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On Initial-Periodic Type Problems for Three-Dimensional Linear Hyperbolic System

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In the rectangular box $\Omega = [0,\omega_1]\times[0,\omega_2]\times[0,\omega_3]$ for the linear hyperbolic system

$$u^{(\mathbf{1})} = \sum_{\boldsymbol{\alpha}<\mathbf{1}} P_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{1}$$

consider the initial-periodic conditions

$$\begin{gathered} u(0,x_2,x_3)=\varphi_1(x_2,x_3), \quad u^{(1,0,0)}(x_1,0,x_3)=\varphi_2^{(1,0)}(x_1,x_3) \\ u(x_1,x_2,x_3+\omega_3)=u(x_1,x_2,x_3) \end{gathered} \tag{2}$$

and

$$\begin{gathered} u(0,x_2,x_3)=\varphi(x_2,x_3), \\ u(x_1,x_2+\omega_2,x_3)=u(x_1,x_2,x_3), \quad u(x_1,x_2,x_3+\omega_3)=u(x_1,x_2,x_3) \end{gathered} \tag{3}$$

Here $\mathbf{x}=(x_1,x_2,x_3)$, $\mathbf{1}=(1,1,1)$ and $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$ are multi-indices,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{\alpha_1+\alpha_2+\alpha_3}u(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\,,$$

$P_{\boldsymbol{\alpha}}\in C(\Omega;\mathbb{R}^{n\times n})$ $(\boldsymbol{\alpha}<\mathbf{1})$, $q\in C(\Omega;\mathbb{R}^n)$, $\varphi_1\in C^{1,1}(\Omega_{23})$, $\varphi_2\in C^{1,1}(\Omega_{13})$, $\Omega_{23}=[0,\omega_2]\times[0,\omega_3]$ and $\Omega_{13}=[0,\omega_1]\times[0,\omega_3]$.

Throughout the paper the following g notations will be used:

$\mathbf{0}=(0,0,0)$, $\mathbf{1}=(1,1,1)$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)<\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\alpha_i\le\beta_i$ $(i=1,2,3)$ and $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)\le\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.

$\|\boldsymbol{\alpha}\|=|\alpha_1|+|\alpha_2|+|\alpha_3|$.

Let $\mathbf{m}=(m_1,m_2,m_3)$ be a multi-index. By $C^{\mathbf{m}}(\Omega;\mathbb{R}^n)$ denote the Banach space of vector functions $u:\Omega\to\mathbb{R}^n$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}\le\mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

By a solution of problem (1), (2) (problem (1), (3)) we understand a classical solution, i.e., a vector-function $u\in C^{\mathbf{1}}(\Omega;\mathbb{R}^n)$ satisfying system (1) and boundary conditions (2) (system (1) and boundary conditions (3)) everywhere in Ω.

Along with system (1) consider its corresponding homogeneous system

$$u^{(\mathbf{1})} = \sum_{\boldsymbol{\alpha}<\mathbf{1}} P_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}, \tag{1_0}$$

and the following boundary value problems

$$\begin{aligned} v^{(0,0,1)} &= P_{110}(x_1,x_2,x_3)v, \\ v(x_1,x_2,x_3+\omega_3) &= v(x_1,x_2,x_3), \end{aligned} \tag{4}$$

$$\begin{aligned} v^{(0,1,0)} &= P_{101}(x_1,x_2,x_3)v, \\ v(x_1,x_2+\omega_2,x_3) &= v(x_1,x_2,x_3), \end{aligned} \tag{5}$$

and

$$\begin{aligned} v^{(0,1,1)} &= P_{110}(x_1,x_2,x_3)v^{(0,1,0)} + P_{101}(x_1,x_2,x_3)v^{(0,0,1)} + P_{100}v, \\ v(x_1,x_2+\omega_2,x_3) &= v(x_1,x_2,x_3), \quad v(x_1,x_2,x_3+\omega_3) = v(x_1,x_2,x_3). \end{aligned} \tag{6}$$

Problem (4) is called an σ-*associated problem* of problem (1), (2).

Problems (4), (5) and (6) are called σ-*associated problems* of problem (1), (3).

Notice that:

Problem (4) is a one-dimensional periodic problem with respect to x_3 variable, depending on two parameters x_1 and x_2;

Problem (5) is a one-dimensional periodic problem with respect to x_2 variable, depending on two parameters x_1 and x_3;

Problem (6) is a two-dimensional periodic problem with respect to x_2 and x_3 variables, depending the parameter x_1.

Theorem 1. *Let problem (4) have only the trivial solution for every $(x_1,x_2) \in [0,\omega_1]\times[0,\omega_2]$. Then problem (1), (2) has a unique solution u admitting the estimate*

$$\|u\|_{C^1(\Omega)} \le M\Big(\|\varphi_1\|_{C^{1,1}(\Omega_{23})} + \|\varphi_2\|_{C^{1,1}(\Omega_{13})} + \|q\|_{C(\Omega)}\Big), \tag{7}$$

where M is a positive number independent of φ_1, φ_2 and q.

Definition 1. Problem (1), (2) is called well-posed, if for every $\varphi_1 \in C^{1,1}(\Omega_{23};\mathbb{R}^n)$, $\varphi_2 \in C^{1,1}(\Omega_{13};\mathbb{R}^n)$ and $q \in C(\Omega;\mathbb{R}^n)$, it is uniquely solvable and its solution admits estimate (7), where M is a positive number independent of φ_1, φ_2 and q.

Theorem 2. *Let problem (1), (2) be well-posed. Then problem (4) has only the trivial solution for every $(x_1,x_2) \in [0,\omega_1]\times[0,\omega_2]$.*

Corollary 1. *Let $P_{110}(x_1,x_2,x_3) = P_{110}(x_1,x_2)$. Then problem (1), (2) is well-posed if and only if*

$$\det\big(I - \exp(\omega_3 P_{110}(x_1x_2))\big) \neq 0 \ \ \textit{for} \ \ (x_1,x_2) \in \Omega_{12}.$$

Corollary 2. *Let*

$$\widehat{P}_{110}(x_1,x_2,x_3) = \frac{1}{2}\big(P_{110}(x_1,x_2,x_3) + P^T_{110}(x_1,x_2,x_3)\big),$$

and let there exist $\sigma \in \{-1,1\}$ $(i=1,2)$ such that

$$\sigma\int_0^{\omega_3} \widehat{P}_{110}(x_1,x_2,s)\,ds \ \ \textit{is positive definite for} \ \ (x_1,x_2) \in \Omega_{12}.$$

Then problem (1), (2) is well-posed.

Consider the system

$$u^{(\mathbf{1})} = P(\mathbf{x})u + q(\mathbf{x}). \tag{8}$$

By Theorem 2, problem $(8),(2)$ is *ill-posed*, since its σ-associated problem

$$v^{(0,0,1)} = 0, \quad v(x_1, x_2, x_3 + \omega_3) = v(x_1, x_2, x_3)$$

has a nontrivial solution $v(x_3) \equiv 1$ for *every* $(x_1, x_2) \in [0, \omega_1] \times [0, \omega_2]$. Being ill-posed, problem $(8),(2)$ still can be uniquely solvable.

Theorem 3. *Let $P \in C^{1,1,0}(\Omega; \mathbb{R}^{n\times n})$, $q \in C^{1,1,0}(\Omega; \mathbb{R}^n)$, $\varphi_1 \in C^{2,1}(\Omega_{23})$, $\varphi_2 \in C^{2,1}(\Omega_{13})$, and let*

$$\det\bigg(\int\limits_0^{\omega_3} P(x_1, x_2, s)\,ds\bigg) \neq 0 \ \textit{ for } \ (x_1, x_2) \in [0, \omega_1] \times [0, \omega_2].$$

Then problem $(8),(2)$ has a unique solution u admitting the estimate

$$\|u\|_{C^{\mathbf{1}}(\Omega)} \leq M\Big(\|\varphi_1\|_{C^{2,1}(\Omega_{23})} + \|\varphi_2\|_{C^{2,1}(\Omega_{13})} + \|q\|_{C^{1,1,0}(\Omega)}\Big),$$

where M is a positive number independent of φ_1, φ_2 and q, if and only if

$$\int\limits_0^{\omega_3} \big(P(0, x_2, s)\varphi_1(x_2, s) + q(0, x_2, s)\big)\,ds = 0 \ \textit{ for } \ x_2 \in [0, \omega_2]$$

and

$$\int\limits_0^{\omega_3} \big(P(x_1, 0, s)\varphi_2(x_1, s) + q(x_1, 0, s)\big)\,ds = 0 \ \textit{ for } \ x_1 \in [0, \omega_1].$$

Theorem 4. *Let the following conditions hold:*

(F_1) *Problem (4) has only the trivial solution for every $(x_1, x_2) \in \Omega_{12}$;*

(F_2) *Problem (5) has only the trivial solution for every $(x_1, x_3) \in \Omega_{13}$;*

(F_3) *Problem (6) has only the trivial solution for every $x_1 \in [0, \omega_1]$.*

Then problem $(1),(3)$ has a unique solution u admitting the estimate

$$\|u\|_{C^{\mathbf{1}}(\Omega)} \leq M\big(\|\varphi\|_{C^{1,1}(\Omega_{23})} + \|q\|_{C(\Omega)}\big), \tag{9}$$

where M is a positive number independent of φ and q.

Definition 2. Problem $(1),(3)$ is called well-posed, if for every $\varphi \in C^{1,1}(\Omega_{23}; \mathbb{R}^n)$ and $q \in C(\Omega; \mathbb{R}^n)$, it is uniquely solvable and its solution admits estimate (9), where M is a positive number independent of φ and q.

Theorem 5. *Let problem $(1),(3)$ be well-posed. Then conditions (F_1), (F_2) and (F_3) hold.*

Corollary 3. *Let*

$$P_{110}(x_1,x_2,x_3) \equiv P_{110}(x_1),$$
$$P_{101}(x_1,x_2,x_3) \equiv P_{101}(x_1),$$
$$P_{100}(x_1,x_2,x_3) \equiv P_{100}(x_1),$$

and let

$$\det\big(I - \exp(\omega_3 P_{110}(x_1))\big) \neq 0 \ \text{ for } \ x_1 \in [0,\omega_1],$$
$$\det\big(I - \exp(\omega_2 P_{101}(x_1))\big) \neq 0 \ \text{ for } \ x_1 \in [0,\omega_1].$$

Then problem $(1),(3)$ *is well-posed* ***if and only if***

$$\det\Big(P_{100}(x_1) + i\,\frac{2\pi}{\omega_3}\,mP_{110}(x_1) + i\,\frac{2\pi}{\omega_2}\,kP_{101}(x_1) + mk\,I\Big) \neq 0 \quad \text{for} \quad x_1 \in [0,\omega_1], \quad m,k \in \mathbb{Z}.$$

Consider the equation

$$u^{(\mathbf{1})} = \sum_{\boldsymbol{\alpha}<\mathbf{1}} p_{\boldsymbol{\alpha}}(x_1,x_2)u^{(\boldsymbol{\alpha})} + q(\mathbf{x}). \tag{10}$$

Corollary 4. *Let*

$$p_{100}(x_1,x_2)\,p_{110}(x_1,x_2)\,p_{101}(x_1,x_2) < 0 \ \text{ for } \ (x_1,x_2) \in \Omega_{12}.$$

Then problem $(10),(2)$ *is well-posed.*

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On Initial-Boundary Value Problems for Quasilinear Hyperbolic Systems of Second Order

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In the rectangle $\Omega=[0,a]\times[0,b]$ consider the nonlinear hyperbolic system

$$u_{xy}=f(x,y,u_x,u_y,u), \tag{1}$$

$$u(0,y)=\varphi(y), \quad h(u_x(x,\,\cdot\,))(x)=\psi'(x), \tag{2}$$

where $f:\Omega\times\mathbb{R}^{3n}\to\mathbb{R}^n$ is a continuous vector function that is continuously differentiable with respect to the first $2n$ phase variables, $\varphi\in C^1([0,b];\mathbb{R}^n)$, $\psi\in C^1([0,a];\mathbb{R}^n)$, and $h:C([0,b];\mathbb{R}^n)\to C([0,a];\mathbb{R}^n)$ is a bounded linear operator.

Let $v=(v_1,\dots,v_n)$, $w=(w_1,\dots,w_n)$ and $z=(z_1,\dots,z_n)$. For a function $f(x,y,v,w,u)$ that is continuously differentiable with respect to v, w and u, set:

$$F_1(x,y,v,w,z)=\frac{\partial f(x,y,v,w,z)}{\partial v}\,, \quad F_2(x,y,v,w,z)=\frac{\partial f(x,y,v,w,z)}{\partial w}\,,$$
$$F_0(x,y,v,w,z)=\frac{\partial f(x,y,v,w,z)}{\partial z}\,,$$
$$P_j[u](x,y)=F_j\big(x,y,u_x(x,y),u_y(x,y),u(x,y)\big) \ \ (j=0,1,2).$$

$C^{1,1}(\Omega;\mathbb{R}^n)$ is the Banach space of continuous vector functions $u:\Omega\to\mathbb{R}^n$, having continuous partial derivatives u_x, u_y, u_{xy}, endowed with the norm

$$\|u\|_{C^{1,1}}=\|u\|_C+\|u_x\|_C+\|u_y\|_C+\|u_{xy}\|_C.$$

$C^1(\Omega;\mathbb{R}^n)$ is the Banach space of continuous vector functions $u:\Omega\to\mathbb{R}^n$, having continuous partial derivatives u_x, u_y, endowed with the norm

$$\|u\|_{C^{1,1}}=\|u\|_C+\|u_x\|_C+\|u_y\|_C.$$

If $u_0\in C(\Omega:\mathbb{R}^n)$ and $r>0$, then

$$\mathbf{B}(u_0;r)=\big\{u\in C(\Omega:\mathbb{R}^n): \ \ \|u-u_0\|\le r\big\}.$$

If $u_0\in C^1(\Omega:\mathbb{R}^n)$ and $r>0$, then

$$\mathbf{B}^1(u_0;r)=\big\{u\in C^1(\Omega:\mathbb{R}^n): \ \ \|u-u_0\|_{C^1}\le r\big\}.$$

Definition 1. Let u_0 be a solution of problem (1), (2), and $r>0$. Problem (1), (2) is said to be (u_0,r)-*well-posed* if:

(i) $u_0(x,y)$ is the unique solution of the problem in the ball $\widetilde{\mathcal{B}}^1(u_0;r)$;

(ii) There exists $\varepsilon_0 > 0$ such that for an arbitrary $\varepsilon > 0$ and $M > 0$ there exists $\delta > 0$ such that for any $\widetilde{f}(x, y, v, w, z)$ that is continuously differentiable with respect to v and w, $\widetilde{\varphi} \in C^1([0, b]; \mathbb{R}^n)$, $\widetilde{\psi} \in C^1([0, a]; \mathbb{R}^n)$, satisfying the inequalities

$$\begin{aligned} \Big\| \frac{\partial \widetilde{f}(x, y, v, w, z)}{\partial v} \Big\| &\le \varepsilon_0 \ \text{ for } \ (x, y, v, w, z) \in \Omega \times \mathbb{R}^{3n} \\ \Big\| \frac{\partial \widetilde{f}(x, y, v, w, z)}{\partial w} \Big\| &\le M \ \text{ for } \ (x, y, v, w, z) \in \Omega \times \mathbb{R}^{3n}, \end{aligned} \tag{3}$$

$$\|\widetilde{f}(x, y, v, w, z)\| \le \delta \ \text{ for } \ (x, y, v, w, z) \in \Omega \times \mathbb{R}^{3n}, \quad \|\widetilde{\varphi}\|_{C^1([0,b])} + \|\widetilde{\psi}\|_{C^1([0,a])} \le \delta, \tag{4}$$

the problem

$$u_{xy} = f(x, y, u_x, u_y, u) + \widetilde{f}(x, y, u_x, u_y, u), \tag{$\widetilde{1}$}$$

$$u(0, y) = \varphi(y) + \widetilde{\varphi}(y), \quad h\big(u_x(x, \cdot)\big)(x) = \psi'(x) + \widetilde{\psi}'(x), \tag{$\widetilde{2}$}$$

has at least one solution in the ball $\mathbf{B}^1(u_0; r)$, and each such solution belongs to the ball $\mathbf{B}^1(u_0; \varepsilon)$.

Definition 2. Let u_0 be a solution of problem $(1), (2)$, and $r > 0$. Problem $(1), (2)$ is said to be *strongly* (u_0, r)*-well-posed* if:

(i) Problem $(1), (2)$ is (u_0, r)-well-posed;

(ii) There exist positive numbers M_0 and δ_0 such that for arbitrary $\delta \in (0, \delta_0)$, $\widetilde{f}(x, y, v, w, z)$ that is continuously differentiable with respect to v and w, $\widetilde{\varphi} \in C^1([0, b]; \mathbb{R}^n)$ and $\widetilde{\psi} \in C^1([0, a]; \mathbb{R}^n)$, satisfying the inequalities (3) and (4), problem $(\widetilde{1})$, $(\widetilde{2})$ has at least one solution in the ball $\mathbf{B}^1(u_0; r)$, and each such solution belongs to the ball $\mathbf{B}^1(u_0; M_0\, \delta)$.

Definition 3. Problem $(1), (2)$ is called *well-posed* (*strongly well-posed*) if it has a unique solution u_0 and it is (u_0, r)-well-posed (strongly (u_0, r)-well-posed) for every $r > 0$.

Consider the boundary value problem for the system of nonlinear ordinary differential equations

$$z' = p(t, z), \quad \ell(z) = c, \tag{5}$$

where $p \in C([0, b] \times \mathbb{R}^n; \mathbb{R}^n)$, $c \in \mathbb{R}^n$ and $\ell : C([0, b]; \mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear operator.

Definition 4. Let z_0 be a solution of problem (5), and $r > 0$. Problem (5) is said to be (z_0, r)-*well-posed* if:

(i) $z_0(t)$ is the unique solution of the problem in the ball $\mathbf{B}(z_0; r)$;

(ii) For an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\widetilde{c}$, and $\widetilde{p} \in C([0, b] \times \mathbb{R}^n)$ satisfying the inequalities

$$\big\|c - \widetilde{c}\big\| < \delta, \quad \|p - \widetilde{p}\|_C < \delta \tag{6}$$

the problem

$$z' = \widetilde{p}(t, z), \quad \ell(z) = \widetilde{c}, \tag{$\widetilde{5}$}$$

has at least one solution in the ball $\mathbf{B}(z_0; r)$, and each such solution belongs to the ball $\mathbf{B}(z_0; \varepsilon)$.

Definition 4 is a slight modification of Definition 3.2 from [1]. Definition 1 is an adaptation of the idea of Definition 4 to problem $(1), (2)$.

Definition 5. Let u_0 be a solution of problem (5), and $r > 0$. Problem (5) is said to be *strongly* (z_0, r)*-well-posed* if:

(i) $z_0(t)$ is the unique solution of the problem in the ball $\mathbf{B}(z_0; r)$;

(ii) There exist positive numbers M and δ_0 such that for arbitrary $\delta \in (0, \delta_0)$, $\widetilde{c}_k$, and $\widetilde{p} \in C([0, b] \times \mathbb{R}^n)$ satisfying inequalities (6), problem $(\widetilde{5})$ has at least one solution in the ball $\mathbf{B}(z_0; r)$, and each such solution belongs to the ball $\mathbf{B}(z_0; M\,\delta)$.

Remark 1. It is obvious that strong well-posedness implies well-posedness. The converse, however, is not true. As an example, consider the problem

$$z' = z^3, \quad z(0) = z(\omega), \tag{7}$$

which is well-posed and has the unique solution $z_0(t) \equiv 0$. The perturbed problem

$$z' = z^3 - \delta, \quad z(0) = z(b)$$

has the unique solution $z_\delta(t) = \delta^{\frac{1}{3}}$. It is clear that there exists no positive number M such that $\delta^{\frac{1}{3}} \le M\delta$ as $\delta \to 0$. Consequently, problem (7) is not strongly well-posed.

Definition 6. A solution z_0 of problem (5) is said to be strongly isolated, if problem (5) is strongly (z_0, r)-well-posed for some $r > 0$.

Remark 2. The concept of a strongly isolated solution of a nonlinear boundary value problem was introduced in [1]. However, our definition of a strongly isolated solution is a modification of Definition 3.1 from [1]. Also, Corollary 3.6 from [1] implies that if the vector function $p(t, z)$ is continuously differentiable with respect to the phase variables, then strong isolation of a solution z_0 is equivalent to the fact that the linear homogeneous problem

$$z' = P(t)z, \quad \ell(z) = 0, \tag{8}$$

has only the trivial solution, where $P(t) = \frac{\partial p}{\partial z}(t, z_0(t))$.

Theorem 1. *Let f be a continuously differentiable function with respect to the phase variables v, w and z, and let u_0 be a solution of problem (1), (2). Then, problem (1), (2) is strongly (u_0, r)-well-posed for some $r > 0$, if and only if the linear homogeneous problem*

$$u_{xy} = P_0(x, y)u + P_1(x, y)u_x + P_2(x, y)u_y, \tag{1_0}$$
$$u(0, y) = 0, \quad h(u_x(x, \cdot))(x) = 0, \tag{2_0}$$

where $P_j(x, y) = P_j[u_0](x, y)$ $(j = 0, 1, 2)$, is well-posed.

Theorem 2. *Problem $(1_0), (2_0)$ is well-posed if and only if the linear homogeneous problem*

$$\frac{dz}{dy} = P_1(x, y)z, \quad h(z)(x) = 0$$

has only the trivial solution for every $x \in [0, a]$.

Remark 3. The sufficiency part of Theorem 2 was proved in [2] (see Theorems 4.1 and 4.1′). Similar theorem for higher order linear hyperbolic equations for proved in [4] (see Theorem 1.1).

Theorem 3. *Let f be a continuously differentiable with respect to the phase variables v, w and z, and let there exist matrix functions $Q_i \in C(\Omega; \mathbb{R}^{n \times n})$ $(i = 1, 2)$ and a positive constant ρ such that:*

(A_1) $\|F_0(x,y,v,w,z)\| + \|F_2(x,y,v,w,z)\| \le \rho$ *for* $(x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n}$*;*

(A_2) $Q_1(x,y) \le F_1(x,y,v,w,z) \le Q_2(x,y)$ *for* $(x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n}$*;*

(A_3) *for every* $x \in [0,a]$ *and arbitrary measurable matrix function* $P : [0,b] \to \mathbb{R}^{n\times n}$ *satisfying the inequalities*

$$Q_1(x,y) \le P(y) \le Q_2(x,y) \ \ \textit{for} \ \ y \in [0,b],$$

problem (8) *has only the trivial solution. Then problem* (1), (2) *is strongly well-posed.*

Theorem 4. *Let* f *be a continuously differentiable function with respect to the phase variables* v*,* w *and* z*, and let* v_0 *be a strongly isolated solution of the problem*

$$v' = p(y,v), \quad h(v)(0) = \psi'(0), \tag{9}$$

where

$$p(y,v) = f\big(0,y,v,\varphi'(y),\varphi(y)\big).$$

Then there exists $\alpha \in (0,a]$ *such that in the rectangle* $\Omega_\alpha = [0,\alpha] \times [0,b]$ *problem* (1), (2) *has a unique solution* u *satisfying the condition*

$$u_x(0,y) = v_0(y) \ \ \textit{for} \ \ y \in [0,b].$$

Remark 4. Conditions of Theorem 4 do not guarantee unique solvability of problem (1), (2). Indeed, consider the problem

$$u_{xy} = \prod_{k=1}^{m} \big(u_x - k\big) + x\, f_0(x,y,u_x,u_y,u), \tag{10}$$

$$u(0,y) = 0, \ \ u^{(1,0)}(x,0) = u^{(1,0)}(x,b), \tag{11}$$

where $f_0 : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function. For this case problem (9) has the form

$$v' = \prod_{k=1}^{m} (v-k), \quad v(0) = v(b).$$

The latter problem has exactly m strongly isolated solutions $v_k = k\pi$ $(k = 1,\dots,m)$. By Theorem 4, for every integer $k \in \{1,\dots,m\}$ there exists $\alpha_k > 0$ such that in $\Omega_{\alpha_k} = [0,\alpha_k] \times [0,b]$, problem (10), (11) has a unique solution u_k satisfying the condition

$$u_k^{(1,0)}(0,y) = k \ \ \text{for} \ \ y \in [0,b].$$

Consider the family of problems

$$z' = p_\lambda(t,z), \quad \ell_\lambda(z) = c_\lambda, \tag{12_λ}$$

where $\lambda \in \Lambda$, $p_\lambda \in C([0,b] \times \mathbb{R}^n; \mathbb{R}^n)$, $\ell_\lambda : C([0,b]) \to \mathbb{R}^n$ are bounded linear functionals, and $c_\lambda \in \mathbb{R}^n$.

Let for $\lambda \in \Lambda$ and $r > 0$, z_λ be a solution of problem (12_λ). The family of problems (12_λ) $(\lambda \in \Lambda)$ is said to be *uniformly strongly* (z_λ, r)*-well-posed*, if:

(i) z_λ is unique in the ball $\mathbf{B}(z_\lambda; r)$;

(ii) There exist positive numbers M and δ_0 *independent of* λ such that for arbitrary $\delta \in (0, \delta_0)$, $\widetilde{c} \in \mathbb{R}^n$, and $\widetilde{p}_\lambda \in C([0,b] \times \mathbb{R}^n; \mathbb{R}^n)$ satisfying the inequalities

$$\|c - \widetilde{c}\| < \delta, \quad \|p_\lambda - \widetilde{p}_\lambda\|_C < \delta,$$

the problem

$$z' = \widetilde{p}_\lambda(t, z), \quad \ell_\lambda(z) = \widetilde{c}_\lambda, \tag{$\widetilde{12}\lambda$}$$

has at least one solution in the ball $\mathbf{B}(z_\lambda; r)$, and each such solution belongs to the ball $\mathbf{B}(z_\lambda; M\delta)$.

A family of solutions $\{z_\lambda\}_{\lambda \in \Lambda}$ is said to be *uniformly strongly isolated* if the family of problems (12_λ) $(\lambda \in \Lambda)$ is *uniformly strongly* (z_λ, r)*-well-posed* for some $r > 0$.

Let $J = [0, \alpha)$, $\alpha \in (0, a]$, ($J = [0, \alpha]$, $\alpha \in (0, a)$), and u be a solution of problem (1), (2) in the rectangle $J \times [0, b]$. u is called *continuable*, if there exists $\alpha_1 \in [\alpha, a]$ ($\alpha_1 \in (\alpha, a]$) and a solution u_1 of problem (1), (2) in $[0, \alpha_1] \times [0, b]$ such that

$$u_1(x, y) = u(x, y) \ \text{ for } \ (x, y) \in [0, \alpha) \times [0, b].$$

Otherwise u is called *non-continuable.*

Theorem 5. *Let u be a a non-continuable solution of problem* (1), (2) *defined on $J \times [0, b]$, and let for every $x_0 \in J$, $v(y) = u_x(x_0, y)$ be a solution of the problem*

$$v' = p[u](x_0, y, v), \quad h(v)(x_0) = \psi(x_0). \tag{13}$$

If the family of solutions $v(y) = u_x(x_0, y)$ $(x_0 \in J)$ is uniformly strongly isolated, then either $J = [0, a]$, or $J = [0, \alpha)$ and

$$\lim_{x \to \alpha} \Big(\|u_x(x, \cdot\,)\|_{C([0,b])} + \|u(x, \cdot\,)\|_{C([0,b])} + \|u_y(x, \cdot\,)\|_{C([0,b])} \Big) = +\infty. \tag{14}$$

Definition 7. Let u be a non-continuable solution of problem (1), (2) in $J \times [0, b]$ and let $\alpha = \sup J$. We say that a measurable matrix function $P : [0, b] \to \mathbb{R}^{n \times n}$ belongs to the set $S_f^\alpha[u]$, if there exists an increasing sequence $x_k \uparrow \alpha$ as $k \to \infty$ such that

$$\lim_{k \to \infty} \int\limits_0^y P_1[u](x_k, t)\, dt = \int\limits_0^y P(t)\, dt$$

uniformly on $[0, b]$.

Corollary. *Let u be a non-continuable solution of problem* (1), (2) *in $J \times [0, b]$, and let $\alpha = \sup J$. If for an arbitrary $P \in S_f^\alpha[u]$ the homogeneous problem*

$$z' = P(t) z, \quad h(z)(\alpha) = 0$$

has only the trivial solution, then either $J = [0, a]$, or $J = [0, \alpha)$ and (14) *holds.*

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On the Criterion of Well-Posedness of the Cauchy Problem with Weight for Systems of Linear Ordinary Differential Equations with Singularities

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Let $I=[a,b]\subset\mathbb{R}$ be a finite and closed interval non-degenerate in the point, $t_0\in\,]a,b[$ and $I_{t_0}=[a,b]\setminus\{t_0\}$, $I_{t_0}^-=[a,t_0[\,$, $I_{t_0}^+=\,]t_0,b]$.

Consider the Cauchy problem with weight for linear system of ordinary differential equations with singularities

$$\frac{dx}{dt}=P(t)\,x+q(t)\ \text{ for a.a. } t\in I_{t_0}, \tag{1}$$

$$\lim_{t\to t_0}(\Phi^{-1}(t)\,x(t))=0, \tag{2}$$

where $P\in L_{loc}(I_{t_0},\mathbb{R}^{n\times n})$, $q\in L_{loc}(I_{t_0},\mathbb{R}^n)$; $\Phi=\operatorname{diag}(\varphi_1,\dots,\varphi_n)$ is a positive diagonal $n\times n$-matrix valued function defined, continuous on $[a,b]\setminus\{t_0\}$ and having an inverse $\Phi^{-1}(t)$ for $t\in[a,b]\setminus\{t_0\}$.

Along with system (1) consider the sequence of singular systems

$$\frac{dx}{dt}=P_m(t)\,x+q_m(t) \tag{3}$$

$(m=1,2,\dots)$ under condition (2), where P_m and q_m are, as above, a matrix- and vector-functions, respectively.

We discuss the question whether the unique solvability of problem (1), (2) guarantees the unique solvability of problem (3), (2) for each sufficiently large m and nearness of its solutions to the solution of problem (1), (2) in the definite sense if matrix-functions P and P_m and vector-functions q and q_m are nearly among themselves.

The singularity of system (1) is considered in the sense that the matrix P and vector q functions, in general, are not integrable at the point t_0. In general, the solution of problem (1), (2) is not continuous at the point t_0 and, therefore, it can not be a solution in the classical sense. But its restriction on every interval from I_{t_0} is a solution of system (1). In connection with this we remind the following example from [6, 7].

Let $\alpha>0$ and $\varepsilon\in\,]0,\alpha[\,$. Then $x(t)=|t|^{\varepsilon-\alpha}\operatorname{sgn}t$ is the unique solution of the problem

$$\frac{dx}{dt}=-\frac{\alpha\,x}{t}+\varepsilon|t|^{\varepsilon-1-\alpha}\,,\quad \lim_{t\to0}(t^\alpha x(t))=0.$$

The function x is not a solution of the equation on the set $I=\mathbb{R}$, however x is a solution to the above equation only on $\mathbb{R}\setminus\{0\}$.

First, the same problem for the differential system (3) have been investigated by I. Kiguradze (see, [6,7]), where only the sufficient conditions are obtained. As to sufficient conditions of well-possedness for functional-differential case they are obtained in [8] (see also the references therein).

The necessary and sufficient conditions of well-posedness of problem (1), (2) has been obtained in [1,2] for the regular case.

To our knowledge, the question on necessary and sufficient conditions of well-posedness of problem (1), (2) with singularity has not been considered up to now. So the problem is actual.

As to the existence of solutions and related problems for system (1), first, they are investigated by V. A. Chechik in the monograph [5]. Similar problems for impulsive differential and so called generalized ordinary differential systems are investigated in [3,4] and for functional-differential case in [8] (see also the references therein).

We present necessary and sufficient conditions, as well effective sufficient conditions for so called Φ-well-posedness of problem (1), (2).

Throughout the paper we use the following notation and definitions.

$\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$; $[a,b]$ and $]a,b[$ $(a,b\in\mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of real $n\times m$ matrices X with the standard norm $\|X\|$.

If $X=(x_{ik})_{i,k=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{ik}|)_{i,k=1}^{n,m}$.

$[X]_\pm=\dfrac{1}{2}\,(|X|\pm X)$, $\mathbb{R}_+^{n\times m}=\big\{(x_{ik})_{i,k=1}^{n,m}:\ \ x_{ik}\geq 0\ \ (i=1,\dots,n,\ \ k=1,\dots,m)\big\}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

$O_{n\times m}$ (or O) is the zero $n\times m$-matrix, 0_n (or 0) is the zero n-vector.

I_n is identity $n\times n$-matrix.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X.

The inequalities between the matrices are understood componentwisely.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

$\mathrm{AC}([a,b];D)$, where $D\subset\mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$.

$\mathrm{AC}_{loc}(J;D)$, where $J\subset\mathbb{R}$, is the set of all matrix-functions $X:J\to D$ whose restrictions to an arbitrary closed interval $[a,b]\subset J$ belong to $\mathrm{AC}([a,b];D)$.

$L_{loc}(I_{t_0};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:I_{t_0}\to D$ whose restrictions on every closed interval $[a,b]$ from I_{t_0} belong to $L([a,b];\mathbb{R}^{n\times m})$.

Under a solution of problem (1), (2) we understand a vector-function $x\in\mathrm{AC}(I_{t_0};\mathbb{R}^n)$ satisfying the equality $x'(t)=P(t)\,x(t)+q(t)$ for a.a. $t\in I_{t_0}$ and condition (2).

Let $P_*=(p_{*ik})_{i,k=1}^n\in L_{loc}(I_{t_0};\mathbb{R}^{n\times n})$. A matrix-function $C_*:I_{t_0}\times I_{t_0}\to\mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous system

$$\frac{dx}{dt}=P_*(t)x, \tag{4}$$

if, for each interval $J\subset I$ and $\tau\in J$, the restriction of the matrix-function $C_*(\,\cdot\,,\tau):I_{t_0}\to\mathbb{R}^{n\times n}$ on J is the fundamental matrix of system (4), satisfying the condition $C_*(\tau,\tau)=I_n$. Therefore, C_* is the Cauchy matrix of system (4) if and only if the restriction of C_* on $J\times J$ is the Cauchy matrix of the system in the regular case.

Definition 1. Problem $(1),(2)$ is said to be Φ-well-posed with respect to the matrix-function P_* if it has a unique solution x_0 and for every sequences of matrix P_m and vector q_m $(m=1,2,\dots)$ functions such that

$$\lim_{t\to t_0}\bigg(\Phi^{-1}(t)\int\limits_{t_0}^{t}(q_m(s)-q(s))\,ds\bigg)=0_n \tag{5}$$

for each sufficiently large m, and conditions

$$\lim_{m\to+\infty}\bigg\|\int\limits_{t_0}^{t}\Phi^{-1}(s)\,|P_m(s)-P(s)|\,\Phi(s)\,ds\bigg\|=0 \ \text{ uniformly on } I \tag{6}$$

and

$$\lim_{m\to+\infty}\bigg(\bigg\|\Phi^{-1}(t)\int\limits_{t_0}^{t}(q_m(s)-q(s))\,ds\bigg\|+\bigg\|\int\limits_{t_0}^{t}\Phi^{-1}(s)\,\bigg|P_*(s)\int\limits_{t_0}^{s}(q_m(\tau)-q(\tau))\,d\tau\bigg|\,ds\bigg\|\bigg)=0 \quad \text{uniformly on } I \tag{7}$$

hold, problem $(3),(2)$ has the unique solution x_m for each sufficiently large m and the condition

$$\lim_{m\to+\infty}\big\|\Phi^{-1}(t)\,(x_m(t)-x_0(t))\big\|=0 \ \text{ uniformly on } I \tag{8}$$

hold.

The introduced definition is more general than the one given in [6,7].

Definition 2. We say that the sequence (P_m,q_m) $(m=1,2,\dots)$ belongs to the set $\mathcal{S}_{P_*}(P,q;\Phi,t_0)$ if problem $(3),(2)$ has a unique solution x_m for each sufficiently large m and condition (8) holds.

Theorem 1. *Let there exist a matrix-function $P_*\in L_{loc}(I_{t_0};\mathbb{R}^{n\times n})$ and constant matrices $B_0,B\in\mathbb{R}_+^{n\times n}$ such that*

$$r(B)<1, \tag{9}$$

and the estimates

$$|C_*(t,\tau)|\le\Phi(t)B_0\Phi^{-1}(\tau) \ \textit{ for } \ t\in I_{t_0}(\delta), \ \ (t-t_0)(\tau-t_0)>0, \ \ |\tau-t_0|\le|t-t_0| \tag{10}$$

and

$$\bigg|\int\limits_{t_0}^{t}\big|C_*(t,s)(P(s)-P_*(s))\big|\,\Phi(s)\,ds\bigg|\le\Phi(t)B \ \textit{ for } \ t\in I_{t_0}(\delta) \tag{11}$$

are fulfilled for some $\delta>0$, where C_* is the Cauchy matrix of system (4). Let, moreover,

$$\lim_{t\to t_0}\bigg\|\int\limits_{t_0}^{t}\Phi^{-1}(t)\,C_*(t,s)q(s)\,ds\bigg\|=0. \tag{12}$$

Then problem $(1),(2)$ is Φ-well-posed with respect to P_*.

Remark 1. Under the conditions of Theorem 1 problem $(1),(2)$ is uniquely solvable (see, [6,7]). In addition, condition (9) is essential and it cannot be neglected by $r(B)\le1$, i.e., in the last case the problem may not be solvable. Corresponding example one can find in [6,7], as well.

Theorem 2. *Let conditions of Theorem 1 be fulfilled and sequences P_m and q_m $(m=1,2,\dots)$ be such that conditions* (6) *and* (7) *hold. Then*

$$((P_m,q_m))_{m=1}^{+\infty}\in\mathcal{S}_{P_*}(P,q;\Phi,t_0). \tag{13}$$

Theorem 3. *Let conditions of Theorem 1 be fulfilled and let there exist a sequence of the non-degenerated matrix-functions $H_m\in\mathrm{AC}_{loc}(I_{t_0};\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ such that*

$$\lim_{t\to t_0}\left(\Phi^{-1}(t)\int\limits_{t_0}^{t}(q_{m*}(s)-q(s))\,ds\right)=0_n \tag{14}$$

for each sufficiently large m. Let, moreover, the conditions

$$\lim_{m\to+\infty}\left\|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t)-I_n\right\|=0, \tag{15}$$

$$\lim_{m\to+\infty}\left\|\int\limits_{t_0}^{t}\Phi^{-1}(s)|P_{m*}(s)-P(s)|\Phi(s)\,ds\right\|=0 \tag{16}$$

and

$$\lim_{m\to+\infty}\left\|\Phi^{-1}(t)\left|\int\limits_{t_0}^{t}(q_{m*}(s)-q(s))\,ds\right|+\left|\int\limits_{t_0}^{t}\Phi^{-1}(s)\left|P_*(s)\int\limits_{t_0}^{s}(q_{m*}(\tau)-q(\tau))\,d\tau\right|ds\right|\right\|=0 \tag{17}$$

be fulfilled uniformly on I_{t_0}, where $P_{m*}(t)\equiv(H_m'(t)+H_m(t)P_m(t))H_m^{-1}(t)$ and $q_{m*}(t)\equiv H_m(t)q_m(t)$ $(m=1,2,\dots)$. Then inclusion (13) *holds.*

Theorem 4. *Let conditions of Theorem 1 be fulfilled. Let, moreover,*

$$\|B_0\|\,\|(I_n-B)^{-1}\|<1 \tag{18}$$

and

$$\limsup_{t\to t_0}\left\|\int\limits_{t_0}^{t}\Phi^{-1}(s)|P_*(s)|\Phi(s)\,ds\right\|<+\infty. \tag{19}$$

Then inclusion (13) *holds if and only if there exist a sequence of matrix-functions $H_m\in\mathrm{AC}_{loc}(I_{t_0};\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ such that condition* (15) *holds uniformly on I, and conditions* (16) *and* (17) *hold uniformly on I_{t_0}, where $P_{m*}(t)$ and q_{m*} $(m=1,2,\dots)$ are defined as in Theorem 3.*

Theorem 4′. *Let conditions of Theorem 1 be fulfilled. Let, moreover,* (18) *and* (19) *hold. Then inclusion* (13) *holds if and only if*

$$\lim_{t\to t_0}\left\|\Phi^{-1}(t)\int\limits_{t_0}^{t}\big(X_0(s)X_m^{-1}(s)q_m(s)-q(s)\big)\,ds\right\|=0\ \ (m=1,2,\dots),$$

and

$$\lim_{m\to+\infty}\|\Phi^{-1}(t)(X_m(t)-X_0(t))\|=0,$$

$$\lim_{m\to+\infty}\bigg\|\Phi^{-1}(t)\bigg|\int\limits_{t_0}^{t}\big(X_0(s)X_m^{-1}(s)q_m(s)-q(s)\big)\,ds\bigg| \\ +\bigg|\int\limits_{t_0}^{t}\Phi^{-1}(s)P_*(s)\bigg|\int\limits_{t_0}^{s}\big(X_0(\tau)X_m^{-1}(\tau)q_m(\tau)-q(\tau)\big)\,d\tau\bigg|\,ds\bigg|\bigg\|=0,$$

hold uniformly on I_{t_0}, where X_0 and X_m $(m=1,2,\dots)$ are the fundamental matrices of systems (1) *and* (3)*, respectively.*

Corollary 1. *Let conditions of Theorem 1 be fulfilled for $q(t)\equiv 0_n$. Let, moreover,* (18) *and* (19) *hold. Then inclusion* (13) *holds if and only if*

$$((P_m,0_n))_{m=1}^{+\infty}\in\mathcal{S}_{P_*}(P,0_n;\Phi,t_0). \tag{20}$$

Remark 2. In Theorem 4′, as in Corollary 1, condition (18) is essential and it cannot be neglected, i.e., if the condition is violated, then the conclusion of the theorem and the corollary is not true. We present an example.

Let $I=[0,1]$, $n=1$, $t_0=0$, $B=0$, $B_0=1$, $\Phi(t)\equiv t$; $P(t)=P_m(t)=P_*(t)\equiv t^{-1}$ $(m=1,2,\dots)$, $q(t)\equiv 0$, $q_m(t)\equiv m^{-1}\cos(m^{-1}\ln t)$ $(m=1,2,\dots)$. Then

$$C_*(t,\tau)\equiv t\tau^{-1},\quad x_0(t)\equiv 0,\quad x_m(t)\equiv t\sin\frac{\ln t}{m}\ \ (m=1,2,\dots).$$

So, all the conditions of Theorem 4′ are fulfilled, except of (18), but condition (8) is not fulfilled uniformly on I. On other hand, this means that condition (20) is fulfilled but condition (13) is violated.

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Application of the Fractional Power Series to Solving Some Fractional Emden–Fowler Type Equations

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1 Introduction

Fractional differential equations have already proved to be valuable tools to the modelling of many physical phenomena [2, 4, 5, 8]. There are many techniques for solving fractional differential equations, in particular, RFPS method (residual fractional power series), which allows us to obtain solutions to initial value problems for Emden–Fowler type equations in the form of fractional power series [7]. These equations have many applications in the fields of radioactivity cooling and in the mean-field treatment of a phase transition in critical adsorption, kinetics of combustion or reactants concentration in chemical reactor and isothermal gas spheres and thermionic currents [1, 9].

2 Problem statement

Definition 2.1 ([4])**.** For $\mu \in \mathbb{R}$ the space C_μ is the space of functions f given on the half-axis $\mathbb{R}_+ \equiv [0, +\infty)$ and represented in the form $f = x^p f_1$ for some $p > \mu$, where the function f_1 is continuous on $\mathbb{R}_+$:

$$C_\mu = \Big\{ f : \;\; f = x^p f_1, \;\; f_1` \in C(\mathbb{R}_+) \;\text{ for some }\; p > \mu \in \mathbb{R} \Big\}.$$

Similarly, the space C_μ^n is the space of functions f given on the half-axis $\mathbb{R}_+$ such that $f^{(n)} \in C_\mu$.

Definition 2.2. For given $x_0 \geq 0$ *the α order Caputo fractional derivative* of function $f \in C_{-1}^n$ such that $f^{(n)}|_{x=x_0} = 0$, where $\alpha \in [n, n+1)$, $n \in \mathbb{N}$, is defined by

$${}_{x_0}^{\;C}D_x^\alpha f \equiv \frac{1}{\Gamma(n-\alpha+1)} \int\limits_{x_0}^{x} (x-t)^{n-\alpha} f^{(n+1)}(t)\,\mathrm{d}t \;\text{ or, respectively, }\; {}_{x_0}^{\;C}D_x^n f \equiv f^{(n)}(x).$$

Definition 2.3. For given $\alpha \geq 0$ *the fractional power series* (FPS) around the center $x_0 \in \mathbb{R}$ is a functional series of the following form:

$$\sum_{n=0}^{+\infty} c_n (x-x_0)^{n\alpha}, \;\; x \geq x_0.$$

Properties of FPS are presented in [3]. Let $\alpha \in (1/2, 1]$, and $D_x^\alpha \equiv {}_0^C D_x^\alpha$. We consider the following initial value problem (IVP):

$$D_x^{2\alpha} u + \frac{a}{x^\alpha} D_x^\alpha u + s(x) g(u) = h(x), \;\; x > 0, \;\; u(0) = \widehat{u}_0, \;\; D_x^\alpha u\big|_{x=0} = 0, \qquad (1)$$

where

$$s(x) \equiv \sum_{n=0}^{+\infty} s_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}, \quad h(x) \equiv \sum_{n=0}^{+\infty} h_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}, \quad g(u) \equiv \sum_{k=0}^{K} a_k u^k, \ \ K \in \mathbb{N}.$$

Using FPS the solution to IVP (1) can be written as

$$u(x) = \sum_{n=0}^{+\infty} u_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \quad \Bigg(U_N(x) \equiv \sum_{n=0}^{N} u_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \Bigg). \tag{2}$$

3 Main results

Theorem. *If the solution to IVP* (1) *is sought in the form of series* (2), *then the following equalities hold:* $u_0 = \widehat{u}_0$, $u_1 = 0$ *and*

$$u_N \bigg(1 + \frac{a\Gamma(1+(N-2)\alpha)}{\Gamma(1+(N-1)\alpha)} \bigg) = h_{N-2} - D_x^{(N-2)\alpha} \bigg(\sum_{n=0}^{N} s_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \sum_{k=0}^{K} a_k U_N^k(x) \bigg) \bigg|_{x=0}, \quad N \geq 2.$$

Corollary 3.1. *If* $s(x) \equiv s \in \mathbb{R}$, $h(x) \equiv 0$ *and* $g(u) \equiv u$, *then the solution to problem* (1) *is given in the form of the following series*

$$u(x) = \widehat{u}_0 + \sum_{n=1}^{+\infty} (-1)^n s^n \widehat{u}_0 \bigg(\prod_{k=1}^{n} \frac{\Gamma(1+(2k-1)\alpha)}{\Gamma(1+(2k-1)\alpha) + a\Gamma(1+2(k-1)\alpha)} \bigg) \frac{x^{2n\alpha}}{\Gamma(1+2n\alpha)},$$

which converges absolutely and uniformly for $x \geq 0$.

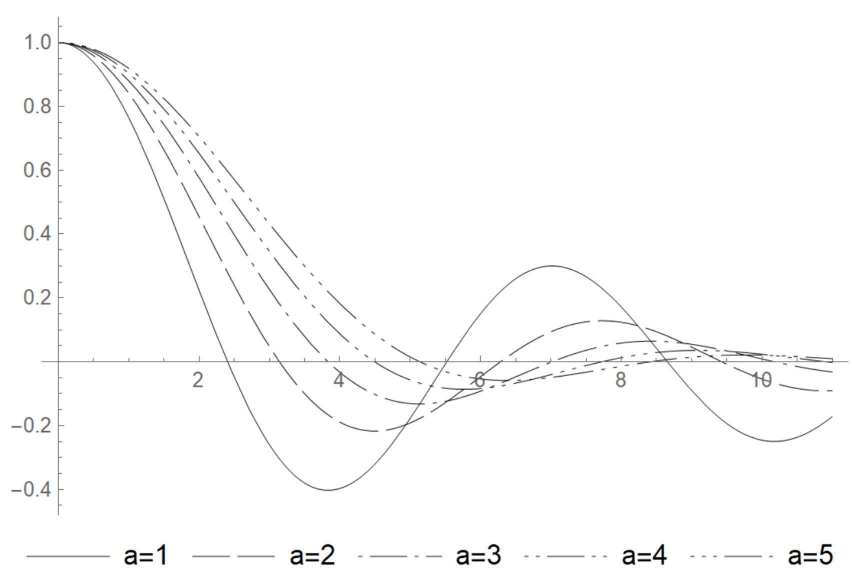

Figure 1. Graphs of the solutions to IVP (1) in case $s(x) \equiv s \in \mathbb{R}$, $h(x) \equiv 0$, $g(u) \equiv u$ and $\alpha = \widehat{u}_0 = s = 1$ and various values of a.

Under the conditions of Corollary 3.1 and in case of integer order differential operator ($\alpha = 1$) we obtain the solutions to IVP (1), if $\widehat{u}_0 = s(x) \equiv 1$ and $a = \overline{1,5}$:

$$u(x) = J_0(x), \ \frac{\sin x}{x}, \ \frac{2J_1(x)}{x}, \ \frac{3\sin x - 3x\cos x}{x^3}, \ \frac{8J_2(x)}{x^2},$$

where $J_a(x)$ are Bessel functions of the first kind. It is noteworthy that in [6] in case $a = 2$ the same solution was obtained by using fractional differential transformation method (FDT).

Corollary 3.2. *If* $s(x) \equiv x^\alpha$ *and*

$$h(x) \equiv \Gamma(1+2\alpha) + \frac{a\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} + x^\alpha(\widehat{u}_0 + x^{2\alpha})^k, \quad g(u) \equiv u^k,$$

where $k \in \mathbb{N}$*, then IVP* (1) *has a solution* $u(x) = \widehat{u}_0 + x^{2\alpha}$*.*

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Double Minimum Control Problem for a Parabolic Equation

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1 Introduction

We consider an extremum problem with weighted integral cost functional for the following parabolic mixed problem

$$u_t = (a(x,t)u_x)_x + b(x,t)u_x + h(x,t)u, \ \ (x,t) \in Q_T = (0,1) \times (0,T), \ \ T > 0, \tag{1.1}$$

$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \ \ 0 < t < T, \tag{1.2}$$

$$u(x,0) = \xi(x), \ \ 0 < x < 1, \tag{1.3}$$

where the real functions a, b and h are smooth in $\overline{Q}_T$, $0 < a_0 \le a(x,t) \le a_1 < \infty$, $\varphi \in W_2^1(0,T)$, $\psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$. Here $W_2^1(0,T)$ is the Sobolev space of weakly differentiable functions with the norm

$$\|u\|^2_{W_2^1(0,T)} = \int_0^T (u'^2 + u^2)\,dt.$$

We study the control problem with pointwise observation: by controlling the temperature φ at the left end of the segment (the functions ψ and ξ are assumed to be fixed), we try to make at some point $x_0 \in (0,1)$ the temperature $u(x_0,t)$ close to the given function $z(t)$ over the entire time interval $(0,T)$. This problem arises in the model of climate control in industrial greenhouses [1, 6]. Note that extremal problems for parabolic equations were considered in [11,15,17,18] (as usual, problems with final or distributed observation). But the results and methods of investigation are not similar to our methods. The proposed paper develops and generalizes the authors' results of [1–8]. Here we study a more general equation with a variable diffusion coefficient a, a convection coefficient b, and a potential h called the depletion potential. We state a problem of double minimization to our functional obtain by finding first minimum of the functional in some class of control functions and iterated minimum by weight. For such problem we prove the existence of a pair of minimizers.

As well as in [13, p. 6], we denote by $V_2^{1,0}(Q_T)$ the Banach space of functions $u \in W_2^{1,0}(Q_T)$ with the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0 \le t \le T} \|u(\,\cdot\,,t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)}$$

such that $t \mapsto u(\,\cdot\,,t)$ is a continuous mapping from $[0,T]$ to $L_2(0,1)$. Let $\widetilde{W}_2^1(Q_T)$ be set of all functions $\eta \in W_2^1(Q_T)$ satisfying the conditions $\eta(\,\cdot\,,T) = 0$, $\eta(0,\,\cdot\,) = 0$.

Definition 1.1. A function $u \in V_2^{1,0}(Q_T)$, satisfying the condition $u(0,t) = \varphi(t)$ and the equality

$$\int\limits_{Q_T} (a(x,t)u_x\eta_x - b(x,t)u_x\eta - h(x,t)u\eta - u\eta_t)\, dx\, dt$$

$$= \int\limits_0^1 \xi(x)\eta(x,0)\, dx + \int\limits_0^T a(1,t)\psi(t)\,\eta(1,t)\, dt \qquad (1.4)$$

for all $\eta \in \widetilde{W_2^1}(Q_T)$, is called a weak solution to problem (1.1)–(1.3).

2 Main Results

Theorem 2.1. *The problem* (1.1)–(1.3) *has a unique weak solution* $u \in V_2^{1,0}(Q_T)$*, which satisfies the inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} \le C_1 \Big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)} \Big) \qquad (2.1)$$

with some constant C_1 *independent of* φ, ψ *and* ξ.

Corollary 2.1. *The solution* u *to problem* (1.1)–(1.3) *continuously depends on the triple* (ξ, φ, ψ) *from* $L_2(0,1) \times W_2^1(0,T) \times W_2^1(0,T)$.

We consider a set of control functions $\varphi \in W_2^1(0,T)$ and a set of objective functions $z \in L_2(0,T)$. These sets are denoted by Φ and Z. Hereafter we suppose that Φ is a non-empty, closed, convex, and bounded set. Consider the weighted integral cost functional

$$J[z,\rho,\varphi] = \int\limits_0^T (u_\varphi(x_0,t) - z(t))^2 \rho(t)\, dt, \quad x_0 \in (0,1), \quad \varphi \in \Phi, \quad z \in Z,$$

where $u_\varphi \in V_2^{1,0}(Q_T)$ is the solution to problem (1.1)–(1.3) with the given control function φ. Here $\rho \in L_\infty(0,T)$ is a real-valued weight function such that $\operatorname{ess}\inf\limits_{t\in(0,T)} \rho(t) > 0$. Assuming the functions z and ρ to be fixed, consider the minimization problem of finding

$$m[z,\rho,\Phi] = \inf_{\varphi\in\Phi} J[z,\rho,\varphi].$$

Theorem 2.2 ([5,8,9])**.** *For any* $z \in L_2(0,T)$ *there exists a unique function* $\varphi_0 \in \Phi$ *such that*

$$m[z,\rho,\Phi] = J[z,\rho,\varphi_0].$$

Take $\widetilde{\widetilde{\rho}} > \widetilde{\rho} > 0$, we consider the set $P \subset L_\infty(0,T)$ of all weight functions ρ with

$$\operatorname{ess}\inf_{t\in(0,T)} \rho(t) \geqslant \widetilde{\rho}, \quad \operatorname{ess}\sup_{t\in(0,T)} \rho(t) \leqslant \widetilde{\widetilde{\rho}}.$$

Let us state for some subset $\widetilde{P} \subset P$ the double minimum problem

$$\mu[z,\widetilde{P},\Phi] = \inf_{\rho\in\widetilde{P}} m[z,\rho,\Phi].$$

Definition 2.1 ([12])**.** Let X be a Banach space. The set $Y \subset X^*$ is called a regularly convex if for any $y \notin Y$ there exists an element $x_0 \in X$ such that

$$\sup_{f\in Y} f(x_0) < y(x_0).$$

Theorem 2.3. *Let the set $\widetilde{P}$ be regularly convex in $L_\infty(0,T)$. Then for all $z \in L_2(0,T)$ there exist functions $\rho_0 \in \widetilde{P}$ and $\varphi_0 \in \Phi$ such that*

$$\mu[z,\widetilde{P},\Phi] = J[z,\rho_0,\varphi_0].$$

3 Proofs

At first we establish the following generalization of the classical maximum principle (see [13, Ch. 3, Par. 7]).

Lemma 3.1. *If $u \in V_2^{1,0}(Q_T)$ is a solution to the problem*

$$\begin{gathered}
u_t = (a(x,t)u_x)_x + b(x,t)u_x + h(x,t)u, \ \ (x,t) \in Q_T,\\
u(0,t) = \varphi(t), \quad u_x(1,t) = 0, \ \ 0 < x < 1, \ \ t > 0,\\
u(x,0) = 0, \ \ 0 < x < 1,
\end{gathered} \tag{3.1}$$

then the inequality

$$\operatorname{ess}\sup_{(x,t)\in Q_T} |u(x,t)| \le C_2 \sup_{t\in[0,T]} |\varphi(t)| \tag{3.2}$$

holds with a constant $C_2 > 0$ depending only on the coefficients of equation (3.1).

Also we will use the following statements to prove Theorem 2.3.

Theorem 3.1 ([12, Theorem 10])**.** *Let X be a separable Banach space. Then a set $Y \subset X^*$ is regularly convex if and only if it is convex and $*$-weakly closed.*

Theorem 3.2 ([10, Ch. 8, § 7])**.** *For any bounded sequence $(\rho_k)_{k\in\mathbb{N}}$ in $L_\infty(0,T)$ there exist a subsequence $(\rho_{k_j})_{j\in\mathbb{N}}$ and a function $\rho_0 \in L_\infty(0,T)$ such that*

$$\lim_{j\to+\infty} \int_0^T \rho_{k_j}(t)\zeta(t)\,dt = \int_0^T \rho_0(t)\zeta(t)\,dt$$

for any function $\zeta \in L_1(0,T)$.

Proof of Theorem 2.3. Put $d = \mu[z,\widetilde{P},\Phi]$. Then there exists a sequence of weight functions $\rho_1,\rho_2,\ldots \in \widetilde{P}$ such that

$$m[z,\rho_k,\Phi] \to d, \ \ k \to \infty. \tag{3.3}$$

So, by (3.3) and Theorem 2.2 there exists a sequence of control functions $\varphi_1,\varphi_2,\dots \in \Phi$ satisfying

$$J[z,\rho_k,\varphi_k] = m[z,\rho_k,\Phi] \to d, \ \ k \to \infty.$$

The functions φ_k belong to Φ, so, the sequence of norms $\|\varphi_k\|_{W_2^1(0,T)}$ is bounded due to boundedness of the set Φ. Therefore, there exists a subsequence $(\varphi_{k_j})_{j\in\mathbb{N}}$ converging weakly in $W_2^1(0,T)$ to some function $\varphi_0 \in \Phi$ due to closeness of the set Φ. Now, by compact embedding of $W_2^1(0,T)$ into $C([0,T])$, the sequence $(\varphi_{k_j})_{j\in\mathbb{N}}$ converges to φ_0 by norm of $C([0,T])$:

$$\|\varphi_{k_j} - \varphi_0\|_{C([0,T])} \to 0, \ \ j \to \infty. \tag{3.4}$$

Further we write φ_k instead if φ_{k_j}.

The next step is to study behavior of the sequence of solutions $u_k = u_{\varphi_k}$, $k = 1, 2, \dots,$ to problem (1.1)–(1.3) in the space $W_2^{1,0}(Q_T)$. Functions u_k are solutions of the following mixed problems:

$$\begin{gathered} u_{kt} = (a(x,t)u_{kx})_x + b(x,t)u_{kx} + h(x,t)u_k, \quad (x,t) \in Q_T, \\ u_k(0,t) = \varphi_k(t), \quad u_{kx}(1,t) = \psi(t), \quad 0 < t < T, \\ u_k(x,0) = \xi(x), \quad 0 < x < 1. \end{gathered}$$

Functions u_{φ_k} satisfy the condition $u_{\varphi_k}(0,t) = \varphi_k(t)$ and by (1.4) the equalities

$$\int\limits_{Q_T} (a(x,t)u_{\varphi_k x}\eta_x - b(x,t)u_{\varphi_k x}\eta - h(x,t)u_{\varphi_k}\eta - u_{\varphi_k}\eta_t)\,dx\,dt = \int\limits_0^1 \xi(x)\eta(x,0)\,dx + \int\limits_0^T a(1,t)\psi(t)\,\eta(1,t)\,dt \tag{3.5}$$

for all $\eta \in \widetilde{W_2^1}(Q_T)$. It follows from (2.1) that

$$\|u_{\varphi_k}\|_{W_2^{1,0}(Q_T)} \le C_3\|u_{\varphi_k}\|_{V_2^{1,0}(Q_T)} \le C_4\Big(\|\varphi_k\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)}\Big) \le C_5$$

with a constant C_5 independent of k. So, there exists a subsequence $u_{\varphi_{k_j}}$ (we denote it by u_j) such that $u_j \to u_0$, $j \to \infty$, weakly for some $u_0 \in W_2^{1,0}(Q_T)$. From (3.5) and the weak convergence of the sequence u_j in $W_2^{1,0}(Q_T)$, it follows that the weak limit (the function u_0) satisfies equality (1.4) for all $\eta \in \widetilde{W_2^1}(Q_T)$. Now, we prove that $u_0|_{x=0} = \varphi_0$. By the Banach–Saks theorem [16, Ch. 2, Sec. 38] we have a subsequence (we denote it by u_j too) such that

$$\|\widehat{u}_k - u_0\|_{W_2^1(Q_T)} \to 0, \quad k \to \infty, \quad \widehat{u}_k = \frac{1}{k}\sum_{j=1}^k u_j. \tag{3.6}$$

Therefore,

$$\|\widehat{u}_k(0,t) - u_0(0,t)\|_{L_2(0,T)} \le C_6\|\widehat{u}_k - u_0\|_{W_2^1(Q_T)} \to 0, \quad k \to \infty. \tag{3.7}$$

But it follows from (3.6), (3.7) that in the $L_2(0,T)$ space we have

$$u_0(0,\,\cdot\,) = s - \lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^k \varphi_j(\,\cdot\,) = w - \lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^k \varphi_j(\,\cdot\,) = w - \lim_{k\to\infty}\varphi_k(\,\cdot\,) = \varphi_0(\,\cdot\,).$$

(If φ_k converges to φ_0 weakly in $W_2^1(0,T)$, then it converges weakly to φ_0 in $L_2(0,T)$ too.) So, the limit function u satisfies $u_0|_{x=0} = \varphi_0$. It means that u is a solution to problem (1.1)–(1.3) with the control function $\varphi = \varphi_0$. Let $v_k = u_{\varphi_k} - u_{\varphi_0}$. Functions v_k are solutions to the following mixed problems:

$$\begin{gathered} v_{kt} = (a(x,t)v_{kx})_x + b(x,t)v_{kx} + h(x,t)v_k, \quad (x,t) \in Q_T, \\ v_k(0,t) = \varphi_k(t) - \varphi_0(t), \quad v_{kx}(1,t) = 0, \quad 0 < t < T, \\ v_k(x,0) = 0, \quad 0 < x < 1. \end{gathered}$$

By inequalities (3.2) and (3.4) we obtain that

$$\|v_k(x_0,t)\|_{L_2(0,T)} \le \sqrt{T}\,\|v_k(x_0,t)\|_{C([0,T])} \to 0, \quad k \to \infty.$$

So, the sequence of functions $\{(u_k(x_0,\,\cdot\,)-z(\,\cdot\,))^2\}_{k=1}^\infty$ converges by norm in the $L_1(0,T)$ space to the function $(u_0(x_0,\,\cdot\,)-z(\,\cdot\,))^2$. Now, by Theorem 3.1 we can extract from the minimizing sequence of weight functions $\rho_k(t)$ a subsequence (we will denote it also $\rho_k(t)$) that *-weakly converges in $L_\infty(0,T)$ to some $\rho_0\in\widetilde{P}$. Combining this with Theorem 3.2, we obtain the following relation:

$$\mu[z,\widetilde{P},\Phi]=\lim_{k\to\infty}\int\limits_0^T(u_{\varphi_k}(x_0,t)-z(t))^2\rho_k(t)\,dt=\int\limits_0^T(u_{\varphi_0}(x_0,t)-z(t))^2\rho_0(t)\,dt=J[z,\rho_0,\varphi_0].$$

Proof of Theorem 2.3 is completed. □

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On Extensibility and Asymptotics of Solutions to the Riccati Equation with Real Roots of its Right Part

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Consider the Riccati equation

$$y' = P(x) + Q(x)y + y^2, \tag{1}$$

where $P(x)$ and $Q(x)$ are continuous functions bounded on $(-\infty; \infty)$. Suppose the equation

$$y^2 + Q(x)y + P(x) = 0$$

has real bounded roots $\alpha_1(x) \in C^1(-\infty, +\infty)$ and $\alpha_2(x) \in C^1(-\infty, +\infty)$. So, equation (1) can be written as

$$y'(x) = (y(x) - \alpha_1(x))(y(x) - \alpha_2(x)). \tag{2}$$

Thus we have

$$Q^2(x) - 4P(x) \geqslant 0.$$

Suppose that either $\alpha_2(x) > \alpha_1(x)$, $x \in (-\infty, +\infty)$, or $\alpha_2(x) = \alpha_1(x)$, $x \in (-\infty, +\infty)$, that $\alpha_1(x)$ and $\alpha_2(x)$ are bounded C^1 functions on $(-\infty, +\infty)$.

We define a function $Y_0(x)$ by

$$Y_0(x) := \Big[\frac{(\alpha_1(x) - \alpha_2(x))^2}{4} + \frac{(\alpha_1(x) + \alpha_2(x))'}{2}\Big].$$

Lemma 1 ([4, Lemma 4.1])**.** *Suppose $x_0 < \omega \le +\infty$. Then there exist $S_* \in [x_0, \omega)$ and a solution $y_*(x)$ to equation* (2) *defined on (S_*, ω) such that any solution $y(t)$ defined on (S, ω) satisfies $S \ge S_*$ and $y(x) \le y_*(x)$ for all $x \in (S, \omega)$.*

Hereafter the solution $y_*(x)$ from the last lemma is called a **principal solution**.

Definition 1 ([5])**.** The functions α_1 and α_2 in equation (2) are said to satisfy the **stabilization conditions** if there exist finite limits

$$\lim_{x\to\pm\infty} \alpha_j(x) =: \alpha_{j,\pm} \in \mathbb{R}, \quad j = 1, 2. \tag{3}$$

Definition 2 ([5])**.** The functions α_1 and α_2 are said to satisfy the **monotone stabilization conditions** if there exists $A > 0$ such that

$$\alpha_1'(x) \neq 0, \ \ \alpha_2'(x) \neq 0 \ \text{ for all } \ x \notin [-A, A]. \tag{4}$$

Definition 3 ([5])**.** A solution $y(x)$ to equation (2) is called **stabilizing** if there exist finite limits

$$\lim_{x\to\pm\infty} y(x) =: y_\pm \in \mathbb{R}.$$

Theorem 1. *Suppose*

$$Q'(x) < \frac{Q^2(x)}{2} - 2P(x) \ \textit{for all} \ x \geq x_0.$$

Then any solution $y(x)$ to equation (2) *with $y(x_0) \leq -\frac{Q(x_0)}{2}$ satisfies also the condition*

$$y(x) < -\frac{Q(x)}{2} \ \textit{for all} \ x > x_0.$$

Corollary 1. *Suppose that $\alpha_1(x) = \alpha_2(x) = \alpha(x)$ for all $x \in (-\infty, +\infty)$ and $\alpha'(x) > 0$ for all $x \geq x_0$. Then any solution $y(x)$ to equation* (2) *with $y(x_0) \leq \alpha(x_0)$ satisfies also the condition $y(x) < \alpha(x)$ for all $x > x_0$.*

Theorem 2. *Any solution to equation* (2) *defined at some $x_0 \in \mathbb{R}$ is bounded below to the right of x_0.*

Theorem 3. *Suppose there exists a constant M such that $\alpha_1(x) \leq \alpha_2(x) \leq M$ for all $x \geq x_0$. Then any solution $y(x)$ to equation* (2) *with $y_0 = y(x_0) > M$ monotonically increases to the right of x_0 and*

$$\lim_{x \to \overline{x}} y(x) = +\infty \ \textit{with} \ x_0 < \overline{x} < x_0 + \frac{1}{y_0 - M}.$$

Note that in the particular case $\alpha_2(x) = \alpha_1(x)$ on $(-\infty, +\infty)$, Theorem 3 yields the first statement of Theorem 5.5 from [2].

Now by using the substitutions $\widehat{x} = -x$, $\widehat{y}(\widehat{x}) = -y(-\widehat{x})$ we transform equation (2) to the form

$$\frac{d}{d\widehat{x}} \widehat{y}(\widehat{x}) = \big(\widehat{y}(\widehat{x}) - \widehat{\alpha}_1(\widehat{x})\big)\big(\widehat{y} - \widehat{\alpha}_2(\widehat{x})\big),$$

where $\widehat{\alpha}_1(\widehat{x}) = -\alpha_1(x)$, $\widehat{\alpha}_2(\widehat{x}) = -\alpha_2(x)$. Thus, we can obtain analogues of Theorems 1–3 and their corollaries for the case $x \leq x_0$. In particular, the following theorem is an analogue of Theorem 3.

Theorem 3′. *If there exists a constant m such that $\alpha_2(x) \geq \alpha_1(x) \geq m$ for all $x \leq x_0$, then every solution $y(x)$ to* (2) *with $y_0 = y(x_0) < m$ is monotonic for $x \geq x_0$ and*

$$\lim_{x \to \overline{x}} y(x) = -\infty, \quad \textit{where} \ x_0 > \overline{x} > x_0 - \frac{1}{m - y_0}.$$

Obtained Theorems 1–3 and 3′ complement results of [2]. We used results of [4,5] and the proof of Lemma 7.1 ([3, p. 365]) to obtain the following theorems.

Theorem 4. *Let $y_3(x) < y_2(x) < y_1(x)$ be different solutions to* (2) *defined at a point x_0 and y_1 be extensible on $[x_0, +\infty)$. Then y_2 and y_3 are also extensible on $[x_0, +\infty)$ with the following properties:*

1) *The ratio $\frac{y_1(x) - y_3(x)}{y_1(x) - y_2(x)}$ is monotonically decreasing on $[x_0, +\infty)$;*

2) *There exists a finite limit $\lim\limits_{x \to +\infty} \frac{y_1(x) - y_3(x)}{y_1(x) - y_2(x)}$;*

3) *If $y_1(x)$ is a principal solution for the interval $(x_0, +\infty)$, then the above limit equals* 1.

Theorem 5. *Let $y_1(x)$, $y_2(x)$ be two different solutions to* (2) *defined on $[x_0, +\infty)$. Let both of them have different finite limits as $x \to +\infty$. Then every solution to (2) defined on $[x_0, +\infty)$ has a finite limit as $x \to +\infty$.*

Theorem 6. *Let $y_1(x) > y_2(x)$ be two different solutions to* (2) *defined on* $[x_0, +\infty)$. *Let both of them have finite limits as $x \to +\infty$. Then every solution to* (2) *defined at the point x_0 with $y(x_0) \le y_1(x_0)$ is extensible on $[x_0, +\infty)$ and has a finite limit as $x \to +\infty$.*

Theorem 7. *Let* (2) *have solutions defined on* $[x_0, +\infty)$. *Let $y_1(x) = y_*(x)$ be the principal solution for the interval $(x_0, +\infty)$. Let $y_1(x)$ and another solution $y_2(x) < y_1(x)$ have finite limits as $x \to +\infty$. Then every solution to* (2) *defined on $[x_0, +\infty)$ and different from $y_*(x)$ has a finite limit as $x \to +\infty$. This limit is equal to the limit of $y_2(x)$ as $x \to +\infty$.*

Further we assume that the functions $\alpha_1(x)$ and $\alpha_2(x)$ are bounded and satisfy (3), (4), and $\alpha_2(x) > \alpha_1(x)$, $x \in (-\infty, +\infty)$. As shown in [5], in this case all bounded solutions are stabilizing (and vice versa), all stabilizing solutions are monotonically stabilizing and y_- equals $\alpha_{1,-}$ or $\alpha_{2,-}$, while y_+ equals $\alpha_{1,+}$ or $\alpha_{2,+}$.

According to [5], all stabilizing solutions to (2) are divided into four types:

type I: $\quad y_- = \alpha_{1,-},\ y_+ = \alpha_{1,+}$.

type II: $\quad y_- = \alpha_{2,-},\ y_+ = \alpha_{1,+}$.

type III: $\quad y_- = \alpha_{2,-},\ y_+ = \alpha_{2,+}$.

type IV: $\quad y_- = \alpha_{1,-},\ y_+ = \alpha_{2,+}$.

Theorem 8. *Suppose $\alpha_{1,+} \neq \alpha_{2,+}$, $\alpha_{1,-} \neq \alpha_{2,-}$, and $Y_0(x) \le 0$ on $\mathbb{R} \setminus [a, b]$. Then all solutions to* (2) *are not stabilizing.*

The last theorem complements Theorem 3.4 from [5].

Theorem 9. *Suppose that $\alpha_{1,+} \neq \alpha_{2,+}$, $\alpha_{1,-} \neq \alpha_{2,-}$, and equation* (2) *has a stabilizing solution of type* II. *Then there exist a unique solution of type* I *and a unique solution of type* III. *Denote them by y_I and y_{III}, respectively. Let $y(x)$ be a solution to* (2). *Then:*

- *if $y_I < y < y_{III}$, then $y(x)$ is a stabilizing solution of type* II*;*
- *if $y > y_{III}$, then there exists $x^* \in \mathbb{R}$ such that $y(x)$ is extensible on the interval $(-\infty, x^*)$ and*
$$\lim_{x \to -\infty} y(x) = \alpha_{2,-}, \quad \lim_{x \to x^*-0} y(x) = +\infty;$$
- *if $y < y_I$, then there exists $x^* \in \mathbb{R}$ such that $y(x)$ is extensible on the interval $(x^*, +\infty)$ and*
$$\lim_{x \to +\infty} y(x) = \alpha_{1,+}, \quad \lim_{x \to x^*+0} y(x) = -\infty.$$

Theorem 10. *Suppose $\alpha_{1,+} \neq \alpha_{2,+}$, $\alpha_{1,-} \neq \alpha_{2,-}$. Then the following conditions are equivalent.*

1) *There exist stabilizing solutions to* (2) *of type* I *and of type* III.
2) *There exist a unique stabilizing solution to* (2) *of type* I *and a unique stabilizing solution to* (2) *of type* III.
3) *There exists a stabilizing solution to* (2) *of type* II.

Theorem 11. *Suppose $\alpha_{1,+} \neq \alpha_{2,+}$, $\alpha_{1,-} \neq \alpha_{2,-}$. Then exactly one of the following statements is true:*

1) *There exists a stabilizing solution to* (2) *of type* II.
2) *There exist a stabilizing solutions to* (2) *of type* I *and a unique stabilizing solution of type* IV.
3) *There exist a stabilizing solution to* (2) *of type* III *and a unique stabilizing solution of type* IV.
4) *All stabilizing solutions, if any, are stabilizing solutions of type* IV.

Theorems 8–11 complement Theorems 2.1–2.4 from [5].

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On Two Generalizations of Exponentially Dichotomous Systems

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1 Introduction. Basic definitions

For a positive integer n, by $\mathcal{M}_n$ we denote the class of linear differential systems

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \geqslant 0, \tag{1.1}$$

whose coefficient matrices $A(\,\cdot\,)\colon [0,+\infty) \to \operatorname{End}\mathbb{R}^n$ are piecewise continuous and bounded on the time half-line $t \geqslant 0$. By $C\mathcal{M}_n$ we denote a subclass of the class $\mathcal{M}_n$, consisting of systems with continuous coefficients on the half-line. We identify system (1.1) with its coefficient matrix and write $A \in \mathcal{M}_n$ or $A \in C\mathcal{M}_n$. The linear space of solutions of system (1.1) is denoted by $\mathcal{X}(A)$.

The following definition is well known.

Definition 1.1. A system in $\mathcal{M}_n$ is said to be *exponentially dichotomous* or called a *system with exponential dichotomy on the half-line* if there exist positive constants c_1, c_2 and ν_1, ν_2 and a decomposition of the space $\mathbb{R}^n$ of initial data (at $t = 0$) into a direct sum of subspaces L_- and L_+ (the case of zero dimension of the subspaces not being excluded) such that the solutions $x(\,\cdot\,)$ of the system satisfy the following two conditions:

(1) if $x(0) \in L_-$, then $\|x(t)\| \leqslant c_1 e^{-\nu_1(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant 0$;

(2) if $x(0) \in L_+$, then $\|x(t)\| \geqslant c_2 e^{\nu_2(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant 0$.

The study of this class of systems was initiated in Perron's paper [13]. It was preceded by fundamental works by Hadamard [10] and Bohl [8], who had developed the same key ideas that later transformed to the concept of exponential dichotomy. The above definition was actually given by Maisel' [11], but it was Massera and Schäffer [12] who stated it explicitly for the first time. Systems with exponential dichotomy, are one of the most comprehensively studied classes of linear differential systems, with, in addition, has important application in related branches of the theory of differential equations (see, e.g., [1]).

The efficiency of the notion of exponential dichotomy is in studying the asymptotics of solutions of nonlinear systems that are exponentially dichotomous in the first approximation and in its applications to dynamical systems has served as a reason for diverse generalizations of this notion within the theory of linear differential systems itself and beyond, e.g. in the theory of evolution operators and in the theory of linear extensions dynamical systems. We do not give any references to papers dealing with such generalizations, because there are far too many of them. We only mention the papers [14] and [4,5], in which the generalizations of exponential dichotomy are closest to the ones considered in the present paper.

We denote class of n-dimensional exponentially dichotomous system on the time half-line by $\mathcal{E}_n$ and the subclass of systems whose coefficient matrices are continuous of the half-line by $C\mathcal{E}_n$. In Definition 1.1, the positive constant factors c_1 and c_2 are the same for all solutions such that $x(0) \in L_-$ and $x(0) \in L_+$ respectively (or, in other words, estimates 1) ¨ 2) are uniform with respect to the constants c_1 and c_2 on L_- and L_+ respectively). In exactly the same way, estimates (1) and (2) are also uniform in the time variable; i.e., they hold for all $t \geqslant s$ starting from zero for all $x(0) \in L_-$ and $x(0) \in L_+$.

The question considered in the present paper is as follows. Is the condition that estimates (1) and (2) be uniform with respect to the constant factors or the time variable a necessary condition for the exponential dichotomy of system (1.1)? If yes, how strongly may the known properties of exponentially dichotomous systems change if these conditions are dropped?

In accordance with the preceding, let us introduce two more definitions.

