j}_{kk}\Big(\sum_{\nu=1}^{m_1} \overline{a}^{1\nu}_{kk}\tau_{1j} + c_p\sum_{i=2}^{m}\sum_{\nu=1}^{m_i} \overline{a}^{i\nu}_{kr}\sqrt{\tau_{1j}}\Big) + \sum_{j=1,\,j\in\{1,\dots,m_1\}/I_k}^{m_1} \overline{a}^{1j}_{kk}\bigg) \\ + \frac{c_p}{\sqrt{2\lambda_k}}\sum_{i=2}^{m}\sum_{j=1}^{m_i} \overline{a}^{ij}_{kk}, \ \ k = 1,\dots,l,$$

$$c_{kr} = \frac{1}{\lambda_k}\,\bigg(\sum_{j\in I_k} \overline{a}^{1j}_{kr}\Big(\sum_{\nu=1}^{m_1} \overline{a}^{1\nu}_{kr}\tau_{1j} + c_p\sum_{i=2}^{m}\sum_{\nu=1}^{m_i} \overline{a}^{i\nu}_{kr}\sqrt{\tau_{1j}}\Big) + \sum_{j=1}^{m_1} \overline{a}^{1j}_{kr}\bigg)$$

$$+\frac{c_p}{\sqrt{2\lambda_k}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}_{kr}^{ij}, \ \ k=1,\dots,l, \ \ r=1,\dots,n, \ \ k\neq r,$$

$$c_{kr}=\frac{1}{\lambda_k h}\Big(h\sup_{\tau\in N_+}\sum_{j=\nu_1(\tau)}^{\tau}\overline{a}_{kr}^{1}(\tau,j)+c_p\sqrt{h}\sum_{i=2}^{m}\sup_{\tau\in N_+}\sum_{j=\nu_i(\tau)}^{\tau}\overline{a}_{kr}^{i}(\tau,j)\Big),$$

$$k=1,\dots,l, \ \ r=1,\dots,n.$$

The above assumptions enable us to formulate the main result of this paper.

Theorem. *If the matrix $\overline{E}-C$ is positive invertible, then System* (0.1) *is exponentially $2p$-stable with respect to the initial data, i.e. in the sense of Definition 0.1. Moreover, the exponential decay rate λ of all solutions can be estimated as*

$$0<\lambda<\min\big\{\lambda_i, \ i=1,\dots,l; \ -\ln(1-\lambda_i h), \ i=l+1,\dots,n\big\}. \tag{1.2}$$

The proof of the theorem is based on the regularization method, also known as a method of model (auxiliary) equations or "N. V. Azbelev's W-method", see the monographs [1, 2] and the references therein. This approach has proven to be efficient in the theory of stochastic differential [4] and difference [5] equations. The main idea of the method is to replace functionals on the space of trajectories of solutions by the so-called "model" equation that already has the necessary property of stability and which is used to regularize the initial equation. Checking stability of the latter amounts, then, to estimating the norm of a certain integral operator or checking a positive invertibility of some matrix. The latter version of the W-method was developed in [6].

2 An example

Consider a semidiscrete system of stochastic equations with constant coefficients and bounded delays of the form:

$$\begin{aligned} d\widehat{x}(t)&=-\sum_{j=1}^{m_1}A_{1j}x(t-h_{1j})\,dt+\sum_{i=2}^{m}\sum_{j=1}^{m_i}A_{ij}x(t-h_{ij})d\mathcal{B}_i(t) \ \ (t\geq 0),\\ \widetilde{x}(s+1)&=\widetilde{x}(s)-A_1\sum_{j=s-d_1}^{s}x(j)h+\sum_{i=2}^{m}A_i\sum_{j=s-d_i}^{s}x(j)\big(\mathcal{B}_i((s+1)h)-\mathcal{B}_i(sh)\big) \ \ (s\in N_+), \end{aligned} \tag{2.1}$$

where $A_{ij}=(a_{kr}^{ij})_{k,r=1}^{l,n}$, $i=1,\dots,m$, $j=1,\dots,m_i$ are real $l\times n$-matrices and $A_i=(a_{kr}^{i})_{k=l+1,r=1}^{n}$, $i=1,\dots,m$ are real $(n-l)\times n$-matrices, and $h_{ij}\geq 0$, $i=1,\dots,m$, $j=1,\dots,m_i$ are real numbers, $h>0$ is some (sufficiently small) real number. Put also $\sum\limits_{j=1}^{m_1}a_{kk}^{1j}=a_k$, $k=1,\dots,l$ an define the entries of the $n\times n$-matrix C as follows:

$$c_{kk}=\frac{1}{a_k}\sum_{j=1}^{m_1}|a_{kk}^{1j}|\Big(\sum_{\nu=1}^{m_1}|a_{kk}^{1\nu}|h_{1j}+c_p\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{kr}^{i\nu}|\sqrt{h_{1j}}\Big)+\frac{c_p}{\sqrt{2a_k}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{kk}^{ij}|, \ \ k=1,\dots,l,$$

$$c_{kr}=\frac{1}{a_k}\Bigg(\sum_{j=1}^{m_1}|a_{kr}^{1j}|\Big(\sum_{\nu=1}^{m_1}|a_{kr}^{1\nu}|h_{1j}+c_p\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{kr}^{i\nu}|\sqrt{h_{1j}}\Big)+\sum_{j=1}^{m_1}|a_{kr}^{1j}|\Bigg)$$

$$+\frac{c_p}{\sqrt{2a_k}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{kr}^{ij}|, \ \ k=1,\dots,l, \ \ r=1,\dots,n, \ \ k\neq r,$$

$$c_{kk} = \frac{c_p(d_i+1)}{a_{kk}^1\sqrt{h}} \sum_{i=2}^{m} |a_{kk}^i|, \quad k = 1+1, \dots, l,$$

$$c_{kr} = \frac{(d_1+1)|a_{kr}^1|}{a_{kk}^1} + \frac{c_p(d_i+1)}{a_{kk}^1\sqrt{h}} \sum_{i=2}^{m} |a_{kr}^i|, \quad k = 1+1, \dots, l, \quad r = 1, \dots, n, \quad k \neq r.$$

Then from Theorem we can deduce the following

Proposition. *If $a_k > 0$, $k = 1, \dots, l$, $a_{kk}^1 > 0$, $k = l+1, \dots, n$, and the matrix $\overline{E} - C$ is positively invertible, then system* (2.1) *is exponentially* $2p$*-stable with respect to the initial data.*

In particular, we obtain

Corollary. *Let $n = 2$, $l = 1$ in system* (2.1) *and let the entries c_{ij}, $i, j = 1, 2$ of the 2×2-matrix C be defined as described right before Proposition. If now $1 - c_{11} > 0$; $(1 - c_{11})(1 - c_{22}) > c_{12}c_{21}$, then system* (2.1) *is exponentially* $2p$*-stable with respect to the initial data.*

The corollary follows from Proposition and from the fact that under the conditions of the corollary the 2×2-matrix $\overline{E} - C$ is positive invertible, since its diagonal minors are positive.

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About the Relation Between Qualitative Properties of the Solutions of Differential Equations and Equations on Time Scales

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Differential equations on time scales were introduced by S. Hilger in [9]. His approach gave a possibility to unified theory for both discrete and continuous analysis. The theory was thoroughly stated in [1, 2]. The behavior of the solutions of the dynamic equations, defined on a family of time scales $\mathbb{T}_\lambda$ when graininess function $\mu_\lambda \to 0$ as $\lambda \to 0$ is of interest for study. In this case intervals of the time scale $[t_0, t_1]_\lambda = [t_0, t_1] \cap \mathbb{T}_\lambda$ approach $[t_0, t_1]$ (e.g. in the Hausdorff metric). The question arises about relation between properties of the solutions of equations on time scales and the solutions of boundary equations which are ordinary differential ones. It is obviously, on the finite time intervals it is not complicated to establish the convergence of solutions of dynamical equations to the corresponding solutions of differential equations. However, in case of infinite intervals this problem is not trivial.

This work is devoted to the study of existence of a bounded solution of the differential equation, defined on a family of time scales $\mathbb{T}_\lambda$ provided the graininess function μ_λ converges to zero as $\lambda \to 0$. This work extends the results of [12] about the relation between the existence of bounded solutions of differential equations and the corresponding difference equations to the case of general time scales. The main difficulty here is to obtain estimation between the solutions of differential equation and its analog for the time scale for any $\mathbb{T}_\lambda$. This makes this analysis significantly different from [12], where only special case for $\mathbb{T} = \mathbb{Z}$ was obtained.

Note, the question about existence of the two-sided solutions for dynamical equations on time scales is not trivial by itself. In comparison with the classic theorem about existence of solutions of the system of ordinary differential equations, where local both sides existence with respect to initial point holds, for the equations on time scales it is more complicated. To extend the solution to the left it is necessary the very strong regression condition holds [5]. Here we got the existence of the two-sided global bounded solution without using regression condition.

The proof of the theorems requires continuous dependence of the solutions on initial data uniformly over all time scales. It does not influence, for example, from [10], where investigation was on the fixed scale. This question is not trivial due to the topological complexity of the time scale.

The relation between properties also of the solutions of the system of ordinary differential equation and the solutions of equations on Eulerian time scales was studied before.

The paper [3] showed that the solutions of differential and the corresponding difference equations have the same oscillatory properties. The relation between stability and attractors of differential and difference equations was studied in [7]. The relation between optimal control of the systems of ordinary differential equations and dynamical equations on time scales considered in [4, 6, 11].

Let present few concepts from the monograph [1], which are used here.

Time scale $\mathbb{T}$ is an arbitrary, non-empty, closed subset of the real axis. For every $A \subset \mathbb{R}$, we denote $A_{\mathbb{T}} = A \cap \mathbb{T}$.

Define the forward and backward jump operators as $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$ (supplemented by $\inf \varnothing := \sup \mathbb{T}$ and $\sup \varnothing := \inf \mathbb{T}$).

The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is called left-dense (LD) (left-scattered (LS), right-dense (RD) or right-scattered (RS)) if $\rho(t) = t$ ($\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$) hold. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise, we set $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}^d$ is said to be Δ-differentiable at $t \in \mathbb{T}^k$ if the limit

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^d$.

Consider the following system of differential equations

$$\frac{dx}{dt} = X(t, x), \tag{1}$$

$t \in \mathbb{R}$, $x \in D$, D is a domain in $\mathbb{R}^d$.

Consider the set of time scales $\mathbb{T}_\lambda$ and system (1) defined on $\mathbb{T}_\lambda$

$$x_\lambda^{\Delta}(t) = X(t, x_\lambda), \tag{2}$$

where $t \in \mathbb{T}_\lambda$, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^d$, $x_\lambda^{\Delta}(t)$ be delta derivative x^{Δ} for a function $x(t)$ defined on $\mathbb{T}_\lambda$, $\inf \mathbb{T}_\lambda = -\infty$, $\sup \mathbb{T}_\lambda = \infty$, $\lambda \in \Lambda \subset \mathbb{R}$, and $\lambda = 0$ is a limit point of Λ.

Assume that the function $X(t, x)$ is continuously differentiable and bounded together with its partial derivatives, i.e. $\exists\, C > 0$ such that

$$|X(t, x)| + \Big| \frac{\partial X(t, x)}{\partial t} \Big| + \Big\| \frac{\partial X(t, x)}{\partial x} \Big\| \le C \tag{3}$$

for $t \in \mathbb{R}$, $x \in D$, where $\frac{\partial X}{\partial x}$ is the corresponding Jacobi matrix.

Let $\mu_\lambda = \sup\limits_{t \in \mathbb{T}_\lambda} \mu_\lambda(t)$, where the graininess function $\mu_\lambda \colon \mathbb{T}_\lambda \to [0, \infty)$. Obviously, if $\mu_\lambda(t) \to 0$ when $\lambda \to 0$, then $\mathbb{T}_\lambda$ coincide (for example, in the Hausdorff metric) to a continuous time scale $\mathbb{T}_0 = \mathbb{R}$ (according classification [8]).

The following theorem holds.

Theorem 1. *Let system* (1) *has a bounded on* $\mathbb{R}$*, asymptotically stable uniformly in* $t_0 \in \mathbb{R}$ *solution* $x(t)$*, which lies in the domain* D *with some* ρ*-neighborhood. Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda < \lambda_0$ *system* (2) *has a bounded on* $\mathbb{T}_\lambda$ *solution* $x_\lambda(t)$*.*

Theorem 2. *If there exists* $\lambda_0 > 0$ *such that for all* $\lambda < \lambda_0$ *system* (2) *has an asymptotically stable uniformly in* $t_0 \in \mathbb{T}_\lambda$ *and* λ *bounded on axis solution* $x_\lambda(t)$*, which lies in the domain* D *with some* ρ*-neighborhood, then system* (1) *has a bounded on axis solution.*

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The Boundary Value Problem for a Semilinear Hyperbolic System

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In the space $\mathbb{R}^{n+1}$ of the variables $x=(x_1,\dots,x_n)$ and t, we consider the semilinear hyperbolic system of the form

$$\square^2 u_i + f_i(u_1,u_2,\dots,u_N) = F_i(x,t),\ i=1,\dots,N, \tag{1}$$

where $f=(f_1,\dots,f_N)$, $F=(F_1,\dots,F_N)$ are the given, and $u=(u_1,\dots,u_N)$ is an unknown real vector function, $n\geq 2$, $N\geq 2$, $\square := \frac{\partial^2}{\partial t^2}-\Delta$, $\Delta := \sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$.

For the system (1) we consider the boundary value problem: find in the cylindrical domain $D_T := \Omega\times(0,T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, a solution $u=u(x,t)$ of that system according to the boundary conditions

$$u\big|_{\partial D_T}=0,\quad \frac{\partial u}{\partial \nu}\Big|_{\partial D_T}=0, \tag{2}$$

where $\nu=(\nu_1,\dots,\nu_n,\nu_{n+1})$ is the unit vector of the outer normal to ∂D_T.

Let

$$\overset{\circ}{C}{}^k(\overline{D}_T,\partial D_T) := \Big\{u\in C^k(\overline{D}_T):\ u\big|_{\partial D_T}=\frac{\partial u}{\partial\nu}\Big|_{\partial D_T}=0\Big\},\ \ k\geq 2.$$

Assume $u\in \overset{\circ}{C}{}^4(\overline{D}_T,\partial D_T)$ is a classical solution of the problem (1), (2). Multiplying both parts of the system (1) scalarly by an arbitrary vector function $\varphi=(\varphi_1,\dots,\varphi_N)\in \overset{\circ}{C}{}^2(\overline{D}_T,\partial D_T)$ and integrating the obtained equation by parts over the domain D_T, we obtain

$$\int\limits_{D_T}\square u\,\square\varphi\,dx\,dt + \int\limits_{D_T} f(u)\varphi\,dx\,dt = \int\limits_{D_T} F\varphi\,dx\,dt. \tag{3}$$

When deducing (3), we have used the equality

$$\int\limits_{D_T}\square u\,\square\varphi\,dx\,dt = \int\limits_{\partial D_T}\frac{\partial\varphi}{\partial N}\square u\,ds - \int\limits_{\partial D_T}\varphi\,\frac{\partial}{\partial N}\square u\,ds + \int\limits_{D_T}\varphi\square^2 u\,dx\,dt,$$

where

$$\frac{\partial}{\partial N} = \nu_{n+1}\frac{\partial}{\partial t} - \sum_{i=1}^{n}\nu_i\frac{\partial}{\partial x_i}$$

is the derivative with respect to the conormal, as well as the equalities

$$\frac{\partial\varphi}{\partial N}\Big|_{\Gamma} = -\frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma},\quad \frac{\partial\varphi}{\partial N}\Big|_{\partial D_T\setminus\Gamma} = \frac{\partial\varphi}{\partial\nu}\Big|_{\partial D_T\setminus\Gamma},\quad \Gamma := \partial\Omega\times(0,T),\ \ \varphi\big|_{\partial D_T} = \frac{\partial\varphi}{\partial\nu}\Big|_{\partial D_T}=0.$$

Introduce the Hilbert space $\overset{\circ}{W}{}^1_{2,\square}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\square}(D_T)} = \int\limits_{D_T} \Big[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u}{\partial x_i}\Big)^2 + (\square\, u)^2\Big]\, dx\, dt \tag{4}$$

of the classical space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$. It follows from (4) that if $u \in \overset{\circ}{W}{}^1_{2,\square}(D_T)$, then $u \in \overset{\circ}{W}{}^1_2(D_T)$ and $\square\, u \in L_2(D_T)$. Here $W^1_2(D_T)$ is the well-known Sobolev space consisting of the elements of $L_2(D_T)$, having the first order generalized derivatives from $L_2(D_T)$ and $\overset{\circ}{W}{}^1_2(D_T) = \{u \in W^1_2(D_T) : u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the trace theory.

Below, on the nonlinear vector function $f = (f_1, \dots, f_N)$ from (1) we impose the following requirement

$$f \in C(\mathbb{R}^n), \ \ |f(u)| \le M_1 + M_2|u|^\alpha, \ \ u \in \mathbb{R}^n, \tag{5}$$

where $|\cdot|$ is the norm of the space $\mathbb{R}^n$ and $M_i = const \ge 0, \ i = 1, 2$, and

$$0 \le \alpha = const < \frac{n+1}{n-1}\,. \tag{6}$$

Remark 1. The embedding operator $I : W^1_2(D_T) \to L_q(D_T)$ is a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$ and $n > 1$. At the same time, the Nemytsky operator $K : L_q(D_T) \to L_2(D_T)$, acting according to the formula $K(u) = f(u)$, where $u = (u_1, \dots, u_N) \in L_2(D_T)$ and the vector function $f = (f_1, \dots, f_N)$ satisfies the condition (5), is continuous and bounded if $q \ge 2\alpha$. Therefore, if $\alpha < \frac{n+1}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Thus, in this case the operator

$$K_0 = KI : W^1_2(D_T) \to L_2(D_T)$$

is continuous and compact. Moreover, from $u \in W^1_2(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u^m \to u$ in the space $W^1_2(D_T)$, then $f(u^m) \to f(u)$ in the space $L_2(D_T)$.

Definition 1. Let the vector function f satisfy the conditions (5) and (6), $F \in L_2(D_T)$. The vector function $u \in \overset{\circ}{W}{}^1_{2,\square}(D_T)$ is said to be a weak generalized solution of the problem (1), (2), if for any vector function $\varphi = (\varphi_1, \dots, \varphi_N) \in \overset{\circ}{W}{}^1_{2,\square}(D_T)$ the integral equality (3) is valid, i.e.

$$\int\limits_{D_T} \square\, u\, \square\, \varphi\, dx\, dt + \int\limits_{D_T} f(u)\varphi\, dx\, dt = \int\limits_{D_T} F\varphi\, dx\, dt \ \ \forall\, \varphi \in \overset{\circ}{W}{}^1_{2,\square}(D_T). \tag{7}$$

Notice that in view of Remark 1 the integral $\int\limits_{D_T} f(u)\varphi \ dx\, dt$ in the equality (7) is defined correctly, since from $u \in \overset{\circ}{W}{}^1_{2,\square}(D_T)$ it follows $f(u) \in L_2(D_T)$ and, therefore, $f(u)\varphi \in L_1(D_T)$.

It is not difficult to verify that if the solution u of the problem (1), (2) belongs to the class $\overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T)$ in the sense of Definition 1, then it will also be a classical solution of this problem.

Consider the following condition

$$\lim_{|u|\to\infty} \inf \frac{uf(u)}{||u||^2_{\mathbb{R}^N}} \ge 0. \tag{8}$$

Theorem 1. *Let the conditions* (5), (6) *and* (8) *be fulfilled. Then for any $F \in L_2(D_T)$ the problem* (1), (2) *has at least one weak generalized solution $u \in \overset{\circ}{W}{}^1_{2,\square}(D_T)$.*

Remark 2. If the conditions of Theorem 1 are fulfilled and the Nemytsky operator $K(u)=f(u):\mathbb{R}^N\to\mathbb{R}^N$ is monotonic, i.e.

$$(K(u)-K(v))\cdot(u-v)\geq 0 \ \ \forall\, u,v\in\mathbb{R}^N, \tag{9}$$

then there will hold the uniqueness of the solution of this problem.

Thus, the following theorem is valid.

Theorem 2. *Let the conditions* (5), (6) *and* (8), (9) *be fulfilled. Then for any* $F\in L_2(D_T)$ *the problem* (1), (2) *has a unique weak generalized solution in the space* $\overset{\circ}{W}{}^1_{2,\square}(D_T)$.

Remark 3. The condition (9) will be fulfilled if $f\in C^1(\mathbb{R}^N)$ and the matrix $A=(\frac{\partial f_i}{\partial u_j})_{i,j=1}^N$ is defined non-negatively, i.e.

$$\sum_{i,j=1}^N \frac{\partial f_i}{\partial u_j}\,(u)\xi_i\xi_j\geq 0 \ \ \forall\,\xi=(\xi_1,\dots,\xi_N), \ \ u=(u_1,\dots,u_N)\in\mathbb{R}^N.$$

As the examples show, if the conditions imposed on the nonlinear vector function f are violated, then the problem (1), (2) may not have a solution. For example, if

$$f_i(u_1,\dots,u_N)=\sum_{j=1}^N a_{ij}|u_j|^{\beta_{ij}}+b_i, \ \ i=1,\dots,N,$$

where constant numbers a_{ij},β_{ij} and b_i satisfy inequalities

$$a_{ij}>0, \ \ 1<\beta_{ij}<\frac{n+1}{n-1}\,, \ \ \sum_{i=1}^N b_i>0,$$

then the condition (8) will be violated and the problem (1), (2) will not have a solution $u\in\overset{\circ}{W}{}^1_{2,\square}(D_T)$ for $F=\mu F^o$, where $F^o=(F_1^o,\dots,F_N^o)\in L_2(D_T)$, $G=\sum\limits_{i=1}^N F_i^o\leq 0$; $\|G\|_{L_2(D_T)}\neq 0$ for $\mu>\mu_0=\mu_0(G,\beta_{ij})=const>0$.

Approximate Solution of Optimal Control Problem for Differential Inclusion with Fast-Oscillating Coefficients on Semi-Axis

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1 Introduction

There are many approaches for investigation of control problems for differential equations and inclusions, in particular, asymptotic methods are widely applied. It is worth to emphasize the averaging method, for which M. M. Krylov and M. M. Bogolyubov proposed the strict mathematical justification. In works of V. A. Plotnikov and works of his school (see, for example, [12]) there is the strict justification of the averaging method in application to control problems. In monograph [13] one ca find a justification of the averaging method, in particular, for ordinary differential inclusions, partial differential inclusions, inclusions with Hukuhara derivative. In the paper [11] the time averaging was performed firstly, where time is clearly included in the system, at that the control function was considered a parameter and averaging was not performed on it. Moreover, the authors had to impose a condition of asymptotic stability for the control function. In the paper [5] the approach from [11] is applied to the solvability of the optimal control problem on finite interval, but however, the rather strict condition of asymptotic stability is removed. In the paper [6] similar results to [5] are obtained on semi-axis. In the paper [7] authors apply the averaging method to solve the optimal control problem with fast-oscillating variables which is linear by control on a finite interval; at that the system of differential inclusions with Lipschitz right-hand side by phase variable. Optimal control problems on semi-axis in different perturbed problems are studied in [3, 4, 8–11, 14, 15].

In this work we apply the averaging method to investigate the optimal control problem with fast oscillating variables for the system of differential inclusions on semi-axis. In particular, we prove the solvability of original problem as well as averaged problem using the direct method of calculus of variations. We justify the convergence of optimal controls and optimal trajectories of solutions of original problem to optimal control and optimal trajectory of solutions of averaged problem. We show that optimal control of averaged problem is asymptotically optimal for the original exact problem.

2 Statement of the problem and the main results

Let us consider an optimal control problem for the system of differential inclusions on semi-axis with a small parameter and fast-oscillating coefficients

$$\dot{x} \in f\Big(\frac{t}{\varepsilon}, x\Big) + f_1(x)u(t), \quad x(0, u(0)) = x_0 \tag{2.1}$$

with the quality criterion

$$J_\varepsilon[x,u]=\int\limits_0^\infty \big(e^{-jt}A(t,x_\varepsilon(t))+u^2(t)\big)\,dt\to\inf. \tag{2.2}$$

Here $\varepsilon>0$ is a small parameter, $j>0$ is a fixed constant that defines discount, x is a phase vector in $\mathbb{R}^d$, $u(t)$ is m-measurable control vector which takes values in some set $U\subset\mathbb{R}^m$.

Let there be an uniformly by $x\in\mathbb{R}^d$ averaged value for a multi-valued function

$$\lim_{s\to\infty}\frac{1}{s}\int\limits_0^s f(t,x)\,dt=f_0(x), \tag{2.3}$$

where the integral of multi-valued function we consider in the sense of Aumann [1], and the limit of multi-valued function we consider in the sense of Hausdorf.

The optimal control problem on semi-axis (2.1), (2.2) is matched by the average control problem:

$$\dot y\in f_0(y)+f_1(y)u(t),\quad y(0,u(0))=x_0 \tag{2.4}$$

with the quality criterion

$$J_0[x,u]=\int\limits_0^\infty \big(e^{-jt}A(t,y(t))+u^2(t)\big)\,dt\to\inf. \tag{2.5}$$

Let for the problem (2.1), (2.2) and the corresponding average problem (2.4), (2.5) the next conditions are satisfied:

Condition 2.1. We consider m-measurable vector-functions $u(\cdot)\in L_2([0,\infty))$, which takes values in closed convex set $U\subset\mathbb{R}^m$ as admissible controls, and we consider that $0\in U$ as well.

Condition 2.2. The function $A(t,s)$ is defined for $t\geq 0, x\in\mathbb{R}^d, u\in U$, measurable by t and continuous by x, at that

$$\exists\, C>0:\ \ A(t,x)\geq -C,$$

and satisfies the next growth condition by $x\in\mathbb{R}^d$:

$$\exists\, K>0:\ \ |A(t,x)|\leq K\big(1+|x|^p\big)$$

for each $t\geq 0$ and $x\in\mathbb{R}^d$, $p\geq 0$.

Condition 2.3. The multi-valued function $f(t,x)$ $(f:Q=\{t\geq 0,\ x\in\mathbb{R}^d\}\to conv(\mathbb{R}^d))$ is defined and continuous in the Hausdorf metrics over the set of variables in Q, and matrix-valued function $f_1(x)$ is continuous by $x\in\mathbb{R}^d$ and the next conditions are fulfilled:

(1) $f(t,x)$ satisfies the linear growth condition by x with constants L_1 and L_2 in the domain Q, namely,

$$\|f(t,x)\|_+:=\sup_{\xi\in F(t,x)}\|\xi\|\leq L_1+L_2|x|\qquad \forall(t,x)\in Q;$$

$f_1(x)$ satisfies the linear growth condition by x in the domain $\mathbb{R}^d$ with constants L_3 and L_4, namely,

$$|f_1(x)|\leq L_3+L_4|x|,$$

where

$$j>L_2p; \tag{2.6}$$

(2) $f(t,x)$ and $f_1(x)$ satisfy Lipschitz condition by x uniformly by t in the definition domain with $K_1, K_2 > 0$, respectively.

Condition 2.4. The averaged value of multi-valued function f in the sense of limit (2.3) is a single-valued continuous function.

Taking into account conditions for parameters of problem we can obtain the result concerning solvability of problem (2.1), (2.2). Namely, we have the next

Lemma 1. *Under Conditions 2.1, 2.2, 2.3 there exists a solution of the optimal control problem* (2.1), (2.2).

Taking into account the previous Lemma, we can obtain the similar result about the solvability of the averaged problem (2.4), (2.5).

In the next result we show the convergence of optimal controls, optimal trajectories and optimal values of quality criterion of the original problem (2.1), (2.2) to corresponding parameters of the averaged problem (2.4), (2.5).

Theorem 1. *Let* $(x^*_\varepsilon(t), u^*_\varepsilon(t))$ *be the solution of the problem* (2.1), (2.2). *Then for some solution* $(y^*(t), u^*(t))$ *of problem* (2.4), (2.5) *we have:*

(1) $J^*_\varepsilon \to J^*_0$, $\varepsilon \to 0$ *and* $J^*_\varepsilon = \inf\limits_{x,u} \in \Xi_1 J_\varepsilon[x,u]$, $J^*_0 = \inf\limits_{(x,u)\in\Xi_2} J_0[x,u]$, Ξ_1, Ξ_2 *are sets of admissible pairs for problems* (2.1), (2.2) *and* (2.4), (2.5), *respectively.*

(2) *for each* $\eta > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(\eta)$ *such that* $0 < \varepsilon < \varepsilon_0$ *we have*

$$\big|J^*_\varepsilon - J[x^*_\varepsilon, u^*]\big| < \eta, \tag{2.7}$$

where x^*_ε *is the solution of Cauchy problem* (2.1);

(3) *there exists a sequence* $\varepsilon_n \to 0$, $n \to \infty$ *such that*

$$x^*_{\varepsilon_n} \to y(t) \tag{2.8}$$

uniformly on each interval $[0,T]$ *for any* $T > 0$, *and*

$$u^*_{\varepsilon_n} \xrightarrow{w} u^* \tag{2.9}$$

weakly in $L_2([0,\infty))$.

If, moreover, there exists a unique solution of the averaged problem (2.4), (2.5), *then the convergences* (2.8), (2.9) *take place for all* $\varepsilon \to 0$.

Acknowledgements

This work is supported by National Research Foundation of Ukraine (NRFU) (project # F81/41743).

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Optimal Conditions for the Unique Solvability of Two-Point Boundary Value Problems for Third Order Linear Singular Differential Equations

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On a finite open interval $]a,b[$, we consider the linear differential equation

$$u''' = p(t)u + q(t) \tag{1}$$

with the boundary conditions

$$u(a+) = 0, \quad u'(a+) = 0, \quad \sum_{i=0}^{k} \ell_i u^{(i)}(b-) = 0. \tag{2}$$

Here

$$k \in \{0,1,2\}, \ \ \ell_i \geq 0 \ \ (i = 0, \dots, k), \ \ \ell_k > 0,$$

while p and $q :]a,b[\to \mathbb{R}$ are measurable functions such that

$$\int_a^b (t-a)^2 (b-t)^{2-k} |p(t)| \, dt < +\infty, \quad \int_a^b (t-a)(b-t)^{2-k} |q(t)| \, dt < +\infty. \tag{3}$$

We are mainly interested in the case where the functions p and q have nonintegrable singularities at the boundary points of the interval $]a,b[$, i.e. the case, where

$$\int_a^b \big(|p(t)| + |q(t)|\big) \, dt = +\infty.$$

However, the results below on the unique solvability of problem $(1), (2)$ are new also for the regular case when the functions p and q are integrable on $[a,b]$.

To formulate the above mentioned results, we need the following notation.

$$\Delta_k(t) = \sum_{i=0}^{k} \frac{(b-t)^{2-i}}{(2-i)!} \, \ell_i \Big/ \sum_{i=0}^{k} \frac{(b-a)^{2-i}}{(2-i)!} \, \ell_i,$$

$$g_k(t,s) = \begin{cases} \dfrac{1}{2} \left(\Delta_k(s)(t-a)^2 - (t-s)^2 \right) & \text{for } \ a \leq s < t \leq b, \\[2mm] \dfrac{1}{2} \, \Delta_k(s)(t-a)^2 & \text{for } \ a \leq t \leq s \leq b, \end{cases}$$

$$r_0(\alpha) = 1, \quad r_1(\alpha) = \frac{\ell_0(b-a) + (\alpha+3)\ell_1}{\ell_0(b-a) + 2\ell_1} \, ,$$

$$r_2(\alpha)=\frac{\ell_0(b-a)^2+(\alpha+3)\ell_1(b-a)+(\alpha+3)(\alpha+2)\ell_2}{\ell_0(b-a)^2+2\ell_1(b-a)+2\ell_2}\,,$$
$$p_k(t;\alpha)=\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{r_k(\alpha)(b-a)^{\alpha+1}-(t-a)^{\alpha+1}}\,(t-a)^{\alpha-2}\ \text{ for }\ 0<t<b,\ \ \alpha>-1,$$
$$p_-(t)\equiv\big(|p(t)|-p(t)\big)/2.$$

In [1] it is stated that problem (1), (2) is uniquely solvable if and only if the homogeneous problem

$$u'''=p(t)u \tag{1_0}$$

under the boundary conditions (2) has only a trivial solution. Based on this fact the following theorem is proved.

Theorem. *Let there exist a continuous function $w:]a,b[\to]a,b[$ such that along with* (3) *the following conditions*

$$\sup\bigg\{\int\limits_a^b\frac{g_k(t,s)}{w(t)}\,w(s)p_-(s)\,ds:\ a<t<b\bigg\}<1, \tag{4}$$

$$\liminf_{t\to a}\frac{w(t)}{(t-a)^2}>0,\quad \liminf_{t\to b}\frac{w(t)}{(b-t)^{m_k}}>0 \tag{5}$$

hold, where $m_k=(1-k+|1-k|)/2$. Then problem (1), (2) *has a unique solution.*

Corollary 1. *If for some $\alpha>-1$ along with* (3) *the conditions*

$$p(t)\geq -p_k(t;\alpha)\ \ \text{for}\ \ a<t<b, \tag{6}$$
$$\operatorname{mes}\big\{t\in\,]a,b[:\ p(t)>-p_k(t;\alpha)\big\}>0 \tag{7}$$

hold, then problem (1), (2) *has a unique solution.*

Corollary 2. *If along with* (3) *the condition*

$$\int\limits_a^b(t-a)^2\Delta_k(t)p_-(t)\,dt<2 \tag{8}$$

holds, then problem (1), (2) *has a unique solution.*

Remark 1. In the above formulated theorem, inequality (4) is unimprovable and it cannot be replaced by the nonstrict inequality

$$\sup\bigg\{\int\limits_a^b\frac{g_k(t,s)}{w(t)}\,w(s)p_-(s)\,ds:\ a<t<b\bigg\}\leq 1. \tag{9}$$

Indeed, if

$$p(t)\equiv -p_k(t;\alpha),\quad w(t)\equiv\big(r_k(\alpha)(b-a)^{\alpha+1}-(b-t)^{\alpha+1}\big)(t-a)^2,$$

where $\alpha>-1$, then inequalities (5) are satisfied, while inequality (4) is violated instead of which inequality (9) holds. On the other hand, in this case the homogeneous problem (1_0), (2) has a nontrivial solution $u(t)\equiv w(t)$ and, consequently, problem (1), (2) is not uniquely solvable no matter how the function q is.

Remark 2. The strict inequality (7) in Corollary 1 cannot be replaced by the nonstrict one since if $p(t) \equiv -p_k(t;\alpha)$, then the homogeneous problem $(1_0),(2)$ has a nontrivial solution.

Remark 3. In the case, where $k \in \{1,2\}$, the strict inequality (8) in Corollary 2 cannot be replaced by the condition

$$\int_a^b (t-a)^2 \Delta_k(t) p_-(t)\,dt < 2+\varepsilon \tag{10}$$

no matter how small $\varepsilon > 0$ is. Indeed, if $p(t) \equiv -p_k(t;\alpha)$ and $\alpha > 0$ is so large that

$$r_k(\alpha) > 1 + \frac{2}{\varepsilon}\,,$$

then inequality (8) is violated but inequality (9) holds. On the other hand, as we already mentioned above, in this case the homogeneous problem $(1_0),(2)$ has a nontrivial solution.

Particular cases of the boundary conditions (2) are the Dirichlet boundary conditions

$$u(a+) = 0, \quad u'(a+) = 0, \quad u(b-) = 0, \tag{2_0}$$

and the Nicoletti boundary conditions

$$u(a+) = 0, \quad u'(a+) = 0, \quad u'(b-) = 0, \tag{2_1}$$

$$u(a+) = 0, \quad u'(a+) = 0, \quad u''(b-) = 0. \tag{2_2}$$

For problem $(1),(2_k)$ $(k=0,1,2)$, a pair of conditions (6), (7) has one of the following three forms:

$$p(t) \geq -\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{(b-a)^{\alpha+1}-(t-a)^{\alpha+1}}\,(t-a)^{\alpha-2} \ \text{ for } \ a<t<b, \tag{6_0}$$

$$\operatorname{mes}\Big\{t\in\,]a,b[\,:\ (t-a)^{2-\alpha}p(t) > -\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{(b-a)^{\alpha+1}-(t-a)^{\alpha+1}}\Big\} > 0; \tag{7_0}$$

$$p(t) \geq -\frac{2(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+3)(b-a)^{\alpha+1}-2(t-a)^{\alpha+1}}\,(t-a)^{\alpha-2} \ \text{ for } \ a<t<b, \tag{6_1}$$

$$\operatorname{mes}\Big\{t\in\,]a,b[\,:\ (t-a)^{2-\alpha}p(t) > -\frac{2(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+3)(b-a)^{\alpha+1}-2(t-a)^{\alpha+1}}\Big\} > 0; \tag{7_1}$$

$$p(t) \geq -\frac{2(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+2)(\alpha+3)(b-a)^{\alpha+1}-2(t-a)^{\alpha+1}}\,(t-a)^{\alpha-2} \ \text{ for } \ a<t<b, \tag{6_2}$$

$$\operatorname{mes}\Big\{t\in\,]a,b[\,:\ (t-a)^{2-\alpha}p(t) > -\frac{2(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+2)(\alpha+3)(b-a)^{\alpha+1}-2(t-a)^{\alpha+1}}\Big\} > 0. \tag{7_2}$$

Corollary 3. *Let for some $k \in \{0,1,2\}$ along with (3) conditions (6_k) and (7_k) be satisfied. Then problem $(1),(2_k)$ has a unique solution.*

Corollary 4. *If for some $k \in \{0,1,2\}$ along with (3) the condition*

$$\int_a^b (t-a)^2 (b-t)^{2-k} p_-(t)\,dt < 2(b-a)^{2-k} \tag{11}$$

is satisfied, then problem $(1),(2_k)$ has a unique solution.

Remark 4. The strict inequality (7_k) in Corollary 3 cannot be replaced by the nonstrict one, while inequality (11) in Corollary 4 for some $k \in \{1, 2\}$ cannot be replaced by the inequality

$$\int\limits_a^b (t-a)^2 (b-t)^{2-k} p_-(t)\, dt < (2+\varepsilon)(b-a)^{2-k}$$

no matter how small $\varepsilon > 0$ is.

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[1] I. Kiguradze, On the unique solvability of two-point boundary value problems for third order linear differential equations with singularities. *Trans. A. Razmadze Math. Inst.* **175** (2021), no. 3, 375–390.

Second Order Nonoscillatory Singular Linear Differential Equations

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In the present report, we give necessary and sufficient conditions for the nonoscillation and strong nonoscillation of second order singular linear homogeneous differential equations. These results are new for differential equations with continuous coefficients as well and generalize the classical results by Lyapunov [6], Hartman and Wintner [2], and Vallée Poussin [7]. They also generalize the theorems on the nonoscillation of singular differential equations given in the papers [1, 3, 5] and in our report [4].

On a finite open interval $]a,b[$, we consider the differential equation

$$u''(t)=p_1(t)u(t)+p_2(t)u'(t), \tag{1}$$

where $p_1,p_2:\,]a,b[\,\to\mathbb{R}$ are measurable functions, satisfying one of the following three conditions:

$$\int_a^b \big((t-a)|p_1(t)|+|p_2(t)|\big)\,dt<+\infty, \tag{2_1}$$

$$\int_a^b \big((b-t)|p_1(t)|+|p_2(t)|\big)\,dt<+\infty, \tag{2_2}$$

$$\int_a^b \big((t-a)(b-t)|p_1(t)|+|p_2(t)|\big)\,dt<+\infty. \tag{3}$$

We do not exclude the case, where

$$\int_a^b |p_1(t)|\,dt=+\infty,$$

i.e. the case when the function p_1 has nonintegrable singularity at least at one of the boundary points of the interval $]a,b[$. In such case equation (1) is said to be singular.

A continuously differentiable function $u:\,]a,b[\,\to\mathbb{R}$ is said to be **a solution to equation** (1) if its first derivative is absolutely continuous on every closed interval contained in $]a,b[$ and equation (1) is satisfied almost everywhere in $]a,b[$.

We assume that the values of a solution to equation (1) and its derivative at the points a and b are the corresponding one-sided limits of those functions if such limits exist.

It is well-known [5] that if condition (3) is satisfied, then any solution to equation (1) has finite right and left limits at the points a and b, and if the right limit (the left limit) of this solution is zero at the point a (at the point b), then its first derivative has a finite right (left) limit at the point a (at the point b).

Definition 1. Equation (1) is said to be **nonoscillatory** on the interval $[a,b]$ if its every solution has no more than one zero on that interval.

Definition 2. Equation (1) is said to be **strongly nonoscillatory from the right** (**strongly nonoscillatory from the left**) on the interval $[a,b]$ if for any $t_0 \in [a,b[$ (for any $t_0 \in]a,b]$), an arbitrary nontrivial solution u to equation (1), satisfying the condition

$$u(t_0)=0,$$

satisfies also the inequality

$$u'(t)\neq 0 \ \text{ for } \ t_0<t\leq b \quad \big(u'(t)\neq 0 \ \text{ for } \ a\leq t<t_0\big).$$

We use the following notation.

$$[x]_+=\frac{|x|+x}{2}, \quad [x]_-=\frac{|x|-x}{2} \ \text{ for } \ x\in\mathbb{R}.$$

If $w:]a,b[\to\mathbb{R}$ is a differentiable function, then

$$\begin{aligned} h_1(p_1,p_2,w)(t)&=[p_1(t)]_-w(t)+[p_2(t)]_-w'(t),\\ h_2(p_1,p_2,w)(t)&=[p_1(t)]_-w(t)-[p_2(t)]_+w'(t). \end{aligned}$$

Theorem 1$_1$. *Let condition* (2_1) *hold and let there exist a continuously differentiable function* $w:[a,b]\to[0,+\infty[$ *such that*

$$w(a)=0, \ \ w'(t)>0, \ \int\limits_t^b h_1(p_1,p_2,w)(s)\,ds\leq w'(t) \ \textit{ for } \ a\leq t<b, \tag{4_1}$$

$$\limsup_{t\to b}\int\limits_t^b \frac{h_1(p_1,p_2,w)(s)}{w'(t)}\,ds<1. \tag{5_1}$$

Then the differential equation (1) *is strongly nonoscillatory from the right on* $[a,b]$.

Theorem 1$_2$. *Let condition* (2_2) *hold and let there exist a continuously differentiable function* $w:[a,b]\to[0,+\infty[$ *such that*

$$w(b)=0, \ \ w'(t)<0, \ \int\limits_a^t h_2(p_1,p_2,w)(s)\,ds\leq |w'(t)| \ \textit{ for } \ a<t\leq b, \tag{4_2}$$

$$\limsup_{t\to a}\int\limits_a^t \frac{h_2(p_1,p_2,w)(s)}{|w'(t)|}\,ds<1. \tag{5_2}$$

Then the differential equation (1) *is strongly nonoscillatory from the left on* $[a,b]$.

Theorem 2_1. *Let along with condition* (2_1) *the condition*

$$p_1(t) \le 0, \quad p_2(t) \le 0 \ \text{ for } \ a < t < b \tag{6_1}$$

hold. Then for the differential equation (1) *to be strongly nonoscillatory from the right on* $[a,b]$, *necessary and sufficient is the existence of such a continuously differentiable function* $w : [a,b] \to [0,+\infty[$, *which satisfies conditions* (4_1) *and* (5_1).

Theorem 2_2. *Let along with condition* (2_2) *the condition*

$$p_1(t) \le 0, \quad p_2(t) \ge 0 \ \text{ for } \ a < t < b \tag{6_2}$$

hold. Then for the differential equation (1) *to be strongly nonoscillatory from the left on* $[a,b]$, *necessary and sufficient is the existence of such a continuously differentiable function* $w : [a,b] \to [0,+\infty[$, *which satisfies conditions* (4_2) *and* (5_2).

Theorems 1_1 and 1_2 yield unimprovable effective conditions guaranteeing the strong nonoscillation from the right and left of the differential equation (1) on the interval $[a,b]$. Namely, the following statements are valid.

Corollary 1_1. *Let along with condition* (2_1) *one of the following three conditions hold:*

$$\int\limits_a^b \big((t-a)[p_1(t)]_- + [p_2(t)]_-\big)\,dt \le 1, \tag{7_1}$$

$$p_1(t) \ge -\frac{\lambda_1(t-a)^\alpha}{(\alpha+3)(b-a)^{\alpha+2}-(t-a)^{\alpha+2}}\,, \quad p_2(t) \ge -\lambda_2(t-a)^{\alpha+1} \ \text{ for } \ a<t<b, \tag{8_1}$$

$$p_1(t) \ge -\ell_1, \quad p_2(t) \ge -\ell_2 \ \text{ for } \ a<t<b, \tag{9_1}$$

where $\alpha > -2$, *while* λ_i *and* ℓ_i $(i=1,2)$ *are nonnegative constants such that*

$$\frac{\lambda_1}{\alpha+3} + (b-a)^{\alpha+2}\lambda_2 < \alpha+2, \tag{10}$$

$$\int\limits_0^{+\infty} \frac{dx}{\ell_1+\ell_2x+x^2} > b-a. \tag{11}$$

Then the differential equation (1) *is strongly nonoscillatory from the right on* $[a,b]$.

Corollary 1_2. *Let along with condition* (2_2) *one of the following three conditions hold:*

$$\int\limits_a^b \big((b-t)[p_1(t)]_- + [p_2(t)]_+\big)\,dt \le 1, \tag{7_2}$$

$$p_1(t) \ge -\frac{\lambda_1(b-t)^\alpha}{(\alpha+3)(b-a)^{\alpha+2}-(b-t)^{\alpha+2}}\,, \quad p_2(t) \le \lambda_2(b-t)^{\alpha+2} \ \text{ for } \ a<t<b, \tag{8_2}$$

$$p_1(t) \ge -\ell_1, \quad p_2(t) \le \ell_2 \ \text{ for } \ a<t<b, \tag{9_2}$$

where $\alpha > -2$, *while* λ_i *and* ℓ_i $(i=1,2)$ *are nonnegative constants satisfying inequalities* (10) *and* (11). *Then the differential equation* (1) *is strongly nonoscillatory from the left on* $[a,b]$.

Remark 1. In the right-hand sides of inequalities (7_1) and (7_2), 1 cannot be replaced by $1+\varepsilon$ no matter how small $\varepsilon > 0$ is, and the strict inequalities (10) and (11) cannot be replaced by the non-strict ones.