Definition 1.2. A system in $\mathcal{M}_n$ is said to be *weakly exponentially dichotomous on the half-line* if there exist positive constants ν_1 and ν_2 and a decomposition of the space $\mathbb{R}^n$ of initial data (at $t = 0$) into a direct sum of subspaces L_- and L_+ (the case of zero dimension of the subspaces not being excluded) such that the solutions $x(\,\cdot\,)$ of the system satisfy the following two conditions:

($1'$) if $x(0) \in L_-$, then $\|x(t)\| \leqslant c_1(x)e^{-\nu_1(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant 0$;

($2'$) if $x(0) \in L_+$, then $\|x(t)\| \geqslant c_2(x)e^{\nu_2(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant 0$.

Here $c_1(x)$ and $c_2(x)$ are positive constants generally depending (as hinted in their notation) on the choice of the solution $x(\,\cdot\,)$.

Thus, the definition of weakly exponentially dichotomous systems differs from the definition of exponentially dichotomous systems only in that the condition for the estimates to be uniform in the respective constant factors is dropped.

Definition 1.3. A system in $\mathcal{M}_n$ is called *almost exponentially dichotomous on the half-line* if there exist positive constants c_1, c_2 and ν_1, ν_2 and a decomposition of the space $\mathbb{R}^n$ of initial data (at $t = 0$) into a direct sum of subspaces L_- and L_+ (the case of zero dimension of the subspaces not being excluded) such that the solutions $x(\,\cdot\,)$ of the system satisfy the following two conditions:

($1''$) if $x(0) \in L_-$, then $\|x(t)\| \leqslant c_1e^{-\nu_1(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant t_x$;

($2''$) if $x(0) \in L_+$, then $\|x(t)\| \geqslant c_2e^{\nu_2(t-s)}\|x(s)\|$ for all $t \geqslant s \geqslant t_x$.

Here t_x is a nonnegative number generally depending (as hinted in their notation) on the choice of the solution $x(\,\cdot\,)$.

Although conditions ($1''$) and ($2''$) imply the uniformity of the estimates in the constant factors c_1 and c_2, this is true not for all $t \geqslant s \geqslant 0$ (as the case for exponentially dichotomous systems) but only for $t \geqslant s$ greater than some t_x, which depends on the solution $x(\,\cdot\,)$.

The subspaces L_- and L_+ from Definitions 1.1–1.3 are called, respectively, *stable* and *unstable subspaces*, and the numbers $-\nu_1$ and ν_2 from Definitions 1.1–1.3 are called *dichotomy exponents*.

We denote the class of n-dimensional weakly exponentially dichotomous systems by $W\mathcal{E}_n$ and the class of n-dimensional almost exponentially dichotomous systems by $A\mathcal{E}_n$, with $CW\mathcal{E}_n$ and $CA\mathcal{E}_n$ being their respective subclasses consisting of systems whose coefficient matrices are continuous on the half-line. We have the relations $\mathcal{E}_1 = A\mathcal{E}_1 = W\mathcal{E}_1$. The class $W\mathcal{E}_n$ was introduced in paper [6], in which the authors used the constructions in [3] to prove that, in particular, for $n \geqslant 2$ one has the proper inclusion $\mathcal{E}_n \subset W\mathcal{E}_n$. Inclusion $A\mathcal{E}_n \subset W\mathcal{E}_n$ is obvious (that $A\mathcal{E}_n \neq W\mathcal{E}_n$ if $n \geqslant 2$ is stated below).

2 Main results

Lemma. *If the system is weakly exponentially dichotomous, then its stable subspaces L_- is uniquely determined and coincides with subspace $\mathcal{Z}_A$ of initial (at $t = 0$) vectors of solutions vanishing at infinity, and the subspaces L_+ can be selected to be any subspaces complementing the subspace L_- to $\mathbb{R}^n$.*

A linear subspace of the space $\mathcal{X}(A)$ is called *lineal.* If L is a linear subspace of $\mathbb{R}^n$, then by $L(A;\,\cdot\,)$ we denote the lineal formed by solutions of the system $A \in \mathcal{M}_n$ with initial (at $t = 0$) vectors from the subspace L; herewith $L(A;t)$ is a linear subspace of $\mathbb{R}^n$, formed by the vectors $x(t)$ of those solutions $x(\,\cdot\,)$, for which $x(0) \in L$. The lineals $L_-(A;\,\cdot\,)$ and $L_+(A;\,\cdot\,)$ of the system $A \in W\mathcal{E}_n$ are called *stable* and *unstable* lineals, respectively. For each $t \geqslant 0$, the subspaces $L_-(A;t)$ and $L_+(A;t)$ disjoint, i.e. $L_-(A;t) \cap L_+(A;t) = \{\mathbf{0}\}$.

By $A\mathcal{E}_n^m$ and $W\mathcal{E}_n^m$ we denote the subclasses of the classes $A\mathcal{E}_n$ and $W\mathcal{E}_n$, respectively, consisting of systems that have the dimension of their subspace L_- equal to m $(0 \leqslant m \leqslant n)$, by $CA\mathcal{E}_n^m$ and $CW\mathcal{E}_n^m$ we denote those subclasses of the classes $A\mathcal{E}_n^m$ and $W\mathcal{E}_n^m$, respectively, whose coefficients are continuous. By lemma, classes $W\mathcal{E}_n^m$, $m = \overline{0,n}$, are pairwise disjoint ($W\mathcal{E}_n^{m_1} \cap W\mathcal{E}_n^{m_2} = \varnothing$ if $m_1 \neq m_2$); i.e., $W\mathcal{E}_n = \bigsqcup\limits_{m=0}^{n} W\mathcal{E}_n^m$. Since $A\mathcal{E}_n^m = W\mathcal{E}_n^m \cap A\mathcal{E}_n$, it follows that the classes $A\mathcal{E}_n^m$, $m = \overline{0,n}$, are disjoint as well. Moreover, we have the obvious inclusions $\mathcal{E}_n^m \subset A\mathcal{E}_n^m \subset W\mathcal{E}_n^m$ and $C\mathcal{E}_n^m \subset CA\mathcal{E}_n^m \subset CW\mathcal{E}_n^m$ $m = \overline{0,n}$, where $\mathcal{E}_n^m$ is the subclass of $\mathcal{E}_n$ consisting of systems that have the dimension of their subspace L_- equal to m $(0 \leqslant m \leqslant n)$, and $C\mathcal{E}_n^m$ is a subclass of the class $\mathcal{E}_n^m$, whose systems have continuous coefficients.

In [2] the following theorem was proved.

Theorem 2.1.

(1) *For $(n,m) = (1,0)$, $(n,m) = (1,1)$ and $(n,m) = (2,1)$, we have the relations $\mathcal{E}_n^m = A\mathcal{E}_n^m = W\mathcal{E}_n^m$.*

(2) *For the remaining pairs (n,m) of integer $n \in \mathbb{N}$ and $0 \leqslant m \leqslant n$, the proper inclusions $\mathcal{E}_n^m \subset A\mathcal{E}_n^m \subset W\mathcal{E}_n^m$ hold and, moreover, there are the proper inclusions $C\mathcal{E}_n^m \subset CA\mathcal{E}_n^m \subset CW\mathcal{E}_n^m$.*

Since the definitions of the classes of weakly and almost dichotomous systems are quite close to the definition of the class of exponentially dichotomous systems, then, despite the result of Theorem 2.1, it seems plausible that the main properties of weakly exponentially dichotomous systems differ slightly from the properties of exponentially dichotomous systems. The report shows that this natural assumption is generally wrong.

Let us present the main properties of exponentially dichotomous systems.

(a) Recall that some property of points in a metric space is called *rough* in this space if the points possessing it form an open set. It is well known (see, for example, [9, p. 260]) that in the metric space $(\mathcal{M}_n, \operatorname{dist_u})$ with metric $\operatorname{dist_u}(A,B) = \sup\limits_{t \geqslant 0} \|A(t) - B(t)\|$ of uniform convergence on the half-line the property of a system to be exponentially dichotomous is rough, i.e. the set $\mathcal{E}_n$ is open in the space $(\mathcal{M}_n, \operatorname{dist_u})$. We also recall that the *edge* of a set in the topological space is called the set-theoretic difference between this set and its interior.

(b) If the system A is exponentially dichotomous, then the conjugate to it system $-A^*$ is also exponentially dichotomous; moreover, if $A \in \mathcal{E}_n^m$ and $-\nu_1$, ν_2 are dichotomy exponents of the system A, then $-A^* \in \mathcal{E}_n^{n-m}$ and $-\nu_2$, ν_1 are dichotomy exponents of the system $-A^*$. The above statement about systems, which are conjugate to exponentially dichotomous systems, follows easily, for example, from [15, p. 14, Theorem 1.1]. In particular, the class $\mathcal{E}_n$ of exponentially dichotomous systems is invariant under conjugation.

(c) For a system $A \in \mathcal{E}_n$, $n \geqslant 2$, let us consider its a stable lineal $L_-(A;\,\cdot\,)$ and an unstable lineal $L_+(A;\,\cdot\,)$ (we assume that both of them are different from the zero lineal). As noted above, for every $t \in \mathbb{R}_+$ the subspaces $L_-(A;t)$ and $L_+(A;t)$ are disjoint, so for every $t \geqslant 0$ the inequality $\angle\{L_-(A;t), L_+(A;t)\} > 0$ hold. It is well known (see, for example, [15, p. 10, Lemma 1.1]) that

$$\inf_{t\geqslant 0} \angle\bigl\{L_-(A;t), L_+(A;t)\bigr\} > 0, \tag{2.1}$$

i.e. for stable and unstable lineals of exponentially dichotomous systems, the angles between their corresponding subspaces are separated from zero on a half-line. Note that some strengthening of property (2.1) for exponentially dichotomous systems was established in [7].

Property (2.1) of finite-dimensional exponentially dichotomous systems is so important that when generalizing [9, p. 233–234], [4, p. 131] the concept of exponential dichotomy on linear differential systems in a Banach space, in order to preserve the main features of the theory, this property has to be included in the definition of exponentially dichotomous systems in Banach spaces as an independent condition.

The listed above properties of the class of exponentially dichotomous systems: roughness, invariance under the conjugation operation, and separation of the angles between the stable and any unstable lineals of solutions, do not hold for classes of weakly and almost exponentially dichotomous systems, as the following theorems show.

Theorem 2.2. *For any integer $n \geqslant 2$ in the metric space $\mathcal{M}_n$ with the topology of uniform convergence on the half-line, the interior of the set of weakly (almost) exponentially dichotomous systems coincides with the set of exponentially dichotomous systems, i.e.,* $\operatorname{int} W\mathcal{E}_n = \mathcal{E}_n$ *(respectively* $\operatorname{int} A\mathcal{E}_n = \mathcal{E}_n$*) for any $n \geqslant 2$.*

Theorem 2.2 and some simple considerations imply the following corollary.

Corollary. *In a metric space $\mathcal{M}_n$, $n \geqslant 2$, with the topology of uniform convergence on the half-line, the set $W\mathcal{E}_n$ (the set $A\mathcal{E}_n$) is neither open nor closed, all its points is limit points, and its edge* $\operatorname{ed} W\mathcal{E}_n$ $(\operatorname{ed} A\mathcal{E}_n)$ *are exactly weakly (almost) exponentially dichotomous systems that are not exponentially dichotomous.*

This corollary, in particular, shows that the properties of a system to be weakly or almost exponentially dichotomous are not rough.

Theorem 2.2 and the corollary remain valid if the space $\mathcal{M}_n$ in them is replaced by its subspace $C\mathcal{M}_n$, and the subsets $W\mathcal{E}_n$, $A\mathcal{E}_n$, and $\mathcal{E}_n$ by the subsets $CW\mathcal{E}_n$, $CA\mathcal{E}_n$, and $C\mathcal{E}_n$, respectively.

The non-invariance of the classes $W\mathcal{E}_n$ and $A\mathcal{E}_n$, if $n \geqslant 2$, under conjugation is stated by the following theorem.

Theorem 2.3. *For any $n \geqslant 2$ there exists a continuous n-dimensional a weakly (almost) exponentially dichotomous system such that its conjugate system is not weakly (almost) exponentially dichotomous.*

In the general case, the non-separation from zero of the angle between the stable $L_-(\,\cdot\,)$ and some unstable $L_+(\,\cdot\,)$ lineals of a weakly (almost) exponentially dichotomous system is established by the following theorem.

Theorem 2.4. *For any integer $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$ in the class $CA\mathcal{E}_n^m$ there exists a system such that the angle between its stable lineal $L_-(\,\cdot\,)$ and some unstable lineal $L_+(\,\cdot\,)$ is not separated from zero, i.e.* $\inf\limits_{t\geqslant 0} \angle\bigl(L_-(t), L_+(t)\bigr) = 0$.

Note that the restrictions $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$ in the statement of Theorem 2.4 are essential: if m is equal to 0 or n, then one of the lineals $L_-(\,\cdot\,)$ or $L_+(\,\cdot\,)$ is zero and the angle $\angle\big(L_-(t), L_+(t)\big)$ is undefined; if $n=2$, then for $m=1$ the system is exponentially dichotomous, which means that Theorem 2.4 is not true for it.

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Lyapunov Irregularity Coefficient and Exponential Stability Index of a Linear Parametric System as a Vector Function of the Parameter

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1 Introduction. Statement of the problem

For a given positive integer n, let $\mathcal{M}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \stackrel{\text{def}}{=} [0, +\infty), \tag{1.1}$$

with piecewise continuous and bounded on the half-line $\mathbb{R}_+$ coefficient matrices $A(\,\cdot\,) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$. In what follows, we identify system (1.1) with its coefficient matrix and hence write $A \in \mathcal{M}_n$. For a system $A \in \mathcal{M}_n$, let $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ denote its Lyapunov exponents [7, p. 567], [5, p. 6], $\mathrm{es}(A)$ its exponential stability index, i.e., the dimension of the linear subspace of solutions to system (1.1) that have negative Lyapunov exponents, and $\sigma_{\mathrm{L}}(A)$ its Lyapunov irregularity coefficient [7, p. 563], [5, p. 10], i.e., the quantity

$$\sigma_{\mathrm{L}}(A) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \lambda_i(A) - \varliminf_{t\to+\infty} \frac{1}{t} \int\limits_0^t \mathrm{tr} A(\tau)\, d\tau,$$

tr being the trace of a matrix. By virtue of the Lyapunov inequality [7, p. 562], the quantity $\sigma_{\mathrm{L}}(A)$ is nonnegative.

The Lyapunov irregularity coefficient is one of the most important asymptotic characteristics of systems in the class $\mathcal{M}_n$. The condition $\sigma_{\mathrm{L}}(A) = 0$ singles out in $\mathcal{M}_n$ the subclass $\mathcal{R}_n$ of Lyapunov regular systems, historically the first class of systems for which the problem of conditional stability by the first approximation was solved in the affirmative [7, p. 578]. Moreover, this coefficient is used to state sufficient conditions characterizing the response of a system $A \in \mathcal{M}_n$ to both exponentially decaying linear perturbations and higher-order nonlinear perturbations. For example, the Lyapunov exponents of a system $A \in \mathcal{M}_n$ are preserved under linear exponentially decaying perturbations $Q(\,\cdot\,)$, whenever the estimate $\|Q(t)\| \leqslant C\exp(-\sigma t)$, $t \in \mathbb{R}_+$, holds with some constants $C > 0$ and $\sigma > \sigma_{\mathrm{L}}(A)$ [3]. If for a higher-order perturbation $f(t,x)$ ($\|f(t,x)\| \leqslant \mathrm{const}\|x\|^m$, $t \in \mathbb{R}_+$, $m = const > 1$) of a system $A \in \mathcal{M}_n$ its order $m > 1$ satisfies the estimate $(m-1)\lambda_n(A) + \sigma_{\mathrm{L}}(A) < 0$, then the trivial solution of the perturbed system is stable (the Lyapunov–Massera theorem [7, pp. 578–579], [8]).

It was a long-standing conjecture that the Lyapunov exponents of Lyapunov regular systems are invariant under perturbations vanishing at infinity. The conjecture was based essentially on the fundamental result by Lyapunov which claims that if a nonlinear system (with natural restrictions on the right-hand side) has a regular first approximation system and the latter is conditionally exponentially stable, then so is the zero solution of the original system (with the same dimension of the stable manifold and asymptotic exponent) [7, pp. 576–578]. Nevertheless, in the paper [10] R. È. Vinograd provided an example of systems $A, B \in \mathcal{R}_2$ satisfying

$$\lambda_1(A) = \lambda_2(A) = 0, \quad \lambda_1(B) = -1, \quad \lambda_2(B) = 1, \quad \lim_{t\to+\infty} \|A(t) - B(t)\| = 0.$$

From this result it follows, in particular, that the exponential stability index $\mathrm{es}(\,\cdot\,)$ – a function taking exactly $n+1$ values – is not upper semicontinuous even on the set $\mathcal{R}_n$ of Lyapunov regular systems with the topology of uniform convergence of coefficients on the semiaxis.

Let M be a metric space. Consider a family

$$\dot{x} = A(t, \mu)x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+, \tag{1.2}$$

of linear differential systems depending on a parameter $\mu \in M$ such that for each $\mu \in M$ the matrix-valued function $A(\,\cdot\,, \mu) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is continuous and bounded (for every μ, generally, by a different constant). Therefore, fixing a value of the parameter $\mu \in M$ in family (1.2) we obtain a linear differential system with continuous coefficients bounded on the semiaxis. We denote by $\mathrm{es}(\mu; A)$ its exponential stability index and by $\sigma_{\mathrm{L}}(\mu; A)$ its Lyapunov irregularity coefficient.

It is customary to consider a family of matrix-valued functions $A(\,\cdot\,, \mu)$, $\mu \in M$, under one of the following two natural assumptions: that the family is continuous either **a)** in the compact-open topology, or **b)** in the uniform topology. The condition **a)** is equivalent to the fact that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ of points from M converges to a point μ_0, then the sequence of functions $A(t, \mu_k)$ of the variable $t \in \mathbb{R}_+$ converges to the function $A(t, \mu_0)$ as $k \to +\infty$ uniformly on each segment $[0, T] \subset \mathbb{R}_+$, while the condition **b)** is equivalent to the fact that this convergence is uniform over the whole semiaxis $\mathbb{R}_+$. Denote the class of families (1.2) that are continuous in the compact-open topology by $\mathcal{C}^n(M)$ and the class of those that are continuous in the uniform topology by $\mathcal{U}^n(M)$. It is clear that a proper inclusion $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$ holds. In the sequel, we will identify families (1.2) with the matrix-valued functions $A(\,\cdot\,,\,\cdot\,)$ defining them and therefore write $A \in \mathcal{C}^n(M)$ or $A \in \mathcal{U}^n(M)$.

Along with the class $\mathcal{U}^n(M)$ we consider its subclass $\mathcal{U}\mathcal{Z}^n_{\mathcal{R}}(M)$, which is defined as follows. For a number $n \in \mathbb{N}$ and a metric space M, denote by $\mathcal{Z}_n(M)$ the class of jointly continuous matrix-valued functions $Q(\,\cdot\,,\,\cdot\,) : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ that vanish at infinity uniformly over $\mu \in M$ (the last means that $\sup\limits_{\mu\in M} \|Q(t, \mu)\| \to 0$ as $t \to +\infty$). The class $\mathcal{U}\mathcal{Z}^n_{\mathcal{R}}(M)$ comprises families

$$\dot{x} = (B(t) + Q(t, \mu))x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+, \tag{1.3}$$

where $B \in \mathcal{R}_n$ and $Q \in \mathcal{Z}_n(M)$. Denoting the coefficient matrix of family (1.3) by $A(t, \mu)$ and, as above, identifying it with the family itself, we will write $A \in \mathcal{U}\mathcal{Z}^n_{\mathcal{R}}(M)$.

Problem. For any $n \in \mathbb{N}$ and metric space M obtain a complete function-theoretic description for each of the function classes:

$$\mathfrak{T}\big[\mathcal{C}^n(M)\big] \stackrel{\mathrm{def}}{=} \Big\{\big(\sigma_{\mathrm{L}}(\,\cdot\,; A), \mathrm{es}(\,\cdot\,; A)\big) : \; A \in \mathcal{C}^n(M)\Big\},$$
$$\mathfrak{T}\big[\mathcal{U}^n(M)\big] \stackrel{\mathrm{def}}{=} \Big\{\big(\sigma_{\mathrm{L}}(\,\cdot\,; A), \mathrm{es}(\,\cdot\,; A)\big) : \; A \in \mathcal{U}^n(M)\Big\},$$
$$\mathfrak{T}\big[\mathcal{U}\mathcal{Z}^n_{\mathcal{R}}(M)\big] \stackrel{\mathrm{def}}{=} \Big\{\big(\sigma_{\mathrm{L}}(\,\cdot\,; A), \mathrm{es}(\,\cdot\,; A)\big) : \; A \in \mathcal{U}\mathcal{Z}^n_{\mathcal{R}}(M)\Big\}.$$

2 Preceding results

Let us recall that a function $f : M \to \mathbb{R}$ is said [4, pp. 266–267] to be of the class $({}^*, G_\delta)$ if for each $r \in \mathbb{R}$, the preimage $f^{-1}([r, +\infty))$ of the half-interval $[r, +\infty)$ is a G_δ-set of the metric space M. In particular, the class $({}^*, G_\delta)$ is a proper subclass of the second Baire class [4, p. 294]. Recall also that a function $m : M \to \mathbb{R}$ is called a majorant of a function $f : M \to \mathbb{R}$ if $f(x) \le m(x)$ for all $x \in M$.

A complete description of the classes

$$\mathfrak{S}[\mathcal{U}^n(M)] \stackrel{\text{def}}{=} \big\{\sigma_{\mathrm{L}}(\,\cdot\,; A) :\ A \in \mathcal{U}^n(M)\big\} \ \text{ and } \ \mathfrak{S}[\mathcal{UZ}^n_{\mathcal{R}}(M)] \stackrel{\text{def}}{=} \big\{\sigma_{\mathrm{L}}(\,\cdot\,; A) :\ A \in \mathcal{UZ}^n_{\mathcal{R}}(M)\big\},$$

i.e., the classes made up of the first elements of pairs in the classes $\mathfrak{T}[\mathcal{U}^n(M)]$ and $\mathfrak{T}[\mathcal{UZ}^n_{\mathcal{R}}(M)]$, respectively, is obtained in the paper [2] and is as follows: the classes $\mathfrak{S}[\mathcal{U}^n(M)]$ and $\mathfrak{S}[\mathcal{UZ}^n_{\mathcal{R}}(M)]$ coincide with one another and consist of functions $M \to \mathbb{R}_+$ of the class $({}^*, G_\delta)$ that have a continuous majorant.

A description of the class $\mathfrak{S}[\mathcal{C}^n(M)] \stackrel{\text{def}}{=} \{\sigma_{\mathrm{L}}(\,\cdot\,; A) :\ A \in \mathcal{C}^n(M)\}$ follows from the result of the paper [8]: for any $n \in \mathbb{N}$ and metric space M the class $\mathfrak{S}[\mathcal{C}^n(M)]$ consists of all functions $M \to \mathbb{R}_+$ of the class $({}^*, G_\delta)$. This description can also be immediately drawn from a more general result obtained in the paper [11], which is a complete description of the class $\{(\sigma_{\mathrm{L}}(\,\cdot\,; A), \sigma_{\mathrm{P}}(\,\cdot\,; A)) :\ A \in \mathcal{C}^n(M)\}$ of vector functions composed of the Lyapunov irregularity coefficient σ_{L} and the Perron one σ_{P} [2, p. 10] for families in $\mathcal{C}^n(M)$: for any $n \ge 2$ and metric space M a vector function $(\sigma_1, \sigma_2) : M \to \mathbb{R}^2_+$ belongs to the above mentioned class if and only if the functions σ_1 and σ_2 are $({}^*, G_\delta)$ and for all $\mu \in M$, the inequalities $0 \leqslant \sigma_2(\mu) \leqslant \sigma_1(\mu) \leqslant n\sigma_2(\mu)$ hold. (Recall that the Perron irregularity coefficient $\sigma_{\mathrm{P}}(A)$ of a system $A \in \mathcal{M}_n$ is defined by the equality

$$\sigma_{\mathrm{P}}(A) \stackrel{\text{def}}{=} \max_{1 \le i \le n} \big\{\lambda_i(A) + \lambda_{n-i+1}(-A^T)\big\};$$

$\sigma_{\mathrm{P}}(\,\cdot\,; A)$ stands for the Perron irregularity coefficient of family (1.2).)

A description of the classes $\{\mathrm{es}(\,\cdot\,; A) :\ A \in \mathcal{C}^n(M)\}$ and $\{\mathrm{es}(\,\cdot\,; A) :\ A \in \mathcal{U}^n(M)\}$ is obtained in the paper [1]: both classes consist of functions $f : M \to \{0, \ldots, n\}$ such that the function $(-f)$ is of the class $({}^*, G_\delta)$.

3 The main result

Theorem 3.1. *For any $n \ge 1$ and metric space M a pair of functions (σ, s), where $\sigma : M \to \mathbb{R}_+$ and $s : M \to \{0, \ldots, n\}$, belongs to the class $\mathfrak{T}[\mathcal{C}^n(M)]$ if and only if the functions σ and $(-s)$ are of the class $({}^*, G_\delta)$.*

Unfortunately, the authors of the report failed to completely solve the above stated problem on description of the classes $\mathfrak{T}[\mathcal{U}^n(M)]$ and $\mathfrak{T}[\mathcal{UZ}^n_{\mathcal{R}}(M)]$. Below we consider a simplified version of the problem.

Following the report [9], which treats an analogous quantity, we call the *indicator of total exponential instability* of system (1.1) the quantity $\mathrm{ti}(A)$ defined by

$$\mathrm{ti}\,(A) = \begin{cases} 1, & \text{if } \ \lambda_1(A) \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

The next theorem completely describes the classes of pairs of functions

$$\mathfrak{U}\big[\mathcal{U}^n(M)\big] \stackrel{\text{def}}{=} \Big\{\big(\sigma_{\mathrm{L}}(\,\cdot\,; A), \mathrm{ti}(\,\cdot\,; A)\big) :\ A \in \mathcal{U}^n(M)\Big\},$$
$$\mathfrak{U}\big[\mathcal{UZ}^n_{\mathcal{R}}(M)\big] \stackrel{\text{def}}{=} \Big\{\big(\sigma_{\mathrm{L}}(\,\cdot\,; A), \mathrm{ti}(\,\cdot\,; A)\big) :\ A \in \mathcal{UZ}^n_{\mathcal{R}}(M)\Big\}.$$

Theorem 3.2. *For any $n \geq 2$ and metric space M the equality $\mathfrak{U}[\mathcal{U}^n(M)] = \mathfrak{U}[\mathcal{U}\mathcal{Z}^n_{\mathcal{R}}(M)]$ is valid. A pair of functions (σ, t), where $\sigma : M \to \mathbb{R}_+$ and $t : M \to \{0, 1\}$, belongs to the above defined classes if and only if the functions σ and t are of the class $({}^*, G_\delta)$ and the function σ has a continuous majorant.*

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Slowly Varying Solutions of Essentially Nonlinear Differential Equations of Second Order

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The differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')\exp(R(|\ln|yy'||)), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ are continuous functions, $Y_i \in \{0, \pm\infty\}$ $(i=0,1)$, Δ_{Y_i} is a onesided neighborhood of Y_i, every function $\varphi_i(z)$ $(i = 0,1)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order σ_i, $\sigma_0+\sigma_1 \ne 1$, $\sigma_1 \ne 0$, the function $R :]0,+\infty[\to]0,+\infty[$ is continuously differentiable and regularly varying on infinity of the order μ, $0<\mu<1$, the derivative function of the function R is monotone, is considered in the work.

Definition. The solution y of equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$ if it is defined on $[t_0,\omega[\subset [a,\omega[$ and

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

A lot of works (see, for example, [2, 3]) have been devoted to the establishing asymptotic representations of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equations of the form (1), in which $R \equiv 0$. The $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) are regularly varying functions as $t \uparrow \omega$ of index $\frac{\lambda_0}{\lambda_0-1}$ if $\lambda_0 \in R \setminus \{0,1\}$. The asymptotic properties and necessary and sufficient conditions of existence of such solutions of equation (1) have been received in [1].

The case $\lambda_0 = 0$ is one of cases of the most difficulty because in this cases such solutions are slowly varying functions as $t \uparrow \omega$. Some results about asymptotic properties and existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) in this special case are presented in the work.

We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0,+\infty[$ satisfies the condition S, if for any continuous differentiable function $L : \Delta_{Y_i} \to]0,+\infty[$ such that

$$\lim_{\substack{z\to Y_i \\ z\in\Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the next equality

$$\Theta(zL(z)) = \Theta(z)(1+o(1)) \ \text{ is true as } \ z \to Y \ \ (z \in \Delta_Y)$$

holds.

We need the next subsidiary notations

$$\pi_\omega(t)=\begin{cases} t & \text{as } \ \omega=+\infty,\\ t-\omega & \text{as } \ \omega<+\infty, \end{cases}\quad \Theta_i(z)=\varphi_i(z)|z|^{-\sigma_i}\ \ (i=0,1),$$

$$I(t)=\alpha_0\int\limits_{A_\omega}^{t}p(\tau)d\tau,\quad A_\omega=\begin{cases} a, & \text{if } \displaystyle\int\limits_a^\omega p(\tau)\,d\tau=+\infty,\\ \omega, & \text{if } \displaystyle\int\limits_a^\omega p(\tau)\,d\tau<+\infty. \end{cases}$$

In the case

$$\lim_{t\uparrow\omega}\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}=Y_1,$$

we put

$$J(t)=\int\limits_{B_\omega}^{t}\Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega=\begin{cases} b, & \text{if } \displaystyle\int\limits_b^\omega \Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\ \omega, & \text{if } \displaystyle\int\limits_b^\omega \Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty, \end{cases}$$

$$N(t)=\frac{(1-\sigma_1)I(t)|(1-\sigma_1)I(t)\Theta_1(\frac{y_1^0}{|\pi_\omega(t)|})|^{\frac{1}{\sigma_1-1}}}{I'(t)R'(|\ln|\pi_\omega(t)||)}\,.$$

Theorem 1. *Let in equation* (1) *the function $\varphi_1(y')$ satisfy the condition S and the next condition take place*

$$\lim_{t\uparrow\omega}\frac{R(|\ln|\pi_\omega(t)||)J(t)}{\pi_\omega(t)\ln|\pi_\omega(t)|J'(t)}=0. \tag{2}$$

Then for the existence of $P_\omega(Y_0,Y_1,0)$-solutions of equation (1) *the next conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega}y_0^0|J(t)|^{\frac{1-\sigma_1}{1-\sigma_0-\sigma_1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{J'(t)}{y_1^0|J(t)|}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\sigma_1-1,$$

$$\frac{I(t)}{y_1^0(1-\sigma_1)}>0\ \ \text{as}\ \ t\in\,]a,\omega[\,,\qquad \frac{y_0^0y_1^0(1-\sigma_1)J(t)}{1-\sigma_0-\sigma_1}>0\ \ \text{as}\ \ t\in\,]b,\omega[\,.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$:

$$\frac{y(t)}{|\exp(R(|\ln|y(t)y'(t)||))\varphi_0(y(t))|^{\frac{1}{1-\sigma_1}}}=\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1|^{\frac{1}{1-\sigma_1}}J(t)[1+o(1)],$$

$$\frac{y(t)}{y'(t)}=\frac{(1-\sigma_0-\sigma_1)J(t))}{(1-\sigma_1)J'(t)}\,[1+o(1)].$$

Theorem 2. *Let in Theorem 1 condition* (2) *do not hold but the function* φ_1 *satisfy the condition* S*,* p *be a twice continuously differentiable function, and the next condition takes place*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)N'(t)}{R'(|\ln|\pi_\omega(t)||)N(t)}=0.$$

Then for the existence of $P_\omega(Y_0,Y_1,0)$*-solutions of equation* (1)*, for which there exists a finite or infinite limit* $\lim\limits_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}$*, the next conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega}y_0^0\exp\big(R(|\ln|\pi_\omega(t)||)\big)^{\frac{\sigma_1-1}{1-\sigma_0-\sigma_1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{-\alpha_0}{\pi_\omega(t)}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\frac{\sigma_1-1}{\alpha_0},$$
$$\alpha_0y_1^0\pi_\omega(t)<0,\quad \alpha_0(1-\sigma_1)(1-\sigma_0-\sigma_1)y_0^0R'\big(|\ln|\pi_\omega(t)||\big)>0.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$*:*

$$\frac{y(t)}{|\varphi_0(y(t))\exp(R(|\ln|y(t)y'(t)||))|^{\frac{1}{1-\sigma_1}}}=\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}N(t)[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{I'(t)R'(|\ln|\pi_\omega(t)||)}{(1-\sigma_0-\sigma_1)(1-\sigma_1)I(t)}[1+o(1)].$$

Theorem 3. *Let in Theorem 1 conditions* (2) *do not hold but the function* φ_1 *satisfy the condition* S*,* p *be a twice continuously differentiable function, and the next condition takes place*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)N'(t)}{R'(|\ln|\pi_\omega(t)||)N(t)}=M\in R\setminus\{0,1\}.$$

Then for the existence of $P_\omega(Y_0,Y_1,0)$*-solutions of equation* (1)*, for which there exists a finite or infinite limit* $\lim\limits_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}$*, the next conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega}y_0^0\big(\exp(R(|\ln|\pi_\omega(t)||))\big)^{\frac{\sigma_1-1}{1-\sigma_0-\sigma_1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{-\alpha_0}{\pi_\omega(t)}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\frac{\sigma_1-1}{\alpha_0},$$
$$\alpha_0y_1^0\pi_\omega(t)<0,\quad \alpha_0(1-M)(1-\sigma_1)(1-\sigma_0-\sigma_1)y_0^0R'\big(|\ln|\pi_\omega(t)||\big)>0.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$*:*

$$\frac{y(t)}{|\varphi_0(y(t))\exp(R(|\ln|y(t)y'(t)||))|^{\frac{1}{1-\sigma_1}}}=\frac{1-\sigma_0-\sigma_1}{(1-\sigma_1)(1-M)}N(t)[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{I'(t)R'(|\ln|\pi_\omega(t)||)(1-M)}{(1-\sigma_0-\sigma_1)(1-\sigma_1)I(t)}[1+o(1)].$$

Let consider some more specific class of differential equations of the form (1) and use Theorems 1, 2 and 3. The differential equation

$$y''=mt^{\sigma_1-2}\exp(k\ln^\gamma t)|y|^{\sigma_0}|y'|^{\sigma_1}\exp\big((|\ln|yy'||)^\mu\big) \tag{3}$$

on the interval $[t_0,+\infty[$ $(t_0>0)$, where $m\in]-\infty,0[$, $k\in]0,+\infty[$, $\gamma,\mu\in]0;1[$, $\sigma_0,\sigma_1\in\mathbb{R}$, $\sigma_0+\sigma_1\neq1$, $\sigma_1\neq1$, is the equation of the form (1), where

$$\alpha_0=\operatorname{sign}m=-1,\ \ p(t)=mt^{\sigma_1-2}\exp(k\ln^\gamma t),\ \ \varphi_0=|y|^{\sigma_0},\ \ \varphi_1=|y|^{\sigma_1},\ \ R(z)=z^\mu.$$

This function φ_1 satisfies the condition S. Let consider the case when $\omega=Y_0=Y_1=+\infty$.

Using Theorem 1 we obtain that if $\mu-\gamma<0$, then for the existence $P_{+\infty}(+\infty,+\infty,0)$-solutions of equation (3) the following condition

$$1-\sigma_0-\sigma_1>0 \tag{4}$$

is necessary and sufficient.

Moreover, for each such solution the following asymptotic representations take place as $t\to+\infty$:

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}}\exp\Big(\frac{|\ln|y(t)y'(t)||^{\mu}}{\sigma_1-1}\Big)=\frac{1-\sigma_0-\sigma_1}{\gamma k}\,\exp\Big(\frac{k\ln^{\gamma}t}{1-\sigma_1}\Big)\ln^{1-\gamma}t[1+o(1)],$$
$$\frac{y(t)}{y'(t)}=\frac{(1-\sigma_0-\sigma_1)\gamma k}{(1-\sigma_1)^2}\,\frac{\ln^{\gamma-1}t}{t}\,[1+o(1)].$$

Let us now consider the case $\mu-\gamma>0$. In this case by Theorem 2 we obtain that for $\mu-\gamma>0$ for existence of $P_{+\infty}(+\infty,+\infty,0)$-solutions to equation (3) condition (4) is necessary and sufficient. Moreover, each such solution satisfies the next asymptotic representations as $t\to+\infty$:

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}}\exp\Big(\frac{|\ln|y(t)y'(t)||^{\mu}}{\sigma_1-1}\Big)=\frac{1-\sigma_0-\sigma_1}{\mu(1-\sigma_1)}\,\exp\Big(\frac{k\ln^{\gamma}t}{1-\sigma_1}\Big)\ln^{1-\mu}t[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{\mu}{\sigma_0+\sigma_1-1}\,t^{\sigma_1-2}\ln^{\gamma-1}t[1+o(1)].$$

Let us now consider the case $\mu=\gamma$. By Theorem 3 we obtain that for existence of $P_{+\infty}(+\infty,+\infty,0)$-solutions to equation (3) condition (4) together with the condition

$$(1-\sigma_1-k)(1-\sigma_1)>0$$

is necessary and sufficient. Moreover, each such solution satisfies the next asymptotic representations as $t\to+\infty$:

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}}\exp\Big(\frac{|\ln|y(t)y'(t)||^{\mu}}{\sigma_1-1}\Big)=\frac{1-\sigma_0-\sigma_1}{\mu(1-\sigma_1-k)}\,\exp\Big(\frac{k\ln^{\gamma}t}{1-\sigma_1}\Big)\ln^{1-\mu}t[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{\mu(1-\sigma_1-k)}{(\sigma_0+\sigma_1-1)(1-\sigma_1)}\,t^{\sigma_1-2}\ln^{\gamma-1}t[1+o(1)].$$

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Nonlinear Integro-Differential Fredholm Type Boundary Value Problems Not Solved with Respect to the Derivative with Delay

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The study of the linear differential-algebraic boundary value problems is connected with numerous applications of corresponding mathematical models in the theory of nonlinear oscillations, mechanics, biology, radio engineering, the theory of the motion stability. Thus, the actual problem is the transfer of the results obtained in the articles and monographs of S. Campbell, A. M. Samoilenko and O. A. Boichuk on the nonlinear boundary value problems to the integro-differential boundary value problem of Fredholm type not solved with respect to the derivative, in particular, finding the necessary and sufficient conditions of the existence of the desired solutions of the nonlinear integro-differential boundary value problem not solved with respect to the derivative with delay. We found the conditions of the existence and constructive scheme for finding the solutions of the nonlinear integro-differential boundary value problem not solved with respect to the derivative with delay.

We investigate the problem of finding solutions [3]

$$y(t) \in \mathbb{D}^2[0,T], \ \ y'(t) \in \mathbb{L}^2[0,T]$$

of the linear Noetherian ($n \neq v$) boundary value problem for a system of linear integro-differential equations of Fredholm type not solved with respect to the derivative with delay $[1,3,10]$

$$A(t)y'(t) = B(t)y(t) + C(t)y(h(t)) + \Phi(t)\int\limits_{\Delta}^{T} F(y(s), y(h(s)), y'(s), s)\, ds + f(t), \tag{1}$$

$$y(t) = \varphi(t) \in \mathbb{C}^1[0,\Delta], \ \ \ell y(\,\cdot\,) = \alpha, \ \ \alpha \in \mathbb{R}^{v}. \tag{2}$$

We seek a solution of the boundary value problem $(1),(2)$ in a small neighborhood of the solution

$$y_0(t) \in \mathbb{D}^2[0,T], \ \ y_0'(t) \in \mathbb{L}^2[0,T]$$

of the Noetherian generating problem

$$A(t)y_0'(t) = B(t)y_0(t) + C(t)y(h(t)), \ \ \ell y_0(\,\cdot\,) = \alpha \tag{3}$$

in the case when the matrix $A(t)$ has a variable rank in $[\Delta, T]$. Here

$$A(t), B(t) \in \mathbb{L}^2_{m\times n}[0,T] := \mathbb{L}^2[0,T] \otimes \mathbb{R}^{m\times n}, \ \ \Phi(t) \in \mathbb{L}^2_{m\times q}[0,T],$$
$$f(t) \in \mathbb{L}^2[0,T], \ \ h(t) : [\Delta, T] \to [0,\Delta].$$

We assume that the matrix $A(t)$ is, generally speaking, rectangular: $m \neq n$. Nonlinear vector-function $F(y(t), y(h(t)), y'(t), t)$ is twice continuously differentiable with respect to the unknowns $y(t)$ and with respect to the derivative $y'(t)$ in a small neighborhood of the solution

$$y_0(t) \in \mathbb{C}[0,T], \ \ y_0(t) \in \mathbb{D}^2[0;T], \ \ y_0'(t) \in \mathbb{L}^2[\Delta;T], \ \ T := (q+1)\,\Delta, \ \ q \in \mathbb{N}$$

to the generating problem (3);

$$\ell y(\,\cdot\,) : \mathbb{D}^2[0,T] \to \mathbb{R}^p$$

is a linear bounded vector functional defined on a space $\mathbb{D}^2[0,T]$. The problem of finding solutions of the boundary value problem (1), (2) in case $A(t) = I_n$ was solved by A. M. Samoilenko and A. A. Boichuk [11]. Thus, the boundary value problem (1), (2) is a generalization of the problem solved by A. M. Samoilenko and A. A. Boichuk and also is a generalization of the Noetherian boundary value problems for systems of differential-algebraic equations [4, 7, 8].

Solution of the generating problem (3) can be determined as solution of the problem

$$A(t)y_0'(t) = B(t)y_0(t) + g(t), \quad g(t) := C(t)\varphi(h(t)) + f(t). \tag{4}$$

Let the differential-algebraic system (4) with the constant-rank matrix $A(t)$ satisfy the conditions of the theorem from the paper [7, p. 15]. Then, in the case of the p-order degeneration, the differential-algebraic system (4) has a solution which can be written in the form

$$y_0(t, c_{\rho_{p-1}}) = X_p(t)c_{\rho_{p-1}} + K\big[g(s), \nu_p(s)\big](t), \ \ t \in [\Delta;T], \ \ c_{\rho_{p-1}} \in \mathbb{R}^{\rho_{p-1}}.$$

There $K[g(s), \nu_p(s)](t)$ is generalized Green's operator of the Cauchy problem for the differential-algebraic system (4) where $\nu_p(t)$ is an arbitrary continuous vector function. Substituting the general solution of the Cauchy problem for the differential-algebraic system (4), namely,

$$y_0(t) := K_\Delta\big[f(s), \varphi(s), \nu_p(s)\big](t), \ \ t \in [\Delta;T]$$

into the boundary condition (1), we arrive at the linear algebraic equation

$$P_{X_p^*}(\Delta)\Big\{\varphi(\Delta) - K\big[g(s), \nu_p(s)\big](\Delta)\Big\} = 0, \tag{5}$$

where $P_{X_p^*}(\Delta)$ is an orthoprojector,

$$\begin{aligned} &K_\Delta\big[f(s), \varphi(s), \nu_p(s)\big](t) \\ &\qquad := X_p(t)X_p^+(\Delta)\Big\{\varphi(\Delta) - K\big[g(s), \nu_p(s)\big](\Delta)\Big\} + K\big[g(s), \nu_p(s)\big](t), \ \ t \in [\Delta;T]. \end{aligned}$$

Linear bounded vector functional $\ell y(\,\cdot\,)$ present in the form

$$\ell y(\,\cdot\,) = \ell_0 y(\,\cdot\,) + \ell_1 y(\,\cdot\,) : \mathbb{C}[0;T] \to \mathbb{R}^\upsilon,$$

where

$$\ell_0 y(\,\cdot\,) := \int\limits_0^\Delta dW(t)\,y(t) : \mathbb{C}[0;\Delta] \to \mathbb{R}^\upsilon, \quad \ell_1 y(\,\cdot\,) := \int\limits_\Delta^T dW(t)\,y(t) : \mathbb{C}[\Delta;T] \to \mathbb{R}^\upsilon;$$

$W(t)$ is an $(\upsilon \times n)$ matrix whose entries are functions of bounded variation on $[0,T]$, and the integral used to represent linear functionals is understood in the Riemann–Stieltjes sense [3]. Let the differential-algebraic system (4) with the constant-rank matrix $A(t)$ satisfy the conditions of the theorem from the paper [7, p. 15]. Only if condition

$$\ell_0\,\varphi(\,\cdot\,) + \ell_1 K_\Delta\big[f(s), \varphi(s), \nu_p(s)\big](\,\cdot\,) = \alpha$$

is satisfied, the general solution of the differential-algebraic system (4)

$$y_0(t) = G\big[f(s), \varphi(s); \alpha\big](t), \quad t \in [\Delta; T]$$

determines the solution of the nonlinear differential-algebraic boundary-value problem (1), (2), where [7]

$$G[f(s), \varphi(s); \alpha](t) := K_\Delta\big[f(s), \varphi(s), \nu_p(s)\big](t), \quad t \in [\Delta; T].$$

We found the conditions of the existence and constructive scheme for finding the solutions of the nonlinear integro-differential boundary value problem (1), (2) not solved with respect to the derivative with delay.

Conditions for the solvability of the linear boundary-value problem for systems of differential-algebraic equations (3) with the variable rank of the leading-coefficient matrix and the corresponding solution construction procedure have been found in the paper [9]. In the case of nonsolvability, the nonsingular integro-differential boundary value problems can be regularized analogously [6, 12].

The proposed scheme of studies of the nonlinear integro-differential boundary value problems of Fredholm type not solved with respect to the derivative with delay (1), (2) is a generalization of the Noetherian boundary-value problems for systems of differential-algebraic equations [2, 4, 5, 7–9].

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Fredholm Boundary-Value Problem for the System of Fractional Differential Equations with Caputo Derivative

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In the space $C[a,b]$, $-\infty < a < b < +\infty$, we consider a linear boundary-value problem for the system of fractional differential equations

$$
{}^{C}\mathrm{D}^{\alpha}_{a+}x(t) = A(t)x(t) + f(t), \tag{1}
$$

$$
lx(\,\cdot\,) = q, \tag{2}
$$

where ${}^{C}\mathrm{D}^{\alpha}_{a+}$ is the left Caputo fractional derivative of order α $(0 < \alpha < 1)$ [6, 7, 14]

$$
{}^{C}\mathrm{D}^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(m-\alpha)} \int\limits_a^t \frac{x^{(m)}(s)}{(t-s)^{\alpha-m+1}}\,\mathrm{d}s,
$$

$A(t)$ is an $(n\times n)$-matrix and $f(t)$ is an n-vector, whose components are real functions continuous on $[a,b]$, $l = \operatorname{col}(l_1, l_2, \dots, l_p) : C[a,b] \to \mathbb{R}^p$ is bounded linear vector functional, $l_\nu : C[a,b] \to \mathbb{R}$, $\nu = \overline{1,p}$, $q = \operatorname{col}(q_1, q_2, \dots, q_p) \in \mathbb{R}^p$.

Using the results [1, 2, 5, 15], obtained as a generalization of the classical methods of the theory of periodic boundary-value problems in the theory of oscillations (see [10–13]), we consider the questions of finding necessary and sufficient conditions of solvability and determine a general form of solutions of the boundary-value problem for the systems of fractional differential equations (1), (2). Let us first consider the general solution of system (1) of the form

$$
x(t) = X(t)c + \overline{x}(t) \ \ \forall\, c \in \mathbb{R}^n, \tag{3}
$$

where $X(t)$ is the fundamental solution $(n\times n)$-matrix of the homogeneous system (1) $(f = 0)$, whose column vectors constitute a fundamental system of solutions to the homogeneous system (1) and $\overline{x}(t)$ is an arbitrary special solution of the inhomogeneous system (1). The required special solution $\overline{x}(t)$ can be chosen as a solution of the system of linear Volterra integral equation of the second kind

$$
\overline{x}(t) = g(t) + \int\limits_a^t K(t,s)\overline{x}(s)\,ds, \tag{4}
$$

$$
g(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t \frac{f(s)}{(t-s)^\gamma}\,ds, \quad K(t,s) = \frac{A(s)}{\Gamma(\alpha)(t-s)^\gamma}\,, \tag{5}
$$

$0 < \gamma = 1 - \alpha < 1$.

The solution of the system of equations (4) can be found by different methods. We apply the approach described in [3, 4]. In the Hilbert space $L_2[a,b]$, we show that system (4) with unbounded kernel $K(t,s)$ (5) can be reduced to an equivalent system with square summable kernel. To do this, we consider iterated kernels $K_m(t,s)$, $m\in\mathbb{N}$, given by the recurrence relations

$$K_{m+1}(t,s)=\int\limits_s^t K(t,\xi)K_m(\xi,s)\,\mathrm{d}\xi,\quad K_1(t,s)=K(t,s).$$

The iterated kernels $K_m(t,s)$ have the same structure as weakly singular kernel $K(t,s)$ (5) but the number γ is replaced with the number $1-m(1-\gamma)$ which is negative for sufficiently large m. Therefore (see [9, p. 34]), for all m by which the condition

$$m>\frac{1}{2(1-\gamma)} \tag{6}$$

is satisfied, the kernels $K_m(t,s)$ are square summable.

System (4) can be reduced to a similar system with the kernel $K_m(t,s)$

$$\overline{x}(t)=g^m(t)+\int\limits_a^t K_m(t,s)\overline{x}(s)\,ds, \tag{7}$$

$$g^m(t)=g(t)+\sum_{l=1}^{m-1}\int\limits_a^t K_l(t,s)g(s)\,ds.$$

We apply the approach described in [3, 4] to the study of system (7) and show that it can be reduced to the system

$$\Lambda z=g, \tag{8}$$

where the vectors z, g and the block matrix Λ have the form

$$z=\operatorname{col}(x_1,x_2,\dots,x_i,\dots),\quad g=\operatorname{col}(g_1,g_2,\dots,g_i,\dots),$$

$$\Lambda=\begin{pmatrix}\Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1i} & \cdots\\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2i} & \cdots\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \Lambda_{i1} & \Lambda_{i2} & \cdots & \Lambda_{ii} & \cdots\\ \vdots & \vdots & \dots & \vdots & \ddots\end{pmatrix},\quad \Lambda_{ij}=\begin{cases} I_n-A_{ij}, & \text{if } i=j;\\ -A_{ij}, & \text{if } i\neq j,\end{cases}$$

$$x_i=\int\limits_a^b x(t)\varphi_i(t)\,dt,\quad g_i=\int\limits_a^b g^m(t)\varphi_i(t)\,dt, \tag{9}$$

$$A_{ij}=\int\limits_a^b\int\limits_a^t K_m(t,s)\varphi_i(t)\varphi_j(s)\,dt\,ds, \tag{10}$$

$\{\varphi_i(t)\}_{i=1}^{\infty}$ is a complete orthonormal system of functions in $L_2[a,b]$.

Here, I_n is the identity matrix of dimensions n, the operator $\Lambda:\ell_2\to\ell_2$ appearing on the left-hand side of the operator equation (8) has the form $\Lambda=I-A$, where $I:\ell_2\to\ell_2$ is the identity operator and $A:\ell_2\to\ell_2$ is a compact Volterra operator (see [8]). Hence, $P_\Lambda=P_\Lambda^*=O$, $\Lambda^+=\Lambda^{-1}$. According to [5], the homogeneous equation (8) ($g=0$) possesses a unique solution $z=0$ and the inhomogeneous equation (8) possesses a unique solution of the form $z=\Lambda^{-1}g$.

According to the Riesz–Fischer theorem, one can find an element $\overline{x} \in L_2[a,b]$ such that the quantities x_i, $i = \overline{1,\infty}$ are the Fourier coefficients of this element. Thus, the following representation is true:

$$\overline{x}(t) = \sum_{i=1}^{\infty} x_i \varphi_i(t) = \Phi(t) z = \Phi(t) \Lambda^{-1} g, \tag{11}$$

where

$$\Phi(t) = (\varphi_1(t), \varphi_2(t), \dots \varphi_i(t), \dots).$$

The element $\overline{x}(t)$ given by relations (11) is the required solution of system (7).

We now return to the problem on the existence of a solution of the boundary-value problem (1), (2) and determine a structure of this solution. Substituting (3) in condition (2), we obtain the following algebraic system for vector c:

$$Qc = b, \tag{12}$$

where a $(p \times n)$-matrix Q and a p-vector b having the forms

$$Q = (lX)(\,\cdot\,), \quad b = q - (l\,\overline{x})(\,\cdot\,). \tag{13}$$

According to the criterion for solvability of system (12) (see [5, p. 65]), the following assertion is true.

Theorem. *The homogeneous boundary-value problem* (1), (2) ($f(t) = 0$, $q = 0$) *possesses a d_2-parameter family of solutions*

$$x(t) = X(t) P_{Q_{d_2}} c_{d_2} \ \ \forall\, c_{d_2} \in \mathbb{R}^{d_2}.$$

The inhomogeneous boundary-value problem (1), (2) *is solvable if and only if d_1 linearly independent conditions*

$$P_{Q^*_{d_1}} b = 0, \quad d_1 = p - \operatorname{rank} Q$$

are satisfied and possesses a d_2-parameter family of solutions $x \in C[a,b]$ of the form

$$x(t) = X(t) P_{Q_{d_2}} c_{d_2} + X(t) Q^{+} b + \overline{x}(t) \ \ \forall\, c_{d_2} \in \mathbb{R}^{d_2}.$$

Here, $P_{Q_{d_2}}$ is an $(r \times d_2)$-matrix formed by a complete system of d_2 linearly independent columns of the matrix projector P_Q, where P_Q is the projector onto the kernel of the matrix Q, Q^+ is the pseudoinverse Moore–Penrose $(n \times p)$-matrix for the matrix Q, and $P_{Q^*_{d_1}}$ is a $(d_1 \times p)$-matrix formed by the complete system of d_1 linearly independent rows of the matrix projector P_{Q^*}, where P_{Q^*} is the projector onto the cokernel of the matrix Q.

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On the Solvability of the Cauchy Problem for Second Order Functional Differential Equations with an Alternating Coefficient

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There are many papers devoted to the solvability of the Cauchy problem in the non-Volterra case [1–15]. If the functional operators in the equation don't satisfy the delay conditions, the solvability of the Cauchy problem requires some smallness of these functional operators.

We consider the Cauchy problem for functional differential equations with an alternating coefficient

$$\begin{cases} \ddot{x}(t) = a(t-t_0)x(h(t)) + f(t), \quad t \in [0,1], \\ x(0) = c_0, \quad \dot{x}(0) = c_1, \end{cases} \tag{1}$$

where $a \in \mathbb{R}$, $t_0 \in [0,1]$, $h : [0,1] \to [0,1]$ is a measurable function, $f \in \mathbf{L}[0,1]$, c_0, $c_1 \in \mathbb{R}$. We say that a function $x : [0,1] \to \mathbb{R}$ is a solution of problem (1) if x and the derivative $\dot{x}$ are absolutely continuous on the interval $[0,1]$ and x satisfies the functional differential equation of the problem almost everywhere on $[0,1]$ and satisfies the initial conditions $x(0) = c_0$ and $\dot{x}(0) = c_1$.

Using ideas of [8, 9], we obtain necessary and sufficient conditions for the Cauchy problem

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [0,1], \\ x(0) = c_0, \quad \dot{x}(0) = c_1 \end{cases} \tag{2}$$

to be uniquely solvable for all linear positive operators T^+, $T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that

$$\begin{aligned} (T^+\mathbf{1}\,)(t) &= \begin{cases} a(t-t_0) & \text{if } a(t-t_0) \geq 0, \\ 0, & \text{otherwise,} \end{cases} \\ (T^-\mathbf{1}\,)(t) &= \begin{cases} -a(t-t_0) & \text{if } a(t-t_0) < 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

Here $\mathbf{1} : [0,1] \to \mathbb{R}$, $\mathbf{1}\,(t) = 1$, is the unit function, $\mathbf{C}[0,1]$ and $\mathbf{L}[0,1]$ are the spaces of all continuous and integrable functions with the standard norms respectively, an operator is called *positive* if it maps each non-negative function into almost everywhere non-negative one.

We also need the following notation.

Let $t_{0*} \approx 0,47$ be a solution of the equation

$$\frac{6}{t_0^2(3-t_0)} = \frac{6}{2-3t_0}\,,$$

$t_0^* \approx 0,54$ be a solution of the equation

$$\frac{24}{(3t_0-1)^2} = \frac{6}{(1-t_0)^3}\,.$$

Denote

$$
\begin{aligned}
q_1 &= q_1(t_0,t_1,t_3) = (t_0-t_1)^3 - 3(1-t_1)(t_0-t_3)^2 + 3t_0 - 1,\\
q_2 &= q_2(t_0,t_1,t_3) = t_1^2(3-t_0-2t_3)(t_0-t_3)^2(3t_0-t_1) - (3t_0-1)(t_1-t_3)^2(3t_0-t_1-2t_3),\\
r_1 &= r_1(t_0,t_1,t_3) = \frac{t_1^2(3t_0-t_1)+3(t_0-t_3)^2(t_1-1)}{6}\,,\\
r_2 &= r_2(t_0,t_1,t_3) = \frac{(t_1(t_0+2t_3)(3t_0-t_1)+(3t_1-t_0-2t_3)(1+t_1-3t_0))(t_0-t_3)^2(t_1-1)}{36}\,,
\end{aligned}
$$

$$
A^+(t_0) = \begin{cases} \dfrac{6}{(1-t_0)^3} & \text{if } \ t_0 \in [0,t_0^*],\\[2ex] \displaystyle\min_{0<t_3\le t_1<t_0} \frac{3(q_1+\sqrt{q_1^2+4\,q_2})}{q_2} & \text{if } \ t_0 \in (t_0^*,1], \end{cases}
$$

$$
A^-(t_0) = \begin{cases} \displaystyle\min_{0<t_3\le t_1<t_0} \left\{ \frac{3(r_1-\sqrt{r_1^2-4\,r_2})}{r_2}\,,\ \frac{6}{t_0^2(3-t_0)} \right\} & \text{if } \ t_0 \in [0,t_{0*}),\\[2ex] \dfrac{6}{t_0^2(3-t_0)} & \text{if } \ t_0 \in [t_{0*},1]. \end{cases}
$$

Theorem. *Problem* (2) *is uniquely solvable for all linear positive operator* T^+*,* T^- : $\mathbf{C}[0,1] \to \mathbf{L}[0,1]$ *satisfied conditions* (3) *if and only if*

$$
-A^-(t_0) < a < A^+(t_0). \tag{4}
$$

Corollary. *Problem* (1) *is uniquely solvable for every measurable function* $h:[0,1]\to[0,1]$ *if and only if condition* (4) *holds.*

Example. For $t_0 \in [1/5, t_0^*]$, we have

$$
A^-(t_0) = \frac{6}{t_0^2(3-t_0)}\,, \quad A^+(t_0) = \frac{6}{(1-t_0)^3}\,.
$$

In particular, the problem

$$
\begin{cases} \ddot{x}(t) = a\Big(t-\dfrac{1}{2}\Big)x(h(t)) + f(t), & t \in [0,1],\\ x(0) = c_0, \quad \dot{x}(0) = c_1 \end{cases}
$$

is uniquely solvable for every measurable function $h:[0,1]\to[0,1]$ if and only if

$$
-\frac{48}{5} < a < 48.
$$

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The Asymptotic Representation of $P_\omega(Y_0, Y_1, 1)$-Solutions of Second Order Differential Equations with the Product of Regularly and Rapidly Varying Functions in its Right-Hand Side

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We consider the following differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y')\varphi_1(y). \tag{1}$$

In this equation the constant α_0 is responsible for the sign of the equation, functions $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$ and $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i \in \{0, 1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is the some one-sided neighborhood of Y_i.

We also suppose that function φ_1 is a regularly varying as $y \to Y_1$ function of the index σ_1 [7, pp. 10–15], function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y') \neq 0 \ \text{ as } \ y' \in \Delta_{Y_0}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \varphi_0(y') \in \{0, +\infty\}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \frac{\varphi_0(y')\varphi_0''(y')}{(\varphi_0'(y'))^2} = 1. \tag{2}$$

It follows from conditions (2) that the following statements are true

$$\frac{\varphi_0'(y')}{\varphi_0(y')} \sim \frac{\varphi_0''(y')}{\varphi_0'(y')} \ \text{ as } \ y' \in \Delta_{Y_0}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \frac{y'\varphi_0'(y')}{\varphi_0(y')} = \pm\infty. \tag{3}$$

Also it follows from the above conditions (3) that the function φ_0 and its first-order derivative are rapidly varying functions as the argument tends to Y_0 [1].

So (1) is the second order differential equation that contains in the right-hand side the product of a regularly varying function of unknown function and a rapidly varying function of the first derivative of the unknown function.

In the previous works (see, for example [2]) we obtained results for the second order differential equation containing a rapidly varying function of unknown function and a regularly varying function of its first derivative.

For equation (1) we consider the following class of solutions.

Definition 1. The solution y of the equation (1), that is defined on the interval $[t_0, \omega[\subset [a, \omega[$, is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$, if the following conditions take place

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

In the work we establish the necessary and sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the equation (1) in case $\lambda_0 = 1$ and find asymptotic representations of such solutions and its first order derivatives as $t \uparrow \omega$.

According to the properties of such $P_\omega(Y_0, Y_1, 1)$-solutions (see, for example, [4]) we have that

$$\lim_{t\uparrow\omega} \frac{y'(t)}{y(t)} = \lim_{t\uparrow\omega} \frac{y''(t)}{y'(t)}\,,$$

and

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \pm\infty, \quad \pi_\omega(t) = \begin{cases} t & \text{as } \omega = +\infty, \\ t-\omega & \text{as } \omega < +\infty, \end{cases}$$

So we have that each such $P_\omega(Y_0, Y_1, 1)$-solution and its first-order derivative are rapidly varying functions as $t \uparrow \omega$ and this case of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions is the most difficult.

Let the solution y of equation (1) is a $P_\omega(Y_0, Y_1, 1)$-solution. Note that the function $y(t(y'))$, where $t(y')$ is an inverse function to $y'(t)$, is a regularly varying function of the index 1 as $y' \to Y_0$ $(y' \in \Delta_{Y_0})$.

Indeed, the following statement is true

$$\lim_{y'\to Y_1} \frac{y'(y(t(y')))'}{y(t(y'))} = \lim_{y'\to Y_1} \frac{(y'(t(y')))^2}{y(t(y'))y''(t(y'))} = 1.$$

Definition 2. Let $Y \in \{0, \infty\}$, Δ_Y be some one-sided neighborhood of Y. A continuous-differentiable function $L : \Delta_Y \to]0; +\infty[$ is called [6, pp. 2-3] a normalized slowly varying function as $z \to Y$ $(z \in \Delta_Y)$ if the next statement is true

$$\lim_{\substack{y\to Y\\ y\in\Delta_Y}} \frac{yL'(y)}{L(y)} = 0.$$

Definition 3. We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0; +\infty[$ satisfies the condition S as $z \to Y$, if for any continuous differentiable normalized slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to]0; +\infty[$ the next relation is valid

$$\theta(zL(z)) = \theta(z)(1 + o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

Definition 4. Let's define that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L_0 : \Delta_Y \to]0; +\infty[$ satisfies the condition S_1 as $z \to Y$ if for any finite segment $[a; b] \subset]0; +\infty[$ the next inequality is true

$$\limsup_{\substack{z\to Y\\ z\in\Delta_Y}} \Big| \ln|z| \cdot \Big(\frac{L(\lambda z)}{L(z)} - 1\Big)\Big| < +\infty \ \text{ for all } \ \lambda \in [a; b].$$

Note that

$$\Phi_0(y') = \operatorname{sign} y_1^0 \int\limits_{B_0}^{y} |s|^{\frac{1}{\sigma_1 - 2}} \varphi_0^{\frac{1}{\sigma_1 - 2}}(s)\, ds, \quad B_0 = \begin{cases} y_1^0, & \text{if } \displaystyle\int\limits_{y_0^1}^{Y_0} |s|^{\frac{1}{\sigma_1 - 2}} \varphi_0^{\frac{1}{\sigma_1 - 2}}(s)\, ds = \pm\infty, \\ Y_0, & \text{if } \displaystyle\int\limits_{y_1^0}^{Y_0} |s|^{\frac{1}{\sigma_1 - 2}} \varphi_0^{\frac{1}{\sigma_1 - 2}}(s)\, ds = const, \end{cases}$$

$$\theta_1(z) = \varphi_1(z)|z|^{-\sigma_1}, \quad Z_0 = \lim_{\substack{y'\to Y_0\\ y'\in\Delta_{Y_0}}} \Phi_0(y'), \quad \Phi_1(y') = \int\limits_{B_0}^{y'} \Phi_0(s)\, ds, \quad Z_1 = \lim_{\substack{y'\to Y_0\\ y\in\Delta_{Y_0}}} \Phi_1(y),$$

$$I_0(t)=\int\limits_{A_0}^{t} p^{\frac{1}{2-\sigma_1}}(\tau)\,d\tau,\quad A_0=\begin{cases} a, & \text{if } \int\limits_a^{\omega} p^{\frac{1}{2-\sigma_1}}(\tau)\,d\tau=+\infty,\\ \omega, & \text{if } \int\limits_a^{\omega} p^{\frac{1}{2-\sigma_1}}(\tau)\,d\tau<+\infty \end{cases}$$

in the case $\lim\limits_{t\uparrow\omega} I_0(t)=Z_0$ and $\operatorname{sign} I_0(t)=\operatorname{sign}\Phi_0(y)$, let

$$I_1(t)=\int\limits_{A_1}^{t} \frac{1}{\Phi_0^{-1}(I_0(\tau))))}\,d\tau,\quad A_1=\begin{cases} b, & \text{if } \int\limits_b^{\omega} \frac{1}{\Phi_0^{-1}(I(\tau))))}\,d\tau=\pm\infty,\\ \omega, & \text{if } \int\limits_b^{\omega} \frac{1}{\Phi_0^{-1}(I_0(\tau))))}\,d\tau=const, \quad b\in[a;\omega[\,, \end{cases}$$

$$I_2(t)=-\int\limits_{A_2}^{t} \Big(\frac{I_0(\tau)}{I_1(\tau)}\Big)\,d\tau,\quad A_2=\begin{cases} b, & \text{if } \int\limits_b^{\omega} \Big(\frac{I_0(\tau)}{I_1(\tau)}\Big)\,d\tau=\pm\infty,\\ \omega, & \text{if } \int\limits_b^{\omega} \Big(\frac{I_0(\tau)}{I_1(\tau)}\Big)\,d\tau=const. \end{cases}$$

Note 1. The following statements are true:

1)
$$\Phi_0(z)=(\sigma_1-1)\,\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(z)}{\varphi_0'(z)}\,[1+o(1)] \ \text{ as } \ z\to Y_0 \ \ (z\in\Delta_{Y_0}).$$

From this we have
$$\operatorname{sign}(\varphi_0'(z)\Phi_0(z))=\operatorname{sign}(\sigma_1-1) \ \text{ as } \ z\in\Delta_{Y_0}.$$

2)
$$\Phi_1(z)=\frac{\Phi_0^2(z)}{z\Phi_0'(z)}\,[1+o(1)] \ \text{ as } \ z\to Y_1 \ \ (z\in\Delta_{Y_0}).$$

From this we have
$$\operatorname{sign}(\Phi_1(z))=y_0^1 \ \text{ as } \ z\in\Delta_{Y_0}.$$

3) The functions Φ_0^{-1} and Φ_1^{-1} exist and are slowly varying functions as inverse to rapidly varying functions as the arguments tend to Y_0 functions.

4) The function Φ_1' (Φ_1^{-1}) is a regularly varying function of the index 1 as the argument tends to Y_0.

Note 2. The function $\theta_1(y(t(y')))$ is a slowly varying function for $y'\to Y_0$ $(y'\in\Delta_{Y_0})$ as a composition of regularly and slowly varying functions as $y'\to Y_0$ $(y'\in\Delta_{Y_0})$.

Let's consider the function $\theta_1(y(I_1^{-1}(z)))$, where $I_1^{-1}(z)$ is the function inverse to the function $I_1(t)$, and it can be proved that $\theta_1(y(I_1^{-1}(z)))$ is a slowly varying function as $z\to Z_1$.

Indeed,

$$
\begin{aligned}
\lim_{z\to Z_1}\frac{z(\theta_1(y(I_1^{-1}(z))))'}{\theta_1(y(I_1^{-1}(z)))} &= \lim_{z\to Z_0}\left(\frac{z\theta_1'(y(I_1^{-1}(z)))}{\theta_1(y(I_1^{-1}(z)))}\cdot\frac{y'(I_1^{-1}(z))}{I_1'(I_0^{-1}(z))}\right)\\
&= \lim_{z\to Z_0}\left(\frac{y(I_1^{-1}(z))\theta_1'(y(I_1^{-1}(z)))}{\theta_1(y(I_1^{-1}(z)))}\cdot\frac{y(I_0^{-1}(z))\cdot y'(y'^{-1}(y'(I_1^{-1}(z))))}{(y(y'^{-1}(y'(I_1^{-1}(z)))))^2}\right.\\
&\qquad\left.\times\frac{\tilde{\Phi}(y'(I_1^{-1}(z)))}{y'(I_1^{-1}(z))\widetilde{\Phi}'(y'(I_1^{-1}(z)))}\cdot\frac{z\widetilde{\Phi}'(y'(I_1^{-1}(z)))}{I_1'(I_1^{-1}(z))\tilde{\Phi}(y'(I_1^{-1}(z)))}\right)=0.
\end{aligned}
$$

Let the function Φ_1^{-1} satisfy the condition S, and we have that

$$y'(t)=\Phi_1^{-1}(I_1(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

The following theorem takes place.

Theorem 1. *Let $\sigma_1\in R\setminus\{1\}$, the function θ_1 satisfy the condition S, and the functions θ_1 and $\Phi_1^{-1}\cdot\frac{\Phi_1'}{\Phi_1}(\Phi_1^{-1})$ satisfy the condition S_1. Then for the existence of $P_\omega(Y_0,Y_1,1)$-solutions of equation (1) it is necessary, and if the following condition takes place*

$$(\sigma_1-2)\cdot y_0^0 I_0(t)\cdot I_2(t)>0 \ \ \textit{as} \ \ t\in[a;\omega[\,, \tag{4}$$

and there is a finite or infinite limit

$$\frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|},$$

then it is sufficient the fulfillment of the next conditions

$$y_0^0\alpha_0>0,\quad \lim_{t\uparrow\omega}\Phi_1^{-1}(I_2(t))=Y_0,\quad \lim_{t\uparrow\omega}I_2(t)=Z_1, \tag{5}$$

$$\lim_{t\uparrow\omega}\frac{\Phi_1'(\Phi_1^{-1}(I_2(t)))}{I_1(t)I_2'(t)}=-1, \tag{6}$$

$$y_0^0\cdot I_1(t)<0 \ \ \textit{as} \ \ t\in]b;\omega[\,,\quad \lim_{t\uparrow\omega}\frac{-1}{I_1(t)}=Y_1, \tag{7}$$

$$\lim_{t\uparrow\omega}\frac{I_2(t)\cdot I_0'(t)\cdot\theta_1^{\frac{1}{2-\sigma_1}}(-\frac{1}{I_1(t)})}{\Phi_1'(\Phi_1^{-1}(I_2(t)))I_2'(t)}=1. \tag{8}$$

Moreover, for each such solution the next asymptotic representations as $t\uparrow\omega$ take place:

$$y'(t)=\Phi_1^{-1}(I_1(t))[1+o(1)],\quad y(t)=\frac{I_2'(t)I_1(t)}{I_2(t)\Phi_1'(\Phi_1^{-1}(I_1(t)))}[1+o(1)]. \tag{9}$$

During the proof of Theorem 1, equation (1) is reduced by a special transformation to the equivalent system of quasilinear differential equations. The limit matrix of coefficients of this system has real eigenvalues of different signs.

We obtain that for this system of differential equations all the conditions of Theorem 2.2 in [5] take place. According to this theorem, the system has a one-parameter family of solutions $\{z_i\}_{i=1}^2:[x_1,+\infty[\to\mathbb{R}^2$ $(x_1\geq x_0)$, that tends to zero as $x\to+\infty$.

Any solution of the family gives raise to such a solution y of equation (1) that, together with its first derivative, admit the asymptotic images (9) as $t\uparrow\omega$. From these images and conditions (5)–(8) it follows that these solutions are $P_\omega(Y_0,Y_1,1)$-solutions.

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On the Solution of Nonlinear Boundary Value Problems with a Small Parameter in a Special Critical Case

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The study of weakly nonlinear boundary value problems for systems of ordinary differential equations is a traditional direction for the Kyiv school of nonlinear oscillations [3, 11]. A special critical case for such problems occurs when the equation defining the generating solution turns into an identity [4, 11]. Necessary and sufficient conditions for the solvability of weakly nonlinear boundary value problems in a special critical case are found in the work [4].

1 Statement of the problem

We study the problem of constructing a solution

$$z(t,\varepsilon): \ \ z(\,\cdot\,,\varepsilon) \in \mathbb{C}^1[a,b], \ \ z(t,\,\cdot\,) \in \mathbb{C}[0,\varepsilon_0]$$

to the boundary value problem [3, 4, 11]

$$\frac{dz}{dt} = A(t)z + f(t) + \varepsilon Z(z,t,\varepsilon), \ \ \ell z(\,\cdot\,,\varepsilon) = \alpha + \varepsilon J(z(\,\cdot\,,\varepsilon),\varepsilon). \tag{1.1}$$

We look for the solution of problem (1.1) in a small neighborhood of the solution of the generating Noetherian ($m \neq n$) boundary value problem

$$\frac{dz_0}{dt} = A(t)z_0 + f(t), \ \ \ell z_0(\,\cdot\,) = \alpha, \ \ \alpha \in \mathbb{R}^m. \tag{1.2}$$

Here $A(t)$ is an $(n\times n)$-dimensional matrix and $f(t)$ is an n-dimensional column-vector, the elements of which are real functions continuous on the segment $[a,b]$, $\ell z(\,\cdot\,)$ is a linear bounded vector functional

$$\ell z(\,\cdot\,): \mathbb{C}[a,b] \to \mathbb{R}^m.$$

Nonlinearities $Z(z,t,\varepsilon)$ and

$$J(z(\,\cdot\,,\varepsilon),\varepsilon): \mathbb{C}[a,b] \to \mathbb{R}^m$$

of the boundary value problem (1.1) are assumed to be twice continuously differentiable with respect to the unknown z in a small neighborhood of the generating solution and by a small parameter ε in a small positive neighborhood of zero. In addition, we consider the vector function $Z(z,t,\varepsilon)$ to be continuous with respect to the independent variable t on the segment $[a,b]$. We study the critical case ($P_{Q^*} \neq 0$), and we assume that the condition

$$P_{Q_d^*}\big\{\alpha - \ell K[f(s)](\,\cdot\,)\big\} = 0 \tag{1.3}$$

is fulfilled. In this case, the generating problem has an $(r = n - n_1)$-parametric family of solutions

$$z_0(t, c_0) = X_r(t)c_0 + G[f(s); \alpha](t), \quad c_0 \in \mathbb{R}^r.$$

Here $X(t)$ is a normal $(X(a) = I_n)$ fundamental matrix of a homogeneous part of the generating system (1.2), $Q := \ell X(\,\cdot\,)$ is an $(m \times n)$-dimensional matrix,

$$\operatorname{rank} Q = n_1, \quad X_r(t) = X(t)P_{Q_r},$$

P_{Q_r} is an $(n \times r)$-matrix formed from r linearly independent columns of an $(n \times n)$-orthoprojector matrix

$$P_Q : \mathbb{R}^n \to \mathbb{N}(Q),$$

$P_{Q_d^*}$ is an $(r \times n)$-matrix formed from r linearly independent columns of an orthoprojector

$$P_{Q^*} : \mathbb{R}^m \to \mathbb{N}(Q^*),$$

and

$$G[f(s); \alpha](t) = X(t)Q^+\big\{\alpha - \ell K[f(s)](\,\cdot\,)\big\} + K[f(s)](t)$$

is a generalized Green operator of the generating boundary value problem,

$$K[f(s)](t) = X(t)\int\limits_a^t X^{-1}(s)f(s)\,ds$$

is Green's operator of the Cauchy problem of the generating system, Q^+ is the pseudo-inverse Moore–Penrose matrix [3]. To find the necessary conditions for the existence of solutions

$$z(t, \varepsilon) = z_0(t, c_0) + x(t, \varepsilon)$$

of problem (1.1) in the critical case, the equation for the generating constants

$$F_0(c_0) := P_{Q_d^*}\Big\{J(z_0(\,\cdot\,, c_0), 0) - \ell K\big[Z(z_0(s, c_0), s, 0)\big](\,\cdot\,)\Big\} = 0$$

is traditionally used [3, 4, 8, 11]. Let us consider a less studied case when the equation for the generating constants turns into the identity [2, 4, 11]:

$$F_0(c_0) \equiv 0, \quad c_0 \in \mathbb{R}^r. \tag{1.4}$$

The boundary value problem (1.1) under condition (1.4) according to I. G. Malkin's classification [11, p. 139] represents a special critical case, since the traditional scheme of analysis and construction of solutions [3, 8] for such problems is not applicable. In articles [2, 5], the equation for generating constants is constructed, which determines the necessary conditions for the existence of solutions to problem (1.1) in the special critical case. The sufficient condition for the existence of solutions to problem (1.1) in a special critical case is the simplicity of the roots of this equation [2, 5]. We have found conditions for the existence of solutions to problem (1.1) in the special critical case in the presence of multiple roots of such an equation [2, 5].

2 Equations for generating functions

To find the necessary conditions for the existence of solutions $z(t,\varepsilon)$ to problem (1.1) in a small neighborhood of the solution of generating problem (1.2) in article [7], the following equation is proposed:

$$\mathfrak{F}(c_0(\varepsilon)) := P_{Q_d^*}\Big\{J(z_0(\,\cdot\,,c_0(\varepsilon)),\varepsilon) - \ell K\big[Z(z_0(s,c_0(\varepsilon)),s,\varepsilon)\big](\,\cdot\,)\Big\} = 0.$$

Consider the case when the equation for generating constants turns into an identity:

$$F_0(c_0) \equiv 0, \quad \mathfrak{F}(c_0(\varepsilon)) \not\equiv 0.$$

The solution of the nonlinear boundary value problem (1.1) in a particularly critical case is naturally sought in the vicinity of the solution

$$z_0(t,c_0(\varepsilon)) = X_r(t)c_0(\varepsilon) + G_1(t,c_0(\varepsilon))$$

of the modified generating boundary value problem

$$\begin{gathered}\frac{dz_0(t,c_0(\varepsilon))}{dt} = A(t)z_0(t,c_0(\varepsilon)) + f(t) + \varepsilon\, Z\big(z_0(t,c_0(\varepsilon)),t,0\big),\\ \ell z_0(\,\cdot\,,c_0(\varepsilon)) = \alpha + \varepsilon\, J\big(z_0(\,\cdot\,,c_0(\varepsilon)),0\big);\end{gathered} \tag{2.1}$$

here

$$G_1(t,c_0(\varepsilon)) := G\Big[f(t) + \varepsilon\, Z\big(z_0(s,c_0(\varepsilon)),s,0\big); \alpha + \varepsilon\, J\big(z_0(\,\cdot\,,c_0(\varepsilon)),0\big)\Big](t).$$

Under condition (1.3), the equality

$$z_0(t,c_0(0)) = z_0(t,c_0)$$

holds, therefore, in the special critical case ($F_0(c_0) \equiv 0$), for any value of $c_0 \in \mathbb{R}^r$, the generating boundary value problem (2.1) is solvable. The necessary and sufficient condition for the solvability of the boundary value problem (1.1) in the special critical case has the form

$$F(c(\varepsilon)) := P_{Q_d^*}\Big\{J\big(z_0(\,\cdot\,,c_0) + x(\,\cdot\,,\varepsilon),\varepsilon\big) - \ell K\big[Z(z_0(s,c_0) + x(s,\varepsilon),s,\varepsilon)\big](\,\cdot\,)\Big\} = 0. \tag{2.2}$$

Directing in equality (2.2)

$$z(t,\varepsilon) \to z_0(t,c_0(\varepsilon))$$

with a fixed value of ε, we obtain the necessary condition for the solvability of boundary value problem (1.1)

$$\mathcal{F}_0(c_0(\varepsilon),\varepsilon) := P_{Q_d^*}\Big\{J(z_0(\,\cdot\,,c_0(\varepsilon)),\varepsilon) - \ell K\big[Z(z_0(s,c_0(\varepsilon)),s,\varepsilon)\big](\,\cdot\,)\Big\} = 0. \tag{2.3}$$

In this way, the following lemma is proved.

Lemma. *Suppose that for the boundary value problem* (1.1) *there is a special critical case and condition* (1.3) *of the solvability of the generating problem is used. Let us also assume that in a small neighborhood of the generating solution $z_0(t,c_0^*(\varepsilon))$ problem* (1.1) *has a solution*

$$z(t,\varepsilon): \ z(\,\cdot\,,\varepsilon) \in \mathbb{C}^1[a,b], \quad z(t,\,\cdot\,) \in \mathbb{C}[0,\varepsilon_0].$$

Then the vector $c_0^*(\varepsilon) \in \mathbb{R}^r$ satisfies equation (2.3).

Equation (2.3) defines the generating solutions $z_0(t, c_0^*(\varepsilon))$, in the small neighborhood of which the sought solutions of boundary value problem (1.1) for the special critical case can be found. By analogy with weakly nonlinear boundary value problems in critical case [3], equation (2.3) will be called the equation for generating functions of the boundary value problem (1.1) in the special critical case. In contrast to article [7], equation (2.3) is built on the basis of the auxiliary boundary value problem (2.1), and not the original generating boundary value problem (1.2), which will be obtained from this problem at $\varepsilon = 0$. To find the solution $c_0^*(\varepsilon) \in \mathbb{R}^r$ of the nonlinear equation (2.3), the Newton–Kantorovich method can be used [1, 6, 9]. The smoothness of the vector $c_0^*(\varepsilon) \in \mathbb{R}^r$ significantly affects the form of the sought solution of the boundary value problem (1.1).

The proposed scheme of studies of the nonlinear boundary value problem (1.1) for the special critical case is a generalization of the results for boundary-value problems for systems of differential equations [1, 3, 4, 8, 10–12].

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Control Problem of Asynchronous Spectrum of a Linear Periodic System with a Degenerate Block of Mean Value of Coefficient Matrix

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We consider the linear control system

$$\dot{x} = A(t)x + Bu, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

where $A(t)$ is a continuous periodic $n \times n$-matrix with a modulus of frequencies Mod, B is the constant $r \times n$-matrix, u is the input. Various problems of control theory of linear systems have been studied in many works (see, for example, [6]). In this works it is assumed, as a rule, that the set of frequencies of the solution and the system itself coincide.

At the same time, as shown by X. Masser [5], Ya. Kurzveil and O. Veivoda [4], etc., the system of ordinary differential periodic (almost periodic) equations can have solutions, the intersection of the frequency module of which with the frequency module of the system is trivial. Later such solutions were named strongly irregular, and their frequency spectrum is asynchronous, and describable oscillations are asynchronous. Note that in the case of a periodic system the irregularity means the incommensurability of the periods of the solution and the system.

In what follows, as a control of $u(\,\cdot\,)$ in system (1) we will use continuous on real axis of periodic r-vector-functions, set of exponentials of which $\operatorname{Exp}(u)$ is contained in the frequency modulus $\operatorname{Mod}(A)$ coefficient matrices.

Then, as applied to system (1), the control problem of asynchronous spectrum with a target set L is as follows: select this program control

$$u = U(t)$$

from the indicated admissible set, so that the system

$$\dot{x} = A(t)x + Bu(t)$$

has a strongly irregular periodic solution with a given spectrum frequency L (target set).

The solvability of the formulated problem for system (1) with program control and zero mean value of the matrix coefficients were studied in the work [3]. In this report, we give a solution of the problem of control of the asynchronous spectrum for system (1), the average value of the matrix of coefficients of which has a degenerate non-zero left upper diagonal block, and the rest of the blocks are zero.

Let $P = (p_{ij})$, $i = \overline{1,n}$, $j = \overline{1,m}$, – some matrix and $1 \leq k_1 < \cdots < k_s \leq n$, $1 \leq l_1 < \cdots < l_q \leq m$ – two ordered sequences of natural numbers. Let $P_{k_1 \cdots k_s}^{l_1 \cdots l_q}$ be a block of the matrix P, standing at the intersection of rows with numbers $k_1, \dots, k_s$ and columns with numbers $l_1, \dots, l_q$.

Let $P = (p_{ij})$, $i = \overline{1,n}$, $j = \overline{1,m}$, be some square matrix and $1 \leq k_1 < \cdots < k_s \leq n$, $1 \leq l_1 < \cdots < l_q \leq m$ be two ordered sequences of natural numbers. By $P_{k_1 \cdots k_s}^{l_1 \cdots l_q}$ we denote the $s \times q$-matrix, standing at the intersection of rows with numbers $k_1, \dots, k_s$ and columns with numbers $l_1, \dots, l_q$ of matrix P.

For continuous on $\mathbb{R}$ ω-periodic real-valued matrix $F(t)$, we determine the mean value $\widehat{F} = \frac{1}{\omega}\int\limits_0^\omega F(t)dt$ and the oscillating part $\widetilde{F}(t) = F(t) - \widehat{F}$. Let $\mathrm{Mod}(F)$ be a frequency modulus of the matrix $F(t)$, i.e. the set of all possible linear combinations with integer coefficients of Fourier exponents of this matrix. By $\mathrm{rank}_{\mathrm{col}}\, F$ we denote the column rank of the matrix $F(t)$, i.e. largest number of linearly independent columns. Similarly, it is also possible to determine the row rank of a matrix. Let us note that in the general case, the row and column ranks of the matrix $F(t)$ are not required match. We will talk that $F(t)$ is a matrix of incomplete column rank, if the column rank is less than number of columns.

Further, we assume that the rank of the constant rectangular matrix B under control is not the maximum and row with numbers $k_1, \dots, k_d$, $1 \le k_1 < \dots < k_d \le n$ zero, i.e.,

$$\mathrm{rank}\, B = r_1 < r, \;\; B^{1\cdots r}_{k_1\cdots k_d} = 0 \;\; (d = n - r_1). \tag{2}$$

The last restriction is not a loss of generality of reasoning, so we can achieve this with the help of a linear system transformations (1) using elementary algorithms matrix row transformations.

We also assume that the mean value of the coefficient matrix is a result of permuting rows and columns, we can represent in the form

$$\begin{pmatrix} \widehat{A}^{k_1\cdots k_d}_{k_1\cdots k_d} & \widehat{A}^{k_{d+1}\cdots k_n}_{k_1\cdots k_d} \\ \widehat{A}^{k_1\cdots k_d}_{k_{d+1}\cdots k_n} & \widehat{A}^{k_{d+1}\cdots k_n}_{k_{d+1}\cdots k_n} \end{pmatrix} = \begin{pmatrix} \widehat{A}^{k_1\cdots k_d}_{k_1\cdots k_d} & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{A}^{k_1\cdots k_d}_{k_1\cdots k_d} = \mathrm{diag}\,(\widehat{a}_{k_1\,k_1}, \dots, \widehat{a}_{k_d\,k_d}), \tag{3}$$

and $\widehat{a}_{k_1\,k_1} \cdots \widehat{a}_{k_d\,k_d} = 0$. The last condition means that among the diagonal elements of the block $\widehat{A}^{k_1\cdots k_d}_{k_1\cdots k_d}$ are null. It is possible to assume that they are at the beginning of the diagonal

$$\widehat{a}_{k_{1+i-1}\,k_{1}+i-1} = 0, \;\; i = \overline{1,m}, \;\; 1 \le m < d, \tag{4}$$

and the rest of the elements are non-zero. In the opposite case this can be achieved with the help of linear non-degenerative transformation system (1), which is equivalent to permuting the first d equations in the required order.

Let $k_{d+1}, \dots, k_n$, $1 \le k_{d+1} < \dots < k_n \le n$ be the numbers of non-zero rows of a matrix B. Then, taking into account the numbering of zero and non-zero rows of this matrix to simplify the recording, we take the following notations:

$$A_{11}(t) = A^{k_1\cdots k_d}_{k_1\cdots k_d}(t), \quad A_{12}(t) = A^{k_{d+1}\cdots k_n}_{k_1\cdots k_d}(t).$$

Through $\widetilde{A}^{(1)}_{11}(t)$ we denote $d \times m$-matrix composed of the first m columns of the $d \times d$-block $\widetilde{A}_{11}(t)$. Let's construct $d \times (m + r_1)$-matrix $\widetilde{A}_*(t) = \Big[\widetilde{A}^{(1)}_{11}(t) \quad A_{12}(t)\Big]$.

We have the following

Theorem. *For the linear systems* (1), (2)–(4), *the problem of control of the asynchronous spectrum with target set L is solvable if and only if $L = \{0\}$ and the inequality*

$$\mathrm{rank}_{\mathrm{col}}\, \widetilde{A}_*(t) < r_1 + m$$

is true.

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Asymptotic Representation for Unbounded Solutions of Higher Order Differential Equations

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Consider the higher order nonlinear equation

$$u^{(n)} + q(t)u^{(n-2)} + r(t)|u|^{\lambda} \operatorname{sgn} u = 0, \quad n \geq 3, \tag{1}$$

where the functions r and q are continuous for $t \geq 1$, q is positive and $\lambda > 0$.

We study equation (1) as a perturbation of the linear differential equation

$$y^{(n)} + q(t)y^{(n-2)} = 0, \quad n \geq 3. \tag{2}$$

Some contributions on the proximity of solutions of two differential equations can be found in the quoted monograph [9], in the papers [1–3, 5] and references therein, in which this problem has been studied in various directions for a large variety of equations. Here we present a survey on some results concerning this topic, which are obtained by the authors and others in the last ten years, see [2–5].

An important role on this problem is played by the second order linear equation

$$h'' + q(t)h = 0. \tag{3}$$

Prototypes of (3) are equations with $q(t) \equiv 1$ and $q(t) \equiv 0$. When $q(t) \equiv 1$, then (3) is oscillatory and this case has been considered in [8]. More precisely, in [8] it was shown that, if r is positive and sufficient large in some sense, then for n even every proper solution of

$$u^{(n)} + u^{(n-2)} + r(t)|u|^{\lambda} \operatorname{sgn} u = 0 \tag{4}$$

is oscillatory, and for n odd every proper solution of (4) is oscillatory, or is vanishing at infinity together with its derivatives, or admits the asymptotic representation

$$x(t) = c(1 + \sin(t - \varphi)) + \varepsilon(t),$$

where c, φ are suitable constants and ε is a continuous function for $t \geq 0$ which vanishes at infinity. According to [8], such equation is said to have *property* A', see also [9] for more details.

On the other hand, if $q(t) \equiv 0$, then (3) is nonoscillatory. This case has been studied in [7], where it is proved that if $-r(t) = \varrho(t) > 0$ is sufficient small in some sense, then the equation

$$u^{(n)} = \varrho(t)|u|^{\lambda} \operatorname{sgn} u, \quad \lambda > 1, \tag{5}$$

has an $(n-1)$ parametric family of so-called rapidly increasing solutions, satisfying the condition

$$\lim_{t\to\infty} |u^{(n-1)}(t)| = \infty,$$

see also [9] for more details.

When (3) is oscillatory, the asymptotic representation of solutions to (1) has been studied by authors in [3,5] and the main results have been summarized in [6]. Here, we continue such a study by considering the opposite case, that is the case in which (3) is nonoscillatory. Using some results from [2, Theorem 1], we obtain the following

Theorem 1. *Let the second order differential equation* (3) *be nonoscillatory and*

$$\int_1^\infty tq(t)\,dt = \infty. \tag{6}$$

Assume that for some real number $m \in [0, n-1]$,

$$\int_1^\infty t^{n+m\lambda}\,|r(t)|\,dt < \infty. \tag{7}$$

Then for any solution y *to* (2) *such that* $y(t) = O(t^m)$, *there exists a solution* u *to (1) such that for large* t

$$u^{(i)}(t) = y^{(i)}(t) + \varepsilon_i(t), \quad i = 0, 1, \ldots, n-1, \tag{8}$$

where all ε_i *are functions of bounded variation and* $\lim_{t\to\infty} \varepsilon_i(t) = 0$.

The proof is based on the induction method, an iterative process and suitable estimates for solutions to (2). A similar approach has been used in [3], but using completely different estimations for solutions of (2).

Now consider the special case of (1), i.e. the equation

$$u^{(n)}(t) + \frac{\sigma}{t^2}\,u^{(n-2)}(t) + r(t)|u|^\lambda \operatorname{sgn} u = 0, \quad n \ge 3, \tag{9}$$

where $\sigma \in (0, 1/4)$. Obviously, (6) is satisfied and the corresponding second order equation is the Euler equation

$$h''(t) + \frac{\sigma}{t^2}\,h(t) = 0,$$

which is nonoscillatory and whose solutions are known, see, e.g. [10, p. 45]. Using suitable estimations for solutions of (2), we have the following theorem see [2, Corollary 3].

Theorem 2. *Let* $\sigma \in (0, 1/4)$ *and assume that*

$$\int_1^\infty t^{n-1+\gamma\lambda}\,|r(t)|\,dt < \infty,$$

where

$$\gamma = n - 2^{-1}\big(3 + \sqrt{1-4\sigma}\,\big).$$

Then for any polynomial Q *with* $\deg Q \le n-3$, *there exist solutions* u *of* (9) *such that for large* t

$$u^{(i)}(t) = \big(c_1\Gamma_1(t) + c_2\Gamma_2(t) + Q(t)\big)^{(i)} + \varepsilon_i(t), \quad i = 0, \ldots, n-1,$$

where

$$\Gamma_1(t)=\int\limits_1^t (t-s)^{n-3}s^{\mu}\,ds\,O(t^{\beta}),\quad \Gamma_2(t)=\int\limits_1^t (t-s)^{n-3}s^{v}\,ds\,O(t^{\gamma}),$$
$$\mu=2^{-1}\big(1-\sqrt{1-4\sigma}\,\big),\quad \nu=2^{-1}\big(1+\sqrt{1-4\sigma}\,\big),$$

c_1, c_2 *are constants and functions* ε_i *are of bounded variation for large* t *and* $\lim\limits_{t\to\infty}\varepsilon_i(t)=0$.

The following example illustrates Theorem 1.

Example 1. Let $\lambda>0$ and consider the nonlinear equation for $t\ge 1$

$$u^{(4)}+\frac{1}{t^2\log et}\,u^{(2)}=\frac{e^{-t}(t^2\log et+1)}{(1+e^{-t})^{\lambda}t^2\log et}\,|u|^{\lambda}\operatorname{sgn}u. \tag{10}$$

A solution of (10) is

$$u(t)=t+e^{-t}. \tag{11}$$

Setting

$$q(t)=\frac{1}{t^2\log et}\,,\quad r(t)=\frac{e^{-t}(t^2\log et+1)}{(1+e^{-t})^{\lambda}t^2\log et}\,,$$

we get that (3) is nonoscillatory and (6) is valid. Moreover, we have for any $\sigma>0$

$$\int\limits_1^{\infty} t^{\sigma}r(t)\,dt<\infty.$$

Thus, all the assumptions of Theorem 1 are verified with $m=1$ and so equation (10) has a solution u such that for any large t

$$u^{(i)}(t)=y^{(i)}(t)+\varepsilon_i(t),\ \ i=0,1,2,3,$$

where ε_i are functions of bounded variation such that $\lim\limits_{t\to\infty}\varepsilon_i(t)=0$ and $y(t)=t$, as the solution (11) illustrates.

Finally, consider the fourth-order differential equation with deviating argument

$$x^{(4)}(t)+q(t)x''(t)+r(t)|x(\varphi(t))|^{\lambda}\operatorname{sgn}x(\varphi(t))=0,\ \ \lambda>0, \tag{12}$$

where φ is a nonegative continuous function for $t\ge 1$ and $\varphi(1)=1$, $\lim\limits_{t\to\infty}\varphi(t)=\infty$. From [3, Theorem 1], if q is a continuously differentiable bounded away from zero function, i.e. $q(t)\ge q_0>0$ for large t, such that

$$\int\limits_1^{\infty}|q'(t)|\,dt<\infty, \tag{13}$$

and

$$\int\limits_1^{\infty}t^{\lambda+1}|r(t)|\,dt<\infty, \tag{14}$$

then (12) with $\varphi(t)=t$ has a solution x such that

$$x^{(i)}(t)=t^i+\varepsilon_i(t),\ \ i=0,1,2,3,$$

where functions ε_i are of bounded variation for large t and $\lim_{t\to\infty} \varepsilon_i(t) = 0$. In [4] this result has been improved for a more general equation than (12), without the assumption $\varphi(t) = t$. More precisely, by means of a topological method jointly with certain integral inequalities, the following asymptotic representation of unbounded solutions of (12) has been given, see [4, Corollary 4.1].

Theorem 3. *Let $r(t) \neq 0$ for large t. If q is a continuously differentiable bounded away from zero function satisfying* (13)*, then* (12) *has an asymptotic linear solution x, i.e. a solution x satisfying*

$$\lim_{t\to\infty} |x(t)| = \infty, \quad \lim_{t\to\infty} x'(t) = c_x \neq 0, \tag{15}$$

if and only if

$$\int_{t_0}^{\infty} |r(t)|\varphi^{\lambda}(t)\,dt < \infty. \tag{16}$$

Theorem 3 illustrates the dependence of asymptotic linear solutions from the behavior of the deviating argument φ as $t \to \infty$. Moreover, in view of (14) and (16), when $\varphi(t) = t$, Theorem 2 improves the quoted result in [3, Theorem 1]. The following example illustrates this fact.

Example 2. Consider the equation

$$x^{(4)}(t) + x''(t) + \frac{1}{(t+1)^2}\,|x(t^{1/2})|^{3/2}\operatorname{sgn} x(t^{1/2}) = 0, \quad t \geq 1. \tag{17}$$

By Theorem 3 equation (17) has unbounded asymptotic linear solutions. On the other hand, the corresponding equation

$$x^{(4)}(t) + x''(t) + \frac{1}{(t+1)^2}\,|x(t)|^{3/2}\operatorname{sgn} x(t) = 0, \quad t \geq 1, \tag{18}$$

does not have solutions x satisfying (15). Indeed, by contradiction, let x be an eventually positive solution x of (18) satisfying (15). Since we have for some $T \geq 1$

$$\int_{T}^{\infty} \frac{x^{3/2}(t)}{(t+1)^2}\,dt = \infty,$$

from (18) we get

$$\lim_{t\to\infty} (x'''(t) + x'(t)) = -\infty,$$

which gives a contradiction with (15).

Since the function q considered in Theorem 3 is bounded away from zero, the corresponding second order equation (3) is oscillatory. Thus, in view of the above mentioned result for equation (5), it is natural to ask under which assumptions on deviating argument φ the above results continue to hold for (12) or, more generally, for (1) when q is small so that (3) is nonoscillatory.

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Asymptotic Representations of Rapid Varying Solutions of Differential Equations Asymptotically Close to the Equations with Regularly Varying Nonlinearities

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The differential equation

$$y^n = f(t, y, y', \dots, y^{n-1}) \tag{1}$$

is considered. Here $n \geq 2$, $f : [\alpha, \omega[\, \times \Delta_{Y_0} \times \Delta_{Y_1} \times \cdots \times \Delta_{Y_{n-1}} \to \mathbb{R}$ is some continuous function, $-\infty < \alpha < \omega \leq +\infty$, Y_j equals to zero, or to $+\infty$, Δ_{Y_j} is some one-sided neighborhood of Y_j, $j = 0, 1, \dots, n-1$.

The asymptotic estimations for singular, quickly varying, and Kneser solutions of equation (1) are described in the monograph by I. T. Kiguradze, T. A. Chanturia [4].

Definition 1. The solution y of equation (1), defined on the interval $[t_0, \omega[\, \subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if the next conditions take place

$$y^{(j)}(t) \in \Delta_{Y_j} \text{ as } t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y^{(j)}(t) = Y_j \ \ (j = 0, 1, \dots, n-1), \quad \lim_{t \uparrow \omega)} \frac{[y^{n-1}(t)]^2}{y^{n-2}(t) y^n(t)} = \lambda_0.$$

The asymptotic behavior of such solutions earlier has been investigated in the works by V. M. Evtukhov and A. M. Klopot [1–3, 5] for the differential equation

$$y^n = \sum_{n-1}^{m} a_i p_i(t) \prod_{j=0}^{n-1} \varphi_{ij}(y^{(j)}),$$

where $n \geq 2$, $\alpha_i \in \{-1; 1\}$, $p_i : [\alpha, \omega[\, \to]0, +\infty[$ is a continuous function $i = 1, \dots, m$, $-\infty < \alpha < \omega \leq +\infty$, $\varphi_{ij} : \Delta_{Y_j} \to]0, +\infty[$ is a continuous regularly varying as $y^{(j)} \to Y_j$ function of order

$$\sigma_j, \ \ j = 0, 1, \dots, n-1 \ (i - 1, \dots, m).$$

The aim of the paper is to establish the necessary and sufficient conditions of the existence of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, 1)$-solutions of equation (1) and to find the asymptotic representations of such solutions and their derivatives to the order $n-1$ including.

Every $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, 1)$-solution of the differential equation (1) has (see, for example, [1]) the next a priori asymptotic properties

$$\frac{y'(t)}{y(t)} \sim \frac{y''(t)}{y'(t)} \sim \cdots \sim \frac{y^n(t)}{y^{n-1}(t)} \text{ as } t \uparrow \omega, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) y'(t)}{y(t)} = \pm\infty,$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t - \omega & \text{if } \omega < +\infty. \end{cases}$$

Definition 2. The function f in the differential equation (1) is called a function, that satisfies the condition $(RN)_1$, if there exist a number $\alpha_0 \in \{-1; 1\}$, a continuous function $p : [\alpha, \omega[\to]0, +\infty[$ continuous varying as $z \to Y_j$ $(j = 0, 1, \dots, n-1)$, functions $\varphi_j : \Delta_{Y_j} \to]0, +\infty[$ $(j = 0, 1, \dots, n-1)$ of orders σ_j $(j = 0, 1, \dots, n-1)$, such that for all continuously differentiable functions $z_j : [\alpha, \omega[\to \Delta_{Y_j}$ $(j = 0, 1, \dots, n-1)$, satisfying the conditions

$$\lim_{t \uparrow \omega} z_j(t) = Y_j, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) z_j'(t)}{z_j(t)} = \pm\infty \ \ (j = 0, 1, \dots, n-1),$$
$$\lim_{t \uparrow \omega} \frac{z_{n-1}'(t) z_j(t)}{z_{n-1}(t) z_j'(t)} = 1 \ \ (j = 1, \dots, n-1),$$

the next representation takes place

$$f(t, z_0(t), z_1(t), \dots, z_{n-1}(t)) = \alpha_0 p(t) \prod_{j=0}^{n-1} \varphi_j(z_j(t))[1 + o(1)] \ \text{ as } \ t \uparrow \omega.$$

Furthermore, we will use the following notations.

$$\gamma = 1 - \sum_{j=0}^{n-1} \sigma_j, \quad \mu_n = \sum_{j=0}^{n-2} \sigma_j (n - j - 1),$$

$$\nu_j = \begin{cases} 1 & \text{if } Y_j = +\infty, \text{ or } Y_j = 0 \text{ and } \Delta_{Y_j} \text{ is the right neighborhood of zero,} \\ -1 & \text{if } Y_j = +\infty, \text{ or } Y_j = 0 \text{ and } \Delta_{Y_j} \text{ is the left neighborhood of zero} \end{cases}$$
$$(j = 0, 1, \dots, n-1),$$

$$J_0(t) = \int_{A_0}^{t} p(s)\, ds, \quad J_{00}(t) = \int_{A_{00}}^{t} J_0(s)\, ds,$$

where

$$A_0 = \begin{cases} \alpha & \text{if } \displaystyle\int_\alpha^\omega p(s)\, ds = +\infty, \\ \omega & \text{if } \displaystyle\int_\alpha^\omega p(s)\, ds < +\infty, \end{cases} \qquad A_{00} = \begin{cases} \alpha & \text{if } \displaystyle\int_\alpha^\omega |J_0(s)|\, ds = +\infty, \\ \omega & \text{if } \displaystyle\int_\alpha^\omega |J_0(s)|\, ds < +\infty. \end{cases}$$

Theorem. *Let the function f satisfy the condition $(RN)_1$ and $\gamma \neq 0$. Then for the existence of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, 1)$-solutions of equation* (1) *the next conditions are necessary:*

$$\frac{p(t)}{J_0(t)} \sim \frac{J_0(t)}{J_{00}(t)} \ \ \textit{as} \ \ t \uparrow \omega,$$
$$\lim_{t \uparrow \omega} \frac{\pi_w(t) p(t)}{J_0(t)} = \pm\infty, \quad \nu_j \lim_{t \uparrow \omega} |J_0(t)|^{1/\gamma} = Y_j \ \ (j = 0, 1, \dots, n-1),$$

and for $t \in]\alpha, \omega[$, the next inequalities take place

$$\alpha_0 V_{n-1} \gamma J_0(t) > 0, \ \ \nu_j \nu_{n-1} (\gamma J_0(t))^{n-j-1} > 0 \ \ (j = 0, 1, \dots, n-2).$$

As the algebraic p equation

$$(1 + p)^n = \sum_{j=0}^{n-1} \sigma_j (1 + p^j \tag{2}$$

has no roots with zero real part, the conditions are also sufficient for the existence of such solutions of equation (1)*. Moreover, for such solutions the next asymptotic representations*

$$y^j(t) = \Big(\frac{\gamma J_{00}(t)}{J_0(t)}\Big)^{n-j-1} y^{n-1}(t)[1+o(1)] \ \ (j = 0,1,\dots,n-2), \tag{3}$$

$$\frac{|y^{(n-1)}(t)|^{\gamma}}{\prod\limits_{j=0}^{n-1} L_j\big(\frac{\gamma J_{00}(t)}{J_0(t)}\big)^{n-j-1} y^{n-1}(t)} = \gamma J_0(t)\Big|\frac{\gamma J_{00}(t)}{J_0(t)}\Big|^{\mu_n}[1+0(1)] \tag{4}$$

take place as $t \uparrow \omega$*. Here*

$$L_j(y^{(j)}) = |y^{(j)}|^{-\sigma_j}\varphi_j(y^{(j)}(t)) \ \ (j = 0,1,\dots,n-1).$$

There exists m*-parametric family of such solutions, if among the roots of equation* (2) *there exists* m *roots (taking into account multiple roots), the real parts of which have the sign that is among opposite to the sign of* $\alpha_0 V_{n-1}$*.*

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The Asymptotic of Unboudedly Continuable to the Right Solutions of the Ordinary Differential Equation of Second Order

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We consider the second order ordinary differential equation of the form:

$$F(t,y,y',y'')=\sum_{k=1}^{n}p_k(t)y^{\alpha_k}|y'|^{\beta_k}|y''|^{\gamma_k}=0, \tag{1}$$

$n\in\mathbb{N}$, $n\geq 2$, $\alpha_k,\beta_k,\gamma_k\in\mathbb{R}$, $\sum\limits_{k=1}^{n}|\gamma_k|\neq 0$, $p_k\in\mathrm{C}([a;+\infty),a>0;\mathbb{R})$ $(k=\overline{1,n})$, $p_i(t)\neq 0$ $(i=\overline{1,s}$ for some $2\leq s\leq n)$.

We investigate the question of the existence and asymptotic behavior (as $t\to+\infty$) of unboudedly continuable to the right solutions (R-solutions) $y(t)$ of equation (1) and the derivatives $y'(t)$, $y''(t)$ of these solutions.

Earlier in [3] we have considered a similar question of the asymptotic behavior of solutions of equation of the form (1) when $\sum\limits_{k=1}^{n}|\gamma_k|=0$, that is when equation (1) is a first order differential equation.

The main result is obtained under the assumption that there exists a function $v\in\mathrm{C}^2([t_1;+\infty),t_1>a;\mathbb{R})$ which possesses the following properties:

(A) $v(t)>0$, $v''(t)\neq 0$ on $[t_1;+\infty)$,

$$\lim_{t\to+\infty}v(t)=0\vee+\infty;$$

(B)

$$\lim_{t\to+\infty}\frac{v''(t)v(t)}{(v'(t))^2}=\mu\ \ (0\neq\mu\in\mathbb{R});$$

(C)

$$\lim_{t\to+\infty}\frac{p_i(t)v^{\alpha_i}(t)|v'(t)|^{\beta_i}|v''(t)|^{\gamma_i}}{p_1(t)v^{\alpha_1}(t)|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}}=c_i\ \ (0\neq c_i\in\mathbb{R},\ \ i=\overline{1,s}),$$

$$\sum_{i=1}^{s}\gamma_ic_i\neq 0,$$

$$\lim_{t\to+\infty}\frac{p_j(t)v^{\alpha_j}(t)|v'(t)|^{\beta_j}|v''(t)|^{\gamma_j}}{p_1(t)v^{\alpha_1}(t)|v'(t)|^{\beta_1}|v''(t)|^{\gamma_1}}=0\ \ (j=\overline{s+1,n}).$$

The following lemma is valid.

Lemma. *Let in the relation*

$$\Phi(t,x_1,x_2,x_3)=0, \tag{2}$$

$(t,x_1,x_2,x_3)\in H$, $H=[a;+\infty)\times\prod\limits_{k=1}^{3}H_k$, $H_k=[-h_k;h_k]$, $a\in\mathbb{R}$, $h_k>0$ $(k=1,2,3)$, *the function* $\Phi:H\to\mathbb{R}$ *satisfy the conditions:*

1) $\Phi,\frac{\partial\Phi}{\partial x_1},\frac{\partial\Phi}{\partial x_2},\frac{\partial^2\Phi}{\partial x_3^2}\in \mathrm{C}(H;\mathbb{R})$;

2)
$$\lim_{t\to+\infty}\sup_{(x_1;x_2)\in H_1\times H_2}|\Phi(t,x_1,x_2,0)|=0;$$

3)
$$\lim_{t\to+\infty}\frac{\partial\Phi}{\partial x_3}(t,0,0,0)=A_1\neq 0;$$

4)
$$\sup_D\Big|\frac{\partial^2\Phi}{\partial x_3^2}(t,x_1,x_2,x_3)\Big|=A_2<+\infty.$$

Then in some domain $H^*=H_0\times H_3^*$, $H_0=[t_0;+\infty)\times\prod\limits_{k=1}^{2}H_k^*$, $H_k^*=[-h_k^*;h_k^*]$ $(k=1,2,3)$, *where* t_0 *and* h_k^* *satisfy the inequality* $t_0\geq a$, $0<h_k^*\leq h_k$, $\frac{4A_2h_3^*}{|A_1|}<1$, *relation* (2) *defines a unique function* $x_3:H_0\to\mathbb{R}$ *that satisfies the conditions:*

$$x_3,\frac{\partial x_3}{\partial x_1},\frac{\partial x_3}{\partial x_2}\in\mathrm{C}(H_0;\mathbb{R}),\quad \Phi(t,x_1,x_2,x_3(t,x_1,x_2))\equiv 0,\quad \lim_{t\to+\infty}x_3(t,0,0)=0$$

and

$$x_3(t,x_1,x_2)\sim-\frac{\Phi(t,x_1,x_2,0)}{\frac{\partial\Phi}{\partial x_3}(t,x_1,x_2,0)}.$$

The following theorem was obtained using the above lemma and the results from [1,2,4].

Theorem. *Let there exist a function* $v\in \mathrm{C}^2([t_1;+\infty),t_1>a;\mathbb{R})$ *which possesses the properties* (A)–(C)*. Then for the R-solution* $y(t)$ *of the differential equation* (1) *with the asymptotic representation*

$$y^{(k)}(t)\sim v^{(k)}(t)\ \ (k=\overline{0,2}) \tag{3}$$

to exist it is necessary, and if the roots λ_1, λ_2 *of the algebraic equation*

$$\lambda^2+\Bigg(1+\frac{m\sum\limits_{i=1}^{s}(\beta_i+\gamma_i)c_i}{\sum\limits_{i=1}^{s}\gamma_ic_i}\Bigg)\lambda+\frac{m\sum\limits_{i=1}^{s}(\alpha_i+\beta_i+\gamma_i)c_i}{\sum\limits_{i=1}^{s}\gamma_ic_i}=0$$

have the property $\operatorname{Re}\lambda_k\neq 0$ $(k=1,2)$*, then it is also sufficient that* $\sum\limits_{i=1}^{s}c_i=0$.

Moreover, if $\operatorname{sign}(\operatorname{Re}\lambda_1)\neq\operatorname{sign}(\operatorname{Re}\lambda_2)$*, then there exists a one-parametric set of R-solutions with the asymptotic representation* (3)*; if in some suburb of* $+\infty$

$$\operatorname{sign}(\operatorname{Re}\lambda_1)=\operatorname{sign}(\operatorname{Re}\lambda_2)\neq\operatorname{sign}(v'(t)),$$

then there exists a two-parametric set of R-solutions with the asymptotic representation (3)*.*

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On a Lower Estimate for the First Eigenvalue of a Sturm–Liouville Problem

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1 Introduction

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all locally integrable on $(0,1)$ functions with non-negative values such that the following integral conditions hold:

$$\int_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx = 1, \quad \gamma \neq 0, \tag{3}$$

$$\int_0^1 x(1-x)Q(x)\,dx < \infty. \tag{4}$$

A function y is a *solution* of problem (1), (2) if it is absolutely continuous on the segment $[0,1]$, satisfies (2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1) holds almost everywhere in the interval $(0,1)$.

In Theorem 1 [2], it was proved that if condition (4) does not hold, then for any $0 \le p \le \infty$, there is no non–trivial solution y of equation (1) with properties $y(0) = 0$, $y'(0) = p$.

If $\gamma < 0$, $\alpha \le 2\gamma - 1$ or $\beta \le 2\gamma - 1$, then the set $T_{\alpha,\beta,\gamma}$ is empty; for other values α, β, γ, $\gamma \neq 0$, the set $T_{\alpha,\beta,\gamma}$ is not empty [4, Chapter 1, § 2, Theorem 3]. Since for $\gamma < 0$, $\alpha \le 2\gamma - 1$ or $\beta \le 2\gamma - 1$ there is no function Q satisfying (3) and (4) taken together, then problem (1)–(4) is not considered for these parameters.

This work gives estimates for

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

Consider the functional

$$R[Q,y] = \frac{\int_0^1 y'^2\,dx - \int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}.$$

If condition (4) is satisfied, then the functional $R[Q,y]$ is bounded below in $H_0^1(0,1)$ [3]. It was proved [2,3] that for any $Q \in T_{\alpha,\beta,\gamma}$,

$$\lambda_1(Q) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y].$$

For any $Q \in T_{\alpha,\beta,\gamma}$, we have

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y] \le \inf_{y \in H_0^1(0,1)\setminus\{0\}} \frac{\int\limits_0^1 y'^2\,dx}{\int\limits_0^1 y^2\,dx} = \pi^2.$$

2 Main results

Theorem 2.1. *If $\gamma > 1$, $\alpha, \beta < 2\gamma - 1$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_0^1(0,1)$, $u > 0$ on $(0,1)$ such that $m_{\alpha,\beta,\gamma} = R[Q_*, u]$. Moreover, u satisfies the equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{\gamma+1}{\gamma-1}} \tag{5}$$

and the integral condition

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{2\gamma}{\gamma-1}}\,dx = 1. \tag{6}$$

Theorem 2.2.

(1) *If $\gamma = 1$, $\alpha, \beta \leqslant 0$, then $m_{\alpha,\beta,\gamma} \geqslant \frac{3}{4}\pi^2$.*

(2) *If $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$ or $\alpha \leqslant 0 < \beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

(3) *If $\gamma = 1$, $0 < \alpha, \beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

(4) *If $\gamma > 1$, $\alpha, \beta \leqslant \gamma$, then $m_{\alpha,\beta,\gamma} = 0$.*

(5) *If $\gamma > 1$, $\gamma < \alpha \leqslant 2\gamma - 1$ or $\gamma < \beta \leqslant 2\gamma - 1$, then $m_{\alpha,\beta,\gamma} \leqslant 0$.*

(6) *If $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$, $0 < \gamma < 1$, $-\infty < \alpha, \beta < +\infty$ or if $\gamma \geqslant 1$, $\alpha > 2\gamma - 1$ or $\beta > 2\gamma - 1$, then $m_{\alpha,\beta,\gamma} = -\infty$.*

Let us show that if $\gamma \geqslant 1$, $\alpha > 2\gamma - 1$, $-\infty < \beta < \infty$, then we have $m_{\alpha,\beta,\gamma} = -\infty$ (the case $\gamma \geqslant 1$, $\beta > 2\gamma - 1$, $-\infty < \beta < \infty$ is similar).

Consider the functions $Q_\varepsilon \in T_{\alpha,\beta,\gamma}$ and $y_0 \in H_0^1(0,1)$:

$$Q_\varepsilon(x) = \begin{cases} (\alpha+1)^{\frac{1}{\gamma}} \varepsilon^{-\frac{\alpha+1}{\gamma}} (1-x)^{-\frac{\beta}{\gamma}}, & x \in [0,\varepsilon], \\ 0, & x \in (\varepsilon, 1], \end{cases}$$

$$y_0(x) = \begin{cases} x^\theta, & x \in \Big[0, \dfrac{1}{2}\Big], \\ (1-x)^\theta, & x \in \Big(\dfrac{1}{2}, 1\Big], \quad \theta > \dfrac{1}{2}\,. \end{cases}$$

We have

$$\int\limits_0^1 Q_\varepsilon(x) y_0^2\,dx \geqslant L \cdot \varepsilon^{2\theta+1-\frac{\alpha+1}{\gamma}},$$

where L is a constant. Since $\alpha > 2\gamma - 1$, there is a number $\theta > \frac{1}{2}$ such that $2\theta + 1 < \frac{\alpha+1}{\gamma}$.

Thus,

$$\lambda_1(Q_\varepsilon) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q_\varepsilon, y] \leqslant R[Q_\varepsilon, y_0],$$

$$\inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q) \leqslant \lim_{\varepsilon\to 0} \lambda_1(Q_\varepsilon) \leqslant \lim_{\varepsilon\to 0} R[Q_\varepsilon, y_0] = -\infty.$$

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Periodic-Type Solutions for Differential Equations with Positively Homogeneous Operators

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1 Introduction

In the paper [1], we show how a rather abstract Fredholm-type result from [2] can be successfully applied to study ω-periodic solutions to the following second order differential equation with a $(\lambda+1)$-Laplacian and maxima:

$$\big(|u'(t)|^{\lambda}\operatorname{sgn}u'(t)\big)' = g(t)\max\big\{|u(s)|^{\lambda}\operatorname{sgn}u(s):\ \mu(t)\le s\le\tau(t)\big\}+f_0(t), \tag{1.1}$$

where $f_0, g \in L([0,\omega];\mathbb{R})$, $\lambda > 0$, and $\mu,\tau : [0,\omega] \to [0,\omega]$ are measurable functions satisfying $\mu(t) \le \tau(t)$ for almost all t belonging to the period segment $[0,\omega]$. Two of our main results stated in Section 2, Corollaries 2.3 and 2.4, present easily verifiable conditions for the existence of at least one ω-periodic solution to the equation (1.1) for each perturbation $f_0(t)$. Importantly, the leading coefficient $g(t)$ in (1.1) can oscillate: in such a case, we will assume that either positive or negative part of $g(t)$ dominates the part of $g(t)$ having the opposite sign, see Corollaries 2.3 and 2.4 for the precise formulations. Note that the uniqueness of periodic solutions is not analysed in the present work. Nevertheless, it is known that even the first order periodic equation with the right-hand side as in (1.1) and constant coefficient $g(t)$ can have multiple (or even infinite number of) subharmonic periodic solutions for a class of sine-like forcing terms $f_0(t)$. We leave the aforementioned uniqueness problem for equation (1.1) as an interesting open question.

Now, our approach allows to consider more general objects in the form of two-dimensional system of functional differential equations

$$u_1'(t) = f_1(u_1,u_2)(t), \tag{1.2}$$

$$u_2'(t) = f_2(u_1,u_2)(t), \ \ t\in[0,\omega], \tag{1.3}$$

subjected to the periodic-type boundary value conditions

$$u_1(\omega)-u_1(0) = h_1(u_1,u_2), \quad u_2(\omega)-u_2(0) = h_2(u_1,u_2). \tag{1.4}$$

Here $f_i : C([0,\omega];\mathbb{R}) \times C([0,\omega];\mathbb{R}) \to L([0,\omega];\mathbb{R})$ $(i = 1,2)$ are continuous operators satisfying Carathéodory conditions, i.e., for every $r > 0$ there exists $q_r \in L([0,\omega];\mathbb{R}_+)$ such that

$$|f_1(u_1,u_2)(t)| + |f_2(u_1,u_2)(t)| \le q_r(t) \ \text{ for a.e. } \ t \in [0,\omega] \ \text{ whenever } \ \|u_1\|_C + \|u_2\|_C \le r,$$

and $h_i : C([0,\omega];\mathbb{R}) \times C([0,\omega];\mathbb{R}) \to \mathbb{R}$ $(i = 1,2)$ are continuous functionals bounded on every ball by a constant, i.e., for every $r > 0$ there exists $M_r > 0$ such that

$$|h_1(u_1,u_2)| + |h_2(u_1,u_2)| \le M_r \ \text{ whenever } \ \|u_1\|_C + \|u_2\|_C \le r.$$

By a solution to the system $(1.2),(1.3)$ we understand a vector-valued function $(u_1,u_2) \in C([0,\omega];\mathbb{R}) \times C([0,\omega];\mathbb{R})$ with absolutely continuous components that satisfy the equalities (1.2) and (1.3) almost everywhere in $[0,\omega]$. By a solution to the problem (1.2)–(1.4) we understand a solution to $(1.2),(1.3)$ which satisfies (1.4).

Before presenting our main results in Section 2, let us introduce basic notation used in this work:

$\mathbb{R}$ is a set of all real numbers;

$C([0,\omega];\mathbb{R})$ is a Banach space of continuous functions $u : [0,\omega] \to \mathbb{R}$ endowed with the norm

$$\|u\|_C = \max\big\{|u(t)| : \ \ t \in [0,\omega]\big\};$$

$L([0,\omega];\mathbb{R})$ is a Banach space of Lebesgue integrable functions $u : [0,\omega] \to \mathbb{R}$ endowed with the norm

$$\|u\|_L = \int_0^\omega |u(t)|\,dt;$$

if $g \in L([0,\omega];\mathbb{R})$ then $[g]_+$, resp. $[g]_-$, denotes the non-negative, resp. nonpositive, part of the function g, i.e.,

$$[g]_+(t) \stackrel{def}{=} \frac{|g(t)| + g(t)}{2}, \quad [g]_-(t) \stackrel{def}{=} \frac{|g(t)| - g(t)}{2} \ \text{ for a.e. } \ t \in [0,\omega];$$

$\mathscr{P}(\lambda)$, where $\lambda > 0$, is a set of all continuous nondecreasing operators $p : C([0,\omega];\mathbb{R}) \to L([0,\omega];\mathbb{R})$ satisfying Carathéodory conditions which are positively homogeneous with a degree λ, i.e., for every $c > 0$ and $u \in C([0,\omega];\mathbb{R})$ the following identity holds:

$$p(cu)(t) = c^\lambda p(u)(t) \ \text{ for a.e. } \ t \in [0,\omega].$$

Let $\mu,\tau : [0,\omega] \to [0,\omega]$ be measurable functions. Then, for every $t \in [0,\omega]$, we put $I(\mu(t),\tau(t)) = [\mu(t),\tau(t)]$ if $\mu(t) \le \tau(t)$ and $I(\mu(t),\tau(t)) = \varnothing$ otherwise.

$\mathscr{S}$ is a set of all mappings $S : [0,\omega] \to 2^{[0,\omega]}$ such that $S(t)$ is a union of at most countable number of intervals $(\mu_k(t),\tau_k(t))$, where $\mu_k,\tau_k : [0,\omega] \to [0,\omega]$ are measurable functions satisfying $\mu_k(t) \le \tau_k(t)$ for almost all $t \in [0,\omega]$.

Note that the function $t \mapsto \sup\{|u(s)|^\lambda \operatorname{sgn} u(s) : \ \ s \in S(t)\}$ is measurable whenever $u \in C([0,\omega];\mathbb{R})$, $S \in \mathscr{S}$, and $\lambda > 0$ (we put $\sup\varnothing = -\infty$).

For given $p \in \mathscr{P}(\lambda)$ and a number $\delta \in [0,1]$ we define the operator $p(\,\cdot\,;\delta) : C([0,\omega];\mathbb{R}) \to L([0,\omega];\mathbb{R})$ and a non-negative numbers $\widehat{P}(\delta)$ and $P(\delta)$ in the following way:

$$p(u;\delta)(t) \stackrel{def}{=} (1-\delta)p(u)(t) - \delta p(-u)(t) \ \text{ for a.e. } \ t \in [0,\omega], \quad \widehat{P}(\delta) \stackrel{def}{=} \int_0^\omega p(1;\delta)(t)\,dt,$$

$$P(\delta) \stackrel{def}{=} \max\left\{ \int_x^y p(1;\delta)(t)\,dt + \int_y^{x+\omega} p(1;1-\delta)(t)\,dt : \ \ x \in [0,\omega], \ \ y \in [x,x+\omega] \right\},$$

where

$$p(1;\nu)(t)=p(1;\nu)(t-\omega)\ \text{ for a.e. } t\in(\omega,2\omega],\ \ \nu=\delta,1-\delta.$$

Obviously, $\widehat{P}(\delta)\le P(\delta)$ and $-p(-u;\delta)\equiv p(u;1-\delta)$ for every $u\in C([0,\omega];\mathbb{R})$ and $\delta\in[0,1]$. It can be also easily verified that

$$P(\delta)=P(1-\delta)\ \text{ for }\ \delta\in[0,1]. \tag{1.5}$$

Furthermore, for given $p_0\in\mathscr{P}(\lambda_1)$ and $p_1,p_2\in\mathscr{P}(\lambda_2)$ we define the following functions:

$$q_1(t,\rho)\stackrel{def}{=}\sup\Big\{|f_1(u_1,u_2)(t)-p_0(u_2)(t)|:\ \|u_1\|_C\le\rho,\ \|u_2\|_C\le\rho^{\lambda_2}\Big\} \tag{1.6}$$
$$\text{for a.e. } t\in[0,\omega],$$

$$q_2(t,\rho)\stackrel{def}{=}\sup\Big\{|f_2(u_1,u_2)(t)-p_1(u_1)(t)+p_2(u_1)(t)|:\ \|u_1\|_C\le\rho^{\lambda_1},\ \|u_2\|_C\le\rho\Big\} \tag{1.7}$$
$$\text{for a.e. } t\in[0,\omega],$$

$$\eta_k(\rho)\stackrel{def}{=}\sup\Big\{|h_k(u_1,u_2)|:\ \|u_k\|_C\le\rho,\ \|u_{3-k}\|_C\le\rho^{\lambda_{3-k}}\Big\}\ \ (k=1,2). \tag{1.8}$$

2 Main results

Now we can formulate our main results. The proofs of the results slightly differ depending on the values of λ_i. Therefore it is convenient formulate assertions for two separate cases. Thus, Theorem 2.1 deals with the case when $\lambda_2\ge1$, Theorem 2.2 can be applied in the case when $\lambda_2<1$.

Theorem 2.1. *Let $\lambda_1,\lambda_2>0$, $\lambda_1\lambda_2=1$, and let there exist $p_0\in\mathscr{P}(\lambda_1)$ and $p_1,p_2\in\mathscr{P}(\lambda_2)$ such that*

$$\lim_{\rho\to+\infty}\int_0^\omega\frac{q_k(s,\rho)}{\rho}\,ds=0,\qquad \lim_{\rho\to+\infty}\frac{\eta_k(\rho)}{\rho}=0\ \ (k=1,2), \tag{2.1}$$

where q_k and η_k are given by (1.6)–(1.8). *Let, moreover, $\lambda_2\ge1$, $p_0(1)\not\equiv0$, $p_0(-1)\not\equiv0$, and let there exist $i\in\{1,2\}$ such that, for every $\delta\in[0,1]$, the following inequalities hold:*

$$\frac{P_0(\delta)}{2^{1+\lambda_1}}P_i^{\lambda_1}(\delta)<1,\quad \widehat{P}_i^{\lambda_1}(\delta)<\Big(1-\frac{P_0(\delta)}{2^{1+\lambda_1}}\,\widehat{P}_i^{\lambda_1}(\delta)\Big)\widehat{P}_{3-i}^{\lambda_1}(\delta), \tag{2.2}$$

$$\frac{P_0^{\lambda_2}(\delta)}{2^{2+\lambda_2}}\,P_{3-i}(\delta)<2^{\lambda_2}-1+\sqrt{1-\frac{P_0^{\lambda_2}(\delta)}{2^{1+\lambda_2}}\,P_i(\delta)}\,. \tag{2.3}$$

Then the problem (1.2)–(1.4) *has at least one solution.*

Theorem 2.2. *Let $\lambda_1,\lambda_2>0$, $\lambda_1\lambda_2=1$, and let there exist $p_0\in\mathscr{P}(\lambda_1)$ and $p_1,p_2\in\mathscr{P}(\lambda_2)$ such that* (2.1) *is fulfilled where q_k and η_k are given by* (1.6)–(1.8). *Let, moreover, $\lambda_2<1$, $p_0(1)\not\equiv0$, $p_0(-1)\not\equiv0$, and let there exist $i\in\{1,2\}$ such that, for every $\delta\in[0,1]$, the following inequalities hold:*

$$\frac{P_0(\delta)}{4}\,P_i^{\lambda_1}(\delta)<1,\quad \widehat{P}_i^{\lambda_1}(\delta)<\Big(1-\frac{P_0(\delta)}{2^{1+\lambda_1}}\,\widehat{P}_i^{\lambda_1}(\delta)\Big)\widehat{P}_{3-i}^{\lambda_1}(\delta), \tag{2.4}$$

$$\frac{P_0^{\lambda_2}(\delta)}{2^{2\lambda_2+1}}\,P_{3-i}(\delta)<1+\sqrt{1-\frac{P_0^{\lambda_2}(\delta)}{4^{\lambda_2}}\,P_i(\delta)}\,. \tag{2.5}$$

Then the problem (1.2)–(1.4) *has at least one solution.*

In the case when the operator $p \in \mathscr{P}(\lambda)$ is homogeneous on the constant functions, i.e., if $p(-1) \equiv -p(1)$, then the numbers $\widehat{P}(\delta)$, $P(\delta)$ take more simple form. More precisely, they do not depend on δ anymore and

$$\widehat{P}(\delta) = P(\delta) = \int\limits_0^\omega p(1)(t)\,dt.$$

The typical operator having the above-described property is an operator defined by means of suprema of the function u over certain subsets of its domain:

$$p(u)(t) \stackrel{def}{=} g(t) \sup\big\{|u(s)|^\lambda \operatorname{sgn} u(s) : \quad s \in S(t)\big\},$$

where $g \in L([0,\omega];\mathbb{R})$ and $S \in \mathscr{S}$. Therefore, considering the system

$$u_1'(t) = g_0(t) \sup\big\{|u_2(s)|^{\lambda_1} \operatorname{sgn} u_2(s) : \quad s \in S_0(t)\big\} + \widetilde{f}_1(u_1,u_2)(t), \tag{2.6}$$

$$\begin{aligned} u_2'(t) = {} & g_1(t) \sup\big\{|u_1(s)|^{\lambda_2} \operatorname{sgn} u_1(s) : \quad s \in S_1(t)\big\} \\ & - g_2(t) \sup\big\{|u_1(s)|^{\lambda_2} \operatorname{sgn} u_1(s) : \quad s \in S_2(t)\big\} + \widetilde{f}_2(u_1,u_2)(t), \end{aligned} \tag{2.7}$$

where $g_i \in L([0,\omega];\mathbb{R}_+)$, $S_i \in \mathscr{S}$ $(i = 0,1,2)$, and $\widetilde{f}_1, \widetilde{f}_2 : C([0,\omega];\mathbb{R}) \times C([0,\omega];\mathbb{R}) \to L([0,\omega];\mathbb{R})$ are continuous operators satisfying Carathéodory conditions, from Theorems 2.1 and 2.2 we derive the following assertions:

Corollary 2.1. *Let $\lambda_1, \lambda_2 > 0$, $\lambda_1\lambda_2 = 1$, and let (2.1) be fulfilled where*

$$q_k(t,\rho) \stackrel{def}{=} \sup\Big\{|\widetilde{f}_k(u_1,u_2)(t)| : \quad \|u_k\|_C \le \rho, \ \ \|u_{3-k}\|_C \le \rho^{\lambda_{3-k}}\Big\} \ \textit{for a.e. } t \in [0,\omega] \tag{2.8}$$

and η_k are given by (1.8). Let, moreover, $\lambda_2 \ge 1$ and $g_i(t) \ge 0$ $(i = 0,1,2)$ for almost every $t \in [0,\omega]$, $g_0 \not\equiv 0$, and let there exist $i \in \{1,2\}$ such that the following inequalities hold:

$$\frac{\|g_0\|_L}{2^{1+\lambda_1}}\, \|g_i\|_L^{\lambda_1} < 1, \quad \|g_i\|_L^{\lambda_1} < \Big(1 - \frac{\|g_0\|_L}{2^{1+\lambda_1}}\, \|g_i\|_L^{\lambda_1}\Big) \|g_{3-i}\|_L^{\lambda_1},$$

$$\frac{\|g_0\|_L^{\lambda_2}}{2^{2+\lambda_2}}\, \|g_{3-i}\|_L < 2^{\lambda_2} - 1 + \sqrt{1 - \frac{\|g_0\|_L^{\lambda_2}}{2^{1+\lambda_2}}\, \|g_i\|_L}\,.$$

Then the problem (2.6), (2.7), (1.4) has at least one solution.

Corollary 2.2. *Let $\lambda_1, \lambda_2 > 0$, $\lambda_1\lambda_2 = 1$, and let (2.1) be fulfilled where q_k and η_k are given by (2.8) and (1.8), respectively. Let, moreover, $\lambda_2 < 1$ and $g_i(t) \ge 0$ $(i = 0,1,2)$ for almost every $t \in [0,\omega]$, $g_0 \not\equiv 0$, and let there exist $i \in \{1,2\}$ such that the following inequalities hold:*

$$\frac{\|g_0\|_L}{4}\, \|g_i\|_L^{\lambda_1} < 1, \quad \|g_i\|_L^{\lambda_1} < \Big(1 - \frac{\|g_0\|_L}{2^{1+\lambda_1}}\, \|g_i\|_L^{\lambda_1}\Big) \|g_{3-i}\|_L^{\lambda_1},$$

$$\frac{\|g_0\|_L^{\lambda_2}}{2^{2\lambda_2+1}}\, \|g_{3-i}\|_L < 1 + \sqrt{1 - \frac{\|g_0\|_L^{\lambda_2}}{4^{\lambda_2}}\, \|g_i\|_L}\,.$$

Then the problem (2.6), (2.7), (1.4) has at least one solution.

Now, consider the particular case of equation (1.1) where $f_0, g \in L([0,\omega];\mathbb{R})$, $\lambda > 0$, and $\mu, \tau : [0,\omega] \to [0,\omega]$ are measurable functions satisfying $\mu(t) \le \tau(t)$ for almost all $t \in [0,\omega]$. Obviously, in such a case, we can invoke our previous results setting $g_0 \equiv 1$, $g_1 \equiv [g]_+$, $g_2 \equiv [g]_-$, $\lambda_1 = 1/\lambda$, $\lambda_2 = \lambda$, and $S_0(t) = \{t\}$, $S_1(t) = S_2(t) = [\mu(t), \tau(t)]$ for almost all $t \in [0,\omega]$. Thus, Corollaries 2.1 and 2.2 yields the following assertions dealing with the equation (1.1).

Corollary 2.3. *Let $\lambda \geq 1$ and let there exist $\sigma \in \{-1, 1\}$ such that*

$$\big\|[\sigma g]_+\big\|_L < \frac{2^{1+\lambda}}{\omega^\lambda}\,,$$

$$\frac{\|[\sigma g]_+\|_L}{(1 - \omega 2^{1+1/\lambda} \|[\sigma g]_+\|_L^{1/\lambda})^\lambda} < \|[\sigma g]_-\|_L < \frac{2^{2+\lambda}}{\omega^\lambda} \bigg(2^\lambda - 1 + \sqrt{1 - \frac{\omega^\lambda}{2^{1+\lambda}} \big\|[\sigma g]_+\big\|_L} \,\bigg).$$

Then the equation (1.1) *has at least one solution u that satisfies $u(0) = u(\omega)$, $u'(0) = u'(\omega)$.*

Corollary 2.4. *Let $0 < \lambda < 1$ and let there exist $\sigma \in \{-1, 1\}$ such that*

$$\big\|[\sigma g]_+\big\|_L < \Big(\frac{4}{\omega}\Big)^\lambda,$$

$$\frac{\|[\sigma g]_+\|_L}{(1 - \frac{\omega}{2^{1+1/\lambda}} \|[\sigma g]_+\|_L^{1/\lambda})^\lambda} < \big\|[\sigma g]_-\big\|_L < \frac{2^{2\lambda+1}}{\omega^\lambda} \bigg(1 + \sqrt{1 - \Big(\frac{\omega}{4}\Big)^\lambda \big\|[\sigma g]_+\big\|_L} \,\bigg).$$

Then the equation (1.1) *has at least one solution u that satisfies $u(0) = u(\omega)$, $u'(0) = u'(\omega)$.*

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Anti-Perron Effect of Changing All Positive Characteristic Exponents to Negative in the Linear Case

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We consider the linear differential systems

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq t_0, \tag{1_n}$$

with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ which are the first approximation for perturbed linear systems

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq t_0, \tag{2_n}$$

and also with infinitely differentiable $n \times n$-matrices $Q(t)$.

O. Perron [7] (see also [6, pp. 50–51]) established in the two-dimensional case the existence of systems (1_2) with exponents $\lambda_1(A) \leq \lambda_2(A) < 0$ and with an infinitely differentiable vector function

$$f(t, y) : (t, y) \in [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}^2,$$

satisfying the condition

$$\|f(t, y)\| \leq C_f \|y\|^m, \ \ y \in \mathbb{R}^2, \ \ t \geq t_0, \tag{4}$$

for $m = 2$ such that all nontrivial solutions of the perturbed system

$$\dot{y} = A(t)y + f(t, y), \ \ y \in \mathbb{R}^2, \ \ t \geq t_0 \tag{5}$$

are infinitely extendable to the right, and their Lyapunov exponents form the set $\{\lambda_2(A), \lambda\}$ with some number $\lambda > 0$. This effect of changing the negative exponents of linear approximation (1_2) to positive ones for solutions of the perturbed system (5) with an m-perturbation (4) of an arbitrary order $m > 1$ was studied in a series of our works, including those with S. K. Korovin, and ended (see [2, 3]) with a complete description of Suslin's sets of collections $\Lambda_+(A, f)$ and $\Lambda_-(A, f)$, respectively, of the positive and negative exponents of all nontrivial solutions of system (4), including the necessary case $\Lambda_-(A, f) = \varnothing$.

For possible applications (dealing with the transformation of "absolutely unstable" differential systems into exponentially or conditionally stable ones), of greater interest is the opposite anti-Perron effect (6) of changing by small perturbations (linear, both vanishing at infinity and exponentially decreasing; nonlinear of higher order of smallness) all positive characteristic exponents of linear approximation (1_n) into negative ones for the solutions of the perturbed system. In [4], this effect is investigated for exponentially decreasing linear perturbations: it is proved that the linear systems (1_n) with all positive exponents and the perturbed system (2_n) with an infinitely differentiable $n\times n$-matrix $Q(t)$ satisfying the condition

$$\|Q(t)\| \le C_Q e^{-\sigma t}, \ \ \sigma>0, \ \ t\ge t_0, \tag{6}$$

and with the characteristic exponents

$$\lambda_1(A+Q)\le \cdots \le \lambda_{n-1}(A+Q)<0<\lambda_n(A+Q) \tag{7}$$

exist.

At the same time, the question formulated in this paper on the existence of system (2_n) with perturbation (6) and with a negative higher exponent $\lambda_n(A+Q)$, remains open. Is it possible under a more general perturbation $Q(t)\to 0$, $t\to+\infty$ to realize simultaneously all the necessary inequalities $\lambda_i(A)>0$, $\lambda_i(A+Q)<0$, $i=\overline{1,n}$?

An affirmative answer contains the following

Theorem. *For any parameters*

$$\lambda_n\ge\cdots\ge\lambda_1>0, \quad \mu_1\le\cdots\le\mu_n<0, \ \ 2\le n\in\mathbb{N},$$

there exist:

1) *a linear system (1_n) with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_i(A)=\lambda_i$, $i=\overline{1,n}$;*
2) *an infinitely differentiable $n\times n$-matrix $Q(t)\to 0$ as $t\to+\infty$ such that the perturbed system (2_n) has characteristic exponents $\lambda_i(A+Q)=\mu_i$, $i=\overline{1,n}$.*

The *proof* of this theorem reduces to the proofs of its two particular variants, respectively, in two-dimensional and three-dimensional cases. In addition, just as in [4], first of all, we construct a piecewise constant and bounded in the interval $[t_0,+\infty)$ matrix $A(t)$ of coefficients of system (1_n) with exponents $\lambda_i(A)=\lambda_i$, $i=\overline{1,n}$, and also the necessary piecewise constant $n\times n$-perturbation matrix $Q(t)\to 0$, $t\to+\infty$ such that the perturbed system (2_n) has characteristic exponents

$$\lambda_i(A+Q)=\mu_i, \ \ i=\overline{1,n}.$$

Next, using the corresponding Gelbaum–Olmsted functions [1, p. 54], we redefine the matrices $A(t)$ and $Q(t)$ in the intervals of very small length containing their discontinuity points in such a way that they become infinitely differentiable and still remain bounded on the semi-axis $[t_0,+\infty)$ (as in the Perron effect itself), while retaining [5] the values of the original and perturbed systems.

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Investigation and Approximate Solution of One Nonlinear Degenerate Integro-Differential Equation of Parabolic Type

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Many natural processes in mathematical modeling can be described by the initial-boundary value problems posed for nonlinear parabolic differential and integro-differential models (see, for example, [3, 4, 8, 10, 13, 14, 16–18] and the references therein). Investigation and construction of algorithms for approximate solutions to these problems are the actual sphere of contemporary mathematical physics and numerical analysis.

A lot of scientific works are dedicated to the investigation and numerical resolution of integro-differential models (see, for example, [3, 8, 10, 14, 16, 18] and the references therein).

One type of integro-differential nonlinear parabolic model is obtained at the mathematical simulation of processes of electromagnetic field penetration in the substance. Based on the Maxwell system [12], this model at first appeared and was studied in [5]. Based on the works [1, 2, 5], the models of such type are investigated in many works (see, for example, [6, 7, 9, 11, 15] and the references therein). Equations and systems of such types still yield to the investigation for special cases. In this direction, the latest and rather complete bibliography can be found in the following monographs [8, 10].

Many scientific papers are devoted to the construction and investigation of discrete analogs of the above-mentioned integro-differential models and for problems similar to them. There are still many open questions in this direction.

The present work is dedicated to the investigation and approximate resolution of the initial-boundary value problem for the following equation

$$\frac{\partial U}{\partial t}+A_1U+A_2U+A_3U=f(x,t),$$

where

$$A_1U=-\frac{\partial}{\partial x}\bigg\{\bigg[\int\limits_0^t\Big(\frac{\partial U}{\partial x}\Big)^2\,d\tau\bigg]\,\frac{\partial U}{\partial x}\bigg\},$$

$$A_2U=-\bigg[\int\limits_0^1\int\limits_0^t\Big(\frac{\partial U}{\partial x}\Big)^2\,d\tau\,dx\bigg]\,\frac{\partial^2 U}{\partial x^2}\,,$$

$$A_3U=-\frac{\partial}{\partial x}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2\,\frac{\partial U}{\partial x}\Big].$$

The purpose of this note is to analyze such type of degenerate equation. In [9] unique solvability and convergence of the semi-discrete scheme with respect to the spatial derivative and finite difference scheme for $\partial U/\partial t+A_1U+A_3U=f(x,t)$ equation are studied. The present work is dedicated to studying such questions for $\partial U/\partial t+A_2U+A_3U=f(x,t)$.

So, the investigated problem has the following form. In the rectangle $Q=(0,1)\times(0,T)$ where T is a fixed positive constant, we consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t}-\left[\int_0^1\int_0^t\left(\frac{\partial U}{\partial x}\right)^2 d\tau\, dx\right]\frac{\partial^2 U}{\partial x^2}-\frac{\partial}{\partial x}\left[\left(\frac{\partial U}{\partial x}\right)^2\frac{\partial U}{\partial x}\right]=f(x,t), \tag{1}$$

$$(0,t)=U(1,t)=0,\ \ t\in[0,T], \tag{2}$$

$$U(x,0)=U_0(x),\ \ x\in[0,1]. \tag{3}$$

Here $f=f(x,t)$, $U_0=U_0(x)$ are given functions of their arguments and $U=U(x,t)$ is an unknown function. It is necessary to mention that (1) is a degenerate type parabolic equation with integro-differential and p-Laplacian term ($p=4$).

Using one modification of the compactness method developed in [17] (see also [16]) the following uniqueness and existence statement takes place.

Theorem 1. *If $f\in W_2^1(Q)$, $f(x,0)=0$, $U_0,V_0\in \overset{\circ}{W}{}_2^1(0,1)$, then there exists the unique solution U of problem* (1)–(3) *satisfying the following properties:*

$$U\in L_4(0,T;\overset{\circ}{W}{}_4^1(0,1)\cap W_2^2(0,1)),\quad \frac{\partial U}{\partial t}\in L_2(Q),\quad \sqrt{T-t}\,\frac{\partial^2 U}{\partial t\partial x}\in L_2(Q).$$

Here usual well-known spaces are used.

In order to describe the space-discretization for problem (1)–(3), let us introduce nets:

$$\omega_h=\big\{x_i=ih,\ i=1,2,\dots,M-1\big\},\quad \overline{\omega}_h=\{x_i=ih,\ i=0,1,\dots,M\big\}$$

with $h=1/M$. The boundaries are specified by $i=0$ and $i=M$. The semi-discrete approximation at (x_i,t) is designed by $u_i=u_i(t)$. The exact solution of problem (1)–(3) at (x_i,t) is denoted by $U_i=U_i(t)$ and is assumed to exist and be smooth enough.

Approximating the space derivatives by the differences:

$$u_{x,i}=\frac{u_{i+1}-u_i}{h}\,,\quad u_{\overline{x},i}=\frac{u_i-u_{i-1}}{h}\,,\quad u_{\overline{x}x,i}=\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}\,,$$

let us correspond to problem (1)–(3) the following semi-discrete scheme:

$$\frac{du_i}{dt}-h\sum_{l=1}^{M}\int_0^t(u_{\overline{x},l})^2d\tau u_{\overline{x}x,i}-\big[(u_{\overline{x},i})^2u_{\overline{x},i}\big]_{x,i}=f(x_i,t),\ \ i=1,2,\dots,M-1, \tag{4}$$

$$u_0(t)=u_M(t)=0,\ \ t\in[0,T], \tag{5}$$

$$u_i(0)=U_{0,i},\ \ i=0,1,\dots,M, \tag{6}$$

which approximates problem (1)–(3) on smooth solutions with the first order of accuracy with respect to h.

Problem (4)–(6) is a Cauchy problem for a nonlinear system of ordinary integro-differential equations. It is not difficult to obtain the following estimate for (4)–(6)

$$\|u\|_h^2+\int_0^t\|u_{\overline{x}}]|_h^2\,d\tau<C,$$

where

$$\|u\|_h^2 = (u,u)_h, \quad (u,v)_h = \sum_{i=1}^{M-1} u_i v_i h, \quad \|u_{\overline{x}}]|_h^2 = (u_{\overline{x}}, u_{\overline{x}}]_h, \quad (u_{\overline{x}}, v_{\overline{x}}]_h = \sum_{i=1}^{M} u_{\overline{x},i} v_{\overline{x},i} h.$$

So, the semi-discrete scheme (4)–(6) is stable with respect to initial data and the right-hand side of equation (4).

Here and below in Theorem 2 by C a generic positive constant independent of the mesh parameter h is denoted. This estimate gives us the global existence of a solution to problem (4)–(6).

Using an approach of the work [7], here in Theorem 2 and below in Theorem 3 for the investigation of the finite-difference scheme, the convergence of the approximate solutions is proved.

The following statement takes place.

Theorem 2. *If problem* (1)–(3) *has a sufficiently smooth solution $U = U(x,t)$, then the solution $u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$ of the semi-discrete scheme* (4)–(6) *tends to the solution $U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t))$ as $h \to 0$ and the following estimate is true*

$$\|u(t) - U(t)\|_h \le Ch.$$

In order to describe the fully discrete analog of problem (1)–(3), let us construct a grid on the rectangle $\overline{Q}$. For using the time-discretization in equation (1), the net is introduced as follows $\omega_\tau = \{t_j = j\tau,\ j = 0,1,\dots,J\}$, with $\tau = T/J$ and $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau$, $u_i^j = u(x_i, t_j)$.

Let us correspond to problem (1)–(3) the following implicit finite difference scheme:

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \Big[\tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (u_l^k)^2\Big] u_{\overline{x}x,i}^{j+1} - \Big\{\big[(u_{\overline{x},i}^{j+1})^2\big] u_{\overline{x},i}^{j+1}\Big\}_{x,i} = f_i^{j+1}, \tag{7}$$

$$i = 1,2,\dots,M-1, \quad j = 0,1,\dots,J-1,$$

$$u_0^j = u_M^j = 0, \quad j = 0,1,\dots,J, \tag{8}$$

$$u_i^0 = U_{0,i}, \quad i = 0,1,\dots,M. \tag{9}$$

So, the system of nonlinear algebraic equations (7)–(9) is obtained, which approximates problem (1)–(3) on the sufficiently smooth solution by the order $O(\tau + h)$.

As for the semi-discrete scheme (4)–(6), we easily obtain the estimate

$$\max_{0 \le j\tau \le T} \|u^j\|_h^2 + \sum_{k=1}^{J} \|u_{\overline{x}}^k]|_h^2 \tau < C,$$

which guarantees the stability and solvability of scheme (7)–(9). It is proved also that system (7)–(9) has a unique solution. Here and below C is a positive constant independent from time and spatial steps τ and h.

The following main conclusion is valid for scheme (7)–(9).

Theorem 3. *If problem* (1)–(3) *has a sufficiently smooth solution $U = U(x,t)$, then the solution $u^j = (u_1^j, u_2^j, \dots, u_{M-1}^j)$, $j = 1,2,\dots,J$ of the difference scheme* (7)–(9) *tends to the solution $U^j = (U_1^j, U_2^j, \dots, U_{M-1}^j)$, $j = 1,2,\dots,J$ as $\tau \to 0$, $h \to 0$ and the following estimate is true*

$$\|u^j - U^j\|_h \le C(\tau + h), \quad j = 1,2,\dots,J.$$

Note that for solving the difference scheme (7)–(9) Newton's iterative process is used and various numerical experiments are done. These experiments agree with theoretical research.

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On the Periodicity of the Riemann Function of Second Order General Type Linear Hyperbolic Equations

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It is well known that the Riemann function $R(\xi,\eta;\xi_1,\eta_1)$ of second order general type linear hyperbolic equations (operators)

$$Lu := u_{\xi\eta} + a(\xi,\eta)u_\xi + b(\xi,\eta)u_\eta + c(\xi,\eta)u = 0$$

is defined by the following way (see, for example, [1–3])

$$L^*R := R_{\xi\eta} - (aR)_\xi - (bR)_\eta + cR = 0, \tag{1}$$

$$R\big|_{\xi=\xi_1} = \exp\bigg\{\int\limits_{\eta_1}^{\eta} a(\xi_1,\eta_2)\,d\eta_2\bigg\}, \quad R\big|_{\eta=\eta_1} = \exp\bigg\{\int\limits_{\xi_1}^{\xi} b(\xi_2,\eta_1)\,d\xi_2\bigg\}. \tag{2}$$

It is also well known that problem (1), (2) is equivalently reduced to the Volterra-type integral equation of the second kind, which, as is well known too, is unconditionally and uniquely solvable for any right-hand side

$$R(\xi,\eta;\xi_1,\eta_1) - \int\limits_{\xi_1}^{\xi} R(\xi_2,\eta;\xi_1,\eta_1)b(\xi_2,\eta)\,d\xi_2 - \int\limits_{\eta_1}^{\eta} R(\xi,\eta_2;\xi_1,\eta_1)a(\xi,\eta_2)\,d\eta_2 + \int\limits_{\xi_1}^{\xi} d\xi_2 \int\limits_{\eta_1}^{\eta} R(\xi_2,\eta_2;\xi_1,\eta_1)c(\xi_2,\eta_2)\,d\eta_2 = 1. \tag{3}$$

In this paper, we will discuss the periodicity of the Riemann function. There is proved the following

Theorem 1. *For the periodicity of the Riemann function in the following sense*

$$R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2) = R(\xi,\eta;\xi_1,\eta_1) \tag{4}$$

it is necessary and sufficient that the following conditions

$$a(\xi+T_1,\eta+T_2) = a(\xi,\eta), \quad b(\xi+T_1,\eta+T_2) = b(\xi,\eta), \quad c(\xi+T_1,\eta+T_2) = c(\xi,\eta) \tag{5}$$

hold.

Proof. Sufficiency. Let us show that along with the function $R(\xi,\eta;\xi_1,\eta_1)$ the solution of the equation (3) is also the function $R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)$, with respect to the variables ξ and η. Consider the following expression

$$R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)-\int\limits_{\xi_1}^{\xi}R(\xi_2+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)b(\xi_2,\eta)\,d\xi_2$$
$$-\int\limits_{\eta_1}^{\eta}R(\xi+T_1,\eta_2+T_2;\xi_1+T_1,\eta_1+T_2)a(\xi,\eta_2)\,d\eta_2$$
$$+\int\limits_{\xi_1}^{\xi}d\xi_2\int\limits_{\eta_1}^{\eta}R(\xi_2+T_1,\eta_2+T_2;\xi_1+T_1,\eta_1+T_2)c(\xi_2,\eta_2)\,d\eta_2.$$

Using transformation of variables ($\xi_2':=\xi_2+T_1$, $\eta_2':=\eta_2+T_2$), the last expression can be rewritten as follows

$$R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)-\int\limits_{\xi_1+T_1}^{\xi+T_1}R(\xi_2',\eta+T_2;\xi_1+T_1,\eta_1+T_2)b(\xi_2'-T_1,\eta)\,d\xi_2'$$
$$-\int\limits_{\eta_1+T_2}^{\eta+T_2}R(\xi+T_1,\eta_2';\xi_1+T_1,\eta_1+T_2)a(\xi,\eta_2'-T_2)\,d\eta_2'$$
$$+\int\limits_{\xi_1+T_1}^{\xi+T_1}d\xi_2'\int\limits_{\eta_1+T_2}^{\eta+T_2}R(\xi_2',\eta_2';\xi_1+T_1,\eta_1+T_2)c(\xi_2'-T_1,\eta_2'-T_2)\,d\eta_2',$$

which using the transformation of variables $\xi_2':=\xi_2+T_1$, $\eta_2':=\eta_2+T_2$ can be rewritten as follows

$$R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)-\int\limits_{\xi_1+T_1}^{\xi+T_1}R(\xi_2,\eta+T_2;\xi_1+T_1,\eta_1+T_2)b(\xi_2,\eta+T_2)\,d\xi_2$$
$$-\int\limits_{\eta_1+T_2}^{\eta+T_2}R(\xi+T_1,\eta_2;\xi_1+T_1,\eta_1+T_2)a(\xi+T_1,\eta_2)\,d\eta_2$$
$$+\int\limits_{\xi_1+T_1}^{\xi+T_1}d\xi_2\int\limits_{\eta_1+T_2}^{\eta+T_2}R(\xi_2,\eta_2;\xi_1+T_1,\eta_1+T_2)c(\xi_2,\eta_2)\,d\eta_2=1. \quad (6)$$

From (3) and (6) by virtue of the uniqueness theorem for the solution of the Volterra type integral equation (3), we have that equality (4) is true.