Theorem 3. *Let condition* (3) *hold and there exist a number* $t_0 \in\,]a,b[$ *and continuously differentiable functions* $w_1 : [a,t_0] \to [0,+\infty[\,$, $w_2 : [t_0,b] \to [0,+\infty[$ *such that*

$$w_1(a)=0, \quad w_1'(t)>0, \quad \int\limits_t^{t_0} h_1(p_1,p_2,w_1)(s)\,ds \le w_1'(t) \ \ \text{for} \ \ a \le t < t_0, \tag{12}$$

$$w_2(b)=0, \quad w_2'(t)<0, \quad \int\limits_{t_0}^{t} h_2(p_1,p_2,w_2)(s)\,ds < |w_2'(t)| \ \ \text{for} \ \ t_0 < t \le b, \tag{13}$$

$$\limsup_{t\to t_0} \int\limits_t^{t_0} \frac{h_1(p_1,p_2,w_1)(s)}{w_1'(t)}\,ds + \limsup_{t\to t_0} \int\limits_{t_0}^{t} \frac{h_2(p_1,p_2,w_2)(s)}{|w_2'(t)|}\,ds < 2. \tag{14}$$

Then the differential equation (1) *is nonoscillatory on* $[a,b]$.

Theorem 4. *If*

$$p_1(t) \le 0, \quad p_2(t)=0 \ \ \text{for} \ \ a<t<b, \quad \int\limits_a^b (t-a)|p_1(t)|\,dt < +\infty,$$

then for the differential equation (1) *to be nonoscillatory on* $[a,b]$, *necessary and sufficient is the existence of such a number* $t_0 \in\,]a,b[$ *and continuously differentiable functions* $w_1 : [a,t_0] \to [0,+\infty[\,$, $w_2 : [t_0,b] \to [0,+\infty[\,$, *which satisfy conditions* (12)–(14).

Corollary 2. *Let along with inequality* (3), *for some* $t_0 \in\,]a,b[$ *one of the following three conditions hold:*

$$\int\limits_a^{t_0} \big((t-a)[p_1(t)]_- + [p_2(t)]_-\big)\,dt \le 1, \quad \int\limits_{t_0}^{b} \big((b-t)[p_1(t)]_- + [p_2(t)]_+\big)\,dt \le 1, \tag{15}$$

$$\begin{gathered} p_1(t) \ge -\frac{\lambda_{11}}{(2t_0-a-t)(t-a)}\,, \quad p_2(t) \ge -\lambda_{12} \ \ \text{for} \ \ a<t<t_0, \\ p_1(t) \ge -\frac{\lambda_{21}}{(b+t-2t_0)(b-t)}\,, \quad p_2(t) \le \lambda_{22} \ \ \text{for} \ \ t_0<t<b, \end{gathered} \tag{16}$$

$$p_1(t) \ge -\ell_{11}, \ \ p_2(t) \ge -\ell_{12} \ \ \text{for} \ \ a<t<t_0, \quad p_1(t) \ge -\ell_{21}, \ \ p_2(t) \le \ell_{22} \ \ \text{for} \ \ t_0<t<b, \tag{17}$$

where λ_{ik} *and* ℓ_{ik} $(i,k=1,2)$ *are nonnegative constants such that*

$$\lambda_{11} + 2(t_0-a)\lambda_{12} < 2, \quad \lambda_{21} + 2(b-t_0)\lambda_{22} < 2, \tag{18}$$

$$\int\limits_0^{+\infty} \frac{dx}{\ell_{11}+\ell_{12}x+x^2} > t_0-a, \quad \int\limits_0^{+\infty} \frac{dx}{\ell_{21}+\ell_{22}x+x^2} > b-t_0. \tag{19}$$

Then the differential equation (1) *is nonoscillatory on* $[a,b]$.

Remark 2. In the right-hand sides of inequalities (15), 1 cannot be replaced by $1+\varepsilon$ no matter how small $\varepsilon>0$ is, and the strict inequalities (18) and (19) cannot be replaced by the non-strict ones.

If

$$\int_a^b \left(\frac{(t-a)(b-t)}{b-a}\,[p_1(t)]_- + |p_2(t)|\right) dt \leq 1, \tag{20}$$

then there exists $t_0 \in \,]a,b[$ such that inequalities (15) are satisfied.

On the other hand, it is obvious that if for some nonnegative constants ℓ_1 and ℓ_2 the inequalities

$$p_1(t) \geq -\ell_1, \quad |p_2(t)| \leq \ell_2 \ \text{ for } \ a < t < b, \tag{21}$$

$$\int_0^{+\infty} \frac{dx}{\ell_1 + \ell_2 x + x^2} > (b-a)/2 \tag{22}$$

are satisfied, then inequalities (17) and (19) hold as well, where $\ell_{11} = \ell_{21} = \ell_1$, $\ell_{12} = \ell_{22} = \ell_2$, $t_0 = (a+b)/2$. Therefore, the following statements are valid.

Corollary 3. *If*

$$\int_a^b (t-a)(b-t)|p_1(t)|\,dt < +\infty \tag{23}$$

and inequality (20) *holds, then the differential equation* (1) *is nonoscillatory on* $[a,b]$.

Corollary 4. *Let there exist nonnegative constants* ℓ_1 *and* ℓ_2 *such that along with* (23) *conditions* (21) *and* (22) *are satisfied. Then the differential equation* (1) *is nonoscillatory on* $[a,b]$.

In the case, where p_1 and p_2 are continuous on $[a,b]$ functions, Corollary 3 implies the theorems by Lyapunov [6] and Hartman–Wintner [2], while Corollary 4 yields the Vallée Poussin theorem [7].

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On a Dirichlet Type Boundary Value Problem in an Orthogonally Convex Cylinder for a Class of Linear Partial Differential Equations

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Let $\Omega=(0,\omega_1)\times(0,\omega_2)\times(0,\omega_3)$ be an open rectangular box, and let

$$E=\big\{(x_1,x_2,x_3)\in\Omega:\ (x_1,x_2)\in\mathbf{D},\ x_3\in(0,\omega_3)\big\}$$

be an *orthogonally convex* cylinder with a piecewise smooth base inscribed in Ω. In view of the orthogonal convexity of the cylinder E, its base $\mathbf{D}$ admits the representations

$$\mathbf{D}=\big\{x_1\in(0,\omega_1),\ x_2\in(\gamma_1(x_1),\gamma_2(x_1))\big\}=\big\{x_2\in(0,\omega_2),\ x_1\in(\eta_1(x_2),\eta_2(x_2))\big\}.$$

In the domain E consider the boundary value problem

$$u^{(\mathbf{2})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}),\tag{1}$$

$$u(\eta_k(x_2),x_2,x_3)=\psi_1(\eta_k(x_2),x_2,x_3),\quad u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=\psi_2(x_1,\gamma_k(x_1),x_3),$$
$$u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=\psi_3(x_1,x_2,(k-1)\omega_3)\ \ (k=1,2).\tag{2}$$

Here $\mathbf{x}=(x_1,x_2,x_3)$, $\mathbf{2}=(2,2,2)$, $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index, $u^{(\boldsymbol{\alpha})}(\mathbf{x})=\frac{\partial^{\alpha_1+\alpha_2+\alpha_3}u(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$, $p_{\boldsymbol{\alpha}}\in C(\overline{E})$ $(\boldsymbol{\alpha}<\mathbf{2})$, $q\in C(\overline{E})$, $\psi_i\in C(\overline{E})$ $(i=1,2,3)$ and $\overline{E}$ is the closure of E.

By a solution of problem (1),(2) we understand a *classical* solution, i.e., a function $u\in C^{2,2,2}(E)$ having continuous on $\overline{E}$ partial derivatives $u^{(2,0,0)}$ and $u^{(2,2,0)}$, and satisfying equation (1) and the boundary conditions (2) everywhere in E and ∂E, respectively.

Throughout the paper the following notations will be used:

$\mathbf{0}=(0,0,0)$, $\mathbf{1}=(1,1,1)$, $\boldsymbol{\alpha}_i=(0,\dots,\alpha_i,\dots,0)$, $\boldsymbol{\alpha}_{ij}=\boldsymbol{\alpha}_i+\boldsymbol{\alpha}_j$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)<\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\alpha_i\le\beta_i$ $(i=1,2,3)$ and $\boldsymbol{\alpha}\ne\boldsymbol{\beta}$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)\le\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.

$\|\boldsymbol{\alpha}\|=|\alpha_1|+|\alpha_2|+|\alpha_3|$.

$\boldsymbol{\Xi}=\{\boldsymbol{\sigma}\big|\ \mathbf{0}<\boldsymbol{\sigma}<\mathbf{1}\}$.

$\boldsymbol{\Upsilon}_{\mathbf{2}}=\big\{\boldsymbol{\alpha}<\mathbf{2}:\ \alpha_i=2$ for some $i\in\{1,2,3\}\big\}$.

$\boldsymbol{O}_{\mathbf{2}}=\big\{\boldsymbol{\alpha}<\mathbf{2}:\ \|\boldsymbol{\alpha}\|$ is odd$\big\}$.

$\operatorname{supp}\boldsymbol{\alpha}=\{i\mid\ \alpha_i>0\}$.

$\mathbf{x}_{\boldsymbol{\alpha}}=(\chi(\alpha_1)x_1,\chi(\alpha_2)x_2,\chi(\alpha_3)x_3)$, where $\chi(\alpha)=0$ if $\alpha=0$, and $\chi(\alpha)=1$ if $\alpha>0$.

$\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}=\mathbf{x}-\mathbf{x}_{\boldsymbol{\alpha}}$.

$\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(x_{i_1},\dots,x_{i_l})$, where $\{i_1,\cdots i_l\}=\operatorname{supp}\boldsymbol{\alpha}$. Furthermore, $\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(\mathbf{x}_{\boldsymbol{\alpha}},\widehat{\mathbf{0}}_{\boldsymbol{\alpha}})$, and $\mathbf{x}$ will be identified with $(\mathbf{x}_{\boldsymbol{\alpha}},\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})$, or with $(\mathbf{x}_{\boldsymbol{\alpha}},\mathbf{x}_{\widehat{\boldsymbol{\alpha}}})$.

$\Omega_{\boldsymbol{\sigma}}=[0,\omega_{i_1}]\times\cdots\times[0,\omega_{i_l}]$, where $\{i_1,\cdots i_l\}=\operatorname{supp}\boldsymbol{\sigma}$.

$\Omega_{ij}=(0,\omega_i)\times(0,\omega_j)$ $(1\le i<j\le3)$.

Along with problem (1), (2) consider the corresponding homogeneous problem

$$u^{(\mathbf{2})}=\sum_{\alpha<\mathbf{2}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}, \tag{1_0}$$

$$u(\eta_k(x_2),x_2,x_3)=0,\ \ u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=0,\ \ u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=0\ \ (k=1,2). \tag{2_0}$$

For each $\boldsymbol{\sigma}\in\boldsymbol{\Xi}$ in the domain $\Omega_{\boldsymbol{\sigma}}$ consider the homogeneous boundary value problem depending on the parameter $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}}\in\Omega_{\widehat{\boldsymbol{\sigma}}}$:

$$v^{(2,0,0)}=p_{022}(\mathbf{x}_1,\widehat{\mathbf{x}}_1)v+p_{122}(\mathbf{x}_1,\widehat{\mathbf{x}}_1)v^{(1,0,0)}, \tag{1_{100}}$$
$$v(\eta_1(\mathbf{x}_2),\widehat{\mathbf{x}}_1)=0,\quad v(\eta_2(\mathbf{x}_2),\widehat{\mathbf{x}}_1)=0; \tag{2_{100}}$$

$$v^{(0,2,0)}=p_{202}(\mathbf{x}_2,\widehat{\mathbf{x}}_2)v+p_{212}(\mathbf{x}_2,\widehat{\mathbf{x}}_2)v^{(0,1,0)}, \tag{1_{010}}$$
$$v(\gamma_1(\mathbf{x}_1),\widehat{\mathbf{x}}_2)=0,\quad v(\gamma_2(\mathbf{x}_1),\widehat{\mathbf{x}}_2)=0; \tag{2_{010}}$$

$$v^{(0,0,2)}=p_{220}(\mathbf{x}_3,\widehat{\mathbf{x}}_3)v+p_{221}(\mathbf{x}_3,\widehat{\mathbf{x}}_3)v^{(0,0,1)}, \tag{1_{001}}$$
$$v(0,\widehat{\mathbf{x}}_3)=0,\quad v(\omega_3,\widehat{\mathbf{x}}_3)=0; \tag{2_{001}}$$

$$v^{(\mathbf{2}_{12})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{12}}p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{12}}(\mathbf{x}_{12},\widehat{\mathbf{x}}_{12})v^{(\boldsymbol{\alpha})}, \tag{1_{110}}$$
$$v(\eta_k(\mathbf{x}_2),\widehat{\mathbf{x}}_{12})=0,\quad v^{(2,0,0)}(\gamma_k(\mathbf{x}_1),\widehat{\mathbf{x}}_{12})=0\quad (k=1,2); \tag{2_{110}}$$

$$v^{(\mathbf{2}_{13})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{13}}p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{13}}(\mathbf{x}_{13},\widehat{\mathbf{x}}_{13})v^{(\boldsymbol{\alpha})}, \tag{1_{101}}$$
$$v(\eta_k(\mathbf{x}_2),\widehat{\mathbf{x}}_{13})=0,\quad v^{(2,0,0)}((k-1)\omega_3,\widehat{\mathbf{x}}_{13})=0\ \ (k=1,2); \tag{2_{101}}$$

$$v^{(\mathbf{2}_{23})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{23}}p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{23}}(\mathbf{x}_{23},\widehat{\mathbf{x}}_{23})v^{(\boldsymbol{\alpha})}, \tag{1_{011}}$$
$$v(\gamma_k(\mathbf{x}_1),\widehat{\mathbf{x}}_{23})=0,\quad v^{(2,0,0)}((k-1)\,\omega_3,\widehat{\mathbf{x}}_{23})=0\ \ (k=1,2). \tag{2_{011}}$$

Definition 1. Problem $(1_{\boldsymbol{\sigma}}),(2_{\boldsymbol{\sigma}})$ $(\boldsymbol{\sigma}\in\boldsymbol{\Xi})$ is called $\boldsymbol{\sigma}$-associated problem of problem (1), (2).

Two-dimensional versions of problem (1), (2) were studied in [1], [2], where problems were considered in orthogonally convex smooth domains.

Orthogonal convexity of a domain is essential and cannot be relaxed. Examples attesting the paramount importance of the orthogonal convexity of a domain were introduced in Remarks 1 and 2 of [2]. Similar examples can be easily constructed for the three-dimensional case.

As follows from Remark 5 below, the C^2 regularity of functions η_k $(k=1,2)$ is essential for solvability of problem (1), (2) in a classical sense. However, C^2 regularity of functions η_k $(k=1,2)$ on the closed interval $[0,\omega_2]$ is impossible for smooth domains. Therefore we study the case of a piecewise smooth domain $\mathbf{D}$ separately from the case of a smooth domain $\mathbf{D}$. *Surprisingly*, some piecewise domains are better suited for the solvability of problem (1), (2), then domains with a C^∞ boundary.

Set

$$\mathbf{D}_{0,\delta}=[\eta_1(0)-\delta,\eta_1(0)+\delta]\times[0,\delta],\quad E_{0,\delta}=\mathbf{D}_{0,\delta}\times[0,\omega_3],$$
$$\mathbf{D}_{\omega_2,\delta}=[\eta_1(\omega_2)-\delta,\eta_1(\omega_2)+\delta]\times[\omega_2-\delta,\omega_2],\quad E_{\omega_2,\delta}=\mathbf{D}_{\omega_2,\delta}\times[0,\omega_3],$$
$$\varphi_{1k}(x_2,x_3)=\psi_1(\eta_k(x_2),x_2,x_3),\quad \varphi_{2k}(x_1,x_3)=\psi_2(x_1,\gamma_k(x_1),x_3),$$
$$\varphi_{3k}(x_1,x_2)=\psi_3(x_1,x_2,(k-1)\,\omega_3)\ \ (k=1,2). \tag{3}$$

Theorem 1. *Let*

$$\eta_k \in C^2([0,\omega_2]) \ \ (k=1,2), \tag{4}$$

$p_{\boldsymbol{\alpha}} \in C(\overline{E})$ $(\boldsymbol{\alpha} < \mathbf{2})$, $q \in C(\overline{\Omega})$, $\psi_1 \in C^{2,2,2}(\overline{E})$, $\psi_2 \in C^{0,2,2}(\overline{E})$, $\psi_1 \in C^{0,0,2}(\overline{E})$, *and let each* $\boldsymbol{\sigma}$*-associated problem* $(1_{\boldsymbol{\sigma}}),(2_{\boldsymbol{\sigma}})$ *have only the trivial solution for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$. *Then problem* $(1),(2)$ *has the Fredholm property, i.e.:*

- (i) *problem* $(1_0),(2_0)$ *has a finite dimensional space of solutions;*
- (ii) *if problem* $(1_0),(2_0)$ *has only the trivial solution, then problem* $(1),(2)$ *is uniquely solvable, its solution belongs to* $C^{2,2,2}(\overline{E})$ *and admits the estimate*

$$\|u\|_{C^{2,2,2}(\overline{E})} \le M\bigg(\|q\|_{C(\overline{E})} + \sum_{k=1}^{2}\Big(\|\varphi_{1k}\|_{C^{2,2}(\overline{\Omega}_{2,3})} + \|\varphi_{2k}\|_{C^{0,2}(\overline{\Omega}_{1,3})} + \|\varphi_{3k}\|_{C(\overline{\mathbf{D}})}\Big)\bigg), \tag{5}$$

where M is a positive constant independent of φ_{1k}, φ_{2k}, φ_{3k} $(k=1,2)$ *and* q.

Theorem 2. *Let*

$$\gamma_k \in C^2((0,\omega_1)), \quad \eta_k \in C^2((0,\omega_2)) \ \ (k=1,2), \tag{6}$$

$$p_{\boldsymbol{\alpha}} \in C^{0,2,0}(\overline{E}) \ \ (\alpha_2 = 2, \ \boldsymbol{\alpha} < \mathbf{2}), \tag{7}$$

$p_{\boldsymbol{\alpha}} \in C(\overline{E})$ $(\boldsymbol{\alpha} < \mathbf{2})$, $q \in C(\overline{\Omega})$, $\psi_1 \in C^{2,2,2}(\overline{E})$, $\psi_2 \in C^{0,2,2}(\overline{E})$, $\psi_1 \in C^{0,0,2}(\overline{E})$, *and let each* $\boldsymbol{\sigma}$*-associated problem* $(1_{\boldsymbol{\sigma}}),(2_{\boldsymbol{\sigma}})$ *have only the trivial solution for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$. *Then problem* $(1),(2)$ *has the Fredholm property, i.e.:* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$. *Then problem* $(1),(2)$ *has the Fredholm property, i.e.:*

- (i) *problem* $(1_0),(2_0)$ *has a finite dimensional space of solutions;*
- (ii) *if problem* $(1_0),(2_0)$ *has only the trivial solution, then problem* $(1),(2)$ *is uniquely solvable, its solution belongs to* $C^{2,2,2}(E)$ *and admits the estimate*

$$\begin{aligned}\|u\|_{C(\overline{E})} + \|u^{(2,0,0)}\|_{C(\overline{E})} + \|u^{(2,0,2)}\|_{C(\overline{E})}& \\ \le M\bigg(\|q\|_{C(\overline{E})} + \sum_{k=1}^{2}\Big(\|\varphi_{1k}\|_{C^{0,2}(\overline{\Omega}_{2,3})} &+ \|\varphi_{2k}\|_{C^{0,2}(\overline{\Omega}_{1,3})} + \|\varphi_{3k}\|_{C(\overline{\mathbf{D}})}\Big)\bigg),\end{aligned} \tag{8}$$

where M is a positive constant independent of φ_{1k}, φ_{2k}, φ_{3k} $(k=1,2)$ *and* q.

Furthermore, if:

(F_1) $\mathbf{D}$ *is* ***strongly convex*** *near the points* $(\eta_1(0),0)$ *and* $\eta_2(\omega_2,\omega_2)$, *i.e.*

$$\gamma_1''(\eta_1(0)) > 0 \ \ \textit{and} \ \ \gamma_2''(\eta_2(\omega_2)) < 0; \tag{9}$$

(F_2) $\gamma_1 \in C^5([\eta_1(0)-\delta, \eta_1(0)+\delta])$ *and* $\gamma_2 \in C^5([\eta_1(\omega_2)-\delta, \eta_1(\omega_2)+\delta])$ *for some* $\delta > 0$;

(F_3) $\psi_1 \in C^{5,0,0}(E_{0,\delta} \cup E_{\omega_2,\delta})$ *for some* $\delta > 0$;

(F_4) $\psi_2 \in C^{1,0,0}(E_{0,\delta} \cup E_{\omega_2,\delta})$ *for some* $\delta > 0$;

(F_5) $\psi_3 \in C^{3,0,0}(\mathbf{D}_{0,\delta} \cup \mathbf{D}_{\omega_2,\delta})$ *for some* $\delta > 0$;

(F_6) $p_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} < \mathbf{2})$, $q \in C^{3,0,0}(E_{0,\delta} \cup E_{\omega_2,\delta})$ *for some* $\delta > 0$, *then every solution of problem* (1), (2) *belongs to* $C^{2,2,2}(\overline{E})$.

Consider the equation

$$
\begin{aligned}
u^{(2,2,2)} = p_{220}(x_3)u^{(2,2,0)} + p_{202}(x_2)u^{(2,0,2)} + p_{022}(x_1)u^{(0,2,2)} \\
+ p_{200}(x_2,x_3)u^{(2,0,0)} + p_{020}(x_1,x_3)u^{(0,2,0)} + p_{002}(x_1,x_2)u^{(0,0,2)} \\
+ p_{201}(x_2)u^{(2,0,1)} + p_{102}(x_2)u^{(1,0,2)} + p_{021}(x_1)u^{(0,2,1)} + p_{012}(x_1)u^{(0,1,2)} \\
+ p_{111}u^{(1,1,1)} + p_{100}(x_2,x_3)u^{(1,0,0)} + p_{010}(x_1,x_3)u^{(0,1,0)} + p_{001}(x_1,x_2)u^{(0,0,1)} \\
+ p_{000}(x_1,x_2,x_3)u + q(x_1,x_2,x_3). \qquad (10)
\end{aligned}
$$

Theorem 3. *Let condition* (4) *hold, let the domain* $\mathbf{D}$ *be convex, i.e.*

$$(-1)^{k-1}\eta_k''(x_2) \geq 0 \ \textit{for} \ x_2 \in (0,\omega_2) \ \ (k=1,2), \qquad (11)$$

and let

$$p_{220}(x_3) \geq 0, \ \ p_{202}(x_2) \geq 0, \ \ p_{022}(x_1) \geq 0, \qquad (12)$$

$$p_{200}(x_2,x_3) \leq 0, \ \ p_{020}(x_1,x_3) \leq 0, \ \ p_{002}(x_1,x_2) \leq 0, \qquad (13)$$

$$p_{000}(x_1,x_2,x_3) \geq 0. \qquad (14)$$

Then problem (10), (2) *is uniquely solvable, and its solution admits estimate* (5).

Theorem 4. *Let conditions* (6) *and inequalities* (11)–(14) *hold. Then problem* (10), (2) *is uniquely solvable, its solution belongs to* $C^{2,2,2}(\overline{\Omega})$ *and admits estimate* (8).
Furthermore, if conditions (F_1)–(F_6) *hold, then the solution of problem* (1), (2) *belongs to* $C^{2,2,2}(\overline{E})$.

Remark 1. Condition (F_1) on the strong convexity of $\mathbf{D}$ is essential for the existence of a classical solution of problem (1), (2), and it cannot be replaced by strict convexity. Indeed, consider the problem

$$u^{(2,2,2)} = 0, \qquad (15)$$

$$u(\eta_k(x_2),x_2,x_3)=0; \ \ u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=2x_3^2; \ \ u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=0 \ \ (k=1,2) \qquad (16)$$

in the domain $E = D \times (0,\omega_3)$, where $\mathbf{D} = \{(x_1,x_2) : \ (x_1-1)^4 + (x_2-1)^4 < 1\}$. It is clear that D is strictly convex, but not strongly convex, since

$$\gamma_k(x_1) = 1 + (-1)^k \sqrt[4]{1-(x_1-1)^4} \ \ (k=1,2), \quad \gamma_k''(x_1) > 0 \ \text{ for } \ x_1 \in (0,1) \cup (1,2),$$

and

$$\gamma_k''(1) = 0 \ \ (k=1,2).$$

As a result, the unique solution $u(\mathbf{x}) = \big((x_1-1)^2 - \sqrt{1-(x_2-1)^4}\,\big)x_3^2$ of problem (15), (16) does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,1,2)}$ is discontinuous along the rectangle $x_2 = 1$, $(x_1,x_3) \in [0,2] \times [0,\omega_3]$.

Remark 2. Consider the problem

$$u^{(2,2,2)} = 0, \qquad (17)$$

$$u(\eta_k(x_2),x_2,x_3) = \psi_1(\eta_k(x_2),x_2,x_3); \quad u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3) = 0; \\ u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3) = 0 \ \ (k=1,2) \qquad (18)$$

in the domain $E=D\times(0,\omega_3)$, where $\mathbf{D}=\{(x_1,x_2):(x_1-1)^2+(x_2-1)^2<1\}$, and

$$\psi_1(x_1,x_2,x_3)=\begin{cases}0 & \text{for } 0\le x_1\le 1\\ (x_1-1)^{4+\alpha} & \text{for } 1\le x_1\le 2\end{cases}.$$

It is clear that D is strongly convex domain with the C^∞ boundary, and $\psi_1\in C^4(\overline{E})$ but $\psi_1\notin C^5(\overline{E})$ if $\alpha\in[0,1)$. As a result, the unique solution of problem (17), (18)

$$u(\mathbf{x})=\frac{x_1-\eta_2(x_2)}{\eta_1(x_2)-\eta_2(x_2)}\,\psi_1\Big(1+\sqrt{1-(x_2-1)^2}\,\Big)=\frac{\sqrt{1-(x_2-1)^2}-x_1}{2}\,\big(1-(x_2-1)^2\big)^{2+\frac{\alpha-1}{2}}$$

does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,2,0)}$ and $u^{(1,2,0)}$ are discontinuous along the line segments $(1,0,x_3)$ and $(1,2,x_3)$, $x_3\in[0,\omega_3]$.

Remark 3. Consider the problem

$$u^{(2,2,2)}=0, \tag{19}$$

$$u(\eta_k(x_2),x_2,x_3)=0;\;\; u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=x_3^2;\;\; u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=0\;\;(k=1,2) \tag{20}$$

in the domain $E=D\times(0,\omega_3)$, where $\mathbf{D}$ is a strongly convex C^2 domain inscribed in the rectangle $[0,2]\times[0,1]$ such that

$$\gamma_2(x_1)=1-(x_1-1)^2+|x_1-1|^{4+\alpha}\;\;\text{for}\;\;\frac12<x_1<\frac32\,.$$

It is clear that if $\alpha\in[0,1)$, then $\gamma_2\in C^4([1-\delta,1+\delta])$ but $\gamma_2\notin C^5([1-\delta,1+\delta])$ for any $\delta>0$. Also,

$$\eta_1(x_2)=1+(1-x_2)^{\frac12}\Big(1+c\,(1-x_2)^{\frac{2+\alpha}{2}}+o((1-x_2)^{\frac32})\Big)\;\;\text{for}\;\;x_2\in[1-\delta,1],$$
$$\eta_1(x_2)=1-(1-x_2)^{\frac12}\Big(1-c\,(1-x_2)^{\frac{2+\alpha}{2}}+o((1-x_2)^{\frac32})\Big)\;\;\text{for}\;\;x_2\in[1-\delta,1]$$

for some $\delta>0$, where c is a nonzero constant. As a result, problem (19), (20) has a unique solution

$$u(\mathbf{x})=(x_1-\eta_1(x_2))(x_2-\eta_1(x_2))=x^2-x(\eta_1(x_2)+\eta_2(x_2))-\eta_1(x_2)\eta_2(x_2),$$

which does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,2,0)}$ and $u^{(1,2,0)}$ are discontinuous along the line segment $(1,1,x_3)$, $x_3\in[0,\omega_3]$.

Remark 4. Consider the problem

$$u^{(2,2,2)}=2|x_1-1|^\alpha\operatorname{sgn}(x_1-1), \tag{21}$$

$$u(\eta_k(x_2),x_2,x_3)=0\quad u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=|x_1-1|^\alpha\operatorname{sgn}(x_1-1)x_3(x_3-\omega_3);$$
$$u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=0\;\;(k=1,2) \tag{22}$$

in the domain $E=D\times(0,\omega_3)$, where $\mathbf{D}$ is a strongly convex C^2 domain inscribed in the rectangle $[0,2]\times[0,1]$ such that

$$\gamma_2(x_1)=1-(x_1-1)^2\;\;\text{for}\;\;\frac12<x_1<\frac32\,.$$

It is clear that if $\alpha\in(2,3)$, then $\psi_2(x_1,x_2,x_3)=|x_1|^\alpha\operatorname{sgn}x_1x_3(x_3-\omega_3)\in C^{1,0,0}(E_{\omega_2,\delta}\cup E_{\omega_2,\delta})$ for some $\delta>0$. Thus conditions $(F_1)-(F_5)$ hold, while condition (F_6) is violated.

As a result, problem (21), (22) has a unique solution

$$u(\mathbf{x}) = \frac{|x_1 - 1|^{2+\alpha}\,\mathrm{sgn}(x_1 - 1) - (x_1 - 1)(1 - x_2)^{\frac{1+\alpha}{2}}}{(1+\alpha)(2+\alpha)}\, x_3(x_3 - \omega_2) \ \text{ for } \ \frac{1}{2} < x_1 < \frac{3}{2}\,,$$

which does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,2,0)}$ and $u^{(1,2,0)}$ are discontinuous along the line segment $(1, 1, x_3)$, $x_3 \in (0, \omega_3)$.

Remark 5. As we see, the functions γ_k $(k = 1, 2)$ can be piecewise smooth: γ_k may be nondifferentiable at points $\eta_1((k-1)\omega_2)$ and $\eta_2((k-1)\omega_2)$ (if they differ) $(k = 1, 2)$. On the other hand, C^2 smoothness of the functions η_k is essential and cannot be relaxed. Indeed, let $\alpha \in (1, 2)$ be an arbitrary number,

$$\eta_k(x_2) = 1 + (-1)^k \sqrt{1 - \Big|x_2 - \frac{1}{2}\Big|^{\alpha}} \ \ (k = 1, 2),$$

and let u be a solution of the problem

$$u^{(2,2,2)} = 0, \tag{23}$$

$$u(\eta_k(x_2), x_2, x_3) = 0; \ \ u^{(2,0,0)}(x_1, \gamma_k(x_1), x_3) = x_3^2; \ \ u^{(2,2,0)}(x_1, x_2, (k-1)\omega_3) = 0 \ \ (k = 1, 2). \tag{24}$$

Then

$$u^{(0,0,2)}(x_1, x_2, x_3) = x_1^2 - 2x_1 + |x_2 - 1|^{\alpha}.$$

Consequently, $u^{(0,1,2)}(x_1, x_2, x_3)$ is continuous on $\overline{E}$, however $u^{(0,2,2)}(x_1, x_2, x_3)$ is discontinuous along the line segment $0 \le x_1 \le 2$, $x_2 = 1$ since $\alpha \in (1, 2)$. Thus, problem (23), (24) is not solvable in a classical sense due to the fact that the functions η_k are not twice differentiable at $x_2 = 1$.

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Gaussian Process for Heat Equation Numerical Solution

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The purpose of the present study is to develop a Machine Learning (ML) approach to the solution of the partial differential equations (PDEs). Due to this being our first attempt in this direction, in this note, we consider the simple heat equation.

In the domain $\Omega = (0,1) \times (0,T)$, $T = const > 0$, let us consider the initial-boundary value problem for the heat equation:

$$\begin{gathered} \frac{\partial U(x,t)}{\partial t} - a\,\frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t), \ \ (x,t) \in \Omega, \\ U(-1,t) = U(1,t) = 0, \ \ t \in [0,T], \\ U(x,0) = U_0(x), \ \ x \in [-1,1], \end{gathered} \tag{1}$$

where a is a positive constant and U_0 is a given function.

Our aim is to find the approximate solution $u(t,x)$ at $t > 0$ of problem (1). Introducing the uniform grid for the time variable $t_n = \tau \cdot n$, $\tau = T/N$ and applying Euler scheme, we get

$$u_n(x) = u_{n-1}(x) + a\tau\,\frac{d^2 u_{n-1}(x)}{dx^2} + \tau f_n(x), \ \ n = 1, \dots, N, \tag{2}$$

where N is a positive integer and $u_n(x) = u(t_n, x)$.

Although there are many methods for solving, even more, complex PDEs (see, for example, [2,3] and the references therein), our purpose, as we already mentioned, is to apply one of the well-known ML methods for solving problem (1). In particular, our goal is to design the Gaussian Process (GP) [6,9] for the heat equation to predict the solution [7,8].

The GP is an extension of Multivariate Gaussian Distribution. In turn, the multivariate Gaussian distribution is a generalization of the one-dimensional normal distribution to higher dimensions. For example, if there are inputs from two-dimensional space, then for any cross-section over the fixed one-dimensional input we get Gaussian distribution along each axis (see, Figure 1).

Probability density function (pdf) in two-dimensional space is given as follows:

$$pdf(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\, e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}}.$$

In general, the pdf of the Multivariate Gaussian distribution in d dimensions is defined by the following formula:

$$pdf(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}}\, e^{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)},$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_d)$ is mean vector of $\mathbf{x} = (x_1, x_2, \dots, x_d)$ and Σ^{-1} is the inverse of the $d \times d$ positively defined covariance matrix $\Sigma = cov[\mathbf{x}]$, which is constructed by one of the so-called covariance functions [6,9].

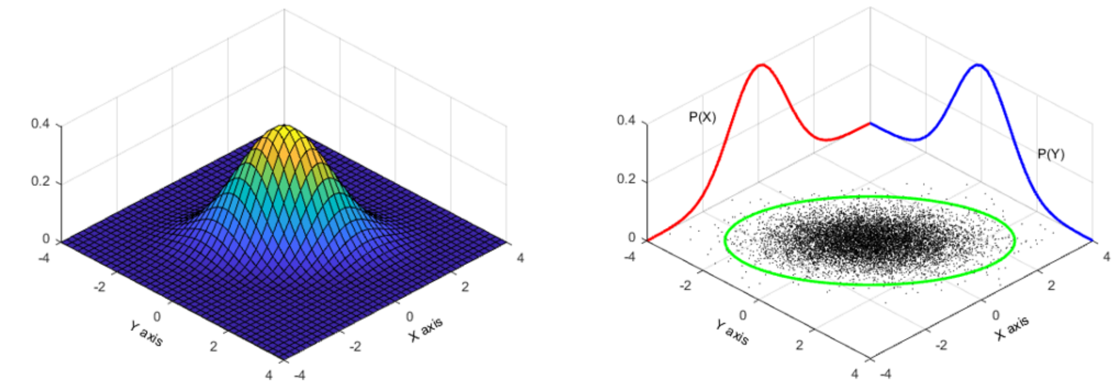

Figure 1. 2D multivariate Gaussian distribution and its cross-section projections.

GP presents one of the most important ML approaches based on a particularly effective method for placing a prior distribution over the space of functions [1, 6, 9]. GP can serve as an effective algorithm for function approximation. As an example let us consider samples from the GP, mean function, and some observation points where the values of the approximated function are known. Figures 2 depict 5 sample functions from the prior distribution over functions specified by a particular Gaussian with two (left) and four (right) observation points. Sample functions are plotted as dashed lines, the mean function is shown as a black solid line, observed points represented as red crosses, and the shaded region denotes uncertainty region. As it can be seen the uncertainty region is narrowing when the number of observed points is increasing. The equations for obtaining the mean function, which can be considered as the function approximation can be derived from the Sherman–Morrison–Woodbury formula [1, 5, 6, 9].

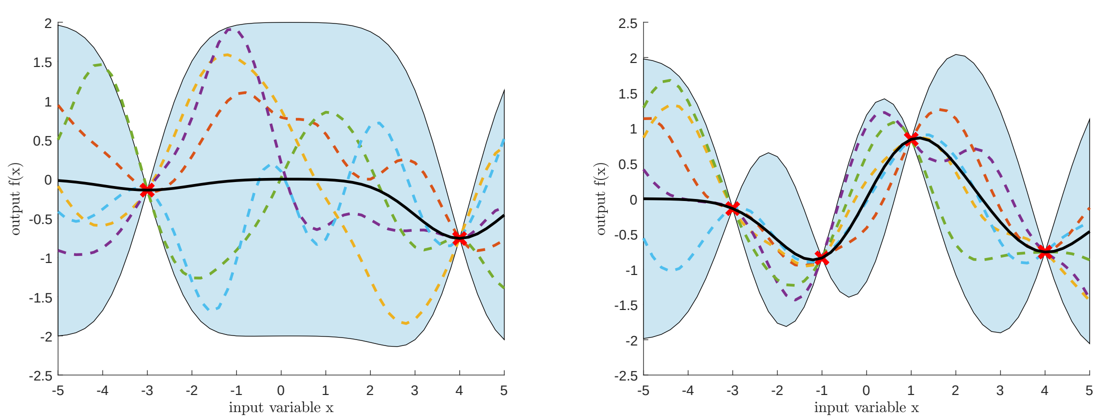

Figure 2. Fife samples from Gaussian posterior (dashed) and its mean (solid black) with the dataset of two and four points (red crosses). For colored figures, please refer to the online version.

Coming back to problem (1), let us set GP prior on u_{n-1} according to [6, 9]

$$u_{n-1}(x) \sim \mathcal{GP}(0, k_{n-1,n-1}(x, x', \theta)), \tag{3}$$

where $k_{n-1,n-1}$ is the kernel (covariance function) of the GP and θ represents the vector of the hyper-parameters of the covariance function [6–9].

Note that there are many different types of covariance functions. In this study, the neural network covariance function is used [9]

$$k(x,x',\theta)=\frac{2}{\pi}\,\sin^{-1}\bigg(\frac{2(\sigma_0^2+\sigma^2xx')}{\sqrt{(1+2(\sigma_0^2+\sigma^2x^2))(1+2(\sigma_0^2+\sigma^2x'^2))}}\bigg), \tag{4}$$

where θ is two component hyper-parameter vector $\theta=(\sigma_0,\sigma)$.

The hyper-parameter θ can be trained by applying the initial (x_0,U_0), boundary (x_n^b,u_n^b) and already collected training data (x_{n-1},u_{n-1}) and Negative Log Marginal Likelihood resulting from [7,8]

$$\begin{bmatrix} u_n^b\\ u_{n-1}^b\end{bmatrix}\sim\mathcal{N}(0,K),$$

where

$$K=\begin{bmatrix} k_{n,n}(x_n^b,x_n^b) & k_{n,n-1}(x_n^b,x_{n-1})\\ & k_{n-1,n-1}(x_{n-1},x_{n-1})\end{bmatrix}.$$

To predict approximation at new point x_n^*, the following conditional distribution can be used

$$u_n(x_n^*)|\begin{bmatrix} u_n^b\\ u_{n-1}\end{bmatrix}\sim\mathcal{N}\bigg(q^TK^{-1}\begin{bmatrix} u_n^b\\ u_{n-1}\end{bmatrix},k_{n,n}(x_n^*,x_n^*)-q^TK^{-1}q\bigg),$$

where

$$q^T=\begin{bmatrix} k_{n,n}(x_n^*,x_n^b) & k_{n,n-1}(x_n^*,x_{n-1})\end{bmatrix}.$$

It is known that linear operations on GP give again GP and thus, taking into account the Euler scheme (2) together with GP prior assumption (3) allows to conclude that u_n and u_{n-1} are jointly Gaussian with the following GP [4,6–9]

$$\begin{bmatrix} u_n\\ u_{n-1}\end{bmatrix}\sim\mathcal{GP}\bigg(0,\begin{bmatrix} k_{n,n} & k_{n,n-1}\\ & k_{n-1,n-1}\end{bmatrix}\bigg),$$

where covariance functions are defined using the (4):

$$\begin{aligned}
k_{n,n}&=k,\\
k_{n,n-1}&=k-a\tau\,\frac{d^2}{dx'^2}\,k-\tau f_n(x'),\\
k_{n-1,n-1}&=k-a\tau\,\frac{d^2}{dx'^2}\,k-\tau f_n(x')-a\tau\,\frac{d^2}{dx^2}\,k-\tau f_n(x)+a^2\tau^2\,\frac{d^2}{dx^2}\,\frac{d^2}{dx'^2}\,k-a\tau^2f_n(x').
\end{aligned}$$

For the test experiment we chouse the right-hand side of problem (1) in such a way that the exact solution is $U(x,t)=-\exp(-0.01\pi t)\sin(\pi x)$ with the initial solution $U_0(x)=-\sin(\pi x)$.

Figures 3 show a pretty good agreement between numerical and exact solutions for different time values.

In the end, let us note that our future work is aimed to apply the mentioned methodology to the PDEs with nonlinear diffusion coefficients as well as for the spatial multi-dimensional cases.

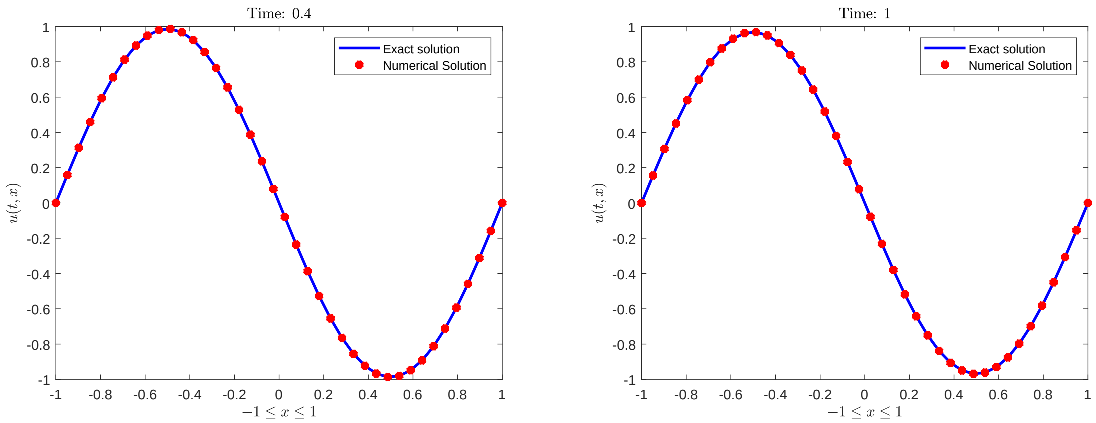

Figure 3. Exact and Numerical solutions at $t = 0.4$ and 1.

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On The Behavior of Solutions to Third Order Differential Equations with General Power-Law Nonlinearities and Positive Potentia

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1 Introduction

Consider solutions to the third order differential equation with general power-law nonlinearities

$$y''' + p(x, y, y', y'')|y|^{k_0}|y'|^{k_1} \cdots |y''|^{k_2} \operatorname{sgn}(yy'y'') = 0, \tag{1.1}$$

with positive real nonlinearity exponents k_0, k_1, k_2 and positive continuous in x and Lipschitz continuous in u_0, u_1, u_2 bounded function $p(u_0, u_1, u_2)$.

The results on qualitative behavior and asymptotic estimates of positive increasing solutions for higher order nonlinear differential equations were obtained by I. T. Kiguradze and T. A. Chanturia in [9]. Questions on qualitative and asymptotic behavior of solutions to higher order Emden–Fowler differential equations ($k_1 = \cdots = k_{n-1} = 0$) were studied by I. V. Astashova in [1, 2, 5, 6].

Equation (1.1) in the case $k_0 > 0$, $k_0 \neq 1$, $k_1 = k_2 = 0$, was studied by I. Astashova in [2, Chapters 6–8]. In particular, asymptotic classification of solutions to such equations was given in [4, 6], and proved in [3]. For third order and higher order differential equations, nonlinear with respect to derivatives of solutions, the asymptotic behavior of certain types of solutions was studied by V. M. Evtukhov, A. M. Klopot in [7, 8]. Qualitative properties of solutions to (1.1) in the case $p(x, y, y', y'') < 0$ were studied in [10].

2 Main results

Since solutions to equation (1.1) are not always unique, in order to obtain the full classification the following notion of μ-solutions is used.

Definition ([1]). A solution $y\colon (a, b) \to \mathbb{R}$, $-\infty \le a < b \le +\infty$ to an ordinary differential equation is a *μ-solution,* if

(1) the equation has no other solutions equal to y on some subinterval (a, b) and not equal to y at some point in (a, b);

(2) the equation either has no solution equal to y on (a, b) and defined on another interval containing (a, b) or has at least two such solutions which differ from each other at points arbitrary close to the boundary of (a, b).

Theorem 2.1. *Let the function $p(u_0, u_1, u_2)$ be continuous, Lipschitz continuous in u_0, u_1, u_2 and satisfying the inequalities $0 < m \le p(u_0, u_1, u_2) \le M$. Then any μ-solution $y(x)$ to equation* (1.1) *according to its qualitative behavior belongs to one of the following types:*

(1) *constant function $y(x) \equiv y_0$;*

(2) *linear function* $y(x) = ax + b$, $a \neq 0$;

(3) *function with exactly one extremum.*

Remark. Let the function $p(u_0, u_1, u_2)$ be continuous, Lipschitz continuous in u_0, u_1, u_2 and satisfying the inequalities $0 < m \le p(x, u, v, w) \le M$. Then the replacemenets $x \mapsto -x$ and $y(x) \mapsto -y(x)$ reduce equation (1.1) to the equation

$$z''' + \widetilde{p}(x, z, z', z'')|z|^{k_0}|z'|^{k_1}|z''|^{k_2} \operatorname{sgn}(zz'z'') = 0,$$

with the function $\widetilde{p}(u_0, u_1, u_2)$ also continuous, Lipschitz continuous in u_0, u_1, u_2 and satisfying the inequalities $0 < m \le p(u_0, u_1, u_2) \le M$.

Thus, it is sufficient to consider the behavior of the solutions with positive initial data near the right boundaries of their domains. In the case of a constant potential $p(u_0, u_1, u_2)$ the following results of the behavior of solutions was obtained.

Theorem 2.2. *Let* $k_2 - k_0 \neq 2$ *and* $p(u_0, u_1, u_2) \equiv p_0 > 0$. *Then any* μ*-solution* $y(x)$ *to* (1.1), *satisfying at some point* x_0 *the conditions* $y(x_0) \ge 0$, $y'(x_0) \ge 0$, $y''(x_0) > 0$ *has the following behavior near the right boundary of its domain:*

(1) *if* $0 < k_2 \le 1$, *then there exists* $x^* < +\infty$ *such that* $y(x), y'(x) \to const$, $y''(x) \to 0$ *as* $x \to x^* - 0$;

(2) *if* $1 < k_2 \le 2$, *then* $y(x) \to +\infty$, $y'(x) \to const$, $y''(x) \to 0$ *as* $x \to +\infty$;

(3) *if* $2 < k_2 < 2 + k_0$, *then* $y(x) \to +\infty$, $y'(x) \to const$, $y''(x) \to 0$ *as* $x \to +\infty$ *or* $y(x)$, $y'(x) \to +\infty$, $y''(x) \to 0$ *as* $x \to +\infty$;

(4) *if* $k_2 > 2 + k_0$, *then* $y(x)$, $y'(x) \to +\infty$, $y''(x) \to 0$ *as* $x \to +\infty$.

Acknowledgement

The reported study was supported by RSF, project # 20-11-20272.

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On k-Point Approximations for the Izobov Sigma-Exponent

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Consider a linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with piecewise continuous and bounded coefficient matrix A such that

$$\|A(t)\| \leq M < +\infty \ \text{ for all } \ t \geq 0.$$

We denote the Cauchy matrix of (1) by X_A and the highest Lyapunov exponent of (1) by $\lambda_n(A)$. Together with system (1) consider a perturbed system

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

with piecewise continuous and bounded perturbation matrix Q such that

$$\|Q(t)\| \leq N_Q \exp(-\sigma t), \ \ t \geq 0. \tag{3}$$

Denote the higher exponent of (2) by $\lambda_n(A+Q)$.

Let $\mathfrak{M}_\sigma(A)$ be the set of all perturbations Q satisfying condition (3) and having the appropriate dimensions. Any $Q \in \mathfrak{M}_\sigma$ is said to be a sigma-perturbation and the number

$$\nabla_\sigma(A) := \sup\big\{\lambda_n(A+Q) : \ Q \in \mathfrak{M}_\sigma(A)\big\}$$

is called [4], [6, p. 225], [5, p. 214] the highest sigma-exponent or the Izobov exponent of system (1). It was proved in [4] that the Izobov exponent can be evaluated by means of the following algorithm:

$$\nabla_\sigma(A) = \varlimsup_{m\to\infty} \frac{\xi_m(\sigma)}{m}$$
$$\xi_m(\sigma) = \max_{i<m} \big(\ln\|X_A(m,i)\| + \xi_i(\sigma) - \sigma i\big), \quad \xi_1 = 0, \ \ i \in \mathbb{N}.$$

It was proved in [1,7] that $\nabla_\sigma(A)$ is a convex monotonically decreasing function on $[0,+\infty[$ such that

$$\nabla_\sigma(A) = \lambda_n(A) \ \text{ for all } \ \sigma > \sigma_0(A),$$

where $2M \geq \sigma_0(A) \geq 0$ is a critical value of σ for system (1).

Some alternative representation for $\nabla_\sigma(A)$ was given in [10]. Let $\mathcal{D}(m)$ be the set of all nonempty $d \subset \{1,\dots,m-1\} \subset \mathbb{N}$. Further we assume that for each $d \in \mathcal{D}(m)$ the elements of d are arranged in the increasing order, so that $d_1 < d_2 < \dots < d_s$ and $d = \{d_1, d_2, \dots, d_s\}$, where $s = |d|$ is the number of elements of the set d. We also put

$$\|d\| := d_1 + \dots + d_s \ \text{ for } \ d \in \mathcal{D}(m)$$

and
$$\|d\| := 0 \ \text{ for } \ d = \varnothing.$$
In addition, for the sake of convenience we assume that $d_0 = 0$ and $d_{s+1} = m$ for each $d \in \mathcal{D}_0(m) := \mathcal{D}(m) \cup \{\varnothing\}$. Note that we do not include these additional elements into the set d. Under the above assumptions, let us define the quantity $\Xi(m, d)$ as
$$\Xi(m, d) := \sum_{i=0}^{s} \ln \|X_A(d_{i+1}, d_i)\|,$$
where $m \in \mathbb{N}$, $d \in \mathcal{D}(m)$ and $s := |d|$. From [2, 10] we can assert that
$$\xi_m(\sigma) = \max_{d \in \mathcal{D}_0(m)} \big(\Xi(m, d) - \sigma \|d\|\big).$$
Thus, we have
$$\nabla_\sigma(A) = \overline{\lim_{m \to \infty}}\, m^{-1} \max_{d \in \mathcal{D}_0(m)} \big(\Xi(m, d) - \sigma \|d\|\big). \tag{4}$$

The main advantage of the $\nabla_\sigma(A)$ is attainability. By virtue of this property we are sure that for each $\varepsilon > 0$ there exists a perturbed system (2) with $Q \in \mathfrak{M}_\sigma$ such that
$$\lambda_n(A + Q) > \nabla_\sigma(A) - \varepsilon.$$
When constructing perturbations Q to provide the values of $\lambda_n(A+Q)$ close to $\nabla_\sigma(A)$, we urgently need to know some (or all) sequences $d(m) \in \mathcal{D}_0(m)$, $m \in \mathbb{N}$, such that
$$\nabla_\sigma(A) = \lim_{m \to \infty} m^{-1} \big(\Xi(m, d(m)) - \sigma \|d(m)\|\big). \tag{5}$$
A primary information on this issue is given by the following statement (see Property A in [4] and Lemma 9.1 in [5, p. 215]).

Proposition 1. *If $b \in \mathcal{D}_0(m)$ satisfies the condition*
$$\xi_m(\sigma) = \Xi(m, b) - \sigma \|b\|,$$
then
$$b_{i+1} - b_i \geq \frac{\sigma}{2M}\, b_i \ \ \textit{for each} \ \ i \in \{1, \dots, s\},$$
where $b = \{b_1, \dots, b_s\}$, $s = |b|$.

Based on the theory of characteristic vectors (see [3,8]) and some results on Malkin estimates [9] we can assert that some information on the sequences $d(m)$ in (5) can be extracted from the slopes of supporting lines to the graph of $\nabla_\sigma(A)$. Since the results of this type are not available yet, here we consider some simplified version of the problem hoping to use it later to clarify the general case. To this end, we limit the number of partition points d_i in (4) on each segment $[0, m]$ by some number $k \in \mathbb{N}$.

Let $\mathcal{D}^k(m) \subset \mathcal{D}(m)$, $k \in \mathbb{N}$, be the set of all $d \in \mathcal{D}(m)$ such that $|d| \leq k$. Let us also put $\mathcal{D}_0^k(m) := \mathcal{D}^k(m) \cup \{\varnothing\}$.

Definition 1. The number
$$\nabla_\sigma^k(A) = \overline{\lim_{m \to \infty}}\, m^{-1} \max_{d \in \mathcal{D}_0^k(m)} \big(\Xi(m, d) - \sigma \|d\|\big)$$
is said to be the k-point approximation for $\nabla_\sigma(A)$.

The introduced concept inherits some basic properties of $\nabla_\sigma(A)$.

Proposition 2. *For each $k \in \mathbb{N}$ the following statements are valid.*

(i) $\nabla_\sigma(A) \geq \nabla_\sigma^k(A) \geq \lambda_n(A)$.

(ii) $\nabla_\sigma^k(A)$ *is a convex monotonically decreasing function on* $[0, +\infty[$ *such that*

$$\nabla_\sigma(A) = \lambda_n(A) \ \text{ for all } \ \sigma > \sigma_0(A).$$

(iii) *If $b \in \mathcal{D}_0^k(m)$ satisfies the condition*

$$\Xi(m, b) - \sigma\|b\| = \max_{d \in \mathcal{D}_0^k(m)} \big(\Xi(m, d) - \sigma\|d\|\big), \tag{6}$$

then

$$\sigma\|b\| \leq 2Mm.$$

Note that (i) is an immediate consequence of Definition 1 and (iii) is a weakened analog of Proposition 1.

For each $\sigma > 0$, let us denote the set of all $b \in \mathcal{D}_0^k(m)$ satisfying condition (6) by $\mathcal{B}_\sigma^k(m)$. Obviously, $\mathcal{B}_\sigma^k(m)$ contains more then one element in general. Now take $b \in \mathcal{B}_\sigma^k(m)$ with maximal and minimal value of $m^{-1}\|b\|$ and denote them by τ_m and β_m, respectively. Finally, put

$$\mathrm{B}_\sigma(A) = \varliminf_{m \to \infty} \beta_m, \quad \mathrm{T}_\sigma(A) = \varlimsup_{m \to \infty} \tau_m.$$

Recall that a supporting line to the graph of some convex function $f : [0, +\infty[\to \mathbb{R}$ at a point $(s, f(s))$, where $s \in [0, +\infty[$, intersects the graph of the f at $(s, f(s))$ and lies beneath the graph everywhere in the domain of the function f. If f is differentiable at $s \in [0, +\infty[$, then there exists a unique supporting line at $(s, f(s))$ and this line coincides with the tangent drawn at the same point. We denote the set of slopes of supporting lines drawn at $s \in [0, +\infty[$ to the graph of f by $\mathcal{S}_s(f)$. It can be easily seen that each $\mathcal{S}_s(f)$ is a segment of the real axis.

Theorem. *The set $\mathcal{S}_\sigma(\nabla_\sigma^k(A))$ coincides with the segment* $[\mathrm{B}_\sigma(A), \mathrm{T}_\sigma(A)]$.

A similar statement is supposed to be valid for the original sigma-exponent. However, it has not yet been possible to find the reasonable representation for bounds of $\mathcal{S}_\sigma(\nabla_\sigma(A))$.

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On an Interpolation Boundary Value Problem

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1 Introduction

In the general case, the multipoint boundary value problem for a differential system

$$\mathcal{L}x = f \tag{1.1}$$

considered on the segment $[0,T]$ is a problem with boundary conditions $\ell x = \beta$ of the form

$$\ell x \equiv \sum_{i=0}^{m} \Lambda_i x(t_i) = \beta,$$

where $\{t_i\}$, $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T$, is a fixed collection of points from $[0,T]$, Λ_i, $i = 0, \dots, m$, are given $(N \times n)$-matrices, $\beta \in R^N$. Here we consider a special case of boundary conditions that correspond to the interpolation problem as a problem of trajectories taking prescribed values at the given points:

$$x(t_i) = \alpha_i, \ \ i = 0, \dots, m. \tag{1.2}$$

The problem under consideration can be interpreted as a part of the routing problem (see, for instance, [2]) namely as the task of implementing the route. Similar problems arise in Economic Dynamics [4,6] where α_i, $i = 1, \dots, m$, are given values of indicators to a modeled economic system at the time moments t_i.

In the case with no constraints with respect to the right-hand side f, for any collection α_i, $i = 0, \dots, m$, there exists $f : [0,T] \to R^n$ that provides the solvability of (1.1), (1.2). In contrast to this, if f is constrained by the inequalities

$$a_i \leqslant f_i(t) \leqslant b_i, \ \ i = 1, \dots, n, \ \ t \in [0,T], \tag{1.3}$$

there arises the task to describe a set of α_i, $i = 0, \dots, m$, for which (1.1), (1.2) is solvable.

Below we propose a way of constructing a hypercube P_N in R^N, $N = mn$ such that the condition $\alpha = \operatorname{col}(\alpha_1, \dots, \alpha_m) \in P_N$ is a sufficient condition to the solvability of (1.1), (1.2) in the sense that there exists an f with (1.3) such that the corresponding trajectory takes the values prescribed by (1.2).

First we descript a class of functional differential equations with linear Volterra operators and appropriate spaces where those are considered. The main relationships that allow to obtain sufficient conditions of the solvability to the problem under consideration are proposed. An illustrative example of application of the main theorem is presented.

2 Main constructions

We consider a quite broad class of functional differential systems with aftereffect and follow the notation and basic statements of the general functional differential theory in the part concerning linear systems with aftereffect [1, 3].

Let $L^n = L^n[0,T]$ be the Lebesgue space of all summable functions $z : [0,T] \to R^n$ defined on a finite segment $[0,T]$ with the norm

$$\|z\|_{L^n} = \int\limits_0^T |z(t)|\,dt,$$

where $|\cdot|$ is a norm in R^n. Denote by $AC^n = AC^n[0,T]$ the space of absolutely continuous functions $x : [0;T] \to R^n$ with the norm

$$\|x\|_{AC^n} = |x(0)| + \|\dot{x}\|_{L^n}.$$

In what follows we will use some results from [1, 3].

We consider the case of the system (1.1) with a linear bounded Volterra operator $\mathcal{L} : AC^n \to L^n$ such that the general solution of the equation (1.1) has the form

$$x(t) = X(t)x(0) + \int\limits_0^t C(t,s)f(s)\,ds, \tag{2.1}$$

where $X(t)$ is the fundamental matrix to the homogeneous equation $\mathcal{L}x = 0$, $C(t,s)$ is the Cauchy matrix. A broad class of operators $\mathcal{L}$ with the property (2.1) is described, for instance, in [5].

The properties of the Cauchy matrix used below are studied in detail in [3]. Without loss of generality we put in the sequel $x(0) = \alpha_0 = 0$ and $a_i < 0$, $b_i > 0$, $i = 1,\dots,n$.

Denote by E_n the identity $(n \times n)$-matrix, e_n^i stands for the i-th (from above) row of E_n. We define

$$P_0 = \prod_{i=1,\dots,n}^{n} [a_i, b_i].$$

Fix a positive integer K and put $\Delta = T/K$.

Let us describe the main steps and constructions on the way to sufficient solvability conditions for (1.1), (1.2) with constraints (1.3).

Define $(n \times N)$-matrix $M(s) = \operatorname{col}(M_1(s),\dots,M_m(s))$ by the equalities

$$M_i(s) = \chi_i(s)C(t_i,s), \quad i = 1,\dots,m, \tag{2.2}$$

where $\chi_i(s)$ is the characteristic function of the segment $[0,t_i]$.

For any $i = 1,\dots,N$ and $j = 1,\dots,K$ consider the following two linear programming problems

$$e_N^i M(\Delta\cdot j)v \to \max, \quad v \in P_0 \ \text{ and } \ e_N^i M(\Delta\cdot j)v \to \min, \quad v \in P_0. \tag{2.3}$$

Denote by v_{ij}^+ and v_{ij}^- solutions to the above problems, respectively,

$$v_{ij}^+ = \operatorname{argmax}\big(e_N^i M(\Delta\cdot j)v,\ v \in P_0\big) \ \text{ and } \ v_{ij}^- = \operatorname{argmin}\big(e_N^i M(\Delta\cdot j)v,\ v \in P_0\big). \tag{2.4}$$

Next define

$$d_i = e_N^i \int\limits_0^T M(s) \sum_{j=1}^K v_{ij}^+ \chi_{[\Delta(j-1),\Delta j]}(s)\,ds, \quad i = 1, \dots, N, \tag{2.5}$$

$$c_i = e_N^i \int\limits_0^T M(s) \sum_{j=1}^K v_{ij}^- \chi_{[\Delta(j-1),\Delta j]}(s)\,ds, \quad i = 1, \dots, N, \tag{2.6}$$

$$\mathcal{D}_k = \operatorname{diag}(d_1, \dots, d_k), \quad \mathcal{C}_k = \operatorname{diag}(c_1, \dots, c_k), \quad k = 1, \dots, N. \tag{2.7}$$

Introduce the following matrices:

$$\begin{gathered} A = \begin{pmatrix} \mathcal{D}_{N-1} & 0 \\ \mathcal{C}_{N-1} & 0 \\ \mathcal{D}_{N-1} & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} F_{N-1}^i \\ d_N e_N^N \end{pmatrix}, \quad i = 1, \dots, 2(N-1), \\ B_i = \begin{pmatrix} F_{N-1}^i \\ c_N e_N^N \end{pmatrix}, \quad i = 2(N-1)+1, \dots, 4(N-1). \end{gathered} \tag{2.8}$$

Here F_{N-1}^i is the i-th group of $N-1$ consecutive rows of matrix A. Each B_i gives a collection of N points in R^N that define the corresponding hyperplane and the corresponding halfspace with the zero point. The intersection of all above hyperplanes is the polyhedron of all attainable points in the considered interpolation problem. Now our task is to construct a hypercube that is a subset of the polyhedron.

For each of $4(N-1)$ mentioned hyperplanes we define the distance ρ_k from the hyperplane to the origin. Namely, let

$$p_1^k z_1 + p_2^k z_2 + \cdots + p_N^k z_n + q^k = 0$$

be the equation of the k-th hyperplane. Then we have

$$\rho_k = \frac{|q_k|}{\sqrt{\sum\limits_{i=1}^N (p_i^k)^2}}, \quad k = 1, \dots, 4(N-1). \tag{2.9}$$

It is clear that the ball $S(0,\rho)$ with the radius $\rho = \min(\rho_k,\ k = 1, \dots, 4(N-1))$ centered by the origin is a subset of the polyhedron defined by the all above hyperplanes, and, for any $\alpha = \operatorname{col}(\alpha_1, \dots, \alpha_m) \in S(0,\rho)$, the problem (1.1), (1.2), (1.3) is solvable. Finally we define the cube

$$P_N = \Big\{ z \in R^N : \ \max |z_i| \leqslant \frac{\rho}{\sqrt{N}}, \ i = 1, \dots, N \Big\}.$$

Thus we obtain

Theorem. *Let the set P_N be defined by the relationships* (2.2)–(2.9). *Then the interpolation problem* (1.1), (1.2), (1.3) *is solvable for any $\alpha \in P_N$.*

3 An example

Following [5], consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-1) + f_1(t), \\ \dot{x}_2(t) &= -x_2(t) + f_2(t); \end{aligned} \quad t \in [0,3] \tag{3.1}$$

where $x_2(s)=0$ if $s<0$, with the initial conditions

$$x_1(0)=0, \quad x_2(0)=0, \tag{3.2}$$

and the boundary conditions

$$x_1(2)=\alpha_1, \;\; x_2(2)=\alpha_2, \;\; x_1(3)=\alpha_3, \;\; x_2(3)=\alpha_4. \tag{3.3}$$

The right-hand sides $f_1(t)$ and $f_2(t)$ are constrained by the inequalities

$$-0.1\leqslant f_1(t)\leqslant 0.4, \;\; -0.2\leqslant f_2(t)\leqslant 0.5. \tag{3.4}$$

Here we have

$$C(t,s)=\begin{pmatrix} 1 & \displaystyle\int_s^t \chi_{[1,3]}(\tau)\chi_{[0,\tau-1]}(s)\exp(1-\tau+s)\,d\tau \\ 0 & \exp(s-t) \end{pmatrix},$$

After calculation the integral in C_{12} for $t=2$ and $t=3$ we obtain

$$C(2,s)=\begin{pmatrix} 1 & \begin{cases} 1-e^{s-1}, & s\in[0,1] \\ 0 & \text{otherwise} \end{cases} \\ 0 & \exp(s-2) \end{pmatrix}, \quad C(3,s)=\begin{pmatrix} 1 & \begin{cases} 1-e^{s-2}, & s\in[0,2] \\ 0 & \text{otherwise} \end{cases} \\ 0 & \exp(s-3) \end{pmatrix},$$

The elements $M_{ij}(s)$ of $M(s)$ are defined by the equalities

$$M_{11}(s)=\chi_{[0,2]}(s), \;\; M_{12}(s)=\chi_{[0,1]}(s)(1-e^{(s-1)}), \quad M_{21}(s)=0, \quad M_{22}(s)=\chi_{[0,2]}(s)e^{(s-2)},$$
$$M_{31}(s)=1, \quad M_{32}(s)=\chi_{[0,2]}(s)(1-e^{(s-2)}), \quad M_{41}(s)=0, \quad M_{42}(s)=e^{(s-3)}.$$

For the case of $K=20$, calculations by the rules (2.4), (2.5), (2.6) bring the values

$$d_1=0.5116, \;\; d_2=1.7360, \;\; d_3=0.4393, \;\; d_4=0.9409,$$
$$c_1=-0.2046, \;\; c_2=-0.5143, \;\; c3=-0.1757, \;\; c_4=-0.2594.$$

Here and below all real values are displayed to four places of decimals.

For all of 12 hyperplanes the corresponding distances $\rho_1,\dots,\rho_{12}$ are as follows:

$$\rho_1=0.3091, \;\; \rho_2=0.1809, \;\; \rho_3=0.1715, \;\; \rho_4=0.1278, \;\; \rho_5=0.1559, \;\; \rho_6=0.1629,$$
$$\rho_7=0.2033, \;\; \rho_8=0.1503, \;\; \rho_9=0.1447, \;\; \rho_{10}=0.1155, \;\; \rho_{11}=0.1350, \;\; \rho_{12}=0.1394.$$

Thus $\rho=0.1155$, and the inequality $\max\{\alpha_1,\dots,\alpha_4\}\leqslant 0.0577$ provides the solvability of (3.1)–(3.4).

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Disconjugacy and Green's Functions Sign for Some Two-Point Boundary Value Problems for Fourth Order Ordinary Differential Equations

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1 Introduction

Here we consider the question of the disconjugacy on the interval $I := [a,b] \subset [0,+\infty[$ of the fourth order linear ordinary differential equation

$$u^{(4)}(t) = p(t)u(t) - \mu u(t) \ \text{ for } \ t \in I, \tag{1.1}$$

when $p : I \to \mathbb{R}$ is Lebesgue integrable function, $\mu \in \mathbb{R}$, and the question of the Green's functions sign for equation (1.1) under one of the following two-point boundary conditions

$$u(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0,1,2), \tag{1.2$_1$}$$

$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0,1), \tag{1.2$_2$}$$

$$u^{(i)}(a) = 0 \ \ (i = 0,1,2), \quad u(b) = 0. \tag{1.2$_3$}$$

There are established the optimal sufficient conditions of disconjugacy of equation (1.1) when the coefficient p is not necessarily constant sign function. On the basis of these results we prove the necessary and sufficient conditions of non-negativity (non-positivity) of Green's function for problems $(1.1),(1.2_\ell)$ $(\ell \in \{1,2,3\})$, which are formulated in a terminology of eigenvalues of problems under the consideration.

Here we use the following notations.

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_0^- =]-\infty,\ 0]\,$, $\mathbb{R}_0^+ = [0,\ +\infty[\,$.

$C(I;\mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : \ t \in I\}$.

$\widetilde{C}^3(I;\mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives.

$L(I;\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$.

For arbitrary $x, y \in L(I;\,\mathbb{R})$, the notation

$$x(t) \preccurlyeq y(t) \quad \big(x(t) \succcurlyeq y(t)\big) \ \text{ for } \ t \in I,$$

means that $x \le y$ $(x \ge y)$ and $x \ne y$.

Also we use the notation $[x]_\pm = \frac{|x| \pm x}{2}$.

By a solution of equation (1.1) we understand a function $u \in \widetilde{C}^3(I;\mathbb{R})$ which satisfies equation (1.1) a.e. on I.

Also we need the following definition.

Definition 1.1. Equation

$$u^{(4)}(t) = p(t)u(t) \tag{1.3}$$

is said to be disconjugate (non-oscillatory) on I, if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity. Otherwise we say that equation (1.3) is oscillatory on I.

The given study is based on our previous results from the paper [3] and the results of A. Cabada and R. Enguica from [1]. For the formulation of our main results we need the following definitions and propositions from our previous paper.

Definition 1.2. We will say that $p \in D_+(I)$ if $p \in L(I;\mathbb{R}_0^+)$, and problem $(1.3),(1.2_2)$ has a solution u such that

$$u(t) > 0 \ \text{ for } \ t \in \,]a, b[\,.$$

Definition 1.3. We will say that $p \in D_-(I)$ if $p \in L(I;\mathbb{R}_0^-)$, and problem $(1.3),(1.2_3)$ has a solution u such that

$$u(t) > 0 \ \text{ for } \ t \in \,]a,\, b[\,.$$

The propositions below are Theorems 2, 4, and 6, respectively, from the paper [3].

Proposition 1.1. *Let $p \in L(I;\mathbb{R}_0^+)$. Then equation* (1.3) *is disconjugate on I iff there exists $p^* \in D_+(I)$ such that*

$$p(t) \preccurlyeq p^*(t) \ \text{ on } I.$$

Proposition 1.2. *Let $p \in L(I;\mathbb{R}_0^-)$. Then equation* (1.3) *is disconjugate on I iff there exists $p_* \in D_-(I)$ such that*

$$p(t) \succcurlyeq p_*(t) \ \text{ on } I.$$

Proposition 1.3. *Let $p_* \in D_-(I)$ and $p^* \in D_+(I)$. Then for an arbitrary function $p \in L(I;\, \mathbb{R})$ such that*

$$p_*(t) \preccurlyeq -[p(t)]_- \,, \quad [p(t)]_+ \preccurlyeq p^*(t) \ \text{ on } I, \tag{1.4}$$

equation (1.3) *is disconjugate on I.*

Remark 1.1. From Proposition 1.1 (Proposition 1.2) it is clear that the structure of the set $D_+(I)$ $(D_-(I))$ is such that if $x, y \in D_+(I) (x, y \in D_-(I))$, then none of the inequalities $x \preccurlyeq y$ and $y \preccurlyeq x$ hold.

Remark 1.2. If λ_1 (λ_2) is the first positive eigenvalue of the problem

$$\begin{gathered} u^{(4)}(t) = \lambda^4 u(t), \quad u^{(i)}(0) = 0, \quad u^{(i)}(1) = 0 \ \ (i = 0, 1) \\ (u^{(4)}(t) = -\lambda^4 u(t), \quad u^{(i)}(0) = 0 \ \ (i = 0, 1, 2), \quad u(1) = 0), \end{gathered}$$

then

$$\frac{\lambda_1^4}{(b-a)^4} \in D_+(I) \quad \Big(-\frac{\lambda_2^4}{(b-a)^4} \in D_-(I) \Big).$$

Also it is well-known (see [1] or [2]), that $\lambda_1 \approx 4.73004$ and $\lambda_2 \approx 5.553$.

2 Main results

First we consider the results on disconjugacy of equation (1.1) on the interval $[a,b]$.

Theorem 2.1. *Let $p \in L(I;\mathbb{R})$, conditions*

$$\begin{aligned}\alpha_p &:= \inf_{p^* \in D_+(I)} \Big\{ \operatorname{ess\,sup}_{t\in I} \big\{ p(t) - p^*(t) \big\} \Big\} \leq 0,\\ \beta_p &:= \sup_{p_* \in D_-(I)} \Big\{ \operatorname{ess\,inf}_{t\in I} \big\{ p(t) - p_*(t) \big\} \Big\} \geq 0,\end{aligned} \tag{2.1}$$

hold, and

$$\alpha_p \neq \beta_p.$$

Then equation (1.1) *is disconjugate on I if $\mu \in]\alpha_p, \beta_p[$.*

Remark 2.1. Theorem 2.1 is optimal in the sense that there exists $p \in L(I;\mathbb{R})$ such that if $\mu = \alpha_p$ or $\mu = \beta_p$, then equation (1.1) is oscillatory on I. Indeed, let $p(t) \equiv \frac{\lambda_1^4 - \lambda_2^4}{2(b-a)^4}$, where due to Remark 1.2 we have $\frac{\lambda_1^4}{(b-a)^4} \in D_+(I)$ and $-\frac{\lambda_2^4}{(b-a)^4} \in D_-(I)$. Then from Remark 1.1 it immediately follows that

$$\alpha_p = p(t) - \frac{\lambda_1^4}{(b-a)^4} \ \text{ and } \ \beta_p = p(t) + \frac{\lambda_2^4}{(b-a)^4}\,.$$

Therefore if $\mu = \alpha_p$ or $\mu = \beta_p$, then equation (1.1) is oscillatory on I.

Corollary 2.1. *Let $p \in D_+(I)$. Then $\beta_p > \frac{\lambda_2^4}{(b-a)^4}$, and equation* (1.1) *is disconjugate on I if $\mu \in]0, \beta_p[$.*

Corollary 2.2. *Let $p \in D_-(I)$. Then $\alpha_p < -\frac{\lambda_1^4}{(b-a)^4}$, and equation* (1.1) *is disconjugate on I if $\mu \in]\alpha_p, 0[$.*

From the last two corollaries we immediately have

Corollary 2.3. *Let $\mu \in]0, \frac{\lambda_1^4}{(b-a)^4}]$ $\big(\mu \in [-\frac{\lambda_2^4}{(b-a)^4}, 0[\big)$. Then equation* (1.1) *is disconjugate on I for an arbitrary $p \in D_+(I)$ $(p \in D_-(I))$.*

Remark 2.2. Corollaries 2.1 and 2.2 are optimal.

As it is well-known the disconjugacy is only a sufficient condition in order to ensure the constant sign of Green's function of problems $(1.3), (1.2_\ell)$ $(\ell \in \{1,2,3\})$. For this reason we introduce here theorems with necessary and sufficient conditions which guarantee that Green's function of problems $(1.1), (1.2_2)$ or $(1.1), (1.2_3)$ will be the constant sign function. Also we will find such coefficients p, and such values of the parameter μ, for which Green's functions of the mentioned problems are constant sign functions but equation (1.1) is oscillatory on I (see Remark 2.3).

Theorem 2.2. *Let $p \in D_+(I) \cap C(I;\mathbb{R})$. Then:*

(a) *Green's function of problem $(1.1), (1.2_2)$ is non-negative on $I \times I$ iff $\mu \in]0, \mu_p]$, where $\mu_p := \min\{\mu_1^*, \mu_3^*\}$, μ_ℓ^* $(\ell = 1,3)$ is the first positive eigenvalue of problem $(1.1), (1.2_\ell)$;*

(b) *The estimation $\mu_p \geq \beta_p > \frac{\lambda_2^4}{(b-a)^4}$ is valid.*

Theorem 2.3. *Let $p \in D_-(I) \cap C(I;\mathbb{R})$. Then:*

(a) *Green's function of problem* $(1.1),(1.2_2)$ *is non-negative on* $I\times I$ *iff* $\mu\in\,]\mu_p,0]$*, where* μ_p *is the biggest negative eigenvalue of problem* $(1.1),(1.2_2)$*;*

(b) *The estimation* $\mu_p\le\alpha_p<-\frac{\lambda_1^4}{(b-a)^4}$ *is valid.*

Theorem 2.4. *Let* $p\in D_+(I)\cap C(I;\mathbb{R})$*. Then:*

(a) *Green's function of problem* $(1.1),(1.2_1)$ *is non-positive on* $I\times I$ *iff* $\mu\in[0,\mu_p[$*, where* μ_p *is the first positive eigenvalue of problem* $(1.1),(1.2_1)$*;*

(b) *The estimation* $\mu_p\ge\beta_p>\frac{\lambda_2^4}{(b-a)^4}$ *is valid.*

Theorem 2.5. *Let* $p\in D_-(I)\cap C(I;\mathbb{R})$*. Then:*

(a) *Green's function of problem* $(1.1),(1.2_1)$ *is non-positive on* $I\times I$ *iff* $\mu\in[\mu_p,0[$*, where* μ_p *is the biggest negative eigenvalue of problem* $(1.1),(1.2_2)$*;*

(b) *The estimation* $\mu_p\le\alpha_p<-\frac{\lambda_1^4}{(b-a)^4}$ *is valid.*

Remark 2.3. In Theorems 2.2 and 2.4 (Theorems 2.3 and 2.5) from the definition of the number μ_p it is clear that equation (1.1) is oscillatory on I if $\mu=\mu_p$. Therefore from Corollary 2.1 (Corollary 2.2) it immediately follows that

$$\mu_p\ge\beta_p>\frac{\lambda_2^4}{(b-a)^4}\quad\Big(\mu_p\le\alpha_p<-\frac{\lambda_1^4}{(b-a)^4}\Big).$$

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On Solvability of Inhomogeneous Boundary-Value Problems in Fractional Sobolev Spaces

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1 Introduction

Boundary-value problems for systems of ordinary differential equations arise in many problems of analysis and its applications. Unlike Cauchy problems, the solutions to such problems may not exist or may not be unique. Thus, it is interesting to investigate the nature of the solvability of inhomogeneous boundary-value problems in the functional Sobolev and Sobolev–Slobodetskiy spaces and the dependence of their solutions on the parameter. For Fredholm boundary-value problems, similar issues have been investigated in papers [1, 2, 4, 5, 7]. The case of underdefined or overdefined boundary-value problems in Sobolev spaces was investigated in paper [3].

2 Statement of the problem

Let a finite interval $(a,b)\subset\mathbb{R}$ and parameters

$$\{m,l\}\subset\mathbb{N},\ \ s\in(1,\infty)\setminus\mathbb{N},\ \ 1\le p<\infty$$

be given. By $W_p^n:=W_p^n([a,b];\mathbb{C})$ we denote a complex Sobolev space and set $W_p^0:=L_p$. By

$$(W_p^n)^m:=W_p^n([a,b];\mathbb{C}^m)\ \text{ and }\ (W_p^n)^{m\times m}:=W_p^n([a,b];\mathbb{C}^{m\times m})$$

we denote the Sobolev spaces of vector functions and matrix functions, respectively, with elements from the function space W_p^n. By $\|\cdot\|_{n,p}$ we denote the norms in these spaces. They are defined as the sums of the corresponding norms of all elements of a vector-valued or matrix-valued function in W_p^n. The space of functions (scalar functions, vector functions, or matrix functions) in which the norm is introduced is always clear from the context. For $m=1$ all these spaces coincide. It is known that W_p^n are separable Banach spaces.

We denote by $W_p^s:=W_p^s([a,b];\mathbb{C})$ where $1\le p<\infty$ and $s>1$, is not integer, the Sobolev–Slobodetskiy space of all complex-valued functions belonging to Sobolev space $W_p^{[s]}$ and satisfying the condition

$$\|f\|_{s,p}:=\|f\|_{[s],p}+\biggl(\int\limits_a^b\int\limits_a^b\frac{|f^{[s]}(x)-f^{[s]}(y)|^p}{|x-y|^{1+\{s\}p}}\,dx\,dy\biggr)^{1/p}<+\infty,$$

where $[s]$ is the integer part, and $\{s\}$ is the fractional part of the number s. Here, we recall that $\|\cdot\|_{[s],p}$ is the norm in the Sobolev space $W_p^{[s]}$. This equality defines the norm $\|f\|_{s,p}$ in the space W_p^s.

Consider a linear boundary-value problem on a finite interval (a,b) for the system of m first-order scalar differential equations

$$(Ly)(t) := y'(t) + A(t)y(t) = f(t), \ \ t \in (a,b), \tag{2.1}$$

$$By = c, \tag{2.2}$$

where the matrix function $A(\cdot)$ belongs to the space $(W_p^s)^{m\times m}$, the vector function $f(\cdot)$ belongs to the space $(W_p^s)^m$, the vector c belongs to the space $\mathbb{C}^l$, and B is a linear continuous operator

$$B : (W_p^{s+1})^m \to \mathbb{C}^l. \tag{2.3}$$

The boundary condition (2.2) consists of l scalar boundary conditions for the system of m differential equations of the first order. We represent vectors and vector functions in the form of columns. A solution to the boundary-value problem (2.1), (2.2) is understood as a vector function $y \in (W_p^{s+1})^m$, satisfying equation (2.1) for $s > 1 + 1/p$ everywhere and, for $s \le 1 + 1/p$, almost everywhere on (a,b) and equality (2.2) specifying l scalar boundary conditions. The solutions to equation (2.1) fill the space $(W_p^{s+1})^m$, if its right-hand side $f(\cdot)$ runs through the space $(W_p^s)^m$. Hence, the boundary condition (2.2) is the most general condition for this equation and includes all known types of classical boundary conditions, namely, the Cauchy problem, two- and multipoint problems, integral and mixed problems, and numerous nonclassical problems. The last class of problems may contain derivatives of integer or fractional order k of required vector–functions, where $0 < k < s + 1$.

The main purpose of this work is to establish whether the boundary-value problem (2.1), (2.2) has the Fredholm property; to find its index and the dimension of the cokernel and the kernel of the operator of an inhomogeneous boundary-value problem in terms of the properties of a special rectangular numerical matrix and to investigate its stability. In the case of Sobolev spaces of integer order, similar results were obtained in [6].

3 Main results

We rewrite the inhomogeneous boundary-value problem (2.1), (2.2) in the form of a linear operator equation

$$(L,B)y = (f,c),$$

where (L,B) is a linear operator in the pair of Banach spaces

$$(L,B) : (W_p^{s+1})^m \to (W_p^s)^m \times \mathbb{C}^l. \tag{3.1}$$

Let X and Y be Banach spaces. Recall that a linear continuous operator $T : X \to Y$ is called a Fredholm operator, if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If the operator is a Fredholm one, then its range $T(X)$ is closed in Y, and the index

$$\operatorname{ind} T := \dim \ker T - \dim \frac{Y}{T(X)}$$

is finite (see, e.g., [6, Lemma 19.1.1]).

Theorem 3.1. *The linear operator* (3.1) *is a bounded Fredholm operator with index* $m - l$.

Denote by $Y(\cdot) \in (W_p^s)^{m\times m}$ the unique solution to a linear homogeneous matrix equation

$$Y'(t) + A(t)Y(t) = O_m, \ \ t \in (a,b) \tag{3.2}$$

with the initial condition

$$Y(a) = I_m. \tag{3.3}$$

Here, O_m are zero matrices, and I_m are identity $(m \times m)$ matrices. The unique solution to the Cauchy problem $(3.2), (3.3)$ belongs to the space $(W_p^{s+1})^{m\times m}$.

By $[BY]$ we denote a numerical matrix of dimension $(m \times l)$ whose i-th column is a result of the action of the operator B from (2.3) on i-th column of the matrix function $Y(\cdot)$, $i \in \{1, \dots, m\}$.

Definition. A rectangular numerical matrix

$$M(L,B) = [BY] \in \mathbb{C}^{m\times l} \tag{3.4}$$

is called the characteristic matrix for the inhomogeneous boundary-value problem $(2.1), (2.2)$.

Here, m is the number of scalar differential equations of system (2.1), and l is the number of scalar boundary conditions.

Theorem 3.2. *The dimensions of the kernel and cokernel of operator* (3.1) *are equal to the dimensions of the kernel and cokernel of the characteristic matrix* (3.4)*, respectively:*

$$\begin{aligned}\dim\ker(L,B) &= \dim\ker(M(L,B)),\\ \dim\operatorname{coker}(L,B) &= \dim\operatorname{coker}(M(L,B)).\end{aligned}$$

A criterion for the invertibility of the operator (L,B) follows from Theorem 3.2, i.e., the condition under which problem $(2.1), (2.2)$ possesses a unique solution, and this solution continuously depends on the right-hand sides of the differential equation and the boundary condition.

Corollary 3.1. *Operator (L,B) is invertible if and only if $l = m$ and the square matrix $M(L,B)$ is nondegenerate.*

4 Application

In addition to problem $(2.1), (2.2)$ we consider the sequence of inhomogeneous boundary-value problems

$$L(k)y(t,k) := y'(t,k) + A(t,k)y(t,k) = f(t,k), \ \ t \in (a,b), \tag{4.1}$$

$$B(k)y(\cdot,k) = c(k), \ \ k \in \mathbb{N}, \tag{4.2}$$

where the matrix functions $A(\cdot,k)$, the vector functions $f(\cdot,k)$, the vectors $c(k)$ and linear continuous operators $B(k)$ satisfy the above conditions for problem $(2.1), (2.2)$.

With the boundary-value problem $(4.1), (4.2)$ we associate a sequence of linear continuous operators

$$(L(k),B(k)) : (W_p^{s+1})^m \to (W_p^s)^m \times \mathbb{C}^l$$

and a sequence of characteristic matrices

$$M(L(k),B(k)) = [B(k)Y(\cdot,k)] \subset \mathbb{C}^{m\times l},$$

depending on the parameter $k \in \mathbb{N}$.

We now formulate a sufficient condition for the convergence of the characteristic matrices $M(L(k),B(k))$ to the matrix $M(L,B)$.

Theorem 4.1. *If the sequence of operators $(L(k), B(k))$ converges strongly to the operator (L, B), for $k \to \infty$, then the sequence of characteristic matrices $M(L(k), B(k))$ converges to the matrix $M(L, B)$.*

Corollary 4.1. *Under the assumptions from Theorem 4.1, the following inequalities hold*

$$\dim\ker(L(k), B(k)) \leq \dim\ker(L, B),$$
$$\dim\operatorname{coker}(L(k), B(k)) \leq \dim\operatorname{coker}(L, B)$$

for sufficiently large k.

In particular:

1. If $l = m$ and the operator(L, B) is invertible, then the operators $(L(k), B(k))$ are also invertible for large k;
2. If the boundary-value problem $(2.1), (2.2)$ has a solution for any values of the right-hand sides, then the boundary-value problems $(4.1), (4.2)$ also have a solution for large k;
3. If the boundary-value problem $(2.1), (2.2)$ has a unique solution, then problems $(4.1), (4.2)$ also have a unique solution for each sufficiently large k.

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Necessary and Sufficient Conditions of Disconjugacy for Fourth Order Linear Ordinary Differential Equations

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1 Introduction

In this study we consider the question of the disconjugacy on the interval $I := [a,b] \subset [0,+\infty[$ of the fourth order linear ordinary differential equation

$$u^{(4)}(t) = p(t)u(t), \tag{1.1}$$

where $p : I \to \mathbb{R}$ is a Lebesgue integrable function.

The disconjugacy results obtained in this study complete Kondrat'ev's second comparison theorem for $n = 4$, and significantly improve some other known results (see Remarks 2.1, 2.2, 2.4).

Here we use the following notations.

$\mathbb{R} =]-\infty,+\infty[\,$, $\mathbb{R}_0^- =]-\infty,0]$, $\mathbb{R}_0^+ = [0,+\infty[\,$.

$C(I;\mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$.

$\widetilde{C}^3(I;\mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives.

$L(I;\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$.

For arbitrary $x, y \in L(I;\mathbb{R})$, the notation

$$x(t) \preccurlyeq y(t) \quad \big(x(t) \succcurlyeq y(t)\big) \ \text{ for } \ t \in I$$

means that $x \le y$ $(x \ge y)$ and $x \ne y$. Also we use the notation $[x]_\pm = (|x| \pm x)/2$.

By a solution of equation (1.1) we understand a function $u \in \widetilde{C}^3(I;\mathbb{R})$ which satisfies equation (1.1) a. e. on I.

For the formulation of our results we need the following definitions.

Definition 1.1. Equation (1.1) is said to be disconjugate (non oscillatory) on I, if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity. Otherwise we say that equation (1.1) is oscillatory on I.

Definition 1.2. We will say that $p \in D_+(I)$ if $p \in L(I;\mathbb{R}_0^+)$, and equation (1.1), under the conditions

$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0,1), \tag{1.2}$$

has a solution u such that $u(t) > 0$ $t \in]a,b[\,$.

Definition 1.3. We will say that $p \in D_-(I)$ if $p \in L(I;\mathbb{R}_0^-)$, and equation (1.1), under the conditions

$$u(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0,1,2), \tag{1.3}$$

has a solution u, such that $u(t) > 0$ $t \in]a,b[$.

Remark 1.1. Let $p \in L(I;\mathbb{R}_0^+)$ $(p \in L(I;\mathbb{R}_0^-))$, and consider the equation

$$u^{(4)}(t) = \lambda^4 p(t)u(t) \ \text{ for } \ t \in I. \tag{1.4}$$

Then the set $D_+(I)$ $(D_-(I))$ can be interpreted as a set of functions $p : I \to \mathbb{R}_0^+$ $(\mathbb{R}_0^-)$ for which $\lambda = 1$ is the first eigenvalue of problem (1.4), (1.2) ((1.4), (1.3)).

2 Main results

2.1 Disconjugacy of equation (1.1) with non-negative coefficient

Theorem 2.1. *Let $p \in L(I;\mathbb{R}_0^+)$. Then equation* (1.1) *is disconjugate on I iff there exists $p^* \in D_+(I)$ such that*

$$p(t) \preccurlyeq p^*(t) \ \textit{ for } \ t \in I. \tag{2.1}$$

Let $\lambda_1 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = \lambda^4 u(t), \quad u^{(i)}(0) = 0, \quad u^{(i)}(1) = 0 \ \ (i = 0,1), \tag{2.2}$$

then due to Remark 1.1 we have $\frac{\lambda_1^4}{(b-a)^4} \in D_+(I)$, and the following corollary is true.

Corollary 2.1. *Equation* (1.1) *is disconjugate on I if*

$$0 \le p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \ \textit{ for } \ t \in I, \tag{2.3}$$

and is oscillatory on I if

$$p(t) \ge \frac{\lambda_1^4}{(b-a)^4} \ \textit{ for } \ t \in I. \tag{2.4}$$

Remark 2.1. It is well-known that the first eigenvalue λ_1 of problem (2.2) is the first positive root of the equation $\cos\lambda \cdot \cosh\lambda = 1$, and $\lambda_1 \approx 4.73004$ (see [3]). Also in Theorem 3.1 of paper [3] it was proved that the equation $u^{(4)} = \lambda^4 u$ is disconjugate on $[0,1]$ if $0 \le \lambda < \lambda_1$.

Even if both conditions (2.3) and (2.4) are violated, the question on the disconjugacy of equation (1.1) can be answered by the following theorem.

Theorem 2.2. *Let $p \in L(I;\mathbb{R}_0^+)$, and there exists $M \in \mathbb{R}_0^+$ such that*

$$M\,\frac{b-a}{2} + \int\limits_a^b [p(s) - M]_+ \, ds \le \frac{192}{(b-a)^3}\,. \tag{2.5}$$

Then equation (1.1) *is disconjugate on I.*

2.2 Disconjugacy of equation (1.1) with non-positive coefficient

Theorem 2.3. *Let $p \in L(I;\mathbb{R}_0^-)$. Then equation* (1.1) *is disconjugate on I iff there exists $p_* \in D_-(I)$ such that*

$$p(t) \succcurlyeq p_*(t) \ \ \text{for} \ \ t \in I. \tag{2.6}$$

Let $\lambda_2 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = -\lambda^4 u(t), u^{(i)}(0) = 0 \ \ (i = 0, 1, 2), \quad u(1) = 0, \tag{2.7}$$

then due to Remark 1.1 we have $-\frac{\lambda_2^4}{(b-a)^4} \in D_-(I)$, and the following corollary is true.