Necessity. From the first equality of (2) and the periodicity condition (4), we obtain

$$\exp\bigg\{\int\limits_{\eta_1}^{\eta}a(\xi_1,\eta_2)\,d\eta_2\bigg\}=R(\xi_1,\eta;\xi_1,\eta_1)=R(\xi_1+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)$$

$$= \exp\left\{ \int\limits_{\eta_1+T_2}^{\eta+T_2} a(\xi_1+T_1,\eta_2)\,d\eta_2 \right\} = \exp\left\{ \int\limits_{\eta_1}^{\eta} a(\xi_1+T_1,\eta_2'+T_2)\,d\eta_2' \right\}$$

and, consequently,

$$\int\limits_{\eta_1}^{\eta} a(\xi_1,\eta_2)\,d\eta_2 = \int\limits_{\eta_1}^{\eta} a(\xi_1+T_1,\eta_2'+T_2)\,d\eta_2'.$$

By differentiating the last equality with respect to the variable η, we get the first equality of (5). Analogously can be obtained the second equality of (5).

Let now obtain the third equality of (5). Indeed, from (1), taking into account the fact that $R(\xi_1,\eta_1;\xi_1,\eta_1)=1$, we have

$$\begin{aligned} c(\xi_1,\eta_1) = -R_{\xi\eta}(\xi_1,\eta_1;\xi_1,\eta_1) + a(\xi_1,\eta_1)R_\xi(\xi_1,\eta_1;\xi_1,\eta_1) \\ + a_\xi(\xi_1,\eta_1) + b(\xi_1,\eta_1)R_\eta(\xi_1,\eta_1;\xi_1,\eta_1) + b_\eta(\xi_1,\eta_1). \end{aligned} \tag{7}$$

Further, from (2) we obtain

$$R_\xi(\xi_1,\eta_1;\xi_1,\eta_1) = b(\xi_1,\eta_1), \quad R_\eta(\xi_1,\eta_1;\xi_1,\eta_1) = a(\xi_1,\eta_1). \tag{8}$$

Considering equalities (7) and (8), we get

$$c(\xi_1,\eta_1) = -R_{\xi\eta}(\xi_1,\eta_1;\xi_1,\eta_1) + 2a(\xi_1,\eta_1)b(\xi_1,\eta_1) + a_\xi(\xi_1,\eta_1) + b_\eta(\xi_1,\eta_1).$$

Therefore, due to the first and second of (5) and (4) equalities, the third equality of (5) is obtained. □

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Global Stability of Nonlinear Delay Itô Equations and N. V. Azbelev's W-Method

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The classical stability analysis based on Lyapunov functions is the main tool in the theory of ordinary differential equations. However, applications of this method to functional differential equations often encounters serious difficulties. A successful alternative, known as the "N. V. Azbelev W-method", is based on searching auxiliary equations instead of Lyapunov functionals. The W-method is also efficient in studying various classes of stochastic delay differential equations.

However, application of the W-method to nonlinear functional equations remains less efficient, even if N .V. Azbelev and P. M. Simonov formulated some general results for nonlinear deterministic functional differential equations in their monograph [2].

In this work we study global Lyapunov stability of solutions of systems of nonlinear differential Itô equations with delays. We describe a nonlinear modification of the W-method based on the theory of inverse-positive matrices and provide sufficient conditions for the moment stability of solutions in terms of the coefficients for rather general classes of Itô equations.

Let $\mathcal{T}=(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By E we denote the expectation on this probability space.

We study the moment exponential stability of solutions to the following system of nonlinear Itô differential equations with delay:

$$\begin{aligned} dx(t)=&-\sum_{j=1}^{N}A^j(t)x(h_j(t))\,dt+F\big(t,x(h_1^0(t)),\dots,x(h_{m_0}^0(t))\big)\,dt\\ &+\sum_{i=1}^{m}G^i\big(t,x(h_1^i(t),\dots,x(h_{m_i}^i(t)))\big)\,d\mathcal{B}_i(t)\ \ (t\geq 0) \end{aligned}\tag{0.1}$$

with respect to the initial data

$$x(t)=\varphi(t)\ \ (t<0), \tag{0.1a}$$
$$x(0)=b, \tag{0.1b}$$

where $x=(x_1,\dots,x_n)^T$ is an unknown n-dimensional random process on the interval $(-\infty,\infty)$ called a solution to problem $(0.1),(0.1a),(0.1b)$.

We assume that problem $(0.1),(0.1a),(0.1b)$ satisfies the following

Conditions 1:

- $A^j = (a^j_{sl})^n_{s,l=1}$ are $n\times n$-matrices, whose entries are progressively measurable (with respect to the stochastic basis $\mathcal{T}$), scalar stochastic processes, the trajectories of which are almost surely (a.s.) locally integrable for all $j = 1,\dots,N$.

- $F(\,\cdot\,,u) = (F_1(\,\cdot\,,u_1,\dots,u_{m_0}),\dots,F_n(\,\cdot\,,u_1,\dots,u_{m_0}))^T$ are progressively measurable, n-dimensional stochastic processes on the interval $[0,\infty)$ with a.s. locally integrable trajectories for all $u\in R^{m_0}$, and $F(t,\,\cdot\,)$ are $P\times\mu$-almost everywhere continuous functions on R^{m_0}, satisfying the condition $F(\,\cdot\,,0)=0$.

- For all $i = 1,\dots,m$ the functions $G^i(\,\cdot\,,u) = (G^i_1(\,\cdot\,,u_1,\dots,u_{m_i}),\dots,G^i_n(\,\cdot\,,u_1,\dots,u_{m_i}))^T$ ($u\in R^{m_i}$) are progressively measurable, n-dimensional stochastic processes on the interval $[0,\infty)$ with a.s. locally square integrable trajectories, and $G^i(t,\,\cdot\,)$ are $P\times\mu$-almost everywhere continuous functions on R^{m_i}, satisfying the condition $G^i(\,\cdot\,,0)=0$.

- h_j, $j = 1,\dots,N$, h^i_j, $i = 0,\dots,m$, $j = 1,\dots,m_i$ are Borel measurable functions on $[0,\infty)$ such that $h_j(t)\le t$, $j = 1,\dots,N$, $h^i_j(t)\le t$, $i = 0,\dots,m$, $j = 1,\dots,m_i$ $(t\ge 0)$ μ-almost everywhere.

- φ is an $\mathcal{F}_0$-measurable n-dimensional stochastic process on the interval$(-\infty,0)$.

- b is an $\mathcal{F}_0$-measurable n-dimensional random variable.

- For any initial conditions $(0.1a)$ and $(0.1b)$, which satisfy the above requirements, there exists a unique strong global solution $x(t,b,\varphi)$ to problem $(0.1),(0.1b)$, i.e., a solution defined on the initial stochastic basis and on the whole interval $(-\infty,\infty)$.

The moment exponential stability is defined in

Definition 0.1. System (0.1) is called exponentially q-stable with respect to the initial data if there are positive numbers c, λ such that all solutions $x(t,b,\varphi)$ $(t\in(-\infty,\infty))$ of the initial value problem $(0.1),(0.1a),(0.1b)$ satisfy the estimate

$$\big(E|x(t,b,\varphi)|^q\big)^{1/q}\le c\exp\{-\lambda t\}\Big(\big(E|b|^q\big)^{1/q}+\operatorname*{ess\,sup}_{\varsigma<0}\big(E|\varphi(\varsigma)|^q\big)^{1/q}\Big)\ \ (t\ge 0).$$

The next definition is used in the main result of the paper.

Definition 0.2. An invertible matrix $B = (b_{ij})^m_{i,j=1}$ is called inverse-positive if all entries of the matrix B^{-1} are nonnegative.

According to [3], the matrix B will be inverse-positive if $b_{ij}\le 0$ for $i,j = 1,\dots,m$, $i\ne j$ and all diagonal minors of the matrix B are positive. In particular, matrices with strict diagonal dominance and non-positive off-diagonal entriess are inverse-positive.

1 Sufficient stability conditions

As we have already mentioned, we study the moment stability of system (0.1) with respect to the initial data by the W-method, which is based on auxiliary systems. Therefore, along with system (0.1) we consider the following system of linear differential equations with random coefficients:

$$d\widehat{x}(t) = (-B(t)\widehat{x}(t)+f_0(t))\,dt+\sum_{i=1}^{n}f_i(t)\,d\mathcal{B}_i(t)\ \ (t\ge 0),\tag{1.1}$$

where $\widehat{x}=(\widehat{x}_1,\dots,\widehat{x}_n)^T$ is an unknown n-dimensional stochastic process on $(-\infty,\infty)$, $B(t)$ is an $n\times n$-matrix, the entries of which are scalar, progressively measurable stochastic processes on $[0,\infty)$ with the a.s. locally integrable trajectories, while $f_0(t)$, $f_i(t)$ $(i=1,\dots,n)$ are n-dimensional, progressively measurable stochastic processes on $[0,\infty)$ with the a.s. locally square integrable trajectories.

Lemma. *The solutions $\widehat{x}(t)$ of system* (1.1) *can be represented as*

$$\widehat{x}(t)=\widehat{X}(t)\widehat{x}(0)+\int\limits_0^t \widehat{X}(t,\varsigma)f_0(\varsigma)\,d\varsigma+\sum_{i=1}^n\int\limits_0^t \widehat{X}(t,\varsigma)f_i(\varsigma)\,d\mathcal{B}_i(\varsigma)\ \ (t\ge 0),$$

where $\widehat{X}(t,\varsigma)$ $(t\ge 0,\ 0\le\varsigma\le t)$ is the $n\times n$-matrix, the columns of which are solutions of the system $d\widehat{x}(t)=B(t)\widehat{x}(t)\,dt$ $(t\ge 0)$, satisfying $\widehat{X}(t,t)=\overline{E}$ $(t\ge 0)$, while $\widehat{X}(t)\equiv\widehat{X}(t,0)$.

By using the auxiliary system (1.1) and the stated lemma, we can rewrite problem (0.1), $(0.1a)$, $(0.1b)$ in the following equivalent form, where the unknown n-dimensional stochastic process $\overline{x}(t)$ replaces the solution $x(t)$ of system (0.1):

$$\overline{x}(t)=\widehat{X}(t)b+(\Theta(\overline{x}+\overline{\varphi}))(t)\ \ (t\ge 0),$$

where

$$\begin{aligned}(\Theta(\overline{x}+\overline{\varphi}))(t)=&\int\limits_0^t \widehat{X}(t,\varsigma)\Big[B(\varsigma)\overline{x}(\varsigma)-\sum_{j=1}^N A^j(\varsigma)(\overline{x}(h_j(\varsigma))+\overline{\varphi}(h_j(\varsigma)))\Big]\,d\varsigma\\ &+\int\limits_0^t \widehat{X}(t,\varsigma)F\Big(\varsigma,\overline{x}(h_1^0(\varsigma))+\overline{\varphi}(h_1^0(\varsigma)),\dots,\overline{x}(h_{m_0}^0(\varsigma))+\overline{\varphi}(h_{m_0}^0(\varsigma))\Big)\,d\varsigma\\ &+\sum_{i=1}^m\int\limits_0^t \widehat{X}(t,\varsigma)G^i\Big(\varsigma,\overline{x}(h_1^i(\varsigma))+\overline{\varphi}(h_1^i(\varsigma)),\dots,\overline{x}(h_{m_i}^i(\varsigma))+\overline{\varphi}(h_{m_i}^i(s))\Big)\,d\mathcal{B}_i(\varsigma).\end{aligned}$$

Given $1\le q<\infty$, $\lambda>0$ and a stopping time η we introduce the following vectors:

- $\overline{x}(q,\lambda)=(\overline{x}_1(q,\lambda),\dots,\overline{x}_n(q,\lambda))^T$, where
$$\overline{x}_i(q,\lambda)=\sup_{t\ge 0}\big(E|e^{\lambda t}\overline{x}_i(t)|^q\big)^{1/q},\ \ i=1,\dots,n;$$
- $\overline{x}^{\,\eta}(q,\lambda)=(\overline{x}_1^{\eta}(q,\lambda),\dots,\overline{x}_n^{\eta}(q,\lambda))^T$, where
$$\overline{x}_i^{\eta}(q,\lambda)=\sup_{t\ge 0}\big(E|e^{\lambda t}\overline{x}_i^{\eta}(t)|^q\big)^{1/q},\ \ i=1,\dots,n.$$

Assume that using some auxiliary equation (1.1) we obtain the following estimate:

$$E_n\overline{x}^{\,\eta}(q,\lambda)\le C\overline{x}^{\,\eta}(q,\lambda)+c\Big(\big(E|b|^q\big)^{1/q}+\operatorname*{ess\,sup}_{\varsigma<0}\ \big(E|\varphi(\varsigma)|^q\big)^{1/q}\Big)e_n, \tag{1.2}$$

where C is some nonnegative $n\times n$-matrix, $c\ge 0$, E_n is the identity $n\times n$-matrix, $e_n=(1,\dots,1)^T$ is the n-dimensional vector, and $0\le\eta\le\infty$ is an arbitrary stopping time.

We remind that the stopping time [4] is a random variable $\eta:\Omega\to[0,\infty]$ satisfying the property $\{\omega\in\Omega:\ \eta(\omega)\le t\}\in\mathcal{F}_t$ for any $t\ge 0$, while the "stopped" stochastic process $z^\eta(t)$ is defined by $z^\eta(t)\equiv z(t\wedge\eta)$, where $t\wedge\eta=\min\{t;\eta\}$.

Theorem 1.1. *Assume that* $1 \le q < \infty$ *and* ***Conditions 1*** *are satisfied. Assume further that estimate* (1.2) *is satisfied for all admissible* b, φ *and any stopping time* $0 \le \eta \le \infty$.

Then system (0.1) *is exponentially* q*-stable with respect to the initial data if the matrix* $E_n - C$ *is inverse-positive.*

To be able to formulate the main result we need

Conditions 2:

- λ is some positive number;
- There exist nonnegative numbers τ_j, $j = 1, \dots, N$, τ_{ij}, $i = 0, \dots, m$, $j = 1, \dots, m_i$ such that $0 \le t - h_j(t) \le \tau_j$, $j = 1, \dots, N$, $0 \le t - h_j^i(t) \le \tau_{ij}$, $i = 0, \dots, m$, $j = 1, \dots, m_i$ $(t \ge 0)$ μ-almost everywhere.
- There exist nonnegative numbers $\overline{F}_{sl}^{j}$, $j = 1, \dots, m_0$, $s, l = 1, \dots, n$ such that

$$\big|F_s(t, u_1, \dots, u_{m_0})\big| \le \sum_{j=1}^{m_0} \sum_{l=1}^{n} \overline{F}_{sl}^{j} |u_j^l|, \quad s = 1, \dots, n, \quad t \ge 0, \quad P \times \mu\text{-almost everywhere.}$$

- There exist nonnegative numbers $\overline{G}_{sl}^{ij}$, $i = 1, \dots, m$, $j = 1, \dots, m_i$, $s, l = 1, \dots, n$ such that

$$\big|G_s^i(t, u_1, \dots, u_{m_i})\big| \le \sum_{j=1}^{m_i} \sum_{l=1}^{n} \overline{G}_{sl}^{ij} |u_j^l|,$$

$$s = 1, \dots, n, \quad i = 1, \dots, m, \quad t \ge 0, \quad P \times \mu\text{-almost everywhere.}$$

- There are subsets $I_s \subset \{1, \dots, N\}$ $(s = 1, \dots, n)$, positive numbers λ_s, $s = 1, \dots, n$ and nonnegative numbers $\overline{a}_{sl}^{j}$, $j = 1, \dots, N$, $s, l = 1, \dots, n$ such that

$$\sum_{j \in I_s} a_{ss}^j(t) \ge \lambda_s, \quad s = 1, \dots, n,$$

$$|a_{sl}^j(t)| \le \overline{a}_{sl}^{j}, \quad j = 1, \dots, N, \quad s, l = 1, \dots, n, \quad t \ge 0, \quad P \times \mu\text{-almost everywhere.}$$

Stability conditions will be formulated in terms of the special $n \times n$-matrix C, whose entries are defined as follows:

$$c_{ss} = \frac{1}{\lambda_s} \left[\sum_{j \in I_s} \overline{a}_{ss}^{j} \tau_j \Big(\sum_{j=1}^{N} \overline{a}_{ss}^{j} + \overline{F}_{ss} + \frac{c_p}{\sqrt{\tau_j}} \overline{G}_{ss} \Big) + \sum_{j=1, j \notin I_s}^{N} \overline{a}_{ss}^{j} + \overline{F}_{ss} \right] + \frac{c_p}{\sqrt{2\lambda_s}} \overline{G}_{ss}, \quad s = 1, \dots, n,$$

$$c_{sl} = \frac{1}{\lambda_s} \left[\sum_{j \in I_s} \overline{a}_{ss}^{j} \tau_j \Big(\sum_{j=1}^{N} \overline{a}_{sl}^{j} + \overline{F}_{sl} + \frac{c_p}{\sqrt{\tau_j}} \overline{G}_{sl} \Big) + \sum_{j=1}^{N} \overline{a}_{sl}^{j} + \overline{F}_{sl} \right] + \frac{c_p}{\sqrt{2\lambda_s}} \overline{G}_{sl}, \quad s, l = 1, \dots, n, \quad s \ne l,$$

where

$$\overline{F}_{sl} = \sum_{j=1}^{m_0} \overline{F}_{sl}^{j}, \quad \overline{G}_{sl} = \sum_{i=1}^{m} \sum_{j=1}^{m_i} \overline{G}_{sl}^{ij}, \quad s, l = 1, \dots, n.$$

Here the constant c_p comes from the estimate

$$\left(E \left| \int_0^t f(\varsigma)\, d\mathcal{B}(\varsigma) \right|^{2p} \right)^{1/(2p)} \le c_p \left(E \Big(\int_0^t |f(\varsigma)|^2 d\varsigma \Big)^p \right)^{1/(2p)}, \tag{1.3}$$

where $f(\varsigma)$ is an arbitrary scalar, progressive measurable stochastic process and $\mathcal{B}(\varsigma)$ is the scalar Wiener process. Estimate (1.3) follows from the inequality mentioned in [4, p. 65], where the expressions for c_p can be found as well.

Theorem 1.2. *Let $1 \le p < \infty$ and* ***Conditions 1-2*** *be satisfied. If the matrix $E_n - C$ is inverse positive, then system* (0.1) *is exponentially* $2p$*-stable with respect to initial data for any* $0 < \lambda < \min\{\lambda_s,\ s = 1, \dots, n\}$.

2 An example

Let us fix a number $1 \le p < \infty$ and consider the system of nonlinear Itô equations

$$dx(t) = -\sum_{j=1}^{N} A^j x(t-h_j)\,dt + \sum_{j=1}^{m_0} A^{0j} x^{\alpha_j^0}(t-h_j^0)\,dt + \sum_{i=1}^{m}\sum_{j=1}^{m_i} A^{ij} x^{\alpha_j^i}(t-h_j^i)\,d\mathcal{B}_i(t) \ \ (t \ge 0), \quad (2.1)$$

where $A^j = (a_{sl}^j)_{s,l=1}^n$, $j = 1, \dots, N$, $A^{ij} = (a_{sl}^{ij})_{s,l=1}^n$, $i = 0, \dots, m$, $j = 1, \dots, m_i$ are real $n \times n$-matrices, $h_j \ge 0$, $j = 1, \dots, N$, $h_j^i \ge 0$, $i = 0, \dots, m$, $j = 1, \dots, m_i$ are real numbers, and α_j^i, $i = 0, \dots, m$, $j = 1, \dots, m_i$ are real numbers satisfying the inequalities $0 < \alpha_j^i \le 1$, $i = 0, \dots, m$, $j = 1, \dots, m_i$.

Assume that

$$\sum_{j=1}^{N} a_{ss}^j = \lambda_s > 0, \quad s = 1, \dots, n$$

and

$$\overline{F}_{sl} = \sum_{j=1}^{m_0} |a_{sl}^{0j}|, \quad \overline{G}_{sl} = \sum_{i=1}^{m}\sum_{j=1}^{m_i} |a_{sl}^{ij}|, \quad s, l = 1, \dots, n,$$

and the $n \times n$-matrix $E_n - C$ is inverse-positive, where C consists of the following entries:

$$c_{sl} = \frac{1}{\lambda_s}\bigg[\sum_{j=1}^{N} |a_{ss}^j| h_j \Big(\sum_{j=1}^{N} |a_{sl}^j| + \overline{F}_{sl} + \frac{c_p}{\sqrt{h_j}}\,\overline{G}_{sl}\Big) + \sum_{j=1}^{N} |a_{sl}^j| + \overline{F}_{sl}\bigg] - \frac{c_p}{\sqrt{\lambda_s}}\,\overline{G}_{sl}, \quad s, l = 1, \dots, n.$$

Then from Theorem 1.2 it follows that the nonlinear system (2.1) is exponentially $2p$-stable with respect to the initial data.

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Robust Stability for the Attractors of Nonlinear Wave Equations

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1 Introduction

In the paper, we consider the qualitative behavior of a nonlinear wave equation with a non-smooth interaction function that recognizes external bounded disturbances. It is proved that the global attractor of the multivalued semiflow generated by the solutions of the undisturbed problem is stable in the sense of ISS with respect to the disturbances.

The qualitative behavior of infinite-dimensional evolutionary systems without uniqueness, i.e., when, along with global solvability, non-unity of the solution of the initial boundary value problem is also possible, began to be actively studied within the framework of the theory of attractors from the end of the 90s of the last century [9, 14, 15, 17, 21]. It turned out that for broad classes of evolutionary objects, under fairly general conditions for the parameters, it is possible to establish the existence in the phase space of a compact uniformly attracting set (be) the global attractor. Its stability in relation to disturbances has been studied in works [1–4, 7, 8, 10, 12]. The theory of input to state stability (ISS), which characterizes the deviation of solutions of a perturbed problem from an asymptotically stable equilibrium position [6, 16, 19, 20], was applied to infinite-dimensional dissipative systems with a nontrivial attractor in works [5, 11, 18]. In particular, the property of local input to state stability (local ISS) and the property of asymptotic gain (AG) for semi-linear parabolic and wave equations, provided that the Cauchy problem is correct, have been established.

In this paper, for the first time, the AG property was obtained for the global attractor of a dynamic system without uniqueness (m-semiflow), generated by the solutions of a nonlinear wave equation with a non-smooth interaction function.

2 Setting of the problem and the main results

In a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the problem

$$\begin{cases} y_{tt} + \alpha y_t - \triangle y + f(y) = 0, \quad t > 0, \\ y\big|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $\alpha > 0$, $f \in \mathbb{C}(\mathbb{R})$,

$$\exists\, c > 0 \ \ \forall\, s \in \mathbb{R} \quad |f(s)| \le c\big(1 + |s|^{\frac{n}{n-2}}\big), \tag{2.2}$$

$$\varliminf_{s\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{2.3}$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\triangle$ in $H_0^1(\Omega)$. Then it is known [1] that in the phase space

$$X = H_0^1(\Omega) \times L^2(\Omega)$$

problem (2.1) for every $z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in X$ has a (perhaps non-unique) solution $z(\,\cdot\,) = \begin{pmatrix} y(\,\cdot\,) \\ y_t(\,\cdot\,) \end{pmatrix} \in \mathbb{C}([0,+\infty); X)$, $z(0) = z_0$, and all solutions (2.1) generate a multivalued semiflow (m-semiflow) $G : \mathbb{R}_+ \times X \mapsto 2^X$,

$$G(t, z_0) = \Big\{ z(t) : \ z(\,\cdot\,) \text{ is the solution of (2.1)}, \ z(0) = z_0 \Big\},$$

for which there is a global attractor in X.

Definition 2.1 ([14])**.** Let G be a m-semiflow, i.e.,

$$\forall\, x \in X, \ \ \forall\, t, s \geq 0 \quad G(0, x) = x, \ \ G(t+s, x) \subset G(t, G(s, t)).$$

A compact set $\Theta \subset X$ is called a global attractor G, if:

(1) $\Theta \subset G(t, \Theta)$ $\forall\, t \geq 0$;

(2) for any bounded set $B \subset X$,

$$\operatorname{dist}(G(t, B), \Theta) \to 0, \ \ t \to \infty,$$

where here and in the future

$$G(t, B) = \bigcup_{z \in B} G(t, z),$$
$$\operatorname{dist}(A, B) = \sup_{z_1 \in A} \inf_{z_2 \in B} \|z_1 - z_2\|_X.$$

Now consider the disturbed problem

$$\begin{cases} y_{tt} + \alpha y_t - \triangle y + f(y) = h(x) \cdot u(t), \quad t > 0, \\ y\big|_{\partial\Omega} = 0, \end{cases} \tag{2.4}$$

where $h \in L^2(\Omega)$, $u \in L^\infty(0, +\infty)$ is the input (disturbing) signal.

Let's mark

$$S_u(t, 0, z_0) = \Big\{ z(t) : \ z(\,\cdot\,) \text{ is the solution of (2.4)}, \ z(0) = z_0 \Big\}.$$

The main result of the work is the establishment of the asymptotic gain (AG) property in relation to the attractor Θ of the unperturbed ($u \equiv 0$) system [18]:

$$\exists\, \gamma \in \mathcal{K} \ \ \forall\, z_0 \in X, \ \ \forall\, u \in U \subseteq L^\infty(0, +\infty) : \quad \varlimsup_{t \to \infty} \operatorname{dist}(S_u(t, 0, z_0), \Theta) \leq \gamma(\|u\|_\infty),$$

where U is some translationally invariant set of input signals, $\mathcal{K}$ is the class of continuous, monotonically increasing functions with $\gamma(0) = 0$ [13],

$$\|u\|_\infty = \operatorname*{ess\,sup}_{t \geq 0} |u(t)|.$$

3 Robust stability and attractors of multivalued semiflows

Let $(X, \|\cdot\|_X)$ be the Banach space, $\mathbb{R}_{\geq} = \{(t,\tau): \ t \geq \tau \geq 0\}$, Σ be the arbitrary translation-invariant set, i.e.,

$$\forall\, \sigma \in \Sigma, \ \ \forall\, h \geq 0: \quad \sigma(\,\cdot\, + h) \in \Sigma.$$

Definition 3.1 ([2])**.** A family of multivalued mappings $\{S_\sigma : \ \mathbb{R}_{\geq} \times X \mapsto 2^X\}_{\sigma\in\Sigma}$ is called a family of m-semi-processes if $\forall\, \sigma \in \Sigma$, $\forall\, x \in X$, $\forall\, t \geq s \geq \tau \geq 0$, $\forall\, h \geq 0$:

$$S_\sigma(\tau,\tau,x) = x,$$
$$S_\sigma(t,\tau,x) \subset S_\sigma(t,s,S_\sigma(s,\tau,x)),$$
$$S_\sigma(t+h,\tau+h,x) \subset S_{\sigma(\,\cdot\,+h)}(t,\tau,x).$$

Let's mark

$$S_\Sigma = \bigcup_{\sigma\in\Sigma} S_\sigma.$$

Definition 3.2 ([2])**.** A compact set $\Theta_\Sigma \subset X$ is called a uniform attractor $\{S_\sigma\}_{\sigma\in\Sigma}$ if for any bounded set $B \subset X$,

$$\operatorname{dist}\big(S_\Sigma(t,0,B), \Theta_\Sigma\big) \to 0, \ \ t \to \infty$$

and Θ_Σ is the minimal set in the class of such sets.

Remark 3.1. If $\Sigma = \{0\}$, then for $G(t,x) := S_0(t,0,x)$ we have the properties:

$$G(0,x) = S_0(0,0,x) = x,$$
$$G(t+s,x) = S_0(t+s,0,x) \subset S_0(t+s,s,S_0(s,0,x)) \subset S_0(t,0,S_0(s,0,x)) = G(t,G(s,x)),$$

so G is the m-semiflow.

The following lemma guarantees the existence of a uniform attractor in $\{S_\sigma\}_{\sigma\in\Sigma}$.

Lemma 3.1 ([2])**.** *Let $\{S_\sigma\}_{\sigma\in\Sigma}$ be the family of m-semi-processes, Σ be the translation-invariant subset of some metric space and the next conditions be fulfilled:*

(1) *there exists a bounded set $B_0 \subset X$ such that for any bounded set $B \subset X$ exists $T = T(B)$ such that*
$$\forall\, t \geq T \quad S_\Sigma(t,0,B) \subset B_0;$$

(2) $\forall\, \{\sigma_n\} \subset \Sigma$, $\forall\, t_n \nearrow \infty$, $\forall$ *limited sequence $\{x_n\} \subset X$ sequence $\{\xi_n \in S_{\sigma_n}(t_n,0,x_n)\}_{n\geq1}$ is precompact.*

Then $\{S_\sigma\}_{\sigma\in\Sigma}$ has a uniform attractor Θ_Σ.

If, in addition, the next condition is satisfied:

(3) *the mapping $\Sigma \times X \ni (\sigma,x) \mapsto S_\sigma(t,0,x) \subset X$ has a closed graph, then*
$$\Theta_\Sigma \subset S_\Sigma(t,0,\Theta_\Sigma).$$

Remark 3.2. In condition 1) it can be assumed that $B_0 = \{x \in X| \ \|x\|_X \leq R_0\}$.

Remark 3.3. For the $\Sigma = \{0\}$ conditions 1)-3) have the form:

$$\forall\, t \geq T \quad G(t,B) \subset B_0,$$

every sequence $\xi_n \in G(t_n,B)$ is precompact, the mapping $x \mapsto G(t,x)$ has a closed graph; and guarantee [14] that $\Theta := \Theta_{\{0\}}$ is a global attractor m-semiflow G.

Theorem 3.1. *Let for each $u \in U \subset L^\infty(\mathbb{R}_+)$ there exist a translation-invariant set $\Sigma(u)$ such that the family m-semi-processes $\{S_\sigma\}_{\sigma\in\Sigma(u)}$ satisfies conditions (1)–(3) of Lemma 3.1,*

$$\Sigma(0) = \{0\}, \ \ \forall\, u \in U \ \ u \in \Sigma(u),$$

$\forall\, r_0 > 0$ there exists the set B_0 such that condition (1) of Lemma 3.1 is fulfilled $\forall\,\|u\|_\infty \le r_0$, i.e.,

$$\exists\, T = T(r_0, B) \ \ \forall\, t \ge T \qquad \bigcup_{\|u\|_\infty \le r_0} S_{\Sigma(u)}(t, 0, B) \subset B_0, \tag{3.1}$$

and in addition, the next conditions are met

(1)
$$\|u_k\|_\infty \to 0, \ \ t_k \to \infty \implies \xi_k \in S_{\Sigma(u_k)}(t_k, 0, B_0)$$

is precompact,

(2)
$$\|u_k\|_\infty \to 0, \ \ x_k \to x, \ \ \xi_k \in S_{\Sigma(u_k)}(t, 0, x_k), \ \ \xi_k \to \xi \implies \xi \in S_0(t, 0, x).$$

Then
$$\exists\, \gamma \in \mathcal{K} \ \ \forall\, x \in X, \ \ \forall\, u \in U \qquad \varlimsup_{t\to\infty} \operatorname{dist}(S_u(t, 0, x), \Theta) \le \gamma(\|u\|_\infty).$$

4 Application for the disturbed wave equation

We consider a perturbed problem (2.4). Let's strengthen condition (2.3) to the following:

$\exists\, c_1, c_2, c_3 > 0$ such that for $F(s) := \int\limits_0^s f(p)\,dp$ for all $s \in \mathbb{R}$ next inequalities are fulfilled

$$F(s) \ge -ms^2 - c_1, \quad f(s)\cdot s - c_2F(s) + ms^2 \ge c_3, \tag{4.1}$$

where $m \in (0, \lambda_1)$ is small enough.

Under conditions (2.2), (4.1) it is known [2] that $\forall\,\tau \ge 0$, $\forall\, z_\tau \in X$, $\forall\, u \in L^2_{loc}(\mathbb{R}_+)$ problem (2.4) has at least one solution $z \in \mathbb{C}([\tau, +\infty); X):\ z(\tau) = z_\tau$. Moreover, the family of mappings $\{S_u:\ \mathbb{R}_\ge \times X \mapsto 2^X\}$ such that

$$S_u(t, \tau, z_\tau) = \Big\{z(t):\ z(\,\cdot\,) \text{ is the solution of (2.4) and } z(\tau) = z_\tau\Big\} \tag{4.2}$$

generates a family of m-semiprocesses for any translation-invariant $U \subset L^2_{loc}(\mathbb{R}_+)$. In addition, for every solution (2.4) $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ the next evaluation is fair:

$$\|y_t(t)\|^2_{L^2} + \|y(t)\|^2_{H^1_0} \le c_4\bigg(\Big(\|y_t(\tau)\|^2_{L^2} + \|y(\tau)\|^{\frac{2n-2}{n-2}}_{H^1_0}\Big)\cdot e^{-\delta(t-\tau)} + 1 + \int\limits_\tau^t |u(p)|^2 e^{-\delta(t-p)}\,dp\bigg)$$
$$\forall\, t \ge \tau \ge 0,$$

where $c_4 > 0$, $\delta > 0$ do not depend on z.

In particular, if $\sup\limits_{t\geq 0}\int\limits_t^{t+1} |u(p)|^2\,dp < \infty$, then $\forall\, t \geq \tau \geq 0$,

$$\|z(t)\|_X^2 \leq c_5\biggl(\|z(\tau)\|_X^{\frac{2n-2}{n-2}} \cdot e^{-\delta(t-\tau)} + 1 + \sup_{t\geq 0}\int\limits_t^{t+1} |u(p)|^2\,dp\biggr). \tag{4.3}$$

As U, we choose all functions from $L^\infty(\mathbb{R}_+)$ for which

$$\sup_{t\geq 0}\int\limits_t^{t+1} |u(s+l)-u(s)|^2\,ds \leq \varkappa(|l|), \tag{4.4}$$

where $\varkappa$ may depend on u and $\varkappa(p)\to 0$, $p\to 0+$.

It is known [2] that $\forall\, u\in U$ the set

$$\Sigma(u) := cl_{L^2_{loc}}\bigl\{u(\,\cdot\,+h)|,\ h\geq 0\bigr\}$$

is translation invariant and compact in $L^2_{loc}(\mathbb{R}_+)$, $u\in\Sigma(u)$, $\Sigma(0)=\{0\}$ and, in addition,

$$\sup_{t\geq 0}\int\limits_t^{t+1} |v(s)|^2\,ds \leq \sup_{t\geq 0}\int\limits_t^{t+1} |u(s)|^2\,ds \leq \|u\|_\infty^2\ \ \forall\, v\in\Sigma(u). \tag{4.5}$$

If condition (4.4) is fulfilled, the family of m-semi-processes $\{S_v\}_{v\in\Sigma(u)}$, defined in (4.2), satisfies conditions (1)–(3) of Lemma 3.1, and therefore has a uniform attractor $\Theta_{\Sigma(u)}$. At the same time, due to (4.3) and (4.5), condition (3.1) is fulfilled.

Theorem 4.1. *Let the parameters of the disturbed problem* (2.4) *satisfy conditions* (2.2), (4.1), *and* (4.4). *Then*

$$\exists\,\gamma\in\mathcal{K}\ \ \forall\, z_0\in X,\ \ \forall\, u\in U\qquad \varlimsup_{t\to\infty}\operatorname{dist}(S_u(t,0,z_0),\Theta)\leq\gamma(\|u\|_\infty),$$

where Θ *is the global attractor of the undisturbed problem* (2.1).

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Averaging Method to the Optimal Control Problem of a Non-Linear Differential Inclusion with Fast-Oscillating Coefficients on Finite Interval

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1 Introduction

The intensive development of science and technology regularly stimulates the search for effective methods for control of various natural, economic, social, and technical processes. Mathematical models of such situations are problems of optimal control of various classes of evolutionary systems. Considerable attention is paid to mathematical models of processes in the form of differential equations and inclusions with a small parameter. For their solution, asymptotic methods are widely used, in particular, the averaging method, the strict mathematical justification of which was proposed by M. M. Krylov and M. M. Bogolyubov. In works of V. A. Plotnikov and works of his school (see, for example, [12]) there is the strict justification of the averaging method in application to control problems.

It is known that the averaging method is one of the most effective tools for solving various optimal control problems for differential equations [4, 5, 8, 9] as well as for differential inclusions with fast oscillating coefficients [6, 7, 13]. The Krasnoselski-Krein theorem [8] and its multi-valued analogue [11] play an essential role for investigation of above-mentioned problems. The concept of integral continuity plays a key role in investigation of the considered optimal control problem using averaging method, since the existence of the limit when we pass to aversged coefficients is equivalent to the integral continuity.

In the present paper we consider the optimal control problem of a non-linear system of differential inclusions with fast oscillating parameters. First, we prove the existence of solutions for the initial perturbed optimal control problem and corresponding problem with averaged coefficients. Then we prove that optimal control of the problem with averaging coefficients can be considered as "approximately" optimal for the initial perturbed one.

2 Setting of the problem and main results

Let us consider an optimal control problem

$$\begin{cases} \dot{x}(t) \in X\Big(\dfrac{t}{\varepsilon}, x(t), u(t)\Big), \quad t \in (0,T), \\ x(0) = x_0, \quad u(\,\cdot\,) \in U, \\ J[x,u] = \displaystyle\int_0^T L(t, x(t), u(t))\, dt + \Phi(x(T)) \to \inf . \end{cases} \tag{2.1}$$

Here $\varepsilon > 0$ is a small parameter, $x : [0,T] \to \mathbb{R}$ is an unknown phase variable, $u : [0,T] \to \mathbb{R}^m$ is an unknown control function, $X : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \operatorname{conv}(\mathbb{R}^n)$ is a multi-valued function, $U \subset L^2(0,T)$ is a fixed set.

Assume that uniformly with respect to x for every $u \in \mathbb{R}^m$

$$\operatorname{dist}_H \bigg(\frac{1}{s} \int\limits_0^s X(\tau, x, u)\, d\tau, Y(x, u) \bigg) \to 0, \quad s \to \infty, \tag{2.2}$$

where limits for multi-valued function are considered in the sense of [1, 3], dist_H is the Hausdorff metric, $Y : \mathbb{R}^n \times \mathbb{R}^m \to \operatorname{conv}(\mathbb{R}^n)$, and the integral of multi-valued function is considered in the sense of Aumann [2]. We consider the following problem with averaged right hand side:

$$\begin{cases} \dot{y}(t) \in Y(y(t), u(t)), \\ y(0) = x_0, \quad u(\,\cdot\,) \in U, \\ J[x, u] = \displaystyle\int\limits_0^T L(t, y(t), u(t))\, dt + \Phi(x(T)) \to \inf. \end{cases} \tag{2.3}$$

Under the natural assumptions on X, L, Φ, U we will show that problems (2.1) and (2.3) have solutions $\{\overline{x}_\varepsilon, \overline{u}_\varepsilon\}$ and $\{\overline{y}, \overline{u}\}$, respectively,

$$\overline{J}_{\varepsilon_n} \to \overline{J}, \quad \varepsilon_n \to 0,$$

where $\overline{J}_{\varepsilon_n} := J[\overline{x}_{\varepsilon_n}, \overline{u}_{\varepsilon_n}]$, $\overline{J} := [\overline{y}, \overline{u}]$, and up to a subsequence

$$\begin{aligned} &\overline{u}_{\varepsilon_n} \to \overline{u} \ \text{ in } L^2(0, T), \\ &\overline{x}_{\varepsilon_n} \to \overline{y} \ \text{ in } \mathbb{C}([0, T]). \end{aligned}$$

In what follows we consider the problem of finding an approximate solution of (2.1) by transition to averaged coefficients. We note that the transition to the averaging parameters can essentially simplify the problem.

Let us consider some assumptions and notations regarding parameters of our problem.

Let $Q = \{t \geq 0,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m\}$ and assume the following assumptions hold.

Condition 2.1. Mapping $t, x, u \mapsto X(t, x, u)$ is continuous in Hausdorff metric.

Condition 2.2. Multi-valued function $X(t, x, u)$ satisfies the growth property: $\exists\, M > 0$ such that

$$\|X(t, x, u)\|_+ \leq M(1 + \|x\|) \ \ \forall\, (t, x, u) \in Q,$$

where

$$\|X(t, x, u)\|_+ = \sup_{\xi \in X(t, x, u)} \|\xi\|,$$

$\|\xi\|$ is the Euclidian norm of $\xi \in \mathbb{R}^n$;

Condition 2.3. Multi-valued function $X(t, x, u)$ satisfies the Lipschitz condition: $\exists\, \lambda > 0$ such that

$$\operatorname{dist}_H \big(X(t, x_1, u_1), X(t, x_2, u_2) \big) \leq \lambda \big(\|x_1 - x_2\| + \|u_1 - u_2\| \big).$$

Condition 2.4. Mapping $(x, u) \mapsto L(t, x, u)$ is a continuous one, moreover, function $t \mapsto L(t, x, u)$ is measurable $\forall\, x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and

$$|L(t, x, u)| \leq c(t) \big(1 + \|u\| \big),$$

where $c(\,\cdot\,) \in L^2(0, T)$ is a given function.

Condition 2.5. $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

Condition 2.6. $U \subset L^2(0,T)$ is a compact set.

Condition 2.7. Let D and D' be two domains in $\mathbb{R}^n$. We suppose that the embedding $D'+\delta B \subset D$ is fulfilled for some $\delta > 0$, where $B = \{x \in \mathbb{R}^n : \ \|x\| \le 1\}$.

Let us note that under Conditions 2.1–2.3 for all $u \in L^2(0,T)$ the Cauchy problem

$$\begin{cases} \dot{x} \in X\Big(\dfrac{t}{\varepsilon}, x, u\Big), \quad t \in (0,T) \\ x(0) = x_0 \end{cases} \tag{2.4}$$

has a solution, that is there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ satisfying the inclusion (2.4) a.e.

Under condition (2.2) the multi-valued mapping Y satisfies the Conditions 2.1–2.3, hence $\forall\, u \in L^2(0,T)$ the Cauchy problem

$$\begin{cases} \dot{y} \in Y(y,u), \quad t \in (0,T) \\ y(0) = x_0 \end{cases} \tag{2.5}$$

has a solution.

Taking into account conditions for parameters of problem we can show the existence of solutions for the initial perturbed optimal control problem and corresponding problem with averaged coefficients. Namely, we have the next

Theorem 2.1. *Under Conditions 2.1–2.6 problem* (2.1) (*resp. problem* (2.3)) *has the solution* $\{\overline{x}_\varepsilon, \overline{u}_\varepsilon\}$ (*resp.* $\{\overline{y}, \overline{u}\}$).

It worth noting the multi-valued analogue of Krasnoselski–Krein theorem [8, 10, 11, 13] plays an essential role for investigation of the above-mentioned problems. Let us make sure that optimal control of the problem with averaging coefficients can be considered as "approximately" optimal for the initial perturbed one.

Theorem 2.2. *Suppose that for all* $u(\,\cdot\,) \in U$ *problem* (2.5) *has a unique solution. Under Conditions 2.1–2.6 and* (2.2) *we have*

$$\overline{J}_{\varepsilon_n} = J[\overline{x}_{\varepsilon_n}, \overline{u}_{\varepsilon_n}] \to \overline{J} := J[\overline{y}, \overline{u}] \ \ \textit{as} \ \ \varepsilon_n \to 0$$

and up to a subsequence

$$\begin{aligned} &\overline{u}_{\varepsilon_n} \to \overline{u} \ \ \textit{in} \ L^2(0,T), \ \ \varepsilon_n \to 0, \\ &\overline{x}_{\varepsilon_n} \to \overline{y} \ \ \textit{in} \ C(0,T), \ \ \varepsilon_n \to 0, \end{aligned}$$

where $\{\overline{x}_{\varepsilon_n}, \overline{u}_{\varepsilon_n}\}$ *is the solution of* (2.1) *and* $\{\overline{y}, \overline{u}\}$ *is the solution of* (2.3).

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The Critical Case of the Matrix Differential Equations' Systems

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This paper considers a system of M linear matrix differential equations with coefficients, depicted in the form of absolutely and uniformly convergent Fourier series with slowly variable in a certain sense coefficients and with the frequency (class F). This system is close to the block-diagonal system with slowly changing coefficients. We are looking for a transformation with coefficients of a similar type which brings this system to purely block-diagonal form. Regarding the coefficients of this transformation, chews a quasi-linear system of matrix differential equations, which decays on M independent subsystems, each of which has the form of some auxiliary nonlinear systems. We obtained conditions of existence of the desired transformation for this auxiliary system in a critical case.

1 Basic notation and definitions

Let

$$G(\varepsilon_0) = \Big\{(t;\varepsilon): \ \ t \in \mathbb{R}, \ \ \varepsilon \in [0;\varepsilon_0), \ \ \varepsilon_0 \in \mathbb{R}^*\Big\}.$$

Definition 1.1. Let's say that the function $p(t;\varepsilon)$ belongs to the class $S(m;\varepsilon_0)$ if the following conditions are true

(1) $p\colon G(\varepsilon_0) \to \mathbb{C}$;

(2) $p(t;\varepsilon) \in C^m(G(\varepsilon_0))$ for t;

(3)
$$\frac{d^k p(t;\varepsilon)}{dt^k} = \varepsilon^k p_k(t;\varepsilon) \ \ (0 \leqslant k \leqslant m),$$

where

$$\|p\|_{S(m;\varepsilon_0)} \overset{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |p_k(t;\varepsilon)| < +\infty.$$

Definition 1.2. Let's say that the function $f(t;\varepsilon;\theta(t;\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$ ($m \in \mathbb{N} \cup \{0\}$), if this function can be represented in the following form:

$$f(t;\varepsilon;\theta(t;\varepsilon) = \sum_{n=-\infty}^{+\infty} f_n(t;\varepsilon) \exp(in\theta(t;\varepsilon)),$$

where

(1) $f_n(t;\varepsilon) \in S(m;\varepsilon_0)$ ($n \in \mathbb{Z}$);

(2)
$$\|f\|_{F(m;\varepsilon_0;\theta)} \overset{def}{=} \sum_{n=-\infty}^{+\infty} \|f_n\|_{S(m;\varepsilon_0)} < +\infty;$$

(3)

$$\theta(t;\varepsilon)=\int\limits_0^t\varphi(\tau;\varepsilon)\,d\tau,\ \ \varphi\in\mathbb{R}^*,\ \ \varphi\in S(m;\varepsilon_0),\ \ \inf_{G(\varepsilon_0)}\varphi(t;\varepsilon)=\varphi_0>0.$$

Definition 1.3. Let's say that the matrix $A(t;\varepsilon)=(a_{jk}(t;\varepsilon))_{j,k=\overline{1,N}}$ belongs to the class $S_2(m;\varepsilon_0)$ $(m\in\mathbb{N}\cup\{0\})$, in case $a_{jk}\in S(m;\varepsilon_0)$ $(j,k=\overline{1,N})$.

Let's define the norm

$$\|A(t;\varepsilon)\|_{S_2(m;\varepsilon_0)}\overset{def}{=}\max_{1\leqslant j\leqslant N}\sum_{k=1}^N\|a_{jk}(t;\varepsilon)\|_{S(m;\varepsilon_0)}.$$

Definition 1.4. Let's say that the matrix $B(t;\varepsilon;\theta)=(b_{jk}(t;\varepsilon;\theta))_{j,k=\overline{1,N}}$ belongs to the class $F_2(m;\varepsilon_0;\theta)$ $(m\in\mathbb{N}\cup\{0\})$, in case $b_{jk}(t;\varepsilon;\theta)\in F(m;\varepsilon_0;\theta)$ $(j,k=\overline{1,N})$.

Let's define the norm

$$\|B(t;\varepsilon;\theta)\|_{F_2(m;\varepsilon_0;\theta)}\overset{def}{=}\max_{1\leqslant j\leqslant N}\sum_{k=1}^N\|b_{jk}(t;\varepsilon;\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

Note that in case $B_1\in F_2(m;\varepsilon_0;\theta)$, $B_2\in F_2(m;\varepsilon_0;\theta)$, the following conditions are true:

(1) $B_1+B_2, B_1B_2\in F_2(m;\varepsilon_0;\theta)$,

(2) $\|B_1+B_2\|_{F_2(m;\varepsilon_0;\theta)}\le\|B_1\|_{F_2(m;\varepsilon_0;\theta)}+\|B_2\|_{F_2(m;\varepsilon_0;\theta)}$,

(3) $\|B_1B_2\|_{F_2(m;\varepsilon_0;\theta)}\le 2^m\|B_1\|_{F_2(m;\varepsilon_0;\theta)}\cdot\|B_2\|_{F_2(m;\varepsilon_0;\theta)}$.

2 Statement of the problem

The following system of linear matrix equations is considered

$$\frac{dX_j}{dt}=A_j(t,\varepsilon)X_j+\mu\sum_{k=1}^M B_{jk}(t,\varepsilon,\theta)X_k,\ \ j=\overline{1,M},\tag{2.1}$$

where X_j are unknown square matrices of the order N, belonging to some closed bounded region $D\subset\mathbb{C}^{N\times N}$, $\mathbb{C}^{N\times N}$ is the space of complex-valued matrices of dimension $(N\times N)$. Also, let $A_j(t,\varepsilon)\in S_2(m;\varepsilon_0)$, $B_{kj}(t,\varepsilon,\theta)\in F_2(m;\varepsilon_0;\theta)$, $\mu\in(0,1)$ be real parameter.

We are looking for the transformation

$$X_j=Y_j+\sum_{\substack{k=1\\k\neq j}}^M Q_{jk}(t,\varepsilon,\theta(t,\varepsilon),\mu)Y_k,\ \ j=\overline{1,M},\tag{2.2}$$

in which $Q_{jk}(t,\varepsilon,\theta(t,\varepsilon),\mu)$ $(j,k=\overline{1,M})$ are unknown square matrices of dimension $N\times N$ that belong to the class $F_2(m_1;\varepsilon_1;\theta)$ $(m_1\le m_0;\ \varepsilon_1\le\varepsilon_0)$ which brings system (2.1) to the form

$$\frac{dY_j}{dt}=V_j(t,\varepsilon,\theta,\mu)Y_j,\tag{2.3}$$

where $V_j(t,\varepsilon,\theta,\mu)\in F_2(m_1;\varepsilon_0;\theta)$.

Using transformation (2.2) with respect to unknown functions $Q_{jk}(t,\varepsilon,\theta,\mu)$ $(j=\overline{1,M})$ we will get the system

$$\frac{dQ_{jk}}{dt} = A_j(t,\varepsilon)Q_{jk} - Q_{jk}A_k(t,\varepsilon) + \mu(B_{jj}(t,\varepsilon,\theta)Q_{jk} - Q_{jk}B_{kk}(t,\varepsilon,\theta))$$
$$+ \mu B_{jk}(t,\varepsilon,\theta) + \mu \sum_{\substack{s=1\\ s\neq j,\, s\neq k}}^{M} B_{js}(t,\varepsilon,\theta)Q_{sk} - \mu Q_{jk}\sum_{\substack{s=1\\ s\neq k}}^{M} B_{ks}(t,\varepsilon,\theta)Q_{sk}, \quad j,k=\overline{1,M}, \quad j\neq k. \quad (2.4)$$

So, system (2.1) turns into

$$\frac{dY_j}{dt} = V_j(t,\varepsilon,\theta,\mu)Y_j = \Big(\mu B_{jj}(t,\varepsilon,\theta) + \Lambda(t,\varepsilon) + \sum_{\substack{s=1\\ s\neq j}}^{M} B_{js}(t,\varepsilon,\theta)Q_{sj}\Big)Y_j, \quad j=\overline{1,M}. \quad (2.5)$$

The following lemma takes place.

Lemma 2.1. *Let the matrices $A_j(t,\varepsilon)$ $(j=\overline{1,M})$ in system (2.4) be such that there are matrices $L_j(t,\varepsilon)$ $(j=\overline{1,M})$, for which the following conditions are true:*

(1) $L_j(t,\varepsilon)\in S_2(m;\varepsilon)$ $(j=\overline{1,M})$;

(2) $|\det(L_j(t,\varepsilon))| \geq a_0 > 0$ $(j=\overline{1,M})$;

(3)
$$L_j^{-1}(t,\varepsilon)A_j(t,\varepsilon)L_j(t,\varepsilon) = \triangle_j(t,\varepsilon) \ \ (j=\overline{1,M}),$$
in which $\triangle_j(t,\varepsilon)$ $(j=\overline{1,M})$ – lower triangular matrices of the Nth order of the class $S_2(m;\varepsilon_0)$.

Then using the transformation

$$Q_{jk} = L_j(t,\varepsilon)Y_{jk}L_k^{-1}(t,\varepsilon) \ \ (j,k=\overline{1,M}, \ \ j\neq k), \quad (2.6)$$

system (2.4) is reduced to the next system

$$\frac{dY_{jk}}{dt} = \triangle_j(t,\varepsilon)Y_{jk} - Y_{jk}\triangle_k(t,\varepsilon) - L^{-1}\frac{dL_j}{dt}Y_{jk} - Y_{jk}L_k^{-1}(t,\varepsilon)\frac{dL_k}{dt}$$
$$+ \mu(L_j^{-1}(t,\varepsilon)B_{jj}(t,\varepsilon,\theta)L_j(t,\varepsilon)Y_{jk} - Y_{jk}L_k^{-1}(t,\varepsilon)B_{kk}(t,\varepsilon,\theta)L_k(t,\varepsilon))$$
$$+ \mu L_j^{-1}(t,\varepsilon)B_{jk}(t,\varepsilon,\theta)L_k(t,\varepsilon) + \mu \sum_{\substack{s=1\\ s\neq j,\, s\neq k^c}}^{M} L_j^{-1}(t,\varepsilon)B_{js}(t,\varepsilon,\theta)L_s(t,\varepsilon)Y_{sk}$$
$$- \mu Y_{jk}\sum_{\substack{s=1\\ s\neq k^c}}^{M} L_k^{-1}B_{ks}(t,\varepsilon,\theta)L_s(t,\varepsilon)Y_{sk}, \quad j,k=\overline{1,M} \ \ (j\neq k). \quad (2.7)$$

3 Main results

Lemma 3.1. *Let the following system of matrix differential equations be given:*

$$\frac{dY_j}{dt} = D_{j1}(t,\varepsilon)Q_{jk} - Q_{jk}D_{j2}(t,\varepsilon) + \mu F_j(t,\varepsilon,\theta) + \mu\sum_{s=1}^{M} P_{js1}(t,\varepsilon,\theta)Y_sP_{js2}(t,\varepsilon,\theta)$$
$$- \mu Y_j\sum_{s=1}^{M} R_{js1}(t,\varepsilon,\theta)Y_sR_{js2}(t,\varepsilon,\theta) - \varepsilon H_{j1}(t,\varepsilon)Y_j - \varepsilon Y_jH_{j2}(t,\varepsilon), \quad j=\overline{1,M}, \quad (3.1)$$

where $D_{j1}(t,\varepsilon)=(d^{j1}_{\alpha\beta}(t,\varepsilon))_{\alpha,\beta=\overline{1,N}}$, $D_{j2}(t,\varepsilon)=(d^{j2}_{\alpha\beta}(t,\varepsilon))_{\alpha,\beta=\overline{1,N}}$ *– lower triangular matrices of the class* $S_2(m;\varepsilon_0)$, $F_j(t,\varepsilon,\theta)$, $P_{js1}(t,\varepsilon,\theta)$, $P_{js2}(t,\varepsilon,\theta)$, $R_{js1}(t,\varepsilon,\theta)$, $R_{js2}(t,\varepsilon,\theta)$ *is in the class* $F_2(m;\varepsilon_0;\theta)$, $H_{j1}(t,\varepsilon)$, $H_{j2}(t,\varepsilon)$ *are in the class* $S_2(m-1;\varepsilon_0)$*,*$\mu\in(0,1)$ *is a real parameter. And let the conditions be fulfilled:*

(1^0)

$$\inf_{G(\varepsilon_0)}\big|d^{j1}_{\alpha\beta}(t,\varepsilon)-d^{k1}_{\alpha\beta}(t,\varepsilon)-in\varphi(t,\varepsilon)\big|\ge b_0>0,$$
$$\inf_{G(\varepsilon_0)}\big|d^{j2}_{\alpha\beta}(t,\varepsilon)-d^{k2}_{\alpha\beta}(t,\varepsilon)-in\varphi(t,\varepsilon)\big|\ge b_0>0\ \ \forall\,n\in\mathbb{Z},\ \ j,k=\overline{1,N},\ \ j\neq k.$$

(2^0)

$$d^{j1}_{\alpha\beta}(t,\varepsilon)-d^{k2}_{\alpha\beta}(t,\varepsilon)=i\omega_{jk}(t,\varepsilon),\ \ \omega_{jk}(t,\varepsilon)\in\mathbb{R},$$
$$\inf_{G(\varepsilon_0)}|\omega_{jk}(t,\varepsilon)-n\varphi(t,\varepsilon)|\ge b_0>0\ \ \forall\,n\in\mathbb{Z},\ \ j,k=\overline{1,N}.$$

Then there exist constants $\mu_1\in(0;\mu_0),\varepsilon_2\in(0;\mu_0)$ *such that for all* $\mu\in[0;\mu_2)$ *and for all* $\varepsilon\in(0,\varepsilon_2)$*, system* (3.1) *has a partial solution of the class* $F_2(m-1;\varepsilon_2;\theta)$*.*

Condition (2^0) shows that in this case we are dealing with critical by chance, as opposed to work [8], in which it is assumed that

$$\big|\mathrm{Re}\,(d^{j1}_{\alpha\beta}(t,\varepsilon)-d^{k2}_{\alpha\beta}(t,\varepsilon))\big|\ge\gamma>0\ \ (j=\overline{1,M},\ \ k=\overline{1,N}).$$

The next theorem takes place.

Theorem 3.1. *Let system* (2.4) *satisfy the conditions of Lemma 3.1, and let system* (2.7)*, obtained by transformation* (2.6)*, for each* $k=\overline{1,M}$ *satisfy all the conditions of Lemma 3.1. Then there exist* $\mu_4\in(0;1)$*,* $\varepsilon_4(\mu)\in(0;\varepsilon_0)$ *such that for all* $\mu\in(0;\varepsilon_4)$ *and for all* $\varepsilon\in(0;\varepsilon_4(\mu))$ *there exists the transformation of the form* (2.2)*, in which the coefficients* $Q_{jk}(t,\varepsilon,\theta(t,\varepsilon),\mu)$ *belong to the class* $F_2(m-1;\varepsilon_4(\mu);\theta)$*, that brings system* (2.1) *to the form* (2.3)*, in which* $V_j(t,\varepsilon,\theta,\mu)$ *are determined by formulas* (2.5)*.*

For matrix systems of this type, such a result was not obtained before. In previous works [9] a matrix differential equation was considered:

$$\frac{dX}{dt}=A(t,\varepsilon)X-XB(t,\varepsilon)+P(t;\varepsilon_0;\theta)+\mu\Phi(t;\varepsilon_0;\theta;X),\tag{3.2}$$

where X is an unknown square matrix of order N, that belongs to some closed limited area $D\subset\mathbb{C}^{N\times N}$, where $\mathbb{C}^{N\times N}$ is the space of complex-valued matrices of dimention $N\times N$, $A(t;\varepsilon),B(t,\varepsilon)\in S_2(m;\varepsilon_0)$, $P(t;\varepsilon_0;\theta)\in F_2(m;\varepsilon_0;\theta)$. It is also assumed that $\Phi(t;\varepsilon_0;\theta;X)$ is a matrix-function that belongs to the class $F_2(m;\varepsilon_0;\theta)$ with respect to m,ε_0,θ and is continuous over X in D. μ is a real parameter.

For equation (3.2) in the critical case, the issue of the presence of partial class solutions was studied $F(m_1;\varepsilon_1;\theta)$ $(m_1\le m;\ \varepsilon_1\le\varepsilon_0)$.

The results of the works [1–7, 10] were used for obtaining our results.

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The Boundary Value Problem for One Class of Nonlinear Systems of Partial Differential Equations

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In Euclidean space $\mathbb{R}^{n+1}$ of variables $x=(x_1,\dots,x_n)$ and t consider a nonlinear system of partial differential equations of the form

$$L_f u := \frac{\partial^{4k}u}{\partial t^{4k}} - \sum_{i,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + f(u) = F, \tag{1}$$

where $f=(f_1,\dots,f_N)$, $F=(F_1,\dots,F_N)$ are the given and $u=(u_1,\dots,u_N)$ is an unknown vector functions, $N\geq 2$; A_{ij} are the given constant quadratic matrices of order N, besides $A_{ij}=A_{ji}$, $i,j=1,\dots,n$, $n\geq 2$, k is a natural number.

For system (1) consider the following boundary value problem: in cylindrical domain $D_T := \Omega\times(0,T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, find a solution $u=u(x,t)$ to system (1.1) according to the following boundary conditions

$$u\big|_{\Gamma}=0, \tag{2}$$

$$\frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T}=0, \quad i=0,\dots,2k-1, \tag{3}$$

where $\Gamma:=\partial\Omega\times(0,T)$ is a lateral face of the cylinder D_T, $\Omega_0:\ x\in\Omega,\ t=0$ and $\Omega_T:\ x\in\Omega,\ t=T$ are upper and lower bases of this cylinder, respectively.

Denote by $C^{2,4k}(\overline{D}_T)$ the space of continuous in $\overline{D}_T$ vector functions $u=(u_1,\dots,u_N)$, having continuous in $\overline{D}_T$ partial derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i\partial x_j}$, $\frac{\partial^l u}{\partial t^l}$, $i,j=1,\dots,n$; $l=1,\dots,4k$. Let

$$C_0^{2,4k}(\overline{D}_T,\partial D_T) := \Big\{u\in C^{2,4k}(\overline{D}_T):\ \ u\big|_{\Gamma}=0,\ \ \frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T}=0,\ \ i=0,\dots,2k-1\Big\}.$$

Consider the Hilbert space $W_0^{1,2k}(D_T)$, which is obtained by completion with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)} = \int\limits_{D_T}\bigg[|u|^2+\sum_{i=1}^{2k}\Big|\frac{\partial^i u}{\partial t^i}\Big|^2+\sum_{i=1}^{n}\Big|\frac{\partial u}{\partial x_i}\Big|^2\bigg]\,dx\,dt \tag{4}$$

of the classical space $C_0^{2,4k}(\overline{D}_T,\partial D_T)$, where $|\cdot|$ is the norm in the space $\mathbb{R}^N$.

Remark 1. From (4) it follows that if $u\in W_0^{1,2k}(D_T)$, then $u\in \overset{\circ}{W}{}_2^1(D_T)$ and $\frac{\partial^i u}{\partial t^i}\in L_2(D_T)$, $i=1,\dots,2k$. Here $W_2^1(D_T)$ is a well-known Sobolev space consisting of elements from $L_2(D_T)$ and having generalized partial derivatives of the first order from $L_2(D_T)$, and

$$\overset{\circ}{W}{}_2^1(D_T)=\big\{u\in W_2^1(D_T):\ \ u|_{\partial D_T}=0\big\},$$

where the equality $u|_{\partial D_T}=0$ must be understood in the sense of the trace theory.

Below, we impose on nonlinear vector function $f=(f_1,\dots,f_N)$ from (1) the following requirements

$$f\in C(\mathbb{R}^N),\ \ |f(u)|\le M_1+M_2|u|^\alpha,\ \ u\in\mathbb{R}^N, \tag{5}$$

where $M_i=const\ge 0$, $i=1,2$, and

$$0\le\alpha=const<\frac{n+1}{n-1}\,. \tag{6}$$

Remark 2. The embedding operator $I:W_2^1(D_T)\to L_q(D_T)$ represents a linear continuous compact operator for $1<q<\frac{2(n+1)}{n-1}$, $n>1$. At the same time the Nemitsky operator $K:L_q(D_T)\to L_2(D_T)$, acting by the formula $Ku=f(u)$, where $u=(u_1,\dots,u_N)\in L_q(D_T)$ and vector function $f=(f_1,\dots,f_N)$ satisfies condition (5), is continuous and bounded if $q\ge 2\alpha$. Therefore, if $\alpha<\frac{n+1}{n-1}$, then there exists such number q that $1<q<\frac{2(n+1)}{n-1}$ and $q\ge 2\alpha$. Therefore, in this case the operator

$$K_0=KI:W_2^1(D_T)\to L_2(D_T)$$

is continuous and compact. Then from $u\in W_2^1(D_T)$ it follows that $f(u)\in L_2(D_T)$ and, if $u^m\to u$ in the space $W_2^1(D_T)$, then $f(u^m)\to f(u)$ in $L_2(D_T)$.

Remark 3. Let $u\in C_0^{2,4k}(\overline{D}_T,\partial D_T)$ be a classical solution of problem (1)–(3). Multiplying scalarly both parts of system (1) by an arbitrary vector function $\varphi\in C_0^{2,4k}(\overline{D}_T,\partial D_T)$ and integrating by parts the obtained equality on the domain D_T, we have

$$\int\limits_{D_T}\Big[\frac{\partial^{2k}u}{\partial t^{2k}}\,\frac{\partial^{2k}\varphi}{\partial t^{2k}}+\sum_{i,j=1}^n A_{ij}\,\frac{\partial u}{\partial x_i}\,\frac{\partial\varphi}{\partial x_j}\Big]\,dx\,dt+\int\limits_{D_T}f(u)\varphi\,dx\,dt=\int\limits_{D_T}F\varphi\,dx\,dt \tag{7}$$
$$\forall\,\varphi\in C_0^{2,4k}(\overline{D}_T,\partial D_T).$$

We consider equality (7) as a basis for defining a weak generalized solution of problem (1)–(3).

Definition 1. Let the vector function f satisfy conditions (5), (6) and $F\in L_2(D_T)$. A vector function $u\in W_0^{1,2k}(D_T)$ is called a weak generalized solution of problem (1)–(3) if the integral equality (7) is valid for any vector function $\varphi\in W_0^{1,2k}(D_T)$, i.e.,

$$\int\limits_{D_T}\Big[\frac{\partial^{2k}u}{\partial t^{2k}}\cdot\frac{\partial^{2k}\varphi}{\partial t^{2k}}+\sum_{i,j=1}^n A_{ij}\,\frac{\partial u}{\partial x_i}\,\frac{\partial\varphi}{\partial x_j}\Big]\,dx\,dt+\int\limits_{D_T}f(u)\varphi\,dx\,dt=\int\limits_{D_T}F\varphi\,dx\,dt \tag{8}$$
$$\forall\,\varphi\in W_0^{1,2k}(D_T).$$

Note that due to Remark 2, the integral $\int\limits_{D_T}f(u)\varphi\,dx\,dt$ in equality (8) is defined correctly, since from $u\in W_0^{1,2k}(D_T)$ it follows $f(u)\in L_2(D_T)$ and, therefore, $f(u)\varphi\in L_1(D_T)$.

It is easy to verify that if the solution u of problem (1)–(3) belongs to the class $C_0^{2,4k}(D_T,\partial D_T)$ in the sense of Definition 1, then it will also be a classical solution of this problem.

Below we assume that the operator

$$\sum_{i,j=1}^n A_{ij}\,\frac{\partial^2}{\partial x_i\partial x_j} \tag{9}$$

is strictly elliptic, i.e., the matrix $Q(\xi)=\sum\limits_{i,j=1}^n A_{ij}\xi_i\xi_j$ is positively defined for each $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n\setminus\{(0,\dots,0)\}$:

$$(Q(\xi)\eta,\eta)_{\mathbb{R}^N}>0\quad\forall\,\eta\in\mathbb{R}^N\setminus\{(0,\dots,0)\}, \tag{10}$$

where $(\,\cdot\,,\,\cdot\,)_{\mathbb{R}^N}$ is a standard scalar product in the euclidian space $\mathbb{R}^N$. Note that in the scalar case the operator from (9) represents an elliptic operator and in this case the linear part of the operator L_f from (1), i.e. L_0 is semielliptic.

At fulfilment of condition (10) in the space $C_0^{2,4k}(\overline{D}_T, \partial D_T)$, together with the scalar product

$$(u,v)_o = \int\limits_{D_T} \bigg[uv + \sum_{i=1}^{2k} \frac{\partial^i u}{\partial t^i} \frac{\partial^i v}{\partial t^i} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \bigg] \, dx \, dt \tag{11}$$

with norm $||\cdot||_0 = ||\cdot||_{W_0^{1,2k}(D_T)}$, defined by the right-hand side of equality (4), let us introduce the following scalar product

$$(u,v)_1 = \int\limits_{D_T} \bigg[\frac{\partial^{2k} u}{\partial t^{2k}} \frac{\partial^{2k} v}{\partial t^{2k}} + \sum_{i,j=1}^{n} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \bigg] \, dx \, dt \tag{12}$$

with norm

$$\|u\|_1^2 = \int\limits_{D_T} \bigg[\Big| \frac{\partial^{2k} u}{\partial t^{2k}} \Big|^2 + \sum_{i,j=1}^{n} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \bigg] \, dx \, dt, \tag{13}$$

where $u, v \in C_0^{2,4k}(\overline{D}_T, \partial D_T)$.

It is proved the validity of the following inequalities

$$c_1 \|u\|_0 \le ||u||_1 \le c_2 \|u\|_0 \quad \forall\, u \in C_0^{2,4k}(\overline{D}_T, \partial D_T)$$

with positive constants c_1 and c_2, not dependent on u. Hence it follows that if we complete the space $C_0^{2,4k}(\overline{D}_T, \partial D_T)$ under norm (13), then in view of (11) we obtain the same Hilbert space $W_0^{1,2k}(D_T)$ with equivalent scalar products (11) and (12). Further, it can be proved the unique solvability of the linear problem correspondent to (1)–(3), i.e., when $f = 0$: for any $F \in L_2(D_T)$ there exists the unique solution $u = L_0^{-1} F \in W_0^{1,2k}(D_T)$ of this problem, where the linear operator

$$L_0^{-1} : L_2(D_T) \to W_0^{1,2k}(D_T)$$

is continuous. Thus, the nonlinear problem (1)–(3) is reduced to the following functional equation

$$u = L_0^{-1} \big[-f(u) + F \big] \tag{14}$$

in the Hilbert space $W_0^{1,2k}(D_T)$.

At fulfillment of the condition

$$\lim_{|u| \to \infty} \inf \frac{u f(u)}{|u|^2} \ge 0 \tag{15}$$

it can be proved the a priori estimate of the solution $u \in W_0^{1,2k}(D_T)$ of equation (14), whence due to Remark 2 we have the solvability of this equation, and, therefore, of problem (1)–(3) in the space $W_0^{1,2k}(D_T)$. Therefore the following theorem is valid.

Theorem 1. *Let conditions* (5), (6), (10) *and* (15) *be fulfilled. Then for any* $F \in L_2(D_T)$ *problem* (1)–(3) *has at least one generalized solution* u *in the space* $W_0^{1,2k}(D_T)$.

Remark 4. If conditions (5), (6) and (10) are fulfilled and the mapping $f(u) : \mathbb{R}^N \to \mathbb{R}^N$ satisfies the condition

$$(f(u) - f(v))(u - v) \ge 0 \quad \forall\, u, v \in \mathbb{R}^N, \tag{16}$$

then the solution of this problem is unique.

Thus, the following theorem is valid.

Theorem 2. *Let conditions* (5), (6), (10) *and* (15), (16) *be fulfilled. Then for any* $F \in L_2(D_T)$ *problem* (1)–(3) *has a unique weak generalized solution* u *in the space* $W_0^{1,2k}(D_T)$.

As the examples show, if the conditions imposed on the nonlinear vector function f are violated, then problem (1)–(3) may not have a solution. Indeed, consider the particular case of system (1), when it is split in the main part, i.e., $A_{ij} = a_{ij} I_N$, where I_N is a unit matrix of order N, and a_{ij} are numbers such that the operator $\sum\limits_{i,j=1}^{n} a_{ij}\partial^2/\partial x_i \partial x_j$ is a scalar elliptic operator.

Consider the following requirement imposed on the vector function f: there exist numbers $l_1, \dots, l_N$, $\sum\limits_{i=1}^{N} |l_i| \neq 0$ such that

$$\sum_{i=1}^{N} l_i f_i(u) \leq -d_0 \Big| \sum_{i=1}^{N} l_i u_i \Big|^{\beta} \quad \forall\, u \in \mathbb{R}^N, \;\; 1 < \beta = const < \frac{n+1}{n-1}\,, \tag{17}$$

where $d_0 = const > 0$. Let the domain Ω be given by the equation $\partial\Omega : \omega(x) = 0$, where $\nabla_x \omega|_{\partial\Omega} \neq 0$, $\omega|_{\Omega} > 0$, $\nabla_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and $\omega \in C^2(\mathbb{R}^n)$.

Theorem 3. *Let the vector function* f *satisfy conditions* (5), (6), (10) *and* (17). *Let* $F^0 = (F_1^0, \dots, F_N^0) \in L_2(D_T)$, $G = \sum\limits_{i=1}^{N} l_i F_i^0 \geq 0$ *and* $||G||_{L_2(D_T)} \neq 0$. *Then there exists a number* $\mu_0 = \mu_0(G, \beta) > 0$ *such that for* $\mu > \mu_0$ *problem* (1)–(3) *cannot have a weak generalized solution in the space* $W_0^{1,2k}(D_T)$ *for* $F = \mu F_0$.

Remark 5. Consider one class of vector functions f:

$$f_i(u_1, \dots, u_N) = \sum_{j=1}^{N} a_{ij} |u_j|^{\beta_{ij}} + b_i, \;\; i = 1, \dots, N,$$

where constants a_{ij}, β_{ij} and b_i satisfy the inequalities

$$a_{ij} > 0, \;\; 1 < \beta_{ij} < \frac{n+1}{n-1}\,, \;\; \sum_{k=1}^{N} b_k > 0, \;\; , i, j = 1, \dots, N.$$

It is easy to verify that this class satisfies condition (17).

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On the Set of Solutions of the Cauchy Problem for Higher Order Non-Lipshitzian Ordinary Differential Equations

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In the present report, the initial value problem

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}), \tag{1}$$

$$u^{(i-1)}(a) = 0 \;\; (i = 1, \dots, n) \tag{2}$$

is considered, where n is an arbitrary natural number, $-\infty < a < b < +\infty$, while $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function. We are interested in the case where the function f with respect to the phase variables does not satisfy the Lipshitz condition in the neighborhood of the point $(0, \dots, 0) \in \mathbb{R}^n$. In this case, as far as we know, the questions on the unique and multivalued solvability of problem $(1), (2)$ remain actually open. The structure of a set of solutions of that problem is insufficiently studied as well (see, e.g., [1–5] and the references therein). The results given below fill to some extent this gap. Those cover the case where the function f admits one of the following four representations:

$$f(t, x_1, \dots, x_n) = f_0(t, x_1, \dots, x_n) + \sum_{i=1}^{n} g_i(t)|x_i|^{\lambda_i}, \tag{3}$$

$$f(t, x_1, \dots, x_n) = f_0(t, x_1, \dots, x_n) + \sum_{i=1}^{n} g_i(t)\omega(|x_i|), \tag{4}$$

$$f(t, x_1, \dots, x_n) = f_0(t, x_1, \dots, x_n) + \sum_{i=1}^{n} g_i(t)|x_i|^{\lambda_i} + g(t), \tag{5}$$

$$f(t, x_1, \dots, x_n) = f_0(t, x_1, \dots, x_n) + \sum_{i=1}^{n} g_i(t)\omega(|x_i|) + g(t). \tag{6}$$

Here $\lambda_i \in]0, 1[\;\; (i = 1, \dots, n)$,

$$\omega(x) = \begin{cases} \dfrac{1}{\ln(1 + 1/x)} & \text{for } \; x > 0, \\[2mm] 0 & \text{for } \; x = 0, \end{cases}$$

while $f_0 : [a, b] \times \mathbb{R}^n \to \mathbb{R}_+$, $g_i : [a, b] \to \mathbb{R}_+$ $(i = 1, \dots, n)$, $g : [a, b] \to \mathbb{R}_+$ are continuous functions. It is also assumed that the function f_0 on the set $[a, b] \times \mathbb{R}^n$ satisfies one of the following two conditions:

$$f_0(t, 0, \dots, 0) = 0, \;\; f_0(t, x_1, \dots, x_n) \leq r\Big(1 + \sum_{i=1}^{n} |x_i|\Big), \tag{7}$$

$$f_0(t, 0, \dots, 0) = 0, \;\; \big|f_0(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)\big| \leq r \sum_{i=1}^{n} |x_i - y_i|, \tag{8}$$

where r is a positive constant.

We use the following notation.

$\mathbb{R}_+ = [0, +\infty[$;

$$\mathcal{D}^n(]a,b[\,;g) = \bigg\{(t,x_1,\dots,x_n) \in]a,b[\times\mathbb{R}^n: \ \ x_i \geq \frac{1}{(n-i)!}\int\limits_a^t (t-s)^{n-i} g(s)\,ds \ \ (i=1,\dots,n)\bigg\};$$

$S_f([a,b];t_0)$, where $t_0 \in [a,b[$, is the set of solutions of problem (1), (2) defined on the interval $[a,b]$ and satisfying the conditions

$$u^{(i-1)}(t) = 0 \ \text{ for } \ a \leq t \leq t_0, \quad u^{(i-1)}(t) > 0 \ \text{ for } \ t_0 < t \leq b \ \ (i=1,\dots,n);$$

$S_f([a,b])$ is the set of all nontrivial solutions of problem (1), (2) on the interval $[a,b]$.