Corollary 2.2. *Equation* (1.1) *is disconjugate on I if*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \le 0 \ \ \text{for} \ \ t \in I, \tag{2.8}$$

and is oscillatory on I if

$$p(t) \le -\frac{\lambda_2^4}{(b-a)^4} \ \ \text{for} \ \ t \in I. \tag{2.9}$$

Remark 2.2. In Theorem 4.1 of [3] the following is proved: Let λ_2 be the first positive root of the equation $\tanh \frac{\lambda}{\sqrt{2}} = \tan \frac{\lambda}{\sqrt{2}}$ ($\lambda_2 \approx 5.553$). Then the equation $u^{(4)} = -\lambda^4 u$ is disconjugate on $[0, 1]$ if $0 \le \lambda < \lambda_2$.

Even if both conditions (2.8) and (2.9) are violated, the question on the disconjugacy of equation (1.1) can be answered by the following

Theorem 2.4. *Let $p \in L(I;\mathbb{R}_0^-)$ be such that there exists $M \in \mathbb{R}_0^+$ with*

$$M\frac{495}{1024}(b-a) + \int\limits_a^b [p(s) + M]_- \, ds \le \frac{110}{(b-a)^3}\,. \tag{2.10}$$

Then equation (1.1) *is disconjugate on I.*

2.3 Disconjugacy of equation (1.1) with not necessarily constant sign coefficient

Theorem 2.5. *Let $p_* \in D_-(I)$ and $p^* \in D_+(I)$. Then for an arbitrary function $p \in L(I;\mathbb{R})$ such that*

$$p_*(t) \preccurlyeq -[p(t)]_-, \quad [p(t)]_+ \preccurlyeq p^*(t) \ \ \text{for} \ \ t \in I, \tag{2.11}$$

equation (1.1) *is disconjugate on I.*

The theorem is optimal in the sense that inequalities (2.11) can not be replaced by the condition $p_* \le p \le p^*$.

Remark 2.3. Let $p_1, p_2 : [a, b] \to \mathbb{R}$ be continuous functions such that the equations

$$u^{(4)}(t) = p_1(t)u(t), \quad u^{(4)}(t) = p_2(t)u(t), \tag{2.12}$$

are disconjugate on I, then due to Kondrat'ev's second comparison theorem, if $p_1 \le p \le p_2$, then equation (1.1) is disconjugate too. Here coefficients p_1 and p_2 should not necessarily be constant sign functions, while in Theorem 2.5 for the permissible coefficients p_1 and p_2, equations (2.12) should not necessarily be disconjugate and continuous. For this reason, if

$$p(t) = \lambda_1^4 \Big[\cos \frac{2\pi t}{n} \Big]]_+ - \lambda_2^4 \Big[\cos \frac{2\pi t}{n} \Big]_-,$$

then from Theorem 2.5 it follows the disconjugacy of equation (1.1) on $[0,1]$ for all $n \in N$ (see Corollary 2.4), while this fact does not follow from Kondrat'ev's theorem.

Corollary 2.3. *Let $p_* \in D_-(I)$, $p^* \in D_+(I)$, and*

$$\operatorname{mes}\Big\{ t \in I \mid \ p_*(t) \cdot p^*(t) \neq 0 \Big\} > 0.$$

Then equation (1.1) *with $p = p_* + p^*$ is disconjugate on I.*

From Theorem 2.5 with

$$p_* := -\frac{\lambda_2^4}{(b-a)^4} \ \text{ and } \ p^* := \frac{\lambda_1^4}{(b-a)^4}$$

we obtain

Corollary 2.4. *et $\lambda_1 > 0$ and $\lambda_2 > 0$ be the first eigenvalues of problems* (2.2) *and* (2.7)*, respectively, and the function $p \in L(I;\mathbb{R})$ admits the inequalities*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \ \textit{ for } \ t \in I.$$

Then equation (1.1) *is disconjugate on I.*

Remark 2.4. If we take into account that $\lambda_1^4 \approx 501$ and $\lambda_2^4 \approx 951$, then it is clear that Corollary 2.4 significantly improves Coppel's well-known condition

$$\max_{t \in [a,b]} |p(t)| \leq \frac{128}{(b-a)^4}\,,$$

proved in [1], which for $p \in C(I;\mathbb{R})$ guarantees the disconjugacy of equation (1.1) on I.

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Matrix Boundary-Value Problems for Differential Equations with p-Laplacian

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The boundary-value problems for differential equations with p-Laplacian arise while studying the radial solutions of nonlinear partial differential equations. A feature of such various boundary-value problems for differential, including difference equations with p-Laplacian is the lack of uniqueness of the solution.

In this thesis, we consider the boundary-value problem for the linear system of differential equations with matrix p-Laplacian, which is reduced to the traditional differential-algebraic system with an unknown in the form of the vector function. We considered two cases of the obtained differential-algebraic system, in particular, the cases of solvability and unsolvability of the differential-algebraic system with respect to the derivative. For both cases, we obtained a sufficient condition for the solvability of the matrix boundary-value problem for the differential equation with p-Laplacian, in which connection its general solution determines the general solution for the homogeneous part of the matrix differential equation with p-Laplacian and the Green operator of the original matrix boundary-value problem.

The relevance of studying the boundary-value problems for differential equations with p-Laplacian is associated with numerous applications of such problems in the theory of elasticity, the theory of plasma, and astrophysics. The purpose of this thesis is to generalize various boundary-value problems for differential equations with p-Laplacian, which preserves the features of the solution of such problems, namely, the lack of uniqueness of the solution, and, in this case, the dependence of the desired solution of the arbitrary function.

We have studied the problem on the construction of solutions [1–3, 5, 6]

$$Z(t) \in \mathbb{C}^2_{\alpha\times\beta}[a,b] := \mathbb{C}^2[a,b] \otimes \mathbb{R}^{\alpha\times\beta}$$

of the linear system of differential equations

$$\mathcal{P}Z(t) = A(t)Z(t) + F(t) \tag{1}$$

with the boundary condition

$$\mathcal{L}Z(\,\cdot\,) = \mathcal{A}, \;\; \mathcal{A} \in \mathbb{R}^{\lambda\times\mu} \tag{2}$$

and with a matrix p-Laplacian $\mathcal{P}Z(t) := ((R(t)Z(t))'S(t))'$. Here,

$$R(t) \in \mathbb{C}^2_{\gamma\times\alpha}[a,b] := \mathbb{C}^2[a,b] \otimes \mathbb{R}^{\gamma\times\alpha}, \;\; S(t) \in \mathbb{C}^2_{\beta\times\delta}[a,b] := \mathbb{C}^2[a,b] \otimes \mathbb{R}^{\beta\times\delta},$$
$$F(t) \in \mathbb{C}_{\gamma\times\delta}[a,b] := \mathbb{C}^1[a,b] \otimes \mathbb{R}^{\gamma\times\delta},$$

$\mathcal{L}Z(\,\cdot\,)$ — is a linear bounded matrix functional: $\mathcal{L}Z(\,\cdot\,) : \mathbb{C}^2[a;b] \to \mathbb{R}^{\lambda\times\mu}$. Generally speaking, we assume that $\alpha \neq \beta \neq \gamma \neq \delta \neq \lambda \neq \mu$ are any natural numbers. By $\Xi^{(j)} \in \mathbb{R}^{\alpha\times\beta}$, $j = 1, 2, \dots, \alpha\cdot\beta$ we denote the natural basis of the space $\mathbb{R}^{\alpha\times\beta}$. In this case, the problem of determination of solutions of equation (1) can be reduced to a problem of determination of a vector $z(t) \in \mathbb{C}^2_{\alpha\beta}[a;b]$, whose components $z_j(t) \in \mathbb{C}^2[a;b]$ define the expansion of the matrix

$$Z(t) = \sum_{j=1}^{\alpha\beta} \Xi^{(j)} z_j(t), \ \ z_j(t) \in \mathbb{C}^1[a;b], \ \ j = 1, 2, \dots, \alpha \cdot \beta$$

in vectors $\Xi^{(j)} \in \mathbb{R}^{\alpha\times\beta}$ of the basis of the space $\mathbb{R}^{\alpha\times\beta}$. We now define the operator

$$\mathcal{M}[\mathcal{A}] : \ \mathbb{R}^{m\times n} \to \mathbb{R}^{m\cdot n}$$

as an operator that puts the matrix $\mathcal{A} \in \mathbb{R}^{m\times n}$ in correspondence to a vector-column $\mathcal{B} := \mathcal{M}[\mathcal{A}] \in \mathbb{R}^{m\cdot n}$ formed from n columns of the matrix A. We also introduce the inverse operator

$$\mathcal{M}^{-1}[\mathcal{B}] : \mathbb{R}^{m\cdot n} \to \mathbb{R}^{m\times n}$$

that puts the vector $\mathcal{B} \in \mathbb{R}^{m\cdot n}$ in correspondence to a matrix $\mathcal{A} \in \mathbb{R}^{m\times n}$. We present the product $A(t)Z(t)$ in the form

$$A(t)Z(t) := A(t) \sum_{k=1}^{\alpha\beta} \Xi^{(k)} z_k(t), \ \ \mathcal{M}\big[A(t)Z(t)\big] = \check{A}(t) \cdot z(t),$$

where

$$\check{A}(t) := \big[\check{A}_k(t)\big]_{k=1}^{\alpha\beta} \in \mathbb{C}_{\gamma\delta\times\alpha\beta}[a,b], \ \ \check{A}_k(t) = \mathcal{M}\big[A(t)\Xi^{(k)}\big], \ \ k = 1, 2, \dots, \alpha \cdot \beta.$$

We now define the matrices

$$B(t), C(t), D(t) \in \mathbb{C}_{\gamma\delta\times\alpha\beta}[a,b]$$

in the following way:

$$\frac{\partial}{\partial z''} \mathcal{MP}Z(t) := B(t)z(t), \ \ \frac{\partial}{\partial z'} \mathcal{MP}Z(t) := C(t)z(t), \ \ \frac{\partial}{\partial z} \mathcal{MP}Z(t) := D(t)z(t).$$

The problem of construction of solutions of Eq. (1) can be reduced to a problem of determination of the vector $z(t) \in \mathbb{C}^2_{\alpha\beta}[a;b]$ that is defined by the system

$$B(t)z'' + C(t)z' + D(t)z = \check{A}(t)z + f(t), \ \ f(t) := \mathcal{M}F(t).$$

Changing the variables

$$y_1 := z, \ \ y_2 := y_1',$$

we get the problem of determination of the vector

$$y(t) \in \mathbb{C}^2_{2\alpha\beta}[a;b]$$

defined by the differential-algebraic system of equations [2, 3, 5]

$$U(t)y' = V(t)y + \check{f}(t), \tag{3}$$

where

$$U(t) := \begin{pmatrix} I_{\alpha\beta} & O_{\alpha\beta} \\ C(t) & B(t) \end{pmatrix}, \quad V(t) := \begin{pmatrix} O_{\alpha\beta} & I_{\alpha\beta} \\ \check{A}(t) - D(t) & O_{\gamma\delta\times\alpha\beta} \end{pmatrix}, \quad \check{f}(t) := \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Thus, under the condition [2, 3, 5]

$$P_{U^*(t)} V(t) = 0, \quad P_{U^*(t)} \check{f}(t) = 0, \tag{4}$$

we have proved the sufficient condition of solvability of the Cauchy problem for the matrix differential equation with p-Laplacian (1).

Lemma. *Under conditions* (4) *the matrix Cauchy problem* $Z(a)=\mathfrak{A}$ *for the matrix differential equation with p-Laplacian* (1) *is uniquely solvable for any initial value of* $\mathfrak{A}\in\mathbb{R}^{\mu\times\nu}$. *Under conditions* (4)*, the general solution*

$$\begin{gathered} Z(t,c)=W(t,c)+\mathcal{K}\big[\mathfrak{F}(s,\varphi(s))\big](t),\ \ c\in\mathbb{R}^{2\,\alpha\cdot\beta},\\ W(t,c):=\mathcal{M}^{-1}\big[\mathcal{J}_{\alpha\beta}X(t)c\big],\quad \mathcal{J}_{\alpha\beta}:=\big[I_{\alpha\beta}\quad O_{\alpha\beta}\big]\in\mathbb{R}^{\alpha\cdot\beta\times2\,\alpha\cdot\beta},\\ \mathcal{K}\big[\mathfrak{F}(s,\varphi(s))\big](t):=\mathcal{M}^{-1}\big\{\mathcal{J}_{\alpha\beta}\,K\big[\mathfrak{F}(s,\varphi(s))\big](t)\big\}, \end{gathered}$$

of the Cauchy problem

$$Z(a)=\mathfrak{A}$$

for the matrix differential equation with p-Laplacian (1) *defines a generalized Green's operator* $\mathcal{K}[\mathfrak{F}(s,\varphi(s))](t)$ *of the Cauchy problem* $Z(a)=0$ *for the matrix differential equation with p-Laplacian* (1) *and the general solution* $W(t,c)$ *of the Cauchy problem* $Z(a)=\mathfrak{A}$ *or the homogeneous part of the matrix differential equation with p-Laplacian* (1)*.*

Thus, in the critical case under conditions (4) and in the case of fulfillment of the solvability condition

$$P_{\mathcal{Q}_d^*}\mathcal{M}\big\{\mathfrak{A}-\mathcal{L}\mathcal{K}\big[\mathfrak{F}(s,\varphi(s))\big](\,\cdot\,)\big\}=0 \tag{5}$$

the solution of the matrix boundary-value problem with p-Laplacian (1), (2) takes the form

$$Z(t,c_r)=W(t,c_r)+G\big[\mathfrak{F}(s,\varphi(s));\mathfrak{A}\big](t),\quad W(t,c_r):=\mathcal{M}^{-1}\big[\mathcal{J}_{\alpha\beta}\,X(t)P_{\mathcal{Q}_r}c_r\big], \tag{6}$$

where

$$G\big[\mathfrak{F}(s,\varphi(s));\mathfrak{A}\big](t):=\mathcal{M}^{-1}\Big\{\mathcal{J}_{\alpha\beta}\,X(t)\mathcal{Q}^{+}\mathcal{M}\big\{\mathfrak{A}-\mathcal{L}\mathcal{K}\big[\mathfrak{F}(s,\varphi(s))\big](\,\cdot\,)\big\}\Big\}+\mathcal{K}\big[\mathfrak{F}(s,\varphi(s))\big](t).$$

Hence, we have proved the sufficient condition of solvability of the matrix boundary-value problem for the differential equation with p-Laplacian (1), (2).

Theorem. *In the critical case* ($P_{\mathcal{Q}^*}\neq0$) *under conditions* (4) *and* (5)*, solution* (6) *of the matrix boundary-value problem with p-Laplacian* (1)*,* (2) *determines the generalized Green's operator* $G[\mathfrak{F}(s,\varphi(s));\mathfrak{A}](t)$ *of the matrix boundary-value problem with p-Laplacian* (1)*,* (2) *and the general solution* $W(t,c_r)$ *for the homogeneous part of the differential equation with p-Laplacian* (1)*,* (2)*.*

Assume that condition (4) is not satisfied [2, 3, 5], i.e., $P_{U^*(t)}V(t)\neq0$ or $P_{U^*(t)}\check{f}(t)\neq0$. Then the problem of determination of solutions of Eq. (1) after the change of the variables

$$y_1:=z:=\Omega x_1,\ \ y_2:=y_1':=\Omega x_1':=\Omega x_2,\ \ \Omega\in\mathbb{R}^{\alpha\beta\times\alpha\beta},\ \ x(t):=\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}$$

leads to the problem of determination of the vector $x(t)\in\mathbb{C}^2_{\alpha\beta}[a;b]$ defined by the differential-algebraic system of equations [2, 3, 5]

$$\check{U}(t)x'=\check{V}(t)x+\check{f}(t); \tag{7}$$

here,

$$\check{U}(t):=\begin{pmatrix}\Omega & O_{\alpha\beta}\\ C(t)\Omega & B(t)\Omega\end{pmatrix},\quad \check{V}(t):=\begin{pmatrix}O_{\alpha\beta} & \Omega\\ (\check{A}(t)-D(t))\Omega & O_{\gamma\delta\times\alpha\beta}\end{pmatrix}.$$

Under the conditions [2, 3, 5]

$$P_{\check{U}^*(t)}\check{V}(t)=0,\quad P_{\check{U}^*(t)}\check{f}(t)=0, \tag{8}$$

system (7) is solvable with respect to the derivative [2, 3, 5]:

$$x' = \check{W}(t)x + \mathfrak{F}_1(t, \varphi(t)), \tag{9}$$

here,

$$\check{W}(t) := \check{U}^{+}(t)\check{V}(t), \quad \mathfrak{F}_1(t, \varphi(t)) := \check{U}^{+}(t)\check{f}(t) + P_{\check{U}_\varrho}(t)\varphi(t).$$

Thus, we have proved the sufficient condition of solvability of the matrix Cauchy problem for the differential equation with p-Laplacian (1).

Corollary 1. *Under conditions* (8)*, the matrix Cauchy problem $Z(a) = \mathfrak{A}$ for the matrix differential equation with p-Laplacian* (1) *is uniquely solvable for any initial value of $\mathfrak{A} \in \mathbb{R}^{\mu\times\nu}$. Under conditions* (8)*, the general solution*

$$Z(t, c) = \check{W}(t, c) + \mathcal{K}\big[\mathfrak{F}_1(s, \varphi(s))\big](t), \ \ c \in \mathbb{R}^{2\,\alpha\cdot\beta},$$
$$\check{W}(t, c) := \mathcal{M}^{-1}\big[\Omega\,\mathcal{J}_{\alpha\beta}\check{X}(t)c\big], \quad \mathcal{K}\big[\mathfrak{F}_1(s, \varphi(s))\big](t) := \mathcal{M}^{-1}\big\{\mathcal{J}_{\alpha\beta}\,K\big[\Omega\,\mathfrak{F}_1(s, \varphi(s))\big](t)\big\}.$$

of the Cauchy problem $Z(a) = \mathfrak{A}$ for the matrix differential equation with p-Laplacian (1) *determines the generalized Green's operator $\mathcal{K}[\mathfrak{F}_1(s, \varphi(s))](t)$ of the Cauchy problem $Z(a) = 0$ for the matrix differential equation with p-Laplacian* (1) *and the general solution $\check{W}(t, c)$ of the Cauchy problem $Z(a) = \mathfrak{A}$ for the homogeneous part of the matrix differential equation with p-Laplacian* (1)*.*

Hence, in the critical case under conditions (8) and in case of fulfillment of the condition of solvability

$$P_{\mathcal{Q}_d^*}\mathcal{M}\big\{\mathfrak{A} - \mathcal{L}\mathcal{K}\big[\mathfrak{F}_1(s, \varphi(s))\big](\,\cdot\,)\big\} = 0, \tag{10}$$

the solution of the matrix boundary-value problem with p-Laplacian (1), (2) takes the form

$$Z(t, c_r) = \check{W}(t, c_r) + G\big[\mathfrak{F}_1(s, \varphi(s)); \mathfrak{A}\big](t), \quad \check{W}(t, c_r) := \mathcal{M}^{-1}\big[\Omega\,\mathcal{J}_{\alpha\beta}\,\check{X}(t)P_{\mathcal{Q}_r}c_r\big], \tag{11}$$

where

$$G\big[\mathfrak{F}_1(s, \varphi(s)); \mathfrak{A}\big](t) := \mathcal{M}^{-1}\Big\{\Omega\,\mathcal{J}_{\alpha\beta}\,\check{X}(t)\mathcal{Q}^{+}\mathcal{M}\big\{\mathfrak{A} - \mathcal{L}\mathcal{K}\big[\mathfrak{F}(s, \varphi(s))\big](\,\cdot\,)\big\}\Big\} + \mathcal{K}\big[\mathfrak{F}_1(s, \varphi(s))\big](t).$$

Thus, we have proved the sufficient condition of solvability of the matrix boundary-value problem for differential equation with p-Laplacian (1), (2).

Corollary 2. *In the critical case* ($P_{\mathcal{Q}^*} \neq 0$) *under conditions* (8) *and* (10)*, solution* (11) *of the matrix boundary-value problem with p-Laplacian* (1)*,* (2) *determines the generalized Green's operator $G[\mathfrak{F}_1(s, \varphi(s)); \mathfrak{A}](t)$ of the matrix boundary-value problem with p-Laplacian* (1)*,* (2) *and the general solution $\check{W}(t, c_r)$ for the homogeneous part of the differential equation with p-Laplacian* (1)*,* (2)*.*

The research scheme proposed in the thesis can be transferred on the nonlinear matrix boundary value problems for differential equations with p-Laplacian, on the linear matrix boundary-value problems for difference equations, and on the matrix boundary-value problems for functional differential equations with p-Laplacian in abstract spaces, in particular, on the matrix boundary-value problems for differential equations with argument deviation. The proposed scheme of research of the linear system of differential equations with matrix p-Laplacian in the article was illustrated in details with examples.

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Sign-indefinite Logistic Models with Flux-Saturated Diffusion: Existence and Multiplicity Results

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Let us consider the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda a(x) f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the diffusion is driven by the mean curvature operator $-\operatorname{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big)$. In equation (1), $\lambda > 0$ is a parameter measuring diffusivity and

(H_1) $\Omega \subset \mathbb{R}^N$ is a bounded domain, with a C^2 boundary $\partial\Omega$ in case $N \geq 2$;

(H_2) $a : \overline{\Omega} \to \mathbb{R}$ is a continuous function such that $\max\limits_{\overline{\Omega}} a > 0$;

(H_3) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying, for some constant $L > 0$, $f(0) = f(L) = 0$, and $f(s) > 0$ for every $s \in \,]0, L[\,$.

Assumption (H_2) on the weight a introduces spatial heterogeneities within the model and allows that a changes sign in Ω. Assumption (H_3) basically requires that the reaction term af is of logistic-type. As it is well-known, logistic maps play a pivotal role in the modeling theory of various disciplines, with special prominence in biology, ecology, genetics. Unlike the classical theory based on the Fick–Fourier's law, where the flux depends linearly on ∇u, here the diffusion is governed by the bounded flux $\nabla u/\sqrt{1+|\nabla u|^2}$, which is approximately linear for small gradients but approaches saturation for large ones.

Following our recent paper [3], we aim here to synthetically describe and clarify the effects of a flux-saturated diffusion in logistic growth models featuring spatial heterogeneities. This study is motivated by the investigations on reaction processes with saturating diffusion started in [4], in order to correct the non-physical gradient-flux relations at high gradients. This specific mechanism of diffusion, of which the mean curvature operator provides a paradigmatic example, may determine spatial patterns exhibiting abrupt transitions at the boundary or between adjacent profiles, up to the formation of discontinuities. This makes the mathematical analysis of the problem (1) more delicate and sophisticated than the study of the corresponding semilinear model, the use of some tools of geometric measure theory being in particular required. Indeed, it is an established fact that the space of bounded variation functions is the natural setting for dealing with this problem. The precise notion of bounded variation solution of (1) used in this paper has been basically introduced in [1] and is recalled below for completeness.

Notation. For every $v \in BV(\Omega)$, $Dv = D^a v\,\mathrm{d}x + D^s v$ is the Lebesgue–Nikodym decomposition of the Radon measure Dv in its absolutely continuous part $D^a v\,\mathrm{d}x$ and its singular part $D^s v$ with respect to the N-dimensional Lebesgue measure $\mathrm{d}x$ in $\mathbb{R}^N$, $|Dv|$ denotes the total variation of the measure Dv, and $\frac{Dv}{|Dv|}$ stands for the density of Dv with respect to its total variation. Further, $|\Omega|$ is the Lebesgue measure of Ω, while $\mathcal{H}_{N-1}$ represents the $(N-1)$-dimensional Hausdorff measure, and $|\partial\Omega|$ is the $\mathcal{H}_{N-1}$-measure of $\partial\Omega$. Moreover, for all functions $u, v : \Omega \to \mathbb{R}$, we write: $u \geq v$ if $\operatorname{ess\,inf}(u - v) \geq 0$; $u > v$ if $u \geq v$ and $\operatorname{ess\,sup}(u - v) > 0$; $u \gg v$ if, for a.e. $x \in \Omega$, $u(x) - v(x) \geq \operatorname{dist}(x, \partial\Omega)$.

Definition. By a *bounded variation solution* of (1) we mean a function $u \in BV(\Omega)$, with $f(u) \in L^N(\Omega)$, which satisfies

$$\int\limits_{\Omega} \frac{D^a u\, D^a \phi}{\sqrt{1 + |D^a u|^2}}\,\mathrm{d}x + \int\limits_{\Omega} \frac{Du}{|Du|}\,\frac{D\phi}{|D\phi|}\,|D^s\phi| + \int\limits_{\partial\Omega} \operatorname{sgn}(u)\phi\,\mathrm{d}\mathcal{H}_{N-1} = \lambda \int\limits_{\Omega} af(u)\phi\,\mathrm{d}x \tag{2}$$

for every $\phi \in BV(\Omega)$ such that $|D^s\phi|$ is absolutely continuous with respect to $|D^s u|$ and $\phi(x) = 0$ $\mathcal{H}_{N-1}$-a.e. on the set $\{x \in \partial\Omega :\ u(x) = 0\}$. A bounded variation solution u is said *positive* if $u > 0$.

Remark 1. If a bounded variation solution u of (1) belongs to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for some $p > N$, then it satisfies the differential equation in (1) for a.e. $x \in \Omega$ and the boundary condition for all $x \in \partial\Omega$. Therefore, u is a *strong solution* of (1). The L^p-regularity theory then entails that $u \in W^{2,q}(\Omega)$ for all $q > N$. Conversely, it is evident that any strong solution is a bounded variation solution. Note that bounded variation solutions, unlike the strong ones, may not satisfy the Dirichlet boundary conditions.

Remark 2. It is clear that, for any given $\lambda > 0$, $u = 0$ is a bounded variation solution of (1), while $u = L$ is not. Indeed, if L were a solution, taking $\phi = 1$ as test function in (2) would yield $\int\limits_{\partial\Omega} 1\,\mathrm{d}\mathcal{H}_{N-1} = |\partial\Omega| = 0$, which is a contradiction.

We are now going to present the main results obtained in [3]. Here, for the sake of clarity, our statements are set out in a simplified form, while referring to [3] for some variants or extensions that rely on slightly more general but less neat conditions.

The first result only exploits the structural assumptions (H_1), (H_2), and (H_3). It provides us with the existence of a number $\lambda_* \geq 0$ such that, for all $\lambda > \lambda_*$, the problem (1) has a maximum solution u_λ, with $0 < u_\lambda < L$. The asymptotic behavior of u_λ, as $\lambda \to +\infty$, is described too, and the bifurcation of the solutions from the trivial line $\{(\lambda, 0) :\ \lambda \geq 0\}$ at the point $(0, 0)$ is ascertained in the case $\lambda_* = 0$. Figure 1 illustrates two admissible bifurcations diagrams.

Theorem 1. *Assume (H_1), (H_2) and (H_3). Then there exists $\lambda_* \geq 0$ such that for all $\lambda \in]\lambda_*, +\infty[$ the problem (1) admits a maximum bounded variation solution u_λ, with $0 < u_\lambda < L$, which satisfies*

$$\lim_{\lambda \to +\infty} (\operatorname{ess\,sup} u_\lambda) = L. \tag{3}$$

Moreover, if $\lambda_* = 0$, then

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{BV} = 0. \tag{4}$$

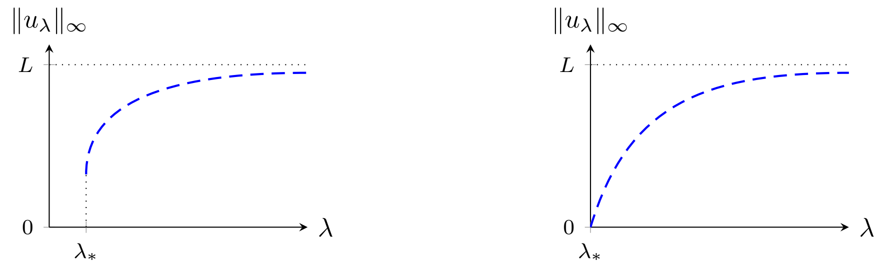

Figure 1. Admissible bifurcation diagrams for the problem (1) under the structural assumptions (H_1), (H_2), and (H_3), in case $\lambda_* > 0$ (left) or $\lambda_* = 0$ (right). Dashed curves indicate bounded variation solutions.

The specific features displayed by the bifurcation diagrams of the problem (1) are determined by the slope at 0 of the function f, as expressed by the following conditions:

(H_4) there exists $\lim\limits_{s\to 0^+} \frac{f(s)}{s} = +\infty$ (*sublinear growth at* 0);

(H_5) there exists $\lim\limits_{s\to 0^+} \frac{f(s)}{s} = \kappa \in]0,+\infty[$ (*linear growth at* 0);

(H_6) there exists $\lim\limits_{s\to 0^+} \frac{f(s)}{s} = 0$ (*superlinear growth at* 0).

When f has a sublinear growth at zero, a bifurcation from the trivial line occurs at the point $(0,0)$, and the existence of positive bounded variation solutions of the problem (1) is guaranteed for all $\lambda > 0$. In addition, positive strong solutions exist provided that λ is small enough.

Theorem 2. *Assume (H_1), (H_2), (H_3), and (H_4). Then for all $\lambda > 0$ the problem* (1) *admits at least one bounded variation solution $u_\lambda \in BV(\Omega)$, with $0 < u_\lambda < L$, which satisfies* (3) *and* (4). *Moreover, there exists $\lambda^* > 0$ such that, for all $\lambda \in]0,\lambda^*[$, solutions u_λ can be selected so that $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for any $p > N$, it is a strong solution and it satisfies*

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_{W^{2,p}} = 0. \tag{5}$$

When f grows linearly at zero the bifurcation occurs from the trivial line at the point $(\lambda_1, 0)$, where λ_1 is the principal eigenvalue of the linear weighted problem

$$\begin{cases} -\Delta\varphi = \lambda a(x)\kappa\varphi & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω satisfies (H_1), κ comes from (H_5), and a satisfies (H_2). It is a classical fact that λ_1 is positive and simple, with a positive eigenfunction φ_1. The L^p-regularity theory and a standard bootstrap argument entail that $\varphi_1 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $p > N$, while the strong maximum principle and the Hopf boundary point lemma yield $\varphi_1 \gg 0$. In this case the solvability of the problem (1) is guaranteed for all $\lambda > \lambda_1$. In addition, for λ close to λ_1 strong solutions do exist.

Theorem 3. *Assume (H_1), (H_2), (H_3), and (H_5). Then for all $\lambda > \lambda_1$ the problem* (1) *admits at least one bounded variation solution u_λ, with $0 < u_\lambda < L$, which satisfies* (3). *Moreover, suppose that*

(H_7) *f is of class C^2*

and fix any $p > N$. Then there exists a neighborhood $\mathcal{U}$ of $(\lambda_1, 0)$ in $\mathbb{R} \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that solutions u_λ can be selected so that $(\lambda, u_\lambda) \in \mathcal{U}$, u_λ is a strong solution and it satisfies

$$\lim_{\lambda\to\lambda_1} \|u_\lambda\|_{W^{2,p}} = 0 \ \ \text{and} \ \ \lim_{\lambda\to\lambda_1} \frac{u_\lambda}{\|u_\lambda\|_{C^1}} = \varphi_1. \tag{6}$$

Finally, there exists $\eta > 0$ such that the following assertions hold:

(i) *if $f''(0) < 0$, then for all $\lambda \in \,]\lambda_1, \lambda_1 + \eta[$ there is at least one strong solution $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying* (6)*;*

(ii) *if $f''(0) > 0$, then for all $\lambda \in \,]\lambda_1 - \eta, \lambda_1[$ there is at least one strong solution $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying* (6)*.*

Remark 3. For the standard logistic model $f(s) = s(L - s)$, the condition $f''(0) = -2 < 0$ holds and therefore the bifurcation is supercritical.

When f exhibits a superlinear growth at zero, the existence of multiple solutions can be detected if, for instance, conditions (H_2) and (H_6) are strengthened as follows. Let us set

$$\Omega^+ = \big\{x \in \Omega : \ a(x) > 0\big\}, \quad \Omega^- = \big\{x \in \Omega : \ a(x) < 0\big\}, \quad \Omega^0 = \big\{x \in \Omega : \ a(x) = 0\big\},$$

and replace (H_2) with

(H_8) $a \in C^2(\overline{\Omega})$, $\Omega^+ \neq \varnothing$, $\Omega^- \neq \varnothing$, $\Omega^0 = \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega$, and $\nabla a(x) \neq 0$ for all $x \in \Omega^0$,

as well as (H_6) with

(H_9) there exists $q > 1$, with $q < \frac{N+2}{N-2}$ if $N \geq 3$, such that

$$\lim_{s\to 0^+} \frac{f(s)}{s^q} = 1.$$

Then, for λ sufficiently large, the problem (1) has at least two positive bounded variation solutions, the smaller being strong.

Theorem 4. *Assume (H_1), (H_3), (H_8), and (H_9). Then there exists $\lambda_* \geq 0$ such that for all $\lambda \in \,]\lambda_*, +\infty[$, the problem* (1) *admits at least one bounded variation solution u_λ and one strong solution $v_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, for any $p > N$, such that $0 \ll v_\lambda < u_\lambda < L$. In addition, u_λ satisfies* (3)*, while v_λ satisfies*

$$\lim_{\lambda\to+\infty} \|v_\lambda\|_{W^{2,p}} = 0. \tag{7}$$

Figure 2 illustrates three qualitatively different bifurcation diagrams corresponding, respectively, to Theorems 2, 3, and 4.

Unexpectedly enough, the existence of multiple solutions can always be detected in the standard logistic model, whenever the carrying capacity L is sufficiently large, even in the case where the weight function a is a positive constant (cf. Remark 4 below). We state such a multiplicity result for the simplest one-dimensional prototype of the problem (1), that is,

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1 + (u')^2}}\Big)' = \lambda a f(u) \ \ \text{in} \ \ ]0, 1[\,, \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{8}$$

Theorem 5. *Assume (H_3),*

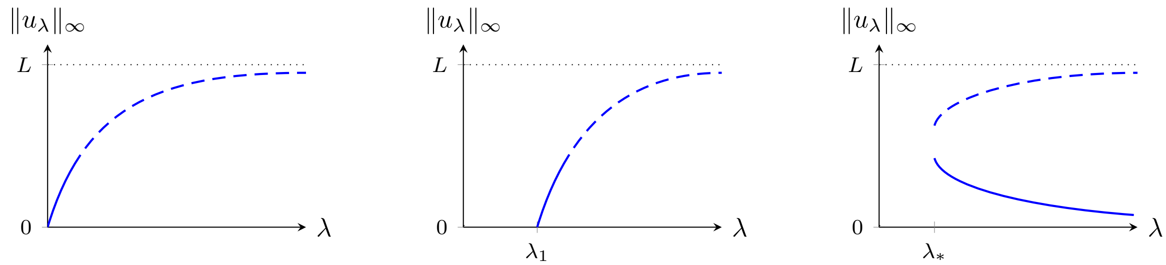

Figure 2. Admissible qualitative bifurcation diagrams for the problem (1), according to the growth of f at 0: either sublinear (left), or linear (center), or superlinear (right). Dashed curves indicate bounded variation solutions, solid curves represent strong solutions.

(H_{10}) *$a \in C^0([0,1])$ satisfies $a > 0$,*

and

(H_{11}) *there exist $r, R \in]0, L[$, with $r < R$, such that*

$$\frac{2F(r)}{r^2}\Big(1+\sqrt{1+r^2}\Big) < \frac{F(R)}{R}\,,$$

where $F(s) = \int\limits_0^s f(t)\,\mathrm{d}t$ is the potential of f. Then there exist $\lambda_\sharp$ and $\lambda^\sharp$, with $0 \le \lambda_\sharp < \lambda^\sharp$, such that for all $\lambda \in]\lambda_\sharp, \lambda^\sharp[$ the problem (8) admits at least two bounded variation solutions u_λ, v_λ such that $0 < u_\lambda < v_\lambda < L$.

It is worth stressing that the assumptions of Theorem 5 do not prevent f from being concave in $[0, L]$: this fact witnesses the peculiarity of this multiplicity result, which is specific of the quasilinear problem (1) and has no similarity with the semilinear case, where the concavity of f always guarantees the uniqueness of the positive solution, as proven in [2] even for sign-changing weights a.

Remark 4. For the standard logistic model, where $f(s) = s(L - s)$, condition (H_{11}) is satisfied if, for instance, $L > \frac{32}{3} \approx 10.67$.

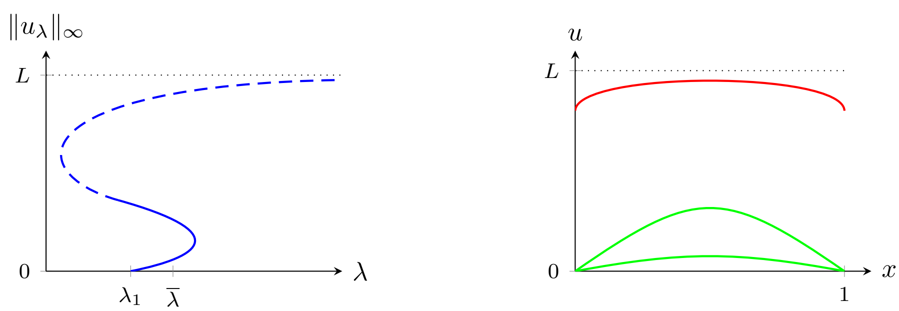

Figure 3. On the left, an admissible bifurcation diagram is depicted with reference to Example 1: the dashed curve indicates bounded variation solutions, the solid curve represents strong solutions. On the right, the profiles of the three detected solutions at $\lambda = \overline{\lambda}$ are shown: in green the regular ones, in red the singular one.

Example 1. A numerical study of the problem (8), with $a = 1$, $f(s) = s(L - s)$ and $L = 11 > \frac{32}{3}$, reveals the existence of three positive solutions in a (small) right neighborhood of the bifurcation point $\lambda_1 = \frac{\pi^2}{L} \approx 0.8972$, in particular at $\overline{\lambda} = 0.8975$, and of two positive solutions in a left neighborhood of λ_1. This is in complete agreement with (i) the bifurcation result stated in Theorem 3 and Remark 3, which predicts the bifurcation branch emanates from λ_1 pointing to the right; (ii) the multiplicity conclusions of Theorem 5, which guarantee the existence of two solutions in an interval of the λ-axis located on the left of λ_1. Hence an S-shaped bifurcation diagram is expected as shown by the picture on the left in Figure 3.

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Robust Stability of Global Attractors for Reaction-Diffusion System w.r.t. Disturbances

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1 Introduction

Asymptotic stability of an equilibrium is a fundamental property of evolutionary processes and plays important role for many applications. It is well-known that a globally asymptotically stable equilibrium of a linear and finite dimensional system is robust in the sense that for any essentially bounded external disturbance entering to the system the corresponding solution remains bounded for all times and that it tends to a ball around the equilibrium when time goes to infinity. The size of this ball depends on the disturbance norm only. For nonlinear systems this is in general not true and leads to the notion of Input-to-State Stability (ISS), introduced by E. D. Sontag [12, 13] for finite dimensional systems. This notion is also suitable to study robustness of equilibria in case of infinite dimensional systems [2]. During the last decade many authors tried to extend the ISS theory to this class of systems. Many of these extensions were developed for systems given in terms of partial differential equations (PDEs), see, for example, the works [4, 5, 7, 11]. It should be noted that almost all ISS-like results for PDEs were developed for the case of single equilibrium point of the unperturbed system. It is well known that many nonlinear systems possess a nontrivial global attractor instead. In this work we study the question of robustness of such an attracting set with respect to external disturbances. Existence and different properties of global attractors were studied in many books [1,10,14] and papers [3,6]. We are interested in the following question: given a system possessing a global attractor, what can we say about attracting sets for solutions if some perturbation h enters to this system? In this work, we consider such a question for perturbed reaction-diffusion system. Using a general scheme suggested in [4], we prove local ISS and asymptotic gain (AG) properties w.r.t. global attractors for dissipative RD system.

2 Statement of the problem and the main results

In a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the following parabolic problem (named Reaction-Diffusion system)

$$\begin{cases} u_t = a\Delta u - f(u) + h(x) + d(t,x), \quad x \in \Omega, \ \ t > 0, \\ u\big|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $u = u(t,x) = (u^1(t,x), \dots, u^N(t,x))$ is an unknown vector-function, $f = (f^1, \dots, f^N)$, $h = (h^1, \dots, h^N)$ are given functions, a is a real $N \times N$ matrix with positive symmetric part $\frac{1}{2}(a + a^*) \geq \mu I$, $\mu > 0$, $d = (d^1, \dots, d^N)$ is an external disturbances.

We assume that the following properties hold:

$$h \in (L^2(\Omega))^N, \ \ f \in C^1(\mathbb{R}^N; \mathbb{R}^N),$$

$\exists\, C_1, C_2 > 0$, $C_3 > 0$, $\gamma_i > 0$, $p_i \geq 2$, $i = 1, \dots, N$ such that $\forall\, v \in \mathbb{R}^N$

$$\sum_{i=1}^N |f^i(v)|^{\frac{p_i}{p_i-1}} \leq C_1\Big(1 + \sum_{i=1}^N |v^i|^{p_i}\Big), \quad \sum_{i=1}^N f^i(v)v^i \geq \sum_{i=1}^N \gamma_i |v^i|^{p_i} - C_2,$$
$$\forall\, w \in \mathbb{R}^N \ \ (Df(v)w, w) \geq -C_3 \sum_{i=1}^N |w^i|^2.$$

In further arguments we will use standard functional spaces

$$H = (L^2(\Omega))^N \ \text{ and } \ V = (H_0^1(\Omega))^N.$$

Let us denote

$$p = (p_1, \dots, p_N), \ \ L^p(\Omega) = L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega).$$

It is known [1] that under such assumptions for every disturbances $d \in L^\infty(\mathbb{R}_+; (L^2(\Omega))^N)$ the problem (2.1) is globally uniquely resolvable in a weak sense in the phase space H, i.e., for every $u_0 \in H$ there exists a unique function $u = u(t,x) \in L^2_{loc}(0,+\infty; V) \cap L^p_{loc}(0,+\infty; L^p(\Omega))$ such that for all $T > 0$, $v \in V \cap L^p(\Omega)$,

$$\frac{d}{dt} \int_\Omega u(t,x)v(x)\,dx + \int_\Omega \Big(a\nabla u(t,x)\nabla v(x) + f(u(t,x))v(x) - h(x)v(x) - d(t,x)v(x)\Big)\,dx = 0$$

in the sense of scalar distributions on $(0,T)$, and $u(0,x) = u_0(x)$.

Due to the inclusion $u \in C([0,+\infty); H)$, the last equality makes sense.

Let us consider the unperturbed system ($d \equiv 0$)

$$\begin{cases} u_t = a\Delta u - f(u) + h(x), \quad x \in \Omega, \ \ t > 0, \\ u\big|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

It is known [1] that the corresponding semigroup $S : \mathbb{R}_+ \times H \mapsto H$

$$S(t, u_0) = u(t), \ \text{ where } u(\,\cdot\,) \text{ is a global weak solution of (2.2)}, \ \ u(0) = u_0$$

possesses a global attractor Θ in H, i.e., there exists a compact set $\Theta \subset H$ such that the following properties hold:

(i) $\Theta = S(t, \Theta)$, $t \geq 0$;

(ii) for any bounded $B \subset H$

$$\operatorname{dist}(S(t,B), \Theta) \to 0 \ \text{ as } \ t \to \infty,$$

where for given $A, B \subset H$ we denote

$$\operatorname{dist}(A,B) = \sup_{x\in A} \inf_{y \in B} \|x - y\|_H.$$

This property guarantees that any solution to (2.2) approaches Θ as $t \to \infty$. We are interested in the long term behavior of the corresponding solutions in the case the system (2.1) is perturbed by external perturbation d.

Given an initial state $u_0 := u(0) \in H$ and a perturbing signal $d \in L^\infty(\mathbb{R}_+; H)$, the corresponding unique solution to (2.1) is denoted by $u(t, u_0, d)$. Due to the disturbance we have no guarantee, in general, that this solution will converge to Θ as $t \to \infty$. It turns out that the global attractor is robust under perturbation, i.e., its attractivity properties are affected only slightly by disturbances of small magnitude. This robustness property can be expressed in the ISS framework as follows: there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any $u_0 \in H$ and $d \in L^\infty(\mathbb{R}_+; H)$,

$$\|u(t, u_0, d)\|_\Theta \le \beta\big(\|u_0\|_\Theta, t\big) + \gamma\big(\|d\|_\infty\big), \quad t \ge 0, \tag{2.3}$$

where the well-known classes $\mathcal{K}$ stands for the class of continuous strictly increasing functions on $[0, +\infty)$ vanishing at the origin, $\mathcal{KL}$ is the set of continuous functions defined on $[0, +\infty)^2$ which are of class $\mathcal{K}$ in the first argument and strictly decreasing to zero in the second one,

$$\|d\|_\infty = \operatorname*{ess\,sup}_{t \ge 0} \|d(t)\|_H,$$
$$\|u\|_\Theta = \inf_{\theta \in \Theta} \|u - \theta\|_H.$$

It should be noted that in the general case $\Theta \ne \{0\}$. Moreover, its structure is very complicated [6]. Therefore, estimates like (2.3) cannot be obtained by using direct a priori estimates.

In this paper we prove that local variant of this property holds for the problem (2.1).

Unfortunately, this property is in general not guaranteed even for the case $\Theta = \{0\}$, see, for example, [2].

In this paper we prove this property for the problem (2.1) at least locally. More precisely, we prove the following result

Theorem. *Under the mentioned above assumptions the problem* (2.1)

(i) *is local ISS with respect to Θ, i.e., there exists $r > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any $\|u_0\|_H \le r$ and any $\|d\|_\infty \le r$,*

$$\|u(t, u_0, d)\|_\Theta \le \beta\big(\|u_0\|_\Theta, t\big) + \gamma\big(\|d\|_\infty\big), \quad t \ge 0; \tag{2.4}$$

(ii) *satisfies the asymptotic gain (AG) property with respect to Θ, that is there exists $\gamma \in \mathcal{K}$ such that for any $u_0 \in H$ and any $d \in L^\infty(\mathbb{R}_+; H)$ it holds*

$$\limsup_{t \to \infty} \|u(t, u_0, d)\|_\Theta \le \gamma\big(\|d\|_\infty\big). \tag{2.5}$$

To prove the local ISS property, the Laypunov technique is used. To establish the AG property, the uniform attractors theory for non-autonomous systems [1] is used.

Acknowledgements

This work is supported by the German Research Foundation (DFG) and National Research Foundation of Ukraine (NRFU) through the joint German–Ukrainian grant *"Stability and robustness of attractors of nonlinear infinite-dimensional systems with respect to disturbances"* (project # DA 767/12-1 and # F81/41743).

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Conditional Partial Integrals of Polynomial Hamiltonian Systems

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1 Introduction

Consider a canonical Hamiltonian odinary differential system with n degrees of freedom

$$\frac{dq_i}{dt} = \partial_{p_i} H(q,p), \quad \frac{dp_i}{dt} = -\partial_{q_i} H(q,p), \ \ i = 1,\dots,n, \tag{1.1}$$

where $q = (q_1,\dots,q_n) \in \mathbb{R}^n$ and $p = (p_1,\dots,p_n) \in \mathbb{R}^n$ are the generalized coordinates and momenta, $t \in \mathbb{R}$, and the Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a polynomial of degree $h \geqslant 2$.

In this paper, using the Darboux theory of integrability [4] and the notion of conditional partial integral [5,7], we will study the existence of additional first integrals of the Hamiltonian system (1.1).

The Darboux theory of integrability (or the theory of partial integrals) was established by the French mathematician *Jean-Gaston Darboux* [4] in 1878, which provided a link between the existence of first integrals and invariant algebraic curves (or partial integrals) for polynomial autonomous differential systems. For the polynomial differential systems, the Darboux theory of integrability is one of the best theories for studying the existence of first integrals (see, for example, [5, 6, 11, 14, 15]). Note that the Darboux theory of integrability is related to the Poincaré problem [13], which asks to find the upper bound of invariant algebraic curves of planar polynomial differential systems. The Darboux theory of integrability is also involved in the study of Hilbert's 16-th problem (see, for example, the paper by Yu. Ilyashenko [9]). For the current state of the theory of integrability of differential systems see the monographs [2, 6, 8, 10, 11, 14, 15] and the references therein.

To avoid ambiguity, we give the following notation and definitions.

The *Poisson bracket* of functions $u, v \in C^1(G)$ on a domain $G \subset \mathbb{R}^{2n}$ is the function

$$\big[u(q,p), v(q,p)\big] = \sum_{i=1}^{n} \Big(\partial_{q_i} u(q,p)\, \partial_{p_i} v(q,p) - \partial_{p_i} u(q,p)\, \partial_{q_i} v(q,p)\Big) \ \text{ for all } \ (q,p) \in G.$$

A function $F \in C^1(G)$ is called a *first integral on the domain* G of the Hamiltonian system (1.1) if the functions F and H are in involution, i.e.,

$$\big[F(q,p), H(q,p)\big] = 0 \ \text{ for all } \ (q,p) \in G \subset \mathbb{R}^{2n}.$$

The Hamiltonian differential system (1.1) is *completely integrable* (in the Liouville sense) if it has n functionally independent first integrals which are in involution. Notice that the Hamiltonian H is a first integral of the Hamiltonian differential system (1.1).

A set of functionally independent on a domain $G \subset \mathbb{R}^{2n}$ first integrals $F_l \in C^1(G)$, $l = 1, \dots, k$, of the Hamiltonian system (1.1) is called a *basis of first integrals* (or *integral basis*) on the domain G of system (1.1) if any first integral $F \in C^1(G)$ of system (1.1) can be represented on G in the form

$$F(q,p) = \Phi\big(F_1(q,p), \dots, F_k(q,p)\big) \ \text{ for all } \ (q,p) \in G,$$

where Φ is some continuously differentiable function. The number k is said to be the *dimension* of basis of first integrals on the domain G for the Hamiltonian differential system (1.1).

The autonomous Hamiltonian differential system (1.1) on a domain without equilibrium points has an integral basis (autonomous) of dimension $2n - 1$ [1, pp. 167 – 169].

A real polynomial w is a *partial integral* of the Hamiltonian system (1.1) if the Poisson bracket

$$\big[w(q,p), H(q,p)\big] = w(q,p)M(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n},$$

where the polynomial M (*cofactor* of the partial integral w) is such that $\deg M \leqslant h - 2$.

Suppose w be a partial integral of the Hamiltonian differential system (1.1). Then the algebraic hypersurface $\{(q,p) : w(q,p) = 0\}$ is invariant by the flow of the Hamiltonian differential system (1.1) and if the cofactor M of the partial integral w is zero, then w is a polynomial first integral.

An exponential function $\omega(q,p) = \exp v(q,p)$ for all $(q,p) \in \mathbb{R}^{2n}$ with some real polynomial v is called a *conditional partial integral* of the Hamiltonian system (1.1) if the Poisson bracket

$$\big[v(q,p), H(q,p)\big] = S(q,p) \ \text{ for all } \ (q,p) \in \mathbb{C}^{2n},$$

where the polynomial S (*cofactor* of the conditional partial integral ω) is such that $\deg S \leqslant h - 2$.

We stress that a conditional partial integral is a special case of exponential factor (or exponential partial integral) [3, 5, 11] for the polynomial Hamiltonian ordinary differential system (1.1).

2 Main results

Suppose the Hamiltonian differential system (1.1) has real partial integrals w_l with the cofactors M_l, $l = 1, \dots, s$, respectively, such that the Poisson brackets

$$\big[w_l(q,p), H(q,p)\big] = w_l(q,p)M_l(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \quad \deg M_l \leqslant h - 2, \ \ l = 1, \dots, s. \tag{2.1}$$

And moreover, the polynomial Hamiltonian system (1.1) has conditional partial integrals

$$\omega_\nu(q,p) = \exp v_\nu(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \ \ \nu = 1, \dots, m, \tag{2.2}$$

with polynomials v_ν, $\nu = 1, \dots, m$, such that the following identities hold

$$\big[v_\nu(q,p), H(q,p)\big] = S_\nu(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \quad \deg S_\nu \leqslant h - 2, \ \ \nu = 1, \dots, m. \tag{2.3}$$

Theorem 2.1. *Let the exponential functions* (2.2) *be conditional partial integrals of the polynomial Hamiltonian differential system* (1.1). *Then the scalar function*

$$F(q,p) = \sum_{\nu=1}^{m} \beta_\nu v_\nu(q,p) \ \textit{ for all } \ (q,p) \in \mathbb{R}^{2n}, \ \ \beta_\nu \in \mathbb{R}, \ \ \nu = 1, \dots, m, \ \ \sum_{\nu=1}^{m} |\beta_\nu| \neq 0, \tag{2.4}$$

is an additional first integral of the Hamiltonian system (1.1) *if and only if*

$$\sum_{\nu=1}^{m} \beta_\nu S_\nu(q,p) = 0 \ \textit{ for all } \ (q,p) \in \mathbb{R}^{2n}. \tag{2.5}$$

Proof. Taking into account the identities (2.3) and bilinearity of Poisson brackets, we calculate the Poisson bracket of the function (2.4) and the Hamiltonian H:

$$\big[F(q,p),H(q,p)\big]=\Big[\sum_{\nu=1}^{m}\beta_\nu v_\nu(q,p),H(q,p)\Big]=\sum_{\nu=1}^{m}\beta_\nu\big[v_\nu(q,p),H(q,p)\big]=\sum_{\nu=1}^{m}\beta_\nu S_\nu(q,p).$$

Therefore, by definition of first integral, the function (2.4) is a first integral of the polynomial Hamiltonian ordinary differential system (1.1) if and only if the identity (2.5) is true. ☐

Theorem 2.2. *Suppose the polynomial Hamiltonian differential system* (1.1) *has the conditional partial integrals* (2.2) *such that the identities* (2.3) *are true under the conditions*

$$S_\nu(q,p)=\mu_\nu M(q,p)\ \ \textit{for all}\ \ (q,p)\in\mathbb{R}^{2n},\ \ \mu_\nu\in\mathbb{R},\ \ \nu=1,\dots,m,\ \ \deg M\leqslant h-2. \tag{2.6}$$

Then the scalar function (2.4) *is an additional first integral of the Hamiltonian differential system* (1.1) *if real numbers* β_ν *are a solution to the linear equation* $\sum\limits_{\nu=1}^{m}\mu_\nu\beta_\nu=0$ *under* $\sum\limits_{\nu=1}^{m}|\beta_\nu|\neq 0$.

Proof. If the representations (2.6) are true and numbers β_ν are a solution to $\sum\limits_{\nu=1}^{m}\mu_\nu\beta_\nu=0$, then

$$\sum_{\nu=1}^{m}\beta_\nu S_\nu(q,p)=\sum_{\nu=1}^{m}\beta_\nu\mu_\nu M(q,p)=0.$$

This implies that the condition (2.5) is true. Therefore, by Theorem 2.1, the function (2.4) is an additional first integral of the Hamiltonian system (1.1). ☐

From Theorem 2.2 under $m=2$, $\mu_1=\mu_2\neq 0$, we get the following statement.

Corollary 2.1. *If the polynomial Hamiltonian differential system* (1.1) *has the conditional partial integrals* (2.2) *under the condition* $m=2$ *such that the identity holds*

$$\frac{[v_1(q,p),H(q,p)]}{[v_2(q,p),H(q,p)]}=\frac{v_1(q,p)}{v_2(q,p)}\ \ \textit{for all}\ \ (q,p)\in G\subset\mathbb{R}^{2n},$$

then an additional first integral of the polynomial Hamiltonian system (1.1) *is the function*

$$F:(q,p)\to\ v_1(q,p)-v_2(q,p)\ \ \textit{for all}\ \ (q,p)\in\mathbb{R}^{2n}.$$

From Theorem 2.2 under $m=2$, $\mu_1=-\mu_2\neq 0$, we obtain the following statement.

Corollary 2.2. *If the polynomial Hamiltonian differential system* (1.1) *has the conditional partial integrals* (2.2) *under the condition* $m=2$ *such that the identity holds*

$$\frac{[v_1(q,p),H(q,p)]}{[v_2(q,p),H(q,p)]}=-\frac{v_1(q,p)}{v_2(q,p)}\ \ \textit{for all}\ \ (q,p)\in G\subset\mathbb{R}^{2n},$$

then an additional first integral of the polynomial Hamiltonian system (1.1) *is the function*

$$F:(q,p)\to\ v_1(q,p)+v_2(q,p)\ \ \textit{for all}\ \ (q,p)\in\mathbb{R}^{2n}.$$

Corollary 2.3. *Under the conditions of Theorem 2.2, we get the scalar functions*

$$F_{\xi\zeta}(q,p)=\beta_\xi v_\xi(q,p)+\beta_\zeta v_\zeta(q,p) \ \text{ for all } \ (q,p)\in\mathbb{R}^{2n}, \ \ \xi,\zeta=1,\dots,m, \ \ \zeta\neq\xi,$$

are first integrals of the polynomial Hamiltonian system (1.1)*, where numbers* β_ξ *and* β_ζ *are solutions to the linear homogeneous equations*

$$\mu_\xi\beta_\xi+\mu_\zeta\beta_\zeta=0$$

under

$$|\beta_\xi|+|\beta_\zeta|\neq 0, \ \ \xi,\zeta=1,\dots,m, \ \ \zeta\neq\xi.$$

Theorem 2.3. *Suppose the Hamiltonian system* (1.1) *has the partial integrals* w_l, $l=1,\dots,s$, *such that the identities* (2.1) *hold with* $M_l(q,p)=\lambda_l M(q,p)$ *for all* $(q,p)\in\mathbb{R}^{2n}$, $\lambda_l\in\mathbb{R}$, $l=1,\dots,s$, *and the conditional partial integrals* (2.2) *such that the identities* (2.3) *under* (2.6) *are true. Then*

$$F_{\xi\zeta}(q,p)=w_\xi^{\gamma_\xi}(q,p)\exp\big(\beta_\zeta v_\zeta(q,p)\big) \ \text{ for all } \ (q,p)\in G\subset\mathbb{R}^{2n}, \ \ \xi=1,\dots,s, \ \ \zeta=1,\dots,m, \tag{2.7}$$

are first integrals of system (1.1)*, where numbers* γ_ξ *and* β_ζ *are solutions to the equations*

$$\lambda_\xi\gamma_\xi+\mu_\zeta\beta_\zeta=0 \ \text{ under the conditions } \ |\gamma_\xi|+|\beta_\zeta|\neq 0, \ \ \xi=1,\dots,s, \ \ \zeta=1,\dots,m. \tag{2.8}$$

Proof. Using the functional identities (2.1) and (2.3), we obtain

$$\begin{aligned}
&\big[F_{\xi\zeta}(q,p),H(q,p)\big]\\
&\qquad=\big[w_\xi^{\gamma_\xi}(q,p),H(q,p)\big]\cdot\exp\big(\beta_\zeta v_\zeta(q,p)\big)+w_\xi^{\gamma_\xi}(q,p)\cdot\big[\exp\big(\beta_\zeta v_\zeta(q,p)\big)\big]\\
&\qquad=\gamma_\xi w_\xi^{\gamma_\xi-1}(q,p)\exp\big(\beta_\zeta v_\zeta(q,p)\big)\big[w_\xi(q,p),H(q,p)\big]\\
&\qquad\qquad\qquad+\beta_\zeta w_\xi^{\gamma_\xi}(q,p)\exp\big(\beta_\zeta v_\zeta(q,p)\big)\big[v_\zeta(q,p),H(q,p)\big]\\
&\qquad=\big(\lambda_\xi!\gamma_\xi+\mu_\zeta!\beta_\zeta\big)M(q,p)w_\xi^{\gamma_\xi}(q,p)\exp\big(\beta_\zeta v_\zeta(q,p)\big)\\
&\qquad\qquad\qquad\qquad\qquad\text{for all } \ (q,p)\in G, \ \ \xi=1,\dots,s, \ \ \zeta=1,\dots,m.
\end{aligned}$$

If the real numbers γ_ξ and β_ζ are solutions to the linear equations (2.8), then the functions (2.7) are additional first integrals of the polynomial Hamiltonian differential system (1.1). □

For example, the polynomial Hamiltonian differential system given by [12]

$$H(q,p)=\frac{1}{2}\,(p_1^2+p_2^2)+2q_2p_1p_2-q_1 \ \text{ for all } \ (q,p)\in\mathbb{R}^4 \tag{2.9}$$

has the polynomial partial integral $w(q,p)=p_2$ with cofactor $M(q,p)=-2p_1$ and the conditional partial integral $\omega(q,p)=\exp p_1^2$ with cofactor $S(q,p)=2p_1$. By Theorem 2.3, we can build the additional first integral of the Hamiltonian system (2.9): $F(q,p)=p_2\exp p_1^2$ for all $(q,p)\in\mathbb{R}^4$. The functionally independent first integrals H and F of the Hamiltonian system (2.9) are in involution. Therefore, the Hamiltonian system (2.9) is completely integrable (in the Liouville sense).

Acknowledgements

Research was supported by Horizon2020-2017-RISE-777911 project.

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Description of the Linear Perron Effect Under Parametric Perturbations of a System with Unbounded Coefficients

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For a given $n \in \mathbb{N}$ let us denote by $\widetilde{\mathcal{M}}_n$ the set of linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0, +\infty). \tag{1}$$

with continuous matrix-valued functions $A : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$, which we identify with the corresponding linear systems. The subset of $\widetilde{\mathcal{M}}_n$ that consists of systems (1) with bounded coefficients will be denoted by $\mathcal{M}_n$.

In [8] O. Perron constructed a two-dimensional system $A \in \mathcal{M}_2$ with negative Lyapunov exponents and an exponentially decaying at infinity continuous matrix-valued function Q such that the Lyapunov exponents of the perturbed system $A + Q$ are greater than those of the original system A. Perron's studies (see also [9]) have become a starting point of deeper researches on dependency of the Lyapunov exponents on perturbations of different classes.

The phenomenon of abrupt change of the Lyapunov exponents of a system in $\mathcal{M}_n$ under a small perturbation was called in the monograph [6, Ch. 4] the *Perron effect*. Since the paper [5], this term is being used only for the case when perturbations do not decrease the Lyapunov exponents of the original system. Unlike [5, 6, 9], which consider higher-order perturbations, we study the Perron effect under linear perturbations and hence call it *linear* [2].

Let us recall that *the characteristic exponent* [1, p. 25] of a vector-function $f : P \to \mathbb{R}^n$, where P is an unbounded subset of the semi-axis $\mathbb{R}_+$, is the quantity (we assume that $\ln 0 = -\infty$)

$$\lambda[f] = \varlimsup_{P \ni t \to +\infty} \ln \|f(t)\|^{1/t},$$

and the *Lyapunov exponents* [7] of a system $A \in \widetilde{\mathcal{M}}_n$ are the quantities

$$\lambda_i(A) = \inf_{L \in G_i(S(A))} \sup_{x \in L} \lambda[x], \ \ i = 1, \dots, n,$$

$S(A)$ being the space of solutions of system (1) and $G_i(S(A))$ the set of i-dimensional subspaces of $S(A)$.

The spectrum of the Lyapunov exponents of system (1) is the n-tuple $\Lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$. As coefficients of systems under consideration are not supposed to be bounded, the Lyapunov exponents of these systems are points of the extended real line $\overline{\mathbb{R}} \equiv \mathbb{R} \sqcup \{-\infty, +\infty\}$ with the standard order and topology.

As a more general case, for an arbitrary metric space M, let us consider a parametric family of linear differential systems

$$\dot{x} = A(t, \mu)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{2}$$

depending on a parameter $\mu \in M$ such that for each fixed $\mu \in M$ system (2) has continuous coefficients. Fixing $i = 1, \dots, n$ and assigning to each $\mu \in M$ the i-th Lyapunov exponent of system (2) we obtain the function $\lambda_i(A,\,\cdot\,): M \to \overline{\mathbb{R}}$ which is called *the i-th Lyapunov exponent of family* (2). Accordingly, the function $\Lambda(A,\,\cdot\,) = (\lambda_1(A,\,\cdot\,),\dots,\lambda_n(A,\,\cdot\,))$ is called *the spectrum of the Lyapunov exponents of family* (2).

Henceforth we will consider parametric families of linear differential systems of the form

$$\dot{x} = \big(A(t) + Q(t,\mu)\big)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0,+\infty),$$

where $Q(\,\cdot\,,\,\cdot\,): \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ is called a *parametric perturbation* of system (1).

As previously, let M be a metric space. For a given $\theta : \mathbb{R}_+ \to \mathbb{R}$ we denote by $\mathcal{Q}_n^\theta(M)$ the class of jointly continuous functions $Q(\,\cdot\,,\,\cdot\,): \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ such that each function Q for some $C_Q > 0$ satisfies the condition

$$\sup_{\mu\in M} \|Q(t,\mu)\| \leqslant C_Q e^{-\theta(t)t}, \ \ t \in \mathbb{R}_+.$$

For each $A \in \widetilde{\mathcal{M}}_n$, let

$$\mathcal{Q}_n^\theta[A](M) = \Big\{ Q \in \mathcal{Q}_n^\theta(M) \mid \ \ \forall\, i = 1,\dots,n, \ \forall\, \mu \in M \ \ \lambda_i(A+Q,\mu) \geqslant \lambda_i(A) \Big\}. \tag{3}$$

Put simply, the set $\mathcal{Q}_n^\theta[A](M)$ is the subset of $\mathcal{Q}_n^\theta(M)$ consisting of those perturbations that don't decrease the Lyapunov exponents of the original system A. Note that for each system $A \in \mathcal{M}_n$ the class $\mathcal{Q}_n^\theta[A](M)$ is nonempty since it contains the matrix $Q \equiv 0$.

It is of interest to describe in terms of the descriptive set theory the set of pairs composed of the spectrum of the Lyapunov exponents of a system A and that of a family of perturbed systems $A+Q$, where $A \in \widetilde{\mathcal{M}}_n$ and $Q \in \mathcal{Q}_n^\theta[A](M)$, i.e. the set

$$\Pi\mathcal{Q}_n^\theta(M) = \Big\{ (\Lambda(A), \Lambda(A+Q,\,\cdot\,)) \mid \ \ A \in \widetilde{\mathcal{M}}_n, \ Q \in \mathcal{Q}_n^\theta[A](M) \Big\}. \tag{4}$$

Let us recall some necessary set theory notation. We say [3, p. 267] that the function $f : M \to \overline{\mathbb{R}}$ belongs to the class $({}^*, G_\delta)$ if for each $r \in \mathbb{R}$, the inverse image $f^{-1}\big([r,+\infty]\big)$ of the semi-interval $[r,+\infty]$ is a G_δ-set in M (i.e. it can be represented as countable intersection of open sets). In particular, the class $({}^*, G_\delta)$ is a subclass of the second Baire class [3, p. 294].

The sought description of set (4) is contained in the following

Theorem 1. *For any metric space M, number $n \geqslant 2$ and continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}$ the pair $\big(l, f(\,\cdot\,)\big)$, where $l = (l_1,\dots,l_n) \in (\overline{\mathbb{R}})^n$ and $f(\,\cdot\,) = (f_1(\,\cdot\,),\dots,f_n(\,\cdot\,)) : M \to (\overline{\mathbb{R}})^n$, belongs to the set $\Pi\mathcal{Q}_n^\theta(M)$ if and only if the following conditions are satisfied:*

1) *$l_1 \leqslant \dots \leqslant l_n$;*

2) *$f_1(\mu) \leqslant \dots \leqslant f_n(\mu)$ for each $\mu \in M$;*

3) *$f_i(\mu) \geqslant l_i$ for all $\mu \in M$ and $i = 1,\dots,n$;*

4) *for each $i = 1,\dots,n$ the function $f_i(\,\cdot\,) : M \to \overline{\mathbb{R}}$ belongs to the class $({}^*, G_\delta)$.*

Note that a similar result for systems with bounded coefficients is obtained in [2].

As an important application of the stated theorem, consider the following problem. Let Φ be the set of all continuous functions $\varphi : \mathbb{R}_+ \to (0,+\infty)$. For an arbitrary metric space M and a subset $\Psi \subset \Phi$ let $\mathcal{Q}_n[\Psi](M)$ denote the class consisting of continuous matrix-valued functions $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ satisfying the condition

$$\lim_{t\to+\infty} (\psi(t))^{-1} \sup_{\mu\in M} \|Q(t,\mu)\| = 0 \ \text{ for each } \ \psi \in \Psi.$$

Next, for each $A \in \widetilde{\mathcal{M}}_n$, let $\mathcal{Q}_n[\Psi, A](M)$ denote the subset of $\mathcal{Q}_n[\Psi](M)$ that consists of parametric perturbations that don't decrease the Lyapunov exponents of the system A, i.e.

$$\mathcal{Q}_n[\Psi, A](M) = \Big\{Q \in \mathcal{Q}_n[\Psi](M) \mid \ \forall\, i = 1, \dots, n, \ \forall\, \mu \in M \ \ \lambda_i(A + Q, \mu) \geqslant \lambda_i(A)\Big\}.$$

The class $\mathcal{Q}_n[\Psi, A](M)$ is nonempty for the same reasons as class (3) is.

The problem is to describe the set of all pairs $(\Lambda(A), \Lambda(A + Q, \cdot\,))$, where $A \in \widetilde{\mathcal{M}}_n$ and $Q \in \mathcal{Q}_n[\Psi, A](M)$, i.e. the set

$$\Pi\mathcal{Q}_n[\Psi](M) = \Big\{(\Lambda(A), \Lambda(A + Q, \cdot\,)) \mid \ \ A \in \widetilde{\mathcal{M}}_n, \ Q \in \mathcal{Q}_n[\Psi, A](M)\Big\},$$

for given $n \in \mathbb{N}$, metric space M, and set $\Psi \subset \Phi$.

The solution to this problem for a countable set Ψ is stated in the following

Theorem 2. *For any metric space M, $n \geqslant 2$ and countable set $\Psi \subset \Phi$ the pair $(l, f(\,\cdot\,))$, where $l \in (\overline{\mathbb{R}})^n$ and $f(\,\cdot\,) : M \to (\overline{\mathbb{R}})^n$, belongs to the set $\Pi\mathcal{Q}_n[\Psi](M)$ if and only if conditions* 1)–4) *of Theorem 1 are met.*

The last result shows that all theoretically possible pairs of the spectrum of an original and parametrically perturbed systems (with an additional condition that all the exponents of a perturbed system are not less than those of the original one) can be obtained even in the class of perturbations that decay arbitrarily fast at infinity. This situation is specific for systems with unbounded coefficients since the Lyapunov exponents of a system with bounded coefficients are invariant under perturbations that decay faster than any exponent [4, § 8.1].

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On Non-Linear Boundary Value Problems for Iterative Differential Equations

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We study the general form boundary value problem

$$\frac{dx(t)}{dt} = f(t, x(t), x(x(t))), \quad t \in [a, b], \tag{1}$$

for the system of so called iterative differential equations (see, e.g., [1,5] and the references therein) under the non-linear boundary conditions

$$\Phi(x(t), x(x(t))) = d, \tag{2}$$

where $f \in C([a, b] \times D \times D; \mathbb{R}^n)$, $d \in \mathbb{R}^n$ is a given vector, Φ is a continuous n-dimensional vector functional and there exist some $n \times n$ matrices K_1, K_2 with non-negative entries such that for all $t \in [a, b]$, $u_i, v_i \in D$, $i = 1, 2$ the inequality

$$\big|f(t, u_1, u_2) - f(t, v_1, v_2)\big| \le K_1|u_1 - v_1| + K_2|u_2 - v_2| \tag{3}$$

holds.

The domain $D \sqsubseteq [a, b]^n$ will be defined in Eqs. (10) and (11).

We deal only with such solutions

$$x : [a, b] \to D \sqsubseteq [a, b]^n, \tag{4}$$

of problem (1), (2), which belong to the set

$$S := \Big\{x \in C([a, b]; D) : \ |x(t_1) - x(t_2)| \le L|t_1 - t_2|, \ \forall\, t_1, t_2 \in [a, b]\Big\}, \tag{5}$$

where L is a given diagonal matrix with non-negative entries $L = diag(L_1, \dots, L_n)$. On the base of conditions (3) and (5), we obtain

$$\big|f(t, u_1, u_2) - f(t, v_1, v_2)\big| \le K_1|u_1 - v_1| + K_2L|u_1 - v_1| = [K_1 + K_2L]\,|u_1 - v_1|, \tag{6}$$

$t \in [a,b]$. Thus, we prescribed some restrictions for the values of the derivative of the possible solutions similarly to that of [5] and [1].

To study the BVP (1), (2) we will use an approach similar to [2]. Note that this technique can be applied also in the case when, instead of (5), the condition

$$S := \Big\{ x \in C([a,b];[a_1,b_1]^n): \ |x(t_1)-x(t_2)| \le L|t_1-t_2|, \ \forall\, t_1,t_2 \in [a_1,b_1] \Big\}$$

is fulfilled and in addition there are given some initial functions

$$\beta \in C([a_1,a],D), \quad \gamma \in C([b,b_1],D).$$

For vectors $x = col(x_1,\dots,x_n) \in \mathbf{R}^n$ the obvious notation $|x| = col(|x_1|,\dots,|x_n|)$ is used and the inequalities between vectors are understood componentwise. The same convention is adopted for operations like "max" and "min".

$\mathbf{I}_n$ and $\mathbf{0}_n$ are the unit and zero matrices of dimension n, respectively. $r(K)$ is the maximal (in modulus) eigenvalue of the matrix K.

For any non-negative vector $\rho \in \mathbf{R}^n$ under the componentwise ρ-neighbourhood of a point $z \in \mathbf{R}^n$, we understand the set

$$O_\rho(z) := \big\{ \xi \in \mathbf{R}^n : \ |\xi - z| \le \rho \big\}. \tag{7}$$

Similarly, the ρ-neighbourhood of a domain $\Omega \subset \mathbf{R}^n$ is defined as

$$O_\rho(\Omega) := \bigcup_{z\in\Omega} O_\rho(z). \tag{8}$$

A particular kind of vector ρ will be specified below in relation (11).

Let us choose certain compact convex sets $D_a \subset \mathbb{R}^n$, $D_b \subset \mathbb{R}^n$ and define the set

$$D_{a,b} := (1-\theta)z + \theta\eta, \ \ z \in D_a, \ \ \eta \in D_b, \ \ \theta \in [0,1], \tag{9}$$

moreover, according to (8) its ρ- neighbourhood

$$D = O_\rho(D_{a,b}) \tag{10}$$

with a non-negative vector $\rho = col(\rho_1,\dots,\rho_n) \in \mathbb{R}^n$, such that

$$\rho \ge \frac{b-a}{2}\, \delta_{[a,b],D\times D}(f), \tag{11}$$

where $\delta_{[a,b],D\times D}(f)$ denotes the half of the oscillation of the function f over $[a,b]\times D\times D$, i.e.,

$$\delta_{[a,b],D\times D}(f) := \frac{\max\limits_{(t,x,y)\in[a,b]\times D\times D} f(t,x,y) - \min\limits_{(t,x,y)\in[a,b]\times D\times D} f(t,x,y)}{2}. \tag{12}$$

Instead of the original boundary value problem (1), (2), we will consider the following auxiliary two-point parametrized boundary value problem

$$\frac{dx(t)}{dt} = f(t,x(t),x(x(t))), \ \ t \in [a,b], \tag{13}$$

$$x(a) = z, \quad x(b) = \eta, \tag{14}$$

where z and η are treated as free parameters.

Let us connect with problem (13), (14) the sequence of functions

$$
\begin{aligned}
x_{m+1}(t,z,\eta)=z+\int\limits_a^t f\big(s,x_m(s,z,\eta),x_m(x_m(s,z,\eta),z,\eta)\big)\,ds \\
-\frac{t-a}{b-a}\int\limits_a^b f\big(s,x_m(s,z,\eta),x_m(x_m(s,z,\eta),z,\eta)\big)\,ds \\
+\frac{t-a}{b-a}\,[\eta-z],\ \ t\in[a,b],\ \ m=0,1,2,\dots, \quad (15)
\end{aligned}
$$

satisfying (14) for arbitrary $z,\eta\in\mathbb{R}^n$, where

$$
x_0(t,z,\eta)=z+\frac{t-a}{b-a}\,[\eta-z]=\Big(1-\frac{t-a}{b-a}\Big)z+\frac{t-a}{b-a}\,\eta,\ \ t\in[a,b]. \tag{16}
$$

It is easy to see from (16) that $x_0(t,z,\eta)$ is a linear combination of vectors z and η, when $z\in D_a$ and $\eta\in D_b$.

The following statement establishes the uniform convergence of sequence (15) to some parameterized limit function.

Theorem 1. *Let conditions* (6), (11) *be fulfilled, moreover, for the matrix*

$$
Q=\frac{3(b-a)}{10}\,K,\ \ K=K_1+K_2L \tag{17}
$$

the inequality

$$
r(Q)<1 \tag{18}
$$

hold.

Then, for all fixed $(z,\eta)\in D_a\times D_b$:

1. *The functions of sequence* (15) *belonging to the domain D of form* (10) *are continuously differentiable on the interval $[a,b]$ and satisfy conditions* (14).
2. *The sequence of functions* (15) *for $t\in[a,b]$ uniformly converges as $m\to\infty$ with respect to the domain $(t,z,\eta)\in[a,b]\times D_a\times D_b$ to the limit function*

$$
x_\infty(t,z,\eta)=\lim_{m\to\infty}x_m(t,z,\eta), \tag{19}
$$

satisfying conditions (14).

3. *The function $x_\infty(t,z,\eta)$ for all $t\in[a,b]$ is a unique continuously differentiable solution of the integral equation*

$$
x(t)=z+\int\limits_a^t f(s,x(s),x(x(s)))\,ds-\frac{t-a}{b-a}\int\limits_a^b f(s,x(s),x(x(s)))\,ds+\frac{t-a}{b-a}\,[\eta-z], \tag{20}
$$

i.e., it is the solution to the Cauchy problem for the modified system of integro-differential equations:

$$
\begin{gathered}
\frac{dx}{dt}=f(t,x(t),x(x(t)))+\frac{1}{b-a}\,\Delta(z,\eta),\\
x(a)=z
\end{gathered} \tag{21}
$$

where $\Delta(z,\eta): D_a\times D_b\to\mathbb{R}^n$ *is a mapping given by the formula*

$$\Delta(z,\eta)=[\eta-z]-\int\limits_a^b f\big(s,x_\infty(s,z,\eta),x_\infty(x_\infty(s,z,\eta),z,\eta)\big)\,ds. \tag{22}$$

4. *The error estimation*

$$\big|x_\infty(t,z,\eta)-x_m(t,z,\eta)\big|\leqslant\frac{10}{9}\,\alpha_1(t)Q^m(1_n-Q)^{-1}\delta_{[a,b],D\times D}(f),\ \ t\in[a,b],\ \ m\geq 0 \tag{23}$$

holds, where

$$\alpha_1(t)=2(t-a)\Big(1-\frac{t-a}{b-a}\Big)\leq\frac{b-a}{2}\,,\ \ t\in[a,b].$$

The following statement gives a relation of the parameterized limit function $x_\infty(t,z,\eta)$ to the solution of the original boundary value problem (1), (2).

Theorem 2. *Under the assumptions of Theorem 1, the limit function*

$$x_\infty(t,z,\eta)=\lim_{m\to\infty}x_m(t,z,\eta)$$

of sequence (15) *is a solution of the boundary value problem* (1), (2) *with property* (5) *if and only if the pair of parameters* (z,η) *satisfies the system of* $2n$ *algebraic equations*

$$\begin{gathered}\Delta(z,\eta):=[\eta-z]-\int\limits_a^b f\big(s,x_\infty(s,z,\eta),x_\infty(x_\infty(s,z,\eta),z,\eta)\big)\,ds=0,\\ \Phi(z,\eta):=\Phi(x_\infty(t,z,\eta)),\quad x_\infty(x_\infty(t,z,\eta))-d=0.\end{gathered} \tag{24}$$

We apply the above techniques to the following model BVP in $\mathbf{R}^2$

$$\begin{aligned}\frac{dx_1(t)}{dt}&=\big[x_1(x_1(t))\big]^2-\frac18\,x_2(t)+\frac12=f_1\big(x_1,x_2,x_1(x_1(t)),x_2(x_2(t))\big),\ \ t\in[a,b]=\Big[0,\frac12\Big],\\ \frac{dx_2(t)}{dt}&=x_2(x_2(t))-\frac t2\,x_1(t)\cdot x_2(t)+t=f_2\big(x_1,x_2,x_1(x_1(t)),x_2(x_2(t))\big),\end{aligned} \tag{25}$$

with the iterative integral boundary conditions

$$\begin{aligned}\Phi_1\big(x(t),x(x(t))\big)&=\int\limits_0^{1/2}\big[x_1(s)+x_2(s)\big]\,ds=\frac{1}{12}\,,\\ \Phi_2\big(x(t),x(x(t))\big)&=\int\limits_0^{1/2}\big[x_1(x_1(s))\big]^2\,ds=\frac{1}{384}\,.\end{aligned} \tag{26}$$

Clearly, problem (25), (26) is a particular case of (1), (2) with $a=0$, $b=\frac12$, $d=col(\frac18,\frac{1}{384})$. It is easy to check that $x_1(t)=\frac t2$, $x_2(t)=\frac{t^2}{2}$ is a continuously differentiable solution to problem (25), (26).

One can check that all the conditions of Theorem 1 for this example are fulfilled for the following choosing and computation of corresponding sets, vectors, matrices:

$$D_a=D_b=\big\{(x_1,x_2):\ -0.05\leq x_1\leq 0.3,\ -0.05\leq x_2\leq 0.2\big\},\ \ D_{a,b}=D_a=D_b, \tag{27}$$

$$\rho := col(0.15, 0.15), \;\; O\rho(D_{a,b}) = D = \big\{(x_1, x_2) : \; -0.2 \le x_1 \le 0.45, \; -0.2 \le x_2 \le 0.35\big\},$$

$$K_1 = \begin{bmatrix} 0 & \frac{1}{8} \\ 0.25 & 0.25 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$K = K_1 + K_2 L = \begin{bmatrix} 1 & \frac{1}{8} \\ 0.25 & 1.25 \end{bmatrix}, \quad Q = \frac{3(b-a)}{10}\, K = \begin{bmatrix} 0.15 & 0.01875 \\ 0.0375 & 0.1875 \end{bmatrix}, \;\; r(Q) \approx 0.2 < 1,$$

$$\delta_{[a,b], D\times D}(f) ::= \begin{bmatrix} 0.176875 \\ 0.415 \end{bmatrix}, \quad \rho = \begin{bmatrix} 0.15 \\ 0.15 \end{bmatrix} \ge \frac{b-a}{2}\, \delta_{[a,b], D\times D}(f) = \begin{bmatrix} 0.03546875 \\ 0.08125 \end{bmatrix}.$$

In the case of Maple computations for iterative systems it is more appropriate to use instead of (15) a scheme with polynomial interpolation, [3,4], when instead of (15), we introduce the sequence $\{x^{q+1}_{m+1}(t,z,\eta)\}_{m=0}^{\infty}$ of vector polynomials $x^{q+1}_{m+1}(t,z,\eta) = col(x^{q+1}_{m+1,1}(t,z,\eta), x^{q+1}_{m+1,2}(t,z,\eta))$ of degree $(q+1)$

$$\begin{aligned} x^{q+1}_{m+1,j}(t,z,\eta) &:= a_{m+1,j,0}(z,\eta) + a_{m+1,j,1}(z,\eta)t + a_{m+1,j,2}(z,\eta)t^2 + \dots + a_{m+1,j,q+1}(z,\eta)t^{q+1} \\ &= z + \int\limits_a^t \Big[A_{m,j,0}(z,\eta) + A_{m,j,1}(z,\eta)t + A_{m,j,2}(z,\eta)t^2 + \dots + A_{m,j,q}(z,\eta)t^q\Big]\,dt \\ &\quad - \frac{t-a}{b-a} \int\limits_a^b \Big[A_{m,j,0}(z,\eta) + A_{m,j,1}(z,\eta)t + A_{m,j,2}(z,\eta)t^2 + \dots + A_{m,j,q}(z,\eta)t^q\Big]\,dt \\ &\quad + \frac{t-a}{b-a}\,[\eta_j - zj], \;\; t \in [a,b], \;\; m = 0,1,2,\dots, \;\; j = 1,2, \end{aligned} \tag{28}$$

where

$$A_{m,j,0}(z,\eta) + A_{m,j,1}(z,\eta)t + A_{m,j,2}(z,\eta)t^2 + \dots + A_{m,j,q}(z,\eta)t^q, \;\; j = 1,2$$

are the Lagrange interpolation polynomials of degree q on the Chebyshev nodes, translated from $(-1,1)$ to the interval (a,b), corresponding to the functions

$$f_j\Big(t, x^{q+1}_{m,1}(t,z,\eta), x^{q+1}_{m,2}(t,z,\eta), x^{q+1}_{m,1}\big(x^{q+1}_{m,1}(t,z,\eta)\big), x^{q+1}_{m,2}\big(x^{q+1}_{m,2}(t,z,\eta)\big)\Big), \;\; j = 1,2$$

respectively in (25). Note that the coefficients of the interpolation polynomials depend on the parameters z and η. On the basis of (28), instead of (24) let us define the mth approximate polynomial determining system, which consists of four algebraic equations when $j = 1,2$,

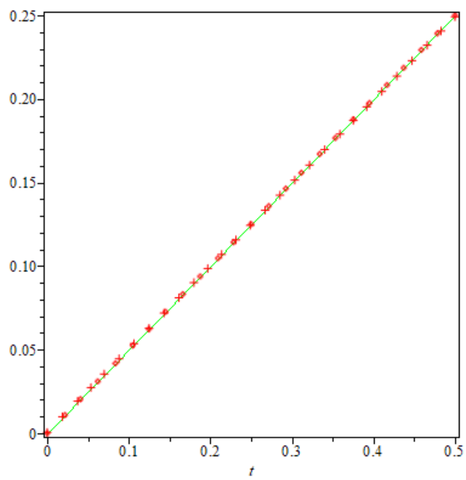
$$\begin{aligned} \Delta^q_{m,j}(z,\eta) &:= [\eta_j - z_j] \\ &\quad - \int\limits_a^b \Big[A_{m,j,0}(z,\eta) + A_{m,j,1}(z,\eta)t + A_{m,j,2}(z,\eta)t^2 + \dots + A_{m,j,q}(z,\eta)t^q\Big]\,dt = 0, \end{aligned} \tag{29}$$

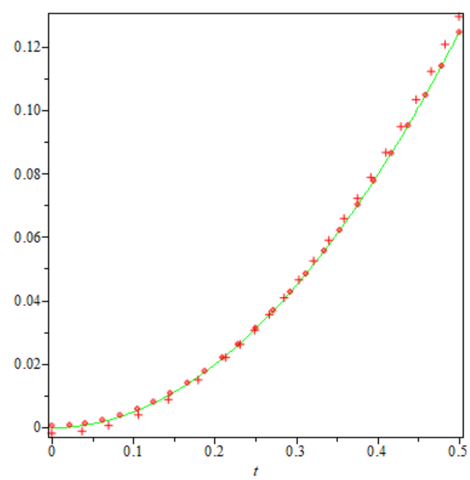
$$\Phi^q_{m,j}(z,\eta) := \Phi_j\Big(x^{q+1}_{m,1}(t,z,\eta), x^{q+1}_{m,2}(t,z,\eta), x^{q+1}_{m,1}\big(x^{q+1}_{m,1}(t,z,\eta)\big), x^{q+1}_{m,2}\big(x^{q+1}_{m,2}(t,z,\eta)\big)\Big) - d_j = 0.$$

By choosing $q = 3$, using (28) and solving (29) (applying Maple 14) we obtain the approximate numerical values for the introduced parameters given in table.

The graphs of the zeroth ($\times$), sixth ($\diamond$) approximation and the exact solution (solid line) to problem (25), (26) are shown in figure.

	z_1	z_2	η_1	η_2
$m=0$	$0.5332693 \cdot 10^{-3}$	-0.194303210^{-2}	0.2491448305	0.1294598825
$m=3$	$1.4024463 \cdot 10^{-7}$	$0.4841504 \cdot 10^{-3}$	0.2500002840	0.1245609255
$m=6$	$1.3907241 \cdot 10^{-7}$	$0.4841505 \cdot 10^{-3}$	0.2500002846	0.1245609251
Exact	0	0	0.25	0.125





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Investigation of the Oscillation, Rotation and Wandering Radial Indicators of a Differential System by the First Approximation

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For a given zero neighborhood G in the Euclidean space $\mathbb{R}^n$, we consider a nonlinear, generally speaking, differential system of the form

$$\dot{x} = f(t,x), \quad f(t,0) = 0, \ \ t \in \mathbb{R}_+ \equiv [0,+\infty), \ \ x \in G, \tag{1}$$

where the right-hand side satisfies the condition $f, f'_x \in C(\mathbb{R}_+ \times G)$ and the zero solution is allowed. We associate with system (1) the linear homogeneous system of its *first approximation*

$$\dot{x} = A(t)x, \quad A(t) \equiv f'_x(t,0), \ \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R}^n, \tag{2}$$

for which we do not require here the uniformity in $t \in \mathbb{R}_+$ of the natural (pointwise) smallness of the nonlinear addition

$$h(t,x) \equiv f(t,x) - A(t)x = o(x), \ \ x \to 0.$$

Let $x_f(\,\cdot\,,x_0)$ be a non-extendable solution of system (1) with the initial condition $x_f(0,x_0) = x_0$. By $S_*(f)$ and S_A we denote the set of all nonzero solutions to system (1) and, accordingly, the set of all solutions to system (2).

Definition 1. Let us list three basic [1] *functional* $\mathrm{K}(t,u)$ defined on the pairs $t \in \mathbb{R}_+$ and $u : [0,t] \to \mathbb{R}^n$ (taking the value $+\infty$ whenever the function is not defined on the entire segment $[0,t]$), corresponding to *indicators*

$$\varkappa = \nu, \theta, \rho, \ \ \text{respectively, for} \ \ \mathrm{K} = \mathrm{N}, \Theta, \mathrm{P}, \tag{3}$$

and describing the following properties of solutions:

1) *oscillation* ($\varkappa = \nu$), if $\mathrm{K}(t,u) = \mathrm{N}(t,u)$ is the number (multiplied by π) zeros of the function $P_1 u$ on the interval $(0,t]$, where P_1 is an orthogonal projector onto a fixed line, and if at least one of these zeros is *multiple* (that is, it is also a zero and derivative $(P_1 u)^{\cdot}$), then we assume $\mathrm{N}(t,u) = +\infty$;

2) *rotation* (*oriented*, $\varkappa = \theta$), if $\mathrm{K}(t,u) = \Theta(t,u) \equiv |\varphi(t,P_2 u)|$ is module of *oriented* angle $\varphi(t,P_2 u)$ (continuous in t, with initial condition $\varphi(0,P_2 u) = 0$) between the vector $P_2 u(t)$ and the initial vector $P_2 u(0)$, where P_2 is the orthogonal projector onto a fixed two-dimensional plane, and if $P_2 u(\tau) = 0$ for at least one $\tau \in [0,t]$, then we assume $\Theta(t,u) = +\infty$;

3) *wandering* ($\varkappa = \rho$), if

$$\mathrm{K}(t,u) = \mathrm{P}(t,u) \equiv \int\limits_0^t \big|\big(u(\tau)/|u(\tau)|\big)^{\cdot}\big|\,d\tau, \ \ u(\tau) \neq 0, \ \ \tau \in [0,t].$$

There are also known the other functionals that are responsible for the *non-oriented* or *frequency rotation* [1], *k-th rank rotation* [2], and *plane rotation* [3].