Theorem 1. *Let*

$$f(t,0,\dots,0) = 0 \ \ \textit{for} \ \ a \leq t \leq b,$$

and let on the set $[a,b]\times\mathbb{R}^n$ *one of the following two conditions*

$$\sum_{i=1}^n g_i(t)|x_i|^{\lambda_i} \leq f(t,x_1,\dots,x_n) \leq r\Big(1+\sum_{i=1}^n |x_i|\Big),$$
$$\sum_{i=1}^n g_i(t)\omega(|x_i|) \leq f(t,x_1,\dots,x_n) \leq r\Big(1+\sum_{i=1}^n |x_i|\Big)$$

be satisfied, where $\lambda_i \in]0,1[$ $(i=1,\dots,n)$ *and* $r>0$ *are constants, and* $g_i : [a,b]\to\mathbb{R}_+$ $(i=1,\dots,n)$ *are continuous functions such that*

$$\sum_{i=1}^n g_i(t) > 0 \ \ \textit{for} \ \ a < t < b. \tag{9}$$

Then

$$S_f([a,b];t_0) \neq \varnothing \ \ \textit{for} \ \ a \leq t_0 < b, \quad S_f([a,b]) = \bigcup_{a\leq t_0<b} S_f([a,b];t_0). \tag{10}$$

Corollary 1. *If the function* f *admits representation* (3) *or* (4), *then for condition* (10) *to be satisfied it is sufficient that inequalities* (7) *and* (9) *hold.*

Theorem 2. *Let there exist continuous functions* $g : [a,b]\to\mathbb{R}_+$ *and* $h_i :]a,b[\to\mathbb{R}_+$ $(i=1,\dots,n)$ *such that the function* f *on the set* $[a,b]\times\mathbb{R}^n$ *admits the estimate*

$$f(t,x_1,\dots,x_n) \geq g(t),$$

while on the set $\mathcal{D}^n(]a,b[\,;g)$ *satisfies the Lipschitz condition*

$$\big|f(t,x_1,\dots,x_n) - f(t,y_1,\dots,y_n)\big| \leq \sum_{i=1}^n h_i(t)|x_i-y_i|.$$

If, moreover,

$$\int\limits_a^b (t-a)^{n-i} h_i(t)\,dt < +\infty \ \ (i=1,\dots,n),$$

then problem (1), (2) *has a unique solution.*

Corollary 2. *Let the function f admit representation* (5) *and let there exist a nonnegative constant α such that along with* (8) *the conditions*

$$\liminf_{t\to a} \frac{g(t)}{(t-a)^\alpha} > 0, \tag{11}$$

$$\int\limits_a^b (t-a)^{(n-i+1)\lambda_i-(1-\lambda_i)\alpha-1} g_i(t)\,dt < +\infty \ \ (i=1,\dots,n) \tag{12}$$

are satisfied. Then problem (1), (2) *is uniquely solvable and its solution satisfies the inequalities*

$$u^{(i-1)}(t) > 0 \ \ \text{for} \ \ a < t \le b \ \ (i=1,\dots,n). \tag{13}$$

Remark 1. In view of the continuity of the functions $g_i : [a,b] \to \mathbb{R}_+$ $(i=1,\dots,n)$, for condition (12) to be satisfied it is sufficient that the constant α satisfy the inequality

$$\alpha < \min\Big\{ \frac{(n-i+1)\lambda_i}{1-\lambda_i} \ : \ \ i=1,\dots,n \Big\}. \tag{14}$$

Corollary 3. *Let the function f admit representation* (6) *and let there exist a nonnegative constant α such that along with* (8) *and* (11)*, the conditions*

$$\int\limits_a^b (t-a)^{-\alpha} g_i(t)dt < +\infty \ \ \ (i=1,\dots,n) \tag{15}$$

are satisfied. Then problem (1), (2) *is uniquely solvable and its solution satisfies inequalities* (13).

As an example, consider the differential equations

$$u^{(n)} = \sum_{i=1}^n g_i(t)|u^{(i-1)}|^{\lambda_i}, \tag{16}$$

$$u^{(n)} = \sum_{i=1}^n g_i(t)|u^{(i-1)}|^{\lambda_i} + g(t), \tag{17}$$

$$u^{(n)} = \sum_{i=1}^n g_i(t)\omega(|u^{(i-1)}|), \tag{18}$$

$$u^{(n)} = \sum_{i=1}^n g_i(t)\omega(|u^{(i-1)}|) + g(t), \tag{19}$$

where $\lambda_i \in]0,1[$ $(i=1,\dots,n)$, while $g_i : [a,b] \to \mathbb{R}_+$ $(i=1,\dots,n)$, $g : [a,b] \to \mathbb{R}_+$ are continuous functions.

From Corollaries 1 and 2 it follows

Corollary 4. *Let conditions* (9) *and* (11) *hold, where α is a nonnegative constant satisfying inequality* (14)*. Then problem* (16), (2) *has a continuum of solutions, while problem* (17), (2) *has a unique solution.*

From Corollaries 1 and 3 follows

Corollary 5. *Let conditions* (9), (11) *and* (15) *hold, where α is a nonnegative constant. Then problem* (18), (2) *has a continuum of solutions, while problem* (19), (2) *is uniquelly solvable.*

Therefore, a multivalued solvable initial value problem can be made uniquely solvable by using an arbitrarily small perturbation of the equation under consideration.

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On a Dirichlet Type Boundary Value Problem in an Orthogonally Convex Piecewise Smooth Cylinder for a Class of Quasilinear Partial Differential Equations

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In the orthogonally convex cylinder $E=\{(x_1,x_2,x_3)\in\Omega:\ (x_1,x_2)\in D,\ x_3\in(0,\omega_3)\}$, where

$$\begin{aligned}D&=\Big\{(x_1,x_2)\in\Omega:\ \ x_1\in(0,\omega_1),\ \ x_2\in(\gamma_1(x_1),\gamma_2(x_1))\Big\}\\&=\Big\{(x_1,x_2)\in\Omega:\ \ x_2\in(0,\omega_2),\ \ x_1\in(\eta_1(x_2),\eta_2(x_2))\Big\},\end{aligned}$$

consider the boundary value problem

$$u^{(\mathbf{2})}=f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u]\big), \tag{1}$$

$$u\,\nu_1\Big|_{\partial E}=\nu_1(\mathbf{x})\psi_1(\mathbf{x}),\quad u^{(2,0,0)}\,\nu_2\Big|_{\partial E}=\nu_2(\mathbf{x})\psi_2(\mathbf{x}),\quad u^{(2,2,0)}\,\nu_3\Big|_{\partial E}=\nu_3(\mathbf{x})\psi_3(\mathbf{x}). \tag{2}$$

Here $\mathbf{x}=(x_1,x_2,x_3)$, ∂E is the boundary of E, and $\boldsymbol{\nu}(\mathbf{x})=(\nu_1(\mathbf{x}),\nu_2(\mathbf{x}),\nu_3(\mathbf{x}))$ is the outward unit normal vector at point $\mathbf{x}\in\partial E$, $\mathbf{2}=(2,2,2)$, $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index,

$$\mathcal{D}^{\mathbf{2}}[u]=(u^{(\boldsymbol{\alpha})})_{\boldsymbol{\alpha}\le\mathbf{2}},\quad \widetilde{\mathcal{D}}^{\mathbf{2}}[u]=(u^{(\boldsymbol{\alpha})})_{\boldsymbol{\alpha}<\mathbf{2}},\quad u^{(\boldsymbol{\alpha})}(\mathbf{x})=\frac{\partial^{\alpha_1+\alpha_2+\alpha_3}u(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\,,$$

$f(\mathbf{x},\mathbf{z})$ is a continuous function on $\overline{\mathbf{E}}\times\mathbb{R}^{23}$, $\mathbf{z}=(z_{000},z_{100},z_{010},z_{001},\dots,z_{221},z_{212},z_{122})$, $\psi_i\in C(\overline{E})$ $(i=1,2,3)$ and $\overline{E}$ is the closure of E.

By a solution of problem (1),(2) we understand a *classical* solution, i.e., a function $u\in C^{2,2,2}(E)$ having continuous on $\overline{E}$ partial derivatives $u^{(2,0,0)}$ and $u^{(2,2,0)}$, and satisfying equation (1) and the boundary conditions (2) everywhere in E and ∂E, respectively.

Throughout the paper the following notations will be used:

$\mathbf{0}=(0,0,0)$, $\mathbf{1}=(1,1,1)$, $\boldsymbol{\alpha}_i=(0,\dots,\alpha_i,\dots,0)$, $\boldsymbol{\alpha}_{ij}=\boldsymbol{\alpha}_i+\boldsymbol{\alpha}_j$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)<\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\alpha_i\le\beta_i$ $(i=1,2,3)$ and $\boldsymbol{\alpha}\ne\boldsymbol{\beta}$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)\le\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.

$\|\boldsymbol{\alpha}\|=|\alpha_1|+|\alpha_2|+|\alpha_3|$.

$\boldsymbol{\Xi}=\big\{\boldsymbol{\sigma}\big|\ \mathbf{0}<\boldsymbol{\sigma}<\mathbf{1}\big\}$.

$\boldsymbol{\Upsilon}_{\mathbf{2}}=\big\{\boldsymbol{\alpha}<\mathbf{2}:\ \ \alpha_i=2\ \text{ for some }\ i\in\{1,2,3\}\big\}$.

The variables $z_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha}\in\boldsymbol{\Upsilon}_{\mathbf{2}})$ are called ***principal phase variables*** of the function $f(\mathbf{x},\mathbf{z})$.

$\mathbf{z}=\big(z_{\boldsymbol{\alpha}}\big)_{\boldsymbol{\alpha}<\mathbf{2}}$;$f_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{z})=\frac{\partial f(\mathbf{x},\mathbf{z})}{\partial z_{\boldsymbol{\alpha}}}$.

$\operatorname{supp}\boldsymbol{\alpha}=\{i:\ \alpha_i>0\}$.

$\mathbf{x}_{\boldsymbol{\alpha}}=(\chi(\alpha_1)\,x_1,\chi(\alpha_2)\,x_2,\chi(\alpha_3)x_3)$, where $\chi(\alpha)=0$ if $\alpha=0$, and $\chi(\alpha)=1$ if $\alpha>0$.

$\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}=\mathbf{x}-\mathbf{x}_{\boldsymbol{\alpha}}$.

$\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(x_{i_1},\dots,x_{i_l})$, where $\{i_1,\dots,i_l\}=\operatorname{supp}\boldsymbol{\alpha}$. Furthermore, $\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(\mathbf{x}_{\boldsymbol{\alpha}},\widehat{\mathbf{0}}_{\boldsymbol{\alpha}})$, and $\mathbf{x}$ will be identified with $(\mathbf{x}_{\boldsymbol{\alpha}},\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})$, or with $(\mathbf{x}_{\boldsymbol{\alpha}},\mathbf{x}_{\widehat{\boldsymbol{\alpha}}})$.

$\Omega_{\boldsymbol{\sigma}}=[0,\omega_{i_1}]\times\cdots\times[0,\omega_{i_l}]$, where $\{i_1,\dots,i_l\}=\operatorname{supp}\boldsymbol{\sigma}$.

$\Omega_{ij}=(0,\omega_i)\times(0,\omega_j)$ $(1\le i<j\le 3)$.

$C^{\mathbf{m}}(\overline{E})$ is the Banach space of functions $u:\overline{E}\to\mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}\le\mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\overline{E})}=\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\|u^{(\boldsymbol{\alpha})}\|_{C(\overline{E})}.$$

$\widetilde{C}^{\mathbf{m}}(\overline{E})$ is the Banach space of functions $u:\overline{E}\to\mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}<\mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\overline{E})}=\sum_{\boldsymbol{\alpha}<\mathbf{m}}\|u^{(\boldsymbol{\alpha})}\|_{C(\overline{E})}.$$

If $u_0\in C^{\mathbf{m}}(\overline{E})$ and $r>0$, then $\mathbf{B}^{\mathbf{m}}(u_0;r)=\big\{u\in C^{\mathbf{m}}(\overline{E}):\ \|u-u_0\|_{C^{\mathbf{m}}}\le r\big\}$.

If $u_0\in \widetilde{C}^{\mathbf{m}}(\overline{E})$ and $r>0$, then $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_0;r)=\big\{u\in \widetilde{C}^{\mathbf{m}}(\overline{E}):\ \|u-u_0\|_{\widetilde{C}^{\mathbf{m}}}\le r\big\}$.

The boundary conditions (2) can be written int the following way

$$\begin{gathered}u(\eta_k(x_2),x_2,x_3)=\varphi_{1k}(x_2,x_3),\quad u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=\varphi_{2k}(x_1,x_3),\\ u^{(2,2,0)}(x_1,x_2,(k-1)\,\omega_3)=\varphi_{3k}(x_1,x_2)\ \ (k=1,2),\end{gathered}\tag{3}$$

where

$$\begin{gathered}\varphi_{1k}(x_2,x_3)=\psi_1(\eta_k(x_2),x_2,x_3),\quad \varphi_{2k}(x_1,x_3)=\psi_2(x_1,\gamma_k(x_1),x_3),\\ \varphi_{3k}(x_1,x_2)=\psi_3(x_1,x_2,(k-1)\,\omega_3)\ \ (k=1,2).\end{gathered}\tag{4}$$

Along with problem (1), (3) consider the linear homogeneous problem

$$u^{(\mathbf{2})}=\sum_{\alpha<\mathbf{2}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})},\tag{1_0}$$

$$u(\eta_k(x_2),x_2,x_3)=0,\ \ u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=0,\ \ u^{(2,2,0)}(x_1,x_2,(k-1)\,\omega_3)=0\ \ (k=1,2).\tag{3_0}$$

For each $\boldsymbol{\sigma}\in\boldsymbol{\Xi}$ in the domain $\Omega_{\boldsymbol{\sigma}}$ consider the homogeneous boundary value problem depending on the parameter $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}}\in\Omega_{\widehat{\boldsymbol{\sigma}}}$:

$$v^{(2,0,0)}=p_{022}(\mathbf{x}_1,\widehat{\mathbf{x}}_1)v+p_{122}(\mathbf{x}_1,\widehat{\mathbf{x}}_1)v^{(1,0,0)},\tag{1_{100}}$$

$$v(\eta_1(\mathbf{x}_2),\widehat{\mathbf{x}}_1)=0,\quad v(\eta_2(\mathbf{x}_2),\widehat{\mathbf{x}}_1)=0;\tag{2_{100}}$$

$$v^{(0,2,0)}=p_{202}(\mathbf{x}_2,\widehat{\mathbf{x}}_2)v+p_{212}(\mathbf{x}_2,\widehat{\mathbf{x}}_2)v^{(0,1,0)},\tag{1_{010}}$$

$$v(\gamma_1(\mathbf{x}_1),\widehat{\mathbf{x}}_2)=0,\quad v(\gamma_2(\mathbf{x}_1),\widehat{\mathbf{x}}_2)=0;\tag{3_{010}}$$

$$v^{(0,0,2)}=p_{220}(\mathbf{x}_3,\widehat{\mathbf{x}}_3)v+p_{221}(\mathbf{x}_3,\widehat{\mathbf{x}}_3)v^{(0,0,1)},\tag{1_{001}}$$

$$v(0,\widehat{\mathbf{x}}_3)=0,\quad v(\omega_3,\widehat{\mathbf{x}}_3)=0;\tag{3_{001}}$$

$$v^{(\mathbf{2}_{12})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{12}}p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{12}}(\mathbf{x}_{12},\widehat{\mathbf{x}}_{12})v^{(\boldsymbol{\alpha})},\tag{1_{110}}$$

$$v(\eta_k(\mathbf{x}_2),\widehat{\mathbf{x}}_{12})=0,\quad v^{(2,0,0)}(\gamma_k(\mathbf{x}_1),\widehat{\mathbf{x}}_{12})=0\ \ (k=1,2);\tag{3_{110}}$$

$$v^{(\mathbf{2}_{13})} = \sum_{\boldsymbol{\alpha}<\mathbf{2}_{13}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{13}}(\mathbf{x}_{13},\widehat{\mathbf{x}}_{13})v^{(\boldsymbol{\alpha})}, \tag{1_{101}}$$

$$v(\eta_k(\mathbf{x}_2),\widehat{\mathbf{x}}_{13}) = 0, \quad v^{(2,0,0)}((k-1)\,\omega_3,\widehat{\mathbf{x}}_{13}) = 0 \ \ (k=1,2); \tag{3_{101}}$$

$$v^{(\mathbf{2}_{23})} = \sum_{\boldsymbol{\alpha}<\mathbf{2}_{23}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{23}}(\mathbf{x}_{23},\widehat{\mathbf{x}}_{23})v^{(\boldsymbol{\alpha})}, \tag{1_{011}}$$

$$v(\gamma_k(\mathbf{x}_1),\widehat{\mathbf{x}}_{23}) = 0, \quad v^{(2,0,0)}((k-1)\,\omega_3,\widehat{\mathbf{x}}_{23}) = 0 \ \ (k=1,2). \tag{3_{011}}$$

Definition 1. Problem $(1_{\boldsymbol{\sigma}}),(3_{\boldsymbol{\sigma}})$ $(\boldsymbol{\sigma}\in\boldsymbol{\Xi})$ is called $\boldsymbol{\sigma}$-associated problem of problem $(1_0),(3_0)$.

Along with problem $(1),(2)$ consider the perturbed problem

$$u^{(\mathbf{2})} = f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u]\big) + \widetilde{f}\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u]\big), \tag{5}$$

$$\begin{aligned} &u(\eta_k(x_2),x_2,x_3) = \varphi_{1k}(x_2,x_3) + \widetilde{\varphi}_{1k}(x_2,x_3),\\ &\qquad u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3) = \varphi_{2k}(x_1,x_3) + \widetilde{\varphi}_{2k}(x_1,x_3),\\ &\qquad\qquad u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3) = \varphi_{3k}(x_1,x_2) + \widetilde{\varphi}_{3k}(x_1,x_2) \ \ (k=1,2), \end{aligned} \tag{6}$$

where

$$\begin{aligned} &\widetilde{\varphi}_{1k}(x_2,x_3) = \widetilde{\psi}_1(\eta_k(x_2),x_2,x_3), \quad \widetilde{\varphi}_{2k}(x_1,x_3) = \widetilde{\psi}_2(x_1,\gamma_k(x_1),x_3),\\ &\qquad\qquad \widetilde{\varphi}_{3k}(x_1,x_2) = \widetilde{\psi}_3(x_1,x_2,(k-1)\,\omega_3) \ \ (k=1,2). \end{aligned} \tag{7}$$

A vector function $(\widetilde{f};\widetilde{\psi}_1,\widetilde{\psi}_2,\widetilde{\psi}_3)$ is said to be an *admissible perturbation* if $\widetilde{f}\in C(\Omega\times\mathbb{R}^{23})$ is locally Lipschitz continuous with respect to the *principal* phase variables, $\widetilde{\psi}_1\in C^{2,2,2}(\overline{E})$, $\widetilde{\psi}_2\in C^{0,2,2}(\overline{E})$ and $\widetilde{\psi}_2\in C^{0,0,2}(\overline{E})$.

Definition 2. Let u_0 be a solution of problem $(1),(2)$, and $r>0$. We say that problem $(1),(2)$ is (u_0,r)-*well-posed*, if:

(I) $u_0(\mathbf{x})$ is the unique solution of problem $(1),(2)$ in the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;r)$;

(II) there exist positive constant δ_0 and an increasing continuous $\varepsilon:[0,\delta_0]\to[0,+\infty)$ such that $\varepsilon(0)=0$ and for any $\delta\in(0,\delta_0]$ and an arbitrary admissible perturbation $(\widetilde{f};\widetilde{\psi}_1,\widetilde{\psi}_2,\widetilde{\psi}_3)$ satisfying the conditions

$$|\widetilde{f}_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{z})| \le \delta_0 \ \text{ for } \ (\mathbf{x},\mathbf{z})\in\Omega\times\mathbb{R}^{23} \ \ (\boldsymbol{\alpha}\in\boldsymbol{\Upsilon}_{\mathbf{m}}), \tag{8}$$

$$|\widetilde{f}(\mathbf{x},\mathbf{z})| \le \delta \ \text{ for } \ (\mathbf{x},\mathbf{z})\in\Omega\times\mathbb{R}^{23}, \tag{9}$$

$$\|\widetilde{\psi}_1\|_{C^{2,2,2}(\overline{E})} + \|\widetilde{\psi}_2\|_{C^{0,2,2}(\overline{E})} + \|\widetilde{\psi}_3\|_{C^{0,0,2}(\overline{E})} \le \delta, \tag{10}$$

problem $(4),(5)$ has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;\varepsilon(\delta))$.

Definition 3. Let u_0 be a solution of problem $(1),(2)$, and $r>0$. We say that problem $(1),(2)$ is *strongly* (u_0,r)-*well-posed*, if:

(I) $u_0(\mathbf{x})$ is the unique solution of problem $(1),(2)$ in the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0,r)$;

(II) there exist a positive constants δ_0 and M such that for any $\delta\in(0,\delta_0]$ and an arbitrary admissible perturbation $(\widetilde{f};\widetilde{\psi}_1,\widetilde{\psi}_2,\widetilde{\psi}_3)$ satisfying conditions (7)–(9), problem $(4),(5)$ has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;M\delta)$.

Definition 4. Problem $(1),(2)$ is called well-posed (strongly well-posed), if it is (u_0,r)-well-posed (strongly (u_0,r)-well-posed) for every $r>0$.

Definition 5. A solution u_0 of problem $(1),(2)$ is called *strongly isolated*, if problem $(1),(2)$ is strongly (u_0,r)-well-posed for some $r>0$.

Theorem 1. *Let*

$$\eta_k \in C^2([0,\omega_2]) \;\; (k=1,2), \tag{11}$$

let the function $f(\mathbf{x},\mathbf{z})$ be continuously differentiable with respect to the phase variables, and let there exist functions $P_{i\boldsymbol{\alpha}}(\mathbf{x})\in C(\overline{E})$ $(\boldsymbol{\alpha}<\mathbf{2};\ i=1,2)$ such that:

(E_1)

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le f_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{z}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \;\; \text{for} \;\; (\mathbf{x},\mathbf{Z})\in \overline{E}\times\mathbb{R}^{23} \quad (\boldsymbol{\alpha}<\mathbf{2}); \tag{12}$$

(E_2) *For every $\boldsymbol{\sigma}\in\Xi\cup\{\mathbf{1}\}$,*[1] *$\widehat{\mathbf{x}}_{\boldsymbol{\sigma}}\in\overline{E}_{\widehat{\boldsymbol{\sigma}}}$ and arbitrary measurable functions $p_{\boldsymbol{\alpha}}\in L^\infty(E_{\boldsymbol{\sigma}})$ $(\boldsymbol{\alpha}<\mathbf{2}_{\boldsymbol{\sigma}})$ satisfying the inequalities*

$$P_{1\,\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{\boldsymbol{\sigma}}}(\mathbf{y},\widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) \le p_{\boldsymbol{\alpha}}(\mathbf{y}) \le P_{2\,\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{\boldsymbol{\sigma}}}(\mathbf{y},\widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) \;\; \text{for} \;\; \mathbf{y}\in E_{\boldsymbol{\sigma}} \;\; (\boldsymbol{\alpha}<\mathbf{2}_{\boldsymbol{\sigma}}), \tag{13}$$

the $\boldsymbol{\sigma}$-associated problem $(1_{\boldsymbol{\sigma}})$, $(3_{\boldsymbol{\sigma}})$ has only the trivial solution in $AC^{\mathbf{1}}(\overline{E}_{\boldsymbol{\sigma}})$;

(E_3) *the problem*

$$u^{(\mathbf{2})} = \sum_{\boldsymbol{\alpha}<\mathbf{2}} P_{1\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})},$$

$$u(\eta_k(x_2),x_2,x_3)=0, \;\; u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=0, \;\; u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=0 \;\; (k=1,2),$$

is well-posed. Then problem $(1),(3)$ is strongly well-posed, and its solution belongs to $C^{2,2,2}(\overline{E})$.

Theorem 2. *Let condition (11) hold, the function $f(\mathbf{x},\mathbf{z})$ be continuously differentiable with respect to the phase variables, and let u_0 be a solution of problem $(1),(3)$. Then problem $(1),(3)$ is strongly (u_0,r)-well-posed for some $r>0$ if and only if the linear homogeneous problem $(1_0),(3_0)$ is well-posed, where*

$$p_{\boldsymbol{\alpha}}(\mathbf{x}) = f_{\boldsymbol{\alpha}}(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u_0(\mathbf{x})]) \;\; (\boldsymbol{\alpha}<\mathbf{2}).$$

Consider the equations

$$u^{(\mathbf{2})} = f(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u]) + q(\mathbf{x},\mathcal{D}^{\mathbf{1}}[u]), \tag{14}$$

$$u^{(\mathbf{2})} = \big(p_1(\mathbf{x})u^{(1,0,0)}\big)^{(1,0,0)} + \big(p_2(\mathbf{x})u^{(0,1,0)}\big)^{(0,1,0)} + \big(p_3(\mathbf{x})u^{(0,0,1)}\big)^{(0,0,1)} + p_0(\mathbf{x},u), \tag{15}$$

$$u^{(\mathbf{2})} = \sum_{\boldsymbol{\alpha}<\mathbf{2}} \rho_{\boldsymbol{\alpha}}\big(\mathbf{x},\mathcal{D}^{\mathbf{1}}[u]\big)\,u^{(\boldsymbol{\alpha})} + q\big(\mathbf{x},\mathcal{D}^{\mathbf{1}}[u]\big), \tag{16}$$

$$\begin{aligned} u^{(\mathbf{2})} = {} & \big(p_1(\mathbf{x},u)u^{(1,0,0)}\big)^{(1,0,0)} + \big(p_2(\mathbf{x},u)u^{(0,1,0)}\big)^{(0,1,0)} \\ & + \big(p_3(\mathbf{x},u)u^{(0,0,1)}\big)^{(0,0,1)} + p_0(\mathbf{x},u) + q\big(\mathbf{x},\mathcal{D}^{\mathbf{1}}[u]\big). \end{aligned} \tag{17}$$

Theorem 3. *Let the function f satisfy all of the conditions of Theorem 1, and let $q\in C(\Omega\times\mathbb{R}^8)$ be such that*

$$\lim_{\|\mathbf{z}\|\to+\infty} \frac{|q(\mathbf{x},\mathbf{z})|}{\|\mathbf{z}\|} = 0 \;\; \text{uniformly on } \overline{E}. \tag{18}$$

Then problem $(14),(3)$ is solvable and its every solution belongs to $C^{2,2,2}(\overline{E})$.

[1] If $\boldsymbol{\sigma}=\mathbf{1}$, then by $(1_{\boldsymbol{\sigma}}),(3_{\boldsymbol{\sigma}})$ we understand the homogeneous problem (1_0), (3_0).

Corollary. *Let condition* (11) *hold,*

$$(-1)^{k-1}\eta_k''(x_2) \geq 0 \ \ for \ \ x_2 \in (0,\omega_2) \ \ (k=1,2), \tag{19}$$

and let $p_1 \in C^{1,0,0}(\overline{E})$, $p_2 \in C^{0,1,0}(\overline{E})$, $p_3 \in C^{0,0,1}(\overline{E})$, $p_0 \in C(\overline{E} \times \mathbb{R})$ *satisfy the inequalities*

$$p_1(\mathbf{x}) \leq 0, \quad p_2(\mathbf{x}) \leq 0, \quad p_3(\mathbf{x}) \leq 0 \ \ for \ \ \mathbf{x} \in \overline{E}, \tag{20}$$

$$\big(p_0(\mathbf{x}, z_1) - p_0(\mathbf{x}, z_2)\big)(z_1 - z_2) \geq 0 \ \ for \ \ (\mathbf{x}_1, x_2, z) \in \overline{E} \times \mathbb{R}. \tag{21}$$

Then problem (15), (3) *is strongly well-posed and its solution belongs to* $C^{2,2,2}(\overline{E})$.

Theorem 4. *Let conditions* (11) *and* (18) *hold, and let there exist functions* $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C(\overline{E})$ ($\boldsymbol{\alpha} < \mathbf{2}$; $i = 1, 2$) *satisfying conditions* (E_2) *and* (E_2) *of Theorem 1 such that:*

$$P_{1\alpha}(\mathbf{x}) \leq \rho_\alpha(\mathbf{x}, \mathbf{z}) \leq P_{2\alpha}(\mathbf{x}) \ \ for \ \ (\mathbf{x}, \mathbf{z}) \in \overline{E} \times \mathbb{R}^8 \ \ (\boldsymbol{\alpha} < \mathbf{2}). \tag{22}$$

Then problem (16), (3) *is solvable and its every solution belongs to* $C^{2,2,2}(\overline{E})$.

Theorem 5. *Let conditions* (11), (18) *and* (19) *hold, and let* $p_k \in C^1(\overline{E} \times \mathbb{R})$ $(k = 1, 2, 3)$ *satisfy the inequalities*

$$p_k(\mathbf{x}, z) \leq 0 \ \ for \ \ (\mathbf{x}, z) \in \overline{E} \times \mathbb{R} \ \ (k = 1, 2, 3),$$
$$p_0(\mathbf{x}, z)z \geq 0 \ \ for \ \ (\mathbf{x}, z) \in \overline{\Omega} \times \mathbb{R}.$$

Then problem (17), (3_0) *is solvable and its every solution belongs to* $C^{2,2,2}(\overline{E})$.

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On a Periodic Type Boundary Value Problem for a Second Order Linear Hyperbolic System

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In the rectangle $\Omega=[0,\omega_1]\times[0,\omega_2]$ consider the problem

$$u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y+q(x,y), \tag{1}$$

$$u(0,y)=Au(\omega_1,y)+\varphi(y), \quad u(x,0)=Bu(x,\omega_2)+\psi(x), \tag{2}$$

where $P_j\in C(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, $q\in C(\Omega;\mathbb{R}^n)$, $A,B\in\mathbb{R}^{n\times n}$, $\varphi\in C^1([0,\omega_2];\mathbb{R}^n)$ and $\psi\in C^1([0,\omega_1];\mathbb{R}^n)$.

Problem (1), (2) is not well-posed, since for its solvability the vector functions φ and ψ should satisfy some compatibility condition. For example, if

$$AB=BA, \tag{3}$$

then for solvability of problem (1), (2) it is necessary that

$$\varphi(0)-B\varphi(\omega_2)=\psi(0)-A\psi(\omega_1). \tag{4}$$

Indeed, for an arbitrary $u\in C(\Omega;\mathbb{R}^n)$, in view of equality (3), we have

$$h\circ\ell(u)=\ell\circ h(u), \tag{5}$$

where

$$\ell(z)=z(0)-Az(\omega_1), \quad h(z)=z(0)-Bz(\omega_2).$$

Consequently, if $u(x,y)$ satisfies condition (2), then equality (5) implies

$$\psi(0)-A\psi(\omega_1)=\ell\circ h(u)=h\circ\ell(u)=\varphi(0)-B\varphi(\omega_2).$$

Notice that, if $u\in C^{1,1}(\Omega;\mathbb{R}^n)$ satisfies condition (2), then

$$h(u_x(x,\,\cdot\,))=\psi'(x).$$

Therefore,

$$u(0,y)=Au(\omega_1,y)+\varphi(y), \quad u_x(x,0)=Bu_x(x,\omega_2)+\psi'(x). \tag{6}$$

Along with system (1) and conditions (2) and (6) consider their corresponding homogeneous system and conditions

$$u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y, \tag{1_0}$$

$$u(0,y)=Au(\omega_1,y), \quad u(x,0)=Bu(x,\omega_2) \tag{2_0}$$

and

$$u(0,y)=Au(\omega_1,y), \quad u_x(x,0)=Bu_x(x,\omega_2). \tag{6_0}$$

Let $Y(y;x)$ be the fundamental matrix of the differential system

$$\frac{dz}{dy} = P_1(x,y)z,$$

satisfying the initial condition

$$Y(0;x) = I,$$

where I is $n \times n$ identity matrix. By $X(x;y)$ denote the fundamental matrix of the differential system

$$\frac{dz}{dx} = P_2(x,y)z,$$

satisfying the initial condition

$$X(0;y) = I.$$

If problem

$$\frac{dz}{dx} = P_2(x,y)z, \quad z(0) - Az(\omega_1) = 0,$$

has only the trivial solution, then by $G_1(x,s;y)$ denote its Green's matrix, and if problem

$$\frac{dz}{dy} = P_1(x,y)z, \quad z(0) - Bz(\omega_2) = 0$$

has only the trivial solution, then by $G_2(y,t;x)$ denote its Green's matrix.

Theorem 1. *Let the problem*

$$z' = 0, \quad z(0) = A\,z(\omega_1) \tag{7}$$

have only the trivial solution, and let the following inequalities hold:

$$\det\big(I - Y(\omega_2;x)\,B\big) \neq 0 \ \ for \ \ x \in [0,\omega_1], \tag{8}$$

$$\det\big(I - X(\omega_1;y)\,A\big) \neq 0 \ \ for \ \ y \in [0,\omega_2]. \tag{9}$$

Then problem $(1),(6)$ *has the Fredholm property. Furthermore, if problem* $(1_0),(6_0)$ *has only the trivial solution, then problem* $(1),(6)$ *has a unique solution* u u *admitting the estimate*

$$\|u\|_{C^{1,1}(\Omega)} \leq M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C^1([0,\omega_1])}\Big), \tag{10}$$

where M *is a positive number independent of* φ, ψ *and* q.

Definition. Problem $(1),(6)$ is called well-posed if for every $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C^1([0,\omega_1];\mathbb{R}^n)$ and $q \in C(\Omega;\mathbb{R}^n)$ it has a unique solution u admitting estimate (10), where M is a positive number independent of φ, ψ and q.

Theorem 2. *If problem* $(1),(6)$ *is well-posed, then problem* $(7),(8)$ *has only the trivial solution and inequalities* (9) *and* (10) *hold.*

Theorem 3. *Let inequalities* (9) *and* (10) *hold, and let the matrices* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times n}$ *satisfy condition* (3). *Then:*

(i) *the space of solutions of problem* $(1_0),(2_0)$ *is finite dimensional;*

(ii) *if the homogeneous problem* $(1_0),(2_0)$ *has only the trivial solution, then problem* $(1),(2)$ *is uniquely solvable if and only if the compatibility condition* (4) *holds.*

Corollary 1. *Let* $P_1(x,y) \equiv P_1(x)$, $P_2(x,y) \equiv P_2(y)$, *let the problem* (7) *have only the trivial solution, and let*

$$\det\big(I - \exp(\omega_2 P_1(x))\, B\big) \neq 0 \ \ for \ \ x \in [0,\omega_1], \tag{11}$$

$$\det\big(I - \exp(\omega_1 P_2(y))\, A\big) \neq 0 \ \ for \ \ y \in [0,\omega_2]. \tag{12}$$

Then problem (1), (6) *has the Fredholm property.*

Corollary 2. *Let problem* (7) *have only the trivial solution, and let there exist* $\sigma_i \in \{-1,1\}$ $(i=1,2)$ *such that*

$$\begin{aligned} \sigma_1(A^T A - I) & \ \ \textit{is positive semi-definite},\\ \sigma_1 P_1(x,y) & \ \ \textit{is positive definite for} \ \ (x,y)\in\Omega \end{aligned}$$

and

$$\begin{aligned} \sigma_2\big(B^T B - I\big) & \ \ \textit{is positive semi-definite},\\ \sigma_2 P_2(x,y) & \ \ \textit{is positive definite for} \ \ (x,y)\in\Omega. \end{aligned}$$

Then problem (1) (6) *has the Fredholm property.*

Theorem 4. *Let conditions* (8) *and* (9) *hold, let problem* (7) *have only the trivial solution, let* $\Gamma \in \mathbb{R}_+^{n\times n}$ *be a nonnegative matrix with the* ***spectral radius less than 1***, *and let either*

$$P_1 \in C^{1,0}(\Omega;\mathbb{R}^{n\times n}), \ \ P_1(0,y) = P_1(\omega_1,y), \ \ P_1(\omega_1,y)\, A = A\, P_1(\omega_1,y), \tag{13}$$

and

$$\int_0^{\omega_2}\int_0^{\omega_1} \Big| G_2(y,t;x) G_1(x,s;t) \Big(P_0(s,t) + P_2(s,t) P_1(s,t) - \frac{\partial}{\partial s} P_1(s,t)\Big)\Big|\, ds\, dt \leq \Gamma, \tag{14}$$

or

$$P_2 \in C^{0,1}(\Omega;\mathbb{R}^{n\times n}), \ \ P_2(x,0) = P_2(x,\omega_2), \ \ P_2(x,\omega_2)\, B = B\, P_2(x,\omega_2), \tag{15}$$

and

$$\int_0^{\omega_1}\int_0^{\omega_2} \Big| G_1(x,s;y) G_2(y,t;s) \Big(P_0(s,t) + P_1(s,t) P_2(s,t) - \frac{\partial}{\partial t} P_2(s,t)\Big)\Big|\, dt\, ds \leq \Gamma. \tag{16}$$

Then problem (1) (6) *is uniquely solvable.*

Consider the system

$$u_{xy} = P_0(x,y)u + u_x + u_y + q(x,y). \tag{17}$$

Theorem 5. *Let problem* (7) *have only the trivial solution,*

$$\begin{gathered} P_0(x,y) = P_0^T(x,y) \ \ \textit{for} \ \ x,y)\in\Omega,\\ A^T A - I \ \ \textit{be positive semi-definite},\\ B^T B - I \ \ \textit{be positive semi-definite},\\ I - A^T A - B^T B + B^T A^T A B \ \ \textit{be positive semi-definite}, \end{gathered}$$

and let ***one of the following three conditions*** *hold:*

(i) $P_0 \in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$ *and*

$$\begin{aligned} P_0(\omega_1,y)-A^T P_0(0,y)A &\ \ \textit{is positive semi-definite for } y\in[0,\omega_2],\\ P_0(x,y)+\frac{1}{2}\,\frac{\partial P_0(x,y)}{\partial x} &\ \ \textit{is negative semi-definite for } (x,y)\in\Omega,\\ \int\limits_0^{\omega_1} P_0(s,y)\,ds &\ \ \textit{is negative definite for } y\in[0,\omega_2]; \end{aligned}$$

(ii) $P_0 \in C^{0,1}(\Omega;\mathbb{R}^{n\times n})$ *and*

$$\begin{aligned} P_0(x,\omega_2)-B^T P_0(x,\omega_2)B &\ \ \textit{is positive semi-definite for } x\in[0,\omega_1],\\ P_0(x,y)+\frac{1}{2}\,\frac{\partial P_0(x,y)}{\partial y} &\ \ \textit{is negative semi-definite for } (x,y)\in\Omega,\\ \int\limits_0^{\omega_2} P_0(x,t)\,dt &\ \ \textit{is negative definite for } x\in[0,\omega_1]; \end{aligned}$$

(iii) $P_0 \in C^{1}(\Omega;\mathbb{R}^{n\times n})$ *and*

$$\begin{aligned} P_0(\omega_1,y)-A^T P_0(0,y)A &\ \ \textit{is positive semi-definite for } y\in[0,\omega_2],\\ P_0(x,\omega_2)-B^T P_0(x,\omega_2)B &\ \ \textit{is positive semi-definite for } x\in[0,\omega_1],\\ P_0(x,y)+\frac{1}{4}\Big(\frac{\partial P_0(x,y)}{\partial x}+\frac{\partial P_0(x,y)}{\partial y}\Big) &\ \ \textit{is negative semi-definite for } (x,y)\in\Omega,\\ \int\limits_0^{\omega_1}\int\limits_0^{\omega_2} P_0(s,t)\,dt\,ds &\ \ \textit{is negative definite.} \end{aligned}$$

Then problem (17), (6) *is uniquely solvable.*

Consider the case, where $P_i(x,y)\equiv P_i$ $(i=0,1,2)$ and $A=I$, i.e. consider the problem

$$u_{xy}=P_0u+P_1u_x+P_2u_y+q(x,y), \tag{18}$$
$$u(0,y)=u(\omega_1,y)+\varphi(y), \quad u(x,0)=Bu(x,\omega_2)+\psi(x). \tag{19}$$

Theorem 6. *Let*

$$\det\big(I-\exp(\omega_2P_1)B\big)\neq 0,$$
$$\det\big(I-\exp(\omega_1P_2)\big)\neq 0,$$

and let the compatibility condition

$$\varphi(0)-B\varphi(\omega_2)=\psi(0)-\psi(\omega_1)$$

hold. Then problem (18), (19) *is uniquely solvable if and only if*

$$\det\big(I-\exp(\omega_1\Lambda_k)\,B\big)\neq 0 \ \ \textit{for } k\in\mathbb{Z},$$

where

$$\Lambda_k=\Big(i\,\frac{2\pi}{\omega_1}\,kI-P_2\Big)\Big(P_0+i\,\frac{2\pi}{\omega_1}\,kP_1\Big).$$

Consider the case $n = 1$. For the equation

$$u_{xy} = p_0(y)u + p_1(y)u_x + p_2(y)u_y + q(x, y) \tag{20}$$

consider the boundary conditions

$$u(0, y) = u(\omega_1, y), \quad u(x, 0) = bu(x, \omega_2). \tag{21}$$

Theorem 7. *Let the following inequalities hold:*

$$p_0(y)\, p_1(y)\, p_2(y) < 0 \;\; \textit{for} \;\; y \in [0, \omega] \tag{22}$$

and

$$(1 - b)\, p_1(y) \geq 0 \;\; \textit{for} \;\; y \in [0, \omega].$$

Then problem (20), (21) *is uniquely solvable. In particular, if* $b = 1$*, then the doubly periodic problem* (20), (21) *is uniquely solvable if inequality* (22) *holds.*

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Approximate Solution for Heat Equation Applying Neural Network

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Our goal in the proposed study is to apply Neural Network (NN) capabilities to the approximate solution of the partial differential equations (PDEs). Neural networks are one of the popular approach to approximate multi-variable nonlinear functions. It has been also successfully applied to different kinds of real-world problems arising in finance, healthcare, signature verification and facial recognition, weather forecasting, etc. In this note, we consider the simple heat equation and propose its approximate solution by using NN.

In the domain $\Omega = (0,1) \times (0,T)$, $T = const > 0$, let us consider the initial-boundary value problem for the heat equation:

$$\begin{gathered} \frac{\partial U(x,t)}{\partial t} - a\,\frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t), \ \ (x,t) \in \Omega, \\ U(0,t) = U(1,t) = 0, \ \ t \in [0,T], \\ U(x,0) = U_0(x), \ \ x \in [0,1], \end{gathered} \tag{1}$$

where a is a positive constant and U_0 is a given function.

Qualitative and quantitative properties, as well as the numerical solution for problem (1) and its even more complicated nonlinear analogs, are well-studied in the literature (see, for example, [2–5,9] and the references therein). Our purpose, as we already mentioned, is to study a different approach to solving PDEs by means of Machine Learning methods, in particular, to train the neural network so that the trained surrogate model could predict the solution of PDE at any arbitrary point $(x,t) \in \Omega$. Neural Networks could contain several layers. It necessarily contains input and output layers and could have any number of inner layers called hidden layers (see, for example, Fig. 1). In each layer, the user can choose the number of neurons (green circles).

The neural network constructs approximation for the solution of problem (1)

$$u(x,t,\theta) \approx U(x,t),$$

where $u(x,t,\theta)$ denotes the function obtained from a NN, and θ is the variable combining all NN parameters which should be optimized during the training of the NN. In general, training of the NN requires a large amount of training data, representing the NN's input. However, applying the NN to the PDEs approximate solution has the advantage due to it tacks into account the physics and therefore shortens the size of the training data (see, for example, [1,6–8]).

The state-of-the-art machine learning software algorithms, provide automatic differentiation capabilities for functions realized by neural networks, the approximate solution $u(x,t,\theta)$, which in turn allows the residual of the nonlinear problem (1) to be evaluated at a set of training points.

$$R(x,t,\theta) = \frac{\partial u(x,t,\theta)}{\partial t} - a\,\frac{\partial^2 u(x,t,\theta)}{\partial x^2} - f(x,t). \tag{2}$$

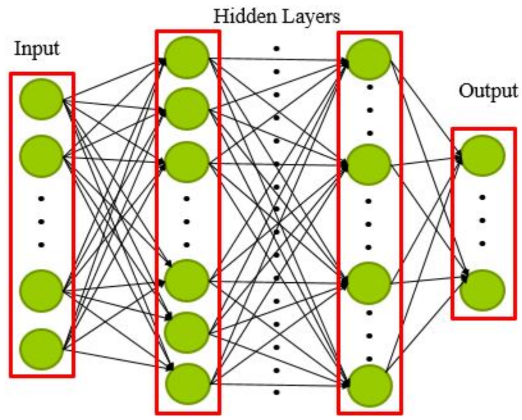

Figure 1. Architecture of the general Neural Network.

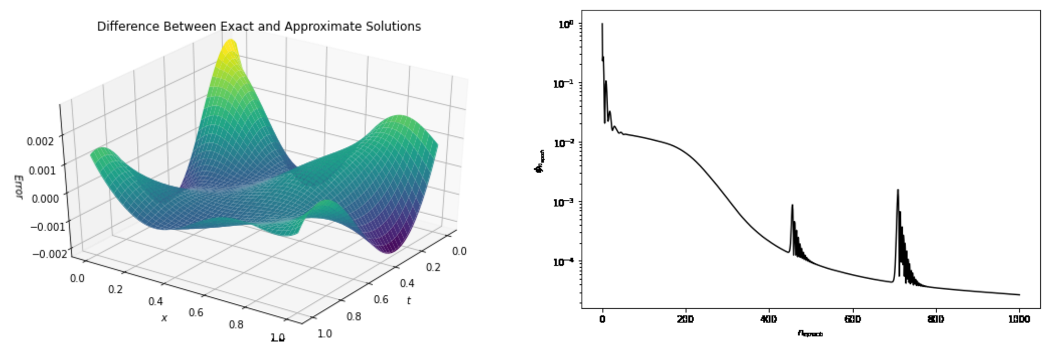

Figure 2. Difference between exact and numerical solutions and learning rate (1000 epochs).

Let us construct the cost function $\mathcal{F}(x,t,\theta)$ which should be minimized by a neural network during the training. The cost function should incorporate residual (2) as well as initial and boundary conditions as follows:

$$\mathcal{F}(x,t,\theta) = Err_{residual}(x,t,\theta) + Err_{initial}(x,t,\theta) + Err_{boindary}(x,t,\theta),$$

where

$$Err_{residual}(x_i^r, t_i^r, \theta) = \frac{1}{N_r}\sum_{i=1}^{N_r} \left|R(x_i^r, t_i^r, \theta)\right|^2,$$

$$Err_{initial}(x_i^0, t_i^0, \theta) = \frac{1}{N_0}\sum_{i=1}^{N_0} \left|u(x_i^0, t_i^0, \theta) - U_0(x_i^0)\right|^2,$$

$$Err_{boindary}(x_i^b, t_i^b, \theta) = \frac{1}{N_b}\sum_{i=1}^{N_r} \left|u(x_i^b, t_i^b, \theta)\right|^2.$$

The number of the training points is denoted by N_r, while (x_i^r, t_i^r) represents a set of training data. Collection of the following points (x_i^0, t_i^0), (x_i^b, t_i^b) are used for initial and boundary conditions respectively.

Below the results of the numerical experiments are given. For the training of the neural network, the library of scientific computing NumPy and the library of machine learning TensorFlow are used. For the test experiments the Jupyter Notebook implementation, proposed in [1], was modified and applied to the problem (1). The right-hand side of the problem (1) was chosen in such a way that the exact solution is $U(x,t) = x(1-x)\exp(-x-t)$ with the corresponding initial solution $U_0(x) = x(1-x)\exp(-x)$.

For the initial line and the boundaries we set $N_0 = N_b = 25$ and for the inner points, training data of size $N_r = 1000$ was chosen.

For the neural network architecture, we set five hidden layers and 10 neurons in each layer. The surfaces in Fig. 3 depict exact and numerical solutions when for NN training 1000 epochs (iterations) were used.

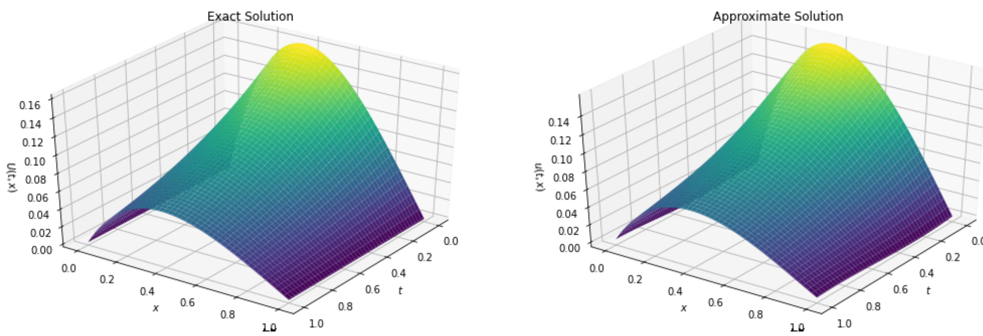

Figure 3. Exact and numerical solutions (1000 epochs).

The difference between exact and numerical solutions is given in Fig. 2 (left). In the same figure, the NN learning rate is given on right.

We also ran NN training for 5000 epochs. The results for the difference between exact and numerical solutions and the learning rate are given in Fig. 4.

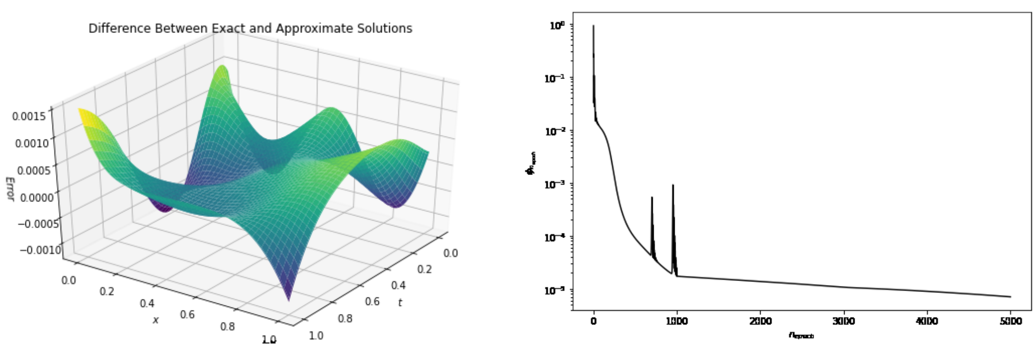

Figure 4. Difference between exact and numerical solutions and learning rate (5000 epochs).

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Optimal Regularity Results for the One-Dimensional Prescribed Curvature Equation Via the Strong Maximum Principle

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1 Introduction

This contribution is based on our recent paper [1], where we establish a novel, extended, version of the strong maximum principle for a general class of second order ordinary differential equations

$$v'' = g(t, v, v'),$$

in the absence of any assumption of continuity or monotonicity on the function g, and where, exploiting this tool, we provide some optimal regularity results for the bounded variation solutions, positive and nodal, of the non-autonomous curvature equation

$$-\bigg(\frac{u'}{\sqrt{1+(u')^2}}\bigg)' = f(x, u), \tag{1.1}$$

f being an arbitrary function prescribing the curvature of the graph of u.

The analysis carried out in [1] allows us, through a completely different technical device, to extend most of the results we previoulsly obtained in [2–5], for the positive bounded variation solutions of (1.1) under homogeneous Neumann boundary condition and the structural assumption $f(x, s) = h(x)k(s)$, to more general classes of equations and to, possibly non-homogeneous, Dirichlet, Neumann, Robin, or even periodic boundary value problems. Furthermore, we are able to produce a new interpretation of the assumptions used in our previous works, clarifying their meaning and displaying some deep, though previously hidden, connections with the strong maximum principle.

2 A variant of the strong maximum principle

The main result of this section is the following version of the strong maximum principle for second order ordinary differential equations with possibly discontinuous and non-monotone right-hand sides. In this respect, the Keller–Osserman assumption (G) stated below is independent of the conditions required by the classical Vázquez strong maximum principle in [6] and by its extensions given by Pucci and Serrin in [7], where G' is always supposed to be continuous and increasing. Accordingly, this result yields, in the one-dimensional setting, a completion and a sharpening of its counterparts in [6] or [7]; its proof, delivered in [1], being also more delicate than in the classical situations.

Theorem 2.1. *Let $g : (\alpha, \omega) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function and let $v \in W^{2,1}_{\mathrm{loc}}(\alpha, \omega) \cap W^{1,1}(\alpha, \omega)$ be a non-trivial non-negative solution of the differential equation*

$$v''(t) = g(t, v(t), v'(t)) \;\; \textit{for almost all} \;\; t \in (\alpha, \omega).$$

Assume that:

(G) *there exist a constant $\varepsilon > 0$ and an absolutely continuous function $G : [0, \varepsilon] \to \mathbb{R}$ such that*

$$0 \le g(t, v(t), v'(t)) \le G'(v(t)) \;\; \textit{for almost all} \;\; t \in (\alpha, \omega) \\ \textit{for which} \;\; 0 < v(t) \le \varepsilon \;\; \textit{and} \;\; |v'(t)| \le \varepsilon,$$

and either

$$G(s) = 0 \;\; \textit{for all} \;\; s \in (0, \varepsilon],$$

or

$$G(s) > 0 \;\; \textit{for all} \;\; s \in (0, \varepsilon] \;\; \textit{and} \;\; \int_0^{\varepsilon} \frac{1}{\sqrt{G(s)}}\, ds = +\infty. \tag{2.1}$$

Then, v is strongly positive in the sense that the following properties hold true:

- *$v(t) > 0$ for all $t \in (\alpha, \omega)$;*
- *$v'(\alpha^+) > 0$ if $v(\alpha) = 0$ and $v'(\alpha^+)$ exists;*
- *$v'(\omega^-) < 0$ if $v(\omega) = 0$ and $v'(\omega^-)$ exists.*

3 Optimal regularity results for the prescribed curvature equation

In this section we discuss the regularity properties of the bounded variation solutions of the one-dimensional non-autonomous prescribed curvature equation

$$-\left(\frac{u'}{\sqrt{1 + (u')^2}}\right)' = f(x, u), \quad a < x < b, \tag{3.1}$$

where $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ is any given function. We begin by recalling the notion of bounded variation solution of equation (3.1). To this end, for any $v \in BV(a, b)$, we denote by $Dv = D^a v\, dx + D^s v$ the Lebesgue–Nikodym decomposition, with respect to the Lebesgue measure dx in $\mathbb{R}$, of the Radon measure Dv in its absolutely continuous part $D^a v\, dx$, with density function $D^a v$, and its singular part $D^s v$. Further, $\frac{D^s v}{|D^s v|}$ stands for the density function of $D^s v$ with respect to its absolute variation $|D^s v|$. Finally, for every $x_0 \in [a, b)$, $v(x_0^+)$ denotes the right trace of v at x_0 and, for every $x_0 \in (a, b]$, $v(x_0^-)$ denotes the left trace of v at x_0.

Definition 3.1. A function $u \in BV(a, b)$ is a bounded variation solution of (3.1) if $f(\,\cdot\,, u(\,\cdot\,)) \in L^1(a, b)$ and

$$\int_a^b \frac{D^a u(x) D^a \phi(x)}{\sqrt{1 + (D^a u(x))^2}}\, dx + \int_a^b \frac{D^s u}{|D^s u|}(x)\, D^s \phi = \int_a^b f(x, u(x)) \phi(x)\, dx$$

for all $\phi \in BV(a, b)$ such that $|D^s \phi|$ is absolutely continuous with respect to $|D^s u|$ and $\phi(a^+) = \phi(b^-) = 0$.

We begin with a partial regularity result: it establishes that a bounded variation solution u of (3.1) can lose its regularity at the endpoints, but never at the interior points, of the intervals where the function $f(\,\cdot\,,u(\,\cdot\,))$ has a definite sign; whereas, u can be singular at an interior point of its domain if such a point separates two adjacent intervals where $f(\,\cdot\,,u(\,\cdot\,))$ changes sign. In both cases, the derivative u' blows up, but, in the latter one, u can further exhibit a jump discontinuity.

Theorem 3.1. *Let u be a bounded variation solution of equation* (3.1)*. Then, the following statements hold.*

(i) *If $f(x,u(x))\geq 0$ for almost all $x\in(a,b)$, then u is concave and either $u\in W^{2,1}(a,b)$, or $u\in W^{2,1}_{\rm loc}[a,b)\cap W^{1,1}(a,b)$ and $u'(b^-)=-\infty$, or $u\in W^{2,1}_{\rm loc}(a,b]\cap W^{1,1}(a,b)$ and $u'(a^+)=+\infty$, or $u\in W^{2,1}_{\rm loc}(a,b)\cap W^{1,1}(a,b)$, $u'(a^+)=+\infty$, and $u'(b^-)=-\infty$. In all cases, u satisfies equation* (3.1) *for almost all $x\in(a,b)$.*

(ii) *If $f(x,u(x))\leq 0$ for almost all $x\in(a,b)$, then u is convex and either $u\in W^{2,1}(a,b)$, or $u\in W^{2,1}_{\rm loc}[a,b)\cap W^{1,1}(a,b)$ and $u'(b^-)=+\infty$, or $u\in W^{2,1}_{\rm loc}(a,b]\cap W^{1,1}(a,b)$ and $u'(a^+)=-\infty$, or $u\in W^{2,1}_{\rm loc}(a,b)\cap W^{1,1}(a,b)$, $u'(a^+)=-\infty$, and $u'(b^-)=+\infty$. In all cases, u satisfies equation* (3.1) *for almost all $x\in(a,b)$.*

(iii) *If there is $c\in(a,b)$ such that $f(x,u(x))\geq 0$ for almost all $x\in(a,c)$ and $f(x,u(x))\leq 0$ for almost all $x\in(c,b)$, then $u_{|(a,c)}$ is concave, $u_{|(c,b)}$ is convex, and either $u\in W^{2,1}_{\rm loc}(a,b)\cap W^{1,1}(a,b)$, or $u_{|(a,c)}\in W^{2,1}_{\rm loc}(a,c)\cap W^{1,1}(a,c)$, $u_{|(c,b)}\in W^{2,1}_{\rm loc}(c,b)\cap W^{1,1}(c,b)$, $u(c^-)\geq u(c^+)$, and $u'(c^-)=-\infty=u'(c^+)$. Moreover, in case $u(c^-)>u(c^+)$, we have that*

$$D^s u=(u(c^+)-u(c^-))\,\delta_c,$$

where δ_c stands for the Dirac measure concentrated at c. In any circumstances, u satisfies equation (3.1) *for almost all $x\in(a,b)$.*

(iiii) *If there is $c\in(a,b)$ such that $f(x,u(x))\leq 0$ for almost all $x\in(a,c)$ and $f(x,u(x))\geq 0$ for almost all $x\in(c,b)$, then $u_{|(a,c)}$ is convex, $u_{|(c,b)}$ is concave, and either $u\in W^{2,1}_{\rm loc}(a,b)\cap W^{1,1}(a,b)$, or $u_{|(a,c)}\in W^{2,1}_{\rm loc}(a,c)\cap W^{1,1}(a,c)$, $u_{|(c,b)}\in W^{2,1}_{\rm loc}(c,b)\cap W^{1,1}(c,b)$, $u(c^-)\leq u(c^+)$, and $u'(c^-)=+\infty=u'(c^+)$. Moreover, in case $u(c^-)<u(c^+)$,* (3.1) *holds. In any circumstances, u satisfies equation* (3.1) *for almost all $x\in(a,b)$.*

Our next two results, Theorems 3.2 and 3.3, establish the complete regularity of the bounded variation solutions u of (3.1). Precisely, Theorem 3.2 guarantees the regularity at the endpoints of any interval where the sign of $f(\,\cdot\,,u(\,\cdot\,))$ is constant, by imposing at these points a suitable control, expressed by any of the conditions (j)–(jjjj), on the decay rate to zero of $f(\,\cdot\,,u(\,\cdot\,))$ Theorem 3.3, instead, guarantees the regularity of u at any interior point, z, separating two adjacent interval where $f(\,\cdot\,,u(\,\cdot\,))$ changes sign, by imposing a similar decay property to $f(\,\cdot\,,u(\,\cdot\,))$ either on the left, or on the right, of z, as expressed by the conditions (h) or (hh). From [3–5] we also know that these assumptions on the decay rate of $f(\,\cdot\,,u(\,\cdot\,))$ are optimal, in the sense that, if they fail at some point, the derivative u' might blow-up there, and the solution u might even develop a jump discontinuity.

The proof of Theorems 3.2 and 3.3 presented in [1] is completely new and it relies on the use of the strong maximum principle as expressed by Theorem 2.1. Our approach, besides being far more general and versatile, displays the following striking fact: it turns out that the assumption yielding the regularity of a solution u of (3.1), through a control on the decay rate to zero of $f(\,\cdot\,,u(\,\cdot\,))$ at some point z, is precisely the Keller–Osserman condition (2.1) required by Theorem 2.1 so that the strong maximum principle holds for the differential equation

$$\left(\frac{v'}{\sqrt{1+(v')^2}}\right)' = f(z+v,t) \iff v'' = f(z+v,t)(1+(v')^2)^{\frac{3}{2}}, \tag{3.2}$$

satisfied by the shift $v = w - z$ of a local inverse w of u. Note that, as f is not assumed to satisfy any regularity condition, the right-hand side of (3.2), that is, the function

$$g(t,s,\xi) := f(z+s,t)(1+\xi^2)^{\frac{3}{2}},$$

may be discontinuous, besides in t, in the state variable s as well. Note that this could happen even if f were a Carathéodory function and thus g would be continuous in t and ξ, but just Lebesgue measurable with respect to s. Essentially, we establish that the validity of the strong maximum principle for equation (3.2) yields the regularity for the solutions of (3.1). As a consequence, the bounded variation solutions of (3.1) can develop singularities only when the conclusions of the strong maximum principle fail for (3.2). This appears to be a quite remarkable achievement that illuminates and clarify the otherwise apparently exotic conditions we introduced in [3].

Theorem 3.2. *Let u be a bounded variation solution of* (3.1)*. Then the following assertions hold.*

(j) *If $f(x,u(x)) \geq 0$ for almost all $x \in (a,b)$ and there exist $\delta > 0$ and $\mu \in L^1(a,a+\delta)$ such that*

- *$f(x,u(x)) \leq \mu(x)$ for almost all $x \in (a,a+\delta)$,*
- *$M(x) := \int_a^x \mu(t)\,dt > 0$ for all $x \in (a,a+\delta]$, and $\int_a^{a+\delta} \frac{1}{\sqrt{M(x)}}\,dx = +\infty$,*

then $u \in W^{2,1}_{\mathrm{loc}}[a,b) \cap W^{1,1}(a,b)$.

(jj) *If $f(x,u(x)) \geq 0$ for almost all $x \in (a,b)$ and there exist $\delta > 0$ and $\mu \in L^1(b-\delta,b)$ such that*

- *$f(x,u(x)) \leq \mu(x)$ for almost all $x \in (b-\delta,b)$,*
- *$M(x) := \int_x^b \mu(t)\,dt > 0$ for all $x \in [b-\delta,b)$, and $\int_{b-\delta}^{b} \frac{1}{\sqrt{M(x)}}\,dx = +\infty$,*

then $u \in W^{2,1}_{\mathrm{loc}}(a,b] \cap W^{1,1}(a,b)$.

(jjj) *If $f(x,u(x)) \leq 0$ for almost all $x \in (a,b)$ and there exist $\delta > 0$ and $\nu \in L^1(a,a+\delta)$ such that*

- *$f(x,u(x)) \geq \nu(x)$ for almost all $x \in (a,a+\delta)$,*
- *$N(x) := \int_a^x \nu(t)\,dt < 0$ for all $x \in (a,a+\delta]$, and $\int_a^{a+\delta} \frac{1}{\sqrt{-N(x)}}\,dx = +\infty$,*

then $u \in W^{2,1}_{\mathrm{loc}}[a,b) \cap W^{1,1}(a,b)$.

(jjjj) *If $f(x,u(x)) \leq 0$ for almost all $x \in (a,b)$ and there exist $\delta > 0$ and $\nu \in L^1(b-\delta,b)$ such that*

- *$f(x,u(x)) \geq \nu(x)$ for almost all $x \in (b-\delta,b)$,*
- *$N(x) := \int_x^b \nu(t)\,dt < 0$ for all $x \in [b-\delta,b)$, and $\int_{b-\delta}^{b} \frac{1}{\sqrt{-N(x)}}\,dx = +\infty$,*

then $u \in W^{2,1}_{\mathrm{loc}}(a,b] \cap W^{1,1}(a,b)$.

Theorem 3.3. *Let u be a bounded variation solution of equation* (3.1). *Then the following statements hold.*

(h) *If there is $c \in (a,b)$ such that $f(x,u(x)) \geq 0$ for almost all $x \in (a,c)$ and $f(x,u(x)) \leq 0$ for almost all $x \in (c,b)$ and either there exist $\delta > 0$ and $\mu \in L^1(c-\delta,c)$ such that*

- $f(x,u(x)) \leq \mu(x)$ *for almost all* $x \in (c-\delta,c)$,
- $M(x) := \displaystyle\int\limits_x^c \mu(t)\,dt > 0$ *for all* $x \in [c-\delta,c)$, *and* $\displaystyle\int\limits_{c-\delta}^{c} \frac{1}{\sqrt{M(x)}}\,dx = +\infty$,

or there exist $\delta > 0$ and $\nu \in L^1(c,c+\delta)$ such that

- $f(x,u(x)) \geq \nu(x)$ *for almost all* $x \in (c,c+\delta)$,
- $N(x) := \displaystyle\int\limits_c^x \nu(t)\,dt < 0$ *for all* $x \in (c,c+\delta]$, *and* $\displaystyle\int\limits_{c}^{c+\delta} \frac{1}{\sqrt{-N(x)}}\,dx = +\infty$,

then $u \in W^{2,1}_{\mathrm{loc}}(a,b) \cap W^{1,1}(a,b)$.

(hh) *If there is $c \in (a,b)$ such that $f(x,u(x)) \leq 0$ for almost all $x \in (a,c)$ and $f(x,u(x)) \geq 0$ for almost all $x \in (c,b)$ and either there exist $\delta > 0$ and $\nu \in L^1(c-\delta,c)$ such that*

- $f(x,u(x)) \geq \nu(x)$ *for almost all* $x \in (c-\delta,c)$,
- $N(x) := \displaystyle\int\limits_x^c \nu(t)\,dt < 0$ *for all* $x \in [c-\delta,c)$, *and* $\displaystyle\int\limits_{c-\delta}^{c} \frac{1}{\sqrt{-N(x)}}\,dx = +\infty$,

or there exist $\delta > 0$ and $\mu \in L^1(c,c+\delta)$ such that

- $f(x,u(x)) \leq \mu(x)$ *for almost all* $x \in (c,c+\delta)$,
- $M(x) := \displaystyle\int\limits_c^x \nu(t)\,dt > 0$ *for all* $x \in (c,c+\delta]$, *and* $\displaystyle\int\limits_{c}^{c+\delta} \frac{1}{\sqrt{M(x)}}\,dx = +\infty$,

then $u \in W^{2,1}_{\mathrm{loc}}(a,b) \cap W^{1,1}(a,b)$.

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Characteristic Vectors for Normed Partitions of Cauchy Matrices

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For any map $y:\mathbb{R}_+\to\mathbb{R}^n$, where $\mathbb{R}_+=\{t\in\mathbb{R}:\ t\geq 0\}$, we can calculate the Lyapunov exponent $\lambda[y]$ as

$$\lambda[y]=\overline{\lim_{t\to+\infty}}\,\frac{1}{t}\,\ln\|y(t)\|. \tag{1}$$

It is well known that Lyapunov exponents play an important role in qualitative theory of differential equations and stability theory, see [2] or [8]. For maps defined on some subsets of $\mathbb{R}^m$ with $m>1$, such as solutions of total differential equations, we can not define the Lyapunov exponent by (1) without substantial improvements. Some appropriate definitions for the required analogs of Lyapunov exponents in multivariate case has been proposed by E. I. Grudo [5] and M. V. Kozhero [9].

Now the following asymptotic characteristics are used for solutions of total differential equations: strong exponents [9], (weak) characteristic exponents [9], [4, p. 115], and characteristic functionals (vectors) [5], [4, p. 108], [3, p. 82]. Each of these notions is a straightforward generalization of classical Lyapunov exponent and coincides with it when $m=1$.

The results concerning these asymptotic characteristics are summarized by I. V. Gaishun in monographs [3] and [4], where general and asymptotic theory of total differential equations are systematically presented. Some additional information on these issues can be found in [12].

Let $K\subset\mathbb{R}^n$ be a closed convex cone such that $K\cap(-K)=\{0\}$. A linear functional (in fact, a row vector) $\mu\in(\mathbb{R}^n)^*$ is said to be positive on K if $\mu(x)\geq 0$ for all $x\in K$. The set K^+ of all positive on K linear functionals is called the dual cone of K.

Take any $y:K\to\mathbb{R}^m$.

Definition 1. A linear functional $\lambda\in(\mathbb{R}^n)^*$ is said to be a characteristic functional of y with respect to the cone K if

$$\limsup_{\|x\|\to+\infty}\|x\|^{-1}\big(\lambda x+\ln\|y(x)\|\big)=0$$

and

$$\limsup_{\|x\|\to+\infty}\|x\|^{-1}\big(\lambda x+\mu x+\ln\|y(x)\|\big)>0$$

for all $\mu\in K^+$, $\mu\neq 0$.

The set of all characteristic functionals is called the characteristic set of y. We denote it by $\mathcal{M}[y]$.

Definition 2. The (weak) characteristic exponent of y is the function $\chi[y]:K\setminus\{0\}\to\mathbb{R}$ defined by

$$\chi[y](x):=\overline{\lim_{t\to+\infty}}\,\frac{1}{t\|x\|}\,\ln\|y(tx)\|.$$

There exist an interrelation between (weak) characteristic exponents and characteristic functionals. In [10] (see also [12]) it was proved that if $\ln\|y\|$ is a Lipshitz function, then $\mathcal{M}[y]=\mathcal{M}[\exp\psi[y]]$, where $\psi[y](x)=\|x\|\chi[y](x)$ is the modified characteristic exponent of y.

It occurs that the above asymptotic characteristics are useful not only in the study of total differential equations, but also in the theory of linear ordinary differential systems. To demonstrate this fact, consider a linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{2}$$

with piecewise continuous and bounded coefficient matrix A such that $\|A(t)\| \leq M < +\infty$ for all $t \geq 0$. We denote the Cauchy matrix of (2) by X_A and the highest Lyapunov exponent of (2) by $\lambda_n(A)$.

In [16], see also [15, p. 379] and [2, p. 236], I. G. Malkin has used estimations of the form

$$\|X_A(t,s)\| \leq D\exp(\alpha(t-s) + \beta s), \quad t \geq s \geq 0, \quad D > 0, \quad \alpha, \beta \in \mathbb{R}, \tag{3}$$

in order to investigate asymptotic stability of the trivial solution to a system

$$\dot{y} = A(t)y + f(t,y), \quad y \in \mathbb{R}^n, \quad t \geq 0,$$

with a nonlinear perturbation $f(t,y)$ of a higher order.

An ordered pair $(\alpha, \beta) \in \mathbb{R}^2$ is called a Malkin estimation for system (2) if there exists a number $D = D(\alpha, \beta) > 0$ such that (3) holds. A pair $(\alpha, \beta) \in \mathbb{R}^2$ is said to be a minimal Malkin estimation [11] if $(\alpha+\xi, \beta+\eta) \in E(A)$ for all $\xi > 0$, $\eta > 0$, and $(\alpha+\xi, \beta+\eta) \notin E(A)$ for all $\xi \leq 0$, $\eta \leq 0$, $\xi^2 + \eta^2 \neq 0$.

It can be easily seen that the set of minimal Malkin estimations for system (2) coincides with the set of Grudo characteristic vectors for the function $\|X_A(t,s)\|$ with respect to the cone $C = \{(t,s) \in \mathbb{R}^2 : \ t \geq s \geq 0\}$. Using this fact, in [11] we have given an alternative description for the set of minimal Malkin estimations in terms of the function

$$\overline{\lim_{s\to+\infty}} \frac{1}{(\theta-1)s} \ln \|X_A(\theta s, s)\|. \tag{4}$$

Definition 3. Let τ be an increasing sequence $t_0 < t_1 < \cdots < t_{s+1}$ of $s+2$ real numbers. The expression

$$P_A(\tau) = \prod_{i=0}^{s} \|X_A(t_{i+1}, t_i)\|$$

is said to be a normed partition of the Cauchy matrix for system (2).

Normed partitions are common in Lyapunov exponents theory. Formulae for calculating the central (see [2, p. 99], [8, p. 43])

$$\Omega(A) = \lim_{T\to+\infty} \overline{\lim_{m\to\infty}} \frac{1}{mT} \sum_{k=1}^{m} \ln \|X_A(kT, kT-T)\|$$

as well as the exponential exponent (see [7], [8, p. 52])

$$\nabla_0(A) = \lim_{\theta\to1+0} \overline{\lim_{m\to\infty}} \frac{1}{\theta^m} \sum_{k=1}^{m} \ln \|X_A(\theta^k, \theta^{k-1})\|, \tag{5}$$

contain the expressions of the form

$$\Xi_A(\tau) = \sum_{i=0}^{s} \ln \|X_A(t_{i+1}, t_i)\| = \ln P_A(\tau)$$

with some appropriate τ. The highest sigma-exponent (or the Izobov exponent) of system (2) (see [6], [8, p. 225])

$$\nabla_\sigma(A) = \overline{\lim_{m\to\infty}} \frac{\xi_m(\sigma)}{m},$$
$$\xi_m(\sigma) = \max_{i<m} \big(\ln \|X_A(m,i)\| + \xi_i(\sigma) - \sigma i\big), \ \ \xi_1 = 0, \ \ i \in \mathbb{N},$$

can be represented in an equivalent form [1] (see also [14]) as

$$\nabla_\sigma(A) = \overline{\lim_{m\to\infty}} m^{-1} \max_{\tau\in\mathcal{D}_0(m)} \big(\Xi_A(\tau) - \sigma\|\tau\|_{\mathrm{i}}\big), \tag{6}$$

where $\mathcal{D}_0(m)$ is the set of all increasing sequences $0 = t_0 < t_1 < \ldots < t_{s+1} = m$ of integer numbers with at least two terms and $\|\tau\|_{\mathrm{i}} = t_1 + \cdots + t_s$. Note that $\tau \in \mathcal{D}_0(m)$ may have different numbers of elements.

Let $t_0 = 0$. Fix some $k \in \mathbb{N}$ and consider sequences $0 < t_1 < \cdots < t_{k+1}$ of real numbers with $k+1$ elements as vectors $(t_1,\ldots,t_{k+1}) \in \mathbb{R}^{k+1}$. Taking $K = \{\tau = (t_1,\ldots,t_{k+1}) \in \mathbb{R}^{k+1} : \ 0 \le t_1 < \cdots \le t_{k+1}\}$, we define the set $\mathcal{M}[P_A]$ and the function

$$\Psi_A(\tau) = \psi[P_A](\tau) = \overline{\lim_{t\to+\infty}} \frac{1}{t} \ln P_A(t\tau)$$

according to Definitions 1 and 2. By [10] (see also [12]) we have the following statements.

Proposition 1. *The equality*

$$\mathcal{M}[P_A] = \mathcal{M}[\exp \Psi_A]$$

holds.

Proposition 2. *Let $\lambda \in \mathcal{M}[\Psi_A]$. If for some sequence of vectors $\tau_j \in K \subset \mathbb{R}^{k+1}$, such that $\|\tau_j\| \to \infty$ and $\tau_j\|\tau_j\|^{-1} \to \xi \in \mathbb{R}^{k+1}$ as $j \to \infty$, we have*

$$\lim_{j\to\infty} \|\tau_j\|^{-1}(\lambda\tau_j + \ln P_A(\tau_j)) = 0,$$

then $\lambda\xi + \Psi_A(\xi) = 0$ and $\lambda\xi + \Psi_A(\xi) \ge 0$ for all $\xi \in K$.

We cannot use these results to calculate $\nabla_\sigma(A)$, since in (6) the length of τ can increase indefinitely as m increases. However, we can apply Propositions 1 and 2 to obtain some information on finite-point approximations of $\nabla_\sigma(A)$.

Let $\mathcal{D}_0^k(m)$ be a subset of $\mathcal{D}_0(m)$ containing sequences with at most k elements.

Definition 4 ([13])**.** The number

$$\nabla_\sigma^k(A) = \overline{\lim_{m\to\infty}} m^{-1} \max_{\tau\in\mathcal{D}_0^k(m)} \big(\Xi_A(\tau) - \sigma\|\tau\|_{\mathrm{i}}\big)$$

is said to be the k-point approximation for $\nabla_\sigma(A)$.

Proposition 3. *If $(\sigma,\mu) \in \mathbb{R}^2$ is an extreme point for the epigraph of $\nabla_\sigma^k(A)$, then the vector $(-\sigma,\ldots,-\sigma,-\mu) \in (\mathbb{R}^{k+1})^*$ is a characteristic vector for P_A.*

Corollary. *If $(\sigma,\mu) \in \mathbb{R}^2$ is an extreme point for the epigraph of $\nabla_\sigma^k(A)$, then*

$$\sigma\sum_{i=1}^k \xi_i + \mu\xi_{k+1} \le \Psi_A(\xi)$$

for all $\xi \in K$ and there exists some $\xi^0 \in K$ such that

$$\sigma\sum_{i=1}^k \xi_i^0 + \mu\xi_{k+1}^0 = \Psi_A(\xi^0).$$

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On the Continuous Dependence of Solutions to Linear Boundary Value Problems on Boundary Conditions

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1 Introduction

The general questions of the continuous dependence of solutions to boundary value problems on parameters as applied to functional differential equations are studied in [1, 3, 4, 8], see also the references to Section 1.5 in [1].

We consider a quite broad class of functional differential systems with aftereffect and follow the notation and basic statements of the general theory of functional differential equations in the part concerning linear systems with aftereffect [1, 4].

Let $L^n = L^n[0, T]$ be the Lebesgue space of all summable functions $z : [0, T] \to R^n$ defined on a finite segment $[0, T]$ with the norm

$$\|z\|_{L^n} = \int\limits_0^T |z(t)| \, dt,$$

where $| \cdot |$ is a norm in R^n. Below we use $\| \cdot \|$ for the matrix norm agreed with $| \cdot |$.

Denote by $AC^n = AC^n[0, T]$ the space of absolutely continuous functions $x : [0; T] \to R^n$ with the norm

$$\|x\|_{AC^n} = |x(0)| + \|\dot{x}\|_{L^n}.$$

In the sequel we will use some results from [1, 4].

The system

$$\mathcal{L}x = f \tag{1.1}$$

with a linear bounded Volterra operator $\mathcal{L} : AC^n \to L^n$ is considered under the assumption that the general solution of equation (1.1) has the form

$$x(t) = X(t)x(0) + \int\limits_0^t C(t, s)f(s) \, ds, \tag{1.2}$$

where $X(t)$ is the fundamental matrix to the homogeneous equation $\mathcal{L}x = 0$, $C(t, s)$ is the Cauchy matrix. A broad class of operators $\mathcal{L}$ with property (1.2) is described, for instance, in [5].

We consider the boundary value problems (BVPs)

$$\mathcal{L}x = f, \quad \ell_0 x = 0, \tag{1.3}$$

and

$$\mathcal{L}x = f, \quad \ell x = 0, \tag{1.4}$$

where $\ell_0, \ell : AC^n \to R^n$ are linear bounded vector-functional, assuming (1.3) to be uniquely solvable, i.e. $\det \ell_0 X \neq 0$. We will consider the question of the continuous dependence of solutions on the boundary conditions in terms of the proximity of ℓ to ℓ_0 and the proximity of the solution x of BVP (1.4) to the solution x_0 of BVP (1.3).

2 Two theorems

First we give a theorem that follows from the theorem on the invertible operator (see, for instance, Theorem 3.6.3 [2]).