Definition 2 ([4])**.** For each functional described in Definition 1, we define:

(a) *weak* and *strong lower linear* indicators (3) of the solution $x \in S_*(f)$ defined on the whole semiaxis $\mathbb{R}_+$ – by the formulas

$$\check{\varkappa}^\circ(x) \equiv \varliminf_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\mathrm{K}(t,Lx), \quad \check{\varkappa}^\bullet(x) \equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varliminf_{t\to+\infty} t^{-1}\mathrm{K}(t,Lx); \tag{4}$$

(b) *weak* and *strong lower radial* indicators (3) of the Cauchy problem for system (1) with the initial value $x_0 \in G$ – by the formulas

$$\begin{aligned} \check{\varkappa}_r^\circ(f,x_0) &\equiv \varliminf_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\check{\mathrm{K}}_r(f,x_0,t,L), \\ \check{\varkappa}_r^\bullet(f,x_0) &\equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varliminf_{t\to+\infty} t^{-1}\check{\mathrm{K}}_r(f,x_0,t,L), \end{aligned} \tag{5}$$

where

$$\check{\mathrm{K}}_r(f,x_0,t,L) = \varliminf_{\mu\to+0} \mathrm{K}(t,Lx_f(\,\cdot\,,\mu x_0)); \tag{6}$$

(c) *weak* and *strong lower spherical* indicators (3) of the Cauchy problem for system (1) with the initial value $x_0 \in G$ – by the formulas

$$\begin{aligned} \check{\varkappa}_s^\circ(f,x_0) &\equiv \varliminf_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\check{\mathrm{K}}_s(f,x_0,t,L), \\ \check{\varkappa}_s^\bullet(f,x_0) &\equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varliminf_{t\to+\infty} t^{-1}\check{\mathrm{K}}_s(f,x_0,t,L), \end{aligned} \tag{7}$$

where

$$\mathrm{K}_s(f,x_0,t,L) \equiv \mathrm{K}(t,Lx_{f_s}(\,\cdot\,,x_0)), \quad f_s(t,x) \equiv P_x^\perp f(t,x),$$

$P_x^\perp$ is a projector onto a hyperplane orthogonal to x, and the modified system

$$\dot{x} = f_s(t,x), \quad (t,x) \in \mathbb{R}_+ \times G, \tag{8}$$

is also called *spherical* (with respect to system (1));

(d) *weak* and *strong upper* indicators – *linear* $\widehat{\varkappa}^\circ(x)$, $\widehat{\varkappa}^\bullet(x)$, *radial* $\widehat{\varkappa}_r^\circ(f,x_0)$, $\widehat{\varkappa}_r^\bullet(f,x_0)$ and *spherical* $\widehat{\varkappa}_s^\circ(f,x_0)$, $\widehat{\varkappa}_s^\bullet(f,x_0)$ – by the same formulas (3), (5) and (7), respectively, but with the replacement in formulas (3)–(7) of all lower limits for $t \to +\infty$ and for $\mu \to +0$ by upper ones;

(e) *exact* or *absolute* varieties of the same indicators that arise when the corresponding values of the lower and upper indicators or, respectively, weak and strong ones coincide: in the first case, we will omit the checkmark and the cap in their designation, and in the second one – an empty and full circle.

Everywhere below, the letters $\varkappa$ or K mean any (corresponding) of the indicators or functionals (3), and the top icons ~ or $*$ are any of the icons $\check{}$, $\hat{}$ or $\circ$, $\bullet$, respectively.

The introduction of radial and spherical indicators (as well as *ball* ones [4]) is due to the fact that some solutions of the nonlinear system (1) may be defined not on the entire time semiaxis.

On the one hand, for linear systems, the linear and nonlinear (radial and spherical) indicators are indistinguishable.

Theorem 1. *If system* (1) *is linear homogeneous and* $G = \mathbb{R}^n$*, then for any solution* $x \in S_*(f)$ *the equalities hold*

$$\widetilde{\mathrm{K}}_r(f, x(0), t, L) = \mathrm{K}_s(f, x(0), t, L) = \mathrm{K}(f, Lx), \quad t \in \mathbb{R}_+, \quad \operatorname{Aut} \mathbb{R}^n,$$
$$\widetilde{\varkappa}_r^*(f, x(0)) = \widetilde{\varkappa}_s^*(f, x(0)) = \widetilde{\varkappa}^*(x).$$

On the other hand, in the nonlinear (even if autonomous) case, that coincidence is no longer observed.

Theorem 2. *If* $n = 2$ *and* $G = \mathbb{R}^2$*, then for each of the following four lines of relations separately*

$$\begin{aligned} 0 = \varkappa_r(f, x(0)) < \varkappa_s(f, x(0)) < \varkappa(x) = +\infty, \\ 0 = \varkappa_r(f, x(0)) = \varkappa(x) < \varkappa_s(f, x(0)) < +\infty, \\ 1 = \varkappa_r(f, x(0)) > \varkappa_s(f, x(0)) > \varkappa(x) = 0, \\ 1 = \varkappa_r(f, x(0)) = \varkappa(x) > \varkappa_s(f, x(0)) > 0, \end{aligned}$$

there exists an autonomous system (1) *such that any solution* $x \in S_*(f)$ *is defined on* $\mathbb{R}_+$*, and all linear, radial and spherical indicators are exact, absolute and satisfy the relations of that particular line.*

The radial wandering indicators completely coincide with the corresponding linear ones of the first approximation system.

Theorem 3. *For any system* (1) *and any nonzero solution* $x \in S_A$ *to the system of its first approximation* (2)*, the equalities hold*

$$\check{\mathrm{P}}_r(f, x(0), t, L) = \widehat{\mathrm{P}}_r(f, x(0), t, L) = \mathrm{P}(Lx, t), \quad t \in \mathbb{R}_+, \quad L \in \operatorname{Aut} \mathbb{R}^n,$$
$$\widetilde{\rho}_r^*(f, x(0)) = \widetilde{\rho}^*(x).$$

In the two-dimensional case, a similar coincidence is observed also for the rotation indicators.

Theorem 4. *If* $n = 2$*, then for any system* (1) *and any nonzero solution* $x \in S_A$ *to the system of its first approximation* (2)*, the equalities hold*

$$\check{\Theta}_r(f, x(0), t, L) = \widehat{\Theta}_r(f, x(0), t, L) = \Theta(Lx, t), \quad t \in \mathbb{R}_+, \quad L \in \operatorname{Aut} \mathbb{R}^n,$$
$$\widetilde{\theta}_r^*(f, x(0)) = \widetilde{\theta}^*(x).$$

However, already in the three-dimensional (and even autonomous) case, the rotational radial indicators, as well as the oscillation ones, generally speaking, do not match the linear ones.

Theorem 5. *For* $n = 3$ *and* $G = \mathbb{R}^3$ *there exists an autonomous system* (1) *such that for any nonzero solution* $x \in S_A$ *of the system of its first approximation* (2) *the solution* $x_f(\,\cdot\,, x(0))$ *is also defined on* $\mathbb{R}_+$*, and all the rotational and oscillation indicators are exact, absolute and for some two-dimensional subspace* $S \subset S_A$ *satisfy the relations*

$$0 = \theta_r(f, x(0)) = \nu_r(f, x(0)) \leqslant \theta(x) = \nu(x) = \begin{cases} 1, & x \in S \setminus \{0\}; \\ 0, & x \notin S. \end{cases}$$

For the linear and nonlinear radial indicators of oscillation, a similar mismatch is observed already in the two-dimensional (albeit only in a non-autonomous) case.

Theorem 6. *For $n = 2$ and $G = \mathbb{R}^2$ there exists a system* (1) *such that for any solution $x \in S_A$ of the system of its first approximation* (2) *the solution $x_f(\,\cdot\,, x(0))$ is also defined on $\mathbb{R}_+$, and all the linear and radial oscillation indicators are exact, absolute and satisfy the relations*

$$0 = \nu_r(f, x(0)) < \nu(x) = 1.$$

For the spherical indicators, however, no analogs of Theorems 3 and 4 above are valid (that follows from Theorems 2 and 3).

Theorem 7. *If $n = 2$ and $G = \mathbb{R}^2$, then for each of the following two lines of relations separately*

$$0 = \varkappa(x) < \varkappa_s(f, x(0)) < +\infty,$$
$$1 = \varkappa(x) > \varkappa_s(f, x(0)) > 0,$$

there exists an autonomous system (1) *such that for any nonzero solution $x \in S_A$ of the system of its first approximation* (2) *the solution $x_f(\,\cdot\,, x(0))$ is also defined on $\mathbb{R}_+$, and all the linear and spherical indicators are exact, absolute and satisfy the relations of that particular line.*

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On the Representation of a Solution for the Perturbed Controlled Differential Equation with the Discontinuous Initial Condition Considering Perturbation of the Initial Moment

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In the paper, for the perturbed controlled nonlinear differential equation with the constant delay in the phase coordinates and in controls a formula of the analytic representation of a solution is obtained in the left semi-neighborhood of the endpoint of the main interval. The novelty here is the effect in the formula related with perturbation of the initial moment. Analogous formulas without perturbation of the initial moment and without delay in controls are given in [1, 3, 4].

Let $I = [0, T]$ be a finite interval and let $\tau_2 > \tau_1 > 0$, and $\theta > 0$ be given numbers; suppose that $O \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^r$ are open sets. Let n-dimensional function $f(t, x, y, u, v)$ be continuous on $I \times O^2 \times U^2$ and continuously differentiable with respect to x, y, u and v. Let Φ be a set of continuously differentiable initial functions $\varphi : I_1 = [-\tau_2, T] \to O$ and let Ω be a set of piecewise-continuous and bounded control functions $u(t) \in U$, $t \in I_2 = [-\theta, T]$.

In the space $\mathbb{R}^n$ to each element $\mu = (t_0, \tau, x_0, \varphi, u) \in \Lambda = [0, T) \times [\tau_1, \tau_2] \times O \times \Phi \times \Omega$ we assign the delay controlled differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau), u(t), u(t-\theta)), \quad t \in [t_0, T] \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \quad t \in [-\tau_2, t_0), \qquad x(t_0) = x_0. \tag{2}$$

Condition (2) is called the discontinuous initial condition because, in general, $x(t_0) \neq \varphi(x_0)$.

Definition. Let $\mu = (t_0, \tau, x_0, \varphi, u) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in I_1$ is called a solution of equation (1) with the initial condition (2) or a solution corresponding to μ and defined on the interval I_1 if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, T]$ and satisfies equation (1) almost everywhere on $[t_0, T]$.

It is clear that the solution $x(t) = x(t; \mu)$, $t \in I_1$, in general, at the point t_0 is discontinuous. Let us introduce notations

$$|\mu| = |t_0| + |\tau| + |x_0| + \|\varphi\|_1 + \|u\|, \quad \Lambda_\varepsilon(\mu_0) = \big\{\mu \in \Lambda : \ |\mu - \mu_0| \le \varepsilon\big\},$$

where
$$\|\varphi\|_1=\sup\big\{|\varphi(t)|+|\dot\varphi(t)|:\ t\in I_1\big\},\quad \|u\|=\sup\big\{|u(t)|:\ t\in I_2\big\},$$
$\varepsilon>0$ is a fixed number and $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,u_0)\in\Lambda$ is a fixed element; furthermore,
$$\delta t_0=t_0-t_{00},\ \ \delta x_0=x_0-x_{00},\ \ \delta\varphi(t)=\varphi(t)-\varphi_0(t),\ \ \delta u(t)=u(t)-u_0(t),$$
$$\delta\mu=\mu-\mu_0=(\delta t_0,\delta\tau,\delta x_0,\delta\varphi,\delta u),\quad |\delta\mu|=|\delta t_0|+|\delta\tau|+|\delta x_0|+\|\delta\varphi\|_1+\|\delta u\|.$$

Remark. Let $x(t;\mu_0)$ be the solution corresponding to $\mu_0\in\Lambda$ and defined on the interval I_1, i.e. $x(t;\mu_0)$ is the solution of the problem
$$\dot x(t)=f\big(t,x(t),x(t-\tau_0),u_0(t),u_0(t-\theta)\big),\ \ t\in[t_{00},T],$$
$$x(t)=\varphi_0(t),\ \ t\in[-\tau_2,t_{00}),\quad x(t_{00})=x_{00}.$$

Then, there exists a number $\varepsilon_1>0$ such that to each element $\mu\in\Lambda_{\varepsilon_1}(\mu_0)$ there corresponds the solution $x(t;\mu)$ defined on the interval I_1, i.e. the perturbed problem (1), (2) has the solution $x(t;\mu)$, $t\in I_1$ [2, p. 18].

Theorem 1. *Let $x_0(t):=x(t;\mu_0)$ be the solution corresponding to the element $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,u_0)\in\Lambda$ and defined on the interval I_1, with $t_{00}+\tau_0<T$. Let $\delta>0$ and $\varepsilon_2\in(0,\varepsilon_1)$ be numbers such that $t_{00}+\tau_0+\varepsilon_2<T-\delta$. Then, for arbitrary*
$$\mu\in\Lambda^-_{\varepsilon_2}(\mu_0)=\big\{\mu=(t_0,\tau,x_0,\varphi,u)\in\Lambda_{\varepsilon_2}(\mu_0):\ 0\le t_0\le t_{00}\big\}$$
on the interval $[T-\delta,T]$, the following representation holds
$$x(t;\mu)=x_0(t)+\delta x^-(t;\delta\mu)+o(t;\delta\mu),\tag{3}$$
where
$$\begin{aligned}
\delta x^-(t;\delta\mu)&=-\big[Y(t_{00};t)f_0^-+Y(t_{00}+\tau_0;t)f_1^-\big]\delta t_0-Y(t_{00}+\tau_0;t)f_1^-\delta\tau+\beta(t;\delta\mu),\\
\beta(t;\delta\mu)&=Y(t_{00};t)\delta x_0\\
&\quad+\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)\,ds+\bigg[\int\limits_{t_{00}}^{t}Y(s;t)f_y[s]\dot x_0(s-\tau_0)\,ds\bigg]\delta\tau\\
&\quad+\int\limits_{t_{00}}^{t}Y(s;t)f_u[s]\delta u(s)\,ds+\int\limits_{t_{00}}^{t}Y(s;t)f_v[s]\delta u(s-\theta)\,ds,
\end{aligned}\tag{4}$$
$$\begin{aligned}
f_0^-&=f\big(t_{00},x_{00},\varphi_0(t_{00}-\tau_0),u_0(t_{00}-),u_0(t_{00}-\theta-)\big),\\
f_1^-&=f\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),x_{00},u_0(t_{00}+\tau_0-),u_0(t_{00}+\tau_0-\theta-)\big)\\
&\qquad-f\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00}),u_0(t_{00}+\tau_0-),u_0(t_{00}+\tau_0-\theta-)\big),\\
f_y[s]&=f_y\big(s,x_0(s),x_0(s-\tau_0),u_0(s),u_0(t_{00}+\tau_0-\theta-)\big),
\end{aligned}$$
$$\lim_{|\delta\mu|\to0}\frac{|o(t;\delta\mu)|}{|\delta\mu|}=0\ \ \textit{uniformly for}\ \ t\in[T-\delta,T].$$
Next, $Y(s;t)$ is the $n\times n$-matrix function satisfying the equation
$$Y_s(s;t)=-Y(s;t)f_x[s]-Y(s+\tau_0;t)f_y[s+\tau_0],\ \ s\in[t_{00},t]$$
and the condition
$$Y(s;t)=\begin{cases}H & \textit{for}\ \ s=t,\\ \Theta & \textit{fors}>t;\end{cases}$$
H is the identity matrix and Θ is the zero matrix.

Some comments

The function $\delta x^-(t;\delta\mu)$ is called the first variation of a solution $x_0(t)$, $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$. The expression (4) is called the variation formula of a solution. The addend

$$-\big[Y(t_{00};t)f_0^- + Y(t_{00}+\tau_0;t)f_1^-\big]\delta t_0$$

in (4) *is the effect of perturbation of the initial moment* t_{00}. Namely, here f_1^- is the effect of the discontinuous initial condition (2). The addend

$$-\bigg[Y(t_{00}+\tau_0;t)f_1^- + \int\limits_{t_{00}}^{t} Y(s;t)f_y[s]\dot x_0(s-\tau_0)\,ds\bigg]\delta\tau$$

in (4) is the effect of perturbation of the delay τ_0. The expression

$$Y(t_{00};t)\delta x_0 + \int\limits_{t_{00}-\tau_0}^{t_{00}} Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)\,ds$$

in (4) is the effect of perturbations of the initial vector x_{00} and the initial function $\varphi_0(t)$. The expression

$$\int\limits_{t_{00}}^{t} Y(s;t)\big[f_u[s]\delta u(s)+f_v[s]\delta u(s-\theta)\big]\,ds$$

in (4) is the effect of perturbation of the control function $u_0(t)$.

Formula (3) allows us to obtain an approximate solution of the perturbed problem (1), (2) in the analytical form on the interval $T-\delta,T]$. In fact, for a small $|\delta\mu|$ from (3) it follows

$$x(t;\mu)\approx x_0(t)+\delta x^-(t;\delta\mu),\ \ t\in[T-\delta,T].$$

Theorem 2. *Let $x_0(t):=x(t;\mu_0)$ be the solution corresponding to the element $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,u_0)\in\Lambda$ and defined on the interval I_1, with $t_{00}+\tau_0<T$. Let $\delta>0$ and $\varepsilon_2\in(0,\varepsilon_1)$ be numbers such that $t_{00}+\tau_0+\varepsilon_2<T-\delta$. Then, for arbitrary*

$$\mu\in\Lambda^+_{\varepsilon_2}(\mu_0)=\big\{\mu=(t_0,\tau,x_0,\varphi,u)\in\Lambda_{\varepsilon_2}(\mu_0):\ t_{00}\le t_0<T\big\}$$

on the interval $[T-\delta,T]$, the following representation holds

$$x(t;\mu)=x_0(t)+\delta x^+(t;\delta\mu)+o(t;\delta\mu),$$

where

$$\delta x^+(t;\delta\mu)=-\big[Y(t_{00};t)f_0^+ + Y(t_{00}+\tau_0;t)f_1^+\big]\delta t_0 - Y(t_{00}+\tau_0;t)f_1^+\delta\tau+\beta(t;\delta\mu).$$

Theorem 3. *Let $x_0(t):=x(t;\mu_0)$ be the solution corresponding to the element $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,u_0)\in\Lambda$ and defined on the interval I_1, with $t_{00}+\tau_0<T$. Let the functions $u_0(t)$ and $u_0(t-\theta)$ are continuous at the points t_{00} and $t_{00}+\tau_0$. Besides, let $\delta>0$ and $\varepsilon_2\in(0,\varepsilon_1)$ be numbers such that $t_{00}+\tau_0+\varepsilon_2<T-\delta$. Then, for arbitrary $\mu\in\Lambda_{\varepsilon_2}(\mu_0)$ on the interval $[T-\delta,T]$, the following representation holds*

$$x(t;\mu)=x_0(t)+\delta x(t;\delta\mu)+o(t;\delta\mu),$$

where

$$\begin{aligned}
\delta x(t;\delta\mu)&=-\big[Y(t_{00};t)f_0+Y(t_{00}+\tau_0;t)f_1\big]\delta t_0-Y(t_{00}+\tau_0;t)f_1\delta\tau+\beta(t;\delta\mu),\\
f_0&=f\big(t_{00},x_{00},\varphi_0(t_{00}-\tau_0),u_0(t_{00}),u_0(t_{00}-\theta)\big),\\
f_1&=f\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),x_{00},u_0(t_{00}+\tau_0),u_0(t_{00}+\tau_0-\theta)\big)\\
&\qquad-f\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00}),u_0(t_{00}+\tau_0),u_0(t_{00}+\tau_0-\theta)\big).
\end{aligned}$$

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On the Critical Case in the Theory of the Matrix Differential Equations

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In the mathematical description of various phenomena and processes arising in mathematical physics, electrical engineering, economics, have to deal with matrix differential equations. Therefore, such equations are relevant as for mathematicians and specialists in other fields of natural sciences. This article considers quasi-linear matrix differential equations with coefficients depicted in the form of absolutely and uniformly convergent Fourier series with slow variable in a sense coefficients and frequency (class F). The differences of the diagonal elements of the matrices of the linear part are pure imaginary, that is, we are dealing with a critical case. But between these diagonal elements assume certain relations that indicate the absence of resonance between the natural frequencies of the system and frequency of external excitation force. The problem is considered establishing signs of existence in such an equation of solutions class F. By means of a number of transformations the equation is reduced to the equation in noncritical case, and the solution of the class F of this equation is sought by the method of successive approximations using the principle compression reflections. Then based on the properties of the solutions of the transformed equation, conclusions are drawn about the properties of the initial equation.

1 Basic notation and definitions

Let

$$G(\varepsilon_0) = \big\{t, \varepsilon :\ t \in \mathbf{R},\ \varepsilon \in (\mathbf{0}, \varepsilon_\mathbf{0}),\ \varepsilon_\mathbf{0} \in \mathbf{R}^+\big\}.$$

Definition 1.1. We say that a function $p(t, \varepsilon)$ belongs to the class $S(m; \varepsilon_0)$, $m \in \mathbf{N} \cup \{\mathbf{0}\}$, if:

(1) $p : G(\varepsilon_0) \to \mathbf{C}$;

(2) $p(t, \varepsilon) \in C^m(G(\varepsilon_0))$ at t;

(3) $\dfrac{d^k p(t, \varepsilon)}{dt^k} = \varepsilon^k p_k(t, \varepsilon)$ $(0 \le k \le m)$,

$$\|p\|_{S(m;\varepsilon_0)} \overset{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |p_k(t, \varepsilon)| < +\infty.$$

Definition 1.2. We say that a function $f(t, \varepsilon, \theta(t, \varepsilon))$ belongs to the class $F(m; \varepsilon_0; \theta)$ $(m \in \mathbf{N} \cup \{0\})$, if

$$f(t, \varepsilon, \theta(t, \varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) \exp(in\,\theta(t, \varepsilon)),$$

and

(1) $f_n(t, \varepsilon) \in S(m, \varepsilon_0)$ $(n \in \mathbf{Z})$;

(2)
$$\|f\|_{F(m;\varepsilon_0;\theta)} \overset{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m;\varepsilon_0)} < +\infty;$$

(3)
$$\theta(t,\varepsilon) = \int_0^t \varphi(\tau,\varepsilon)\,d\tau, \quad \varphi \in \mathbf{R}^+, \ \ \varphi \in (m,\varepsilon_0), \ \ \inf_{G(\varepsilon_0)} \varphi(t,\varepsilon) = \varphi_0 > 0.$$

Definition 1.3. We say that a matrix $A(t,\varepsilon) = (a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$ belongs to the class $S_2(m;\varepsilon_0)$ $(m \in \mathbf{N} \cup \{0\})$, if $a_{jk} \in S(m;\varepsilon_0)$ $(j,k = \overline{1,N})$.

We define the norm
$$\|A(t,\varepsilon)\|_{S_2(m;\varepsilon_0)} = \max_{1\le j\le N} \sum_{k=1}^{N} \|a_{jk}(t,\varepsilon)\|_{S(m;\varepsilon_0)}.$$

Definition 1.4. We say that a matrix $B(t,\varepsilon,\theta) = (b_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,N}}$ belongs to the class $F_2(m;\varepsilon_0;\theta)$ $(m \in \mathbf{N} \cup \{0\})$, if $b_{jk}(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$ $(j,k = \overline{1,N})$.

We define the norm
$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} = \max_{1\le j\le N} \sum_{k=1}^{N} \|b_{jk}(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

2 Statement of the problem

Consider the matrix differential equation:
$$\frac{dX}{dt} = A(t,\varepsilon)X - XB(t,\varepsilon) + P(t,\varepsilon,\theta) + \mu\Phi(t,\varepsilon,\theta,X), \tag{2.1}$$
where X is an unknown square matrix of order N, belonging to some closed bounded domain $D \subset \mathbf{C}^{N\times N}$, where $\mathbf{C}^{N\times N}$ is the space of complex-valued matrices of dimension $N \times N$, $A(t,\varepsilon)$, $B(t,\varepsilon)$ belongs to the class $S_2(m;\varepsilon_0)$, $P(t,\varepsilon,\theta)$ belongs to the class $F_2(m;\varepsilon_0;\theta)$. $\Phi(t,\varepsilon,\theta,X)$ is a matrix function belonging to the class ${}_2(m;\varepsilon_0;\theta)$ with respect t,ε,θ and continuos with respect X in D. μ are real parameter.

We denote $\lambda_j^1(t,\varepsilon)$, $\lambda_j^2(t,\varepsilon)$ $(j = \overline{1,N})$ – eigenvalues, respectively, of matrices $A(t,\varepsilon)$, $B(t,\varepsilon)$, for which the following conditions are satisfied:

1^0.
$$\inf_{G(\varepsilon_0)} \big|\lambda_j^1(t,\varepsilon) - \lambda_k^1(t,\varepsilon) - in\,\varphi(t,\varepsilon)\big| \ge b_0 > 0,$$
$$\inf_{G(\varepsilon_0)} \big|\lambda_j^2(t,\varepsilon) - \lambda_k^2(t,\varepsilon) - in\,\varphi(t,\varepsilon)\big| \ge b_0 > 0 \ \ \forall\, n \in \mathbf{Z}, \ \ j,k = \overline{1,N}, \ \ j \ne k.$$

2^0.
$$\lambda_j^1(t,\varepsilon) - \lambda_k^2(t,\varepsilon) = i\omega_{jk}(t,\varepsilon), \ \ \omega_{jk}(t,\varepsilon) \in \mathbf{R},$$
$$\inf_{G(\varepsilon_0)} \big|\omega_{jk}(t,\varepsilon) - n\varphi(t,\varepsilon)\big| \ge b_0 > 0 \ \ \forall\, n \in \mathbf{Z}, \ \ j,k = \overline{1,N}.$$

We study the problem on the existence of particular solutions of classes $F_2(m_1;\varepsilon_1;\theta)$, $m_1 \le m$, $\varepsilon_1 \le \varepsilon_0$ of equation (2.1). The condition 2^0 shows that in this case we are dealing with a critical case.

3 Auxiliary results

Lemma 3.1. *Let*

$$\frac{dx}{dt}=\lambda(t,\varepsilon)x+u(t,\varepsilon,\theta(t,\varepsilon)) \tag{3.1}$$

be a given scalar linear non-homogeneous first-order differential equation, where $\lambda(t,\varepsilon)\in S(m;\varepsilon)$, $\inf\limits_{G(\varepsilon_0)}|\operatorname{Re}\lambda(t,\varepsilon)|=\gamma>0$, *and* $u(t,\varepsilon,\theta)\in F(m;\varepsilon_0;\theta)$. *Then equation* (3.1) *has a unique particular solution* $x(t,\varepsilon,\theta)\in F(m;\varepsilon_0;\theta)$. *This solution is given by the formula:*

$$x(t,\varepsilon,\theta(t,\varepsilon))=\int\limits_T^t u(\tau,\varepsilon,\theta(\tau,\varepsilon))\exp\bigg(\int\limits_\tau^t\lambda(s,\varepsilon)\,ds\bigg)\,d\tau,$$

where

$$T=\begin{cases}-\infty & \text{if } \operatorname{Re}\lambda(t,\varepsilon)\le-\gamma<0,\\ +\infty & \text{if } \operatorname{Re}\lambda(t,\varepsilon)\ge\gamma>0.\end{cases}$$

Moreover, there exists $K_0\in(0,+\infty)$ *such that*

$$\|x(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}\le K_0\|u(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

Lemma 3.2. *Let equation* (2.1) *satisfy the next conditions:*
1) there exist matrices $L_1(t,\varepsilon),L_2(t,\varepsilon)\in S_2(m;\varepsilon_0)$ *such that*

(a) $|\det L_k(t,\varepsilon)|\ge a_0>0$ $(k=1,2)$;

(b) $L_1^{-1}(t,\varepsilon)A(t,\varepsilon)L_1(t,\varepsilon)=D_1(t,\varepsilon)=(d^1_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$;

(c) $L_2(t,\varepsilon)B(t,\varepsilon)L_2^{-1}(t,\varepsilon)=D_2(t,\varepsilon)=(d^2_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$,

where D_1, D_2 *are lower triangular matrices, belonging to the class* $S_2(m;\varepsilon_0)$,

$$d^1_{jj}(t,\varepsilon)=\lambda^1_j(t,\varepsilon),\quad d^2_{kk}(t,\varepsilon)=\lambda^2_k(t,\varepsilon).$$

Then by using the transformation

$$X=L_1(t,\varepsilon)YL_2(t,\varepsilon)$$

equation (2.1) leads to the form:

$$\frac{dY}{dt}=D_1(t,\varepsilon)Y-YD_2(t,\varepsilon)-\varepsilon H_1(t,\varepsilon)Y-\varepsilon YH_2(t,\varepsilon)+F_1(t,\varepsilon,\theta)+\mu\Phi_1(t,\varepsilon,\theta,Y), \tag{3.2}$$

where

$$\begin{gathered}H_1(t,\varepsilon)=\frac{1}{\varepsilon}\,L_1^{-1}(t,\varepsilon)\,\frac{dL_1(t,\varepsilon)}{dt}\,,\quad H_2(t,\varepsilon)=\frac{1}{\varepsilon}\,\frac{dL_2(t,\varepsilon)}{dt}\,L_2^{-1}(t,\varepsilon),\\ F_1(t,\varepsilon,\theta)=L_1^{-1}(t,\varepsilon)F(t,\varepsilon,\theta)L_2^{-1}(t,\varepsilon),\\ \Phi_1(t,\varepsilon,\theta,Y)=L_1^{-1}(t,\varepsilon)\Phi(t,\varepsilon,\theta,L_1(t,\varepsilon)YL_2(t,\varepsilon))L_2^{-1}(t,\varepsilon).\end{gathered}$$

Lemma 3.3. *Let a linear matrix equation be given*

$$\frac{dX}{dt}=\Big(D_1(t,\varepsilon)+\sum_{l=1}^q B_{1l}(t,\varepsilon,\theta)\mu^l\Big)X-X\Big(D_2(t,\varepsilon)+\sum_{l=1}^q B_{2l}(t,\varepsilon,\theta)\mu^l\Big), \tag{3.3}$$

$D_1(t,\varepsilon), D_2(t,\varepsilon)$ – the same as in Lemma 3.2, $B_{1l}(t,\varepsilon,\theta)$, $B_{2l}(t,\varepsilon,\theta)$ $(l=\overline{1,q})$ belong to the class $F_2(m;\varepsilon_0;\theta)$, $\mu\in(0,1)$ is a small real parameter. Then for sufficiently small values μ there exists transformation

$$X=\Big(E+\sum_{l=1}^{q}Q_{1l}(t,\varepsilon,\theta)\mu^l\Big)Y\Big(E+\sum_{l=1}^{q}Q_{2l}(t,\varepsilon,\theta)\mu^l\Big),$$

where $Q_{1l}(t,\varepsilon,\theta)$, $Q_{2l}(t,\varepsilon,\theta)$ $(l=\overline{1,q})$ belong to the class $F_2(m;\varepsilon_0;\theta)$, which leads equation (3.2) *to the form*

$$\begin{aligned}\frac{dY}{dt}=\Big(D_1(t,\varepsilon)+\sum_{l=1}^{q}U_{1l}(t,\varepsilon)\mu^l+\varepsilon\sum_{l=1}^{q}V_{1l}(t,\varepsilon,\theta)\mu^l+\mu^{q+1}W_1(t,\varepsilon,\theta,\mu)\Big)Y-\\ -Y\Big(D_2(t,\varepsilon)+\sum_{l=1}^{q}U_{2l}(t,\varepsilon)\mu^l+\varepsilon\sum_{l=1}^{q}V_{2l}(t,\varepsilon,\theta)\mu^l+\mu^{q+1}W_2(t,\varepsilon,\theta,\mu)\Big),\end{aligned}$$

where $U_{1l}(t,\varepsilon)$, $U_{2l}(t,\varepsilon)$ $(l=\overline{1,q})$ are diagonal matrices which belong to the class $S_2(m;\varepsilon_0)$, V_{1l}, V_{2l}, W_1, W_2 $(l=\overline{1,q})$ are square matrices which belong to the class $F(m-1;\varepsilon_0;\theta)$.

Lemma 3.4. *Let a matrix-function $\Phi_1(t,\varepsilon,\theta,Y)$ in equation* (3.2) *have in D^* continuous derivatives with respect to Y in the sense of Frechet up to order $2q+1$ inclusive, and if $Y\in F_2(m;\varepsilon_0;\theta)$, then these derivatives are also from the class $F_2(m;\varepsilon_0;\theta)$. Then there exists $\mu_0\in(0,1)$ such that for all $\mu_1\in(0,\mu_0)$ there exists the transformation*

$$Y=\Psi_1(t,\varepsilon,\theta,\mu)+\Psi_2(t,\varepsilon,\theta,\mu)Z\Psi_3(t,\varepsilon,\theta,\mu), \tag{3.4}$$

where $Z\in D^{**}\subset\mathbf{C}^{N\times N}$, $\Psi_1(t,\varepsilon,\theta,\mu),\Psi_2(t,\varepsilon,\theta,\mu),\Psi_3(t,\varepsilon,\theta,\mu)\in F_2(m;\varepsilon_0;\theta)$, which leads equation (3.2) *to the form:*

$$\begin{aligned}\frac{dZ}{dt}=\Big(D_1(t,\varepsilon)+\sum_{l=1}^{q}U_{1l}(t,\varepsilon)\mu^l\Big)Z-Z\Big(D_2(t,\varepsilon)+\sum_{l=1}^{q}U_{2l}(t,\varepsilon)\mu^l\Big)\\ +\varepsilon K(t,\varepsilon,\theta,\mu)+\mu^{2q}C(t,\varepsilon,\theta,\mu)+\varepsilon V_1(t,\varepsilon,\theta,\mu)Z-\varepsilon ZV_2(t,\varepsilon,\theta,\mu)\\ +\mu^{q+1}\big(R_1(t,\varepsilon,\theta,\mu)Z-ZR_2(t,\varepsilon,\theta,\mu)\big)+\mu\Phi_2(t,\varepsilon,\theta,Z,\mu),\end{aligned} \tag{3.5}$$

where $K\in F_2(m-1;\varepsilon_0;\theta)$, $U_{1l},U_{2l}\in S_2(m;\varepsilon_0)$, $R_1,R_2,C\in F_2(m;\varepsilon_0;\theta)$, $V_1,V_2\in F(m-1;\varepsilon_0;\theta)$, matrix-function Φ_2 belong to class $F_2(m;\varepsilon_0;\theta)$ with respect t, ε, θ, continuously differentiable in the sense of Frechet with respect Z and contains terms of at least second order with respect to Z.

4 Basic results

Theorem 4.1. *Let equation* (3.5) *be such that there exists $q_0\in\mathbf{N}$ $(1\le q_0\le N)$ such that*

$$\inf_{G(\varepsilon_0)}\Big|\operatorname{Re}\big((U_{1q_0}(t,\varepsilon))_{jj}-(U_{2q_0}(t,\varepsilon))_{kk}\big)\Big|\ge b_0>0\ \ (j,k=\overline{1,N}),$$

and for all $l=\overline{1,q_0-1}$ (if $q_0>1$):

$$\operatorname{Re}\big((U_{1l}(t,\varepsilon))_{jj}-(U_{2l}(t,\varepsilon))_{kk}\big)\equiv0\ \ (j,k=\overline{1,N}).$$

Then there exists $\mu_3\in(0,1)$, $\varepsilon_1(\mu)\in(0,\varepsilon_0)$ such that for all $\mu\in(0,\mu_3)$, $\varepsilon\in(0,\varepsilon_1(\mu))$ there exist a particular solution of equation (3.5) *which belongs to the class $F_2(m-1;\varepsilon_1(\mu);\theta)$.*

Theorem 4.2. *Let equation* (2.1) *be such that the following conditions are met:*

(1) *conditions* 1^0, 2^0*;*

(2) *conditions of Lemma 3.2;*

(3) *equation* (3.2) *satisfies the conditions of Lemma 3.4;*

(4) *equation* (3.5) *satisfies the conditions of Theorem 4.1.*

Then there exist $\mu_4 \in (0,1)$, $\varepsilon_4(\mu) \in (0,\varepsilon_0)$ *such that for all* $\mu \in (0,\mu_4)$ *and for all* $\varepsilon \in (0,\varepsilon_4(\mu))$ *there exist a particular solution of equation* (2.1) *which belongs to the class* $F_2(m-1;\varepsilon_4(\mu);\theta)$.

Cauchy's Problem for Singular Perturbed Systems of Differential Equations with Nonstable First-Order Turning Point

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A class of singularly perturbed differential equations (SPDE) with turning points is an effective model for the studies of various physical phenomena. There is a wide spectrum of papers devoted to the investigation of such problems and to the construction of the uniform asymptotic of the solution. This spectrum of SPDE is represented by R. Langer, W. Wasow, C. Lin, S. Lomov etc. Generalization on the class of systems of SPDE in the above-mentioned direction of the research is a relevant problem also nowadays.

In the paper [5], a system of SPDE with a stable turning point has been considered. In this case we have used the apparatus of Airy-Dorodnitsyn functions [1, 3]. An unstable turning point assumes a use of the following Airy-Langer functions:

$$TW = W'' \equiv W''(t) - tW(t) = 0.$$

Let us consider a system of SPDE with a stable turning point (SSPDE):

$$\varepsilon Y'(x,\varepsilon) - A(x,\varepsilon)Y(x,\varepsilon) = H(x), \tag{0.1}$$

where

$$A(x,\varepsilon) = A_0(x) + \varepsilon A_1(x),$$

is a known matrix where

$$\mathbf{A_0}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b(x) & -a(x) & 0 \end{pmatrix}, \quad \mathbf{A_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

when $\varepsilon \to 0$, $x \in [-4, 0]$, $Y(x,\varepsilon) \equiv Y_k(x,\varepsilon) = col(y_1(x,\varepsilon), y_2(x,\varepsilon), y_3(x,\varepsilon))$ is an unknown vector function, $H(x) = col(0, 0, h(x))$ is a given vector function.

The system to be studied here (0.1) will be investigated under the following conditions:

(1) $\widetilde{a}(x), b(x), h(x) \in C^\infty[-4; 0]$;

(2) $a(x) \equiv x\widetilde{a}(x)$, $\widetilde{a}(x) = 3x$, $b(x) = 3x + 20$, $h(x) = 6x + 2$.

The scalar reduced equation for this matrix will be

$$x\widetilde{a}(x)\omega'(x) + b(x)\omega(x) = h(x). \tag{0.2}$$

The analysis of such kind of problems and construction of uniform asymptotic solution on a given segment with a turning point brings certain difficulties and problems in the construction of asymptotic forms [3].

The characteristic equation that corresponds to the SP system (0.1) is as follows:

$$|A(x,0) - \lambda E| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ -b(x) & -a(x) & -\lambda \end{vmatrix} = -\lambda^3 - x\widetilde{a}(x)\lambda = 0.$$

The roots of this equation are: $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{x\widetilde{a}(x)}$.

1 Regularization of singularly perturbed systems of differential equations

In order to save all essential singular functions that appear in the solution of system (0.1) due to the special point

$$t = \mu^{-2} \cdot \varphi(x),$$

where $\mu = \varepsilon^{\frac{1}{3}}$, exponent p and regularizing function $\varphi(x)$ are to be determined. Instead of $Y(x, \varepsilon)$ function $\widetilde{Y}(x, t, \varepsilon)$ transformation function will be studied, also the transformation will be performed in such a way that the following identity is true

$$\widetilde{Y_k}(x, t, \varepsilon)\big|_{t=\varepsilon^{-p}\varphi(x)} \equiv Y_k(x, \varepsilon),$$

which is the necessary condition for suggested method. The vector equation (0.1) can be written as

$$\widetilde{L}_\varepsilon \widetilde{Y_k}(x, t, \varepsilon) \equiv \mu\varphi' \, \frac{\partial \widetilde{y}(x, t, \varepsilon)}{\partial t} + \mu^3 \, \frac{\partial \widetilde{y}(x, t, \varepsilon)}{\partial x} - A(x, \varepsilon)\widetilde{Y_k}(x, t, \varepsilon) = H(x). \tag{1.1}$$

We describe the space of functions in which it will be possible to construct a uniform asymptotic solution of the transformed system (1.1)

$$D_{1k} = \alpha_{1k}(x)\mathrm{Ai(t)} + \beta_{1k}(x)\mathrm{Ai'(t)}, \quad D_{2k} = \alpha_{2k}(x)\mathrm{Bi(t)} + \varepsilon^{\gamma}\beta_{2k}(x)\mathrm{Bi'(t)},$$
$$D_{3k} = f_k(x)\nu(t) + \varepsilon^{\gamma} g_k(x)\nu'(t), \quad D_{4k} = \omega_k(x),$$

where $\alpha_{ik}(x), \beta_{ik}(x), f_k(x), g_k(x), \omega_k(x) \in C^{\infty}[-4, 0]$.

Here functions Ai(t), Bi(t) are the Airy-Langer functions, $\nu(t)$ is an essentially special function [3].

The element of this space has the form

$$\widetilde{Y_k}(x, t, \varepsilon) = \sum_{i=1}^{2} \big[\alpha_{ik}(x)U_i(t) + \beta_{ik}(x)U_i'(t)\big] + f_k(x)\nu(t) + \varepsilon^{\gamma} g_k(x)\nu'(t) + \omega_k(x).$$

Denote the Airy-Langer functions as $U_1(t) \equiv \mathrm{Ai(t)}$, $U_2(t) \equiv \mathrm{Bi(t)}$.

Now we have to investigate how the transformed operator $\widetilde{L}_\varepsilon$ acts on the elements of the Space of non-resonant solutions D_{1k} and D_{2k}. Let us write the obtained result in the form of the following vector equations

$$\begin{aligned} U_i'(t) : \alpha_k(x, \varepsilon)\varphi'(x) - \big[A_0(x) + \mu^3 A_1\big]\beta_k(x, \varepsilon) = -\mu^3 \beta_k'(x, \varepsilon), \\ U_i(t) : \beta_k(x, \varepsilon)\varphi(x)\varphi'(x) - \big[A_0(x) + \mu^3 A_1\big]\alpha_k(x, \varepsilon) = -\mu^3 \alpha_k'(x, \varepsilon). \end{aligned} \tag{1.2}$$

2 Construction of formal solutions of a homogeneous transformation system

The unknown coefficients of the vector equations (1.1) are sought as following vector function series $(i = 1, 2)$:

$$\alpha_k(x, \varepsilon) = \sum_{r=0}^{+\infty} \mu^r \alpha_{kr}(x), \quad \beta_k(x, \varepsilon) = \sum_{r=0}^{+\infty} \mu^r \beta_{kr}(x). \tag{2.1}$$

At the moment, the regularizing function has not yet been defined; therefore, it will be defined as a solution of the problem

$$\varphi(x) = \bigg(\frac{3}{2} \int\limits_0^x \sqrt{-x\widetilde{a}(x)}\, dx\bigg)^{\frac{2}{3}}.$$

The regularizing function of such kind has been considered in [2,4]. Due to such a choice of the regularizing variable $\det\Phi(x)\equiv 0$, there is a nontrivial solution of the homogeneous system (1.1) that is

$$Z_{k0}(x)=colon\Big(0,\frac{1}{\varphi'(x)}\,\beta_{i30}(x),-\varphi(x)\varphi'(x)\beta_{i20}(x),0,\beta_{i20}(x),\beta_{i30}(x)\Big), \tag{2.2}$$

where $\beta_{0ik}(x)$, $i=1,2$, $i=1,2,3$ are arbitrary up to some point and sufficiently smooth functions at $x\in[-4;0]$.

Two linearly independent solutions of the system (1.1) are

$$D_k(x,t,\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^r[\alpha_{ikr}(x)U_i(t)+\varepsilon^{\frac{1}{3}}\beta_{ikr}(x,\varepsilon)U_i'(t)],\ \ i=1,2, \tag{2.3}$$

where $\alpha_{ikr}(x)=col(\alpha_{i1r}(x),\alpha_{i2r}(x),\alpha_{i3r}(x))$ and $\beta_{ikr}(x)=col(\beta_{i1r}(x),\beta_{i2r}(x),\beta_{i3r}(x))$ are known vector-functions.

Thus, gradual solving of systems of equations $t=\varepsilon^{-\frac{2}{3}}\cdot\varphi(x)$, $i=1,2$, then gives two formal solutions of the transformation vector equation

$$D_k(x,\varepsilon^{-\frac{2}{3}}\varphi(x),\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^r\Big[\alpha_{ikr}(x)U_i(\varepsilon^{-\frac{2}{3}}\varphi(x))+\varepsilon^{\frac{1}{3}}\beta_{ikr}(x,\varepsilon)U_i'(\varepsilon^{-\frac{2}{3}}\varphi(x))\Big]. \tag{2.4}$$

The third formal solution of the homogeneous vector equation (0.1) is then constructed as the series

$$\omega(x,\varepsilon)\equiv\sum_{r=0}^{\infty}\varepsilon^r\omega_r(x)\equiv colon\Big(\sum_{r=0}^{\infty}\varepsilon^r\omega_{1r}(x),\sum_{r=0}^{\infty}\varepsilon^r\omega_{2r}(x),\sum_{r=0}^{\infty}\varepsilon^r\omega_{3r}(x)\Big). \tag{2.5}$$

3 Construction of formal partial solutions

To construct a partial solution of the SSPDE (0.1), let us analyze how transformation operator operates $\widetilde{L}_\varepsilon$ on an element from the space of non-resonant solutions D_{3r} and D_{4r}. The result is written in the form

$$\begin{aligned}
&\widetilde{L}_\varepsilon(f_k(x,\varepsilon)\nu(t)+\mu g_k(x,\varepsilon)\nu'(t)+\omega_k(x,\varepsilon))\\
&\quad=\mu f_k(x,\varepsilon)\varphi'(x)\nu(t)+g_k(x,\varepsilon)\varphi'(x)\varphi(x)\nu(t)-A(x,\varepsilon)f_k(x,\varepsilon)\nu(t)-\mu A(x,\varepsilon)g_k(x,\varepsilon)\nu'(t)\\
&\qquad+\mu^3f_k'(x)\nu(t)+\mu^4g_k'(x)\nu'(t)+\mu^2\varphi'(x)g_k(x)\pi^{-1}+\mu^3\omega'(x)-A(x,\varepsilon)\omega_k(x)=H(x).
\end{aligned}$$

Therefore, the partial solution of the transformation vector equation (1.1) is then defined as the series

$$\widetilde{Y}_{\text{part.}}(x,t,\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^r\big[f_{kr}(x)\nu(t)+\varepsilon^{\frac{1}{3}}g_{kr}(x)\nu'(t)\big]+\sum_{r=0}^{\infty}\varepsilon^r\bar{\omega}_{kr}(x).$$

Narrowing the solution, when $t=\varepsilon^{-\frac{2}{3}}\cdot\varphi(x)$, the series

$$\widetilde{Y}_{\text{part.}}(x,t,\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^r\bigg[f_{kr}(x)\nu(\varepsilon^{\frac{2}{3}}\cdot\varphi(x))+\varepsilon^{\frac{1}{3}}g_{kr}(x)\,\frac{d\nu(\varepsilon^{-\frac{2}{3}}\cdot\varphi(x))}{d(\varepsilon^{-\frac{2}{3}}\cdot\varphi(x))}\bigg]+\sum_{r=0}^{\infty}\varepsilon^r\bar{\omega}_{kr}(x), \tag{3.1}$$

is a formal partial solution of the SSPDE (0.1).