Theorem 2.1. *Let the inequality*

$$\Delta = \|\ell_0 X - \ell X\| \cdot \|(\ell_0 X)^{-1}\| < 1 \tag{2.1}$$

be fulfilled. Then BVP (1.4) *is uniquely solvable and the estimate*

$$\begin{aligned}\|x_0 - x\|_{AC^n} \le \|X\|_{AC^{n\times n}} \cdot \frac{\Delta}{1-\Delta} \cdot \|(\ell_0 X)^{-1}\| \cdot \|\ell\|_{AC^n \to R^n} \cdot \|Cf\|_{AC^n} \\ + \|X\|_{AC^{n\times n}} \cdot \|(\ell_0 X)^{-1}\| \cdot \|\ell_0 - \ell\|_{AC^n \to R^n} \cdot \|Cf\|_{AC^n}\end{aligned}$$

holds.

Results of the constructive study of boundary value problems, based on conditions like (2.1), are presented systematically in [1, 7], see also [6]. Condition (2.1) often turns out to be quite rigid. To formulate the next theorem based on another approach, we introduce additional notation:

$$\begin{gathered}\ell_0 X = \Gamma^0 = (\gamma_{ij}^0)_{i,j=1,\dots,n}; \quad \ell X = \Gamma = \big([\gamma_{ij}^b,\, \gamma_{ij}^u]\big)_{i,j=1,\dots,n}; \quad \gamma_{ij}^0 \in [\gamma_{ij}^b, \gamma_{ij}^u]; \\ (\ell_0 X)^{-1} = B^0 = (\beta_{ij}^0)_{i,j=1,\dots,n}; \quad (\ell X)^{-1} = B = \big([\beta_{ij}^b,\, \beta_{ij}^u]\big)_{i,j=1,\dots,n}; \\ M = \max\big(\det\Gamma:\ \gamma_{ij} \in [\gamma_{ij}^b, \gamma_{ij}^u],\ i,j = 1,\dots,n\big); \\ \mu = \min\big(\det\Gamma:\ \gamma_{ij} \in [\gamma_{ij}^b, \gamma_{ij}^u],\ i,j = 1,\dots,n\big).\end{gathered}$$

For an $(n\times n)$-matrix $\mathcal{A}$ with interval-valued elements $[a_{ij}, b_{ij}]$ we define $\|\mathcal{A}\|_I$ by the equality

$$\|\mathcal{A}\|_I = \big\|(\alpha_{ij})_{i,j=1,\dots,n}\big\|,$$

where $\alpha_{ij} = \max(|a_{ij}|, |b_{ij}|)$.

Theorem 2.2. *Let the inequality*

$$M \cdot \mu > 0$$

be fulfilled. Then BVP (1.4) *is uniquely solvable and the estimate*

$$\begin{aligned}\|x_0 - x\|_{AC^n} \le \|X\|_{AC^{n\times n}} \cdot \|B_0 - B\|_I \cdot \|\ell\|_{AC^n \to R^n} \cdot \|Cf\|_{AC^n} \\ + \|X\|_{AC^{n\times n}} \cdot \|B_0\| \cdot \|\ell_0 - \ell\|_{AC^n \to R^n} \cdot \|Cf\|_{AC^n}\end{aligned}$$

holds.

This theorem allows to cover a set of boundary value problems (1.4) for which condition (2.1) is not fulfilled.

3 An example

Consider the boundary value problem

$$\dot{x}(t) = Fx(t) + f(t), \ \ t \in [0,1], \ \ \ell_{01}x \equiv ax_1(0) + bx_2(1) = 0, \ \ \ell_{02}x \equiv cx_1(1) + dx_2(0) = 0. \tag{3.1}$$

Here

$$F=\begin{pmatrix}0.5 & -0.1\\ -0.2 & 0.6\end{pmatrix},\quad A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}.$$

For definiteness, let the norm in R^2 be defined by the equality $|x|=\max(|x_1|,|x_2|)$, hence for $B=(b_{ij})$ we have

$$\|B\|=\max\big(|b_{11}|+|b_{12}|,|b_{21}|+|b_{22}|\big).$$

For the case the matrix ℓ_0X is defined by the equality

$$\ell_0X=\begin{pmatrix}0.304 & 3.680\\ 4.997 & 3.478\end{pmatrix},\quad (\ell_0X)^{-1}=\begin{pmatrix}-0.200 & 0.212\\ 0.288 & -0.018\end{pmatrix},\quad \|(\ell_0X)^{-1}\|=0.413.$$

Thus by virtue of Theorem 2.1 problem (3.1) is uniquely solvable and, together with it, any problem

$$\dot{x}=Fx+f,\ \ \ell x=0 \tag{3.2}$$

with ℓ such that $\|\ell X-\ell_0X\|<2.421$ is uniquely solvable too.

Let us show that Theorem 2.2 makes it possible to go beyond this inequality. Immerse the matrix ℓ_0X into the family $\Gamma=\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix}$ with $\gamma_{11}\in[0.2,0.4]$, $\gamma_{12}\in[3.5,3.8]$, $\gamma_{21}\in[4.5,5.5]$, $\gamma_{22}\in[3.4,8]$.

Further

$$\max\Big(\det\Gamma:\ \gamma_{11}\in[0.2,0.4],\ \gamma_{12}\in[3.5,3.8],\ \gamma_{21}\in[4.5,5.5],\ \gamma_{22}\in[3.4,8]\Big)=-12.55,$$
$$\min\Big(\det\Gamma:\ \gamma_{11}\in[0.2,0.4],\ \gamma_{12}\in[3.5,3.8],\ \gamma_{21}\in[4.5,5.5],\ \gamma_{22}\in[3.4,8]\Big)=-20.22,$$

therefore, the determinant of any matrix from the family Γ differs from zero. It should be noted that in terms of the parameters a, b, c, d of ℓ_0 it means the unique solvability for all the problems (3.2) with $a\in[0.862,1.119]$, $b\in[1.902,2.065]$, $c\in[2.701,3.301]$, $d\in[3.870,8.574]$.

Let us take the element $\Gamma_1=\begin{pmatrix}0.2 & 3.5\\ 5.5 & 8\end{pmatrix}$ from Γ and calculate

$$\|\ell_0X-\Gamma_1\|=5.025>\frac{1}{\|(\ell_0X)^{-1}\|}=2.421.$$

As for estimating difference of a solution x_0 to (3.1) and a solution x to an arbitrary problem from (3.2) with $\ell X\in\Gamma$, first we calculate

$$\Gamma^{-1}=\begin{pmatrix}[-0.637,-0.168] & [0.188,0.279]\\ [0.272,0.359] & [-0.032,-0.010]\end{pmatrix}$$

with $\|\Gamma^{-1}\|_I\leq 0.805$ and

$$\ell_0X-\Gamma^{-1}=\begin{pmatrix}[0.032,0.437,] & [-0.067,0.024]\\ [-0.016,0.071] & [-0.008,0.014]\end{pmatrix},$$

hence $\|(\ell_0X)^{-1}-\Gamma^{-1}\|_I\leq 0.504$. Having in mind the representation

$$x_0-x=X\big[(\ell_0X)^{-1}\ell-(\ell X)^{-1}\ell_0\big]Cf=X\big[(\ell_0X)^{-1}-(\ell X)^{-1}\big]\ell Cf+X\big[(\ell_0X)^{-1}(\ell_0-\ell)\big]Cf,$$

we obtain

$$\|x_0-x\|_{AC_2}\leq 0.504\|X\|_{AC^{2\times 2}}\cdot\|\ell\|_{AC^2\to R^2}\cdot\|Cf\|_{AC^2}+0.414\|X\|\cdot\|\ell_0-\ell\|_{AC^2\to R^2}\cdot\|Cf\|_{AC^2}$$

and, taking into account the estimate $\|X\|_{AC^{2\times 2}} \le 2.188$,

$$\|x_0 - x\|_{AC^2} \le 1.103\,\|\ell\|_{AC^2\to R^2} \cdot \|Cf\|_{AC^2} + 0.906\,\|\ell_0 - \ell\|_{AC^2\to R^2} \cdot \|Cf\|_{AC^2}.$$

Note again that, in this example, the statements of Theorem 2.2 cover the set of problems including those that do not belong to the set defined by Theorem 2.1.

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On Limit Theorems for Solutions of Boundary-Value Problems

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The question of finding the conditions for the convergence of solutions of systems of ordinary differential equations arises in many problems of modern analysis and its applications. It were deeply investigated in the case of the solutions of Cauchy's problems for the system of first-order differential equations. More complicated case of linear boundary-value problems was studied by I. T. Kiguradze [2,3] and his followers [1,4–8].

On a finite interval $(a,b)\subset\mathbb{R}$, we consider the systems of $m\in\mathbb{N}$ linear differential equations of the first order

$$y'(t,n)+A(t,n)y(t,n)=f(t,n) \tag{1}$$

with inhomogeneous boundary conditions

$$B(n)y(\,\cdot\,,n)=c(n), \tag{2}$$

where

$$B(n):C([a,b];\mathbb{C}^m)\to\mathbb{C}^m,\ \ n\in\mathbb{N}\cup\{0\}$$

is a linear continuous operator.

We suppose that the matrix-valued functions $A(\,\cdot\,,n)\in L_1([a,b];\mathbb{C}^{m\times m})$, the vector-valued functions $f(\,\cdot\,,n)\in L_1([a,b];\mathbb{C}^m)$, and the vectors $c(n)\in\mathbb{C}^m$.

The solution of the system of differential equations (1) is understood as a vector-valued function $y(\,\cdot\,)\in W_1^1([a,b];\mathbb{C}^m)$ absolutely continuous on the compact interval $[a,b]$ satisfying the vector equation (1) almost everywhere. The inhomogeneous boundary condition (2) is correctly defined on the solutions of system (1) and cover all classical types of boundary condition. It was shown (see, e.g., [7]) that the boundary-value problem (1), (2) is a Fredholm problem with zero index. For the unique solvability of this problem everywhere, it is necessary and sufficient to guarantee that the corresponding homogeneous boundary-value problem has only a trivial solution.

Assume that the solution of problem (1), (2), with $n=0$, is uniquely defined. Then the following problems are of high importance:

Under what conditions imposed on the left-hand sides of problems (1), (2) their solutions $y(\,\cdot\,,n)$ exist and are unique for sufficiently large $n\in\mathbb{N}$? What additional conditions imposed on the left- and right-hand sides of problems (1), (2) guarantee the limit equality

$$\|y(\,\cdot\,,n)-y(\,\cdot\,,0)\|_\infty\to 0,\ \ n\to\infty, \tag{3}$$

where $\|\cdot\|_\infty$ – sup-norm on the compact interval $[a,b]$.

For the first time, these problems were investigated by Kiguradze [3] in the case of real-valued functions.

We introduce the notation:

$$R_A(\,\cdot\,,n) := A(\,\cdot\,,n) - A(\,\cdot\,,0) \in L_1([a,b];\mathbb{C}^{m\times m}),$$

$$F(\,\cdot\,,n) := \begin{pmatrix} f_1(\,\cdot\,,n) & 0 & \dots & 0 \\ f_2(\,\cdot\,,n) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_m(\,\cdot\,,n) & 0 & \dots & 0 \end{pmatrix} \in L_1([a,b];\mathbb{C}^{m\times m}),$$

$$R_F(\,\cdot\,,n) = F(\,\cdot\,,n) - F(\,\cdot\,,0),$$

$$R_F^{\vee}(t,n) := \int_a^t R_F(s,n)\,ds, \quad R_A^{\vee}(t,n) := \int_a^t R_A(s,n)\,ds.$$

Put also $\|\cdot\|_1$ is the norm in Lebesgue space of vector-valued functions (matrix-valued functions) on the interval $[a,b]$.

Further we assume that all asymptotic relations are considered as $n \to \infty$.

Theorem (Kiguradze [3])**.** *Suppose that*

(0) *the homogeneous boundary-value problem* (1), (2)*, with* $n = 0$*, has only the trivial solution;*

(I) $\|R_A^{\vee}(\,\cdot\,,n)\|_\infty \to 0$*;*

(II) $\|R_A(\,\cdot\,,n)\|_1 = O(1)$*;*

(III) $B(n)y \to B(0)y$*,* $y(\,\cdot\,) \in C([a,b];\mathbb{C}^m)$*.*

Then, for sufficiently large n*, problem* (1), (2) *possesses a unique solution. In addition, if the right-hand sides of problems satisfy the following conditions*

(IV) $c(n) \to c(0)$*;*

(V) $\|R_F^{\vee}(\,\cdot\,,n)\|_\infty \to 0$,

then the unique solutions of problems (1), (2) *satisfy the limit equality* (3)*.*

The examples show that all the conditions of Kiguradze's Theorem are essential and none of them can be omitted. However, some conditions can be weakened.

Denote by $\mathcal{M}^m := \mathcal{M}(a,b;m)$, $m \in \mathbb{N}$ class of sequences of the matrix functions $R(\,\cdot\,,n) : \mathbb{N} \to L_1([a,b];\mathbb{C}^{m\times m})$ such that solution $Z(\,\cdot\,,n)$ of the Cauchy problem

$$Z'(\,\cdot\,,n) + R(\,\cdot\,,n)Z(\,\cdot\,,n) = O, \quad Z(a,n) = I_m$$

satisfies the limit equality

$$\|Z(\,\cdot\,,n) - I_m\|_\infty \to 0,$$

where I_m is an identity $(m\times m)$-matrix.

Put

$$A_F(\,\cdot\,,n) := \begin{pmatrix} A(\,\cdot\,,n) & F(\,\cdot\,,n) \\ O_m & O_m \end{pmatrix} \in L_1([a,b];\mathbb{C}^{2m\times 2m}),$$

$$R_{A_F}(\,\cdot\,,n) := A_F(\,\cdot\,,n) - A_F(\,\cdot\,,0) \in L_1([a,b];\mathbb{C}^{2m\times 2m}),$$

where O_m is a zero $(m\times m)$–matrix.

Theorem 1. *In Kiguradze's Theorem, conditions* (I), (II) can be replaced by one condition

$$R_A(\,\cdot\,,n)\in\mathcal{M}^m, \tag{4}$$

if condition (V) *is replaced by the following*

$$R_{A_F}(\,\cdot\,,n)\in\mathcal{M}^{2m}. \tag{5}$$

Conditions (4), (5) *are very general but not constructive because there are no explicit descriptions of the classes $\mathcal{M}^m$ and $\mathcal{M}^{2m}$.*

However, the results of [4] contain explicit sufficient conditions that the sequence of matrix-valued functions belongs to the class $\mathcal{M}^m$ or $\mathcal{M}^{2m}$. These sufficient conditions are more convenient to use. Therefore, from Theorem 1 follows a number of constructive statements that generalize or complement Kiguradze's Theorem.

Theorem 2. *In Kiguradze's Theorem, condition* (II) *can be replaced by the one more general condition*

(II*) $\|R_A(\,\cdot\,,n)R_A^\vee(\,\cdot\,,n)\|_1\to 0$,

with the additional condition

(VI*) $\|R_A(\,\cdot\,,n)R_F^\vee(\,\cdot\,,n)\|_1\to 0$.

This theorem generalizes Kiguradze's result, since it does not contain the requirement of boundedness of the norms of coefficients of systems.

The advantages of Theorem 2 over Kyguradze's Theorem become more noticeable if we consider their applications to systems of linear differential equations of the higher order of the form

$$y^{(r)}(t,n)+A_{r-1}(t,n)y^{(r-1)}(t,n)+\dots+A_0(t,n)y(t,n)=f(t,n) \tag{6}$$

with inhomogeneous boundary conditions

$$B_j(n)y(\,\cdot\,,n)=c_j(n),\ \ j\in\{1,\dots,r\}:=[r],\ \ n\in\mathbb{N}\cup\{0\}, \tag{7}$$

where $B_j(n):C^{(r-1)}([a,b];\mathbb{C}^m)\to\mathbb{C}^m$ are linear continuous operators with $j\in[r]$.

Assume that the matrix-valued functions $A_{j-1}(\,\cdot\,,n)$, the vector-valued functions $f(\,\cdot\,,n)$ and the vectors $c_j(n)$ satisfy the conditions presented above for problem (1), (2).

A solution of the system of differential equations (6), (7) is understood as a vector-valued function $y(\,\cdot\,,n)\in W_1^r([a,b];\mathbb{C}^m)$ satisfying the equation almost everywhere. The inhomogeneous boundary conditions (7) are correctly defined on the solutions of system (6) and cover all classical types of boundary conditions.

Each of these problems can be reduced to the general inhomogeneous boundary-value problem for the system of equations of the first order. For applied to these problems, Kiguradze's Theorem takes the following form.

Theorem 3. *Suppose that the solutions of problem* (6), (7) *are uniquely defined and*

(I$'$) $\|R^\vee_{A_{j-1}}(\,\cdot\,,n)\|_\infty\to 0$;

(II$'$) $\|R_{A_{j-1}}(\,\cdot\,,n)\|_1=O(1)$;

(III$'$) $B_j(n)y\to B_j(0)y$, $y\in C^{(r-1)}([a,b];\mathbb{C}^m)$.

Then, for sufficiently large n problems (6), (7) *possess the unique solutions. Moreover, if*

(IV′) $c_j(n) \to c_j(0)$,

(V′) $\|R_F^{\vee}(\,\cdot\,, n)\|_\infty \to 0$,

then the unique solutions of problems (6), (7) *satisfy the limit equality*

$$\left\|y^{(j-1)}(\,\cdot\,, 0) - y^{(j-1)}(\,\cdot\,, n)\right\|_\infty \to 0.$$

In this case, from Theorem 2 follows the next result.

Theorem 4. *In Theorem 3, condition* (II′) can be replaced by the condition

(II**) $\|R_{A_{r-1}}(\,\cdot\,, n)R^{\vee}_{A_{j-1}}(\,\cdot\,, n)\|_1 \to 0$,

if the additional condition is fulfilled

(VI**) $\|R_{A_{r-1}}(\,\cdot\,, n)R_F^{\vee}(\,\cdot\,, n)\|_1 \to 0$, $n \to \infty$.

The condition (VI**) *is fulfilled if conditions* (II′), (V′) *hold.*

Note also that conditions (II**), (VI**) are obviously fulfilled if

$$\|R_{A_{r-1}}(\,\cdot\,, n)\|_1 = O(1).$$

At the same time, there are no restrictions on the sequence $\{\|R_{A_{j-1}}(\,\cdot\,, n)\|_1 : \ n \geq 1\}$, with $j \in [r-1]$.

These and other results are presented in more detail in [6–8].

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Two Point Boundary Value Problems for the Fourth Order Ordinary Differential Equations

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We study on the interval $I := [a, b]$ the fourth order ordinary differential equations

$$u^{(4)}(t) = p(t)u(t) + q(t), \tag{0.1}$$

and

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)) + h(t), \tag{0.2}$$

under the boundary conditions

$$u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0 \;\; (j = 0, 1), \tag{0.3$_1$}$$

$$u^{(j)}(a) = 0 \;\; (j = 0, 1, 2), \quad u(b) = 0, \tag{0.3$_2$}$$

where $p, h \in L(I; \mathbb{R})$, $f \in K(I \times \mathbb{R}; \mathbb{R})$.

By a solution of problem $(0.2), (1.3_i)$ $(i \in \{1, 2\})$ we understand a function $u \in \widetilde{C}^3(I; \mathbb{R})$, which satisfies equation (0.2) a.e. on I, and conditions (1.3_i).

Throughout the paper we use the following notations.

$C(I; \mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm

$$\|u\|_C = \max\big\{|u(t)| : \; t \in I\big\}.$$

$\widetilde{C}^{(3)}(I; \mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives.

$L(I; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm

$$\|p\|_L = \int_a^b |p(s)|\, ds.$$

$K(I \times \mathbb{R}; \mathbb{R})$ is the set of functions $f : I \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory conditions, i.e., $f(\,\cdot\,, x) : I \to \mathbb{R}$ is a measurable function for all $x \in \mathbb{R}$, $f(t, \cdot\,) : \mathbb{R} \to \mathbb{R}$ is a continuous function for almost all $t \in I$, and for arbitrary $r > 0$ the inclusion

$$f^*_r(t) := \sup\big\{|f(t, x)| : \;\; |x| \le r\big\} \in L(I; \mathbb{R}^+_0)$$

holds.

For arbitrary $x, y \in L(I;\mathbb{R})$, the notation

$$x(t) \preccurlyeq y(t) \ (x(t) \succcurlyeq y(t)) \ \text{ for } \ t \in I,$$

means that $x \leq y$ $(x \geq y)$ and $x \neq y$.

We also use the notation $[x]_{\pm} = (|x| \pm x)/2$.

The aim of our work is to study the solvability of the above mentioned problems. We have proved the unimprovable sufficient conditions of the unique solvability for the linear problem, which show that the solvability of problem $(0.1), (0.3_1)$ $((0.1), (0.3_2))$ depends only on the nonnegative (non positive) part of the coefficient p if this nonnegative (non positive) part is small enough. On the basis of these results for the nonlinear problems, sufficient conditions of solvability have been proved in non resonance and resonance cases in which nonlinearities can have the linear growth.

Below we present some definitions from the work [2] which we need for the formulation of our results.

Definition 0.1. Equation

$$u^{(4)}(t) = p(t)u(t) \ \text{ for } \ t \in I \tag{0.4}$$

is said to be disconjugate (non-oscillatory) on I if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity.

Definition 0.2. We will say that $p \in D_+(I)$ if $p \in L(I;\mathbb{R}_0^+)$, and problem $(0.4), (0.3_1)$ has a solution u such that

$$u(t) > 0 \ \text{ for } \ t \in \,]a, b[\,. \tag{0.5}$$

Definition 0.3. We will say that $p \in D_-(I)$ if $p \in L(I;\mathbb{R}_0^-)$, and problem $(0.4), (0.3_2)$ has a solution u such that inequality (0.5) holds.

1 Linear problems

The proofs of the following results of the unique solvability of problems $(0.1), (0.3_1)$ and $(0.1), (0.3_2)$ are based on the results from the papers [1] and [2].

Theorem 1.1. *Let $i \in \{1, 2\}$ and the function $p_0 \in L(I;\mathbb{R})$ be such that the equation*

$$u^{(4)}(t) = [p_0(t)]_+ u(t) \ \textit{ if } \ i = 1,$$
$$u^{(4)}(t) = -[p_0(t)]_- u(t) \ \textit{ if } \ i = 2,$$

is diconjugate on I. Then if the inequality

$$(-1)^{i-1}[p(t) - p_0(t)] \leq 0 \ \textit{ for } \ t \in I$$

holds, problem $(0.1), (0.3_i)$ is uniquely solvable.

From the last theorem with $p_0 = [p]_+$ or $p_0 = -[p]_-$ it immediately follows:

Corollary 1.1. *Let there exist $p^* \in D_+(I)$ $(p_* \in D_-(I))$ such that the inequality*

$$[p(t)]_+ \preccurlyeq p^*(t) \quad (\,-[p(t)]_- \succcurlyeq p_*(t)\,) \ \textit{ for } \ t \in I \tag{1.1}$$

holds. Then problem $(0.1), (0.3_1)$ $((0.1), (0.3_2))$ is uniquely solvable.

Remark. Condition (1.1) in Corollary 1.1 is optimal in the sense that the inequality $\preccurlyeq$ $(\succcurlyeq)$ can not be replaced by the inequality $\leq$ $(\geq)$.

2 Nonlinear Problem at the non resonance case

On the basis of our results for the linear problems for the nonlinear problems in non resonance case, i.e. when problem $(0.4),(0.3_i)$ has only the trivial solution, in [3] we have proved the following solvability theorem:

Theorem 2.1. *Let $i\in\{1,2\}$ and there exist $r\in\mathbb{R}^+$ and $g\in L(I;\mathbb{R}_0^+)$ such that a.e. on I the inequality*

$$-g(t)|x|\le(-1)^{i-1}f(t,x)\operatorname{sgn}x\le\delta(t,|x|)\ \ \textit{for}\ \ |x|>r$$

holds, where the function $\delta\in K(I\times\mathbb{R}_0^+;\mathbb{R}_0^+)$ is nondecreasing in the second argument and

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\int\limits_a^b\delta(s,\rho)\,ds=0.$$

Then if the equation

$$\begin{aligned}u^{(4)}(t)&=[p(t)]_+u(t)\ \ \textit{if}\ \ i=1,\\ u^{(4)}(t)&=-[p(t)]_-u(t)\ \ \textit{if}\ \ i=2\end{aligned}$$

is disconjugate, problem $(0.2),(0.3_i)$ has at least one solution.

3 Nonlinear Problem at the resonance

On the basis of Corollary 1.1 and Theorem 2.1 we proved the following Landesman–Laser type sufficient conditions of solvability of problem $(0.4),(0.3_i)$ at the resonance case. It is well known that problem $(0.4),(0.3_i)$ is unique solvable if $(-1)^{i+1}p(t)\le0$. Therefor when we speak about problem (0.2), (0.3_i) at the resonance case we must assume that the condition

$$(-1)^{i+1}p(t)>0\ \ \text{for}\ \ t\in I \tag{3.1}$$

holds.

Theorem 3.1. *Let $i\in\{1,2\}$ the constant $r>0$ and the functions $f^-,f^+,g\in L(I;\mathbb{R}_0^+)$, $p\in L(I;\mathbb{R})$, be such that the conditions* (3.1),

$$p\in D_+(I)\ \ \textit{if}\ \ i=1,\quad p\in D_-(I)\ \ \textit{if}\ \ i=2, \tag{3.2}$$

and

$$\begin{aligned}f^-(t)\le(-1)^{i-1}f(t,x)\le g(t)|x|\ \ &\textit{for}\ \ x<-r,\ \ t\in I,\\ -g(t)|x|\le(-1)^{i-1}f(t,x)\le-f^+(t)\ \ &\textit{for}\ \ x>r,\ \ t\in I\end{aligned}$$

hold. Moreover, let w be a nontrivial solution of homogeneous problem $(0.4),(0.3_i)$ and there exists $\varepsilon>0$ such that the condition

$$-\int\limits_a^b f^-(s)|w(s)|\,ds+\varepsilon\gamma_r\|w\|_C\le(-1)^{i-1}\int\limits_a^b h(s)|w(s)|\,ds\le\int\limits_a^b f^+(s)|w(s)|\,ds-\varepsilon\gamma_r\|w\|_C,$$

holds, where

$$\gamma_r=\int\limits_a^b f_r^*(s)\,ds.$$

Then for an arbitrary function $h\in L(I;\mathbb{R})$ problem $(0.2),(0.3_i)$ is solvable.

Also is true the following existence and uniqueness theorem.

Theorem 3.2. *Let $i \in \{1,2\}$, condition* (3.1), (3.2) *holds and $f(t,0) \equiv 0$. Moreover, let there exist functions $\eta : \mathbb{R}^2 \to]0,+\infty[$, and $g, \ell \in L(I;\mathbb{R}_0^+)$ such that $\ell \not\equiv 0$ and the condition*

$$-g(t)|x_1 - x_2| \le (-1)^{i-1}(f(t,x_1) - f(t,x_2))\,\mathrm{sgn}(x_1 - x_2) \le -\ell(t)\eta(x_1,x_2)|x_1 - x_2|,$$

for $t \in I$, $x_1, x_2 \in \mathbb{R}$ holds, where

$$\lim_{|\rho| \to +\infty} |\rho|\eta(\rho,0) = +\infty.$$

Then for an arbitrary function $h \in L(I;\mathbb{R})$ problem $(0.2), (0.3_i)$ is uniquely solvable.

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Nonlinear Autonomous Boundary-Value Problem for Differential Algebraic System

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We suppose that A and B are $(m\times n)$-measurable matrices and $Z(z,\varepsilon)$ is an n measurable vector function. We will call a weakly nonlinear autonomous periodic differential-algebraic boundary-value problem the problem of finding solutions [6]

$$z(t,\varepsilon):\ \ z(\,\cdot\,,\varepsilon)\in\mathbb{C}^1[a,b(\varepsilon)],\ \ z(t,\,\cdot\,)\in\mathbb{C}[0,\varepsilon_0],\ \ b(0):=b^*$$

of the differential-algebraic system

$$A\,z'=B\,z+\varepsilon Z(z,\varepsilon), \tag{1}$$

satisfying the boundary condition

$$\ell z(\,\cdot\,,\varepsilon)=\alpha. \tag{2}$$

Here, $\ell z(\,\cdot\,,\varepsilon)$ is a linear bounded vector functional

$$\ell z(\,\cdot\,,\varepsilon):\mathbb{C}[a,b(\varepsilon)]\to\mathbb{R}^q.$$

We seek solutions of problem (1), (2) in a small neighborhood of the solution $z_0(t)\in\mathbb{C}^1[a,b^*]$ of the generating Noether $(q\neq n)$ differential-algebraic boundary-value problem

$$A\,z_0'=B\,z_0,\ \ \ell z_0(\,\cdot\,)=\alpha\in\mathbb{R}^q. \tag{3}$$

We assume that the vector function $Z(z,\varepsilon)$ is a continuously differentiable with respect to the unknown $z(t,\varepsilon)$ in a small neighborhood of the solution of the generating problem and continuously differentiable with respect to the small parameter ε in a small positive neighborhood of zero. The matrix A is generally assumed to be rectangular $m\neq n$, or square, but degenerate. Under the condition

$$P_{A^*}=0 \tag{4}$$

the generating system (3) is reduced to the traditional system of ordinary differential equations [2]

$$z_0'=A^+B\,z_0+P_{A_{\rho_0}}\nu_0(t). \tag{5}$$

Moreover, A^+ is a pseudoinverse (by Moore–Penrose) matrix, P_{A^*} is a matrix orthoprojector

$$P_{A^*}:\mathbb{R}^m\to\mathbb{N}(A^*),$$

$P_{A_{\rho_0}}$ is an $(n \times \rho_0)$ matrix formed by ρ_0 linearly independent columns of the $(n \times n)$ matrix orthoprojector

$$P_A : \mathbb{R}^n \to \mathbb{N}(A),$$

$\nu_0(t) \in \mathbb{R}^{\rho_0}$ is an arbitrary continuous vector function. Under the condition (4) system (1) will be called nondegenerate. Suppose that the boundary-value problem for system (3) corresponds to a critical case

$$P_{Q^*} \neq 0, \quad Q := \ell X_0(\,\cdot\,).$$

In the critical case for a fixed vector function $\nu_0(t) \in \mathbb{C}[a, b^*]$ under the condition

$$P_{Q_d^*}\Big\{\alpha - \ell K\big[P_{A_{\rho_0}}\nu_0(s)\big](\,\cdot\,)\Big\} = 0 \tag{6}$$

the generating problem (3) has an r parametric family of solutions [3]

$$z_0(t, c_r) = X_r(t)c_r + G\big[P_{A_{\rho_0}}\nu_0(s)\big](t), \quad c_r \in \mathbb{R}^r.$$

Here, $X_0(t)$ is the normal $(X_0(a) = I_n)$ fundamental matrix of the homogeneous part of the differential system (5). Moreover,

$$G\big[P_{A_{\rho_0}}\nu_0(s)\big](t) := X_0(t)Q^+\ell K\big[P_{A_{\rho_0}}\nu_0(s)\big](\,\cdot\,) + K\big[P_{A_{\rho_0}}\nu_0(s)\big](t)$$

is the generalized Green's operator of the generating periodic differential-algebraic boundary-value problem (3) and

$$K\big[P_{A_{\rho_0}}\nu_0(s)\big](t) := X_0(t)\int_a^t X_0^{-1}(s)\, P_{A_{\rho_0}}\nu_0(s)\, ds$$

is the generalized Green's operator of the Cauchy problem $z(a) = 0$ for the differential-algebraic system (3). The matrix $P_{Q_d^*}$ formed by d linearly independent rows of the matrix orthoprojector P_{Q^*}, and the matrix P_{Q_r} formed by r linearly independent columns of the matrix orthoprojector P_Q. Under condition (4) system (1) is reduced to the traditional system of the ordinary differential equations

$$z' = A^+B\, z + P_{A_{\rho_0}}\nu_0(t) + \varepsilon\, A^+Z(z, \varepsilon). \tag{7}$$

The boundary-value problem for the nondegenerate differential-algebraic system (6) differs significantly from similar nonautonomous boundary-value problems depending on an arbitrary vector function $\nu_0(t) \in \mathbb{C}[a, b^*]$. In exceptional cases, the autonomous boundary-value problem (1), (2) is solvable on a segment of fixed length.

As is known [7], an autonomous boundary-value problem for system (7) differs significantly from similar nonautonomous boundary-value problems. Unlike the latter, the right end $b(\varepsilon)$ of the interval $[a, b(\varepsilon)]$, on which we are finding solution of the nonlinear boundary-value problem for system (7), is unknown and must be defined in the process of constructing the solution itself. Let's use the technique [6, 7] which consists in defining the unknown function

$$b(\varepsilon) = b^* + \varepsilon(b^* - a)\beta(\varepsilon)$$

in terms of the new unknown

$$\beta(\varepsilon) \in \mathbb{C}[0, \varepsilon_0], \quad \beta(0) := \beta^*.$$

The function $\beta(\varepsilon)$ is to be determined in the process of finding a solution of the boundary-value problem for system (7). The essence of the reception is to replace the independent variable

$$t = a + (\tau - a)\big(1 + \varepsilon\beta(\varepsilon)\big)$$

and finding a solution for the nonlinear boundary-value problem (2), (7) and the function $\beta(\varepsilon)$ as a function of a small parameter. In the critical case, under the condition (6) for a fixed function $\nu_0(\tau)$ the condition of solving of the nonlinear boundary-value problem (2), (7) takes the form [6]

$$P_{Q_d^*}\Big\{(1+\varepsilon\beta(\varepsilon))\,\alpha-\ell K\Big[\beta(\varepsilon)\big(A^+B\,z(s,\varepsilon)+P_{A_{\rho_0}}\nu_0(s)\big)+(1+\varepsilon\beta(\varepsilon))\,A^+Z(z(s,\varepsilon),\varepsilon)\Big](\,\cdot\,)\Big\}=0. \quad (8)$$

Using the continuously of the nonlinear vector function $Z(z(t,\varepsilon),\varepsilon)$ on ε in a small positive neighborhood of zero, we pass to the boundary for $\varepsilon \to 0$ in equality (8) and obtain the necessary condition

$$F(\check{c}_0):=P_{Q_d^*}\Big\{\alpha-\ell K\Big[\beta^*\big(A^+B\,z_0(s,c_r^*)+P_{A_{\rho_0}}\nu_0(s)\big)+A^+Z(z_0(s,c_r^*),0)\Big](\,\cdot\,)\Big\}=0 \qquad (9)$$

for the existence of a solution of the boundary-value problem (1), (2) in a critical case. Here,

$$\check{c}_0 := \begin{pmatrix} c_r^* \\ \beta^* \end{pmatrix} \in \mathbb{R}^{r+1}.$$

Thus, the following lemma is proved.

Lemma. *Suppose that the autonomous differential-algebraic boundary-value problem* (1), (2) *for a fixed constant $\nu_0 \in \mathbb{R}^{\rho_0}$ under conditions* (4) *and* (6) *corresponds to the critical case $P_{Q^*} \neq 0$ and has the solution $z(t,\varepsilon)$, that for $\varepsilon = 0$ is transformed into generating $z(t,0) = z_0(t,c_r^*)$. Then the vector $\check{c}_0$ satisfies to equation* (9)*.*

The first r components $c_r^* \in \mathbb{R}^r$ of the root of equation (9) determine the amplitude of the generating solution $z_0(t,c_r^*)$ in a small neighborhood of which can exist the desired solution of the original problem (1), (2). In addition, from equation (9) can be found the value β^* which determines the first approximation to the unknown function

$$b_1(\varepsilon) = b^* + \varepsilon(b^* - a)\beta^*.$$

If equation (9) has no real roots, then the original differential-algebraic problem (1), (2) does not have the desired solutions. Equation (9) will be further called the equation for generating constants of the autonomous nonlinear differential-algebraic boundary-value problem (1), (2). The statement of the lemma generalizes the corresponding results of [1, 5] onto the case of the autonomous nonlinear differential-algebraic boundary-value problem (1), (2), namely, for the case of $A \neq I_n$. As is known [1, 5, 6], the nondegenerate differential-algebraic problem (1) (2) is solvable when the roots of the equation for generating constants (9) are simple. Proposed in the article scheme of study of the nonlinear autonomous boundary-value problem for a nondegenerate system of differential-algebraic equations can be transferred, analogously to [3], onto degenerate systems of differential-algebraic equations. The above-proposed scheme of study of the nonlinear autonomous boundary value problem for a nondegenerate system of differential-algebraic equations can be transferred, analogously to [4], onto systems of differential-algebraic equations with a matrix of variable rank at the derivative, and analogously to [8], onto nonlinear boundary-value problems not solved with respect to the derivative.

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Differential-Algebraic Boundary-Value Problems with Pulse Perturbations with Constant Rank of a Leading Coefficient Matrix

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We construct conditions for the existence of a solution of linear boundary-value problem for a system of differential-algebraic equations with pulse perturbations with constant rank of a leading coefficient matrix.

The problem of constructing solutions [2, 12]

$$z(t) \in \mathbb{C}^1\big\{[a,b] \setminus \{\tau_i\}_I\big\}, \ \ i = 1, 2, \dots, q$$

of the linear differential-algebraic system

$$A(t)z'(t) = B(t)z(t) + f(t), \ \ t \neq \tau_i, \tag{0.1}$$

subject to the boundary condition [5]

$$\ell z(\,\cdot\,) = \alpha, \ \ \alpha \in \mathbb{R}^k. \tag{0.2}$$

was studied. Here,

$$A(t), B(t) \in \mathbb{C}_{m\times n}[a,b]$$

are continuous matrices,

$$f(t) \in \mathbb{C}[a,b]$$

is a continuous vector function; $\ell z(\,\cdot\,)$ is a linear bounded vector functional

$$\ell z(\,\cdot\,) := \sum_{i=0}^{q} \ell_i z(\,\cdot\,) : \mathbb{C}^1\big\{[a,b] \setminus \{\tau_i\}_I\big\} \to \mathbb{R}^k,$$

in addition

$$\ell_i z(\,\cdot\,) : \mathbb{C}^1[\tau_i, \tau_{i+1}[\, \to \mathbb{R}^k, \ \ i = 0, \dots, p-1, \ \ \tau_0 := a,$$

and

$$\ell_q z(\,\cdot\,) : \mathbb{C}^1[\tau_p, b] \to \mathbb{R}^k$$

are linear bounded functionals. The differential-algebraic boundary-value problem (0.1), (0.2) generalizes the traditional formulation of Noetherian boundary-value problems for systems of differential equations with pulse perturbations [2, 5, 6, 11, 12]. The differential-algebraic boundary-value problem (0.1), (0.2) also generalizes the statements of various boundary-value problems for systems of differential-algebraic equations [3, 4].

1 Solvability conditions of a differential-algebraic system with impulse perturbations

Suppose that for the differential-algebraic system (0.1) with a matrix $A(t)$ of constant rank, the requirements of the theorem see, [7, p. 15] are fulfilled. We fix an arbitrary continuous vector function $\nu_p(t) \in \mathbb{C}_{\rho_p}[a,b]$. Substituting the general solution

$$z(t,c) := \begin{cases} X_p(t)\, c_0 + K\big[f(s), \nu_p(s)\big](t), & t \in [a;\tau_1[\,, \\ X_p(t)\, c_1 + K\big[f(s), \nu_p(s)\big](t), & t \in [\tau_1;\tau_2[\,, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ X_p(t)\, c_q + K\big[f(s), \nu_p(s)\big](t), & t \in [\tau_p; b] \end{cases}$$

of the Cauchy problem $z(a) = c$ for the differential-algebraic equation (0.1) into the boundary condition (0.2), we arrive at the linear algebraic equation

$$Q\, c = \alpha - \ell K f(\,\cdot\,). \tag{1.1}$$

Here, P_{Q^*} is orthoprojector

$$\mathbb{R}^k \to \mathbb{N}(Q^*)$$

and matrix $P_{Q_d^*}$ is formed from d independent lines of the orthoprojector P_{Q^*}, in addition,

$$Q := \big(\ell_0 X_p(\,\cdot\,) \ell_1 X_p(\,\cdot\,) \cdots \ell_q X_p(\,\cdot\,)\big) \in \mathbb{R}^{k \times \rho_p(q+1)}.$$

Equation (1.1) is solvable if and only if [1, 2]

$$P_{Q_d^*}\big\{\alpha - \ell K\big[f(s), \nu_p(s)\big](\,\cdot\,)\big\} = 0. \tag{1.2}$$

Under condition (1.2) and only under it, the general solution of equation (0.1)

$$c = Q^+\big\{\alpha - \ell K\big[f(s), \nu_p(s)\big](\,\cdot\,)\big\} + P_{Q_r}\, c_r, \quad c_r \in \mathbb{R}^r$$

determines the general solution of the boundary-value problem (0.1), (0.2)

$$z(t, c_r) = X_r(t) c_r + X(t) Q^+\big\{\alpha - \ell K\big[f(s), \nu_p(s)\big](\,\cdot\,)\big\} + K\big[f(s), \nu_p(s)\big](t), \quad c_r \in \mathbb{R}^r.$$

Here, P_Q is an orthoprojector matrix

$$\mathbb{R}^{\rho_p(q+1)} \to \mathbb{N}(Q);$$

the matrix $P_{Q_r} \in \mathbb{R}^{\rho_p(q+1) \times r}$ is composed of r linearly independent columns of the orthoprojector

$$P_Q := \begin{pmatrix} P_Q^{(0)} \\ P_Q^{(1)} \\ \dots\dots \\ P_Q^{(q)} \end{pmatrix} \in \mathbb{R}^{\rho_p(q+1) \times \rho_p(q+1)},$$

in addition, $c_0, c_1, \dots, c_q \in \mathbb{R}^{\rho_p}$ are constants

$$c := \operatorname{col}(c_0, \dots, c_q) := Q^+\big\{\alpha - \ell K\big[f(s), \nu_p(s)\big](\,\cdot\,)\big\} \in \mathbb{R}^{\rho_p(q+1)}.$$

Thus, the following lemma is proved.

Lemma. *Suppose that the differential-algebraic equation* (0.1) *satisfies the requirements of the theorem in the article* [7, p. 15]. *Under condition* (1.2) *and only under it, for a fixed continuous vector function*

$$\nu_p(t) \in \mathbb{C}_{\rho_p}[a,b],$$

general solution of the differential-algebraic boundary-value problem (0.1), (0.2)

$$z(t,c_r) = X_r(t)\, c_r + G\big[f(s);\nu_p(s);\alpha\big](t), \quad c_r \in \mathbb{R}^r$$

defines the generalized Green's operator of the differential-algebraic boundary-value problem (0.1), (0.2)

$$G\big[f(s);\nu_p(s);\alpha\big](t) := \begin{cases} X_p(t)c_0 + K\big[f(s),\nu_p(s)\big](t), & t \in [a,\tau_1[\,, \\ X_p(t)c_1 + K\big[f(s),\nu_p(s)\big](t), & t \in [\tau_1,\tau_2[\,, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ X_p(t)c_q + K\big[f(s),\nu_p(s)\big](t), & t \in [\tau_p, b]. \end{cases}$$

Here,

$$X_r(t) = \begin{cases} X_p(t)P_Q^{(0)}, & t \in [a,\tau_1[\,, \\ X_p(t)P_Q^{(1)}, & t \in [\tau_1,\tau_2[\,, \\ \dots\dots\dots\dots\dots\dots & \\ X_p(t)P_Q^{(q)}, & t \in [\tau_p, b]. \end{cases}$$

Note that the matrix differential-algebraic boundary-value problem with pulse perturbations, studied in the article [10], is reduced to the form (0.1), (0.2), while in the articles [9–11] the case of a non-degenerate system of the form (0.1) was studied. We also note the essentiality of the requirement of constancy of the rank of the matrix under the derivative [7, 8].

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Optimal Conditions for the Solvability of the Cauchy Weighted Problem for Higher Order Singular in Time and Phase Variables Ordinary Differential Equations

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The theory of the Cauchy problem for ordinary differential equations and systems with non-integrable singularities in the time variable was constructed in the early 1970s (see, e.g., [1] and the references therein). However, the investigation of this problem for singular in phase variables differential equations was started later (see [2]). In [3], unimprovable in a certain sense conditions are established guaranteeing, respectively, the solvability, unique solvability and unsolvability of the Cauchy weighted problem for singular in time and phase variables ordinary delayed differential equations. The results below are refinements of the theorems proved in [3] on the solvability and unsolvability of the Cauchy weighted problem for differential equations without delay.

We use the following notation.

$\mu! = 1$ for $\mu \in]-1,0]$ and $\mu! = \prod\limits_{i=0}^{m}(i+\mu_0)$ for $\mu = m+\mu_0$, where $\mu_0 \in]0,1[$ and m is a nonnegative integer;

$\mathbb{R}_+ = [0,+\infty[\,$, $\mathbb{R}_{0+} =]0,+\infty[\,$;

If n is a natural number, $\alpha \in \mathbb{R}_{0+}$, $x \in \mathbb{R}_{0+}$, and $q :]a,b[\to \mathbb{R}_+$ is a continuous function, satisfying the condition

$$\int\limits_a^t q(s)\,ds < +\infty \ \text{ for } \ a < t < b,$$

then

$$\mathcal{D}_*^{n,\alpha}(\,]a,b[\,;x) = \Big\{(t,x_1,\dots,x_n) \in]a,b[\,\times\mathbb{R}_{0+}^n : \ \ x_i \geq \frac{\alpha!}{(n-i+\alpha)!}\,(t-a)^{n-i+\alpha}x \ \ (i=1,\dots,n)\Big\},$$

$$\mathcal{D}^{n,\alpha}(\,]a,b[\,;x;q)$$
$$= \Big\{(t,x_1,\dots,x_n) \in]a,b[\,\times\mathbb{R}_{0+}^n : \ \ Q^{(i-1)}(t) \leq x_i \leq \frac{\alpha!}{(n-i+\alpha)!}\,(t-a)^{n-i+\alpha}x \ \ (i=1,\dots,n)\Big\},$$

where

$$Q(t) = \frac{1}{(n-i)!}\int\limits_a^t (t-s)^{n-1}q(s)\,ds.$$

Consider the differential equation

$$u^{(n)} = f(t,u,\dots,u^{(n-1)}) \tag{1}$$

with the weighted initial conditions

$$\limsup_{t\to a} \frac{u^{(i-1)}(t)}{(t-a)^{n-i+\alpha}} < +\infty \quad (i=1,\dots,n), \tag{2}$$

where $f :]a,b[\, \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is a continuous function, and α is a positive constant.

We are interested in the case where the function f has singularities in both time and phase variables, i.e. the case, where

$$\int\limits_a^t f(s,x_1,\dots,x_n)\,ds = +\infty \ \text{ for } \ a<t<b, \ \ (x_1,\dots,x_n)\in\mathbb{R}^n_{0+},$$

$$\lim_{x_1+\dots+x_n\to 0} f(t,x_1,\dots,x_n) = +\infty \ \text{ for } \ a<t<b.$$

By a solution of Eq. (1) it is naturally understood an n-times continuously differentiable function $u :]a,b[\, \to \mathbb{R}$, satisfying this equation together with the inequalities

$$u^{(i-1)}(t) > 0 \quad (i=1,\dots,n)$$

in the interval $]a,b[$.

Theorem 1. *Let the function f in the domain $]a,b[\, \times \mathbb{R}^n_{0+}$ admit the estimate*

$$f(t,x_1,\dots,x_n) \geq q(t),$$

where $q :]a,b[\, \to \mathbb{R}_+$ is a continuous function, satisfying the condition

$$x_0 = \limsup_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t q(s)\,ds\bigg) < +\infty.$$

Let, moreover, there exist continuous functions p and $q_0 :]a,b[\, \to \mathbb{R}_+$ such that

$$\limsup_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t p(s)\,ds\bigg) < 1, \quad \limsup_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t q_0(s)\,ds\bigg) < +\infty,$$

and on the set $\mathcal{D}^{n,\alpha}(]a,b[\,;x;q)$ the inequality

$$f(t,x_1,\dots,x_n) \leq p(t)x + q_0(t)$$

holds for any $x > x_0$. Then problem (1), (2) has at least one solution.

The restrictions imposed on the function f in the above theorem are optimal in a certain sense. The following theorem is valid.

Theorem 2. *Let the function f in the domain $\mathcal{D}^{n,\alpha}_*(]a,b[\,;x)$ admit the estimate*

$$f(t,x_1,\dots,x_n) \geq p(t)x + q(t),$$

where p and $q :]a,b[\, \to \mathbb{R}_+$ are continuous functions, satisfying the conditions

$$\int\limits_a^t p(s)\,ds < +\infty, \ \ \int\limits_a^t q(s)\,ds < +\infty \ \textit{ for } \ a<t<b, \quad \liminf_{t\to a}\bigg((t-a)^{-\alpha}\int\limits_a^t q(s)\,ds\bigg) > 0.$$

Let, moreover, either

$$\limsup_{t\to a}\bigg((t-a)^{-\alpha}\int_a^t q(s)\,ds\bigg)=+\infty,$$

or there exist $b_0\in\,]a,b[$ *such that*

$$\int_a^t p(s)\,ds\geq (t-a)^{\alpha}\ \ \text{for}\ \ a\leq t\leq b_0.$$

Then problem (1), (2) *has no solution.*

The two corollaries below of Theorems 1 and 2 concern the case where the function f in the domain $]a,b[\,\times\mathbb{R}^n_{0+}$ admits one of the following two estimates

$$q(t)\leq f(t,x_1,\dots,x_n)\leq \sum_{i=1}^m\Big(p_i(t)\prod_{k=1}^n x_k^{\gamma_{ik}}+q_i(t)\prod_{k=1}^n x_k^{-\lambda_{ik}}\Big)+q_0(t), \tag{3}$$

$$f(t,x_1,\dots,x_n)\geq \sum_{i=1}^m\Big(p_i(t)\prod_{k=1}^n x_k^{\gamma_{ik}}+q_i(t)\prod_{k=1}^n x_k^{-\lambda_{ik}}\Big)+q(t), \tag{4}$$

or Eq. (1) has the form

$$u^{(n)}=\sum_{i=1}^m\Big(p_i(t)\prod_{k=1}^n (u^{(k-i)})^{\gamma_{ik}}+q_i(t)\prod_{k=1}^n (u^{(k-i)})^{-\lambda_{ik}}\Big)+q_0(t). \tag{5}$$

Here and in what follows we assume that m is an arbitrary natural number, γ_{ik}, λ_{ik} ($i=1,\dots,m$; $k=1,\dots,n$) are nonnegative constants, satisfying the conditions

$$\sum_{k=1}^n\gamma_{ik}=1,\quad \sum_{k=1}^n\lambda_{ik}>0\ \ (i=1,\dots,m),$$

and $p_i:\,]a,b[\,\to\mathbb{R}_+$ ($i=1,\dots,m$), $q_j:\,]a,b[\,\to\mathbb{R}_+$ ($j=0,\dots,m$) and $q:\,]a,b[\,\to\mathbb{R}_+$ are continuous functions.

Let

$$\ell_i=\prod_{k=1}^n\Big(\frac{\alpha!}{(n-k+\alpha)!}\Big)^{\gamma_{ik}},\quad \mu_i=\sum_{k=1}^n(n-k+\alpha)\gamma_{ik},\quad \nu_i=\sum_{k=1}^n(n-k+\alpha)\lambda_{ik}\ \ (i=1,\dots,m).$$

Corollary 1. *If along with estimate* (3) *the conditions*

$$\limsup_{t\to a}\bigg(\sum_{i=1}^m\ell_i(t-a)^{-\alpha}\int_a^t (s-a)^{\mu_i}p_i(s)\,ds\bigg)<1, \tag{6}$$

$$\limsup_{t\to a}\bigg((t-a)^{-\alpha}\int_a^t\Big(q_0(s)+\sum_{i=1}^m(s-a)^{-\nu_i}q_i(s)\Big)\,ds\bigg)<+\infty, \tag{7}$$

$$\liminf_{t\to a}\bigg((t-a)^{-\alpha}\int_a^t q(s)\,ds\bigg)>0,\quad \limsup_{t\to a}\bigg((t-a)^{-\alpha}\int_a^t q(s)\,ds\bigg)<+\infty$$

hold, then problem (1), (2) *has at least one solution.*

Corollary 2. *Let the function f admit estimate* (4) *and let, moreover, either the condition*

$$\limsup_{t\to a}\left((t-a)^{-\alpha}\int\limits_a^t\Big(q_0(s)+\sum_{i=1}^m(s-a)^{-\nu_i}q_i(s)\Big)\,ds\right)=+\infty \tag{8}$$

hold or there exist numbers $b_0\in\,]a,b[\,$, $\delta>0$ such that in the interval $]a,b_0[$ the following inequalities are satisfied:

$$\sum_{i=1}^m\ell_i\int\limits_a^t(s-a)^{\mu_i}p_i(s)\,ds\geq(t-a)^{\alpha},\qquad\int\limits_a^t q_0(s)\,ds\geq\delta(t-a). \tag{9}$$

Then problem (1), (2) *has no solution.*

The above corollaries imply the following statements for problem (5), (2).

Corollary 3. *If along with inequalities* (6), (7), *the inequality*

$$\liminf_{t\to a}\left((t-a)^{-\alpha}\int\limits_a^t q_0(s)\,ds\right)>0 \tag{10}$$

holds, then problem (5), (2) *has at least one solution. If condition* (8) *is satisfied or for some $b_0\in\,]a,b[$ and $\delta>0$ inequalities* (9) *hold, then problem* (5), (2) *has no solution.*

Corollary 4. *Let inequality* (10) *hold and let there exist numbers $b_0\in\,]a,b[$ and $\ell\geq0$ such that in the interval $]a,b[$ the following equality*

$$\sum_{i=1}^m\ell_i(t-a)^{\mu_i}p_i(t)=\ell(t-a)^{\alpha-1}$$

is satisfied. Then for the solvability of problem (5), (2) *it is necessary and sufficient that, along with* (7), *the condition*

$$\ell<\alpha$$

be satisfied.

Remark. Let

$$p_i(t)\equiv p_{i0}(t-a)^{-\sum\limits_{k=1}^n(n-k)\gamma_{ik}-1},\quad q_i(t)\equiv q_{i0}(t-a)^{\nu_{0i}}\ \ (i=1,\dots,m),$$
$$q(t)=q_{00}(t-a)^{\alpha-1},$$

where p_{i0}, q_{i0} $(i=1,\dots,m)$, q_{00} are positive constants and

$$\nu_{0i}>\nu-1\ \ (i=1,\dots,m).$$

Then, according to Corollary 4, for the solvability of problem (5), (2) it is necessary and sufficient that the inequality

$$\sum_{i=1}^m\ell_ip_{0i}<\alpha$$

be satisfied.

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Non-Autonomous First Integrals of Autonomous Polynomial Hamiltonian Systems

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1 Introduction

We consider a canonical Hamiltonian ordinary differential system with n degrees of freedom

$$\frac{dq_i}{dt} = \partial_{p_i} H(q,p), \quad \frac{dp_i}{dt} = -\partial_{q_i} H(q,p), \ \ i = 1, \dots, n, \tag{1.1}$$

where $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ and $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ are the generalized coordinates and momenta, $t \in \mathbb{R}$, and the Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a polynomial of degree $h \geqslant 2$.

In this paper, using the Darboux theory of integrability [3, 4] and the notion of partial integral (multiple partial integral, conditional partial integral) [5, 8–11], we study the existence of additional non-autonomous first integrals of the autonomous polynomial Hamiltonian system (1.1).

The Darboux theory of integrability (or the theory of partial integrals) was established by the French mathematician *Jean-Gaston Darboux* [3] in 1878, which provided a link between the existence of first integrals and invariant algebraic curves (or partial integrals) for polynomial autonomous differential systems. For the polynomial differential systems, the Darboux theory of integrability is one of the best theories for studying the existence of first integrals (see [4, 6, 12]).

To avoid ambiguity, we give the following notation and definitions.

The *Poisson bracket* of functions $u, v \in C^1(G)$ on a domain $G \subset \mathbb{R}^{2n}$ is the function

$$\big[u(q,p), v(q,p)\big] = \sum_{i=1}^{n} \big(\partial_{q_i} u(q,p)\, \partial_{p_i} v(q,p) - \partial_{p_i} u(q,p)\, \partial_{q_i} v(q,p)\big) \ \text{ for all } \ (q,p) \in G.$$

We say that [4, p. 20] the linear differential operator of first order

$$\mathfrak{B}(t,q,p) = \partial_t + \sum_{i=1}^{n} \Big(\partial_{p_i} H(q,p)\, \partial_{q_i} - \partial_{q_i} H(q,p)\, \partial_{p_i}\Big) \ \text{ for all } \ (t,q,p) \in \mathbb{R}^{2n+1}$$

is the *operator of differentiation by virtue of the Hamiltonian system* (1.1).

A function $F \in C^1(D)$ is called a *first integral on the domain* $D \subset \mathbb{R}^{2n+1}$ of the Hamiltonian system (1.1) if $\mathfrak{B}\, F(t,q,p) = 0$ or

$$\partial_t F(t,q,p) + \big[F(t,q,p), H(q,p)\big] = 0 \ \text{ for all } \ (t,q,p) \in D.$$

A function $F \in C^1(G)$ is an *autonomous first integral* of the Hamiltonian system (1.1) if the functions F and H are in involution, i.e., $[F(q,p),H(q,p)] = 0$ for all $(q,p) \in G \subset \mathbb{R}^{2n}$. Notice that the Hamiltonian H is an autonomous first integral of the Hamiltonian differential system (1.1).

A set of functionally independent on $D \subset \mathbb{R}^{2n+1}$ first integrals $F_l \in C^1(D)$, $l = 1,\dots,k$, of the Hamiltonian system (1.1) is called a *basis of first integrals* (or *integral basis*) on the domain D of system (1.1) if any first integral $F \in C^1(D)$ of system (1.1) can be represented on D in the form

$$F(t,q,p) = \Phi\big(F_1(t,q,p),\dots,F_k(t,q,p)\big) \ \text{ for all } \ (t,q,p) \in D,$$

where Φ is some continuously differentiable function. The number k is said to be the *dimension* of basis of first integrals on the domain D for the Hamiltonian differential system (1.1).

The Hamiltonian differential system (1.1) on an neighbourhood of any point from the domain D has a basis of first integrals of dimension $2n$ (see, for example, [4, p. 54]). Besides, the autonomous Hamiltonian differential system (1.1) on a domain G without equilibrium points has an autonomous integral basis of dimension $2n-1$ [1, pp. 167–169].

A polynomial w is a *partial integral* of the Hamiltonian system (1.1) if the Poisson bracket

$$\big[w(q,p),H(q,p)\big] = w(q,p)M(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \tag{1.2}$$

where the polynomial M (*cofactor* of the partial integral w) such that $\deg M \leqslant h-2$.

A partial integral w with cofactor M of the Hamiltonian system (1.1) is said to be *multiple* with multiplicity

$$\varkappa = 1 + \sum_{\xi=1}^{\varepsilon} r_\xi$$

if there exist natural numbers f_ξ and polynomials

$$Q_{f_\xi g_\xi}, \ \ g_\xi = 1,\dots,r_\xi, \ \ \xi = 1,\dots,\varepsilon,$$

such that on the domain $G \subset \{(q,p): \ w(q,p) \neq 0\}$ the identities hold

$$\left[\frac{Q_{f_\xi g_\xi}(q,p)}{w^{f_\xi}(q,p)}\,, H(q,p)\right] = R_{f_\xi g_\xi}(q,p), \ \ g_\xi = 1,\dots,r_\xi, \ \ \xi = 1,\dots,\varepsilon, \tag{1.3}$$

where the polynomials $R_{f_\xi g_\xi}$ have degrees at most $h-2$. Note that a similar point of view on multiplicity of partial integrals was presented by J. Llibre and X. Zhang in [7].

An exponential function $\omega(q,p) = \exp v(q,p)$ for all $(q,p) \in \mathbb{R}^{2n}$ with some real polynomial v is called a *conditional partial integral* of the Hamiltonian system (1.1) if the Poisson bracket

$$\big[v(q,p),H(q,p)\big] = S(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \tag{1.4}$$

where the polynomial S (*cofactor* of the conditional partial integral ω) such that $\deg S \leqslant h-2$.

We stress that a conditional partial integral is a special case of exponential factor (or exponential partial integral) [2, 5, 6] for the polynomial Hamiltonian ordinary differential system (1.1).

2 Main results

The general results of this paper are formulated in Theorems 2.1–2.3.

Theorem 2.1. *If the Hamiltonian system* (1.1) *has the partial integral w with cofactor*

$$M(q,p) = \lambda \ \textit{for all} \ (q,p) \in \mathbb{R}^{2n}, \ \ \lambda \in \mathbb{C} \setminus \{0\}, \tag{2.1}$$

then an non-autonomous first integral of the autonomous Hamiltonian system (1.1) *is the function*

$$F(t,q,p) = \ w(q,p) \exp\big(-\lambda t\big) \ \textit{for all} \ (t,q,p) \in \mathbb{R}^{2n+1}.$$

Proof. Using the identity (1.2) under the condition (2.1), we have

$$\begin{aligned}\mathfrak{B}\, F(t,q,p) = \partial_t F(t,q,p) + \big[F(t,q,p), H(q,p)\big] = F(t,q,p)\, \partial_t\big(-\lambda t\big) \\ + \exp\big(-\lambda t\big)\, \big[w(q,p), H(q,p)\big] = 0 \ \text{ for all } \ (t,q,p) \in \mathbb{R}^{2n+1}.\end{aligned}$$

Therefore the function F is a first integral of the autonomous Hamiltonian system (1.1). □

For example, the autonomous polynomial Hamiltonian differential system given by

$$H(q,p) = \frac{1}{2}\big(p_1^2 + p_2^2 - q_1^2 - q_2^2\big) \ \text{ for all } \ (q,p) \in \mathbb{R}^4 \tag{2.2}$$

has the polynomial partial integrals

$$w_1(q,p) = q_1 - p_1, \quad w_2(q,p) = q_2 - p_2, \quad w_3(q,p) = q_1 + p_1, \quad w_4(q,p) = q_2 + p_2$$

with cofactors

$$M_1(q,p) = M_2(q,p) = -1, \quad M_3(q,p) = M_4(q,p) = 1 \ \text{ for all } \ (q,p) \in \mathbb{R}^4.$$

By Theorem 2.1, we can build the non-autonomous first integrals of the Hamiltonian system (2.2)

$$\begin{gathered} F_1(t,q,p) = (q_1 - p_1)e^{t}, \ \ F_2(t,q,p) = (q_2 - p_2)e^{t}, \\ F_3(t,q,p) = (q_1 + p_1)e^{-t}, \ \ F_4(t,q,p) = (q_2 + p_2)e^{-t}. \end{gathered}$$

The functionally independent non-autonomous first integrals $F_1, \ldots, F_4$ are an integral basis (non-autonomous) of the autonomous Hamiltonian system (2.2) on the space $\mathbb{R}^5$.

Theorem 2.2. *Suppose the polynomial Hamiltonian differential system* (1.1) *has the partial integral w with multiplicity*

$$\varkappa = 1 + \sum_{\xi=1}^{\varepsilon} r_\xi.$$

If the identity (1.3) *under some numbers $\xi \in \{1, \ldots, \varepsilon\}$ and $g_\xi \in \{1, \ldots, r_\xi\}$ such that the polynomial*

$$R_{f_\xi g_\xi}(q,p) = \lambda \ \textit{for all} \ (q,p) \in G \subset \mathbb{R}^{2n}, \ \ \lambda \in \mathbb{R}, \tag{2.3}$$

then an non-autonomous first integral of the autonomous Hamiltonian system (1.1) *is the function*

$$F(t,q,p) = K_{f_\xi g_\xi}(q,p) \ - \ \lambda t \ \textit{for all} \ (t,q,p) \in \mathbb{R} \times G.$$

Proof. Taking into account the identity (1.3) under the condition (2.3), we obtain

$$\mathfrak{B}\,F(t,q,p)=\partial_t F(t,q,p)+\big[F(t,q,p),H(q,p)\big]=-\partial_t\big(\lambda t\big)+\big[K_{f_\xi g_\xi}(q,p),H(q,p)\big]=0.$$

For example, the autonomous polynomial Hamiltonian differential system given by [9]

$$H(q,p)=-q_1^2+6q_1q_2+(2p_1+p_2)q_1+2q_2p_2+3p_2^2 \ \text{ for all } \ (q,p)\in\mathbb{R}^4 \tag{2.4}$$

has the multiple partial integral $w_1(q,p)=3q_1+2p_2$ for all $(q,p)\in\mathbb{R}^4$ with

$$M_1(q,p)=-2,\quad K_{1,11}(q,p)=\frac{17q_1+12q_2+8p_1}{32(3q_1+2p_2)}\,,\quad R_{1,11}(q,p)=1,$$

and the multiple partial integral $w_2(q,p)=q_1$ for all $(q,p)\in\mathbb{R}^4$ with

$$M_2(q,p)=2,\quad K_{2,11}(q,p)=\frac{2q_2+3p_2}{16q_1}\,,\quad R_{2,11}(q,p)=-1.$$

Using Theorems 2.1 and 2.2, we can construct the basis (non-autonomous) of first integrals on a domain $\mathbb{R}\times G$, $G\subset G_1\cap G_2$, for the autonomous polynomial Hamiltonian system (2.4)

$$F_1(t,q,p)=(3q_1+2p_2)\,e^{2t},\quad F_2(t,q,p)=\frac{17q_1+12q_2+8p_1}{16(3q_1+2p_2)}-t,\quad G_1\subset\big\{(q,p):\ 3q_1+2p_2\neq 0\big\},$$
$$F_3(t,q,p)=q_1\,e^{-2t},\quad F_4(t,q,p)=\frac{2q_2+3p_2}{8q_1}+t,\quad G_2\subset\big\{(q,p):\ q_1\neq 0\big\}.$$

Notice also that the functionally independent autonomous first integrals (see [9])

$$\begin{aligned}
W_1(q,p)&=(3q_1+2p_2)\exp\Big(\frac{17q_1+12q_2+8p_1}{16(3q_1+2p_2)}\Big)\ \text{ for all } \ (q,p)\in G_1,\\
W_2(q,p)&=q_1\exp\Big(\frac{2q_2+3p_2}{8q_1}\Big)\ \text{ for all } \ (q,p)\in G_2,\\
W_3(q,p)&=\frac{17q_1+12q_2+8p_1}{32(3q_1+2p_2)}+\frac{2q_2+3p_2}{16q_1}\ \text{ for all } \ (q,p)\in G
\end{aligned}$$

of system (2.4) are an autonomous integral basis of the Hamiltonian system (2.4) on any domain G. □

Theorem 2.3. *Suppose the polynomial Hamiltonian differential system* (1.1) *has the conditional partial integral* ω. *If the identity* (1.4) *such that the polynomial*

$$S(q,p)=\lambda \ \textit{ for all } \ (q,p)\in\mathbb{R}^{2n},\ \ \lambda\in\mathbb{R}\setminus\{0\}, \tag{2.5}$$

then the Hamiltonian system (1.1) *has the non-autonomous first integral*

$$F(t,q,p)=v(q,p)-\lambda t \ \textit{ for all } \ (t,q,p)\in\mathbb{R}^{2n+1}.$$

Proof. Using the identity (1.4) under the condition (2.5), we get

$$\mathfrak{B}\,F(t,q,p)=\partial_t F(t,q,p)+\big[F(t,q,p),H(q,p)\big]=\partial_t\big(-\lambda t\big)+\big[v(q,p),H(q,p)\big]=0.$$ □

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Necessary Solvability Conditions for Non-Linear Integral Boundary Value Problems

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We study the following non-linear integral boundary value problem

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad t \in [a, b], \qquad \int\limits_a^b g(s, x(s))\, ds = d, \tag{1}$$

where $f \in C([a, b] \times D; \mathbb{R}^n)$, $g \in C([a, b] \times D; \mathbb{R}^n)$, $d \in \mathbb{R}^n$ is a given vector and the domain $D \subset \mathbb{R}^n$ will be specified later (See, (7), (8)). Moreover, we suppose that $f \in Lip(K, D)$, $g \in Lip(K_g, D)$, i.e., f and g locally Lipsichitzian

$$\begin{gathered} |f(t, u) - f(t, v)| \le K|u - v|, \ \text{ for all } \ \{u, v\} \subset D \ \text{ and } \ t \in [a, b], \\ |g(t, u) - g(t, v)| \le K_g|u - v|, \ \text{ for all } \ \{u, v\} \subset D \ \text{ and } \ t \in [a, b]. \end{gathered} \tag{2}$$

To study the BVP (1) we will use an approach similar to that of [1].

For vectors $x = col(x_1, \dots, x_n) \in \mathbf{R}^n$ the notation $|x| = col(|x_1|, \dots, |x_n|)$ is used and the inequalities between vectors are understood componentwise. The same convention is adopted for operations like "max" and "min". For any non-negative vector $\rho \in \mathbf{R}^n$ under the componentwise ρ-neighbourhood of a point $z \in \mathbf{R}^n$ we understand the set

$$O_\rho(z) := \big\{\xi \in \mathbf{R}^n : \ \ |\xi - z| \le \rho\big\}. \tag{3}$$

Similarly, the ρ-neighbourhood of a domain $\Omega \subset \mathbf{R}^n$ is defined as

$$O_\rho(\Omega) := \bigcup_{z \in \Omega} O_\rho(z). \tag{4}$$

A particular kind of vector ρ will be specified below in relations (7), (8).

I_n is the identity matrix of dimension n. $r(K)$ is the maximal, in modulus, eigenvalue of the matrix K. We also assume that

$$r(Q) < 1, \ \ Q = \frac{3(b - a)}{10}\, K. \tag{5}$$

Let us choose certain compact convex sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, and define the set

$$D_{a,b} := (1-\theta)z + \theta\eta, \quad z \in D_a, \quad \eta \in D_b, \quad \theta \in [0,1] \tag{6}$$

and according to (4) its ρ-neighbourhood

$$D = O_\rho(D_{a,b}) \tag{7}$$

with a non-negative vector $\rho = col(\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ such that

$$\rho \geq \frac{b-a}{2}\, \delta_{[a,b],D}(f), \tag{8}$$

where $\delta_{[a,b],D}(f)$ denotes the 1/2 of oscillation of function f over $[a,b] \times D \times D$

$$\delta_{[a,b],D}(f) := \frac{\max\limits_{(t,x)\in[a,b]\times D} f(t,x) - \min\limits_{(t,x)\in[a,b]\times D} f(t,x)}{2}. \tag{9}$$

Instead of the original boundary value problem (1) we will consider the family of auxiliary two-point parametrized boundary value problems

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad t \in [a,b], \tag{10}$$

$$x(a) = z, \quad x(b) = \eta, \tag{11}$$

where z and η are treated as free parameters.

Let us connect with problem (10), (11) the sequence of functions

$$\begin{aligned} x_{m+1}(t,z,\eta) = z &+ \int_a^t f(s, x_m(s,z,\eta))\, ds \\ &- \frac{t-a}{b-a}\int_a^b f(s, x_m(s,z,\eta))\, ds + \frac{t-a}{b-a}\,[\eta - z], \quad t \in [a,b], \quad m = 0,1,2,\ldots, \end{aligned} \tag{12}$$

satisfying (11) for arbitrary $z, \eta \in \mathbb{R}^n$, where

$$x_0(t,z,\eta) = z + \frac{t-a}{b-a}\,[\eta - z] = \Big(1 - \frac{t-a}{b-a}\Big)z + \frac{t-a}{b-a}\,\eta, \quad t \in [a,b]. \tag{13}$$

It is easy to see from (13) that $x_0(t,z,\eta)$ is a linear combination of vectors z and η, when $z \in D_a$, $\eta \in D_b$.

We have previously proved the following statements.

Theorem 1 (Uniform convergence)**.** *Let conditions* (2), (5), (8) *be fulfilled.*

Then, for all fixed $(z,\eta) \in D_a \times D_b$ *we have*

1. *The functions of sequence* (12) *belonging to the domain* D *of form* (7) *are continuously differentiable on the interval* $[a,b]$ *and satisfy conditions* (11)*.*

2. *The sequence of functions* (12) *for* $t \in [a,b]$ *converges uniformly as* $m \to \infty$ *with respect to the domain* $(t,z,\eta) \in [a,b] \times D_a \times D_b$ *to the limit function*

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta), \tag{14}$$

satisfying conditions (11)*.*

3. *The function* $x_\infty(t,z,\eta)$ *for all* $t\in[a,b]$ *is a unique continuously differentiable solution of the integral equation*

$$x(t)=z+\int\limits_a^t f(s,x(s))\,ds-\frac{t-a}{b-a}\int\limits_a^b f(s,x(s))\,ds+\frac{t-a}{b-a}\,[\eta-z], \tag{15}$$

i.e., it is the solution to the Cauchy problem for the modified system of integro-differential equations

$$\frac{dx}{dt}=f(t,x(t))+\frac{1}{b-a}\,\Delta(z,\eta),\ \ x(a)=z, \tag{16}$$

where $\Delta(z,\eta):D_a\times D_b\to\mathbb{R}^n$ *is a mapping given by the formula*

$$\Delta(z,\eta)=[\eta-z]-\int\limits_a^b f(s,x_\infty(s,z,\eta))\,ds. \tag{17}$$

4. *The error estimation*

$$\big|x_\infty(t,z,\eta)-x_m(t,z,\eta)\big|\leqslant\frac{10}{9}\,\alpha_1(t)Q^m(1_n-Q)^{-1}\delta_{[a,b],D}(f),\ \ t\in[a,b],\ \ m\geq 0 \tag{18}$$

holds, where

$$\alpha_1(t)=2(t-a)\Big(1-\frac{t-a}{b-a}\Big)\leq\frac{b-a}{2}\,,\ \ t\in[a,b].$$

Theorem 2 (Relation $x_\infty(t,z,\eta)$ to the solution of the original boundary value problem (1))**.** *Under the assumptions of Theorem 1, the limit function* $x_\infty(t,z,\eta)=\lim\limits_{m\to\infty}x_m(t,z,\eta)$ *of sequence* (12) *is a solution to the integral boundary value problem* (1) *if and only if the pair of vector-parameters* (z,η) *satisfies the system of* $2n$ *determining algebraic equations*

$$\Delta(z,\eta):=[\eta-z]-\int\limits_a^b f(s,x_\infty(s,z,\eta))\,ds=0,\quad \Lambda(z,\eta)=\int\limits_a^t g(s,x_\infty(s,z,\eta))\,ds=d. \tag{19}$$

On the base of m*th approximate determining equations*

$$\Delta_m(z,\eta):=[\eta-z]-\int\limits_a^b f(s,x_m(s,z,\eta))\,ds=0,\quad \Lambda_m(z,\eta)=\int\limits_a^t g(s,x_m(s,z,\eta))\,ds=d \tag{20}$$

introduce the mapping $H_m:D_a\times D_b\to\mathbb{R}^{2n}$

$$H_m(z,\eta)=\begin{bmatrix}[\eta-z]-\int\limits_a^b f(s,x_\infty(s,z,\eta))\,ds\\ \Lambda_m(z,\eta)-d\end{bmatrix}. \tag{21}$$

Theorem 3 (Sufficient conditions for the solvability of the integral boundary value problem (1))**.** *Assume that the conditions of Theorem 1 hold. Moreover, one can specify an* $m\geq 1$ *and set*

$\Omega \subset \mathbb{R}^{2n}$ of the form $\Omega := D_1 \times D_2$, where $D_1 \sqsubseteq D_a$, $D_2 \sqsubseteq D_b$ are certain bounded open sets, such that the mapping H_m, satisfies the relation

$$|H_m(z,\eta)| \rhd_{\partial\Omega} \begin{bmatrix} \dfrac{10(b-a)^2}{27}\, KQ^m(I_n-Q)^{-1}\delta_{[a,b],D}(f) \\ \dfrac{5(b-a)}{9}\, K_g Q^m(I_n-Q)^{-1}\delta_{[a,b],D}(f) \end{bmatrix} \tag{22}$$

on the boundary $\partial\Omega$, where the binary relation $\rhd_{\partial\Omega}$ in (22) means that for all $(z,\eta) \in \partial\Omega$ at least one of the components $k(z,\eta)$ of the vector $H_m(z,\eta)$ is greater than the corresponding component of the right hand side vector in (22). (One can see, that the number $k(z,\eta)$ of components depends on the point $(z,\eta) \in \partial\Omega$.)

If, in addition, the Brouwer's degree of the mapping H_m does not equal to zero, i.e.,

$$\deg(H_m, \Omega, 0) \neq 0, \tag{23}$$

then there exists a pair (z^*, η^*) from $D_1 \times D_2$ for which the function $x^*(\cdot) = x_\infty(\cdot, z^*, \eta^*)$ is a continuously differentiable solution to the boundary value problem (1), where $x_\infty(t, z^*, \eta^*) = \lim\limits_{m\to\infty} x_m(t, z^*, \eta^*)$, $t \in [a,b]$.

In order to verify condition (22) of Theorem 3 one has to use the recurrence formula (12) to compute the function $x_m(\cdot, z, \eta)$ analytically, depending on the parameters z and η, at every point $(z,\eta) \in \partial\Omega$, verify whether at least one component of the $2n$-dimensional vector $|H_m(z,\eta)|$ is strictly greater than the corresponding component of the vector at right hand side of (22). Verification of the validity of (23) is a rather difficult problem in general. But in the smooth case, it follows directly from the definition of the topological degree, that if the Jacobian matrix of the function H_m in (21) is non-singular at its isolated zero (z_m^0, η_m^0), i.e.,

$$\det \frac{\partial}{\partial(z,\eta)}\, H_m(z_m^0, \eta_m^0) \neq 0,$$

then inequality (23) holds. The symbol $\frac{\partial}{\partial(z,\eta)}$ means the derivative with respect to the vector of variables $(z_1, \ldots, z_n, \eta_1, \ldots, \eta_n)$.

We proved the following lemma about the continuous dependence of the limit function $x_\infty(\cdot, z, \eta)$ and determining functions $\Delta(z,\eta), \Lambda(z,\eta)$ defined in (19) with respect to parameters $(z,\eta) \in D_a \times D_b$.

Lemma 1. *Let the assumptions of Theorem 1 be satisfied for the integral boundary value problem (1). Then for arbitrary pairs of parameters $(z', \eta') \in D_a \times D_b$ and $(z'', \eta'') \in D_a \times D_b$, the limit functions $x'_\infty(\cdot, z', \eta')$, $x''_\infty(\cdot, z'', \eta'')$ of sequence (12) for $t \in [a,b]$ satisfy the following Lipschitz-type condition*

$$\left|x'_\infty(\cdot, z', \eta') - x''_\infty(\cdot, z'', \eta'')\right| \leq \Big[I_n + \frac{10}{9}\,\alpha_1(\cdot) K(I_n-Q)^{-1}\Big]\,\big[\,|z'-z''| + |\eta'-\eta''|\,\big]. \tag{24}$$

Formulas (19) determine well defined functions $\Delta(z,\eta) : \mathbf{R}^{2n} \to \mathbf{R}^n$ and $\Lambda(z,\eta) : \mathbf{R}^{2n} \to \mathbf{R}^n$, which in addition satisfy the following Lipschitz-type estimates

$$\left|\Delta(z',\eta') - \Delta(z'',\eta'')\right| \leq \Big[I_n + \Big((b-a)K + \frac{10}{27}\,(b-a)^2 K(I_n-Q)^{-1}\Big)\Big]\,\big[\,|z'-z''| + |\eta'-\eta''|\,\big],$$

$$\left|\Lambda(z',\eta') - \Lambda(z'',\eta'')\right| \leq \Big[\Big((b-a)K_g + \frac{10}{27}\,K_g(b-a)^2 K(I_n-Q)^{-1}\Big)\Big]\,\big[\,|z'-z''| + |\eta'-\eta''|\,\big].$$

The following statement gives a condition which is necessary for the domain

$$\Omega = G_a \times G_b, \ \ G_a \sqsubseteq D_a, \ \ G_b \sqsubseteq D_b \tag{25}$$

to contain a pair of parameters (z^*, η^*) determining the solution

$$x(\,\cdot\,) = x_\infty(\,\cdot\,, z^*, \eta^*) = \lim_{m\to\infty} x_m(\,\cdot\,, z^*, \eta^*)$$

of the given integral boundary value problem (1).

Theorem 4. *Let the assumptions of Theorem 1 be satisfied for the integral boundary value problem* (1)*. Then for domain* (25) *to contain a pair of parameters* (z^*, η^*) *determining the solution* $x(\,\cdot\,)$ *of the given integral boundary value problem at the points* $t = a$ *and* $t = b$

$$x(a) = z^* \ \text{ and } \ x(b) = \eta^*,$$

it is necessary that for all m *and arbitrary* $\widetilde{z} \in G_a$, $\widetilde{\eta} \in G_b$ *to be true for the approximate determining functions the following inequalities*

$$\begin{aligned}
\Delta_m(\widetilde{z}, \widetilde{\eta}) \le \sup_{z\in G_a,\ \eta\in G_b} &\bigg[I_n + \Big((b-a)K + \frac{10}{27}\,(b-a)^2K(I_n-Q)^{-1}\Big)\bigg]\,\big[\,|z'-z''| + |\eta'-\eta''|\,\big]\\
&+ \frac{10}{27}\,(b-a)^2KQ^m(1_n-Q)^{-1}\delta_{[a,b],D}(f),\\
\Lambda_m(\widetilde{z}, \widetilde{\eta}) \le \sup_{z\in G_a,\ \eta\in G_b} &\bigg[\Big((b-a)K_g + \frac{10}{27}\,K_g(b-a)^2K(I_n-Q)^{-1}\Big)\bigg]\,\big[\,|z'-z''| + |\eta'-\eta''|\,\big]\\
&+ \frac{10}{27}\,(b-a)^2K_gQ^m(1_n-Q)^{-1}\delta_{[a,b],D}(f).
\end{aligned}$$

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Definition of Total Wandering and Total Nonwandering of a Differential System and their Study at the First Approximation

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For a given zero neighborhood G in the Euclidean space $\mathbb{R}^n$, we consider a nonlinear, generally speaking, differential system of the form

$$\dot{x} = f(t,x), \quad f(t,0) = 0, \ \ t \in \mathbb{R}_+ \equiv [0,+\infty), \ \ x \in G, \tag{1}$$

where the right-hand side satisfies the condition $f, f'_x \in C(\mathbb{R}_+ \times G)$ and the zero solution is allowed.

We associate with system (1) the linear homogeneous system of its *first approximation*

$$\dot{x} = A(t)x \equiv f_{\prime}(t,x), \quad A(t) \equiv f'_x(t,0), \ \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R}^n, \tag{2}$$

for which we do not require here the uniformity in $t \in \mathbb{R}_+$ of the natural (pointwise) smallness of the nonlinear addition

$$h(t,x) \equiv f(t,x) - A(t)x = o(x), \ \ x \to 0.$$

Denote by $x_f(\,\cdot\,, x_0)$ and $S_*(f)$ or $S_\delta(f)$ a *non-extendable* solution of system (1) with the initial condition $x_f(0,x_0) = x_0$ and sets of solutions with initial values x_0, satisfying the conditions $|x_0| \neq 0$ or, respectively, $0 < |x_0| < \delta$.

Definition 1. *Wandering functional* $\mathrm{P}(u,t)$, defined for numbers $t \in \mathbb{R}_+$ and continuously-differentiable functions $u : [0,t] \to \mathbb{R}^n \setminus \{0\}$, is given by the formula

$$\mathrm{P}(t,u) \equiv \int\limits_0^t \Big| \Big(\frac{u(\tau)}{|u(\tau)|} \Big)^{\cdot} \Big| \, d\tau, \ \ \tau \in [0,t],$$

adding that whenever the function u is not defined on the entire segment $[0,t]$, it takes the value $+\infty$. For each system (1), momentum $t \in \mathbb{R}_+$, and non-degenerate transformation $L \in \operatorname{Aut}\mathbb{R}^n$ we define the values of the *lower* and the *upper ball wandering functionals*, given respectively by the equalities

$$\check{\mathrm{P}}_b(f,t,L) \equiv \varliminf_{x_0 \to 0} \mathrm{P}(t, Lx_f(\,\cdot\,,x_0)), \quad \hat{\mathrm{P}}_b(f,t,L) \equiv \varlimsup_{x_0 \to 0} \mathrm{P}(t, Lx_f(\,\cdot\,,x_0)). \tag{3}$$

Lower weak $\check{\rho}_b^{\circ}(f)$ and *strong* $\check{\rho}_b^{\bullet}(f)$ *ball wandering indicators* of system (1) are given by the formulas

$$\check{\rho}_b^{\circ}(f) \equiv \varliminf_{t \to +\infty} \inf_{L \in \operatorname{Aut}\mathbb{R}^n} t^{-1} \check{\mathrm{P}}_b(f,t,L), \quad \check{\rho}_b^{\bullet}(f) \equiv \inf_{L \in \operatorname{Aut}\mathbb{R}^n} \varliminf_{t \to +\infty} t^{-1} \check{\mathrm{P}}_b(f,t,L), \tag{4}$$

and *upper weak* $\hat{\rho}_b^{\circ}(f)$ and *strong* $\hat{\rho}_b^{\bullet}(f)$ *ball wandering indicators* – by the same formulas (4) respectively, but with the upper limits at $t \to +\infty$ instead of the lower ones

$$\hat{\rho}_b^{\circ}(f) \equiv \varlimsup_{t \to +\infty} \inf_{L \in \operatorname{Aut}\mathbb{R}^n} t^{-1} \hat{\mathrm{P}}_b(f,t,L), \quad \hat{\rho}_b^{\bullet}(f) \equiv \inf_{L \in \operatorname{Aut}\mathbb{R}^n} \varlimsup_{t \to +\infty} t^{-1} \hat{\mathrm{P}}_b(f,t,L). \tag{5}$$

The indicators $\check{\rho}_b^{\circ}(f)$ and $\hat{\rho}_b^{\bullet}(f)$ turn out to be respectively the smallest and the largest of four ball wandering indicators (4), (5) of system (1) introduced in Definition 1.

Other functionals are also known that are responsible for similar properties of solutions not related to their norm (see, for example, [1–3]): the *oscillation* or the *oriented, non-oriented, frequency* and *flat rotation*, as well as the *rotation* of the given *rank*. In addition to the ball indicators, we can also consider the *spherical* or the *radial* ones [4].

The total wandering of a differential system defined below (near its zero solution, which we will not mention further for brevity) remotely resembles Lyapunov stability. In contrast to stability, wandering does not mean that all solutions that start close enough to zero remain forever in its given neighborhood, but that their average (in time) angular velocity is positive and even separated from zero (uniformly in all these solutions at once). However, in the nonlinear case, the matter is complicated by the fact that the solutions mentioned may not be defined on the entire time semiaxis. The situation is similar with complete nonwandering.

Definition 2. We say that system (1) has:

1) *complete wandering* if there exist $\varepsilon > 0$ and $T \in \mathbb{R}_+$ such that for each $L \in \operatorname{Aut}\mathbb{R}^n$ and $t > T$ the estimate holds
$$\check{\mathrm{P}}(f,t,L) > \varepsilon t;$$

2) *complete nonwandering* if for any $\varepsilon > 0$ there exist $T \in \mathbb{R}_+$ and $L \in \operatorname{Aut}\mathbb{R}^n$, that for every $t > T$ the estimate holds
$$\hat{\mathrm{P}}(f,t,L) < \varepsilon t.$$

Whether a system is completely wandering or nonwandering is uniquely determined by the signs of its corresponding ball wandering indicators.

Theorem 1. *The complete wandering and the complete nonwandering of system* (1) *are equivalent to the positiveness of its lower ball wandering indicator*
$$\check{\rho}_b^{\circ}(f) > 0$$
and, respectively, to the equality to zero of its upper ball wandering indicator
$$\hat{\rho}_b^{\bullet}(f) = 0.$$

All the ball wandering indicators of a system coincide with the corresponding indicators of the system of its first approximation (which are calculated much easier, since in the case of a linear system in formulas (3) the lower and upper limits at $x_0 \to 0$ can be replaced by the exact lower and upper bounds over all $x_0 \neq 0$, respectively).

Theorem 2. *For any system* (1) *and system* (2) *of its first approximation, the equalities hold*
$$\tilde{\rho}_b^{*}(f) = \tilde{\rho}_b^{*}(f_\prime), \quad \tilde{} = \check{}\,,\hat{}\,, \quad * = \circ, \bullet.$$

Thus, both the complete wandering and the complete nonwandering of a nonlinear system are uniquely determined by the system of its first approximation.

Theorem 3. *The complete wandering and the complete nonwandering of system* (1) *are equivalent to the positiveness of the indicator of system* (2) *of its first approximation*
$$\check{\rho}_b^{\circ}(f_\prime) > 0$$
and, accordingly, to the equality to zero of the indicator of system (2)
$$\hat{\rho}_b^{\bullet}(f_\prime) = 0.$$

Definition 3 ([4])**.** For a system (1) and for its nonzero solution $x \in S_*(f)$ defined on the whole semiaxis $\mathbb{R}_+$, we define:

(a) *lower weak* and *strong wandering indicators* – by the formulas

$$\check{\rho}^\circ(x) \equiv \varliminf_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\mathrm{P}(t, Lx), \quad \check{\rho}^\bullet(x) \equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varliminf_{t\to+\infty} t^{-1}\mathrm{P}(t, Lx); \tag{6}$$

(b) *upper weak* and *strong wandering indicators* – by the same formulas (6) respectively, but with the upper limits at $t \to +\infty$ instead of the lower ones

$$\hat{\rho}^\circ(x) \equiv \varlimsup_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\mathrm{P}(t, Lx), \quad \hat{\rho}^\bullet(x) \equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varlimsup_{t\to+\infty} t^{-1}\mathrm{P}(t, Lx); \tag{7}$$

(c) *exact* or *absolute* varieties of indicators (4)–(7) that arise when the corresponding values of the lower and upper indicators or, respectively, weak and strong ones coincide: in the first case, we will omit the checkmark and the cap in their designation, and in the second one – an empty and full circle.

Surprisingly, the presence of a complete wandering system does not mean that it has at least one solution with a positive wandering indicator, and vice versa, the presence of complete nonwandering system does not mean that it has at least one solution with a zero wandering indicator.

Theorem 4. *For $n = 2$, there exist two Lyapunov stable systems* (1)*, which, like all their nonzero solutions defined on the entire semiaxis $\mathbb{R}_+$, have exact absolute wandering indicators: one of these systems has complete wandering, is periodic, and satisfies the conditions*

$$\rho_b(f) = 1 > 0 = \rho(x), \quad x \in S_*(f),$$

while the other system has complete nonwandering and satisfies the conditions

$$\rho_b(f) = 0 < 1 = \rho(x), \quad x \in S_*(f).$$

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Asymptotic Representations of Some Classes of Solutions of Third Order Nonautonomous Ordinary Differential Equations

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We consider the differential equation

$$y''' = \alpha_0 p(t) y |\ln |y||^{\sigma}, \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p : [a, \omega[\,\to\,]0 < +\infty[$ is a continuous function, $-\infty < a < \omega \leq +\infty$[1].

The solution y of equation (1), given and different from zero on the interval $[t_y, \omega[\,\subset [a, \omega[$, is called $P_\omega(\lambda_0)$-solution if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{if } \ 0, \\ \text{if } \ \pm\infty \end{cases} \qquad (k = 0, 1, 2), \quad \lim_{t\uparrow\omega} \frac{(y''(t))^2}{y'''(t) y'(t)} = \lambda_0.$$

In [6], for equation (1) the conditions for the existence of a $P_\omega(\lambda_0)$-solution were established in the non-singular case, when $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, 1\}$, asymptotic representations were also obtained for such solutions and their derivatives up to the second order inclusive. At the same time, the question of the number of solutions with the found asymptotic representations was also clarified.

In [5] Shinkarenko V., Sharay N. for the differential equation (1) investigated questions about the existence and asymptotics of the so-called $P_\omega(Y_0, \lambda_0)-$ solutions at $\lambda_0 = +\infty$.

The purpose of this work is to establish necessary and sufficient conditions for the existence of y for the differential equation (1) $P_\omega(\frac{1}{2})$-solutions, as well as asymptotic representations at $t \uparrow \omega$ for all such solutions and their derivatives up to the second order inclusive.

In special cases, using the results from the work of Evtukhov V. [2, Ch. 3, § 10, pp. 142–144] it follows a corollary on the asymptotic properties of $P_\omega(\frac{1}{2})$-solutions of equation (1). To describe them, we need the following auxiliary notation

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t - \omega, & \text{if } \ \omega < +\infty. \end{cases}$$

Lemma. *For each $P_\omega(\frac{1}{2})$-solutions of the differential equation* (1) *when $t \uparrow \omega$ we have the asymptotic relations*

$$y'(t) = o\Big(\frac{y(t)}{\pi_\omega(t)}\Big), \quad y''(t) \sim -\frac{1}{\pi_\omega(t)}\, y'(t), \quad y'''(t) \sim \frac{2!}{[\pi_\omega(t)]^2}\, y'(t).$$

[1] We assume that $a > 1$ at $\omega = +\infty$ and $\omega - a < 1$ at $\omega < +\infty$.

Using this result and the work of Evtukhov V. and Samoylenko A. [3], the following result is established.

To formulate the main result, we need the auxiliary functions

$$J_A(t) = \int_A^t \pi_\omega(\tau)p(\tau)\,d\tau, \quad I_B(t) = \int_B^t J_A(\tau)\,d\tau,$$

where

$$A = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega |\pi_\omega(\tau)|p(\tau)\,d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega |\pi_\omega(\tau)|p(\tau)\,d\tau < +\infty, \end{cases} \qquad B = \begin{cases} a, & \displaystyle\int_a^\omega |J_A(\tau)|\,d\tau = +\infty, \\ \omega, & \displaystyle\int_a^\omega |J_A(\tau)|\,d\tau < +\infty. \end{cases}$$

Theorem 1. *Let $\sigma \neq 1$. Then, for the existence of $P_\omega(\frac{1}{2})$-solutions of the differential equation* (1) *it is necessary and sufficient that the conditions*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_A'(t)}{J_A(t)} = -1, \quad \lim_{t\uparrow\omega} |I_B(t)|^{\frac{1}{1-\sigma}} = +\infty, \quad \lim_{t\uparrow\omega} \pi_\omega(t)J_A(t)|I_B(t)|^{\frac{\sigma}{1-\sigma}} = 0, \tag{2}$$

hold. Moreover, for each such solution, when $t \uparrow \omega$ the asymptotic representations

$$\ln|y(t)| = \nu_0 \Big|\frac{1-\sigma}{2} I_B(t)\Big|^{\frac{1}{1-\sigma}}[1+o(1)], \tag{3}$$

$$\frac{y'(t)}{y(t)} = -\frac{\alpha_0 J_A(t)}{2}\Big|\frac{1-\sigma}{2} I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}[1+o(1)], \tag{4}$$

$$\frac{y''(t)}{y(t)} = \frac{\alpha_0}{2}\,\frac{J_A(t)}{\pi_\omega(t)}\Big|\frac{1-\sigma}{2} I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}[1+o(1)] \tag{5}$$

hold, where

$$\nu_0 = -\alpha_0 \operatorname{sign}[(1-\sigma)I_B(t)].$$

Furthermore, if conditions (2) *are met, the differential equation* (1) *in the case when $\omega = +\infty$ has a one-parameter family of solutions with representations* (3)–(5)*, if $\sigma < 1$, and in the case when $\omega < \infty$ solutions there is a two-parameter family if $\sigma > 1$ and three-parameter family if $\sigma < 1$.*

Remark 1. It is also shown that under conditions (2) it can be proved that for $\omega = +\infty$ and $\sigma > 1$ the differential equation (1) has a unique solution that admits for $t \uparrow \omega$ the asymptotic representations (3)–(5).

Remark 2. When checking the fulfillment of (2), we can take into account the fact that by virtue of the first of them the second and third ones are equivalent to the conditions

$$\lim_{t\uparrow\omega} \Big|\int_B^t \pi_\omega^2(\tau)p(\tau)\,d\tau\Big|^{\frac{1}{1-\sigma}} = +\infty, \quad \lim_{t\uparrow\omega} \pi_\omega^3(t)p(t)\Big|\int_B^t \pi_\omega^2(\tau)p(\tau)\,d\tau\Big|^{\frac{\sigma}{1-\sigma}} = 0.$$

In conclusion, we pay attention to the fact that Theorem 1 covers the case $\sigma = 0$, i.e. when equation (1) is a linear differential equation of the form

$$y''' = \alpha_0 p(t) y. \tag{6}$$

For this equation, by virtue of Theorem 1, the following assertion holds.

Corollary. *For the existence of $P_\omega(\frac{1}{2})$-solutions of* (6)*, it is necessary and sufficient that the conditions*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_A(t)}{J_A(t)}=-1,\quad \lim_{t\uparrow\omega}|I_B(t)|=+\infty,\quad \lim_{t\uparrow\omega}\pi_\omega(t)J_A(t)=0 \tag{7}$$

hold. Moreover, for each such solution, $t\uparrow\omega$ the asymptotic representations

$$\ln|y(t)|=\nu_0\Big|\frac{1-\sigma}{2}\,I_B(t)\Big|[1+o(1)], \tag{8}$$

$$\frac{y'(t)}{y(t)}=-\frac{\alpha_0J_A(t)}{2}\,[1+o(1)], \tag{9}$$

$$\frac{y''(t)}{y(t)}=\frac{\alpha_0}{2}\,\frac{J_A(t)}{\pi_\omega(t)}\,[1+o(1)] \tag{10}$$

hold, where

$$\nu_0=-\alpha_0\operatorname{sign}[(1-\sigma)I_B(t)].$$

Furthermore, when conditions (7) *are met, the differential equation* (6) *has a one-parameter family of solutions, and in the case $\omega=+\infty$, a two-parameter family of solutions with representations* (8)*–*(10)*.*

The obtained asymptotics are consistent with the already known results for linear differential equations (see [4]).

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On the Functional Integral Equation with the Two Types Controls

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In the paper, the nonlinear controlled functional integral equation corresponding to the quasi-linear neutral functional differential equation with two types controls is constructed. A structure and properties of the integral kernel are established. Equivalence of the functional integral equation and the neutral functional differential equation is established also. We note that theorems formulated below play a principal role in the study of well-posedness of Cauchy's problem for the quasi-linear neutral functional differential equations. In details, about of this investigations for the quasi-linear neutral functional differential equations without control are given in [1-3].

Let $\mathbb{R}^n_x$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T is the sign of transposition; let $I=[t_0,t_1]$ be a fixed interval and let $\tau>0$ be a given number, with $t_0+\tau<t_1$; the $n\times n$-dimensional matrix-function $A(t,x,y,v)$ and the n-dimensional vector-function $f(t,x,y,u)$ are continuous and bounded on the set $I\times\mathbb{R}^n_x\times\mathbb{R}^n_x\times\mathbb{R}^m_v$ and $I\times\mathbb{R}^n_x\times\mathbb{R}^n_x\times\mathbb{R}^r_u$, respectively, and satisfy Lipschptz's condition with respect to (x,y,v) and (x,y,u), i.e. there exist $L_A>0$ and $L_f>0$ such that

$$\big|A(t,x_1,y_1,v_1)-A(t,x_2,y_2,v_2)\big|\le L_A\big(|x_1-x_2|+|y_1-y_2|+|v_1-v_2|\big)$$
$$\forall\, t\in I,\ \ (x_i,y_i,v_i)\in\mathbb{R}^n_x\times\mathbb{R}^n_x\times\mathbb{R}^m_v,\ \ i=1,2,$$

and

$$\big|f(t,x_1,y_1,u_1)-f(t,x_2,y_2,u_2)\big|\le L_f\big(|x_1-x_2|+|y_1-y_2|+|u_1-u_2|\big)$$
$$\forall\, t\in I,\ \ (x_i,y_i,u_i)\in\mathbb{R}^n_x\times\mathbb{R}^n_x\times\mathbb{R}^r_u,\ \ i=1,2.$$

Further, denote by V and Ω the sets of piecewise-continuous control functions $v(t)\in\mathbb{R}^m_v$ with finitely many discontinuous of the first kind and bounded measurable control functions $u(t)\in\mathbb{R}^r_u$, respectively, equipped with the norm

$$\|v\|=\sup\big\{|v(t)|:\ t\in I\big\}\quad\Big(\|u\|=\sup\big\{|u(t)|:\ t\in I\big\}\Big);$$

$\varphi(t)\in\mathbb{R}^n_x$, $t\in[t_0-\tau,t_0]$ is a given continuously differentiable initial function; $x_0\in\mathbb{R}^n_x$ is a given initial vector.

Let us consider the quasi-linear controlled neutral functional differential equation

$$\dot x(t)=A\big(t,x(t),x(t-\tau),v(t)\big)\dot x(t-\tau)+f\big(t,x(t),x(t-\tau),u(t)\big),\ \ t\in I \tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,t_0),\ \ x(t_0)=x_0, \tag{2}$$

where $\widehat\tau=t_0-\tau$.

Definition 1. Let $w=(v(t),u(t))\in W=V\times\Omega$. A function $x(t)=x(t;w)$, $t\in I_1=[\widehat{\tau},t_1]$, is called a solution of equation (1) with the initial condition (2), if it satisfies condition (2) and is absolutely continuous on the interval I and satisfies equation (1) almost everywhere on I.

Theorem 1. *For any $w\in W$ there exists the unique solution $x(t)=x(t;w)$, $t\in I_1$.*

Theorem 2. *The solution $x(t)$, $t\in I_1$ of problem $(1),(2)$ can be represented on the interval I in the following form*

$$\begin{aligned}x(t)=x_0&+\int\limits_{t_0-\tau}^{t_0}Y\big(\xi+\tau;t,x(\,\cdot\,),v(\,\cdot\,)\big)A\big(\xi+\tau,x(\xi+\tau),x(\xi),v(\xi+\tau)\big)\dot{\varphi}(\xi)\,d\xi\\&+\int\limits_{t_0}^{t}Y\big(\xi;t,x(\,\cdot\,),v(\,\cdot\,)\big)f\big(\xi,x(\xi),x(\xi-\tau),u(\xi)\big)\,d\xi,\end{aligned}$$

where

$$x(\xi)=\varphi(\xi),\ \ \xi\in[\widehat{\tau},t_0)$$

and $Y(\xi,t,x(\,\cdot\,),v(\,\cdot\,))$ is the matrix-function satisfying the difference equation

$$Y(\xi;t,x(\,\cdot\,),v(\,\cdot\,))=E+Y\big(\xi+\tau;t,x(\,\cdot\,),v(\,\cdot\,)\big)\cdot A\big(\xi+\tau,x(\xi+\tau),x(\xi),v(\xi+\tau)\big)\tag{3}$$

on (t_0,t) for any fixed $t\in(t_0,t_1]$ and the condition

$$Y(\xi;t,x(\,\cdot\,),v(\,\cdot\,))=\begin{cases}E, & \xi=t,\\ \Theta, & \xi>t.\end{cases}$$

Here, E is the identity matrix and Θ is the zero matrix.

The expression

$$\begin{aligned}y(t)=x_0&+\int\limits_{t_0}^{t_0+\tau}Y\big(\xi;t,y(\,\cdot\,),v(\,\cdot\,)\big)A\big(\xi,y(\xi),y(\xi-\tau),v(\xi)\big)\dot{\varphi}(\xi-\tau)\,d\xi\\&+\int\limits_{t_0}^{t}Y\big(\xi;t,y(\,\cdot\,),v(\,\cdot\,)\big)f\big(\xi,y(\xi),y(\xi-\tau),u(\xi)\big)\,d\xi\end{aligned}\tag{4}$$

with the condition

$$y(\xi)=\varphi(\xi),\ \ \xi\in[\widehat{\tau},t_0)\tag{5}$$

is called the functional integral equation corresponding to problem $(1),(2)$.

Definition 2. Let $w\in W$. A function $y(t)=y(t;w)$, $t\in I_1$, is called a solution of equation (4) with condition (5), if it satisfies condition (5) and is continuous on the interval I and satisfies equation (4) everywhere on I.

Theorem 3. *Let $t\in(t_0,t_1]$ be a fixed point. The solution of the difference equation (3) can be represented by the following formula*

$$Y(\xi;t,x(\,\cdot\,),v(\,\cdot\,))=\chi(\xi;t)E+\sum_{i=1}^{k}\chi(\xi+i\tau;t)\prod_{q=i}^{1}A\big(\xi+q\tau,x(\xi+q\tau),x(\xi+(q-1)\tau),v(\xi+q\tau)\big),$$

where

$$\chi(\xi;t)=\begin{cases}1, & t_0\le\xi\le t,\\ 0, & \xi>t\end{cases}$$

and k is a minimal natural number satisfying the condition

$$t_1-k\tau<t_0.$$

Theorem 4. *Let $s_1,s_2\in(t_0,t_1]$ and $0<s_2-s_1<\tau$. Let $y(t)$, $t\in I$ be a continuous function. Then there exist subintervals $I_1(s_1,s_2)\subset I$ and $I_2(s_1,s_2)\subset I$ such that*

$$\begin{cases}Y\big(\xi;s_1,y(\,\cdot\,),v(\,\cdot\,)\big)=Y\big(\xi;s_2,y(\,\cdot\,),v(\,\cdot\,)\big), & \xi\in I_1(s_1;s_2),\\ Y\big(\xi;s_1,y(\,\cdot\,),v(\,\cdot\,)\big)\neq Y\big(\xi;s_2,y(\,\cdot\,),v(\,\cdot\,)\big), & \xi\in I_2(s_1;s_2),\end{cases}$$

with

$$\lim_{s_2-s_1\to0}\operatorname{mes}I_2(s_1,s_2)\to0.$$

Theorem 5. *Let $y(t)\in R^n$, $t\in[\widehat{\tau},t_1]$ be a given piecewise-continuous function, with $y(\xi)=\varphi(\xi)$, $\xi\in[\widehat{\tau},t_0)$; $v(t)\in V$ and $u(t)\in\Omega$. Then the function*

$$\begin{aligned}z(t)=x_0+\int_{t_0}^{t_0+\tau}&Y\big(\xi;t,y(\,\cdot\,),v(\,\cdot\,)\big)A\big(\xi,y(\xi),y(\xi-\tau),v(\xi)\big)\dot{\varphi}(\xi-\tau)\,d\xi\\&+\int_{t_0}^{t}Y\big(\xi;t,y(\,\cdot\,),v(\,\cdot\,)\big)f\big(\xi,y(\xi),y(\xi-\tau),u(\xi)\big)\,d\xi\end{aligned}$$

is continuous on the interval I.

Theorem 6. *Let $y_i(t)\in\mathbb{R}^n_x$, $t\in I$, $i=1,2$ be continuous functions and $v_i(t)\in V$, $i=1,2$. Then for a fixed $(\xi,t)\in I^2$,*

$$\begin{aligned}&\Big|Y\big(\xi;t,y_1(\,\cdot\,),v_1(\,\cdot\,)\big)-Y\big(\xi;t,y_2(\,\cdot\,),v_2(\,\cdot\,)\big)\Big|\\&\qquad\le L_A\sum_{i=1}^{k}\chi(\xi+i\tau;t)\|A\|^{i-1}\bigg(\sum_{q=i}^{1}\Big[\big|y_1(\xi+q\tau)-y_2(\xi+q\tau)\big|\\&\qquad\qquad+\big|y_1(\xi+(q-1)\tau)-y_2(\xi+(q-1)\tau)\big|+\big|v_1(\xi+q\tau)-v_2(\xi+q\tau)\big|\Big]\bigg),\end{aligned}$$

where

$$\|A\|=\sup\Big\{|A(t,x,y,v)|:\ (t,x,y,v)\in I\times R^n_x\times R^n_x\times R^m_v\Big\}.$$

Theorem 7. *Let $y_i(t)\in\mathbb{R}^n_x$, $t\in I$, $i=0,1,\dots$ be continuous functions and $v_i(t)\in V$, $i=0,1,\dots,$ with*

$$\|y_i-y_0\|\to0,\quad\|v_i-v_0\|\to0,$$

Then

$$\int_{t_0}^{t}Y\big(\xi;t,y_i(\,\cdot\,),v_i(\,\cdot\,)\big)\,d\xi\longrightarrow\int_{t_0}^{t}Y\big(\xi;t,y_0(\,\cdot\,),v_0(\,\cdot\,)\big)\,d\xi$$

uniformly for $t\in I$.

Theorem 8. *The functional integral equation* (4) *with condition* (5) *has the unique solution.*

Theorem 9. *The quasi-linear neutral functional differential equation* (1) *and the functional integral equations* (4) *are equivalent.*

Remark. The analogous theorems for the case, where $A(t,x,y,v) \equiv A(t)$ and functional integral equation (3) depends on the one control function, are proved in [1, 3, 4] and [2], respectively.

Conclusion

On the basis of the given theorems, it can be investigated continuous dependence of a solution of the quasi-linear controlled neutral functional differential equation (1) with respect to perturbations of the initial data. In future work the case, where a controlled functional integral equation contains several variable delays, will be considered.

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Algorithm for Constructing Uniform Asymptotics of a Solution for Problem for Singular Perturbed Systems of Differential Equations with Differential Turning Point

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We consider the following system of differential equations with turning point (SSPDE):

$$\varepsilon Y'(x,\varepsilon) - A(x,\varepsilon)Y(x,\varepsilon) = H(x), \tag{0.1}$$

where

$$A(x,\varepsilon) = A_0(x) + \varepsilon A_1(x),$$

is a known matrix,

$$\mathbf{A_0}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b(x) & -a(x) & 0 \end{pmatrix}, \quad \mathbf{A_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

when $\varepsilon \to 0$, $x \in [-l, l]$, $Y(x,\varepsilon) \equiv Y_k(x,\varepsilon) = \text{colomn}(y_1(x,\varepsilon), y_2(x,\varepsilon), y_3(x,\varepsilon))$ is an unknown vector function, $H(x) = \text{colomn}(0, 0, h(x))$ is a given vector function.

The scalar reduced equation for this matrix will be

$$x\widetilde{a}(x)\omega'(x) + b(x)\omega(x) = h(x).$$

The characteristic equation that corresponds to the SP system (0.1) is as follows:

$$A(x,0) - \lambda E| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ -b(x) & -a(x) & -\lambda \end{vmatrix} = -\lambda^3 - x\widetilde{a}(x)\lambda = 0.$$

The roots of this equation are

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{x\widetilde{a}(x)}\,.$$

The purpose of this work is to construct uniform asymptotic of a solution for a given SSPDE with a stable turning point of the first kind.

1 Regularization of singularly perturbed systems of differential equations

In order to save all essential singular functions that appear in the solution of system (0.1) due to the special point $\varepsilon = 0$, a regularizing variable is introduced $t = \varepsilon^{-p} \cdot \varphi(x)$, where exponent p and regularizing function $\varphi(x)$ are to be determined.

Instead of $Y_k(x,\varepsilon)$ function $\widetilde{Y}_k(x,t,\varepsilon)$ transformation function will be studied, also the transformation will be performed in such a way that the following identity is true

$$\widetilde{Y}(x,t,\varepsilon)\Big|_{t=\varepsilon^{-p}\varphi(x)} \equiv Y(x,\varepsilon),$$

which is the necessary condition for a suggested method. The vector equation (0.1) can be written as

$$\widetilde{L}_\varepsilon \widetilde{Y}_k(x,t,\varepsilon) \equiv \mu\varphi' \, \frac{\partial \widetilde{Y}(x,t,\varepsilon)}{\partial t} + \mu^3 \, \frac{\partial \widetilde{y}(x,t,\varepsilon)}{\partial x} - A(x,\varepsilon)\widetilde{Y}_k(x,t,\varepsilon) = H(x). \tag{1.1}$$

Asymptotic forms of solutions for equation (1.1) are constructed in the form of the series

$$\widetilde{Y}_k(x,t,\varepsilon) = \sum_{i=1}^{2} D_i(x,t,\varepsilon) + f(x,\varepsilon)\nu(t) + \varepsilon^\gamma g(x,\varepsilon)\nu'(t) + \omega(x,\varepsilon),$$

$$\sum_{i=1}^{2} D_i(x,t,\varepsilon) = \begin{pmatrix} \varepsilon^{s1}\alpha_{k1}(x,\varepsilon) \\ \varepsilon^{s2}\alpha_{k2}(x,\varepsilon) \\ \varepsilon^{s3}\alpha_{k3}(x,\varepsilon) \end{pmatrix} U_i(t) + \varepsilon^\gamma \begin{pmatrix} \varepsilon^{k1}\beta_{k1}(x,\varepsilon) \\ \varepsilon^{k2}\beta_{k2}(x,\varepsilon) \\ \varepsilon^{k3}\beta_{k3}(x,\varepsilon) \end{pmatrix} {U_i}'(t),$$

where $U_1(t)$, $U_2(t)$ are the Airy–Langer functions [3] and $\alpha_{ik}(x,\varepsilon)$, $\beta_{ik}(x,\varepsilon)$, $f_k(x,\varepsilon)$, $g_k(x,\varepsilon)$, $\omega_k(x,\varepsilon)$, $k=\overline{1,3}$ are analytic functions with reference to a small parameter and are infinitely differentiable functions of variable $x\in[-l;l]$ which are still to be determined.

First of all, the analysis how transformation operator $\widetilde{L}_\varepsilon$ operates on vector function $D_k(x,t,\varepsilon)$ will be performed, and then the obtained result will be utilized in the homogeneous transformation equation (0.1). The following equation is obtained

$$\begin{aligned}
&\widetilde{L}_\varepsilon(\alpha_{ik}(x,\varepsilon)U_i(t) + \varepsilon^\gamma \beta_{ik}(x,\varepsilon)U_i'(t)) \\
&\quad = \varepsilon^{1-p}\alpha_{ik}(x,\varepsilon)\varphi'(x)U_i'(t) - \varepsilon^{1+\gamma-2p}\beta_{ik}(x,\varepsilon)\varphi'(x)\varphi(x)U_i(t) - A(x,\varepsilon)\alpha_k(x,\varepsilon)U_i(t) \\
&\qquad\qquad - \varepsilon^\gamma A(x,\varepsilon)\beta_{ik}(x,\varepsilon)U_i'(t) + \varepsilon\alpha_{ik}'(x)U_i(t) + \varepsilon^{1+\gamma}\beta_k'(x)U_i'(t) = 0.
\end{aligned}$$

Then, after equating corresponding coefficients of essential singular functions $U_k(t)$, $k=1,2$ and their derivatives, two following vector equations are obtained:

$$U_i'(t):\ \varepsilon^{1-p}\alpha_{ik}(x,\varepsilon)\varphi'(x) - \varepsilon^\gamma\big[A_0(x)+\varepsilon A_1\big]\beta_{ik}(x,\varepsilon) = -\varepsilon^{1+\gamma}\beta_{ik}'(x,\varepsilon), \tag{1.2}$$

$$U_i(t):\ -\varepsilon^{1+\gamma-2p}\beta_{ik}(x,\varepsilon)\varphi(x)\varphi'(x) - \big[A_0(x)+\varepsilon A_1\big]\alpha_{ik}(x,\varepsilon) = -\varepsilon\alpha_{ik}'(x,\varepsilon). \tag{1.3}$$

2 Construction of formal solutions of a homogeneous transformation system

The unknown coefficients of the vector equations (1.2) and (1.3) are sought as following vector function series ($i=1,2$):

$$\alpha_{ik}(x,\varepsilon) = \sum_{r=0}^{+\infty} \mu^r \alpha_{ikr}(x), \quad \beta_{ik}(x,\varepsilon) = \sum_{r=0}^{+\infty} \mu^r \beta_{ikr}(x).$$

To determine vector function components $\alpha_{ikr} = \mathrm{colomn}(\alpha_{i1r}(x), \alpha_{i2r}(x), \alpha_{i3r}(x))$ and $\beta_{ikr}(x) = \mathrm{colomn}(\beta_{i1r}(x), \beta_{i2r}(x), \beta_{i3r}(x))$, the following recurrent systems of equations are obtained:

$$\begin{aligned}
&\Phi(x)Z_{k0}(x) = 0, \ \ r=0,1,2, \\
&\Phi(x)Z_{kr}(x) = FZ_{k(r-3)}(x), \ \ r\ge 3.
\end{aligned} \tag{2.1}$$

At the moment, the regularizing function has not yet been defined; therefore, it will be defined as a solution of the problem

$$\varphi'^2\varphi(x) = x, \quad \varphi(0) = 0,$$

which is the following function

$$\varphi(x) = x.$$

The regularizing function of such kind has been considered in [3, 5].

Due to such a choice of the regularizing variable $\varphi(x)$, there is a nontrivial solution of the homogeneous system $\Phi(x)Z_{kr}(x) = 0$, $r = \overline{0,2}$, that is

$$Z_{ikr}(x) = \text{colomn}\,\big(0, \beta_{i3r}(x), -\beta_{i2r}(x), 0, \beta_{i2r}(x), \beta_{i3r}(x)\big),$$

where $\beta_{ksr}(x)$, $i = 1, 2$, $s = 2, 3$ are arbitrary up to some point and sufficiently smooth function at $x \in [0; l]$.

Solving systems of recurrent equations at the third step, i.e., when $r = 3$, and taking into account the already obtained solution (2.1), the following systems of algebraic equations in $\alpha_{kr}(x)$ and $\beta_{kr}(x)$ are obtained

$$\begin{cases} \alpha_{i13}(x) = \beta_{i20}(x) - \beta'_{i10}(x) \equiv \beta_{i20}(x), \\ \alpha_{i23}(x) - \beta_{i33}(x) = -\beta'_{i20}(x), \\ \alpha_{i33}(x) - \beta_{i13}(x) + \beta_{i23}(x) = -\beta'_{i30}(x), \end{cases} \tag{2.2}$$

and

$$\begin{cases} x\beta_{i13}(x) = -\alpha'_{i10}(x) + \alpha_{i20}(x) \equiv \alpha_{i20}(x) \equiv \beta_{i30}(x), \\ x\beta_{i23}(x) + \alpha_{i33}(x) = \alpha'_{i20}(x) \equiv (\beta_{i30}(x))', \\ x\beta_{i33}(x) + \alpha_{i13}(x) + \alpha_{i23}(x) = \alpha'_{i30}(x) \equiv [-x\beta_{i20}(x)]'. \end{cases} \tag{2.3}$$

Taking into account the fact that the functions are arbitrary, $\beta_{is0}(x) = \beta^0_{is0} \cdot \hat{\beta}_{is0}(x)$, $i = 1, 2$, $s = 2, 3$, where $\beta^0_{is0}(x)$ are an arbitrary constants, $\hat{\beta}_{is0}(x)$ is a partial and sufficiently smooth for all $x \in [-l; l]$ solutions of homogeneous equations. This definition of vector functions $Z_{ik0}(x)$ implies that there are following solutions of inhomogeneous systems of algebraic equations (2.2) and (2.3):

$$Z_{ik3}(x) = \text{colomn}\,\big(z_{i13}, z_{i23}, z_{i33}, z_{i43}, z_{i53}, z_{i63}\big),$$

$$z_{i13} = \beta_{i20}(x), \quad z_{i23} = -\beta'_{i20}(x) + \beta_{i33}(x), \quad z_{i33} = -\beta'_{i30}(x) - \beta_{i23}(x) + \frac{\beta_{i30}}{x}, \quad z_{i43} = \frac{\beta_{i20}(x)}{x},$$

$$z_{i53} = \beta_{i21}(x), \quad z_{i63} = \beta_{i31}(x),$$

where $\beta_{i21}(x)$ and $\beta_{i31}(x)$ are arbitrary up to some point and sufficiently smooth functions for all $x \in [-l; l]$.

Thus, gradual solving of systems of equations (2.2) and (2.3) gives two formal solutions of the transformation vector equation (0.1)

$$D_{ik}(x, \varepsilon^{-\frac{2}{3}}\varphi(x), \varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r \Big[\alpha_{ikr}(x)U_i(\varepsilon^{-\frac{2}{3}}\varphi(x)) + \varepsilon^{\frac{1}{3}}\beta_{ikr}(x,\varepsilon)U'_i(\varepsilon^{-\frac{2}{3}}\varphi(x))\Big].$$

The third formal solution of the homogeneous vector equation (0.1) is then constructed as a series

$$\omega(x, \varepsilon) \equiv \sum_{r=0}^{\infty} \varepsilon^r \omega_r(x) \equiv \text{colon}\,\Big(\sum_{r=0}^{\infty} \varepsilon^r \omega_{1r}(x), \sum_{r=0}^{\infty} \varepsilon^r \omega_{2r}(x), \sum_{r=0}^{\infty} \varepsilon^r \omega_{3r}(x)\Big). \tag{2.4}$$

Substituting solution (2.4) into equation (0.1), the following recurrent system of differential equations can be obtained:

$$\begin{aligned}&A_0(x)\omega_0(x)=0,\\&A_r(x)\omega_r(x)=-A_1(x)\omega_{(r-1)}(x)-\omega'_{(r-1)}(x),\ \ r\geq 1.\end{aligned}$$

Then, solving these systems step by step, the following zero approximation can be constructed

$$\omega_0(x)=\operatorname{colomn}(\omega_{10}(x),\omega_{20}(x),\omega_{30}(x))\equiv\operatorname{colomn}(\omega_{10}^0\cdot x,-\omega_{10}^0,0),$$

that has only one arbitrary constant ω_{01}^0.

3 Construction of formal partial solutions

Similarly to the previous steps, in order to construct asymptotic forms of partial solutions of the inhomogeneous transformation vector equation (0.1), let us analyze how transformation operator operates on an element from the space of non-resonant solutions

$$f(x,\varepsilon)\psi(t)+\varepsilon^\gamma g(x,\varepsilon)\psi'(t)+\overline{\omega}(x,\varepsilon).$$

Consequently, the following systems are obtained

$$\psi'(t):\ f_k(x,\varepsilon)-\big[A_0(x)+\mu^3A_1\big]g_k(x,\varepsilon)=-\mu^3g'_k(x,\varepsilon),\tag{3.1}$$

$$\psi(t):\ xg_k(x,\varepsilon)+\big[A_0(x)+\mu^3A_1\big]f_k(x,\varepsilon)=\mu^3f'_k(x,\varepsilon),\tag{3.2}$$

$$\mu^3\overline{\omega}'(x,\varepsilon)-\big[A_0(x)+\mu^3A_1\big]\overline{\omega}(x,\varepsilon)+\mu^2g_k(x,\varepsilon)=H(x).\tag{3.3}$$

In order to have smooth solutions of systems (3.1)–(3.3), the asymptotic forms of the solutions are constructed as series

$$f_k(x,\varepsilon)=\sum_{r=-2}^{+\infty}\mu^rf_r(x),\quad g_k(x,\varepsilon)=\sum_{r=-2}^{+\infty}\mu^rg_r(x),\quad \overline{\omega}(x,\varepsilon)=\sum_{r=0}^{+\infty}\mu^r\overline{\omega}_r(x).$$

To determine components of the vector functions $f_{kr}=\operatorname{colomn}(f_{1r}(x),f_{2r}(x),f_{3r}(x))$ and $g_{kr}(x)=\operatorname{colomn}(g_{1r}(x),g_{2r}(x),g_{3r}(x))$, the following recurrent systems of equations are obtained:

$$\begin{gathered}\Phi(x)Z_{k0}^{part.}(x)=0,\ \ r=-2,-1,0,\\\Phi(x)Z_{kr}^{part.}(x)=-Z_{k(r-3)}^{part.}(x),\ \ r\geq 1.\end{gathered}$$

Then, to determine the vector functions $\overline{\omega}_r(x)$, the following recurrent systems of equations are obtained as well

$$\begin{gathered}-A_0(x)\overline{\omega}_{kr}(x)=H(x)-g_{k(r-2)}(x),\ \ r=0,\\-A_0(x)\overline{\omega}_{kr}(x)=-g_{k(r-2)}(x),\ \ r=1,2,\\\overline{\omega}'_{k(r-3)}(x)-A_0(x)\overline{\omega}_{kr}(x)=-g_{k(r-3)}(x)+A_1\overline{\omega}_{k(r-3)}(x),\ \ r\geq 3,\end{gathered}$$

where $\overline{\omega}_r(x)=\operatorname{colomn}(\overline{\omega}_{1r}(x),\overline{\omega}_{2r}(x),\overline{\omega}_{3r}(x))$ is an unknown vector function. Doing further iterations, functions $\overline{\omega}_r(x)$, $f_r(x)$, $g_r(x)$, which are sufficiently smooth in the whole domain, are obtained. Therefore, the partial solution of the transformation vector equation (0.1) is then defined as the series

$$\widetilde{Y}_k^{part.}(x,t,\varepsilon)=\sum_{r=-2}^{\infty}\mu^r\big[f_{kr}(x)\nu(t)+\mu g_{kr}(x)\nu'(t)\big]+\sum_{r=0}^{\infty}\mu^r\overline{\omega}_{kr}(x).$$

4 Conclusions

Thus, the transformation vector equation (0.1) has three formal solutions in form of the series

$$\widetilde{Y}(x,t,\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^{r}\Bigg[\sum_{i=1}^{2}\Big[\alpha_{ikr}(x)U_i(\varepsilon^{-\frac{2}{3}}\cdot x)+\varepsilon^{\frac{1}{3}}\beta_{kr}(x)\,\frac{dU_i(\varepsilon^{\frac{2}{3}}\cdot x)}{d(\varepsilon^{-\frac{2}{3}}\cdot x)}\Big]\Bigg] \\ +\sum_{r=-2}^{\infty}\varepsilon^{r}\Big[f_{kr}(x)\nu(\varepsilon^{\frac{2}{3}}\cdot x)+\varepsilon^{\frac{1}{3}}g_{kr}(x)\,\frac{d\nu(\varepsilon^{-\frac{2}{3}}\cdot x)}{d(\varepsilon^{-\frac{2}{3}}\cdot x)}\Big]+\sum_{r=0}^{\infty}\varepsilon^{r}\overline{\omega}_{kr}(x).$$

5 Algorithm for constructing the asymptotics of a solution of the system

Let us write the main result of this paper in the following algorithm:

Step I. An extension of the singularly perturbed problem. In a singularly perturbed system with a turning point next to an independent one variable x introduces a new vector-variable $t=\varepsilon^{-p}\cdot\varphi(x)$. Then instead of the wanted one vector-function $Y(x,\varepsilon)$ a new “extended vector-function” $\widetilde{Y}(x,t,\varepsilon)$ is studied. The expansion is carried out in such a way that the condition as in regularization method

$$\widetilde{Y}(x,t,\varepsilon)\Big|_{t=\varepsilon^{-p}\cdot\varphi(x)}\equiv Y(x,\varepsilon).$$

p and $\varphi(x)$ are determined for each specific case. There is a transition from a problem with one variable to a problem with two variables t and x.

Step II. The space of resonance-free solutions. For regularization, a specific space of functions is introduced, this space is called *the space of resonance-free solutions* and for each specific problem this space has its own specificity

$$\sum_{k=1}^{2}D_k(x,t,\varepsilon),f_k(x,\varepsilon)\psi(t),\varepsilon^{\gamma}g_k(x,\varepsilon)\psi'(t),\omega_k(x,\varepsilon).$$

Step III. Regularization of a singularly perturbed problem. The extended problem is studied in the space of resonance-free solutions and is reduced to an equation in which the small parameter $\varepsilon>0$ enters regularly.

Step IV. The formalism of constructing a solution to the problem. Since the extended problem is regularly perturbed with respect to the small one parameter in the space of resonance-free solutions, then we will look for the solution of the problem in the form of a series

$$\widetilde{Y}(x,t,\mu)=\sum_{r=-2}^{\infty}\mu^{r}Y(x), \tag{5.1}$$

where $\mu=\sqrt[3]{\varepsilon}$ is a small parameter.

We start the construction of the asymptotic series with negative powers of a small parameter in order to obtain uniform asymptotics intersection of the SSPDE. The right part of the system will have a break of the second kind at the turning point. Therefore, in general, it will not belong set of values of the main extended operator $\widetilde{L}_{\varepsilon}$. By substituting series (5.1) in system (1.1), to determine the coefficients of this series, we will get some system of pointwise recurrent equations with initial or boundary conditions.

Step V. Construction of formal solutions of homogeneous extended system. Those obtained in the previous point are recurrent the equation for determining the coefficients of series (5.1) is partial differential equations with point boundary conditions. We will show that this system of equations is asymptotically correct in the space of resonance-free solutions D_k. At this stage, the theory of existence is developed of the iterative equation of the form

$$\Phi(x) \cdot Z_{kr}(x) = F \cdot Z_{kr}(x),$$

where $\Phi(x)$ is the matrix of system (1.1), $Z_{kr}(x)$ is a column vector composed of analytic functions $\theta_1(x,\varepsilon)$. And the first members are being built of the asymptotic solution of the homogeneous problem under consideration.

Step VI. Construction of formal inhomogeneous solutions extended system. In this section, a function is being built for the inhomogeneous problem using a recurrent equation

$$\Phi(x) \cdot Z_{kr}(x) = F \cdot Z_{kr}(x),$$

where $\Phi(x)$ is the matrix of system (1.1), $Z_{kr}(x)$ is a column vector composed of analytic functions $\theta_2(x,\varepsilon)$.

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Bifurcation of Positive Periodic Solutions to Non-Autonomous Undamped Duffing Equations

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The extended abstract concerns the parameter-dependent periodic problem

$$u'' = p(t)u - h(t)|u|^\lambda \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{1}$$

where $p, h, f \in L([0,\omega])$, $h \geq 0$ a.e. on $[0,\omega]$, $\lambda > 1$, and $\mu \in \mathbb{R}$ is a parameter. By a solution to problem (1), as usual, we understand a function $u : [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies the given equation almost everywhere, and meets the periodic conditions. The text is based on the paper [3].

We first note that the differential equation in (1) with $\lambda = 3$ is derived, for example, when approximating non-linearities in the equations of motion of the oscillators in Figs. 1 and 2.

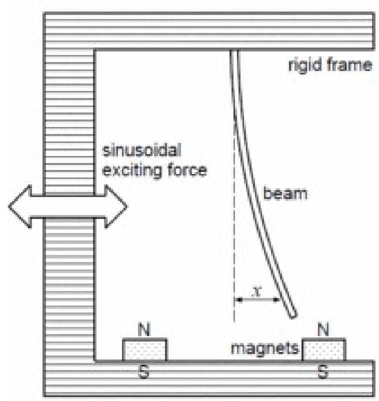

Figure 1. Forced steel beam deflected toward the two magnets[1].

Consider a forced undamped oscillator consisting of a mass body of weight m and a linear spring of characteristic k and non-deformed length ℓ (see Fig. 2). Assume that the mass body moves horizontally without any friction and the spring's base point B oscillates vertically, i.e., d is a positive ω-periodic function. This is a system with a single degree of freedom, described by the coordinate x, whose equation of motion is of the form

$$x'' = \frac{k}{m}\, x\left(\frac{\ell}{\sqrt{d^2(t) + x^2}} - 1\right) + \frac{F(t)}{m}\,. \tag{2}$$

A classical approach to deriving Duffing equation is to approximate the non-linearity in (2) by a third-degree Taylor polynomial centred at 0. We thus get the equation

$$x'' = \frac{k(\ell - d(t))}{md(t)}\, x - \frac{k\ell}{2md^3(t)}\, x^3 + \frac{F(t)}{m}\,, \tag{3}$$

[1]A figure is adopt from `http://www.scholarpedia.org/article/Duffing_oscillator`.

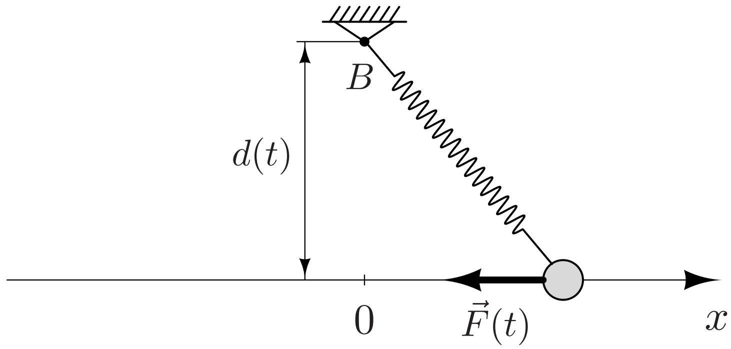

Figure 2. Forced undamped mass-spring oscillator with the so-called geometric non-linearity.

which is a particular case of the differential equation in (1). It is worth mentioning that the results below can be applied, for instance, to the forcing terms

$$F(t) := -f_0, \quad F(t) := A\Big(\sin\frac{2\pi t}{\omega} - \frac{1}{2}\Big),$$

where $f_0, A > 0$ are parameters.

To formulate our results, we need the following definitions.

Definition 1 ([2, Definitions 0.1 and 15.1, Proposition 15.2])**.** We say that a function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^-(\omega)$ if, for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a.e. } t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) \geq u'(\omega),$$

the inequality $u(t) \leq 0$ holds for $t \in [0,\omega]$.

Remark 1. Let $\omega > 0$. If $p(t) := p_0$ for $t \in [0,\omega]$, then one can show by direct calculation that $p \in \mathcal{V}^-(\omega)$ if and only if $p_0 > 0$. For non-constant functions $p \in L([0,\omega])$, efficient conditions guaranteeing the inclusion $p \in \mathcal{V}^-(\omega)$ are provided in [2] (see also [1,4]).

Definition 2 ([2, Definition 16.1])**.** Let $p, f \in L([0,\omega])$. We say that a pair (p,f) belongs to the set $\mathcal{U}(\omega)$, if the problem

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

has a unique solution which is positive.

Remark 2. Let $p \in \mathcal{V}^-(\omega)$. It follows from [2, Theorem 16.2] that $(p,f) \in \mathcal{U}(\omega)$ provided that

$$\int_0^\omega [f(s)]_- \,\mathrm{d}s > \mathrm{e}^{\frac{\omega}{4}\int_0^\omega [p(s)]_+ \,\mathrm{d}s} \int_0^\omega [f(s)]_+ \,\mathrm{d}s. \tag{4}$$

In particular, if

$$f(t) \leq 0 \quad \text{for a.e. } t \in [0,\omega], \quad f(t) \not\equiv 0, \tag{5}$$

then $(p,f) \in \mathcal{U}(\omega)$.

In what follows, we discuss the existence/non-existence as well as the exact multiplicity of positive solutions to problem (1) depending on the choice of the parameter μ provided that $p \in \mathcal{V}^-(\omega)$. Let us show, as a motivation, what happens in the autonomous case of (1). Hence, consider the equation

$$x'' = ax - b|x|^\lambda \operatorname{sgn} x - \mu. \tag{6}$$

In view of Remark 1 and the hypothesis $h \geq 0$ a.e. on $[0, \omega]$, we assume that $a, b > 0$. By direct calculation, the phase portraits of equation (6) can be elaborated depending on the choice of the parameter $\mu \in \mathbb{R}$ (see, Fig. 3) and, thus, one can prove the following proposition concerning the positive periodic solutions to equation (6).

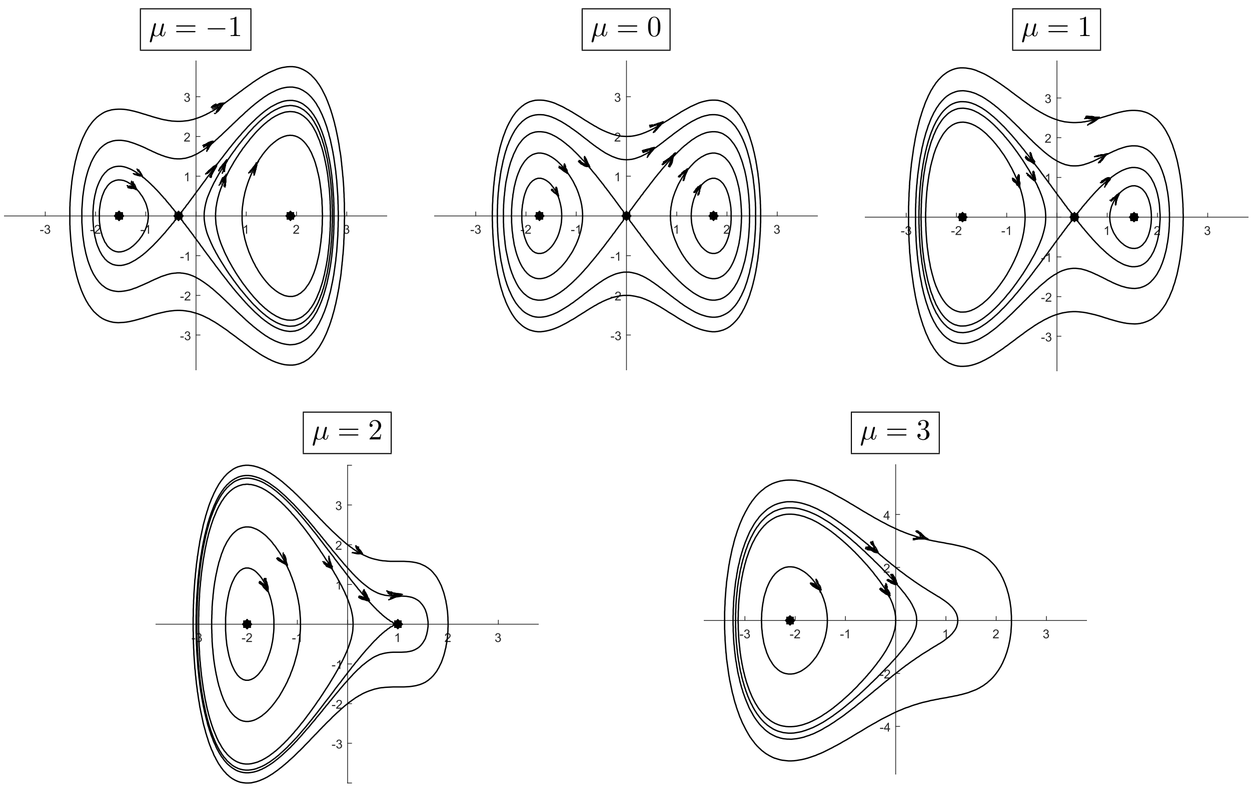

Figure 3. Phase portraits of equation (6) with $a = 3$, $b = 1$, and $\lambda = 3$.

Proposition 1. *Let $\lambda > 1$ and $a, b > 0$. Then, the following conclusions hold:*

(1) *If $\mu \leq 0$, then equation (6) has a unique positive equilibrium (center) and non-constant positive periodic solutions with different periods.*

(2) *If $0 < \mu < \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (6) possesses exactly two positive equilibria $x_2 > x_1$ (x_1 is a saddle and x_2 is a center) and non-constant positive periodic solutions with different periods. Moreover, all the non-constant positive periodic solutions are greater than x_1 and oscillate around x_2.*

(3) *If $\mu = \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (6) has a unique positive equilibrium (cusp) and no non-constant positive periodic solution occurs.*

(4) *If $\mu > \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (6) has no positive periodic solution.*

Proposition 1 shows that, if we consider μ as a bifurcation parameter, then, crossing the value $\mu^* = \frac{(\lambda-1)a}{\lambda}\big(\frac{a}{\lambda b}\big)^{\frac{1}{\lambda-1}}$, a bifurcation of positive periodic solutions to equation (6) occurs. In Fig. 3, the critical value of the bifurcation parameter is $\mu^* = 2$.

Theorem 1 (Main result)**.** *Let $\lambda > 1$, $p \in \mathcal{V}^-(\omega)$, $(p,f) \in \mathcal{U}(\omega)$, $\int\limits_0^\omega f(s)\,\mathrm{d}s < 0$, and*

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \quad h(t) \not\equiv 0. \tag{7}$$

Then, there exists $\mu_0 \in \,]0,+\infty[$ such that the following conclusions hold:

(1) *If $\mu = 0$, then problem (1) has at least one positive solution and, for any couple of distinct positive solutions u_1, u_2 to (1), the conditions*

$$\min\big\{u_1(t) - u_2(t) :\ t \in [0,\omega]\big\} < 0, \quad \max\big\{u_1(t) - u_2(t) :\ t \in [0,\omega]\big\} > 0$$

hold. If, moreover,

$$\mathrm{e}^{-1+\sqrt{1+\omega\int\limits_0^\omega p(s)\,\mathrm{d}s}}\bigg(-1 + \sqrt{1+\omega\int\limits_0^\omega p(s)\,\mathrm{d}s}\bigg) \leq \frac{8}{\lceil\lambda\rceil}\,, \tag{8}$$

where $\lceil\,\cdot\,\rceil$ is the ceiling function, then problem (1) with $\mu = 0$ has a unique positive solution.

(2) *If $0 < \mu < \mu_0$, then problem (1) has solutions u_1, u_2 such that*

$$u_2(t) > u_1(t) > 0 \quad \text{for } t \in [0,\omega]$$

and every non-negative solution u to problem (1) different from u_1 and u_2 satisfies

$$u(t) > u_1(t) \quad \text{for } t \in [0,\omega],$$
$$\min\big\{u(t) - u_2(t) :\ t \in [0,\omega]\big\} < 0, \quad \max\big\{u(t) - u_2(t) :\ t \in [0,\omega]\big\} > 0.$$

(3) *If $\mu = \mu_0$, then problem (1) has a unique positive solution.*

(4) *If $\mu > \mu_0$, then problem (1) has no positive solution.*

Open question. The following question remains open in Theorem 1: What happens in the case of $\mu < 0$?

We now provide lower and upper estimates of the number μ_0 appearing in the conclusion of Theorem 1.

Proposition 2. *Let $\lambda > 1$, $p \in \mathcal{V}^-(\omega)$, h satisfy (7), and f be such that (4) holds. Then, the number μ_0 appearing in the conclusion of Theorem 1 satisfies*

$$\mu_0 \geq \frac{(\lambda-1)[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda\int\limits_0^\omega h(s)\,\mathrm{d}s\big]^{\frac{1}{\lambda-1}}\int\limits_0^\omega [f(s)]_-\,\mathrm{d}s}\,,$$

where Δ is a number depending on the coefficient p only, and

$$\mu_0 < \frac{(\lambda-1)\big[\mathrm{e}^{\frac{\omega}{4}\int\limits_0^\omega [p(s)]_+\,\mathrm{d}s}\int\limits_0^\omega [p(s)]_+\,\mathrm{d}s - \int\limits_0^\omega [p(s)]_-\,\mathrm{d}s\big]^{\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda\int\limits_0^\omega h(s)\,\mathrm{d}s\big]^{\frac{1}{\lambda-1}}\big[\int\limits_0^\omega [f(s)]_-\,\mathrm{d}s - \mathrm{e}^{\frac{\omega}{4}\int\limits_0^\omega [p(s)]_+\,\mathrm{d}s}\int\limits_0^\omega [f(s)]_+\,\mathrm{d}s\big]}\,.$$

Remark 3. Let $\omega > 0$ and put $p(t) := a$, $h(t) := b$, $f(t) := -1$ for $t \in [0, \omega]$, where $a, b > 0$. Then, $p \in \mathcal{V}^-(\omega)$, h and f satisfy (7) and (5), respectively, and conclusions of Theorem 1 extend conclusions (2)–(4) of Proposition 1 for the non-autonomous Duffing equations with a sign-changing forcing term. Moreover, one can show that the number μ_0 appearing in Proposition 2 satisfies

$$\left(\frac{1}{\cosh\frac{\omega\sqrt{a}}{2}}\right)^{\frac{\lambda}{\lambda-1}} \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}} < \mu_0 < \left(\mathrm{e}^{\frac{\omega^2 a}{4}}\right)^{\frac{\lambda}{\lambda-1}} \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}};$$

compare it with the number appearing in Proposition 1.

If the forcing term f is non-positive, then Theorem 1 can be refined as follows.

Corollary. *Let $\lambda > 1$, $p \in \mathcal{V}^-(\omega)$, and conditions* (5), (7), *and* (8) *be satisfied. Then, there exists $\mu_0 \in]0, +\infty[$ such that the following conclusions hold:*

(1) *If $\mu = 0$, then problem* (1) *has a unique positive solution.*

(2) *If $0 < \mu < \mu_0$, then problem* (1) *has exactly two positive solutions u_1, u_2 and these solutions satisfy*
$$u_1(t) \neq u_2(t) \quad \textit{for } \ t \in [0, \omega].$$

(3) *If $\mu = \mu_0$, then problem* (1) *has a unique positive solution.*

(4) *If $\mu > \mu_0$, then problem* (1) *has no positive solution.*

Acknowledgements

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Existence of Solutions to BVPs at Resonance for Mixed Caputo Fractional Differential Equations

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1 Introduction

Let $J=[0,1]$, $X=C(J)\times\mathbb{R}$ and $\|x\|=\max\{|x(t)|:\ t\in J\}$ be the norm in $C(J)$.

We discuss the fractional boundary value problem

$$ {}^c\!D^{\alpha}_{1-}\,{}^c\!D^{\beta}_{0+}x(t)=f(t,x(t)), \tag{1.1}$$

$$ u(0)={}^c\!D^{\beta}_{0+}x(t)\Big|_{t=0}={}^c\!D^{\beta}_{0+}x(t)\Big|_{t=1}, \tag{1.2}$$

where $\alpha,\beta\in(0,1)$, $f\in C(J\times\mathbb{R})$, ${}^c\!D_{1-}$ and ${}^c\!D_{0+}$ denote the right and the left Caputo fractional derivatives.

Definition 1.1. We say that $x:J\to\mathbb{R}$ is *a solution of equation* (1.1) if $x,{}^c\!D^{\beta}_{0+}x\in C(J)$ and x satisfies (1.1) for $t\in J$. A solution x of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

Let $x:J\to\mathbb{R}$, $\gamma\in(0,1)$ and $\mu\in(0,\infty)$. Then the left ${}^c\!D^{\gamma}_{0+}x$ and the right ${}^c\!D^{\gamma}_{1-}x$ Caputo fractional derivatives of x of order γ are defined respectively by [2,3]

$$ {}^c\!D^{\gamma}_{0+}x(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^t\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s)-x(0))\,\mathrm{d}s$$

and

$$ {}^c\!D^{\gamma}_{1-}x(t)=-\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_t^1\frac{(s-t)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s)-x(1))\,\mathrm{d}s,$$

where Γ is the Euler gamma function.

The left $I^{\mu}_{0+}x$ and the right $I^{\mu}_{1-}x$ Riemann–Liouville fractional integrals of x of order μ are defined respectively by

$$ I^{\mu}_{0+}x(t)=\int\limits_0^t\frac{(t-s)^{\mu-1}}{\Gamma(\mu)}\,x(s)\,\mathrm{d}s\ \text{ and }\ I^{\mu}_{1-}x(t)=\int\limits_t^1\frac{(s-t)^{\mu-1}}{\Gamma(\mu)}x(s)\,\mathrm{d}s.$$

If $x\in C(J)$ and $\gamma\in(0,1)$, then

$$ {}^c\!D^{\gamma}_{0+}I^{\gamma}_{0+}x(t)=x(t),\quad {}^c\!D^{\gamma}_{1-}I^{\gamma}_{1-}x(t)=x(t)\ \text{ for }\ t\in J,$$
$$ I^{\gamma}_{0+}\,{}^c\!D^{\gamma}_{0+}x(t)=x(t)-x(0),\quad I^{\gamma}_{1-}\,{}^c\!D^{\gamma}_{1-}x(t)=x(t)-x(1)\ \text{ for }\ t\in J$$

and

$$I_{0+}^{\gamma_1} I_{0+}^{\gamma_2} x(t) = I_{0+}^{\gamma_1+\gamma_2} x(t), \quad I_{1-}^{\gamma_1} I_{1-}^{\gamma_2} x(t) = I_{1-}^{\gamma_1+\gamma_2} x(t) \ \text{ for } \ t \in J, \ \ \gamma_1, \gamma_2 \in (0, \infty).$$

Problem (1.1), (1.2) is at resonance because $\left\{c(1 + \frac{t^\beta}{\Gamma(\beta+1)}) : \ c \in \mathbb{R}\right\}$ is the set of nontrivial solutions to the homogeneous boundary value problem ${}^c\!D_{1-}^{\alpha}\, {}^c\!D_{0+}^{\beta} x = 0$, (1.2).

2 Operator $\mathcal{H}$ and its properties

Let an operator $\mathcal{H} : X \to X$ be given by the formula

$$\mathcal{H}(x, c) = \left(c\Big(1 + \frac{t^\beta}{\Gamma(\beta+1)}\Big) + I_{0+}^{\beta} I_{1-}^{\alpha} f(t, x(t)), c - I_{1-}^{\alpha} f(t, x(t))\Big|_{t=0} \right).$$

Note that

$$I_{0+}^{\beta} I_{1-}^{\alpha} f(t, x(t)) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big(\int_s^1 \frac{(\tau - s)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, x(\tau))\, \mathrm{d}\tau \Big)\, \mathrm{d}s$$

and

$$I_{1-}^{\alpha} f(t, x(t))\big|_{t=0} = \int_0^1 \frac{s^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, \mathrm{d}s.$$

If $x \in C(J)$ and $0 \le t_1 < t_2 \le 1$, then

$$\big| I_{0+}^{\beta} I_{1-}^{\alpha} x(t) \big| \le \frac{\|x\|}{\Gamma(\beta+1)\Gamma(\alpha+1)}, \quad t \in J, \tag{2.1}$$

$$\Big| I_{0+}^{\beta} I_{1-}^{\alpha} x(t)\big|_{t=t_2} - I_{0+}^{\beta} I_{1-}^{\alpha} x(t)\big|_{t=t_1} \Big| \le \frac{2\|x\|}{\Gamma(\beta+1)\Gamma(\alpha+1)} (t_2 - t_1)^\beta.$$

Lemma 2.1. *$\mathcal{H}$ is a completely continuous operator.*

The following result gives the relation between fixed points of $\mathcal{H}$ and solutions to problem (1.1), (1.2).

Lemma 2.2. *If $(x, c) \in X$ is a fixed point of $\mathcal{H}$, then x is a solution of problem* (1.1), (1.2).

Proof. Let $\mathcal{H}(x, c) = (x, c)$ for some $(x, c) \in X$. Then

$$x(t) = c\Big(1 + \frac{t^\beta}{\Gamma(\beta+1)}\Big) + I_{0+}^{\beta} I_{1-}^{\alpha} f(t, x(t)), \ \ t \in J, \tag{2.2}$$

$$I_{1-}^{\alpha} f(t, x(t))\Big|_{t=0} = 0. \tag{2.3}$$

Applying ${}^c\!D_{0+}^{\beta}$ to (2.2), we get

$$ {}^c\!D_{0+}^{\beta} x(t) = c + I_{1-}^{\alpha} f(t, x(t)), \ \ t \in J. \tag{2.4}$$

Hence ${}^c\!D_{0+}^{\beta} x \in C(J)$, ${}^c\!D_{0+}^{\beta} x(t)|_{t=1} = c$ and (see (2.3)) ${}^c\!D_{0+}^{\beta} x(t)|_{t=0} = c$. We now apply ${}^c\!D_{1-}^{\alpha}$ to (2.4) and have

$$ {}^c\!D_{1-}^{\alpha}\, {}^c\!D_{0+}^{\beta} x(t) = f(t, x(t)), \ \ t \in J.$$

Thus x is a solution of equation (1.1). From

$$ {}^c\!D_{0+}^{\beta} x(t)\Big|_{t=1} = c, \quad {}^c\!D_{0+}^{\beta} x(t)\Big|_{t=0} = c$$

and (see (2.2)) $x(0) = c$ it follows that x satisfies (1.2). Consequently, x is a solution of problem (1.1), (1.2). $\square$

3 Existence result

Theorem 3.1. *Suppose that*

(H_1) *there exists* $M>0$ *such that* $xf(t,x)>0$ *for* $t\in J$ *and* $|x|\geq M$;

(H_2) *there exist positive constants* A, B *and* $\rho\in(0,1)$ *such that* $|f(t,x)|\leq A+B|x|^\rho$ *for* $t\in J$ *and* $x\in\mathbb{R}$.

Then problem (1.1), (1.2) *has at least one solution.*

Proof. Keeping in mind Lemma 2.2, we need to prove that $\mathcal{H}$ admits a fixed point. We prove the existence of a fixed point of $\mathcal{H}$ by the Schaefer fixed point theorem [1, 4]. To this end, let

$$\Omega=\Big\{(x,c)\in X:\ \ (x,c)=\lambda\mathcal{H}(x,c)\ \text{ for some }\ \lambda\in(0,1)\Big\}.$$

Since $\mathcal{H}$ is a completely continuous operator, it follows from the Schaefer fixed point theorem that the boundedness of Ω in X guarantees the existence of a fixed point of $\mathcal{H}$.

Let $(x,c)=\lambda\mathcal{H}(x,c)$ for some $(x,c)\in X$ and $\lambda\in(0,1)$, that is,

$$x(t)=\lambda c\Big(1+\frac{t^\beta}{\Gamma(\beta+1)}\Big)+\lambda I^\beta_{0+}I^\alpha_{1-}f(t,x(t)),\ \ t\in J, \tag{3.1}$$

$$(1-\lambda)c=-\lambda I^\alpha_{1-}f(t,x(t))\Big|_{t=0}. \tag{3.2}$$

We claim that

$$|x(\xi)|<M\ \text{ for some }\ \xi\in J, \tag{3.3}$$

where M is from (H_1). By (3.1), $x(0)=\lambda c$. Suppose that $x>M$ on J. Then $c>0$ and, by (H_1), $I^\alpha_{1-}f(t,x(t))|_{t=0}>0$, contrary to (3.2) because $(1-\lambda)c>0$ and $I^\alpha_{1-}f(t,x(t))|_{t=0}>0$. Similarly, $x<-M$ on J gives contrary to (3.2). Hence (3.3) is valid.

Putting $t=\xi$ in (3.1), we have

$$\lambda c=\frac{1}{1+\xi^\beta/\Gamma(\beta+1)}\,\Big(x(\xi)-\lambda I^\beta_{0+}I^\alpha_{1-}f(t,x(t))\big|_{t=\xi}\Big). \tag{3.4}$$

Thus (see (3.1))

$$x(t)=\frac{1+t^\beta/\Gamma(\beta+1)}{1+\xi^\beta/\Gamma(\beta+1)}\,\Big(x(\xi)-\lambda I^\beta_{0+}I^\alpha_{1-}f(t,x(t))\big|_{t=\xi}\Big)+\lambda I^\beta_{0+}I^\alpha_{1-}f(t,x(t)),\ \ t\in J.$$

Hence (see (H_2), (2.1) and (3.3))

$$|x(t)|\leq\Big(1+\frac{1}{\Gamma(\beta+1)}\Big)\Big(M+\frac{A+B\|x\|^\rho}{\Gamma(\beta+1)\Gamma(\alpha+1)}\Big)+\frac{A+B\|x\|^\rho}{\Gamma(\beta+1)\Gamma(\alpha+1)}\,,\ \ t\in J.$$

In particular,

$$\|x\|\leq W_1+W_2\|x\|^\rho, \tag{3.5}$$

where

$$W_1=M\Big(1+\frac{1}{\Gamma(\beta+1)}\Big)+\frac{A}{\Gamma(\beta+1)\Gamma(\alpha+1)}\Big(2+\frac{1}{\Gamma(\beta+1)}\Big),$$
$$W_2=\frac{B}{\Gamma(\beta+1)\Gamma(\alpha+1)}\,\Big(2+\frac{1}{\Gamma(\beta+1)}\Big).$$

Since

$$\lim_{v\to\infty} \frac{W_1 + W_2 v^\rho}{v} = 0,$$

there exists $S > 0$ such that $W_1 + W_2 v^\rho < v$ or $v > S$. Consequently (see (3.5)), $\|x\| \le S$.

Hence $|f(t, x(t))| \le L$ for $t \in J$, where $L = A + BS^\rho$. In order to give the bound for c, we consider two cases if $\lambda \in (0, 1/2]$ of $\lambda \in (1/2, 1)$. Let $\lambda \in (0, 1/2]$. Then (see (3.2))

$$|c| \le \frac{\lambda}{1-\lambda} \int\limits_0^1 \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, |f(s, x(s))| \, \mathrm{d}s \le \frac{L}{\Gamma(\alpha+1)} \, .$$

Let $\lambda \in (1/2, 1)$. Then (see (3.4))

$$|c| \le \frac{1}{\lambda(1+\xi^\beta/\Gamma(\beta+1))} \Big(|x(\xi)| + \big|I^\beta_{0+} I^\alpha_{1-} f(t, x(t))|_{t=\xi}\big|\Big) \le 2\Big(M + \frac{L}{\Gamma(\beta+1)\Gamma(\alpha+1)}\Big).$$

To summarize, we have $|c| \le D$, where

$$D = \max\left\{\frac{L}{\Gamma(\alpha+1)}\,, 2\Big(M + \frac{L}{\Gamma(\beta+1)\Gamma(\alpha+1)}\Big)\right\}.$$

As a result, Ω is bounded and $\|x\| \le S$, $|c| \le D$ for $(x, c) \in \Omega$. $\square$

Example 3.2. Let $p \in C(J)$, $\rho \in (0, 1)$ and $f(t, x) = p(t) + \sin x + 2|x|^\rho \arctan x$. Then f satisfies conditions (H_1) and (H_2) for $M = \sqrt[\rho]{1 + \|p\|}$ and $A = 1 + \|p\|$, $B = \pi$. By Theorem 3.1, there exists a solution x of the equation

$${}^c\!D^\alpha_{1-}\, {}^c\!D^\beta_{0+} x(t) = p(t) + \sin x(t) + 2|x(t)|^\rho \arctan x(t),$$

satisfying the boundary condition (1.2).