4 Estimation of the remainder terms of the asymptotic solution

In this paper we have considered the case of an unstable turning point. In this case the remainder terms of the solution have characteristic differences in comparison with the case of a stable turning point [3]. Let us write the formal solution of the transformed problem (1.1) in the following form:

$$\alpha_{ikr}(x,\varepsilon) \equiv \alpha_{ikr}(x,\varepsilon) + \varepsilon^{q+1}\xi_{\alpha(q+1)}(x,\varepsilon), \tag{4.1}$$

$$\beta_{ikr}(x,\varepsilon) \equiv \beta_{kr}(x,\varepsilon) + \varepsilon^{q+1}\xi_{\beta(q+1)}(x,\varepsilon), \tag{4.2}$$

where $\alpha_{kq}(x,\varepsilon)$ and $\beta_{kq}(x,\varepsilon)$ are partial q-sums of the series (1.1), $\varepsilon^{1+q}\xi_{\alpha(q+1)}(x,\varepsilon)$ and $\varepsilon^{1+q}\xi_{\beta(q+1)}(x,\varepsilon)$ are the remainder terms.

Let us write the main result of this paper in the following theorem.

Theorem. *Let for the SPDE system* (0.1) *the conditions* (1) *and* (2) *take place. Then for suffciciently small values of the parameter* $\varepsilon > 0$*:*

- *three linearly independent solutions of homogeneous transformed vector equation* (1.1) *can be built in form of series* (2.1) *and* (2.5)*;*
- *narrowing these solutions at* $t = \varepsilon^{-\frac{2}{3}} \cdot \varphi(x)$ *is the formal asymptotic solution of the homogeneous SPDE system* (0.1)*;*
- *partial solution of the nonhomogeneous SPDE system* (0.1) *constructed with the series* (3.1)*;*
- *for the remainder terms of the asymptotic solutions* (4.1)*,* (4.2) *estimations are valid.*

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On Stability and Asymptotic Properties of Solutions of Second-Order Damped Linear Differential Equations with Periodic Non-Constant Coefficients

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Consider the equation

$$\boxed{x'' = p(t)x + g(t)x',} \tag{1}$$

where $p, g : \mathbb{R} \to \mathbb{R}$ are ω-periodic locally Lebesgue integrable functions, $\omega > 0$. By a solution to equation (1), as usual, we understand a function $x : \mathbb{R} \to \mathbb{R}$ which is locally absolutely continuous together with its first derivative and satisfies (1) almost everywhere in $\mathbb{R}$.

We first introduce the following definitions.

Definition 1. We say that the pair (p, g) belongs to the set $\mathcal{V}^-(\omega)$ (resp. $\mathcal{V}^+(\omega)$) if, for any function $u : [0, \omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative and satisfies

$$u''(t) \geq p(t)u(t) + g(t)u'(t) \quad \text{for a.e. } t \in [0, \omega], \quad u(0) = u(\omega), \quad u'(0) \geq u'(\omega),$$

the inequality

$$u(t) \leq 0 \quad \text{for } t \in [0, \omega] \quad \big(\text{resp. } u(t) \geq 0 \quad \text{for } t \in [0, \omega]\big)$$

holds.

Remark 1. In the related literature, the fact that $(p, g) \in \mathcal{V}^-(\omega)$ (resp. $(p, g) \in \mathcal{V}^+(\omega)$) is often called maximum (resp. anti-maximum) principle for the periodic problem

$$x'' = p(t)x + g(t)x'; \quad x(0) = x(\omega), \ x'(0) = x'(\omega). \tag{2}$$

Moreover, the relationship of the classes $\mathcal{V}^-(\omega)$ and $\mathcal{V}^+(\omega)$ with a sign of the Green's function of (2) is known.

Definition 2. We say that the pair (p, g) belongs to the set $\mathcal{V}_0(\omega)$ if the homogeneous problem (2) has a positive solution.

Definition 3. We say that the pair (p, g) belongs to the set $\mathcal{D}$ if any non-trivial solution to equation (1) has at most one zero in $\mathbb{R}$.

The aim of this note is not to provide conditions guaranteeing that the maximum (resp. anti-maximum) principle holds for (2). Let us mention only that such effective conditions are derived, e.g., in [1, 3, 5] (see, also [2, 4] for the case of $g(t) \equiv 0$).

Below we discuss the stability and asymptotic properties of solutions of equation (1), if the pair (p, g) of the coefficients in (1) belongs to each of the above-defined classes.

Theorem 2. *Let $(p, g) \in \mathcal{V}^-(\omega)$. Then, there exist $\mu_1, \mu_2 > 0$ and positive linearly independent solutions x_1, x_2 to equation* (1) *such that*

$$\mu_2 - \mu_1 = \frac{1}{\omega} \int\limits_0^\omega g(s)\, \mathrm{d}s$$

and

$$x_1(t) = \mathrm{e}^{-\mu_1 t} \varphi_1(t), \quad x_2(t) = \mathrm{e}^{\mu_2 t} \varphi_2(t) \quad \textit{for } t \in \mathbb{R},$$

where $\varphi_1, \varphi_2 \in AC^1_{loc}(\mathbb{R})$ are ω-periodic functions; equation (1) *is unstable.*

Proposition 3. *Let $\int\limits_0^\omega g(s)\, \mathrm{d}s \geq 0$ and there exist a positive solution x to equation* (1) *satisfying*

$$x(t) = \mathrm{e}^{-\mu t} \varphi(t) \quad \textit{for } t \in \mathbb{R},$$

where $\mu > 0$ and $\varphi \in AC^1_{loc}(\mathbb{R})$ is an ω-periodic function. Then $(p, g) \in \mathcal{V}^-(\omega)$.

Proposition 4. *Let $\int\limits_0^\omega g(s)\, \mathrm{d}s \leq 0$ and there exist a positive solution y to equation* (1) *satisfying*

$$y(t) = \mathrm{e}^{\nu t} \psi(t) \quad \textit{for } t \in \mathbb{R},$$

where $\nu > 0$ and $\psi \in AC^1_{loc}(\mathbb{R})$ is an ω-periodic function. Then $(p, g) \in \mathcal{V}^-(\omega)$.

Following [4, Definition 13.1], we introduce the definition.

Definition 4. Equation (1) is said to be strongly exponential dichotomic, if there exist $\mu, \nu > 0$ and linearly independent solutions x, y to equation (1) such that the functions

$$t \mapsto \mathrm{e}^{\mu t} x(t), \quad t \mapsto \mathrm{e}^{-\nu t} y(t)$$

are positive and ω-periodic on $\mathbb{R}$.

Corollary 1. *Equation* (1) *is strongly exponential dichotomic if and only if $(p, g) \in \mathcal{V}^-(\omega)$.*

Theorem 5. *Let $(p, g) \in \mathcal{V}_0(\omega)$. Then, the following conclusions hold:*

(1) *If $\int\limits_0^\omega g(s)\, \mathrm{d}s > 0$, then equation* (1) *has linearly independent solutions x_1, x_2 such that x_1 is a positive ω-periodic solution and x_2 is a positive solution satisfying*

$$x(t) = \mathrm{e}^{\mu t} \varphi(t) \quad \textit{for } t \in \mathbb{R},$$

where

$$\mu = \frac{1}{\omega} \int\limits_0^\omega g(s)\, \mathrm{d}s$$

and $\varphi \in AC^1_{loc}(\mathbb{R})$ is an ω-periodic function; equation (1) *is unstable.*

(2) *If $\int\limits_0^\omega g(s)\, \mathrm{d}s = 0$, then equation* (1) *has linearly independent solutions x_1, x_2 such that x_1 is a positive ω-periodic solution and x_2 is a solution, with exactly one zero in $\mathbb{R}$, satisfying*

$$\lim_{t \to -\infty} x_2(t) = -\infty, \quad \lim_{t \to +\infty} x_2(t) = +\infty;$$

equation (1) *is unstable.*

(3) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s < 0$, *then equation* (1) *has linearly independent solutions* x_1, x_2 *such that* x_1 *is a positive* ω*-periodic solution and* x_2 *is a positive solution satisfying*

$$x(t) = \mathrm{e}^{-\mu t}\,\varphi(t) \quad \textit{for } t \in \mathbb{R},$$

where

$$\mu = -\frac{1}{\omega}\int\limits_0^\omega g(s)\,\mathrm{d}s$$

and $\varphi \in AC^1_{loc}(\mathbb{R})$ *is an* ω*-periodic function; equation* (1) *is stable.*

Theorem 6. *Let* $(p,g) \in \mathcal{V}^+(\omega) \cap \mathcal{D} = \operatorname{Int}\mathcal{V}^+(\omega) \cap \mathcal{D}$. *Then,* $\int\limits_0^\omega g(s)\,\mathrm{d}s \neq 0$ *and the following conclusions hold:*

(1) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s > 0$, *then equation* (1) *has a positive solution* x_0 *satisfying*

$$\lim_{t\to-\infty} x_0(t) = 0, \quad \lim_{t\to+\infty} x_0(t) = +\infty$$

and, moreover, every solution x *to equation* (1) *has at most one zeros in* $\mathbb{R}$ *and satisfies*

$$\lim_{t\to-\infty} x(t) = 0, \quad \lim_{t\to+\infty} |x(t)| = +\infty;$$

equation (1) *is unstable.*

(2) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s < 0$, *then equation* (1) *has a positive solution* x_0 *satisfying*

$$\lim_{t\to-\infty} x_0(t) = +\infty, \quad \lim_{t\to+\infty} x_0(t) = 0$$

and, moreover, every solution x *to equation* (1) *has at most one zeros in* $\mathbb{R}$ *and satisfies*

$$\lim_{t\to-\infty} |x(t)| = +\infty, \quad \lim_{t\to+\infty} x(t) = 0;$$

equation (1) *is asymptotically stable.*

Theorem 7. *Let* $(p,g) \in \operatorname{Int}\mathcal{V}^+(\omega) \setminus \mathcal{D}$. *Then, every non-trivial solution to equation* (1) *is oscillatory in the neighbourhood* $+\infty$ *as well as* $-\infty$ *and the following conclusions hold:*

(1) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s > 0$, *then every non-trivial solution* x *to equation* (1) *satisfies*

$$\lim_{t\to-\infty} x(t) = 0, \quad \limsup_{t\to+\infty} x(t) = +\infty, \quad \liminf_{t\to+\infty} x(t) = -\infty;$$

equation (1) *is unstable.*

(2) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s = 0$, *then every solution to equation* (1) *is bounded; equation* (1) *is stable.*

(3) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s < 0$, *then every non-trivial solution* x *to equation* (1) *satisfies*

$$\limsup_{t\to-\infty} x(t) = +\infty, \quad \liminf_{t\to-\infty} x(t) = -\infty, \quad \lim_{t\to+\infty} x(t) = 0;$$

equation (1) *is asymptotically stable.*

Acknowledgement

The research has been supported by the internal grant FSI-S-20-6187 of FME BUT.

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Boundary Value Problems for Implicit Fractional Differential Equations at Resonance

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1 Introduction

Let $T > 0$ be given, $J = [0, T]$, and $\|x\| = \max\{|x(t)| :\ t \in J\}$ be the norm in $C(J)$.

We discuss the implicit fractional differential equation

$$ {}^c\!D^{\alpha} u(t) = a(t)\,{}^c\!D^{\beta} u(t) + f\big(t, u(t), {}^c\!D^{\alpha} u(t)\big) \tag{1.1}$$

together with the boundary condition

$$u(0) = \sum_{k=1}^{n} c_k u(\rho_k), \tag{1.2}$$

where $0 < \beta < \alpha < 1$, $a \in C(J)$, $f \in C(J \times \mathbb{R}^2)$, ${}^c\!D$ denotes the Caputo fractional derivative and $n \in \mathbb{N}$, $0 < \rho_1 < \rho_2 < \cdots < \rho_n \le T$, $c_k > 0$, $\sum\limits_{k=1}^{n} c_k = 1$. Further conditions for a, f will be specified later.

Definition 1.1. We say that $u : J \to \mathbb{R}$ is *a solution of equation* (1.1) if $u, {}^c\!D^{\alpha} u \in C(J)$ and (1.1) holds for $t \in J$. A solution u of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

We recall the definitions of the Caputo fractional derivative and the Riemann–Liouville fractional integral [1, 2]. The *Caputo fractional derivative* ${}^c\!D^{\gamma} x$ of order $\gamma \in (0, 1)$ of a function $x : J \to \mathbb{R}$ is given as

$${}^c\!D^{\gamma} x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s) - x(0))\,\mathrm{d}s,$$

and the *Riemann–Liouville fractional integral* $I^{\gamma} x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$I^{\gamma} x(t) = \int\limits_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\,x(s)\,\mathrm{d}s,$$

where Γ is the Euler gamma function. It is not difficult to verify that

$$0 < \gamma < \mu < 1, \quad x, {}^c\!D^{\mu} x \in C(J) \Longrightarrow {}^c\!D^{\gamma} x(t) = I^{\mu-\gamma}\,{}^c\!D^{\mu} x(t), \ \ t \in J. \tag{1.3}$$

Remark 1.1. It follows from (1.3) that u is a solution of (1.1) if and only if it is a solution of the implicit equation

$$ {}^{c}\!D^{\alpha}u(t) = a(t)I^{\alpha-\beta}\,{}^{c}\!D^{\alpha}u(t) + f\big(t, u(t), {}^{c}\!D^{\alpha}u(t)\big). $$

We note that for $n = 1$, $c_1 = 1$ and $\rho_1 = T$, the boundary condition (1.2) reduces to the periodic condition $x(0) = x(T)$.

Problem (1.1), (1.2) is at resonance, because each constant function u on J is a solution of problem ${}^{c}\!D^{\alpha}u = a(t)\,{}^{c}\!D^{\beta}u$, (1.2).

We suppose that the functions a, f satisfy the conditions:

(H_1) $a(t) \le 0$ for $t \in J$.

(H_2) There exist $D, H \in \mathbb{R}$, $D < H$, such that

$$ \begin{aligned} f(t, D, y) > 0 \ \text{ for } \ t \in J, \ \ y \le 0, \\ f(t, H, y) < 0 \ \text{ for } \ t \in J, \ \ y \ge 0. \end{aligned} $$

(H_3) There exists $L > 0$ such that

$$ \big|f(t, x, y_1) - f(t, x, y_2)\big| \le L|y_1 - y_2| \ \text{ for } \ t \in J, \ \ x \in [C, D], \ \ y_1, y_2 \in \mathbb{R}, $$

and

$$ Q = \frac{\|a\|T^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} + L < 1. $$

The aim of this paper is to study the existence of solutions to problem (1.1), (1.2). The existence result is proved by the initial value method. To this end we first introduce an operator $\mathcal{F} : C(J) \to C(J)$. We prove that for each $c \in [D, H]$ the initial value problem ${}^{c}\!D^{\alpha}x(t) = \mathcal{F}x(t)$, $x(0) = c$ has a solution on J and all its solutions x satisfy $D < x < H$ on $(0, T]$. Let $\mathcal{C}$ be the set of all solutions to this problem for $c \in [D, H]$. We prove that there exists at least one $u \in \mathcal{C}$ satisfying (1.2) and u is a solution of problem (1.1), (1.2).

2 Operator $\mathcal{F}$ and its properties

Let $f^* : J \times \mathbb{R}^2$ be defined as

$$ f^*(t, x, y) = \begin{cases} f(t, H, y) & \text{if } \ x > H, \\ f(t, x, y) & \text{if } \ x \in [D, H], \\ f(t, D, y) & \text{if } \ x < D \end{cases} $$

for $t \in J$ and $y \in \mathbb{R}$. Then $f^* \in C(J \times \mathbb{R}^2)$,

$$ \begin{aligned} f^*(t, x, y) > 0 \ \text{ if } \ t \in J, \ \ x \le D, \ \ y \le 0, \\ f^*(t, x, y) < 0 \ \text{ if } \ t \in J, \ \ x \ge H, \ \ y \ge 0, \end{aligned} \tag{2.1} $$

$$ \big|f^*(t, x, y_1) - f^*(t, x, y_2)\big| \le L|y_1 - y_2|, \ \ t \in J, \ \ x, y_1, y_2 \in \mathbb{R}, \tag{2.2} $$

and

$$ |f^*(t, x, 0)| \le M, \ \ t \in J, \ \ x \in \mathbb{R}, \tag{2.3} $$

where

$$ M = \max\big\{|f(t, x, 0)| : \ t \in J, \ x \in [D, H]\big\}. $$

The following result is proved by the Banach fixed point theorem.

Lemma 2.1. *Let $x \in C(J)$. Then there exists a unique solution $w \in C(J)$ of the equation*

$$w(t) = a(t)I^{\alpha-\beta}w(t) + f^*(t, x(t), w(t)). \tag{2.4}$$

Keeping in mind Lemma 2.1, for each $x \in C(J)$ there exists a unique solution $w \in C(J)$ of equation (2.4). We put $w = \mathcal{F}x$ and have an operator $\mathcal{F} : C(J) \to C(J)$ satisfying the equality

$$\mathcal{F}x(t) = a(t)I^{\alpha-\beta}\mathcal{F}x(t) + f^*(t, x(t), \mathcal{F}x(t)) \ \text{ for } \ t \in J, \ \ x \in C(J). \tag{2.5}$$

The properties of $\mathcal{F}$ are given in the following two results.

Lemma 2.2. *$\mathcal{F} : C(J) \to C(J)$ is a continuous operator and*

$$\|\mathcal{F}x\| \le \frac{M}{1-Q}\,, \ \ x \in C(J),$$

where M is from (2.3) *and Q from* (H_3).

Lemma 2.3. *If $u \in C(J)$ is a solution of the equation*

$$^c\!D^\alpha u(t) = \mathcal{F}u(t), \tag{2.6}$$

then u is a solution of the equation

$$^c\!D^\alpha u(t) = a(t)\,^c\!D^\beta u(t) + f^*\big(t, u(t), {}^c\!D^\alpha u(t)\big). \tag{2.7}$$

Proof. Let $u \in C(J)$ be a solution of (2.6). Then $\mathcal{F}u \in C(J)$ and so $^c\!D^\alpha u \in C(J)$. Hence, by (2.5) and (1.3) (see Remark 1.1),

$$\begin{aligned} {}^c\!D^\alpha u &= a(t)I^{\alpha-\beta}\mathcal{F}u + f^*(t, u, \mathcal{F}u) \\ &= a(t)I^{\alpha-\beta}\,{}^c\!D^\alpha u + f^*(t, u, {}^c\!D^\alpha u) = a(t)\,{}^c\!D^\beta u + f^*(t, u, {}^c\!D^\alpha u). \end{aligned}$$

As a result u is a solution of (2.7). □

3 Initial value problem

We investigate the initial value problem

$$^c\!D^\alpha u(t) = \mathcal{F}u(t), \tag{3.1}$$

$$u(0) = c, \tag{3.2}$$

where $c \in \mathbb{R}$. It is easy to check that $u \in C(J)$ is a solution of problem (3.1), (3.2) if and only if u is a fixed point of the operator $\mathcal{L}_c : C(J) \to C(J)$, $\mathcal{L}_c x(t) = c + I^\alpha \mathcal{F}x(t)$.

The following existence result is proved by the Schauder fixed point theorem.

Lemma 3.1. *Let $c \in \mathbb{R}$. Then problem* (3.1), (3.2) *has at least one solution.*

For $c \in \mathbb{R}$, let $\mathcal{S}(c)$ be the set of all solutions to problem (3.1), (3.2). By Lemma 3.1, $\mathcal{S}(c) \ne \varnothing$.

Lemma 3.2. *Let $c \in [D, H]$ and $x \in \mathcal{S}(c)$. Then*

$$D < x(t) < H \ \ \textit{for} \ \ t \in (0, T]. \tag{3.3}$$

Proof. By Lemma 2.3, ${}^c\!D^\alpha x(t) = a(t){}^c\!D^\beta x(t) + f^*(t, x(t), {}^c\!D^\alpha x(t))$, $t \in J$. Suppose that $\max\{x(t) : t \in J\} = x(\xi) \geq H$ for some $\xi \in (0, T]$. Then it follows from the maximum principle for the Caputo fractional derivative [3, Lemma 2.1] that

$${}^c\!D^\gamma x(t)\big|_{t=\xi} \geq \frac{x(\xi) - x(0)}{\Gamma(1-\gamma)}\,\xi^{-\gamma} = \frac{x(\xi) - c}{\Gamma(1-\gamma)}\,\xi^{-\gamma} \geq 0 \ \text{ for } \ \gamma \in (0,1).$$

Hence (see (H_1) and (2.1)),

$$a(\xi){}^c\!D^\beta x(t)\big|_{t=\xi} + f^*\big(\xi, x(\xi), {}^c\!D^\alpha x(t)\big|_{t=\xi}\big) < 0,$$

contrary to ${}^c\!D^\alpha x(t)|_{t=\xi} \geq 0$ and

$${}^c\!D^\alpha x(t)\big|_{t=\xi} = a(\xi){}^c\!D^\beta x(t)\big|_{t=\xi} + f^*\big(\xi, u(\xi), {}^c\!D^\alpha x(t)\big|_{t=\xi}\big).$$

Therefore $x < H$ on $(0, T]$. Similarly, for $x > D$ on this interval. □

4 Problem (1.1), (1.2)

Let

$$\mathcal{C} = \bigcup_{c \in [D,H]} \mathcal{S}(c).$$

Then $\mathcal{C} \neq \varnothing$, $\mathcal{C}$ is a compact subset in $C(J)$ and, by Lemma 3.2,

$$D < x(t) < H \ \text{ for } \ t \in (0, T], \ \ x \in \mathcal{C}. \tag{4.1}$$

Theorem 4.1. *Let* (H$_1$)–(H$_3$) *hold. Then problem* (1.1), (1.2) *has at least one solution* u *and* $D < u(t) < H$ *for* $t \in (0, T]$.

Proof. Suppose that

$$x(0) \neq \sum_{k=1}^{n} c_k x(\rho_k) \ \text{ for } \ x \in \mathcal{C}. \tag{4.2}$$

Let

$$\mathcal{C}_c^+ = \Big\{x \in \mathcal{S}(c) :\ x(0) < \sum_{k=1}^{n} c_k x(\rho_k)\Big\},$$
$$\chi^+ = \big\{c \in [D, H] :\ \mathcal{C}_c^+ \neq \varnothing\big\}, \quad \chi^- = [D, H] \setminus \chi^+.$$

We observe that if $c \in \chi^-$ and $x \in \mathcal{S}(c)$, then $x(0) > \sum\limits_{k=1}^{n} c_k x(\rho_k)$. Since, by (4.1), $\mathcal{S}(D) = \mathcal{C}_D^+$ and $\mathcal{C}_H^+ = \varnothing$, we have $D \in \chi^+$ and $H \in \chi^-$. Hence χ^+ and χ^- are nonempty sets. We can prove that χ^+ and χ^- are closed in $[D, H]$. Hence the compact interval $[D, H]$ is the union of two nonempty, closed and disjoint subsets χ^+, χ^-, which is impossible. Thus assumption (4.2) is false, and therefore there exists $u \in \mathcal{C}$ such that $u(0) = \sum\limits_{k=1}^{n} c_k u(\rho_k)$. Since, by (4.1), $D < u < H$ on $(0, T]$, we have

$$f^*\big(t, u(t), {}^c\!D^\alpha u(t)\big) = f\big(t, u(t), {}^c\!D^\alpha u(t)\big)$$

for $t \in J$. As a result u is a solution of problem (1.1), (1.2). □

Example 4.1. Let $T = 1$, $n \in \mathbb{N}$, $r \in C(J)$, $a(t) = t(t-1)$ and

$$f(t,x,y) = r(t) - x^{2n-1} + \sin x - (1/2)\arctan y.$$

Then $\|a\| = 1/4$ and the functions a, f satisfy (H_1)–(H_3) for $H > \sqrt[2n-1]{\|r\|+1}$, $D = -H$ and $L = 1/2$ since $\Gamma(v) > 4/5$ for $v \in [1,2]$. By Theorem 4.1, there exists a solution u of the equation

$${}^cD^\alpha u = t(t-1)\,{}^cD^\beta u + r(t) - u^{2n-1} + \sin u - \frac{1}{2}\arctan {}^cD^\alpha u,$$

satisfying the boundary condition (1.2) and $D < u < H$ on $(0,1]$.

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Weak Solutions for Coupled Stochastic Functional-Differential Equations in Infinite-Dimensional Spaces

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In this paper, we are concerned with the existence and uniqueness of weak and strong solutions to stochastic functional differential equations in a Hilbert space of the form

$$\begin{cases} du(t) = \big[Au(t) + f(u_t, y_t)\big]\,dt + \sigma(u_t, y_t)\,dW(t),\\ dy(t) = g(u_t, y_t)\,dt, \quad t \geq 0,\\ u(t) = \phi(t), \quad y(t) = \psi(t), \quad t \in [-h, 0], \quad h > 0. \end{cases} \tag{0.1}$$

Here $u_t = u(t+\theta)$, $y_t = y(t+\theta)$, $\theta \in [-h, 0]$, A is an infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t): \ t \geq 0\}$ in a separable Hilbert space H, $W(t)$ is a Q-Wiener process on a separable Hilbert space K, $u(t)$ is a state process, the functionals f and g map the space of functions continuous on $[-h, 0]$ into H, σ maps the same space into a special space of Hilbert–Schmidt operators. Finally, $\phi, \psi : [-h, 0] \to H$ are the initial condition functions.

Functional differential equations are mathematical models of processes whose evolution depends on their previous states. The paired stochastic equations of type (0.1) arise in various applications; for instance, the bidomain equation (defibrillator model), the Hodgkin–Huxley equation for nerve axons, the nuclear reactor dynamics equation, etc. These equations are characterized by the fact that one of them is a partial differential equation (infinite-dimensional), and the other is an ordinary one (finite-dimensional). The nonlinearities in such equations do not satisfy the Lipschitz condition, which complicates the proof of the existence and uniqueness. However, as a rule, the right-hand sides of these equations satisfy some monotonicity conditions, which makes it possible to apply Galerkin approximations. This method is the main technique for obtaining the existence and uniqueness of weak solutions in this paper.

1 Preliminaries and main results

Let K and H be two separable Hilbert spaces and let $V \subset H$ be a reflexive Banach space with the dual space H'. By identifying H with its dual H', we have $V \subset H \cong H' \subset V'$, where the inclusions are assumed to be continuous and dense. (V, H, V') is called a Gelfand triple. Let the norms in V, H and V' be denoted by $\|\cdot\|_V$, $\|\cdot\|$ and $\|\cdot\|_{V'}$, respectively. The inner product in H and the duality scalar product between V and V' will be denoted by $(\cdot\,,\cdot)$, and $\langle\cdot\,,\cdot\rangle$. The norm and inner product in K will be denoted by $\|\cdot\|_K$ and $(\cdot\,,\cdot)$, respectively.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with the normal filtration $\{\mathcal{F}_t :\ t \geq 0\}$ generated by the Q-Wiener process W on $(\Omega, \mathcal{F}, P)$ with the linear bounded covariance operator such that $\operatorname{tr} Q < \infty$. We assume that there exist a complete orthonormal system $\{e_k\}$ in K and a sequence of nonnegative real numbers λ_k such that

$$Qe_k = \lambda_k e_k, \ \ k = 1, 2, \dots, \ \text{ and } \ \sum_{k=1}^{\infty} \lambda_k < \infty.$$

The Wiener process admits the expansion

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \beta_k(t) e_k,$$

where $\beta_k(t)$ are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, P)$.

Let $U_0 = Q^{\frac{1}{2}}(U)$ and $L_2^0 = L_2(U_0, H)$ be the space of all Hilbert-Schmidt operators from U_0 to H with the inner product $(\Phi, \Psi)_{L_2^0} = \operatorname{tr}[\Phi Q \Psi^*]$ and the norm $\|\Phi\|_{L_2^0}$, respectively. $C := C([-h, 0]; H)$ is the space of continuous mappings from $[-h, 0]$ to H equipped with the norm $\|u\|_C = \sup\limits_{[-h,0]} \|u(\theta)\|$, and $L_V^2 := L^2((-h, 0); V)$ is the space of V-valued mappings with the norm

$$\|u\|_{L_V^2}^2 = \int_{-h}^{0} \|u(t)\|_V^2 \, dt.$$

We impose the following **conditions on the operator A**:

(A1) A is a linear operator with domain $D(A)$ dense in H such that $A : V \to V'$.

(A2) For any $u, v \in V$ there exists $\alpha > 0$ such that

$$|\langle Au, v\rangle| \leq \alpha \|u\|_V \cdot \|v\|_V.$$

(A3) A satisfies the coercivity condition: there exist constants $\beta > 0$ and γ such that

$$\langle Av, v\rangle \leq -\beta \|v\|_V^2 + \gamma \|v\|^2, \ \ \forall\, v \in V.$$

Conditions on nonlinearities:

(N1) f and g are mappings from $C \cap L_V^2 \times C$ to H, and σ is a mapping from $C \cap L_V^2 \times C$ to L_2^0.

(N2) (Growth condition) There exist positive constants $\alpha > 0$ and $\gamma \geq 1$ such that

$$\|f(\phi, \psi)\| + \|g(\phi, \psi)\| \leq \alpha \bigg(1 + \Big(\int_{-h}^{0} \|\phi\|_V \, dt\Big)^{\gamma} + \|\phi\|_C^{\gamma} + \|\psi\|_C^{\gamma}\bigg)$$

and

$$\|\sigma(\phi, \psi)\|_{L_2^0}^2 \leq \alpha \big(1 + \|\phi\|_C^2 + \|\psi\|_C^2\big).$$

(N3) (Local Lipschitz condition) For any $N > 0$ there exists a constant $K_N > 0$ such that

$$\|f(\phi,\psi) - f(\phi_1,\psi_1)\|^2 + \|g(\phi,\psi) - g(\phi_1,\psi_1)\|^2 + \|\sigma(\phi,\psi) - \sigma(\phi_1,\psi_1)\|^2_{L_2^0} \le K_N\big(\|\phi-\phi_1\|_C^2 + \|\psi-\psi_1\|_C^2\big)$$

for any $\phi,\phi_1 \in C \cap L_V^2$ and $\psi,\psi_1 \in C$ with $\|\phi\|_C^2 + \|\psi\|_C^2 < N$, $\|\phi_1\|_C^2 + \|\psi_1\|_C^2 < N$.

(N4) (Coercivity condition) There exist constants $\beta > 0$, λ and C_1 such that

$$\langle A\phi(0),\phi(0)\rangle + (f(\phi,\psi),\phi(0)) + (g(\phi,\psi),\psi(0)) + \|\sigma(\phi,\psi)\|^2_{L_2^0} \le -\beta\|\phi(0)\|_V^2 + \lambda\big(\|\phi\|_C^2 + \|\psi\|_C^2\big) + C_1.$$

(N5) (Monotonicity condition) For any $\phi,\phi_1 \in C \cap L_V^2$ and $\psi,\psi_1 \in C$, we have

$$\begin{aligned} 2\big\langle A(\phi(0)-\phi_1(0),\phi(0)-\phi_1(0))\big\rangle &+ 2\big(f(\phi,\psi) - f(\phi_1,\psi_1),\phi(0)-\phi_1(0)\big) \\ &+ 2\big(g(\phi,\psi) - g(\phi_1,\psi_1),\psi(0)-\psi_1(0)\big) + \big\|\sigma(\phi,\psi) - \sigma(\phi_1,\psi_1)\big\|^2_{L_2^0} \\ &\le \delta\big(\|\phi-\phi_1\|_C^2 + \|\psi-\psi_1\|_C^2\big) \end{aligned}$$

for some constant $\delta > 0$.

Let $\phi(t) \in C \cap L_V^2$ and $\psi(t) \in C$, $t \in [-h,0]$.
Let $\Omega_T = [0,T] \times \Omega$.

Definition. We call an $\mathcal{F}_t$-adapted random process $(u(t),y(t)) \in V \times H$ a weak solution of the initial problem (0.1) on $[0,T]$ if:

(1) $u(t) = \phi(t),\ y(t) = \psi(t),\ t \in [-h,0]$;

(2) $u \in L^2(\Omega_T,V),\ y \in L^2(\Omega_T,H)$;

(3) for any $v \in V$ and $z \in H$, the equations

$$(u(t),v) = (u(0),v) + \int_0^t \big(\langle Au(s),v\rangle + (f(u_s,y_s),v)\big)\,ds + \int_0^t (\sigma(u_s,y_s)\,dW(s),v),$$

$$(y(t),z) = (y(0),z) + \int_0^t (g(u_s,y_s),z)\,dz$$

hold a.s. for each $t \in [0,T]$.

Theorem 1.1 (Existence and uniqueness)**.** *Suppose that conditions* (A1)–(A3) *and* (N1)–(N5) *hold. Then, for every $\phi \in C \cap L_V^2$ and $\psi \in C$, the initial problem* (0.1) *has a unique weak solution $(u(t),y(t))$ on $[0,T]$ such that*

$$u \in L^2(\Omega;C([0,T];H)) \cap L^2(\Omega_T,V),\ \ y \in L^2(\Omega,C([0,T];H)).$$

Moreover, the energy equation holds:

$$\begin{aligned} \|u(t)\|^2 + \|y(t)\|^2 &= \|u(0)\|^2 + \|y(0)\|^2 \\ &+ 2\int_0^t \Big(\langle Au(s),u(s)\rangle + (f(u_s,y_s),u(s)) + (g(u_s,y_s),y(s))\Big)\,ds \\ &+ \int_0^t \|\sigma(u_s,y_s)\|^2_{L_2^0}\,ds + 2\int_0^t \big(\sigma(u_s,y_s)\,dW(s),u(s)\big). \end{aligned} \tag{1.1}$$

2 Proof of the main result

In this section, we provide the sketch of the proof of Theorem 1.1.

Proof.
Uniqueness. Suppose that $(u(t),y(t))$ and $(u^1(t),y^1(t))$ are two weak solutions of the initial problem (0.1). Then, in view of (1.1) and condition (N5), we can easily show that

$$\mathbf{E}\|u(t)-u^1(t)\|^2+\mathbf{E}\|y(t)-y^1(t)\|^2=2\mathbf{E}\int\limits_0^t\big\langle A(u(s)-u^1(s)),u(s)-u^1(s)\big\rangle\,ds$$
$$+2\mathbf{E}\int\limits_0^t\Big[\big(f(u_s,y_s)-f(u_s^1,y_s^1),u(s)-u^1(s)\big)+\big(g(u_s,y_s)-g(u_s^1,y_s^1),y(s)-y^1(s)\big)\Big]\,ds$$
$$+\mathbf{E}\int\limits_0^t\big\|\sigma(u_s,y_s)-\sigma(u_s^1,y_s^1)\big\|^2_{L_2^0}\,ds\le\delta\mathbf{E}\int\limits_0^t\big(\|u_s-u_s^1\|_C^2+\|y_s-y_s^1\|_C^2\big)\,ds. \quad (2.1)$$

In what follows, we will need the following obvious statement.

Lemma. *The following inequality holds:*

$$\mathbf{E}\sup_{t\in[0,T]}\big(\|u_t\|_C^2+\|y_t\|_C^2\big)\le\mathbf{E}\big(\|\phi\|_C^2+\|\psi\|_C^2\big)+\mathbf{E}\sup_{t\in[0,T]}\big(\|u(t)\|^2+\|y(t)\|^2\big). \quad (2.2)$$

So, taking into account (2.2), from (2.1) we obtain

$$\sup_{s\in[0,T]}\mathbf{E}\big(\|u(s)-u^1(s)\|^2+\|y(s)-y^1(s)\|^2\big)\le\delta\int\limits_0^t\sup_{\tau\in[0,s]}\mathbf{E}\big(\|u_\tau-u_\tau^1\|_C^2+\|y_\tau-y_\tau^1\|_C^2\big)\,ds$$
$$\le\delta\int\limits_0^t\sup_{\tau\in[0,s]}\mathbf{E}\big(\|u(\tau)-u^1(\tau)\|^2+\|y(\tau)-y^1(\tau)\|^2\big)\,ds,$$

which, by Gronwall's inequality, yields

$$\mathbf{E}\big(\|u(s)-u^1(s)\|^2+\|y(s)-y^1(s)\|^2\big)=0,\ \ \forall\,t\in[0,T],$$

which establishes the uniqueness.

Existence. We will prove the existence by using Galerkin approximations.

Step 1. Finite-dimensional case. Approximate solutions.

Let $\{v_k\}$ be a complete orthonormal basis for H with $v_k\in V$, and let $H_n=\operatorname{span}\{v_1,\dots,v_n\}$. Suppose that $P_n:H\to H_n$ is an orthogonal projector such that

$$P_nh=\sum_{k=1}^n(h,v_k)v_k\ \text{ for }\ h\in H.$$

We extend P_n to the projection operator $P_n':V'\to V_n'$ defined as

$$P_n'w=\sum_{k=1}^n\langle w,v_k\rangle v_k\ \text{ for }\ w\in V'.$$

Obviously, $V_n = H_n = V_n'$.

Let $K_n = \operatorname{span}\{e_1, \dots, e_n\}$. We denote by Π_n a projection operator from K to K_n such that

$$\Pi_n a = \sum_{k=1}^{n} (a, e_k) e_k.$$

Let us introduce the following notation:

$$A^n u = P_n' A u, \quad f^n(\phi, \psi) = P_n f(\phi, \psi), \quad g^n(\phi, \psi) = P_n g(\phi, \psi), \quad \sigma^n(\phi, \psi) = P^n \sigma(\phi, \psi),$$

for $u \in V, \phi \in C \cap L_V^2$, and $\psi \in C$.

We consider the approximate equations to equations (0.1):

$$\begin{aligned} du^n(t) &= \big[A^n u^n(t) + f^n(u_t^n, y_t^n)\big]\,dt + \sigma^n(u_t^n, y_t^n)\,dW^n(t), \\ dy^n(t) &= g^n(u_t^n, y_t^n)\,dt, \\ &\qquad u^n(t) = P_n\phi(t), \quad y^n(t) = P_n\psi(t), \ \ t \in [-h, 0], \end{aligned} \tag{2.3}$$

for $t \in [0, T]$, where $W^n(t) = \Pi_n W(t)$.

The above equations can be regarded as Itô equations in $\mathbb{R}^n$. It can be shown that, under conditions (N1)–(N5), the coefficients f^n, σ^n and g^n of these equations are locally bounded and Lipschitz continuous and monotone. Hence, (2.3) has a unique solution $(u^n(t), y^n(t))$ in V_n on any finite time interval $[0, T]$. Moreover, it satisfies the property $u^n \in L^2(\Omega, C([0, T]; H)) \cap L^2(\Omega_T, V)$ and $y^n \in L^2(\Omega, C([0, T]; H))$.

Step 2. A priori estimate.

Next, we will establish a priori estimates with some positive constant A:

$$\mathbf{E} \sup_{t \in [0,T]} \big(\|u^n(t)\|^2 + \|y^n(t)\|^2\big) + \int_0^T \mathbf{E}\|u^n(s)\|_V^2\,ds \le A. \tag{2.4}$$

Step 3. Weak limits.

It follows from (2.4) that there exists subsequences, denoted for convenience by u^n and y^n such that $u^n \to u$ weakly in $L^2(\Omega_T, V)$ and $y^n \to y$ weakly in $L^2(\Omega_T, V)$. Next, we justify the passage to the limit in the finite-dimensional equation, which proves the theorem. $\square$

Acknowledgment

The work was partially supported by the National Research Foundation of Ukraine # F81/41743 and Ukrainian Government Scientific Research Grant # 210BF38-01.

Asymptotic Behaviour of Special Types of Solutions of Third Order Differential Equations, Asymptotically Close to Linear

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Consider the third order differential equation

$$y''' = \alpha_0 p(t) y L(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \leq +\infty$, $L : \Delta_{Y_0} \to]0, +\infty[$ is a continuous function slowly varying as $y \to Y_0$, Y_0 is equal to either zero or $\pm\infty$, and Δ_{Y_0} is a one-sided neighborhood of Y_0.

In the case where $L(y) \equiv 1$, Eq. (1) is a linear third-order differential equation. The asymptotic behavior of its solutions as $t \to +\infty$ (the case $\omega = +\infty$) is investigated in details (see, for example, the monograph [6, § 6, p . 175–194]).

Eq. (1) is a special case of the n-th order equation with regularly varying nonlinearity which was studied in work [2] (see also [3, 4]). However, the results of this work did not cover the case of an equation that is asymptotically close to linear. Some results on the asymptotic behavior of solutions of equation (1) were obtained in [5].

A second-order differential equation with a similar right-hand side was studied in paper [1].

The purpose of this work is to establish necessary and sufficient conditions for existence, as well as asymptotic representations of $P_\omega(Y_0, \lambda_0)$-solutions of the differential equation (1) in special cases, when $\lambda_0 \in \{0, 1, \pm\infty\}$.

Definition 1. The solution y of Eq. (1) is called $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the conditions

$$y : [t_0, \omega[\to \Delta_{Y_0}, \quad \lim_{t \uparrow \omega} y(t) = Y_0,$$

$$\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \pm\infty \end{cases} \quad (k = 1, 2), \quad \lim_{t \uparrow \omega} \frac{[y''(t)]^2}{y'''(t) y'(t)} = \lambda_0.$$

According to the properties of slowly varying functions (see [7]), for any function $L : \Delta_{Y_0} \to]0, +\infty[$, slowly varying as $y \to Y_0$, there exists a continuously differentiable, slowly varying as $y \to Y_0$ function $L_0 : \Delta_{Y_0} \to]0, +\infty[$ such that

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{L(y)}{L_0(y)} = 1 \ \text{ and } \ \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y L_0'(y)}{L_0(y)} = 0. \tag{2}$$

We set

$$\Delta_{Y_0} = \Delta_{Y_0}(b) = \begin{cases} [b, Y_0[, & \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0, \\ ]Y_0, b], & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0, \end{cases}$$

where a number $b \in \Delta_{Y_0}$ is such that

$$|b| < 1 \ \text{ as } \ Y_0 = 0, \quad b > 1 \ \ (b < -1) \ \text{ as } \ Y_0 = +\infty \ \ (Y_0 = -\infty).$$

We introduce the following notation

$$\mu_0 = \operatorname{sign} b, \quad \mu_1 = \begin{cases} \mu_0, & \text{if } Y_0 = \pm\infty, \\ -\mu_0, & \text{if } Y_0 = 0, \end{cases}$$

that define respectively the signs of the $P_\omega(Y_0, \lambda_0)$-solution and its first derivative in some left neighborhood of ω. We also need the following functions

$$\Phi(y) = \int\limits_B^y \frac{ds}{sL^{\frac{1}{3}}(s)}, \quad B = \begin{cases} b, & \text{if } \int\limits_b^{Y_0} \frac{ds}{sL^{\frac{1}{3}}(s)} = \pm\infty, \\ Y_0, & \text{if } \int\limits_b^{Y_0} \frac{ds}{sL^{\frac{1}{3}}(s)} = const, \end{cases}$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t - \omega, & \text{if } \omega < +\infty, \end{cases} \quad I_1(t) = \int\limits_{A_1}^t p(\tau)\,d\tau, \quad I_2(t) = \int\limits_{A_2}^t p^{\frac{1}{3}}(\tau)\,d\tau,$$

where each of the integration limits $A_i \in \{\omega; a\}$ $(i = 1, 2)$ is chosen so that the corresponding integral tends either to zero or $\pm\infty$ as $t \uparrow \omega$.

The function Φ is strictly monotone and differentiable on Δ_{Y_0}. For it there is a continuously differentiable and strictly monotone inverse function $\Phi^{-1} : \Delta_Z(c) \to \Delta_{Y_0}$, for which

$$\lim_{z \to Z} \Phi^{-1}(z) = Y_0, \quad Z = \lim_{y \to Y_0} \Phi(y),$$

where

$$\Delta_Z = \begin{cases} [c, Z[\,, & \text{if } \mu_0 > 0, \\ ]Z, c], & \text{if } \mu_0 < 0, \end{cases} \quad c = \Phi(b).$$

Theorem 1. *Let the function $L(\Phi^{-1}(z))$ be a regularly varying as $z \to Z$ of index γ. Then for the existence of $P_\omega(Y_0, 1)$-solutions of equation* (1) *it is necessary and, if function $p : [a, \omega[\, \to]0, +\infty[$ is continuously differentiable and there is the finite or equal $\pm\infty$*

$$\lim_{t \uparrow \omega} \frac{\big(p^{\frac{1}{3}}(t) L_0^{\frac{1}{3}}(\Phi^{-1}(\alpha_0 I_2(t)))\big)'}{p^{\frac{2}{3}}(t) L_0^{\frac{2}{3}}(\Phi^{-1}(\alpha_0 I_2(t)))}\,, \tag{3}$$

where $L_0 : \Delta_{Y_0} \to]0, +\infty[$ – is continuously differentiable and slowly varying function as $y \to Y_0$ with properties (2), *then it is sufficient that*

$$\lim_{t \uparrow \omega} \pi_\omega(t) p^{\frac{1}{3}}(t) L^{\frac{1}{3}}\big(\Phi^{-1}(\alpha_0 I_2(t))\big) = \infty, \quad \alpha_0 \lim_{t \uparrow \omega} I_2(t) = Z \tag{4}$$

and the following inequalities

$$\alpha_0 \mu_0 \mu_1 > 0, \quad \mu^* I_2(t) > 0 \ \text{ when } \ t \in]a, \omega[\tag{5}$$

are satisfied, where $\mu^* = \mu_0 \mu_1 \operatorname{sign} \Phi(y)$ when $y \in \Delta_{Y_0}$. Moreover, each of these solutions admits the following asymptotic representations

$$\begin{gathered} \Phi(y(t)) = \alpha_0 I_2(t)[1 + o(1)] \ \text{ as } \ t \uparrow \omega, \\ \frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \alpha_0 p^{\frac{1}{3}}(t) L^{\frac{1}{3}}\big(\Phi^{-1}(\alpha_0 I_2(t))\big)[1 + o(1)] \ \ (k = 1, 2) \ \text{ as } \ t \uparrow \omega. \end{gathered} \tag{6}$$

If conditions (4), (5) *are satisfied and there is the finite or equal* $\pm\infty$ *limit* (3), *then for* $\alpha_0 = 1$ *there exists a three-parameter family of* $P_\omega(Y_0, 1)$*-solutions with the asymptotic representations* (6) *in the case when* $\mu^* > 0$ *and a two-parameter family in the case when* $\mu^* < 0$, *and for* $\alpha_0 = -1$ *– a one-parameter family of such solutions in the case when* $\mu^* > 0$.

The next three theorems are devoted to the cases when $\lambda_0 = \pm\infty$, $\lambda_0 = 0$. They are established on the condition that slowly varying function L at $y \to Y_0$ satisfies the S conditions.