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Existence of Bounded Solutions of a Dynamic Equation

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1 Basic concepts of the theory of time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real line $\mathbb{R}$. Assume $\mathbb{T}$ has the topology that it inherits from $\mathbb{R}$ with the standard topology.

Since the object of our study is the oscillations of solutions of dynamic equations, we will assume $\sup\mathbb{T}=\infty$. For any interval $[a,b]\subset\mathbb{R}$ we define $[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}$.

For a time scale $\mathbb{T}$, the *forward jump operator* $\sigma(t):\mathbb{T}\to\mathbb{T}$ is defined as $\sigma(t)=\inf\{s\in\mathbb{T}:\ s>t\}$; the *backward jump operator* $\rho(t):\mathbb{T}\to\mathbb{T}$ is defined as $\rho(t)=\sup\{s\in\mathbb{T}:\ s<t\}$. The *graininess function* $\mu:\mathbb{T}\to[0,1)$ is defined as $\mu(t):=\sigma(t)-t$.

A point $t\in\mathbb{T}$ is called *right-dense* if $t>\inf\mathbb{T}$ and $\rho(t)=t$. A point $t\in\mathbb{T}$ is called *left-dense* if $t<\sup\mathbf{T}$ and $\sigma(t)=t$. Points that are right- and left-dense at the same time are called *dense*.

If $\sigma(t)>t$ ($\rho(t)<t$), we say that t is *right-scattered* (*left-scattered*). Points that are right- and left-scattered at the same time are called *isolated points*.

If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k=\mathbb{T}\setminus\{M\}$; otherwise, $\mathbb{T}^k=\mathbb{T}$.

A function $f:\mathbb{T}\to\mathbb{R}^d$ is called Δ-*differentiable* at $t\in\mathbb{T}^k$ if there exists the finite in $\mathbb{R}^d$ limit

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(\sigma(t))-f(t)}{\sigma-t}\,,$$

and the number $f^{\Delta}(t)$ is called the Δ-*derivative* at the point t.

We cite some known results [1]:

(a) If $t\in\mathbb{T}^k$ is a right-dense point of a time scale $\mathbb{T}$, then f is Δ-differentiable at t iff the limit

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists in $\mathbb{R}^d$.

(b) If $t\in\mathbb{T}^k$ is a right-scattered point of a time scale $\mathbb{T}$ and f is continuous at t, then f is Δ-differentiable at t and

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}\,.$$

2 Problem statement and auxiliary results

We consider the system of differential equations

$$\frac{dx}{dt} = X(t,x), \tag{2.1}$$

where $x \in D$, $D \subset \mathbb{R}^d$, and the corresponding system of dynamic equations

$$x_\lambda^\Delta = X(t, x_\lambda), \tag{2.2}$$

where $t \in \mathbb{T}_\lambda$, $\lambda \in \Lambda \subset \mathbb{R}$, $\lambda = 0$ is a limit point of the set Λ, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^d$, and $x_\lambda^\Delta(t)$ is the Δ-derivative of $x_\lambda(t)$ in $\mathbb{T}_\lambda$.

Assume that $X(t,x)$ is continuously differentiable and bounded with its partial derivatives, i.e. there exists $C > 0$ such that

$$|X(t,x)| + \left|\frac{\partial X(t,x)}{\partial t}\right| + \left\|\frac{\partial X(t,x)}{\partial x}\right\| \le C \tag{2.3}$$

for $t \in \mathbb{T}_\lambda$ and $x \in D$. Here $\frac{\partial X}{\partial x}$ is the corresponding Jacobian matrix, $|\cdot|$ is the Euclidian norm of a vector, and $\|\cdot\|$ is the norm of a matrix.

Let $\mu_\lambda := \sup\limits_{t \in \mathbb{T}_\lambda} \mu_\lambda(t)$, where $\mu_\lambda : \mathbb{T}_\lambda \to [0, \infty)$ is the graininess function. If $\mu_\lambda \to 0$ as $\lambda \to 0$, then $\mathbb{T}_\lambda$ approaches the continuous time scale $\mathbb{T}_0 = \mathbb{R}$. Therefore, it is natural to expect that, under certain conditions, the existence of a bounded solution of equation (2.1) implies the existence of a bounded solution of equation (2.2) on the time scale $\mathbb{T}_\lambda$.

Let $t_0, t_0 + T \in \mathbb{T}_\lambda$, and let $x(t)$ and $x_\lambda(t)$ be solutions of (2.1) and (2.2) on $[t_0, t_0 + T]$ and on $[t_0, t_0+T]_{\mathbb{T}_\lambda}$, respectively, with initial conditions $x(t_0) = x_0$, $x_\lambda(t_0) = x_{\lambda 0}$.

Lemma 2.1 ([3])**.** *If x_λ and $x(t)$ are the corresponding solutions of* (2.2) *and* (2.1)*, then the inequality*

$$|x(t) - x_\lambda(t)| \le \mu(\lambda) K(T) \tag{2.4}$$

holds for $t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}$. Here

$$\mu(\lambda) = \sup_{t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}} \mu_\lambda(t), \quad K(T) = \max\{L_1, L_2\},$$

$$L_1 = \mu_\lambda\Big(\Pi e^C C_1 + \frac{1}{4}\,\Pi C_1^2 e^C\Big), \quad L_2 = \mu_\lambda\Big(\Pi e^C\Big(C_1 + \frac{C_1^2}{4}\Big) + 3C_1\Big).$$

Under condition (2.3), the following statement holds.

Lemma 2.2 ([3])**.** *A solution x_λ of system* (2.2) *continuously depends on the initial data until the moment it leaves the region D.*

We also give the definition of the exponential stability for solutions of dynamic equations on time scales which is similar to the definition of the exponential stability for solutions of differential equations [2].

Definition 2.1. A solution $x_\lambda(t)$ of system (2.2), defined on $\mathbb{T}_\lambda$, is called exponentially stable, uniformly in t_0, if there exist $\delta > 0$, $N > 0$ and $\alpha > 0$ such that for any solution $y_\lambda(t)$ of system (2.2), satisfying

$$|x_\lambda(t_0) - y_\lambda(t_0)| < \delta,$$

the inequality

$$|x_\lambda(t) - y_\lambda(t)| \le N e^{-\alpha(t - t_0)} \big|x_\lambda(t_0) - y_\lambda(t_0)\big|$$

holds for $t \ge t_0$. Here the constants δ, N and α are independent of t_0.

3 Main results

We found the minimum conditions on the graininess function μ_0 under which the existence of a bounded solution of the dynamical system (2.2) on the corresponding time scale $\mathbb{T}_{\lambda_0}$ implies the existence of a bounded solution of this system on any scale whose graininess function is less than μ_0.

Theorem 3.1. *Let the following conditions be satisfied:*

(1) *$X(t,x)$ is defined and continuously differentiable for $t \in \mathbb{R}$, $x \in D$, where D is a domain in the space $\mathbb{R}^d$, and satisfies condition* (2.3)*.*

(2) *There exists $\mu_0 > 0$ such that system* (2.2) *has a bounded on $\mathbb{T}_{\lambda_0}$ and exponentially stable, uniformly in t_0, solution $x_{\lambda_0}(t)$, which belongs to D together with some its ρ-neighborhood.*

Then, if the inequalities

$$\mu_0 K\Big(\frac{\ln 4N}{\alpha}+1\Big) \le \frac{\delta}{8}, \tag{3.1}$$

$$\frac{3N\delta}{2} < \rho, \tag{3.2}$$

$$\mu_0 \le \frac{\rho}{4C} \tag{3.3}$$

hold, where δ, N and α are the constants from Definition 2.1 and C is from condition (2.3)*, then, for all μ_λ satisfying $\mu_\lambda < \mu_0$, system* (2.2) *has a solution bounded on $\mathbb{T}_\lambda$.*

Proof. Without loss of generality, we set $t_0 = 0$ and $x_\lambda(0) = x(0)$.

It follows from condition (2) of this theorem that, for $\mu_\lambda = \mu_0$, system (2.2) has an exponentially stable, uniformly in t_{k_0}, solution x_{λ_0}, which belongs to D together with some its ρ-neighborhood. Hence, there exists a constant $C_0 > 0$ such that

$$|x_{\lambda_0}(t_k)| \le C_0 \ \text{ for an arbitrary } t_k \in \mathbb{T}_{\lambda 0}.$$

Let t_{k_0} be the smallest number on the time scale $\mathbb{T}_{\lambda_0}$, defined by the graininess function μ_0, such that $t_{k_0} \ge \frac{\ln 4N}{\alpha}$. Clearly, $t_{k_0} \le \frac{\ln 4N}{\alpha} + 1$.

Now we fix $0 < \mu_\lambda < \mu_0$ and denote by x_λ solutions of system (2.2) on the corresponding time scale $\mathbb{T}_\lambda$.

Let us consider points $t \in [0, t_{k_0}]_{\mathbb{T}_\lambda}$. For every t one can indicate the smallest number t_k on the scale $\mathbb{T}_{\lambda_0}$ such that

$$0 \le t_k - t \le \mu_0.$$

Let x_λ be a solution of system (2.2) such that $x_\lambda(0) = x_{\lambda_0}(0)$. We denote by $x(t)$ the solution of system (2.1) with the initial data $x(0) = x_{\lambda_0}(0)$. We can show that $x(t)$ can be continued to the interval $[0, t_{k_0}]$. Indeed, in view of inequality (3.3) and Lemma 2.1, it follows from Picard's theorem that $x(t)$ is defined at each point $n\mu_0 \le t_{k_0}$, $n \in \mathbb{N}$, and takes on the values which belong to D together with their $\frac{\rho}{2}$-neighborhoods. Thus, the solution $x(t)$ is continued to the whole interval $[0, t_{k_0}]$ and belongs to D together with its $\frac{\rho}{2}$-neighborhood. It follows from (2.4) and (3.1) that the solution x_λ of system (2.2) is defined for all $t \in \mathbb{T}_\lambda$ that do not exceed $t_{k_0} \in \mathbb{T}_{\lambda_0}$, and belongs to the domain D.

Further, we partition the axis into the intervals $[nt_{k_0}, (n+1)t_{k_0}]$ and denote by t_n the largest numbers in $\mathbb{T}_\lambda$ such that $t_n \le nt_{k_0}$. Let us examine how the solution $x_\lambda(t)$ of equation (2.2), starting at t_n, $n \in \mathbb{N}$, behaves on $[t_n, (n+1)t_{k_0})]_{\mathbb{T}_\lambda}$.

Let us now construct a solution of equation (2.2) which is bounded on the whole axis of the timescale $\mathbb{T}_\lambda$.

Let x_{λ,t^*} be such a solution of equation (2.2) which starts at a point t^* of $\mathbb{T}_\lambda$, $t \geq t^*$, and $x_{\lambda,t^*}(t^*) = x_\lambda(t^*)$.

For each t^* we choose the smallest non-negative number $\tilde{t}_{\lambda_0} \in \mathbb{T}_{\lambda_0}$ such that $t^* \leq \tilde{t}_{\lambda_0} \leq t^* + \mu_0$.

We now consider a solution $x_{\lambda,t^*}(t)$ such that $|x_{\lambda,t^*}(t^*) - x_{\lambda_0}(\tilde{t}_{\lambda_0})| \leq \frac{3\delta}{4}$, where x_{λ_0} is a bounded solution of equation (2.2) on the timescale $\mathbb{T}_{\lambda_0}$ with the graininess function $\mu_\lambda = \mu_0$, which is indicated in the statement of this theorem.

We partition the left semi-axis of the timescale $\mathbb{T}_\lambda$ into the intervals $[-nt_{k_0}, -(n+1)t_{k_0}]$, $n \to -\infty$. For each point $-nt_{k_0}$ we choose the largest $t_n \in \mathbb{T}_\lambda$ such that

$$t_n \leq -nt_{k_0} \leq t_n + \mu_0.$$

The point t_0 is chosen in the same way.

Let us now consider the set of solutions x_{λ,t_n} of equation (2.2), whose initial data satisfy the inequality

$$\left|x_{\lambda,t_n}(t_n) - x_{\lambda_0}(-nt_{k_0})\right| \leq \frac{3\delta}{4}.$$

Obviously, these solutions satisfy conditions 1°–3°. Let S_n be the set of values of these solutions at t_n. Each S_n is the image of the ball of radius $\frac{3\delta}{4}$ centered at the point $x_{\lambda_0}(-nt_{k_0})$, generated by the mapping x_{λ,t_n}. By Lemma 2.2 and conditions 1°-3°, each set S_n is a nonempty subset of S_{n-1} and a compact.

Let us denote $z = \bigcap_n S_n$ and consider the solution x_{λ,t_1} of equation (2.2) with the initial condition $x_{\lambda,t_1}(t_1) = z$. This solution can be continued to the left to the point t_n, at which it belongs to the $\frac{3\delta}{4}$-neighborhood of $x_{\lambda_0}(t_n)$ for every natural n. It means that this solution is defined for all t satisfying the inequality in 3°. Hence, it is bounded. This proves that system (2.2) has a bounded solution, defined on $\mathbb{T}_\lambda$. □

The following statement provides the conditions for the existence of a solution of system (2.2) bounded on $\mathbb{T}_\lambda$ given the existence of such a solution of the corresponding system (2.1).

Theorem 3.2. *Let the following conditions be satisfied:*

(1) *$X(t,x)$ is defined and continuously differentiable for $t \in \mathbb{R}$, $x \in D$, where D is a domain in $\mathbb{R}^d$, and satisfies condition* (2.3)*.*

(2) *System* (2.1) *has a bounded on $\mathbb{R}$ and exponentially stable, uniformly in $t_0 \in \mathbb{R}$, solution $x(t)$, which belongs to D together with some its ρ-neighborhood.*

Then there exists μ_0 such that for all $0 < \mu_\lambda \leq \mu_0$ system (2.2) *has a solution $x_\lambda(t)$ bounded on $\mathbb{T}_\lambda$. Moreover,*

$$\sup_{t \in \mathbb{T}_\lambda} |x_\lambda(t) - x(t)| \to 0, \quad \mu_\lambda \to 0.$$

The existence of $\mu_0 > 0$, such that for all $0 < \mu_\lambda \leq \mu_0$ system (2.2) has a solution $x_\lambda(t)$ bounded on $\mathbb{T}_\lambda$, follows from Theorem 2.3 [3].

We also obtained the opposite result.

Theorem 3.3. *Let the following conditions be satisfied:*

(1) *the function $X(t,x)$ satisfies condition (1) of Theorem 3.1;*

(2) *there exists* $\mu_0 > 0$ *such that system* (2.2) *with initial data at the point* $t_0 = 0$ *has a solution, bounded on* $\mathbb{T}_{\lambda_0}$ *and uniformly in* k_0 *exponentially stable, which belongs to* D *with some its* ρ*-neighborhood.*

Then, if inequalities (3.1)–(3.3) *hold, then system* (2.1) *has a solution bounded on* $\mathbb{R}$.

The proof of this theorem is based on the reasoning in the proof of Theorem 3.1.

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Localization Property of Solutions for Parabolic PDE

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1 Introduction

We investigate the Cauchy–Dirichlet problem for a wide class of quasi-linear parabolic equations with the model representative:

$$u_t - \triangle u + g(t)|u|^{q-1}u = 0, \ \ 0 < q < 1,$$

where the continuous absorption potential $g(t)$ is positive for $t > 0$ and degenerates at $t = 0$: $g(0) = 0$. For an arbitrary boundary regime (without any subordination conditions), a certain type of weakened localization is obtained. Under some restriction from below on the degeneration of the potential, the strong localization holds for an arbitrary boundary regime (including regimes that do not satisfy any conditions of subordination).

It is well-known that, in case of non-degenerate absorption potential $g(t,x)$, i.e., when

$$g(t,x) \geq c_0 > 0 \ \ \forall\,(t,x) \in (0,T] \times \overline{\Omega},$$

an arbitrary energy solution of the considered problem has the finite-speed propagation property for solution's support:

$$\zeta(t) := \sup\big\{|x| :\ x \in \operatorname{supp} u(t,\,\cdot\,)\big\} < 1 + c(t), \ \text{ where } \ c(t) \to 0 \ \text{ as } \ t \to 0.$$

In particular, this implies the localization of solution (see, e.g., [3,5] and the references therein):

$$\zeta(t) := \sup\big\{|x| : \ \ x \in \operatorname{supp} u(t,\,\cdot\,)\big\} < c_1 = c_1(T_1) < l \ \ \forall\, t :\ 0 \leq t < T_1 = T_1(l) \leq T. \tag{1.1}$$

For various quasi- and semi-linear parabolic equations, the localization of solutions' supports were studied by many authors (see, e.g., [3,7] and the references therein). It seems that Kalashnikov [8] was the first who investigated the localization property for the first initial-boundary problem for a 1-D heat equation. More precisely, he considered problem (2.1)–(2.4) in the domain $[1,+\infty)$ with $n = 1$,

$$a_i(t,x,s,\xi) = \xi, \ \ \xi \in \mathbb{R}^1,$$
$$g(t,x) = g_0(t) \in C^1([1,+\infty)) \cap L_\infty([1,+\infty)), \ \ g_0(0) = 0, \ \ g_0(t) > 0 \ \ \forall\, t > 0$$

and

$$u(t,1)=f(t)\in C^1([1,+\infty))\cap L_\infty([1,+\infty)).$$

Under the assumption

$$g_0(t)^{-1}\cdot f(t)\to 0 \ \text{ as } \ t\to 0, \tag{1.2}$$

he proved that solutions possess weak localization property for t separated from 0:

$$\sup\big\{\zeta(t):\ 0<\delta\le t<T\big\}<c_1=c_1(\delta)<\infty \ \ \forall\,\delta>0.$$

On the other side, following G. I. Barenblatt's conjecture on an initial jump of the free boundary, Kalashnikov in [8] proved that

$$\inf\big\{\zeta(t):\ 0<t<t_*\big\}\ge c_2=c_2(t_*)>0, \tag{1.3}$$

if potential $g_0(t)$ decreases fast enough when $t\to 0$. In particular, the free boundary has an initial jump (1.3), when

$$g_0(t)=t^{\frac12}\exp\Big(-\frac{1}{t^2}\Big),\quad f_0(t)=t\exp\Big(-\frac{1}{t^2}\Big).$$

The analysis of [8] concerns only the case of strongly degenerating boundary regimes $f(t)$ (see condition (1.2)). Method [12] involutes arbitrary $f(t)$, which are strongly degenerate, weakly degenerate as well as non-degenerate as $t\to 0$. Also, note that the barrier technique of [8] can be applied only to equations that admit the comparison theorems. Our approach is adaptation and combination of a variant of local energy method and an estimate method of Saint–Venant's principle type. These methods are the result of a long evolution of ideas coming from the theory of linear elliptic and parabolic equations. The essence of the energy method consists of special inequalities links different energy norms of solutions. This method was developed and used in [2, 4, 5, 9, 11, 12]. The second approach is a technique of parameter's introduction. This method was offered by G. A. Iosif'jan and O. A. Oleinik [6].

2 Setting of the problem and the main results

Let $Q_T=(0,T)\times\Omega$, $0<T<\infty$, $\Omega\subset\{x\in\mathbb{R}^n:\ x|>1\}$ be a bounded domain in $\mathbb{R}^n$, $n\geqslant 1$, with C^1-boundary $\partial\Omega=\partial_0\Omega\cup\partial_1\Omega$, where

$$\partial_0\Omega=\big\{x\in\mathbb{R}^n:\ |x|=1\big\},\quad \partial_1\Omega\subset\big\{x\in\mathbb{R}^n:\ |x|>l\big\},\ \text{ where } \ l=\mathit{const}>1. \tag{2.1}$$

The aim of this brief communication is to investigate the behavior of weak solutions of the following initial-boundary problem:

$$u_t-\sum_{i=1}^n\big(a_i(t,x,u,\nabla_x u)\big)_{x_i}+g(t,x)|u|^{q-1}u=0 \ \text{ in } Q_T,\ \ 0<q<1, \tag{2.2}$$

$$u(t,x)=f(t,x) \ \text{ on } (0,T)\times\partial_0\Omega,\quad u(t,x)=0 \ \text{ on } (0,T)\times\partial_1\Omega, \tag{2.3}$$

$$u(0,x)=0\ \ \forall\,x\in\Omega. \tag{2.4}$$

Here the functions $a_i(t,x,s,\xi)$ $(i=1,\dots,n)$ are continuous in all arguments and satisfy the following conditions for $(t,x,s,\xi)\in(0,T)\times\Omega\times\mathbb{R}^1\times\mathbb{R}^n$:

$$|a_i(t,x,s,\xi)|\le d_1|\xi|,\ \ d_1=\mathit{const}<\infty,$$

$$\sum_{i=1}^n\big(a_i(t,x,s,\xi)-a_i(t,x,s,\eta)\big)(\xi_i-\eta_i)\ge d_0|\xi-\eta|^2,\ \ d_0=\mathit{const}>0.$$

The absorption potential $g(t,x)$ is continuous nonnegative function such that

$$g(t,x) > 0 \ \ \forall\,(t,x) \in (0,T] \times \overline{\Omega}; \quad g(0,x) = 0 \ \ \forall\ x \in \overline{\Omega}. \tag{2.5}$$

Without loss of generality assume that the function $f(t,x)$ in (2.3) is defined in the domain $(0,T)\times\Omega$ and

$$f(t,\,\cdot\,) \in L_2(0,T;H^1(\Omega,\partial_1\Omega)) \cap H^1(0,T;L_2(\Omega)).$$

Following [1], by a weak solution of problem (2.1)–(2.4) we understand the function

$$u(t,\,\cdot\,) \in f(t,\,\cdot\,) + L_2\big(0,T;H^1(\Omega,\partial\Omega)\big)$$

such that

$$u_t(t,\,\cdot\,) \in L_2(0,T;(H^1(\Omega,\partial\Omega))^*),$$

and u satisfies (2.3), (2.4) and the integral identity

$$\int\limits_{(0,T)} \langle u_t,\xi\rangle\,dt + \int\limits_{(0,T)\times\Omega} \sum_{i=1}^{n} a_i(t,x,u,\nabla_x u)\xi_{x_i}\,dx\,dt + \int\limits_{(0,T)\times\Omega} g(t,x)|u|^{q-1}u\xi\,dx\,dt = 0$$
$$\forall\,\xi \in L_2(0,T;H^1(\Omega,\partial\Omega)).$$

With boundary regime $f(t,x)$ from (2.3), we associate the function:

$$F(t) := \sup_{0\leqslant s\leqslant t} \int\limits_{\Omega} f(s,x)^2\,dx + \int\limits_{0}^{t}\int\limits_{\Omega} \left(|\nabla_x f|^2 + g(t,x)|f(t,x)|^{q+1}\right)dx\,dt + \int\limits_{0}^{t}\int\limits_{\Omega} |f_t(t,x)|^2\,dx\,dt \tag{2.6}$$

which will be used in all of our main results.

Theorem 1 (Theorem [Weakened localization for an arbitrary regime)**.** *Let the absorption potential g from* (2.2) *satisfy condition* (2.5). *Then an arbitrary energy solution $u(t,x)$ to problem* (2.1)–(2.4) *possesses the weakened localization property. That is, there exists $\zeta_1(t) \in C(0,\infty)$ such that*

$$\zeta(t) \le \min(\zeta_1(t), cL_1) \ \ \textit{for all} \ \ t > 0,$$

where $\zeta(\,\cdot\,)$ is the compactification radius defined from (1.1) *and* $L_1 = \operatorname{diam}\Omega$.

The function $\zeta_1(t)$ may go to infinity as $t \to 0$. That is, an infinite initial jump of the support is possible.

Theorem 2 (Strong localization for an arbitrary regime)**.** *Let the function $F(\,\cdot\,)$ be from* (2.6), *the absorption potential g from* (2.2) *have a nonnegative monotonic minorant:*

$$g(t,x) \ge g_\omega(t) := \exp\Big(-\frac{\omega(t)}{t}\Big) \ \ \forall\, t > 0,$$

where $\omega(t)$ is a nonnegative nondecreasing function such that $\omega(t) \to 0$ as $t \to 0$. Then an arbitrary energy solution $u(t,x)$ of problem (2.1)–(2.4) *possesses the strong localization property and the following upper estimate holds:*

$$\zeta(t) \le 1 + \frac{t}{2} + c_1\Big\{t\ln(c_2F(t)) + c_3 t\ln t^{-1} + c_4\omega\Big(\frac{t}{2}\Big)\Big\}^{\frac{1}{2}} \ \ \forall\, t < T.$$

Let us notice that in both theorems we do not impose any conditions on the function $F(\,\cdot\,)$ from (2.6).

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On the Optimization Problem for One Class of Controlled Functional Differential Equation with the Mixed Initial Condition

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In the paper, necessary conditions of optimality of the initial and final moments, delay parameters, the initial vector and initial functions, the control function are obtained for the optimization problem containing the nonlinear functional differential equation with constant delays in the phase coordinates and controls.

Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$. Suppose that $P\subset\mathbb{R}^k$, $Q\subset\mathbb{R}^m$, $V\subset\mathbb{R}^r$ are convex and open sets, with $k+m=n$, $x=(p,q)^T\in O=(P,Q)^T$. Let $a_{11}<a_{12}<a_{21}<a_{22}$, $\tau_2>\tau_1>0$, $\sigma_2>\sigma_1>0$, $\theta_2>\theta_1>0$ be given numbers, with $a_{21}-a_{12}>\tau_2$; let $I=[a_{11},a_{22}]$, $I_1=[\widehat{\tau},a_{12}]$ and $I_2=[a_{11}-\theta_2,a_{22}]$, where $\widehat{\tau}=a_{11}-\max\{\tau_2,\sigma_2\}$. Furthermore, let the n-dimensional function $f(t,x,p,q,u,v)$ be continuous on $I\times O\times P\times Q\times V^2$, and continuously differentiable with respect to (x,p,q,u,v). Denote by $AC_\varphi(I_1,P)$ the space of absolutely continuous functions $\varphi:I_1\to\mathbb{R}^k$, with $|\dot{\varphi}(t)|\le const$. Let us introduce the sets:

$$\Phi=AC_\varphi(I_1,K),\ \ G=AC_g(I_1,M),\ \ \Omega=AC_u(I_2,U),$$

where $K\subset P$, $M\subset Q$ and $U\subset V$ are convex and compact sets. To any element

$$\begin{aligned}w=(t_0,t_1,\tau,\sigma,\theta,p_0,\varphi,g,u)\in W\\ =(a_{11},a_{12})\times(a_{21},a_{22})\times(\tau_1,\tau_2)\times(\sigma_1,\sigma_2)\times(\theta_1,\theta_2)\times P_0\times\Phi\times G\times\Omega,\end{aligned}$$

where $P_0\subset P$ is a convex and compact set, we assign the nonlinear controlled functional differential equation with delays in the phase coordinates and controls

$$\dot{x}(t)=f\big(t,x(t),p(t-\tau),q(t-\sigma),u(t),u(t-\theta)\big),\ \ t\in[t_0,t_1] \tag{1}$$

with the mixed initial condition

$$\begin{cases}x(t)=(p(t),q(t))^T=(\varphi(t),g(t))^T, & t\in[\widehat{\tau},t_0),\\ x(t_0)=(p_0,g(t_0))^T.\end{cases} \tag{2}$$

Condition (2) is said to be the mixed initial condition, because it consists of two parts: the first part is $p(t)=\varphi(t)$, $t\in[\widehat{\tau},t_0)$, $p(t_0)=p_0$, the discontinuous part, since in general $p(t_0)\ne\varphi(t_0)$; discontinuity at the initial moment may be related to the instant change in a dynamic process, for example, changes of investment and environment etc; the second part is $q(t)=g(t)$, $t\in[\widehat{\tau},t_0]$, the continuous part, since always $q(t_0)=g(t_0)$.

Definition 1. Let $w = (t_0, t_1, \tau, \sigma, \theta, p_0, \varphi, g, u) \in W$. A function $x(t) = x(t; w) \in O, t \in [\widehat{\tau}, t_1]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element w, if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let the scalar-valued functions $z^i(t_0, t_1, \tau, \sigma, \theta, p, x)$, $i = \overline{0, l}$, be continuously differentiable on $[a_{11}, a_{12}] \times [a_{21}, a_{22}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times [\theta_1, \theta_2] \times P \times O$.

Definition 2. An element $w = (t_0, t_1, \tau, \sigma, \theta, p_0, \varphi, g, u) \in W$ is said to be admissible if the corresponding solution $x(t) = x(t; w)$ satisfies the boundary conditions

$$z^i(t_0, t_1, \tau, \sigma, \theta, p_0, x(t_1)) = 0, \;\; i = \overline{1, l}. \tag{3}$$

By W_0 we denote the set of admissible elements.

Definition 3. An element $w_0 = (t_{00}, t_{10}, \tau_0, \sigma_0, \theta_0, p_{00}, \varphi_0, g_0, u_0) \in W_0$ is said to be optimal if for an arbitrary element $w \in W_0$ the inequality

$$z^0\big(t_{00}, t_{10}, \tau_0, \sigma_0, \theta_0, p_{00}, x_0(t_{10})\big) \leq z^0\big(t_0, t_1, \tau, \sigma, \theta, p_0, x(t_1)\big), \tag{4}$$

where $x_0(t) = x(t; w_0)$, holds.

(1)–(4) is called the optimization problem for the functional differential equation (1) with the mixed initial condition (2).

Theorem 1. *Let w_0 be an optimal element and let $x_0(t) = (p_0(t), q_0(t))^T$, $t \in [\widehat{\tau}, t_{10}]$ be the corresponding solution. The function $\dot{g}_0(t)$ is continuous at the point t_{00}. Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution $\psi(t) = (\psi_1(t), \dots, \psi_n(t))$ of the equation*

$$\dot{\psi}(t) = -\psi(t) f_{0x}[t] - \psi(t + \tau_0)\big(f_{0p}[t + \tau_0], \Theta_{n\times m}\big) - \psi(t + \sigma_0)\big(\Theta_{n\times k}, f_{0q}[t + \sigma_0]\big), \;\; t \in (t_{00}, t_{10})$$

with the initial condition

$$\psi(t_{10}) = \pi Z_{0x}, \;\; \psi(t) = 0, \;\; t > t_{10},$$

where $\Theta_{n\times m}$ is the $n \times m$ zero matrix and $Z = (z^0, \dots, z^l)^T$,

$$Z_{0x} = \frac{\partial Z(t_{00}, t_{10}, \tau_0, \sigma_0, \theta_0, p_{00}, x_0(t_{10}))}{\partial x},$$
$$f_{0x}[t] = f_x\big(t, x_0(t), p_0(t - \tau_0), q_0(t - \sigma_0), u_0(t), u_0(t - \theta_0)\big),$$

such that the following conditions hold:

1) *the condition for the initial moment t_{00}*

$$\pi Z_{0t_0} + \big(\pi Z_{0q} + \big(\psi_{k+1}(t_{00}), \dots, \psi_n(t_{00})\big)\big)\dot{q}_0(t_{00}) = \psi(t_{00}) f_0[t_{00}] + \psi(t_{00} + \tau_0) f_1,$$

where

$$\begin{aligned} f_0[t] &= f\big(t, x_0(t), p_0(t - \tau_0), q_0(t - \sigma_0), u_0(t), u_0(t - \theta_0)\big), \\ f_1 &= f\Big(t_{00} + \tau_0, x_0(t_{00} + \tau_0), p_{00}, q_0(t_{00} + \tau_0 - \sigma_0), u_0(t_{00} + \tau_0), u_0(t_{00} + \tau_0 - \theta_0)\Big) \\ &\quad - f\Big(t_{00} + \tau_0, x_0(t_{00} + \tau_0), \varphi_0(t_{00}), q_0(t_{00} + \tau_0 - \sigma_0), u_0(t_{00} + \tau_0), u_0(t_{00} + \tau_0 - \theta_0)\Big); \end{aligned}$$

2) *the condition for the final moment t_{10}*

$$\pi Z_{0t_1} = -\psi(t_{10}) f_0[t_{10}];$$

3) *the condition for the delay* τ_0

$$\pi Z_{0\tau} = \psi(t_{00}+\tau_0)f_1 + \int\limits_{t_{00}}^{t_{10}} \psi(t) f_{0p}[t]\dot p_0(t-\tau_0)\,dt;$$

4) *the condition for the delay* σ_0

$$\pi Z_{0\sigma} = \int\limits_{t_{00}}^{t_{10}} \psi(t) f_{0q}[t]\dot q_0(t-\sigma_0)\,dt;$$

5) *the condition for the delay* θ_0

$$\pi Z_{0\theta} = \int\limits_{t_{00}}^{t_{10}} \psi(t) f_{0v}[t]\dot u_0(t-\theta_0)\,dt;$$

6) *the condition for the vector* p_{00}

$$\big(\pi Z_{0p} + (\psi_1(t_{00}),\dots,\psi_k(t_{00}))\big)p_{00} = \max_{p_0\in P_0}\big(\pi Z_{0p} + (\psi_1(t_{00}),\dots,\psi_k(t_{00}))\big)p_0;$$

7) *the condition for the initial function* $\varphi_0(t)$

$$\int\limits_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0) f_{0p}[t+\tau_0]\varphi_0(t)\,dt = \max_{\varphi\in\Phi}\int\limits_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0) f_{0p}[t+\tau_0]\varphi(t)\,dt;$$

8) *the condition for the initial function* $g_0(t)$

$$\begin{aligned}\big(\psi_{k+1}(t_{00}),\dots,\psi_n(t_{00})\big)g_0(t_{00}) &+ \int\limits_{t_{00}-\sigma_0}^{t_{00}} \psi[t+\sigma_0] f_{0q}[t+\sigma_0]g_0(t)\,dt\\ &= \max_{g\in G}\left[(\psi_{k+1}(t_{00}),\dots,\psi_n(t_{00}))g(t_0) + \int\limits_{t_{00}-\sigma_0}^{t_{00}} \psi(t+\sigma_0) f_{0q}[t+\sigma_0]g(t)\,dt\right];\end{aligned}$$

9) *the condition for the control function* $u_0(t)$

$$\int\limits_{t_{00}}^{t_{10}} \psi(t)\big[f_{0u}[t]u_0(t) + f_{0v}[t]u_0(t-\theta_0)\big]\,dt = \max_{u\in\Omega}\int\limits_{t_{00}}^{t_{10}} \psi(t)\big[f_{0u}[t]u(t) + f_{0v}[t]u(t-\theta_0)\big]\,dt.$$

Theorem 1 is proved by the scheme given in [2]. A problem with the mixed initial without optimization of delay parameters was considered in [1]. Now we consider a particular case of problem (1)–(4):

$$\begin{aligned}\dot x(t) &= (\dot p(t),\dot q(t))^T\\ &= A(t)x(t) + B(t)p(t-\tau) + C(t)q(t-\sigma) + D(t)u(t) + E(t)u(t-\theta),\ \ t\in[t_0,t_1],\end{aligned}\tag{5}$$

$$\begin{cases} x(t) = (p(t),q(t))^T = (\varphi(t),g(t))^T, \quad t\in[\widehat\tau,t_0),\\ x(t_0) = (p_0,g(t_0))^T.\end{cases}\tag{6}$$

$$z^i(\tau,\sigma,\theta,x(t_1)) = 0,\ \ i=\overline{1,l}\tag{7}$$

$$z^0(\tau,\sigma,\theta,x(t_1)) \to \min.\tag{8}$$

Here $A(t)$, $B(t)$, $C(t)$, $D(t)$ and $E(t)$ are the continuous matrix functions with dimensions $n \times n$, $n \times k$, $n \times m$, $n \times r$ and $n \times r$, respectively; t_0, t_1 are fixed moments; $\varphi(t)$, $g(t)$ are fixed initial functions; p_0 is a fixed initial function. In this case we have

$$w = (\tau, \sigma, \theta, u) \in W = (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times (\theta_1, \theta_2) \times \Omega \ \text{ and } \ w_0 = (\tau_0, \sigma_0, \theta_0, u_0);$$
$$Z(\tau, \sigma, \theta, x) = (z^0(\tau, \sigma, \theta, x), \dots, z^l(\tau, \sigma, \theta, x))^T, \quad Z_{0x} = \frac{\partial Z(\tau_0, \sigma_0, \theta_0, x_0(t_1))}{\partial x}\,.$$

Theorem 2. *Let w_0 be an optimal element for problem* (5)–(8)*. Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution $\psi(t) = (\psi_1(t), \dots, \psi_n(t))$ of the equation*

$$\dot\psi(t) = -\psi(t)A(t) - \psi(t+\tau_0)(B(t+\tau_0), \Theta_{n\times m}) - \psi(t+\sigma_0)(\Theta_{n\times k}, C(t+\sigma_0)), \ \ t \in (t_0, t_1)$$

with the initial condition

$$\psi(t_1) = \pi Z_{0x}, \ \ \psi(t) = 0, \ \ t > t_1,$$

such that the following conditions hold:

10) *the condition for the delay τ_0*

$$\pi Z_{0\tau} = \psi(t_0+\tau_0)[p_0 - \varphi(t_0)] + \int\limits_{t_0}^{t_{10}} \psi(t)B[t]\dot p_0(t-\tau_0)\,dt;$$

11) *the condition for the delay σ_0*

$$\pi Z_{0\sigma} = \int\limits_{t_0}^{t_1} \psi(t)C(t)\dot q_0(t-\sigma_0)\,dt;$$

12) *the condition for the delay θ_0*

$$\pi Z_{0\theta} = \int\limits_{t_0}^{t_1} \psi(t)E(t)\dot u_0(t-\theta_0)\,dt;$$

13) *the condition for the control function $u_0(t)$*

$$\int\limits_{t_0}^{t_1} \psi(t)\big[D(t)u_0(t) + E(t)[t]u_0(t-\theta_0)\big]\,dt = \max_{u\in\Omega} \int\limits_{t_0}^{t_1} \psi(t)\big[D(t)u(t) + E(t)u(t-\theta_0)\big]\,dt.$$

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Duality for Stieltjes Integral Equations

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1 Linear functionals and operators on the spaces of regulated functions

Let $-\infty < a < b < \infty$. $\mathbb{R}^{n\times n}$ is the space of real $n\times n$-matrices. Let BV^n and G^n be the spaces of n-vector valued functions with bounded variation on $[a,b]$ or regulated on $[a,b]$, respectively. (By regulated functions we understand functions having only discontinuities of the first kind.) Similarly, $\mathrm{BV}^{n\times n}$ and $\mathrm{G}^{n\times n}$ are spaces of of $n\times n$-matrix valued functions having the corresponding properties. The function $R:[a,b]\to\mathbb{R}^{n\times n}$ is said to be *summable* if it vanishes except for a countable set and $\sum\limits_{a\le t\le b}\|R(t)\|<\infty$.

Theorem 1.1. *If Φ is a continuous linear operator from G into $\mathbb{R}^n$ then there exist $K,\widetilde{K}\in BV^{n\times n}$, $M\in\mathbb{R}^{n\times n}$ and a summable function $R:[a,b]\to\mathbb{R}^{n\times n}$ such that*

$$\Phi(x)=M\,x(a)+\int\limits_a^b K\,\mathrm{d}x-\sum_{a\le t<b}R(t)\,\Delta^+x(t)\ \ \textit{for}\ \ x\in G^n$$

and

$$\Phi(x)=M\,x(a)+\int\limits_a^b \widetilde{K}\,\mathrm{d}x+\sum_{a<t\le b}R(t)\,\Delta^-x(t)\ \ \textit{for}\ \ x\in G^n.$$

Remark 1.1. $R(t)=\Phi(\chi_{[t]})$ and $\widetilde{K}(t)=K(t)-R(t)$ for $t\in[a,b]$.

The representation of linear bounded functionals in the space of left-continuous regulated functions is considerably simpler, as shown by the following older result from 1989, cf. [3]. ($\mathrm{G}^n_{\mathrm{L}}$ stands for the space of n-vector valued functions regulated on $[a,b]$, left-continuous on $(a,b]$ and right-continuous at a.)

Theorem 1.2. *Φ is a linear bounded operator from G^n_L into $\mathbb{R}^n$ if and only if there is $M\in\mathbb{R}^{n\times n}$ and an $n\times n$-matrix valued function K of bounded variation on $[a,b]$ such that*

$$\Phi(x)=M\,x(a)+\int\limits_a^b K\,\mathrm{d}[x]\ \ \textit{for}\ \ x\in G^n_L.$$

Later Š. Schwabik [7] generalized this result and described a general form of bounded linear operators on $\mathrm{G}^n_{\mathrm{L}}$.. In what follows $\mathcal{K}^{n\times n}_{\mathrm{L}}$ stands for the set of functions $K:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ such that:

- $K(t,\cdot\,)\in\mathrm{BV}^{n\times n}$ for $t\in[a,b]$;

- the abstract function $t \in [a,b] \mapsto K(t,\cdot\,) \in \mathrm{BV}^{n\times n}$ is regulated on $[a,b]$ and left-continuous on $(a,b]$.

Theorem 1.3. *$\mathcal{L}$ is a linear compact operator on $G^n_{\scriptscriptstyle L}$ if and only if there are a regulated function $A:[a,b]\to\mathbb{R}^{n\times n}$ and a function B from the class $\in \mathcal{K}^{n\times n}_{\scriptscriptstyle L}$ such that*

$$(\mathcal{L}x)(t) = A(t)\,x(a) + \int_a^b B(t,s)\,\mathrm{d}[x(s)] \ \ \textit{for}\ \ x \in G^n_{\scriptscriptstyle L}\ \ \textit{and}\ \ t\in[a,b].$$

2 Bray theorem

Remark 2.1. Let $\mathcal{K}^{n\times n}$ be the set of functions $K:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ such that: $K(t,\cdot\,)\in \mathrm{BV}^{n\times n}$ for $t\in[a,b]$ and the mapping $t\in[a,b]\mapsto K(t,\cdot\,)\in\mathrm{BV}^{n\times n}$ is regulated on $[a,b]$. If $K\in\mathcal{K}^{n\times n}$, then

- $K(\,\cdot\,,s)\in \mathrm{G}^{n\times n}$ for all $s\in[a,b]$ and
$$g(t) := \int_a^b \mathrm{d}_s K(t,s)\,x(s) \in \mathrm{G}^n \ \ \text{for all}\ \ x\in\mathrm{G}^n;$$
- $\operatorname{var}_a^b K(t,\cdot\,)\le \varkappa<\infty$ for all $t\in[a,b]$ and
$$h^*(s) := \int_a^b y^*(t)\,\mathrm{d}_s K(t,s) \in \mathrm{BV}^n \ \ \text{for all}\ \ y\in\mathrm{BV}^n;$$
- $K(\,\cdot\,,s)$ is left-continuous for all $s\in[a,b]$ and $g\in \mathrm{G}^n_{\mathrm{L}}$ for all $x\in\mathrm{G}^n$ whenever $K\in\mathcal{K}^{n\times n}_{\mathrm{L}}$.

A crucial tool for deriving the explicit form of the dual operator $\mathcal{L}^*$ to $\mathcal{L}$ is the next Fubini type theorem called usually the Bray theorem, cf. [5].

Theorem 2.1. *If $K\in\mathcal{K}^{n\times n}$, then*

$$\int_a^b y^*(t)\,\mathrm{d}_t\Big[\int_a^b K(t,s)\,\mathrm{d}[x(s)]\Big] = \int_a^b\Big(\int_a^b y^*(t)\,\mathrm{d}_t[K(t,s)]\Big)\,\mathrm{d}[x(s)]$$

holds for any $x\in G^n$ and any $y\in BV^n$.

3 Linear integral equations in $\mathrm{G}^n_{\mathrm{L}}$

If $\mathcal{L}:\mathrm{G}^n_{\mathrm{L}}\to\mathrm{G}^n_{\mathrm{L}}$ is linear compact operator and $f\in\mathrm{G}^n_{\mathrm{L}}$, then $x-\mathcal{L}\,x=f$ can be rewritten as

$$x(t) - A(t)\,x(a) - \int_a^b B(t,s)\,\mathrm{d}[x(s)] = f(t), \ \ t\in[a,b],$$

where $A\in\mathrm{G}^{n\times n}_{\mathrm{L}}$ and $B\in\mathcal{K}^{n\times n}_{\mathrm{L}}$. Obviously, Fredholm–Stieltjes integral equations, Volterra–Stieltjes integral equations, and generalized linear differential equations are special cases. Adjoint operator $\mathcal{L}^*$ maps $\mathrm{BV}^n\times\mathbb{R}^n$ into $\mathrm{BV}^n\times\mathbb{R}^n$. In view of Bray Theorem we have, cf. [5].

Theorem 3.1. $\mathcal{L}^* : (y,\gamma) \in BV^n \times \mathbb{R}^n \to (\mathcal{L}_1^*(y,\gamma), \mathcal{L}_2^*(y,\gamma)) \in BV^n \times \mathbb{R}^n$, *where*

$$\big(\mathcal{L}_1^*(y,\gamma)\big)(t) = B^*(a,t)\,\gamma + \int_a^b \mathrm{d}_s[B^*(s,t)]\,y(s) \ \ \textit{for} \ \ t \in [a,b],$$

$$\mathcal{L}_2^*(y,\gamma) = A^*(a)\,\gamma + \int_a^b \mathrm{d}[A^*(s)]\,y(s).$$

Analogously. cf. [4], we can treat the boundary value problem

$$\left.\begin{aligned} & x(t) - x(a) - \int_a^t \mathrm{d}[A]\,x = f(t) - f(a) \ \ \text{on } [a,b], \\ & M\,x(a) + \int_a^b K\,\mathrm{d}[x] = r, \end{aligned}\right\} \tag{BVP}$$

where $A \in \mathrm{BV}_{\mathrm{L}}^{n\times n}$, $f \in \mathrm{G}_{\mathrm{L}}^n$ and $r \in \mathbb{R}^n$, and corresponding operator $\mathcal{L}\colon \mathrm{G}_{\mathrm{L}}^n \to \mathrm{G}_{\mathrm{L}}^n \times \mathbb{R}^n$. The adjoint $\mathcal{L}^*$ of $\mathcal{L}$ maps $(\mathrm{BV}^n \times \mathbb{R}^n) \times \mathbb{R}^n$ into $\mathrm{BV}^n \times \mathbb{R}^n$. Next theorem has been proved in [4].

Theorem 3.2. *Let $B(a) = A(a)$, $B(b) = A(b)$ and $B(t) = A(t+)$ on (a,b). Then $(y,\gamma,\delta) \in \mathcal{N}(\mathcal{L}^*)$ if and only if*

$$\left.\begin{aligned} & y^*(t) - y^*(b) - \int_t^b y^*(s)\,\mathrm{d}[B(s)] = \delta^*\,(K(t) - K(b)) \ \ \textit{on } [a,b], \\ & y^*(a) + \delta^*\,(K(a) - M) = 0, \quad y^*(b) + \delta^*\,K(b) = 0, \end{aligned}\right\} \tag{BVP*}$$

Moreover, (BVP) *has a solution if and only if*

$$\int_a^b y^*\,\mathrm{d}[f] + \delta^*\,r = 0 \ \ \textit{for all solutions} \ \ (y,\delta) \ \ \textit{of} \ (\text{BVP}^*).$$

Remark 3.1. Let $t_0 \in [a,b]$, $A \in \mathrm{BV}^{n\times n}$, $\det[I + \Delta^+ A(t)] \neq 0$ for $t \in [a,t_0)$ and $\det[I - \Delta^- A(t)] \neq 0$ for $t \in (t_0,b]$. Then, there is a unique $X : [a,b] \to \mathbb{R}^{n\times n}$ such that

$$X(t) = I + \int_{t_0}^t \mathrm{d}[A]\,X \ \ \text{for} \ \ t \in [a,b].$$

This X is then called the *generalized exponential* and denoted $X(t) = \exp_{dA}(t,t_0)$.

4 Alternative approach based on the Lagrange identity

Besides the functional analytical tool, there is an alternative way to obtain the duality theory. This approach is based on the Lagrange identity. It is well known, cf. [6], that the classical Lagrange identity can be extended to generalized linear differential systems as follows: Let $A \in \mathrm{BV}_{\mathrm{L}}^{n\times n}$, $B(a) = A(a)$, $B(b) = A(b)$ and $B(t) = A(t+)$ on (a,b). Then

$$\int_a^b y^*(t)\,\mathrm{d}\Big[x(t) - \int_a^t \mathrm{d}[A]\,x\Big] + \int_a^b \mathrm{d}\Big[y^*(s) - \int_s^b y^*\,\mathrm{d}[B]\Big]\,x(s) = y^*(b)\,x(b) - y^*(a)\,x(a)$$

for all $x \in \mathrm{G}^n_{\mathrm{L}}$ and $y \in \mathrm{BV}^n$ right-continuous on $[a,b)$. The proof easily follows from the integration-by-parts theorem for Kurzweil–Stieltjes integrals. Notice that this theorem can be slightly modified as follows, cf. [1].

Theorem 4.1 (Integration by parts)**.** *Let $f,g \in G^n$ and let at least one of them has a bounded variation on $[a,b]$. Then*

$$\int_a^b f^*(t-)\,\mathrm{d}[g(t)] + \int_a^b \mathrm{d}[f^*(t)]\,g(t+) = f^*(b)\,g(b) - f^*(a)\,g(a),$$

where $f(a-) = f(a)$ and $g(b+) = g(b)$.

As a result, we can reformulate the Lagrange formula under less restrictive continuity requirements. To this aim consider operators

$$(L\,x)(t) := x(t) - x(t_0) - \int_{t_0}^t \mathrm{d}[A(s)]\,x(s-) \ \text{ and } \ (L^*\,y)(t) := y^*(t) - y^*(t_0) + \int_{t_0}^t y^*(s+)\,\mathrm{d}[A(s)]$$

under the conventions

$$x(s-) = x(s) \ \text{ if } \ s = \min\{t,t_0\} \ \text{ and } \ y(s+) = y(s) \ \text{ if } \ s = \max\{t,t_0\}$$

in the integrals. More exactly:

$$(L\,x)(t) := \begin{cases} x(t) - x(t_0) + \displaystyle\int_t^{t_0} \mathrm{d}[A(s)]\,\big(x(t)\,\chi_{[t]}(s) + x(s-)\chi_{(t,t_0]}(s)\big) & \text{if } t \le t_0, \\ x(t) - x(t_0) - \displaystyle\int_{t_0}^{t} \mathrm{d}[A(s)]\,\big(x(t_0)\,\chi_{[t_0]}(s) + x(s-)\chi_{(t_0,t]}(s)\big) & \text{if } t \ge t_0, \end{cases}$$

$$(L^*\,y)(t) := \begin{cases} y^*(t) - y^*(t_0) + \displaystyle\int_{t_0}^{t} \big(y^*(s+)\,\chi_{[t,t_0)}(s) + y^*(t_0)\,\chi_{[t_0]}(s)\big)\,\mathrm{d}[A(s)] & \text{if } t \le t_0, \\ y^*(t) - y^*(t_0) - \displaystyle\int_{t}^{t_0} \big(y^*(s+)\,\chi_{[t_0,t)}(s) + y^*(t)\,\chi_{[t](s)}\big)\,\mathrm{d}[A(s)] & \text{if } t \ge t_0. \end{cases}$$

The related equations $L\,x = 0$ and $L^*\,y = 0$ are, of course, no longer generalized ODEs, but special cases of Stieltjes integral equations. The modified version of the Lagrange identity, cf. [1], then reads as follows:

Theorem 4.2 (Lagrange Identity)**.** *Let $A \in BV^{n\times n}$, $x,y \in G^n$, $x(t-) = x(t)$ if $t = \min\{t_0,T\}$ and $y(t+) = y(t)$ if $t = \max\{t_0,T\}$. Then for each $t_0 \in [a,b]$ and $T \in [a,b]$ we have*

$$\int_{t_0}^T y^*(t+)\,\mathrm{d}[(L\,x)(t)] + \int_{t_0}^T \mathrm{d}[(L^*\,y)(t)]\,x(t-) = y^*(T)\,x(T) - y^*(t_0)\,x(t_0).$$

Remark 4.1. The above result no longer holds if we abandon the convention concerning the endpoints.

Corollary. *If $L\,x = 0$ and $L^*\,y = 0$ on $[a,b]$, then $y^*\,x$ is constant on $[a,b]$.*

In other words, the equations

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A(s)]\,x(s-) \quad \text{on } [a,b], \tag{E}$$

$$y^*(t) = y^*(t_0) - \int_{t_0}^{t} y^*(s+)\,\mathrm{d}[A(s)] \;\text{ on } [a,b] \tag{E^*}$$

can be considered to be mutually dual.

Remark 4.2. If we restrict to $t_0 = a$, everything becomes considerably simpler. In particular, in such a case we get

$$\begin{aligned} L\,x = 0 \;\text{ on } [a,b] &\implies x(t-) = \big[I + \Delta^- A(t)\big]^{-1} x(t) \;\text{ if } t \in (a,b] \text{ and } \det[I + \Delta^- A(t)] \neq 0,\\ L^*\,y = 0 \;\text{ on } [a,b] &\implies y^*(t+) = y^*(t)\,\big[I + \Delta A(t)\big]^{-1}(I + \Delta^- A(t))\\ &\qquad\qquad \text{if } t \in [a,b) \text{ and } \det\big[I + \Delta A(t)\big] \neq 0. \end{aligned}$$

Therefore, if $\det[I + \Delta^- A(t)] \neq 0$ and $\det[I + \Delta A(t)] \neq 0$, then $L\,x = 0$ if and only if

$$x(t) = x(a) + \int_a^t \mathrm{d}[K]\,x \;\text{ on } [a,b], \;\text{ where } K(s) = \int_a^s \mathrm{d}[A(\tau)]\,\big[I + \Delta^- A(\tau)\big]^{-1}.$$

Analogously, $L^*\,y = 0$ if and only if

$$y^*(t) = y^*(a) - \int_a^t y^*\,\mathrm{d}[L] \;\text{ on } [a,b], \;\text{ where } L(s) = \int_a^s [I + \Delta A(\tau)]^{-1}\big[I + \Delta^- A(\tau)\big]\,\mathrm{d}[A(\tau)].$$

Concluding comments

The present contribution is closely related to the recent paper [1]. Some of its results have been here extended from the scalar case to the n-dimensional case and functional analytical background has been recalled. On the other hand, in [1] Stieltjes differential equations and dynamical equations on time scale were considered. For more details, see [1]. To a large extent, the properties of the Kurzweil-Stieltjes integral are utilized. For more details, see the monograph [2].

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Exact Baire Class of the Local Entropy Considered as a Function of a Point in the Phase Space

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Following [1], we give the definition of the local entropy that will be necessary in what follows. Let X be a compact metric space with a metric d and $f : X \to X$ a continuous map. Along with the original metric d, we define an additional system of metrics on X:

$$d_n^f(x, y) = \max_{0 \le i \le n-1} d(f^i(x), f^i(y)), \quad x, y \in X, \quad n \in \mathbb{N},$$

where f^i, $i \in \mathbb{N}$, is the i-th iteration of f, $f^0 \equiv \mathrm{id}_X$. Given a point $x \in X$, for any $n \in \mathbb{N}$, $r > 0$ and $\rho > 0$, denote by $N_d(f, r, n, x, \rho)$ the maximum number of points in the ball $B_d(x, \rho) = \{y \in X : \ d(x, y) < \rho\}$, pairwise d_n^f-distances between which are greater than r. Then *the local entropy* of the mapping f at the point x is defined by the formula

$$h_d(f, x) = \lim_{r \to 0} \lim_{\rho \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \ln N_d(f, r, n, x, \rho).$$

Recall one more formula for calculating the local entropy. For any r, $\rho > 0$ and $n \in \mathbb{N}$ a set $A \subset B_d(x, \rho)$ is called an (f, r, n, x, ρ)-cover of the ball $B_d(x, \rho)$, if for any point $y \in B_d(x, \rho)$ there is a point $z \in A$ such that $d_n^f(z, y) < r$. Let $S_d(f, r, n, x, \rho)$ denote the minimum number of elements in an (f, r, n, x, ρ)-cover, then the local entropy can be calculated by the formula

$$h_d(f, x) = \lim_{r \to 0} \lim_{\rho \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \ln S_d(f, r, n, x, \rho). \tag{1}$$

For a fixed continuous mapping $f : X \to X$, consider the function

$$x \mapsto h_d(f, x). \tag{2}$$

As the following example shows, function (2) can be discontinuous on the space X. Let $X = [-1, 1]$ and define a mapping $f : X \to X$ by

$$f(x) = \begin{cases} 0, & \text{if } \ x \in [-1, 0), \\ 4x(1 - x), & \text{if } \ x \in [0, 1]. \end{cases}$$

Then $h_d(f, x) = 0$ for $x \in [-1, 0)$ and $h_d(f, 0) = \ln 2$, hence function (2) has a discontinuity at zero.

Recall that continuous functions on a metric space $\mathcal{M}$ are called functions of the zeroth Baire class, and for every natural number p, functions of the p-th Baire class are those that are pointwise limits of sequences of functions in the $(p - 1)$-th class.

There are many, not equivalent to each other, interpretations as to which properties are typical and which are not. Here we recall the notion of typicality introduced and studied by R.-L. Baire. A property of a point in a topological space is called Baire typical if the set of points possessing this property contains an everywhere dense G_δ-set.

Theorem 1 ([2])**.** *For any continuous mapping $f : X \to X$, function (2) belongs to the second Baire class and is lower semicontinuous at a Baire typical point of the space X.*

Proof. Let us transform formula (1) to the form

$$h_d(f,x)=\lim_{m\to\infty}\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\,\frac{1}{n}\,\ln S_d\Big(f,\frac{1}{m},n,x,\frac{1}{k}\Big), \tag{3}$$

and for a fixed natural number m consider the function

$$x\mapsto \varphi_m(x)=\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\,\frac{1}{n}\,\ln S_d\Big(f,\frac{1}{m},n,x,\frac{1}{k}\Big).$$

For any $k>0$ and any point $y\in B_d(x,1/k)$, there exists $l_k>0$ such that for all $l\geqslant l_k$ the inclusion

$$B_d\Big(y,\frac{1}{l}\Big)\subset B_d\Big(x,\frac{1}{k}\Big)$$

holds, which implies the inequality

$$S_d\Big(f,\frac{1}{m},n,y,\frac{1}{l}\Big)\leqslant S_d\Big(f,\frac{1}{m},n,x,\frac{1}{k}\Big),\ \ m,n\in\mathbb{N}.$$

Consequently,

$$\lim_{l\to\infty}\overline{\lim_{n\to\infty}}\,\frac{1}{n}\,\ln S_d\Big(f,\frac{1}{m},n,y,\frac{1}{l}\Big)\leqslant \overline{\lim_{n\to\infty}}\,\frac{1}{n}\,\ln S_d\Big(f,\frac{1}{m},n,x,\frac{1}{k}\Big).$$

Since the point $y\in B_d(x,1/k)$ is arbitrary, we obtain the inequality

$$\sup_{y\in B_d(x,1/k)}\lim_{l\to\infty}\overline{\lim_{n\to\infty}}\,\frac{1}{n}\,\ln S_d\Big(f,\frac{1}{m},n,y,\frac{1}{l}\Big)\leqslant \overline{\lim_{n\to\infty}}\,\frac{1}{n}\ln S_d\Big(f,\frac{1}{m},n,x,\frac{1}{l}\Big).$$

Passing in the last inequality to the limit as $k\to+\infty$, we obtain the inequality

$$\overline{\lim_{y\to x}}\,\varphi_m(x)\leqslant\varphi_m(x),$$

which establishes upper semicontinuity of the function $x\mapsto\varphi_m(x)$ at the point x. Hence the function $x\mapsto\varphi_m(x)$ belongs to the first Baire class on the space X. Thus, from (3) we get the following representation of the local entropy of the continuous mapping f at the point x:

$$h_d(f,x)=\lim_{m\to\infty}\varphi_m(x),\quad \varphi_1(x)\leqslant\varphi_2(x)\leqslant\varphi_3(x)\leqslant\cdots,$$

which implies that the function $x\mapsto h_d(f,x)$ belongs to the second Baire class on the space X.

By the Baire theorem on functions of the first class, for each $m\in\mathbb{N}$, the set of points of continuity G_m for the function $x\mapsto\varphi_m(x)$ is an everywhere dense G_δ-set. The intersection of all G_m is again an everywhere dense set, each point of which is a point of continuity for all functions $x\mapsto\varphi_m(x)$, $m\in\mathbb{N}$. Let $x\in\bigcap\limits_{m\in\mathbb{N}}G_m$ and $\varepsilon>0$. By definition of the limit, $\varphi_m(x)\geqslant h_d(f,x)-\varepsilon$ for all sufficiently large m. Fixing such m, find a neighborhood $B_d(x,\delta)$ of the point x such that for every $y\in B_d(x,\delta)$ we have $\varphi_m(y)\geqslant\varphi_m(x)-\varepsilon$. Since the sequence (φ_m) is nondecreasing, it follows that $h_d(f,y)\geqslant\varphi_m(y)$ for all $y\in B_d(x,\delta)$, hence $\varphi_m(y)\geqslant h_d(f,x)-2\varepsilon$. Therefore, at each point of the set $\bigcap\limits_{m\in\mathbb{N}}G_m$ the function $x\mapsto h_d(f,x)$ is lower semicontinuous. $\square$

On the set of sequences $x=(x_1,x_2,\dots)$, $x_k\in\{0,1\}$, introduce a metric

$$d_{\Omega_2}(x,y)=\begin{cases}0, & \text{if } x=y,\\ \dfrac{1}{\min\{i:\ x_i\neq y_i\}}, & \text{if } x\neq y.\end{cases}$$

The resulting compact metric space will be denoted by Ω_2. Note that the space Ω_2 is homeomorphic to the Cantor set on the segment $[0,1]$ with the metric induced by the natural metric of the real line.

Theorem 2 ([2])**.** *If $X=\Omega_2\times\Omega_2$ with the metric*

$$d\big((x,\alpha),(y,\beta)\big)=\max\big\{d_{\Omega_2}(x,y),d_{\Omega_2}(\alpha,\beta)\big\},$$

then there is a continuous mapping $f:X\to X$ such that function (2) *is everywhere discontinuous and does not belong to the first Baire class on the space X.*

Proof. Define a mapping $f:\Omega_2\times\Omega_2\to\Omega_2\times\Omega_2$ as follows:

$$f\big((x_1,x_2,x_3,\dots),(\alpha_1,\alpha_2,\alpha_3,\dots)\big)=\Big((x_{1+\alpha_1},x_{2+\alpha_2},x_{3+\alpha_3},\dots),(\alpha_1,\alpha_2,\alpha_3,\dots)\Big).$$

Denote by $\mathcal{P}_0$ the set of sequences from Ω_2 for which all but a finite number of terms are equal to zero, and by $\mathcal{P}_1$ the set of sequences from Ω_2 for which all but a finite number of terms are equal to one.

Lemma 1. *For any point $(x,\alpha)\in\Omega_2\times P_0$, the equality $h_d(f,(x,\alpha))=0$ is valid.*

Proof. If $(\alpha_1,\alpha_2,\alpha_3,\dots)\in P_0$, then there is a natural number p_0 such that $\alpha_p=0$ for all $p\geqslant p_0$. Therefore, for any $m\geqslant p_0$ and $(y,\beta)\in B_d((x,\alpha),\frac{1}{m+1})$,

$$\begin{aligned}&f(y,\beta)\\&\quad=\Big(\Big(x_{1+\alpha_1},\dots,x_{p_0+\alpha_{p_0}},x_{p_0},\dots,x_m,y_{m+1+\beta_{m+1}},y_{m+2+\beta_{m+2}},\dots\Big),(\alpha_1,\dots,\alpha_m,\beta_{m+1},\dots)\Big),\end{aligned}$$

therefore d_n^f-distance between any two points of the ball $B_d((x,\alpha),\frac{1}{m+1})$ does not exceed $\frac{1}{m+1}$. Thus, for any $k>m$ we have

$$N_d\Big(f,\frac{1}{m},n,(x,\alpha),\frac{1}{k}\Big)=1,$$

and hence

$$h_d(f,(x,\alpha))=0.\qquad\square$$

Lemma 2. *For any point $(x,\alpha)\in\Omega_2\times P_1$, the inequality $h_d(f,(x,\alpha))\geqslant\ln 2$ is valid.*

Proof. If $(\alpha_1,\alpha_2,\alpha_3,\dots)\in P_1$, then there is a natural number p_0 such that $\alpha_p=1$ for all $p\geqslant p_0$ and hence for any point $(x,\alpha)\in\Omega_2\times P_1$ we have the equality

$$f(x,\alpha)=\big((x_{1+\alpha_1},\dots,x_{p_0-1+\alpha_{p_0-1}},x_{p_0+1},x_{p_0+2},\dots),\alpha\big).$$

In the ball $B_d((x,\alpha),\frac{1}{p})$ for each natural number $n\geqslant p+2$, consider the set $A_{n,p}$ of points of the form

$$\big((x_1,\dots,x_p,y_{p+1},\dots,y_n,0,0,\dots),\alpha\big),\quad\text{where}\quad y_i\in\{0,1\},\quad i=p+1,\dots,n.$$

Since the d_n^f-distance between any two points from $A_{n,p}$ is not less than $\frac{1}{p+1}$, then the quantity $N_d(f, \frac{1}{p}, (x, \alpha), \frac{1}{p})$ is at least the cardinality of the set $A_{n,p}$. Thus we have

$$h_d(f,(x,\alpha)) = \lim_{r\to 0}\lim_{p\to\infty}\overline{\lim_{n\to\infty}}\,\frac{1}{n}\ln N_d\Big(f,r,n,(x,\alpha),\frac{1}{p}\Big)$$
$$\geqslant \lim_{p\to\infty}\overline{\lim_{n\to\infty}}\,\frac{1}{n}\ln N_d\Big(f,\frac{1}{p_0},n,(x,\alpha),\frac{1}{p}\Big) \geqslant \lim_{n\to\infty}\frac{(n-p)\ln 2}{n} = \ln 2. \quad \square$$

Completion of the proof of Theorem 2. Suppose that the function $(x, \alpha) \mapsto h_d(f,(x,\alpha))$ belongs to the first Baire class on the space $\Omega_2 \times \Omega_2$, then, by the Baire theorem on functions of the first class, in the space $\Omega_2 \times \Omega_2$ there must be points of continuity of the function $(x, \alpha) \mapsto h_d(f,(x,\alpha)$. On the other hand, the sets $\Omega_2 \times P_0$ and $\Omega_2 \times P_1$ are everywhere dense in the space $\Omega_2 \times \Omega_2$. Therefore, by virtue of Lemmas 1 and 2, each point of the space $\Omega_2 \times \Omega_2$ is a discontinuity point of the function $(x, \alpha) \mapsto h_d(f,(x,\alpha))$. Thus, the function $(x, \alpha) \mapsto h_d(f,(x,\alpha))$ is everywhere discontinuous and does not belong to the first Baire class on the space $\Omega_2 \times \Omega_2$. $\square$

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Quaternionic Exponentially Dichotomous Operators

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1 Introduction

The notion of quaternions that is a noncommutative extension of complex numbers is a mathematical concept introduced by Irish mathematician Hamilton in 1843 and it has been widely applied to both pure and applied mathematics and physics. For instance, Adler studied the quaternionic quantum mechanics and quantum fields in 1995 (see [1]). Since the quaternionic algebra has a significant feature that its multiplication does not follow commutative law and it refers to applied dynamic equations (see [5–8]) and many mathematical and physical research fields, many momentous studies based on quaternionic theory have been hot topics.

In 1998, Colombo and Sabadini studied the quaternionic functional calculus of Fueter-regular function based on Cauchy formula (see [4]). In [2], by using the theory of S-spectrum, Cerejeiras et al. studied the slice hyper-holomorphism of S-resolvent operator for the perturbation problem of quaternionic normal operator in Hilbert space and the conditions to ensure the existence of nontrivial hyper-invariant subspace of quaternionic linear operator were given. In the book [3], Colombo et al. systematically presented the discovery of the S-spectrum and of the S-functional calculus in the introduction and how hypercomplex analysis methods were used to identify the appropriate notion of quaternionic spectrum whose existence was suggested by quaternionic quantum mechanics. In 2022, based on S-spectral theory, Wang, Qin and Agarwal introduced the notion of the quaternionic exponentially dichotomous operator and obtained its integral representation formula.

2 Quaternionic exponentially dichotomous operators

In this section, we will present a notion of the quaternionic exponentially dichotomous operators of the quaternionic version and some fundamental results which are important to discuss the quaternionic evolution equations. For more details, one may consult [9].

2.1 Quaternionic bisemigroups and direct sum decomposition of quaternionic Banach space

Definition 2.1 ([9]). Let X be a quaternionic Banach space, by a (strongly continuous) bisemigroup we mean a function $E(\cdot) : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X)$ having the following properties:

(1) If $t,s>0$, we have $E(t)E(s)=E(t+s)$ and for $t,s<0$ we have $E(t)E(s)=-E(t+s)$.

(2) For every $x\in X$ the function $E(\,\cdot\,)x:\mathbb{R}\setminus\{0\}\to X$ is continuous, apart from a jump discontinuity in $t=0$. That is,

$$\lim_{t\to 0^{\pm}}\|E(t)x-E(0^{\pm})x\|_X=0,\ \ x\in X.$$

(3) $E(0^+)x-E(0^-)x=x$ for every $x\in X$.

(4) There exist $M,\lambda>0$ such that $\|E(t)\|_{\mathcal{B}(X)}\leqslant Me^{-\lambda|t|}$ for $t\in\mathbb{R}\setminus\{0\}$.

From Definition 2.1, any quaternionic strongly continuous semigroup $\{E(t)\}_{t\geqslant 0}$ having a negative exponential growth bound extends to a uniformly continuous bisemigroup when defining $E(t)=0_{\mathcal{B}(X)}$ for $t<0$. Notice properties (1) and (3) in Definition 2.1, we can obtain the following proposition.

Proposition 2.1 ([9])**.** *Let $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ be a strongly continuous bisemigroup and $P=-E(0^-)$, then the following holds*

$$\begin{cases}E(t)\big[\operatorname{Ker}P\big]\subset\operatorname{Ker}P, & t>0,\\ E(t)\big[\operatorname{Im}P\big]\subset\operatorname{Im}P, & t<0.\end{cases}$$

Proposition 2.1 implies that $E(0^+)$ and $-E(0^-)$ are bounded complementary, we may introduce the concept of the constituent semigroup of a bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ as follows.

Definition 2.2 ([9])**.** Let $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ be a strongly continuous bisemigroup, then we call the operator $P=-E(0^-)$ the separating projection of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$. The restriction of $E(t)$ to $\operatorname{Ker}P$ is a quaternionic strongly continuous semigroup on $\operatorname{Ker}P$, while the restriction of $-E(-t)$ to $\operatorname{Im}P$ is a strongly continuous semigroup on $\operatorname{Im}P$. These two semigroups are called the constituent semigroups of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$.

Definition 2.2 indicates that we can describe the exponential growth bounds of $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ through the exponential growth bounds of its corresponding constituent semigroups, hence we introduce the following notion.

Definition 2.3 ([9])**.** Let $E_j:[0,\infty)\to X_j (j=1,2)$ be the quaternionic strongly continuous semigroups, and both have a negative exponential growth bound, we define the strongly continuous bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ on $X=X_1\oplus X_2$ by

$$E(t)=\begin{cases}E_1(t)\oplus 0_{X_2}, & t>0,\\ 0_{X_1}\oplus\big(-E_2(-t)\big), & t<0,\end{cases}$$

which has $\{E_1(t)\}_{t\geqslant 0}$ and $\{E_2(t)\}_{t\geqslant 0}$ as its constituent semigroups. For the pair of exponential growth bounds of a bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$, we denote the pair of (necessarily negative) exponential growth bounds of its constituent semigroups by:

$$\big\{\lambda_+(E),\lambda_-(E)\big\}.$$

For the exponential growth bound $\lambda(E)$ of a bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$, we denote it by

$$\lambda(E)\overset{\text{def}}{=}\max\big\{\lambda_-(E),\lambda_+(E)\big\}<0.$$

Definition 2.4 ([9])**.** Let $T_+(\operatorname{Ker} P \to \operatorname{Ker} P)$ and $-T_-(\operatorname{Im} P \to \operatorname{Im} P)$ stand for the infinitesimal generators of the constituent semigroups of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ on X, then the linear quaternionic operator $T(X \to X)$ defined by

$$\mathcal{D}(T) = \big\{x_+ \oplus x_- : \ \ x_+ \in \mathcal{D}(T_+),\ x_- \in \mathcal{D}(T_-)\big\},$$
$$T(x_+ \oplus x_-) = T_+(x_+) - T_-(x_-)$$

is called the (infinitesimal) generator of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$, since $T(X \to X)$ is closed and densely defined, then we define the constituent Laplace transform formulas as follows:

$$S_R^{-1}(s, T_+)x_+ = \int_0^\infty e^{-st}E(t)x_+\,dt, \ \ x_+ \in \operatorname{Ker} P, \ \ \operatorname{Re}(s) > \lambda_+(E),$$
$$S_R^{-1}(-s, -T_-)x_- = -\int_0^\infty e^{st}E(-t)x_-\,dt, \ \ x_- \in \operatorname{Im} P, \ \ \operatorname{Re}(-s) > \lambda_-(E),$$

where both of $\lambda_\pm(E) < 0$, which imply the Laplace transform formula

$$S_R^{-1}(s, T)x = \int_{-\infty}^\infty e^{-st}E(t)x\,dt, \ \ \lambda_+(E) < \operatorname{Re}(s) < -\lambda_-(E), \tag{1}$$

where the (Bochner) integral converges absolutely in the norm of X. Now we will write $E(t, T)$ for the strongly continuous bisemigroup with infinitesimal generator T.

Remark 2.1. From Definition 2.4, there exists a quaternionic district in the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ about the 2-dimensional sphere $\mathbb{S}$ contained in the S-resolvent set of the infinitesimal generator T of $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$.

2.2 Quaternionic exponentially dichotomous operators and integral representation

We will present the concept of a quaternionic exponentially dichotomous operator.

Definition 2.5 ([9])**.** A closed and densely defined linear quaternionic operator $T(X \to X)$ on a quaternionic Banach space X is called exponentially dichotomous if it is the infinitesimal generator of a strongly continuous bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ on X.

Proposition 2.2 ([9])**.** *Let X be an quaternionic Banach space, $T(X \to X)$ be an exponentially dichotomous quaternionic operator. Then T has precisely one separating projection P of the bisemigroup $E(t, T)$.*

Definition 2.6 ([3])**.** Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \cap \mathbb{R} \neq \varnothing$ and suppose that $f \in \mathcal{R}^L_{\overline{\sigma}_S(T)}$ (resp. $f \in \mathcal{R}^R_{\overline{\sigma}_S(T)}$). Let us consider $k \in \mathbb{R}$ and the function $\Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ defined by $p = \Phi(s) = (s - k)^{-1}$, $\Phi(\infty) = 0$, $\Phi(k) = \infty$. Now consider

$$\phi(p) := f(\Phi^{-1}(p))$$

and the bounded linear operator defined by

$$A := (T - k\mathcal{I})^{-1} \ \text{ for some } \ k \in \rho_S(T) \cap \mathbb{R}.$$

We define, in both cases, the operator $f(T)$ as

$$f(T) = \phi(A). \tag{2}$$

Now we introduce the following slice symmetric domain of the quaternionic Banach space.

Definition 2.7 ([9])**.** We define a slice symmetric domain $D_{\eta-I\xi}^{\eta+I\xi}$ as follows:

$$D_{\eta-I\xi}^{\eta+I\xi} := \Big\{ s \in \mathbb{C}_I : \ \ \operatorname{Re}(s) = \eta, \ \ I \in \mathbb{S}, \ \ |\operatorname{Im}(s)| \leqslant \xi \Big\}.$$

Moreover, if $\xi = \infty$, we denote the ∞-symmetric domain by $D_{\eta-I\infty}^{\eta+I\infty}$.

To present an integral representation for the separating projection P, we established the following lemma.

Lemma 2.1 ([9])**.** *Let $\Psi(X \to X)$ be a closed linear quaternionic operator on the two-side quaternionic Banach space X such that $\rho_S(\Psi) \cap \mathbb{R} \neq \varnothing$ and assume that $\sigma_S(\Psi) \subset \{s \in \mathbb{H} : \ \operatorname{Re}(s) > \eta\}$, where η is some positive real number. Let $f \in \mathcal{R}^R_{\sigma_S(\Psi)}$ and $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ for any $I \in \mathbb{S}$, Φ and ϕ are the same as in Definition 2.6. Then*

$$f(\Psi)x = \frac{1}{2\pi} \oint\limits_{\gamma_\eta} \phi(p)\, dp_I S_R^{-1}(p, A)x, \ \ x \in \mathcal{D}(\Psi^2),$$

where $A = (\Psi - k\mathcal{I})^{-1}$ with $k \in \rho_S(\Psi) \cap \mathbb{R}$ such that $|k| < \eta$, $p = \Phi(s)$, $\gamma_\eta = \{s \in \mathbb{C}_I : |s - (\eta - k)^{-1}/2| = (\eta - k)^{-1}/2\}$ and whenever $x \in \mathcal{D}(\Psi^2)$ and $S_R^{-1}(s, \Psi)$ is bounded on $\operatorname{Re}(s) \leqslant \eta$.

Theorem 2.1 ([9])**.** *Let $T(X \to X)$ be a quaternionic exponentially dichotomous operator such that $\rho_S(T) \cap \mathbb{R} \neq \varnothing$ and suppose the exponential growth bound $\lambda(E) < 0$, and let P be its separating projection, Φ, p and A are as in Definition 2.6. Then*

$$Px = \frac{1}{2\pi} \oint\limits_{\gamma_\eta} dp_I S_R^{-1}(p, A)x, \ \ x \in \mathcal{D}(T^2), \tag{3}$$

where $dp_I = dp/I$, $|k| < \eta < -\lambda(E)$ and γ_η, k, η are as in Lemma 2.1.

Remark 2.2. Let $E(t, T)$ be a bisemigroup, for any $k \in \rho_S(T) \cap \mathbb{R}$ and $|\eta| < \lambda(E)$, noticing that $\Phi\big(D_{\eta-I\infty}^{\eta+I\infty}\big)$ is a circle with the center $p_0 = \Phi(\eta)/2$ and radius $(\eta - k)^{-1}/2$, we denote it by γ_η.

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