Definition 2. The slowly varying as $y \to Y$ function $L : \Delta_Y \to]0, +\infty[$, where Y is equal to either zero or $\pm\infty$, and Δ_Y is a one-sided neighborhood of Y satisfies the S, if

$$L\big(\mu e^{[1+o(1)]\ln|y|}\big) = L(y)[1+o(1)] \ \text{ as } \ y \to Y \ \ (y \in \Delta_Y),$$

where $\mu = \operatorname{sign} y$.

Theorem 2. *Let L satisfy S. Then for the existence of $P_\omega(Y_0, \pm\infty)$-solutions of equation* (1) *it is necessary and sufficient that*

$$\mu_0\mu_1\pi_\omega(t) > 0 \ \textit{ when } \ t \in]a, \omega[\,, \qquad \mu_0 \lim_{t\uparrow\omega} |\pi_\omega(t)| = Y_0, \tag{7}$$

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^3(t)L(\mu_0\pi_\omega^2(t)) = 0, \qquad \int\limits_{a_1}^{\omega} p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau = +\infty, \tag{8}$$

where $a_1 \in [a, \omega[$ is such that $\mu_0\pi_\omega^2(t) \in \Delta_{Y_0}$ when $t \in [a_1, \omega[$. Moreover, each of solutions admits the following asymptotic representations

$$\ln|y(t)| = 2\ln|\pi_\omega(t)| + \frac{\alpha_0}{2}\int\limits_{a_1}^{t} p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau\,[1+o(1)] \ \ \textit{as} \ \ t \uparrow \omega, \tag{9}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{3-k}{\pi_\omega(t)}\,[1+o(1)] \ \ (k=1,2) \ \ \textit{as} \ \ t \uparrow \omega. \tag{10}$$

If conditions (7), (8) *are satisfied, then there is a three-parameter family of* $P_\omega(Y_0, \pm\infty)$*-solutions with the asymptotic representations* (9), (10) *in the case of* $\omega = +\infty$, *and a one-parametric family of these solutions with the same representations when* $\omega < +\infty$.

Theorem 3. *Let L satisfy S and conditions* (7), (8) *hold. In addition, let the function $p : [a, \omega[\to]0, +\infty[$ be continuous and differentiable and such that there is a finite or equal $\pm\infty$*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega p'(t)}{p(t)}\,.$$

Then for each $P_\omega(Y_0, \pm\infty)$*-solutions of the differential equation* (1) *the place asymptotic representations*

$$\ln|y(t)| = 2\ln|\pi_\omega(t)| + \frac{\alpha_0}{2}\int\limits_{a_1}^{t} p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau\,[1+o(1)] \ \ \textit{as} \ \ t \uparrow \omega,$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{1}{\pi_\omega(t)}\Big[3-k+\frac{\alpha_0}{2}\,p(\tau)\pi_\omega^3(\tau)L(\mu_0\pi_\omega^2(\tau))[1+o(1)]\Big] \ \ (k=1,2) \ \ \textit{as} \ \ t \uparrow \omega$$

take palce.

Theorem 4. *Let L satisfy S. Then for the existence of $P_\omega(Y_0, 0)$-solutions of equation* (1) *for which there is a finite or equal $\pm\infty$*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'''(t)}{y''(t)}\,,$$

it is necessary and sufficient that

$$\mu_0\mu_1\pi_\omega(t) > 0, \ \ \textit{where } \ t \in \,]a,\omega[\,, \quad \mu_0 \lim_{t\uparrow\omega} |\pi_\omega(t)| = Y_0, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)p(t)}{I_1(t)} = -2, \tag{11}$$

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^3(t)L(\mu_0|\pi_\omega(t)|) = 0, \quad \int\limits_{a_1}^{\omega} p(\tau)\pi_\omega^2(\tau)L(\mu_0|\pi_\omega(\tau)|)\,d\tau = +\infty, \tag{12}$$

where $a_1 \in [a,\omega[$ is such that $\mu_0|\pi_\omega(t)| \in \Delta_{Y_0}$ when $t \in [a_1,\omega[\,$. Moreover, each of solutions admits the following asymptotic representations

$$\ln|y(t)| = \ln|\pi_\omega(t)| - \alpha_0 \int\limits_{a_1}^{t} p(\tau)\pi_\omega^2(\tau)L\big(\mu_0|\pi_\omega(\tau)|\big)\,d\tau\,[1+o(1)] \ \ \textit{as} \ \ t\uparrow\omega, \tag{13}$$

$$\frac{y'(t)}{y(t)} = \frac{1+o(1)}{\pi_\omega(t)}\,, \quad \frac{y''(t)}{y'(t)} = -\alpha_0 p(t)\pi_\omega^2(t)L\big(\mu_0|\pi_\omega(t)|\big)\,[1+o(1)] \ \ \textit{as} \ \ t\uparrow\omega. \tag{14}$$

If conditions (11), (12) *are satisfied, then there exists a two-parameter family of $P_\omega(Y_0,0)$-solutions with the asymptotic representations* (13), (14).

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On the Existence of an Optimal Element in Control Problems with Several Constant Delays

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In the paper, Filippov's type theorems on the existence of an optimal element [2] are given for the nonlinear optimal control problems with delays in the phase coordinates and commensurable delays in controls. Unlike considered in [1,3,5–9], here under element is implied the collection of the final moment t_1, the delay parameters τ_i, $i=1,\dots,s$ containing in the phase coordinates, the initial vector x_0, the piecewise-continuous initial function $\varphi(t)$ and measurable control function $u(t)$.

Let $I=[t_0,T]$ be a fixed interval; let $\tau_{2i}>\tau_{1i}>0$, $i=1,\dots,s$ and $\theta_p>\cdots>\theta_1>0$ be given numbers. Suppose that $O\subset\mathbb{R}^n$ is an open set and $\Phi\subset O$ and $U\subset\mathbb{R}^r$ are compact sets; the function $f(t,x_0,x_1,\dots,x_s,u_0,u_1,\dots,u_p)$ is continuous on the set $I\times O^{1+s}\times U^{1+p}$ and continuously differentiable with respect to $x_i\in O$, $i=0,1,\dots,s$; denote by Δ the set of piecewise-continuous functions $\varphi:[t_0-\tau,t_0]\to\Phi$, where $\tau=\max\{\tau_{21},\dots,\tau_{2s}\}$, satisfying the conditions:

(a) for each function $\varphi(t)\in\Delta$ there exists a partition $t_0-\tau=\xi_0<\xi_1<\cdots<\xi_{k+1}=t_0$ of the interval $[t_0-\tau,t_0]$ such that the restriction of the function $\varphi(t)$ satisfies Lipschitz's condition on the open interval (ξ_i,ξ_{i+1}), $i=0,1,\dots,k$, i.e.

$$|\varphi(s_1)-\varphi(s_2)|\le L|s_1-s_2|,\ \ \forall\, s_1,s_2\in(\xi_i,\xi_{i+1}),\ \ i=0,1,\dots,k;$$

(b) the numbers k and L do not depend on $\varphi(t)$. By Ω we denote the set of measurable functions $u:[t_0-\theta_p,T]\to U$. Let

$$g_i: I\times[\tau_{11},\tau_{21}]\times\cdots\times[\tau_{1s},\tau_{2s}]\times X_0\times O\to\mathbb{R}^1,\ \ i=0,1,\dots,l$$

be continuous functions, where $X_0\subset O$ is a compact set. In the space $\mathbb{R}^n$ to each element

$$w=(t_1,\tau_1,\dots,\tau_s,x_0,\varphi(t),u(t))\in W=(t_0,T]\times[\tau_{11},\tau_{21}]\times\cdots\times[\tau_{1s},\tau_{2s}]\times X_0\times\Delta\times\Omega$$

we assign the differential equation with delays in the phase coordinates and controls

$$\dot x(t)=f\big(t,x(t),x(t-\tau_1),\dots,x(t-\tau_s),u(t),u(t-\theta_1),\dots,u(t-\theta_p)\big),\ \ t\in[t_0,t_1] \tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[t_0-\tau,t_0),\quad x(t_0)=x_0. \tag{2}$$

Definition 1. Let $w=(t_1,\tau_1,\dots,\tau_s,x_0,\varphi(t),u(t))\in W$. A function $x(t)=x(t;w)\in O$, $t\in[t_0-\tau,t_1]$, is called a solution corresponding to the element w, if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies Eq. (1) almost everywhere on $[t_0,t_1]$.

Definition 2. An element $w \in W$ is said to be admissible if there exists the corresponding solution $x(t) = x(t; w)$, satisfying the condition

$$g(t_1, \tau_1, \dots, \tau_s, x_0, x(t_1)) = 0, \quad \text{where} \;\; g = (g_1, \dots, g_l). \tag{3}$$

By W_0 we denote the set of admissible elements. Now we consider the functional

$$J(w) = g_0(t_1, \tau_1, \dots, \tau_s, x_0, x(t_1; w)).$$

Definition 3. An element $w_0 = (t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0(t), u_0(t)) \in W_0$ is said to be optimal if

$$J(w_0) = \inf_{w \in W_0} J(w). \tag{4}$$

(1)–(4) is called the control problem with delays in the phase coordinates and controls.

Theorem 1. *There exists an optimal element $w_0 \in W_0$ if the following conditions holds:*

1) *there exist a number $h > 0$ such that $\theta_i = ih$, $i = 1, \dots, p$ – commensurability of delays θ_i, $i = 1, \dots, p$;*

2) *$T = t_0 + mh$, where m is a natural number with $m \geq p$;*

3) *$W_0 \neq \varnothing$;*

4) *there exists a number $M > 0$ such that for an arbitrary $w \in W_0$,*

$$|x(t; w)| \leq M, \quad t \in [t_0, t_1];$$

5) *for each fixed $t \in [t_0, t_0 + h]$ and $z_i = (x_{0i}, x_{1i}, \dots, x_{si}) \in O^{1+s}$, $i = 0, 1, \dots, m-1$ the set*

$$V_f(t; z_0, z_1, \dots, z_{m-1}) := \left\{ \begin{pmatrix} f(t, z_0, u_0, u_{-1}, \dots, u_{-p}) \\ f(t+h, z_1, u_1, u_0, u_{-1}, \dots, u_{-p+1}) \\ \vdots \\ f(t+ph, z_p, u_p, u_{p-1}, \dots, u_0) \\ f(t+(p+1)h, z_{p+1}, u_{p+1}, u_p, \dots, u_1) \\ \vdots \\ f(t+(m-2)h, z_{m-2}, u_{m-2}, u_{m-3}, \dots, u_{m-p-2}) \\ f(t+(m-1)h, z_{m-1}, u_{m-1}, u_{m-2}, \dots, u_{m-p-1}) \end{pmatrix} : \; u_i \in U, \; i = -p, \dots, -1, 0, 1, \dots, m-1 \right\}$$

is convex.

Remark. Let U be a convex set and

$$f(t, x_0, x_1, \dots, x_s, u_0, u_1, \dots, u_p) = \sum_{i=0}^{p} A_i(t, x_0, x_1, \dots, x_s) u_i.$$

Then the condition 5) of Theorem 1 holds.

Now we consider an optimal problem with the integral functional and with fixed ends

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t), u(t-\theta_1), \dots, u(t-\theta_p)\big), \quad t \in I, \tag{5}$$

$$x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0), \quad x(t_0) = x_0, \quad x(t_1) = x_1, \tag{6}$$

$$\int_{t_0}^{t_1} f_0\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t), u(t-\theta_1), \dots, u(t-\theta_p)\big)\, dt \longrightarrow \min. \tag{7}$$

Here $f_0(t, x_0, x_1, \dots, x_s, u_0, u_1, \dots, u_p) : I \times O^{1+s} \times U^{1+p} \to \mathbb{R}^1$ is a continuous function, $x_0, x_1 \in O$ are fixed points. For the problem (5)–(7) by Q_0 we denote the set of admissible elements

$$q = (t_1, \tau_1, \dots, \tau_s, \varphi(t), u(t)) \in Q = (t_0, T] \times [\tau_{11}, \tau_{21}] \times \cdots \times [\tau_{1s}, \tau_{2s}] \times \Delta \times \Omega$$

and by

$$q_0 = (t_{10}, \tau_{10}, \dots, \tau_{s0}, \varphi_0(t), u_0(t))$$

we denote an optimal element(see Definitions 2 and 3).

Theorem 2. *There exists an optimal element $q_0 \in Q_0$ if the conditions 1) and 2) of Theorem 1 hold. Moreover: $Q_0 \neq \oslash$; there exists a number $M > 0$ such that for an arbitrary $q \in Q_0$ we have $|x(t;q)| \leq M$, $t \in [t_0, t_1]$; for each fixed $t \in [t_0, t_0 + h]$ and $z_i = (x_{0i}, \dots, x_{si}) \in O^{1+s}, i = 0, 1, \dots, m-1$ the set $V_F(t; z_0, z_1, \dots, z_{m-1})$ is convex, where $F = (f_0, f)$.*

It is clear that Theorem 2 is valid also for a problem with the free right end. Below we give an example which shows that for the existence of an optimal element the convexity of the set V_F is essential.

Example. Consider the optimal control problem

$$\dot{x}(t) = -x\big(t - \sqrt{2}\big) + u(t) + u^2(t-1), \quad t \in [0, 2],$$

$$x(t) = 0, \quad t \in \big[-\sqrt{2}, 0\big), \quad x(0) = 0; \quad u(t) = 1, \quad t \in [-1, 0), \quad u(t) \in U = [-1, 1], \quad t \in [0, 2],$$

$$\int_0^2 [x(t) - t]^2\, dt \to \min.$$

Here under an element is implied only control function $u(t) \in \Omega$. For a given $i = 2, 3, \dots$ we shall decompose the interval $[0, 1]$ into intervals I_j, $j = 2, \dots, i$, of length $1/i$. Define the control $u_i(t)$, $t \in [0, 2]$:

$$\begin{aligned} u_i(t) &= v_i(t), \quad t \in [0, 1]; \\ u_i(t) &= 0, \quad t \in \big(1, \sqrt{2}\big]; \\ u_i(t) &= t - \sqrt{2}\,, \quad t \in \big(\sqrt{2}, 2\big], \end{aligned}$$

here $v_i(t)$ is a control oscillating between $+1$ and -1, i.e.

$$v_i(t) = 1, \quad t \in I_1, \quad v_i(t) = -1, \quad t \in I_2,$$

etc. Furthermore,

$$\lim_{i \to \infty} x(t; u_i) = x_0(t) = t \ \text{ uniformly in } \ [0, 2]$$

and by Gamkrelidze's approximation lemma [4] the sequence of Dirac measures $\delta_{v_i(t)}$, $i = 2, 3, \dots$, $t \in [0,1]$ weakly converges to $\frac{1}{2}\,\delta_{-1} + \frac{1}{2}\,\delta_{+1}$. It is easy to observe that the trajectory $x_0(t)$, $t \in [0,2]$, minimizing the functional corresponds to the control

$$u_0(t) = \begin{cases} 0, & t \in [0,1], \\ 1, & t \in \big(1, \sqrt{2}\,\big], \\ t + 1 - \sqrt{2}\,, & t \in \big(\sqrt{2}, 2\big]. \end{cases}$$

But $u_0(t) \notin [-1,1]$, $t \in (\sqrt{2}, 2]$, i.e. it is not an admissible control. Consequently, in the considered example, there is no optimal element since the set

$$V_F(t; z_0, z_1) := \left\{ \begin{pmatrix} F(t, z_0, u_0, u_{-1}) \\ F(t+1, z_1, u_1, u_0) \end{pmatrix} : \ u_i \in [-1,1], \ i = -1, 0, 1 \right\}$$

is not convex. Here $z_i = (x_{0i}, x_{1i})$, $i = 0, 1$ and

$$F(t, z_0, u_0, u_{-1}) = \begin{pmatrix} (x_{00} - t)^2 \\ -x_{10} + u_0 + u_{-1}^2 \end{pmatrix}, \quad F(t+1, z_1, u_1, u_0) = \begin{pmatrix} (x_{01} - t - 1)^2 \\ -x_{11} + u_1 + u_0^2 \end{pmatrix}.$$

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Linear Differential Equations with the Hukuhara Derivative that Preserve Sets of Constant Width

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Solutions of ordinary differential equations with the Hukuhara derivative [1], [2, p. 14] are compact convex sets for every value of the independent variable. The geometric characteristics of these sets can be considered as functions of the independent variable. The study of these functions is of particular interest. For example, in the paper [4] the radii of inscribed and circumscribed spheres of solutions of linear time-invariant differential equations were considered and their Lyapunov exponents were calculated. The paper [3] gives a complete description of linear time-invariant differential equations with the Hukuhara derivative that preserve polytopes, i.e., of equations such that any solution of them that is a polytope for the initial value of the independent variable remains a polytope for all subsequent values.

This report examines other geometric characteristics of solutions. But before formulating the obtained result, let us give some necessary definitions. By $\Omega(\mathbb{R}^d)$ we denote the family of all nonempty bounded subsets of the space $\mathbb{R}^d$. The set of all nonempty convex compact subsets of the space $\mathbb{R}^d$ is denoted by $K_c(\mathbb{R}^d)$.

Definition 1. A set $X \subset K_c(\mathbb{R}^d)$ is called a *set of constant width* if the length of the orthogonal projection of X onto an arbitrary line equals the same value $w(X)$ that is called the *width* of X.

Definition 2. A set $Z \stackrel{\text{def}}{=} \{x + y : \ x \in X, \ y \in Y\}$ is called the *Minkowski sum* of two subsets X, $Y \subset \mathbb{R}^d$.

Generally speaking, for arbitrary real matrices A and B consisting of d columns and a set $X \subset \mathbb{R}^d$, we have $(A+B)X \neq AX + BX$.

Definition 3 ([1])**.** A set $Z \subset \mathbb{R}^d$ is called the *Hukuhara difference* of X, $Y \subset \mathbb{R}^d$ and denoted by $Z = X - Y$, if $X = Y + Z$.

By $B \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : \ \|x\| \leq 1\}$ we denote the closed ball of unit radius centered at the origin.

Definition 4. The *Hausdorff distance* $h(\,\cdot\,,\,\cdot\,)$ on the set $\Omega(\mathbb{R}^d)$ is the function

$$h(X,Y) \stackrel{\text{def}}{=} \inf\big\{r \geq 0 : \ X \subset Y + rB, \ Y \subset X + rB\big\}, \quad X, Y \in \Omega(\mathbb{R}^d).$$

According to Hahn Theorem, the pair $\big(K_c(\mathbb{R}^d), h\big)$ is a complete metric space. By $I \subset \mathbb{R}$ we denote an arbitrary open interval that may be unbounded.

Definition 5 ([1])**.** A mapping $X : I \to K_c(\mathbb{R}^d)$ is called *differentiable by Hukuhara* at a point $t_0 \in I$ if there exist limits

$$\lim_{\Delta t \to +0} \frac{X(t_0 + \Delta t) - X(t_0)}{\Delta t}, \quad \lim_{\Delta t \to +0} \frac{X(t_0) - X(t_0 - \Delta t)}{\Delta t}$$

and these limits are equal to each other. In this case, the common value of these limits, which is obviously a convex compact set, is denoted by $D_H X(t_0)$ and called the Hukuhara derivative of the mapping X at the point t_0.

Consider the linear differential equation

$$D_H X = \sum_{i=1}^{n} A_i(t)X, \quad X(t) \in K_c(\mathbb{R}^d), \ \ t \geq 0, \tag{1}$$

with piecewise continuous $d \times d$-matrices of coefficients $A_i(\,\cdot\,)$, $1 \leq i \leq n$. We say that equation (1) *preserves sets of constant width* if for any its solution $X(\,\cdot\,)$, such that $X(0)$ is a set of constant width, it follows that $X(t)$ is a set of constant width for all $t \geq 0$. Naturally, the problem arises of obtaining a necessary and sufficient condition for equation (1) to preserve sets of constant width. The complete solution of the problem is given by the following theorem.

Theorem. *Equation* (1) *preserves sets of constant width if and only if there exist piecewise continuous function* $\alpha_1(\,\cdot\,), \alpha_2(\,\cdot\,), \dots, \alpha_n(\,\cdot\,) : [0, \infty) \to \mathbb{R}_{\geq 0}$*, such that the following equalities hold*

$$A_i(t)^T A_i(t) = \alpha_i(t)E, \ \ 1 \leq i \leq n, \ \ t \geq 0.$$

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The Exact Baire Class of the Asymptotic ε-Capacity of a Family of Non-Autonomous Dynamical Systems Continuously Depending on a Parameter

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In this paper we consider a parametric family of non-autonomous dynamical systems defined on a compact metric space and continuously depending on a parameter from some topological space. For any such family, we study the asymptotic ε-capacity of its dynamical systems as a function of the parameter from the standpoint of the Baire classification.

As a measure of "massiveness" of a compact metric space (X,d), A. N. Kolmogorov in the paper [1] introduced the notion of ε-capacity which is defined as the maximum number of ε-distinguishable elements in X. Using this notion, we give the definition of the topological entropy of a non-autonomous dynamical system [2].

Let $\mathcal{F}\equiv(f_1,f_2,\dots)$ be a sequence of continuous mappings from X to X. For any positive integer n, denote by F_n the subsequence $(f_n,f_{n+1},\dots)$ of the sequence $\mathcal{F}$. Along with the original metric d we define on X an additional system of metrics

$$d_k^{F_n}(x,y)=\max_{0\le i\le k-1} d(f_n^{\circ i}(x),f_n^{\circ i}(y)),$$
$$\big(f_n^{\circ i}\equiv f_{n+(i-1)}\circ\cdots\circ f_n,\ f_n^{\circ 0}\equiv \mathrm{id}_X\big),\ \ x,y\in X,\ \ k,n\in\mathbb{N}.$$

For any $k\in\mathbb{N}$ and $\varepsilon>0$, we denote by $N_d(F_n,\varepsilon,k)$ the maximum number of points in X such that their pairwise $d_k^{F_n}$-distances are greater than ε. Such a set of points will be called (F_n,ε,k)-separated. Then the *ε-capacity* $h_d(\mathcal{F},\varepsilon)$ and *asymptotic ε-capacity* $h_d^*(\mathcal{F},\varepsilon)$ of the non-autonomous dynamical system $(X,\,\mathcal{F})$ are defined by the equalities

$$h_d(\mathcal{F},\varepsilon)=\varlimsup_{k\to\infty}\frac{1}{k}\,\ln N_d(F_1,\varepsilon,k), \tag{1}$$
$$h_d^*(\mathcal{F},\varepsilon)=\sup_{n\in\mathbb{N}}\lim_{\varepsilon\to 0}\limsup_{k\to\infty}\frac{1}{k}\,\ln N_d(F_n,\varepsilon,k). \tag{2}$$

It follows directly from formulas (1) and (2) that

$$h_d(\mathcal{F},\varepsilon)\le h_d^*(\mathcal{F},\varepsilon)$$

holds for any sequence $\mathcal{F}$. As the following example shows, quantities (1) and (2) may not coincide. Let us equip the set Ω_2 of two-sided sequences

$$x=(\dots,x_{-2},x_{-1},x_0,x_1,x_2,\dots),\ x_k\in\{0,\,1\},$$

with the metric

$$d_{\Omega_2}(x,y)=\begin{cases}0 & \text{if } x=y;\\ 2^{-\min\{|i|:\,x_i\ne y_i\}} & \text{if } x\ne y.\end{cases}$$

Note that the resulting metric space (Ω_2, d_{Ω_2}) is compact and homeomorphic to the Cantor set on the segment $[0,1]$ with the metric induced by the standard metric of the real line. Let $\sigma : \Omega_2 \to \Omega_2$ stand for the left shift by one element:

$$\sigma((\dots, x_{-1}, x_0, x_1, \dots)) = (\dots, x_0, x_1, x_2, \dots),$$

and $\chi : \Omega_2 \to \Omega_2$ be the map that takes any element from Ω_2 to the sequence of zeros:

$$\chi((\dots, x_{-1}, x_0, x_1, \dots)) = (\dots, 0, 0, 0, \dots).$$

Then for the sequence $\mathcal{F} \equiv (\chi, \sigma, \sigma, \dots)$ and $\varepsilon < 1/2$ we have

$$h_d(\mathcal{F}, \varepsilon) = 0 < \ln 2 = h_d^*(\mathcal{F}, \varepsilon).$$

Note that in this example the equality

$$h_d^*(\mathcal{F}, \varepsilon) = \limsup_{k\to\infty} \frac{1}{k} \ln N_d(F_2, \varepsilon, k)$$

holds from which we obtain

$$h_d^*(\mathcal{F}, \varepsilon) = \max_{n\in\mathbb{N}} \limsup_{k\to\infty} \frac{1}{k} \ln N_d(F_n, \varepsilon, k).$$

In the general case, as the following example shows, the supremum over n in formula (2) cannot be replaced by the maximum.

Let Λ_2 be the set of infinite matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_{ij} \in \{0,1\}$; on this set we introduce the metric

$$d_{\Lambda_2}(A, B) = \begin{cases} 0 & \text{if } A = B; \\ 2^{-\min\{\max\{i,j\} :\, a_{ij}\neq b_{ij}\}} & \text{if } A \neq B. \end{cases}$$

Consider the sequence $\mathcal{F} \equiv (f_1, f_2, \dots)$ of continuous mappings $\Lambda_2 \to \Lambda_2$ defined by

$$f_1\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right) = \begin{pmatrix} a_{11+1} & a_{12+1} & a_{13+1} & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$f_2\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right) = \begin{pmatrix} a_{11+1} & a_{12+1} & a_{13+1} & \dots \\ a_{21+2} & a_{22+2} & a_{23+2} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \dots .$$

It follows that for all $n \in \mathbb{N}$ and $\varepsilon < 1/2$, the inequality

$$+\infty = h_d^*(\mathcal{F}) > \max_{1\le m\le n} \overline{\lim_{k\to\infty}} \frac{1}{k} \ln N_d(F_n, \varepsilon, k)$$

holds.

For a given metric space $\mathcal{M}$ and a sequence of continuous mappings

$$\mathcal{F} \equiv (f_1, f_2, \dots), \quad f_i : \mathcal{M} \times X \to X, \tag{3}$$

we form the functions

$$\mu \longmapsto h_d\big(\mathcal{F}(\mu, \,\cdot\,), \varepsilon\big), \tag{4}$$

$$\mu \longmapsto h_d^*\big(\mathcal{F}(\mu, \,\cdot\,), \varepsilon\big). \tag{5}$$

It was proved in [3] that for any metric space $\mathcal{M}$, compact metric space X and sequence of mappings (3) function (4) belongs to the second Baire class and in general does not belong to the first Baire class. Recall that the zeroth Baire class on a topological space $\mathcal{M}$ consists of all continuous functions, and for any positive integer p functions of the p-th Baire class are the functions that are pointwise limits of sequences of functions belonging to the $(p-1)$-th class.

In the same paper [3] it was proved for a complete metric space $\mathcal{M}$ that the set of points of upper semicontinuity of function (4) is an everywhere dense G_δ-set.

In this paper similar results are obtained for function (5).

Theorem 1. *For any sequence of mappings* (3)*, function* (5) *belongs to the second Baire class. Furthermore, its set of points of upper (lower) semicontinuity is a G_δ-set (an $F_{\sigma\delta}$-set).*

Theorem 2. *If $\mathcal{M} = X = \Omega_2$, then there exists a sequence of mappings* (3) *such that for any* $\varepsilon \in (0; 1/4]$ *function* (5) *does not belong to the first Baire class on the space $\mathcal{M}$.*

Theorem 3. *If a space $\mathcal{M}$ is complete, then the set of points of upper semicontinuity of function* (5) *is an everywhere dense G_δ-set.*

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The Exponential Solution to Quaternion Dynamic Equations Based on a New Quaternion Hyper-Complex Space with Hyper Argument

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Abstract

In this short communication, we will introduce the notion of quaternion hyper argument to construct the non-commutative quaternion hyper argument space. By virtue of the structure of the Hilger complex plane and hyper argument space theory, we establish a theoretical framework of the quaternion hyper-complex space in which the new quaternion hyper-complex exponent, the hyper-complex logarithm are introduced. Note that the quaternion exponential functions introduced here is a solution of the linear homogeneous dynamic equation $x^{\Delta}(t) = f(t)x(t)$ under the non-commutative quaternion function f.

1 Quaternion hyper argument space and calculus

The notion of quaternion was introduced by Hamilton in 1843, which provides a type of hyper-complex numbers and extends the filed $\mathbb{C}$ of the complex numbers to a novel non-commutative division ring under the addition and multiplication operation. The study quaternion dynamic equations becomes a hot topic and some basic results were established on time scales by Wang and Agarwal et al. (see [1–6]).

In the literature [4], some important notions of the hyper-complex polar form of the quaternion numbers and a notion of the quaternion hyper argument are presented as follows.

Definition 1.1 ([4])**.** Let $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{Q}$, $\cos^{\mathbb{Q}}, \sin^{\mathbb{Q}} : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{C}$, we define the quaternion polar form of q by

$$q := |q| e^{\arg^{\mathbb{Q}}(q)} = |q| e^{\Theta} = |q| \big[\cos^{\mathbb{Q}} \Theta + \sin^{\mathbb{Q}} \Theta j \big],$$

where

$$\Theta = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}}, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}, \ \ \theta^{(1)}, \theta^{(2)} \in (-\pi, \pi], \ \ \theta^{(3)} \in \Big[0, \frac{\pi}{2}\Big],$$

$$\cos^{\mathbb{Q}} \Theta = \cos\theta^{(1)} \cos\theta^{(3)} + \sin\theta^{(1)} \cos\theta^{(3)} i, \quad \sin^{\mathbb{Q}} \Theta = \cos\theta^{(2)} \sin\theta^{(3)} + \sin\theta^{(2)} \sin\theta^{(3)} i,$$

and $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ satisfy the following conditions:

(i) $\cos\theta^{(1)} = \dfrac{q_0}{\sqrt{q_0^2+q_1^2}}$ and $\sin\theta^{(1)} = \dfrac{q_1}{\sqrt{q_0^2+q_1^2}}$ if $q_0+q_1 i \neq 0$; $\theta^{(1)}=0$ if $q_0+q_1 i = 0$;

(ii) $\cos\theta^{(2)} = \dfrac{q_2}{\sqrt{q_2^2+q_3^2}}$ and $\sin\theta^{(2)} = \dfrac{q_3}{\sqrt{q_2^2+q_3^2}}$ if $q_2 j+q_3 k \neq 0$; $\theta^{(2)}=0$ if $q_2 j+q_3 k = 0$;

(iii) $\cos\theta^{(3)} = \dfrac{\sqrt{q_0^2+q_1^2}}{|q|}$ and $\sin\theta^{(3)} = \dfrac{\sqrt{q_2^2+q_3^2}}{|q|}$ if $q \neq 0$; $\theta^{(1)}=\theta^{(2)}=\theta^{(3)}=0$ if $q=0$,

we call Θ the quaternion hyper-principle argument. Generally, we define the quaternion hyper argument $\operatorname{Arg}^{\mathbb{Q}}(q)$ of q satisfying

$$e^{\operatorname{Arg}^{\mathbb{Q}}(q)} := e^{\Upsilon} = \cos^{\mathbb{Q}}\Upsilon + \sin^{\mathbb{Q}}\Upsilon j,$$

where $\Upsilon = (\alpha^{(1)},\alpha^{(2)},\alpha^{(3)})^{\mathbb{Q}} \in \Gamma_q$ and

$$\Gamma_q = \Big\{\Upsilon \mid \ \cos^{\mathbb{Q}}\Theta + \sin^{\mathbb{Q}}\Theta j = \cos^{\mathbb{Q}}\Upsilon + \sin^{\mathbb{Q}}\Upsilon j\Big\}.$$

Remark 1.1. Let

$$q = |q|e^{(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}}, \quad p = |p|e^{(\gamma^{(1)},\gamma^{(2)},\gamma^{(3)})^{\mathbb{Q}}},$$

then

$$\arg^{\mathbb{Q}}(qp) \neq \big(\theta^{(1)}+\gamma^{(1)},\theta^{(2)}+\gamma^{(2)},\theta^{(3)}+\gamma^{(3)}\big)^{\mathbb{Q}}$$

in general.

Remark 1.2. Let

$$\arg^{\mathbb{Q}}(q) = (\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}, \quad \arg^{\mathbb{Q}}(\overline{q}) = (\gamma^{(1)},\gamma^{(2)},\gamma^{(3)})^{\mathbb{Q}},$$

then

$$\theta^{(1)}+\gamma^{(1)}=0, \quad |\theta^{(2)}-\gamma^{(2)}|=\pi \ \text{ and } \ \theta^{(3)}=\gamma^{(3)}.$$

Remark 1.3. Note that the quaternion hyper-principle argument

$$\arg^{\mathbb{Q}}(q) = (\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}$$

is unique for each fixed q.

Remark 1.4. Let

$$\arg^{\mathbb{Q}}(q) = (\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}} \ \text{ and } \ \operatorname{Arg}^{\mathbb{Q}}(q) = (\alpha^{(1)},\alpha^{(2)},\alpha^{(3)})^{\mathbb{Q}}, \ \ n_1,n_2,n_3\in\mathbb{Z},$$

then

$$\alpha^{(1)} = \theta^{(1)}+2n_1\pi, \quad \alpha^{(2)} = \theta^{(2)}+2n_2\pi, \quad \alpha^{(3)} = \theta^{(3)}+2n_3\pi,$$

or

$$\alpha^{(1)} = \theta^{(1)}+2n_1\pi, \quad \alpha^{(2)} = \theta^{(2)}+2n_2\pi+\pi, \quad \alpha^{(3)} = -\theta^{(3)}+2n_3\pi,$$

or

$$\alpha^{(1)} = \theta^{(1)}+2n_1\pi+\pi, \quad \alpha^{(2)} = \theta^{(2)}+2n_2\pi+\pi, \quad \alpha^{(3)} = \theta^{(3)}+2n_3\pi+\pi,$$

etc., this indicates that Γ_q is an infinite set.

Remark 1.5. Note that

$$\Big\{q \mid \ q\in\mathbb{Q},\ \arg^{\mathbb{Q}}(q)=(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}},\ \theta^{(3)}=0\Big\}=\mathbb{C}$$

and

$$\Big\{q \mid \ q\in\mathbb{Q},\ \arg^{\mathbb{Q}}(q)=(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}},\ \theta^{(3)}=0,\ \theta^{(1)}=0 \text{ or } \pi\Big\}=\mathbb{R}.$$

Moreover,

$$\begin{gathered}
e^{(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}}=e^{\theta^{(1)}i} \ \text{ if } \ \theta^{(3)}=0;\\
q\in\mathbb{R} \ \text{ and } \ e^{(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}}=1 \ \text{ if } \ \theta^{(3)}=\theta^{(1)}=0;\\
e^{(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}}=e^{\theta^{(2)}i}j \ \text{ if } \ \theta^{(3)}=\frac{\pi}{2}\,.
\end{gathered}$$

Remark 1.6. Note that for $\arg^{\mathbb{Q}}(q)=(\theta^{(1)},\theta^{(2)},\theta^{(3)})^{\mathbb{Q}}$, it follows that

$$q=a+bj=|a|e^{\theta^{(1)}i}+|b|e^{\theta^{(2)}i}j=|q|\big(e^{\theta^{(1)}i}\cos\theta^{(3)}+e^{\theta^{(2)}i}\sin\theta^{(3)}j\big),$$

where $a,b\in\mathbb{C}$.

2 The quaternion hyper-complex space

Definition 2.1 ([4])**.** Let $h>0$, $\mathbb{Q}=\mathbb{C}_1\times\mathbb{C}_2$, $q=(q_0+q_1i)+(q_2+q_3i)j\in\mathbb{Q}$, $q_0+q_1i\in\mathbb{C}_1$ and $q_2+q_3i\in\mathbb{C}_2$. Then $\mathbb{C}_1$ is called the sub-complex plane of the quaternion hyper-complex space, and $\mathbb{C}_2$ is called the imaginary-complex plane of the quaternion hyper-complex space. Moreover, we define the Hilger quaternion number set as

$$\mathbb{Q}_h:=\Big\{q\in\mathbb{Q}:\ q\neq-\frac{1}{h}\Big\}.$$

Let $q=a+bj\in\mathbb{Q}_h,\ a,b\in\mathbb{C},\ \theta^{(1)}=\mathrm{Im}_h(a),\ \theta^{(2)}=\mathrm{Im}_h(b),\ \theta^{(3)}=\mathrm{Im}_h(|a|+|b|j)$, then the schematic diagram of the quaternion hyper-complex space is showed by **Figure** 1. For $h=0$, then $\mathbb{Q}_0=\mathbb{Q}$.

Now, let

$$\chi_h(q)=\begin{cases}\dfrac{\ln|1+hq|}{h} & \text{for } \ h>0,\\[2mm] q_0 & \text{for } \ h=0,\end{cases}\qquad \mathbb{A}_h(q)=\begin{cases}\dfrac{1}{h}\cdot\arg^{\mathbb{Q}}(1+hq) & \text{for } \ h>0,\\[2mm] \displaystyle\lim_{h\to}\frac{1}{h}\cdot\arg^{\mathbb{Q}}(1+hq) & \text{for } \ h=0,\end{cases}$$

we introduce the hyper-complex cylinder transformation $\xi_h^{\mathbb{Q}}:\mathbb{Q}_h\to\mathbb{Z}_h^{\mathbb{Q}}$ by

$$\xi_h^{\mathbb{Q}}(q)=\chi_h(q)+\mathbb{A}_h(q)=\begin{cases}\dfrac{\ln|1+hq|}{h}+\dfrac{1}{h}\cdot\arg^{\mathbb{Q}}(1+hq) & \text{for } \ h>0,\\[2mm] q_0+\displaystyle\lim_{h\to0}\frac{1}{h}\cdot\arg^{\mathbb{Q}}(1+hq) & \text{for } \ h=0,\end{cases}$$

where

$$\mathbb{Z}_h^{\mathbb{Q}}=\Big\{q\in\mathbb{Q}:\ \theta^{(1)},\theta^{(2)}\in\Big(-\frac{\pi}{h}\,,\frac{\pi}{h}\Big],\theta^{(3)}\in\Big[0,\frac{\pi}{2}\Big]\Big\}.$$

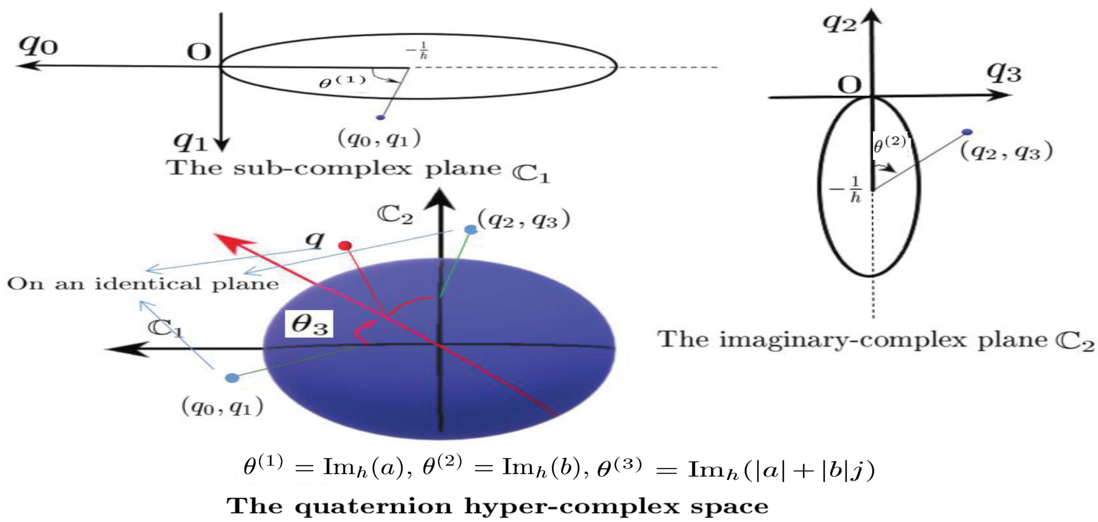

Figure 1. The geometric diagram of the quaternion hyper-complex space.

Remark 2.1. Let $h > 0$, the Hilger complex numbers $\mathbb{C}_h = \{z \in \mathbb{C} \mid \ z \neq -\frac{1}{h}\}$, then $\mathbb{C}_h \subset \mathbb{Q}_h$. In fact, let $p, q \in \mathbb{Q}_h$ and $\arg^{\mathbb{Q}}(q) = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}}$, $\arg^{\mathbb{Q}}(p) = (\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})^{\mathbb{Q}}$, we have

$$\begin{aligned}\arg^{\mathbb{Q}}(q) \oplus_{\mathbb{Q}} \arg^{\mathbb{Q}}(p) &= \theta^{(1)} i + \gamma^{(1)} i,\\ \arg^{\mathbb{Q}}(q) \ominus_{\mathbb{Q}} \arg^{\mathbb{Q}}(p) &= \theta^{(1)} i - \gamma^{(1)} i,\\ b \cdot \arg^{\mathbb{Q}}(q) &= b\theta^{(1)} i,\end{aligned}$$

where $b \in \mathbb{R}$ and $\theta^{(2)} = \theta^{(3)} = \gamma^{(2)} = \gamma^{(3)} = 0$, it means that the operations $\oplus_{\mathbb{Q}}$ and $\ominus_{\mathbb{Q}}$ will turn into the classical operations $+$ and $-$ when $\theta^{(2)} = \theta^{(3)} = \gamma^{(2)} = \gamma^{(3)} = 0$, by Remark 1.5, we can obtain $\mathbb{C}_h \subset \mathbb{Q}_h$.

Next, we will introduce the quaternion hyper-complex logarithm in the quaternion hyper-complex space.

Definition 2.2 ([4])**.** Let $q \in \mathbb{Q}$, $q \neq 0$. We define the quaternion hyper-complex logarithm by

$$\mathrm{Log}^{\mathbb{Q}}(q) := \ln|q| + \arg^{\mathbb{Q}}(q).$$

Remark 2.2. Note that $e^{\mathrm{Log}^{\mathbb{Q}}(q)} = q$ for any nonzero quaternion number $q \in \mathbb{Q}$. In fact,

$$e^{\mathrm{Log}^{\mathbb{Q}}(q)} = e^{\ln|q| + \arg^{\mathbb{Q}}(q)} = e^{\ln|q|} e^{\arg^{\mathbb{Q}}(q)} = |q| e^{\arg^{\mathbb{Q}}(q)} = q.$$

Remark 2.3. Let $q, p \in \mathbb{Q}$,

$$\arg^{\mathbb{Q}}(q) = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}} \ \text{ and } \ \arg^{\mathbb{Q}}(p) = (\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})^{\mathbb{Q}},$$

then

$$\mathrm{Log}^{\mathbb{Q}}(qp) = \ln|q| + \ln|p| + (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})^{\mathbb{Q}} \oplus_{\mathbb{Q}} (\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})^{\mathbb{Q}}.$$

3 The quaternion hyper-complex exponential function and dynamic equation on time scales

Definition 3.1 ([4])**.** Let $t,s\in\mathbb{T}$, $f:\mathbb{T}\to\mathbb{Q}$, $1+\mu(t)f(t)\neq 0$ for any $t\in\mathbb{T}^\kappa$, then we define $\widehat{x}(t,t_0)$ and $\widetilde{x}(t,t_0)$ as follows:

(i) $\widehat{x}(t,s):=e^{\int\limits_s^t \frac{\ln|1+\mu(\tau)f(\tau)|}{\mu(\tau)}\,\Delta\tau+\int\limits_s^t \frac{1}{\mu(\tau)}\cdot\arg^{\mathbb{Q}}(1+\mu(\tau)f(\tau))\,\Delta\tau}$ if $\mu(\tau)>0$ for any $\tau\in[s,t]_{\mathbb{T}}$.

(ii) If $\lim\limits_{u\to 0}\frac{1}{u}\cdot\arg^{\mathbb{Q}}(1+uf(t))=\Theta(t)$ and $\Theta(t)$ is an integrable quaternion hyper argument function, then we define

$$\widetilde{x}(t,s):=e^{\int\limits_s^t f_0(\tau)\,d\tau+\int\limits_s^t \Theta(\tau)\,d\tau}$$

if $\mu(\tau)=0$ for any $\tau\in[s,t]_{\mathbb{T}}$, where $f(t)=f_0(t)+f_1(t)i+f_2(t)j+f_3(t)k$.

Generally, Based on the hyper-complex cylinder transformation $\xi^{\mathbb{Q}}_{\mu(t)}:\mathbb{Q}_h\to\mathbb{Z}^{\mathbb{Q}}_h$ by

$$\begin{aligned}\xi^{\mathbb{Q}}_{\mu(t)}(f(t))&=\chi_{\mu(t)}(f(t))+\mathbb{A}_{\mu(t)}(f(t))\\&=\begin{cases}\dfrac{\ln|1+\mu(t)f(t)|}{\mu(t)}+\dfrac{1}{\mu(t)}\cdot\arg^{\mathbb{Q}}(1+\mu(t)f(t)) & \text{for } \ \mu(t)>0,\\ f_0(t)+\Theta(t) & \text{for } \ \mu(t)=0,\end{cases}\end{aligned}$$

we define the quaternion hyper-complex exponential function by

$$e^{\mathbb{Q}}_f(t,s):=e^{\int\limits_s^t \xi^{\mathbb{Q}}_{\mu(\tau)}(f(\tau))\,\Delta\tau}=e^{\int\limits_s^t \chi_{\mu(\tau)}(f(\tau))\,\Delta\tau+\int\limits_s^t \mathbb{A}_{\mu(\tau)}(f(\tau))\,\Delta\tau}.$$

The following result is valid.

Theorem 3.1 ([4])**.** *Let $s,r,t\in\mathbb{T}$, $f:\mathbb{T}\to\mathbb{Q}$, $1+\mu(t)f(t)\neq 0$ for any $t\in\mathbb{T}^\kappa$. Then*

(i) $e^{\mathbb{Q}}_f(s,s)=1$;

(ii) $e^{\mathbb{Q}}_f(t,r)e^{\mathbb{Q}}_f(r,s)=e^{\mathbb{Q}}_f(t,s)$;

(iii) $(e^{\mathbb{Q}}_f(t,s))^\Delta=f(t)e^{\mathbb{Q}}_f(t,s)$;

(iv) $(e^{\mathbb{Q}}_f(s,t))^\Delta=e^{\mathbb{Q}}_f(s,t)(1+\mu(t)f(t))^{-1}[-f(t)]$ *if t is a right scattered point on $\mathbb{T}$.*

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